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FINITE DIFFERENCE METHOD AND THE LAME'S EQUATION IN HEREDITARY SOLID MECHANICS .

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** Abstract : The Lame's differential equation is solved by the finite-difference method .
** Subjects: Viscoelasticity , Hereditary Solid Mechanics , The Differential equation .

NOTE: This worksheet demonstrates the use of Maple for calculating the solution of Lame's differential equation .

The authors expect that this worksheet will only be used for teaching and educational purposes ..

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Co.H Tran - Phong .T .Ngo - **FINITE DIFFERENCE METHOD AND THE LAME'S EQUATION IN HEREDITARY SOLID MECHANICS .** ***Use Maple 9.5***

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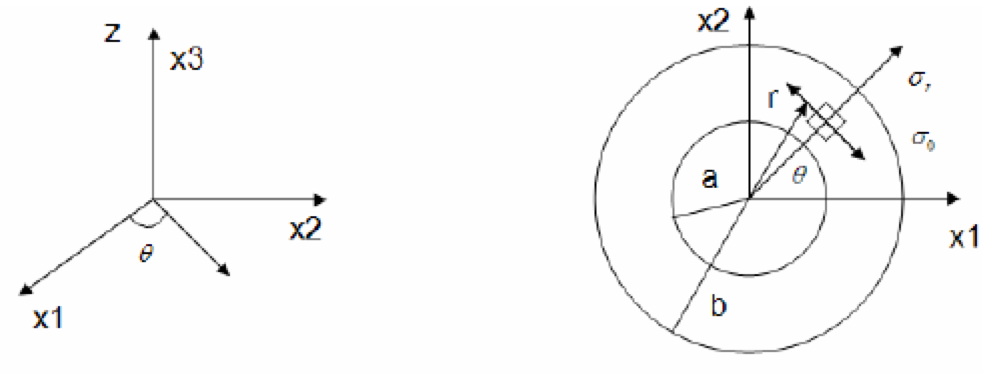
A. THE DISPLACEMENT DIFFERENTIAL EQUATION :

The Lame's equation of the plane-deformation problem in the cylinder made of orthotropic viscoelastic composite material does not have constant modules .
The modules Et , Er and Ert will be replaced with the functions Er(t) , Et(t) and Ert(t) respectively .



*a**. ***The plane-deformation problem*** *:* Bai toan bien dang phang cua ong tru truc huong composite dan nhot :
We examine an orthotropic viscoelastic composite material cylinder which has the horizontal section within limit of 2 circles : r = a , r = b (a < b) . Choosing the cylindrical coordinates r , θ , z (the axial z is along with the cylinder) . The components of stress and deformation are functions of r , t respectively .

Xet ong tru có tiet dien ngang gioi han boi 2 duong tron dong tam co ban kinh r = a , r = b (a < b) , ong tru duoc làm bang vat lieu có tính truc huong . Chon he toa do tru r , θ , z (truc z huong doc theo ong tru . Các thanh phan bien dang và ung suat tuong ung la ε_r , ε_θ , σ_r , σ_θ la cac ham theo r , t .

The two components of deformation-tensor : (2 thanh phan cua tensor bien dang la :)

$$\varepsilon_r\ (r,t) = \frac{\partial u(r,t)}{\partial r} \quad ; \quad \varepsilon_\theta(r,t) = \frac{u(r,t)}{r}$$

and the differential equation of equilibrium is : (phuong trinh vi phan can bang)

$$r\frac{\partial \sigma_r(r,t)}{\partial r} + \sigma_r(r,t) - \sigma_\theta(r,t) = 0$$

when t = 0 , boundary conditions : (khi t = 0 , cac dieu kien bien :) $\sigma_r(a,0) = -P \;\; ; \;\; \sigma_r(b,0) = -Q$

b* . The displacement - differential equation : Phuong trinh vi phan chuyen vi :

The differential equation of the cylinder displacement in the case of viscoelastic plane-deformation : (Phuong trinh chuyen vi ong trong truong hop bien dang phang dan nhot)

$$r^2\frac{\partial^2 u(r,t)}{\partial r^2} + r\frac{\partial u(r,t)}{\partial r} - \frac{\hat{E}_\theta}{\hat{E}_r}u(r,t) = 0$$

Here $\frac{\hat{E}_\theta}{\hat{E}_r}$ is $\frac{\left(\frac{t}{To}\right)^{\left(\frac{2}{5}\right)}}{100. + \left(\frac{t}{To}\right)^{\left(\frac{1}{2}\right)}}$

(To : const)

B. FINITE DIFFERENCE METHOD :

The boundary conditions of the problem are given at two edges (Dieu kien bien cua bai toan duoc cho o 2 canh) : r = a and r = b . (a = 1 , b = 2) ?

