Review of the Real Number System

1

Social Security is the largest source of income for elderly Americans. It is projected that in about 30 years, however, there will be twice as many older Americans as there are today, and the excess revenues now accumulating in Social Security's trust funds will be exhausted. (*Source:* Social Security Administration.)

To supplement their retirement incomes, more and more Americans have begun investing in mutual funds, pension plans, and other means of savings. In Section 1.3, we relate the concepts of this chapter to the percent of U.S. households investing in mutual funds.

1.1 Basic Concepts

OBJECTIVES

1. Write sets using set notation.
2. Use number lines.
3. Know the common sets of numbers.
4. Find additive inverses.
5. Use absolute value.
6. Use inequality symbols.

Study Skills Workbook
Activity 2: Your Textbook

In this chapter we review some of the basic symbols and rules of algebra.

OBJECTIVE 1 Write sets using set notation. A **set** is a collection of objects called the **elements** or **members** of the set. In algebra, the elements of a set are usually numbers. Set braces, { }, are used to enclose the elements. For example, 2 is an element of the set $\{1, 2, 3\}$. Since we can count the number of elements in the set $\{1, 2, 3\}$, it is a *finite set*.

In our study of algebra, we refer to certain sets of numbers by name. The set

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$

is called the **natural numbers** or the **counting numbers.** The three dots show that the list continues in the same pattern indefinitely. We cannot list all of the elements of the set of natural numbers, so it is an *infinite set*.

When 0 is included with the set of natural numbers, we have the set of **whole numbers,** written

$$W = \{0, 1, 2, 3, 4, 5, 6, \ldots\}.$$

A set containing no elements, such as the set of whole numbers less than 0, is called the **empty set,** or **null set,** usually written $\emptyset$.

CAUTION
Do not write $\{\emptyset\}$ for the empty set; $\{\emptyset\}$ is a set with one element, $\emptyset$. Use only the notation $\emptyset$ for the empty set.

Work Problem 1 at the Side.

In algebra, letters called **variables** are often used to represent numbers or to define sets of numbers. For example,

$$\{x \mid x \text{ is a natural number between 3 and 15}\}$$

(read "the set of all elements x such that x is a natural number between 3 and 15") defines the set

$$\{4, 5, 6, 7, \ldots, 14\}.$$

The notation $\{x \mid x$ is a natural number between 3 and 15$\}$ is an example of **set-builder notation.**

$$\{\, x \mid x \text{ has property } P \,\}$$

the set of — all elements x — such that — x has a given property P

EXAMPLE 1 Listing the Elements in Sets

List the elements in each set.

(a) $\{x \mid x$ is a natural number less than 4$\}$
The natural numbers less than 4 are 1, 2, and 3. This set is $\{1, 2, 3\}$.

(b) $\{y \mid y$ is one of the first five even natural numbers$\} = \{2, 4, 6, 8, 10\}$

(c) $\{z \mid z$ is a natural number greater than or equal to 7$\}$
The set of natural numbers greater than or equal to 7 is an infinite set, written with three dots as $\{7, 8, 9, 10, \ldots\}$.

Work Problem 2 at the Side.

1 Consider the set
$$\left\{0, 10, \frac{3}{10}, 52, 98.6\right\}.$$

(a) Which elements of the set are natural numbers?

(b) Which elements of the set are whole numbers?

2 List the elements in each set.

(a) $\{x \mid x$ is a whole number less than 5$\}$

(b) $\{y \mid y$ is a whole number greater than 12$\}$

ANSWERS
1. **(a)** 10 and 52 **(b)** 0, 10, and 52
2. **(a)** $\{0, 1, 2, 3, 4\}$ **(b)** $\{13, 14, 15, \ldots\}$

Video

EXAMPLE 2 Using Set-Builder Notation to Describe Sets

Use set-builder notation to describe each set.

(a) $\{1, 3, 5, 7, 9\}$

There are often several ways to describe a set with set-builder notation. One way is $\{y \mid y$ is one of the first five odd natural numbers$\}$.

(b) $\{5, 10, 15, \ldots\}$

This set can be described as $\{x \mid x$ is a multiple of 5 greater than 0$\}$.

Work Problem 3 at the Side.

OBJECTIVE 2 Use number lines. A good way to get a picture of a set of numbers is by using a **number line.** To construct a number line, choose any point on a horizontal line and label it 0. Next, choose a point to the right of 0 and label it 1. The distance from 0 to 1 establishes a scale that can be used to locate more points, with positive numbers to the right of 0 and negative numbers to the left of 0. The number 0 is neither positive nor negative. A number line is shown in Figure 1.

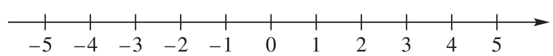


Figure 1

The set of numbers identified on the number line in Figure 1, including positive and negative numbers and 0, is part of the set of **integers,** written

$$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Each number on a number line is called the **coordinate** of the point that it labels, while the point is the **graph** of the number. Figure 2 shows a number line with several selected points graphed on it.

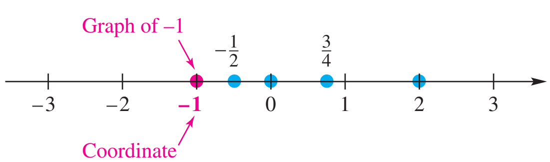


Figure 2

Work Problem 4 at the Side.

The fractions $-\frac{1}{2}$ and $\frac{3}{4}$, graphed on the number line in Figure 2, are examples of **rational numbers.** Rational numbers can be written in decimal form, either as terminating decimals such as $\frac{3}{5} = .6$, $\frac{1}{8} = .125$, or $\frac{11}{4} = 2.75$, or as repeating decimals such as $\frac{1}{3} = .33333\ldots$ or $\frac{3}{11} = .272727\ldots$. A repeating decimal is often written with a bar over the repeating digit(s). Using this notation, $.2727\ldots$ is written $.\overline{27}$.

Decimal numbers that neither terminate nor repeat are *not* rational, and thus are called **irrational numbers.** Many square roots are irrational numbers; for example, $\sqrt{2} = 1.4142136\ldots$ and $-\sqrt{7} = -2.6457513\ldots$ repeat indefinitely without pattern. (Some square roots *are* rational: $\sqrt{16} = 4, \sqrt{100} = 10$, and so on.) Another irrational number is π, the ratio of the circumference of a circle to its diameter.

Some of the rational and irrational numbers discussed above are graphed on the number line in Figure 3 on the next page. The rational numbers together with the irrational numbers make up the set of **real numbers.** Every point on a number line corresponds to a real number, and every real number corresponds to a point on the number line.

3 Use set-builder notation to describe each set.

(a) $\{0, 1, 2, 3, 4, 5\}$

(b) $\{7, 14, 21, 28, \ldots\}$

4 Graph the elements of each set.

(a) $\{-4, -2, 0, 2, 4, 6\}$

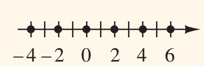
(b) $\left\{-1, 0, \frac{2}{3}, \frac{5}{2}\right\}$

(c) $\left\{5, \frac{16}{3}, 6, \frac{13}{2}, 7, \frac{29}{4}\right\}$

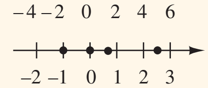
ANSWERS

3. **(a)** One answer is $\{x \mid x$ is a whole number less than $6\}$. **(b)** One answer is $\{x \mid x$ is a multiple of 7 greater than $0\}$.

4. **(a)**



(b)



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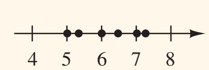


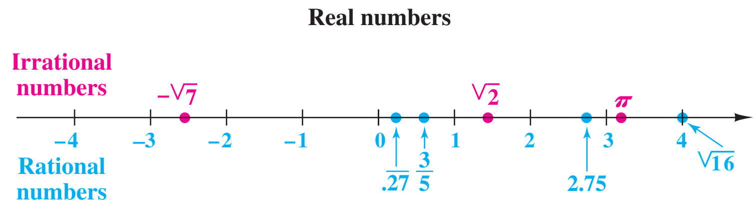


Figure 3

OBJECTIVE 3 Know the common sets of numbers. The sets of numbers listed below will be used throughout the rest of this text.

Sets of Numbers

Natural numbers or counting numbers	$\{1, 2, 3, 4, 5, 6, \ldots\}$
Whole numbers	$\{0, 1, 2, 3, 4, 5, 6, \ldots\}$
Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\left\{\frac{p}{q} \,\middle\|\, p \text{ and } q \text{ are integers, } q \neq 0\right\}$ *Examples:* $\frac{4}{1}, 1.3, -\frac{9}{2}, \frac{16}{8}$ or 2, $\sqrt{9}$ or 3, $.\overline{6}$
Irrational numbers	$\{x \mid x$ is a real number that is not rational$\}$ *Examples:* $\sqrt{3}, -\sqrt{2}, \pi$
Real numbers	$\{x \mid x$ is represented by a point on a number line$\}$*

The relationships among these sets of numbers are shown in Figure 4; in particular, notice that the set of real numbers includes both the rational and irrational numbers. ***Every real number is either rational or irrational.*** Also, notice that the integers are elements of the set of rational numbers and that whole numbers and natural numbers are elements of the set of integers.

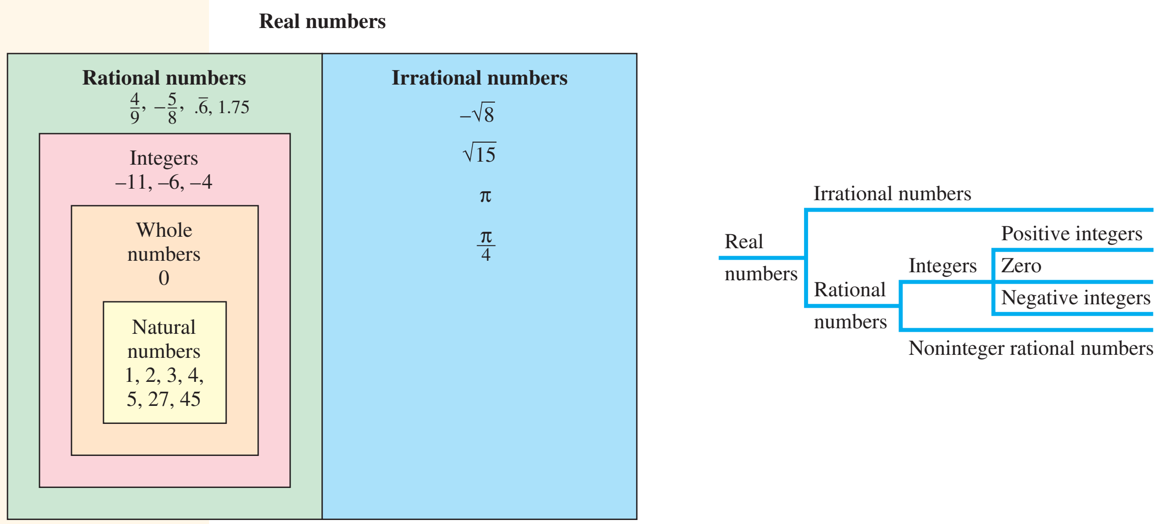


Figure 4 The Real Numbers

*An example of a number that is not a coordinate of a point on a number line is $\sqrt{-1}$. This number, called an *imaginary number,* is discussed in **Section 9.7.**

EXAMPLE 3 Identifying Examples of Number Sets

Which numbers in

$$\left\{-8, -\sqrt{2}, -\frac{9}{64}, 0, .5, \frac{2}{3}, 1.\overline{12}, \sqrt{3}, 2\right\}$$

are elements of each set?

(a) Integers -8, 0, and 2 are integers.

(b) Rational numbers $-8, -\frac{9}{64}, 0, .5, \frac{2}{3}, 1.\overline{12}$, and 2 are rational numbers.

(c) Irrational numbers $-\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

(d) Real numbers
All the numbers in the given set are real numbers.

Work Problem 5 at the Side.

EXAMPLE 4 Determining Relationships between Sets of Numbers

Decide whether each statement is *true* or *false*.

(a) All irrational numbers are real numbers.
This is true. As shown in Figure 4, the set of real numbers includes all irrational numbers.

(b) Every rational number is an integer.
This statement is false. Although some rational numbers are integers, other rational numbers, such as $\frac{2}{3}$ and $-\frac{1}{4}$, are not.

Work Problem 6 at the Side.

OBJECTIVE 4 Find additive inverses. Look again at the number line in Figure 1. For each positive number, there is a negative number on the opposite side of 0 that lies the same distance from 0. These pairs of numbers are called **additive inverses, negatives,** or **opposites** of each other. For example, 5 is the additive inverse of -5, and -5 is the additive inverse of 5.

Additive Inverse

For any real number a, the number $-a$ is the additive inverse of a.

Change the sign of a number to get its additive inverse. ***The sum of a number and its additive inverse is always 0.***

The symbol "$-$" can be used to indicate any of the following:

1. a negative number, such as -9 or -15;
2. the additive inverse of a number, as in "-4 is the additive inverse of 4";
3. subtraction, as in $12 - 3$.

In the expression $-(-5)$, the symbol "$-$" is being used in two ways: the first $-$ indicates the additive inverse of -5, and the second indicates a negative number, -5. Since the additive inverse of -5 is 5, then $-(-5) = 5$. This example suggests the following property.

For any real number a, $\qquad -(-a) = a.$

5 Select all the sets from the following list that apply to each number.
Whole number
Rational number
Irrational number
Real number

(a) -6

(b) 12

(c) $.\overline{3}$

(d) $-\sqrt{15}$

(e) π

(f) $\frac{22}{7}$

(g) 3.14

6 Decide whether the statement is *true* or *false*. If *false*, tell why.

(a) All whole numbers are integers.

(b) Some integers are whole numbers.

(c) Every real number is irrational.

ANSWERS

5. (a) rational, real
(b) whole, rational, real
(c) rational, real
(d) irrational, real
(e) irrational, real
(f) rational, real
(g) rational, real

6. (a) true **(b)** true
(c) false; Some real numbers are irrational, but others are rational numbers.

7 Give the additive inverse of each number.

(a) 9

(b) -12

(c) $-\frac{6}{5}$

(d) 0

(e) 1.5

Numbers written with positive or negative signs, such as $+4$, $+8$, -9, and -5, are called **signed numbers.** A positive number can be called a signed number even though the positive sign is usually left off. The following table shows the additive inverses of several signed numbers. Note that 0 is its own additive inverse.

Number	Additive Inverse
6	-6
-4	4
$\frac{2}{3}$	$-\frac{2}{3}$
-8.7	8.7
0	0

Work Problem 7 at the Side.

OBJECTIVE 5 Use absolute value. Geometrically, the **absolute value** of a number a, written $|a|$, is the distance on the number line from 0 to a. For example, the absolute value of 5 is the same as the absolute value of -5 because each number lies five units from 0. See Figure 5. That is,

$$|5| = 5 \quad \text{and} \quad |-5| = 5.$$

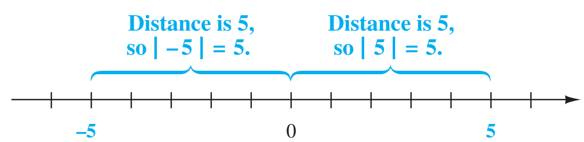


Figure 5

CAUTION
Because absolute value represents distance, and distance is always positive (or 0), ***the absolute value of a number is always positive (or 0).***

The formal definition of absolute value follows.

Absolute Value

$$|a| = \begin{cases} a & \text{if } a \text{ is positive or } 0 \\ -a & \text{if } a \text{ is negative} \end{cases}$$

The second part of this definition, $|a| = -a$ if a is negative, requires careful thought. If a is a *negative* number, then $-a$, the additive inverse or opposite of a, is a positive number, so $|a|$ is positive. For example, if $a = -3$, then

$$|a| = |-3| = -(-3) = 3. \qquad |a| = -a \text{ if } a \text{ is negative.}$$

EXAMPLE 5 Evaluating Absolute Value Expressions

Find the value of each expression.

(a) $|13| = 13$ **(b)** $|-2| = -(-2) = 2$

(c) $|0| = 0$

Continued on Next Page

ANSWERS
7. **(a)** -9 **(b)** 12 **(c)** $\frac{6}{5}$ **(d)** 0 **(e)** -1.5

(d) $-|8|$
Evaluate the absolute value first. Then find the additive inverse.

$$-|8| = -(8) = -8$$

(e) $-|-8|$
Work as in part (d): $|-8| = 8$, so

$$-|-8| = -(8) = -8.$$

(f) $|-2| + |5|$
Evaluate each absolute value first, and then add.

$$|-2| + |5| = 2 + 5 = 7$$

(g) $-|5 - 2| = -|3| = -3$

Work Problem 8 at the Side.

Absolute value is useful when comparing size without regard to sign.

EXAMPLE 6 Comparing Rates of Change in Industries

The projected annual rates of employment change (in percent) in some of the fastest growing and most rapidly declining industries from 1994 through 2005 are shown in the table.

Industry (1994–2005)	Percent Rate of Change
Health services	5.7
Computer and data processing services	4.9
Child day care services	4.3
Footware, except rubber and plastic	−6.7
Household audio and video equipment	−4.2
Luggage, handbags, and leather products	−3.3

Source: U.S. Bureau of Labor Statistics.

What industry in the list is expected to see the greatest change? the least change?

We want the greatest *change*, without regard to whether the change is an increase or a decrease. Look for the number in the list with the largest absolute value. That number is found in footware, since $|-6.7| = 6.7$. Similarly, the least change is in the luggage, handbags, and leather products industry: $|-3.3| = 3.3$.

Work Problem 9 at the Side.

OBJECTIVE 6 Use inequality symbols. The statement $4 + 2 = 6$ is an **equation;** it states that two quantities are equal. The statement $4 \neq 6$ (read "4 is not equal to 6") is an **inequality,** a statement that two quantities are *not* equal. When two numbers are not equal, one must be less than the other. The symbol $<$ means "is less than." For example,

$$8 < 9, \quad -6 < 15, \quad -6 < -1, \quad \text{and} \quad 0 < \frac{4}{3}.$$

The symbol $>$ means "is greater than." For example,

$$12 > 5, \quad 9 > -2, \quad -4 > -6, \quad \text{and} \quad \frac{6}{5} > 0.$$

In each case, ***the symbol "points" toward the smaller number.***

8 Find the value of each expression.

(a) $|6|$

(b) $|-3|$

(c) $-|5|$

(d) $-|-2|$

(e) $-|-7|$

(f) $|-6| + |-3|$

(g) $|-9| - |-4|$

(h) $-|9 - 4|$

9 Refer to the table in Example 6. Of the household audio/video equipment industry and computer/data processing services, which will show the greater change (without regard to sign)?

ANSWERS

8. (a) 6 **(b)** 3 **(c)** −5 **(d)** −2 **(e)** −7 **(f)** 9 **(g)** 5 **(h)** −5

9. Computer/data processing services

10 Insert $<$ or $>$ in each blank to make a true statement.

(a) 3 _______ 7

(b) 9 _______ 2

(c) -4 _______ -8

(d) -2 _______ -1

(e) 0 _______ -3

The number line in Figure 6 shows the numbers 4 and 9, and we know that $4 < 9$. On the graph, 4 is to the left of 9. The lesser of two numbers is always to the left of the other on a number line.

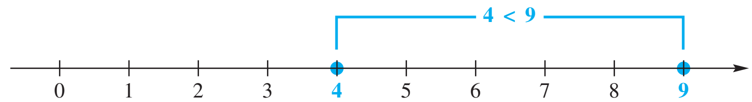


Figure 6

Inequalities on a Number Line

On a number line,

$\boldsymbol{a < b}$ if a is to the left of b; $\boldsymbol{a > b}$ if a is to the right of b.

We can use a number line to determine order. As shown on the number line in Figure 7, -6 is located to the left of 1. For this reason, $-6 < 1$. Also, $1 > -6$. From the same number line, $-5 < -2$, or $-2 > -5$.

Figure 7

CAUTION

Be careful when ordering negative numbers. Since -5 is to the left of -2 on the number line in Figure 7, $\boldsymbol{-5 <} -2$, or $-2 \boldsymbol{> -5}$. In each case, the symbol points to -5, the smaller number.

Work Problem 10 at the Side.

The following table summarizes results about positive and negative numbers in both words and symbols.

Words	Symbols
Every negative number is less than 0.	If a is negative, then $a < 0$.
Every positive number is greater than 0.	If a is positive, then $a > 0$.
0 is neither positive nor negative.	

In addition to the symbols $\neq$, $<$, and $>$, the symbols $\leq$ and $\geq$ are often used.

INEQUALITY SYMBOLS

Symbol	Meaning	Example
$\neq$	is not equal to	$3 \neq 7$
$<$	is less than	$-4 < -1$
$>$	is greater than	$3 > -2$
$\leq$	is less than or equal to	$6 \leq 6$
$\geq$	is greater than or equal to	$-8 \geq -10$

Answers

10. (a) $<$ **(b)** $>$ **(c)** $>$ **(d)** $<$ **(e)** $>$

The following table shows several inequalities and why each is true.

Inequality	Why It Is True
$6 \leq 8$	$6 < 8$
$-2 \leq -2$	$-2 = -2$
$-9 \geq -12$	$-9 > -12$
$-3 \geq -3$	$-3 = -3$
$6 \cdot 4 \leq 5(5)$	$24 < 25$

Notice the reason why $-2 \leq -2$ is true. With the symbol $\leq$, if *either* the $<$ part *or* the $=$ part is true, then the inequality is true. This is also the case with the $\geq$ symbol.

In the last line, recall that the dot in $6 \cdot 4$ indicates the product 6×4, or 24, and **5(5)** means **5×5**, or 25. Thus, the inequality $6 \cdot 4 \leq 5(5)$ becomes $24 \leq 25$, which is true.

Work Problem 11 at the Side.

11 Answer *true* or *false*.

(a) $-2 \leq -3$

(b) $8 \leq 8$

(c) $-9 \geq -1$

(d) $5 \cdot 8 \leq 7 \cdot 7$

(e) $3(4) > 2(6)$

Answers

11. (a) false **(b)** true **(c)** false **(d)** true **(e)** false

1.2 Operations on Real Numbers

In this section we review the rules for adding, subtracting, multiplying, and dividing real numbers.

OBJECTIVES

1. Add real numbers.
2. Subtract real numbers.
3. Multiply real numbers.
4. Find the reciprocal of a number.
5. Divide real numbers.

OBJECTIVE 1 Add real numbers. Number lines can be used to illustrate addition and subtraction of real numbers. To add two real numbers on a number line, start at 0. Move right (the *positive* direction) to add a positive number or left (the *negative* direction) to add a negative number. See Figure 8.

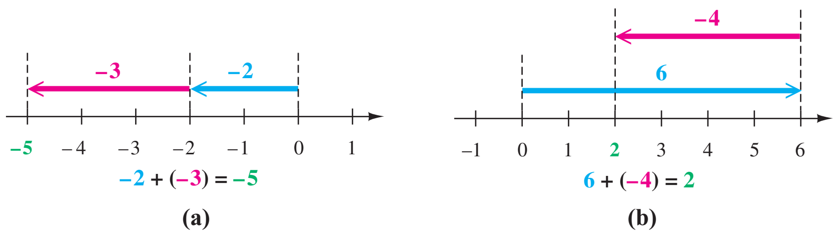


Figure 8

This procedure for adding real numbers can be generalized in the following rules.

Adding Real Numbers

Like signs To add two numbers with the *same* sign, add their absolute values. The sign of the answer (either + or −) is the same as the sign of the two numbers.

Unlike signs To add two numbers with *different* signs, subtract the smaller absolute value from the larger. The sign of the answer is the same as the sign of the number with the larger absolute value.

Recall that the answer to an addition problem is called the **sum.**

EXAMPLE 1 Adding Two Negative Numbers

Find each sum.

(a) $-12 + (-8)$

First find the absolute values.

$$|-12| = 12 \qquad \text{and} \qquad |-8| = 8$$

Because -12 and -8 have the *same* sign, add their absolute values. Both numbers are negative, so the answer is negative.

$$-12 + (-8) = -(12 + 8) = -(20) = -20$$

(b) $-6 + (-3) = -(|-6| + |-3|) = -(6 + 3) = -9$

(c) $-1.2 + (-.4) = -(1.2 + .4) = -1.6$

(d) $-\frac{5}{6} + \left(-\frac{1}{3}\right) = -\left(\frac{5}{6} + \frac{1}{3}\right) = -\left(\frac{5}{6} + \frac{2}{6}\right) = -\frac{7}{6}$

Work Problem 1 at the Side.

1 Find each sum.

(a) $-2 + (-7)$

(b) $-15 + (-6)$

(c) $-1.1 + (-1.2)$

(d) $-\frac{3}{4} + \left(-\frac{1}{2}\right)$

ANSWERS

1. (a) -9 **(b)** -21 **(c)** -2.3 **(d)** $-\frac{5}{4}$

2 Find each sum.

(a) $12 + (-1)$

(b) $3 + (-7)$

(c) $-17 + 5$

(d) $-\frac{3}{4} + \frac{1}{2}$

(e) $-1.5 + 3.2$

ANSWERS

2. **(a)** 11 **(b)** -4 **(c)** -12 **(d)** $-\frac{1}{4}$ **(e)** 1.7

EXAMPLE 2 Adding Numbers with Different Signs

Find each sum.

(a) $-17 + 11$

First find the absolute values.

$$|-17| = 17 \quad \text{and} \quad |11| = 11$$

Because -17 and 11 have *different* signs, subtract their absolute values.

$$17 - 11 = 6$$

The number -17 has a larger absolute value than 11, so the answer is negative.

$$-17 + 11 = \underset{\substack{\uparrow \\ \text{Negative because } |-17| > |11|}}{-6}$$

(b) $4 + (-1)$

Subtract the absolute values, 4 and 1. Because 4 has the larger absolute value, the sum must be positive.

$$4 + (-1) = 4 - 1 = \underset{\substack{\uparrow \\ \text{Positive because } |4| > |-1|}}{3}$$

(c) $-9 + 17 = 17 - 9 = 8$

(d) $-16 + 12$

The absolute values are 16 and 12. Subtract the absolute values. The negative number has the larger absolute value, so the answer is negative.

$$-16 + 12 = -(16 - 12) = -4$$

(e) $-\frac{4}{5} + \frac{2}{3}$

Write each number with a common denominator.

$$\frac{4}{5} = \frac{4 \cdot 3}{5 \cdot 3} = \frac{12}{15} \quad \text{and} \quad \frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}$$

$$-\frac{4}{5} + \frac{2}{3} = -\frac{12}{15} + \frac{10}{15}$$

$$= -\left(\frac{12}{15} - \frac{10}{15}\right) \qquad -\tfrac{12}{15} \text{ has the larger absolute value.}$$

$$= -\frac{2}{15} \qquad \text{Subtract.}$$

(f) $-2.3 + 5.6 = 3.3$

Work Problem 2 at the Side.

OBJECTIVE 2 Subtract real numbers. Recall that the answer to a subtraction problem is called the **difference.** Thus, the difference between 6 and 4 is 2. To see how subtraction should be defined, compare the following two statements.

$$6 - 4 = 2$$
$$6 + (-4) = 2$$

The second statement is pictured on the number line in Figure 8(b) at the beginning of this section. Similarly, $9 - 3 = 6$ and $9 + (-3) = 6$ so that $9 - 3 = 9 + (-3)$. These examples suggest the following rule for subtraction.

Audio

Subtraction

For all real numbers a and b,

$$a - b = a + (-b).$$

In words, change the sign of the second number (subtrahend) and add.

You Try It

Video

EXAMPLE 3 Subtracting Real Numbers

Find each difference.

Change to addition.
Change sign of second number (subtrahend).

(a) $6 - 8 = 6 + (-8) = -2$

Changed
Sign changed

(b) $-12 - 4 = -12 + (-4) = -16$

(c) $-10 - (-7) = -10 + [-(-7)]$ This step is often omitted.

$= -10 + 7$

$= -3$

(d) $-2.4 - (-8.1) = -2.4 + 8.1 = 5.7$

(e) $\frac{8}{3} - \left(-\frac{5}{3}\right) = \frac{8}{3} + \frac{5}{3} = \frac{13}{3}$

Work Problem 3 at the Side.

When working a problem that involves both addition and subtraction, add and subtract in order from left to right. Work inside brackets or parentheses first.

EXAMPLE 4 Adding and Subtracting Real Numbers

Perform the indicated operations.

(a) $-8 + 5 - 6 = (-8 + 5) - 6$ Work from left to right.

$= -3 - 6$

$= -3 + (-6)$

$= -9$

(b) $15 - (-3) - 5 - 12 = (15 + 3) - 5 - 12$

$= 18 - 5 - 12$

$= 13 - 12$

$= 1$

(c) $-4 - (-6) + 7 - 1 = (-4 + 6) + 7 - 1$

$= 2 + 7 - 1$

$= 9 - 1$

$= 8$

Continued on Next Page

3 Find each difference.

(a) $9 - 12$

(b) $-7 - 2$

(c) $-8 - (-2)$

(d) $-6.3 - (-11.5)$

(e) $12 - (-5)$

ANSWERS

3. **(a)** -3 **(b)** -9 **(c)** -6 **(d)** 5.2 **(e)** 17

(d) $-9 - [-8 - (-4)] + 6 = -9 - [-8 + 4] + 6$ Work inside the brackets.

$= -9 - [-4] + 6$

$= -9 + 4 + 6$

$= -5 + 6$

$= 1$

Work Problem 4 at the Side.

OBJECTIVE 3 Multiply real numbers. The answer to a multiplication problem is called the **product.** For example, 24 is the product of 8 and 3. The rules for finding signs of products of real numbers are given below.

Multiplying Real Numbers

Like signs The product of two numbers with the *same* sign is positive.

Unlike signs The product of two numbers with *different* signs is negative.

EXAMPLE 5 Multiplying Real Numbers

Find each product.

(a) $-3(-9) = 27$ Same sign; product is positive.

(b) $-.5(-.4) = .2$

(c) $-\frac{3}{4}\left(-\frac{5}{3}\right) = \frac{5}{4}$

(d) $6(-9) = -54$ Different signs; product is negative.

(e) $-.05(.3) = -.015$

(f) $\frac{2}{3}(-3) = -2$

(g) $-\frac{5}{8}\left(\frac{12}{13}\right) = -\frac{15}{26}$

Work Problem 5 at the Side.

OBJECTIVE 4 Find the reciprocal of a number. Earlier, subtraction was defined in terms of addition. Now, division is defined in terms of multiplication. The definition of division depends on the idea of a **multiplicative inverse** or *reciprocal;* two numbers are *reciprocals* if they have a product of 1.

Reciprocal

The **reciprocal** of a nonzero number a is $\frac{1}{a}$.

Calculator Tip Reciprocals (in decimal form) can be found with a calculator that has a key labeled $1/x$ or x^{-1}. For example, a calculator shows that the reciprocal of 25 is .04.

---

4 Perform the indicated operations.

(a) $-6 + 9 - 2$

(b) $12 - (-4) + 8$

(c) $-6 - (-2) - 8 - 1$

(d) $-3 - [(-7) + 15] + 6$

5 Find each product.

(a) $-7(-5)$

(b) $-.9(-15)$

(c) $-\frac{4}{7}\left(-\frac{14}{3}\right)$

(d) $7(-2)$

(e) $-.8(.006)$

(f) $\frac{5}{8}(-16)$

(g) $-\frac{2}{3}(12)$

ANSWERS

4. **(a)** 1 **(b)** 24 **(c)** -13 **(d)** -5

5. **(a)** 35 **(b)** 13.5 **(c)** $\frac{8}{3}$ **(d)** -14 **(e)** $-.0048$ **(f)** -10 **(g)** -8

The table gives several numbers and their reciprocals.

Number	Reciprocal
$-\frac{2}{5}$	$-\frac{5}{2}$
-6	$-\frac{1}{6}$
$\frac{7}{11}$	$\frac{11}{7}$
.05	20
0	**None**

$-\frac{2}{5}\left(-\frac{5}{2}\right) = 1$

$-6\left(-\frac{1}{6}\right) = 1$

$\frac{7}{11}\left(\frac{11}{7}\right) = 1$

$.05(20) = 1$

There is no reciprocal for 0 because there is no number that can be multiplied by 0 to give a product of 1.

CAUTION
A number and its additive inverse have *opposite* signs; however, a number and its reciprocal always have the *same* sign.

Work Problem 6 at the Side.

OBJECTIVE 5 Divide real numbers. The result of dividing one number by another is called the **quotient.** For example, when 45 is divided by 3, the quotient is 15. To define division of real numbers, we first write the quotient of 45 and 3 as $\frac{45}{3}$, which equals 15. The same answer will be obtained if 45 and $\frac{1}{3}$ are multiplied, as follows.

$$45 \div 3 = \frac{45}{3} = 45 \cdot \frac{1}{3} = 15$$

This suggests the following definition of division of real numbers.

Audio

Division

For all real numbers a and b (where $b \neq 0$),

$$a \div b = \frac{a}{b} = a \cdot \frac{1}{b}.$$

In words, multiply the first number by the reciprocal of the second number.

There is no reciprocal for the number 0, so ***division by 0 is undefined.*** For example, $\frac{15}{0}$ is undefined and $-\frac{1}{0}$ is undefined.

CAUTION
Division by 0 is undefined. However, dividing 0 by a nonzero number gives the quotient 0. For example,

$$\frac{6}{0} \text{ is undefined, but } \frac{0}{6} = 0 \quad (\text{since } 0 \cdot 6 = 0).$$

Be careful when 0 is involved in a division problem.

Work Problem 7 at the Side.

6 Give the reciprocal of each number.

(a) 15

(b) -7

(c) $\frac{8}{9}$

(d) $-\frac{1}{3}$

(e) .125

7 Divide where possible.

(a) $\frac{9}{0}$

(b) $\frac{0}{9}$

(c) $\frac{-9}{0}$

(d) $\frac{0}{-9}$

Answers

6. (a) $\frac{1}{15}$ **(b)** $-\frac{1}{7}$ **(c)** $\frac{9}{8}$ **(d)** -3 **(e)** 8

7. (a) undefined **(b)** 0 **(c)** undefined **(d)** 0

8 Find each quotient.

(a) $\dfrac{-16}{4}$

(b) $\dfrac{8}{-2}$

(c) $\dfrac{-15}{-3}$

(d) $\dfrac{\frac{3}{8}}{\frac{11}{16}}$

9 Which of the following fractions are equal to $\frac{-3}{5}$?

A. $\dfrac{3}{5}$ **B.** $\dfrac{3}{-5}$

C. $-\dfrac{3}{5}$ **D.** $\dfrac{-3}{-5}$

Answers

8. (a) -4 **(b)** -4 **(c)** 5 **(d)** $\frac{6}{11}$

9. B, C

Since division is defined as multiplication by the reciprocal, the rules for signs of quotients are the same as those for signs of products.

Dividing Real Numbers

Like signs The quotient of two nonzero real numbers with the *same* sign is positive.

Unlike signs The quotient of two nonzero real numbers with *different* signs is negative.

EXAMPLE 6 Dividing Real Numbers

Find each quotient.

(a) $\dfrac{-12}{4} = -12 \cdot \dfrac{1}{4} = -3$ $\qquad \frac{a}{b} = a \cdot \frac{1}{b}$

(b) $\dfrac{6}{-3} = 6\left(-\dfrac{1}{3}\right) = -2$ $\qquad$ The reciprocal of -3 is $-\frac{1}{3}$.

(c) $\dfrac{-30}{-2} = -30\left(-\dfrac{1}{2}\right) = 15$

(d) $\dfrac{\frac{2}{3}}{\frac{5}{9}} = \dfrac{2}{3} \cdot \dfrac{9}{5} = \dfrac{6}{5}$ $\qquad$ The reciprocal of the denominator $\frac{5}{9}$ is $\frac{9}{5}$.

This is a *complex fraction* **(Section 8.3),** a fraction that has a fraction in the numerator, the denominator, or both.

Work Problem 8 at the Side.

The rules for multiplication and division suggest the following results.

Equivalent Forms of a Fraction

The fractions $\dfrac{-x}{y}$, $-\dfrac{x}{y}$, and $\dfrac{x}{-y}$ are equal. (Assume $y \neq 0$.)

Example: $\dfrac{-4}{7} = -\dfrac{4}{7} = \dfrac{4}{-7}$.

The fractions $\dfrac{x}{y}$ and $\dfrac{-x}{-y}$ are equal.

Example: $\dfrac{4}{7} = \dfrac{-4}{-7}$.

The forms $\frac{x}{-y}$ and $\frac{-x}{-y}$ are not used very often.

Every fraction has three signs: the sign of the numerator, the sign of the denominator, and the sign of the fraction itself. Changing any two of these three signs does not change the value of the fraction. Changing only one sign, or changing all three, *does* change the value.

Work Problem 9 at the Side.

1.3 Exponents, Roots, and Order of Operations

Two or more numbers whose product is a third number are **factors** of that third number. For example, 2 and 6 are factors of 12 since $2 \cdot 6 = 12$. Other factors of 12 are 1, 3, 4, 12, -1, -2, -3, -4, -6, and -12.

OBJECTIVE 1 Use exponents. In algebra, we use *exponents* as a way of writing products of repeated factors. For example, the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ is written

$$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{5 \text{ factors of } 2} = 2^5.$$

The number 5 shows that 2 is used as a factor 5 times. The number 5 is the **exponent,** and 2 is the **base.**

$$2^5 \leftarrow \text{Exponent}$$
Base

Read 2^5 as "2 to the fifth power" or simply "2 to the fifth." Multiplying out the five 2s gives

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$$

Audio

Exponential Expression

If a is a real number and n is a natural number,

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors of } a},$$

where n is the **exponent,** a is the **base,** and a^n is an **exponential expression.** Exponents are also called **powers.**

EXAMPLE 1 Using Exponential Notation

Write each expression using exponents.

(a) $4 \cdot 4 \cdot 4$

Here, 4 is used as a factor 3 times, so

$$\underbrace{4 \cdot 4 \cdot 4}_{3 \text{ factors of } 4} = 4^3.$$

Read 4^3 as "4 cubed."

(b) $\frac{3}{5} \cdot \frac{3}{5} = \left(\frac{3}{5}\right)^2$ 2 factors of $\frac{3}{5}$

Read $\left(\frac{3}{5}\right)^2$ as "$\frac{3}{5}$ squared."

(c) $(-6)(-6)(-6)(-6) = (-6)^4$

(d) $(.3)(.3)(.3)(.3)(.3) = (.3)^5$

(e) $x \cdot x \cdot x \cdot x \cdot x \cdot x = x^6$

Work Problem 1 at the Side.

OBJECTIVES

1. Use exponents.
2. Identify exponents and bases.
3. Find square roots.
4. Use the order of operations.
5. Evaluate expressions for given values of variables.

1 Write each expression using exponents.

(a) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

(b) $\frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7} \cdot \frac{2}{7}$

(c) $(-10)(-10)(-10)$

(d) $(.5)(.5)$

(e) $y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y$

ANSWERS

1. (a) 3^5 **(b)** $\left(\frac{2}{7}\right)^4$ **(c)** $(-10)^3$ **(d)** $(.5)^2$ **(e)** y^8

2 Write each expression without exponents.

(a) 5^3

(b) 3^4

(c) $(-4)^5$

(d) $(-3)^4$

(e) $(.75)^3$

(f) $\left(\frac{2}{5}\right)^4$

In parts (a) and (b) of Example 1, we used the terms *squared* and *cubed* to refer to powers of 2 and 3, respectively. The term *squared* comes from the figure of a square, which has the same measure for both length and width, as shown in Figure 9(a). Similarly, the term *cubed* comes from the figure of a cube. As shown in Figure 9(b), the length, width, and height of a cube have the same measure.

(a) $3 \cdot 3 = 3$ squared, or 3^2 **(b)** $6 \cdot 6 \cdot 6 = 6$ cubed, or 6^3

Figure 9

EXAMPLE 2 Evaluating Exponential Expressions

Write each expression without exponents.

(a) $5^2 = 5 \cdot 5 = 25$ 5 is used as a factor 2 times.

(b) $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$ $\frac{2}{3}$ is used as a factor 3 times.

(c) $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$

(d) $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

(e) $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$

Parts (d) and (e) of Example 2 suggest the following generalization.

The product of an *even* number of negative factors is positive.
The product of an *odd* number of negative factors is negative.

Work Problem 2 at the Side.

Calculator Tip Most calculators have a key labeled x^y or y^x that can be used to raise a number to a power. See "An Introduction to Calculators" at the beginning of this book for more information.

OBJECTIVE 2 Identify exponents and bases.

EXAMPLE 3 Identifying Exponents and Bases

Identify the exponent and the base. Then evaluate each expression.

(a) 3^6 The exponent is 6, and the base is 3.

$$3^6 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 729$$

(b) 5^4 The exponent is 4, and the base is 5.

$$5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$$

Continued on Next Page

ANSWERS
2. **(a)** 125 **(b)** 81 **(c)** -1024 **(d)** 81 **(e)** .421875 **(f)** $\frac{16}{625}$

(c) $(-2)^6$

The exponent 6 applies to the number -2, so the base is -2.

$$(-2)^6 = (-2)(-2)(-2)(-2)(-2)(-2) = 64 \qquad \text{The base is } -2.$$

(d) -2^6

Since there are no parentheses, the exponent 6 applies *only* to the number 2, not to -2; the base is 2.

$$-2^6 = -(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = -64 \qquad \text{The base is } 2.$$

CAUTION

As shown in Examples 3(c) and (d), it is important to distinguish between $-a^n$ and $(-a)^n$.

$$-a^n = -1\underbrace{(a \cdot a \cdot a \cdots a)}_{n \text{ factors of } a} \qquad \text{The base is } a.$$

$$(-a)^n = \underbrace{(-a)(-a)\cdots(-a)}_{n \text{ factors of } -a} \qquad \text{The base is } -a.$$

Work Problem 3 at the Side.

OBJECTIVE 3 Find square roots. As we saw in Example 2(a), $5^2 = 5 \cdot 5 = 25$, so 5 squared is 25. The opposite of squaring a number is called taking its **square root.** For example, a square root of 25 is 5. Another square root of 25 is -5 since $(-5)^2 = 25$; thus, 25 has two square roots, 5 and -5.

We write the positive or *principal* square root of a number with the symbol $\sqrt{}$, called a **radical sign.** For example, the positive or principal square root of 25 is written $\sqrt{25} = 5$. The negative square root of 25 is written $-\sqrt{25} = -5$. Since the square of any nonzero real number is positive, ***the square root of a negative number, such as $\sqrt{-25}$, is not a real number.***

Video

EXAMPLE 4 Finding Square Roots

Find each square root that is a real number.

(a) $\sqrt{36} = 6$ since 6 is positive and $6^2 = 36$.

(b) $\sqrt{0} = 0$ since $0^2 = 0$.

(c) $\sqrt{\frac{9}{16}} = \frac{3}{4}$ since $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

(d) $\sqrt{.16} = .4$ since $(.4)^2 = .16$.

(e) $\sqrt{100} = 10$ since $10^2 = 100$.

(f) $-\sqrt{100} = -10$ since the negative sign is outside the radical sign.

(g) $\sqrt{-100}$ is not a real number since the negative sign is inside the radical sign. No *real number* squared equals -100.

Notice the difference among the expressions in parts (e), (f), and (g). Part (e) is the positive or principal square root of 100, part (f) is the negative square root of 100, and part (g) is the square root of -100, which is not a real number.

3 Identify the exponent and the base. Then evaluate each expression.

(a) 7^3

(b) $(-5)^4$

(c) -5^4

(d) $-(.9)^5$

ANSWERS

3. (a) 3; 7; 343 **(b)** 4; -5; 625 **(c)** 4; 5; -625 **(d)** 5; .9; $-.59049$

4 Find each square root that is a real number.

(a) $\sqrt{9}$

(b) $\sqrt{49}$

(c) $-\sqrt{81}$

(d) $\sqrt{\dfrac{121}{81}}$

(e) $\sqrt{.25}$

(f) $\sqrt{-9}$

(g) $-\sqrt{-169}$

5 Simplify.

(a) $5 \cdot 9 + 2 \cdot 4$

(b) $4 - 12 \div 4 \cdot 2$

Answers

4. **(a)** 3 **(b)** 7 **(c)** -9 **(d)** $\frac{11}{9}$ **(e)** .5
(f) not a real number
(g) not a real number

5. **(a)** 53 **(b)** -2

CAUTION

The symbol $\sqrt{}$ is used only for the *positive* square root, except that $\sqrt{0} = 0$. The symbol $-\sqrt{}$ is used for the negative square root.

Work Problem 4 at the Side.

Calculator Tip Most calculators have a square root key, usually labeled $\sqrt{x}$, that allows us to find the square root of a number. On some models, the square root key must be used in conjunction with the key marked INV or 2nd.

OBJECTIVE 4 **Use the order of operations.** To simplify an expression such as $5 + 2 \cdot 3$, what should we do first—add 5 and 2, or multiply 2 and 3? When an expression involves more than one operation symbol, we use the following **order of operations.**

Order of Operations

1. Work separately above and below any **fraction bar.**
2. If **grouping symbols** such as **parentheses ( ), square brackets [ ],** or **absolute value bars | |** are present, start with the innermost set and work outward.
3. Evaluate all **powers, roots,** and **absolute values.**
4. Do any **multiplications** or **divisions** in order, working from left to right.
5. Do any **additions** or **subtractions** in order, working from left to right.

EXAMPLE 5 Using the Order of Operations

Simplify.

(a) $5 + \mathbf{2 \cdot 3} = 5 + \mathbf{6}$ Multiply.
$= 11$ Add.

(b) $24 \div 3 \cdot 2 + 6$

Multiplications and divisions are done *in the order in which they appear from left to right,* so divide first.

$$\mathbf{24 \div 3} \cdot 2 + 6 = \mathbf{8} \cdot 2 + 6 \quad \text{Divide.}$$
$$= 16 + 6 \quad \text{Multiply.}$$
$$= 22 \quad \text{Add.}$$

Work Problem 5 at the Side.

EXAMPLE 6 Using the Order of Operations

Simplify.

(a) $4 \cdot 3^2 + 7 - (2 + 8)$

Work inside the parentheses first.

Continued on Next Page

$$4 \cdot 3^2 + 7 - (2 + 8) = 4 \cdot 3^2 + 7 - 10 \quad \text{Add inside parentheses.}$$
$$= 4 \cdot 9 + 7 - 10 \quad \text{Evaluate powers.}$$
$$= 36 + 7 - 10 \quad \text{Multiply.}$$
$$= 43 - 10 \quad \text{Add.}$$
$$= 33 \quad \text{Subtract.}$$

(b) $\frac{1}{2} \cdot 4 + (6 \div 3 - 7)$

Work inside the parentheses, dividing before subtracting.

$$\frac{1}{2} \cdot 4 + (6 \div 3 - 7) = \frac{1}{2} \cdot 4 + (2 - 7) \quad \text{Divide inside parentheses.}$$
$$= \frac{1}{2} \cdot 4 + (-5) \quad \text{Subtract inside parentheses.}$$
$$= 2 + (-5) \quad \text{Multiply.}$$
$$= -3 \quad \text{Add.}$$

Work Problem 6 at the Side.

EXAMPLE 7 Using the Order of Operations

You Try It

Video

Simplify $\dfrac{5 + 2^4}{6\sqrt{9} - 9 \cdot 2}$.

$$\frac{5 + 2^4}{6\sqrt{9} - 9 \cdot 2} = \frac{5 + 16}{6 \cdot 3 - 9 \cdot 2} \quad \text{Evaluate powers and roots.}$$
$$= \frac{5 + 16}{18 - 18} \quad \text{Multiply.}$$
$$= \frac{21}{0} \quad \text{Add and subtract.}$$

Because division by 0 is undefined, the given expression is undefined.

Work Problem 7 at the Side.

Calculator Tip Most calculators follow the order of operations given in this section. You may want to try some of the examples to see whether your calculator gives the same answers. Use the parentheses keys to insert parentheses where they are needed. To work Example 7 with a calculator, put parentheses around the numerator and the denominator.

OBJECTIVE 5 Evaluate expressions for given values of variables.

Any collection of numbers, variables, operation symbols, and grouping symbols, such as

$$6ab, \quad 5m - 9n, \quad \text{and} \quad -2(x^2 + 4y), \quad \text{Algebraic expressions}$$

is called an **algebraic expression.** Algebraic expressions have different numerical values for different values of the variables. We can evaluate such expressions by *substituting* given values for the variables.

Algebraic expressions are used in problem solving. For example, if movie tickets cost \$8 each, the amount in dollars you pay for x tickets can be represented by the algebraic expression $8x$. We can substitute different numbers of tickets to get the costs to purchase those tickets.

6 Simplify.

(a) $(4 + 2) - 3^2 - (8 - 3)$

(b) $6 + \frac{2}{3}(-9) - \frac{5}{8} \cdot 16$

7 Simplify.

(a) $\dfrac{10 - 6 + 2\sqrt{9}}{11 \cdot 2 - 3(2)^2}$

(b) $\dfrac{-4(8) + 6(3)}{3\sqrt{49} - \frac{1}{2}(42)}$

ANSWERS

6. (a) -8 **(b)** -10

7. (a) 1 **(b)** undefined

8 Evaluate each expression if $w = 4$, $x = -12$, $y = 64$, and $z = -3$.

(a) $5x - 2w$

(b) $-6(x - \sqrt{y})$

(c) $\dfrac{5x - 3 \cdot \sqrt{y}}{x - 1}$

(d) $w^2 + 2z^3$

9 Use the expression in Example 9 to approximate the percent of U.S. households investing in mutual funds in 1990 and 2000. Round answers to the nearest tenth.

ANSWERS

8. **(a)** -68 **(b)** 120 **(c)** $\frac{84}{13}$ **(d)** -38

9. 1990: 26.0%; 2000: 48.0%

EXAMPLE 8 Evaluating Expressions

Evaluate each expression if $m = -4$, $n = 5$, $p = -6$, and $q = 25$.

(a) $5m - 9n = 5(-4) - 9(5) = -20 - 45 = -65$ Replace m with -4 and n with 5.

(b) $\dfrac{m + 2n}{4p} = \dfrac{-4 + 2(5)}{4(-6)} = \dfrac{-4 + 10}{-24} = \dfrac{6}{-24} = -\dfrac{1}{4}$

(c)
$$\begin{aligned} -3m^3 - n^2(\sqrt{q}) &= -3(-4)^3 - (5)^2(\sqrt{25}) && \text{Substitute; } m = -4,\ n = 5, \text{ and } q = 25. \\ &= -3(-64) - 25(5) && \text{Evaluate powers and roots.} \\ &= 192 - 125 && \text{Multiply.} \\ &= 67 && \text{Subtract.} \end{aligned}$$

CAUTION

To avoid errors when evaluating expressions, ***use parentheses around any negative numbers that are substituted for variables.***

Work Problem 8 at the Side.

EXAMPLE 9 Evaluating an Expression to Approximate Mutual Fund Investors

An approximation of the percent of U.S. households investing in mutual funds during the years 1980 through 2000 can be obtained by substituting a given year for x in the expression

$$2.2023x - 4356.6$$

and then evaluating. (*Source:* Investment Company Institute.)

(a) Approximate the percent of U.S. households investing in mutual funds in 1980. Round to the nearest tenth.

$$\begin{aligned} 2.2023x - 4356.6 &= 2.2023(1980) - 4356.6 && \text{Let } x = 1980. \\ &\approx 4.0 && \text{Use a calculator.} \end{aligned}$$

Recall that the symbol ≈ means "is approximately equal to." In 1980, about 4.0% of U.S. households invested in mutual funds.

Work Problem 9 at the Side.

(b) Give the results found above and in Problem 9 at the side in a table. How has the percent of households investing in mutual funds changed during these years?

The table follows. The percent of U.S. households investing in mutual funds increased dramatically during these years.

Year	Percent of U.S. Households Investing in Mutual Funds
1980	4.0
1990	26.0
2000	48

Percent in 2000 is twelve times percent in 1980.

1.4 Properties of Real Numbers

OBJECTIVES

1. Use the distributive property.
2. Use the inverse properties.
3. Use the identity properties.
4. Use the commutative and associative properties.
5. Use the multiplication property of 0.

The study of any object is simplified when we know the properties of the object. For example, a property of water is that it freezes when cooled to 0°C. Knowing this helps us to predict the behavior of water.

The study of numbers is no different. The basic properties of real numbers reflect results that occur consistently in work with numbers, so they have been generalized to apply to expressions with variables as well.

OBJECTIVE 1 Use the distributive property. Notice that

$$2(3 + 5) = 2 \cdot 8 = 16$$

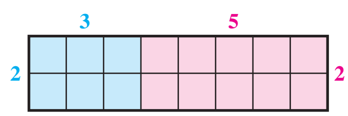
and

$$2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16,$$

so

$$2(3 + 5) = 2 \cdot 3 + 2 \cdot 5.$$

This idea is illustrated by the divided rectangle in Figure 10.



Area of left part is $2 \cdot 3 = 6$.
Area of right part is $2 \cdot 5 = 10$.
Area of total rectangle is $2(3 + 5) = 16$.

Figure 10

Similarly,

$$-4[5 + (-3)] = -4(2) = -8$$

and

$$-4(5) + (-4)(-3) = -20 + 12 = -8,$$

so

$$-4[5 + (-3)] = -4(5) + (-4)(-3).$$

These arithmetic examples are generalized to *all* real numbers as the **distributive property of multiplication with respect to addition,** or simply the **distributive property.**

Distributive Property

For any real numbers a, b, and c,

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca.$$

The distributive property can also be written

$$ab + ac = a(b + c) \quad \text{and} \quad ba + ca = (b + c)a.$$

It can be extended to more than two numbers as well.

$$a(b + c + d) = ab + ac + ad$$

This property is important because it provides a way to rewrite a *product* $a(b + c)$ as a *sum* $ab + ac$, or a *sum* as a *product.*

NOTE
When we rewrite $a(b + c)$ as $ab + ac$, we sometimes refer to the process as "removing" or "clearing" parentheses.

EXAMPLE 1 Using the Distributive Property

Use the distributive property to rewrite each expression.

(a) $3(x + y)$
Use the first form of the property.

$$3(x + y) = 3x + 3y$$

(b) $-2(5 + k) = -2(5) + (-2)(k)$
$= -10 - 2k$

(c) $4x + 8x$
Use the second form of the property.

$$4x + 8x = (4 + 8)x = 12x$$

(d) $3r - 7r = 3r + (-7r)$ Definition of subtraction
$= [3 + (-7)]r$ Distributive property
$= -4r$

(e) $5p + 7q$
Because there is no common number or variable here, we cannot use the distributive property to rewrite the expression.

(f) $6(x + 2y - 3z) = 6x + 6(2y) + 6(-3z)$
$= 6x + 12y - 18z$

As illustrated in Example 1(d), the distributive property can also be used for subtraction, so

$$a(b - c) = ab - ac.$$

Work Problem 1 at the Side.

The distributive property can be used to mentally perform calculations.

EXAMPLE 2 Using the Distributive Property for Calculation

Calculate $38 \cdot 17 + 38 \cdot 3$.

$$38 \cdot 17 + 38 \cdot 3 = 38(17 + 3) \quad \text{Distributive property}$$
$$= 38(20)$$
$$= 760$$

Work Problem 2 at the Side.

OBJECTIVE 2 Use the inverse properties. In **Section 1.1** we saw that the additive inverse of a number a is $-a$ and that the sum of a number and its additive inverse is 0. For example, 3 and -3 are additive inverses, as are -8 and 8. The number 0 is its own additive inverse.

1 Use the distributive property to rewrite each expression.

(a) $8(m + n)$

(b) $-4(p - 5)$

(c) $3k + 6k$

(d) $-6m + 2m$

(e) $2r + 3s$

(f) $5(4p - 2q + r)$

2 Use the distributive property to calculate each expression.

(a) $14 \cdot 5 + 14 \cdot 85$

(b) $78 \cdot 33 + 22 \cdot 33$

ANSWERS
1. **(a)** $8m + 8n$ **(b)** $-4p + 20$ **(c)** $9k$ **(d)** $-4m$ **(e)** cannot be rewritten **(f)** $20p - 10q + 5r$
2. **(a)** 1260 **(b)** 3300

In **Section 1.2,** we saw that two numbers with a product of 1 are reciprocals. As mentioned there, another name for reciprocal is *multiplicative inverse*. This is similar to the idea of an additive inverse. Thus, 4 and $\frac{1}{4}$ are multiplicative inverses, as are $-\frac{2}{3}$ and $-\frac{3}{2}$. (Recall that reciprocals have the same sign.) We can extend these properties of arithmetic, the **inverse properties** of addition and multiplication, to the real numbers of algebra.

Inverse Properties

For any real number a, there is a single real number $-a$ such that

$$\boldsymbol{a + (-a) = 0} \quad \text{and} \quad \boldsymbol{-a + a = 0.}$$

The inverse "undoes" addition with the result 0.

For any *nonzero* real number a, there is a single real number $\frac{1}{a}$ such that

$$\boldsymbol{a \cdot \frac{1}{a} = 1} \quad \text{and} \quad \boldsymbol{\frac{1}{a} \cdot a = 1.}$$

The inverse "undoes" multiplication with the result 1.

Work Problem 3 at the Side.

OBJECTIVE 3 **Use the identity properties.** The number 0 can be added to any number to get that number. That is, adding 0 leaves the identity of a number unchanged. Thus, 0 is the **identity element for addition** or the **additive identity.** Similarly, multiplying by 1 leaves the identity of any number unchanged, so 1 is the **identity element for multiplication** or the **multiplicative identity.** The following **identity properties** summarize this discussion and extend these properties from arithmetic to algebra.

Identity Properties

For any real number a, $\quad \boldsymbol{a + 0 = 0 + a = a.}$

Start with a number a; add 0. The answer is "identical" to a.

Also, $\quad \boldsymbol{a \cdot 1 = 1 \cdot a = a.}$

Start with a number a; multiply by 1. The answer is "identical" to a.

EXAMPLE 3 **Using the Identity Property $1 \cdot a = a$**

Simplify each expression.

(a) $12m + m = 12m + 1m$ Identity property

$= (12 + 1)m$ Distributive property

$= 13m$ Add inside parentheses.

(b) $y + y = 1y + 1y$ Identity property

$= (1 + 1)y$ Distributive property

$= 2y$ Add inside parentheses.

(c) $-(m - 5n) = -1(m - 5n)$ Identity property

$= -1(m) + (-1)(-5n)$ Distributive property

$= -m + 5n$ Multiply.

Work Problem 4 at the Side.

3 Complete each statement.

(a) $4 + \underline{\qquad} = 0$

(b) $-7.1 + \underline{\qquad} = 0$

(c) $-9 + 9 = \underline{\qquad}$

(d) $5 \cdot \underline{\qquad} = 1$

(e) $-\frac{3}{4} \cdot \underline{\qquad} = 1$

(f) $7 \cdot \frac{1}{7} = \underline{\qquad}$

4 Simplify each expression.

(a) $p - 3p$

(b) $r + r + r$

(c) $-(3 + 4p)$

(d) $-(k - 2)$

ANSWERS

3. **(a)** -4 **(b)** 7.1 **(c)** 0 **(d)** $\frac{1}{5}$ **(e)** $-\frac{4}{3}$ **(f)** 1

4. **(a)** $-2p$ **(b)** $3r$ **(c)** $-3 - 4p$ **(d)** $-k + 2$

Expressions such as $12m$ and $5n$ from Example 3 are examples of *terms*. A **term** is a number or the product of a number and one or more variables. Terms with exactly the same variables raised to exactly the same powers are called **like terms.** Some examples of like terms are

$$5p \text{ and } -21p \qquad -6x^2 \text{ and } 9x^2. \qquad \text{Like terms}$$

Some examples of unlike terms are

$$3m \text{ and } 16x \qquad 7y^3 \text{ and } -3y^2. \qquad \text{Unlike terms}$$

The numerical factor in a term is called the **numerical coefficient,** or just the **coefficient.** For example, in the term $9x^2$, the coefficient is 9.

OBJECTIVE 4 Use the commutative and associative properties. Simplifying expressions as in parts (a) and (b) of Example 3 is called **combining like terms.** Only like terms may be combined. To combine like terms in an expression such as

$$-2m + 5m + 3 - 6m + 8,$$

we need two more properties. From arithmetic, we know that

$$3 + 9 = 12 \quad \text{and} \quad 9 + 3 = 12.$$

Also,

$$3 \cdot 9 = 27 \quad \text{and} \quad 9 \cdot 3 = 27.$$

Furthermore, notice that

$$(5 + 7) + (-2) = 12 + (-2) = 10$$

and

$$5 + [7 + (-2)] = 5 + 5 = 10.$$

Also,

$$(5 \cdot 7)(-2) = 35(-2) = -70$$

and

$$(5)[7 \cdot (-2)] = 5(-14) = -70.$$

These arithmetic examples can now be extended to algebra.

Audio

Commutative and Associative Properties

For any real numbers a, b, and c,

$$\left.\begin{aligned} a + b &= b + a \\ \text{and} \qquad ab &= ba. \end{aligned}\right\} \text{Commutative properties}$$

Interchange the order of the two terms or factors.

$$\left.\begin{aligned} \text{Also,} \qquad a + (b + c) &= (a + b) + c \\ \text{and} \qquad a(bc) &= (ab)c. \end{aligned}\right\} \text{Associative properties}$$

Shift parentheses among the three terms or factors; order stays the same.

The commutative properties are used to change the *order* of the terms or factors in an expression. Think of *commuting* from home to work and then from work to home. The associative properties are used to *regroup* the terms or factors of an expression. Remember, to *associate* is to be part of a group.

EXAMPLE 4 Using the Commutative and Associative Properties

Simplify $-2m + 5m + 3 - 6m + 8$.

$-2m + 5m + 3 - 6m + 8$

$= (-2m + 5m) + 3 - 6m + 8$ Order of operations

$= (-2 + 5)m + 3 - 6m + 8$ Distributive property

$= 3m + 3 - 6m + 8$

By the order of operations, the next step would be to add $3m$ and 3, but they are unlike terms. To get $3m$ and $-6m$ together, use the associative and commutative properties. Begin by inserting parentheses and brackets according to the order of operations.

$[(3m + 3) - 6m] + 8$

$= [3m + (3 - 6m)] + 8$ Associative property

$= [3m + (-6m + 3)] + 8$ Commutative property

$= [(3m + [-6m]) + 3] + 8$ Associative property

$= (-3m + 3) + 8$ Combine like terms.

$= -3m + (3 + 8)$ Associative property

$= -3m + 11$ Add.

In practice, many of the steps are not written down, but you should realize that the commutative and associative properties are used whenever the terms in an expression are rearranged and regrouped to combine like terms.

EXAMPLE 5 Using the Properties of Real Numbers

Simplify each expression.

(a) $5y - 8y - 6y + 11y$

$= (5 - 8 - 6 + 11)y$ Distributive property

$= 2y$ Combine like terms.

(b) $3x + 4 - 5(x + 1) - 8$

$= 3x + 4 - 5x - 5 - 8$ Distributive property

$= 3x - 5x + 4 - 5 - 8$ Commutative property

$= -2x - 9$ Combine like terms.

(c) $8 - (3m + 2) = 8 - 1(3m + 2)$ Identity property

$= 8 - 3m - 2$ Distributive property

$= 6 - 3m$ Combine like terms.

(d) $(3x)(5)(y) = [(3x)(5)]y$ Order of operations

$= [3(x \cdot 5)]y$ Associative property

$= [3(5x)]y$ Commutative property

$= [(3 \cdot 5)x]y$ Associative property

$= (15x)y$ Multiply.

$= 15(xy)$ Associative property

$= 15xy$

As previously mentioned, many of these steps are not usually written out.

Work Problem 5 at the Side.

5 Simplify each expression.

(a) $12b - 9b + 4b - 7b + b$

(b) $-3w + 7 - 8w - 2$

(c) $-3(6 + 2t)$

(d) $9 - 2(a - 3) + 4 - a$

(e) $(4m)(2n)$

ANSWERS

5. **(a)** b **(b)** $-11w + 5$ **(c)** $-18 - 6t$ **(d)** $19 - 3a$ **(e)** $8mn$

CAUTION

Be careful. Notice that the distributive property does not apply in Example 5(d), because there is no addition involved.

$$(3x)(5)(y) \neq (3x)(5) \cdot (3x)(y)$$

OBJECTIVE 5 Use the multiplication property of 0. The additive identity property gives a special property of 0, namely that $a + 0 = a$ for any real number a. The **multiplication property of 0** gives a special property of 0 that involves multiplication: The product of any real number and 0 is 0.

Multiplication Property of 0

For any real number a,

$$\boldsymbol{a \cdot 0 = 0} \quad \text{and} \quad \boldsymbol{0 \cdot a = 0.}$$

Work Problem 6 at the Side.

6 Complete each statement.

(a) $197 \cdot 0 =$ _______

(b) $0\left(-\frac{8}{9}\right) =$ _______

(c) $0 \cdot$ _______ $= 0$

ANSWERS

6. (a) 0 **(b)** 0 **(c)** any real number

Linear Equations and Applications

2

Television, first operational in the 1940s, has become the most widespread form of communication in the world. In 2003, 106.7 million homes, 98% of all U.S. households, owned at least one TV set, and average viewing time among all viewers exceeded 30 hours per week. Favorite prime-time television programs were *CSI* and *Friends*, which concluded a highly successful 10-year run with a finale episode on May 6, 2004. (*Source:* Nielsen Media Research; *Microsoft Encarta Encyclopedia 2002*.)

In Section 2.2 we discuss the concept of *percent*—one of the most common everyday applications of mathematics—and use it in Exercises 45–50 to determine additional information about televisions in U.S. households.

2.1 Linear Equations in One Variable

OBJECTIVES

1. **Decide whether a number is a solution of a linear equation.**
2. **Solve linear equations using the addition and multiplication properties of equality.**
3. **Solve linear equations using the distributive property.**
4. **Solve linear equations with fractions or decimals.**
5. **Identify conditional equations, contradictions, and identities.**

Study Skills Workbook
Activity 2: Your Textbook

Audio

1 Are the given numbers solutions of the given equations?

(a) $3k = 15$; 5

(b) $r + 5 = 4$; 1

(c) $-8m = 12$; $\frac{3}{2}$

ANSWERS
1. **(a)** yes **(b)** no **(c)** no

In the previous chapter we began to use *algebraic expressions*. Some examples of algebraic expressions are

$$8x + 9, \quad y - 4, \quad \text{and} \quad \frac{x^3y^8}{z}. \qquad \text{Algebraic expressions}$$

Equations and inequalities compare algebraic expressions, just as a balance scale compares the weights of two quantities. Many applications of mathematics lead to *equations,* statements that two algebraic expressions are equal. A *linear equation in one variable* involves only real numbers and one variable raised to the first power. Examples are

$$x + 1 = -2, \quad x - 3 = 5, \quad \text{and} \quad 2k + 5 = 10. \qquad \text{Linear equations}$$

It is important to be able to distinguish between algebraic expressions and equations. ***An equation always contains an equals sign, while an expression does not.***

Linear Equation in One Variable

A **linear equation in one variable** can be written in the form

$$Ax + B = C,$$

where A, B, and C are real numbers, with $A \neq 0$.

A linear equation is also called a **first-degree equation** since the greatest power on the variable is one. Some examples of equations that are not linear (that is, *nonlinear*) are

$$x^2 + 3y = 5, \quad \frac{8}{x} = -22, \quad \text{and} \quad \sqrt{x} = 6. \qquad \text{Nonlinear equations}$$

OBJECTIVE 1 Decide whether a number is a solution of a linear equation. If the variable in an equation can be replaced by a real number that makes the statement true, then that number is a **solution** of the equation. For example, 8 is a solution of the equation $x - 3 = 5$, since replacing x with 8 gives a true statement, $8 - 3 = 5$. An equation is *solved* by finding its **solution set,** the set of all solutions. The solution set of the equation $x - 3 = 5$ is $\{8\}$.

Work Problem 1 at the Side.

Equivalent equations are equations that have the same solution set. To solve an equation, we usually start with the given equation and replace it with a series of simpler equivalent equations. For example,

$$5x + 2 = 17, \quad 5x = 15, \quad \text{and} \quad x = 3 \qquad \text{Equivalent equations}$$

are all equivalent since each has the solution set $\{3\}$.

OBJECTIVE 2 Solve linear equations using the addition and multiplication properties of equality. Two important properties that are used in producing equivalent equations are the **addition property of equality** and the **multiplication property of equality.**

Audio

Addition and Multiplication Properties of Equality

Addition Property of Equality
For all real numbers A, B, and C, the equations

$$A = B \quad \text{and} \quad A + C = B + C$$

are equivalent.

In words, the same number may be added to each side of an equation without changing the solution set.

Multiplication Property of Equality
For all real numbers A and B, and for $C \neq 0$, the equations

$$A = B \quad \text{and} \quad AC = BC$$

are equivalent.

In words, each side of an equation may be multiplied by the same nonzero number without changing the solution set.

Because subtraction and division are defined in terms of addition and multiplication, respectively, these properties can be extended:

The same number may be subtracted from each side of an equation, and each side of an equation may be divided by the same nonzero number, without changing the solution set.

You Try It

Animation

Video

EXAMPLE 1 Using the Addition and Multiplication Properties to Solve a Linear Equation

Solve $4x - 2x - 5 = 4 + 6x + 3$.

The goal is to get x alone on one side of the equation. First, combine like terms on each side of the equation to get

$$2x - 5 = 7 + 6x.$$

Next, use the addition property to get the terms with x on the same side of the equation and the remaining terms (the numbers) on the other side. One way to do this is to first subtract $6x$ from each side.

$$\begin{aligned}
2x - 5 - 6x &= 7 + 6x - 6x && \text{Subtract } 6x.\\
-4x - 5 &= 7 && \text{Combine like terms.}\\
-4x - 5 + 5 &= 7 + 5 && \text{Add 5.}\\
-4x &= 12 && \text{Combine like terms.}\\
\frac{-4x}{-4} &= \frac{12}{-4} && \text{Divide by } -4.\\
x &= -3 && \text{Proposed solution}
\end{aligned}$$

Check by substituting -3 for x in the *original* equation.

Check:

$$\begin{aligned}
4x - 2x - 5 &= 4 + 6x + 3 && \text{Original equation}\\
4(-3) - 2(-3) - 5 &\stackrel{?}{=} 4 + 6(-3) + 3 && \text{Let } x = -3.\\
-12 + 6 - 5 &\stackrel{?}{=} 4 - 18 + 3 && \text{Multiply.}\\
-11 &= -11 && \text{True}
\end{aligned}$$

The true statement indicates that $\{-3\}$ is the solution set.

Work Problem 2 at the Side.

2 Solve and check.

(a) $3p + 2p + 1 = -24$

(b) $3p = 2p + 4p + 5$

(c) $4x + 8x = 17x - 9 - 1$

(d) $-7 + 3t - 9t = 12t - 5$

ANSWERS

2. (a) $\{-5\}$ **(b)** $\left\{-\frac{5}{3}\right\}$ **(c)** $\{2\}$ **(d)** $\left\{-\frac{1}{9}\right\}$

3 Solve and check.

(a) $5p + 4(3 - 2p) = 2 + p - 10$

(b) $3(z - 2) + 5z = 2$

(c) $-2 + 3(x + 4) = 8x$

(d) $6 - (4 + m) = 8m - 2(3m + 5)$

The steps to solve a linear equation in one variable are as follows.

Solving a Linear Equation in One Variable

Step 1 **Clear fractions.** Eliminate any fractions by multiplying each side by the least common denominator.

Step 2 **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms as needed.

Step 3 **Isolate the variable terms on one side.** Use the addition property to get all terms with variables on one side of the equation and all numbers on the other.

Step 4 **Isolate the variable.** Use the multiplication property to get an equation with just the variable (with coefficient 1) on one side.

Step 5 **Check.** Substitute the proposed solution into the original equation.

OBJECTIVE 3 Solve linear equations using the distributive property.

In Example 1 we did not use Step 1 or the distributive property in Step 2 as given in the box. Many equations require one or both of these steps.

EXAMPLE 2 Using the Distributive Property to Solve a Linear Equation

Solve $2(k - 5) + 3k = k + 6$.

Step 1 Since there are no fractions in this equation, Step 1 does not apply.

Step 2 Use the distributive property to simplify and combine terms on the left side of the equation.

$$2(k - 5) + 3k = k + 6$$

$$2k - 10 + 3k = k + 6 \qquad 2(k - 5) = 2(k) - 2(5) = 2k - 10$$

$$5k - 10 = k + 6 \qquad \text{Combine like terms.}$$

Step 3 Next, use the addition property of equality.

$$5k - 10 - k = k + 6 - k \qquad \text{Subtract } k.$$

$$4k - 10 = 6 \qquad \text{Combine like terms.}$$

$$4k - 10 + 10 = 6 + 10 \qquad \text{Add 10.}$$

$$4k = 16 \qquad \text{Combine like terms.}$$

Step 4 Use the multiplication property of equality to get just k on the left.

$$\frac{4k}{4} = \frac{16}{4} \qquad \text{Divide by 4.}$$

$$k = 4$$

Step 5 Check by substituting 4 for k in the original equation.

Check:

$$2(k - 5) + 3k = k + 6 \qquad \text{Original equation}$$

$$2(4 - 5) + 3(4) \stackrel{?}{=} 4 + 6 \qquad \text{Let } k = 4.$$

$$2(-1) + 12 \stackrel{?}{=} 10$$

$$10 = 10 \qquad \text{True}$$

The solution checks, so the solution set is $\{4\}$.

Work Problem 3 at the Side.

ANSWERS

3. (a) $\{5\}$ (b) $\{1\}$ (c) $\{2\}$ (d) $\{4\}$

NOTE
Notice in Examples 1 and 2 that the equals signs are aligned in columns. Do not use more than one equals sign in a horizontal line of work when solving an equation.

OBJECTIVE 4 Solve linear equations with fractions or decimals. When fractions or decimals appear as coefficients in equations, our work can be made easier if we multiply each side of the equation by the least common denominator (LCD) of all the fractions. This is an application of the multiplication property of equality, and it produces an equivalent equation with integer coefficients.

Video

EXAMPLE 3 Solving a Linear Equation with Fractions

Solve $\frac{x+7}{6}+\frac{2x-8}{2}=-4$.

Step 1 Start by eliminating the fractions. Multiply each side by the LCD, 6.

$$6\left(\frac{x+7}{6}+\frac{2x-8}{2}\right)=6(-4)$$

Step 2

$$\begin{aligned}
6\left(\frac{x+7}{6}\right)+6\left(\frac{2x-8}{2}\right)&=6(-4) && \text{Distributive property}\\
(x+7)+3(2x-8)&=-24 && \text{Multiply.}\\
x+7+3(2x)-3(8)&=-24 && \text{Distributive property}\\
x+7+6x-24&=-24 && \text{Multiply.}\\
7x-17&=-24 && \text{Combine like terms.}
\end{aligned}$$

Step 3

$$\begin{aligned}
7x-17+17&=-24+17 && \text{Add 17.}\\
7x&=-7 && \text{Combine like terms.}
\end{aligned}$$

Step 4

$$\begin{aligned}
\frac{7x}{7}&=\frac{-7}{7} && \text{Divide by 7.}\\
x&=-1
\end{aligned}$$

Step 5 Check by substituting -1 for x in the original equation.

$$\begin{aligned}
\frac{x+7}{6}+\frac{2x-8}{2}&=-4 && \text{Original equation}\\
\frac{-1+7}{6}+\frac{2(-1)-8}{2}&=-4 \quad ? && \text{Let } x=-1.\\
\frac{6}{6}+\frac{-10}{2}&=-4 \quad ?\\
1-5&=-4 \quad ?\\
-4&=-4 && \text{True}
\end{aligned}$$

The solution checks, so the solution set is $\{-1\}$.

Work Problem 4 at the Side.

4 Solve and check.

(a) $\frac{2p}{7}-\frac{p}{2}=-3$

(b) $\frac{k+1}{2}+\frac{k+3}{4}=\frac{1}{2}$

ANSWERS
4. (a) $\{14\}$ **(b)** $\{-1\}$

5 Solve and check.

(a) $.04x + .06(20 - x) = .05(50)$

(b) $.10(x - 6) + .05x = .06(50)$

In **Sections 2.2** and **2.3** we solve problems involving interest rates and concentrations of solutions. These problems involve percents that are converted to decimals. The equations that are used to solve such problems involve decimal coefficients. We can clear these decimals by multiplying by a power of 10, such as $10^1 = 10$, $10^2 = 100$, and so on, that will allow us to obtain integer coefficients.

You Try It

EXAMPLE 4 Solving a Linear Equation with Decimals

Solve $.06x + .09(15 - x) = .07(15)$.

Because each decimal number is given in hundredths, multiply each side of the equation by 100. A number can be multiplied by 100 by moving the decimal point two places to the right.

$$
\begin{aligned}
.06x + .09(15 - x) &= .07(15) && \\
.06x + .09(15 - x) &= .07(15) && \text{Multiply by 100.} \\
6x + 9(15 - x) &= 7(15) && \\
6x + 9(15) - 9(x) &= 7(15) && \text{Distributive property} \\
6x + 135 - 9x &= 105 && \text{Multiply.} \\
-3x + 135 &= 105 && \text{Combine like terms.} \\
-3x + 135 - 135 &= 105 - 135 && \text{Subtract 135.} \\
-3x &= -30 && \text{Combine like terms.} \\
\frac{-3x}{-3} &= \frac{-30}{-3} && \text{Divide by } -3. \\
x &= 10 &&
\end{aligned}
$$

Check by substituting 10 for x in the original equation.

$$
\begin{aligned}
\textit{Check:} \quad .06x + .09(15 - x) &= .07(15) && \text{Original equation} \\
.06(10) + .09(15 - 10) &\stackrel{?}{=} .07(15) && \text{Let } x = 10. \\
.06(10) + .09(5) &\stackrel{?}{=} .07(15) && \\
.6 + .45 &\stackrel{?}{=} 1.05 && \\
1.05 &= 1.05 && \text{True}
\end{aligned}
$$

The solution set is $\{10\}$.

Work Problem 5 at the Side.

NOTE

Because of space limitations, we will not always show the check when solving an equation. To be sure that your solution is correct, ***you should always check your work.***

OBJECTIVE 5 Identify conditional equations, contradictions, and identities. All of the preceding equations had solution sets containing one element; for example, $2(k - 5) + 3k = k + 6$ has solution set $\{4\}$. Some linear equations, however, have no solutions, while others have an infinite number of solutions. The table on the next page gives the names of these types of equations.

ANSWERS

5. **(a)** $\{-65\}$ **(b)** $\{24\}$

Type of Linear Equation	Number of Solutions	Indication When Solving
Conditional	One	Final line is $x =$ a number. (See Example 5(a).)
Contradiction	None; solution set $\emptyset$	Final line is false, such as $-15 = -20$. (See Example 5(c).)
Identity	Infinite; solution set {all real numbers}	Final line is true, such as $0 = 0$. (See Example 5(b).)

EXAMPLE 5 Recognizing Conditional Equations, Identities, and Contradictions

Solve each equation. Decide whether it is a *conditional equation,* an *identity,* or a *contradiction*.

(a)

$$\begin{aligned} 5x - 9 &= 4(x - 3) & \\ 5x - 9 &= 4x - 12 & \text{Distributive property} \\ 5x - 9 - 4x &= 4x - 12 - 4x & \text{Subtract } 4x. \\ x - 9 &= -12 & \text{Combine like terms.} \\ x - 9 + 9 &= -12 + 9 & \text{Add 9.} \\ x &= -3 & \end{aligned}$$

The solution set, $\{-3\}$, has only one element, so $5x - 9 = 4(x - 3)$ is a conditional equation.

(b) $5x - 15 = 5(x - 3)$

Use the distributive property to clear parentheses on the right side.

$$\begin{aligned} 5x - 15 &= 5(x - 3) & \\ 5x - 15 &= 5x - 15 & \text{Distributive property} \\ 5x - 15 - 5x + 15 &= 5x - 15 - 5x + 15 & \text{Subtract } 5x\text{; add 15.} \\ \mathbf{0} &= \mathbf{0} & \text{True} \end{aligned}$$

The final line, the *true* statement $0 = 0$, indicates that the solution set is {all real numbers}, and the equation $5x - 15 = 5(x - 3)$ is an identity. (Notice that the first step yielded $5x - 15 = 5x - 15$, which is true for all values of x. We could have identified the equation as an identity at that point.)

(c)

$$\begin{aligned} 5x - 15 &= 5(x - 4) & \\ 5x - 15 &= 5x - 20 & \text{Distributive property} \\ 5x - 15 - 5x &= 5x - 20 - 5x & \text{Subtract } 5x. \\ \mathbf{-15} &= \mathbf{-20} & \text{False} \end{aligned}$$

Since the result, $-15 = -20$, is *false,* the equation has no solution. The solution set is $\emptyset$, so the equation $5x - 15 = 5(x - 4)$ is a contradiction.

Work Problem 6 at the Side.

6 Solve each equation. Decide whether it is a *conditional equation,* an *identity,* or a *contradiction*. Give the solution set.

(a) $5(x + 2) - 2(x + 1) = 3x + 1$

(b) $\dfrac{x + 1}{3} + \dfrac{2x}{3} = x + \dfrac{1}{3}$

(c) $5(3x + 1) = x + 5$

ANSWERS

6. **(a)** contradiction; $\emptyset$
(b) identity; {all real numbers}
(c) conditional; $\{0\}$

2.2 Formulas

A **mathematical model** is an equation or inequality that describes a real situation. Models for many applied problems already exist; they are called *formulas*. A **formula** is a mathematical equation in which variables are used to describe a relationship. Some formulas that we will be using are

$$d = rt, \quad I = prt, \quad \text{and} \quad P = 2L + 2W. \qquad \text{Formulas}$$

A list of some common formulas used in algebra is given inside the covers of this book.

OBJECTIVES

1. Solve a formula for a specified variable.
2. Solve applied problems using formulas.
3. Solve percent problems.

OBJECTIVE 1 Solve a formula for a specified variable. In some applications, the appropriate formula may be solved for a different variable than the one to be found. For example, the formula $I = prt$ says that interest on a loan or investment equals principal (amount borrowed or invested) times rate (percent) times time at interest (in years). To determine how long it will take for an investment at a stated interest rate to earn a predetermined amount of interest, it would help to first solve the formula for t. This process is called **solving for a specified variable** or **solving a literal equation.**

The steps used in the following examples are very similar to those used in solving linear equations from **Section 2.1.** ***When you are solving for a specified variable, the key is to treat that variable as if it were the only one; treat all other variables like numbers (constants).***

1 Solve $I = prt$ for each given variable.

(a) p

(b) r

You Try It

Animation

EXAMPLE 1 Solving for a Specified Variable

Solve the formula $I = prt$ for t.

We solve this formula for t by treating I, p, and r as constants (having fixed values) and treating t as the only variable. We first write the formula so that the variable for which we are solving, t, is on the left side. Then we use the properties of the previous section as follows.

$$prt = I$$

$$(pr)t = I \qquad \text{Associative property}$$

$$\frac{(pr)t}{pr} = \frac{I}{pr} \qquad \text{Divide by } pr.$$

$$t = \frac{I}{pr}$$

The result is a formula for t, time in years.

Work Problem 1 at the Side.

ANSWERS

1. (a) $p = \frac{I}{rt}$ (b) $r = \frac{I}{pt}$

2 **(a)** Solve the formula

$$P = a + b + c$$

for a.

(b) Solve the formula

$$m = 2k + 3b$$

for k.

To solve for a specified variable, follow these steps.

Solving for a Specified Variable

Step 1 Transform so that all terms containing the specified variable are on one side of the equation and all terms without that variable are on the other side.

Step 2 If necessary, use the distributive property to combine the terms with the specified variable.* The result should be the product of a sum or difference and the variable.

Step 3 Divide each side by the factor that is the coefficient of the specified variable.

EXAMPLE 2 Solving for a Specified Variable

Solve the formula $P = 2L + 2W$ for W.

This formula gives the relationship between perimeter of a rectangle, P, length of the rectangle, L, and width of the rectangle, W. See Figure 1.

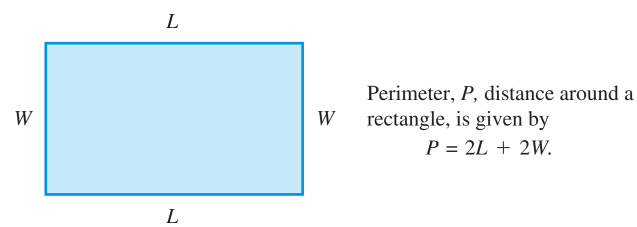


Figure 1

Solve the formula for W by isolating W on one side of the equals sign. To begin, subtract $2L$ from each side.

$$P = 2L + 2W$$

Step 1 $\quad P - 2L = 2L + 2W - 2L \quad$ Subtract $2L$.

$$P - 2L = 2W$$

Step 2 is not needed here.

Step 3 $\quad \dfrac{P - 2L}{2} = \dfrac{2W}{2} \quad$ Divide by 2.

$$\frac{P - 2L}{2} = W \quad \text{or} \quad W = \frac{P - 2L}{2}$$

Work Problem 2 at the Side.

CAUTION

In Step 3 of Example 2, you cannot simplify the fraction by dividing 2 into the term $2L$. The subtraction in the numerator must be done before the division.

$$\frac{P - 2L}{2} \neq P - L$$

ANSWERS

2. **(a)** $a = P - b - c$ **(b)** $k = \dfrac{m - 3b}{2}$

*Using the distributive property to write $ab + ac$ as $a(b + c)$ is called *factoring*. See **Chapter 7.**

EXAMPLE 3 Solving a Formula with Parentheses

The formula for the perimeter of a rectangle is sometimes written in the equivalent form $P = 2(L + W)$. Solve this form for W.

One way to begin is to use the distributive property on the right side of the equation to get $P = 2L + 2W$, which we would then solve as in Example 2. Another way to begin is to divide by the coefficient 2.

$$P = 2(L + W)$$

$$\frac{P}{2} = L + W \qquad \text{Divide by 2.}$$

$$\frac{P}{2} - L = W \quad \text{or} \quad W = \frac{P}{2} - L \qquad \text{Subtract } L.$$

We can show that this result is equivalent to our result in Example 2 by multiplying L by $\frac{2}{2}$.

$$\frac{P}{2} - \frac{2}{2}(L) = W \qquad \tfrac{2}{2} = 1, \text{ so } L = \tfrac{2}{2}(L).$$

$$\frac{P}{2} - \frac{2L}{2} = W$$

$$\frac{P - 2L}{2} = W \qquad \text{Subtract fractions.}$$

The final line agrees with the result in Example 2.