Now we choose the number of mesh points (Ta chon so diem luoi) N = 20 . The interval over which we approximate this equation is (Doan xap xi cua phuong trinh la) [a, b] . And the step size for this interval is (Va kich thuoc buoc nhay cho doan nay la) $h := \frac{1}{20}$

The difference operators are (Cac toan tu sai phan la) Uj and Ujj , And we have two boundary conditions equations (Va ta co 2 phuong trinh dieu kien bien) : $e_0 := u_{0,t} = 1$; $e_{20} := u_{20,t} = -1$. For determining the values at the interior mesh points we obtain the N-1 equations (De xac dinh cac gia tri cho cac diem trong , ta thu duoc N - 1 phuong trinh) , then by replacing u '(x) and u "(x) (Va thay the u'(x) va u"(x)) :

$$U1 := (k, t) \rightarrow \frac{1\,(u_{k+1,t} - u_{k-1,t})}{2\,h} \qquad ; \qquad U2 := (k, t) \rightarrow \frac{u_{k+1,t} - 2\,u_{k,t} + u_{k-1,t}}{h^2}$$

We arrange this system of N+1 equations in the form of matrix equation (Sap xep he thong gom N+1 phuong trinh nay) . The matrix of it has N+1 rows(Ma tran chinh co N+1 hang). The first row is fixed with the boundary condition at r = a (Hang dau duoc xep cho dieu kien bien tai r = a) . Obviously the last row is fixed with the boundary condition at r = b (Hien nhien hang cuoi cung duoc xep cho dieu kien bien tai r = b) . Now, we join these rows by listing them out , then construct the matrix symboled A . (Lien ket cac hang nay lai , va xay dung nen ma tran A) .

The unknown values will be written as a vector (cac gia tri chua biet se duoc viet dang vector) $u_j, j = 1 .. N$ and the right hand side of the equations is a column vector B (va ve phai phuong trinh la 1 vector cot B) . Solving the matrix equation for u (Giai phuong trinh ma tran tim nghiem u) . Then we find $\varepsilon_\theta(r,t) := \frac{u(r,t)}{r}$ with $E_\theta(r,t) := \left(\frac{100}{\left(\frac{t}{To}\right)^{.1}} + 1\right) Ee$ and the expression of $\sigma_\theta(r,t) := \frac{u(r,t)\,E_\theta(r,t)}{r}$;

C. NUMERICAL SOLUTION :

Use Maple 9.5

```
> restart:with(plots):with(PDETools):with(LinearAlgebra):
 m:=(100.+(t/To)^(1/10))*(t/To)^(2/5)/(100.+(t/To)^(1/2)); To:=1;
lame_cyl:=r^2*diff(u(r,t),r$2)+r*diff(u(r,t),r)+m*u(r,t)=0; bound_con:=u(1,t)=1,u(2,t)=-1; a:=1; b:=
N:=20; h:=(b-a)/N;R:=k->a+k*h; U1:=(k,t)->(u[k+1,t]-u[k-1,t])/(2*h); U2:=(k,t)->(u[k+1,t]-2*u[k,t]+
e[0]:=u[0,t]=rhs(bound_con[1]);e[N]:=u[N,t]= rhs(bound_con[2]);
for k from 1 to N-1 do e[k]:=eval(lame_cyl,{r=R(k),u(r,t)=u[k,t], diff(u(r,t),r)=U1(k,t),diff(u(r,t),r$2)
row[0]:=[rhs(bound_con[1]),seq(0,j=1..N-1)];
row[1]:=[coeff(lhs(e[1]),u[0,t]),coeff(lhs(e[1]),u[1,t]),coeff(lhs(e[1]),u[2,t]),seq(0,j=1..N-3)];
 for n from 2 to N-1 do row[n]:=[seq(0,j=1..n-2),coeff(lhs(e[n]),u[n-1,t]),coeff(lhs(e[n]),u[n,t]),coeff(lh
row[N]:=[seq(0,j=1..N-1),rhs(bound_con[2])];
row_matrix:=[row[0],seq(row[n],n=1..N-2),row[N]];A:=(row_matrix);
 U:=Vector([seq(u[j,t],j=1..N)]);B:=Vector([rhs(bound_con[1]),seq(rhs(e[j]),j=1..N-2),rhs(bound_con
print("Ham epsilon[theta](1,t)");plot3d(-U[N-1]/r,r=1..1.000001,t=0..100 );print("Ham epsilon[theta
u(r,t):=U[N-1];;Ee:=0.5;;epsilon[theta](r,t):=u(r,t)/r;E[theta](r,t) := (100/(t/To)^.1+1)*Ee; sigma[the
sigma[theta](r,t)));sigma[b](t):=normal(subs(r=2,sigma[theta](r,t)));;;with(plottools):with(plots):plo
style=[point,line],symbol=diamond,color=[red,black],thickness=[1],legend=[`sigma[bn](t)`,`sigma[b]
```

Warning, the name changecoords has been redefined

$$m := \frac{\left(100. + \left(\frac{t}{To}\right)^{\left(\frac{1}{10}\right)}\right)\left(\frac{t}{To}\right)^{\left(\frac{2}{5}\right)}}{100. + \sqrt{\frac{t}{To}}}$$

$$To := 1$$

$$lame\_cyl := r^2\left(\frac{\partial^2}{\partial r^2}u(r,t)\right) + r\left(\frac{\partial}{\partial r}u(r,t)\right) + \frac{\left(100. + t^{\left(\frac{1}{10}\right)}\right) t^{\left(\frac{2}{5}\right)} u(r,t)}{100. + \sqrt{t}} = 0$$

$$bound\_con := u(1,t) = 1,\ u(2,t) = -1$$

$$a := 1$$

$$b := 2$$

$$N := 20$$

$$h := \frac{1}{20}$$

$$R := k \rightarrow a + k\, h$$

$$U1 := (k, t) \rightarrow \frac{1}{2} \frac{u_{k+1,\, t} - u_{k-1,\, t}}{h}$$

$$U2 := (k, t) \rightarrow \frac{u_{k+1,\, t} - 2\, u_{k,\, t} + u_{k-1,\, t}}{h^2}$$