Work Problem 3 at the Side.

3 Solve the formula

$$y = \frac{1}{2}(x + 3)$$

for x.

A rectangular solid has the shape of a box, but is solid. See Figure 2. The labels H, W, and L represent the height, width, and length of the figure, respectively. The surface area of any solid three-dimensional figure is the total area of its surface. For a rectangular solid, the surface area A is

$$A = 2HW + 2LW + 2LH.$$

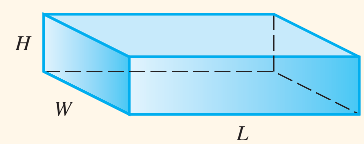


Figure 2

EXAMPLE 4 Using the Distributive Property to Solve for a Specified Variable

You Try It

Video

Given the surface area, height, and width of a rectangular solid, write a formula for the length.

To solve for the length L, treat L as the only variable and treat all other variables as constants.

$$A = 2HW + 2LW + 2LH$$

$$A - 2HW = 2LW + 2LH \qquad \text{Subtract } 2HW.$$

$$A - 2HW = L(2W + 2H) \qquad \text{Distributive property}$$

$$\frac{A - 2HW}{2W + 2H} = L \quad \text{or} \quad L = \frac{A - 2HW}{2W + 2H} \qquad \text{Divide by } 2W + 2H.$$

4 Solve the formula

$$A = 2HW + 2LW + 2LH$$

for W.

CAUTION

Be careful when working a problem like Example 4 to use the distributive property correctly. We must write the expression so that the specified variable is a ***factor;*** then we can divide by its coefficient in the final step.

Work Problem 4 at the Side.

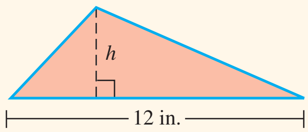
ANSWERS

3. $x = 2y - 3$

4. $W = \dfrac{A - 2LH}{2H + 2L}$

5 Solve each problem.

(a) A triangle has an area of 36 in.^{2} (square inches) and a base of 12 in. Find its height.



(b) The distance is 500 mi and the time is 20 hr. Find the rate.

(c) In 2003, Gil de Ferran won the Indianapolis 500 (mile) race with a speed of 156.291 mph. (*Source: World Almanac and Book of Facts*, 2004.) Find his time to the nearest thousandth.

ANSWERS

5. **(a)** 6 in. **(b)** 25 mph **(c)** 3.199 hr

OBJECTIVE 2 **Solve applied problems using formulas.** The distance formula, $d = rt$, relates d, the distance traveled, r, the rate or speed, and t, the travel time.

EXAMPLE 5 Finding Average Speed

Janet Branson found that on average it took her $\frac{3}{4}$ hr each day to drive a distance of 15 mi to work. What was her average speed?

Find the speed r by solving $d = rt$ for r.

$$d = rt$$

$$\frac{d}{t} = \frac{rt}{t} \qquad \text{Divide by } t.$$

$$\frac{d}{t} = r \quad \text{or} \quad r = \frac{d}{t}$$

Notice that only Step 3 was needed to solve for r in this example. Now find the speed by substituting the given values of d and t into this formula.

$$r = \frac{15}{\frac{3}{4}} \qquad \text{Let } d = 15,\ t = \tfrac{3}{4}.$$

$$r = 15 \cdot \frac{4}{3} \qquad \text{Multiply by the reciprocal of } \tfrac{3}{4}.$$

$$r = 20$$

Her average speed was 20 mph. (That is, at times she may have traveled a little faster or slower than 20 mph, but overall her speed was 20 mph.)

Work Problem 5 at the Side.

OBJECTIVE 3 **Solve percent problems.** An important everyday use of mathematics involves the concept of percent. Percent is written with the symbol %. The word **percent** means “per one hundred.” One percent means “one per one hundred” or “one one-hundredth.”

$$\mathbf{1\% = .01 \quad or \quad 1\% = \frac{1}{100}}$$

Solving a Percent Problem

Let a represent a partial amount of b, the base, or whole amount. Then the following formula can be used in solving a percent problem.

$$\frac{\textbf{amount}}{\textbf{base}} = \frac{a}{b} = \textbf{percent (represented as a decimal)}$$

For example, if a class consists of 50 students and 32 are males, then the percent of males in the class is

$$\frac{\text{amount}}{\text{base}} = \frac{a}{b} = \frac{32}{50} \qquad \text{Let } a = 32,\ b = 50.$$

$$= .64$$

$$= 64\%.$$

Video

EXAMPLE 6 Solving Percent Problems

(a) A 50-L mixture of acid and water contains 10 L of acid. What is the percent of acid in the mixture?

The given amount of the mixture is 50 L, and the part that is acid is 10 L. Let x represent the percent of acid. Then, the percent of acid in the mixture is

$$x = \frac{10}{50} = .20 \quad \text{or} \quad 20\%.$$

(b) If a savings account balance of \$3550 earns 8% interest in one year, how much interest is earned?

Let x represent the amount of interest earned (that is, the part of the whole amount invested). Since 8% = .08, the equation is

$$\frac{x}{3550} = .08 \qquad \frac{a}{b} = \text{percent}$$

$$x = .08(3550) \qquad \text{Multiply by 3550.}$$

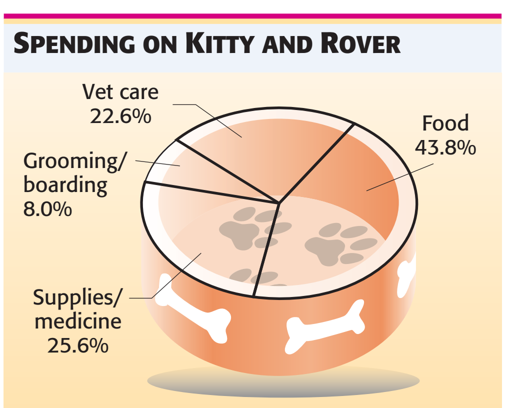
$$x = 284.$$

The interest earned is \$284.

Work Problem 6 at the Side.

EXAMPLE 7 Interpreting Percents from a Graph

In 2003, people in the United States spent an estimated \$29.7 billion on their pets. Use the graph in Figure 3 to determine how much of this amount was spent on pet food.



Source: American Pet Products Manufacturers Association Inc.

Figure 3

According to the graph, 43.8% was spent on food. Let x represent this amount in billions of dollars.

$$\frac{x}{29.7} = .438 \qquad 43.8\% = .438$$

$$x = .438(29.7) \qquad \text{Multiply by 29.7.}$$

$$x = 13.0 \qquad \text{Nearest tenth}$$

Therefore, about \$13.0 billion was spent on pet food.

Work Problem 7 at the Side.

6 Solve each problem.

(a) A mixture of gasoline and oil contains 20 oz, 1 oz of which is oil. What percent of the mixture is oil?

(b) An automobile salesman earns an 8% commission on every car he sells. How much does he earn on a car that sells for \$12,000?

7 Refer to Figure 3. How much was spent on pet supplies/medicine? Round your answer to the nearest tenth.

ANSWERS
6. **(a)** 5% **(b)** \$960
7. \$7.6 billion

2.3 Applications of Linear Equations

OBJECTIVES

1. Translate from words to mathematical expressions.
2. Write equations from given information.
3. Distinguish between expressions and equations.
4. Use the six steps in solving an applied problem.
5. Solve percent problems.
6. Solve investment problems.
7. Solve mixture problems.

OBJECTIVE 1 Translate from words to mathematical expressions. Producing a mathematical model of a real situation often involves translating verbal statements into mathematical statements. Although the problems we will be working with are simple ones, the methods we use will also apply to more difficult problems later.

Audio

PROBLEM-SOLVING HINT

Usually there are key words and phrases in a verbal problem that translate into mathematical expressions involving addition, subtraction, multiplication, and division. Translations of some commonly used expressions follow.

TRANSLATING FROM WORDS TO MATHEMATICAL EXPRESSIONS

Verbal Expression	Mathematical Expression (where x and y are numbers)
Addition	
The **sum** of a number and 7	$x + 7$
6 **more than** a number	$x + 6$
3 **plus** a number	$3 + x$
24 **added to** a number	$x + 24$
A number **increased by** 5	$x + 5$
The **sum** of two numbers	$x + y$
Subtraction	
2 **less than** a number	$x - 2$
12 **minus** a number	$12 - x$
A number **decreased by** 12	$x - 12$
A number **subtracted from** 10	$10 - x$
The **difference between** two numbers	$x - y$
Multiplication	
16 **times** a number	$16x$
A number **multiplied by** 6	$6x$
$\frac{2}{3}$ **of** a number (used with fractions and percent)	$\frac{2}{3}x$
Twice (2 times) a number	$2x$
The **product** of two numbers	xy
Division	
The **quotient** of 8 and a number	$\frac{8}{x}$ $(x \neq 0)$
A number **divided by** 13	$\frac{x}{13}$
The **ratio** of two numbers or the **quotient** of two numbers	$\frac{x}{y}$ $(y \neq 0)$

Work Problem 1 at the Side.

Study Skills Workbook
Activity 4: Note Taking

1 Translate each verbal expression as a mathematical expression. Use x as the variable.

(a) 9 added to a number

(b) The difference between 7 and a number

(c) Four times a number

(d) The quotient of 7 and a nonzero number

ANSWERS

1. (a) $9 + x$ or $x + 9$ (b) $7 - x$ (c) $4x$ (d) $\frac{7}{x}$

2 Translate each verbal sentence into an equation. Use x as the variable.

(a) The sum of a number and 6 is 28.

(b) If twice a number is decreased by 3, the result is 17.

(c) The product of a number and 7 is twice the number plus 12.

(d) The quotient of a number and 6, added to twice the number, is 7.

3 Decide whether each is an *expression* or an *equation*.

(a) $5x - 3(x + 2) = 7$

(b) $5x - 3(x + 2)$

Answers

2. **(a)** $x + 6 = 28$ **(b)** $2x - 3 = 17$ **(c)** $7x = 2x + 12$ **(d)** $\frac{x}{6} + 2x = 7$

3. **(a)** equation **(b)** expression

CAUTION

Because subtraction and division are not commutative operations, be careful to correctly translate expressions involving them. For example, "2 less than a number" is translated as $x - 2$, *not* $2 - x$. "A number subtracted from 10" is expressed as $10 - x$, *not* $x - 10$.

For division, the number *by which* we are dividing is the denominator, and the number *into which* we are dividing is the numerator. For example, "a number divided by 13" and "13 divided into x" both translate as $\frac{x}{13}$. Similarly, "the quotient of x and y" is translated as $\frac{x}{y}$.

OBJECTIVE 2 Write equations from given information. The symbol for equality, $=$, is often indicated by the word ***is***. In fact, any words that indicate the idea of "sameness" translate to $=$.

EXAMPLE 1 Translating Words into Equations

Translate each verbal sentence into an equation.

Verbal Sentence	Equation
Twice a number, **decreased by** 3, **is** 42.	$2x - 3 = 42$
If the **product of a number and 12** is decreased by 7, the result **is** 105.	$12x - 7 = 105$
The **quotient of a number and the number plus 4 is** 28.	$\frac{x}{x + 4} = 28$
The **quotient of a number and 4,** plus the number, **is** 10.	$\frac{x}{4} + x = 10$

Work Problem 2 at the Side.

OBJECTIVE 3 Distinguish between expressions and equations. An expression translates as a phrase. An equation includes the $=$ symbol and translates as a sentence.

EXAMPLE 2 Distinguishing between Expressions and Equations

Decide whether each is an *expression* or an *equation*.

(a) $2(3 + x) - 4x + 7$

There is no equals sign, so this is an expression.

(b) $2(3 + x) - 4x + 7 = -1$

Because of the equals sign, this is an equation.

Note that the expression in part (a) simplifies to the expression $-2x + 13$, and the equation in part (b) has solution 7.

Work Problem 3 at the Side.

OBJECTIVE 4 Use the six steps in solving an applied problem. While there is no one method that will allow us to solve all types of applied problems, the following six steps are helpful.*

*__Appendix A__ Strategies for Problem Solving introduces additional methods and tips for solving applied problems.

Solving an Applied Problem

Step 1 **Read** the problem, several times if necessary, until you *understand* what is given and what is to be found.

Step 2 **Assign a variable** to represent the unknown value, using diagrams or tables as needed. Write down what the variable represents. Express any other unknown values in terms of the variable.

Step 3 **Write an equation** using the variable expression(s).

Step 4 **Solve** the equation.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

EXAMPLE 3 Solving a Geometry Problem

The length of a rectangle is 1 cm more than twice the width. The perimeter of the rectangle is 110 cm. Find the length and the width of the rectangle.

Step 1 **Read** the problem. We must find the length and width of the rectangle. The length is 1 cm more than twice the width, and the perimeter is 110 cm.

Step 2 **Assign a variable.** Let W = the width; then $1 + 2W$ = the length. Make a sketch, as in Figure 4.

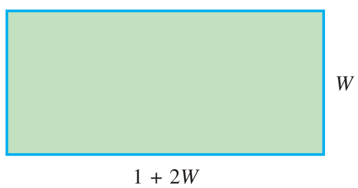


Figure 4

Step 3 **Write an equation.** The perimeter of a rectangle is given by the formula $P = 2L + 2W$.

$$P = 2L + 2W$$

$$110 = 2(1 + 2W) + 2W \qquad \text{Let } L = 1 + 2W \text{ and } P = 110.$$

Step 4 **Solve** the equation obtained in Step 3.

$$110 = 2 + 4W + 2W \qquad \text{Distributive property}$$

$$110 = 2 + 6W \qquad \text{Combine like terms.}$$

$$110 - 2 = 2 + 6W - 2 \qquad \text{Subtract 2.}$$

$$108 = 6W$$

$$\frac{108}{6} = \frac{6W}{6} \qquad \text{Divide by 6.}$$

$$18 = W$$

Step 5 **State the answer.** The width of the rectangle is 18 cm and the length is $1 + 2(18) = 37$ cm.

Step 6 **Check** the answer by substituting these dimensions into the words of the original problem.

Work Problem 4 at the Side.

4 Solve the problem.

The length of a rectangle is 5 cm more than its width. The perimeter is five times the width. What are the dimensions of the rectangle?

Answers

4. width: 10 cm; length: 15 cm

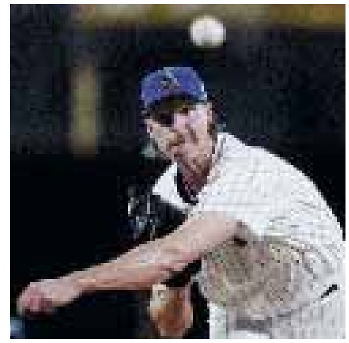
5 Solve the problem.

At the end of the 2003 baseball season, Sammy Sosa and Barry Bonds had a lifetime total of 1197 home runs. Bonds had 119 more than Sosa. How many home runs did each player have? (*Source: World Almanac and Book of Facts*, 2004.)

EXAMPLE 4 Finding Unknown Numerical Quantities

Animation

Two outstanding major league pitchers in recent years are Randy Johnson and Pedro Martinez. In 2002, they combined for a total of 573 strikeouts. Johnson had 95 more strikeouts than Martinez. How many strikeouts did each pitcher have? (*Source: World Almanac and Book of Facts*, 2004.)



Step 1 **Read** the problem. We are asked to find the number of strikeouts each pitcher had.

Step 2 **Assign a variable** to represent the number of strikeouts for one of the men.

Let s = the number of strikeouts for Pedro Martinez.

We must also find the number of strikeouts for Randy Johnson. Since he had 95 more strikeouts than Martinez,

$s + 95$ = the number of strikeouts for Johnson.

Step 3 **Write an equation.** The sum of the numbers of strikeouts is 573, so

$$\underset{s}{\underset{\downarrow}{\text{Martinez's strikeouts}}} \quad + \quad \underset{(s+95)}{\underset{\downarrow}{\text{Johnson's strikeouts}}} \quad = \quad \underset{573.}{\underset{\downarrow}{\text{Total}}}$$

Step 4 **Solve** the equation.

$$\begin{aligned} s + (s + 95) &= 573 \\ 2s + 95 &= 573 && \text{Combine like terms.} \\ 2s + 95 - \mathbf{95} &= 573 - \mathbf{95} && \text{Subtract 95.} \\ 2s &= 478 \\ \frac{2s}{\mathbf{2}} &= \frac{478}{\mathbf{2}} && \text{Divide by 2.} \\ s &= 239 \end{aligned}$$

Step 5 **State the answer.** We let s represent the number of strikeouts for Martinez, so Martinez had 239. Also,

$$\mathbf{s} + 95 = \mathbf{239} + 95 = 334$$

is the number of strikeouts for Johnson.

Step 6 **Check.** 334 is 95 more than 239, and the sum of 239 and 334 is 573. The conditions of the problem are satisfied, and our answer checks.

CAUTION

A common error in solving applied problems is forgetting to answer all the questions asked in the problem. In Example 4, we were asked for the number of strikeouts *each* player had, so there was an extra step at the end in order to find the number Johnson had.

Work Problem 5 at the Side.

ANSWERS

5. Sosa: 539; Bonds: 658

OBJECTIVE 5 Solve percent problems. Recall from **Section 2.2** that percent means "per one hundred," so 5% means .05, 14% means .14, and so on.

EXAMPLE 5 Solving a Percent Problem

In 2002 there were 301 long-distance area codes in the United States. This was an increase of 250% over the number when the area code plan originated in 1947. How many area codes were there in 1947? (*Source:* SBC Telephone Directory.)

Step 1 **Read** the problem. We are given that the number of area codes increased by 250% from 1947 to 2002, and there were 301 area codes in 2002. We must find the original number of area codes.

Step 2 **Assign a variable.** Let x represent the number of area codes in 1947.

$$250\% = 250(.01) = 2.5,$$

so $2.5x$ represents the number of codes added since then.

Step 3 **Write an equation** from the given information.

$$\begin{array}{ccccc} \text{the number in 1947} & + & \text{the increase} & = & 301 \\ \downarrow & & \downarrow & & \downarrow \\ x & + & 2.5x & = & 301 \end{array}$$

Step 4 **Solve** the equation.

$$\begin{aligned} 1x + 2.5x &= 301 && \text{Identity property} \\ 3.5x &= 301 && \text{Combine like terms.} \\ x &= 86 && \text{Divide by 3.5.} \end{aligned}$$

Step 5 **State the answer.** There were 86 area codes in 1947.

Step 6 **Check** that the increase, $301 - 86 = 215$, is 250% of 86.

CAUTION

Avoid two common errors that occur in solving problems like the one in Example 5.

1. Do not try to find 250% of 301 and subtract that amount from 301. The 250% should be applied to *the amount in 1947, not the amount in 2002*.
2. Do not write the equation as

$$x + 2.5 = 301. \qquad \text{Incorrect}$$

The percent must be multiplied by some amount; in this case, the amount is the number of area codes in 1947, giving $2.5x$.

Work Problem 6 at the Side.

OBJECTIVE 6 Solve investment problems. We use linear equations to solve certain investment problems. The investment problems in this chapter deal with *simple interest*. In most real-world applications, *compound interest* (covered in a later chapter) is used.

6 Solve each problem.

(a) A number increased by 15% is 287.5. Find the number.

(b) Michelle Raymond was paid \$162 for a week's work at her part-time job after 10% deductions for taxes. How much did she make before the deductions were made?

ANSWERS

6. (a) 250 **(b)** \$180

EXAMPLE 6 Solving an Investment Problem

After winning the state lottery, Mark LeBeau has \$40,000 to invest. He will put part of the money in an account paying 4% interest and the remainder into stocks paying 6% interest. His accountant tells him that the total annual income from these investments should be \$2040. How much should he invest at each rate?

Step 1 **Read** the problem again. We must find the two amounts.

Step 2 **Assign a variable.**

Let $x =$ the amount to invest at 4%;

$40{,}000 - x =$ the amount to invest at 6%.

The formula for interest is $I = prt$. Here the time, t, is 1 year. Make a table to organize the given information.

Rate (as a decimal)	Principal	Interest
.04	x	$.04x$
.06	$40{,}000 - x$	$.06(40{,}000 - x)$
	40,000	2040 ← Totals

Step 3 **Write an equation.** The last column of the table gives the equation.

$$\underbrace{.04x}_{\text{interest at 4\%}} + \underbrace{.06(40{,}000 - x)}_{\text{interest at 6\%}} = \underbrace{2040}_{\text{total interest}}$$

Step 4 **Solve** the equation. We do so without clearing decimals.

$$.04x + .06(40{,}000) - .06x = 2040 \quad \text{Distributive property}$$
$$.04x + 2400 - .06x = 2040 \quad \text{Multiply.}$$
$$-.02x + 2400 = 2040 \quad \text{Combine like terms.}$$
$$-.02x = -360 \quad \text{Subtract 2400.}$$
$$x = 18{,}000 \quad \text{Divide by } -.02.$$

Step 5 **State the answer.** Mark should invest \$18,000 at 4%. At 6%, he should invest \$40,000 − \$18,000 = \$22,000.

Step 6 **Check** by finding the annual interest at each rate; they should total \$2040.

$$.04(\$18{,}000) = \$720 \quad \text{and} \quad .06(\$22{,}000) = \$1320$$
$$\$720 + \$1320 = \$2040, \quad \text{as required.}$$

Work Problem 7 at the Side.

PROBLEM-SOLVING HINT
In Example 6, we chose to let the variable represent the amount invested at 4%. Students often ask, "Can I let the variable represent the other unknown?" The answer is yes. The equation will be different, but in the end the two answers will be the same.

7 Solve each problem.

(a) A woman invests \$72,000 in two ways—some at 5% and some at 3%. Her total annual interest income is \$3160. Find the amount she invests at each rate.

(b) A man has \$34,000 to invest. He invests some at 5% and the balance at 4%. His total annual interest income is \$1545. Find the amount he invests at each rate.

ANSWERS
7. **(a)** \$50,000 at 5%; \$22,000 at 3%
(b) \$18,500 at 5%; \$15,500 at 4%

OBJECTIVE 7 Solve mixture problems. Mixture problems involving rates of concentration can be solved with linear equations.

EXAMPLE 7 Solving a Mixture Problem

A chemist must mix 8 L of a 40% acid solution with some 70% solution to get a 50% solution. How much of the 70% solution should be used?

Step 1 **Read** the problem. The problem asks for the amount of 70% solution to be used.

Step 2 **Assign a variable.** Let x = the number of liters of 70% solution to be used. The information in the problem is illustrated in Figure 5.

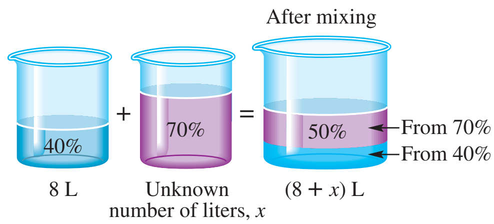


Figure 5

Use the given information to complete the following table.

Percent (as a decimal)	Number of Liters	Liters of Pure Acid
.40	8	$.40(8) = 3.2$
.70	x	$.70x$
.50	$8 + x$	$.50(8 + x)$

Sum must equal

The numbers in the right column were found by multiplying the strengths and the numbers of liters. The number of liters of pure acid in the 40% solution plus the number of liters of pure acid in the 70% solution must equal the number of liters of pure acid in the 50% solution.

Step 3 **Write an equation.**

$$3.2 + .70x = .50(8 + x)$$

Step 4 **Solve.**

$$3.2 + .70x = 4 + .50x \quad \text{Distributive property}$$
$$.20x = .8 \quad \text{Subtract 3.2 and } .50x.$$
$$x = 4 \quad \text{Divide by .20.}$$

Step 5 **State the answer.** The chemist should use 4 L of the 70% solution.

Step 6 **Check.** 8 L of 40% solution plus 4 L of 70% solution is

$$8(.40) + 4(.70) = \mathbf{6\ L}$$

of acid. Similarly, 8 + 4 or 12 L of 50% solution has

$$12(.50) = \mathbf{6\ L}$$

of acid in the mixture. The total amount of pure acid is 6 L both before and after mixing, so the answer checks.

Work Problem 8 at the Side.

8 Solve each problem.

(a) How many liters of a 10% solution should be mixed with 60 L of a 25% solution to get a 15% solution?

(b) How many pounds of candy worth \$8 per lb should be mixed with 100 lb of candy worth \$4 per lb to get a mixture that can be sold for \$7 per lb?

ANSWERS

8. (a) 120 L **(b)** 300 lb

9 Solve each problem.

(a) How much pure acid should be added to 6 L of 30% acid to increase the concentration to 50% acid?

(b) How much water must be added to 20 L of 50% antifreeze solution to reduce it to 40% antifreeze?

ANSWERS
9. **(a)** 2.4 L **(b)** 5 L

PROBLEM-SOLVING HINT
When pure water is added to a solution, remember that water is 0% of the chemical (acid, alcohol, etc.). Similarly, pure chemical is 100% chemical.

Video

EXAMPLE 8 Solving a Mixture Problem When One Ingredient Is Pure

The octane rating of gasoline is a measure of its antiknock qualities. For a standard fuel, the octane rating is the percent of isooctane. How many liters of pure isooctane should be mixed with 200 L of 94% isooctane, referred to as 94 octane, to get a mixture that is 98% isooctane?

Step 1 **Read** the problem. The problem asks for the amount of pure isooctane.

Step 2 **Assign a variable.** Let $x =$ the number of liters of pure (100%) isooctane. Complete a table with the given information. Recall that $100\% = 100(.01) = 1$.

Percent (as a decimal)	Number of Liters	Liters of Pure Isooctane
1	x	x
.94	200	$.94(200)$
.98	$x + 200$	$.98(x + 200)$

Step 3 **Write an equation.** The equation comes from the last column of the table, as in Example 7.

$$x + .94(200) = .98(x + 200)$$

Step 4 **Solve.**

$$\begin{aligned} x + .94(200) &= .98x + .98(200) && \text{Distributive property} \\ x + 188 &= .98x + 196 && \text{Multiply.} \\ .02x &= 8 && \text{Subtract } .98x \text{ and } 188. \\ x &= 400 && \text{Divide by } .02. \end{aligned}$$

Step 5 **State the answer.** 400 L of isooctane are needed.

Step 6 **Check** by showing that $400 + .94(200) = .98(400 + 200)$.

Work Problem 9 at the Side.

2.4 Further Applications of Linear Equations

There are three common applications of linear equations that we did not discuss in **Section 2.3**: money problems, uniform motion problems, and problems involving the angles of a triangle.

OBJECTIVE 1 Solve problems about different denominations of money. These problems are very similar to the simple interest problems in **Section 2.3.**

PROBLEM-SOLVING HINT
In problems involving money, use the fact that

denomination × number of monetary units of the same kind = total monetary value.

For example, 30 dimes have a monetary value of $.10(30) = $3. Fifteen five-dollar bills have a value of $5(15) = $75.

EXAMPLE 1 Solving a Money Denomination Problem

For a bill totaling $5.65, a cashier received 25 coins consisting of nickels and quarters. How many of each type of coin did the cashier receive?

Step 1 **Read** the problem. The problem asks that we find the number of nickels and the number of quarters the cashier received.

Step 2 **Assign a variable.**

Let x represent the number of nickels;
then $25 - x$ represents the number of quarters.

We can organize the information in a table.

Denomination	Number of Coins	Total Value
$.05	x	$.05x$
$.25	$25 - x$	$.25(25 - x)$
	25	5.65

← Totals

Step 3 **Write an equation.** From the last column of the table,

$$.05x + .25(25 - x) = 5.65.$$

Step 4 **Solve.**

$$5x + 25(25 - x) = 565 \quad \text{Multiply by 100.}$$
$$5x + 625 - 25x = 565 \quad \text{Distributive property}$$
$$-20x = -60 \quad \text{Subtract 625; combine terms.}$$
$$x = 3 \quad \text{Divide by } -20.$$

Step 5 **State the answer.** The cashier has 3 nickels and $25 - 3 = 22$ quarters.

Step 6 **Check.** The cashier has $3 + 22 = 25$ coins, and the value of the coins is $\$.05(3) + \$.25(22) = \$5.65$, as required.

Work Problem 1 at the Side.

OBJECTIVES

1. Solve problems about different denominations of money.
2. Solve problems about uniform motion.
3. Solve problems involving the angles of a triangle.

1 Solve the problem.
At the end of a day, a cashier had 26 coins consisting of dimes and half-dollars. The total value of these coins was $8.60. How many of each type did he have?

ANSWERS
1. 11 dimes, 15 half-dollars

CAUTION
Be sure that your answer is reasonable when working problems like Example 1. Because you are dealing with a number of coins, the correct answer can neither be negative nor a fraction.

OBJECTIVE 2 Solve problems about uniform motion.

PROBLEM-SOLVING HINT
Uniform motion problems use the distance formula, $d = rt$. In this formula, ***when rate (or speed) is given in miles per hour, time must be given in hours.*** To solve such problems, ***draw a sketch*** to illustrate what is happening in the problem, and ***make a table*** to summarize the given information.

EXAMPLE 2 Solving a Motion Problem (Motion in Opposite Directions)

Two cars leave the same place at the same time, one going east and the other west. The eastbound car averages 40 mph, while the westbound car averages 50 mph. In how many hours will they be 300 mi apart?

Step 1 **Read** the problem. We must find the time it takes for the two cars to be 300 mi apart.

Step 2 **Assign a variable.** A sketch shows what is happening in the problem: The cars are going in *opposite* directions. See Figure 6.

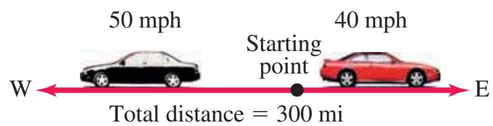


Figure 6

Let x represent the time traveled by each car. Organize the information in a table. ***Fill in each distance by multiplying rate by time*** using the formula $d = rt$. The sum of the two distances is 300.

	Rate	Time	Distance
Eastbound Car	40	x	$40x$
Westbound Car	50	x	$50x$
			300 ← Total

Step 3 **Write an equation.** $40x + 50x = 300$

Step 4 **Solve.**

$$90x = 300 \qquad \text{Combine like terms.}$$

$$x = \frac{300}{90} = \frac{10}{3} \qquad \text{Divide by 90; lowest terms}$$

Step 5 **State the answer.** The cars travel $\frac{10}{3} = 3\frac{1}{3}$ hr, or 3 hr and 20 min.

Step 6 **Check.** The eastbound car traveled $40\left(\frac{10}{3}\right) = \frac{400}{3}$ mi, and the westbound car traveled $50\left(\frac{10}{3}\right) = \frac{500}{3}$ mi, for a total of $\frac{400}{3} + \frac{500}{3} = \frac{900}{3} = 300$ mi, as required.

CAUTION
It is a common error to write 300 as the distance for *each* car in Example 2. Three hundred miles is the *total* distance traveled.

As in Example 2, in general, the equation for a problem involving motion in opposite directions is of the form

partial distance + partial distance = total distance.

Work Problem 2 at the Side.

EXAMPLE 3 Solving a Motion Problem (Motion in the Same Direction)

Jeff can bike to work in $\frac{3}{4}$ hr. By bus, the trip takes $\frac{1}{4}$ hr. If the bus travels 20 mph faster than Jeff rides his bike, how far is it to his workplace?

Step 1 **Read** the problem. We must find the distance between Jeff's home and his workplace.

Step 2 **Assign a variable.** Although the problem asks for a distance, it is easier here to let x be Jeff's speed when he rides his bike to work. Then the speed of the bus is $x + 20$. For the trip by bike,

$$d = rt = x \cdot \frac{3}{4} = \frac{3}{4}x,$$

and by bus,

$$d = rt = (x + 20) \cdot \frac{1}{4} = \frac{1}{4}(x + 20).$$

Summarize this information in a table.

	Rate	Time	Distance
Bike	x	$\frac{3}{4}$	$\frac{3}{4}x$
Bus	$x + 20$	$\frac{1}{4}$	$\frac{1}{4}(x + 20)$

Same

Step 3 **Write an equation.** The key to setting up the correct equation is to realize that the distance in each case is the same. See Figure 7.

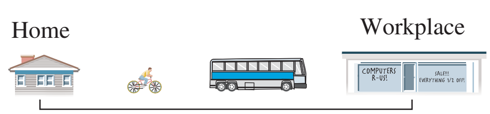


Figure 7

$$\frac{3}{4}x = \frac{1}{4}(x + 20) \qquad \text{The distance is the same.}$$

Step 4 **Solve.**

$$4\left(\frac{3}{4}x\right) = 4\left(\frac{1}{4}\right)(x + 20) \qquad \text{Multiply by 4.}$$

$$3x = x + 20 \qquad \text{Multiply; identity property}$$

$$2x = 20 \qquad \text{Subtract } x.$$

$$x = 10 \qquad \text{Divide by 2.}$$

Continued on Next Page

2 Solve the problem.

Two cars leave the same location at the same time. One travels north at 60 mph and the other south at 45 mph. In how many hours will they be 420 mi apart?

ANSWERS
2. 4 hr

3 Solve the problem.

Elayn begins jogging at 5:00 A.M., averaging 3 mph. Clay leaves at 5:30 A.M., following her, averaging 5 mph. How long will it take him to catch up to her? (*Hint:* 30 min $= \frac{1}{2}$ hr.)

4 Solve the problem.

One angle in a triangle is 15° larger than a second angle. The third angle is 25° larger than twice the second angle. Find the measure of each angle.

Step 5 **State the answer.** The required distance is given by

$$d = \frac{3}{4}x = \frac{3}{4}(10) = \frac{30}{4} = 7.5 \text{ mi.}$$

Step 6 **Check** by finding the distance using

$$d = \frac{1}{4}(x + 20) = \frac{1}{4}(10 + 20) = \frac{30}{4} = 7.5 \text{ mi,}$$

the same result.

As in Example 3, the equation for a problem involving motion in the same direction is often of the form

one distance = other distance.

PROBLEM-SOLVING HINT

In Example 3 it was easier to let the variable represent a quantity other than the one that we were asked to find. This is the case in some problems. It takes practice to learn when this approach is best, and practice means working lots of problems!

Work Problem 3 at the Side.

OBJECTIVE 3 Solve problems involving the angles of a triangle. An important result of Euclidean geometry (the geometry of the Greek mathematician Euclid) is that the sum of the angle measures of any triangle is 180°. This property is used in the next example.

EXAMPLE 4 Finding Angle Measures

Find the value of x, and determine the measure of each angle in Figure 8.

Step 1 **Read** the problem. We are asked to find the measure of each angle.

Step 2 **Assign a variable.** Let x represent the measure of one angle.

Step 3 **Write an equation.** The sum of the three measures shown in the figure must be 180°.

$$x + (x + 20) + (210 - 3x) = 180$$

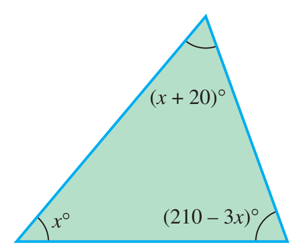


Figure 8

Step 4 **Solve.**

$$\begin{aligned} -x + 230 &= 180 && \text{Combine like terms.} \\ -x &= -50 && \text{Subtract 230.} \\ x &= 50 && \text{Divide by } -1. \end{aligned}$$

Step 5 **State the answer.** One angle measures 50°, another measures $x + 20 = 50 + 20 = 70°$, and the third measures $210 - 3x = 210 - 3(50) = 60°$.

Step 6 **Check.** Since $50° + 70° + 60° = 180°$, the answer is correct.

Work Problem 4 at the Side.

ANSWERS

3. $\frac{3}{4}$ hr or 45 min

4. 35°, 50°, and 95°

3.1 Linear Inequalities in One Variable

OBJECTIVES

1. Graph intervals on a number line.
2. Solve linear inequalities using the addition property.
3. Solve linear inequalities using the multiplication property.
4. Solve linear inequalities with three parts.
5. Solve applied problems using linear inequalities.

Study Skills Workbook
Activity 8: Study Cards

Solving inequalities is closely related to solving equations. In this section we introduce properties for solving inequalities.

Inequalities are algebraic expressions related by

$<$ "is less than,"
$\leq$ "is less than or equal to,"
$>$ "is greater than,"
$\geq$ "is greater than or equal to."

We solve an inequality by finding all real number solutions for it. For example, the solution set of $x \leq 2$ includes *all* real numbers that are less than or equal to 2, not just the integers less than or equal to 2. For example, -2.5, -1.7, -1, $\frac{1}{2}$, $\sqrt{2}$, $\frac{7}{4}$, and 2 are real numbers less than or equal to 2 and are therefore solutions of $x \leq 2$.

OBJECTIVE 1 Graph intervals on a number line. A good way to show the solution set of an inequality is by graphing. We graph all the real numbers satisfying $x \leq 2$ by placing a square bracket at 2 on a number line and drawing an arrow extending from the bracket to the left (to represent the fact that all numbers less than 2 are also part of the graph). The graph is shown in Figure 1.

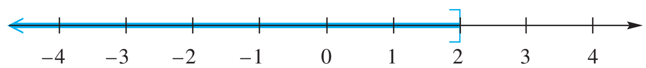


Figure 1

The set of numbers less than or equal to 2 is an example of an **interval** on the number line. To write intervals, we use **interval notation.** For example, using this notation, the interval of all numbers less than or equal to 2 is written $(-\infty, 2]$. The negative infinity symbol $-\infty$ does not indicate a number. It is used to show that the interval includes all real numbers less than 2. As on the number line, the square bracket indicates that 2 is included in the solution set. ***A parenthesis is always used next to the infinity symbol.*** The set of real numbers is written in interval notation as $(-\infty, \infty)$.

Video

EXAMPLE 1 Graphing Intervals Written in Interval Notation on Number Lines

Write each inequality in interval notation and graph it.

(a) $x > -5$

The statement $x > -5$ says that x can represent any number greater than -5, but x cannot equal -5. This interval is written $(-5, \infty)$. We show this solution set on a number line by placing a parenthesis at -5 and drawing an arrow to the right, as in Figure 2. The parenthesis at -5 shows that -5 is *not* part of the graph.

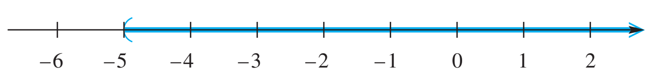


Figure 2

Continued on Next Page

(b) $-1 \leq x < 3$

This statement is read "-1 is less than or equal to x *and* x is less than 3." Thus, we want the set of numbers that are *between* -1 and 3, with -1 included and 3 excluded. In interval notation, we write the solution set as $[-1, 3)$, using a square bracket at -1 because it is part of the graph and a parenthesis at 3 because it is not part of the graph. The graph is shown in Figure 3.

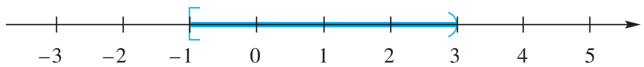


Figure 3

Work Problem 1 at the Side.

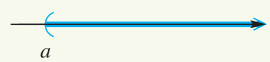
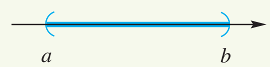
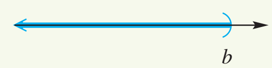
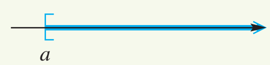
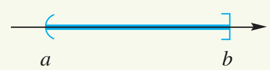
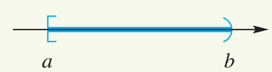
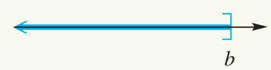
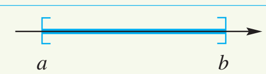
1 Write each inequality in interval notation and graph it.

(a) $x < -1$

(b) $x \geq -3$

(c) $-4 \leq x < 2$

We now summarize the various types of intervals.

Type of Interval	Set	Interval Notation	Graph
Open interval	$\{x \mid a < x\}$	(a, ∞)	
	$\{x \mid a < x < b\}$	(a, b)	
	$\{x \mid x < b\}$	$(-\infty, b)$	
	$\{x \mid x$ is a real number$\}$	$(-\infty, \infty)$	
Half-open interval	$\{x \mid a \leq x\}$	$[a, \infty)$	
	$\{x \mid a < x \leq b\}$	$(a, b]$	
	$\{x \mid a \leq x < b\}$	$[a, b)$	
	$\{x \mid x \leq b\}$	$(-\infty, b]$	
Closed interval	$\{x \mid a \leq x \leq b\}$	$[a, b]$	

An **inequality** says that two expressions are *not* equal. Solving inequalities is similar to solving equations.

Linear Inequality

A **linear inequality in one variable** can be written in the form

$$Ax + B < C,$$

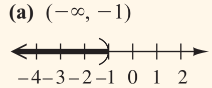
where A, B, and C are real numbers, with $A \neq 0$.

(Throughout this section we give definitions and rules only for $<$, but they are also valid for $>$, $\leq$, and $\geq$.) Examples of linear inequalities include

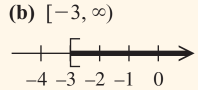
$$x + 5 < 2, \quad t - 3 \geq 5, \quad \text{and} \quad 2k + 5 \leq 10. \qquad \text{Linear inequalities}$$

ANSWERS

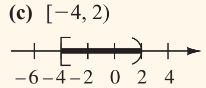
1. (a) $(-\infty, -1)$



(b) $[-3, \infty)$



(c) $[-4, 2)$



2 Solve each inequality, check your solutions, and graph the solution set.

(a) $p + 6 < 8$

(b) $8x < 7x - 6$

OBJECTIVE 2 Solve linear inequalities using the addition property. We solve an inequality by finding all numbers that make the inequality true. Usually, an inequality has an infinite number of solutions. These solutions, like solutions of equations, are found by producing a series of simpler equivalent inequalities. **Equivalent inequalities** are inequalities with the same solution set. We use the **addition** and **multiplication properties of inequality** to produce equivalent inequalities.

Addition Property of Inequality

For all real numbers A, B, and C, the inequalities

$$A < B \quad \text{and} \quad A + C < B + C$$

are equivalent.

In words, adding the same number to each side of an inequality does not change the solution set.

EXAMPLE 2 Using the Addition Property of Inequality

Solve $x - 7 < -12$, and graph the solution set.

$$x - 7 < -12$$
$$x - 7 + 7 < -12 + 7 \qquad \text{Add 7.}$$
$$x < -5$$

Check: Substitute -5 for x in the *equation* $x - 7 = -12$. The result should be a true statement.

$$x - 7 = -12$$
$$-5 - 7 \stackrel{?}{=} -12 \qquad \text{Let } x = -5.$$
$$-12 = -12 \qquad \text{True}$$

This shows that -5 is the boundary point. Now we test a number on each side of -5 to verify that numbers *less than* -5 make the *inequality* true. We choose -4 and -6.

$$x - 7 < -12$$

$-4 - 7 \stackrel{?}{<} -12$ Let $x = -4$.	$-6 - 7 \stackrel{?}{<} -12$ Let $x = -6$.
$-11 < -12$ False	$-13 < -12$ True
-4 is not in the solution set.	-6 is in the solution set.

The check confirms that $(-\infty, -5)$, graphed in Figure 4, is the solution set.

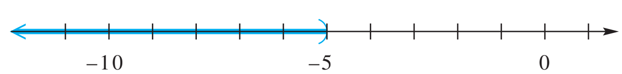


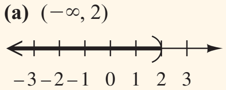
Figure 4

Work Problem 2 at the Side.

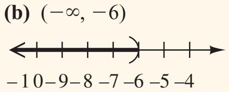
As with equations, the addition property of inequality can be used to *subtract* the same number from each side of an inequality. For example, to solve the inequality $x + 4 > 10$, we subtract 4 from each side to get $x > 6$.

ANSWERS

2. (a) $(-\infty, 2)$



(b) $(-\infty, -6)$



EXAMPLE 3 Using the Addition Property of Inequality

Solve $14 + 2m \leq 3m$, and graph the solution set.

$$14 + 2m \leq 3m$$
$$14 + 2m - \mathbf{2m} \leq 3m - \mathbf{2m} \qquad \text{Subtract } 2m.$$
$$14 \leq m \qquad \text{Combine like terms.}$$

The inequality $14 \leq m$ (14 is less than or equal to m) can also be written $m \geq 14$ (m is greater than or equal to 14). Notice that in each case, the inequality symbol points to the lesser number, 14.

Check:

$$14 + 2\mathbf{m} = 3\mathbf{m}$$
$$14 + 2(\mathbf{14}) \stackrel{?}{=} 3(\mathbf{14}) \qquad \text{Let } m = 14.$$
$$42 = 42 \qquad \text{True}$$

So 14 satisfies the equality part of $\leq$. Choose 10 and 15 as test points.

$$14 + 2\mathbf{m} < 3\mathbf{m}$$

$$14 + 2(\mathbf{10}) \stackrel{?}{<} 3(\mathbf{10}) \qquad \text{Let } m = 10.$$
$$34 < 30 \qquad \text{False}$$

10 is not in the solution set.

$$14 + 2(\mathbf{15}) \stackrel{?}{<} 3(\mathbf{15}) \qquad \text{Let } m = 15.$$
$$44 < 45 \qquad \text{True}$$

15 is in the solution set.

The check confirms that $[14, \infty)$ is the solution set. See Figure 5.

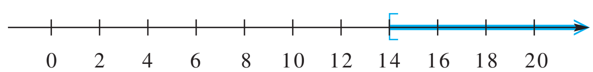


Figure 5

Work Problem 3 at the Side.

CAUTION

To avoid errors, rewrite an inequality such as $14 \leq m$ as $m \geq 14$ so that the variable is on the left, as in Example 3.

OBJECTIVE 3 Solve linear inequalities using the multiplication property. Solving an inequality such as $3x \leq 15$ requires dividing each side by 3 using the multiplication property of inequality. To see how this property works, start with the true statement

$$-2 < 5.$$

Multiply each side by, say, 8.

$$-2(\mathbf{8}) < 5(\mathbf{8}) \qquad \text{Multiply by 8.}$$
$$-16 < 40 \qquad \text{True}$$

The result is true. Start again with $-2 < 5$, and multiply each side by -8.

$$-2(\mathbf{-8}) < 5(\mathbf{-8}) \qquad \text{Multiply by } -8.$$
$$16 < -40 \qquad \text{False}$$

The result, $16 < -40$, is false. To make it true, we must change the direction of the inequality symbol to get

$$16 \mathbf{>} -40. \qquad \text{True}$$

Work Problem 4 at the Side.

3 Solve $2k - 5 \geq 1 + k$, check, and graph the solution set.

4 Multiply both sides of each inequality by -5. Then insert the correct symbol, either $<$ or $>$, in the first blank, and fill in the other blanks in part (b).

(a) $7 < 8$

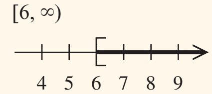
-35 ______ -40

(b) $-1 > -4$

5 ______ ______

ANSWERS

3. $[6, \infty)$



4. (a) $>$ **(b)** $<$; 20

5 Solve each inequality, check, and graph the solution set.

(a) $2x < -10$

(b) $-7k \geq 8$

(c) $-9m < -81$

As these examples suggest, multiplying each side of an inequality by a *negative* number reverses the direction of the inequality symbol. The same is true for dividing by a negative number since division is defined in terms of multiplication.

Multiplication Property of Inequality

For all real numbers A, B, and C, with $C \neq 0$,

(a) the inequalities

$$A < B \quad \text{and} \quad AC < BC$$

are equivalent **if $C > 0$;**

(b) the inequalities

$$A < B \quad \text{and} \quad AC > BC$$

are equivalent **if $C < 0$.**

In words, each side of an inequality may be multiplied (or divided) by a *positive* number without changing the direction of the inequality symbol. ***Multiplying (or dividing) by a negative number requires that we reverse the inequality symbol.***

Audio

You Try It

Video

EXAMPLE 4 Using the Multiplication Property of Inequality

Solve each inequality, and graph the solution set.

(a) $5m \leq -30$

Use the multiplication property to divide each side by 5. ***Since 5 > 0, do not reverse the inequality symbol.***

$$5m \leq -30$$

$$\frac{5m}{5} \leq \frac{-30}{5} \qquad \text{Divide by 5.}$$

$$m \leq -6$$

Check that the solution set is the interval $(-\infty, -6]$, graphed in Figure 6.

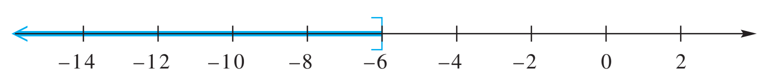


Figure 6

(b) $-4k \leq 32$

Divide each side by -4. ***Since −4 < 0, reverse the inequality symbol.***

$$-4k \leq 32$$

$$\frac{-4k}{-4} \geq \frac{32}{-4} \qquad \text{Divide by } -4 \text{ and reverse the symbol.}$$

$$k \geq -8$$

Check the solution set. Figure 7 shows the graph of the solution set, $[-8, \infty)$.

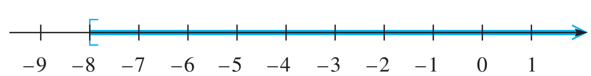
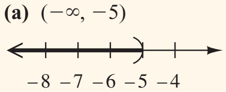


Figure 7

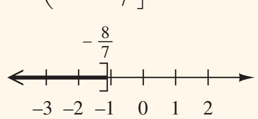
Work Problem 5 at the Side.

ANSWERS

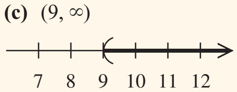
5. (a) $(-\infty, -5)$



(b) $\left(-\infty, -\frac{8}{7}\right]$



(c) $(9, \infty)$



The steps used in solving a linear inequality are given below.

Solving a Linear Inequality

Step 1 **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms as needed.

Step 2 **Isolate the variable terms on one side.** Use the addition property of inequality to get all terms with variables on one side of the inequality and all numbers on the other side.

Step 3 **Isolate the variable.** Use the multiplication property of inequality to change the inequality to the form $x < k$ or $x > k$.

CAUTION

Reverse the direction of the inequality symbol only when multiplying or dividing each side of an inequality by a negative number.

You Try It

EXAMPLE 5 Solving a Linear Inequality Using the Distributive Property

Solve $-3(x + 4) + 2 \geq 7 - x$, and graph the solution set.

Step 1 $-3x - 12 + 2 \geq 7 - x$ Distributive property

$-3x - 10 \geq 7 - x$

Step 2 $-3x - 10 + x \geq 7 - x + x$ Add x.

$-2x - 10 \geq 7$

$-2x - 10 + 10 \geq 7 + 10$ Add 10.

$-2x \geq 17$

Step 3 $\dfrac{-2x}{-2} \leq \dfrac{17}{-2}$ Divide by -2; change $\geq$ to $\leq$.

$x \leq -\dfrac{17}{2}$

Figure 8 shows the graph of the solution set, $(-\infty, -\frac{17}{2}]$.

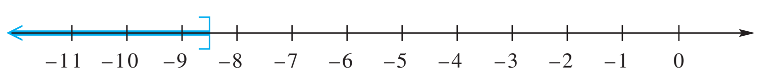


Figure 8

NOTE

In Example 5, if, after distributing, we add $3x$ to both sides of the inequality, we have

$-3x - 10 + 3x \geq 7 - x + 3x$ Add 3x.

$-10 \geq 2x + 7$

$-10 - 7 \geq 2x + 7 - 7$ Subtract 7.

$-17 \geq 2x$

$-\dfrac{17}{2} \geq x.$ Divide by 2.

The result "$-\frac{17}{2}$ is greater than or equal to x" means the same thing as "x is less than or equal to $-\frac{17}{2}$." Thus, the solution set is the same.

6 Solve, check, and graph the solution set of each inequality.

(a) $5 - 3(m - 1) \leq 2(m + 3) + 1$

(b) $\frac{1}{4}(m + 3) + 2 \leq \frac{3}{4}(m + 8)$

EXAMPLE 6 Solving a Linear Inequality with Fractions

Solve $-\frac{2}{3}(r - 3) - \frac{1}{2} < \frac{1}{2}(5 - r)$, and graph the solution set.

To clear fractions, multiply each side by the least common denominator, 6.

$$-\frac{2}{3}(r - 3) - \frac{1}{2} < \frac{1}{2}(5 - r)$$

$$\mathbf{6}\left[-\frac{2}{3}(r - 3) - \frac{1}{2}\right] < \mathbf{6}\left[\frac{1}{2}(5 - r)\right] \qquad \text{Multiply by 6.}$$

$$\mathbf{6}\left[-\frac{2}{3}(r - 3)\right] - \mathbf{6}\left(\frac{1}{2}\right) < \mathbf{6}\left[\frac{1}{2}(5 - r)\right] \qquad \text{Distributive property}$$

$$-4(r - 3) - 3 < 3(5 - r)$$

Step 1 $$-4r + 12 - 3 < 15 - 3r \qquad \text{Distributive property}$$

$$-4r + 9 < 15 - 3r$$

Step 2 $$-4r + 9 \mathbf{+ 3r} < 15 - 3r \mathbf{+ 3r} \qquad \text{Add } 3r.$$

$$-r + 9 < 15$$

$$-r + 9 \mathbf{- 9} < 15 \mathbf{- 9} \qquad \text{Subtract 9.}$$

$$-r < 6$$

Step 3 $$\mathbf{-1}(-r) \mathbf{>} \mathbf{-1}(6) \qquad \text{Multiply by } -1\text{; change } < \text{ to } >.$$

$$r > -6$$

Check that the solution set is $(-6, \infty)$. See Figure 9.

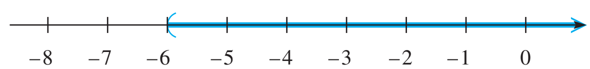


Figure 9

Work Problem 6 at the Side.

OBJECTIVE 4 Solve linear inequalities with three parts. For some applications, it is necessary to work with an inequality such as

$$3 < x + 2 < 8,$$

where $x + 2$ is *between* 3 and 8. To solve this inequality, we subtract 2 from each of the three parts of the inequality, giving

$$3 \mathbf{- 2} < x + 2 \mathbf{- 2} < 8 \mathbf{- 2}$$

$$1 < x < 6.$$

Thus, x must be between 1 and 6 so that $x + 2$ will be between 3 and 8. The solution set, $(1, 6)$, is graphed in Figure 10.

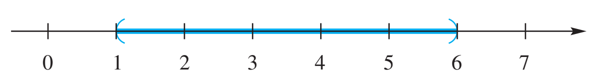


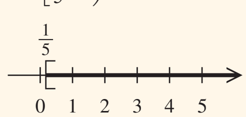
Figure 10

CAUTION

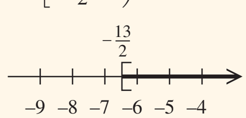
When inequalities have three parts, the order of the parts is important. It would be *wrong* to write an inequality as $8 < x + 2 < 3$, since this would imply that $8 < 3$, a false statement. In general, three-part inequalities are written so that the symbols point in the same direction, and both point toward the lesser number.

ANSWERS

6. (a) $\left[\frac{1}{5}, \infty\right)$



(b) $\left[-\frac{13}{2}, \infty\right)$



You Try It

Video

EXAMPLE 7 Solving a Three-Part Inequality

Solve $-2 \leq -3k - 1 \leq 5$, and graph the solution set.

Begin by adding 1 to each of the three parts to isolate the variable term in the middle.

$$-2 + 1 \leq -3k - 1 + 1 \leq 5 + 1 \qquad \text{Add 1 to each part.}$$

$$-1 \leq -3k \leq 6$$

$$\frac{-1}{-3} \geq \frac{-3k}{-3} \geq \frac{6}{-3} \qquad \text{Divide each part by } -3\text{; reverse the inequality symbols.}$$

$$\frac{1}{3} \geq k \geq -2$$

$$-2 \leq k \leq \frac{1}{3} \qquad \text{Rewrite in order based on the number line.}$$

Check that the solution set is $[-2, \frac{1}{3}]$, as shown in Figure 11.

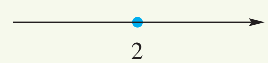
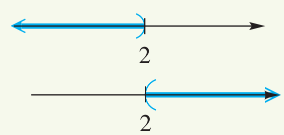
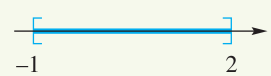
Figure 11

Work Problem 7 at the Side.

Examples of the types of solution sets to be expected from solving linear equations and linear inequalities are shown below.

Audio

SOLUTIONS OF LINEAR EQUATIONS AND INEQUALITIES

Equation or Inequality	Typical Solution Set	Graph of Solution Set
Linear equation $5x + 4 = 14$	$\{2\}$	
Linear inequality $5x + 4 < 14$ or $5x + 4 > 14$	$(-\infty, 2)$ $(2, \infty)$	
Three-part inequality $-1 \leq 5x + 4 \leq 14$	$[-1, 2]$	

OBJECTIVE 5 Solve applied problems using linear inequalities. In addition to the familiar "is less than" and "is greater than," other expressions such as "is no more than" and "is at least" also indicate inequalities. The table below shows how to interpret these expressions.

Word Expression	Interpretation
a is at least b	$a \geq b$
a is no less than b	$a \geq b$
a is at most b	$a \leq b$
a is no more than b	$a \leq b$

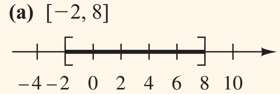
7 Solve, check, and graph the solution set of each inequality.

(a) $-3 \leq x - 1 \leq 7$

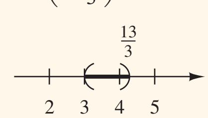
(b) $5 < 3x - 4 < 9$

ANSWERS

7. (a) $[-2, 8]$



(b) $\left(3, \frac{13}{3}\right)$



8 Solve the problem.

A rental company charges \$5 to rent a leaf blower, plus \$1.75 per hr. Dona Kenly can spend no more than \$26 to blow leaves from her driveway and pool deck. What is the *maximum* amount of time she can use the rented leaf blower?

9 Solve the problem.

Wade has grades of 92, 90, and 84 on his first three tests. What grade must he make on his fourth test in order to keep an average of at least 90?

Answers
8. 12 hr
9. at least 94

In Examples 8 and 9, we use the six problem-solving steps from **Section 2.3,** changing Step 3 to "Write an inequality" instead of "Write an equation."

EXAMPLE 8 Using a Linear Inequality to Solve a Rental Problem

A rental company charges \$15 to rent a chain saw, plus \$2 per hr. Jay Jenkins can spend no more than \$35 to clear some logs from his yard. What is the *maximum* amount of time he can use the rented saw?

Step 1 **Read** the problem again.

Step 2 **Assign a variable.** Let h = the number of hours he can rent the saw.

Step 3 **Write an inequality.** He must pay \$15, plus \$$2h$, to rent the saw for h hours, and this amount must be *no more than* \$35.

$$\overbrace{15 + 2h}^{\text{Cost of renting}} \quad \overbrace{\leq}^{\text{is no more than}} \quad \overbrace{35}^{\text{35 dollars.}}$$

Step 4 **Solve.**

$$2h \leq 20 \qquad \text{Subtract 15.}$$
$$h \leq 10 \qquad \text{Divide by 2.}$$

Step 5 **State the answer.** He can use the saw for a maximum of 10 hr. (He may use it for less time, as indicated by the inequality $h \leq 10$.)

Step 6 **Check.** If Jay uses the saw for **10** hr, he will spend $15 + 2(\mathbf{10}) =$ 35 dollars, the maximum amount.

Work Problem 8 at the Side.

EXAMPLE 9 Finding an Average Test Score

Helen has scores of 88, 86, and 90 on her first three algebra tests. An average score of at least 90 will earn an A in the class. What possible scores on her fourth test will earn her an A average?

Let x represent the score on the fourth test. Her average score must be at least 90. To find the average of four numbers, add them and then divide by 4.

$$\overbrace{\frac{88 + 86 + 90 + x}{4}}^{\text{Average}} \quad \overbrace{\geq}^{\text{is at least}} \quad \overbrace{90}^{\text{90.}}$$

$$\frac{264 + x}{4} \geq 90 \qquad \text{Add the scores.}$$
$$264 + x \geq 360 \qquad \text{Multiply by 4.}$$
$$x \geq 96 \qquad \text{Subtract 264.}$$

She must score **96** or more on her fourth test.

Check: $$\frac{88 + 86 + 90 + \mathbf{96}}{4} = \frac{360}{4} = 90$$

A score of 96 or more will give an average of at least 90, as required.

Work Problem 9 at the Side.

3.2 Set Operations and Compound Inequalities

OBJECTIVES

1. Find the intersection of two sets.
2. Solve compound inequalities with the word *and*.
3. Find the union of two sets.
4. Solve compound inequalities with the word *or*.

The table shows symptoms of an overactive thyroid and an underactive thyroid.

Underactive Thyroid	Overactive Thyroid
Sleepiness, s	Insomnia, i
Dry hands, d	Moist hands, m
Intolerance of cold, c	Intolerance of heat, h
Goiter, g	Goiter, g

Source: The Merck Manual of Diagnosis and Therapy, 16th Edition, Merck Research Laboratories, 1992.

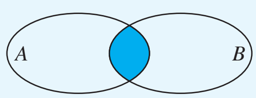
Let N be the set of symptoms for an underactive thyroid, and let O be the set of symptoms for an overactive thyroid. Suppose we are interested in the set of symptoms that are found in *both* sets N *and* O. In this section we discuss the use of the words *and* and *or* as they relate to sets and inequalities.

OBJECTIVE 1 Find the intersection of two sets. The intersection of two sets is defined using the word *and*.

Intersection of Sets

For any two sets A and B, the **intersection** of A and B, symbolized $A \cap B$, is defined as follows:

$$A \cap B = \{x \mid x \text{ is an element of } A \textbf{ and } x \text{ is an element of } B\}.$$



EXAMPLE 1 Finding the Intersection of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cap B$.

The set $A \cap B$ contains those elements that belong to both A *and* B: the numbers 2 and 4. Therefore,

$$A \cap B = \{1, 2, 3, 4\} \cap \{2, 4, 6\} = \{2, 4\}.$$

Work Problem 1 at the Side.

1 List the elements in each set.

(a) $A \cap B$, if $A = \{3, 4, 5, 6\}$ and $B = \{5, 6, 7\}$

(b) $N \cap O$ (Refer to the thyroid table.)

A **compound inequality** consists of two inequalities linked by a connective word such as *and* or *or*. Examples of compound inequalities are

$$x + 1 \le 9 \quad \textbf{and} \quad x - 2 \ge 3$$

and

$$2x > 4 \quad \textbf{or} \quad 3x - 6 < 5.$$

Compound inequalities

OBJECTIVE 2 Solve compound inequalities with the word *and*. Use the following steps.

Solving a Compound Inequality with *and*

Step 1 Solve each inequality in the compound inequality individually.

Step 2 Since the inequalities are joined with *and*, the solution set of the compound inequality will include all numbers that satisfy both inequalities in Step 1 (the intersection of the solution sets).

Answers

1. (a) $\{5, 6\}$ (b) $\{g\}$

2 Solve each compound inequality, and graph the solution set.

(a) $x < 10$ and $x > 2$

(b) $x + 3 \leq 1$ and $x - 4 \geq -12$

3 Solve $2x \geq x - 1$ and $3x \geq 3 + 2x$, and graph the solution set.

EXAMPLE 2 Solving a Compound Inequality with *and*

Solve the compound inequality $x + 1 \leq 9$ and $x - 2 \geq 3$.

Step 1 Solve each inequality in the compound inequality individually.

$$x + 1 \leq 9 \quad \text{and} \quad x - 2 \geq 3$$
$$x + 1 - 1 \leq 9 - 1 \quad \text{and} \quad x - 2 + 2 \geq 3 + 2$$
$$x \leq 8 \quad \text{and} \quad x \geq 5$$

Step 2 Because the inequalities are joined with the word *and*, the solution set will include all numbers that satisfy both inequalities in Step 1 at the same time. Thus, the compound inequality is true whenever $x \leq 8$ and $x \geq 5$ are both true. The top graph in Figure 12 shows $x \leq 8$, and the bottom graph shows $x \geq 5$.

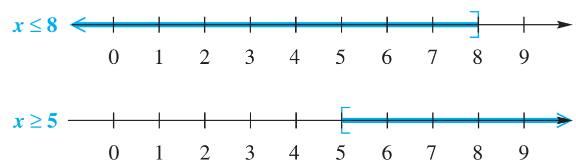


Figure 12

Find the intersection of the two graphs in Figure 12 to get the solution set of the compound inequality. The intersection of the two graphs in Figure 13 shows that the solution set in interval notation is [5, 8].

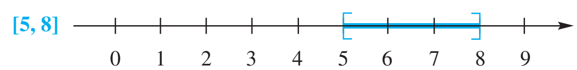


Figure 13

Work Problem 2 at the Side.

EXAMPLE 3 Solving a Compound Inequality with *and*

Solve the compound inequality $-3x - 2 > 5$ and $5x - 1 \leq -21$.

Step 1 Solve each inequality separately.

$$-3x - 2 > 5 \quad \text{and} \quad 5x - 1 \leq -21$$
$$-3x > 7 \quad \text{and} \quad 5x \leq -20$$
$$x < -\frac{7}{3} \quad \text{and} \quad x \leq -4$$

The graphs of $x < -\frac{7}{3}$ and $x \leq -4$ are shown in Figure 14.

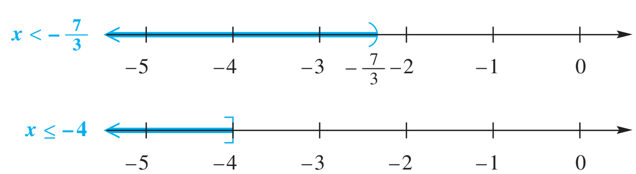


Figure 14

Step 2 Now find all values of x that satisfy both conditions; that is, the real numbers that are less than $-\frac{7}{3}$ and also less than or equal to -4. As shown by the graph in Figure 15, the solution set is $(-\infty, -4]$.

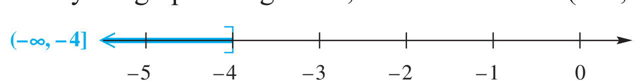
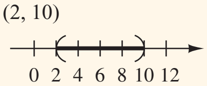


Figure 15

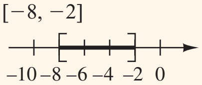
Work Problem 3 at the Side.

ANSWERS

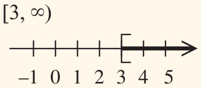
2. (a) (2, 10)



(b) $[-8, -2]$



3. $[3, \infty)$



EXAMPLE 4 Solving a Compound Inequality with *and*

Solve $x + 2 < 5$ and $x - 10 > 2$.

First solve each inequality separately.

$$x + 2 < 5 \quad \text{and} \quad x - 10 > 2$$
$$x < 3 \quad \text{and} \quad x > 12$$

The graphs of $x < 3$ and $x > 12$ are shown in Figure 16.

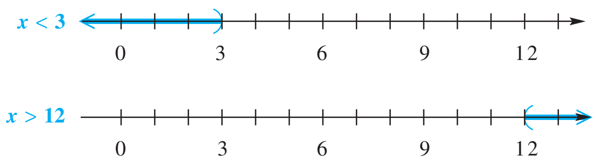


Figure 16

There is no number that is both less than 3 *and* greater than 12, so the given compound inequality has no solution. The solution set is $\emptyset$. See Figure 17.

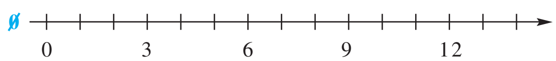


Figure 17

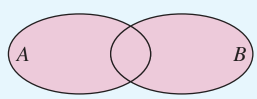
Work Problem 4 at the Side.

OBJECTIVE 3 Find the union of two sets. The union of two sets is defined using the word *or*.

Union of Sets

For any two sets A and B, the **union** of A and B, symbolized $A \cup B$, is defined as follows:

$$A \cup B = \{x \mid x \text{ is an element of } A \textbf{ or } x \text{ is an element of } B\}.$$



EXAMPLE 5 Finding the Union of Two Sets

Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$. Find $A \cup B$.

Begin by listing all the elements of set A: 1, 2, 3, 4. Then list any additional elements from set B. In this case the elements 2 and 4 are already listed, so the only additional element is 6. Therefore,

$$A \cup B = \{1, 2, 3, 4\} \cup \{2, 4, 6\}$$
$$= \{1, 2, 3, 4, 6\}.$$

The union consists of all elements in either A *or* B (or both).

NOTE

Although the elements 2 and 4 appeared in both sets A and B, they are written only once in $A \cup B$.

Work Problem 5 at the Side.

4 Solve.

(a) $x < 5$ and $x > 5$

(b) $x + 2 > 3$ and $2x + 1 < -3$

5 List the elements in each set.

(a) $A \cup B$, if $A = \{3, 4, 5, 6\}$ and $B = \{5, 6, 7\}$

(b) $N \cup O$ from the thyroid table at the beginning of this section

ANSWERS

4. **(a)** $\emptyset$ **(b)** $\emptyset$

5. **(a)** $\{3, 4, 5, 6, 7\}$ **(b)** $\{s, d, c, g, i, m, h\}$

OBJECTIVE 4 **Solve compound inequalities with the word *or*.** Use the following steps.

Solving a Compound Inequality with *or*

Step 1 Solve each inequality in the compound inequality individually.

Step 2 Since the inequalities are joined with *or,* the solution set includes all numbers that satisfy either one of the two inequalities in Step 1 (the union of the solution sets).

EXAMPLE 6 Solving a Compound Inequality with *or*

Solve $6x - 4 < 2x$ or $-3x \leq -9$.

Step 1 Solve each inequality separately.

$$6x - 4 < 2x \quad \text{or} \quad -3x \leq -9$$
$$4x < 4$$
$$x < 1 \quad \text{or} \quad x \geq 3$$

The graphs of these two inequalities are shown in Figure 18.

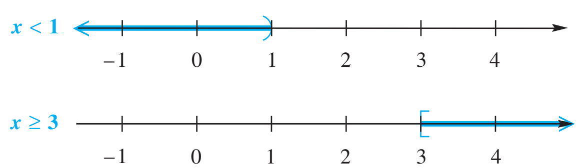


Figure 18

Step 2 Since the inequalities are joined with *or*, find the union of the two solution sets. The union is shown in Figure 19 and is written

$$(-\infty, 1) \cup [3, \infty).$$

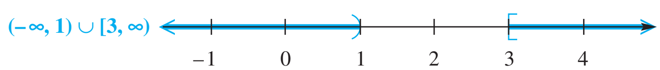


Figure 19

CAUTION

When inequalities are used to write the solution set in Example 6, it *must* be written as

$$x < 1 \quad \text{or} \quad x \geq 3,$$

which keeps the numbers 1 and 3 in their order on the number line. Writing $3 \leq x < 1$ would imply that $3 \leq 1$, which is ***FALSE***. There is no other way to write the solution set of such a union.

Work Problem 6 at the Side.

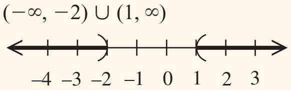
6 Give each solution set in both interval and graph forms.

(a) $x + 2 > 3$ or $2x + 1 < -3$

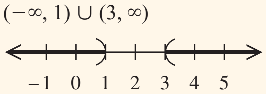
(b) $x - 1 > 2$ or $3x + 5 < 2x + 6$

ANSWERS

6. (a) $(-\infty, -2) \cup (1, \infty)$



(b) $(-\infty, 1) \cup (3, \infty)$



EXAMPLE 7 Solving a Compound Inequality with *or*

Solve $-4x + 1 \geq 9$ or $5x + 3 \geq -12$.

First, solve each inequality separately.

$$\begin{aligned} -4x + 1 &\geq 9 & \text{or} \quad 5x + 3 &\geq -12 \\ -4x &\geq 8 & \text{or} \quad 5x &\geq -15 \\ x &\leq -2 & \text{or} \quad x &\geq -3 \end{aligned}$$

The graphs of these two inequalities are shown in Figure 20.

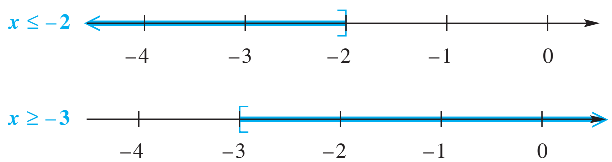


Figure 20

By taking the union, we obtain every real number as a solution, since every real number satisfies at least one of the two inequalities. The set of all real numbers is written in interval notation as $(-\infty, \infty)$ and graphed as in Figure 21.

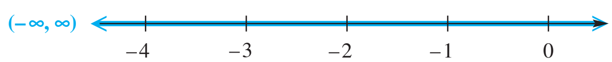


Figure 21

Work Problem 7 at the Side.

EXAMPLE 8 Applying Intersection and Union

The five highest domestic grossing films (adjusted for inflation) are listed in the table.

FIVE ALL-TIME HIGHEST GROSSING FILMS

Film	Admissions	Gross Income
Gone with the Wind	200,605,313	\$972,900,000
Star Wars	178,119,595	\$863,900,000
The Sound of Music	142,415,376	\$690,700,000
E.T.	135,987,938	\$659,500,000
The Ten Commandments	131,000,000	\$635,400,000

Source: New York Times Almanac, 2001.

List the elements of the following sets.

(a) The set of top-five films with admissions greater than 180,000,000 *and* gross income greater than \$800,000,000

The only film that satisfies both conditions is *Gone with the Wind*, so the set is

{*Gone with the Wind*}.

(b) The set of top-five films with admissions less than 170,000,000 *or* gross income greater than \$700,000,000

Here, a film that satisfies at least one of the conditions is in the set. This set includes all five films:

{*Gone with the Wind, Star Wars, The Sound of Music, E.T., The Ten Commandments*}.

Work Problem 8 at the Side.

7 Solve.

(a) $2x + 1 \leq 9$ or $2x + 3 \leq 5$

(b) $3x - 2 \leq 13$ or $x + 5 \geq 7$

8 From Example 8, list the elements that satisfy each set.

(a) The set of films with admissions greater than 130,000,000 and gross income less than \$500,000,000

(b) The set of films with admissions greater than 130,000,000 or gross income less than \$500,000,000

ANSWERS

7. (a) $(-\infty, 4]$ **(b)** $(-\infty, \infty)$

8. (a) $\emptyset$ **(b)** {*Gone with the Wind, Star Wars, The Sound of Music, E.T., The Ten Commandments*}

3.3 Absolute Value Equations and Inequalities

OBJECTIVES

1. **Use the distance definition of absolute value.**
2. **Solve equations of the form $|ax + b| = k$, for $k > 0$.**
3. **Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.**
4. **Solve absolute value equations that involve rewriting.**
5. **Solve equations of the form $|ax + b| = |cx + d|$.**
6. **Solve special cases of absolute value equations and inequalities.**

In a production line, quality is controlled by randomly choosing items from the line and checking to see how selected measurements vary from the optimum measure. These differences are sometimes positive and sometimes negative, so they are expressed with absolute value. For example, a machine that fills quart milk cartons might be set to release 1 qt plus or minus 2 oz per carton. Then the number of ounces in each carton should satisfy the *absolute value inequality* $|x - 32| \leq 2$, where x is the number of ounces.

OBJECTIVE 1 Use the distance definition of absolute value. In **Section 1.1** we saw that the absolute value of a number x, written $|x|$, represents the distance from x to 0 on the number line. For example, the solutions of $|x| = 4$ are 4 and -4, as shown in Figure 22.

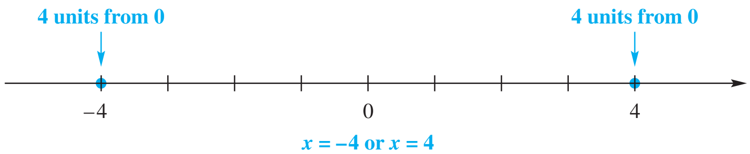


Figure 22

Because absolute value represents distance from 0, it is reasonable to interpret the solutions of $|x| > 4$ to be all numbers that are *more* than 4 units from 0. The set $(-\infty, -4) \cup (4, \infty)$ fits this description. Figure 23 shows the graph of the solution set of $|x| > 4$. Because the graph consists of two separate intervals, the solution set is described using *or* as $x < -4$ or $x > 4$.

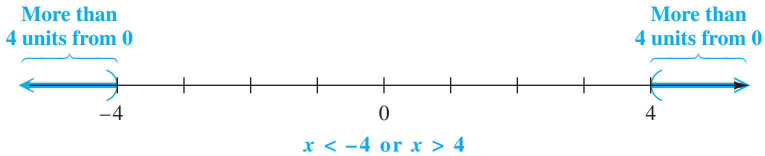


Figure 23

The solution set of $|x| < 4$ consists of all numbers that are *less* than 4 units from 0 on the number line. Another way of thinking of this is to think of all numbers *between* -4 and 4. This set of numbers is given by $(-4, 4)$, as shown in Figure 24. Here, the graph shows that $-4 < x < 4$, which means $x > -4$ *and* $x < 4$.

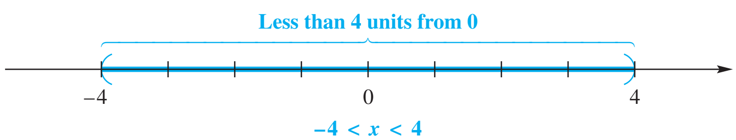


Figure 24

Work Problem 1 at the Side.

The equation and inequalities just described are examples of **absolute value equations and inequalities.** They involve the absolute value of a variable expression and generally take the form

$$|ax + b| = k, \quad |ax + b| > k, \quad \text{or} \quad |ax + b| < k,$$

where k is a positive number. From Figures 22–24, we see that

$|x| = 4$ has the same solution set as $x = -4$ **or** $x = 4$,
$|x| > 4$ has the same solution set as $x < -4$ **or** $x > 4$,
$|x| < 4$ has the same solution set as $x > -4$ **and** $x < 4$.

1 Graph the solution set of each equation or inequality.

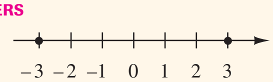
(a) $|x| = 3$

(b) $|x| > 3$

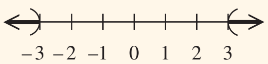
(c) $|x| < 3$

ANSWERS

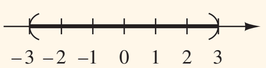
1. (a)



(b)



(c)



2 Solve each equation, check, and graph the solution set.

(a) $|x + 2| = 3$

(b) $|3x - 4| = 11$

Thus, we can solve an absolute value equation or inequality by solving the appropriate compound equation or inequality.

Solving Absolute Value Equations and Inequalities

Let k be a positive real number, and p and q be real numbers.

1. To solve $|ax + b| = k$, solve the compound equation

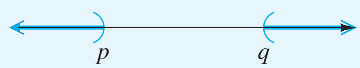
$$ax + b = k \quad \text{or} \quad ax + b = -k.$$

The solution set is usually of the form $\{p, q\}$, which includes two numbers.

2. To solve $|ax + b| > k$, solve the compound inequality

$$ax + b > k \quad \text{or} \quad ax + b < -k.$$

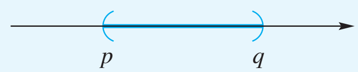
The solution set is of the form $(-\infty, p) \cup (q, \infty)$, which consists of two separate intervals.



3. To solve $|ax + b| < k$, solve the three-part inequality

$$-k < ax + b < k.$$

The solution set is of the form (p, q), a single interval.



OBJECTIVE 2 **Solve equations of the form $|ax + b| = k$, for $k > 0$.** Remember that because absolute value refers to distance from the origin, an absolute value equation will have two parts.

EXAMPLE 1 **Solving an Absolute Value Equation**

Solve $|2x + 1| = 7$.

For $|2x + 1|$ to equal 7, $2x + 1$ must be 7 units from 0 on the number line. This can happen only when $2x + 1 = 7$ or $2x + 1 = -7$. This is the first case in the preceding summary. Solve this compound equation as follows.

$$\begin{aligned} 2x + 1 &= 7 &\quad \text{or} \quad 2x + 1 &= -7 \\ 2x &= 6 &\quad \text{or} \quad 2x &= -8 \\ x &= 3 &\quad \text{or} \quad x &= -4 \end{aligned}$$

Check by substituting 3 and then -4 in the original absolute value equation to verify that the solution set is $\{-4, 3\}$. The graph is shown in Figure 25.

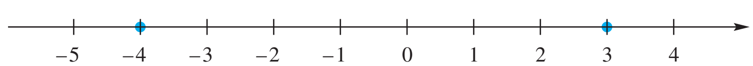
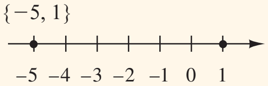


Figure 25

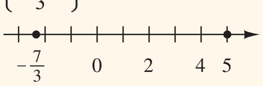
Work Problem 2 at the Side.

ANSWERS

2. **(a)** $\{-5, 1\}$



(b) $\left\{-\frac{7}{3}, 5\right\}$



NOTE
Some people prefer to write the compound statements in parts 1 and 2 of the summary on the previous page as the equivalent forms

$$ax + b = k \quad \text{or} \quad -(ax + b) = k$$

and $$ax + b > k \quad \text{or} \quad -(ax + b) > k.$$

These forms produce the same results.

OBJECTIVE 3 **Solve inequalities of the form $|ax + b| < k$ and of the form $|ax + b| > k$, for $k > 0$.**

EXAMPLE 2 Solving an Absolute Value Inequality with >

Solve $|2x + 1| > 7$.

By part 2 of the summary, this absolute value inequality is rewritten as

$$2x + 1 > 7 \quad \text{or} \quad 2x + 1 < -7,$$

because $2x + 1$ must represent a number that is *more* than 7 units from 0 on either side of the number line. Now, solve the compound inequality.

$$\begin{aligned} 2x + 1 &> 7 &\text{or}\quad 2x + 1 &< -7 \\ 2x &> 6 &\text{or}\quad 2x &< -8 \\ x &> 3 &\text{or}\quad x &< -4 \end{aligned}$$

Check these solutions. The solution set is $(-\infty, -4) \cup (3, \infty)$. See Figure 26. Notice that the graph consists of two intervals.

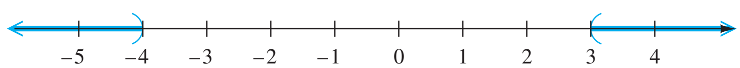


Figure 26

Work Problem 3 at the Side.

EXAMPLE 3 Solving an Absolute Value Inequality with <

Solve $|2x + 1| < 7$.

The expression $2x + 1$ must represent a number that is less than 7 units from 0 on either side of the number line. Another way of thinking of this is to realize that $2x + 1$ must be between -7 and 7. As part 3 of the summary shows, this is written as the three-part inequality

$$-7 < 2x + 1 < 7.$$

$$-8 < 2x < 6 \qquad \text{Subtract 1 from each part.}$$

$$-4 < x < 3 \qquad \text{Divide each part by 2.}$$

Check that the solution set is $(-4, 3)$, so the graph consists of the single interval shown in Figure 27.

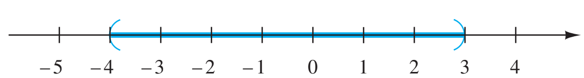


Figure 27

Work Problem 4 at the Side.

3 Solve each inequality, check, and graph the solution set.

(a) $|x + 2| > 3$

(b) $|3x - 4| \geq 11$

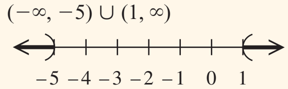
4 Solve each inequality, check, and graph the solution set.

(a) $|x + 2| < 3$

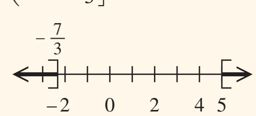
(b) $|3x - 4| \leq 11$

ANSWERS

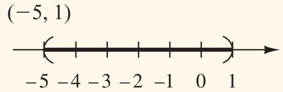
3. (a) $(-\infty, -5) \cup (1, \infty)$



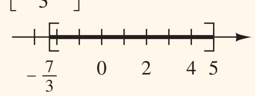
(b) $\left(-\infty, -\frac{7}{3}\right] \cup [5, \infty)$



4. (a) $(-5, 1)$



(b) $\left[-\frac{7}{3}, 5\right]$



5 **(a)** Solve $|5x + 2| - 9 = -7$.

(b) Solve $|x + 2| - 3 > 2$, and graph the solution set.

(c) Solve, and graph the solution set.

$|3x + 2| + 4 \leq 15$

Look back at Figures 25, 26, and 27, with the graphs of $|2x + 1| = 7$, $|2x + 1| > 7$, and $|2x + 1| < 7$. If we find the union of the three sets, we get the set of all real numbers. This is because for any value of x, $|2x + 1|$ will satisfy one and only one of the following: it is equal to 7, greater than 7, or less than 7.

CAUTION

When solving absolute value equations and inequalities of the types in Examples 1, 2, and 3, remember the following.

1. The methods described apply when the constant is alone on one side of the equation or inequality and is *positive*.
2. Absolute value equations and absolute value inequalities of the form $|ax + b| > k$ translate into "or" compound statements.
3. Absolute value inequalities of the form $|ax + b| < k$ translate into "and" compound statements, which may be written as three-part inequalities.
4. An "or" statement *cannot* be written in three parts. It would be incorrect to use $-7 > 2x + 1 > 7$ in Example 2, because this would imply that $-7 > 7$, which is *false*.

OBJECTIVE 4 Solve absolute value equations that involve rewriting. Sometimes an absolute value equation or inequality requires some rewriting before it can be set up as a compound statement, as shown in the next example.

EXAMPLE 4 Solving an Absolute Value Equation That Requires Rewriting

Solve the equation $|x + 3| + 5 = 12$.

First, rewrite so that the absolute value expression is alone on one side of the equals sign by subtracting 5 from each side.

$$|x + 3| + 5 - 5 = 12 - 5 \qquad \text{Subtract 5.}$$
$$|x + 3| = 7$$

Now use the method shown in Example 1.

$$x + 3 = 7 \quad \text{or} \quad x + 3 = -7$$
$$x = 4 \quad \text{or} \quad x = -10$$

Check that the solution set is $\{4, -10\}$ by substituting 4 and then -10 into the original equation.

We use a similar method to solve an absolute value *inequality* that requires rewriting.

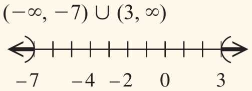
Work Problem 5 at the Side.

OBJECTIVE 5 Solve equations of the form $|ax + b| = |cx + d|$. By definition, for two expressions to have the same absolute value, they must either be equal or be negatives of each other.