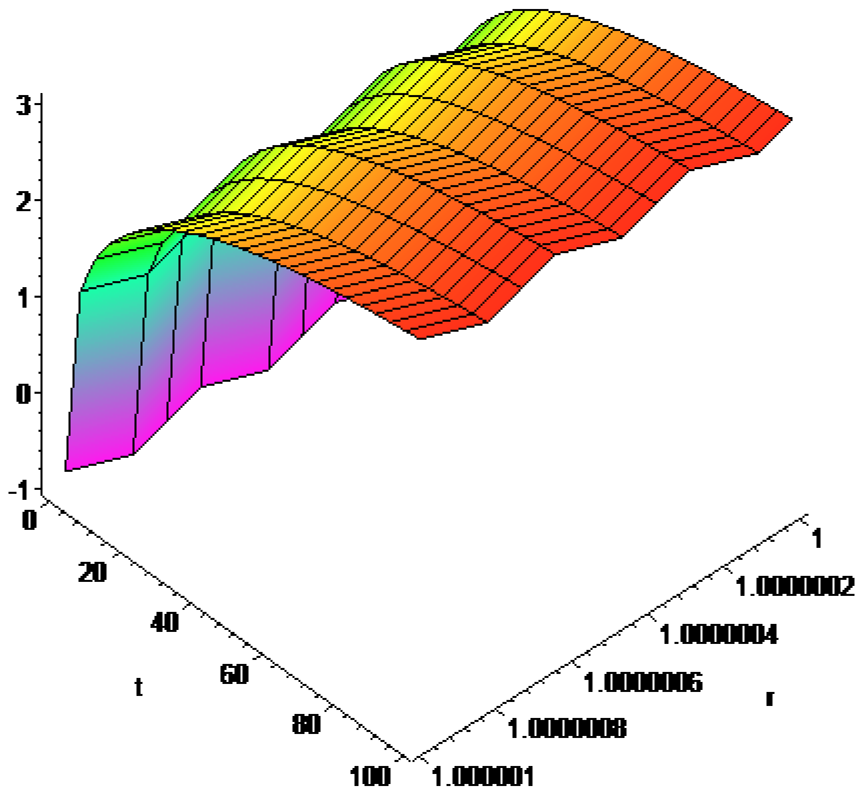
$$e_0 := u_{0,\, t} = 1$$

$$e_{20} := u_{20,\, t} = -1$$

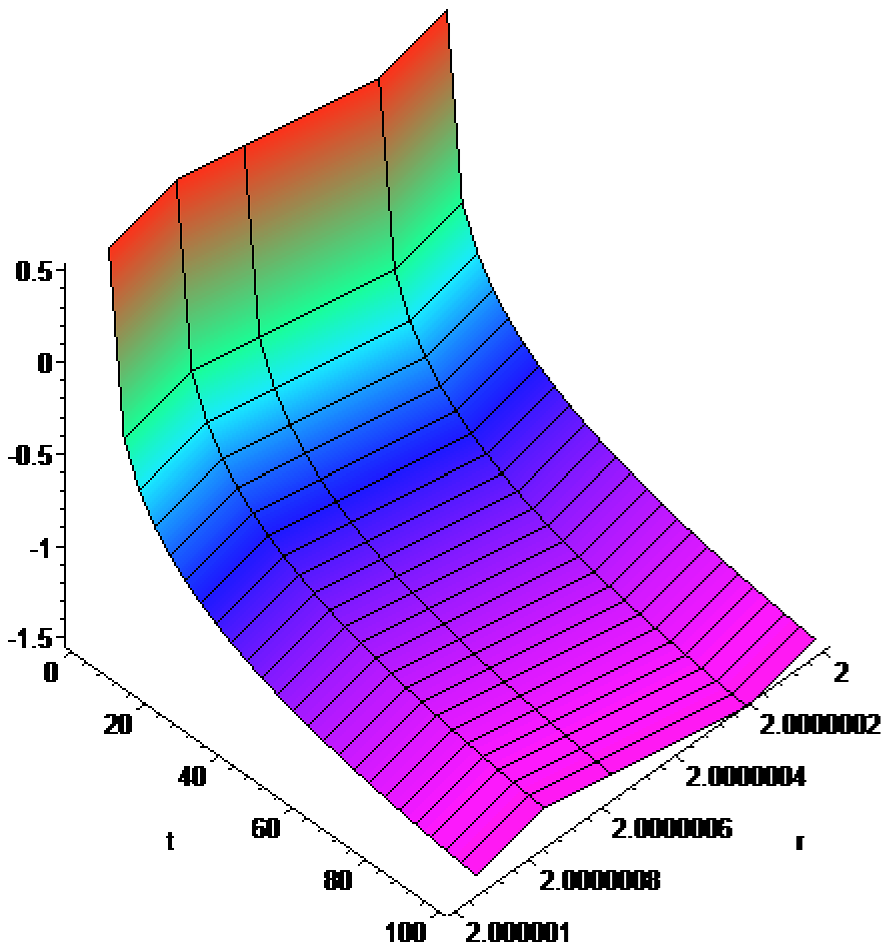
$$e_1 := \frac{903}{2} u_{2,\, t} - 882\, u_{1,\, t} + \frac{861}{2} u_{0,\, t} + \frac{\left(100. + t^{\left(\frac{1}{10}\right)}\right) t^{\left(\frac{2}{5}\right)} u_{1,\, t}}{100. + \sqrt{t}} = 0$$

$$e_2 := 495\, u_{3,\, t} - 968\, u_{2,\, t} + 473\, u_{1,\, t} + \frac{\left(100. + t^{\left(\frac{1}{10}\right)}\right) t^{\left(\frac{2}{5}\right)} u_{2,\, t}}{100. + \sqrt{t}} = 0 \quad \ldots..$$

"Ham epsilon[theta](1,t)"