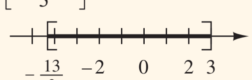
ANSWERS

5. (a) $\left\{-\frac{4}{5}, 0\right\}$

(b) $(-\infty, -7) \cup (3, \infty)$



(c) $\left[-\frac{13}{3}, 3\right]$



Audio

Solving $|ax + b| = |cx + d|$

To solve an absolute value equation of the form

$$|ax + b| = |cx + d|,$$

solve the compound equation

$$ax + b = cx + d \quad \text{or} \quad ax + b = -(cx + d).$$

You Try It

Video

EXAMPLE 5 Solving an Equation with Two Absolute Values

Solve the equation $|z + 6| = |2z - 3|$.

This equation is satisfied either if $z + 6$ and $2z - 3$ are equal to each other, or if $z + 6$ and $2z - 3$ are negatives of each other. Thus,

$$z + 6 = 2z - 3 \quad \text{or} \quad z + 6 = -(2z - 3).$$

Solve each equation.

$$\begin{aligned} z + 6 &= 2z - 3 &\quad \text{or} \quad z + 6 &= -2z + 3 \\ 9 &= z & 3z &= -3 \\ & & z &= -1 \end{aligned}$$

Check that the solution set is $\{9, -1\}$.

Work Problem 6 at the Side.

OBJECTIVE 6 **Solve special cases of absolute value equations and inequalities.** When a typical absolute value equation or inequality involves a *negative* constant or *0* alone on one side, as in Examples 6 and 7, we use the properties of absolute value to solve. Keep the following in mind.

Special Cases for Absolute Value

1. The absolute value of an expression can never be negative: $|a| \geq 0$ for all real numbers a.
2. The absolute value of an expression equals 0 only when the expression is equal to 0.

EXAMPLE 6 Solving Special Cases of Absolute Value Equations

Solve each equation.

(a) $|5r - 3| = -4$

See Case 1 in the preceding box. Since the absolute value of an expression can never be negative, there are no solutions for this equation. The solution set is $\emptyset$.

(b) $|7x - 3| = 0$

See Case 2 in the preceding box. The expression $7x - 3$ will equal 0 *only* if

$$7x - 3 = 0.$$

The solution of this equation is $\frac{3}{7}$. Thus, the solution set of the original equation is $\left\{\frac{3}{7}\right\}$, with just one element. Check by substitution.

Work Problem 7 at the Side.

6 Solve each equation.

(a) $|k - 1| = |5k + 7|$

(b) $|4r - 1| = |3r + 5|$

7 Solve each equation.

(a) $|6x + 7| = -5$

(b) $\left|\frac{1}{4}x - 3\right| = 0$

Answers

6. (a) $\{-1, -2\}$ **(b)** $\left\{-\frac{4}{7}, 6\right\}$

7. (a) $\emptyset$ **(b)** $\{12\}$

8 Solve.

(a) $|x| > -1$

(b) $|x| < -5$

(c) $|x + 2| \leq 0$

EXAMPLE 7 **Solving Special Cases of Absolute Value Inequalities**

Solve each inequality.

(a) $|x| \geq -4$

The absolute value of a number is always greater than or equal to 0. Thus, $|x| \geq -4$ is true for *all* real numbers. The solution set is $(-\infty, \infty)$.

(b) $|x + 6| - 3 < -5$

Add 3 to each side to get the absolute value expression alone on one side.

$$|x + 6| < -2$$

There is no number whose absolute value is less than -2, so this inequality has no solution. The solution set is $\emptyset$.

(c) $|x - 7| + 4 \leq 4$

Subtracting 4 from each side gives

$$|x - 7| \leq 0.$$

The value of $|x - 7|$ will never be less than 0. However, $|x - 7|$ will equal 0 when $x = 7$. Therefore, the solution set is $\{7\}$.

Work Problem 8 at the Side.

ANSWERS

8. **(a)** $(-\infty, \infty)$ **(b)** $\emptyset$ **(c)** $\{-2\}$

4.1 The Rectangular Coordinate System

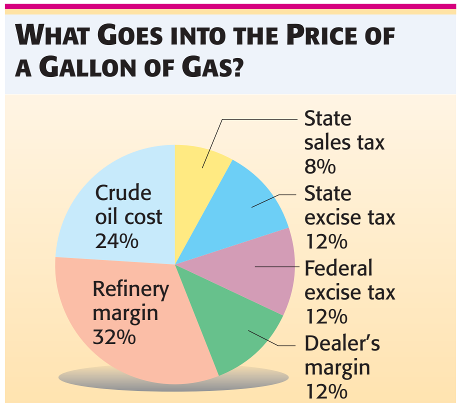
OBJECTIVES

1. **Plot ordered pairs.**
2. **Find ordered pairs that satisfy a given equation.**
3. **Graph lines.**
4. **Find *x*- and *y*-intercepts.**
5. **Recognize equations of horizontal and vertical lines.**

There are many ways to present information graphically. The circle graph (or pie chart) in Figure 1(a) shows the cost breakdown for a gallon of regular unleaded gasoline in California. What contributes most to the cost?

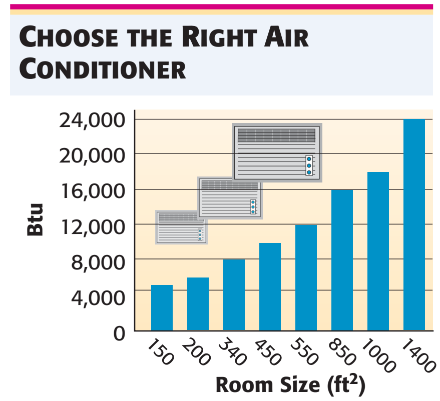
Figure 1(b) shows a bar graph in which the heights of the bars represent the Btu (British thermal units) required to cool different-sized rooms. How many Btu are needed to cool a 1400 ft^2 room?

The line graph in Figure 1(c) shows personal spending (in billions of dollars) on medical care in the United States from 1997 through 2002. About how much was spent on medical care in 2002?



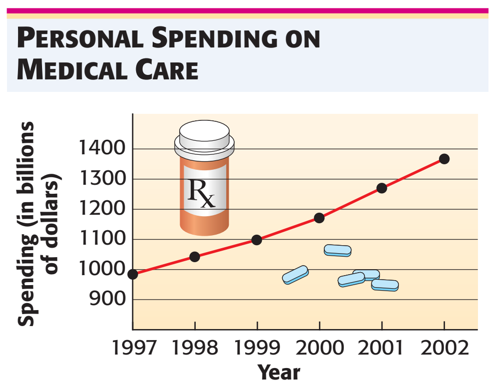
Source: California Energy Commission.

(a)



Source: Carey, Morris and James, *Home Improvement for Dummies*, IDG Books.

(b)

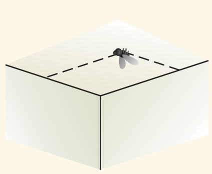


Source: Bureau of Economic Analysis.

(c)

Figure 1

The line graph in Figure 1(c) presents information based on a method for locating a point in a plane developed by René Descartes, a 17th-century French mathematician. Legend has it that Descartes, who was lying in bed ill, was watching a fly crawl about on the ceiling near a corner of the room. It occurred to him that the location of the fly on the ceiling could be described by determining its distances from the two adjacent walls. See the figure in the margin. In this chapter we use this insight to plot points and graph linear equations in two variables whose graphs are straight lines.



Locating a fly on a ceiling

OBJECTIVE 1 Plot ordered pairs. Each of the pairs of numbers (3, 1), (−5, 6), and (4, −1) is an example of an **ordered pair;** that is, a pair of numbers written within parentheses in which the order of the numbers is important. We graph an ordered pair using two perpendicular number lines that intersect at their 0 points, as shown in Figure 2. The common 0 point is called the **origin.** The position of any point in this plane is determined by referring to the horizontal number line, the ***x*-axis,** and the vertical number line, the ***y*-axis.** The first number in the ordered pair indicates the position relative to the *x*-axis, and the second number indicates the position relative to the *y*-axis. The *x*-axis and the *y*-axis make up a **rectangular** (or **Cartesian,** for Descartes) **coordinate system.**

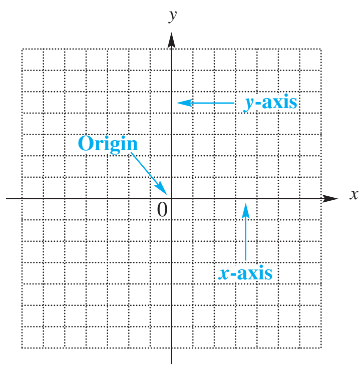


Figure 2

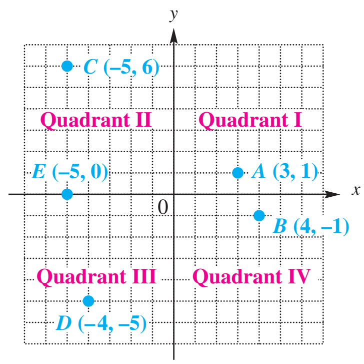


Figure 3

To locate, or **plot,** the point on the graph that corresponds to the ordered pair (3, 1), we move three units from 0 to the right along the *x*-axis, and then one unit up parallel to the *y*-axis. The point corresponding to the ordered pair (3, 1) is labeled *A* in Figure 3. Additional points are labeled *B*–*E*. The phrase "the point corresponding to the ordered pair (3, 1)" is often abbreviated as "the point (3, 1)." The numbers in the ordered pairs are called **components** and are the **coordinates** of the corresponding point.

We can relate this method of locating ordered pairs to the line graph in Figure 1(c). We move along the horizontal axis to a year, then up parallel to the vertical axis to find medical spending for that year. Thus, we can write the ordered pair (2002, 1370) to indicate that in 2002, personal spending on medical care was about $1370 billion.

CAUTION
The parentheses used to represent an ordered pair are also used to represent an open interval (introduced in **Section 3.1**). The context of the discussion tells whether ordered pairs or open intervals are being represented.

The four regions of the graph, shown in Figure 3, are called **quadrants I, II, III,** and **IV,** reading counterclockwise from the upper-right quadrant. The points on the *x*-axis and *y*-axis do not belong to any quadrant. For example, point *E* in Figure 3 belongs to no quadrant.

Work Problem 1 at the Side.

1 Plot each point. Name the quadrant (if any) in which each point is located.

(a) (−4, 2)

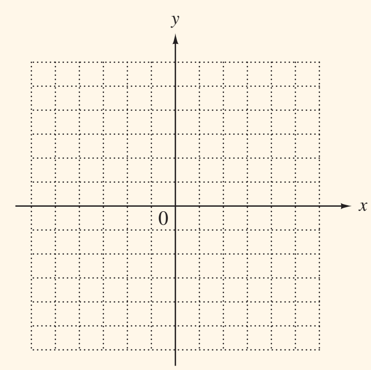
(b) (3, −2)

(c) (−5, −6)

(d) (4, 6)

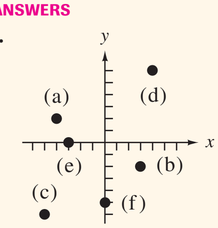
(e) (−3, 0)

(f) (0, −5)



ANSWERS

1.



(a) II **(b)** IV **(c)** III **(d)** I
(e) no quadrant **(f)** no quadrant

OBJECTIVE 2 Find ordered pairs that satisfy a given equation. Each solution to an equation with two variables, such as $2x + 3y = 6$, will include two numbers, one for each variable. To keep track of which number goes with which variable, we write the solutions as ordered pairs. (If x and y are used as the variables, the x-value is given first.) For example, we can show that $(6, -2)$ is a solution of $2x + 3y = 6$ by substitution.

$$\begin{aligned} 2x + 3y &= 6 \\ 2(6) + 3(-2) &= 6 \quad ? \quad \text{Let } x = 6, y = -2. \\ 12 - 6 &= 6 \quad ? \\ 6 &= 6 \quad \text{True} \end{aligned}$$

Because the pair of numbers $(6, -2)$ makes the equation true, it is a solution. On the other hand, $(5, 1)$ is not a solution of the equation $2x + 3y = 6$ because

$$\begin{aligned} 2x + 3y &= 2(5) + 3(1) \\ &= 10 + 3 \\ &= 13, \quad \textit{not} \quad 6. \end{aligned}$$

To find ordered pairs that satisfy an equation, select any number for one of the variables, substitute it into the equation for that variable, and then solve for the other variable. Two other ordered pairs satisfying $2x + 3y = 6$ are $(0, 2)$ and $(3, 0)$. Since any real number could be selected for one variable and would lead to a real number for the other variable, linear equations in two variables have an infinite number of solutions.

2 **(a)** Complete each ordered pair for $3x - 4y = 12$.

$(0, \quad)$

$(\quad, 0)$

$(\quad, -2)$

$(-4, \quad)$

(b) Find one other ordered pair that satisfies the equation.

Video

EXAMPLE 1 Completing Ordered Pairs

Complete each ordered pair for $2x + 3y = 6$.

(a) $(-3, \quad)$
We are given $x = -3$. We substitute into the equation to find y.

$$\begin{aligned} 2x + 3y &= 6 \\ 2(-3) + 3y &= 6 \quad \text{Let } x = -3. \\ -6 + 3y &= 6 \\ 3y &= 12 \\ y &= 4 \end{aligned}$$

The ordered pair is $(-3, 4)$.

(b) $(\quad, -4)$
Replace y with -4 in the equation to find x.

$$\begin{aligned} 2x + 3y &= 6 \\ 2x + 3(-4) &= 6 \quad \text{Let } y = -4. \\ 2x - 12 &= 6 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

The ordered pair is $(9, -4)$.

Work Problem 2 at the Side.

ANSWERS

2. **(a)** $(0, -3)$, $(4, 0)$, $\left(\frac{4}{3}, -2\right)$, $(-4, -6)$
(b) Many answers are possible; for example, $\left(-6, -\frac{15}{2}\right)$.

OBJECTIVE 3 Graph lines. The **graph of an equation** is the set of points corresponding to all ordered pairs that satisfy the equation. It gives a "picture" of the equation. Most equations in two variables are satisfied by an infinite number of ordered pairs, so their graphs include an infinite number of points.

To graph an equation, we plot a number of ordered pairs that satisfy the equation until we have enough points to suggest the shape of the graph. For example, to graph $2x + 3y = 6$, we plot all the ordered pairs found in Objective 2 and Example 1 on the previous page. These points, shown in a table of values and plotted in Figure 4(a), appear to lie on a straight line. If all the ordered pairs that satisfy the equation $2x + 3y = 6$ were graphed, they would form the straight line shown in Figure 4(b).

x	y
-3	4
0	2
3	0
6	-2
9	-4

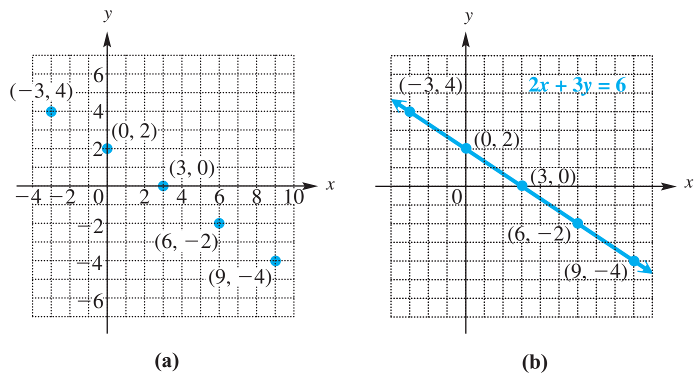


Figure 4

Work Problem 3 at the Side.

The equation $2x + 3y = 6$ is called a **first-degree equation** because it has no term with a variable to a power greater than one.

The graph of any first-degree equation in two variables is a straight line.

Since first-degree equations with two variables have straight-line graphs, they are called *linear equations in two variables*.

Linear Equation in Two Variables

A **linear equation in two variables** can be written in the form

$$Ax + By = C,$$

where A, B, and C are real numbers (A and B not both 0). This form is called **standard form.**

OBJECTIVE 4 Find *x*- and *y*-intercepts. A straight line is determined if any two different points on the line are known, so finding two different points is enough to graph the line. Two useful points for graphing are the x- and y-intercepts. The ***x*-intercept** is the point (if any) where the line intersects the x-axis; likewise, the ***y*-intercept** is the point (if any) where the line intersects the y-axis.* See Figure 5.

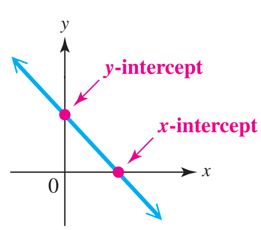
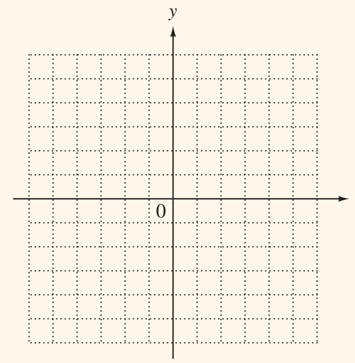


Figure 5

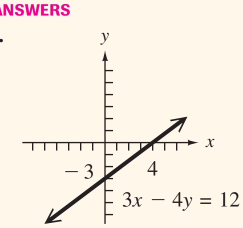
* Some texts define an intercept as a number, not a point.

3 Graph $3x - 4y = 12$. Use the points from Problem 2 in the margin on the previous page.

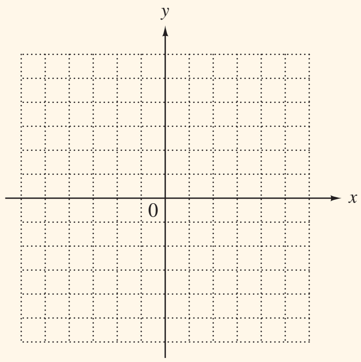


ANSWERS

3.



4 Find the intercepts, and graph $2x - y = 4$.



The y-value of the point where a line intersects the x-axis is 0. Similarly, the x-value of the point where a line intersects the y-axis is 0. This suggests a method for finding the x- and y-intercepts.

Finding Intercepts

When graphing the equation of a line,

let $y = 0$ to find the x-intercept;

let $x = 0$ to find the y-intercept.

EXAMPLE 2 Finding Intercepts

Find the x- and y-intercepts of $4x - y = -3$, and graph the equation.

We find the x-intercept by letting $y = 0$.

$$4x - y = -3$$
$$4x - 0 = -3 \qquad \text{Let } y = 0.$$
$$4x = -3$$
$$x = -\frac{3}{4} \qquad x\text{-intercept is } \left(-\tfrac{3}{4}, 0\right).$$

For the y-intercept, let $x = 0$.

$$4x - y = -3$$
$$4(0) - y = -3 \qquad \text{Let } x = 0.$$
$$-y = -3$$
$$y = 3 \qquad y\text{-intercept is } (0, 3).$$

The intercepts are the two points $\left(-\frac{3}{4}, 0\right)$ and $(0, 3)$. We show these ordered pairs in the table next to Figure 6 and use these points to draw the graph.

x	y
$-\frac{3}{4}$	0
0	3

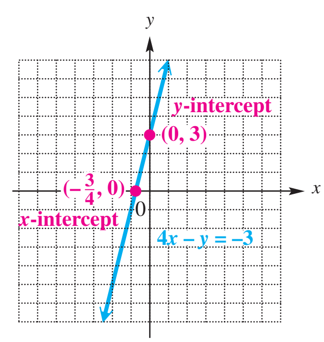


Figure 6

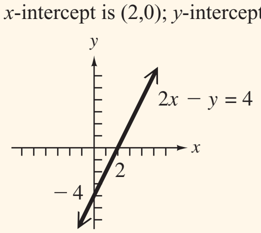
NOTE

While two points, such as the two intercepts in Figure 6, are sufficient to graph a straight line, ***it is a good idea to use a third point to guard against errors.*** Verify by substitution that $(-1, -1)$ also lies on the graph of $4x - y = -3$.

Work Problem 4 at the Side.

ANSWERS

4. x-intercept is $(2,0)$; y-intercept is $(0, -4)$.



OBJECTIVE 5 Recognize equations of horizontal and vertical lines. A graph can fail to have an x-intercept or a y-intercept, which is why the phrase "if any" was added when discussing intercepts.

EXAMPLE 3 Graphing a Horizontal Line

Graph $y = 2$.

Since y is always 2, there is no value of x corresponding to $y = 0$, so the graph has no x-intercept. The y-intercept is $(0, 2)$. The graph in Figure 7, shown with a table of ordered pairs, is a horizontal line.

x	y
-1	2
0	2
3	2

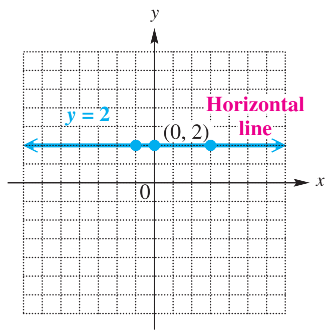


Figure 7

Work Problem 5 at the Side.

EXAMPLE 4 Graphing a Vertical Line

Graph $x + 1 = 0$.

The x-intercept is $(-1, 0)$. The standard form $1x + 0y = -1$ shows that every value of y leads to $x = -1$, so no value of y makes $x = 0$. The only way a straight line can have no y-intercept is if it is vertical, as in Figure 8.

x	y
-1	-4
-1	0
-1	5

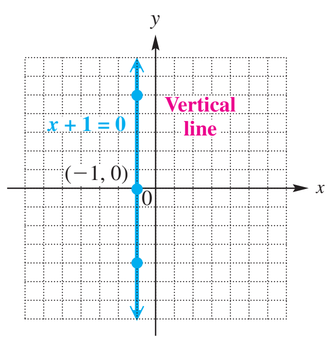


Figure 8

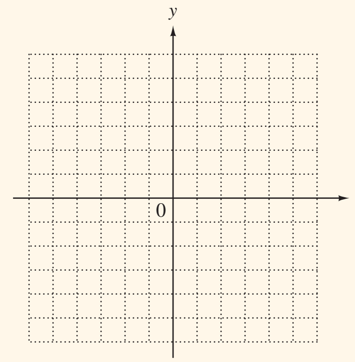
Work Problem 6 at the Side.

CAUTION

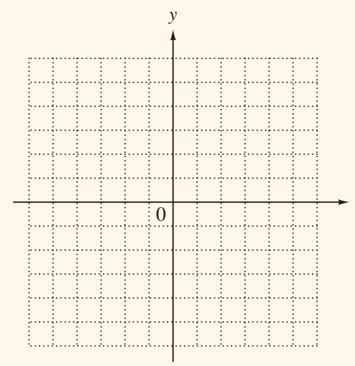
To avoid confusing equations of horizontal and vertical lines remember that

1. An equation with only the variable x will always intersect the *x-axis* and thus will be *vertical*.
2. An equation with only the variable y will always intersect the *y-axis* and thus will be *horizontal*.

5 Find the intercepts, and graph $y + 4 = 0$.

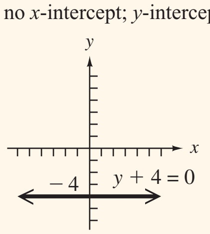


6 Find the intercepts, and graph the line $x = 2$.

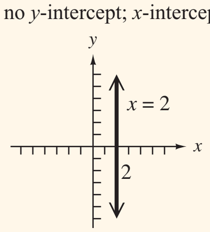


ANSWERS

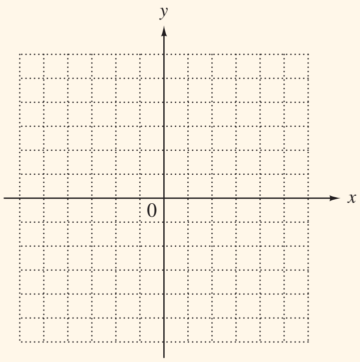
5. no x-intercept; y-intercept is $(0, -4)$.



6. no y-intercept; x-intercept is $(2, 0)$.



7 Find the intercepts, and graph the line $3x - y = 0$.



Some lines have both the x- and y-intercepts at the origin.

EXAMPLE 5 Graphing a Line That Passes through the Origin

Graph $x + 2y = 0$.

Find the x-intercept by letting $y = 0$.

$$x + 2y = 0$$

$$x + 2(0) = 0 \qquad \text{Let } y = 0.$$

$$x + 0 = 0$$

$$x = 0 \qquad x\text{-intercept is } (0, 0).$$

To find the y-intercept, let $x = 0$.

$$x + 2y = 0$$

$$0 + 2y = 0 \qquad \text{Let } x = 0.$$

$$y = 0 \qquad y\text{-intercept is } (0, 0).$$

Both intercepts are the same ordered pair, (0, 0). (This means that the graph goes through the origin.) To find another point to graph the line, choose any nonzero number for x, say $x = 4$, and solve for y.

$$x + 2y = 0$$

$$4 + 2y = 0 \qquad \text{Let } x = 4.$$

$$2y = -4$$

$$y = -2$$

This gives the ordered pair $(4, -2)$. These two points lead to the graph shown in Figure 9. As a check, verify that $(-2, 1)$ also lies on the line.

x	y
-2	1
0	0
4	-2

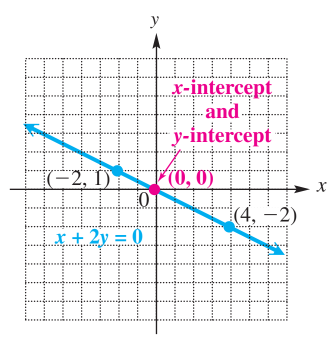


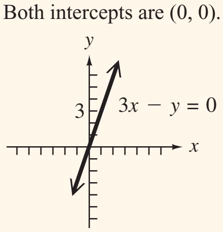
Figure 9

To find the additional point to graph, we could have chosen any number (except 0) for y instead of x.

Work Problem 7 at the Side.

Answers

7. Both intercepts are (0, 0).



4.2 Slope

OBJECTIVES

1. Find the slope of a line given two points on the line.
2. Find the slope of a line given an equation of the line.
3. Graph a line given its slope and a point on the line.
4. Use slopes to determine whether two lines are parallel, perpendicular, or neither.
5. Solve problems involving average rate of change.

Slope (steepness) is used in many practical ways. The slope of a highway (sometimes called the *grade*) is often given as a percent. For example, a 10% (or $\frac{10}{100} = \frac{1}{10}$) slope means the highway rises 1 unit for every 10 horizontal units. Stairs and roofs have slopes too, as shown in Figure 10.

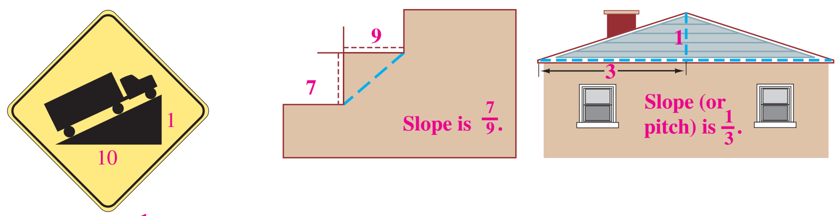


Figure 10

In each example mentioned, slope is the ratio of vertical change, or **rise,** to horizontal change, or **run.** A simple way to remember this is to think "slope is rise over run."

OBJECTIVE 1 Find the slope of a line given two points on the line.

To obtain a formal definition of the slope of a line, we designate two different points on the line. To differentiate between the points, we write them as (x_1, y_1) and (x_2, y_2). See Figure 11. (The small numbers 1 and 2 in these ordered pairs are called *subscripts.* Read (x_1, y_1) as "*x*-sub-one, *y*-sub-one.")

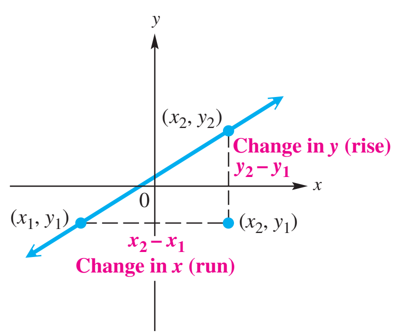


Figure 11

As we move along the line in Figure 11 from (x_1, y_1) to (x_2, y_2), the *y*-value changes (vertically) from y_1 to y_2, an amount equal to $y_2 - y_1$. As *y* changes from y_1 to y_2, the value of *x* changes (horizontally) from x_1 to x_2 by the amount $x_2 - x_1$. The ratio of the change in *y* to the change in *x* (the rise over the run) is called the *slope* of the line, with the letter *m* traditionally used for slope.

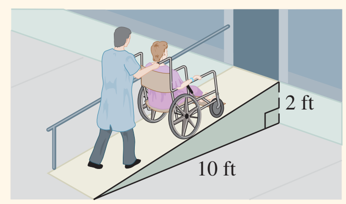
Slope Formula

The **slope** of the line through the distinct points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_1 \neq x_2).$$

Work Problem 1 at the Side.

1 Use the information given for the walkway in the figure to find the following.



(a) The rise

(b) The run

(c) The slope

ANSWERS

1. **(a)** 2 ft **(b)** 10 ft **(c)** $\frac{2}{10}$ or $\frac{1}{5}$

2 Find the slope of the line through each pair of points.

(a) $(-2, 7), (4, -3)$

(b) $(1, 2), (8, 5)$

(c) $(8, -4), (3, -2)$

EXAMPLE 1 Finding the Slope of a Line

Find the slope of the line through the points $(2, -1)$ and $(-5, 3)$.

If $(2, -1) = (x_1, y_1)$ and $(-5, 3) = (x_2, y_2)$, then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - (-1)}{-5 - 2} = \frac{4}{-7} = -\frac{4}{7}.$$

See Figure 12. On the other hand, if the pairs are reversed so that $(2, -1) = (x_2, y_2)$ and $(-5, 3) = (x_1, y_1)$, the slope is

$$m = \frac{-1 - 3}{2 - (-5)} = \frac{-4}{7} = -\frac{4}{7},$$

the same answer.

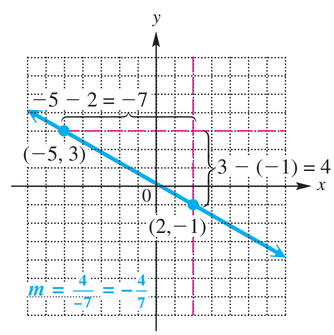


Figure 12

Example 1 suggests that the slope is the same no matter which point we consider first. Also, using similar triangles from geometry, we can show that the slope is the same no matter which two different points on the line we choose.

CAUTION

When calculating slope, ***be careful to subtract the y-values and the x-values in the same order.***

Correct	**Incorrect**
$\dfrac{y_2 - y_1}{x_2 - x_1}$ or $\dfrac{y_1 - y_2}{x_1 - x_2}$	$\cancel{\dfrac{y_2 - y_1}{x_1 - x_2}}$ or $\cancel{\dfrac{y_1 - y_2}{x_2 - x_1}}$

Also, remember that ***the change in y is the numerator and the change in x is the denominator.***

Work Problem 2 at the Side.

OBJECTIVE 2 Find the slope of a line given an equation of the line. When an equation of a line is given, one way to find the slope is to use the definition of slope by first finding two different points on the line.

ANSWERS

2. (a) $-\frac{5}{3}$ **(b)** $\frac{3}{7}$ **(c)** $-\frac{2}{5}$

EXAMPLE 2 Finding the Slope of a Line

Find the slope of the line $4x - y = -8$.

The intercepts can be used as the two different points needed to find the slope. Let $y = 0$ to find that the x-intercept is $(-2, 0)$. Then let $x = 0$ to find that the y-intercept is $(0, 8)$. Use these two points in the slope formula. The slope is

$$m = \frac{\text{rise}}{\text{run}} = \frac{8 - 0}{0 - (-2)} = \frac{8}{2} = 4.$$

Work Problem 3 at the Side.

EXAMPLE 3 Finding the Slopes of Horizontal and Vertical Lines

Find the slope of each line.

(a) $y = 2$

Figure 7 in **Section 4.1** shows that the graph of $y = 2$ is a horizontal line. To find the slope, select two different points on the line, such as $(3, 2)$ and $(-1, 2)$, and use the slope formula.

$$m = \frac{\text{rise}}{\text{run}} = \frac{2 - 2}{3 - (-1)} = \frac{0}{4} = 0$$

In this case, the *rise* is 0, so the slope is 0.

(b) $x = -1$

As shown in Figure 8 (**Section 4.1**), the graph of $x = -1$ (or $x + 1 = 0$) is a vertical line. Two points that satisfy the equation $x = -1$ are $(-1, 5)$ and $(-1, -4)$. Use these two points to find the slope.

$$m = \frac{\text{rise}}{\text{run}} = \frac{-4 - 5}{-1 - (-1)} = \frac{-9}{0}$$

Since division by 0 is undefined, the slope is undefined. This is why the definition of slope includes the restriction $x_1 \neq x_2$.

Generalizing from Example 3, we can make the following statements about horizontal and vertical lines.

Slopes of Horizontal and Vertical Lines

The slope of a horizontal line is 0.

The slope of a vertical line is undefined.

Work Problem 4 at the Side.

3 Find the slope of each line.

(a) $2x + y = 6$

(b) $3x - 4y = 12$

4 Find the slope of each line.

(a) $x = -6$

(b) $y + 5 = 0$

ANSWERS

3. (a) -2 **(b)** $\frac{3}{4}$ **4. (a)** undefined **(b)** 0

5 Find the slope of the graph of $2x - 5y = 8$.

The slope of a line can also be found directly from its equation. Look again at the equation $4x - y = -8$ from Example 2. Solve this equation for y.

$$4x - y = -8 \quad \text{Equation from Example 2}$$
$$-y = -4x - 8 \quad \text{Subtract } 4x.$$
$$y = 4x + 8 \quad \text{Multiply by } -1.$$

Notice that the slope, **4**, found using the slope formula in Example 2 is the same number as the coefficient of x in the equation $y = 4x + 8$. We will see in the next section that this always happens, *as long as the equation is solved for y*.

You Try It

EXAMPLE 4 Finding the Slope from an Equation

Find the slope of the graph of $3x - 5y = 8$.

Solve the equation for y.

$$3x - 5y = 8$$
$$-5y = -3x + 8 \quad \text{Subtract } 3x.$$
$$y = \frac{3}{5}x - \frac{8}{5} \quad \text{Divide by } -5.$$

The slope is given by the coefficient of x, so the slope is $\frac{3}{5}$.

Work Problem 5 at the Side.

OBJECTIVE 3 Graph a line given its slope and a point on the line. Example 5 shows how to graph a straight line by using the slope and one point on the line.

You Try It

Video

EXAMPLE 5 Using the Slope and a Point to Graph Lines

Graph each line.

(a) With slope $\frac{2}{3}$ through the point $(-1, 4)$

First locate the point $P(-1, 4)$ on a graph as shown in Figure 13. Then use the slope to find a second point. From the slope formula,

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{2}{3},$$

so move *up* 2 units and then 3 units to the *right* to locate another point on the graph (labeled R). The line through $P(-1, 4)$ and R is the required graph.

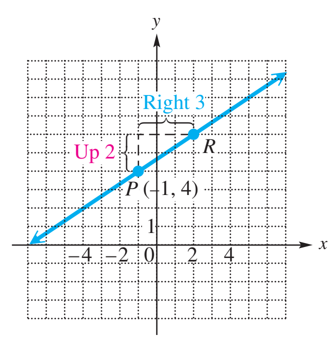


Figure 13

Continued on Next Page

ANSWERS

5. $\frac{2}{5}$

(b) Through (3, 1) with slope -4

Start by locating the point $P(3, 1)$ on a graph. Find a second point R on the line by writing -4 as $\frac{-4}{1}$ and using the slope formula.

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{-4}{1}$$

Move *down* 4 units from (3, 1), and then move 1 unit to the *right*. Draw a line through this second point R and $P(3, 1)$, as shown in Figure 14.

The slope also could be written as

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{4}{-1}.$$

In this case the second point R is located *up* 4 units and 1 unit to the *left*. Verify that this approach produces the same line.

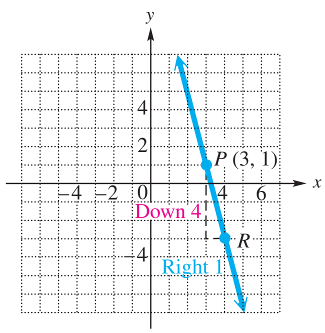


Figure 14

Work Problem 6 at the Side.

In Example 5(a), the slope of the line is the *positive* number $\frac{2}{3}$. The graph of the line in Figure 13 goes up (rises) from left to right. The line in Example 5(b) has a *negative* slope, -4. As Figure 14 shows, its graph goes down (falls) from left to right. These facts suggest the following generalization.

A positive slope indicates that the line goes *up* (rises) from left to right.
A negative slope indicates that the line goes *down* (falls) from left to right.

Figure 15 shows lines of positive, 0, negative, and undefined slopes.

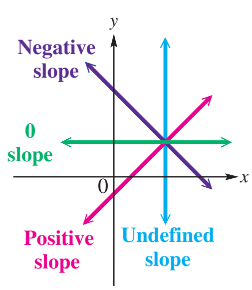
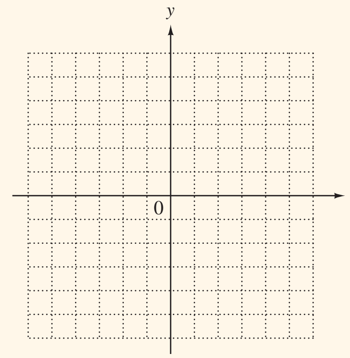


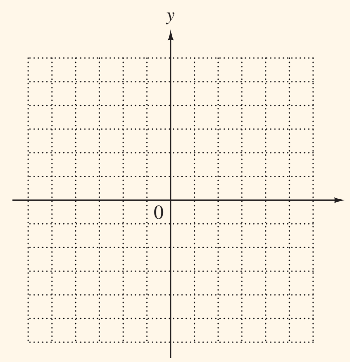
Figure 15

6 Graph each line.

(a) Through $(1, -3)$; $m = -\frac{3}{4}$

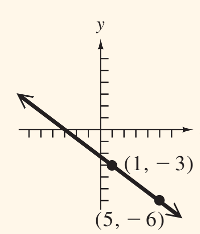


(b) Through $(-1, -4)$; $m = 2$

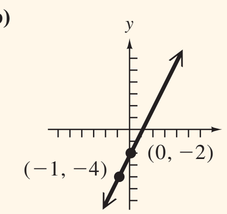


ANSWERS

6. (a)



(b)



OBJECTIVE 4 Use slopes to determine whether two lines are parallel, perpendicular, or neither. The slopes of a pair of parallel or perpendicular lines are related in a special way. The slope of a line measures the steepness of the line. Since parallel lines have equal steepness, their slopes must be equal; also, lines with the same slope are parallel.

Slopes of Parallel Lines

Two nonvertical lines with the same slope are parallel.

Two nonvertical parallel lines have the same slope.

EXAMPLE 6 Determining whether Two Lines Are Parallel

Are the lines L_1, through $(-2, 1)$ and $(4, 5)$, and L_2, through $(3, 0)$ and $(0, -2)$, parallel?

$$\text{The slope of } L_1 \text{ is} \quad m_1 = \frac{5-1}{4-(-2)} = \frac{4}{6} = \frac{2}{3}.$$

$$\text{The slope of } L_2 \text{ is} \quad m_2 = \frac{-2-0}{0-3} = \frac{-2}{-3} = \frac{2}{3}.$$

Because the slopes are equal, the two lines are parallel.

To see how the slopes of perpendicular lines are related, consider a nonvertical line with slope $\frac{a}{b}$. If this line is rotated 90°, the vertical change and the horizontal change are reversed and the slope is $-\frac{b}{a}$, since the horizontal change is now negative. See Figure 16. Thus, the slopes of perpendicular lines have product -1 and are negative reciprocals of each other. For example, if the slopes of two lines are $\frac{3}{4}$ and $-\frac{4}{3}$, then the lines are perpendicular because $\frac{3}{4}\left(-\frac{4}{3}\right) = -1$.

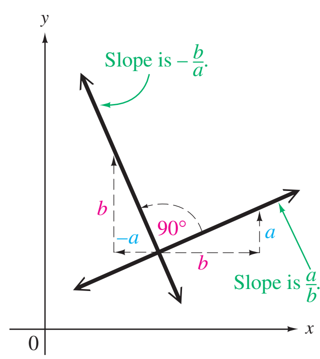


Figure 16

Slopes of Perpendicular Lines

If neither is vertical, perpendicular lines have slopes that are negative reciprocals; that is, their product is -1. Also, lines with slopes that are negative reciprocals are perpendicular.

EXAMPLE 7 Determining whether Two Lines Are Perpendicular

Are the lines with equations $2y = 3x - 6$ and $2x + 3y = -6$ perpendicular?

Find the slope of each line by first solving each equation for y.

$2y = 3x - 6$	$2x + 3y = -6$
$y = \underset{\text{Slope}}{\underbrace{\frac{3}{2}}}x - 3$	$3y = -2x - 6$
	$y = \underset{\text{Slope}}{\underbrace{-\frac{2}{3}}}x - 2$

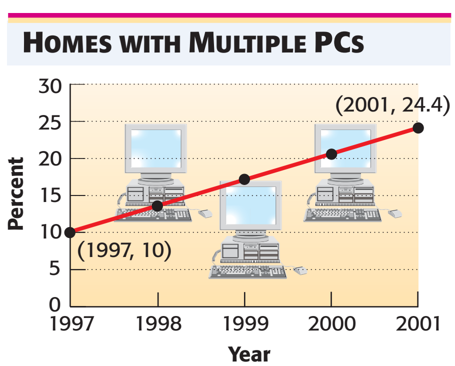
Since the product of the slopes of the two lines is $\frac{3}{2}\left(-\frac{2}{3}\right) = -1$, the lines are perpendicular.

Work Problem 7 at the Side.

OBJECTIVE 5 Solve problems involving average rate of change. We know that the slope of a line is the ratio of the change in y (vertical) to the change in x (horizontal). This idea can be applied to real-life situations. The slope gives the *average rate of change* in y per unit of change in x, where the value of y depends on the value of x.

EXAMPLE 8 Interpreting Slope as Average Rate of Change

The graph in Figure 17 approximates the percent of U.S. households owning multiple personal computers in the years 1997–2001. Find the average rate of change in percent per year.



Source: The Yankee Group.

Figure 17

To determine the average rate of change, we need two pairs of data. From the graph, if $x = 1997$, then $y = 10$ and if $x = 2001$, then $y = 24.4$, so we have the ordered pairs (1997, 10) and (2001, 24.4). By the slope formula,

$$\text{average rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{24.4 - 10}{2001 - 1997} = \frac{14.4}{4} = 3.6.$$

This means that the number of U.S. households owning multiple computers *increased* by 3.6% each year from 1997 to 2001.

Work Problem 8 at the Side.

7 Write *parallel, perpendicular*, or *neither* for each pair of two distinct lines.

(a) The line through $(-1, 2)$ and $(3, 5)$ and the line through $(4, 7)$ and $(8, 10)$

(b) The line through $(5, -9)$ and $(3, 7)$ and the line through $(0, 2)$ and $(8, 3)$

(c) $2x - y = 4$ and $2x + y = 6$

(d) $3x + 5y = 6$ and $5x - 3y = 2$

8 Use the ordered pairs (1997, 10) and (2000, 20.8), which are plotted in Figure 17, to find the average rate of change. How does it compare to the average rate of change found in Example 8?

ANSWERS

7. (a) parallel (b) perpendicular (c) neither (d) perpendicular
8. 3.6; It is the same.

9 In 1997, 36.4 percent of high school students smoked. In 2001, 28.5 percent of high school students smoked. Find the average rate of change in percent per year. (*Source:* U.S. Centers for Disease Control and Prevention.)

EXAMPLE 9 Interpreting Slope as Average Rate of Change

Animation

Video

In 1997, sales of VCRs numbered 16.7 million. In 2002, sales of VCRs were 13.3 million. Find the average rate of change, in millions, per year. (*Source: The Gazette*, June 22, 2002.)

To use the slope formula, we need two ordered pairs. Here, if $x = 1997$, then $y = 16.7$ and if $x = 2002$, then $y = 13.3$, which gives the ordered pairs $(1997, 16.7)$ and $(2002, 13.3)$. (Note that y is in millions.)

$$\text{average rate of change} = \frac{13.3 - 16.7}{2002 - 1997} = \frac{-3.4}{5} = \mathbf{-.68}$$

The graph in Figure 18 confirms that the line through the ordered pairs falls from left to right and therefore has negative slope. Thus, sales of VCRs *decreased* by .68 million each year from 1997 to 2002.

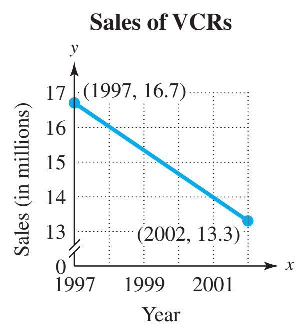


Figure 18

Work Problem 9 at the Side.

ANSWERS

9. -1.975% per yr

4.3 Linear Equations in Two Variables

OBJECTIVES

1. Write an equation of a line given its slope and y-intercept.
2. Graph a line using its slope and y-intercept.
3. Write an equation of a line given its slope and a point on the line.
4. Write an equation of a line given two points on the line.
5. Write an equation of a line parallel or perpendicular to a given line.
6. Write an equation of a line that models real data.

OBJECTIVE 1 Write an equation of a line given its slope and *y*-intercept. In **Section 4.2** we found the slope of a line from the equation of the line by solving the equation for y. For example, we found that the slope of the line with equation

$$y = 4x + 8$$

is 4, the coefficient of x. What does the number 8 represent?

To find out, suppose a line has slope m and y-intercept $(0, b)$. We can find an equation of this line by choosing another point (x, y) on the line, as shown in Figure 19. Using the slope formula,

$$m = \frac{y - b}{x - 0}$$

$$m = \frac{y - b}{x}$$

$$mx = y - b \qquad \text{Multiply by } x.$$

$$mx + b = y \qquad \text{Add } b.$$

$$y = mx + b. \qquad \text{Rewrite.}$$

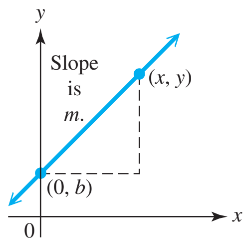


Figure 19

This last equation is called the *slope-intercept form* of the equation of a line, because we can identify the slope m and y-intercept $(0, b)$ at a glance. Thus, in the line with equation

$$y = 4x + 8,$$

the number 8 indicates that the y-intercept is $(0, 8)$.

Slope-Intercept Form

The **slope-intercept form** of the equation of a line with slope m and y-intercept $(0, b)$ is

$$\boldsymbol{y = mx + b.}$$

$$\underset{\text{Slope}}{\uparrow} \qquad \underset{y\text{-intercept is } (0,\, b).}{\uparrow}$$

EXAMPLE 1 Using the Slope-Intercept Form to Find an Equation of a Line

Find an equation of the line with slope $-\frac{4}{5}$ and y-intercept $(0, -2)$.

Here $m = -\frac{4}{5}$ and $b = -2$. Substitute these values into the slope-intercept form.

$$y = mx + b \qquad \text{Slope-intercept form}$$

$$y = -\frac{4}{5}x - 2 \qquad m = -\frac{4}{5};\ b = -2$$

Work Problem 1 at the Side.

1 Write an equation in slope-intercept form for each line with the given slope and y-intercept.

(a) Slope 2; y-intercept $(0, -3)$

(b) Slope $-\frac{2}{3}$; y-intercept $(0, 0)$

ANSWERS

1. **(a)** $y = 2x - 3$ **(b)** $y = -\frac{2}{3}x$

OBJECTIVE 2 Graph a line using its slope and y-intercept. If the equation of a line is written in slope-intercept form, we can use the slope and y-intercept to obtain its graph.

2 Graph each line using its slope and y-intercept.

(a) $y = 2x + 3$

(b) $3x + 4y = 8$

Animation

EXAMPLE 2 Graphing Lines Using Slope and y-Intercept

Graph each line using its slope and y-intercept.

(a) $y = 3x - 6$

Here $m = 3$ and $b = -6$. Plot the y-intercept $(0, -6)$. The slope 3 can be interpreted as

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{3}{1}.$$

From $(0, -6)$, move *up* 3 units and to the *right* 1 unit, and plot a second point at $(1, -3)$. Join the two points with a straight line to obtain the graph in Figure 20.

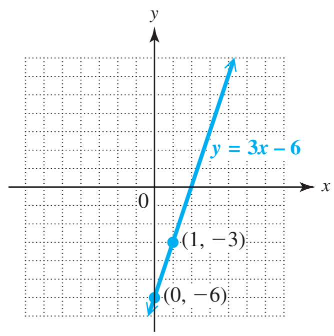


Figure 20

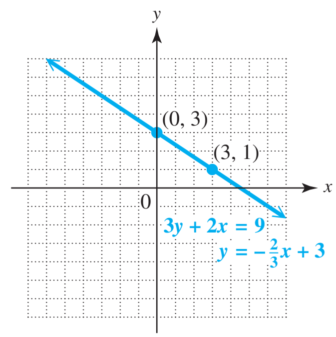


Figure 21

(b) $3y + 2x = 9$

Write the equation in slope-intercept form by solving for y.

$$3y + 2x = 9$$

$$3y = -2x + 9 \qquad \text{Subtract } 2x.$$

$$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x + \underbrace{3}_{y\text{-intercept is } (0,\,3).} \qquad \text{Slope-intercept form}$$

To graph this equation, plot the y-intercept $(0, 3)$. The slope can be interpreted as either $\frac{-2}{3}$ or $\frac{2}{-3}$. Using $\frac{-2}{3}$, move from $(0, 3)$ *down* 2 units and to the *right* 3 units to locate the point $(3, 1)$. The line through these two points is the required graph. See Figure 21. (Verify that the point obtained using $\frac{2}{-3}$ as the slope is also on this line.)

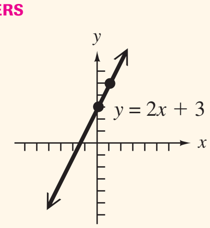
Work Problem 2 at the Side.

NOTE

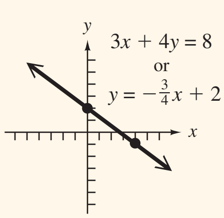
The slope-intercept form of a linear equation is the most useful for several reasons. Every linear equation (of a nonvertical line) has a *unique* (one and only one) slope-intercept form. In **Section 4.5** we study *linear functions,* which are defined using slope-intercept form. Also, this is the form we use when graphing a line with a graphing calculator.

ANSWERS

2. (a)



(b)



OBJECTIVE 3 Write an equation of a line given its slope and a point on the line. Let m represent the slope of a line and (x_1, y_1) represent a given point on the line. Let (x, y) represent any other point on the line. See Figure 22. Then by the slope formula,

$$m = \frac{y - y_1}{x - x_1}$$

$$m(x - x_1) = y - y_1 \quad \text{Multiply by } x - x_1.$$

$$\mathbf{y - y_1 = m(x - x_1)}. \quad \text{Rewrite.}$$

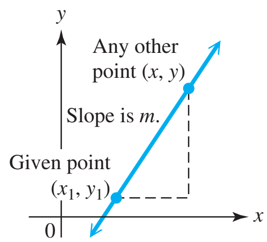


Figure 22

This last equation is the *point-slope form* of the equation of a line.

Point-Slope Form

The **point-slope form** of the equation of a line with slope m passing through the point (x_1, y_1) is

$$\mathbf{y - \underbrace{y_1}_{\text{Given point}} = \overbrace{m}^{\text{Slope}}(x - \underbrace{x_1}_{\text{Given point}})}.$$

To use this form to write the equation of a line, we need to know the coordinates of a point (x_1, y_1) and the slope m of the line.

EXAMPLE 3 Using the Point-Slope Form

Find an equation of the line with slope $\frac{1}{3}$ passing through the point $(-2, 5)$.

Use the point-slope form of the equation of a line, with $(x_1, y_1) = (-2, 5)$ and $m = \frac{1}{3}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$

$$y - 5 = \frac{1}{3}[x - (-2)] \quad y_1 = 5, m = \tfrac{1}{3}, x_1 = -2$$

$$y - 5 = \frac{1}{3}(x + 2)$$

$$3y - 15 = x + 2 \quad \text{Multiply by 3.}$$

$$-x + 3y = 17 \quad \text{Subtract } x\text{; add 15.}$$

In **Section 4.1,** we defined *standard form* for a linear equation as

$$Ax + By = C,$$

where A, B, and C are real numbers. Most often, however, A, B, and C are integers. In this case, let us agree that integers A, B, and C have no common factor (except 1) and $A \geq 0$. For example, the final equation in Example 3, $-x + 3y = 17$, is written in standard form as $x - 3y = -17$.

3 Write an equation of each line in standard form.

(a) Through $(-2, 7)$; $m = 3$

(b) Through $(1, 3)$; $m = -\frac{5}{4}$

NOTE

The definition of "standard form" is not standard from one text to another. Any linear equation can be written in many different (all equally correct) forms. For example, the equation $2x + 3y = 8$ can be written as $2x = 8 - 3y$, $3y = 8 - 2x$, $x + \frac{3}{2}y = 4$, $4x + 6y = 16$, and so on. In addition to writing it in standard form $Ax + By = C$ with $A \geq 0$, let us agree that the form $2x + 3y = 8$ is preferred over any multiples of each side, such as $4x + 6y = 16$. (To write $4x + 6y = 16$ in standard form, divide each side by 2.)

Work Problem 3 at the Side.

OBJECTIVE 4 Write an equation of a line given two points on the line. To find an equation of a line when two points on the line are known, first use the slope formula to find the slope of the line. Then use the slope with either of the given points and the point-slope form of the equation of a line.

EXAMPLE 4 Finding an Equation of a Line Given Two Points

Find an equation of the line through the points $(-4, 3)$ and $(5, -7)$. Write the equation in standard form.

First find the slope by using the slope formula.

$$m = \frac{-7 - 3}{5 - (-4)} = -\frac{10}{9}$$

Use either $(-4, 3)$ or $(5, -7)$ as (x_1, y_1) in the point-slope form of the equation of a line. If we choose $(-4, 3)$, then $-4 = x_1$ and $3 = y_1$.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - 3 = -\frac{10}{9}[x - (-4)]$	$y_1 = 3$, $m = -\frac{10}{9}$, $x_1 = -4$
$y - 3 = -\frac{10}{9}(x + 4)$	
$9y - 27 = -10x - 40$	Multiply by 9; distributive property
$10x + 9y = -13$	Standard form

Verify that if $(5, -7)$ were used, the same equation would result.

Work Problem 4 at the Side.

4 Write an equation in standard form for each line.

(a) Through $(-1, 2)$ and $(5, 7)$

(b) Through $(-2, 6)$ and $(1, 4)$

A horizontal line has slope 0. From the point-slope form, the equation of a horizontal line through the point (a, b) is

$y - y_1 = m(x - x_1)$	Point-slope form
$y - b = 0(x - a)$	$y_1 = b$, $m = 0$, $x_1 = a$
$y - b = 0$	Multiplication property of 0
$y = b.$	Add b.

Notice that the point-slope form does not apply to a vertical line, since the slope of a vertical line is undefined. A vertical line through the point (a, b) has equation $x = a$.

ANSWERS

3. (a) $3x - y = -13$ (b) $5x + 4y = 17$
4. (a) $5x - 6y = -17$ (b) $2x + 3y = 14$

In summary, horizontal and vertical lines have the following special equations.

Equations of Horizontal and Vertical Lines

The horizontal line through the point (a, b) has equation $\boldsymbol{y = b}$.

The vertical line through the point (a, b) has equation $\boldsymbol{x = a}$.

Work Problem 5 at the Side.

OBJECTIVE 5 Write an equation of a line parallel or perpendicular to a given line. As mentioned in the previous section, parallel lines have the same slope and perpendicular lines have slopes with product -1.

EXAMPLE 5 Finding Equations of Lines Parallel or Perpendicular to a Given Line

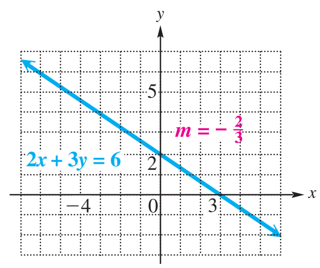
Find the equation in slope-intercept form of the line passing through the point $(-4, 5)$ and **(a)** parallel to the line $2x + 3y = 6$; **(b)** perpendicular to the line $2x + 3y = 6$.

(a) The slope of the line $2x + 3y = 6$ can be found by solving for y.

$$2x + 3y = 6$$

$$3y = -2x + 6 \qquad \text{Subtract } 2x.$$

$$y = \underbrace{-\frac{2}{3}}_{\text{Slope}}x + 2 \qquad \text{Divide by 3.}$$



The slope is given by the coefficient of x, so $m = -\frac{2}{3}$. See the figure. Since parallel lines have the same slope, the required equation of the line through $(-4, 5)$ and parallel to $2x + 3y = 6$ must also have slope $-\frac{2}{3}$. To find this equation, use the point-slope form, with $(x_1, y_1) = (-4, 5)$ and $m = -\frac{2}{3}$.

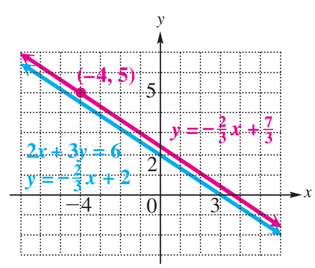
$$y - 5 = -\frac{2}{3}[x - (-4)] \qquad y_1 = 5,\ m = -\tfrac{2}{3},\ x_1 = -4$$

$$y - 5 = -\frac{2}{3}(x + 4)$$

$$y - 5 = -\frac{2}{3}x - \frac{8}{3} \qquad \text{Distributive property}$$

$$y = -\frac{2}{3}x - \frac{8}{3} + \frac{15}{3} \qquad \text{Add } 5 = \tfrac{15}{3}.$$

$$y = -\frac{2}{3}x + \frac{7}{3} \qquad \text{Combine like terms.}$$



We did not clear fractions after the substitution step here because we want the equation in slope-intercept form—that is, solved for y. Both lines are shown in the figure.

Continued on Next Page

5 Write an equation for each line.

(a) Through $(8, -2)$; $m = 0$

(b) The vertical line through $(3, 5)$

ANSWERS

5. **(a)** $y = -2$ **(b)** $x = 3$

6 Write an equation in slope-intercept form of the line passing through the point $(-8, 3)$ and

(a) parallel to the line $2x - 3y = 10$.

(b) perpendicular to the line $2x - 3y = 10$.

(b) In part (a), the given line $2x + 3y = 6$ was written as

$$y = -\frac{2}{3}x + 2,$$

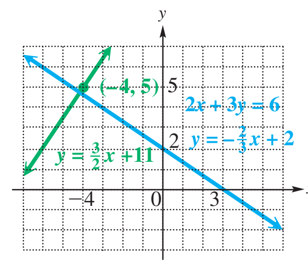
so the line has slope $-\frac{2}{3}$. To be perpendicular to the line $2x + 3y = 6$, a line must have a slope that is the negative reciprocal of $-\frac{2}{3}$, which is $\frac{3}{2}$. Use the point $(-4, 5)$ and slope $\frac{3}{2}$ in the point-slope form to get the equation of the perpendicular line shown in the figure.

$$y - 5 = \frac{3}{2}[x - (-4)] \qquad y_1 = 5,\ m = \tfrac{3}{2},\ x_1 = -4$$

$$y - 5 = \frac{3}{2}(x + 4)$$

$$y - 5 = \frac{3}{2}x + 6 \qquad \text{Distributive property}$$

$$y = \frac{3}{2}x + 11 \qquad \text{Add 5.}$$



Work Problem 6 at the Side.

A summary of the various forms of linear equations follows.

FORMS OF LINEAR EQUATIONS

Equation	Description	When to Use
$y = mx + b$	**Slope-Intercept Form** Slope is m. y-intercept is $(0, b)$.	The slope and y-intercept can be easily identified and used to quickly graph the equation.
$y - y_1 = m(x - x_1)$	**Point-Slope Form** Slope is m. Line passes through (x_1, y_1).	This form is ideal for finding the equation of a line if the slope and a point on the line or two points on the line are known.
$Ax + By = C$	**Standard Form** (A, B, and C integers, $A \geq 0$) Slope is $-\frac{A}{B}$ $(B \neq 0)$. x-intercept is $(\frac{C}{A}, 0)$ $(A \neq 0)$. y-intercept is $(0, \frac{C}{B})$ $(B \neq 0)$.	The x- and y-intercepts can be found quickly and used to graph the equation. Slope must be calculated.
$y = b$	**Horizontal Line** Slope is 0. y-intercept is $(0, b)$.	If the graph intersects only the y-axis, then y is the only variable in the equation.
$x = a$	**Vertical Line** Slope is undefined. x-intercept is $(a, 0)$.	If the graph intersects only the x-axis, then x is the only variable in the equation.

OBJECTIVE 6 Write an equation of a line that models real data. We can use the information presented in this section to write equations of lines that mathematically describe, or *model,* real data if the given set of data changes at a fairly constant rate. In this case, the data fit a linear pattern, and the rate of change is the slope of the line.

ANSWERS

6. (a) $y = \frac{2}{3}x + \frac{25}{3}$ (b) $y = -\frac{3}{2}x - 9$

Video

EXAMPLE 6 **Determining a Linear Equation to Describe Real Data**

Suppose it is time to fill your car with gasoline. At your local station, 89-octane gas is selling for \$1.80 per gal.

(a) Write an equation that describes the cost y to buy x gallons of gas.

Experience has taught you that the total price you pay is determined by the number of gallons you buy multiplied by the price per gallon (in this case, \$1.80). As you pump the gas, two sets of numbers spin by: the number of gallons pumped and the price for that number of gallons.

The table uses ordered pairs to illustrate this situation.

Number of Gallons Pumped	Price of This Number of Gallons
0	0(\$1.80) = \$0.00
1	1(\$1.80) = \$1.80
2	2(\$1.80) = \$3.60
3	3(\$1.80) = \$5.40
4	4(\$1.80) = \$7.20

If we let x denote the number of gallons pumped, then the total price y in dollars can be found by the linear equation

Total price — Number of gallons

$$y = 1.80x.$$

Theoretically, there are infinitely many ordered pairs (x, y) that satisfy this equation, but here we are limited to nonnegative values for x, since we cannot have a negative number of gallons. There is also a practical maximum value for x in this situation, which varies from one car to another. What determines this maximum value?

(b) You can also get a car wash at the gas station if you pay an additional \$3.00. Write an equation that defines the price for gas and a car wash.

Since an additional \$3.00 will be charged, you pay $1.80x + 3.00$ dollars for x gallons of gas and a car wash, described by

$$y = 1.8x + 3. \quad \text{Delete unnecessary 0s.}$$

(c) Interpret the ordered pairs (5, 12) and (10, 21) in relation to the equation from part (b).

The ordered pair (5, 12) indicates that the price of 5 gal of gas and a car wash is \$12.00. Similarly, (10, 21) indicates that the price of 10 gal of gas and a car wash is \$21.00.

Work Problem 7 at the Side.

NOTE

In Example 6(a), the ordered pair (0, 0) satisfied the equation, so the linear equation has the form $y = mx$, where $b = 0$. If a situation involves an initial charge b plus a charge per unit m as in Example 6(b), the equation has the form $y = mx + b$, where $b \neq 0$.

7 **(a)** Suppose it costs \$.10 per minute to make a long-distance call. Write an equation to describe the cost y to make an x-minute call.

(b) Suppose there is a flat rate of \$.20 plus a charge of \$.10 per minute to make a call. Write an equation that gives the cost y for a call of x minutes.

(c) Interpret the ordered pair (15, 1.7) in relation to the equation from part (b).

ANSWERS

7. **(a)** $y = .1x$ (Note: $.10x = .1x$)
(b) $y = .1x + .2$
(c) The ordered pair (15, 1.7) indicates that the price of a 15-minute call is \$1.70.

8 The percent of mothers of children under 1 yr old who participated in the U.S. labor force is shown in the table for selected years.

Year	Percent
1980	38
1984	47
1988	51
1992	54
1998	59

Source: U.S. Bureau of the Census.

(a) Let $x = 0$ represent 1980, $x = 4$ represent 1984, and so on. Use the data for 1980 and 1998 to find an equation that models the data.

(b) Use the equation from part (a) to approximate the percentage of mothers of children under 1 yr old who participated in the U.S. labor force in 2000.

EXAMPLE 7 Finding an Equation of a Line That Models Data

Average annual tuition and fees for in-state students at public 4-year colleges are shown in the table for selected years and graphed as ordered pairs of points in the *scatter diagram* in Figure 23, where $x = 0$ represents 1990, $x = 4$ represents 1994, and so on, and y represents the cost in dollars.

Year	Cost (in dollars)
1990	2035
1994	2820
1996	3151
1998	3486
2000	3774

Source: U.S. National Center for Education Statistics.

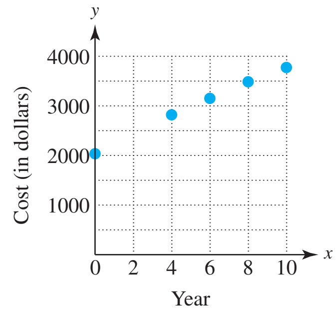


Figure 23

(a) Find an equation that models the data.

Since the points in Figure 23 lie approximately on a straight line, we can write a linear equation that models the relationship between year x and cost y. We choose two data points, (0, 2035) and (10, 3774), to find the slope of the line.

$$m = \frac{3774 - 2035}{10 - 0} = \frac{1739}{10} = 173.9$$

The slope 173.9 indicates that the cost of tuition and fees for in-state students at public 4-year colleges increased by about \$174 per year from 1990 to 2000. We use this slope, the y-intercept (0, **2035**), and the slope-intercept form to write an equation of the line. Thus,

$$y = 173.9x + 2035.$$

(b) Use the equation from part (a) to approximate the cost of tuition and fees at public 4-year colleges in 2002.

The value $x = 12$ corresponds to the year 2002, so we substitute 12 for x in the equation.

$$y = 173.9x + 2035$$
$$y = 173.9(12) + 2035$$
$$y = 4121.8$$

According to the model, average tuition and fees for in-state students at public 4-year colleges in 2002 were about \$4122.

NOTE

In Example 7, if we had chosen different data points, we would have found a slightly different equation. However, all such equations should yield similar results, since the data points are approximately linear.

Work Problem 8 at the Side.

ANSWERS

8. **(a)** $y = 1.17x + 38$ **(b)** 61%

4.4 Linear Inequalities in Two Variables

OBJECTIVES

1. Graph linear inequalities in two variables.
2. Graph the intersection of two linear inequalities.
3. Graph the union of two linear inequalities.

OBJECTIVE 1 Graph linear inequalities in two variables. In **Section 3.1** we graphed linear inequalities in one variable on the number line. We now graph linear inequalities in two variables on a rectangular coordinate system.

Linear Inequality in Two Variables

An inequality that can be written as

$$Ax + By < C \quad \text{or} \quad Ax + By > C,$$

where A, B, and C are real numbers and A and B are not both 0, is a **linear inequality in two variables.**

The symbols $\leq$ and $\geq$ may replace $<$ and $>$ in the definition.

Consider the graph in Figure 24. The graph of the line $x + y = 5$ divides the points in the rectangular coordinate system into three sets:

1. Those points that lie on the line itself and satisfy the equation $\boldsymbol{x + y = 5}$ [like (0, 5), (2, 3), and (5, 0)];
2. Those that lie in the half-plane above the line and satisfy the inequality $\boldsymbol{x + y > 5}$ [like (5, 3) and (2, 4)];
3. Those that lie in the half-plane below the line and satisfy the inequality $\boldsymbol{x + y < 5}$ [like (0, 0) and $(-3, -1)$].

The graph of the line $x + y = 5$ is called the **boundary line** for the inequalities $x + y > 5$ and $x + y < 5$. Graphs of linear inequalities in two variables are *regions* in the real number plane that may or may not include boundary lines.

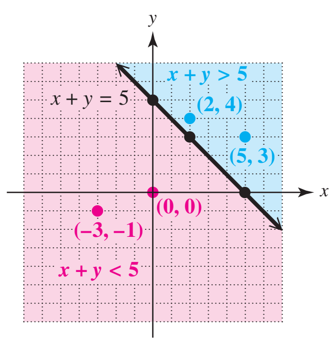


Figure 24

To graph a linear inequality in two variables, follow these steps.

Graphing a Linear Inequality

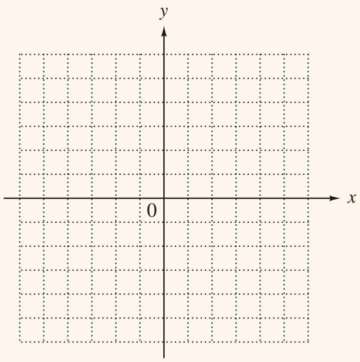
Step 1 **Draw the graph of the straight line that is the boundary.** Make the line solid if the inequality involves $\leq$ or $\geq$; make the line dashed if the inequality involves $<$ or $>$.

Step 2 **Choose a test point.** Choose any point not on the line, and substitute the coordinates of this point in the inequality.

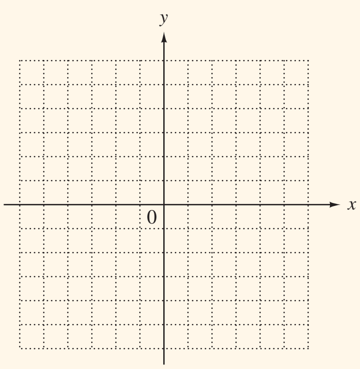
Step 3 **Shade the appropriate region.** Shade the region that includes the test point if it satisfies the original inequality; otherwise, shade the region on the other side of the boundary line.

1 Graph each inequality.

(a) $x + y \leq 4$



(b) $3x + y \geq 6$



EXAMPLE 1 Graphing a Linear Inequality

Graph $3x + 2y \geq 6$.

Step 1 First graph the line $3x + 2y = 6$. The graph of this line, the boundary of the graph of the inequality, is shown in Figure 25.

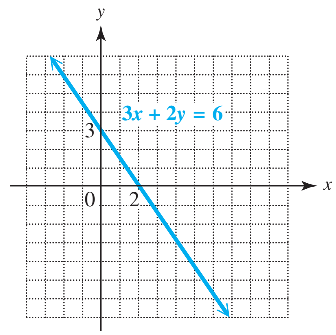


Figure 25

Step 2 The graph of the inequality $3x + 2y \geq 6$ includes the points of the boundary line $3x + 2y = 6$ and either the points *above* the line $3x + 2y = 6$ or the points *below* that line. To decide which, select any point not on the line $3x + 2y = 6$ as a test point. The origin, (0, 0), is often a good choice. Substitute the values from the test point (0, 0) for x and y in the inequality.

$$3x + 2y > 6$$
$$3(0) + 2(0) > 6 \quad ?$$
$$0 > 6 \quad \text{False}$$

Step 3 Because the result is false, (0, 0) does *not* satisfy the inequality, and so the solution set includes all points on the other side of the line. This region is shaded in Figure 26.

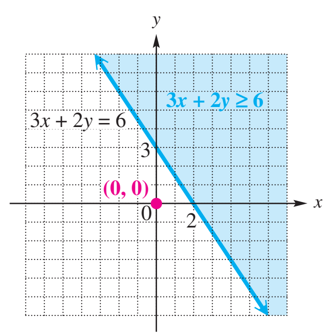


Figure 26

Work Problem 1 at the Side.

If the inequality is written in the form $y > mx + b$ or $y < mx + b$, the inequality symbol indicates which half-plane to shade.

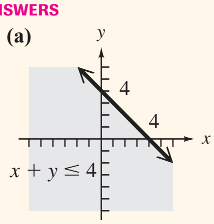
If $y > mx + b$, shade **above** the boundary line.

If $y < mx + b$, shade **below** the boundary line.

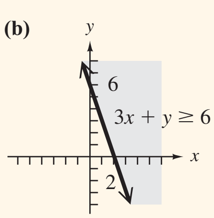
This method works only if the inequality is solved for y.

ANSWERS

1. (a)



(b)



Animation

Video

EXAMPLE 2 Graphing a Linear Inequality

Graph $x - 3y < 4$.

First graph the boundary line, shown in Figure 27. The points of the boundary line do not belong to the inequality $x - 3y < 4$ (because the inequality symbol is $<$, not $\leq$). For this reason, the line is dashed. Now solve the inequality for y.

$$x - 3y < 4$$
$$-3y < -x + 4 \qquad \text{Subtract } x.$$
$$y > \frac{x}{3} - \frac{4}{3} \qquad \text{Multiply by } -\tfrac{1}{3}\text{; change} < \text{to} >.$$

Because of the *is greater than* symbol, shade *above* the line. As a check, choose a test point not on the line, say (1, 2), and substitute for x and y in the original inequality.

$$x - 3y < 4$$
$$1 - 3(2) < 4 \quad ?$$
$$-5 < 4 \qquad \text{True}$$

This result agrees with the decision to shade above the line. The solution set, graphed in Figure 27, includes only those points in the shaded half-plane (not those on the line).

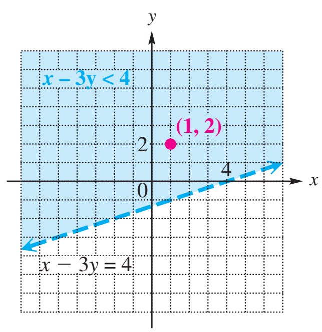


Figure 27

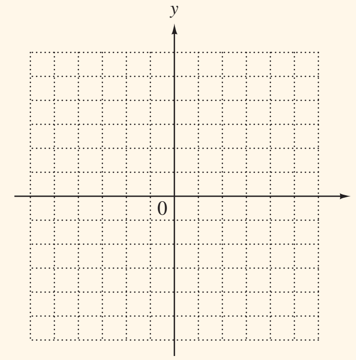
Work Problem 2 at the Side.

OBJECTIVE 2 Graph the intersection of two linear inequalities. In **Section 3.2** we discussed how the words *and* and *or* are used with compound inequalities. In that section, the inequalities had one variable. Those ideas can be extended to include inequalities in two variables.

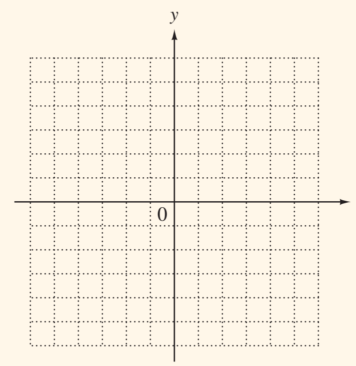
A pair of inequalities joined with the word *and* is interpreted as the intersection of the solution sets of the inequalities. ***The graph of the intersection of two or more inequalities is the region of the plane where all points satisfy all of the inequalities at the same time.***

2 Graph each inequality.

(a) $x - y > 2$

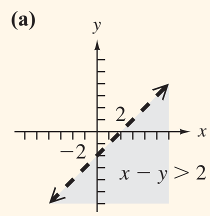


(b) $3x + 4y < 12$

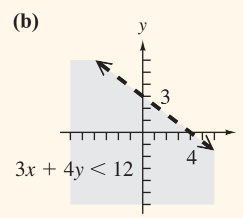


ANSWERS

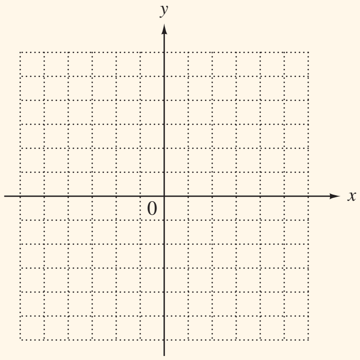
2. (a)



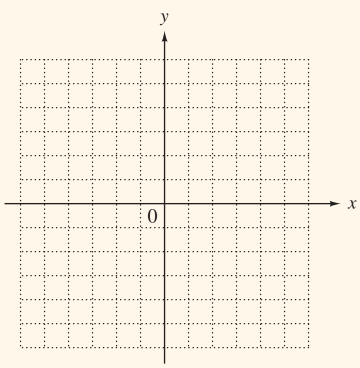
(b)



3 Graph $x - y \leq 4$ and $x \geq -2$.



4 Graph $7x - 3y < 21$ or $x > 2$.



EXAMPLE 3 Graphing the Intersection of Two Inequalities

Graph $2x + 4y \geq 5$ and $x \geq 1$.

To begin, we graph each of the two inequalities $2x + 4y \geq 5$ and $x \geq 1$ separately. The graph of $2x + 4y \geq 5$ is shown in Figure 28(a), and the graph of $x \geq 1$ is shown in Figure 28(b).

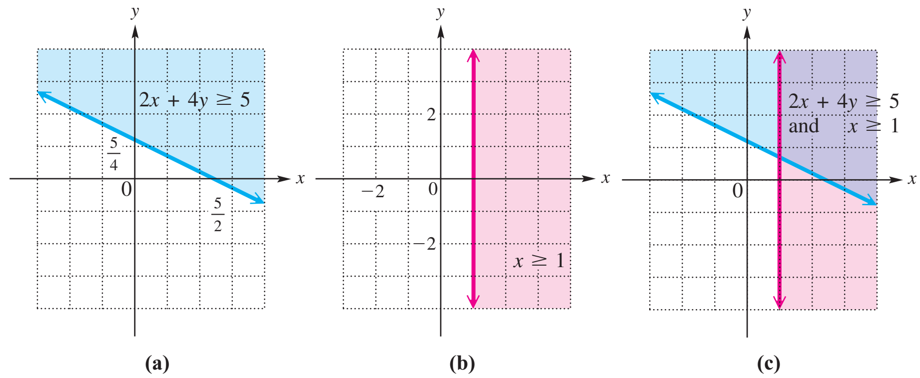


Figure 28

In practice, the two graphs in Figures 28(a) and 28(b) are graphed on the same axes. Then we use heavy shading to identify the intersection of the graphs, as shown in Figure 28(c). To check, we can use a test point from each of the four regions formed by the intersection of the boundary lines. Verify that only ordered pairs in the heavily shaded region satisfy both inequalities.

Work Problem 3 at the Side.

OBJECTIVE 3 Graph the union of two linear inequalities. When two inequalities are joined by the word *or,* we must find the union of the graphs of the inequalities. ***The graph of the union of two inequalities includes all of the points that satisfy either inequality.***

EXAMPLE 4 Graphing the Union of Two Inequalities

Graph $2x + 4y \geq 5$ or $x \geq 1$.

The graphs of the two inequalities are shown in Figures 28(a) and 28(b) in Example 3. The graph of the union is shown in Figure 29.

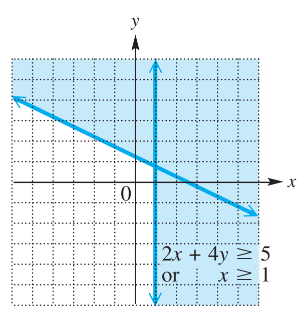
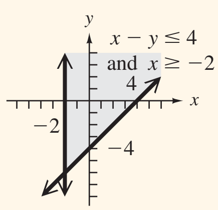


Figure 29

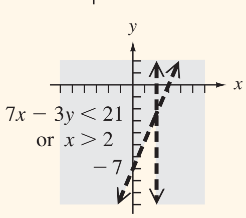
Work Problem 4 at the Side.

ANSWERS

3.



4.



4.5 Introduction to Functions

OBJECTIVES

1. Define and identify relations and functions.
2. Find domain and range.
3. Identify functions defined by graphs and equations.
4. Use function notation.
5. Identify linear functions.

We often describe one quantity in terms of another. Consider the following.

- The amount of your paycheck if you are paid hourly depends on the number of hours you worked.
- The cost at the gas station depends on the number of gallons of gas you pumped into your car.
- The distance traveled by a car moving at a constant speed depends on the time traveled.

We can use ordered pairs to represent these corresponding quantities. For example, we indicate the relationship between the amount of your paycheck and hours worked by writing ordered pairs in which the first number represents hours worked and the second number represents paycheck amount in dollars. Then the ordered pair (**5**, **40**) indicates that when you work **5** hr, your paycheck is $**40**. Similarly, the ordered pairs (10, 80) and (20, 160) show that working 10 hr results in an $80 paycheck and working 20 hr results in a $160 paycheck.

1 What would the ordered pair (40, 320) in the correspondence between number of hours worked and paycheck amount (in dollars) indicate?

Work Problem 1 at the Side.

Since the amount of your paycheck *depends* on the number of hours worked, your paycheck amount is called the *dependent variable,* and the number of hours worked is called the *independent variable*. Generalizing, if the value of the variable y depends on the value of the variable x, then y is the **dependent variable** and x is the **independent variable.**

Independent variable ↓ ↓ Dependent variable

$$(x, y)$$

OBJECTIVE 1 Define and identify relations and functions. Since we can write related quantities using ordered pairs, a set of ordered pairs such as

$$\{(5, 40), (10, 80), (20, 160), (40, 320)\}$$

is called a *relation*.

Relation

A **relation** is any set of ordered pairs.

A special kind of relation, called a *function,* is very important in mathematics and its applications.

Function

A **function** is a relation in which, for each value of the first component of the ordered pairs, there is *exactly one value* of the second component.

Answers

1. It indicates that when you work 40 hr, your paycheck is $320.

2 Determine whether each relation defines a function.

(a) $\{(0, 3), (-1, 2), (-1, 3)\}$

(b) $\{(2, -2), (4, -4), (6, -6)\}$

(c) $\{(-1, 5), (0, 5)\}$

ANSWERS

2. **(a)** not a function **(b)** function **(c)** function

EXAMPLE 1 Determining whether Relations Are Functions

Tell whether each relation defines a function.

$$F = \{(1, 2), (-2, 4), (3, -1)\}$$
$$G = \{(-2, -1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$$
$$H = \{(-4, 1), (-2, 1), (-2, 0)\}$$

Relations F and G are functions, because for each different x-value there is exactly one y-value. Notice that in G, the last two ordered pairs have the same y-value (1 is paired with 2, and 2 is paired with 2). This does not violate the definition of function, since the first components (x-values) are different and each is paired with only one second component (y-value).

In relation H, however, the last two ordered pairs have the *same* x-value paired with *two different* y-values (-2 is paired with both 1 and 0), so H is a relation but not a function. ***In a function, no two ordered pairs can have the same first component and different second components.***

Different y-values

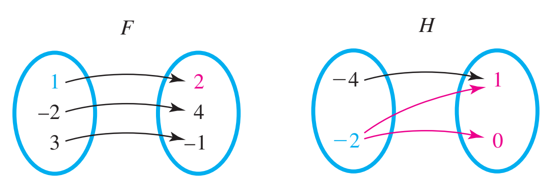
$$H = \{(-4, 1), (\mathbf{-2}, \mathbf{1}), (\mathbf{-2}, \mathbf{0})\} \qquad \text{Not a function}$$

Same x-value

Work Problem 2 at the Side.

In a function, there is *exactly one* value of the dependent variable, the second component, for each value of the independent variable, the first component. This is what makes functions so important in applications.

Relations and functions can also be expressed as a correspondence or *mapping* from one set to another, as shown in Figure 30 for function F and relation H from Example 1. The arrow from 1 to 2 indicates that the ordered pair (1, 2) belongs to F—each first component is paired with exactly one second component. In the mapping for set H, which is not a function, the first component -2 is paired with two different second components, 1 and 0.

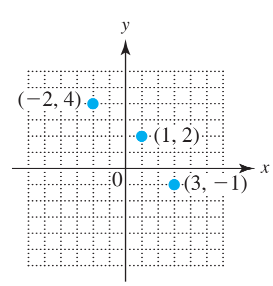


F is a function. H is not a function.

Figure 30

Since relations and functions are sets of ordered pairs, we can represent them using tables and graphs. A table and graph for function F is shown in Figure 31.

x	y
1	2
-2	4
3	-1



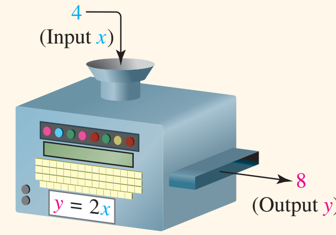
Graph of F

Figure 31

Finally, we can describe a relation or function using a rule that tells how to determine the dependent variable for a specific value of the independent variable. The rule may be given in words, such as "the dependent variable is twice the independent variable." Usually, however, the rule is given as an equation:

$$\underset{\substack{\uparrow\\ \text{Dependent variable}}}{y} = 2\underset{\substack{\uparrow\\ \text{Independent variable}}}{x}.$$

An equation is the most efficient way to define a relation or function.



Function machine

NOTE

Another way to think of a function relationship is to think of the independent variable as an input and the dependent variable as an output. This is illustrated by the input-output (function) machine in the margin for the function defined by $y = 2x$.

OBJECTIVE 2 Find domain and range. For every relation, there are two important sets of elements called the *domain* and *range*.

Domain and Range

In a relation, the set of all values of the independent variable (x) is the **domain.** The set of all values of the dependent variable (y) is the **range.**

EXAMPLE 2 Finding Domains and Ranges of Relations

You Try It

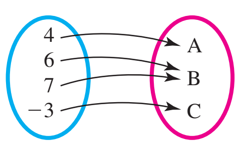
Video

Give the domain and range of each relation. Tell whether the relation defines a function.

(a) $\{(3, -1), (4, 2), (4, 5), (6, 8)\}$

The domain, the set of x-values, is $\{3, 4, 6\}$; the range, the set of y-values, is $\{-1, 2, 5, 8\}$. This relation is not a function because the same x-value 4 is paired with two different y-values, 2 and 5.

(b)



The domain of this relation is

$$\{4, 6, 7, -3\}.$$

The range is

$$\{A, B, C\}.$$

This mapping defines a function—each x-value corresponds to exactly one y-value.

(c)

x	y
-5	2
0	2
5	2

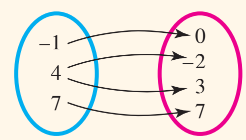
This is a table of ordered pairs, so the domain is the set of x-values, $\{-5, 0, 5\}$, and the range is the set of y-values, $\{2\}$. The table defines a function because each different x-value corresponds to exactly one y-value (even though it is the same y-value).

Work Problem 3 at the Side.

3 Give the domain and range of each relation. Does the relation define a function?

(a) $\{(4, 0), (4, 1), (4, 2)\}$

(b)



(c)

Year	Cell Phone Subscribers (in thousands)
1995	33,786
1996	44,043
1997	55,312
1998	69,209
1999	86,047

Source: Cellular Telecommunications Industry Association.