"Ham epsilon[theta](2,t)"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0
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-1
-1.5
0
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40
60
80
100
t
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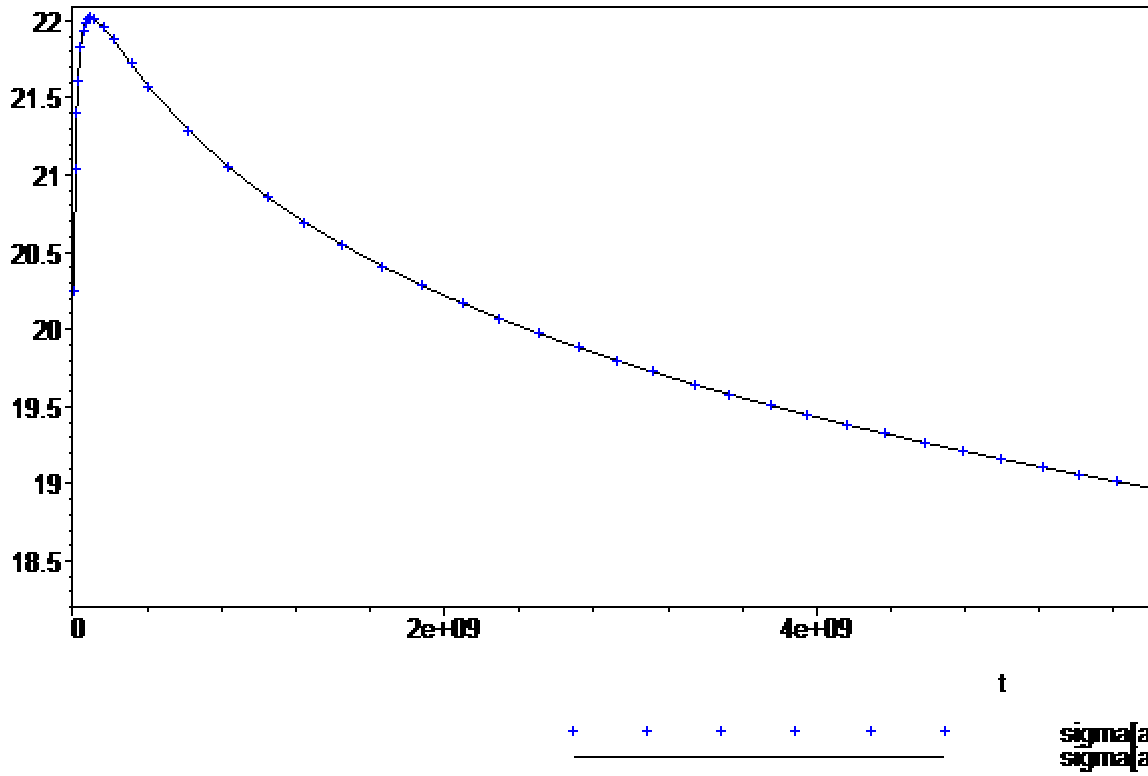
2.0000004
2.0000006
2.0000008
2.000001
r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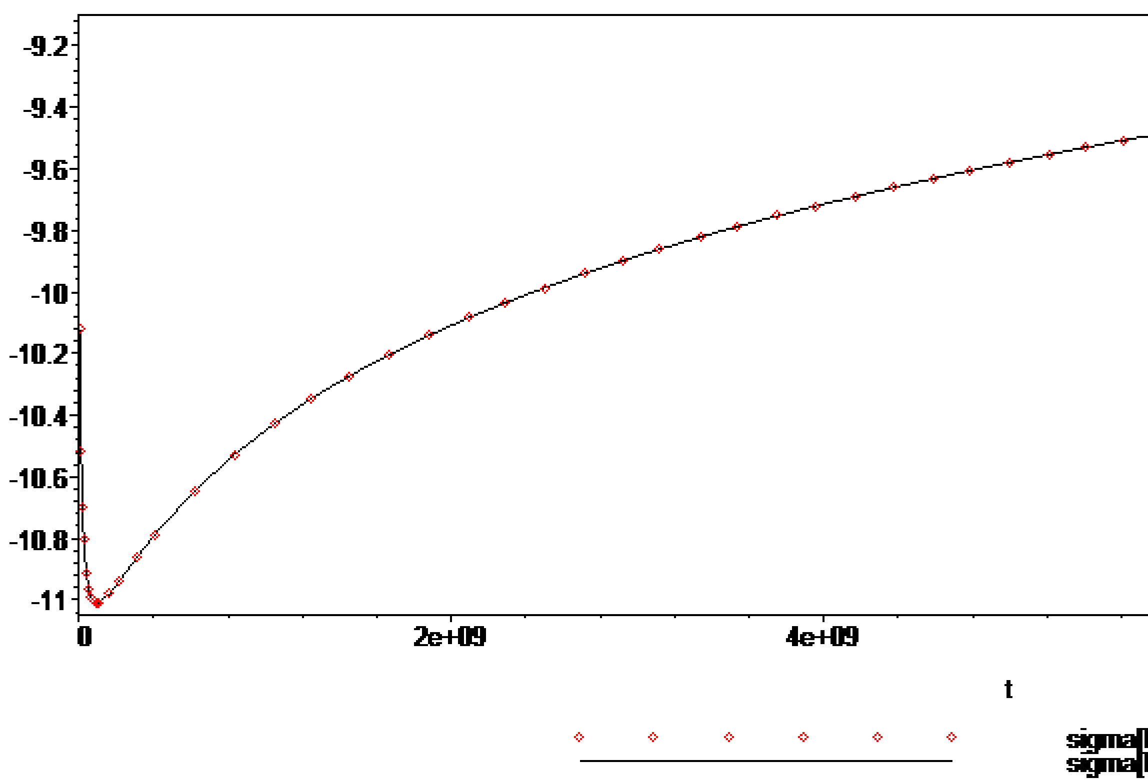
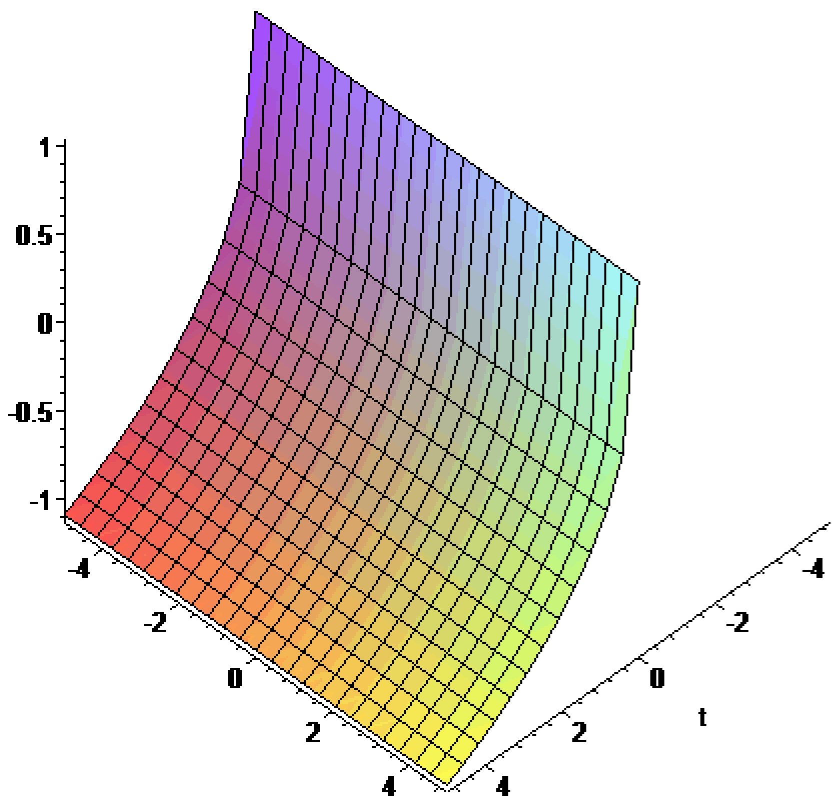
21.5
21
20.5
20
19.5
19
18.5
0
2e+09
4e+09
t


-9.2
-9.4
-9.6
-9.8
-10
-10.2
-10.4
-10.6
-10.8
-11
0
2e+09
4e+09
t

"HAM u(r,t)"

Live



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