ANSWERS

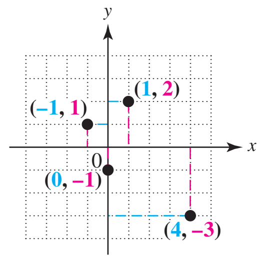
3. (a) domain: $\{4\}$; range: $\{0, 1, 2\}$; No, the relation does not define a function.
(b) domain: $\{-1, 4, 7\}$; range: $\{0, -2, 3, 7\}$; No, the relation does not define a function.
(c) domain: $\{1995, 1996, 1997, 1998, 1999\}$; range: $\{33{,}786, 44{,}043, 55{,}312, 69{,}209, 86{,}047\}$; Yes, the relation defines a function.

The graph of a relation gives a picture of the relation, which can be used to determine its domain and range.

EXAMPLE 3 Finding Domains and Ranges from Graphs

Give the domain and range of each relation.

(a)



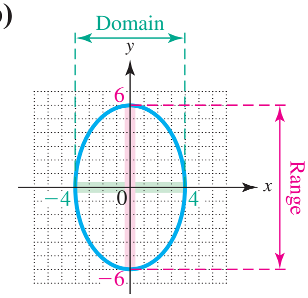
The domain is the set of x-values,

$$\{-1, 0, 1, 4\}.$$

The range is the set of y-values,

$$\{-3, -1, 1, 2\}.$$

(b)

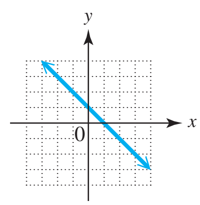


The x-values of the points on the graph include all numbers between -4 and 4, inclusive. The y-values include all numbers between -6 and 6, inclusive. Using interval notation,

the domain is $[-4, 4]$;

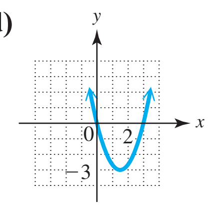
the range is $[-6, 6]$.

(c)



The arrowheads indicate that the line extends indefinitely left and right, as well as up and down. Therefore, both the domain and the range include all real numbers, written $(-\infty, \infty)$.

(d)



The arrowheads indicate that the graph extends indefinitely left and right, as well as upward. The domain is $(-\infty, \infty)$. Because there is a least y-value, -3, the range includes all numbers greater than or equal to -3, written $[-3, \infty)$.

Work Problem 4 at the Side.

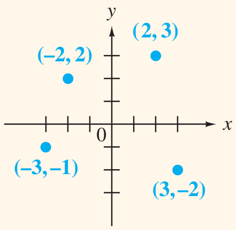
Since relations are often defined by equations, such as $y = 2x + 3$ and $y^2 = x$, we must sometimes determine the domain of a relation from its equation. We assume the following agreement on the domain of a relation.

> **Agreement on Domain**
>
> The domain of a relation is assumed to be all real numbers that produce real numbers when substituted for the independent variable.

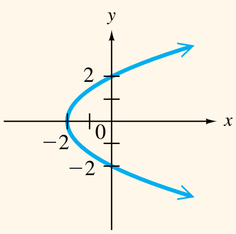
To illustrate this agreement, since any real number can be used as a replacement for x in $y = 2x + 3$, the domain of this function is the set of all real numbers. The function defined by $y = \frac{1}{x}$ has all real numbers except 0 as domain, since y is undefined if $x = 0$. In general, the domain of a function defined by an algebraic expression is all real numbers, except those numbers that lead to division by 0 or an even root of a negative number.

4 Give the domain and range of each relation.

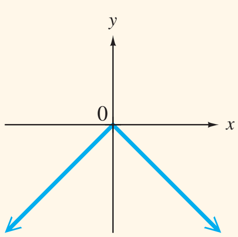
(a)



(b)



(c)



ANSWERS

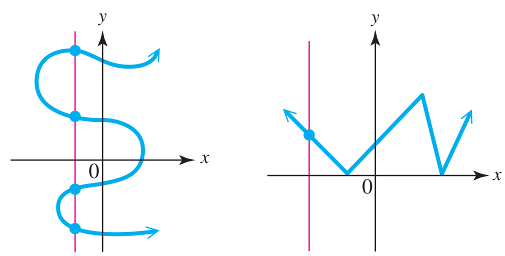
4. (a) domain: $\{-3, -2, 2, 3\}$; range: $\{-2, -1, 2, 3\}$
(b) domain: $[-2, \infty)$; range: $(-\infty, \infty)$
(c) domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

OBJECTIVE 3 Identify functions defined by graphs and equations. Since each value of x in a function corresponds to only one value of y, any vertical line drawn through the graph of a function must intersect the graph in at most one point. This is the *vertical line test* for a function.

Vertical Line Test

If every vertical line intersects the graph of a relation in no more than one point, then the relation represents a function.

For example, the graph shown in Figure 32(a) is not the graph of a function since a vertical line intersects the graph in more than one point. The graph in Figure 32(b) does represent a function.



Not a function – the same x-value corresponds to four different y-values.

(a)

Function – each x-value corresponds to only one y-value.

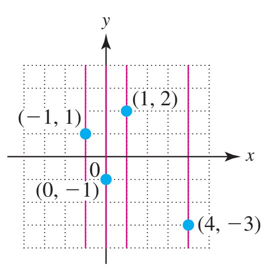
(b)

Figure 32

EXAMPLE 4 Using the Vertical Line Test

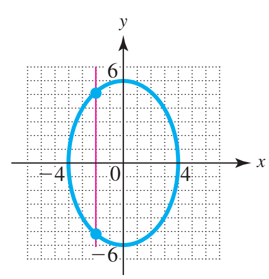
Use the vertical line test to determine whether each relation graphed in Example 3 is a function.

(a)



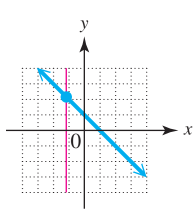
Function

(b)



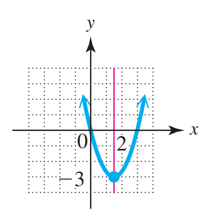
Not a function

(c)



Function

(d)



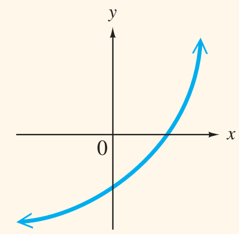
Function

The graphs in (a), (c), and (d) represent functions. The graph of the relation in (b) fails the vertical line test, since the same x-value corresponds to two different y-values; therefore, it is not the graph of a function.

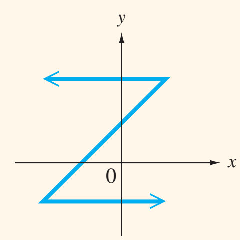
Work Problem 5 at the Side.

5 Use the vertical line test to decide which graphs represent functions.

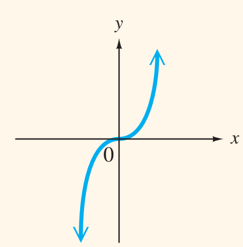
A.



B.



C.



ANSWERS

5. A and C are graphs of functions.

6 Decide whether each relation defines a function, and give the domain.

(a) $y = 6x + 12$

(b) $y \leq 4x$

(c) $y = -\sqrt{3x - 2}$

(d) $y^2 = 25x$

> **NOTE**
> Graphs that do not represent functions are still relations. ***Remember that all equations and graphs represent relations and that all relations have a domain and range.***

It can be more difficult to decide whether a relation defined by an equation is a function. The next example gives some hints that may help.

EXAMPLE 5 Identifying Functions from Their Equations

Decide whether each relation defines a function and give the domain.

(a) $y = x + 4$

In the defining equation, $y = x + 4$, y is always found by adding 4 to x. Thus, each value of x corresponds to just one value of y and the relation defines a function; x can be any real number, so the domain is $(-\infty, \infty)$.

(b) $y = \sqrt{2x - 1}$

For any choice of x in the domain, there is exactly one corresponding value for y (the radical is a nonnegative number), so this equation defines a function. Since the equation involves a square root, the quantity under the radical sign cannot be negative. Thus,

$$\begin{aligned} 2x - 1 &\geq 0 \\ 2x &\geq 1 \\ x &\geq \frac{1}{2}, \end{aligned}$$

and the domain of the function is $\left[\frac{1}{2}, \infty\right)$.

(c) $y^2 = x$

The ordered pairs $(16, 4)$ and $(16, -4)$ both satisfy this equation. Since one value of x, 16, corresponds to two values of y, 4 and -4, this equation does not define a function. Because x is equal to the square of y, the values of x must always be nonnegative. The domain of the relation is $[0, \infty)$.

(d) $y \leq x - 1$

By definition, y is a function of x if every value of x leads to exactly one value of y. Here a particular value of x, say 1, corresponds to many values of y. The ordered pairs $(1, 0)$, $(1, -1)$, $(1, -2)$, $(1, -3)$, and so on, all satisfy the inequality. Thus, ***an inequality never defines a function.*** Any number can be used for x, so the domain is the set of real numbers, $(-\infty, \infty)$.

(e) $y = \dfrac{5}{x - 1}$

Given any value of x in the domain, we find y by subtracting 1, then dividing the result into 5. This process produces exactly one value of y for each value in the domain, so this equation defines a function. The domain includes all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving for x.

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

The domain includes all real numbers *except* 1, written $(-\infty, 1) \cup (1, \infty)$.

Work Problem 6 at the Side.

ANSWERS

6. **(a)** yes; $(-\infty, \infty)$ **(b)** no; $(-\infty, \infty)$
(c) yes; $\left[\frac{2}{3}, \infty\right)$ **(d)** no; $[0, \infty)$

In summary, three variations of the definition of function are given here.

Variations of the Definition of Function

1. A **function** is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
2. A **function** is a set of ordered pairs in which no first component is repeated.
3. A **function** is a rule or correspondence that assigns exactly one range value to each domain value.

OBJECTIVE 4 Use function notation. When a function f is defined with a rule or an equation using x and y for the independent and dependent variables, we say "y is a function of x" to emphasize that y *depends on* x. We use the notation

$$y = f(x),$$

called **function notation,** to express this and read $f(x)$ as "f of x." (In this special notation the parentheses do not indicate multiplication.) The letter f stands for *function*. For example, if $y = 9x - 5$, we can name this function f and write

$$f(x) = 9x - 5.$$

Note that ***$f(x)$ is just another name for the dependent variable y.*** For example, if $y = f(x) = 9x - 5$ and $x = 2$, then we find y, or $f(2)$, by replacing x with 2.

$$\begin{aligned} y = f(2) &= 9 \cdot 2 - 5 \\ &= 18 - 5 \\ &= 13. \end{aligned}$$

For function f, the statement "if $x = 2$, then $y = 13$" is represented by the ordered pair $(2, 13)$ and is abbreviated with function notation as

$$f(2) = 13.$$

Read $f(2)$ as "f of 2" or "f at 2." Also,

$$f(0) = 9 \cdot 0 - 5 = -5 \qquad \text{and} \qquad f(-3) = 9(-3) - 5 = -32.$$

These ideas can be illustrated as follows.

Name of the function

$$y = f(x) = \overbrace{9x - 5}^{\text{Defining expression}}$$

Value of the function   Name of the independent variable

CAUTION

The symbol $f(x)$ *does not* indicate "f times x," but represents the y-value for the indicated x-value. As just shown, $f(2)$ is the y-value that corresponds to the x-value 2.

7 Find $f(-3)$, $f(p)$, and $f(m+1)$.

(a) $f(x) = 6x - 2$

(b) $f(x) = \dfrac{-3x+5}{2}$

(c) $f(x) = \dfrac{1}{6}x - 1$

EXAMPLE 6 Using Function Notation

Let $f(x) = -x^2 + 5x - 3$. Find the following.

(a) $f(2)$

$$f(x) = -x^2 + 5x - 3$$
$$f(2) = -2^2 + 5 \cdot 2 - 3 \quad \text{Replace } x \text{ with } 2.$$
$$f(2) = -4 + 10 - 3$$
$$f(2) = 3$$

Since $f(2) = 3$, the ordered pair $(2, 3)$ belongs to f.

(b) $f(q)$

$$f(x) = -x^2 + 5x - 3$$
$$f(q) = -q^2 + 5q - 3 \quad \text{Replace } x \text{ with } q.$$

The replacement of one variable with another is important in later courses.

Sometimes letters other than f, such as g, h, or capital letters F, G, and H are used to name functions.

EXAMPLE 7 Using Function Notation

Let $g(x) = 2x + 3$. Find and simplify $g(a+1)$.

$$g(x) = 2x + 3$$
$$g(a+1) = 2(a+1) + 3 \quad \text{Replace } x \text{ with } a+1.$$
$$= 2a + 2 + 3$$
$$= 2a + 5$$

Work Problem 7 at the Side.

Functions can be evaluated in a variety of ways, as shown in Example 8.

EXAMPLE 8 Using Function Notation

For each function, find $f(3)$.

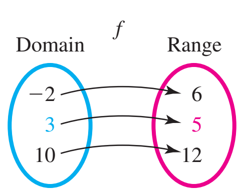
(a) $f(x) = 3x - 7$

$$f(3) = 3(3) - 7$$
$$f(3) = 9 - 7$$
$$f(3) = 2$$

(b) $f = \{(-3, 5), (0, 3), (3, 1), (6, -1)\}$

We want $f(3)$, the y-value of the ordered pair where $x = 3$. As indicated by the ordered pair $(3, 1)$, when $x = 3$, $y = 1$, so $f(3) = 1$.

(c)



The domain element 3 is paired with 5 in the range, so $f(3) = 5$.

Continued on Next Page

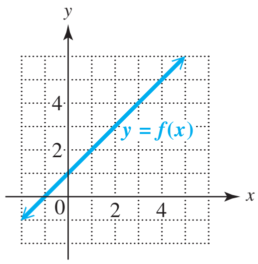
ANSWERS

7. **(a)** -20; $6p - 2$; $6m + 4$

(b) 7; $\dfrac{-3p+5}{2}$; $\dfrac{-3m+2}{2}$

(c) $-\dfrac{3}{2}$; $\dfrac{1}{6}p - 1$; $\dfrac{1}{6}(m+1) - 1$ or $\dfrac{1}{6}m - \dfrac{5}{6}$

(d)



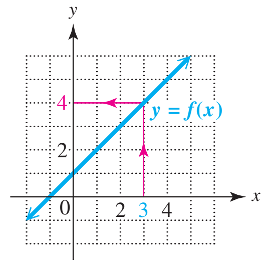


Figure 33

To evaluate $f(3)$, find 3 on the x-axis. See Figure 33. Then move up until the graph of f is reached. Moving horizontally to the y-axis gives 4 for the corresponding y-value. Thus, $f(3) = 4$.

Work Problem 8 at the Side.

If a function f is defined by an equation with x and y, not with function notation, use the following steps to find $f(x)$.

Finding an Expression for $f(x)$

Step 1 Solve the equation for y.

Step 2 Replace y with $f(x)$.

EXAMPLE 9 Writing Equations Using Function Notation

You Try It

Video

Rewrite each equation using function notation. Then find $f(-2)$ and $f(a)$.

(a) $y = x^2 + 1$

This equation is already solved for y. Since $y = f(x)$,

$$f(x) = x^2 + 1.$$

To find $f(-2)$, let $x = -2$.

$$\begin{aligned} f(-2) &= (-2)^2 + 1 \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

Find $f(a)$ by letting $x = a$: $\quad f(a) = a^2 + 1.$

(b) $x - 4y = 5$

First solve $x - 4y = 5$ for y. Then replace y with $f(x)$.

$$\begin{aligned} x - 4y &= 5 \\ x - 5 &= 4y \qquad \text{Add } 4y\text{; subtract 5.} \end{aligned}$$

$$y = \frac{x-5}{4} \quad \text{so} \quad f(x) = \frac{1}{4}x - \frac{5}{4}$$

Now find $f(-2)$ and $f(a)$.

$$f(-2) = \frac{1}{4}(-2) - \frac{5}{4} = -\frac{7}{4} \qquad \text{Let } x = -2.$$

$$f(a) = \frac{1}{4}a - \frac{5}{4} \qquad \text{Let } x = a.$$

Work Problem 9 at the Side.

8 For each function, find $f(-2)$.

(a) $f(x) = -4x - 8$

(b) $f = \{(0, 5), (-1, 3), (-2, 1)\}$

(c)

x	$f(x)$
-4	16
-2	4
0	0
2	4
4	16

9 Rewrite each equation using function notation. Then find $f(-1)$.

(a) $y = \sqrt{x+2}$

(b) $x^2 - 4y = 3$

Answers

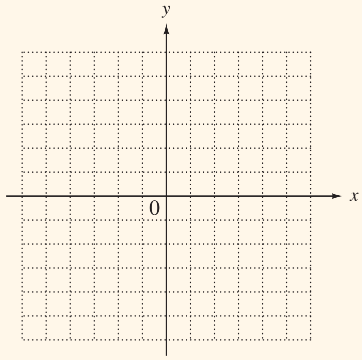
8. **(a)** 0 **(b)** 1 **(c)** 4

9. **(a)** $f(x) = \sqrt{x+2}$; 1

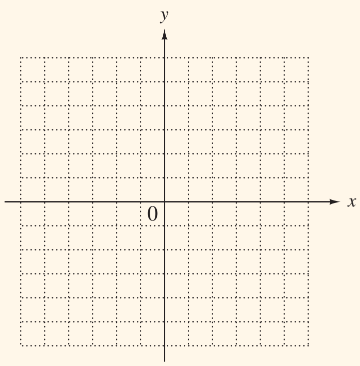
(b) $f(x) = \frac{x^2-3}{4}$ or $f(x) = \frac{1}{4}x^2 - \frac{3}{4}$; $-\frac{1}{2}$

10 Graph each linear function. Give the domain and range.

(a) $f(x) = \frac{3}{4}x - 2$

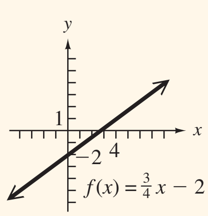


(b) $g(x) = 3$



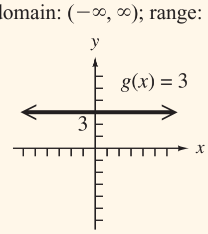
ANSWERS

10. (a)



domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$

(b)



domain: $(-\infty, \infty)$; range: $\{3\}$

OBJECTIVE 5 **Identify linear functions.** Our first two-dimensional graphing was of straight lines. Linear equations (except for vertical lines with equations $x = a$) define *linear functions*.

Linear Function

A function that can be defined by

$$f(x) = mx + b$$

for real numbers m and b is a **linear function.**

Recall from **Section 4.3** that m is the slope of the line and $(0, b)$ is the y-intercept. In Example 9(b), we wrote the equation $x - 4y = 5$ as the linear function defined by

$$f(x) = \underbrace{\frac{1}{4}}_{\text{Slope}}x - \underbrace{\frac{5}{4}}_{y\text{-intercept is } \left(0, -\frac{5}{4}\right)}.$$

To graph this function, plot the y-intercept and use the definition of slope as $\frac{\text{rise}}{\text{run}}$ to find a second point on the line. Draw the straight line through the points to obtain the graph shown in Figure 34.

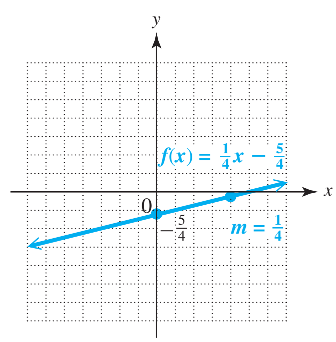


Figure 34

A linear function defined by $f(x) = b$ (whose graph is a horizontal line) is sometimes called a **constant function.** The domain of any linear function is $(-\infty, \infty)$. The range of a nonconstant linear function is $(-\infty, \infty)$, while the range of the constant function defined by $f(x) = b$ is $\{b\}$.

Work Problem 10 at the Side.

4.6 Variation

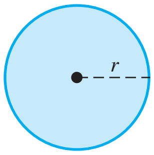
OBJECTIVES

1. Write an equation expressing direct variation.
2. Find the constant of variation, and solve direct variation problems.
3. Solve inverse variation problems.
4. Solve joint variation problems.
5. Solve combined variation problems.

Certain types of functions are very common, especially in business and the physical sciences. These are functions where y depends on a multiple of x, or y depends on a number divided by x. In such situations, y is said to *vary directly as* x (in the first case) or *vary inversely as* x (in the second case). For example, by the distance formula, the distance traveled varies directly as the rate (or speed) and the time. The simple interest formula and the formulas for area and volume are other familiar examples of *direct variation*.

On the other hand, the force required to keep a car from skidding on a curve varies inversely as the radius of the curve. Another example of *inverse variation* is how travel time is inversely proportional to rate or speed.

OBJECTIVE 1 Write an equation expressing direct variation. The circumference of a circle is given by the formula $C = 2\pi r$, where r is the radius of the circle. See the figure. Circumference is always a constant multiple of the radius. (C is always found by multiplying r by the constant 2π.) Thus,



$C = 2\pi r$

As the *radius increases,* the *circumference increases*.

The reverse is also true.

As the *radius decreases,* the *circumference decreases*.

Because of this, the circumference is said to *vary directly* as the radius.

Direct Variation

y* varies directly as *x if there exists some constant k such that

$$y = kx.$$

Also, y is said to be **proportional to** x. The number k is called the **constant of variation.** In direct variation, for $k > 0$, as the value of x increases, the value of y also increases. Similarly, as x decreases, y decreases.

OBJECTIVE 2 Find the constant of variation, and solve direct variation problems. The direct variation equation $y = kx$ defines a linear function, where the constant of variation k is the slope of the line. For example, we wrote the equation

$$y = 1.80x$$

to describe the cost y to buy x gallons of gas in Example 6 of **Section 4.3.** The cost varies directly as, or is proportional to, the number of gallons of gas purchased. That is, as the number of gallons of gas increases, cost increases; also, as the number of gallons of gas decreases, cost decreases. The constant of variation k is 1.80, the cost of 1 gallon of gas.

1 Find the constant of variation, and write a direct variation equation.

(a) Suzanne Alley is paid a daily wage. One month she worked 17 days and earned \$1334.50.

(b) Distance varies directly as time (at a constant speed). A car travels 100 mi at a constant speed in 2 hr.

EXAMPLE 1 Finding the Constant of Variation and the Variation Equation

Gina Linko is paid an hourly wage. One week she worked 43 hr and was paid \$795.50. How much does she earn per hour?

Let h represent the number of hours she works and P represent her corresponding pay. Then, P **varies directly as** h, so

$$P = kh.$$

Here k represents Gina's hourly wage. Since $P = 795.50$ when $h = 43$,

$$795.50 = 43k$$

$$k = 18.50. \qquad \text{Use a calculator.}$$

Her hourly wage is \$18.50, and P and h are related by

$$P = 18.50h.$$

Work Problem 1 at the Side.

2 The charge (in dollars) to customers for electricity (in kilowatt-hours) varies directly as the number of kilowatt-hours used. It costs \$52 to use 800 kilowatt-hours. Find the cost to use 1000 kilowatt-hours.

EXAMPLE 2 Solving a Direct Variation Problem

Hooke's law for an elastic spring states that the distance a spring stretches is proportional to the force applied. If a force of 150 newtons* stretches a certain spring 8 cm, how much will a force of 400 newtons stretch the spring?

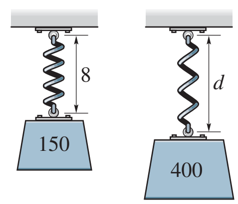


Figure 35

See Figure 35. If d is the distance the spring stretches and f is the force applied, then $d = kf$ for some constant k. Since a force of 150 newtons stretches the spring 8 cm, we can use these values to find k.

$$d = kf \qquad \text{Variation equation}$$

$$8 = k \cdot 150 \qquad \text{Let } d = 8 \text{ and } f = 150.$$

$$k = \frac{8}{150} \qquad \text{Find } k.$$

$$k = \frac{4}{75}$$

Substitute $\frac{4}{75}$ for k in the variation equation $d = kf$ to get

$$d = \frac{4}{75}f.$$

For a force of 400 newtons,

$$d = \frac{4}{75}(400) = \frac{64}{3}. \qquad \text{Let } f = 400.$$

The spring will stretch $\frac{64}{3}$ cm if a force of 400 newtons is applied.

Work Problem 2 at the Side.

*A newton is a unit of measure of force used in physics.

Answers

1. (a) $k = 78.50$; Let E represent her earnings for d days. Then $E = 78.50d$.
   (b) $k = 50$; Let d represent the distance traveled in h hours. Then $d = 50h$.
2. \$65

In summary, use the following steps to solve a variation problem.

Solving a Variation Problem

Step 1 Write the variation equation.

Step 2 Substitute the initial values and solve for k.

Step 3 Rewrite the variation equation with the value of k from Step 2.

Step 4 Substitute the remaining values, solve for the unknown, and find the required answer.

The direct variation equation $y = kx$ is a linear equation. However, other kinds of variation involve other types of equations. For example, one variable can be proportional to a power of another variable.

Direct Variation as a Power

y* varies directly as the *n*th power of *x if there exists a real number k such that

$$\boldsymbol{y = kx^n}.$$

An example of direct variation as a power is the formula for the area of a circle, $A = \pi r^2$. Here, π is the constant of variation, and the area varies directly as the square of the radius.

You Try It

Video

EXAMPLE 3 Solving a Direct Variation Problem

The distance a body falls from rest varies directly as the square of the time it falls (disregarding air resistance). If a skydiver falls 64 ft in 2 sec, how far will she fall in 8 sec?

Step 1 If d represents the distance the skydiver falls and t the time it takes to fall, then d is a function of t, and

$$d = kt^2$$

for some constant k.

Step 2 To find the value of k, use the fact that the skydiver falls 64 ft in 2 sec.

$$\begin{aligned} d &= kt^2 && \text{Variation equation} \\ 64 &= k(2)^2 && \text{Let } d = 64 \text{ and } t = 2. \\ k &= 16 && \text{Find } k. \end{aligned}$$

Step 3 Using 16 for k, the variation equation becomes

$$d = 16t^2.$$

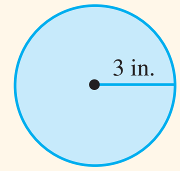
Step 4 Let $t = 8$ to find the number of feet the skydiver will fall in 8 sec.

$$d = 16(8)^2 = 1024 \qquad \text{Let } t = 8.$$

The skydiver will fall 1024 ft in 8 sec.

Work Problem 3 at the Side.

3 The area of a circle varies directly as the square of its radius. A circle with radius 3 in. has area 28.278 in.^{2}.



(a) Write a variation equation and give the value of k.

(b) What is the area of a circle with radius 4.1 in.?

ANSWERS

3. (a) $A = kr^2$; 3.142
(b) 52.817 in.^{2} (to the nearest thousandth)

OBJECTIVE 3 Solve inverse variation problems. In direct variation, where $k > 0$, as x increases, y increases. Similarly, as x decreases, y decreases. Another type of variation is *inverse variation*. With inverse variation, where $k > 0$, as one variable increases, the other variable decreases. For example, in a closed space, volume decreases as pressure increases, as illustrated by a trash compactor. See Figure 36. As the compactor presses down, the pressure on the trash increases; in turn, the trash occupies a smaller space.

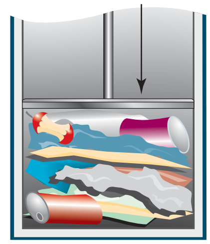


Figure 36

Inverse Variation

y* varies inversely as *x if there exists a real number k such that

$$y = \frac{k}{x}.$$

Also, ***y* varies inversely as the *n*th power of *x*** if there exists a real number k such that

$$y = \frac{k}{x^n}.$$

The inverse variation equation also defines a function. Since x is in the denominator, these functions are *rational functions*. (See **Chapter 8.**) Another example of inverse variation comes from the distance formula. In its usual form, the formula is

$$d = rt.$$

Dividing each side by r gives

$$t = \frac{d}{r}.$$

Here, t (time) varies inversely as r (rate or speed), with d (distance) serving as the constant of variation. For example, if the distance between Chicago and Des Moines is 300 mi, then

$$t = \frac{300}{r}$$

and the values of r and t might be any of the following.

$$\left.\begin{aligned} r &= 50, t = 6 \\ r &= 60, t = 5 \\ r &= 75, t = 4 \end{aligned}\right\} \text{As } r \text{ increases, } t \text{ decreases.} \qquad \left.\begin{aligned} r &= 30, t = 10 \\ r &= 25, t = 12 \\ r &= 20, t = 15 \end{aligned}\right\} \text{As } r \text{ decreases, } t \text{ increases.}$$

If we *increase* the rate (speed) we drive, time *decreases*. If we *decrease* the rate (speed) we drive, time *increases*.

EXAMPLE 4 **Solving an Inverse Variation Problem**

The weight of an object above Earth varies inversely as the square of its distance from the center of Earth. A space shuttle in an elliptical orbit has a maximum distance from the center of Earth (*apogee*) of 6700 mi. Its minimum distance from the center of Earth (*perigee*) is 4090 mi. See Figure 37. If an astronaut in the shuttle weighs 57 lb at its apogee, what does the astronaut weigh at its perigee?

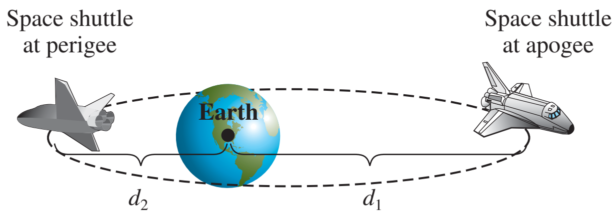


Figure 37

If w is the weight and d is the distance from the center of Earth, then

$$w = \frac{k}{d^2}$$

for some constant k. At the apogee the astronaut weighs 57 lb, and the distance from the center of Earth is 6700 mi. Use these values to find k.

$$57 = \frac{k}{(6700)^2} \quad \text{Let } w = 57 \text{ and } d = 6700.$$

$$k = 57(6700)^2$$

Then the weight at the perigee with $d = 4090$ mi is

$$w = \frac{k}{d^2} = \frac{57(6700)^2}{(4090)^2} \approx 153 \text{ lb.} \quad \text{Use a calculator.}$$

Work Problem 4 at the Side.

OBJECTIVE 4 **Solve joint variation problems.** It is possible for one variable to depend on several others. If one variable varies directly as the *product* of several other variables (perhaps raised to powers), the first variable is said to *vary jointly* as the others.

Joint Variation

y* varies jointly as *x* and *z if there exists a real number k such that

$$y = kxz.$$

CAUTION

Note that *and* in the expression "y varies jointly as ***x and z***" translates as the product

$$y = kxz.$$

The word *and* does not indicate addition here.

4 If the temperature is constant, the volume of a gas varies inversely as the pressure. For a certain gas, the volume is 10 cm^3 when the pressure is 6 kg per cm^2.

(a) Find the variation equation.

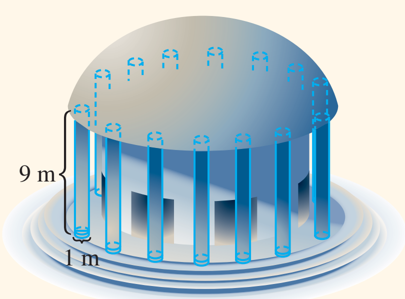
(b) Find the volume when the pressure is 12 kg per cm^2.

ANSWERS

4. **(a)** $V = \frac{60}{P}$ **(b)** 5 cm^3

5 The volume of a rectangular box of a given height is proportional to its width and length. A box with width 2 ft and length 4 ft has volume 12 ft^3. Find the volume of a box with the same height that is 3 ft wide and 5 ft long.

6 The maximum load that a cylindrical column with a circular cross section can hold varies directly as the fourth power of the diameter of the cross section and inversely as the square of the height. A 9-m column 1 m in diameter will support 8 metric tons. How many metric tons can be supported by a column 12 m high and $\frac{2}{3}$ m in diameter?



Load = 8 metric tons

ANSWERS
5. 22.5 ft^3
6. $\frac{8}{9}$ metric ton

EXAMPLE 5 Solving a Joint Variation Problem

The interest on a loan or an investment is given by the formula $I = prt$. Here, for a given principal p, the interest earned I varies jointly as the interest rate r and the time t that the principal is left at interest. If an investment earns \$100 interest at 5% for 2 yr, how much interest will the same principal earn at 4.5% for 3 yr?

We use the formula $I = prt$, where p is the constant of variation because it is the same for both investments. For the first investment,

$$\begin{aligned} I &= prt \\ 100 &= p(.05)(2) && \text{Let } I = 100,\ r = .05, \text{ and } t = 2. \\ 100 &= .1p \\ p &= 1000. && \text{Divide by } .1. \end{aligned}$$

Now we find I when $p = 1000$, $r = .045$, and $t = 3$.

$$I = 1000(.045)(3) = 135 \qquad \text{Let } p = 1000,\ r = .045, \text{ and } t = 3.$$

The interest will be \$135.

Work Problem 5 at the Side.

OBJECTIVE 5 Solve combined variation problems. There are many combinations of direct and inverse variation, called **combined variation.**

EXAMPLE 6 Solving a Combined Variation Problem

Body mass index, or BMI, is used by physicians to assess a person's level of fatness. A BMI from 19 through 25 is considered desirable. BMI varies directly as an individual's weight in pounds and inversely as the square of the individual's height in inches. A person who weighs 118 lb and is 64 in. tall has a BMI of 20. (The BMI is rounded to the nearest whole number.) Find the BMI of a person who weighs 165 lb with a height of 70 in.

Let B represent the BMI, w the weight, and h the height. Then

$$B = \frac{kw}{h^2}. \qquad \begin{aligned} &\leftarrow \text{BMI varies directly as the weight.} \\ &\leftarrow \text{BMI varies inversely as the square of the height.} \end{aligned}$$

To find k, let $B = 20$, $w = 118$, and $h = 64$.

$$\begin{aligned} 20 &= \frac{k(118)}{64^2} \\ k &= \frac{20(64^2)}{118} && \text{Multiply by } 64^2; \text{ divide by } 118. \\ k &\approx 694 && \text{Use a calculator.} \end{aligned}$$

Now find B when $k = 694$, $w = 165$, and $h = 70$.

$$B = \frac{694(165)}{70^2} \approx 23 \qquad \text{Nearest whole number}$$

The person's BMI is 23.

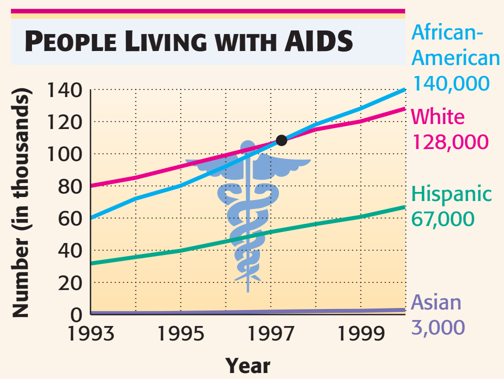
Work Problem 6 at the Side.

Systems of Linear Equations

5

During the last decade of the twentieth century, the number of people living with AIDS in various racial groups in the United States followed linear patterns, as shown in the accompanying graph. At some year during that decade, the number of people in two of these groups was the same. The graph can be used to determine that year and number by finding the coordinates of the point of intersection of the two lines. See Exercises 1 and 2 of the Chapter 5 Test.

The process of determining the point of intersection of two lines is the idea behind solving a *system of linear equations in two variables*. This chapter illustrates methods of finding such points.



Source: U.S. Centers for Disease Control and Prevention.

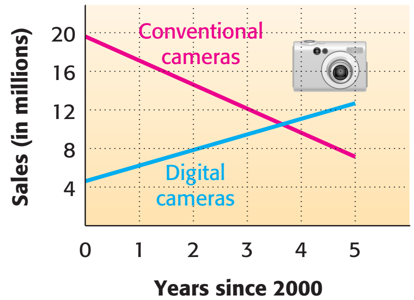
5.1 Systems of Linear Equations in Two Variables

OBJECTIVES

1. **Solve linear systems by graphing.**
2. **Decide whether an ordered pair is a solution of a linear system.**
3. **Solve linear systems (with two equations and two variables) by substitution.**
4. **Solve linear systems (with two equations and two variables) by elimination.**
5. **Solve special systems.**

As technology continues to improve, the sale of digital cameras increases, while that of conventional cameras decreases. This can be seen in Figure 1, which illustrates this growth and decline using a graph. The two straight-line graphs intersect where the two types of cameras had the same sales.

SAY CHEESE!



Source: Consumer Electronics Association.

Figure 1

We could use a linear equation to model the graph of digital camera sales and another linear equation to model the graph of conventional camera sales. Such a set of equations is called a **system of equations,** in this case a **linear system of equations.** The point where the graphs in Figure 1 intersect is a solution of each of the individual equations. It is also the solution of the linear system of equations.

OBJECTIVE 1 Solve linear systems by graphing. The **solution set of a system of equations** contains all ordered pairs that satisfy all the equations of the system *at the same time*. An example of a linear system is

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4. \end{aligned}$$

Linear system of equations

One way to find the solution set of a linear system of equations is to graph each equation and find the point where the graphs intersect.

Video

EXAMPLE 1 Solving a System by Graphing

Solve the system of equations by graphing.

$$\begin{aligned} x + y &= 5 \qquad (1) \\ 2x - y &= 4 \qquad (2) \end{aligned}$$

When we graph these linear equations as shown in Figure 2, the graph suggests that the point of intersection is the ordered pair (3, 2).

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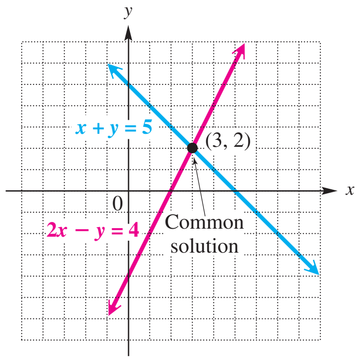

Figure 2

To be sure that (3, 2) is a solution of *both* equations, we check by substituting 3 for x and 2 for y in each equation.

$x + y = 5$	(1)	$2x - y = 4$	(2)
$3 + 2 = 5$?		$2(3) - 2 = 4$?	
$5 = 5$	True	$6 - 2 = 4$?	
		$4 = 4$	True

Since (3, 2) makes both equations true, $\{(3, 2)\}$ is the solution set of the system.

Work Problem 1 at the Side.

Calculator Tip A graphing calculator can be used to solve a system. Each equation must be solved for y before being entered in the calculator. The point of intersection of the graphs, which is the solution of the system, can then be displayed. Consult your owner's manual for details.

OBJECTIVE 2 Decide whether an ordered pair is a solution of a linear system. To decide if an ordered pair is a solution of a system, we substitute the ordered pair in both equations of the system, just as we did when we checked the solution in Example 1.

EXAMPLE 2 Deciding whether an Ordered Pair Is a Solution

Decide whether the given ordered pair is a solution of the given system.

(a) $\begin{aligned} x + y &= 6 \\ 4x - y &= 14 \end{aligned}$; (4, 2)

Replace x with 4 and y with 2 in each equation of the system.

$x + y = 6$		$4x - y = 14$	
$4 + 2 = 6$?		$4(4) - 2 = 14$?	
$6 = 6$	True	$16 - 2 = 14$?	
		$14 = 14$	True

Since (4, 2) makes both equations true, (4, 2) is a solution of the system.

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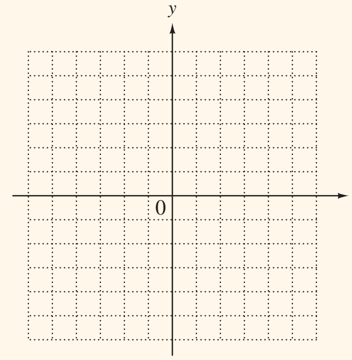
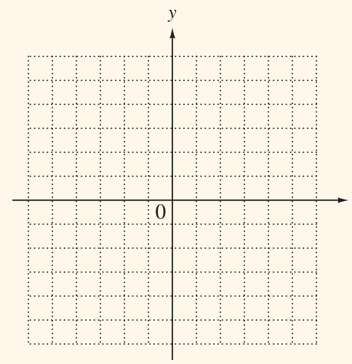
1 Solve each system by graphing.

(a) $\begin{aligned} x - y &= 3 \quad (1) \\ 2x - y &= 4 \quad (2) \end{aligned}$

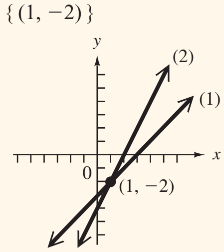


(b) $\begin{aligned} 2x + y &= -5 \quad (1) \\ -x + 3y &= 6 \quad (2) \end{aligned}$

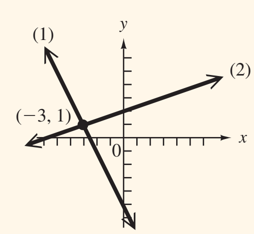


ANSWERS

1. (a) $\{(1, -2)\}$



(b) $\{(-3, 1)\}$



(b) $\begin{aligned} 3x + 2y &= 11 \\ x + 5y &= 36 \end{aligned}$; $(-1, 7)$

$3x + 2y = 11$	$x + 5y = 36$
$3(-1) + 2(7) = 11$?	$-1 + 5(7) = 36$?
$-3 + 14 = 11$?	$-1 + 35 = 36$?
$11 = 11$ True	$34 = 36$ False

The ordered pair $(-1, 7)$ is not a solution of the system, since it does not make *both* equations true.

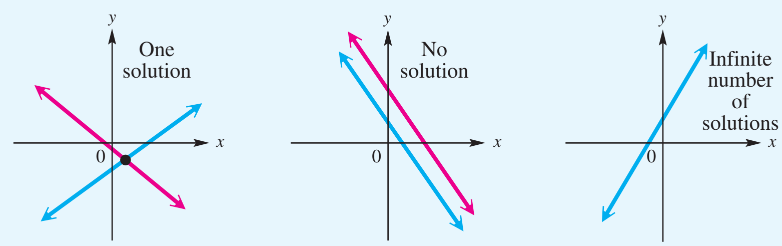
Work Problem 2 at the Side.

Since the graph of a linear equation is a straight line, there are three possibilities for the solution set of a linear system in two variables.

Audio

Graphs of Linear Systems in Two Variables

1. The two graphs intersect in a single point. The coordinates of this point give the only solution of the system. In this case the system is **consistent,** and the equations are **independent.** This is the most common case. See Figure 3(a).
2. The graphs are parallel lines. In this case the system is **inconsistent;** that is, there is no solution common to both equations of the system, and the solution set is $\emptyset$. See Figure 3(b).
3. The graphs are the same line. In this case the equations are **dependent,** since any solution of one equation of the system is also a solution of the other. The solution set is an infinite set of ordered pairs representing the points on the line. See Figure 3(c).



(a) (b) (c)

Figure 3

OBJECTIVE 3 Solve linear systems (with two equations and two variables) by substitution. Since it can be difficult to read exact coordinates, especially if they are not integers, from a graph, we usually use algebraic methods to solve systems. One such method, the **substitution method,** is most useful for solving linear systems in which one equation is solved or can be easily solved for one variable in terms of the other.

EXAMPLE 3 Solving a System by Substitution

Solve the system.

$$2x - y = 6 \qquad (1)$$
$$x = y + 2 \qquad (2)$$

Since equation (2) is solved for x, substitute $y + 2$ for x in equation (1).

Continued on Next Page

2 Are the given ordered pairs solutions of the given systems?

(a) $\begin{aligned} 2x + y &= -6 \\ x + 3y &= 2 \end{aligned}$; $(-4, 2)$

(b) $\begin{aligned} 9x - y &= -4 \\ 4x + 3y &= 11 \end{aligned}$; $(-1, 5)$

ANSWERS
2. (a) yes (b) no

$$
\begin{aligned}
2x - y &= 6 && (1)\\
2(y + 2) - y &= 6 && \text{Let } x = y + 2.\\
2y + 4 - y &= 6 && \text{Distributive property}\\
y + 4 &= 6 && \text{Combine terms.}\\
y &= 2 && \text{Subtract 4.}
\end{aligned}
$$

We found y. Now find x by substituting **2** for y in equation (2).

$$x = y + 2 = 2 + 2 = 4$$

Thus, $x = 4$ and $y = 2$, giving the ordered pair $(4, 2)$. Check this solution in both equations of the original system. The solution set is $\{(4, 2)\}$.

CAUTION

Be careful! Even though we found y first in Example 3, ***the x-coordinate is always written first in the ordered pair solution of a system.***

Work Problem 3 at the Side.

The substitution method is summarized as follows.

Audio

Solving a Linear System by Substitution

Step 1 **Solve one of the equations for either variable.** If one of the variable terms has coefficient 1 or -1, choose it, since the substitution method is usually easier this way.

Step 2 **Substitute** for that variable in the other equation. The result should be an equation with just one variable.

Step 3 **Solve** the equation from Step 2.

Step 4 **Find the other value.** Substitute the result from Step 3 into the equation from Step 1 to find the value of the other variable.

Step 5 **Check** the solution in both of the original equations. Then write the solution set.

You Try It

Video

EXAMPLE 4 **Solving a System by Substitution**

Solve the system.

$$
\begin{aligned}
3x + 2y &= 13 && (1)\\
4x - y &= -1 && (2)
\end{aligned}
$$

Step 1 First solve one of the equations for x or y. Since the coefficient of y in equation (2) is -1, it is easiest to solve for y in equation (2).

$$
\begin{aligned}
4x - y &= -1 && (2)\\
-y &= -1 - 4x && \text{Subtract } 4x.\\
y &= 1 + 4x && \text{Multiply by } -1.
\end{aligned}
$$

Step 2 Substitute $1 + 4x$ for y in equation (1).

$$
\begin{aligned}
3x + 2y &= 13 && (1)\\
3x + 2(1 + 4x) &= 13 && \text{Let } y = 1 + 4x.
\end{aligned}
$$

Continued on Next Page

3 Solve by substitution.

(a) $7x - 2y = -2$
$y = 3x$

(b) $5x - 3y = -6$
$x = 2 - y$

ANSWERS

3. **(a)** $\{(-2, -6)\}$ **(b)** $\{(0, 2)\}$

4 Solve by substitution.

(a) $3x - y = 10$
$2x + 5y = 1$

(b) $4x - 5y = -11$
$x + 2y = 7$

5 Solve by elimination.

(a) $3x - y = -7$
$2x + y = -3$

(b) $-2x + 3y = -10$
$2x + 2y = 5$

Answers
4. (a) $\{(3, -1)\}$ (b) $\{(1, 3)\}$
5. (a) $\{(-2, 1)\}$ (b) $\left\{\left(\frac{7}{2}, -1\right)\right\}$

Step 3 Solve for x.

$$\begin{aligned} 3x + 2(1 + 4x) &= 13 && \text{From Step 2} \\ 3x + 2 + 8x &= 13 && \text{Distributive property} \\ 11x &= 11 && \text{Combine terms; subtract 2.} \\ x &= 1 && \text{Divide by 11.} \end{aligned}$$

Step 4 Now solve for y. From Step 1, $y = 1 + 4x$, so

$$y = 1 + 4(1) = 5. \quad \text{Let } x = 1.$$

Step 5 Check the solution (1, 5) in both equations (1) and (2).

$$\begin{aligned} 3x + 2y &= 13 && (1) \\ 3(1) + 2(5) &= 13 && ? \\ 3 + 10 &= 13 && ? \\ 13 &= 13 && \text{True} \end{aligned} \qquad \begin{aligned} 4x - y &= -1 && (2) \\ 4(1) - 5 &= -1 && ? \\ 4 - 5 &= -1 && ? \\ -1 &= -1 && \text{True} \end{aligned}$$

The solution set is $\{(1, 5)\}$.

Work Problem 4 at the Side.

OBJECTIVE 4 Solve linear systems (with two equations and two variables) by elimination. Another algebraic method, the **elimination method,** involves combining the two equations in a system so that one variable is eliminated. This is done using the following logic:

$$\text{If} \quad a = b \text{ and } c = d, \quad \text{then} \quad a + c = b + d.$$

EXAMPLE 5 Solving a System by Elimination

Solve the system.

$$\begin{aligned} 2x + 3y &= -6 && (1) \\ 4x - 3y &= 6 && (2) \end{aligned}$$

Notice that adding the equations together will eliminate the variable y.

$$\begin{aligned} 2x + 3y &= -6 && (1) \\ 4x - 3y &= 6 && (2) \\ \hline 6x \qquad &= 0 && \text{Add.} \\ x &= 0 && \text{Solve for } x. \end{aligned}$$

To find y, substitute 0 for x in either equation (1) or equation (2).

$$\begin{aligned} 2x + 3y &= -6 && (1) \\ 2(0) + 3y &= -6 && \text{Let } x = 0. \\ 0 + 3y &= -6 \\ 3y &= -6 \\ y &= -2 \end{aligned}$$

The solution of the system is $(0, -2)$. Check by substituting 0 for x and -2 for y in both equations of the original system. The solution set is $\{(0, -2)\}$.

Work Problem 5 at the Side.

By adding the equations in Example 5, we eliminated the variable y because the coefficients of the y-terms were opposites. In many cases the coefficients will *not* be opposites, and we must transform one or both equations so that the coefficients of one pair of variable terms are opposites.

Audio

Solving a Linear System by Elimination

Step 1 **Write both equations in standard form $Ax + By = C$.**

Step 2 **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the x- or y-terms is 0.

Step 3 **Add** the new equations to eliminate a variable. The sum should be an equation with just one variable.

Step 4 **Solve** the equation from Step 3 for the remaining variable.

Step 5 **Find the other value.** Substitute the result of Step 4 into either of the original equations and solve for the other variable.

Step 6 **Check** the solution in both of the original equations. Then write the solution set.

You Try It

Video

EXAMPLE 6 **Solving a System by Elimination**

Solve the system.

$$5x - 2y = 4 \quad (1)$$
$$2x + 3y = 13 \quad (2)$$

Step 1 Both equations are in standard form.

Step 2 Suppose that you wish to eliminate the variable x. One way to do this is to multiply equation (1) by 2 and equation (2) by -5.

$$10x - 4y = 8 \quad \text{2 times each side of equation (1)}$$
$$-10x - 15y = -65 \quad -5 \text{ times each side of equation (2)}$$

Step 3 Now add.

$$\begin{aligned} 10x - 4y &= 8 \\ -10x - 15y &= -65 \\ \hline -19y &= -57 \quad \text{Add.} \end{aligned}$$

Step 4 Solve for y. $\quad y = 3 \quad$ Divide by -19.

Step 5 To find x, substitute 3 for y in either equation (1) or (2). Substituting in equation (2) gives

$$\begin{aligned} 2x + 3y &= 13 && (2) \\ 2x + 3(3) &= 13 && \text{Let } y = 3. \\ 2x + 9 &= 13 \\ 2x &= 4 && \text{Subtract 9.} \\ x &= 2. && \text{Divide by 2.} \end{aligned}$$

Step 6 The solution is (2, 3). To check, substitute 2 for x and 3 for y in both equations (1) and (2).

$$\begin{aligned} 5x - 2y &= 4 && (1) \\ 5(2) - 2(3) &= 4 && ? \\ 10 - 6 &= 4 && ? \\ 4 &= 4 && \text{True} \end{aligned} \qquad \begin{aligned} 2x + 3y &= 13 && (2) \\ 2(2) + 3(3) &= 13 && ? \\ 4 + 9 &= 13 && ? \\ 13 &= 13 && \text{True} \end{aligned}$$

The solution set is $\{(2, 3)\}$.

Work Problem 6 at the Side.

6 Solve by elimination.

(a) $x + 3y = 8$
$2x - 5y = -17$

(b) $6x - 2y = -21$
$-3x + 4y = 36$

(c) $2x + 3y = 19$
$3x - 7y = -6$

Answers

6. **(a)** $\{(-1, 3)\}$ **(b)** $\left\{\left(-\frac{2}{3}, \frac{17}{2}\right)\right\}$
(c) $\{(5, 3)\}$

7 Solve each system.

(a) $\frac{1}{3}x - \frac{1}{2}y = \frac{1}{6}$

$3x - 2y = 9$

(b) $\frac{x}{5} + \frac{2y}{3} = -\frac{8}{5}$

$3x - y = 9$

You Try It

EXAMPLE 7 Solving a System with Fractional Coefficients

Solve the system.

$$5x - 2y = 4 \qquad (1)$$

$$\frac{1}{2}x + \frac{3}{4}y = \frac{13}{4} \qquad (2)$$

If an equation in a system has fractional coefficients, as in equation (2), first multiply by the least common denominator to clear the fractions.

$$4\left(\frac{1}{2}x + \frac{3}{4}y\right) = 4 \cdot \frac{13}{4} \qquad \text{Multiply equation (2) by the LCD, 4.}$$

$$4 \cdot \frac{1}{2}x + 4 \cdot \frac{3}{4}y = 4 \cdot \frac{13}{4} \qquad \text{Distributive property}$$

$$2x + 3y = 13 \qquad \text{Equivalent to equation (2)}$$

The system of equations becomes

$$5x - 2y = 4 \qquad (1)$$

$$2x + 3y = 13, \qquad \text{Equation (2) with fractions cleared}$$

which is identical to the system we solved in Example 6. The solution set is $\{(2, 3)\}$. To confirm this, check the solution in both equations (1) and (2).

Work Problem 7 at the Side.

NOTE

If an equation in a system contains decimal coefficients, it is best to first clear the decimals by multiplying by 10, 100, or 1000, depending on the number of decimal places. Then solve the system. For example, we multiply *each side* of the equation

$$.5x + .75y = 3.25$$

by 100 to get the equivalent equation

$$50x + 75y = 325.$$

OBJECTIVE 5 Solve special systems. As we saw in Figures 3(b) and (c), some systems of linear equations have no solution or an infinite number of solutions.

You Try It

Video

EXAMPLE 8 Solving a System of Dependent Equations

Solve the system.

$$2x - y = 3 \qquad (1)$$

$$6x - 3y = 9 \qquad (2)$$

We multiply equation (1) by -3, and then add the result to equation (2).

$$-6x + 3y = -9 \qquad -3 \text{ times each side of equation (1)}$$

$$6x - 3y = 9 \qquad (2)$$

$$0 = 0 \qquad \text{True}$$

Continued on Next Page

ANSWERS

7\. (a) $\{(5, 3)\}$ (b) $\{(2, -3)\}$

Adding these equations gives the true statement $0 = 0$. In the original system, we could get equation (2) from equation (1) by multiplying equation (1) by 3. Because of this, equations (1) and (2) are equivalent and have the same graph, as shown in Figure 4. The equations are dependent. The solution set is the set of all points on the line with equation $2x - y = 3$, written

$$\{(x, y) \mid 2x - y = 3\}$$

and read "the set of all ordered pairs (x, y), such that $2x - y = 3$."

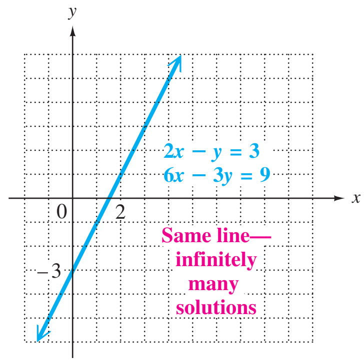


Figure 4

NOTE
When a system has an infinite number of solutions, as in Example 8, either equation of the system could be used to write the solution set. We prefer to use the equation (in standard form) with coefficients that are integers having no common factor (except 1).

Work Problem 8 at the Side.

EXAMPLE 9 Solving an Inconsistent System

Solve the system.

$$\begin{aligned} x + 3y &= 4 \quad (1) \\ -2x - 6y &= 3 \quad (2) \end{aligned}$$

Multiply equation (1) by 2, and then add the result to equation (2).

$$\begin{aligned} 2x + 6y &= 8 \qquad \text{Equation (1) multiplied by 2} \\ -2x - 6y &= 3 \qquad \text{(2)} \\ \hline \mathbf{0} &= \mathbf{11} \qquad \text{False} \end{aligned}$$

The result of the addition step is a false statement, which indicates that the system is inconsistent. As shown in Figure 5, the graphs of the equations of the system are parallel lines. There are no ordered pairs that satisfy both equations, so there is no solution for the system. The solution set is $\emptyset$.

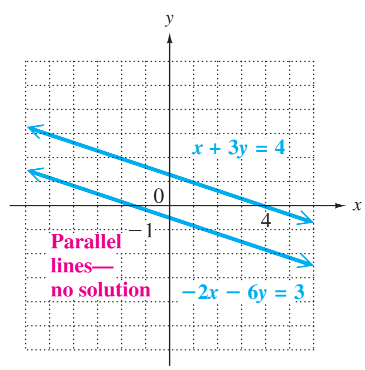
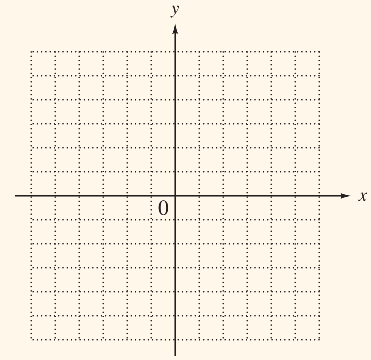


Figure 5

Work Problem 9 at the Side.

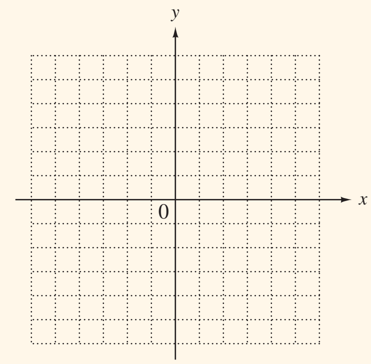
8 Solve the system. Then graph both equations.

$$\begin{aligned} 2x + y &= 6 \quad (1) \\ -8x - 4y &= -24 \quad (2) \end{aligned}$$



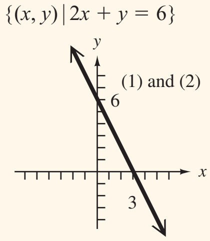
9 Solve the system. Then graph both equations.

$$\begin{aligned} 4x - 3y &= 8 \quad (1) \\ 8x - 6y &= 14 \quad (2) \end{aligned}$$

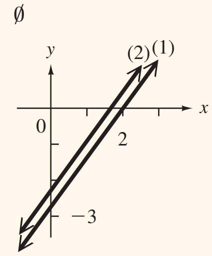


ANSWERS

8. $\{(x, y) \mid 2x + y = 6\}$



9. $\emptyset$



10 Write the equations of Example 8 in slope-intercept form. Use function notation.

11 Write the equations of Example 9 in slope-intercept form. Use function notation.

The results of Examples 8 and 9 are generalized as follows.

Special Cases of Linear Systems

If both variables are eliminated when a system of linear equations is solved, then

1. there is no solution if the resulting statement is *false;*

2. there are infinitely many solutions if the resulting statement is *true*.

Audio

Slopes and y-intercepts can be used to decide if the graphs of a system of equations are parallel lines or if they coincide. In Example 8, writing each equation in slope-intercept form shows that both lines have slope 2 and y-intercept $(0, -3)$, so the graphs are the same line and the system has an infinite number of solutions.

Work Problem 10 at the Side.

In Example 9, both equations have slope $-\frac{1}{3}$, but the y-intercepts are $(0, \frac{4}{3})$ and $(0, -\frac{1}{2})$, showing that the graphs are two distinct parallel lines. Thus, the system has $\emptyset$ as its solution set.

Work Problem 11 at the Side.

Answers

10. Both equations are $f(x) = 2x - 3$.

11. $f(x) = -\frac{1}{3}x + \frac{4}{3}$; $f(x) = -\frac{1}{3}x - \frac{1}{2}$

5.2 Systems of Linear Equations in Three Variables

OBJECTIVES

1. **Understand the geometry of systems of three equations in three variables.**
2. **Solve linear systems (with three equations and three variables) by elimination.**
3. **Solve linear systems where some of the equations have missing terms.**
4. **Solve special systems.**

A solution of an equation in three variables, such as

$$2x + 3y - z = 4,$$

is called an **ordered triple** and is written (x, y, z). For example, the ordered triple $(0, 1, -1)$ is a solution of the equation, because

$$2(0) + 3(1) - (-1) = 0 + 3 + 1 = 4.$$

Verify that another solution of this equation is $(10, -3, 7)$.

In the rest of this chapter, the term *linear equation* is extended to equations of the form

$$Ax + By + Cz + \ldots + Dw = K,$$

where not all the coefficients $A, B, C, \ldots, D$ equal 0. For example,

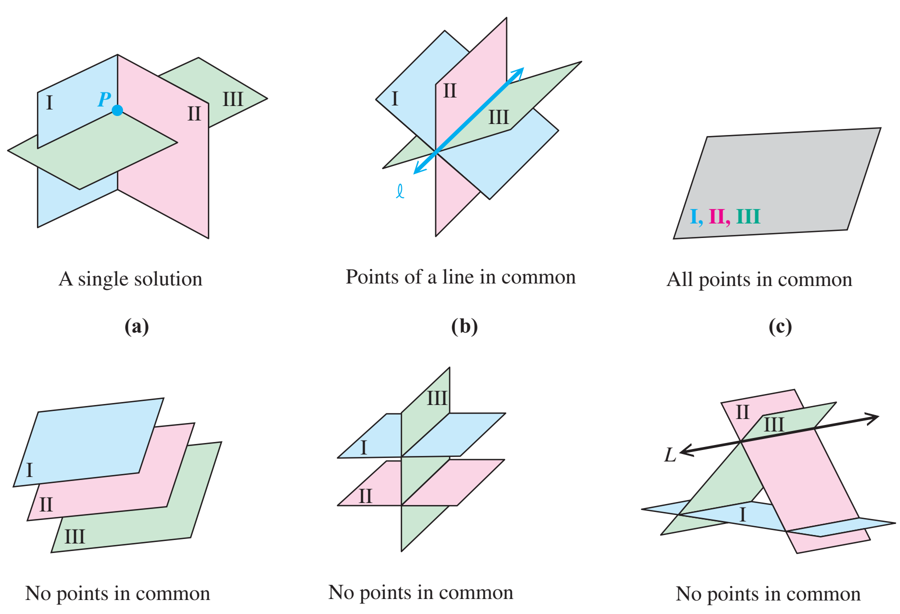
$$2x + 3y - 5z = 7 \quad \text{and} \quad x - 2y - z + 3u - 2w = 8$$

are linear equations, the first with three variables and the second with five variables.

OBJECTIVE 1 Understand the geometry of systems of three equations in three variables. In this section, we discuss the solution of a system of linear equations in three variables, such as

$$\begin{aligned} 4x + 8y + z &= 2 \\ x + 7y - 3z &= -14 \\ 2x - 3y + 2z &= 3. \end{aligned}$$

Theoretically, a system of this type can be solved by graphing. However, the graph of a linear equation with three variables is a *plane,* not a line. Since the graph of each equation of the system is a plane, which requires three-dimensional graphing, this method is not practical. However, it does illustrate the number of solutions possible for such systems, as shown in Figure 6.



A single solution (a)

Points of a line in common (b)

All points in common (c)

No points in common (d)

No points in common (e)

No points in common (f)

Figure 6

Figure 6 on the preceding page illustrates the following cases.

Graphs of Linear Systems in Three Variables

1. The three planes may meet at a single, common point that is the solution of the system. See Figure 6(a).
2. The three planes may have the points of a line in common so that the infinite set of points that satisfy the equation of the line is the solution of the system. See Figure 6(b).
3. The three planes may coincide so that the solution of the system is the set of all points on a plane. See Figure 6(c).
4. The planes may have no points common to all three so that there is no solution of the system. See Figures 6(d), (e), and (f).

OBJECTIVE 2 Solve linear systems (with three equations and three variables) by elimination. Is it possible to solve a system of three equations in three variables such as the one that follows?

$$\begin{aligned} 4x + 8y + z &= 2 \\ x + 7y - 3z &= -14 \\ 2x - 3y + 2z &= 3 \end{aligned}$$

Graphing to find the solution set of such a system is impractical, so these systems are solved with an extension of the elimination method from **Section 5.1,** summarized as follows.

Solving a Linear System in Three Variables

Step 1 **Eliminate a variable.** Use the elimination method to eliminate any variable from any two of the original equations. The result is an equation in two variables.

Step 2 **Eliminate the same variable again.** Eliminate the *same* variable from any *other* two equations. The result is an equation in the same two variables as in Step 1.

Step 3 **Eliminate a different variable and solve.** Use the elimination method to eliminate a second variable from the two equations in two variables that result from Steps 1 and 2. The result is an equation in one variable that gives the value of that variable.

Step 4 **Find a second value.** Substitute the value of the variable found in Step 3 into either of the equations in two variables to find the value of the second variable.

Step 5 **Find a third value.** Use the values of the two variables from Steps 3 and 4 to find the value of the third variable by substituting into any of the original equations.

Step 6 **Check** the solution in all of the original equations. Then write the solution set.

EXAMPLE 1 Solving a System in Three Variables

Solve the system.

$$\begin{aligned} 4x + 8y + z &= 2 && (1) \\ x + 7y - 3z &= -14 && (2) \\ 2x - 3y + 2z &= 3 && (3) \end{aligned}$$

Step 1 As before, the elimination method involves eliminating a variable from the sum of two equations. The choice of which variable to eliminate is arbitrary. Suppose we decide to begin by eliminating z. To do this, we multiply equation (1) by 3 and then add the result to equation (2).

$$\begin{array}{rcl} 12x + 24y + 3z &=& 6 \quad \text{Multiply each side of (1) by 3.} \\ x + 7y - 3z &=& -14 \quad (2) \\ \hline 13x + 31y &=& -8 \quad \text{Add.} \quad (4) \end{array}$$

Step 2 Equation (4) has only two variables. To get another equation without z, we multiply equation (1) by -2 and add the result to equation (3). It is essential at this point to ***eliminate the same variable, z.***

$$\begin{array}{rcl} -8x - 16y - 2z &=& -4 \quad \text{Multiply each side of (1) by } -2. \\ 2x - 3y + 2z &=& 3 \quad (3) \\ \hline -6x - 19y &=& -1 \quad \text{Add.} \quad (5) \end{array}$$

Step 3 Now we solve the resulting system of equations (4) and (5) for x and y.

$$\begin{aligned} 13x + 31y &= -8 && (4) \\ -6x - 19y &= -1 && (5) \end{aligned}$$

This step is possible only if the *same* variable is eliminated in Steps 1 and 2.

$$\begin{array}{rcl} 78x + 186y &=& -48 \quad \text{Multiply each side of (4) by 6.} \\ -78x - 247y &=& -13 \quad \text{Multiply each side of (5) by 13.} \\ \hline -61y &=& -61 \quad \text{Add.} \\ \mathbf{y} &=& \mathbf{1} \end{array}$$

Step 4 We substitute 1 for y in either equation (4) or (5). Choosing (5) gives

$$\begin{aligned} -6x - 19y &= -1 && (5) \\ -6x - 19(1) &= -1 && \text{Let } y = 1. \\ -6x - 19 &= -1 \\ -6x &= 18 \\ \mathbf{x} &= \mathbf{-3}. \end{aligned}$$

Step 5 We substitute -3 for x and 1 for y in any one of the three original equations to find z. Choosing (1) gives

$$\begin{aligned} 4x + 8y + z &= 2 && (1) \\ 4(-3) + 8(1) + z &= 2 && \text{Let } x = -3 \text{ and } y = 1. \\ -4 + z &= 2 \\ \mathbf{z} &= \mathbf{6}. \end{aligned}$$

Continued on Next Page

Step 6 It appears that the ordered triple $(-3, 1, 6)$ is the only solution of the system. We must check that the solution satisfies all three original equations of the system. For equation (1),

$$\begin{aligned} 4x + 8y + z &= 2 && (1) \\ 4(-3) + 8(1) + 6 &= 2 && ? \\ -12 + 8 + 6 &= 2 && ? \\ 2 &= 2. && \text{True} \end{aligned}$$

Work Problem 1 at the Side.

Because $(-3, 1, 6)$ also satisfies equations (2) and (3), the solution set is $\{(-3, 1, 6)\}$.

Work Problem 2 at the Side.

OBJECTIVE 3 Solve linear systems where some of the equations have missing terms. If a linear system has an equation missing a term or terms, one elimination step can be omitted.

EXAMPLE 2 Solving a System of Equations with Missing Terms

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Solve the system.

$$\begin{aligned} 6x - 12y &= -5 && (1) \\ 8y + z &= 0 && (2) \\ 9x - z &= 12 && (3) \end{aligned}$$

Since equation (3) is missing the variable y, a good way to begin the solution is to eliminate y again using equations (1) and (2).

$$\begin{aligned} 12x - 24y &= -10 && \text{Multiply each side of (1) by 2.} \\ 24y + 3z &= 0 && \text{Multiply each side of (2) by 3.} \\ \hline 12x + 3z &= -10 && \text{Add.} \quad (4) \end{aligned}$$

Use this result, together with equation (3), $9x - z = 12$, to eliminate z. Multiply equation (3) by 3. This gives

$$\begin{aligned} 27x - 3z &= 36 && \text{Multiply each side of (3) by 3.} \\ 12x + 3z &= -10 && (4) \\ \hline 39x &= 26 && \text{Add.} \end{aligned}$$

$$x = \frac{26}{39} = \frac{2}{3}.$$

Substituting into equation (3) gives

$$\begin{aligned} 9x - z &= 12 && (3) \\ 9\left(\frac{2}{3}\right) - z &= 12 && \text{Let } x = \tfrac{2}{3}. \\ 6 - z &= 12 \\ z &= -6. \end{aligned}$$

Continued on Next Page

1 Check that the solution $(-3, 1, 6)$ satisfies equations (2) and (3) of Example 1.

(a) $x + 7y - 3z = -14$ (2)

Does the solution satisfy equation (2)?

(b) $2x - 3y + 2z = 3$ (3)

Does the solution satisfy equation (3)?

2 Solve each system.

(a)
$$\begin{aligned} x + y + z &= 2 \\ x - y + 2z &= 2 \\ -x + 2y - z &= 1 \end{aligned}$$

(b)
$$\begin{aligned} 2x + y + z &= 9 \\ -x - y + z &= 1 \\ 3x - y + z &= 9 \end{aligned}$$

ANSWERS

1. **(a)** yes **(b)** yes
2. **(a)** $\{(-1, 1, 2)\}$ **(b)** $\{(2, 1, 4)\}$

Substituting -6 for z in equation (2) gives

$$\begin{aligned} 8y + z &= 0 && (2) \\ 8y - 6 &= 0 && \text{Let } z = -6. \\ 8y &= 6 \\ y &= \frac{3}{4}. \end{aligned}$$

Thus, $x = \frac{2}{3}$, $y = \frac{3}{4}$, and $z = -6$. Check these values in each of the original equations of the system to verify that the solution set of the system is $\{(\frac{2}{3}, \frac{3}{4}, -6)\}$.

Work Problem 3 at the Side.

OBJECTIVE 4 Solve special systems. Linear systems with three variables may be inconsistent or may include dependent equations. The next examples illustrate these cases.

EXAMPLE 3 Solving an Inconsistent System with Three Variables

Solve the system.

$$\begin{aligned} 2x - 4y + 6z &= 5 && (1) \\ -x + 3y - 2z &= -1 && (2) \\ x - 2y + 3z &= 1 && (3) \end{aligned}$$

Eliminate x by adding equations (2) and (3) to get the equation

$$y + z = 0.$$

Now, ***eliminate x again,*** using equations (1) and (3).

$$\begin{aligned} -2x + 4y - 6z &= -2 && \text{Multiply each side of (3) by } -2. \\ 2x - 4y + 6z &= 5 && (1) \\ \hline 0 &= 3 && \text{False} \end{aligned}$$

The resulting false statement indicates that equations (1) and (3) have no common solution. Thus, the system is inconsistent and the solution set is $\emptyset$. The graph of this system would show these two planes parallel to one another.

NOTE
If a false statement results when adding as in Example 3, it is not necessary to go any further with the solution. Since two of the three planes are parallel, it is not possible for the three planes to have any common points.

Work Problem 4 at the Side.

3 Solve each system.

(a) $$\begin{aligned} x - y &= 6 \\ 2y + 5z &= 1 \\ 3x - 4z &= 8 \end{aligned}$$

(b) $$\begin{aligned} 5x - y &= 26 \\ 4y + 3z &= -4 \\ x + z &= 5 \end{aligned}$$

4 Solve each system.

(a) $$\begin{aligned} 3x - 5y + 2z &= 1 \\ 5x + 8y - z &= 4 \\ -6x + 10y - 4z &= 5 \end{aligned}$$

(b) $$\begin{aligned} 7x - 9y + 2z &= 0 \\ y + z &= 0 \\ 8x - z &= 0 \end{aligned}$$

ANSWERS
3. (a) $\{(4, -2, 1)\}$ **(b)** $\{(5, -1, 0)\}$
4. (a) $\emptyset$ **(b)** $\{(0, 0, 0)\}$

5 Solve the system.

$$\begin{aligned} x - y + z &= 4 \\ -3x + 3y - 3z &= -12 \\ 2x - 2y + 2z &= 8 \end{aligned}$$

EXAMPLE 4 Solving a System of Dependent Equations with Three Variables

Solve the system.

$$2x - 3y + 4z = 8 \qquad (1)$$

$$-x + \frac{3}{2}y - 2z = -4 \qquad (2)$$

$$6x - 9y + 12z = 24 \qquad (3)$$

Multiplying each side of equation (1) by 3 gives equation (3). Multiplying each side of equation (2) by -6 also gives equation (3). Because of this, the equations are dependent. All three equations have the same graph, as illustrated in Figure 6(c). The solution set is written

$$\{(x, y, z) \mid 2x - 3y + 4z = 8\}.$$

Although any one of the three equations could be used to write the solution set, we use the equation with coefficients that are integers with no common factor (except 1), as we did in **Section 5.1.**

Work Problem 5 at the Side.

ANSWERS

5. $\{(x, y, z) \mid x - y + z = 4\}$

5.3 Applications of Systems of Linear Equations

OBJECTIVES

1. Solve problems using two variables.
2. Solve money problems using two variables.
3. Solve mixture problems using two variables.
4. Solve distance-rate-time problems using two variables.
5. Solve problems with three variables using a system of three equations.

Many applied problems involve more than one unknown quantity. Although some problems with two unknowns can be solved using just one variable, it is often easier to use two variables. To solve a problem with two unknowns, we must write two equations that relate the unknown quantities. The system formed by the pair of equations can then be solved using the methods of this chapter.

Problems that can be solved by writing a system of equations have been of interest historically. The following problem, which is given in the exercises for this section, first appeared in a Hindu work that dates back to about A.D. 850.

> The mixed price of 9 citrons [a lemonlike fruit shown in the photo] and 7 fragrant wood apples is 107; again, the mixed price of 7 citrons and 9 fragrant wood apples is 101. O you arithmetician, tell me quickly the price of a citron and the price of a wood apple here, having distinctly separated those prices well.

The following steps, based on the six-step problem-solving method first introduced in **Section 2.3,** give a strategy for solving applied problems using more than one variable.

Solving an Applied Problem by Writing a System of Equations

Step 1 **Read** the problem, several times if necessary, until you understand what is given and what is to be found.

Step 2 **Assign variables** to represent the unknown values, using diagrams or tables as needed. *Write down* what each variable represents.

Step 3 **Write a system of equations** that relates the unknowns.

Step 4 **Solve** the system of equations.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

OBJECTIVE 1 Solve problems using two variables. Problems about the perimeter of a geometric figure often involve two unknowns and can be solved using systems of equations.

1 Solve the problem.
The length of the foundation of a rectangular house is to be 6 m more than its width. Find the length and width of the house if the perimeter must be 48 m.

EXAMPLE 1 Finding the Dimensions of a Soccer Field

Unlike football, where the dimensions of a playing field cannot vary, a rectangular soccer field may have a width between 50 and 100 yd and a length between 100 and 130 yd. Suppose that one particular field has a perimeter of 320 yd. Its length measures 40 yd more than its width. What are the dimensions of this field? (*Source: Microsoft Encarta Encyclopedia 2002.*)

Step 1 **Read** the problem again. We must find the dimensions of the field.

Step 2 **Assign variables.** Let $L =$ the length and $W =$ the width. Figure 7 shows a soccer field with these variables as labels.

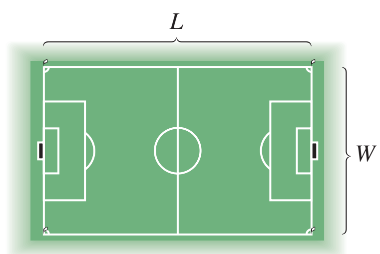


Figure 7

Step 3 **Write a system of equations.** Because the perimeter is 320 yd, we find one equation by using the perimeter formula:

$$2L + 2W = 320.$$

Because the length is 40 yd more than the width, we have

$$L = W + 40.$$

The system is, therefore,

$$\begin{aligned} 2L + 2W &= 320 && (1) \\ L &= W + 40. && (2) \end{aligned}$$

Step 4 **Solve** the system of equations. Since equation (2) is solved for L, we can use the substitution method. We substitute $W + 40$ for L in equation (1), and solve for W.

$$\begin{aligned} 2L + 2W &= 320 && (1) \\ 2(W + 40) + 2W &= 320 && \text{Let } L = W + 40. \\ 2W + 80 + 2W &= 320 && \text{Distributive property} \\ 4W + 80 &= 320 && \text{Combine terms.} \\ 4W &= 240 && \text{Subtract 80.} \\ W &= 60 && \text{Divide by 4.} \end{aligned}$$

Let $W = 60$ in the equation $L = W + 40$ to find L.

$$L = 60 + 40 = 100$$

Step 5 **State the answer.** The length is 100 yd, and the width is 60 yd. Both dimensions are within the ranges given in the problem.

Step 6 **Check.** The perimeter is $2(100) + 2(60) = 320$ yd, and the length, 100 yd, is indeed 40 yd more than the width, since $100 - 40 = 60$. The answer is correct.

Work Problem 1 at the Side.

ANSWERS
1. length: 15 m; width: 9 m

OBJECTIVE 2 Solve money problems using two variables. Professional sport ticket prices increase annually. Average per-ticket prices in three of the four major sports (football, basketball, and hockey) now exceed \$40.00.

You Try It

Video

EXAMPLE 2 Solving a Problem about Ticket Prices

It was reported in March 2004 that during the National Hockey League and National Basketball Association seasons, two hockey tickets and one basketball ticket purchased at their average prices would have cost \$126.77. One hockey ticket and two basketball tickets would have cost \$128.86. What were the average ticket prices for the two sports? (*Source:* Team Marketing Report, Chicago.)

Step 1 **Read** the problem again. There are two unknowns.

Step 2 **Assign variables.** Let h represent the average price for a hockey ticket and b represent the average price for a basketball ticket.

Step 3 **Write a system of equations.** Because two hockey tickets and one basketball ticket cost a total of \$126.77, one equation for the system is

$$2h + b = 126.77.$$

By similar reasoning, the second equation is

$$h + 2b = 128.86.$$

Therefore, the system is

$$\begin{aligned} 2h + b &= 126.77 & &(1)\\ h + 2b &= 128.86. & &(2) \end{aligned}$$

Step 4 **Solve** the system of equations. To eliminate h, multiply equation (2) by -2 and add.

$$\begin{aligned} 2h + b &= 126.77 & &(1)\\ \underline{-2h - 4b} &\underline{= -257.72} & &\text{Multiply each side of (2) by } -2.\\ -3b &= -130.95 & &\text{Add.}\\ b &= 43.65 & &\text{Divide by } -3. \end{aligned}$$

To find the value of h, let $b = 43.65$ in equation (2).

$$\begin{aligned} h + 2\mathbf{b} &= 128.86 & &(2)\\ h + 2(\mathbf{43.65}) &= 128.86 & &\text{Let } b = 43.65.\\ h + 87.30 &= 128.86 & &\text{Multiply.}\\ h &= 41.56 & &\text{Subtract 87.30.} \end{aligned}$$

Step 5 **State the answer.** The average price for one basketball ticket was \$43.65. For one hockey ticket, the average price was \$41.56.

Step 6 **Check** that these values satisfy the conditions stated in the problem.

Work Problem 2 at the Side.

OBJECTIVE 3 Solve mixture problems using two variables. We solved mixture problems in **Section 2.3** using one variable. For many mixture problems we can use more than one variable and a system of equations.

2 Solve the problem.

For recent Major League Baseball and National Football League seasons, based on average ticket prices, three baseball tickets and two football tickets would have cost \$159.50, while two baseball tickets and one football ticket would have cost \$89.66. What were the average ticket prices for the two sports? (*Source:* Team Marketing Report, Chicago.)

ANSWERS

2. baseball: \$19.82; football: \$50.02

3 Solve each problem.

(a) A grocer has some \$4 per lb coffee and some \$8 per lb coffee, which he will mix to make 50 lb of \$5.60 per lb coffee. How many pounds of each should be used?

(b) Some 40% ethyl alcohol solution is to be mixed with some 80% solution to get 200 L of a 50% solution. How many liters of each should be used?

Answers

3. **(a)** 30 lb of \$4; 20 lb of \$8
(b) 150 L of 40%; 50 L of 80%

EXAMPLE 3 Solving a Mixture Problem

How many ounces each of 5% hydrochloric acid and 20% hydrochloric acid must be combined to get 10 oz of solution that is 12.5% hydrochloric acid?

Step 1 **Read** the problem. Two solutions of different strengths are being mixed together to get a specific amount of a solution with an "in-between" strength.

Step 2 **Assign variables.** Let x represent the number of ounces of 5% solution and y represent the number of ounces of 20% solution. Use a table to summarize the information from the problem.

Percent (as a Decimal)	Ounces of Solution	Ounces of Pure Acid
$5\% = .05$	x	$.05x$
$20\% = .20$	y	$.20y$
$12.5\% = .125$	10	$(.125)10$

Figure 8 also illustrates what is happening in the problem.

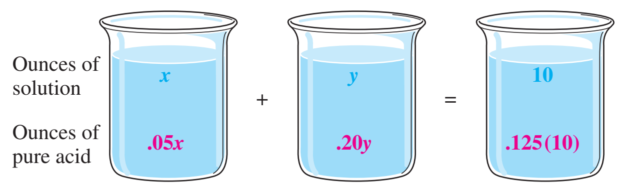


Figure 8

Step 3 **Write a system of equations.** When the x ounces of 5% solution and the y ounces of 20% solution are combined, the total number of ounces is 10, so

$$x + y = 10. \quad (1)$$

The ounces of acid in the 5% solution ($.05x$) plus the ounces of acid in the 20% solution ($.20y$) should equal the total ounces of acid in the mixture, which is $(.125)10$, or 1.25. That is,

$$.05x + .20y = 1.25. \quad (2)$$

Notice that these equations can be quickly determined by reading down in the table or using the labels in Figure 8.

Step 4 **Solve** the system of equations (1) and (2). Eliminate x by first multiplying equation (2) by 100 to clear it of decimals and then multiplying equation (1) by -5.

$$\begin{aligned} 5x + 20y &= 125 && \text{Multiply each side of (2) by 100.} \\ -5x - 5y &= -50 && \text{Multiply each side of (1) by } -5. \\ \hline 15y &= 75 && \text{Add.} \\ y &= 5 \end{aligned}$$

Because $y = 5$ and $x + y = 10$, x is also 5.

Step 5 **State the answer.** The desired mixture will require 5 oz of the 5% solution and 5 oz of the 20% solution.

Step 6 **Check** that these values satisfy both equations of the system.

Work Problem 3 at the Side.

You Try It

Animation

Video

OBJECTIVE 4 Solve distance-rate-time problems using two variables. Motion problems require the distance formula, $d = rt$, where d is distance, r is rate (or speed), and t is time. These applications often lead to systems of equations.

You Try It

Video

EXAMPLE 4 Solving a Motion Problem

A car travels 250 km in the same time that a truck travels 225 km. If the speed of the car is 8 km per hr faster than the speed of the truck, find both speeds.

Step 1 **Read** the problem again. Given the distances traveled, we need to find the speed of each vehicle.

Step 2 **Assign variables.**

Let x = the speed of the car

and y = the speed of the truck.

As in Example 3, a table helps organize the information. Fill in the given information for each vehicle (in this case, distance) and use the assigned variables for the unknown speeds (rates).

	d	r	t
Car	250	x	
Truck	225	y	

To get an expression for time, solve the distance formula, $d = rt$, for t. Since $\frac{d}{r} = t$, the two times can be written as $\frac{250}{x}$ and $\frac{225}{y}$.

Step 3 **Write a system of equations.** The problem states that the car travels 8 km per hr faster than the truck. Since the two speeds are x and y,

$$x = y + 8.$$

Both vehicles travel for the same time, so from the table

$$\frac{250}{x} = \frac{225}{y}.$$

This is not a linear equation. Multiplying each side by xy gives

$$250y = 225x,$$

which is linear. The system is

$$x = y + 8 \quad (1)$$
$$250y = 225x. \quad (2)$$

Step 4 **Solve** the system of equations by substitution. Replace x with $y + 8$ in equation (2).

$$\begin{aligned} 250y &= 225\mathbf{x} && (2) \\ 250y &= 225(\mathbf{y + 8}) && \text{Let } x = y + 8. \\ 250y &= 225y + 1800 && \text{Distributive property} \\ 25y &= 1800 && \text{Subtract } 225y. \\ y &= 72 && \text{Divide by 25.} \end{aligned}$$

Because $x = y + 8$, the value of x is $72 + 8 = 80$.

Continued on Next Page

Step 5 **State the answer.** The car's speed is 80 km per hr, and the truck's speed is 72 km per hr.

Step 6 **Check.** This is especially important since one of the equations had variable denominators.

$$\text{Car: } t = \frac{d}{r} = \frac{250}{80} = \mathbf{3.125}$$

$$\text{Truck: } t = \frac{d}{r} = \frac{225}{72} = \mathbf{3.125}$$

Times are equal.

Since $80 - 72 = 8$, the conditions of the problem are satisfied.

Work Problem 4 at the Side.

4 Solve the problem.

A train travels 600 mi in the same time that a truck travels 520 mi. Find the speed of each vehicle if the train's average speed is 8 mph faster than the truck's.

OBJECTIVE 5 Solve problems with three variables using a system of three equations. To solve such problems, we extend the method used for two unknowns to three variables and three equations.

EXAMPLE 5 Solving a Problem Involving Prices

At Panera Bread, a loaf of honey wheat bread costs \$2.59, a loaf of sunflower bread costs \$2.99, and a loaf of French bread costs \$3.29. On a recent day, three times as many loaves of honey wheat were sold as sunflower. The number of loaves of French bread sold was 5 less than the number of loaves of honey wheat sold. Total receipts for these breads were \$66.07. How many loaves of each type of bread were sold? (*Source:* Panera Bread menu.)

Step 1 **Read** the problem again. There are three unknowns in this problem.

Step 2 **Assign variables** to represent the three unknowns.

Let x = the number of loaves of honey wheat,

y = the number of loaves of sunflower,

and z = the number of loaves of French bread.

Step 3 **Write a system of three equations.** Since three times as many loaves of honey wheat were sold as sunflower,

$$x = 3y, \quad \text{or} \quad x - 3y = 0. \qquad (1)$$

Also,

Number of loaves of French bread	equals	5 less than the number of loaves of honey wheat.
$\downarrow$	$\downarrow$	$\downarrow$
z	$=$	$x - 5,$

so

$$x - z = 5. \qquad (2)$$

Multiplying the cost of a loaf of each kind of bread by the number of loaves of that kind sold and adding gives the total receipts.

$$2.59x + 2.99y + 3.29z = 66.07$$

Multiply each side of this equation by 100 to clear it of decimals.

$$259x + 299y + 329z = 6607 \qquad (3)$$

Step 4 **Solve** the system of three equations using the method shown in **Section 5.2.**

Work Problem 5 at the Side.

5 Solve the system of equations from Example 5.

$$\begin{aligned} x - 3y &= 0 & (1)\\ x - z &= 5 & (2)\\ 259x + 299y + 329z &= 6607 & (3) \end{aligned}$$

Continued on Next Page

ANSWERS

4. train: 60 mph; truck: 52 mph
5. $\{(12, 4, 7)\}$

Step 5 **State the answer.** The solution set is $\{(12, 4, 7)\}$, meaning that 12 loaves of honey wheat, 4 loaves of sunflower, and 7 loaves of French bread were sold.

Step 6 **Check.** Since $12 = 3 \cdot 4$, the number of loaves of honey wheat is three times the number of loaves of sunflower. Also, $12 - 7 = 5$, so the number of loaves of French bread is 5 less than the number of loaves of honey wheat. Multiply the appropriate cost per loaf by the number of loaves sold and add the results to check that total receipts were \$66.07.

Work Problem 6 at the Side.

EXAMPLE 6 Solving a Business Production Problem

A company produces three color television sets, models X, Y, and Z. Each model X set requires 2 hr of electronics work, 2 hr of assembly time, and 1 hr of finishing time. Each model Y requires 1, 3, and 1 hr of electronics, assembly, and finishing time, respectively. Each model Z requires 3, 2, and 2 hr of the same work, respectively. There are 100 hr available for electronics, 100 hr available for assembly, and 65 hr available for finishing per week. How many of each model should be produced each week if all available time must be used?

Step 1 **Read** the problem again. There are three unknowns.

Step 2 **Assign variables.**

Let $x =$ the number of model X produced per week,

$y =$ the number of model Y produced per week,

and $z =$ the number of model Z produced per week.

Organize the information in a table.

	Each Model X	Each Model Y	Each Model Z	Totals
Hours of Electronics Work	2	1	3	100
Hours of Assembly Time	2	3	2	100
Hours of Finishing Time	1	1	2	65

Step 3 **Write a system of three equations.** The x model X sets require $2x$ hr of electronics, the y model Y sets require $1y$ (or y) hr of electronics, and the z model Z sets require $3z$ hr of electronics. Since 100 hr are available for electronics,

$$2x + y + 3z = 100. \qquad (1)$$

Similarly, from the fact that 100 hr are available for assembly,

$$2x + 3y + 2z = 100, \qquad (2)$$

and the fact that 65 hr are available for finishing leads to the equation

$$x + y + 2z = 65. \qquad (3)$$

Notice that by reading across the table, we can easily determine the coefficients and constants in the equations of the system.

Continued on Next Page

6 Solve the problem.

A department store has three kinds of perfume: cheap, better, and best. It has 10 more bottles of cheap than better, and 3 fewer bottles of best than better. Each bottle of cheap costs \$8, better costs \$15, and best costs \$32. The total value of all the perfume is \$589. How many bottles of each are there?

Answers

6. 21 bottles of cheap; 11 of better; 8 of best

Step 4 **Solve** the system

$$\begin{aligned} 2x + y + 3z &= 100 \\ 2x + 3y + 2z &= 100 \\ x + y + 2z &= 65 \end{aligned}$$

to find $x = 15$, $y = 10$, and $z = 20$.

Step 5 **State the answer.** The company should produce 15 model X, 10 model Y, and 20 model Z sets per week.

Step 6 **Check** that these values satisfy the conditions of the problem.

Work Problem 7 at the Side.

7 Solve the problem.

A paper mill makes newsprint, bond, and copy machine paper. Each ton of newsprint requires 3 tons of recycled paper and 1 ton of wood pulp. Each ton of bond requires 2 tons of recycled paper, 4 tons of wood pulp, and 3 tons of rags. A ton of copy machine paper requires 2 tons of recycled paper, 3 tons of wood pulp, and 2 tons of rags. The mill has 4200 tons of recycled paper, 5800 tons of wood pulp, and 3900 tons of rags. How much of each kind of paper can be made from these supplies?

Answers

7. 400 tons of newsprint; 900 tons of bond; 600 tons of copy machine paper

5.4 Solving Systems of Linear Equations by Matrix Methods

OBJECTIVES

1. Define a matrix.
2. Write the augmented matrix for a system.
3. Use row operations to solve a system with two equations.
4. Use row operations to solve a system with three equations.
5. Use row operations to solve special systems.

OBJECTIVE 1 Define a matrix. An ordered array of numbers such as

$$\begin{array}{ccc} & \text{Columns} & \\ \text{Rows} & \begin{bmatrix} 2 & 3 & 5 \\ 7 & 1 & 2 \end{bmatrix} & \text{Matrix} \end{array}$$

is called a **matrix.** The numbers are called **elements** of the matrix. Matrices (the plural of *matrix*) are named according to the number of **rows** and **columns** they contain. The rows are read horizontally, and the columns are read vertically. For example, the first row in the preceding matrix is 2 3 5 and the first column is $\begin{matrix}2\\7\end{matrix}$. This matrix is a 2×3 (read "two by three") matrix because it has 2 rows and 3 columns. The number of rows is given first, and then the number of columns. Two other examples follow.

$$\begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \quad 2 \times 2 \text{ matrix} \qquad\qquad \begin{bmatrix} 8 & -1 & -3 \\ 2 & 1 & 6 \\ 0 & 5 & -3 \\ 5 & 9 & 7 \end{bmatrix} \quad 4 \times 3 \text{ matrix}$$

A **square matrix** is one that has the same number of rows as columns. The 2×2 matrix is a square matrix.

Calculator Tip Figure 9 shows how a graphing calculator displays the preceding two matrices. Work with matrices is made much easier by using technology when available. Consult your owner's manual for details.

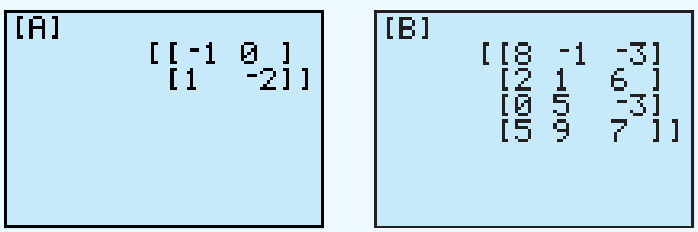


Figure 9

In this section, we discuss a method of solving linear systems that uses matrices. The advantage of this new method is that it can be done by a graphing calculator or a computer, allowing large systems of equations to be solved easily.

OBJECTIVE 2 Write the augmented matrix for a system. To begin, we write an *augmented matrix* for the system. An **augmented matrix** has a vertical bar that separates the columns of the matrix into two groups. For example, to solve the system

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5, \end{aligned}$$

start with the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]. \qquad \text{Augmented matrix}$$

System of equations:

$$x - 3y = 1$$
$$2x + \quad y = -5$$

Augmented matrix:

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Coefficients of the variables ↑ ↑ Constants

Place the coefficients of the variables to the left of the bar, and the constants to the right. The bar separates the coefficients from the constants. The matrix is just a shorthand way of writing the system of equations, so the rows of the augmented matrix can be treated the same as the equations of a system of equations.

We know that exchanging the position of two equations in a system does not change the system. Also, multiplying any equation in a system by a nonzero number does not change the system. Comparable changes to the augmented matrix of a system of equations produce new matrices that correspond to systems with the same solutions as the original system.

The following **row operations** produce new matrices that lead to systems having the same solutions as the original system.

Audio

Matrix Row Operations

1. Any two rows of the matrix may be interchanged.
2. The numbers in any row may be multiplied by any nonzero real number.
3. Any row may be transformed by adding to the numbers of the row the product of a real number and the corresponding numbers of another row.

Examples of these row operations follow.

Row operation 1:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 1 & 0 & 7 \\ 4 & 8 & -3 \\ 2 & 3 & 9 \end{bmatrix}.$$

Interchange row 1 and row 3.

Row operation 2:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 6 & 9 & 27 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 1 by 3.

Row operation 3:

$$\begin{bmatrix} 2 & 3 & 9 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix} \text{ becomes } \begin{bmatrix} 0 & 3 & -5 \\ 4 & 8 & -3 \\ 1 & 0 & 7 \end{bmatrix}.$$

Multiply the numbers in row 3 by -2; add them to the corresponding numbers in row 1.

The third row operation corresponds to the way we eliminated a variable from a pair of equations in the previous sections.

OBJECTIVE 3 Use row operations to solve a system with two equations. Row operations can be used to rewrite a matrix. The goal is a matrix in the form

$$\left[\begin{array}{cc|c} 1 & a & b \\ 0 & 1 & c \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$$

for systems with two or three equations, respectively. Notice that there are 1s down the diagonal from upper left to lower right and 0s below the 1s. A matrix written this way is said to be in **row echelon form.** When these matrices are rewritten as systems of equations, the value of one variable is known, and the rest can be found by substitution. The following examples illustrate this method.

Video

EXAMPLE 1 Using Row Operations to Solve a System with Two Variables

Use row operations to solve the system.

$$\begin{aligned} x - 3y &= 1 \\ 2x + y &= -5 \end{aligned}$$

We start with the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & 1 & -5 \end{array}\right]$$

Now we use the various row operations to change this matrix into one that leads to a system that is easier to solve.

It is best to work by columns. We start with the first column and make sure that there is a 1 in the first row, first column position. There is already a 1 in this position. Next, we get 0 in every position below the first. To get a 0 in row two, column one, we use the third row operation and add to the numbers in row two the result of multiplying each number in row one by -2. (We abbreviate this as $-2R_1 + R_2$.) Row one remains unchanged.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ \underset{\substack{\uparrow \\ \text{Original number} \\ \text{from row two}}}{2} + \underset{\substack{\uparrow \\ -2 \text{ times number} \\ \text{from row one}}}{1(-2)} & 1 + -3(-2) & -5 + 1(-2) \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 7 & -7 \end{array}\right] \quad -2R_1 + R_2$$

The matrix now has a 1 in the first position of column one, with 0 in every position below the first.

Now we go to column two. A 1 is needed in row two, column two. We get this 1 by using the second row operation, multiplying each number of row two by $\frac{1}{7}$.

$$\left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -1 \end{array}\right] \quad \tfrac{1}{7}R_2$$

This augmented matrix leads to the system of equations

$$\begin{aligned} 1x - 3y &= 1 \\ 0x + 1y &= -1 \end{aligned} \quad \text{or} \quad \begin{aligned} x - 3y &= 1 \\ y &= -1. \end{aligned}$$

From the second equation, $y = -1$. We substitute -1 for y in the first equation to get

$$\begin{aligned} x - 3y &= 1 \\ x - 3(-1) &= 1 \\ x + 3 &= 1 \\ x &= -2. \end{aligned}$$

The solution set of the system is $\{(-2, -1)\}$. Check this solution by substitution in both equations of the system.

Work Problem 1 at the Side.

1 Use row operations to solve the system.

$$\begin{aligned} x - 2y &= 9 \\ 3x + y &= 13 \end{aligned}$$

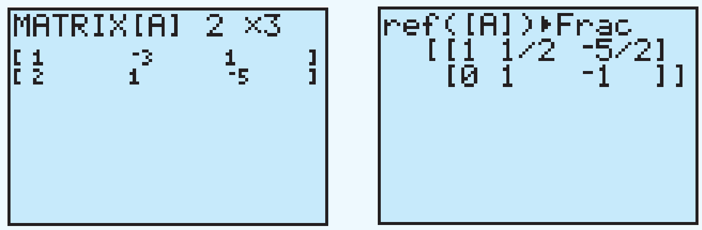
ANSWERS

1. $\{(5, -2)\}$

Calculator Tip If the augmented matrix of the system in Example 1 is entered as matrix A in a graphing calculator (Figure 10(a)) and the row echelon form of the matrix is found (Figure 10(b)), the system becomes

$$x + \frac{1}{2}y = -\frac{5}{2}$$
$$y = -1.$$

While this system looks different from the one we obtained in Example 1, it is equivalent, since its solution set is also $\{(-2, -1)\}$.



(a) (b)

Figure 10

OBJECTIVE 4 Use row operations to solve a system with three equations. As before, we use row operations to get 1s down the diagonal from left to right and all 0s below each 1.

EXAMPLE 2 Using Row Operations to Solve a System with Three Variables

Use row operations to solve the system.

$$\begin{aligned} x - y + 5z &= -6 \\ 3x + 3y - z &= 10 \\ x + 3y + 2z &= 5 \end{aligned}$$

Start by writing the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

This matrix already has 1 in row one, column one. Next get 0s in the rest of column one. First, add to row two the results of multiplying each number of row one by -3. This gives the matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 1 & 3 & 2 & 5 \end{array}\right]. \qquad -3R_1 + R_2$$

Now add to the numbers in row three the results of multiplying each number of row one by -1.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 4 & -3 & 11 \end{array}\right] \qquad -1R_1 + R_3$$

Continued on Next Page

Get 1 in row two, column two by multiplying each number in row two by $\frac{1}{6}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 4 & -3 & 11 \end{array}\right] \quad \tfrac{1}{6}R_2$$

Introduce 0 in row three, column two by adding to row three the results of multiplying each number in row two by -4.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & \frac{23}{3} & -\frac{23}{3} \end{array}\right] \quad -4R_2 + R_3$$

Finally, obtain 1 in row three, column three by multiplying each number in row three by $\frac{3}{23}$.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 1 & -\frac{8}{3} & \frac{14}{3} \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \tfrac{3}{23}R_3$$

This final matrix gives the system of equations

$$\begin{aligned} x - y + 5z &= -6 \\ y - \frac{8}{3}z &= \frac{14}{3} \\ z &= -1. \end{aligned}$$

Substitute -1 for z in the second equation, $y - \frac{8}{3}z = \frac{14}{3}$, to get $y = 2$. Finally, substitute 2 for y and -1 for z in the first equation, $x - y + 5z = -6$, to get $x = 1$. The solution set of the original system is $\{(1, 2, -1)\}$. Check by substitution in the original system.

Work Problem 2 at the Side.

OBJECTIVE 5 Use row operations to solve special systems.

EXAMPLE 3 Recognizing Inconsistent Systems or Dependent Equations

Use row operations to solve each system.

(a) $\begin{aligned} 2x - 3y &= 8 \\ -6x + 9y &= 4 \end{aligned}$

$$\left[\begin{array}{rr|r} 2 & -3 & 8 \\ -6 & 9 & 4 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 4 \\ -6 & 9 & 4 \end{array}\right] \quad \tfrac{1}{2}R_1$$

$$\left[\begin{array}{rr|r} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 28 \end{array}\right] \quad 6R_1 + R_2$$

The corresponding system of equations is

$$\begin{aligned} x - \frac{3}{2}y &= 4 \\ 0 &= 28, \quad \text{False} \end{aligned}$$

which has no solution and is inconsistent. The solution set is $\emptyset$.

Continued on Next Page

2 Use row operations to solve the system.

$$\begin{aligned} 2x - y + z &= 7 \\ x - 3y - z &= 7 \\ -x + y - 5z &= -9 \end{aligned}$$

ANSWERS

2. $\{(2, -2, 1)\}$

3 Use row operations to solve each system.

(a) $\begin{aligned} x - y &= 2 \\ -2x + 2y &= 2 \end{aligned}$

(b) $\begin{aligned} x - y &= 2 \\ -2x + 2y &= -4 \end{aligned}$

ANSWERS

3. **(a)** $\emptyset$ **(b)** $\{(x, y) \mid x - y = 2\}$

(b) $\begin{aligned} -10x + 12y &= 30 \\ 5x - 6y &= -15 \end{aligned}$

$$\left[\begin{array}{rr|r} -10 & 12 & 30 \\ 5 & -6 & -15 \end{array}\right] \quad \text{Write the augmented matrix.}$$

$$\left[\begin{array}{rr|r} 1 & -\frac{6}{5} & -3 \\ 5 & -6 & -15 \end{array}\right] \quad -\tfrac{1}{10}R_1$$

$$\left[\begin{array}{rr|r} 1 & -\frac{6}{5} & -3 \\ 0 & 0 & 0 \end{array}\right] \quad -5R_1 + R_2$$

The corresponding system is

$$\begin{aligned} x - \frac{6}{5}y &= -3 \\ 0 &= 0, \quad \text{True} \end{aligned}$$

which has dependent equations. Using the second equation of the original system, we write the solution set as

$$\{(x, y) \mid 5x - 6y = -15\}.$$

Work Problem 3 at the Side.

Exponents, Polynomials, and Polynomial Functions

6

In 1980 MasterCard International Incorporated first began offering debit cards in an effort to challenge Visa, the leader in credit card transactions at that time. Now immensely popular, debit cards draw money from consumers' bank accounts rather than from established lines of credit. By 2005, it is estimated that 269 million debit cards will be in use. (*Source: Microsoft Encarta Encyclopedia 2002*; HSN Consultants Inc.)

We introduced the concept of function in Section 4.5 and extend our work to include *polynomial functions* in this chapter. In Exercise 11 of Section 6.3, we use a polynomial function to model the number of bank debit cards issued.

6.1 Integer Exponents and Scientific Notation

OBJECTIVES

1. Use the product rule for exponents.
2. Define 0 and negative exponents.
3. Use the quotient rule for exponents.
4. Use the power rules for exponents.
5. Simplify exponential expressions.
6. Use the rules for exponents with scientific notation.

Recall from **Section 1.3** that we use exponents to write products of repeated factors. For example,

$$2^5 \quad \text{is defined as} \quad 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$$

The number 5, the *exponent,* shows that the *base* 2 appears as a factor 5 times. The quantity 2^5 is called an *exponential* or a *power*. We read 2^5 as "2 to the fifth power" or "2 to the fifth."

OBJECTIVE 1 Use the product rule for exponents. There are several useful rules that simplify work with exponents. For example, the product $2^5 \cdot 2^3$ can be simplified as follows.

$$2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2) = 2^8 \qquad 5 + 3 = 8$$

This result, that products of exponential expressions with the *same base* are found by adding exponents, is generalized as the **product rule for exponents.**

Product Rule for Exponents

If m and n are natural numbers and a is any real number, then

$$a^m \cdot a^n = a^{m+n}.$$

In words, when multiplying powers of like bases, keep the same base and add the exponents.

To see that the product rule is true, use the definition of an exponent.

$$a^m = \underbrace{a \cdot a \cdot a \cdots a}_{a \text{ appears as a factor } m \text{ times.}} \qquad a^n = \underbrace{a \cdot a \cdot a \cdots a}_{a \text{ appears as a factor } n \text{ times.}}$$

From this,

$$\begin{aligned} a^m \cdot a^n &= \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ factors}} \cdot \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} \\ &= \underbrace{a \cdot a \cdot a \cdots a}_{(m+n) \text{ factors}} \\ a^m \cdot a^n &= a^{m+n}. \end{aligned}$$

EXAMPLE 1 Using the Product Rule for Exponents

Apply the product rule for exponents, if possible, in each case.

(a) $3^4 \cdot 3^7 = 3^{4+7} = 3^{11}$

(b) $5^3 \cdot 5 = 5^3 \cdot 5^1 = 5^{3+1} = 5^4$

(c) $y^3 \cdot y^8 \cdot y^2 = y^{3+8+2} = y^{13}$

(d) $(5y^2)(-3y^4) = 5(-3)y^2y^4$ Associative and commutative properties
$= -15y^{2+4}$ Multiply; product rule
$= -15y^6$

(e) $(7p^3q)(2p^5q^2) = 7(2)p^3p^5qq^2 = 14p^8q^3$

(f) $x^2 \cdot y^4$
Because the bases are not the same, the product rule does not apply.

Work Problem 1 at the Side.

1 Apply the product rule for exponents, if possible, in each case.

(a) $m^8 \cdot m^6$

(b) $r^7 \cdot r$

(c) $k^4k^3k^6$

(d) $m^5 \cdot p^4$

(e) $(-4a^3)(6a^2)$

(f) $(-5p^4)(-9p^5)$

ANSWERS
1. **(a)** m^{14} **(b)** r^8 **(c)** k^{13}
(d) The product rule does not apply.
(e) $-24a^5$ **(f)** $45p^9$

CAUTION
Be careful in problems like Example 1(a) not to multiply the bases. Notice that $3^4 \cdot 3^7 = 3^{11}$, ***not*** 9^{11}. ***Keep the same base and add the exponents.***

OBJECTIVE 2 Define 0 and negative exponents. So far we have discussed only positive exponents. Now we define 0 as an exponent. Suppose we multiply 4^2 by 4^0. By the product rule, extended to whole numbers,

$$4^2 \cdot 4^0 = 4^{2+0} = 4^2.$$

For the product rule to hold true, 4^0 must equal 1, and so we define a^0 this way for any nonzero real number a.

Audio

Zero Exponent
If a is any nonzero real number, then
$$\mathbf{a^0 = 1.}$$

The expression 0^0 is undefined.*

Video

EXAMPLE 2 Using 0 as an Exponent

Evaluate each expression.

(a) $6^0 = 1$

(b) $(-6)^0 = 1$ Base is -6.

(c) $-6^0 = -(6^0) = -1$ Base is 6.

(d) $-(-6)^0 = -1$

(e) $5^0 + 12^0 = 1 + 1 = 2$

(f) $(8k)^0 = 1, \quad k \neq 0$

Work Problem 2 at the Side.

We should define negative exponents so that the rules for exponents are valid. With this in mind, using the product rule should give

$$8^2 \cdot 8^{-2} = 8^{2+(-2)} = 8^0 = 1.$$

This indicates that 8^{-2} is the reciprocal of 8^2. But $\frac{1}{8^2}$ is the reciprocal of 8^2, and a number can have only one reciprocal. Therefore, it is reasonable to conclude that $8^{-2} = \frac{1}{8^2}$. We can generalize and make the following definition.

Audio

Negative Exponent
For any natural number n and any nonzero real number a,
$$\mathbf{a^{-n} = \frac{1}{a^n}.}$$

With this definition, the expression a^n is meaningful for any integer exponent n and any nonzero real number a.

*In advanced treatments, 0^0 is called an *indeterminate form*.

2 Evaluate each expression.

(a) 29^0

(b) $(-29)^0$

(c) $-(-29)^0$

(d) -29^0

(e) $8^0 - 15^0$

(f) $(-15p^5)^0, \quad p \neq 0$

ANSWERS
2. (a) 1 **(b)** 1 **(c)** -1 **(d)** -1 **(e)** 0 **(f)** 1

3 In parts (a)–(f), write the expressions with only positive exponents. In parts (g) and (h), simplify each expression.

(a) 6^{-3}

(b) 8^{-1}

(c) $(2x)^{-4}, \quad x \neq 0$

(d) $7r^{-6}, \quad r \neq 0$

(e) $-q^{-4}, \quad q \neq 0$

(f) $(-q)^{-4}, \quad q \neq 0$

(g) $3^{-1} + 5^{-1}$

(h) $4^{-1} - 2^{-1}$

CAUTION

A negative exponent does not indicate that an expression represents a negative number. Negative exponents lead to reciprocals.

Expression	Example	
a^{-m}	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	**Not negative**
$-a^{-m}$	$-3^{-2} = -\frac{1}{3^2} = -\frac{1}{9}$	**Negative**

EXAMPLE 3 Using Negative Exponents

In parts (a)–(f), write the expressions with only positive exponents. In parts (g) and (h), simplify each expression.

(a) $2^{-3} = \frac{1}{2^3}$

(b) $6^{-1} = \frac{1}{6^1} = \frac{1}{6}$

(c) $(5z)^{-3} = \frac{1}{(5z)^3}, \quad z \neq 0$

(d) $5z^{-3} = 5\left(\frac{1}{z^3}\right) = \frac{5}{z^3}, \quad z \neq 0$

(e) $-m^{-2} = -\frac{1}{m^2}, \quad m \neq 0$

(f) $(-m)^{-2} = \frac{1}{(-m)^2}, \quad m \neq 0$

(g) $3^{-1} + 4^{-1} = \frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$ $\qquad \frac{1}{3} \cdot \frac{4}{4} = \frac{4}{12}; \frac{1}{4} \cdot \frac{3}{3} = \frac{3}{12}$

(h) $5^{-1} - 2^{-1} = \frac{1}{5} - \frac{1}{2} = \frac{2}{10} - \frac{5}{10} = -\frac{3}{10}$

CAUTION

In Example 3(g), note that $3^{-1} + 4^{-1} \neq (3 + 4)^{-1}$. The expression on the left is equal to $\frac{7}{12}$, as shown in the solution, while the expression on the right is $7^{-1} = \frac{1}{7}$. Similar reasoning can be applied to part (h).

Work Problem 3 at the Side.

EXAMPLE 4 Using Negative Exponents

Evaluate each expression.

(a) $\frac{1}{2^{-3}} = \frac{1}{\frac{1}{2^3}} = 1 \div \frac{1}{2^3} = 1 \cdot \frac{2^3}{1} = 2^3 = 8$

(b) $\frac{2^{-3}}{3^{-2}} = \frac{\frac{1}{2^3}}{\frac{1}{3^2}} = \frac{1}{2^3} \div \frac{1}{3^2} = \frac{1}{2^3} \cdot \frac{3^2}{1} = \frac{3^2}{2^3} = \frac{9}{8}$

Example 4 suggests the following generalizations.

ANSWERS

3. (a) $\frac{1}{6^3}$ (b) $\frac{1}{8}$ (c) $\frac{1}{(2x)^4}$ (d) $\frac{7}{r^6}$ (e) $-\frac{1}{q^4}$ (f) $\frac{1}{(-q)^4}$ (g) $\frac{8}{15}$ (h) $-\frac{1}{4}$

Special Rules for Negative Exponents

If $a \neq 0$ and $b \neq 0$, then $\quad \dfrac{1}{a^{-n}} = a^n \quad$ and $\quad \dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}.$

Work Problem 4 at the Side.

OBJECTIVE 3 Use the quotient rule for exponents. A quotient, such as $\frac{a^8}{a^3}$, can be simplified in much the same way as a product. (In all quotients of this type, assume that the denominator is not 0.) Using the definition of an exponent,

$$\frac{a^{8}}{a^{3}} = \frac{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a \cdot a \cdot a \cdot a = a^{5}.$$

Notice that $8 - 3 = 5$. In the same way,

$$\frac{a^{3}}{a^{8}} = \frac{a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^5} = a^{-5}.$$

Here, $3 - 8 = -5$. These examples suggest the **quotient rule for exponents.**

Quotient Rule for Exponents

If a is any nonzero real number and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}.$$

In words, when dividing powers of like bases, keep the same base and subtract the exponent of the denominator from the exponent of the numerator.

EXAMPLE 5 Using the Quotient Rule for Exponents

Apply the quotient rule for exponents, if possible, and write each result using only positive exponents.

Numerator exponent

Denominator exponent

(a) $\dfrac{3^7}{3^2} = 3^{7-2} = 3^5$

Minus sign

(b) $\dfrac{p^6}{p^2} = p^{6-2} = p^4, \quad p \neq 0$

(c) $\dfrac{k^7}{k^{12}} = k^{7-12} = k^{-5} = \dfrac{1}{k^5}, \quad k \neq 0$

(d) $\dfrac{2^7}{2^{-3}} = 2^{7-(-3)} = 2^{7+3} = 2^{10}$

(e) $\dfrac{8^{-2}}{8^5} = 8^{-2-5} = 8^{-7} = \dfrac{1}{8^7}$

(f) $\dfrac{6}{6^{-1}} = \dfrac{6^1}{6^{-1}} = 6^{1-(-1)} = 6^2$

(g) $\dfrac{z^{-5}}{z^{-8}} = z^{-5-(-8)} = z^3, \quad z \neq 0$

(h) $\dfrac{a^3}{b^4}, \quad b \neq 0$

The quotient rule does not apply because the bases are different.

Work Problem 5 at the Side.

4 Evaluate each expression.

(a) $\dfrac{1}{4^{-3}}$

(b) $\dfrac{3^{-3}}{9^{-1}}$

5 Apply the quotient rule for exponents, if possible, and write each result using only positive exponents.

(a) $\dfrac{4^8}{4^6}$

(b) $\dfrac{x^{12}}{x^3}, \quad x \neq 0$

(c) $\dfrac{r^5}{r^8}, \quad r \neq 0$

(d) $\dfrac{2^8}{2^{-4}}$

(e) $\dfrac{6^{-3}}{6^4}$

(f) $\dfrac{8}{8^{-1}}$

(g) $\dfrac{t^{-4}}{t^{-6}}, \quad t \neq 0$

(h) $\dfrac{x^3}{y^5}, \quad y \neq 0$

ANSWERS

4. **(a)** 64 **(b)** $\frac{1}{3}$

5. **(a)** 4^2 **(b)** x^9 **(c)** $\frac{1}{r^3}$ **(d)** 2^{12} **(e)** $\frac{1}{6^7}$ **(f)** 8^2 **(g)** t^2 **(h)** The quotient rule does not apply.

6 Use one or more power rules to simplify each expression.

(a) $(r^5)^4$

(b) $\left(\frac{3}{4}\right)^3$

(c) $(9x)^3$

(d) $(5r^6)^3$

(e) $\left(\frac{-3n^4}{m}\right)^3, \quad m \neq 0$

OBJECTIVE 4 **Use the power rules for exponents.** The expression $(3^4)^2$ can be simplified as

$$(3^4)^2 = 3^4 \cdot 3^4 = 3^{4+4} = 3^8,$$

where $4 \cdot 2 = 8$. This example suggests the first **power rule for exponents.** The other two power rules can be demonstrated with similar examples.

Power Rules for Exponents

If a and b are real numbers and m and n are integers, then

$$\textbf{(a)}\ (a^m)^n = a^{mn}, \quad \textbf{(b)}\ (ab)^m = a^m b^m, \quad \text{and} \quad \textbf{(c)}\ \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0).$$

In words,

(a) to raise a power to a power, multiply exponents;

(b) to raise a product to a power, raise each factor to that power; and

(c) to raise a quotient to a power, raise the numerator and the denominator to that power.

EXAMPLE 6 Using the Power Rules for Exponents

Use one or more power rules to simplify each expression.

(a) $(p^8)^3 = p^{8 \cdot 3} = p^{24}$

(b) $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$

(c) $(3y)^4 = 3^4 y^4 = 81y^4$

(d) $(6p^7)^2 = 6^2 p^{7 \cdot 2} = 6^2 p^{14} = 36p^{14}$

(e) $\left(\frac{-2m^5}{z}\right)^3 = \frac{(-2)^3 m^{5 \cdot 3}}{z^3} = \frac{(-2)^3 m^{15}}{z^3} = \frac{-8m^{15}}{z^3}, \quad z \neq 0$

Work Problem 6 at the Side.

The reciprocal of a^n is $\frac{1}{a^n} = \left(\frac{1}{a}\right)^n$. Also, by definition, a^n and a^{-n} are reciprocals since

$$a^n \cdot a^{-n} = a^n \cdot \frac{1}{a^n} = 1.$$

Thus, since both are reciprocals of a^n,

$$a^{-n} = \left(\frac{1}{a}\right)^n.$$

Some examples of this result are

$$6^{-3} = \left(\frac{1}{6}\right)^3 \quad \text{and} \quad \left(\frac{1}{3}\right)^{-2} = 3^2.$$

ANSWERS

6. **(a)** r^{20} **(b)** $\frac{27}{64}$ **(c)** $729x^3$ **(d)** $125r^{18}$ **(e)** $\frac{-27n^{12}}{m^3}$

This discussion can be generalized as follows.

Audio

More Special Rules for Negative Exponents

If $a \neq 0$ and $b \neq 0$ and n is an integer, then

$$a^{-n} = \left(\frac{1}{a}\right)^n \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

In words, any nonzero number raised to the negative nth power is equal to the reciprocal of that number raised to the nth power.

EXAMPLE 7 Using Negative Exponents with Fractions

Write each expression with only positive exponents and then evaluate.

(a) $\left(\frac{3}{7}\right)^{-2} = \left(\frac{7}{3}\right)^2 = \frac{49}{9}$ **(b)** $\left(\frac{4}{5}\right)^{-3} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$

Work Problem 7 at the Side.

The definitions and rules of this section are summarized here.

Audio

Definitions and Rules for Exponents

For all integers m and n and all real numbers a and b, the following rules apply.

Product Rule	$a^m \cdot a^n = a^{m+n}$
Quotient Rule	$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$
Zero Exponent	$a^0 = 1 \quad (a \neq 0)$
Negative Exponent	$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$
Power Rules	$(a^m)^n = a^{mn}$
	$(ab)^m = a^m b^m$
	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \quad (b \neq 0)$
Special Rules	$\frac{1}{a^{-n}} = a^n \quad (a \neq 0) \qquad \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n} \quad (a, b \neq 0)$
	$a^{-n} = \left(\frac{1}{a}\right)^n \quad (a \neq 0) \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n \quad (a, b \neq 0)$

OBJECTIVE 5 Simplify exponential expressions. With the rules of exponents developed so far in this section, we can simplify expressions that involve one or more rules, as shown in Example 8 on the next page.

7 Write each expression with only positive exponents and then evaluate.

(a) $\left(\frac{3}{4}\right)^{-3}$

(b) $\left(\frac{5}{6}\right)^{-2}$

ANSWERS

7. (a) $\left(\frac{4}{3}\right)^3; \frac{64}{27}$ **(b)** $\left(\frac{6}{5}\right)^2; \frac{36}{25}$

8 Simplify each expression so that no negative exponents appear in the final result. Assume all variables represent nonzero real numbers.

(a) $5^4 \cdot 5^{-6}$

(b) $x^{-4} \cdot x^{-6} \cdot x^8$

(c) $(5^{-3})^{-2}$

(d) $(y^{-2})^7$

(e) $\dfrac{a^{-3}b^5}{a^4b^{-2}}$

(f) $(3^2k^{-4})^{-1}$

(g) $\left(\dfrac{2y}{x^3}\right)^2\left(\dfrac{4y}{x}\right)^{-1}$

Answers

8. **(a)** $\frac{1}{5^2}$ or $\frac{1}{25}$ **(b)** $\frac{1}{x^2}$ **(c)** 5^6 **(d)** $\frac{1}{y^{14}}$ **(e)** $\frac{b^7}{a^7}$ **(f)** $\frac{k^4}{3^2}$ or $\frac{k^4}{9}$ **(g)** $\frac{y}{x^5}$

EXAMPLE 8 Using the Definitions and Rules for Exponents

Simplify each expression so that no negative exponents appear in the final result. Assume all variables represent nonzero real numbers.

(a) $3^2 \cdot 3^{-5} = 3^{2+(-5)} = 3^{-3} = \dfrac{1}{3^3}$ or $\dfrac{1}{27}$

(b) $x^{-3} \cdot x^{-4} \cdot x^2 = x^{-3+(-4)+2} = x^{-5} = \dfrac{1}{x^5}$

(c) $(4^{-2})^{-5} = 4^{(-2)(-5)} = 4^{10}$

(d) $(x^{-4})^6 = x^{(-4)6} = x^{-24} = \dfrac{1}{x^{24}}$

(e)
$$\begin{aligned}\frac{x^{-4}y^2}{x^2y^{-5}} &= \frac{x^{-4}}{x^2} \cdot \frac{y^2}{y^{-5}}\\ &= x^{-4-2} \cdot y^{2-(-5)}\\ &= x^{-6}y^7\\ &= \frac{y^7}{x^6}\end{aligned}$$

(f)
$$\begin{aligned}(2^3x^{-2})^{-2} &= (2^3)^{-2} \cdot (x^{-2})^{-2}\\ &= 2^{-6}x^4\\ &= \frac{x^4}{2^6} \text{ or } \frac{x^4}{64}\end{aligned}$$

(g)
$$\begin{aligned}\left(\frac{3x^2}{y}\right)^2\left(\frac{4x^3}{y^{-2}}\right)^{-1} &= \frac{3^2(x^2)^2}{y^2} \cdot \frac{y^{-2}}{4x^3} && \text{Combination of rules}\\ &= \frac{9x^4}{y^2} \cdot \frac{y^{-2}}{4x^3} && \text{Power rule}\\ &= \frac{9}{4}x^{4-3}y^{-2-2} = \frac{9x}{4y^4} && \text{Quotient rule; } a^{-n} = \left(\frac{1}{a}\right)^n\end{aligned}$$

NOTE

There is often more than one way to simplify expressions like those in Example 8. For instance, we could simplify Example 8(e) as follows.

$$\frac{x^{-4}y^2}{x^2y^{-5}} = \frac{y^5y^2}{x^4x^2} = \frac{y^7}{x^6} \qquad \text{Use } \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}\text{; product rule}$$

Work Problem 8 at the Side.

OBJECTIVE 6 Use the rules for exponents with scientific notation. The number of one-celled organisms that will sustain a whale for a few hours is 400,000,000,000,000, and the shortest wavelength of visible light is approximately .0000004 m. It is often simpler to write these numbers using *scientific notation*.

In scientific notation, a number is written with the decimal point after the first nonzero digit and multiplied by a power of 10.

Scientific Notation

A number is written in **scientific notation** when it is expressed in the form

$$a \times 10^n$$

where $1 \le |a| < 10$, and n is an integer.

For example, in scientific notation,

$$8000 = 8 \times 1000 = 8 \times 10^3.$$

The following numbers are not in scientific notation.

$.230 \times 10^4$	46.5×10^{-3}
.230 is less than 1.	46.5 is greater than 10.

To write a number in scientific notation, use the following steps. (If the number is negative, ignore the negative sign, go through these steps, and then attach a negative sign to the result.)

Audio

Converting to Scientific Notation

Step 1 **Position the decimal point.** Place a caret, ^, to the right of the first nonzero digit, where the decimal point will be placed.

Step 2 **Determine the numeral for the exponent.** Count the number of digits from the decimal point to the caret. This number gives the absolute value of the exponent on 10.

Step 3 **Determine the sign for the exponent.** Decide whether multiplying by 10^n should make the result of Step 1 larger or smaller. The exponent should be positive to make the result larger; it should be negative to make the result smaller.

It is helpful to remember that for $n \geq 1$, $10^{-n} < 1$ and $10^n \geq 10$.

Animation

EXAMPLE 9 Writing Numbers in Scientific Notation

Write each number in scientific notation.

(a) 820,000

Place a caret to the right of the 8 (the first nonzero digit) to mark the new location of the decimal point.

$$8_\wedge 20{,}000$$

Count from the decimal point, which is understood to be after the last 0, to the caret.

$$8.\underbrace{20{,}000}_{\text{Count 5 places.}}. \leftarrow \text{Decimal point}$$

Since the number 8.2 is to be made larger, the exponent on 10 is positive.

$$820{,}000 = 8.2 \times 10^5$$

(b) .0000072

Count from left to right.

$$.\underbrace{000007}_{\text{6 places}}.2$$

Since the number 7.2 is to be made smaller, the exponent on 10 is negative.

$$.0000072 = 7.2 \times 10^{-6}$$

Work Problem 9 at the Side.

9 Write each number in scientific notation.

(a) 400,000

(b) 29,800,000

(c) -6083

(d) .00172

(e) .0000000503

Answers

9. **(a)** 4×10^5 **(b)** 2.98×10^7 **(c)** -6.083×10^3 **(d)** 1.72×10^{-3} **(e)** 5.03×10^{-8}

10 Write each number in standard notation.

(a) 4.98×10^5

(b) 6.8×10^{-7}

(c) -5.372×10^0

Converting from Scientific Notation

Multiplying a number by a positive power of 10 makes the number larger, so move the decimal point to the right n places if n is positive in 10^n.

Multiplying by a negative power of 10 makes a number smaller, so move the decimal point to the left $|n|$ places if n is negative.

If n is 0, leave the decimal point where it is.

EXAMPLE 10 Converting from Scientific Notation to Standard Notation

Write each number in standard notation.

(a) 6.93×10^7

$$6.9\underbrace{300000}_{7\text{ places}}. \qquad \text{Attach 0s as necessary.}$$

We moved the decimal point 7 places to the right. (We had to attach five 0s.)

$$6.93 \times 10^7 = 69{,}300{,}000$$

(b) 4.7×10^{-6}

$$.\underbrace{000004}_{6\text{ places}}.7$$

We moved the decimal point 6 places to the left.

$$4.7 \times 10^{-6} = .0000047$$

(c) $-1.083 \times 10^0 = -1.083 \times 1 = -1.083$

Work Problem 10 at the Side.

When problems require operations with numbers that are very large and/or very small, and a calculator is not available, we can write the numbers in scientific notation and perform the calculations using the rules for exponents.

11 Evaluate

$$\frac{200{,}000 \times .0003}{.06 \times 4{,}000{,}000}.$$

EXAMPLE 11 Using Scientific Notation in Computation

Evaluate $\dfrac{1{,}920{,}000 \times .0015}{.000032 \times 45{,}000}$.

$$\begin{aligned}
\frac{1{,}920{,}000 \times .0015}{.000032 \times 45{,}000} &= \frac{1.92 \times 10^6 \times 1.5 \times 10^{-3}}{3.2 \times 10^{-5} \times 4.5 \times 10^4} && \text{Express all numbers in scientific notation.}\\
&= \frac{1.92 \times 1.5 \times 10^6 \times 10^{-3}}{3.2 \times 4.5 \times 10^{-5} \times 10^4} && \text{Commutative property}\\
&= \frac{1.92 \times 1.5 \times 10^3}{3.2 \times 4.5 \times 10^{-1}} && \text{Product rule}\\
&= \frac{1.92 \times 1.5}{3.2 \times 4.5} \times 10^4 && \text{Quotient rule}\\
&= .2 \times 10^4 && \text{Simplify.}\\
&= (2 \times 10^{-1}) \times 10^4\\
&= 2 \times 10^3 \quad \text{or} \quad 2000
\end{aligned}$$

Work Problem 11 at the Side.

Answers

10. (a) 498,000 **(b)** .00000068 **(c)** -5.372

11. 2.5×10^{-4} or .00025

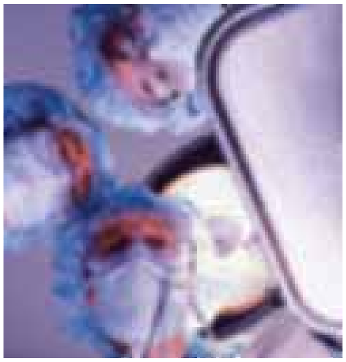
Calculator Tip To enter numbers in scientific notation, you can use the EE or EXP key on a scientific calculator. For instance, to work Example 11 using a popular model calculator with an EE key, enter the following symbols.

1.92 EE 6 × 1.5 EE 3 +/− ÷ (3.2 EE 5 +/− × 4.5 EE 4) =

The EXP key is used in exactly the same way. Notice that the negative exponent −3 is entered by pressing 3, then +/−. (***Keystrokes vary among different models of calculators,*** so you should refer to your owner's manual if this sequence does not apply to your particular model.)

EXAMPLE 12 Using Scientific Notation to Solve Problems

In 1990, the national health care expenditure was \$695.6 billion. By 2000, this figure had risen by a factor of 1.9; that is, it almost doubled in only 10 years. (*Source:* U.S. Centers for Medicare & Medicaid Services.)



(a) Write the 1990 health care expenditure using scientific notation.

$$\begin{aligned} 695.6 \text{ billion} &= 695.6 \times 10^9 \\ &= (6.956 \times 10^2) \times 10^9 \\ &= 6.956 \times 10^{11} \qquad \text{Product rule} \end{aligned}$$

In 1990, the expenditure was $\$6.956 \times 10^{11}$.

(b) What was the expenditure in 2000?
Multiply the result in part (a) by 1.9.

$$\begin{aligned} (6.956 \times 10^{11}) \times 1.9 &= (1.9 \times 6.956) \times 10^{11} && \text{Commutative and associative properties} \\ &= 13.216 \times 10^{11} && \text{Round to three decimal places.} \\ &= 1.3216 \times 10^{12} && \text{Scientific notation} \end{aligned}$$

The 2000 expenditure was about \$1,321,600,000,000 (over \$1 trillion).

Work Problem 12 at the Side.

12 The distance to the sun is 9.3×10^7 mi. How long would it take a rocket, traveling at 3.2×10^3 mph, to reach the sun? (*Hint:* $t = \frac{d}{r}$.)

ANSWERS
12. approximately 2.9×10^4 hr

6.2 Adding and Subtracting Polynomials

OBJECTIVE 1 Know the basic definitions for polynomials. Just as whole numbers are the basis of arithmetic, *polynomials* are fundamental in algebra. To understand polynomials, we must review several words from **Section 1.4.** A **term** is a number, a variable, or the product or quotient of a number and one or more variables raised to powers. Examples of terms include

$$4x, \quad \frac{1}{2}m^5 \left(\text{or } \frac{m^5}{2}\right), \quad -7z^9, \quad 6x^2z, \quad \frac{5}{3x^2}, \quad \text{and} \quad 9. \qquad \text{Terms}$$

The number in the product is called the **numerical coefficient,** or just the **coefficient.*** In the term $8x^3$, the coefficient is 8. In the term $-4p^5$, it is -4. The coefficient of the term k is understood to be 1. The coefficient of $-r$ is -1. In the term $\frac{x}{3}$, the coefficient is $\frac{1}{3}$ since $\frac{x}{3} = \frac{1x}{3} = \frac{1}{3}x$.

Work Problem 1 at the Side.

Recall that any combination of variables or constants (numerical values) joined by the basic operations of addition, subtraction, multiplication, and division (except by 0), or raising to powers or taking roots is called an **algebraic expression.** The simplest kind of algebraic expression is a *polynomial*.

Polynomial

A **polynomial** is a term or a finite sum of terms in which all variables have whole number exponents and no variables appear in denominators or under radicals.

Examples of polynomials include

$$3x - 5, \quad 4m^3 - 5m^2p + 8, \quad \text{and} \quad -5t^2s^3. \qquad \text{Polynomials}$$

Even though the expression $3x - 5$ involves subtraction, it is a sum of terms since it could be written as $3x + (-5)$.

Some examples of expressions that are not polynomials are

$$x^{-1} + 3x^{-2}, \quad \sqrt{9 - x}, \quad \text{and} \quad \frac{1}{x}. \qquad \text{Not polynomials}$$

The first of these is not a polynomial because it has negative integer exponents, the second because it involves a variable under a radical, and the third because it contains a variable in the denominator.

Most of the polynomials used in this book contain only one variable. A polynomial containing only the variable x is called a **polynomial in *x*.** A polynomial in one variable is written in **descending powers** of the variable if the exponents on the variable decrease from left to right. For example,

$$x^5 - 6x^2 + 12x - 5$$

is a polynomial in descending powers of x. The term -5 in this polynomial can be thought of as $-5x^0$, since $-5x^0 = -5(1) = -5$.

Work Problem 2 at the Side.

---

* More generally, any factor in a term is the coefficient of the product of the remaining factors. For example, $3x^2$ is the coefficient of y in the term $3x^2y$, and $3y$ is the coefficient of x^2 in $3x^2y$.

OBJECTIVES

1. Know the basic definitions for polynomials.
2. Find the degree of a polynomial.
3. Add and subtract polynomials.

1 Identify each coefficient.

(a) $-9m^5$

(b) $12y^2x$

(c) x

(d) $-y$

(e) $\frac{z}{4}$

2 Write each polynomial in descending powers.

(a) $-4 + 9y + y^3$

(b) $-3z^4 + 2z^3 + z^5 - 6z$

(c) $-12m^{10} + 8m^9 + 10m^{12}$

Answers

1. (a) -9 **(b)** 12 **(c)** 1 **(d)** -1 **(e)** $\frac{1}{4}$

2. (a) $y^3 + 9y - 4$ **(b)** $z^5 - 3z^4 + 2z^3 - 6z$ **(c)** $10m^{12} - 12m^{10} + 8m^9$

3 Identify each polynomial as a *trinomial, binomial, monomial,* or *none of these.*

(a) $12m^4 - 6m^2$

(b) $-6y^3 + 2y^2 - 8y$

(c) $3a^5$

(d) $-2k^{10} + 2k^9 - 8k^5 + 2k$

4 Give the degree of each polynomial.

(a) $9y^4 + 8y^3 - 6$

(b) $-12m^7 + 11m^3 + m^9$

(c) $-2k$

(d) 10

(e) $3mn^2 + 2m^3n$

ANSWERS

3. **(a)** binomial **(b)** trinomial **(c)** monomial **(d)** none of these

4. **(a)** 4 **(b)** 9 **(c)** 1 **(d)** 0 **(e)** 4

Some polynomials with a specific number of terms are so common that they are given special names. A polynomial with exactly three terms is a **trinomial,** and a polynomial with exactly two terms is a **binomial.** A single-term polynomial is a **monomial.** The table that follows gives examples.

Type of Polynomial	Examples
Monomial	$5x,\quad 7m^9,\quad -8,\quad x^2y^2$
Binomial	$3x^2 - 6,\quad 11y + 8,\quad 5a^2b + 3a$
Trinomial	$y^2 + 11y + 6,\quad 8p^3 - 7p + 2m,\quad -3 + 2k^5 + 9z^4$
None of these	$p^3 - 5p^2 + 2p - 5,\quad -9z^3 + 5c^3 + 2m^5 + 11r^2 - 7r$

Work Problem 3 at the Side.

OBJECTIVE 2 Find the degree of a polynomial. The **degree of a term** with one variable is the exponent on the variable. For example, the degree of $2x^3$ is 3, the degree of $-x^4$ is 4, and the degree of $17x$ (that is, $17x^1$) is 1. The degree of a term in more than one variable is defined to be the sum of the exponents on the variables. For example, the degree of $5x^3y^7$ is 10, because $3 + 7 = 10$.

The greatest degree of any term in a polynomial is called the **degree of the polynomial.** In most cases, we will be interested in finding the degree of a polynomial in one variable. For example, $4x^3 - 2x^2 - 3x + 7$ has degree 3, because the greatest degree of any term is 3 (the degree of $4x^3$).

The table shows several polynomials and their degrees.

Polynomial	Degree
$9x^2 - 5x + 8$	2
$17m^9 + 18m^{14} - 9m^3$	14
$5x$	1, because $5x = 5x^1$
-2	0, because $-2 = -2x^0$ (Any nonzero constant has degree 0.)
$5a^2b^5$	7, because $2 + 5 = 7$
$13xy^4 + x^3y^9 + 7xy$	12, because the degrees of the terms are 5, 12, and 2; 12 is the greatest.

NOTE
The number 0 has no degree, since 0 times a variable to any power is 0.

Work Problem 4 at the Side.

OBJECTIVE 3 Add and subtract polynomials. We use the distributive property to simplify polynomials by combining terms. For example,

$$\begin{aligned} x^3 + 4x^2 + 5x^2 - 1 &= x^3 + (4 + 5)x^2 - 1 \qquad \text{Distributive property} \\ &= x^3 + 9x^2 - 1. \end{aligned}$$

On the other hand, the terms in the polynomial $4x + 5x^2$ cannot be combined. As these examples suggest, only terms containing exactly the same variables to the same powers may be combined. As mentioned in **Section 1.4,** such terms are called **like terms.**

CAUTION
Remember that only like terms can be combined.

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Animation
Video

EXAMPLE 1 Combining Like Terms

Combine terms.

(a) $-5y^3 + 8y^3 - y^3 = (-5 + 8 - 1)y^3 = 2y^3$

(b) $6x + 5y - 9x + 2y = 6x - 9x + 5y + 2y$ Associative and commutative properties

$= -3x + 7y$ Combine like terms.

Since $-3x$ and $7y$ are unlike terms, no further simplification is possible.

(c) $5x^2y - 6xy^2 + 9x^2y + 13xy^2 = 5x^2y + 9x^2y - 6xy^2 + 13xy^2$
$= 14x^2y + 7xy^2$

Work Problem 5 at the Side.

We use the following rule to add two polynomials.

Audio

Adding Polynomials
To add two polynomials, combine like terms.

Polynomials can be added horizontally or vertically.

You Try It
Video

EXAMPLE 2 Adding Polynomials

Add: $(3a^5 - 9a^3 + 4a^2) + (-8a^5 + 8a^3 + 2)$.

Use the commutative and associative properties to rearrange the polynomials so that like terms are together. Then use the distributive property to combine like terms.

$$(3a^5 - 9a^3 + 4a^2) + (-8a^5 + 8a^3 + 2)$$
$$= 3a^5 - 8a^5 - 9a^3 + 8a^3 + 4a^2 + 2$$
$$= -5a^5 - a^3 + 4a^2 + 2 \qquad \text{Combine like terms.}$$

Add these same two polynomials vertically by placing like terms in columns.

$$\begin{array}{r} 3a^5 - 9a^3 + 4a^2 \\ \underline{-8a^5 + 8a^3 + 2} \\ -5a^5 - a^3 + 4a^2 + 2 \end{array}$$

Work Problem 6 at the Side.

In **Section 1.2,** we defined subtraction of real numbers as

$$a - b = a + (-b).$$

That is, we add the first number (minuend) and the negative (or opposite) of the second (subtrahend). We can give a similar definition for subtraction of polynomials by defining the **negative of a polynomial** as that polynomial with the sign of every coefficient changed.

5 Combine terms.

(a) $11x + 12x - 7x - 3x$

(b) $11p^5 + 4p^5 - 6p^3 + 8p^3$

(c) $2y^2z^4 + 3y^4 + 5y^4 - 9y^4z^2$

6 Add, using both the horizontal and vertical methods.

(a) $(12y^2 - 7y + 9) + (-4y^2 - 11y + 5)$

(b) $$\begin{array}{r} -6r^5 + 2r^3 - r^2 \\ \underline{8r^5 - 2r^3 + 5r^2} \end{array}$$

ANSWERS
5. (a) $13x$ **(b)** $15p^5 + 2p^3$
(c) $2y^2z^4 + 8y^4 - 9y^4z^2$
6. (a) $8y^2 - 18y + 14$ **(b)** $2r^5 + 4r^2$

Subtracting Polynomials

To subtract two polynomials, add the first polynomial and the negative of the *second* polynomial.

EXAMPLE 3 Subtracting Polynomials

Subtract: $(-6m^2 - 8m + 5) - (-5m^2 + 7m - 8)$.

Change every sign in the second polynomial and add.

$$\begin{aligned}
&(-6m^2 - 8m + 5) - (-5m^2 + 7m - 8) \\
&\quad = -6m^2 - 8m + 5 + 5m^2 - 7m + 8 && \text{Definition of subtraction} \\
&\quad = -6m^2 + 5m^2 - 8m - 7m + 5 + 8 && \text{Rearrange terms.} \\
&\quad = -m^2 - 15m + 13 && \text{Combine like terms.}
\end{aligned}$$

Check by adding the sum, $-m^2 - 15m + 13$, to the second polynomial. The result should be the first polynomial.

To subtract these two polynomials vertically, write the first polynomial above the second, lining up like terms in columns.

$$\begin{array}{r}
-6m^2 - 8m + 5 \\
\underline{-5m^2 + 7m - 8}
\end{array}$$

Change all the signs in the second polynomial, and add.

$$\begin{array}{rl}
-6m^2 - 8m + 5 & \\
\underline{+\ 5m^2 - 7m + 8} & \text{Change all signs.} \\
-m^2 - 15m + 13 & \text{Add in columns.}
\end{array}$$

Work Problem 7 at the Side.

7 Subtract, using both the horizontal and vertical methods.

(a) $(6y^3 - 9y^2 + 8) - (2y^3 + y^2 + 5)$

(b)
$$\begin{array}{r}
6y^3 - 2y^2 + 5y \\
\underline{-2y^3 + 8y^2 - 11y}
\end{array}$$

Answers

7. **(a)** $4y^3 - 10y^2 + 3$
(b) $8y^3 - 10y^2 + 16y$

6.3 Polynomial Functions

OBJECTIVES

1. Recognize and evaluate polynomial functions.
2. Use a polynomial function to model data.
3. Add and subtract polynomial functions.
4. Graph basic polynomial functions.

OBJECTIVE 1 Recognize and evaluate polynomial functions. In **Chapter 4** we studied linear (first-degree polynomial) functions, defined as $f(x) = mx + b$. Now we consider more general polynomial functions.

Polynomial Function

A **polynomial function of degree n** is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

for real numbers $a_n, a_{n-1}, \ldots, a_1$, and a_0, where $a_n \neq 0$ and n is a whole number.

Another way of describing a polynomial function is to say that it is a function defined by a polynomial in one variable, consisting of one or more terms. It is usually written in descending powers of the variable, and its degree is the degree of the polynomial that defines it.

Suppose that the polynomial $3x^2 - 5x + 7$ defines function f. Then

$$f(x) = 3x^2 - 5x + 7.$$

If $x = -2$, then $f(x) = 3x^2 - 5x + 7$ takes on the value

$$\begin{aligned} f(-2) &= 3(-2)^2 - 5(-2) + 7 && \text{Let } x = -2. \\ &= 3 \cdot 4 + 10 + 7 \\ &= 29. \end{aligned}$$

Thus, $f(-2) = 29$ and the ordered pair $(-2, 29)$ belongs to f.

EXAMPLE 1 Evaluating Polynomial Functions

Let $f(x) = 4x^3 - x^2 + 5$. Find each value.

(a) $f(3)$

$$\begin{aligned} f(x) &= 4x^3 - x^2 + 5 \\ f(3) &= 4 \cdot 3^3 - 3^2 + 5 && \text{Substitute 3 for } x. \\ &= 4 \cdot 27 - 9 + 5 && \text{Order of operations} \\ &= 108 - 9 + 5 \\ &= 104 \end{aligned}$$

(b) $$\begin{aligned} f(-4) &= 4 \cdot (-4)^3 - (-4)^2 + 5 && \text{Let } x = -4\text{; use parentheses.} \\ &= 4 \cdot (-64) - 16 + 5 && \text{Be careful with signs.} \\ &= -267 \end{aligned}$$

While f is the most common letter used to represent functions, recall that other letters such as g and h are also used. The capital letter P is often used for polynomial functions. Note that the function defined as $P(x) = 4x^3 - x^2 + 5$ yields the same ordered pairs as the function f in Example 1.

Work Problem 1 at the Side.

1 Let $f(x) = -x^2 + 5x - 11$. Find each value.

(a) $f(1)$

(b) $f(-4)$

(c) $f(0)$

ANSWERS

1. (a) -7 **(b)** -47 **(c)** -11

2 Use the function in Example 2 to approximate the number of households expected to pay at least one bill on-line each month in 2006.

OBJECTIVE 2 Use a polynomial function to model data. Polynomial functions can be used to approximate data. They are usually valid for small intervals, and they allow us to predict (with caution) what might happen for values just outside the intervals. These intervals are often periods of years, as shown in Example 2.

EXAMPLE 2 Using a Polynomial Model to Approximate Data

The number of U.S. households estimated to see and pay at least one bill on-line each month during the years 2000 through 2006 can be modeled by the polynomial function defined by

$$P(x) = .808x^2 + 2.625x + .502,$$

where $x = 0$ corresponds to the year 2000, $x = 1$ corresponds to 2001, and so on, and $P(x)$ is in millions. Use this function to approximate the number of households expected to pay at least one bill on-line each month in 2005.

Since $x = 5$ corresponds to 2005, we must find $P(5)$.

$$\begin{aligned} P(x) &= .808x^2 + 2.625x + .502 & \\ P(5) &= .808(5)^2 + 2.625(5) + .502 & \quad \text{Let } x = 5. \\ &= 33.827 & \quad \text{Evaluate.} \end{aligned}$$

Thus, in 2005 about 33.83 million households are expected to pay at least one bill on-line each month.

Work Problem 2 at the Side.

OBJECTIVE 3 Add and subtract polynomial functions. The operations of addition, subtraction, multiplication, and division are also defined for functions. For example, businesses use the equation "profit equals revenue minus cost," written using function notation as

$$\underset{\substack{\uparrow \\ \text{Profit} \\ \text{function}}}{P(x)} = \underset{\substack{\uparrow \\ \text{Revenue} \\ \text{function}}}{R(x)} - \underset{\substack{\uparrow \\ \text{Cost} \\ \text{function}}}{C(x)},$$

where x is the number of items produced and sold. Thus, the profit function is found by subtracting the cost function from the revenue function.

We define the following **operations on functions.**

Adding and Subtracting Functions

If $f(x)$ and $g(x)$ define functions, then

$$(f + g)(x) = f(x) + g(x) \qquad \text{Sum function}$$

and

$$(f - g)(x) = f(x) - g(x). \qquad \text{Difference function}$$

In each case, the domain of the new function is the intersection of the domains of $f(x)$ and $g(x)$.

ANSWERS

2. 45.34 million

You Try It

Video

EXAMPLE 3 Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = x^2 - 3x + 7 \quad \text{and} \quad g(x) = -3x^2 - 7x + 7,$$

find **(a)** the sum and **(b)** the difference.

(a) $(f + g)(x) = f(x) + g(x)$ — Use the definition.

$= (x^2 - 3x + 7) + (-3x^2 - 7x + 7)$ — Substitute.

$= -2x^2 - 10x + 14$ — Add the polynomials.

(b) $(f - g)(x) = f(x) - g(x)$ — Use the definition.

$= (x^2 - 3x + 7) - (-3x^2 - 7x + 7)$ — Substitute.

$= (x^2 - 3x + 7) + (3x^2 + 7x - 7)$ — Change subtraction to addition.

$= 4x^2 + 4x$ — Add.

Work Problem 3 at the Side.

Video

EXAMPLE 4 Adding and Subtracting Functions

For the polynomial functions defined by

$$f(x) = 10x^2 - 2x \quad \text{and} \quad g(x) = 2x,$$

find each of the following.

(a) $(f + g)(2)$

$(f + g)(2) = f(2) + g(2)$ — Use the definition.

$= [10(2)^2 - 2(2)] + 2(2)$ — Substitute.

$= 40$

Alternatively, we could first find $(f + g)(x)$.

$(f + g)(x) = f(x) + g(x)$ — Use the definition.

$= (10x^2 - 2x) + 2x$ — Substitute.

$= 10x^2$ — Combine like terms.

Then,

$(f + g)(2) = 10(2)^2$

$= 40.$ — The result is the same.

(b) $(f - g)(x)$ and $(f - g)(1)$

$(f - g)(x) = f(x) - g(x)$ — Use the definition.

$= (10x^2 - 2x) - 2x$ — Substitute.

$= 10x^2 - 4x$ — Combine like terms.

Then,

$(f - g)(1) = 10(1)^2 - 4(1)$ — Substitute.

$= 6.$

Confirm that $f(1) - g(1)$ gives the same result.

Work Problem 4 at the Side.

3 Let

$$f(x) = 3x^2 + 8x - 6$$

and $\quad g(x) = -4x^2 + 4x - 8.$

Find each function.

(a) $(f + g)(x)$

(b) $(f - g)(x)$

4 For

$$f(x) = 18x^2 - 24x$$

and $\quad g(x) = 3x,$

find each of the following.

(a) $(f + g)(x)$ and $(f + g)(-1)$

(b) $(f - g)(x)$ and $(f - g)(1)$

ANSWERS

3. **(a)** $-x^2 + 12x - 14$
   (b) $7x^2 + 4x + 2$
4. **(a)** $18x^2 - 21x$; 39
   (b) $18x^2 - 27x$; -9

OBJECTIVE 4 Graph basic polynomial functions. Functions were introduced in **Section 4.5.** Recall that each input (or x-value) of a function results in one output (or y-value). The simplest polynomial function is the **identity function,** defined by $f(x) = x$. The domain (set of x-values) of this function is all real numbers, $(-\infty, \infty)$, and it pairs each real number with itself. Therefore, the range (set of y-values) is also $(-\infty, \infty)$. Its graph is a straight line, as first seen in **Chapter 4.** (Notice that a *linear function* is a specific kind of polynomial function.) Figure 1 shows its graph and a table of selected ordered pairs.

x	$f(x) = x$
-2	-2
-1	-1
0	0
1	1
2	2

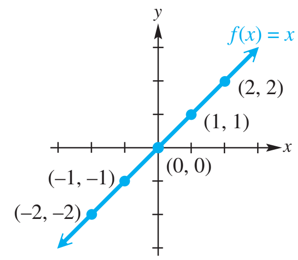


Figure 1

Another polynomial function, defined by $f(x) = x^2$, is the **squaring function.** For this function, every real number is paired with its square. The input can be any real number, so the domain is $(-\infty, \infty)$. Since the square of any real number is nonnegative, the range is $[0, \infty)$. Its graph is a *parabola*. Figure 2 shows the graph and a table of selected ordered pairs.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

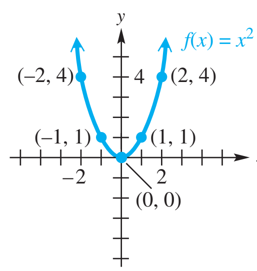


Figure 2

The **cubing function** is defined by $f(x) = x^3$. Every real number is paired with its cube. The domain and the range are both $(-\infty, \infty)$. Its graph is neither a line nor a parabola. See Figure 3 and the table of ordered pairs. (Polynomial functions of degree 3 and greater are studied in detail in more advanced courses.)

x	$f(x) = x^3$
-2	-8
-1	-1
0	0
1	1
2	8

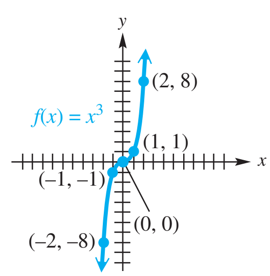


Figure 3

Video

EXAMPLE 5 Graphing Variations of the Identity, Squaring, and Cubing Functions

Graph each function by creating a table of ordered pairs. Give the domain and the range of each function by observing the graphs.

(a) $f(x) = 2x$

To find each range value, multiply the domain value by 2. Plot the points and join them with a straight line. See Figure 4. Both the domain and the range are $(-\infty, \infty)$.

x	$f(x) = 2x$
-2	-4
-1	-2
0	0
1	2
2	4

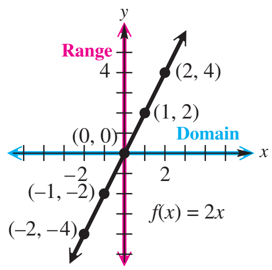


Figure 4

(b) $f(x) = -x^2$

For each input x, square it and then take its opposite. Plotting and joining the points gives a parabola that opens down. See the table and Figure 5. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0]$.

x	$f(x) = -x^2$
-2	-4
-1	-1
0	0
1	-1
2	-4

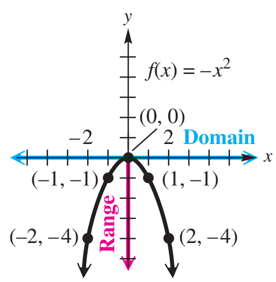


Figure 5

(c) $f(x) = x^3 - 2$

For this function, cube the input and then subtract 2 from the result. The graph is that of the cubing function *shifted* 2 units down. See the table and Figure 6. The domain and the range are both $(-\infty, \infty)$.

x	$f(x) = x^3 - 2$
-2	-10
-1	-3
0	-2
1	-1
2	6

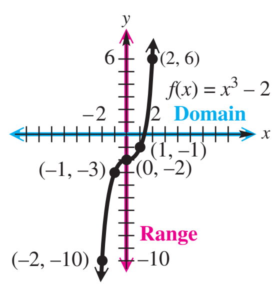
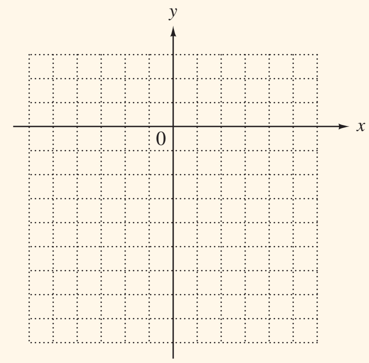


Figure 6

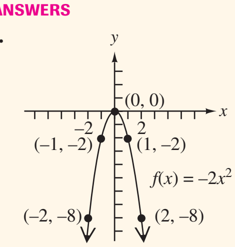
Work Problem 5 at the Side.

5 Graph $f(x) = -2x^2$. Give the domain and the range.



ANSWERS

5.



domain: $(-\infty, \infty)$; range: $(-\infty, 0]$

6.4 Multiplying Polynomials

OBJECTIVE 1 Multiply terms. Recall that the product of the two terms $3x^4$ and $5x^3$ is found by using the commutative and associative properties, along with the rules for exponents.

$$\begin{aligned}(3x^4)(5x^3) &= 3 \cdot 5 \cdot x^4 \cdot x^3\\ &= 15x^{4+3}\\ &= 15x^7\end{aligned}$$

EXAMPLE 1 Multiplying Monomials

Find each product.

(a) $-4a^3(3a^5) = -4(3)a^3 \cdot a^5 = -12a^8$

(b) $2m^2z^4(8m^3z^2) = 2(8)m^2 \cdot m^3 \cdot z^4 \cdot z^2 = 16m^5z^6$

Work Problem 1 at the Side.

OBJECTIVE 2 Multiply any two polynomials. We use the distributive property to extend this process to find the product of any two polynomials.

EXAMPLE 2 Multiplying Polynomials

Find each product.

(a) $-2(8x^3 - 9x^2)$

$$\begin{aligned}-2(8x^3 - 9x^2) &= -2(8x^3) - 2(-9x^2) \quad \text{Distributive property}\\ &= -16x^3 + 18x^2\end{aligned}$$

(b) $5x^2(-4x^2 + 3x - 2) = 5x^2(-4x^2) + 5x^2(3x) + 5x^2(-2)$
$= -20x^4 + 15x^3 - 10x^2$

(c) $(3x - 4)(2x^2 + x)$
Use the distributive property to multiply each term of $2x^2 + x$ by $3x - 4$.

$$(3x - 4)(2x^2 + x) = (3x - 4)(2x^2) + (3x - 4)(x)$$

Here $3x - 4$ has been treated as a single expression so that the distributive property could be used. Now use the distributive property two more times.

$$\begin{aligned}&= 3x(2x^2) + (-4)(2x^2) + (3x)(x) + (-4)(x)\\ &= 6x^3 - 8x^2 + 3x^2 - 4x\\ &= 6x^3 - 5x^2 - 4x\end{aligned}$$

(d) $2x^2(x + 1)(x - 3) = 2x^2[(x + 1)(x) + (x + 1)(-3)]$

$$\begin{aligned}&= 2x^2[x^2 + x - 3x - 3]\\ &= 2x^2(x^2 - 2x - 3)\\ &= 2x^4 - 4x^3 - 6x^2\end{aligned}$$

Work Problem 2 at the Side.

OBJECTIVES

1. Multiply terms.
2. Multiply any two polynomials.
3. Multiply binomials.
4. Find the product of the sum and difference of two terms.
5. Find the square of a binomial.
6. Multiply polynomial functions.

1 Find each product.

(a) $-6m^5(2m^4)$

(b) $8k^3y(9ky^3)$

2 Find each product.

(a) $-2r(9r - 5)$

(b) $3p^2(5p^3 + 2p^2 - 7)$

(c) $(4a - 5)(3a + 6)$

(d) $3x^3(x + 4)(x - 6)$

ANSWERS
1. (a) $-12m^9$ (b) $72k^4y^4$
2. (a) $-18r^2 + 10r$ (b) $15p^5 + 6p^4 - 21p^2$
(c) $12a^2 + 9a - 30$ (d) $3x^5 - 6x^4 - 72x^3$

3 Find each product.

(a)
$$\begin{array}{r} 2m - 5 \\ \underline{3m + 4} \end{array}$$

(b)
$$\begin{array}{r} 5a^3 - 6a^2 + 2a - 3 \\ \underline{2a - 5} \end{array}$$

It is often easier to multiply polynomials by writing them vertically.

EXAMPLE 3 Multiplying Polynomials Vertically

Find each product.

(a) $(5a - 2b)(3a + b)$

$$\begin{array}{rrrl}
 & 5a & -\ 2b & \\
 & 3a & +\ \ b & \\ \hline
 & 5ab & -\ 2b^2 & \longleftarrow b(5a - 2b) \\
15a^2 & -\ 6ab & & \longleftarrow 3a(5a - 2b) \\ \hline
15a^2 & -\ ab & -\ 2b^2 & \text{Combine like terms.}
\end{array}$$

(b) $(3m^3 - 2m^2 + 4)(3m - 5)$

$$\begin{array}{rrrrrl}
 & 3m^3 & -\ 2m^2 & & +\ 4 & \\
 & & 3m & & -\ 5 & \\ \hline
 & -15m^3 & +\ 10m^2 & & -\ 20 & -5(3m^3 - 2m^2 + 4) \\
9m^4 & -\ 6m^3 & & +\ 12m & & 3m(3m^3 - 2m^2 + 4) \\ \hline
9m^4 & -\ 21m^3 & +\ 10m^2 & +\ 12m & -\ 20 & \text{Combine like terms.}
\end{array}$$

Work Problem 3 at the Side.

OBJECTIVE 3 Multiply binomials. When working with polynomials, the product of two binomials occurs repeatedly. There is a shortcut method for finding these products. Recall that a binomial has just two terms, such as $3x - 4$ or $2x + 3$. We can find the product of these binomials using the distributive property as follows.

$$\begin{aligned}
(3x - 4)(2x + 3) &= 3x(2x + 3) - 4(2x + 3) \\
&= 3x(2x) + 3x(3) - 4(2x) - 4(3) \\
&= 6x^2 + 9x - 8x - 12
\end{aligned}$$

Before combining like terms to find the simplest form of the answer, let us check the origin of each of the four terms in the sum. First, $6x^2$ is the product of the two *first* terms.

$$(\mathbf{3x} - 4)(\mathbf{2x} + 3) \qquad 3x(2x) = 6x^2 \qquad \text{First terms}$$

To get $9x$, the *outer* terms are multiplied.

$$(\mathbf{3x} - 4)(2x + \mathbf{3}) \qquad 3x(3) = 9x \qquad \text{Outer terms}$$

The term $-8x$ comes from the *inner* terms.

$$(3x \mathbf{- 4})(\mathbf{2x} + 3) \qquad -4(2x) = -8x \qquad \text{Inner terms}$$

Finally, -12 comes from the *last* terms.

$$(3x \mathbf{- 4})(2x + \mathbf{3}) \qquad -4(3) = -12 \qquad \text{Last terms}$$

The product is found by combining these four results.

$$\begin{aligned}
(3x - 4)(2x + 3) &= 6x^2 + 9x - 8x - 12 \\
&= 6x^2 + x - 12
\end{aligned}$$

To keep track of the order of multiplying these terms, we use the initials FOIL (**F**irst, **O**uter, **I**nner, **L**ast). All the steps of the FOIL method can be done as follows. Try to do as many of these steps as possible mentally.

ANSWERS

3. **(a)** $6m^2 - 7m - 20$
(b) $10a^4 - 37a^3 + 34a^2 - 16a + 15$

First Last
$(3x - 4)(2x + 3)$
Inner
Outer

$6x^2$ -12
$(3x - 4)(2x + 3)$
$-8x$
$9x$
x Add.

EXAMPLE 4 Using the FOIL Method

Use the FOIL method to find each product.

(a) $(4m - 5)(3m + 1)$

First terms	$(\mathbf{4m} - 5)(\mathbf{3m} + 1)$	$4m(3m) = \mathbf{12m^2}$
Outer terms	$(\mathbf{4m} - 5)(3m \mathbf{+ 1})$	$4m(1) = \mathbf{4m}$
Inner terms	$(4m \mathbf{- 5})(\mathbf{3m} + 1)$	$-5(3m) = \mathbf{-15m}$
Last terms	$(4m \mathbf{- 5})(3m \mathbf{+ 1})$	$-5(1) = \mathbf{-5}$

Simplify by combining the four terms.

$$\begin{aligned}(4m - 5)(3m + 1) &= \overset{\text{F}}{12m^2} + \overset{\text{O}}{4m} - \overset{\text{I}}{15m} - \overset{\text{L}}{5} \\ &= 12m^2 - 11m - 5\end{aligned}$$

The procedure can be written in compact form as follows.

$12m^2$ -5
$(4m - 5)(3m + 1)$
$-15m$
$4m$
$-11m$ Add.

Combine these four results to get $12m^2 - 11m - 5$.

(b) $$\begin{aligned}(6a - 5b)(3a + 4b) &= \overset{\text{First}}{18a^2} + \overset{\text{Outer}}{24ab} - \overset{\text{Inner}}{15ab} - \overset{\text{Last}}{20b^2} \\ &= 18a^2 + 9ab - 20b^2\end{aligned}$$

(c) $(2k + 3z)(5k - 3z) = 10k^2 + 9kz - 9z^2$ FOIL

Work Problem 4 at the Side.

OBJECTIVE 4 Find the product of the sum and difference of two terms. Some types of binomial products occur frequently. For example, the product of the sum and difference of the same two terms, x and y, is

$$\begin{aligned}(x + y)(x - y) &= x^2 - xy + xy - y^2 \quad \text{FOIL} \\ &= x^2 - y^2.\end{aligned}$$

Product of the Sum and Difference of Two Terms

The **product of the sum and difference of the two terms x and y** is the difference of the squares of the terms.

$$\mathbf{(x + y)(x - y) = x^2 - y^2}$$

---

4 Use the FOIL method to find each product.

(a) $(3z + 2)(z + 1)$

(b) $(5r - 3)(2r - 5)$

(c) $(4p + 5q)(3p - 2q)$

(d) $(4y - z)(2y + 3z)$

(e) $(8r + 1)(8r - 1)$

Answers

4. **(a)** $3z^2 + 5z + 2$ **(b)** $10r^2 - 31r + 15$ **(c)** $12p^2 + 7pq - 10q^2$ **(d)** $8y^2 + 10yz - 3z^2$ **(e)** $64r^2 - 1$

5 Find each product.

(a) $(m + 5)(m - 5)$

(b) $(x - 4y)(x + 4y)$

(c) $(7m - 2n)(7m + 2n)$

(d) $4y^2(y + 7)(y - 7)$

EXAMPLE 5 Multiplying the Sum and Difference of Two Terms

Find each product.

(a) $(p + 7)(p - 7) = p^2 - 7^2$ $\quad (x + y)(x - y) = x^2 - y^2$
$= p^2 - 49$

(b) $(2r + 5)(2r - 5) = (2r)^2 - 5^2$
$= 2^2r^2 - 25$
$= 4r^2 - 25$

(c) $(6m + 5n)(6m - 5n) = (6m)^2 - (5n)^2$
$= 36m^2 - 25n^2$

(d) $2x^3(x + 3)(x - 3) = 2x^3(x^2 - 9)$
$= 2x^5 - 18x^3$

Work Problem 5 at the Side.

OBJECTIVE 5 Find the square of a binomial. Another special binomial product is the *square of a binomial*. To find the square of $x + y$, or $(x + y)^2$, multiply $x + y$ by itself.

$$(x + y)(x + y) = x^2 + xy + xy + y^2 \quad \text{FOIL}$$
$$= x^2 + 2xy + y^2$$

A similar result is true for the square of a difference.

Square of a Binomial

The **square of a binomial** is the sum of the square of the first term, twice the product of the two terms, and the square of the last term.

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

EXAMPLE 6 Squaring Binomials

Find each product.

(a) $(m + 7)^2 = m^2 + 2 \cdot m \cdot 7 + 7^2$ $\quad (x + y)^2 = x^2 + 2xy + y^2$
$= m^2 + 14m + 49$

(b) $(p - 5)^2 = p^2 - 2 \cdot p \cdot 5 + 5^2$ $\quad (x - y)^2 = x^2 - 2xy + y^2$
$= p^2 - 10p + 25$

(c) $(2p + 3v)^2 = (2p)^2 + 2(2p)(3v) + (3v)^2$
$= 4p^2 + 12pv + 9v^2$

(d) $(3r - 5s)^2 = (3r)^2 - 2(3r)(5s) + (5s)^2$
$= 9r^2 - 30rs + 25s^2$

ANSWERS

5. **(a)** $m^2 - 25$ **(b)** $x^2 - 16y^2$ **(c)** $49m^2 - 4n^2$ **(d)** $4y^4 - 196y^2$

CAUTION
As the products in the formula for the square of a binomial show,

$$(x + y)^2 \neq x^2 + y^2.$$

More generally,

$$(x + y)^n \neq x^n + y^n \qquad (n \neq 1).$$

Work Problem 6 at the Side.

We can use the patterns for the special products with more complicated products, as the following example shows.

You Try It

Animation

Video

EXAMPLE 7 Multiplying More Complicated Binomials

Use special products to find each product.

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$

$= (3p - 2)^2 - (5q)^2$ — Product of sum and difference of terms

$= 9p^2 - 12p + 4 - 25q^2$ — Square both quantities.

(b) $[(2z + r) + 1]^2 = (2z + r)^2 + 2(2z + r)(1) + 1^2$ — Square of a binomial

$= 4z^2 + 4zr + r^2 + 4z + 2r + 1$ — Square again; use the distributive property.

(c) $(x + y)^3 = (x + y)^2(x + y)$

$= (x^2 + 2xy + y^2)(x + y)$ — Square $x + y$.

$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$

$= x^3 + 3x^2y + 3xy^2 + y^3$

(d) $(2a + b)^4 = (2a + b)^2(2a + b)^2$

$= (4a^2 + 4ab + b^2)(4a^2 + 4ab + b^2)$ — Square $2a + b$.

$= 16a^4 + 16a^3b + 4a^2b^2 + 16a^3b + 16a^2b^2 + 4ab^3 + 4a^2b^2 + 4ab^3 + b^4$

$= 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$

Work Problem 7 at the Side.

OBJECTIVE 6 Multiply polynomial functions. In **Section 6.3** we saw how functions can be added and subtracted. Functions can also be multiplied.

Audio

Multiplying Functions

If $f(x)$ and $g(x)$ define functions, then

$$(fg)(x) = f(x) \cdot g(x). \qquad \text{Product function}$$

The domain of the product function is the intersection of the domains of $f(x)$ and $g(x)$.

6 Find each product.

(a) $(a + 2)^2$

(b) $(2m - 5)^2$

(c) $(y + 6z)^2$

(d) $(3k - 2n)^2$

7 Find each product.

(a) $[(m - 2n) - 3] \cdot [(m - 2n) + 3]$

(b) $[(k - 5h) + 2]^2$

(c) $(p + 2q)^3$

(d) $(x + 2)^4$

ANSWERS

6. (a) $a^2 + 4a + 4$
(b) $4m^2 - 20m + 25$
(c) $y^2 + 12yz + 36z^2$
(d) $9k^2 - 12kn + 4n^2$

7. (a) $m^2 - 4mn + 4n^2 - 9$
(b) $k^2 - 10kh + 25h^2 + 4k - 20h + 4$
(c) $p^3 + 6p^2q + 12pq^2 + 8q^3$
(d) $x^4 + 8x^3 + 24x^2 + 32x + 16$

8 For

$$f(x) = 2x + 7$$

and $g(x) = x^2 - 4,$

find $(fg)(x)$ and $(fg)(2)$.

EXAMPLE 8 Multiplying Polynomial Functions

For $f(x) = 3x + 4$ and $g(x) = 2x^2 + x$, find $(fg)(x)$ and $(fg)(-1)$.

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x) && \text{Use the definition.}\\ &= (3x + 4)(2x^2 + x) &&\\ &= 6x^3 + 3x^2 + 8x^2 + 4x && \text{FOIL}\\ &= 6x^3 + 11x^2 + 4x && \text{Combine like terms.}\end{aligned}$$

Then

$$\begin{aligned}(fg)(\mathbf{-1}) &= 6(\mathbf{-1})^3 + 11(\mathbf{-1})^2 + 4(\mathbf{-1}) && \text{Let } x = -1.\\ &= -6 + 11 - 4 &&\\ &= 1. &&\end{aligned}$$

(Another way to find $(fg)(-1)$ is to find $f(-1)$ and $g(-1)$ and then multiply the results. Verify this by showing that $f(-1) \cdot g(-1)$ equals 1. This follows from the definition.)

Work Problem 8 at the Side.

CAUTION

Write the product $f(x) \cdot g(x)$ as $(fg)(x)$, *not* $f(g(x))$, which has a different mathematical meaning as discussed in **Section 12.1.**

ANSWERS

8. $2x^3 + 7x^2 - 8x - 28$; 0

6.5 Dividing Polynomials

OBJECTIVE 1 Divide a polynomial by a monomial. We now discuss polynomial division, beginning with division by a monomial. (Recall that a monomial is a single term, such as $8x$, $-9m^4$, or $11y^2$.)

Dividing by a Monomial

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial, and then write each quotient in lowest terms.

EXAMPLE 1 Dividing a Polynomial by a Monomial

Divide.

(a) $$\frac{15x^2 - 12x + 6}{3} = \frac{15x^2}{3} - \frac{12x}{3} + \frac{6}{3} \quad \text{Divide each term by 3.}$$

$$= 5x^2 - 4x + 2 \quad \text{Write in lowest terms.}$$

Check this answer by multiplying it by the divisor, 3. You should get $15x^2 - 12x + 6$ as the result.

$$\underset{\text{Divisor}}{3}\underbrace{(5x^2 - 4x + 2)}_{\text{Quotient}} = \underbrace{15x^2 - 12x + 6}_{\text{Original polynomial}}$$

(b) $$\frac{5m^3 - 9m^2 + 10m}{5m^2} = \frac{5m^3}{5m^2} - \frac{9m^2}{5m^2} + \frac{10m}{5m^2} \quad \text{Divide each term by } 5m^2.$$

$$= m - \frac{9}{5} + \frac{2}{m} \quad \text{Write in lowest terms.}$$

This result is not a polynomial. (Why?) The quotient of two polynomials need not be a polynomial.

(c) $$\frac{8xy^2 - 9x^2y + 6x^2y^2}{x^2y^2} = \frac{8xy^2}{x^2y^2} - \frac{9x^2y}{x^2y^2} + \frac{6x^2y^2}{x^2y^2}$$

$$= \frac{8}{x} - \frac{9}{y} + 6$$

Work Problem 1 at the Side.

OBJECTIVE 2 Divide a polynomial by a polynomial of two or more terms. This process is similar to that for dividing whole numbers.

EXAMPLE 2 Dividing a Polynomial by a Polynomial

Divide $\dfrac{2m^2 + m - 10}{m - 2}$.

Write the problem, making sure that both polynomials are written in descending powers of the variables.

$$m - 2\overline{)2m^2 + m - 10}$$

Continued on Next Page

OBJECTIVES

1. Divide a polynomial by a monomial.
2. Divide a polynomial by a polynomial of two or more terms.
3. Divide polynomial functions.

1 Divide.

(a) $\dfrac{12p + 30}{6}$

(b) $\dfrac{9y^3 - 4y^2 + 8y}{2y^2}$

(c) $\dfrac{8a^2b^2 - 20ab^3}{4a^3b}$

ANSWERS

1. (a) $2p + 5$ (b) $\dfrac{9y}{2} - 2 + \dfrac{4}{y}$ (c) $\dfrac{2b}{a} - \dfrac{5b^2}{a^2}$

2 Divide.

(a) $\dfrac{2r^2 + r - 21}{r - 3}$

(b) $\dfrac{2k^2 + 17k + 30}{2k + 5}$

Divide the first term of $2m^2 + m - 10$ by the first term of $m - 2$. Since $\dfrac{2m^2}{m} = 2m$, place this result above the division line.

$$\begin{array}{r} \mathbf{2m} \\ m-2\overline{)2m^2+m-10} \end{array} \quad \longleftarrow \text{ Result of } \tfrac{2m^2}{m}$$

Multiply $m - 2$ and $2m$, and write the result below $2m^2 + m - 10$.

$$\begin{array}{rl} 2m & \\ m-2\overline{)2m^2+\ \ m-10} & \\ \underline{2m^2-4m} & \longleftarrow 2m(m-2) = 2m^2 - 4m \end{array}$$

Now subtract by mentally changing the signs on $2m^2 - 4m$ and *adding*.

$$\begin{array}{rl} 2m & \\ m-2\overline{)2m^2+\ \ m-10} & \\ \underline{2m^2-4m} & \\ 5m & \longleftarrow \text{Subtract. The difference is } 5m. \end{array}$$

Bring down -10 and continue by dividing $5m$ by m.

$$\begin{array}{rl} 2m+\mathbf{5} & \longleftarrow \tfrac{5m}{m} = 5 \\ m-2\overline{)2m^2+\ \ m-10} & \\ \underline{2m^2-4m} & \\ 5m-10 & \longleftarrow \text{Bring down } -10. \\ \underline{5m-10} & \longleftarrow 5(m-2) = 5m - 10 \\ 0 & \longleftarrow \text{Subtract. The difference is } 0. \end{array}$$

Finally, $(2m^2 + m - 10) \div (m - 2) = 2m + 5$. Check by multiplying $m - 2$ and $2m + 5$. The result should be $2m^2 + m - 10$.

Work Problem 2 at the Side.

EXAMPLE 3 Dividing a Polynomial with a Missing Term

Divide $3x^3 - 2x + 5$ by $x - 3$.

Make sure that $3x^3 - 2x + 5$ is in descending powers of the variable. Add a term with 0 coefficient as a placeholder for the missing x^2-term.

$$x-3\overline{)3x^3+\mathbf{0x^2}-2x+5} \quad \longleftarrow \text{Missing term}$$

Start with $\dfrac{3x^3}{x} = 3x^2$.

$$\begin{array}{rl} 3x^2 & \longleftarrow \tfrac{3x^3}{x} = 3x^2 \\ x-3\overline{)3x^3+0x^2-2x+5} & \\ \underline{3x^3-9x^2} & \longleftarrow 3x^2(x-3) \end{array}$$

Subtract by mentally changing the signs on $3x^3 - 9x^2$ and adding.

$$\begin{array}{rl} 3x^2 & \\ x-3\overline{)3x^3+0x^2-2x+5} & \\ \underline{3x^3-9x^2} & \\ 9x^2 & \longleftarrow \text{Subtract.} \end{array}$$

Bring down the next term.

$$\begin{array}{rl} 3x^2 & \\ x-3\overline{)3x^3+0x^2-2x+5} & \\ \underline{3x^3-9x^2} & \\ 9x^2-2x & \longleftarrow \text{Bring down } -2x. \end{array}$$

Continued on Next Page

ANSWERS

2. **(a)** $2r + 7$ **(b)** $k + 6$

In the next step, $\dfrac{9x^2}{x} = 9x$.

$$
\begin{array}{r@{}l}
 & 3x^2 + 9x \qquad \longleftarrow \tfrac{9x^2}{x} = 9x \\
x - 3 & \overline{)3x^3 + 0x^2 - 2x + 5} \\
 & \underline{3x^3 - 9x^2} \\
 & \phantom{)3x^3 -{}} 9x^2 - 2x \\
 & \phantom{)3x^3 -{}} \underline{9x^2 - 27x} \qquad \longleftarrow 9x(x-3) \\
 & \phantom{)3x^3 - 9x^2 -{}} 25x + 5 \qquad \longleftarrow \text{Subtract; bring down 5.}
\end{array}
$$

Finally, $\dfrac{25x}{x} = 25$.

$$
\begin{array}{r@{}l}
 & 3x^2 + 9x + 25 \qquad \longleftarrow \tfrac{25x}{x} = 25 \\
x - 3 & \overline{)3x^3 + 0x^2 - 2x + 5} \\
 & \underline{3x^3 - 9x^2} \\
 & \phantom{)3x^3 -{}} 9x^2 - 2x \\
 & \phantom{)3x^3 -{}} \underline{9x^2 - 27x} \\
 & \phantom{)3x^3 - 9x^2 -{}} 25x + 5 \\
 & \phantom{)3x^3 - 9x^2 -{}} \underline{25x - 75} \qquad \longleftarrow 25(x-3) \\
 & \phantom{)3x^3 - 9x^2 - 25x -{}} 80 \qquad \longleftarrow \text{Remainder}
\end{array}
$$

Write the remainder, 80, as the numerator of the fraction $\frac{80}{x-3}$. In summary,

$$\frac{3x^3 - 2x + 5}{x - 3} = 3x^2 + 9x + 25 + \frac{80}{x - 3}.$$

Check by multiplying $x - 3$ and $3x^2 + 9x + 25$ and adding 80. The result should be $3x^3 - 2x + 5$.

CAUTION

Remember to write $\dfrac{\text{remainder}}{\text{divisor}}$ as part of the quotient.

Work Problem 3 at the Side.

EXAMPLE 4 Performing a Division with a Fractional Coefficient in the Quotient

Divide $2p^3 + 5p^2 + p - 2$ by $2p + 2$.

$$
\begin{array}{r@{}l}
 & \phantom{)2p^3 + {}} p^2 + \frac{3}{2}p - 1 \qquad \longleftarrow \tfrac{3p^2}{2p} = \tfrac{3}{2}p \\
2p + 2 & \overline{)2p^3 + 5p^2 + p - 2} \\
 & \underline{2p^3 + 2p^2} \\
 & \phantom{)2p^3 + {}} 3p^2 + p \\
 & \phantom{)2p^3 + {}} \underline{3p^2 + 3p} \\
 & \phantom{)2p^3 + 5p^2 {}} -2p - 2 \\
 & \phantom{)2p^3 + 5p^2 {}} \underline{-2p - 2} \\
 & \phantom{)2p^3 + 5p^2 + p - {}} 0
\end{array}
$$

Since the remainder is 0, the quotient is $p^2 + \frac{3}{2}p - 1$.

Work Problem 4 at the Side.

3 Divide.

$$\frac{3k^3 + 9k - 14}{k - 2}$$

4 Divide $2p^3 + 7p^2 + 9p + 2$ by $2p + 2$.

ANSWERS

3. $3k^2 + 6k + 21 + \dfrac{28}{k - 2}$

4. $p^2 + \dfrac{5}{2}p + 2 + \dfrac{-2}{2p + 2}$

5 Divide.

(a) $\dfrac{3r^5 - 15r^4 - 2r^3 + 19r^2 - 7}{3r^2 - 2}$

(b) $\dfrac{4x^4 - 7x^2 + x + 5}{2x^2 - x}$

6 For

$$f(x) = 2x^2 + 17x + 30$$

and $g(x) = 2x + 5$,

find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(-1)$.

ANSWERS

5. (a) $r^3 - 5r^2 + 3 + \dfrac{-1}{3r^2 - 2}$

(b) $2x^2 + x - 3 + \dfrac{-2x + 5}{2x^2 - x}$

6. $x + 6,\quad x \neq -\frac{5}{2}$; 5

EXAMPLE 5 Dividing by a Polynomial with a Missing Term

You Try It

Divide $6r^4 + 9r^3 + 2r^2 - 8r + 7$ by $3r^2 - 2$.

Write $3r^2 - 2$ as $3r^2 + 0r - 2$ and divide as usual.

$$\begin{array}{r@{}l}
 & 2r^2 + 3r + 2 \\
3r^2 + 0r - 2 & \overline{)6r^4 + 9r^3 + 2r^2 - 8r + 7} \\
 & \underline{6r^4 + 0r^3 - 4r^2} \\
 & \phantom{)6r^4 + {}}9r^3 + 6r^2 - 8r \\
 & \phantom{)6r^4 + {}}\underline{9r^3 + 0r^2 - 6r} \\
 & \phantom{)6r^4 + 9r^3 + {}}6r^2 - 2r + 7 \\
 & \phantom{)6r^4 + 9r^3 + {}}\underline{6r^2 + 0r - 4} \\
 & \phantom{)6r^4 + 9r^3 + 6r^2 {}}-2r + 11
\end{array}$$

Missing term

Since the degree of the remainder, $-2r + 11$, is less than the degree of the divisor, $3r^2 - 2$, the process is now finished. The result is written

$$2r^2 + 3r + 2 + \frac{-2r + 11}{3r^2 - 2}.$$

Work Problem 5 at the Side.

CAUTION

When dividing a polynomial by a polynomial of two or more terms:

1. Be sure the terms in both polynomials are in descending powers.
2. Write any missing terms with 0 placeholders.

OBJECTIVE 3 Divide polynomial functions.

Audio

Dividing Functions

If $f(x)$ and $g(x)$ define functions, then

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}. \quad \text{Quotient function}$$

The domain of the quotient function is the intersection of the domains of $f(x)$ and $g(x)$, excluding any values of x for which $g(x) = 0$.

EXAMPLE 6 Dividing Polynomial Functions

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Video

For $f(x) = 2x^2 + x - 10$ and $g(x) = x - 2$, find $\left(\frac{f}{g}\right)(x)$ and $\left(\frac{f}{g}\right)(-3)$.

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2 + x - 10}{x - 2}$$

This quotient, found in Example 2, with x replacing m, is $2x + 5$, so

$$\left(\frac{f}{g}\right)(x) = 2x + 5, \quad x \neq 2.$$

Then $$\left(\frac{f}{g}\right)(-3) = 2(-3) + 5 = -1. \quad \text{Let } x = -3.$$

(Which is easier to find here: $\left(\frac{f}{g}\right)(-3)$ or $\frac{f(-3)}{g(-3)}$?)

Work Problem 6 at the Side.

Factoring

7

Factoring is used to solve quadratic equations, which have many useful applications. An important one is to express the distance a falling or propelled object travels in a specific time. Such equations are used in astronomy and the space program to describe the motion of objects in space.

In Section 7.4 we use the concepts of this chapter to explore how to find the heights of objects after they are propelled or dropped.

7.1 Greatest Common Factors; Factoring by Grouping

OBJECTIVES

1. **Factor out the greatest common factor.**
2. **Factor by grouping.**

Study Skills Workbook
Activity 8: Study Cards

Writing a polynomial as the product of two or more simpler polynomials is called **factoring** the polynomial. For example, the product of $3x$ and $5x - 2$ is $15x^2 - 6x$, and $15x^2 - 6x$ can be factored as the product $3x(5x - 2)$.

$$3x(5x - 2) = 15x^2 - 6x \qquad \text{Multiplying}$$
$$15x^2 - 6x = 3x(5x - 2) \qquad \text{Factoring}$$

Notice that both multiplying and factoring use the distributive property, but in opposite directions. Factoring "undoes" or reverses multiplying.

OBJECTIVE 1 **Factor out the greatest common factor.** The first step in factoring a polynomial is to find the *greatest common factor* for the terms of the polynomial. The **greatest common factor (GCF)** is the largest term that is a factor of all terms in the polynomial. For example, the greatest common factor for $8x + 12$ is 4, since 4 is the largest term that is a factor of (that is, divides into) both $8x$ and 12. Using the distributive property,

$$\begin{aligned} 8x + 12 &= 4(2x) + 4(3) \\ &= 4(2x + 3). \end{aligned}$$

As a check, multiply 4 and $2x + 3$. The result should be $8x + 12$. Using the distributive property this way is called *factoring out the greatest common factor*.

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $9z - 18$
Since 9 is the GCF, factor 9 from each term.

$$\begin{aligned} 9z - 18 &= 9 \cdot z - 9 \cdot 2 \\ &= 9(z - 2) \end{aligned}$$

Check: $9(z - 2) = 9z - 18$ Original polynomial

(b) $56m + 35p = 7(8m + 5p)$

(c) $2y + 5$ There is no common factor other than 1.

(d) $12 + 24z = 12 \cdot 1 + 12 \cdot 2z$
$= 12(1 + 2z)$ 12 is the GCF.

Check: $12(1 + 2z) = 12(1) + 12(2z)$ Distributive property
$= 12 + 24z$ Original polynomial

CAUTION
In Example 1(d), remember to write the factor 1. ***Always check answers by multiplying.***

Work Problem 1 at the Side.

1 Factor out the greatest common factor.

(a) $7k + 28$

(b) $32m + 24$

(c) $8a - 9$

(d) $5z + 5$

ANSWERS
1. **(a)** $7(k + 4)$ **(b)** $8(4m + 3)$
(c) cannot be factored **(d)** $5(z + 1)$

You Try It

Video

EXAMPLE 2 Factoring Out the Greatest Common Factor

Factor out the greatest common factor.

(a) $9x^2 + 12x^3$

The numerical part of the GCF is 3, the largest number that divides into both 9 and 12. For the variable parts, x^2 and x^3, use the least exponent that appears on x; here the least exponent is 2. The GCF is $3x^2$.

$$9x^2 + 12x^3 = \mathbf{3x^2}(3) + \mathbf{3x^2}(4x)$$
$$= \mathbf{3x^2}(3 + 4x)$$

(b) $32p^4 - 24p^3 + 40p^5 = \mathbf{8p^3}(4p) + \mathbf{8p^3}(-3) + \mathbf{8p^3}(5p^2)$ GCF $= 8p^3$

$$= \mathbf{8p^3}(4p - 3 + 5p^2)$$

(c) $3k^4 - 15k^7 + 24k^9 = 3k^4(\mathbf{1} - 5k^3 + 8k^5)$

Check by multiplying:

$$3k^4(1 - 5k^3 + 8k^5) = 3k^4(1) + 3k^4(-5k^3) + 3k^4(8k^5)$$
$$= 3k^4 - 15k^7 + 24k^9 \quad \text{Original polynomial}$$

(d) $24m^3n^2 - 18m^2n + 6m^4n^3$

The numerical part of the GCF is 6. Here 2 is the least exponent that appears on m, while 1 is the least exponent on n. The GCF is $6m^2n$.

$$24m^3n^2 - 18m^2n + 6m^4n^3 = \mathbf{6m^2n}(4mn) + \mathbf{6m^2n}(-3) + \mathbf{6m^2n}(m^2n^2)$$
$$= \mathbf{6m^2n}(4mn - 3 + m^2n^2)$$

(e) $25x^2y^3 + 30y^5 - 15x^4y^7 = 5y^3(5x^2 + 6y^2 - 3x^4y^4)$

Work Problem 2 at the Side.

A greatest common factor need not be a monomial. The next example shows a binomial greatest common factor.

You Try It

Video

EXAMPLE 3 Factoring Out a Binomial Factor

Factor out the greatest common factor.

(a) $(x - 5)(x + 6) + (x - 5)(2x + 5)$

The greatest common factor here is $x - 5$.

$$\mathbf{(x - 5)}(x + 6) + \mathbf{(x - 5)}(2x + 5) = \mathbf{(x - 5)}[(x + 6) + (2x + 5)]$$
$$= (x - 5)(x + 6 + 2x + 5)$$
$$= (x - 5)(3x + 11)$$

(b) $z^2\mathbf{(m + n)} + x^2\mathbf{(m + n)} = \mathbf{(m + n)}(z^2 + x^2)$

(c) $p(r + 2s) - q^2(r + 2s) = (r + 2s)(p - q^2)$

(d) $\mathbf{(p - 5)}(p + 2) - \mathbf{(p - 5)}(3p + 4)$

$$= \mathbf{(p - 5)}[(p + 2) - (3p + 4)] \quad \text{Factor out } p - 5.$$
$$= (p - 5)[p + 2 - 3p - 4] \quad \text{Be careful with signs.}$$
$$= (p - 5)[-2p - 2] \quad \text{Combine terms.}$$
$$= (p - 5)[\mathbf{-2}(p + 1)] \quad \text{Look for a common factor.}$$
$$= \mathbf{-2}(p - 5)(p + 1)$$

Work Problem 3 at the Side.

2 Factor out the greatest common factor.

(a) $16y^4 + 8y^3$

(b) $14p^2 - 9p^3 + 6p^4$

(c) $15z^2 + 45z^5 - 60z^6$

(d) $4x^2z - 2xz + 8z^2$

(e) $12y^5x^2 + 8y^3x^3$

(f) $5m^4x^3 + 15m^5x^6 - 20m^4x^6$

3 Factor out the greatest common factor.

(a) $(a + 2)(a - 3)$
$+ (a + 2)(a + 6)$

(b) $(y - 1)(y + 3)$
$- (y - 1)(y + 4)$

(c) $k^2(a + 5b) + m^2(a + 5b)$

(d) $r^2(y + 6) + r^2(y + 3)$

ANSWERS

2. **(a)** $8y^3(2y + 1)$ **(b)** $p^2(14 - 9p + 6p^2)$
(c) $15z^2(1 + 3z^3 - 4z^4)$
(d) $2z(2x^2 - x + 4z)$
(e) $4y^3x^2(3y^2 + 2x)$
(f) $5m^4x^3(1 + 3mx^3 - 4x^3)$

3. **(a)** $(a + 2)(2a + 3)$
(b) $(y - 1)(-1)$, or $-y + 1$
(c) $(a + 5b)(k^2 + m^2)$
(d) $r^2(2y + 9)$

4 Factor each polynomial in two ways.

(a) $-k^2 + 3k$

You Try It

(b) $-6r^3 - 5r^2 + 14r$

5 Factor $6p - 6q + rp - rq$.

Answers

4. **(a)** $k(-k + 3)$ or $-k(k - 3)$
(b) $r(-6r^2 - 5r + 14)$ or $-r(6r^2 + 5r - 14)$
5. $(p - q)(6 + r)$

When the coefficient of the term of greatest degree is negative, it is sometimes preferable to factor out the -1 that is understood along with the GCF.

EXAMPLE 4 Factoring Out a Negative Common Factor

Factor $-a^3 + 3a^2 - 5a$ in two ways.

First, a could be used as the common factor, giving

$$\begin{aligned} -a^3 + 3a^2 - 5a &= \mathbf{a}(-a^2) + \mathbf{a}(3a) + \mathbf{a}(-5) && \text{Factor out } a. \\ &= \mathbf{a}(-a^2 + 3a - 5). \end{aligned}$$

Because of the leading negative sign, $-a$ could be used as the common factor.

$$\begin{aligned} -a^3 + 3a^2 - 5a &= -\mathbf{a}(a^2) + (-\mathbf{a})(-3a) + (-\mathbf{a})(5) && \text{Factor out } -a. \\ &= -\mathbf{a}(a^2 - 3a + 5) \end{aligned}$$

Either answer is correct.

NOTE

Example 4 showed two ways of factoring a polynomial. Sometimes there may be a reason to prefer one of these forms over the other, but both are correct. The answer section in this book will usually give the form where the common factor has a positive coefficient.

Work Problem 4 at the Side.

OBJECTIVE 2 Factor by grouping. Sometimes the terms of a polynomial have a greatest common factor of 1, but it still may be possible to factor the polynomial by using a process called *factoring by grouping*. We usually factor by grouping when a polynomial has more than three terms. For example, to factor the polynomial

$$ax - ay + bx - by,$$

group the terms as follows.

Terms with common factors

$$(\mathbf{a}x - \mathbf{a}y) + (\mathbf{b}x - \mathbf{b}y)$$

Then factor $ax - ay$ as $a(x - y)$ and factor $bx - by$ as $b(x - y)$.

$$\begin{aligned} ax - ay + bx - by &= (ax - ay) + (bx - by) \\ &= a(\mathbf{x - y}) + b(\mathbf{x - y}) \end{aligned}$$

The common factor is $x - y$. The final factored form is

$$ax - ay + bx - by = (\mathbf{x - y})(a + b).$$

Work Problem 5 at the Side.

EXAMPLE 5 Factoring by Grouping

Factor $3x - 3y - ax + ay$.

Grouping terms gives

$$(3x - 3y) + (-ax + ay) = 3(x - y) + a(-x + y).$$

There is no simple common factor here. However, if we factor out $-a$ instead of a in the second group of terms, we get

$$\begin{aligned}(3x - 3y) + (-ax + ay) &= 3(x - y) - a(x - y),\\ &= (x - y)(3 - a).\end{aligned}$$

Check by multiplying.

$$\begin{aligned}(x - y)(3 - a) &= 3x - ax - 3y + ay && \text{FOIL}\\ &= 3x - 3y - ax + ay && \text{Rearrange terms.}\end{aligned}$$

This final product is the original polynomial.

Work Problem 6 at the Side.

NOTE

In Example 5, different grouping would lead to the factored form

$$(a - 3)(y - x).$$

Verify by multiplying that this form is also correct.

The steps used in factoring by grouping are listed here.

Factoring by Grouping

Step 1 **Group terms.** Collect the terms into groups so that each group has a common factor.

Step 2 **Factor within the groups.** Factor out the common factor in each group.

Step 3 **Factor the entire polynomial.** If each group now has a common factor, factor it out. If not, try a different grouping.

Always check the factored form by multiplying.

EXAMPLE 6 Factoring by Grouping

Factor $6ax + 12bx + a + 2b$ by grouping.

$$6ax + 12bx + a + 2b = (6ax + 12bx) + (a + 2b) \qquad \text{Group terms.}$$

Now factor $6x$ from the first group, and use the identity property of multiplication to introduce the factor 1 in the second group.

$$\begin{aligned}&= 6x(a + 2b) + 1(a + 2b)\\ &= (a + 2b)(6x + 1) && \text{Factor out } a + 2b.\end{aligned}$$

Again, as in Example 1(d), remember to write the 1. *Check* by multiplying.

Work Problem 7 at the Side.

6 Factor $xy - 2y - 4x + 8$.

7 Factor $2xy + 3y + 2x + 3$.

ANSWERS

6. $(x - 2)(y - 4)$

7. $(2x + 3)(y + 1)$

8 Factor.

(a) $mn + 6 + 2n + 3m$

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(b) $4y - zx + yx - 4z$

ANSWERS

8. (a) $(m + 2)(n + 3)$ **(b)** $(y - z)(x + 4)$

EXAMPLE 7 Rearranging Terms before Factoring by Grouping

Factor $p^2q^2 - 10 - 2q^2 + 5p^2$.

Neither the first two terms nor the last two terms have a common factor except 1. Rearrange and group the terms as follows.

$$(p^2q^2 - 2q^2) + (5p^2 - 10) \quad \text{Rearrange and group the terms.}$$
$$= q^2(p^2 - 2) + 5(p^2 - 2) \quad \text{Factor out the common factors.}$$
$$= (p^2 - 2)(q^2 + 5) \quad \text{Factor out } p^2 - 2.$$

Check: $(p^2 - 2)(q^2 + 5) = p^2q^2 + 5p^2 - 2q^2 - 10$ FOIL

$= p^2q^2 - 10 - 2q^2 + 5p^2$ Original polynomial

CAUTION

In Example 7, do not stop at the step

$$q^2(p^2 - 2) + 5(p^2 - 2).$$

This expression is *not in factored form* because it is a *sum* of two terms, $q^2(p^2 - 2)$ and $5(p^2 - 2)$, not a product.

Work Problem 8 at the Side.

7.2 Factoring Trinomials

OBJECTIVES

1. Factor trinomials when the coefficient of the squared term is 1.
2. Factor trinomials when the coefficient of the squared term is not 1.
3. Use an alternative method for factoring trinomials.
4. Factor by substitution.

OBJECTIVE 1 Factor trinomials when the coefficient of the squared term is 1. We begin by finding the product of $x + 3$ and $x - 5$.

$$(x+3)(x-5) = x^2 - 5x + 3x - 15$$
$$= x^2 - 2x - 15$$

By this result, the factored form of $x^2 - 2x - 15$ is $(x + 3)(x - 5)$.

Multiplying

$$\text{Factored form} \longrightarrow (x+3)(x-5) = x^2 - 2x - 15 \longleftarrow \text{Product}$$

Factoring

Since multiplying and factoring are operations that "undo" each other, factoring trinomials involves using FOIL backwards. As shown here, the x^2-term comes from multiplying x and x, and -15 comes from multiplying 3 and -5.

Product of x and x is x^2.

$$(x+3)(x-5) = x^2 - 2x - 15$$

Product of 3 and -5 is -15.

We find the $-2x$ in $x^2 - 2x - 15$ by multiplying the outer terms, multiplying the inner terms, and adding.

Outer terms: $x(-5) = -5x$

$$(x+3)(x-5)$$

Inner terms: $3 \cdot x = 3x$

Add to get $-2x$.

Based on this example, follow these steps to factor a trinomial $x^2 + bx + c$, where 1 is the coefficient of the squared term.

Factoring $x^2 + bx + c$

Step 1 **Find pairs whose product is *c*.** Find all pairs of integers whose product is the third term of the trinomial, c.

Step 2 **Find pairs whose sum is *b*.** Choose the pair whose sum is the coefficient of the middle term, b.

If there are no such integers, the polynomial cannot be factored. A polynomial that cannot be factored with integer coefficients is a **prime polynomial.**

Some examples of prime polynomials are

$$x^2 + x + 2, \quad x^2 - x - 1, \quad \text{and} \quad 2x^2 + x + 7. \qquad \text{Prime polynomials}$$

EXAMPLE 1 Factoring Trinomials in $x^2 + bx + c$ Form

Factor each polynomial.

(a) $y^2 + 2y - 35$

Step 1 Find pairs of numbers whose product is -35.	*Step 2* Write sums of those numbers.
$-35(1)$	$-35 + 1 = -34$
$35(-1)$	$35 + (-1) = 34$
$7(-5)$	$7 + (-5) = 2$ ← Coefficient of the middle term
$5(-7)$	$5 + (-7) = -2$

The required numbers are 7 and -5, so

$$y^2 + 2y - 35 = (y + 7)(y - 5).$$

Check by finding the product of $y + 7$ and $y - 5$.

(b) $r^2 + 8r + 12$

Look for two numbers with a product of 12 and a sum of 8. Of all pairs of numbers having a product of 12, only the pair 6 and 2 has a sum of 8. Therefore,

$$r^2 + 8r + 12 = (r + 6)(r + 2).$$

Because of the commutative property, it would be equally correct to write $(r + 2)(r + 6)$. *Check* by multiplying.

EXAMPLE 2 Recognizing a Prime Polynomial

Factor $m^2 + 6m + 7$.

Look for two numbers whose product is 7 and whose sum is 6. Only two pairs of integers, 7 and 1 and -7 and -1, give a product of 7. Neither of these pairs has a sum of 6, so $m^2 + 6m + 7$ cannot be factored with integer coefficients and is prime.

Work Problem 1 at the Side.

Factoring a trinomial that has more than one variable uses a similar process.

EXAMPLE 3 Factoring a Trinomial in Two Variables

Factor $p^2 + 6ap - 16a^2$.

Look for two expressions whose product is $-16a^2$ and whose sum is $6a$. The quantities $8a$ and $-2a$ have the necessary product and sum, so

$$p^2 + 6ap - 16a^2 = (p + 8a)(p - 2a).$$

Check: $(p + 8a)(p - 2a) = p^2 - 2ap + 8ap - 16a^2$ FOIL

$= p^2 + 6ap - 16a^2$ Original polynomial

Work Problem 2 at the Side.

Sometimes a trinomial will have a common factor that should be factored out first.

1 Factor each polynomial.

(a) $p^2 + 6p + 5$

(b) $a^2 + 9a + 20$

(c) $k^2 - k - 6$

(d) $b^2 - 7b + 10$

(e) $y^2 - 8y + 6$

2 Factor each polynomial.

(a) $m^2 + 2mn - 8n^2$

You Try It

(b) $z^2 - 7zx + 9x^2$

Video

ANSWERS

1. **(a)** $(p + 1)(p + 5)$ **(b)** $(a + 5)(a + 4)$ **(c)** $(k - 3)(k + 2)$ **(d)** $(b - 5)(b - 2)$ **(e)** prime
2. **(a)** $(m - 2n)(m + 4n)$ **(b)** prime

Animation

EXAMPLE 4 Factoring a Trinomial with a Common Factor

Factor $16y^3 - 32y^2 - 48y$.

Start by factoring out the greatest common factor, $16y$.

$$16y^3 - 32y^2 - 48y = 16y(y^2 - 2y - 3)$$

To factor $y^2 - 2y - 3$, look for two integers whose product is -3 and whose sum is -2. The necessary integers are -3 and 1, so

$$16y^3 - 32y^2 - 48y = 16y(y - 3)(y + 1).$$

CAUTION

When factoring, always look for a common factor first. Remember to write the common factor as part of the answer.

Work Problem 3 at the Side.

OBJECTIVE 2 Factor trinomials when the coefficient of the squared term is not 1. We can use a generalization of the method shown in Objective 1 to factor a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$. To factor $3x^2 + 7x + 2$, for example, we first identify the values of a, b, and c.

$$\begin{array}{ccccc} ax^2 & + & bx & + & c \\ \downarrow & & \downarrow & & \downarrow \\ 3x^2 & + & 7x & + & 2 \end{array}$$

$$a = 3, \quad b = 7, \quad c = 2$$

The product ac is $3 \cdot 2 = 6$, so we must find integers having a product of 6 and a sum of 7 (since the middle term has coefficient $b = 7$). The necessary integers are 1 and 6, so we write $7x$ as $1x + 6x$, or $x + 6x$. Thus,

$$3x^2 + 7x + 2 = 3x^2 + \underbrace{x + 6x}_{x + 6x = 7x} + 2$$

$$= (3x^2 + x) + (6x + 2) \quad \text{Factor by grouping.}$$

$$= x(3x + 1) + 2(3x + 1)$$

$$= (3x + 1)(x + 2).$$

Video

EXAMPLE 5 Factoring a Trinomial in $ax^2 + bx + c$ Form

Factor $12r^2 - 5r - 2$.

Since $a = 12$, $b = -5$, and $c = -2$, the product ac is -24. The two integers whose product is -24 and whose sum is b, -5, are 3 and -8.

$$12r^2 - 5r - 2 = 12r^2 + 3r - 8r - 2 \quad \text{Write } -5r \text{ as } 3r - 8r.$$

$$= 3r(4r + 1) - 2(4r + 1) \quad \text{Factor by grouping.}$$

$$= (4r + 1)(3r - 2) \quad \text{Factor out the common factor.}$$

Work Problem 4 at the Side.

OBJECTIVE 3 Use an alternative method for factoring trinomials. An alternative approach, the method of trying repeated combinations and using FOIL, is especially helpful when the product ac is large.

3 Factor $5m^4 - 5m^3 - 100m^2$.

4 Factor each trinomial.

(a) $3y^2 - 11y - 4$

(b) $6k^2 - 19k + 10$

ANSWERS

3. $5m^2(m - 5)(m + 4)$

4. (a) $(y - 4)(3y + 1)$ **(b)** $(2k - 5)(3k - 2)$

5 Use the method of Example 6 to factor each trinomial.

(a) $10x^2 + 17x + 3$

(b) $16y^2 - 34y - 15$

(c) $8t^2 - 13t + 5$

EXAMPLE 6 Factoring Trinomials in $ax^2 + bx + c$ Form

Factor each polynomial.

(a) $3x^2 + 7x + 2$

To factor this trinomial we use an alternative method. The goal is to find the correct numbers to fill in the blanks.

$$3x^2 + 7x + 2 = (\_\_\_\_x + \_\_\_\_)(\_\_\_\_x + \_\_\_\_)$$

Addition signs are used since all the signs in the trinomial indicate addition. The first two expressions have a product of $3x^2$, so they must be $3x$ and x.

$$3x^2 + 7x + 2 = (3x + \_\_\_\_)(x + \_\_\_\_)$$

The product of the two last terms must be 2, so the numbers must be 2 and 1. There is a choice. The 2 could be used with the $3x$ or with the x. Only one of these choices can give the correct middle term, $\mathbf{7x}$. Use the FOIL method to check each one.

$3x$	$6x$
$(3x + 2)(x + 1)$	$(3x + 1)(x + 2)$
$2x$	x
$3x + 2x = 5x$	$6x + x = \mathbf{7x}$
Wrong middle term	Correct middle term

Therefore, $3x^2 + 7x + 2 = (3x + 1)(x + 2)$. (Compare to the method on the preceding page.)

(b) $12r^2 - 5r - 2$

To reduce the number of trials, we note that the trinomial has no common factor (except 1). This means that neither of its factors can have a common factor. We should keep this in mind as we choose factors. We try 4 and 3 for the two first terms.

$$12r^2 - 5r - 2 = (4r\_\_\_\_)(3r\_\_\_\_)$$

The factors of -2 are -2 and 1 or 2 and -1. Try both possibilities.

	$8r$
$(4r - 2)(3r + 1)$	$(4r - 1)(3r + 2)$
Wrong: $4r - 2$ has a common factor of 2.	$-3r$
This cannot be correct, since 2 is not a factor of $12r^2 - 5r - 2$.	$8r - 3r = 5r$
	Wrong middle term

The middle term on the right is $5r$, instead of the $-5r$ that is needed. We get $-5r$ by interchanging the signs in the factors.

$-8r$

$(4r + 1)(3r - 2)$

$3r$

$-8r + 3r = -5r$

Correct middle term

Thus, $12r^2 - 5r - 2 = (4r + 1)(3r - 2)$. (Compare to Example 5.)

Work Problem 5 at the Side.

ANSWERS

5. **(a)** $(5x + 1)(2x + 3)$
(b) $(8y + 3)(2y - 5)$
(c) $(8t - 5)(t - 1)$

This alternative method of factoring a trinomial $ax^2 + bx + c$, $a \neq 1$, is summarized here.

Factoring $ax^2 + bx + c$

Step 1 **Find pairs whose product is *a*.** Write all pairs of integer factors of the coefficient of the squared term, a.

Step 2 **Find pairs whose product is *c*.** Write all pairs of integer factors of the last term, c.

Step 3 **Choose inner and outer terms.** Use FOIL and various combinations of the factors from Steps 1 and 2 until the necessary middle term is found.

If no such combinations exist, the trinomial is prime.

EXAMPLE 7 Factoring a Trinomial in Two Variables

Factor $18m^2 - 19mx - 12x^2$.

There is no common factor (except 1). Follow the steps to factor the trinomial. There are many possible factors of both 18 and -12. Try 6 and 3 for 18 and -3 and 4 for -12.

$(6m - 3x)(3m + 4x)$	$(6m + 4x)(3m - 3x)$
Wrong: common factor	Wrong: common factors

Since 6 and 3 do not work in this situation, try 9 and 2 instead, with -4 and 3 as factors of -12.

$(9m + 3x)(2m - 4x)$	$(9m - 4x)(2m + 3x)$ — outer: $27mx$; inner: $-8mx$
Wrong: common factors	$27mx + (-8mx) = 19mx$

The result on the right differs from the correct middle term only in sign, so interchange the signs in the factors. *Check* by multiplying.

$$18m^2 - 19mx - 12x^2 = (9m + 4x)(2m - 3x)$$

Work Problem 6 at the Side.

EXAMPLE 8 Factoring $ax^2 + bx + c$, with $a < 0$

Factor $-3x^2 + 16x + 12$.

While it is possible to factor this trinomial directly, it is helpful to first factor out -1. Then proceed as in the earlier examples.

$$\begin{aligned} -3x^2 + 16x + 12 &= -1(3x^2 - 16x - 12) \\ &= -1(3x + 2)(x - 6) \\ &= -(3x + 2)(x - 6) \end{aligned}$$

This factored form can be written in other ways. Two of them are

$$(-3x - 2)(x - 6) \quad \text{and} \quad (3x + 2)(-x + 6).$$

Verify that these both give the original trinomial when multiplied.

Work Problem 7 at the Side.

6 Factor each trinomial.

(a) $7p^2 + 15pq + 2q^2$

(b) $6m^2 + 7mn - 5n^2$

(c) $12z^2 - 5zy - 2y^2$

(d) $8m^2 + 18mx - 5x^2$

7 Factor each trinomial.

(a) $-6r^2 + 13r + 5$

(b) $-8x^2 + 10x - 3$

Answers

6. **(a)** $(7p + q)(p + 2q)$
   (b) $(3m + 5n)(2m - n)$
   (c) $(3z - 2y)(4z + y)$
   (d) $(4m - x)(2m + 5x)$
7. **(a)** $-(2r - 5)(3r + 1)$
   (b) $-(4x - 3)(2x - 1)$

8 Factor each trinomial.

(a) $2m^3 - 4m^2 - 6m$

(b) $12r^4 + 6r^3 - 90r^2$

(c) $30y^5 - 55y^4 - 50y^3$

9 Factor each polynomial.

(a) $6(a - 1)^2 + (a - 1) - 2$

(b) $8(z + 5)^2 - 2(z + 5) - 3$

(c) $15(m - 4)^2 - 11(m - 4) + 2$

10 Factor each trinomial.

(a) $y^4 + y^2 - 6$

(b) $2p^4 + 7p^2 - 15$

(c) $6r^4 - 13r^2 + 5$

ANSWERS

8. (a) $2m(m + 1)(m - 3)$
(b) $6r^2(r + 3)(2r - 5)$
(c) $5y^3(2y - 5)(3y + 2)$

9. (a) $(2a - 3)(3a - 1)$
(b) $(4z + 17)(2z + 11)$
(c) $(3m - 13)(5m - 22)$

10. (a) $(y^2 - 2)(y^2 + 3)$
(b) $(2p^2 - 3)(p^2 + 5)$
(c) $(3r^2 - 5)(2r^2 - 1)$

EXAMPLE 9 Factoring a Trinomial with a Common Factor

Factor $16y^3 + 24y^2 - 16y$.

$$16y^3 + 24y^2 - 16y = 8y(2y^2 + 3y - 2) \qquad \text{GCF} = 8y$$
$$= 8y(2y - 1)(y + 2) \qquad \text{Remember the common factor.}$$

Work Problem 8 at the Side.

OBJECTIVE 4 Factor by substitution. Sometimes we can factor a more complicated polynomial by substituting a variable for an expression.

EXAMPLE 10 Factoring a Polynomial Using Substitution

Factor $2(x + 3)^2 + 5(x + 3) - 12$.

Since the binomial $x + 3$ appears to powers 2 and 1, we let the substitution variable represent $x + 3$. We may choose any letter we wish except x. We choose y to equal $x + 3$.

$$2(x + 3)^2 + 5(x + 3) - 12 = 2y^2 + 5y - 12 \qquad \text{Let } y = x + 3.$$
$$= (2y - 3)(y + 4) \qquad \text{Factor.}$$

Now we replace y with $x + 3$ to get

$$2(x + 3)^2 + 5(x + 3) - 12 = [2(x + 3) - 3][(x + 3) + 4]$$
$$= (2x + 6 - 3)(x + 7)$$
$$= (2x + 3)(x + 7).$$

CAUTION

Remember to make the final substitution of $x + 3$ for y in Example 10.

Work Problem 9 at the Side.

EXAMPLE 11 Factoring a Trinomial in $ax^4 + bx^2 + c$ Form

Factor $6y^4 + 7y^2 - 20$.

The variable y appears to powers in which the larger exponent is twice the smaller exponent. We can let a substitution variable equal the smaller power. Here, we let $m = y^2$.

$$6y^4 + 7y^2 - 20 = 6(y^2)^2 + 7y^2 - 20$$
$$= 6m^2 + 7m - 20 \qquad \text{Subtitute.}$$
$$= (3m - 4)(2m + 5) \qquad \text{Factor.}$$
$$= (3y^2 - 4)(2y^2 + 5) \qquad m = y^2$$

NOTE

Some students feel comfortable factoring polynomials like the one in Example 11 directly, without using the substitution method.

Work Problem 10 at the Side.

7.3 Special Factoring

OBJECTIVES

1. **Factor a difference of squares.**
2. **Factor a perfect square trinomial.**
3. **Factor a difference of cubes.**
4. **Factor a sum of cubes.**

OBJECTIVE 1 Factor a difference of squares. The special products introduced in **Section 6.4** are used in reverse when factoring. Recall that the product of the sum and difference of two terms leads to a **difference of squares,** a pattern that occurs often when factoring.

Difference of Squares

$$x^2 - y^2 = (x + y)(x - y)$$

EXAMPLE 1 Factoring Differences of Squares

Factor each polynomial.

(a) $4a^2 - 64$

There is a common factor of 4.

$$4a^2 - 64 = 4(a^2 - 16) \qquad \text{Factor out the common factor.}$$
$$= 4(a + 4)(a - 4) \qquad \text{Factor the difference of squares.}$$

(b)
$$\begin{array}{ccccccc} & A^2 & - & B^2 & = & (A + B) & (A - B) \\ 16m^2 - 49p^2 = & (4m)^2 & - & (7p)^2 & = & (4m + 7p) & (4m - 7p) \end{array}$$

(c)
$$\begin{array}{ccccccc} & A^2 & - & B^2 & = & (A + B) & (A - B) \\ 81k^2 - (a + 2)^2 = & (9k)^2 & - & (a + 2)^2 & = & (9k + \overbrace{a + 2}) & (9k - [\overbrace{a + 2}]) \\ & & & & = & (9k + a + 2) & (9k - a - 2) \end{array}$$

We could have used the method of substitution here.

(d)
$$x^4 - 81 = (x^2 + 9)(x^2 - 9) \qquad \text{Factor the difference of squares.}$$
$$= (x^2 + 9)(x + 3)(x - 3) \qquad \text{Factor } x^2 - 9.$$

Work Problem 1 at the Side.

1 Factor each polynomial.

(a) $2x^2 - 18$

(b) $9a^2 - 16b^2$

(c) $(m + 3)^2 - 49z^2$

(d) $y^4 - 16$

CAUTION

Assuming no greatest common factor except 1, it is not possible to factor (with real numbers) a sum of squares, such as $x^2 + 9$ in Example 1(d). In particular, $x^2 + y^2 \neq (x + y)^2$, as shown next.

OBJECTIVE 2 Factor a perfect square trinomial. Two other special products from **Section 6.4** lead to the following rules for factoring.

Perfect Square Trinomial

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

ANSWERS

1. **(a)** $2(x + 3)(x - 3)$
(b) $(3a - 4b)(3a + 4b)$
(c) $(m + 3 + 7z)(m + 3 - 7z)$
(d) $(y^2 + 4)(y + 2)(y - 2)$

2 Identify any perfect square trinomials.

(a) $z^2 + 12z + 36$

(b) $2x^2 - 4x + 4$

(c) $9a^2 + 12ab + 16b^2$

3 Factor each polynomial.

(a) $49z^2 - 14zk + k^2$

(b) $9a^2 + 48ab + 64b^2$

(c) $(k + m)^2 - 12(k + m) + 36$

(d) $x^2 - 2x + 1 - y^2$

Because the trinomial $x^2 + 2xy + y^2$ is the square of $x + y$, it is called a **perfect square trinomial.** In this pattern, both the first and the last terms of the trinomial must be perfect squares. In the factored form $(x + y)^2$, twice the product of the first and the last terms must give the middle term of the trinomial. It is important to understand these patterns in terms of words, since they occur with many different symbols (other than x and y).

$4m^2 + 20m + 25$	$p^2 - 8p + 64$
Perfect square trinomial since $20m = 2(2m)(5)$	Not a perfect square trinomial; middle term should be $16p$ or $-16p$.

Work Problem 2 at the Side.

EXAMPLE 2 Factoring Perfect Square Trinomials

Factor each polynomial.

(a) $144p^2 - 120p + 25$

Here $144p^2 = (12p)^2$ and $25 = 5^2$. The sign on the middle term is $-$, so if $144p^2 - 120p + 25$ is a perfect square trinomial, the factored form will have to be

$$(12p - 5)^2.$$

Take twice the product of the two terms to see if this is correct.

$$2(12p)(-5) = -120p$$

This is the middle term of the given trinomial, so

$$144p^2 - 120p + 25 = (12p - 5)^2.$$

(b) $4m^2 + 20mn + 49n^2$

If this is a perfect square trinomial, it will equal $(2m + 7n)^2$. By the pattern described earlier, if multiplied out, this squared binomial has a middle term of $2(2m)(7n) = 28mn$, which *does not equal* $20mn$. Verify that this trinomial cannot be factored by the methods of the previous section either. It is prime.

(c) $(r + 5)^2 + 6(r + 5) + 9 = [(r + 5) + 3]^2$

$$= (r + 8)^2,$$

since $2(r + 5)(3) = 6(r + 5)$, the middle term.

(d) $m^2 - 8m + 16 - p^2$

Since there are four terms, we will use factoring by grouping. The first three terms here form a perfect square trinomial. Group them together, and factor as follows.

$$(m^2 - 8m + 16) - p^2 = (m - 4)^2 - p^2$$

The result is the difference of squares. Factor again to get

$$= (m - 4 + p)(m - 4 - p).$$

Work Problem 3 at the Side.

Perfect square trinomials, of course, can be factored using the general methods shown earlier for other trinomials. The patterns given here provide "shortcuts."

ANSWERS

2. **(a)** perfect square trinomial
(b) not a perfect square trinomial
(c) not a perfect square trinomial
3. **(a)** $(7z - k)^2$ **(b)** $(3a + 8b)^2$
(c) $[(k + m) - 6]^2$ or $(k + m - 6)^2$
(d) $(x - 1 + y)(x - 1 - y)$

OBJECTIVE 3 **Factor a difference of cubes.** A **difference of cubes,** such as $x^3 - y^3$, can be factored as follows.

Audio

Difference of Cubes

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

We could check this pattern by finding the product of $x - y$ and $x^2 + xy + y^2$.

You Try It

Video

EXAMPLE 3 **Factoring Differences of Cubes**

Factor each polynomial.

(a) $m^3 - 8 = m^3 - 2^3$

$$= (m - 2)(m^2 + 2m + 2^2)$$

$$= (m - 2)(m^2 + 2m + 4)$$

Check:

m^3 $\quad -8$

$$(m - 2)(m^2 + 2m + 4)$$

$-2m$

Opposite of the product of the cube roots gives the middle term.

(b) $27x^3 - 8y^3 = (3x)^3 - (2y)^3$

$$= (3x - 2y)[(3x)^2 + (3x)(2y) + (2y)^2]$$

$$= (3x - 2y)(9x^2 + 6xy + 4y^2)$$

(c) $1000k^3 - 27n^3 = (10k)^3 - (3n)^3$

$$= (10k - 3n)[(10k)^2 + (10k)(3n) + (3n)^2]$$

$$= (10k - 3n)(100k^2 + 30kn + 9n^2)$$

Work Problem 4 at the Side.

OBJECTIVE 4 **Factor a sum of cubes.** While the binomial $x^2 + y^2$ (a sum of *squares*) cannot be factored with real numbers, a **sum of cubes,** such as $x^3 + y^3$, is factored as follows.

Audio

Sum of Cubes

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

To verify this result, find the product of $x + y$ and $x^2 - xy + y^2$. Compare this pattern with the pattern for a difference of cubes.

NOTE

The sign of the second term in the binomial factor of a sum or difference of cubes is *always the same* as the sign in the original polynomial. In the trinomial factor, the first and last terms are *always positive;* the sign of the middle term is *the opposite of* the sign of the second term in the binomial factor.

4 Factor each polynomial.

(a) $x^3 - 1000$

(b) $8k^3 - y^3$

(c) $27m^3 - 64$

ANSWERS

4. **(a)** $(x - 10)(x^2 + 10x + 100)$
(b) $(2k - y)(4k^2 + 2ky + y^2)$
(c) $(3m - 4)(9m^2 + 12m + 16)$

5 Factor each polynomial.

(a) $8p^3 + 125$

You Try It

Video

(b) $27m^3 + 125n^3$

(c) $2x^3 + 2000$

Audio

ANSWERS

5. (a) $(2p + 5)(4p^2 - 10p + 25)$
(b) $(3m + 5n)(9m^2 - 15mn + 25n^2)$
(c) $2(x + 10)(x^2 - 10x + 100)$

EXAMPLE 4 Factoring Sums of Cubes

Factor each polynomial.

(a) $r^3 + 27 = r^3 + 3^3$
$$= (r + 3)(r^2 - 3r + 3^2)$$
$$= (r + 3)(r^2 - 3r + 9)$$

(b) $27z^3 + 125 = (3z)^3 + 5^3$
$$= (3z + 5)[(3z)^2 - (3z)(5) + 5^2]$$
$$= (3z + 5)(9z^2 - 15z + 25)$$

(c) $125t^3 + 216s^6 = (5t)^3 + (6s^2)^3$
$$= (5t + 6s^2)[(5t)^2 - (5t)(6s^2) + (6s^2)^2]$$
$$= (5t + 6s^2)(25t^2 - 30ts^2 + 36s^4)$$

(d) $3x^3 + 192 = 3(x^3 + 64)$
$$= 3(x + 4)(x^2 - 4x + 16)$$

CAUTION
A common error is to think that the xy-term has a coefficient of 2 when factoring the sum or difference of cubes. Since there is no coefficient of 2, the trinomials $x^2 + xy + y^2$ and $x^2 - xy + y^2$ cannot be factored further.

Work Problem 5 at the Side.

The special types of factoring in this section are summarized here. ***These should be memorized.***

Special Types of Factoring

Difference of Squares	$x^2 - y^2 = (x + y)(x - y)$
Perfect Square Trinomial	$x^2 + 2xy + y^2 = (x + y)^2$ $x^2 - 2xy + y^2 = (x - y)^2$
Difference of Cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

7.4 Solving Equations by Factoring

OBJECTIVES

1. Learn and use the zero-factor property.
2. Solve applied problems that require the zero-factor property.

The equations that we have solved so far in this book have been linear equations. Recall from **Section 2.1** that in a linear equation, the greatest power of the variable is 1. To solve equations of degree greater than 1, other methods must be developed. One of these methods involves factoring.

OBJECTIVE 1 Learn and use the zero-factor property. Some equations can be solved by factoring. Solving equations by factoring depends on a special property of the number 0, called the **zero-factor property.**

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $\boldsymbol{ab = 0}$, then either $\boldsymbol{a = 0}$ or $\boldsymbol{b = 0}$.

To prove the zero-factor property, we first assume $a \neq 0$. (If a does equal 0, then the property is proved already.) If $a \neq 0$, then $\frac{1}{a}$ exists, and each side of $ab = 0$ can be multiplied by $\frac{1}{a}$ to get

$$\frac{1}{a} \cdot ab = \frac{1}{a} \cdot 0$$

$$b = 0.$$

Thus, if $a \neq 0$, then $b = 0$, and the property is proved.

CAUTION

If $ab = 0$, then $a = 0$ or $b = 0$. However, if $ab = 6$, for example, it is not necessarily true that $a = 6$ or $b = 6$; in fact, it is very likely that *neither* $a = 6$ *nor* $b = 6$. ***The zero-factor property works only for a product equal to 0.***

EXAMPLE 1 Using the Zero-Factor Property to Solve an Equation

Solve $(x + 6)(2x - 3) = 0$.

Here the product of $x + 6$ and $2x - 3$ is 0. By the zero-factor property, this can be true only if

$$\boldsymbol{x + 6 = 0} \quad \text{or} \quad \boldsymbol{2x - 3 = 0}.$$

Solve these two equations.

$$\begin{aligned} x + 6 &= 0 &\quad \text{or} \quad 2x - 3 &= 0 \\ x &= -6 & 2x &= 3 \\ & & x &= \frac{3}{2} \end{aligned}$$

The solutions are $x = -6$ or $x = \frac{3}{2}$.

Continued on Next Page

1 Solve each equation.

(a) $(3x+5)(x+1)=0$

(b) $(3x+11)(5x-2)=0$

Check the solutions 6 and $\frac{3}{2}$ by substitution in the original equation.

If $x=-6$, then

$$(x+6)(2x-3)=0$$
$$(-6+6)[2(-6)-3]=0 \quad ?$$
$$0(-15)=0. \quad \text{True}$$

If $x=\frac{3}{2}$, then

$$(x+6)(2x-3)=0$$
$$\left(\frac{3}{2}+6\right)\left(2\cdot\frac{3}{2}-3\right)=0 \quad ?$$
$$\frac{15}{2}(0)=0. \quad \text{True}$$

Both solutions check; the solution set is $\{-6, \frac{3}{2}\}$.

Work Problem 1 at the Side.

Since the product $(x+6)(2x-3)$ equals $2x^2+9x-18$, the equation of Example 1 has a squared term and is an example of a *quadratic equation*. A quadratic equation has degree 2.

Audio

Quadratic Equation

An equation that can be written in the form

$$ax^2+bx+c=0,$$

where $a \neq 0$, is a **quadratic equation.** This form is called **standard form.**

Quadratic equations are discussed in more detail in **Chapter 10.**

The steps involved in solving a quadratic equation by factoring are summarized below.

Audio

Solving a Quadratic Equation by Factoring

Step 1 **Write in standard form.** Rewrite the equation if necessary so that one side is 0.

Step 2 **Factor** the polynomial.

Step 3 **Use the zero-factor property.** Set each variable factor equal to 0.

Step 4 **Find the solution(s).** Solve each equation formed in Step 3.

Step 5 **Check** each solution in the *original* equation.

Video

EXAMPLE 2 **Solving a Quadratic Equation by Factoring**

Solve each equation.

(a) $2x^2+3x=2$

Step 1 $$2x^2+3x=2$$
$$2x^2+3x-2=0 \quad \text{Standard form}$$

Step 2 $$(x+2)(2x-1)=0 \quad \text{Factor.}$$

Step 3 $$x+2=0 \quad \text{or} \quad 2x-1=0 \quad \text{Zero-factor property}$$

Step 4 $$x=-2 \qquad 2x=1 \quad \text{Solve each equation.}$$
$$x=\frac{1}{2}$$

Continued on Next Page

ANSWERS

1. (a) $\left\{-\frac{5}{3}, -1\right\}$ (b) $\left\{-\frac{11}{3}, \frac{2}{5}\right\}$

Step 5 Check each solution in the original equation.

If $x = -2$, then

$$\begin{aligned} 2x^2 + 3x &= 2 \\ 2(-2)^2 + 3(-2) &= 2 \quad ? \\ 2(4) - 6 &= 2 \quad ? \\ 8 - 6 &= 2 \quad ? \\ 2 &= 2. \quad \text{True} \end{aligned}$$

If $x = \frac{1}{2}$, then

$$\begin{aligned} 2x^2 + 3x &= 2 \\ 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) &= 2 \quad ? \\ 2\left(\frac{1}{4}\right) + \frac{3}{2} &= 2 \quad ? \\ \frac{1}{2} + \frac{3}{2} &= 2 \quad ? \\ 2 &= 2. \quad \text{True} \end{aligned}$$

Because both solutions check, the solution set is $\{-2, \frac{1}{2}\}$.

(b) $4x^2 = 4x - 1$

$$\begin{aligned} 4x^2 - 4x + 1 &= 0 && \text{Standard form} \\ (2x - 1)^2 &= 0 && \text{Factor.} \\ 2x - 1 &= 0 && \text{Zero-factor property} \\ 2x &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

There is only one solution because the trinomial is a perfect square. The solution set is $\{\frac{1}{2}\}$.

Work Problem 2 at the Side.

2 Solve each equation.

(a) $3x^2 - x = 4$

(b) $25x^2 = -20x - 4$

EXAMPLE 3 Solving a Quadratic Equation with a Missing Term

You Try It

Solve $5x^2 - 25x = 0$.

This quadratic equation has a missing term. Comparing it with the standard form $ax^2 + bx + c = 0$ shows that $c = 0$. The zero-factor property can still be used.

$$\begin{aligned} 5x^2 - 25x &= 0 \\ 5x(x - 5) &= 0 && \text{Factor.} \\ 5x = 0 \quad \text{or} \quad x - 5 &= 0 && \text{Zero-factor property} \\ x = 0 \quad \text{or} \quad x &= 5 \end{aligned}$$

The solutions are 0 and 5, as can be verified by substituting in the original equation. The solution set is $\{0, 5\}$.

CAUTION
Remember to include 0 as a solution of the equation in Example 3.

Work Problem 3 at the Side.

3 Solve each equation.

(a) $x^2 = -12x$

(b) $t^2 - 16 = 0$

ANSWERS

2. **(a)** $\left\{-1, \frac{4}{3}\right\}$ **(b)** $\left\{-\frac{2}{5}\right\}$

3. **(a)** $\{-12, 0\}$ **(b)** $\{-4, 4\}$

4 Solve.

$(x+6)(x-2)=-8+x$

EXAMPLE 4 Solving an Equation That Requires Rewriting

Solve $(2q+1)(q+1)=2(1-q)+6$.

$$\begin{aligned}
(2q+1)(q+1) &= 2(1-q)+6 && \\
2q^2+3q+1 &= 2-2q+6 && \text{Multiply on each side.}\\
2q^2+3q+1 &= 8-2q && \text{Add on the right.}\\
2q^2+5q-7 &= 0 && \text{Standard form}\\
(2q+7)(q-1) &= 0 && \text{Factor.}\\
2q+7=0 \quad &\text{or} \quad q-1=0 && \text{Zero-factor property}\\
2q=-7 \quad & \quad\quad\; q=1 &&\\
q=-\frac{7}{2} &&&
\end{aligned}$$

Check that the solution set is $\{-\frac{7}{2}, 1\}$.

Work Problem 4 at the Side.

The zero-factor property can be extended to solve certain polynomial equations of degree 3 or higher, as shown in the next example.

5 Solve.

$3x^3+x^2=4x$

EXAMPLE 5 Solving an Equation of Degree 3

Solve $-x^3+x^2=-6x$.

Start by adding $6x$ to each side to get 0 on the right side.

$$\begin{aligned}
-x^3+x^2+6x &= 0 &&\\
x^3-x^2-6x &= 0 && \text{Multiply by } -1.\\
x(x^2-x-6) &= 0 && \text{Factor out } x.\\
x(x+2)(x-3) &= 0 && \text{Factor the trinomial.}
\end{aligned}$$

Use the zero-factor property, extended to include the three variable factors.

$$x=0 \quad \text{or} \quad x+2=0 \quad \text{or} \quad x-3=0$$
$$x=-2 \qquad\qquad x=3$$

Check that the solution set is $\{-2, 0, 3\}$.

Work Problem 5 at the Side.

OBJECTIVE 2 Solve applied problems that require the zero-factor property. An application may lead to a quadratic equation. We continue to use the six-step problem-solving method introduced in **Section 2.3.**

EXAMPLE 6 Using a Quadratic Equation in an Application

A piece of sheet metal is in the shape of a parallelogram. The longer sides of the parallelogram are each 8 m longer than the distance between them. The area of the parallelogram is 48 m^2. Find the length of the longer sides and the distance between them.

Step 1 **Read** the problem again. There will be two answers.

Step 2 **Assign a variable.** Let x represent the distance between the longer sides. Then $x+8$ is the length of each longer side. See Figure 1.

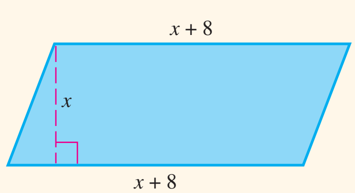


Figure 1

Continued on Next Page

ANSWERS

4. $\{-4, 1\}$

5. $\left\{-\frac{4}{3}, 0, 1\right\}$

Step 3 **Write an equation.** The area of a parallelogram is given by $A = bh$, where b is the length of the longer side and h is the distance between the longer sides. Here $b = x + 8$ and $h = x$.

$$A = bh$$

$$\mathbf{48} = \mathbf{(x + 8)}\mathbf{x} \qquad \text{Let } A = 48,\ b = x + 8,\ h = x.$$

Step 4 **Solve.**

$$48 = x^2 + 8x \qquad \text{Distributive property}$$

$$0 = x^2 + 8x - 48 \qquad \text{Standard form}$$

$$0 = (x + 12)(x - 4) \qquad \text{Factor.}$$

$$x + 12 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Zero-factor property}$$

$$x = -12 \quad \text{or} \quad x = 4$$

Step 5 **State the answer.** A distance cannot be negative, so reject -12 as a solution. The only possible solution is 4, so the distance between the longer sides is 4 m. The length of the longer sides is $4 + 8 = 12$ m.

Step 6 **Check.** The length of the longer sides is 8 m more than the distance between them, and the area is $4 \cdot 12 = 48$ m^2, so the answer checks.

CAUTION

A solution of the equation may not satisfy the physical requirements of the application, as in Example 6. Reject such solutions.

Work Problem 6 at the Side.

A function defined by a quadratic polynomial is called a *quadratic function*. (See **Chapter 10.**) The next example uses such a function.

You Try It

Video

EXAMPLE 7 Using a Quadratic Function in an Application

Quadratic functions are used to describe the height a falling object or a propelled object reaches in a specific time. For example, if a small rocket is launched vertically upward from ground level with an initial velocity of 128 ft per sec, then its height in feet after t seconds is a function defined by

$$h(t) = -16t^2 + 128t,$$

if air resistance is neglected. After how many seconds will the rocket be 220 ft above the ground?

We must let $h(t) = 220$ and solve for t.

$$\mathbf{220} = -16t^2 + 128t \qquad \text{Let } h(t) = 220.$$

$$16t^2 - 128t + 220 = 0 \qquad \text{Standard form}$$

$$4t^2 - 32t + 55 = 0 \qquad \text{Divide by 4.}$$

$$(2t - 5)(2t - 11) = 0 \qquad \text{Factor.}$$

$$2t - 5 = 0 \quad \text{or} \quad 2t - 11 = 0 \qquad \text{Zero-factor property}$$

$$t = 2.5 \quad \text{or} \quad t = 5.5$$

The rocket will reach a height of 220 ft twice: on its way up at 2.5 sec and again on its way down at 5.5 sec.

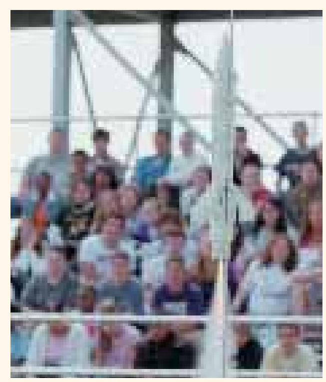
Work Problem 7 at the Side.

6 Solve the problem.

Carl is planning to build a rectangular deck along the back of his house. He wants the area of the deck to be 60 m^2, and the width to be 1 m less than half the length. What length and width should he use?

7 Solve the problem.

How long will it take the rocket in Example 7 to reach a height of 256 ft?



ANSWERS

6. length: 12 m; width: 5 m

7. 4 sec

Rational Expressions and Functions

8

Americans have been car crazy ever since the first automobiles hit the road early in the twentieth century. Today there are about 213.5 million vehicles in the United States driving on 3.4 million miles of paved roadways. There is even a museum devoted exclusively to our four-wheeled passion and its influence on our lives and culture. The Museum of Automobile History in Syracuse, N.Y., features some 200 years of automobile memorabilia, including rare advertising pieces, designer drawings, and Hollywood movie posters. (*Source: Home and Away*, May/June 2002.)

In Exercises 67 and 68 of Section 8.2, we use a *rational expression* to determine the cost of restoring a vintage automobile.

8.1 Rational Expressions and Functions; Multiplying and Dividing

OBJECTIVES

1. Define rational expressions.
2. Define rational functions and describe their domains.
3. Write rational expressions in lowest terms.
4. Multiply rational expressions.
5. Find reciprocals for rational expressions.
6. Divide rational expressions.

OBJECTIVE 1 Define rational expressions. In arithmetic, a rational number is the quotient of two integers, with the denominator not 0. In algebra, a **rational expression** or *algebraic fraction* is the quotient of two polynomials, again with the denominator not 0. For example,

$$\frac{x}{y}, \quad \frac{-a}{4}, \quad \frac{m+4}{m-2}, \quad \frac{8x^2-2x+5}{4x^2+5x}, \quad \text{and} \quad x^5\left(\text{or } \frac{x^5}{1}\right) \qquad \text{Rational expressions}$$

are all rational expressions. In other words, rational expressions are the elements of the set

$$\left\{\frac{P}{Q} \,\middle|\, P \text{ and } Q \text{ are polynomials, with } Q \neq 0\right\}.$$

OBJECTIVE 2 Define rational functions and describe their domains. A function that is defined by a rational expression is called a **rational function** and has the form

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } Q(x) \neq 0.$$

The domain of a rational function includes all real numbers except those that make $Q(x)$, that is, the denominator, equal to 0. For example, the domain of

$$f(x) = \frac{2}{\underbrace{x-5}_{\text{Cannot equal } 0}}$$

includes all real numbers except 5, because 5 would make the denominator equal to 0.

Figure 1 shows a graph of the function defined by

$$f(x) = \frac{2}{x-5}.$$

Notice that the graph does not exist when $x = 5$. It does not intersect the dashed vertical line whose equation is $x = 5$. This line is an *asymptote*. We will discuss graphs of rational functions in more detail in **Section 8.4.**

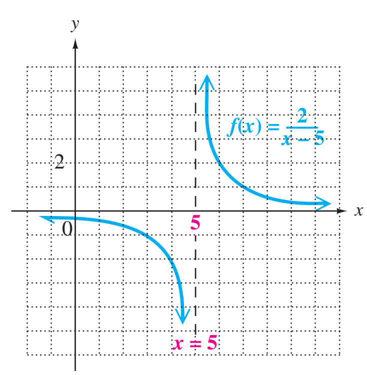


Figure 1

You Try It

EXAMPLE 1 **Finding Numbers That Are Not in the Domains of Rational Functions**

Find all numbers that are not in the domain of each rational function. Then give the domain using set notation.

(a) $f(x) = \dfrac{3}{7x - 14}$

The only values that cannot be used are those that make the denominator 0. To find these values, set the denominator equal to 0 and solve the resulting equation.

$$\begin{aligned} 7x - 14 &= 0 \\ 7x &= 14 \quad \text{Add 14.} \\ x &= 2 \quad \text{Divide by 7.} \end{aligned}$$

The number 2 cannot be used as a replacement for x. The domain of f includes all real numbers except 2, written using set notation as $\{x|x \neq 2\}$.

(b) $g(x) = \dfrac{3 + x}{x^2 - 4x + 3}$

Set the denominator equal to 0, and solve the equation.

$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x - 1)(x - 3) &= 0 \quad \text{Factor.} \\ x - 1 = 0 \quad \text{or} \quad x - 3 &= 0 \quad \text{Zero-factor property} \\ x = 1 \quad \text{or} \quad x &= 3 \end{aligned}$$

The domain of g includes all real numbers except 1 and 3, written $\{x|x \neq 1, 3\}$.

(c) $h(x) = \dfrac{8x + 2}{3}$

The denominator, 3, can never be 0, so the domain of h includes all real numbers, written $(-\infty, \infty)$.

(d) $f(x) = \dfrac{2}{x^2 + 4}$

Setting $x^2 + 4$ equal to 0 leads to $x^2 = -4$. There is no real number whose square is -4. Therefore, any real number can be used, and as in part (c), the domain of f includes all real numbers $(-\infty, \infty)$.

Work Problem 1 at the Side.

OBJECTIVE 3 **Write rational expressions in lowest terms.** In arithmetic, we write the fraction $\frac{15}{20}$ in lowest terms by dividing the numerator and denominator by 5 to get $\frac{3}{4}$. We write rational expressions in lowest terms in a similar way, using the **fundamental property of rational numbers.**

Audio

Fundamental Property of Rational Numbers

If $\frac{a}{b}$ is a rational number and if c is any nonzero real number, then

$$\frac{a}{b} = \frac{ac}{bc}.$$

In words, the numerator and denominator of a rational number may either be multiplied or divided by the same nonzero number without changing the value of the rational number.

1 Find all numbers that are not in the domain of each rational function. Then give the domain using set notation.

(a) $f(x) = \dfrac{x + 4}{x - 6}$

(b) $f(x) = \dfrac{x + 6}{x^2 - x - 6}$

(c) $f(x) = \dfrac{3 + 2x}{5}$

(d) $f(x) = \dfrac{2}{x^2 + 1}$

ANSWERS

1. **(a)** 6; $\{x|x \neq 6\}$ **(b)** $-2, 3$; $\{x|x \neq -2, 3\}$
(c) none; The domain includes all real numbers $(-\infty, \infty)$.
(d) none; The domain includes all real numbers $(-\infty, \infty)$.

In the fundamental property, $\frac{a}{b} = \frac{ac}{bc}$. Since $\frac{c}{c}$ is equivalent to 1, the fundamental property is based on the identity property of multiplication.

A rational expression is a quotient of two polynomials. Since the value of a polynomial is a real number for every value of the variable for which it is defined, any statement that applies to rational numbers will also apply to rational expressions. We use the following steps to write rational expressions in lowest terms.

Writing a Rational Expression in Lowest Terms

Step 1 **Factor** both numerator and denominator to find their greatest common factor (GCF).

Step 2 **Apply the fundamental property.**

EXAMPLE 2 Writing Rational Expressions in Lowest Terms

Write each rational expression in lowest terms.

(a) $\frac{8k}{16} = \frac{k \cdot \mathbf{8}}{2 \cdot \mathbf{8}} = \frac{k}{2} \cdot \mathbf{1} = \frac{k}{2}$ Factor; apply the fundamental property.

(b) $\frac{8 + k}{16}$

The numerator cannot be factored, so this expression cannot be simplified further and is in lowest terms.

(c) $\frac{a^2 - a - 6}{a^2 + 5a + 6} = \frac{(a - 3)\mathbf{(a + 2)}}{(a + 3)\mathbf{(a + 2)}}$ Factor the numerator and the denominator.

$= \frac{a - 3}{a + 3} \cdot \mathbf{1}$ $\frac{a + 2}{a + 2} = 1$

$= \frac{a - 3}{a + 3}$ Lowest terms

(d) $\frac{y^2 - 4}{2y + 4} = \frac{\mathbf{(y + 2)}(y - 2)}{2\mathbf{(y + 2)}}$ Factor the difference of squares in the numerator; factor the denominator.

$= \frac{y - 2}{2}$ Lowest terms

(e) $\frac{x^3 - 27}{x - 3} = \frac{\mathbf{(x - 3)}(x^2 + 3x + 9)}{\mathbf{x - 3}}$ Factor the difference of cubes.

$= x^2 + 3x + 9$ Lowest terms

(f) $\frac{pr + qr + ps + qs}{pr + qr - ps - qs} = \frac{(pr + qr) + (ps + qs)}{(pr + qr) - (ps + qs)}$ Group terms.

$= \frac{r(p + q) + s(p + q)}{r(p + q) - s(p + q)}$ Factor within groups.

$= \frac{\mathbf{(p + q)}(r + s)}{\mathbf{(p + q)}(r - s)}$ Factor by grouping.

$= \frac{r + s}{r - s}$ Lowest terms

CAUTION

Be careful! When using the fundamental property of rational numbers, ***only common factors may be divided***. For example,

$$\frac{y-2}{2} \neq y \quad \text{and} \quad \frac{y-2}{2} \neq y-1$$

because the 2 in $y - 2$ is not a *factor* of the numerator. ***Remember to factor before writing a fraction in lowest terms.***

Work Problem 2 at the Side.

In the rational expression from Example 2(c),

$$\frac{a^2 - a - 6}{a^2 + 5a + 6}, \quad \text{or} \quad \frac{(a-3)(a+2)}{(a+3)(a+2)},$$

a can take any value except -3 or -2 since these values make the denominator 0. In the simplified rational expression

$$\frac{a-3}{a+3},$$

a cannot equal -3. Because of this,

$$\frac{a^2 - a - 6}{a^2 + 5a + 6} = \frac{a-3}{a+3}$$

for all values of a except -3 or -2. From now on such statements of equality will be made with the understanding that they apply only for those real numbers that make neither denominator equal 0. We will no longer state such restrictions.

EXAMPLE 3 Writing Rational Expressions in Lowest Terms

Write each rational expression in lowest terms.

(a) $\dfrac{m-3}{3-m}$

In this rational expression, the numerator and denominator are opposites. The given expression can be written in lowest terms by writing the denominator as $-1(m - 3)$, giving

$$\frac{m-3}{3-m} = \frac{m-3}{-1(m-3)} = \frac{1}{-1} = -1.$$

The numerator could have been rewritten instead to get the same result.

(b)

$$\begin{aligned} \frac{r^2 - 16}{4 - r} &= \frac{(r+4)(r-4)}{4-r} && \text{Factor the difference of squares in the numerator.} \\ &= \frac{(r+4)(r-4)}{-1(r-4)} && \text{Write } 4 - r \text{ as } -1(r-4). \\ &= \frac{r+4}{-1} && \text{Fundamental property} \\ &= -(r+4) \quad \text{or} \quad -r-4 && \text{Lowest terms} \end{aligned}$$

2 Write each rational expression in lowest terms.

(a) $\dfrac{y^2 + 2y - 3}{y^2 - 3y + 2}$

(b) $\dfrac{3y + 9}{y^2 - 9}$

(c) $\dfrac{y + 2}{y^2 + 4}$

(d) $\dfrac{1 + p^3}{1 + p}$

(e) $\dfrac{3x + 3y + rx + ry}{5x + 5y - rx - ry}$

ANSWERS

2. (a) $\dfrac{y+3}{y-2}$ (b) $\dfrac{3}{y-3}$ (c) already in lowest terms (d) $1 - p + p^2$ (e) $\dfrac{3+r}{5-r}$

3 Write each rational expression in lowest terms.

(a) $\dfrac{y-2}{2-y}$

(b) $\dfrac{8-b}{8+b}$

(c) $\dfrac{p-2}{4-p^2}$

As shown in Examples 3(a) and (b), the quotient $\frac{a}{-a}$ $(a \neq 0)$ can be simplified as

$$\frac{a}{-a} = \frac{a}{-1(a)} = \frac{1}{-1} = -1.$$

The following statement summarizes this result.

In general, if the numerator and the denominator of a rational expression are opposites, the expression equals -1.

Based on this result, the following are true:

$$\frac{q-7}{7-q} = -1 \quad \text{and} \quad \frac{-5a+2b}{5a-2b} = -1.$$

Numerator and denominator in each expression are opposites.

However, the expression

$$\frac{r-2}{r+2}$$

Numerator and denominator are *not* opposites.

cannot be simplifed further.

Work Problem 3 at the Side.

OBJECTIVE 4 Multiply rational expressions. To multiply rational expressions, follow these steps.

Multiplying Rational Expressions

Step 1 **Factor** all numerators and denominators as completely as possible.

Step 2 **Apply the fundamental property.**

Step 3 **Multiply** remaining factors in the numerator and remaining factors in the denominator. Leave the denominator in factored form.

Step 4 **Check** to be sure the product is in lowest terms.

EXAMPLE 4 Multiplying Rational Expressions

Multiply.

(a)
$$\frac{5p-5}{p} \cdot \frac{3p^2}{10p-10} = \frac{5(p-1)}{p} \cdot \frac{3p \cdot p}{2 \cdot 5(p-1)} \qquad \text{Factor.}$$

$$= \frac{1}{1} \cdot \frac{3p}{2} \qquad \text{Fundamental property}$$

$$= \frac{3p}{2} \qquad \text{Multiply.}$$

Continued on Next Page

ANSWERS

3. **(a)** -1 **(b)** already in lowest terms **(c)** $\dfrac{-1}{2+p}$

(b) $\dfrac{k^2+2k-15}{k^2-4k+3} \cdot \dfrac{k^2-k}{k^2+k-20} = \dfrac{(k+5)(k-3)}{(k-3)(k-1)} \cdot \dfrac{k(k-1)}{(k+5)(k-4)}$

$= \dfrac{k}{k-4}$

(c) $(p-4) \cdot \dfrac{3}{5p-20} = \dfrac{p-4}{1} \cdot \dfrac{3}{5p-20}$ Write $p-4$ as $\dfrac{p-4}{1}$.

$= \dfrac{p-4}{1} \cdot \dfrac{3}{5(p-4)}$ Factor.

$= \dfrac{3}{5}$ Fundamental property; multiply.

(d) $\dfrac{x^2+2x}{x+1} \cdot \dfrac{x^2-1}{x^3+x^2} = \dfrac{x(x+2)}{x+1} \cdot \dfrac{(x+1)(x-1)}{x^2(x+1)}$ Factor.

$= \dfrac{(x+2)(x-1)}{x(x+1)}$ Multiply; lowest terms.

(e) $\dfrac{x-6}{x^2-12x+36} \cdot \dfrac{x^2-3x-18}{x^2+7x+12} = \dfrac{x-6}{(x-6)^2} \cdot \dfrac{(x+3)(x-6)}{(x+3)(x+4)}$ Factor.

$= \dfrac{1}{x+4}$ Lowest terms

Remember to include 1 in the numerator when all other factors are eliminated using the fundamental property.

Work Problem 4 at the Side.

OBJECTIVE 5 Find reciprocals for rational expressions. The rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are reciprocals of each other if they have a product of 1. The **reciprocal** of a rational expression is defined in the same way: ***Two rational expressions are reciprocals of each other if they have a product of 1.*** Recall that 0 has no reciprocal. The table shows several rational expressions and their reciprocals. In the first two cases, check that the product of the rational expression and its reciprocal is 1.

Rational Expression	Reciprocal
$\dfrac{5}{k}$	$\dfrac{k}{5}$
$\dfrac{m^2-9m}{2}$	$\dfrac{2}{m^2-9m}$
$\dfrac{0}{4}$	undefined

The examples in the table suggest the following procedure.

Finding the Reciprocal

To find the reciprocal of a nonzero rational expression, interchange the numerator and denominator of the expression.

Work Problem 5 at the Side.

4 Multiply.

(a) $\dfrac{2r+4}{5r} \cdot \dfrac{3r}{5r+10}$

(b) $\dfrac{c^2+2c}{c^2-4} \cdot \dfrac{c^2-4c+4}{c^2-c}$

(c) $\dfrac{m^2-16}{m+2} \cdot \dfrac{1}{m+4}$

(d) $\dfrac{x-3}{x^2+2x-15} \cdot \dfrac{x^2-25}{x^2+3x-40}$

5 Find each reciprocal.

(a) $\dfrac{-3}{r}$

(b) $\dfrac{7}{y+8}$

(c) $\dfrac{a^2+7a}{2a-1}$

(d) $\dfrac{0}{-5}$

Answers

4. (a) $\dfrac{6}{25}$ **(b)** $\dfrac{c-2}{c-1}$ **(c)** $\dfrac{m-4}{m+2}$ **(d)** $\dfrac{1}{x+8}$

5. (a) $\dfrac{r}{-3}$ **(b)** $\dfrac{y+8}{7}$ **(c)** $\dfrac{2a-1}{a^2+7a}$
(d) There is no reciprocal.

OBJECTIVE 6 Divide rational expressions. Dividing rational expressions is like dividing rational numbers.

Dividing Rational Expressions

To divide two rational expressions, multiply the first (the dividend) by the reciprocal of the second (the divisor).

EXAMPLE 5 Dividing Rational Expressions

Divide.

(a)

$$\frac{2z}{9} \div \frac{5z^2}{18} = \frac{2z}{9} \cdot \frac{18}{5z^2} \qquad \text{Multiply by the reciprocal of the divisor.}$$

$$= \frac{2z}{9} \cdot \frac{2 \cdot 9}{5z^2} \qquad \text{Factor.}$$

$$= \frac{4}{5z} \qquad \text{Multiply; lowest terms}$$

(b)

$$\frac{8k - 16}{3k} \div \frac{3k - 6}{4k^2} = \frac{8k - 16}{3k} \cdot \frac{4k^2}{3k - 6} \qquad \text{Multiply by the reciprocal.}$$

$$= \frac{8(k - 2)}{3k} \cdot \frac{4k^2}{3(k - 2)} \qquad \text{Factor.}$$

$$= \frac{32k}{9} \qquad \text{Multiply; lowest terms}$$

(c)

$$\frac{5m^2 + 17m - 12}{3m^2 + 7m - 20} \div \frac{5m^2 + 2m - 3}{15m^2 - 34m + 15}$$

$$= \frac{5m^2 + 17m - 12}{3m^2 + 7m - 20} \cdot \frac{15m^2 - 34m + 15}{5m^2 + 2m - 3} \qquad \text{Definition of division}$$

$$= \frac{(5m - 3)(m + 4)}{(m + 4)(3m - 5)} \cdot \frac{(3m - 5)(5m - 3)}{(5m - 3)(m + 1)} \qquad \text{Factor.}$$

$$= \frac{5m - 3}{m + 1} \qquad \text{Lowest terms}$$

Work Problem 6 at the Side.

6 Divide.

(a) $\dfrac{16k^2}{5} \div \dfrac{3k}{10}$

(b) $\dfrac{5p + 2}{6} \div \dfrac{15p + 6}{5}$

(c) $\dfrac{y^2 - 2y - 3}{y^2 + 4y + 4} \div \dfrac{y^2 - 1}{y^2 + y - 2}$

ANSWERS

6. (a) $\dfrac{32k}{3}$ (b) $\dfrac{5}{18}$ (c) $\dfrac{y - 3}{y + 2}$

8.2 Adding and Subtracting Rational Expressions

OBJECTIVES

1. Add and subtract rational expressions with the same denominator.
2. Find a least common denominator.
3. Add and subtract rational expressions with different denominators.

OBJECTIVE 1 Add and subtract rational expressions with the same denominator. The following steps, used to add or subtract rational numbers, are also used to add or subtract rational expressions.

Audio

Adding or Subtracting Rational Expressions

Step 1 **If the denominators are the same,** add or subtract the numerators. Place the result over the common denominator.

If the denominators are different, first find the least common denominator. Write all rational expressions with this LCD, and then add or subtract the numerators. Place the result over the common denominator.

Step 2 **Simplify.** Write all answers in lowest terms.

You Try It

Animation

Video

EXAMPLE 1 Adding and Subtracting Rational Expressions with the Same Denominator

Add or subtract as indicated.

(a) $\dfrac{3y}{5} + \dfrac{x}{5} = \dfrac{3y + x}{5}$ ← Add the numerators. ← Keep the common denominator.

The denominators of these rational expressions are the same, so just add the numerators, and place the sum over the common denominator.

(b) $\dfrac{7}{2r^2} - \dfrac{11}{2r^2} = \dfrac{7 - 11}{2r^2}$ Subtract the numerators; keep the common denominator.

$= \dfrac{-4}{2r^2}$

$= -\dfrac{2}{r^2}$ Lowest terms

(c) $\dfrac{m}{m^2 - p^2} + \dfrac{p}{m^2 - p^2} = \dfrac{m + p}{m^2 - p^2}$ Add the numerators; keep the common denominator.

$= \dfrac{m + p}{(m + p)(m - p)}$ Factor.

$= \dfrac{1}{m - p}$ Lowest terms

(d) $\dfrac{4}{x^2 + 2x - 8} + \dfrac{x}{x^2 + 2x - 8} = \dfrac{4 + x}{x^2 + 2x - 8}$ Add.

$= \dfrac{4 + x}{(x - 2)(x + 4)}$ Factor.

$= \dfrac{1}{x - 2}$ Lowest terms

Work Problem 1 at the Side.

1 Add or subtract.

(a) $\dfrac{3m}{8} + \dfrac{5n}{8}$

(b) $\dfrac{7}{3a} + \dfrac{10}{3a}$

(c) $\dfrac{2}{y^2} - \dfrac{5}{y^2}$

(d) $\dfrac{a}{a + b} + \dfrac{b}{a + b}$

(e) $\dfrac{2y - 1}{y^2 + y - 2} - \dfrac{y}{y^2 + y - 2}$

Answers

1. (a) $\dfrac{3m + 5n}{8}$ (b) $\dfrac{17}{3a}$ (c) $-\dfrac{3}{y^2}$ (d) 1 (e) $\dfrac{1}{y + 2}$

OBJECTIVE 2 Find a least common denominator. We add or subtract rational expressions with different denominators by first writing them with a common denominator, usually the **least common denominator (LCD).**

Finding the Least Common Denominator

Step 1 **Factor** each denominator.

Step 2 **Find the least common denominator.** The LCD is the product of all different factors from each denominator, with each factor raised to the *greatest* power that occurs in any denominator.

EXAMPLE 2 Finding Least Common Denominators

Assume that the given expressions are denominators of fractions. Find the LCD for each group.

(a) $5xy^2$, $2x^3y$

Each denominator is already factored.

$$5xy^2 = 5 \cdot x \cdot y^2$$
$$2x^3y = 2 \cdot x^3 \cdot y$$

Greatest exponent on x is 3.

$$\text{LCD} = 5 \cdot 2 \cdot x^3 \cdot y^2 \quad \leftarrow \text{Greatest exponent on } y \text{ is 2.}$$
$$= 10x^3y^2$$

(b) $k - 3$, k

Each denominator is already factored. The LCD, an expression divisible by *both* $k - 3$ and k, is

$$k(k - 3).$$

It is usually best to leave a least common denominator in factored form.

(c) $y^2 - 2y - 8$, $y^2 + 3y + 2$

Factor the denominators.

$$\left.\begin{aligned} y^2 - 2y - 8 &= (y - 4)(y + 2) \\ y^2 + 3y + 2 &= (y + 2)(y + 1) \end{aligned}\right\} \text{Factor.}$$

The LCD, divisible by both polynomials, is $(y - 4)(y + 2)(y + 1)$.

(d) $8z - 24$, $5z^2 - 15z$

$$\left.\begin{aligned} 8z - 24 &= 8(z - 3) \\ 5z^2 - 15z &= 5z(z - 3) \end{aligned}\right\} \text{Factor.}$$

The LCD is $8 \cdot 5z \cdot (z - 3) = 40z(z - 3)$.

(e) $m^2 + 5m + 6$, $m^2 + 4m + 4$, $2m + 6$

$$\left.\begin{aligned} m^2 + 5m + 6 &= (m + 3)(m + 2) \\ m^2 + 4m + 4 &= (m + 2)^2 \\ 2m + 6 &= 2(m + 3) \end{aligned}\right\} \text{Factor.}$$

The LCD is $2(m + 3)(m + 2)^2$.

Work Problem 2 at the Side.

2 Find the LCD for each group of denominators.

(a) $5k^3s$, $10ks^4$

Audio

(b) $3 - x$, $9 - x^2$

You Try It

Animation

(c) z, $z + 6$

(d) $2y^2 - 3y - 2$, $2y^2 + 3y + 1$

(e) $x^2 - 2x + 1$, $x^2 - 4x + 3$, $4x - 4$

ANSWERS

2. (a) $10k^3s^4$ **(b)** $(3 + x)(3 - x)$
(c) $z(z + 6)$ **(d)** $(y - 2)(2y + 1)(y + 1)$
(e) $4(x - 3)(x - 1)^2$

OBJECTIVE 3 Add and subtract rational expressions with different denominators. Before adding or subtracting two rational expressions, we write each expression with the least common denominator by multiplying its numerator and denominator by the factors needed to get the LCD. This procedure is valid because we are multiplying each rational expression by a form of 1, the identity element for multiplication.

Adding or subtracting rational expressions follows the same procedure as that used for rational numbers. Consider the sum $\frac{7}{15} + \frac{5}{12}$. The LCD for 15 and 12 is 60. Multiply $\frac{7}{15}$ by $\frac{4}{4}$ (a form of 1) and multiply $\frac{5}{12}$ by $\frac{5}{5}$ (another form of 1) so that each fraction has denominator 60. Then add the numerators.

$$\frac{7}{15} + \frac{5}{12} = \frac{7 \cdot 4}{15 \cdot 4} + \frac{5 \cdot 5}{12 \cdot 5} \qquad \text{Fundamental property}$$

$$= \frac{28}{60} + \frac{25}{60}$$

$$= \frac{28 + 25}{60} \qquad \text{Add the numerators.}$$

$$= \frac{53}{60}$$

EXAMPLE 3 Adding and Subtracting Rational Expressions with Different Denominators

Add or subtract as indicated.

(a) $\frac{5}{2p} + \frac{3}{8p}$

The LCD for $2p$ and $8p$ is $8p$. To write the first rational expression with a denominator of $8p$, multiply by $\frac{4}{4}$.

$$\frac{5}{2p} + \frac{3}{8p} = \frac{5 \cdot 4}{2p \cdot 4} + \frac{3}{8p} \qquad \text{Fundamental property}$$

$$= \frac{20}{8p} + \frac{3}{8p}$$

$$= \frac{20 + 3}{8p} \qquad \text{Add the numerators.}$$

$$= \frac{23}{8p}$$

(b) $\frac{6}{r} - \frac{5}{r - 3}$

Write each rational expression with the LCD, $r(r - 3)$.

$$\frac{6}{r} - \frac{5}{r - 3} = \frac{6(r - 3)}{r(r - 3)} - \frac{r \cdot 5}{r(r - 3)} \qquad \text{Fundamental property}$$

$$= \frac{6r - 18}{r(r - 3)} - \frac{5r}{r(r - 3)} \qquad \text{Distributive and commutative properties}$$

$$= \frac{6r - 18 - 5r}{r(r - 3)} \qquad \text{Subtract the numerators.}$$

$$= \frac{r - 18}{r(r - 3)} \qquad \text{Combine terms in the numerator.}$$

Work Problem 3 at the Side.

3 Add or subtract.

(a) $\frac{6}{7} + \frac{1}{5}$

(b) $\frac{8}{3k} - \frac{2}{9k}$

(c) $\frac{2}{y} - \frac{1}{y + 4}$

ANSWERS

3. (a) $\frac{37}{35}$ **(b)** $\frac{22}{9k}$ **(c)** $\frac{y + 8}{y(y + 4)}$

4 Subtract.

(a) $\dfrac{5x+7}{2x+7} - \dfrac{-x-14}{2x+7}$

(b) $\dfrac{2}{r-2} - \dfrac{r}{r-1}$

CAUTION

One of the most common sign errors in algebra occurs when a rational expression with two or more terms in the numerator is being subtracted. In this situation, ***the subtraction sign must be distributed to every term in the numerator of the fraction that follows it.*** Study Example 4 carefully to see how this is done.

EXAMPLE 4 Using the Distributive Property When Subtracting Rational Expressions

Subtract.

(a) $\dfrac{7x}{3x+1} - \dfrac{x-2}{3x+1}$

The denominators are the same for both rational expressions. The subtraction sign must be applied to *both* terms in the numerator of the second rational expression. Notice the careful use of the distributive property here.

$$\frac{7x}{3x+1} - \frac{x-2}{3x+1} = \frac{7x-(x-2)}{3x+1} \quad \text{Subtract the numerators; keep the common denominator.}$$

$$= \frac{7x-x+2}{3x+1} \quad \text{Distributive property; be careful with signs.}$$

$$= \frac{6x+2}{3x+1} \quad \text{Combine terms in the numerator.}$$

$$= \frac{2(3x+1)}{3x+1} \quad \text{Factor the numerator.}$$

$$= 2 \quad \text{Lowest terms}$$

(b) $\dfrac{1}{q-1} - \dfrac{1}{q+1}$

$$= \frac{1(q+1)}{(q-1)(q+1)} - \frac{1(q-1)}{(q+1)(q-1)} \quad \text{The LCD is } (q-1)(q+1)\text{; fundamental property}$$

$$= \frac{(q+1)-(q-1)}{(q-1)(q+1)} \quad \text{Subtract.}$$

$$= \frac{q+1-q+1}{(q-1)(q+1)} \quad \text{Distributive property}$$

$$= \frac{2}{(q-1)(q+1)} \quad \text{Combine terms in the numerator.}$$

Work Problem 4 at the Side.

In some problems, rational expressions to be added or subtracted have denominators that are opposites of each other, such as

$$\frac{y}{y-2} + \frac{8}{2-y}. \quad \text{Denominators are opposites.}$$

The next example illustrates how to proceed in such a problem.

ANSWERS

4. (a) 3 (b) $\dfrac{-r^2+4r-2}{(r-2)(r-1)}$

Video

EXAMPLE 5 Adding Rational Expressions with Denominators That Are Opposites

Add.

$$\frac{y}{y-2} + \frac{8}{2-y}$$

To get a common denominator of $y - 2$, multiply the second expression by -1 in both the numerator and the denominator.

$$\frac{y}{y-2} + \frac{8}{2-y} = \frac{y}{y-2} + \frac{8(-1)}{(2-y)(-1)}$$

$$= \frac{y}{y-2} + \frac{-8}{y-2}$$

$$= \frac{y-8}{y-2} \qquad \text{Add the numerators.}$$

If we had used $2 - y$ as the common denominator and rewritten the first expression, we would have obtained

$$\frac{8-y}{2-y},$$

an equivalent answer. Verify this.

Work Problem 5 at the Side.

You Try It

Video

EXAMPLE 6 Adding and Subtracting Three Rational Expressions

Add and subtract as indicated.

$$\frac{3}{x-2} + \frac{5}{x} - \frac{6}{x^2 - 2x}$$

The denominator of the third rational expression factors as $x(x - 2)$, which is the LCD for the three rational expressions.

$$\frac{3}{x-2} + \frac{5}{x} - \frac{6}{x^2 - 2x}$$

$$= \frac{3x}{x(x-2)} + \frac{5(x-2)}{x(x-2)} - \frac{6}{x(x-2)} \qquad \text{Fundamental property}$$

$$= \frac{3x + 5(x-2) - 6}{x(x-2)} \qquad \text{Add and subtract the numerators.}$$

$$= \frac{3x + 5x - 10 - 6}{x(x-2)} \qquad \text{Distributive property}$$

$$= \frac{8x - 16}{x(x-2)} \qquad \text{Combine terms in the numerator.}$$

$$= \frac{8(x-2)}{x(x-2)} \qquad \text{Factor the numerator.}$$

$$= \frac{8}{x} \qquad \text{Lowest terms}$$

Work Problem 6 at the Side.

5 Add or subtract as indicated.

(a) $\dfrac{8}{x-4} + \dfrac{2}{4-x}$

(b) $\dfrac{9}{2x-9} - \dfrac{4}{9-2x}$

6 Add and subtract as indicated.

$$\frac{4}{x-5} + \frac{-2}{x} - \frac{10}{x^2 - 5x}$$

ANSWERS

5. **(a)** $\dfrac{6}{x-4}$ or $\dfrac{-6}{4-x}$ **(b)** $\dfrac{13}{2x-9}$ or $\dfrac{-13}{9-2x}$

6. $\dfrac{2}{x-5}$

7 Subtract.

$$\frac{-a}{a^2+3a-4} - \frac{4a}{a^2+7a+12}$$

EXAMPLE 7 Subtracting Rational Expressions

Subtract.

$$\frac{m+4}{m^2-2m-3} - \frac{2m-3}{m^2-5m+6}$$

$$= \frac{m+4}{(m-3)(m+1)} - \frac{2m-3}{(m-3)(m-2)}$$ Factor each denominator.

The LCD is $(m-3)(m+1)(m-2)$.

$$= \frac{(m+4)(m-2)}{(m-3)(m+1)(m-2)} - \frac{(2m-3)(m+1)}{(m-3)(m-2)(m+1)}$$ Fundamental property

$$= \frac{(m+4)(m-2)-(2m-3)(m+1)}{(m-3)(m+1)(m-2)}$$ Subtract.

$$= \frac{m^2+2m-8-(2m^2-m-3)}{(m-3)(m+1)(m-2)}$$ Multiply in the numerator.

$$= \frac{m^2+2m-8-2m^2+m+3}{(m-3)(m+1)(m-2)}$$ Distributive property; be careful with signs.

$$= \frac{-m^2+3m-5}{(m-3)(m+1)(m-2)}$$ Combine terms in the numerator.

If we try to factor the numerator, we find that this rational expression is in lowest terms.

Work Problem 7 at the Side.

8 Add.

$$\frac{4}{p^2-6p+9} + \frac{1}{p^2+2p-15}$$

EXAMPLE 8 Adding Rational Expressions

Add.

$$\frac{5}{x^2+10x+25} + \frac{2}{x^2+7x+10}$$

$$= \frac{5}{(x+5)^2} + \frac{2}{(x+5)(x+2)}$$ Factor each denominator.

The LCD is $(x+5)^2(x+2)$.

$$= \frac{5(x+2)}{(x+5)^2(x+2)} + \frac{2(x+5)}{(x+5)^2(x+2)}$$ Fundamental property

$$= \frac{5(x+2)+2(x+5)}{(x+5)^2(x+2)}$$ Add.

$$= \frac{5x+10+2x+10}{(x+5)^2(x+2)}$$ Distributive property

$$= \frac{7x+20}{(x+5)^2(x+2)}$$ Combine terms in the numerator.

Work Problem 8 at the Side.

You Try It

Animation

ANSWERS

7. $\dfrac{-5a^2+a}{(a+4)(a-1)(a+3)}$

8. $\dfrac{5p+17}{(p-3)^2(p+5)}$

8.3 Complex Fractions

OBJECTIVES

1. Simplify complex fractions by simplifying the numerator and denominator. (Method 1)
2. Simplify complex fractions by multiplying by a common denominator. (Method 2)
3. Compare the two methods of simplifying complex fractions.
4. Simplify rational expressions with negative exponents.

A **complex fraction** is an expression having a fraction in the numerator, denominator, or both. Examples of complex fractions include

$$\frac{1+\frac{1}{x}}{2}, \quad \frac{\frac{4}{y}}{6-\frac{3}{y}}, \quad \text{and} \quad \frac{\frac{m^2-9}{m+1}}{\frac{m+3}{m^2-1}}. \qquad \text{Complex fractions}$$

OBJECTIVE 1 Simplify complex fractions by simplifying the numerator and denominator. (Method 1) There are two different methods for simplifying complex fractions.

Simplifying a Complex Fraction: Method 1

Step 1 Simplify the numerator and denominator separately.

Step 2 Divide by multiplying the numerator by the reciprocal of the denominator.

Step 3 Simplify the resulting fraction, if possible.

In Step 2, we are treating the complex fraction as a quotient of two rational expressions and dividing. Before performing this step, be sure that both the numerator and denominator are single fractions.

EXAMPLE 1 Simplifying Complex Fractions by Method 1

Use Method 1 to simplify each complex fraction.

(a) $\dfrac{\frac{x+1}{x}}{\frac{x-1}{2x}}$

Both the numerator and the denominator are already simplified, so divide by multiplying the numerator by the reciprocal of the denominator.

$$\frac{\frac{x+1}{x}}{\frac{x-1}{2x}} = \frac{x+1}{x} \div \frac{x-1}{2x} \qquad \text{Write as a division problem.}$$

$$= \frac{x+1}{x} \cdot \frac{2x}{x-1} \qquad \text{Multiply by the reciprocal of } \tfrac{x-1}{2x}.$$

$$= \frac{2x(x+1)}{x(x-1)} \qquad \text{Multiply.}$$

$$= \frac{2(x+1)}{x-1} \qquad \text{Simplify.}$$

Continued on Next Page

1 Use Method 1 to simplify each complex fraction.

(a) $\dfrac{\frac{a+2}{5a}}{\frac{a-3}{7a}}$

(b) $\dfrac{2+\frac{1}{k}}{2-\frac{1}{k}}$

(c) $\dfrac{\frac{r^2-4}{4}}{1+\frac{2}{r}}$

(b) $\dfrac{2+\frac{1}{y}}{3-\frac{2}{y}} = \dfrac{\frac{2y}{y}+\frac{1}{y}}{\frac{3y}{y}-\frac{2}{y}} = \dfrac{\frac{2y+1}{y}}{\frac{3y-2}{y}}$ Simplify the numerator and denominator. (Step 1)

$= \dfrac{2y+1}{y} \div \dfrac{3y-2}{y}$ Write as a division problem.

$= \dfrac{2y+1}{y} \cdot \dfrac{y}{3y-2}$ Multiply by the reciprocal of $\frac{3y-2}{y}$. (Step 2)

$= \dfrac{2y+1}{3y-2}$ Multiply and simplify. (Step 3)

Work Problem 1 at the Side.

OBJECTIVE 2 Simplify complex fractions by multiplying by a common denominator. (Method 2) The second method for simplifying complex fractions uses the identity property of multiplication.

Simplifying a Complex Fraction: Method 2

Step 1 Multiply the numerator and denominator of the complex fraction by the least common denominator of the fractions in the numerator and the fractions in the denominator of the complex fraction.

Step 2 Simplify the resulting fraction, if possible.

EXAMPLE 2 Simplifying Complex Fractions by Method 2

Use Method 2 to simplify each complex fraction.

(a) $\dfrac{2+\frac{1}{y}}{3-\frac{2}{y}}$

Multiply the numerator and denominator by the LCD of all the fractions in the numerator and denominator of the complex fraction. (This is the same as multiplying by 1.) Here the LCD is y.

$\dfrac{2+\frac{1}{y}}{3-\frac{2}{y}} = \dfrac{2+\frac{1}{y}}{3-\frac{2}{y}} \cdot 1 = \dfrac{\left(2+\frac{1}{y}\right)\cdot y}{\left(3-\frac{2}{y}\right)\cdot y}$ Multiply the numerator and denominator by y, since $\frac{y}{y} = 1$. (Step 1)

$= \dfrac{2\cdot y + \frac{1}{y}\cdot y}{3\cdot y - \frac{2}{y}\cdot y}$ Distributive property

$= \dfrac{2y+1}{3y-2}$ Simplify. (Step 2)

Compare this method with that used in Example 1(b).

Continued on Next Page

Answers

1. (a) $\dfrac{7(a+2)}{5(a-3)}$ (b) $\dfrac{2k+1}{2k-1}$ (c) $\dfrac{r(r-2)}{4}$

(b)
$$\frac{2p+\frac{5}{p-1}}{3p-\frac{2}{p}}=\frac{\left(2p+\frac{5}{p-1}\right)\cdot p(p-1)}{\left(3p-\frac{2}{p}\right)\cdot p(p-1)}$$ Multiply the numerator and denominator by the LCD, $p(p-1)$.

$$=\frac{2p[p(p-1)]+\frac{5}{p-1}\cdot p(p-1)}{3p[p(p-1)]-\frac{2}{p}\cdot p(p-1)}$$ Distributive property

$$=\frac{2p[p(p-1)]+5p}{3p[p(p-1)]-2(p-1)}$$

$$=\frac{2p^3-2p^2+5p}{3p^3-3p^2-2p+2}$$ Multiply; lowest terms

Work Problem 2 at the Side.

OBJECTIVE 3 Compare the two methods of simplifying complex fractions. Choosing whether to use Method 1 or Method 2 to simplify a complex fraction is usually a matter of preference. Some students prefer one method over the other, while other students feel comfortable with both methods and rely on practice with many examples to determine which method they will use on a particular problem.

In the next example, we illustrate how to simplify a complex fraction using both methods so that you can observe the processes and decide for yourself the pros and cons of each method.

Video

EXAMPLE 3 Simplifying Complex Fractions Using Both Methods

Use both Method 1 and Method 2 to simplify each complex fraction.

Method 1

(a)
$$\frac{\frac{2}{x-3}}{\frac{5}{x^2-9}}=\frac{\frac{2}{x-3}}{\frac{5}{(x-3)(x+3)}}=\frac{2}{x-3}\div\frac{5}{(x-3)(x+3)}=\frac{2}{x-3}\cdot\frac{(x-3)(x+3)}{5}=\frac{2(x+3)}{5}$$

Method 2

(a)
$$\frac{\frac{2}{x-3}}{\frac{5}{x^2-9}}=\frac{\frac{2}{x-3}\cdot(x-3)(x+3)}{\frac{5}{(x-3)(x+3)}\cdot(x-3)(x+3)}=\frac{2(x+3)}{5}$$

Continued on Next Page

2 Use Method 2 to simplify each complex fraction.

(a) $\dfrac{\frac{5}{y}+6}{\frac{8}{3y}-1}$

(b) $\dfrac{\frac{1}{y}+\frac{1}{y-1}}{\frac{1}{y}-\frac{2}{y-1}}$

Answers

2. (a) $\dfrac{15+18y}{8-3y}$ **(b)** $\dfrac{2y-1}{-y-1}$ or $\dfrac{1-2y}{y+1}$

3 Use both methods to simplify each complex fraction.

(a) $\dfrac{\dfrac{5}{y+2}}{\dfrac{-3}{y^2-4}}$

(b) $\dfrac{\dfrac{1}{a}-\dfrac{1}{b}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}$

Method 1

(b) $$\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} = \frac{\dfrac{y}{xy}+\dfrac{x}{xy}}{\dfrac{y^2}{x^2y^2}-\dfrac{x^2}{x^2y^2}} = \frac{\dfrac{y+x}{xy}}{\dfrac{y^2-x^2}{x^2y^2}} = \frac{y+x}{xy} \div \frac{y^2-x^2}{x^2y^2} = \frac{y+x}{xy} \cdot \frac{x^2y^2}{(y-x)(y+x)} = \frac{xy}{y-x}$$

Method 2

(b) $$\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} = \frac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2} = \frac{\left(\dfrac{1}{x}\right)x^2y^2+\left(\dfrac{1}{y}\right)x^2y^2}{\left(\dfrac{1}{x^2}\right)x^2y^2-\left(\dfrac{1}{y^2}\right)x^2y^2} = \frac{xy^2+x^2y}{y^2-x^2} = \frac{xy(y+x)}{(y+x)(y-x)} = \frac{xy}{y-x}$$

Work Problem 3 at the Side.

4 Simplify each expression, using only positive exponents in the answer.

(a) $\dfrac{r^{-2}-s^{-1}}{4r^{-1}+s^{-2}}$

(b) $\dfrac{b^{-4}}{b^{-5}+2}$

OBJECTIVE 4 **Simplify rational expressions with negative exponents.** Rational expressions and complex fractions sometimes involve negative exponents. To simplify such expressions, we begin by rewriting the expressions with only positive exponents.

EXAMPLE 4 **Simplifying a Rational Expression with Negative Exponents**

Simplify $\dfrac{m^{-1}+p^{-2}}{2m^{-2}-p^{-1}}$, using only positive exponents in the answer.

First write the expression with only positive exponents.

$$\frac{m^{-1}+p^{-2}}{2m^{-2}-p^{-1}} = \frac{\dfrac{1}{m}+\dfrac{1}{p^2}}{\dfrac{2}{m^2}-\dfrac{1}{p}} \qquad \text{Definition of negative exponent}$$

Note that the 2 in $2m^{-2}$ is *not* raised to the -2 power (since m is the base for the exponent -2), so $2m^{-2} = \dfrac{2}{m^2}$. Simplify the complex fraction using Method 2, multiplying numerator and denominator by the LCD, m^2p^2.

$$\frac{\dfrac{1}{m}+\dfrac{1}{p^2}}{\dfrac{2}{m^2}-\dfrac{1}{p}} = \frac{m^2p^2\left(\dfrac{1}{m}+\dfrac{1}{p^2}\right)}{m^2p^2\left(\dfrac{2}{m^2}-\dfrac{1}{p}\right)} = \frac{m^2p^2\cdot\dfrac{1}{m}+m^2p^2\cdot\dfrac{1}{p^2}}{m^2p^2\cdot\dfrac{2}{m^2}-m^2p^2\cdot\dfrac{1}{p}} = \frac{mp^2+m^2}{2p^2-m^2p}$$

Work Problem 4 at the Side.

Answers

3. (Both methods give the same answers.)
(a) $\dfrac{5(y-2)}{-3}$ (b) $\dfrac{ab}{b+a}$

4. (a) $\dfrac{s^2-r^2s}{4rs^2+r^2}$ (b) $\dfrac{b}{1+2b^5}$

8.4 Equations with Rational Expressions and Graphs

OBJECTIVES

1. Determine the domain of a rational equation.
2. Solve rational equations.
3. Recognize the graph of a rational function.

In **Section 8.1**, we defined the domain of a rational expression as the set of all possible values of the variable. Any value that makes the denominator 0 is excluded.

OBJECTIVE 1 Determine the domain of a rational equation. The **domain of a rational equation** is the intersection (overlap) of the domains of the rational expressions in the equation.

EXAMPLE 1 Determining the Domains of Rational Equations

Find the domain of each equation.

(a) $\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$

The domains of the three rational terms of the equation are, in order, $\{x|x \neq 0\}$, $(-\infty, \infty)$, and $\{x|x \neq 0\}$. The intersection of these three domains is all real numbers except 0, which may be written $\{x|x \neq 0\}$.

(b) $\frac{2}{x-3} - \frac{3}{x+3} = \frac{12}{x^2-9}$

The domains of these three terms are, respectively, $\{x|x \neq 3\}$, $\{x|x \neq -3\}$, and $\{x|x \neq \pm 3\}$. The domain of the equation is the intersection of the three domains, all real numbers except 3 and -3, written $\{x|x \neq \pm 3\}$.

Work Problem 1 at the Side.

1 Find the domain of each equation.

(a) $\frac{3}{x} + \frac{1}{2} = \frac{5}{6x}$

(b) $\frac{4}{x-5} - \frac{2}{x+5} = \frac{1}{x^2-25}$

OBJECTIVE 2 Solve rational equations. The easiest way to solve most equations involving rational expressions is to multiply all terms in the equation by the least common denominator. This step will clear the equation of all denominators. ***We can do this only with equations, not expressions.***

CAUTION

When each side of an equation is multiplied by a *variable* expression, the resulting "solutions" may not satisfy the original equation. ***You must either determine and observe the domain or check all potential solutions in the original equation. It is wise to do both.***

EXAMPLE 2 Solving an Equation with Rational Expressions

Solve $\frac{2}{x} - \frac{3}{2} = \frac{7}{2x}$.

The domain, which excludes 0, was found in Example 1(a).

$$2x\left(\frac{2}{x} - \frac{3}{2}\right) = 2x\left(\frac{7}{2x}\right) \qquad \text{Multiply by the LCD, } 2x.$$

$$2x\left(\frac{2}{x}\right) - 2x\left(\frac{3}{2}\right) = 2x\left(\frac{7}{2x}\right) \qquad \text{Distributive property}$$

$$4 - 3x = 7 \qquad \text{Multiply.}$$

$$-3x = 3 \qquad \text{Subtract 4.}$$

$$x = -1 \qquad \text{Divide by } -3.$$

Continued on Next Page

ANSWERS

1. (a) $\{x|x \neq 0\}$ (b) $\{x|x \neq \pm 5\}$

2 Solve $\frac{3}{20} + \frac{2}{x} = \frac{5}{4x}$.

Check: Replace x with -1 in the original equation.

$$\frac{2}{x} - \frac{3}{2} = \frac{7}{2x} \qquad \text{Original equation}$$

$$\frac{2}{-1} - \frac{3}{2} \stackrel{?}{=} \frac{7}{2(-1)} \qquad \text{Let } x = -1.$$

$$-2 - \frac{3}{2} \stackrel{?}{=} -\frac{7}{2}$$

$$-\frac{7}{2} = -\frac{7}{2} \qquad \text{True}$$

The solution set is $\{-1\}$.

Work Problem 2 at the Side.

3 Solve each equation.

(a) $\frac{3}{x+1} = \frac{1}{x-1} - \frac{2}{x^2-1}$

(b) $\frac{1}{x-3} + \frac{1}{x+3} = \frac{6}{x^2-9}$

You Try It

EXAMPLE 3 Solving an Equation with No Solution

Solve $\frac{2}{x-3} - \frac{3}{x+3} = \frac{12}{x^2-9}$.

Using the result from Example 1(b), we know that the domain excludes 3 and -3, since these values make one or more of the denominators in the equation equal 0. Multiply each side by the LCD, $(x+3)(x-3)$.

$$(x+3)(x-3)\left(\frac{2}{x-3} - \frac{3}{x+3}\right) = (x+3)(x-3)\left(\frac{12}{x^2-9}\right)$$

$$(x+3)(x-3)\left(\frac{2}{x-3}\right) - (x+3)(x-3)\left(\frac{3}{x+3}\right)$$

$$= (x+3)(x-3)\left(\frac{12}{x^2-9}\right) \qquad \text{Distributive property}$$

$$2(x+3) - 3(x-3) = 12 \qquad \text{Multiply.}$$

$$2x + 6 - 3x + 9 = 12 \qquad \text{Distributive property}$$

$$-x + 15 = 12 \qquad \text{Combine terms.}$$

$$-x = -3 \qquad \text{Subtract 15.}$$

$$x = 3 \qquad \text{Divide by } -1.$$

Since 3 is not in the domain, it cannot be a solution of the equation. Substituting 3 in the original equation shows why.

Check:

$$\frac{2}{x-3} - \frac{3}{x+3} = \frac{12}{x^2-9} \qquad \text{Original equation}$$

$$\frac{2}{3-3} - \frac{3}{3+3} \stackrel{?}{=} \frac{12}{3^2-9} \qquad \text{Let } x = 3.$$

$$\frac{2}{0} - \frac{3}{6} \stackrel{?}{=} \frac{12}{0}$$

Since division by 0 is undefined, the given equation has no solution, and the solution set is $\emptyset$.

Work Problem 3 at the Side.

Answers
2. $\{5\}$
3. (a) $\emptyset$ (b) $\emptyset$

You Try It

Animation

EXAMPLE 4 **Solving an Equation with Rational Expressions**

Solve $\dfrac{3}{p^2+p-2} - \dfrac{1}{p^2-1} = \dfrac{7}{2(p^2+3p+2)}$.

Factor each denominator to find the LCD, $2(p-1)(p+2)(p+1)$. The domain excludes 1, -2, and -1. Multiply each side by the LCD.

$$2(p-1)(p+2)(p+1)\left(\frac{3}{(p+2)(p-1)} - \frac{1}{(p+1)(p-1)}\right)$$
$$= 2(p-1)(p+2)(p+1)\left(\frac{7}{2(p+2)(p+1)}\right)$$

$2 \cdot 3(p+1) - 2(p+2) = 7(p-1)$ Distributive property

$6p + 6 - 2p - 4 = 7p - 7$ Distributive property

$4p + 2 = 7p - 7$ Combine terms

$9 = 3p$ Subtract $4p$; add 7.

$3 = p$ Divide by 3.

Note that 3 is in the domain; substitute 3 for p in the original equation to check that the solution set is $\{3\}$.

Work Problem 4 at the Side.

Animation

Video

EXAMPLE 5 **Solving an Equation That Leads to a Quadratic Equation**

Solve $\dfrac{2}{3x+1} = \dfrac{1}{x} - \dfrac{6x}{3x+1}$.

Since the denominator $3x + 1$ cannot equal 0, $-\frac{1}{3}$ is excluded from the domain, as is 0. Multiply each side by the LCD, $x(3x+1)$.

$$x(3x+1)\left(\frac{2}{3x+1}\right) = x(3x+1)\left[\frac{1}{x} - \frac{6x}{3x+1}\right]$$

$$x(3x+1)\left(\frac{2}{3x+1}\right) = x(3x+1)\left(\frac{1}{x}\right) - x(3x+1)\left(\frac{6x}{3x+1}\right)$$ Distributive property

$$2x = 3x + 1 - 6x^2$$

Write this quadratic equation in standard form with 0 on the right side.

$6x^2 - 3x + 2x - 1 = 0$

$6x^2 - x - 1 = 0$ Standard form

$(3x+1)(2x-1) = 0$ Factor.

$3x + 1 = 0$ or $2x - 1 = 0$ Zero-factor property

$x = -\dfrac{1}{3}$ or $x = \dfrac{1}{2}$

Because $-\frac{1}{3}$ is not in the domain of the equation, it is not a solution. Check that the solution set is $\{\frac{1}{2}\}$.

Work Problem 5 at the Side.

4 Solve
$$\frac{4}{x^2+x-6} - \frac{1}{x^2-4} = \frac{2}{x^2+5x+6}.$$

5 Solve
$$\frac{1}{x+4} + \frac{x}{x-4} = \frac{-8}{x^2-16}.$$

Answers

4. $\{-9\}$

5. $\{-1\}$

OBJECTIVE 3 Recognize the graph of a rational function. As mentioned in **Section 8.1,** a function defined by a rational expression is a *rational function*. Because one or more values of x may be excluded from the domain of most rational functions, their graphs are often *discontinuous*. That is, there will be one or more breaks in the graph. For example, we use point plotting and observing the domain to graph the simple rational function defined by

$$f(x) = \frac{1}{x}.$$

The domain of this function includes all real numbers except 0. Thus, there will be no point on the graph with $x = 0$. The vertical line with equation $x = 0$ is called a **vertical asymptote** of the graph. The horizontal line with equation $y = 0$ is called a **horizontal asymptote.** We show some typical ordered pairs in the table for both negative and positive x-values.

x	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-4	-10	10	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$

Notice that the closer positive values of x are to 0, the larger y is. Similarly, the closer negative values of x are to 0, the smaller (more negative) y is. Using this observation, excluding 0 from the domain, and plotting the points in the table, we obtain the graph in Figure 2.

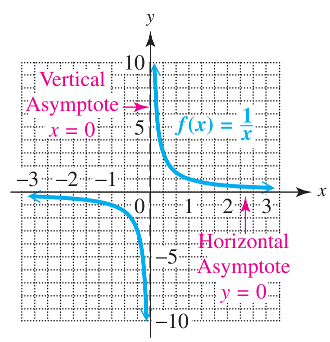


Figure 2

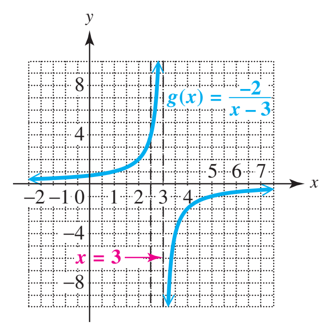


Figure 3

The graph of

$$g(x) = \frac{-2}{x - 3}$$

is shown in Figure 3. Some ordered pairs are shown in the table.

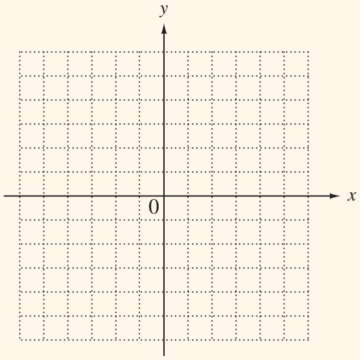
x	-2	-1	0	1	2	2.5	2.75	3.25	3.5	4	5	6
y	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{2}{3}$	1	2	4	8	-8	-4	-2	-1	$-\frac{2}{3}$

There is no point on the graph for $x = 3$ because 3 is excluded from the domain. The dashed line $x = 3$ represents the vertical asymptote and is not part of the graph. As suggested by the points from the table, the graph gets closer to the vertical asymptote as the x-values get closer to 3. Again, $y = 0$ is a horizontal asymptote.

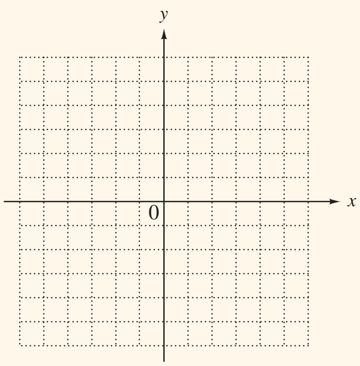
Work Problem 6 at the Side.

6 Graph each rational function, and give the equations of the vertical and horizontal asymptotes.

(a) $f(x) = -\frac{1}{x}$

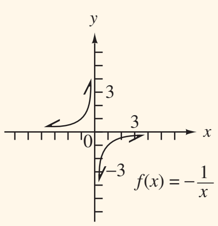


(b) $f(x) = \frac{2}{x + 3}$

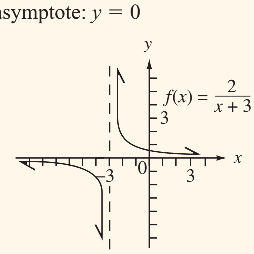


ANSWERS

6. (a) vertical asymptote: $x = 0$; horizontal asymptote: $y = 0$



(b) vertical asymptote: $x = -3$; horizontal asymptote: $y = 0$



8.5 Applications of Rational Expressions

OBJECTIVES

1. Find the value of an unknown variable in a formula.
2. Solve a formula for a specified variable.
3. Solve applications using proportions.
4. Solve applications about distance, rate, and time.
5. Solve applications about work rates.

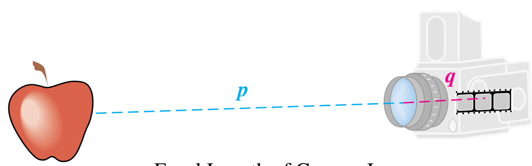
OBJECTIVE 1 Find the value of an unknown variable in a formula. In this section, we work with formulas that contain rational expressions.

EXAMPLE 1 Finding the Value of a Variable in a Formula

In physics, the focal length, f, of a lens is given by the formula

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q},$$

where p is the distance from the object to the lens and q is the distance from the lens to the image. See Figure 4. Find q if $p = 20$ cm and $f = 10$ cm.



Focal Length of Camera Lens

Figure 4

Replace f with 10 and p with 20.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{q} \qquad \text{Let } f = 10,\ p = 20.$$

$$20q \cdot \frac{1}{10} = 20q\left(\frac{1}{20} + \frac{1}{q}\right) \qquad \text{Multiply by the LCD, } 20q.$$

$$20q \cdot \frac{1}{10} = 20q\left(\frac{1}{20}\right) + 20q\left(\frac{1}{q}\right)\text{a} \qquad \text{Distributive property.}$$

$$2q = q + 20 \qquad \text{Multiply.}$$

$$q = 20 \qquad \text{Subtract } q.$$

The distance from the lens to the image is 20 cm.

Work Problem 1 at the Side.

OBJECTIVE 2 Solve a formula for a specified variable. The goal in solving for a specified variable is to isolate it on one side of the equals sign.

EXAMPLE 2 Solving a Formula for a Specified Variable

Solve $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ for p.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$fpq \cdot \frac{1}{f} = fpq\left(\frac{1}{p} + \frac{1}{q}\right) \qquad \text{Multiply by the LCD, } fpq.$$

$$pq = fq + fp \qquad \text{Distributive property}$$

Continued on Next Page

1 Use the formula given in Example 1 to answer each part.

(a) Find p if $f = 15$ and $q = 25$.

(b) Find f if $p = 6$ and $q = 9$.

(c) Find q if $f = 12$ and $p = 16$.

ANSWERS

1. (a) $\frac{75}{2}$ (b) $\frac{18}{5}$ (c) 48

2 Solve $\frac{3}{p} + \frac{3}{q} = \frac{5}{r}$ for q.

Transform the equation so that the terms with p (the specified variable) are on the same side. One way to do this is to subtract fp from each side.

$$pq = fq + fp$$

$$pq - fp = fq \qquad \text{Subtract } fp.$$

$$p(q - f) = fq \qquad \text{Factor out } p.$$

$$p = \frac{fq}{q - f} \qquad \text{Divide by } q - f.$$

Work Problem 2 at the Side.

Video

EXAMPLE 3 Solving a Formula for a Specified Variable

Solve $I = \frac{nE}{R + nr}$ for n.

$$I = \frac{nE}{R + nr}$$

$$(R + nr)I = (R + nr)\frac{nE}{R + nr} \qquad \text{Multiply by } R + nr.$$

$$RI + nrI = nE$$

$$RI = nE - nrI \qquad \text{Subtract } nrI.$$

$$RI = n(E - rI) \qquad \text{Factor out } n.$$

$$\frac{RI}{E - rI} = n \qquad \text{Divide by } E - rI.$$

3 Solve $A = \frac{Rr}{R + r}$ for R.

CAUTION

Refer to the steps in Examples 2 and 3 that factor out the desired variable. This is a step that often gives students difficulty. ***Remember that the variable for which you are solving must be a factor on only one side of the equation.*** Then each side can be divided by the remaining factor in the last step.

Work Problem 3 at the Side.

We can now solve problems that translate into equations with rational expressions. To do so, we continue to use the six-step problem-solving method from **Section 2.3.**

OBJECTIVE 3 Solve applications using proportions. A **ratio** is a comparison of two quantities. The ratio of a to b may be written in any of the following ways:

$$a \text{ to } b, \quad a : b, \quad \text{or} \quad \frac{a}{b}. \qquad \text{Ratio of } a \text{ to } b$$

Ratios are usually written as quotients in algebra. A **proportion** is a statement that two ratios are equal, such as

$$\frac{a}{b} = \frac{c}{d}. \qquad \text{Proportion}$$

Proportions are a useful and important type of rational equation.

ANSWERS

2. $q = \frac{3rp}{5p - 3r}$ or $q = \frac{-3rp}{3r - 5p}$

3. $R = \frac{-Ar}{A - r}$ or $R = \frac{Ar}{r - A}$

EXAMPLE 4 **Solving a Proportion**

In 2002, about 15 of every 100 Americans had no health insurance coverage. The population at that time was about 288 million. How many million Americans had no health insurance? (*Source:* U.S. Bureau of the Census.)

Step 1 **Read** the problem.

Step 2 **Assign a variable.** Let $x =$ the number (in millions) who had no health insurance.

Step 3 **Write an equation.** To get an equation, set up a proportion. The ratio 15 to 100 should equal the ratio x to 288.

$$\frac{15}{100} = \frac{x}{288} \qquad \text{Write a proportion.}$$

Step 4 **Solve.**

$$28{,}800\left(\frac{15}{100}\right) = 28{,}800\left(\frac{x}{288}\right) \qquad \text{Multiply by a common denominator.}$$

$$4320 = 100x \qquad \text{Simplify.}$$

$$x = 43.2 \qquad \text{Divide by 100.}$$

Step 5 **State the answer.** There were 43.2 million Americans with no health insurance in 2002.

Step 6 **Check** that the ratio of 43.2 million to 288 million equals $\frac{15}{100}$.

Work Problem 4 at the Side.

4 Solve the problem.

In 2002, approximately 11.6% (that is, 11.6 of every 100) of the 73,500,000 children under 18 yr of age in the United States had no health insurance. How many such children were uninsured? (*Source:* U.S. Bureau of the Census.)

You Try It

Video

EXAMPLE 5 **Solving a Proportion Involving Rates**

Marissa's car uses 10 gal of gas to travel 210 mi. She has 5 gal of gas in the car, and she still needs to drive 640 mi. If we assume the car continues to use gas at the same rate, how many more gallons will she need?

Step 1 **Read** the problem.

Step 2 **Assign a variable.** Let $x =$ the additional number of gallons of gas.

Step 3 **Write an equation.** To get an equation, set up a proportion.

$$\frac{\text{gallons}}{\text{miles}} \rightarrow \frac{10}{210} = \frac{5 + x}{640} \leftarrow \frac{\text{gallons}}{\text{miles}}$$

Step 4 **Solve.** The LCD is $10 \cdot 21 \cdot 64$.

$$\mathbf{10 \cdot 21 \cdot 64}\left(\frac{10}{210}\right) = \mathbf{10 \cdot 21 \cdot 64}\left(\frac{5 + x}{640}\right)$$

$$64 \cdot 10 = 21(5 + x)$$

$$640 = 105 + 21x \qquad \text{Distributive property}$$

$$535 = 21x \qquad \text{Subtract 105.}$$

$$25.5 \approx x \qquad \text{Divide by 21; round to the nearest tenth.}$$

Step 5 **State the answer.** Marissa will need about 25.5 more gallons of gas.

Step 6 **Check.** The 25.5 gal plus the 5 gal equals 30.5 gal.

$$\frac{30.5}{640} \approx \mathbf{.047} \quad \text{and} \quad \frac{10}{210} \approx \mathbf{.047}$$

Since the ratios are equal, the answer is correct.

Work Problem 5 at the Side.

5 Solve the problem.

In a recent year, the average American family spent 8.2 of every 100 dollars on health care. This amounted to \$3665 per family. To the nearest dollar, what was the average family income at that time? (*Source:* U.S. Health Care Financing Administration, U.S. Bureau of the Census.)

ANSWERS
4. 8,526,000
5. \$44,695

OBJECTIVE 4 Solve applications about distance, rate, and time. The next examples use the distance formula $d = rt$ introduced in **Section 2.2.** A familiar example of a rate is speed, which is the ratio of distance to time, or $r = \frac{d}{t}$.

You Try It

EXAMPLE 6 Solving a Problem about Distance, Rate, and Time

A tour boat goes 10 mi against the current in a small river in the same time that it goes 15 mi with the current. If the speed of the current is 3 mph, find the speed of the boat in still water.

Step 1 **Read** the problem. We must find the speed of the boat in still water.

Step 2 **Assign a variable.**

Let $x =$ the speed of the boat in still water.

When the boat is traveling *against* the current, the current slows the boat down, and the speed of the boat is the difference between its speed in still water and the speed of the current, that is, $x - 3$ mph.

When the boat is traveling *with* the current, the current speeds the boat up, and the speed of the boat is the sum of its speed in still water and the speed of the current, that is, $x + 3$ mph.

Thus, $x - 3 =$ the speed of the boat *against* the current,

and $x + 3 =$ the speed of the boat *with* the current.

Because the time is the same going against the current as with the current, find time in terms of distance and rate (speed) for each situation. Start with the distance formula,

$$d = rt,$$

and divide each side by r to get $t = \frac{d}{r}$. Going against the current, the distance is 10 mi and the rate is $x - 3$, giving

$$t = \frac{d}{r} = \frac{10}{x - 3}.$$

Going with the current, the distance is 15 mi and the rate is $x + 3$, so

$$t = \frac{d}{r} = \frac{15}{x + 3}.$$

This information is summarized in the following table.

	Distance	Rate	Time
Against Current	10	$x - 3$	$\frac{10}{x - 3}$
With Current	15	$x + 3$	$\frac{15}{x + 3}$

Times are equal.

Step 3 **Write an equation.** Because the times are equal,

$$\frac{10}{x - 3} = \frac{15}{x + 3}.$$

Continued on Next Page

Step 4 **Solve.** The LCD is $(x + 3)(x - 3)$.

$$\frac{10}{x-3} = \frac{15}{x+3}$$

$$(x+3)(x-3)\left(\frac{10}{x-3}\right) = (x+3)(x-3)\left(\frac{15}{x+3}\right) \quad \text{Multiply by the LCD.}$$

$$10(x+3) = 15(x-3) \quad \text{Multiply.}$$

$$10x + 30 = 15x - 45 \quad \text{Distributive property}$$

$$30 = 5x - 45 \quad \text{Subtract } 10x.$$

$$75 = 5x \quad \text{Add 45.}$$

$$15 = x \quad \text{Divide by 5.}$$

Step 5 **State the answer.** The speed of the boat in still water is 15 mph.

Step 6 **Check** the answer: $\frac{10}{15-3} = \frac{15}{15+3}$ is true.

Work Problem 6 at the Side.

Video

EXAMPLE 7 Solving a Problem about Distance, Rate, and Time

At O'Hare Airport, Cheryl and Bill are walking to the gate (at the same speed) to catch their flight to Akron, Ohio. Since Bill wants a window seat, he steps onto the moving sidewalk and continues to walk while Cheryl uses the stationary sidewalk. If the sidewalk moves at 1 m per sec and Bill saves 50 sec covering the 300-m distance, what is their walking speed?

Step 1 **Read** the problem. We must find their walking speed.

Step 2 **Assign a variable.** Let x represent their walking speed in meters per second. Thus Cheryl travels at x meters per second and Bill travels at $x + 1$ meters per second. Express their times in terms of the known distances and the variable rates. As in Example 6, start with $d = rt$ and divide each side by r to get $t = \frac{d}{r}$. For Cheryl, the distance is 300 m and the rate is x, so Cheryl's time is

$$t = \frac{d}{r} = \frac{300}{x}.$$

Bill travels 300 m at a rate of $x + 1$, so his time is

$$t = \frac{d}{r} = \frac{300}{x+1}.$$

This information is summarized in the following table.

	Distance	Rate	Time
Cheryl	300	x	$\frac{300}{x}$
Bill	300	$x + 1$	$\frac{300}{x+1}$

Step 3 **Write an equation** using the times from the table.

Bill's time is Cheryl's time less 50 seconds.

$$\frac{300}{x+1} = \frac{300}{x} - 50$$

Continued on Next Page

6 Solve the problem.

A plane travels 100 mi against the wind in the same time that it takes to travel 120 mi with the wind. The wind speed is 20 mph.

(a) Complete this table.

	d	r	t
Against Wind	100	$x - 20$	
With Wind	120	$x + 20$	

(b) Find the speed of the plane in still air.

Answers

6. (a) $\frac{100}{x-20}$; $\frac{120}{x+20}$ **(b)** 220 mph

7 Solve the problem.

Dona Kenly drove 300 mi north from San Antonio, mostly on the freeway. She usually averaged 55 mph, but an accident slowed her speed through Dallas to 15 mph. If her trip took 6 hr, how many miles did she drive at reduced speed?

	d	r	t
Normal Speed	$300 - x$	55	
Reduced Speed	x	15	

Step 4 **Solve.**

$$\frac{300}{x+1} = \frac{300}{x} - 50$$

$$x(x+1)\left(\frac{300}{x+1}\right) = x(x+1)\left(\frac{300}{x} - 50\right) \quad \text{Multiply by the LCD, } x(x+1).$$

$$x(x+1)\left(\frac{300}{x+1}\right) = x(x+1)\left(\frac{300}{x}\right) - x(x+1)(50) \quad \text{Distributive property}$$

$$300x = 300(x+1) - 50x(x+1) \quad \text{Multiply.}$$

$$300x = 300x + 300 - 50x^2 - 50x \quad \text{Distributive property}$$

$$50x^2 + 50x - 300 = 0 \quad \text{Standard form}$$

$$x^2 + x - 6 = 0 \quad \text{Divide by 50.}$$

$$(x+3)(x-2) = 0 \quad \text{Factor.}$$

$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{Zero-factor property}$$

$$x = -3 \quad \text{or} \quad x = 2$$

Discard the negative answer, since speed cannot be negative.

Step 5 **State the answer.** Their walking speed is 2 m per sec.

Step 6 **Check** the answer in the words of the original problem.

Work Problem 7 at the Side.

OBJECTIVE 5 Solve applications about work rates. Problems about work are closely related to distance problems.

Audio

PROBLEM-SOLVING HINT

People work at different rates. If the letters r, t, and A represent the rate at which the work is done, the time required, and the amount of work accomplished, respectively, then $A = rt$. Notice the similarity to the distance formula, $d = rt$.

Amount of work can be measured in terms of jobs accomplished. Thus, if 1 job is completed, $A = 1$, and the formula gives the rate as

$$1 = rt$$

$$r = \frac{1}{t}.$$

To solve a work problem, we begin by using the following fact to express all rates of work.

Audio

Rate of Work

If a job can be accomplished in t units of time, then the rate of work is

$$\frac{1}{t} \textbf{ job per unit of time.}$$

See if you can identify the six problem-solving steps in the next example.

ANSWERS

7. $11\frac{1}{4}$ mi

EXAMPLE 8 Solving a Problem about Work

Letitia and Kareem are working on a neighborhood cleanup. Kareem can clean up all the trash in the area in 7 hr, while Letitia can do the same job in 5 hr. How long will it take them if they work together?

Let x = the number of hours it will take the two people working together. Just as we made a table for the distance formula, $d = rt$, make a table here for $A = rt$, with $A = 1$. Since $A = 1$, the rate for each person will be $\frac{1}{t}$, where t is the time it takes the person to complete the job alone. For example, since Kareem can clean up all the trash in 7 hr, his rate is $\frac{1}{7}$ of the job per hour. Similarly, Letitia's rate is $\frac{1}{5}$ of the job per hour.

	Rate	Time Working Together	Fractional Part of the Job Done
Kareem	$\frac{1}{7}$	x	$\frac{1}{7}x$
Letitia	$\frac{1}{5}$	x	$\frac{1}{5}x$

Since together they complete 1 job, the sum of the fractional parts accomplished by them should equal 1.

$$\underbrace{\frac{1}{7}x}_{\text{Part done by Kareem}} \underbrace{+}_{+} \underbrace{\frac{1}{5}x}_{\text{Part done by Letitia}} \underbrace{=}_{\text{is}} \underbrace{1}_{\text{1 whole job.}}$$

$$35\left(\frac{1}{7}x + \frac{1}{5}x\right) = 35 \cdot 1 \qquad \text{The LCD is 35.}$$

$$5x + 7x = 35$$

$$12x = 35$$

$$x = \frac{35}{12}$$

Working together, Kareem and Letitia can do the entire job in $\frac{35}{12}$ hr, or 2 hr and 55 min. Check this result in the original problem.

Work Problem 8 at the Side.

There is another way to approach problems about work. For instance, in Example 8, x represents the number of hours it will take the two people working together to complete the entire job. In one hour, $\frac{1}{x}$ of the entire job will be completed. Kareem completes $\frac{1}{7}$ of the job in one hour, and Letitia completes $\frac{1}{5}$ of the job, so the sum of their rates should equal $\frac{1}{x}$. Thus,

$$\frac{1}{7} + \frac{1}{5} = \frac{1}{x}.$$

Multiplying each side of this equation by $35x$ gives $5x + 7x = 35$. This is the same equation we got in Example 8 in the third line from the bottom. Thus the solution of the equation is the same using either approach.

8 Solve each problem.

(a) Stan needs 45 min to do the dishes, while Deb can do them in 30 min. How long will it take them if they work together?

	Rate	Time Working Together	Fractional Part of the Job Done
Stan	$\frac{1}{45}$	x	
Deb	$\frac{1}{30}$	x	

(b) Suppose it takes Stan 35 min to do the dishes, and together they can do them in 15 min. How long will it take Deb to do them alone?

ANSWERS

8. **(a)** 18 min **(b)** $26\frac{1}{4}$ min

Roots, Radicals, and Root Functions

9

Tom Skilling is the chief meteorologist for the *Chicago Tribune*. He writes a column titled "Ask Tom Why," where readers question him on a variety of topics. Reader Ted Fleischaker wrote: "I cannot remember the formula to calculate the distance to the horizon. I have a stunning view from my 14th floor condo, 150 feet above the ground. How far can I see?" (See Exercise 125 in Section 9.3.)

In Skilling's answer, he explained the formula for finding the distance d to the horizon in miles,

$$d = 1.224\sqrt{h},$$

where h is the height in feet. Square roots such as this one are often found in formulas. This chapter deals with roots and radicals.

9.1 Radical Expressions and Graphs

OBJECTIVES

1. Find roots of numbers.
2. Find principal roots.
3. Graph functions defined by radical expressions.
4. Find *n*th roots of *n*th powers.
5. Use a calculator to find roots.

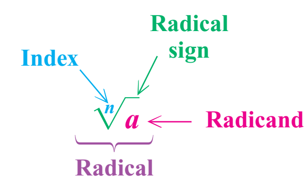
OBJECTIVE 1 Find roots of numbers. In **Section 1.3** we found square roots of positive numbers such as

$$\sqrt{36} = 6, \text{ because } 6 \cdot 6 = 36 \quad \text{and} \quad \sqrt{144} = 12, \text{ because } 12 \cdot 12 = 144.$$

We now extend our discussion of roots to cube roots, fourth roots, and higher roots. In general, $\sqrt[n]{a}$ is a number whose nth power equals a. That is,

$$\sqrt[n]{a} = b \quad \textbf{means} \quad b^n = a.$$

The number a is the **radicand,** n is the **index** or **order,** and the expression $\sqrt[n]{a}$ is a **radical.**



EXAMPLE 1 Simplifying Higher Roots

Simplify.

(a) $\sqrt[3]{27} = 3$, because $3^3 = 27$.

(b) $\sqrt[3]{125} = 5$, because $5^3 = 125$.

(c) $\sqrt[4]{16} = 2$, because $2^4 = 16$.

(d) $\sqrt[5]{32} = 2$, because $2^5 = 32$.

Work Problem 1 at the Side.

1 Simplify.

(a) $\sqrt[3]{8}$

(b) $\sqrt[3]{1000}$

(c) $\sqrt[4]{81}$

(d) $\sqrt[6]{64}$

OBJECTIVE 2 Find principal roots. If n is even, positive numbers have two nth roots. For example, both 4 and -4 are square roots of 16, and 2 and -2 are fourth roots of 16. In such cases, the notation $\sqrt[n]{a}$ represents the positive root, called the **principal root.**

*n*th Root

If n is ***even*** and a is ***positive*** or 0, then

$\sqrt[n]{a}$ represents the principal nth root of a, and

$-\sqrt[n]{a}$ represents the negative nth root of a.

If n is ***even*** and a is ***negative,*** then

$\sqrt[n]{a}$ is not a real number.

If n is ***odd,*** then

there is exactly one nth root of a, written $\sqrt[n]{a}$.

If n is even, then the two nth roots of a are often written together as $\pm\sqrt[n]{a}$, with $\pm$ read "positive or negative."

ANSWERS

1. (a) 2 **(b)** 10 **(c)** 3 **(d)** 2

EXAMPLE 2 **Finding Roots**

Find each root.

(a) $\sqrt{100} = 10$

Because the radicand is positive, there are two square roots, 10 and -10. We want the principal root, which is 10.

(b) $-\sqrt{100} = -10$

Here, we want the negative square root, -10.

(c) $\sqrt[4]{81} = 3$

(d) $\sqrt[6]{-64}$

The index is even and the radicand is negative, so this is not a real number.

(e) $\sqrt[3]{-8} = -2$, because $(-2)^3 = -8$.

Work Problem 2 at the Side.

OBJECTIVE 3 **Graph functions defined by radical expressions.** A **radical expression** is an algebraic expression that contains radicals. For example,

$$3 - \sqrt{x}, \quad \sqrt[3]{x}, \quad \text{and} \quad \sqrt{2x - 1} \qquad \text{Radical expressions}$$

are radical expressions.

In earlier chapters we graphed functions defined by polynomial and rational expressions. Now we examine the graphs of functions defined by the radical expressions $f(x) = \sqrt{x}$ and $f(x) = \sqrt[3]{x}$.

Figure 1 shows the graph of the **square root function** with a table of selected points.

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3

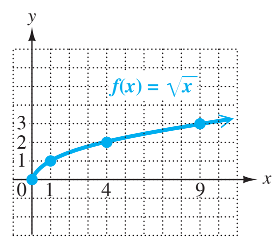


Figure 1

Only nonnegative values can be used for x, so the domain is $[0, \infty)$. Because $\sqrt{x}$ is the principal square root of x, it always has a nonnegative value, so the range is also $[0, \infty)$.

Figure 2 shows the graph of the **cube root function** and a table of selected points.

x	$f(x) = \sqrt[3]{x}$
-8	-2
-1	-1
0	0
1	1
8	2

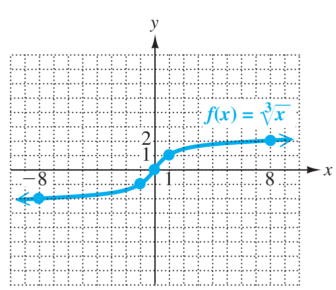


Figure 2

Since any real number (positive, negative, or 0) can be used for x in the cube root function, $\sqrt[3]{x}$ can be positive, negative, or 0. Thus both the domain and the range of the cube root function are $(-\infty, \infty)$.

2 Find each root.

(a) $\sqrt{4}$

(b) $\sqrt[3]{27}$

(c) $-\sqrt{36}$

(d) $\sqrt[4]{625}$

(e) $\sqrt[5]{-32}$

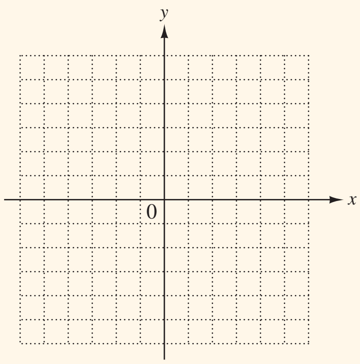
(f) $\sqrt[4]{-16}$

ANSWERS

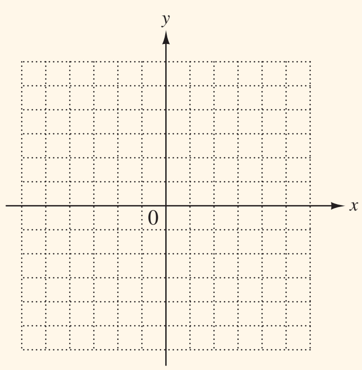
2. **(a)** 2 **(b)** 3 **(c)** -6 **(d)** 5 **(e)** -2 **(f)** not a real number

3 Graph each function by creating a table of values. Give the domain and range.

(a) $f(x) = \sqrt{x} + 2$



(b) $f(x) = \sqrt[3]{x} - 1$



EXAMPLE 3 Graphing Functions Defined with Radicals

Graph each function by creating a table of values. Give the domain and the range.

(a) $f(x) = \sqrt{x - 3}$

A table of values is shown. The x-values were chosen in such a way that the function values are all integers. For the radicand to be nonnegative, we must have $x - 3 \geq 0$, or $x \geq 3$. Therefore, the domain is $[3, \infty)$. Again, function values are positive or 0, so the range is $[0, \infty)$. The graph is shown in Figure 3.

x	$f(x) = \sqrt{x-3}$
3	$\sqrt{3-3} = 0$
4	$\sqrt{4-3} = 1$
7	$\sqrt{7-3} = 2$

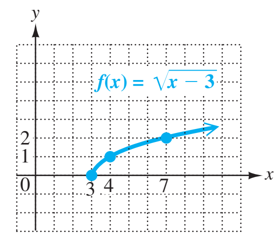


Figure 3

(b) $f(x) = \sqrt[3]{x} + 2$

See the table and Figure 4. Both the domain and the range are $(-\infty\ \infty)$.

x	$f(x) = \sqrt[3]{x} + 2$
-8	$\sqrt[3]{-8} + 2 = 0$
-1	$\sqrt[3]{-1} + 2 = 1$
0	$\sqrt[3]{0} + 2 = 2$
1	$\sqrt[3]{1} + 2 = 3$
8	$\sqrt[3]{8} + 2 = 4$

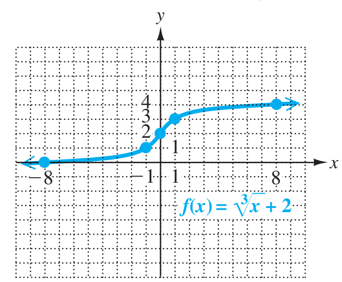


Figure 4

Work Problem 3 at the Side.

OBJECTIVE 4 Find *n*th roots of *n*th powers. A square root of a^2 (where $a \neq 0$) is a number that can be squared to give a^2. This number is either a or $-a$. Since the symbol $\sqrt{a^2}$ represents the *nonnegative* square root, we must write $\sqrt{a^2}$ with absolute value bars, as $|a|$, because a may be a negative number.

> $\sqrt{a^2}$
>
> For any real number a, $\quad \sqrt{a^2} = |a|$.

EXAMPLE 4 Simplifying Square Roots Using Absolute Value

Find each square root that is a real number.

(a) $\sqrt{7^2} = |7| = 7$

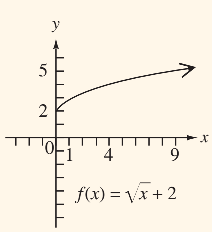
(b) $\sqrt{(-7)^2} = |-7| = 7$

(c) $\sqrt{k^2} = |k|$

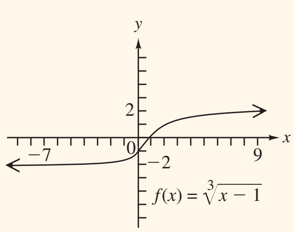
(d) $\sqrt{(-k)^2} = |-k| = |k|$

ANSWERS

3. **(a)** domain: $[0, \infty)$; range: $[2, \infty)$



(b) domain: $(-\infty, \infty)$; range: $(-\infty, \infty)$



Work Problem 4 at the Side.

We can generalize this idea to any nth root.

$\sqrt[n]{a^n}$

If n is an *even* positive integer, $\sqrt[n]{a^n} = |a|$,

and if n is an *odd* positive integer, $\sqrt[n]{a^n} = a$.

In words, use absolute value when n is even; do not use absolute value when n is odd.

EXAMPLE 5 Simplifying Higher Roots Using Absolute Value

Simplify each root.

(a) $\sqrt[6]{(-3)^6} = |-3| = 3$ $\quad$ n is even; use absolute value.

(b) $\sqrt[5]{(-4)^5} = -4$ $\quad$ n is odd.

(c) $-\sqrt[4]{(-9)^4} = -|-9| = -9$

(d) $\sqrt[3]{\frac{8}{27}} = \sqrt[3]{\left(\frac{2}{3}\right)^3} = \frac{2}{3}$

(e) $-\sqrt{m^4} = -|m^2| = -m^2$

No absolute value bars are needed here because m^2 is nonnegative for any real number value of m.

(f) $\sqrt[3]{a^{12}} = a^4$, because $a^{12} = (a^4)^3$.

(g) $\sqrt[4]{x^{12}} = |x^3|$

We use absolute value bars to guarantee that the result is not negative (because x^3 can be either positive or negative, depending on x). If desired, $|x^3|$ can be written as $x^2 \cdot |x|$.

Work Problem 5 at the Side.

OBJECTIVE 5 Use a calculator to find roots. While numbers such as $\sqrt{9}$ and $\sqrt[3]{-8}$ are rational, radicals are often irrational numbers. To find approximations of roots such as $\sqrt{15}$, $\sqrt[3]{10}$, and $\sqrt[4]{2}$, we usually use scientific or graphing calculators. Using a calculator, we find

$$\sqrt{15} \approx 3.872983346, \quad \sqrt[3]{10} \approx 2.15443469, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189207115,$$

where the symbol $\approx$ means "is approximately equal to." In this book we will usually show approximations rounded to three decimal places. Thus, we would write

$$\sqrt{15} \approx 3.873, \quad \sqrt[3]{10} \approx 2.154, \quad \text{and} \quad \sqrt[4]{2} \approx 1.189.$$

Calculator Tip The methods for finding approximations differ among makes and models, and you should always consult your owner's manual for keystroke instructions. Be aware that graphing calculators often differ from scientific calculators in the order in which keystrokes are made.

4 Find each square root that is a real number.

(a) $\sqrt{49}$

(b) $-\sqrt{\frac{36}{25}}$

(c) $\sqrt{(-6)^2}$

(d) $\sqrt{r^2}$

5 Simplify.

(a) $\sqrt[6]{64}$

(b) $-\sqrt[4]{16}$

(c) $\sqrt[3]{\frac{216}{125}}$

(d) $\sqrt[5]{-243}$

(e) $\sqrt[6]{(-p)^6}$

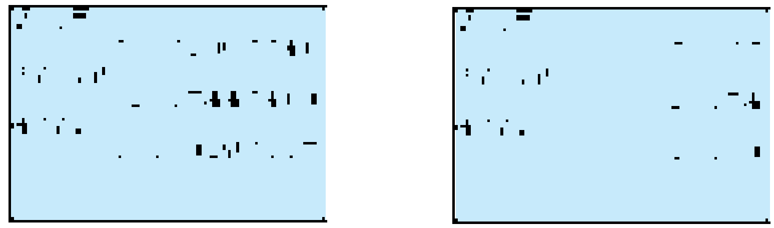
(f) $-\sqrt[6]{y^{24}}$

Answers

4. **(a)** 7 **(b)** $-\frac{6}{5}$ **(c)** 6 **(d)** $|r|$

5. **(a)** 2 **(b)** -2 **(c)** $\frac{6}{5}$ **(d)** -3 **(e)** $|p|$ **(f)** $-y^4$

Figure 5 shows how the preceding approximations are displayed on a TI-83 Plus or TI-84 graphing calculator. In Figure 5(a), eight or nine decimal places are shown, while in Figure 5(b), the number of decimal places is fixed at three.



(a) (b)

Figure 5

There is a simple way to check that a calculator approximation is "in the ballpark." Because 16 is a little larger than 15, $\sqrt{16} = 4$ should be a little larger than $\sqrt{15}$. Thus, 3.873 is a reasonable approximation for $\sqrt{15}$.

EXAMPLE 6 Finding Approximations for Roots

Use a calculator to verify that each approximation is correct.

(a) $\sqrt{39} \approx 6.245$ **(b)** $-\sqrt{72} \approx -8.485$

(c) $\sqrt[3]{93} \approx 4.531$ **(d)** $\sqrt[4]{39} \approx 2.499$

Work Problem 6 at the Side.

6 Use a calculator to approximate each radical to three decimal places.

(a) $\sqrt{17}$

(b) $-\sqrt{362}$

(c) $\sqrt[3]{9482}$

(d) $\sqrt[4]{6825}$

You Try It

EXAMPLE 7 Using Roots to Calculate Resonant Frequency

In electronics, the resonant frequency f of a circuit may be found by the formula

$$f = \frac{1}{2\pi\sqrt{LC}},$$

where f is in cycles per second, L is in henrys, and C is in farads.* Find the resonant frequency f if $L = 5 \times 10^{-4}$ henrys and $C = 3 \times 10^{-10}$ farads. Give your answer to the nearest thousand.

Find the value of f when $L = 5 \times 10^{-4}$ and $C = 3 \times 10^{-10}$.

$$f = \frac{1}{2\pi\sqrt{LC}} \qquad \text{Given formula}$$

$$= \frac{1}{2\pi\sqrt{(5 \times 10^{-4})(3 \times 10^{-10})}} \qquad \text{Substitute for } L \text{ and } C.$$

$$\approx 411{,}000 \qquad \text{Use a calculator.}$$

The resonant frequency f is approximately 411,000 cycles per sec.

Work Problem 7 at the Side.

7 Use the formula in Example 7 to approximate f to the nearest thousand if

$$L = 6 \times 10^{-5}$$
and
$$C = 4 \times 10^{-9}.$$

*Henrys and farads are units of measure in electronics.

Answers

6. **(a)** 4.123 **(b)** −19.026 **(c)** 21.166 **(d)** 9.089
7. 325,000 cycles per sec

9.2 Rational Exponents

OBJECTIVE 1 Use exponential notation for *n*th roots. In mathematics we often formulate definitions so that previous rules remain valid. In **Section 6.1** we defined 0 as an exponent in such a way that the rules for products, quotients, and powers would still be valid. Now we look at exponents that are rational numbers of the form $\frac{1}{n}$, where n is a natural number.

For the rules of exponents to remain valid, the product $(3^{1/2})^2 = 3^{1/2} \cdot 3^{1/2}$ should be found by adding exponents.

$$\begin{aligned}(3^{1/2})^2 &= 3^{1/2} \cdot 3^{1/2} \\ &= 3^{1/2 + 1/2} \\ &= 3^1 \\ &= 3\end{aligned}$$

However, by definition $(\sqrt{3})^2 = \sqrt{3} \cdot \sqrt{3} = 3$. Since both $(3^{1/2})^2$ and $(\sqrt{3})^2$ are equal to 3, we must have

$$3^{1/2} = \sqrt{3}.$$

This suggests the following generalization.

Audio

$a^{1/n}$

If $\sqrt[n]{a}$ is a real number, then

$$a^{1/n} = \sqrt[n]{a}.$$

You Try It

EXAMPLE 1 Evaluating Exponentials of the Form $a^{1/n}$

Evaluate each expression.

(a) $64^{1/3} = \sqrt[3]{64} = 4$

(b) $100^{1/2} = \sqrt{100} = 10$

(c) $-256^{1/4} = -\sqrt[4]{256} = -4$

(d) $(-256)^{1/4} = \sqrt[4]{-256}$ is not a real number because the radicand, -256, is negative and the index is even.

(e) $(-32)^{1/5} = \sqrt[5]{-32} = -2$

(f) $\left(\frac{1}{8}\right)^{1/3} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

CAUTION

Notice the difference between parts (c) and (d) in Example 1. The radical in part (c) is the ***negative fourth root*** of a positive number, while the radical in part (d) is the ***principal fourth root of a negative number, which is not a real number.***

Work Problem 1 at the Side.

OBJECTIVES

1. Use exponential notation for *n*th roots.
2. Define $a^{m/n}$.
3. Convert between radicals and rational exponents.
4. Use the rules for exponents with rational exponents.

1 Evaluate each exponential.

(a) $8^{1/3}$

(b) $9^{1/2}$

(c) $-81^{1/4}$

(d) $(-16)^{1/4}$

(e) $64^{1/3}$

(f) $\left(\frac{1}{32}\right)^{1/5}$

ANSWERS

1. **(a)** 2 **(b)** 3 **(c)** -3 **(d)** not a real number **(e)** 4 **(f)** $\frac{1}{2}$

2 Evaluate each exponential.

(a) $64^{2/3}$

(b) $100^{3/2}$

(c) $-16^{3/4}$

(d) $(-16)^{3/4}$

OBJECTIVE 2 **Define $a^{m/n}$.** We now define a number like $8^{2/3}$. For past rules of exponents to be valid,

$$8^{2/3} = 8^{(1/3)2} = (8^{1/3})^2.$$

Since $8^{1/3} = \sqrt[3]{8}$,

$$8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4.$$

Generalizing from this example, we define $a^{m/n}$ as follows.

Audio

$a^{m/n}$

If m and n are positive integers with m/n in lowest terms, then

$$a^{m/n} = (a^{1/n})^m,$$

provided that $a^{1/n}$ is a real number. If $a^{1/n}$ is not a real number, then $a^{m/n}$ is not a real number.

Video You Try It

EXAMPLE 2 **Evaluating Exponentials of the Form $a^{m/n}$**

Evaluate each exponential.

(a) $36^{3/2} = (36^{1/2})^3 = 6^3 = 216$

(b) $125^{2/3} = (125^{1/3})^2 = 5^2 = 25$

(c) $-4^{5/2} = -(4^{5/2}) = -(4^{1/2})^5 = -(2)^5 = -32$

(d) $(-27)^{2/3} = [(-27)^{1/3}]^2 = (-3)^2 = 9$

Notice how the $-$ sign is used in parts (c) and (d). In part (c), we first evaluate the exponential and then find its negative. In part (d), the $-$ sign is part of the base, -27.

(e) $(-100)^{3/2}$ is not a real number, since $(-100)^{1/2}$ is not a real number.

Work Problem 2 at the Side.

EXAMPLE 3 **Evaluating Exponentials with Negative Rational Exponents**

Evaluate each exponential.

(a) $16^{-3/4}$

By the definition of a negative exponent,

$$16^{-3/4} = \frac{1}{16^{3/4}}.$$

Since $16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$,

$$16^{-3/4} = \frac{1}{16^{3/4}} = \frac{1}{8}.$$

(b) $25^{-3/2} = \dfrac{1}{25^{3/2}} = \dfrac{1}{(\sqrt{25})^3} = \dfrac{1}{5^3} = \dfrac{1}{125}$

Continued on Next Page

ANSWERS

2. **(a)** 16 **(b)** 1000 **(c)** -8 **(d)** not a real number

(c) $\left(\frac{8}{27}\right)^{-2/3} = \frac{1}{\left(\frac{8}{27}\right)^{2/3}} = \frac{1}{\left(\sqrt[3]{\frac{8}{27}}\right)^2} = \frac{1}{\left(\frac{2}{3}\right)^2} = \frac{1}{\frac{4}{9}} = \frac{9}{4}$

We could also use the rule $\left(\frac{b}{a}\right)^{-m} = \left(\frac{a}{b}\right)^m$ here, as follows.

$$\left(\frac{8}{27}\right)^{-2/3} = \left(\frac{27}{8}\right)^{2/3} = \left(\sqrt[3]{\frac{27}{8}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

CAUTION

When using the rule in Example 3(c), we take the reciprocal only of the base, *not* the exponent. Also, be careful to distinguish between exponential expressions like $-16^{1/4}$, $16^{-1/4}$, and $-16^{-1/4}$.

$$-16^{1/4} = -2, \quad 16^{-1/4} = \frac{1}{2}, \quad \text{and} \quad -16^{-1/4} = -\frac{1}{2}.$$

Work Problem 3 at the Side.

We get an alternative definition of $a^{m/n}$ by using the power rule for exponents a little differently than in the earlier definition. If all indicated roots are real numbers, then

$$a^{m/n} = a^{m(1/n)} = (a^m)^{1/n},$$

so

$$a^{m/n} = (a^m)^{1/n}.$$

Audio

$a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.$$

We can now evaluate an expression such as $27^{2/3}$ in two ways:

$$27^{2/3} = (27^{1/3})^2 = 3^2 = 9$$

or

$$27^{2/3} = (27^2)^{1/3} = 729^{1/3} = 9.$$

In most cases, it is easier to use $(a^{1/n})^m$.

This rule can also be expressed with radicals as follows.

Audio

Radical Form of $a^{m/n}$

If all indicated roots are real numbers, then

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

In words, we can raise to the power and then take the root, or take the root and then raise to the power.

For example,

$$8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4, \quad \text{and} \quad 8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4,$$

so

$$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2.$$

3 Evaluate each exponential.

(a) $36^{-3/2}$

(b) $32^{-4/5}$

(c) $\left(\frac{4}{9}\right)^{-5/2}$

ANSWERS

3. (a) $\frac{1}{216}$ **(b)** $\frac{1}{16}$ **(c)** $\frac{243}{32}$

OBJECTIVE 3 Convert between radicals and rational exponents. Using the definition of rational exponents, we can simplify many problems involving radicals by converting the radicals to numbers with rational exponents. After simplifying, we convert the answer back to radical form.

EXAMPLE 4 Converting between Rational Exponents and Radicals

Write each exponential as a radical. Assume that all variables represent positive real numbers. Use the definition that takes the root first.

(a) $13^{1/2} = \sqrt{13}$

(b) $6^{3/4} = (\sqrt[4]{6})^3$

(c) $9m^{5/8} = 9(\sqrt[8]{m})^5$

(d) $6x^{2/3} - (4x)^{3/5} = 6(\sqrt[3]{x})^2 - (\sqrt[5]{4x})^3$

(e) $r^{-2/3} = \dfrac{1}{r^{2/3}} = \dfrac{1}{(\sqrt[3]{r})^2}$

(f) $(a^2 + b^2)^{1/2} = \sqrt{a^2 + b^2}$ Note that $\sqrt{a^2 + b^2} \neq a + b$.

In (g)–(i), write each radical as an exponential. Simplify. Assume that all variables represent positive real numbers.

(g) $\sqrt{10} = 10^{1/2}$

(h) $\sqrt[4]{3^8} = 3^{8/4} = 3^2 = 9$

(i) $\sqrt[6]{z^6} = z$, since z is positive.

Work Problems 4 and 5 at the Side.

OBJECTIVE 4 Use the rules for exponents with rational exponents. The definition of rational exponents allows us to apply the rules for exponents first introduced in **Section 6.1.**

Rules for Rational Exponents

Let r and s be rational numbers. For all real numbers a and b for which the indicated expressions exist:

$$a^r \cdot a^s = a^{r+s} \qquad a^{-r} = \frac{1}{a^r} \qquad \frac{a^r}{a^s} = a^{r-s} \qquad \left(\frac{a}{b}\right)^{-r} = \frac{b^r}{a^r}$$

$$(a^r)^s = a^{rs} \qquad (ab)^r = a^r b^r \qquad \left(\frac{a}{b}\right)^r = \frac{a^r}{b^r} \qquad a^{-r} = \left(\frac{1}{a}\right)^r.$$

EXAMPLE 5 Applying Rules for Rational Exponents

Write with only positive exponents. Assume that all variables represent positive real numbers.

(a) $2^{1/2} \cdot 2^{1/4} = 2^{1/2 + 1/4} = 2^{3/4}$ Product rule

(b) $\dfrac{5^{2/3}}{5^{7/3}} = 5^{2/3 - 7/3} = 5^{-5/3} = \dfrac{1}{5^{5/3}}$ Quotient rule

Continued on Next Page

4 Write each exponential as a radical. Assume that all variables represent positive real numbers. Use the definition that takes the root first.

(a) $5^{2/3}$

(b) $4k^{3/5}$

(c) $(7r)^{4/3}$

(d) $(m^3 + n^3)^{1/3}$

5 Write each radical as an exponential and simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt{y^{10}}$

(b) $\sqrt[3]{27y^9}$

(c) $\sqrt[4]{t^4}$

ANSWERS

4. **(a)** $(\sqrt[3]{5})^2$ **(b)** $4(\sqrt[5]{k})^3$ **(c)** $(\sqrt[3]{7r})^4$ **(d)** $\sqrt[3]{m^3 + n^3}$

5. **(a)** y^5 **(b)** $3y^3$ **(c)** t

(c) $\dfrac{(x^{1/2}y^{2/3})^4}{y} = \dfrac{(x^{1/2})^4(y^{2/3})^4}{y}$ Power rule

$= \dfrac{x^2y^{8/3}}{y^1}$ Power rule

$= x^2y^{8/3-1}$ Quotient rule

$= x^2y^{5/3}$

(d) $\left(\dfrac{x^4y^{-6}}{x^{-2}y^{1/3}}\right)^{-2/3} = \dfrac{(x^4)^{-2/3}(y^{-6})^{-2/3}}{(x^{-2})^{-2/3}(y^{1/3})^{-2/3}}$

$= \dfrac{x^{-8/3}y^4}{x^{4/3}y^{-2/9}}$ Power rule

$= x^{-8/3-4/3}y^{4-(-2/9)}$ Quotient rule

$= x^{-4}y^{38/9}$

$= \dfrac{y^{38/9}}{x^4}$ Definition of negative exponent

The same result is obtained if we simplify within the parentheses first, leading to $(x^6y^{-19/3})^{-2/3}$. Then, apply the power rule. (Show that the result is the same.)

(e) $m^{3/4}(m^{5/4} - m^{1/4}) = m^{3/4} \cdot m^{5/4} - m^{3/4} \cdot m^{1/4}$ Distributive property

$= m^{3/4+5/4} - m^{3/4+1/4}$ Product rule

$= m^{8/4} - m^{4/4}$

$= m^2 - m$

Do not make the common mistake of multiplying exponents in the first step.

Work Problem 6 at the Side.

CAUTION
Use the rules of exponents in problems like those in Example 5. Do not convert the expressions to radical form.

EXAMPLE 6 Applying Rules for Rational Exponents

Rewrite all radicals as exponentials, and then apply the rules for rational exponents. Leave answers in exponential form. Assume that all variables represent positive real numbers.

(a) $\sqrt[3]{x^2} \cdot \sqrt[4]{x} = x^{2/3} \cdot x^{1/4}$ Convert to rational exponents.

$= x^{2/3+1/4}$ Product rule

$= x^{8/12+3/12}$ Write exponents with a common denominator.

$= x^{11/12}$

(b) $\dfrac{\sqrt{x^3}}{\sqrt[3]{x^2}} = \dfrac{x^{3/2}}{x^{2/3}} = x^{3/2-2/3} = x^{5/6}$

(c) $\sqrt{\sqrt[4]{z}} = \sqrt{z^{1/4}} = (z^{1/4})^{1/2} = z^{1/8}$

Work Problem 7 at the Side.

6 Write with only positive exponents. Assume that all variables represent positive real numbers.

(a) $11^{3/4} \cdot 11^{5/4}$

(b) $\dfrac{7^{3/4}}{7^{7/4}}$

(c) $\dfrac{9^{2/3}(x^{1/3})^4}{9^{-1/3}}$

(d) $\left(\dfrac{a^3b^{-4}}{a^{-2}b^{1/5}}\right)^{-1/2}$

(e) $a^{2/3}(a^{7/3} + a^{1/3})$

7 Simplify using the rules for rational exponents. Assume that all variables represent positive real numbers. Leave answers in exponential form.

(a) $\sqrt[5]{m^3} \cdot \sqrt{m}$

(b) $\dfrac{\sqrt[3]{p^5}}{\sqrt{p^3}}$

(c) $\sqrt[4]{\sqrt[3]{x}}$

Answers

6. (a) 11^2 or 121 (b) $\frac{1}{7}$ (c) $9x^{4/3}$ (d) $\dfrac{b^{21/10}}{a^{5/2}}$ (e) $a^3 + a$

7. (a) $m^{11/10}$ (b) $p^{1/6}$ (c) $x^{1/12}$

9.3 Simplifying Radical Expressions

OBJECTIVES

1. Use the product rule for radicals.
2. Use the quotient rule for radicals.
3. Simplify radicals.
4. Simplify products and quotients of radicals with different indexes.
5. Use the Pythagorean formula.
6. Use the distance formula.

OBJECTIVE 1 Use the product rule for radicals. We now develop rules for multiplying and dividing radicals that have the same index. For example, is the product of two nth-root radicals equal to the nth root of the product of the radicands? For example, are $\sqrt{36 \cdot 4}$ and $\sqrt{36} \cdot \sqrt{4}$ equal?

$$\sqrt{36 \cdot 4} = \sqrt{144} = 12$$
$$\sqrt{36} \cdot \sqrt{4} = 6 \cdot 2 = 12$$

Notice that in both cases the result is the same. This is an example of the **product rule for radicals.**

Audio

Product Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a natural number, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}.$$

In words, the product of two radicals is the radical of the product.

We justify the product rule using the rules for rational exponents. Since $\sqrt[n]{a} = a^{1/n}$ and $\sqrt[n]{b} = b^{1/n}$,

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n} \cdot b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}.$$

CAUTION
Use the product rule only when the radicals have the same indexes.

EXAMPLE 1 Using the Product Rule

Multiply. Assume that all variables represent positive real numbers.

(a) $\sqrt{5} \cdot \sqrt{7} = \sqrt{5 \cdot 7} = \sqrt{35}$

(b) $\sqrt{2} \cdot \sqrt{19} = \sqrt{2 \cdot 19} = \sqrt{38}$

(c) $\sqrt{11} \cdot \sqrt{p} = \sqrt{11p}$

(d) $\sqrt{7} \cdot \sqrt{11xyz} = \sqrt{77xyz}$

Work Problem 1 at the Side.

EXAMPLE 2 Using the Product Rule

Multiply. Assume that all variables represent positive real numbers.

(a) $\sqrt[3]{3} \cdot \sqrt[3]{12} = \sqrt[3]{3 \cdot 12} = \sqrt[3]{36}$

(b) $\sqrt[4]{8y} \cdot \sqrt[4]{3r^2} = \sqrt[4]{24yr^2}$

(c) $\sqrt[6]{10m^4} \cdot \sqrt[6]{5m} = \sqrt[6]{50m^5}$

(d) $\sqrt[4]{2} \cdot \sqrt[5]{2}$ cannot be simplified using the product rule for radicals, because the indexes (4 and 5) are different.

Work Problem 2 at the Side.

1 Multiply. Assume that all variables represent positive real numbers.

(a) $\sqrt{5} \cdot \sqrt{13}$

(b) $\sqrt{10y} \cdot \sqrt{3k}$

(c) $\sqrt{\dfrac{5}{a}} \cdot \sqrt{\dfrac{11}{z}}$

2 Multiply. Assume that all variables represent positive real numbers.

(a) $\sqrt[3]{2} \cdot \sqrt[3]{7}$

(b) $\sqrt[6]{8r^2} \cdot \sqrt[6]{2r^3}$

(c) $\sqrt[5]{9y^2x} \cdot \sqrt[5]{8xy^2}$

(d) $\sqrt{7} \cdot \sqrt[3]{5}$

ANSWERS

1. (a) $\sqrt{65}$ **(b)** $\sqrt{30yk}$ **(c)** $\sqrt{\dfrac{55}{az}}$

2. (a) $\sqrt[3]{14}$ **(b)** $\sqrt[6]{16r^5}$ **(c)** $\sqrt[5]{72y^4x^2}$
(d) cannot be simplified using the product rule

3 Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt{\frac{100}{81}}$

(b) $\sqrt{\frac{11}{25}}$

(c) $\sqrt[3]{\frac{18}{125}}$

(d) $\sqrt{\frac{y^8}{16}}$

(e) $\sqrt[3]{\frac{x^2}{r^{12}}}$

Answers

3. (a) $\frac{10}{9}$ (b) $\frac{\sqrt{11}}{5}$ (c) $\frac{\sqrt[3]{18}}{5}$
(d) $\frac{y^4}{4}$ (e) $\frac{\sqrt[3]{x^2}}{r^4}$

OBJECTIVE 2 **Use the quotient rule for radicals.** The **quotient rule for radicals** is similar to the product rule.

Quotient Rule for Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number, then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

In words, the radical of a quotient is the quotient of the radicals.

EXAMPLE 3 Using the Quotient Rule

Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$ **(b)** $\sqrt{\frac{7}{36}} = \frac{\sqrt{7}}{\sqrt{36}} = \frac{\sqrt{7}}{6}$

(c) $\sqrt[3]{-\frac{8}{125}} = \sqrt[3]{\frac{-8}{125}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{125}} = \frac{-2}{5} = -\frac{2}{5}$

(d) $\sqrt[3]{\frac{7}{216}} = \frac{\sqrt[3]{7}}{\sqrt[3]{216}} = \frac{\sqrt[3]{7}}{6}$

(e) $\sqrt[5]{\frac{x}{32}} = \frac{\sqrt[5]{x}}{\sqrt[5]{32}} = \frac{\sqrt[5]{x}}{2}$ **(f)** $\sqrt[3]{\frac{m^6}{125}} = \frac{\sqrt[3]{m^6}}{\sqrt[3]{125}} = \frac{m^2}{5}$

Work Problem 3 at the Side.

OBJECTIVE 3 **Simplify radicals.** We use the product and quotient rules to simplify radicals. A radical is **simplified** if the following four conditions are met.

Conditions for a Simplified Radical

1. The radicand has no factor raised to a power greater than or equal to the index.
2. The radicand has no fractions.
3. No denominator has a radical.
4. Exponents in the radicand and the index of the radical have no common factor (except 1).

EXAMPLE 4 Simplifying Roots of Numbers

Simplify.

(a) $\sqrt{24}$

Check to see whether 24 is divisible by a perfect square (the square of a natural number) such as 4, 9, ... Choose the largest perfect square that divides into 24. The largest such number is 4. Write 24 as the product of 4 and 6, and then use the product rule.

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4} \cdot \sqrt{6} = 2\sqrt{6}$$

Continued on Next Page

(b) $\sqrt{108}$

The number 108 is divisible by the perfect square 36: $\sqrt{108} = \sqrt{36 \cdot 3}$. If this is not obvious, try factoring 108 into its prime factors.

$$\begin{aligned} \sqrt{108} &= \sqrt{2^2 \cdot 3^3} \\ &= \sqrt{\mathbf{2^2} \cdot \mathbf{3^2} \cdot 3} \\ &= \mathbf{2} \cdot \mathbf{3} \cdot \sqrt{3} && \text{Product rule} \\ &= 6\sqrt{3} \end{aligned}$$

(c) $\sqrt{10}$

No perfect square (other than 1) divides into 10, so $\sqrt{10}$ cannot be simplified further.

(d) $\sqrt[3]{16}$

Look for the largest perfect *cube* that divides into 16. The number 8 satisfies this condition, so write 16 as $8 \cdot 2$ (or factor 16 into prime factors).

$$\sqrt[3]{16} = \sqrt[3]{\mathbf{8} \cdot 2} = \sqrt[3]{\mathbf{8}} \cdot \sqrt[3]{2} = \mathbf{2}\sqrt[3]{2}$$

(e)
$$\begin{aligned} -\sqrt[4]{162} &= -\sqrt[4]{\mathbf{81} \cdot 2} && \text{81 is a perfect 4th power.} \\ &= -\sqrt[4]{\mathbf{81}} \cdot \sqrt[4]{2} && \text{Product rule} \\ &= -\mathbf{3}\sqrt[4]{2} \end{aligned}$$

CAUTION

Be careful with which factors belong outside the radical sign and which belong inside. Note in Example 4(b) how $2 \cdot 3$ is written outside because $\sqrt{2^2} = 2$ and $\sqrt{3^2} = 3$. The remaining 3 is left inside the radical.

Work Problem 4 at the Side.

You Try It

EXAMPLE 5 Simplifying Radicals Involving Variables

Simplify. Assume that all variables represent positive real numbers.

(a)
$$\begin{aligned} \sqrt{16m^3} &= \sqrt{\mathbf{16m^2} \cdot m} \\ &= \sqrt{\mathbf{16m^2}} \cdot \sqrt{m} \\ &= \mathbf{4m}\sqrt{m} \end{aligned}$$

No absolute value bars are needed around the m in color because of the assumption that all the variables represent *positive* real numbers.

(b)
$$\begin{aligned} \sqrt{200k^7q^8} &= \sqrt{10^2 \cdot 2 \cdot (k^3)^2 \cdot k \cdot (q^4)^2} && \text{Factor.} \\ &= 10k^3q^4\sqrt{2k} && \text{Remove perfect square factors.} \end{aligned}$$

(c)
$$\begin{aligned} \sqrt[3]{8x^4y^5} &= \sqrt[3]{(8x^3y^3)(xy^2)} && 8x^3y^3 \text{ is the largest perfect cube that divides } 8x^4y^5. \\ &= \sqrt[3]{8x^3y^3} \cdot \sqrt[3]{xy^2} \\ &= 2xy\sqrt[3]{xy^2} \end{aligned}$$

(d)
$$\begin{aligned} -\sqrt[4]{32y^9} &= -\sqrt[4]{(16y^8)(2y)} && 16y^8 \text{ is the largest 4th power that divides } 32y^9. \\ &= -\sqrt[4]{16y^8} \cdot \sqrt[4]{2y} \\ &= -2y^2\sqrt[4]{2y} \end{aligned}$$

4 Simplify.

(a) $\sqrt{32}$

(b) $\sqrt{45}$

(c) $\sqrt{300}$

(d) $\sqrt{35}$

(e) $-\sqrt[3]{54}$

(f) $\sqrt[4]{243}$

ANSWERS

4. **(a)** $4\sqrt{2}$ **(b)** $3\sqrt{5}$ **(c)** $10\sqrt{3}$ **(d)** cannot be simplified further **(e)** $-3\sqrt[3]{2}$ **(f)** $3\sqrt[4]{3}$

5 Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt{25p^7}$

(b) $\sqrt{72y^3x}$

(c) $\sqrt[3]{y^7x^5z^6}$

(d) $-\sqrt[4]{32a^5b^7}$

6 Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt[12]{2^3}$

(b) $\sqrt[6]{t^2}$

7 Simplify $\sqrt{5} \cdot \sqrt[3]{4}$.

ANSWERS
5. (a) $5p^3\sqrt{p}$ (b) $6y\sqrt{2yx}$ (c) $y^2xz^2\sqrt[3]{yx^2}$
(d) $-2ab\sqrt[4]{2ab^3}$
6. (a) $\sqrt[4]{2}$ (b) $\sqrt[3]{t}$
7. $\sqrt[6]{2000}$

NOTE
From Example 5 we see that if a variable is raised to a power with an exponent divisible by 2, it is a perfect square. If it is raised to a power with an exponent divisible by 3, it is a perfect cube. In general, if it is raised to a power with an exponent divisible by n, it is a perfect nth power.

Work Problem 5 at the Side.

The conditions for a simplified radical given earlier state that an exponent in the radicand and the index of the radical should have no common factor (except 1). The next example shows how to simplify radicals with such common factors.

EXAMPLE 6 Simplifying Radicals by Using Smaller Indexes

Video

Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt[9]{5^6}$

We can write this radical using rational exponents and then write the exponent in lowest terms. We then express the answer as a radical.

$$\sqrt[9]{5^6} = 5^{6/9} = 5^{2/3} = \sqrt[3]{5^2} \quad \text{or} \quad \sqrt[3]{25}$$

(b) $\sqrt[4]{p^2} = p^{2/4} = p^{1/2} = \sqrt{p}$ (Recall the assumption that $p > 0$.)

These examples suggest the following rule.

Audio

If m is an integer, n and k are natural numbers, and all indicated roots exist, then

$$\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}.$$

Work Problem 6 at the Side.

OBJECTIVE 4 Simplify products and quotients of radicals with different indexes. Since the product and quotient rules for radicals apply only when they have the same index, we multiply and divide radicals with different indexes by using rational exponents.

EXAMPLE 7 Multiplying Radicals with Different Indexes

Video

Simplify $\sqrt{7} \cdot \sqrt[3]{2}$.

Because the different indexes, 2 and 3, have a least common index of 6, use rational exponents to write each radical as a sixth root.

$$\sqrt{7} = 7^{1/2} = 7^{3/6} = \sqrt[6]{7^3} = \sqrt[6]{343}$$
$$\sqrt[3]{2} = 2^{1/3} = 2^{2/6} = \sqrt[6]{2^2} = \sqrt[6]{4}$$

Therefore,

$$\sqrt{7} \cdot \sqrt[3]{2} = \sqrt[6]{343} \cdot \sqrt[6]{4} = \sqrt[6]{1372}. \qquad \text{Product rule}$$

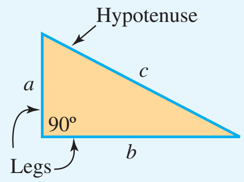
Work Problem 7 at the Side.

OBJECTIVE 5 Use the Pythagorean formula. The **Pythagorean formula** relates the lengths of the three sides of a right triangle.

Pythagorean Formula

If c is the length of the longest side of a right triangle and a and b are the lengths of the shorter sides, then

$$c^2 = a^2 + b^2.$$



The longest side is the **hypotenuse** and the two shorter sides are the **legs** of the triangle. The hypotenuse is the side opposite the right angle.

In **Section 10.1** we will see that an equation such as $x^2 = 7$ has two solutions: $\sqrt{7}$ (the principal, or positive, square root of 7) and $-\sqrt{7}$. Similarly, $c^2 = 52$ has two solutions, $\pm\sqrt{52} = \pm 2\sqrt{13}$. In applications we often choose only the positive square root, as seen in the examples that follow.

EXAMPLE 8 Using the Pythagorean Formula

Use the Pythagorean formula to find the length of the hypotenuse in the triangle in Figure 6.

To find the length of the hypotenuse c, let $a = 4$ and $b = 6$. Then, use the formula.

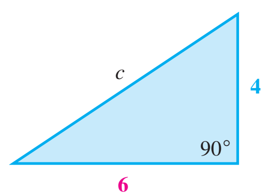


Figure 6

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 6^2 \quad \text{Let } a = 4 \text{ and } b = 6.$$
$$c^2 = 52$$
$$c = \sqrt{52} \quad \text{Choose the principal root.}$$
$$c = \sqrt{4 \cdot 13} \quad \text{Factor.}$$
$$c = \sqrt{4} \cdot \sqrt{13} \quad \text{Product rule}$$
$$c = 2\sqrt{13}$$

The length of the hypotenuse is $2\sqrt{13}$.

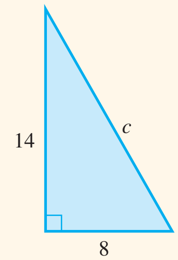
Work Problem 8 at the Side.

OBJECTIVE 6 Use the distance formula. An important result in algebra is derived by using the Pythagorean formula. The *distance formula* allows us to find the distance between two points in the coordinate plane, or the length of the line segment joining those two points. Figure 7 shows the points $(3, -4)$ and $(-5, 3)$. The vertical line through $(-5, 3)$ and the horizontal line through $(3, -4)$ intersect at the point $(-5, -4)$. Thus, the point $(-5, -4)$ becomes the vertex of the right angle in a right triangle. By the Pythagorean formula, the square of the length of the hypotenuse, d, of the right triangle in Figure 7 is equal to the sum of the squares of the lengths of the two legs a and b:

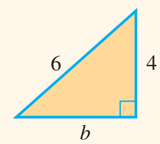
$$d^2 = a^2 + b^2.$$

8 Find the length of the unknown side in each triangle.

(a)



(b)



(*Hint:* Write the Pythagorean formula as $b^2 = c^2 - a^2$ here.)

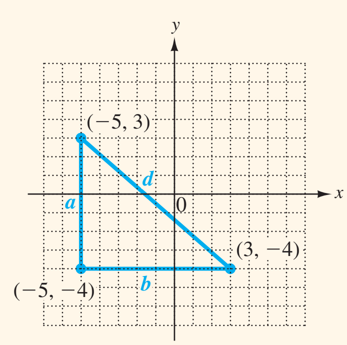


Figure 7

ANSWERS

8. (a) $2\sqrt{65}$ **(b)** $2\sqrt{5}$

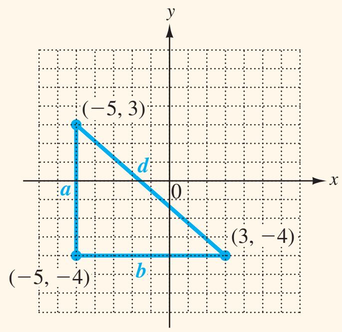


Figure 7 (repeat)

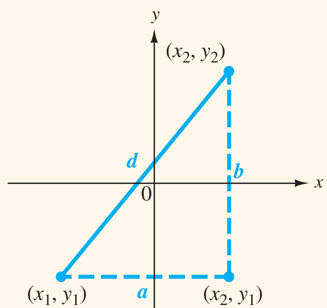


Figure 8

9 Find the distance between each pair of points.

(a) $(2, -1)$ and $(5, 3)$

(b) $(-3, 2)$ and $(0, -4)$

The length a is the difference between the y-coordinates of the endpoints. Since the x-coordinate of both points is -5, the side is vertical, and we can find a by finding the difference between the y-coordinates. We subtract -4 from 3 to get a positive value for a.

$$a = 3 - (-4) = 7$$

Similarly, we find b by subtracting -5 from 3.

$$b = 3 - (-5) = 8$$

Substituting these values into the formula, we have

$$\begin{aligned} d^2 &= 7^2 + 8^2 \qquad \text{Let } a = 7 \text{ and } b = 8 \text{ in } d^2 = a^2 + b^2. \\ d^2 &= 49 + 64 \\ d^2 &= 113 \\ d &= \sqrt{113}. \end{aligned}$$

We choose the principal root since distance cannot be negative. Therefore, the distance between $(-5, 3)$ and $(3, -4)$ is $\sqrt{113}$.

This result can be generalized. Figure 8 shows the two points (x_1, y_1) and (x_2, y_2). Notice that the distance between (x_1, y_1) and (x_2, y_1) is given by

$$a = x_2 - x_1,$$

and the distance between (x_2, y_2) and (x_2, y_1) is given by

$$b = y_2 - y_1.$$

From the Pythagorean formula,

$$\begin{aligned} d^2 &= a^2 + b^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

Choosing the principal square root gives the **distance formula.**

Distance Formula

The distance between the points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 9 Using the Distance Formula

Find the distance between $(-3, 5)$ and $(6, 4)$.

When using the distance formula to find the distance between two points, designating the points as (x_1, y_1) and (x_2, y_2) is arbitrary. Let us choose $(x_1, y_1) = (-3, 5)$ and $(x_2, y_2) = (6, 4)$.

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(6 - (-3))^2 + (4 - 5)^2} \qquad x_2 = 6, y_2 = 4, x_1 = -3, y_1 = 5 \\ &= \sqrt{9^2 + (-1)^2} \\ &= \sqrt{82} \qquad \text{Leave in radical form.} \end{aligned}$$

Work Problem 9 at the Side.

ANSWERS
9. **(a)** 5 **(b)** $\sqrt{45}$ or $3\sqrt{5}$

9.4 Adding and Subtracting Radical Expressions

The examples in the preceding section discussed simplifying radical expressions that involve multiplication and division. Now we show how to simplify radical expressions that involve addition and subtraction.

OBJECTIVE 1 Simplify radical expressions involving addition and subtraction. An expression such as $4\sqrt{2} + 3\sqrt{2}$ can be simplified by using the distributive property.

$$4\sqrt{2} + 3\sqrt{2} = (4 + 3)\sqrt{2} = 7\sqrt{2}$$

As another example, $2\sqrt{3} - 5\sqrt{3} = (2 - 5)\sqrt{3} = -3\sqrt{3}$. This is similar to simplifying $2x + 3x$ to $5x$ or $5y - 8y$ to $-3y$.

CAUTION
Only radical expressions with the same index and the same radicand may be combined. Expressions such as $5\sqrt{3} + 2\sqrt{2}$ or $3\sqrt{3} + 2\sqrt[3]{3}$ cannot be simplified by combining terms.

EXAMPLE 1 Adding and Subtracting Radicals

Add or subtract to simplify each radical expression.

(a) $3\sqrt{24} + \sqrt{54}$

Begin by simplifying each radical; then use the distributive property to combine terms.

$$\begin{aligned} 3\sqrt{24} + \sqrt{54} &= 3\sqrt{4} \cdot \sqrt{6} + \sqrt{9} \cdot \sqrt{6} && \text{Product rule} \\ &= 3 \cdot 2\sqrt{6} + 3\sqrt{6} \\ &= 6\sqrt{6} + 3\sqrt{6} \\ &= 9\sqrt{6} && \text{Combine terms.} \end{aligned}$$

(b)
$$\begin{aligned} 2\sqrt{20x} - \sqrt{45x} &= 2\sqrt{4} \cdot \sqrt{5x} - \sqrt{9} \cdot \sqrt{5x} && \text{Product rule} \\ &= 2 \cdot 2\sqrt{5x} - 3\sqrt{5x} \\ &= 4\sqrt{5x} - 3\sqrt{5x} \\ &= \sqrt{5x}, \quad x \geq 0 && \text{Combine terms.} \end{aligned}$$

(c) $2\sqrt{3} - 4\sqrt{5}$

Here the radicals differ and are already simplified, so $2\sqrt{3} - 4\sqrt{5}$ cannot be simplified further.

Work Problem 1 at the Side.

CAUTION
Do not confuse the product rule with combining like terms. ***The root of a sum does not equal the sum of the roots.*** For example,

$$\sqrt{9 + 16} \neq \sqrt{9} + \sqrt{16}, \text{ since}$$
$$\sqrt{9 + 16} = \sqrt{25} = 5, \quad \text{but} \quad \sqrt{9} + \sqrt{16} = 3 + 4 = 7.$$

OBJECTIVES

1 Simplify radical expressions involving addition and subtraction.

1 Add or subtract to simplify each radical expression.

(a) $3\sqrt{5} + 7\sqrt{5}$

(b) $2\sqrt{11} - \sqrt{11} + 3\sqrt{44}$

(c) $5\sqrt{12y} + 6\sqrt{75y}, \ y \geq 0$

(d) $3\sqrt{8} - 6\sqrt{50} + 2\sqrt{200}$

(e) $9\sqrt{5} - 4\sqrt{10}$

ANSWERS

1. (a) $10\sqrt{5}$ **(b)** $7\sqrt{11}$ **(c)** $40\sqrt{3y}$ **(d)** $-4\sqrt{2}$ **(e)** cannot be simplified further

2 Add or subtract to simplify each radical expression. Assume that all variables represent positive real numbers.

(a) $7\sqrt[3]{81} + 3\sqrt[3]{24}$

You Try It

Video

(b) $-2\sqrt[4]{32} - 7\sqrt[4]{162}$

(c) $\sqrt[3]{p^4q^7} - \sqrt[3]{64pq}$

You Try It

Video

3 Add. Assume that all variables represent positive real numbers.

$$\sqrt{\frac{80}{y^4}} + \sqrt{\frac{81}{y^{10}}}$$

ANSWERS

2. **(a)** $27\sqrt[3]{3}$ **(b)** $-25\sqrt[4]{2}$

(c) $(pq^2 - 4)\sqrt[3]{pq}$

3. $\dfrac{4y^3\sqrt{5} + 9}{y^5}$

EXAMPLE 2 **Adding and Subtracting Radicals with Greater Indexes**

Add or subtract to simplify each radical expression. Assume that all variables represent positive real numbers.

(a)
$$\begin{aligned}
2\sqrt[3]{16} - 5\sqrt[3]{54} &= 2\sqrt[3]{8\cdot 2} - 5\sqrt[3]{27\cdot 2} && \text{Factor.}\\
&= 2\sqrt[3]{8}\cdot\sqrt[3]{2} - 5\sqrt[3]{27}\cdot\sqrt[3]{2} && \text{Product rule}\\
&= 2\cdot 2\cdot\sqrt[3]{2} - 5\cdot 3\cdot\sqrt[3]{2}\\
&= 4\sqrt[3]{2} - 15\sqrt[3]{2}\\
&= -11\sqrt[3]{2} && \text{Combine terms.}
\end{aligned}$$

(b)
$$\begin{aligned}
2\sqrt[3]{x^2y} + \sqrt[3]{8x^5y^4} &= 2\sqrt[3]{x^2y} + \sqrt[3]{(8x^3y^3)x^2y} && \text{Factor.}\\
&= 2\sqrt[3]{x^2y} + 2xy\sqrt[3]{x^2y} && \text{Product rule}\\
&= (2 + 2xy)\sqrt[3]{x^2y} && \text{Distributive property}
\end{aligned}$$

CAUTION

Remember to write the index when working with cube roots, fourth roots, and so on.

Work Problem 2 at the Side.

EXAMPLE 3 **Adding and Subtracting Radicals with Fractions**

Perform the indicated operations. Assume that all variables represent positive real numbers.

(a)
$$\begin{aligned}
2\sqrt{\frac{75}{16}} + 4\frac{\sqrt{8}}{\sqrt{32}} &= 2\frac{\sqrt{25\cdot 3}}{\sqrt{16}} + 4\frac{\sqrt{4\cdot 2}}{\sqrt{16\cdot 2}} && \text{Quotient rule}\\
&= 2\left(\frac{5\sqrt{3}}{4}\right) + 4\left(\frac{2\sqrt{2}}{4\sqrt{2}}\right) && \text{Product rule}\\
&= \frac{5\sqrt{3}}{2} + 2 && \text{Multiply; } \tfrac{\sqrt{2}}{\sqrt{2}} = 1.\\
&= \frac{5\sqrt{3}}{2} + \frac{4}{2} && \text{Write with a common denominator.}\\
&= \frac{5\sqrt{3} + 4}{2}
\end{aligned}$$

(b)
$$\begin{aligned}
10\sqrt[3]{\frac{5}{x^6}} - 3\sqrt[3]{\frac{4}{x^9}} &= 10\frac{\sqrt[3]{5}}{\sqrt[3]{x^6}} - 3\frac{\sqrt[3]{4}}{\sqrt[3]{x^9}} && \text{Quotient rule}\\
&= \frac{10\sqrt[3]{5}}{x^2} - \frac{3\sqrt[3]{4}}{x^3}\\
&= \frac{10x\sqrt[3]{5}}{x^3} - \frac{3\sqrt[3]{4}}{x^3} && \text{Write with a common denominator.}\\
&= \frac{10x\sqrt[3]{5} - 3\sqrt[3]{4}}{x^3}
\end{aligned}$$

Work Problem 3 at the Side.

9.5 Multiplying and Dividing Radical Expressions

OBJECTIVES

1. Multiply radical expressions.
2. Rationalize denominators with one radical term.
3. Rationalize denominators with binomials involving radicals.
4. Write radical quotients in lowest terms.

OBJECTIVE 1 Multiply radical expressions. We multiply binomial expressions involving radicals by using the FOIL (First, Outer, Inner, Last) method from **Section 6.4.** For example, we find the product of the binomials $\sqrt{5} + 3$ and $\sqrt{6} + 1$ as follows.

$$\begin{aligned}(\sqrt{5} + 3)(\sqrt{6} + 1) &= \overbrace{\sqrt{5} \cdot \sqrt{6}}^{\text{First}} + \overbrace{\sqrt{5} \cdot 1}^{\text{Outer}} + \overbrace{3 \cdot \sqrt{6}}^{\text{Inner}} + \overbrace{3 \cdot 1}^{\text{Last}} \\ &= \sqrt{30} + \sqrt{5} + 3\sqrt{6} + 3\end{aligned}$$

This result cannot be simplified further.

EXAMPLE 1 Multiplying Binomials Involving Radical Expressions

Multiply using FOIL.

(a) $$\begin{aligned}(7 - \sqrt{3})(\sqrt{5} + \sqrt{2}) &= \overset{\text{F}}{7\sqrt{5}} + \overset{\text{O}}{7\sqrt{2}} - \overset{\text{I}}{\sqrt{3} \cdot \sqrt{5}} - \overset{\text{L}}{\sqrt{3} \cdot \sqrt{2}} \\ &= 7\sqrt{5} + 7\sqrt{2} - \sqrt{15} - \sqrt{6}\end{aligned}$$

(b) $(\sqrt{10} + \sqrt{3})(\sqrt{10} - \sqrt{3})$

$$\begin{aligned}&= \sqrt{10} \cdot \sqrt{10} - \sqrt{10} \cdot \sqrt{3} + \sqrt{10} \cdot \sqrt{3} - \sqrt{3} \cdot \sqrt{3} \\ &= 10 - 3 \\ &= 7\end{aligned}$$

Notice that this is the kind of product that results in the difference of squares:

$$(x + y)(x - y) = x^2 - y^2.$$

Here, $x = \sqrt{10}$ and $y = \sqrt{3}$.

(c) $$\begin{aligned}(\sqrt{7} - 3)^2 &= (\sqrt{7} - 3)(\sqrt{7} - 3) \\ &= \sqrt{7} \cdot \sqrt{7} - 3\sqrt{7} - 3\sqrt{7} + 3 \cdot 3 \\ &= 7 - 6\sqrt{7} + 9 \\ &= 16 - 6\sqrt{7}\end{aligned}$$

(d) $$\begin{aligned}(5 - \sqrt[3]{3})(5 + \sqrt[3]{3}) &= 5 \cdot 5 + 5\sqrt[3]{3} - 5\sqrt[3]{3} - \sqrt[3]{3} \cdot \sqrt[3]{3} \\ &= 25 - \sqrt[3]{3^2} \\ &= 25 - \sqrt[3]{9}\end{aligned}$$

(e) $$\begin{aligned}(\sqrt{k} + \sqrt{y})(\sqrt{k} - \sqrt{y}) &= (\sqrt{k})^2 - (\sqrt{y})^2 \\ &= k - y, \quad k \geq 0 \text{ and } y \geq 0\end{aligned}$$

1 Multiply using FOIL.

(a) $(2 + \sqrt{3})(1 + \sqrt{5})$

(b) $(2\sqrt{3} + \sqrt{5})(\sqrt{6} - 3\sqrt{5})$

(c) $(4 + \sqrt{3})(4 - \sqrt{3})$

(d) $(\sqrt{6} - \sqrt{5})^2$

(e) $(4 + \sqrt[3]{7})(4 - \sqrt[3]{7})$

(f) $(\sqrt{p} + \sqrt{2})(\sqrt{p} - \sqrt{2})$

NOTE

In Example 1(c) we could have used the formula for the square of a binomial,

$$(x - y)^2 = x^2 - 2xy + y^2,$$

to get the same result.

$$\begin{aligned}(\sqrt{7} - 3)^2 &= (\sqrt{7})^2 - 2(\sqrt{7})(3) + 3^2\\ &= 7 - 6\sqrt{7} + 9\\ &= 16 - 6\sqrt{7}\end{aligned}$$

Work Problem 1 at the Side.

OBJECTIVE 2 Rationalize denominators with one radical term. As defined earlier, a simplified radical expression will have no radical in the denominator. The origin of this agreement no doubt occurred before the days of high-speed calculation, when computation was a tedious process performed by hand. To see this, consider the radical expression $\frac{1}{\sqrt{2}}$. To find a decimal approximation by hand, it would be necessary to divide 1 by a decimal approximation for $\sqrt{2}$, such as 1.414. It would be much easier if the divisor were a whole number. This can be accomplished by multiplying $\frac{1}{\sqrt{2}}$ by 1 in the form $\frac{\sqrt{2}}{\sqrt{2}}$:

$$\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Now the computation would require dividing 1.414 by 2 to obtain .707, a much easier task.

With current technology, either form of this fraction can be approximated with the same number of keystrokes. See Figure 9, which shows how a calculator gives the same approximation for both forms of the expression.

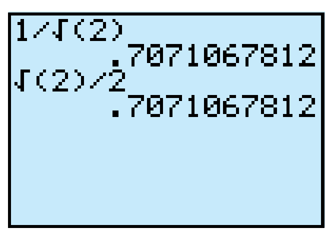


Figure 9

A common way of "standardizing" the form of a radical expression is to have the denominator contain no radicals. The process of removing radicals from a denominator so that the denominator contains only rational numbers is called **rationalizing the denominator.**

EXAMPLE 2 Rationalizing Denominators with Square Roots

Rationalize each denominator.

(a) $\frac{3}{\sqrt{7}}$

Multiply the numerator and denominator by $\sqrt{7}$. This is, in effect, multiplying by 1.

Continued on Next Page

ANSWERS

1. **(a)** $2 + 2\sqrt{5} + \sqrt{3} + \sqrt{15}$
(b) $6\sqrt{2} - 6\sqrt{15} + \sqrt{30} - 15$
(c) 13 **(d)** $11 - 2\sqrt{30}$
(e) $16 - \sqrt[3]{49}$ **(f)** $p - 2, p \geq 0$

$$\frac{3}{\sqrt{7}} = \frac{3 \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}}$$

In the denominator, $\sqrt{7} \cdot \sqrt{7} = \sqrt{7 \cdot 7} = \sqrt{49} = 7$, so

$$\frac{3}{\sqrt{7}} = \frac{3\sqrt{7}}{7}.$$

The denominator is now a rational number.

(b) $\dfrac{5\sqrt{2}}{\sqrt{5}} = \dfrac{5\sqrt{2} \cdot \sqrt{5}}{\sqrt{5} \cdot \sqrt{5}} = \dfrac{5\sqrt{10}}{5} = \sqrt{10}$

(c) $\dfrac{6}{\sqrt{12}}$

Less work is involved if the radical in the denominator is simplified first.

$$\frac{6}{\sqrt{12}} = \frac{6}{\sqrt{4 \cdot 3}} = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}}$$

Now rationalize the denominator by multiplying the numerator and denominator by $\sqrt{3}$.

$$\frac{3}{\sqrt{3}} = \frac{3 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Work Problem 2 at the Side.

EXAMPLE 3 Rationalizing Denominators in Roots of Fractions

Simplify each radical. Assume that all variables represent positive real numbers.

(a)

$$\sqrt{\frac{18}{125}} = \frac{\sqrt{18}}{\sqrt{125}} \quad \text{Quotient rule}$$

$$= \frac{\sqrt{9 \cdot 2}}{\sqrt{25 \cdot 5}} \quad \text{Factor.}$$

$$= \frac{3\sqrt{2}}{5\sqrt{5}} \quad \text{Product rule}$$

$$= \frac{3\sqrt{2} \cdot \sqrt{5}}{5\sqrt{5} \cdot \sqrt{5}} \quad \text{Multiply by } \frac{\sqrt{5}}{\sqrt{5}}.$$

$$= \frac{3\sqrt{10}}{5 \cdot 5} \quad \text{Product rule}$$

$$= \frac{3\sqrt{10}}{25}$$

Continued on Next Page

2 Rationalize each denominator.

(a) $\dfrac{8}{\sqrt{3}}$

(b) $\dfrac{\sqrt{3}}{\sqrt{7}}$

(c) $\dfrac{3}{\sqrt{48}}$

(d) $\dfrac{-16}{\sqrt{32}}$

ANSWERS

2. (a) $\dfrac{8\sqrt{3}}{3}$ **(b)** $\dfrac{\sqrt{21}}{7}$ **(c)** $\dfrac{\sqrt{3}}{4}$ **(d)** $-2\sqrt{2}$

(b) $\sqrt{\dfrac{50m^4}{p^5}}, \quad p > 0$

$$\sqrt{\frac{50m^4}{p^5}} = \frac{\sqrt{50m^4}}{\sqrt{p^5}} \qquad \text{Quotient rule}$$

$$= \frac{5m^2\sqrt{2}}{p^2\sqrt{p}} \qquad \text{Product rule}$$

$$= \frac{5m^2\sqrt{2}\cdot\sqrt{p}}{p^2\sqrt{p}\cdot\sqrt{p}} \qquad \text{Multiply by } \frac{\sqrt{p}}{\sqrt{p}}.$$

$$= \frac{5m^2\sqrt{2p}}{p^2\cdot p} \qquad \text{Product rule}$$

$$= \frac{5m^2\sqrt{2p}}{p^3}$$

Work Problem 3 at the Side.

EXAMPLE 4 Rationalizing Denominators with Cube Roots

Simplify.

(a) $\sqrt[3]{\dfrac{27}{16}}$

Use the quotient rule and simplify the numerator and denominator.

$$\sqrt[3]{\frac{27}{16}} = \frac{\sqrt[3]{27}}{\sqrt[3]{16}} = \frac{3}{\sqrt[3]{8}\cdot\sqrt[3]{2}} = \frac{3}{2\sqrt[3]{2}}$$

To get a rational denominator, multiply the numerator and denominator by a number that will result in a perfect cube in the radicand in the denominator. Since $2 \cdot 4 = 8$, a perfect cube, multiply the numerator and denominator by $\sqrt[3]{4}$.

$$\sqrt[3]{\frac{27}{16}} = \frac{3}{2\sqrt[3]{2}} = \frac{3\cdot\sqrt[3]{4}}{2\sqrt[3]{2}\cdot\sqrt[3]{4}} = \frac{3\sqrt[3]{4}}{2\sqrt[3]{8}} = \frac{3\sqrt[3]{4}}{2\cdot 2} = \frac{3\sqrt[3]{4}}{4}$$

(b) $\sqrt[4]{\dfrac{5x}{z}} = \dfrac{\sqrt[4]{5x}}{\sqrt[4]{z}}\cdot\dfrac{\sqrt[4]{z^3}}{\sqrt[4]{z^3}} = \dfrac{\sqrt[4]{5xz^3}}{\sqrt[4]{z^4}} = \dfrac{\sqrt[4]{5xz^3}}{z}, \quad x \ge 0, z > 0$

CAUTION

In problems like the one in Example 4(a), a typical error is to multiply the numerator and denominator by $\sqrt[3]{2}$, forgetting that

$$\sqrt[3]{2}\cdot\sqrt[3]{2} \ne 2.$$

We need *three* factors of 2 to get 2^3 under the radical. As implied in Example 4(a),

$$\sqrt[3]{2}\cdot\sqrt[3]{2}\cdot\sqrt[3]{2} = 2.$$

Work Problem 4 at the Side.

3 Simplify. Assume that all variables represent positive real numbers.

(a) $\sqrt{\dfrac{8}{45}}$

(b) $\sqrt{\dfrac{72}{y}}$

(c) $\sqrt{\dfrac{200k^6}{y^7}}$

4 Simplify.

(a) $\sqrt[3]{\dfrac{15}{32}}$

(b) $\sqrt[3]{\dfrac{m^{12}}{n}}, \quad n \ne 0$

(c) $\sqrt[4]{\dfrac{6y}{w^2}}, \quad y \ge 0, w \ne 0$

ANSWERS

3. (a) $\dfrac{2\sqrt{10}}{15}$ (b) $\dfrac{6\sqrt{2y}}{y}$ (c) $\dfrac{10k^3\sqrt{2y}}{y^4}$

4. (a) $\dfrac{\sqrt[3]{30}}{4}$ (b) $\dfrac{m^4\sqrt[3]{n^2}}{n}$ (c) $\dfrac{\sqrt[4]{6yw^2}}{w}$

OBJECTIVE 3 Rationalize denominators with binomials involving radicals. Recall the special product

$$(x + y)(x - y) = x^2 - y^2.$$

To rationalize a denominator that contains a binomial expression (one that contains exactly two terms) involving radicals, such as

$$\frac{3}{1 + \sqrt{2}},$$

we must use *conjugates*. The conjugate of $1 + \sqrt{2}$ is $1 - \sqrt{2}$. In general, $x + y$ and $x - y$ are **conjugates.**

Rationalizing a Binomial Denominator

If a radical expression has a sum or difference with square root radicals in the denominator, rationalize the denominator by multiplying both the numerator and denominator by the conjugate of the denominator.

For the expression $\frac{3}{1 + \sqrt{2}}$, we rationalize the denominator by multiplying both the numerator and denominator by $1 - \sqrt{2}$, the conjugate of the denominator.

$$\begin{aligned}\frac{3}{1 + \sqrt{2}} &= \frac{3(1 - \sqrt{2})}{(1 + \sqrt{2})(1 - \sqrt{2})} \\ &= \frac{3(1 - \sqrt{2})}{-1} \qquad \begin{aligned}&(1 + \sqrt{2})(1 - \sqrt{2}) \\ &= 1^2 - (\sqrt{2})^2 \\ &= 1 - 2 = -1\end{aligned} \\ &= \frac{3}{-1}(1 - \sqrt{2}) \\ &= -3(1 - \sqrt{2}) \quad \text{or} \quad -3 + 3\sqrt{2}\end{aligned}$$

EXAMPLE 5 Rationalizing Binomial Denominators

Rationalize each denominator.

(a) $\frac{5}{4 - \sqrt{3}}$

To rationalize the denominator, multiply both the numerator and denominator by the conjugate of the denominator, $4 + \sqrt{3}$.

$$\begin{aligned}\frac{5}{4 - \sqrt{3}} &= \frac{5(4 + \sqrt{3})}{(4 - \sqrt{3})(4 + \sqrt{3})} \\ &= \frac{5(4 + \sqrt{3})}{16 - 3} \\ &= \frac{5(4 + \sqrt{3})}{13}\end{aligned}$$

Notice that the numerator is left in factored form. This makes it easier to determine whether the expression is written in lowest terms.

Continued on Next Page

(b) $$\frac{\sqrt{2}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{(\sqrt{2}-\sqrt{3})(\sqrt{5}-\sqrt{3})}{(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})}$$ Multiply the numerator and denominator by $\sqrt{5}-\sqrt{3}$.

$$= \frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{5-3}$$

$$= \frac{\sqrt{10}-\sqrt{6}-\sqrt{15}+3}{2}$$

(c) $$\frac{3}{\sqrt{5m}-\sqrt{p}} = \frac{3(\sqrt{5m}+\sqrt{p})}{(\sqrt{5m}-\sqrt{p})(\sqrt{5m}+\sqrt{p})}$$

$$= \frac{3(\sqrt{5m}+\sqrt{p})}{5m-p}, \quad 5m \neq p,\ m > 0,\ p > 0$$

Work Problem 5 at the Side.

OBJECTIVE 4 **Write radical quotients in lowest terms.**

EXAMPLE 6 Writing Radical Quotients in Lowest Terms

Write each quotient in lowest terms.

(a) $$\frac{6+2\sqrt{5}}{4}$$

Factor the numerator and denominator, then write in lowest terms.

$$\frac{6+2\sqrt{5}}{4} = \frac{2(3+\sqrt{5})}{2\cdot 2} = \frac{3+\sqrt{5}}{2}$$

Here is an alternative method for writing this expression in lowest terms.

$$\frac{6+2\sqrt{5}}{4} = \frac{6}{4}+\frac{2\sqrt{5}}{4} = \frac{3}{2}+\frac{\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}$$

(b) $$\frac{5y-\sqrt{8y^2}}{6y} = \frac{5y-2y\sqrt{2}}{6y}, \quad y > 0$$ Product rule

$$= \frac{y(5-2\sqrt{2})}{6y}$$ Factor the numerator.

$$= \frac{5-2\sqrt{2}}{6}$$ Lowest terms

Note that the final fraction cannot be simplified further because there is no common factor of 2 in the numerator.

CAUTION
Be careful to factor before writing a quotient in lowest terms.

Work Problem 6 at the Side.

5 Rationalize each denominator.

(a) $\dfrac{-4}{\sqrt{5}+2}$

(b) $\dfrac{15}{\sqrt{7}+\sqrt{2}}$

(c) $\dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{2}-\sqrt{7}}$

Video

(d) $\dfrac{2}{\sqrt{k}+\sqrt{z}}$, $k \neq z,\ k > 0,\ z > 0$

6 Write each quotient in lowest terms.

(a) $\dfrac{15-5\sqrt{3}}{5}$

(b) $\dfrac{24-36\sqrt{7}}{16}$

ANSWERS

5. (a) $-4(\sqrt{5}-2)$ (b) $3(\sqrt{7}-\sqrt{2})$
(c) $\dfrac{-(\sqrt{6}+\sqrt{21}+\sqrt{10}+\sqrt{35})}{5}$
(d) $\dfrac{2(\sqrt{k}-\sqrt{z})}{k-z}$

6. (a) $3-\sqrt{3}$ (b) $\dfrac{6-9\sqrt{7}}{4}$

9.6 Solving Equations with Radicals

An equation that includes one or more radical expressions with a variable is called a **radical equation.** Some examples of radical equations are

$$\sqrt{x-4} = 8, \quad \sqrt{5x+12} = 3\sqrt{2x-1}, \quad \text{and} \quad \sqrt[3]{6+x} = 27.$$

OBJECTIVES

1. Solve radical equations using the power rule.
2. Solve radical equations that require additional steps.
3. Solve radical equations with indexes greater than 2.

OBJECTIVE 1 Solve radical equations using the power rule. The equation $x = 1$ has only one solution. Its solution set is $\{1\}$. If we square both sides of this equation, we get $x^2 = 1$. This new equation has two solutions: -1 and 1. Notice that the solution of the original equation is also a solution of the squared equation. However, the squared equation has another solution, -1, that is *not* a solution of the original equation. When solving equations with radicals, we use this idea of raising both sides to a power. It is an application of the **power rule.**

Audio

Power Rule for Solving Equations with Radicals

If both sides of an equation are raised to the same power, all solutions of the original equation are also solutions of the new equation.

Read the power rule carefully; it does *not* say that all solutions of the new equation are solutions of the original equation. They may or may not be. Solutions that do not satisfy the original equation are called **extraneous solutions;** they must be discarded.

CAUTION

When the power rule is used to solve an equation, ***every solution of the new equation*** **must** ***be checked in the original equation.***

Video

EXAMPLE 1 Using the Power Rule

Solve $\sqrt{3x+4} = 8$.

Use the power rule and square both sides to get

$$(\sqrt{3x+4})^2 = 8^2$$
$$3x + 4 = 64$$
$$3x = 60 \quad \text{Subtract 4.}$$
$$x = 20. \quad \text{Divide by 3.}$$

To check, substitute the potential solution in the *original* equation.

$$\sqrt{3x+4} = 8$$
$$\sqrt{3 \cdot 20 + 4} = 8 \quad ? \quad \text{Let } x = 20.$$
$$\sqrt{64} = 8 \quad ?$$
$$8 = 8 \quad \text{True}$$

Since 20 satisfies the *original* equation, the solution set is $\{20\}$.

Work Problem 1 at the Side.

The solution of the equation in Example 1 can be generalized to give a method for solving equations with radicals.

1 Solve.

(a) $\sqrt{r} = 3$

(b) $\sqrt{5x+1} = 4$

ANSWERS

1. (a) $\{9\}$ **(b)** $\{3\}$

2 Solve.

(a) $\sqrt{x} + 4 = -3$

Audio

Solving an Equation with Radicals

Step 1 **Isolate the radical.** Make sure that one radical term is alone on one side of the equation.

Step 2 **Apply the power rule.** Raise both sides of the equation to a power that is the same as the index of the radical.

Step 3 **Solve.** Solve the resulting equation; if it still contains a radical, repeat Steps 1 and 2.

Step 4 **Check** all potential solutions in the original equation.

CAUTION

Remember Step 4 or you may get an incorrect solution set.

Video You Try It

EXAMPLE 2 **Using the Power Rule**

Solve $\sqrt{5x - 1} + 3 = 0$.

Step 1 To isolate the radical on one side, subtract 3 from each side.

$$\sqrt{5x - 1} = -3$$

Step 2 Now square both sides.

$$(\sqrt{5x - 1})^2 = (-3)^2$$

Step 3

$$5x - 1 = 9$$
$$5x = 10$$
$$x = 2$$

Step 4 Check the potential solution, 2, by substituting it in the original equation.

$$\sqrt{5x - 1} + 3 = 0$$
$$\sqrt{5 \cdot 2 - 1} + 3 \stackrel{?}{=} 0 \quad \text{Let } x = 2.$$
$$3 + 3 = 0 \quad \text{False}$$

This false result shows that 2 is *not* a solution of the original equation; it is extraneous. The solution set is $\emptyset$.

(b) $\sqrt{x - 9} - 3 = 0$

NOTE

We could have determined after Step 1 that the equation in Example 2 has no solution because the expression on the left cannot be negative.

Work Problem 2 at the Side.

OBJECTIVE 2 **Solve radical equations that require additional steps.**

The next examples involve finding the square of a binomial. Recall that

$$(x + y)^2 = x^2 + 2xy + y^2.$$

ANSWERS

2. (a) $\emptyset$ (b) $\{18\}$

EXAMPLE 3 Using the Power Rule; Squaring a Binomial

Solve $\sqrt{4-x} = x + 2$.

Step 1 The radical is alone on the left side of the equation.

Step 2 Square both sides; on the right, $(x+2)^2 = x^2 + 2(x)(2) + 2^2$.

$$(\sqrt{4-x})^2 = (x+2)^2$$
$$4 - x = x^2 \underbrace{+\,4x}_{\text{Twice the product of 2 and } x} + 4$$

Step 3 The new equation is quadratic, so get 0 on one side.

$$0 = x^2 + 5x \qquad \text{Subtract 4 and add } x.$$
$$0 = x(x+5) \qquad \text{Factor.}$$
$$x = 0 \quad \text{or} \quad x + 5 = 0 \qquad \text{Zero-factor property}$$
$$x = -5$$

Step 4 Check each potential solution in the original equation.

If $x = 0$, then		If $x = -5$, then	
$\sqrt{4-x} = x + 2$		$\sqrt{4-x} = x + 2$	
$\sqrt{4-0} = 0 + 2$	?	$\sqrt{4-(-5)} = -5 + 2$	?
$\sqrt{4} = 2$	?	$\sqrt{9} = -3$	?
$2 = 2.$	True	$3 = -3.$	False

The solution set is $\{0\}$. The other potential solution, -5, is extraneous.

CAUTION
When a radical equation requires squaring a binomial as in Example 3, ***remember to include the middle term.***

$$(x+2)^2 = x^2 + 4x + 4$$

Work Problem 3 at the Side.

EXAMPLE 4 Using the Power Rule; Squaring a Binomial

Solve $\sqrt{x^2 - 4x + 9} = x - 1$.

Square both sides; $(x-1)^2 = x^2 - 2(x)(1) + 1^2$ on the right.

$$(\sqrt{x^2 - 4x + 9})^2 = (x-1)^2$$
$$x^2 - 4x + 9 = x^2 \underbrace{-\,2x}_{\text{Twice the product of } x \text{ and } -1} + 1$$
$$-2x = -8 \qquad \text{Subtract } x^2 \text{ and 9; add } 2x.$$
$$x = 4 \qquad \text{Divide by } -2.$$

Check:
$$\sqrt{x^2 - 4x + 9} = x - 1$$
$$\sqrt{4^2 - 4 \cdot 4 + 9} = 4 - 1 \quad ? \qquad \text{Let } x = 4.$$
$$3 = 3 \qquad \text{True}$$

The solution set of the original equation is $\{4\}$.

Work Problem 4 at the Side.

3 Solve.

(a) $\sqrt{3x - 5} = x - 1$

(b) $x + 1 = \sqrt{-2x - 2}$

4 Solve.

$\sqrt{4x^2 + 2x - 3} = 2x + 7$

ANSWERS
3. **(a)** $\{2, 3\}$ **(b)** $\{-1\}$
4. $\{-2\}$

5 **(a)** Verify that 15 is not a solution of the equation in Example 5.

(b) Solve.

$\sqrt{x+1} - \sqrt{x-4} = 1$

6 Solve each equation.

(a) $\sqrt[3]{x^2+3x+12} = \sqrt[3]{x^2}$

(b) $\sqrt[4]{2x+5} + 1 = 0$

EXAMPLE 5 Using the Power Rule; Squaring Twice

Solve $\sqrt{5x+6} + \sqrt{3x+4} = 2$.

Start by isolating one radical on one side of the equation by subtracting $\sqrt{3x+4}$ from each side. Then square both sides.

$$\sqrt{5x+6} = 2 - \sqrt{3x+4}$$
$$(\sqrt{5x+6})^2 = (2-\sqrt{3x+4})^2$$
$$5x+6 = 4 - 4\sqrt{3x+4} + (3x+4)$$

Twice the product of 2 and $-\sqrt{3x+4}$

This equation still contains a radical, so square both sides again. Before doing this, isolate the radical term on the right.

$$5x+6 = 8 + 3x - 4\sqrt{3x+4}$$
$$2x - 2 = -4\sqrt{3x+4} \qquad \text{Subtract 8 and } 3x.$$
$$x - 1 = -2\sqrt{3x+4} \qquad \text{Divide by 2.}$$
$$(x-1)^2 = (-2\sqrt{3x+4})^2 \qquad \text{Square both sides again.}$$
$$x^2 - 2x + 1 = (-2)^2(\sqrt{3x+4})^2 \qquad (ab)^2 = a^2b^2$$
$$x^2 - 2x + 1 = 4(3x+4)$$
$$x^2 - 2x + 1 = 12x + 16 \qquad \text{Distributive property}$$
$$x^2 - 14x - 15 = 0 \qquad \text{Standard form}$$
$$(x+1)(x-15) = 0 \qquad \text{Factor.}$$
$$x+1 = 0 \quad \text{or} \quad x - 15 = 0 \qquad \text{Zero-factor property}$$
$$x = -1 \quad \text{or} \quad x = 15$$

Check each of these potential solutions in the original equation. Only -1 satisfies the equation, so the solution set, $\{-1\}$, has only one element.

Work Problem 5 at the Side.

OBJECTIVE 3 Solve radical equations with indexes greater than 2.

The power rule also works for powers greater than 2.

EXAMPLE 6 Using the Power Rule for a Power Greater than 2

Solve $\sqrt[3]{x+5} = \sqrt[3]{2x-6}$.

Raise both sides to the third power.

$$(\sqrt[3]{x+5})^3 = (\sqrt[3]{2x-6})^3$$
$$x + 5 = 2x - 6$$
$$11 = x \qquad \text{Subtract } x\text{; add 6.}$$

Check this result in the original equation.

$$\sqrt[3]{x+5} = \sqrt[3]{2x-6}$$
$$\sqrt[3]{11+5} \stackrel{?}{=} \sqrt[3]{2\cdot 11-6} \qquad \text{Let } x = 11.$$
$$\sqrt[3]{16} = \sqrt[3]{16} \qquad \text{True}$$

The solution set is $\{11\}$.

Work Problem 6 at the Side.

You Try It

Animation

Video

You Try It

Video

ANSWERS

5. (a) The final step in the check leads to $16 = 2$, which is false.
(b) $\{8\}$
6. (a) $\{-4\}$ (b) $\emptyset$

9.7 Complex Numbers

OBJECTIVES

1. Simplify numbers of the form $\sqrt{-b}$, where $b > 0$.
2. Recognize subsets of the complex numbers.
3. Add and subtract complex numbers.
4. Multiply complex numbers.
5. Divide complex numbers.
6. Find powers of i.

As we saw in **Section 1.1,** the set of real numbers includes many other number sets (the rational numbers, integers, and natural numbers, for example). In this section a new set of numbers is introduced that includes the set of real numbers, as well as numbers that are even roots of negative numbers, like $\sqrt{-2}$.

OBJECTIVE 1 Simplify numbers of the form $\sqrt{-b}$, where $b > 0$. The equation $x^2 + 1 = 0$ has no real number solution since any solution must be a number whose square is -1. In the set of real numbers, all squares are nonnegative numbers because the product of two positive numbers or two negative numbers is positive and $0^2 = 0$. To provide a solution for the equation $x^2 + 1 = 0$, a new number i, the **imaginary unit,** is defined so that

$$i^2 = -1.$$

That is, i is a number whose square is -1, so $i = \sqrt{-1}$. This definition of i makes it possible to define any square root of a negative number as follows.

$\sqrt{-b}$

For any positive number b,

$$\sqrt{-b} = i\sqrt{b}.$$

EXAMPLE 1 Simplifying Square Roots of Negative Numbers

Write each number as a product of a real number and i.

(a) $\sqrt{-100} = i\sqrt{100} = 10i$ **(b)** $-\sqrt{-36} = -i\sqrt{36} = -6i$

(c) $\sqrt{-2} = i\sqrt{2}$

(d) $\sqrt{-8} = \sqrt{-4 \cdot 2} = \sqrt{-4} \cdot \sqrt{2} = 2i\sqrt{2}$

CAUTION

It is easy to mistake $\sqrt{2}i$ for $\sqrt{2i}$, with the i under the radical. For this reason, we usually write $\sqrt{2}i$ as $i\sqrt{2}$, as in the definition of $\sqrt{-b}$.

Work Problem 1 at the Side.

When finding a product such as $\sqrt{-4} \cdot \sqrt{-9}$, we cannot use the product rule for radicals because it applies only to nonnegative radicands. For this reason, we change $\sqrt{-b}$ to the form $i\sqrt{b}$ before performing any multiplications or divisions. For example,

$$\begin{aligned}
\sqrt{-4} \cdot \sqrt{-9} &= i\sqrt{4} \cdot i\sqrt{9} \\
&= i \cdot 2 \cdot i \cdot 3 \\
&= 6i^2 \\
&= 6(-1) \qquad \text{Substitute: } i^2 = -1. \\
&= -6.
\end{aligned}$$

1 Write each number as a product of a real number and i.

(a) $\sqrt{-16}$

(b) $-\sqrt{-81}$

(c) $\sqrt{-7}$

(d) $\sqrt{-32}$

ANSWERS

1. (a) $4i$ (b) $-9i$ (c) $i\sqrt{7}$ (d) $4i\sqrt{2}$

2 Multiply.

(a) $\sqrt{-7} \cdot \sqrt{-7}$

(b) $\sqrt{-5} \cdot \sqrt{-10}$

(c) $\sqrt{-15} \cdot \sqrt{2}$

3 Divide.

(a) $\dfrac{\sqrt{-32}}{\sqrt{-2}}$

(b) $\dfrac{\sqrt{-27}}{\sqrt{-3}}$

(c) $\dfrac{\sqrt{-40}}{\sqrt{10}}$

CAUTION

Using the product rule for radicals *before* using the definition of $\sqrt{-b}$ gives a *wrong* answer. The preceding example shows that

$$\sqrt{-4} \cdot \sqrt{-9} = -6, \text{ but}$$
$$\sqrt{-4(-9)} = \sqrt{36} = 6,$$
so
$$\sqrt{-4} \cdot \sqrt{-9} \neq \sqrt{-4(-9)}.$$

Video

EXAMPLE 2 Multiplying Square Roots of Negative Numbers

Multiply.

(a)
$$\begin{aligned}\sqrt{-3} \cdot \sqrt{-7} &= i\sqrt{3} \cdot i\sqrt{7}\\ &= i^2\sqrt{3 \cdot 7}\\ &= (-1)\sqrt{21} \qquad \text{Substitute: } i^2 = -1.\\ &= -\sqrt{21}\end{aligned}$$

(b)
$$\begin{aligned}\sqrt{-2} \cdot \sqrt{-8} &= i\sqrt{2} \cdot i\sqrt{8}\\ &= i^2\sqrt{2 \cdot 8}\\ &= (-1)\sqrt{16}\\ &= (-1)4\\ &= -4\end{aligned}$$

(c) $\sqrt{-5} \cdot \sqrt{6} = i\sqrt{5} \cdot \sqrt{6} = i\sqrt{30}$

Work Problem 2 at the Side.

The methods used to find products also apply to quotients.

Video

EXAMPLE 3 Dividing Square Roots of Negative Numbers

Divide.

(a) $$\frac{\sqrt{-75}}{\sqrt{-3}} = \frac{i\sqrt{75}}{i\sqrt{3}} = \sqrt{\frac{75}{3}} = \sqrt{25} = 5$$

(b) $$\frac{\sqrt{-32}}{\sqrt{8}} = \frac{i\sqrt{32}}{\sqrt{8}} = i\sqrt{\frac{32}{8}} = i\sqrt{4} = 2i$$

Work Problem 3 at the Side.

OBJECTIVE 2 Recognize subsets of the complex numbers. With the imaginary unit i and the real numbers, a new set of numbers can be formed that includes the real numbers as a subset. The *complex numbers* are defined as follows.

Audio

Complex Number

If a and b are real numbers, then any number of the form $a + bi$ is called a **complex number.**

ANSWERS

2. (a) -7 (b) $-5\sqrt{2}$ (c) $i\sqrt{30}$
3. (a) 4 (b) 3 (c) $2i$

In the complex number $a + bi$, the number a is called the **real part** and b is called the **imaginary part.*** When $b = 0$, $a + bi$ is a real number, so the real numbers are a subset of the complex numbers. Complex numbers with $a = 0$ and $b \neq 0$ are called **pure imaginary numbers.** In spite of their name, these numbers are very useful in applications, particularly in work with electricity.

The relationships among the sets of numbers are shown in Figure 10.

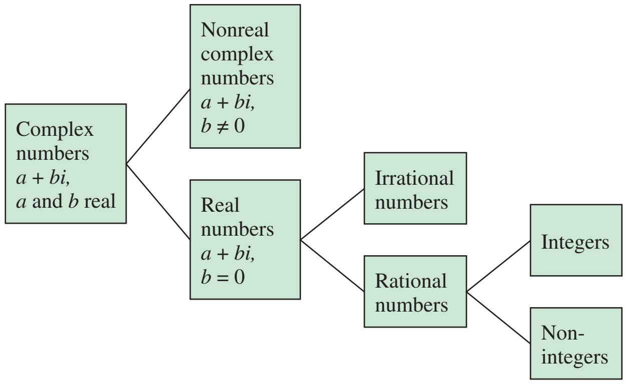


Figure 10

OBJECTIVE 3 Add and subtract complex numbers. The commutative, associative, and distributive properties for real numbers are also valid for complex numbers. Thus, to add complex numbers, we add their real parts and add their imaginary parts.

EXAMPLE 4 Adding Complex Numbers

Add.

(a) $(2 + 3i) + (6 + 4i)$

$= (2 + 6) + (3 + 4)i$ Commutative, associative, and distributive properties

$= 8 + 7i$

(b) $5 + (9 - 3i) = (5 + 9) - 3i$

$= 14 - 3i$

Work Problem 4 at the Side.

We subtract complex numbers by subtracting their real parts and subtracting their imaginary parts.

EXAMPLE 5 Subtracting Complex Numbers

Subtract.

(a) $(6 + 5i) - (3 + 2i) = (6 - 3) + (5 - 2)i$

$= 3 + 3i$

(b) $(7 - 3i) - (8 - 6i) = (7 - 8) + [-3 - (-6)]i$

$= -1 + 3i$

(c) $(-9 + 4i) - (-9 + 8i) = (-9 + 9) + (4 - 8)i$

$= 0 - 4i$

$= -4i$

Work Problem 5 at the Side.

*Some texts define bi as the imaginary part of the complex number $a + bi$.

4 Add.

(a) $(4 + 6i) + (-3 + 5i)$

(b) $(-1 + 8i) + (9 - 3i)$

5 Subtract.

(a) $(7 + 3i) - (4 + 2i)$

(b) $(-6 - i) - (-5 - 4i)$

(c) $8 - (3 - 2i)$

Answers

4. (a) $1 + 11i$ **(b)** $8 + 5i$

5. (a) $3 + i$ **(b)** $-1 + 3i$ **(c)** $5 + 2i$

6 Multiply.

(a) $6i(4 + 3i)$

(b) $(6 - 4i)(2 + 4i)$

(c) $(3 - 2i)(3 + 2i)$

In Example 5(c), the answer was written as $0 - 4i$ and then as just $-4i$. A complex number written in the form $a + bi$, like $0 - 4i$, is in **standard form.** In this section, most answers will be given in standard form, but if a or b is 0, we consider answers such as a or bi to be in standard form.

OBJECTIVE 4 Multiply complex numbers. We multiply complex numbers as we multiply polynomials. Complex numbers of the form $a + bi$ have the same form as binomials, so we multiply two complex numbers in standard form by using the FOIL method for multiplying binomials. (Recall from **Section 6.4** that FOIL stands for *First, Outer, Inner, Last*.)

EXAMPLE 6 Multiplying Complex Numbers

Multiply.

(a) $4i(2 + 3i)$

Use the distributive property.

$$\begin{aligned} 4i(2 + 3i) &= 4i(2) + 4i(3i) \\ &= 8i + 12i^2 \\ &= 8i + 12(-1) \qquad \text{Substitute: } i^2 = -1. \\ &= -12 + 8i \end{aligned}$$

(b) $(3 + 5i)(4 - 2i)$

Use the FOIL method.

$$\begin{aligned} (3 + 5i)(4 - 2i) &= \underbrace{3(4)}_{\text{First}} + \underbrace{3(-2i)}_{\text{Outer}} + \underbrace{5i(4)}_{\text{Inner}} + \underbrace{5i(-2i)}_{\text{Last}} \\ &= 12 - 6i + 20i - 10i^2 \\ &= 12 + 14i - 10(-1) \qquad \text{Substitute: } i^2 = -1. \\ &= 12 + 14i + 10 \\ &= 22 + 14i \end{aligned}$$

(c)
$$\begin{aligned} (2 + 3i)(1 - 5i) &= 2(1) + 2(-5i) + 3i(1) + 3i(-5i) \qquad \text{FOIL} \\ &= 2 - 10i + 3i - 15i^2 \\ &= 2 - 7i - 15(-1) \\ &= 2 - 7i + 15 \\ &= 17 - 7i \end{aligned}$$

Work Problem 6 at the Side.

The two complex numbers $a + bi$ and $a - bi$ are called *complex conjugates* of each other. The product of a complex number and its conjugate is always a real number, as shown here.

$$\begin{aligned} (a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ (a + bi)(a - bi) &= a^2 + b^2 \end{aligned}$$

For example, $(3 + 7i)(3 - 7i) = 3^2 + 7^2 = 9 + 49 = 58$.

OBJECTIVE 5 Divide complex numbers. The quotient of two complex numbers should be a complex number. To write the quotient as a complex number, we need to eliminate i in the denominator. We use conjugates to do this.

ANSWERS

6. **(a)** $-18 + 24i$ **(b)** $28 + 16i$ **(c)** 13

EXAMPLE 7 Dividing Complex Numbers

Find each quotient.

(a) $\dfrac{8 + 9i}{5 + 2i}$

Multiply both the numerator and denominator by the conjugate of the denominator. The conjugate of $5 + 2i$ is $5 - 2i$.

$$\frac{8 + 9i}{5 + 2i} = \frac{(8 + 9i)(5 - 2i)}{(5 + 2i)(5 - 2i)} \qquad \frac{5 - 2i}{5 - 2i} = 1$$

$$= \frac{40 - 16i + 45i - 18i^2}{5^2 + 2^2}$$

$$= \frac{58 + 29i}{29} \qquad \text{Substitute: } i^2 = -1; \text{ combine terms.}$$

$$= \frac{29(2 + i)}{29} \qquad \text{Factor the numerator.}$$

$$= 2 + i \qquad \text{Lowest terms}$$

Notice that this is just like rationalizing a denominator. The final result is in standard form.

(b) $\dfrac{1 + i}{i}$

The conjugate of i is $-i$. Multiply both the numerator and denominator by $-i$.

$$\frac{1 + i}{i} = \frac{(1 + i)(-i)}{i(-i)}$$

$$= \frac{-i - i^2}{-i^2}$$

$$= \frac{-i - (-1)}{-(-1)} \qquad \text{Substitute: } i^2 = -1.$$

$$= \frac{-i + 1}{1}$$

$$= 1 - i$$

Work Problem 7 at the Side.

Calculator Tip In Examples 4–7, we showed how complex numbers can be added, subtracted, multiplied, and divided algebraically. Many current models of graphing calculators can perform these operations. Figure 11 shows how the computations in parts of Examples 4–7 are displayed on a TI-83 Plus or TI-84 calculator. Be sure to use parentheses as shown.

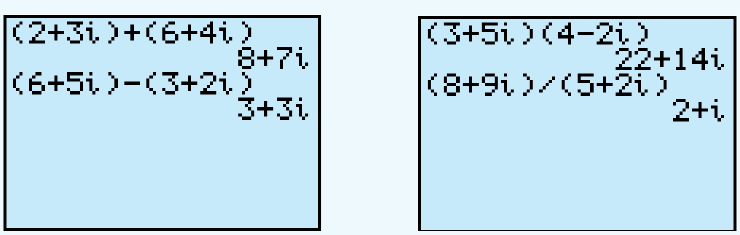


Figure 11

7 Find each quotient.

(a) $\dfrac{2 + i}{3 - i}$

(b) $\dfrac{8 - 4i}{1 - i}$

(c) $\dfrac{5}{3 - 2i}$

(d) $\dfrac{5 - i}{i}$

Answers

7. (a) $\frac{1}{2} + \frac{1}{2}i$ (b) $6 + 2i$ (c) $\frac{15}{13} + \frac{10}{13}i$ (d) $-1 - 5i$

8 Find each power of i.

(a) i^{21}

(b) i^{36}

(c) i^{50}

(d) i^{-9}

OBJECTIVE 6 Find powers of *i*. Because i^2 is defined to be -1, we can find higher powers of i as shown in the following examples.

$$i^3 = i \cdot i^2 = i(-1) = -i \qquad i^6 = i^2 \cdot i^4 = (-1) \cdot 1 = -1$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad i^7 = i^3 \cdot i^4 = (-i) \cdot 1 = -i$$
$$i^5 = i \cdot i^4 = i \cdot 1 = i \qquad i^8 = i^4 \cdot i^4 = 1 \cdot 1 = 1$$

As these examples suggest, the powers of i rotate through the four numbers i, -1, $-i$, and 1. Larger powers of i can be simplified by using the fact that $i^4 = 1$. For example,

$$i^{75} = (i^4)^{18} \cdot i^3 = 1^{18} \cdot i^3 = 1 \cdot i^3 = i^3 = -i.$$

This example suggests a quick method for simplifying larger powers of i.

EXAMPLE 8 Simplifying Powers of *i*

Find each power of i.

(a) $i^{12} = (i^4)^3 = 1^3 = 1$

(b) $i^{39} = i^{36} \cdot i^3 = (i^4)^9 \cdot i^3 = 1^9 \cdot (-i) = -i$

(c) $i^{-2} = \dfrac{1}{i^2} = \dfrac{1}{-1} = -1$

(d) $i^{-1} = \dfrac{1}{i}$

To simplify this quotient, multiply both the numerator and denominator by $-i$, the conjugate of i.

$$\frac{1}{i} = \frac{1(-i)}{i(-i)} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = \frac{-i}{1} = -i$$

Work Problem 8 at the Side.

ANSWERS

8. (a) i (b) 1 (c) -1 (d) $-i$

Quadratic Equations, Inequalities, and Functions

10

Since 1980, the number of multiple births in the United States has increased 59%, primarily due to greater use of fertility drugs and greater numbers of births to women over age 40. The number of higher-order multiple births—that is, births involving triplets or more—has increased over 400%. One of the most publicized higher-order multiple births occurred November 19, 1997, with the birth of the McCaughey septuplets in Des Moines, Iowa. All seven premature babies survived, a first in medical history. (*Source:* American College of Obstetricians and Gynecologists; *The Gazette,* November 19, 2003.)

In Example 6 of Section 10.5, we determine a *quadratic function* that models the number of higher-order multiple births in the United States.

10.1 The Square Root Property and Completing the Square

OBJECTIVES

1. Learn the square root property.
2. Solve quadratic equations of the form $(ax + b)^2 = c$ by using the square root property.
3. Solve quadratic equations by completing the square.
4. Solve quadratic equations with nonreal complex solutions.

We introduced quadratic equations in **Section 7.4.** Recall that a *quadratic equation* is defined as follows.

Quadratic Equation

An equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers, with $a \neq 0$, is a **quadratic equation.** The given form is called **standard form.**

Audio

A quadratic equation is a *second-degree equation,* that is, an equation with a squared term and no terms of higher degree. For example,

$$4m^2 + 4m - 5 = 0 \quad \text{and} \quad 3x^2 = 4x - 8 \qquad \text{Quadratic equations}$$

are quadratic equations, with the first equation in standard form.

Work Problem 1 at the Side.

In **Section 7.4** we used factoring and the zero-factor property to solve quadratic equations.

Zero-Factor Property

If two numbers have a product of 0, then at least one of the numbers must be 0. That is, if $ab = 0$, then $a = 0$ or $b = 0$.

Audio

We solved a quadratic equation such as $3x^2 - 5x - 28 = 0$ using the zero-factor property as follows.

$$3x^2 - 5x - 28 = 0$$

$$(3x + 7)(x - 4) = 0 \qquad \text{Factor.}$$

$$3x + 7 = 0 \quad \text{or} \quad x - 4 = 0 \qquad \text{Zero-factor property}$$

$$3x = -7 \quad \text{or} \quad x = 4 \qquad \text{Solve each equation.}$$

$$x = -\frac{7}{3}$$

The solution set is $\left\{-\frac{7}{3}, 4\right\}$.

Work Problem 2 at the Side.

OBJECTIVE 1 Learn the square root property. Although factoring is the simplest way to solve quadratic equations, not every quadratic equation can be solved easily by factoring. In this section and the next, we develop other methods of solving quadratic equations based on the following property.

Square Root Property

If x and k are complex numbers and $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

1 (a) Which of the following are quadratic equations?

A. $x + 2y = 0$

B. $x^2 - 8x + 16 = 0$

C. $2t^2 - 5t = 3$

D. $x^3 + x^2 + 4 = 0$

(b) Which quadratic equation identified in part (a) is in standard form?

2 Solve each equation by factoring.

(a) $x^2 + 3x + 2 = 0$

(b) $3m^2 = 3 - 8m$

(*Hint:* Remember to write the equation in standard form first.)

Answers

1. (a) B, C (b) B

2. (a) $\{-2, -1\}$ (b) $\left\{-3, \frac{1}{3}\right\}$

The following steps justify the square root property.

$$\begin{aligned} x^2 &= k & & \\ x^2 - k &= 0 & & \text{Subtract } k. \\ (x - \sqrt{k})(x + \sqrt{k}) &= 0 & & \text{Factor.} \\ x - \sqrt{k} = 0 \quad \text{or} \quad x + \sqrt{k} &= 0 & & \text{Zero-factor property} \\ \boldsymbol{x = \sqrt{k}} \quad \text{or} \quad \boldsymbol{x} &\boldsymbol{= -\sqrt{k}} & & \text{Solve each equation.} \end{aligned}$$

Thus, the solutions of the equation $x^2 = k$ are $x = \sqrt{k}$ or $x = -\sqrt{k}$.

CAUTION

If $k \neq 0$, then using the square root property always produces *two* square roots, one positive and one negative.

EXAMPLE 1 Using the Square Root Property

Solve each equation.

(a) $r^2 = 5$

By the square root property, if $r^2 = 5$, then

$$r = \sqrt{5} \quad \text{or} \quad r = -\sqrt{5},$$

and the solution set is $\{\sqrt{5}, -\sqrt{5}\}$.

(b) $4x^2 - 48 = 0$

Solve for x^2.

$$\begin{aligned} 4x^2 - 48 &= 0 & & \\ 4x^2 &= 48 & & \text{Add 48.} \\ x^2 &= 12 & & \text{Divide by 4.} \\ x = \sqrt{12} \quad \text{or} \quad x &= -\sqrt{12} & & \text{Square root property} \\ x = 2\sqrt{3} \quad \text{or} \quad x &= -2\sqrt{3} & & \sqrt{12} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3} \end{aligned}$$

Check: $4x^2 - 48 = 0$ Original equation

$4(2\sqrt{3})^2 - 48 = 0$?	$4(-2\sqrt{3})^2 - 48 = 0$?
$4(12) - 48 = 0$?	$4(12) - 48 = 0$?
$48 - 48 = 0$?	$48 - 48 = 0$?
$0 = 0$ True	$0 = 0$ True

The solution set is $\{2\sqrt{3}, -2\sqrt{3}\}$.

Work Problem 3 at the Side.

NOTE

Recall that solutions such as those in Example 1 are sometimes abbreviated with the symbol $\pm$ (read "positive or negative"); with this symbol the solutions in Example 1 would be written $\pm\sqrt{5}$ and $\pm 2\sqrt{3}$.

3 Solve each equation.

(a) $m^2 = 64$

(b) $p^2 = 7$

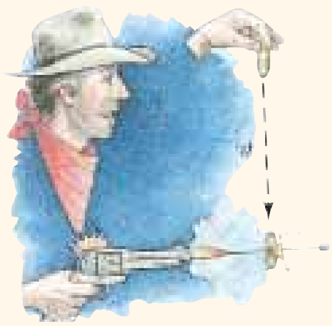
(c) $3x^2 - 54 = 0$

ANSWERS

3. **(a)** $\{8, -8\}$ **(b)** $\{\sqrt{7}, -\sqrt{7}\}$ **(c)** $\{3\sqrt{2}, -3\sqrt{2}\}$

4 Solve the problem.

An expert marksman can hold a silver dollar at forehead level, drop it, draw his gun, and shoot the coin as it passes waist level. If the coin falls about 4 ft, use the formula in Example 2 to find the time that elapses between the dropping of the coin and the shot.



EXAMPLE 2 Using the Square Root Property in an Application

Galileo Galilei (1564–1642) developed a formula for freely falling objects described by

$$d = 16t^2,$$

where d is the distance in feet that an object falls (disregarding air resistance) in t seconds, regardless of weight. Galileo dropped objects from the Leaning Tower of Pisa to develop this formula. If the Leaning Tower is about 180 ft tall, use Galileo's formula to determine how long it would take an object dropped from the tower to fall to the ground. (*Source: Microsoft Encarta Encyclopedia 2002*.)

We substitute 180 for d in Galileo's formula.

$$\begin{aligned} \mathbf{d} &= 16t^2 & & \\ \mathbf{180} &= 16t^2 & & \text{Let } d = 180. \\ 11.25 &= t^2 & & \text{Divide by 16.} \\ t = \sqrt{11.25} \quad \text{or} \quad t &= -\sqrt{11.25} & & \text{Square root property} \end{aligned}$$

Since time cannot be negative, we discard the negative solution. In applied problems, we usually prefer approximations to exact values. Using a calculator, $\sqrt{11.25} \approx 3.4$ so $t \approx 3.4$. The object would fall to the ground in about 3.4 sec.

Work Problem 4 at the Side.

Video

OBJECTIVE 2 Solve quadratic equations of the form $(ax + b)^2 = c$ by using the square root property. To solve more complicated equations using the square root property, such as

$$(x - 5)^2 = 36,$$

substitute $(x - 5)^2$ for x^2 and 36 for k, to get

$$\begin{aligned} x - 5 &= \sqrt{36} & \text{or} \quad x - 5 &= -\sqrt{36} \\ x - 5 &= 6 & \text{or} \quad x - 5 &= -6 \\ x &= \mathbf{11} & \text{or} \quad x &= \mathbf{-1}. \end{aligned}$$

Check: $(x - 5)^2 = 36$ Original equation

$$\begin{aligned} (\mathbf{11} - 5)^2 &= 36 \quad ? & (\mathbf{-1} - 5)^2 &= 36 \quad ? \\ 6^2 &= 36 \quad ? & (-6)^2 &= 36 \quad ? \\ 36 &= 36 \quad \text{True} & 36 &= 36 \quad \text{True} \end{aligned}$$

Video Animation You Try It

EXAMPLE 3 Using the Square Root Property

Solve $(2x - 3)^2 = 18$.

$$\begin{aligned} 2x - 3 = \sqrt{18} \quad &\text{or} \quad 2x - 3 = -\sqrt{18} & & \text{Square root property} \\ 2x = 3 + \sqrt{18} \quad &\text{or} \quad 2x = 3 - \sqrt{18} & & \text{Add 3.} \\ x = \frac{3 + \sqrt{18}}{2} \quad &\text{or} \quad x = \frac{3 - \sqrt{18}}{2} & & \text{Divide by 2.} \\ x = \frac{3 + 3\sqrt{2}}{2} \quad &\text{or} \quad x = \frac{3 - 3\sqrt{2}}{2} & & \sqrt{18} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2} \end{aligned}$$

Continued on Next Page

ANSWERS

4. .5 sec

We show the check for the first solution. The check for the second solution is similar.

Check:

$$(2x - 3)^2 = 18 \quad \text{Original equation}$$

$$\left[2\left(\frac{3 + 3\sqrt{2}}{2}\right) - 3\right]^2 = 18 \quad ?$$

$$(3 + 3\sqrt{2} - 3)^2 = 18 \quad ?$$

$$(3\sqrt{2})^2 = 18 \quad ?$$

$$18 = 18 \quad \text{True}$$

The solution set is $\left\{\frac{3 + 3\sqrt{2}}{2}, \frac{3 - 3\sqrt{2}}{2}\right\}$.

Work Problem 5 at the Side.

OBJECTIVE 3 Solve quadratic equations by completing the square. We can use the square root property to solve *any* quadratic equation by writing it in the form $(x + k)^2 = n$. That is, we must write the left side of the equation as a perfect square trinomial that can be factored as $(x + k)^2$, the square of a binomial, and the right side must be a constant. Rewriting a quadratic equation in this form is called **completing the square.**

Recall that the perfect square trinomial

$$x^2 + 10x + 25$$

can be factored as $(x + 5)^2$. In the trinomial, the coefficient of x (the first-degree term) is 10 and the constant term is 25. Notice that if we take half of 10 and square it, we get the constant term, 25.

$$\left[\frac{1}{2}(\underset{\text{Coefficient of } x}{10})\right]^2 = 5^2 = \underset{\text{Constant}}{25}$$

Similarly, in

$$x^2 + 12x + 36, \quad \left[\frac{1}{2}(12)\right]^2 = 6^2 = 36,$$

and in

$$m^2 - 6m + 9, \quad \left[\frac{1}{2}(-6)\right]^2 = (-3)^2 = 9.$$

This relationship is true in general and is the idea behind completing the square.

Work Problem 6 at the Side.

EXAMPLE 4 Solving a Quadratic Equation by Completing the Square

Solve $x^2 + 8x + 10 = 0$.

This quadratic equation cannot be solved easily by factoring, and it is not in the correct form to solve using the square root property. To solve it by completing the square, we need a perfect square trinomial on the left side of the equation. To get this form, we first subtract 10 from each side.

Continued on Next Page

5 Solve each equation.

(a) $(x - 3)^2 = 25$

(b) $(3k + 1)^2 = 2$

(c) $(2r + 3)^2 = 8$

6 Find the constant to be added to get a perfect square trinomial. In each case, take half the coefficient of the first-degree term and square the result.

(a) $x^2 + 4x +$ ____

(b) $t^2 - 2t +$ ____

(c) $m^2 + 5m +$ ____

(d) $x^2 - \frac{2}{3}x +$ ____

ANSWERS

5. (a) $\{-2, 8\}$

(b) $\left\{\frac{-1 + \sqrt{2}}{3}, \frac{-1 - \sqrt{2}}{3}\right\}$

(c) $\left\{\frac{-3 + 2\sqrt{2}}{2}, \frac{-3 - 2\sqrt{2}}{2}\right\}$

6. (a) 4 **(b)** 1 **(c)** $\frac{25}{4}$ **(d)** $\frac{1}{9}$

7 Solve $n^2 + 6n + 4 = 0$ by completing the square.

$$x^2 + 8x + 10 = 0 \qquad \text{Original equation}$$
$$x^2 + 8x = -10 \qquad \text{Subtract 10.}$$

We must add a constant to get a perfect square trinomial on the left.

$$\underbrace{x^2 + 8x + \_\_\_}_{\text{Needs to be a perfect square trinomial}}$$

To find this constant, we apply the ideas preceding this example—we take half the coefficient of the first-degree term and square the result.

$$\left[\frac{1}{2}(8)\right]^2 = 4^2 = 16 \leftarrow \text{Desired constant}$$

Now we add 16 to *each* side of the equation. (Why?)

$$x^2 + 8x + 16 = -10 + 16$$

Next we factor on the left side and add on the right.

$$(x + 4)^2 = 6$$

We can now use the square root property.

$$x + 4 = \sqrt{6} \quad \text{or} \quad x + 4 = -\sqrt{6}$$
$$x = -4 + \sqrt{6} \quad \text{or} \quad x = -4 - \sqrt{6}$$

Check:

$$x^2 + 8x + 10 = 0 \qquad \text{Original equation}$$
$$(-4 + \sqrt{6})^2 + 8(-4 + \sqrt{6}) + 10 = 0 \quad ? \qquad \text{Let } x = -4 + \sqrt{6}.$$
$$16 - 8\sqrt{6} + 6 - 32 + 8\sqrt{6} + 10 = 0 \quad ?$$
$$0 = 0 \qquad \text{True}$$

The check of the other solution is similar. Thus, $\{-4 + \sqrt{6}, -4 - \sqrt{6}\}$ is the solution set.

Work Problem 7 at the Side.

The procedure from Example 4 can be generalized.

Completing the Square

To solve $ax^2 + bx + c = 0$ $(a \neq 0)$ by completing the square, use these steps.

Step 1 **Be sure the squared term has coefficient 1.** If the coefficient of the squared term is some other nonzero number a, divide each side of the equation by a.

Step 2 **Write the equation in correct form** so that terms with variables are on one side of the equals sign and the constant is on the other side.

Step 3 **Square half the coefficient of the first-degree term.**

Step 4 **Add the square to each side.**

Step 5 **Factor the perfect square trinomial.** One side should now be a perfect square trinomial. Factor it as the square of a binomial. Simplify the other side.

Step 6 **Solve the equation.** Apply the square root property to complete the solution.

ANSWERS

7. $\{-3 + \sqrt{5}, -3 - \sqrt{5}\}$

EXAMPLE 5 Solving a Quadratic Equation with $a = 1$ by Completing the Square

Solve $k^2 + 5k - 1 = 0$.

Since the coefficient of the squared term is 1, begin with Step 2.

Step 2 $\quad k^2 + 5k = 1 \quad$ Add 1 to each side.

Step 3 Take half the coefficient of the first-degree term and square the result.

$$\left[\frac{1}{2}(5)\right]^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

Step 4 $\quad k^2 + 5k + \frac{25}{4} = 1 + \frac{25}{4} \quad$ Add the square to each side of the equation.

Step 5 $\quad \left(k + \frac{5}{2}\right)^2 = \frac{29}{4} \quad$ Factor on the left; add on the right.

Step 6 $\quad k + \frac{5}{2} = \sqrt{\frac{29}{4}} \quad$ or $\quad k + \frac{5}{2} = -\sqrt{\frac{29}{4}} \quad$ Square root property

$$k + \frac{5}{2} = \frac{\sqrt{29}}{2} \quad \text{or} \quad k + \frac{5}{2} = -\frac{\sqrt{29}}{2}$$

$$k = -\frac{5}{2} + \frac{\sqrt{29}}{2} \quad \text{or} \quad k = -\frac{5}{2} - \frac{\sqrt{29}}{2}$$

$$k = \frac{-5 + \sqrt{29}}{2} \quad \text{or} \quad k = \frac{-5 - \sqrt{29}}{2}$$

Check that the solution set is $\left\{\frac{-5 + \sqrt{29}}{2}, \frac{-5 - \sqrt{29}}{2}\right\}$.

Work Problem 8 at the Side.

8 Solve each equation by completing the square.

(a) $x^2 + 2x - 10 = 0$

(b) $r^2 + 3r - 1 = 0$

EXAMPLE 6 Solving a Quadratic Equation with $a \neq 1$ by Completing the Square

Solve $2x^2 - 4x - 5 = 0$.

First divide each side of the equation by 2 to get 1 as the coefficient of the squared term.

$$x^2 - 2x - \frac{5}{2} = 0 \qquad \text{Step 1}$$

$$x^2 - 2x = \frac{5}{2} \qquad \text{Step 2}$$

$$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1 \qquad \text{Step 3}$$

$$x^2 - 2x + 1 = \frac{5}{2} + 1 \qquad \text{Step 4}$$

$$(x - 1)^2 = \frac{7}{2} \qquad \text{Step 5}$$

$$x - 1 = \sqrt{\frac{7}{2}} \quad \text{or} \quad x - 1 = -\sqrt{\frac{7}{2}} \qquad \text{Step 6}$$

Continued on Next Page

ANSWERS

8. (a) $\{-1 + \sqrt{11}, -1 - \sqrt{11}\}$

(b) $\left\{\frac{-3 + \sqrt{13}}{2}, \frac{-3 - \sqrt{13}}{2}\right\}$

$$x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}} \qquad \text{Add 1.}$$

$$x = 1 + \frac{\sqrt{14}}{2} \quad \text{or} \quad x = 1 - \frac{\sqrt{14}}{2} \qquad \text{Rationalize denominators.}$$

Since $1 = \frac{2}{2}$, add the two terms in each solution as follows:

$$1 + \frac{\sqrt{14}}{2} = \frac{2}{2} + \frac{\sqrt{14}}{2} = \frac{2 + \sqrt{14}}{2}$$

$$1 - \frac{\sqrt{14}}{2} = \frac{2}{2} - \frac{\sqrt{14}}{2} = \frac{2 - \sqrt{14}}{2}.$$

Check that the solution set is $\left\{\frac{2 + \sqrt{14}}{2}, \frac{2 - \sqrt{14}}{2}\right\}$.

Work Problem 9 at the Side.

OBJECTIVE 4 Solve quadratic equations with nonreal complex solutions. In the equation $x^2 = k$, if $k < 0$, there will be two nonreal complex solutions.

EXAMPLE 7 Solving Quadratic Equations with Nonreal Complex Solutions

Solve each equation.

(a) $x^2 = -15$

$$x = \sqrt{-15} \quad \text{or} \quad x = -\sqrt{-15} \qquad \text{Square root property}$$

$$x = i\sqrt{15} \quad \text{or} \quad x = -i\sqrt{15} \qquad \sqrt{-1} = i$$

The solution set is $\{i\sqrt{15}, -i\sqrt{15}\}$.

(b) $(t + 2)^2 = -16$

$$t + 2 = \sqrt{-16} \quad \text{or} \quad t + 2 = -\sqrt{-16} \qquad \text{Square root property}$$

$$t + 2 = 4i \quad \text{or} \quad t + 2 = -4i \qquad \sqrt{-16} = 4i$$

$$t = -2 + 4i \quad \text{or} \quad t = -2 - 4i$$

The solution set is $\{-2 + 4i, -2 - 4i\}$.

(c) $x^2 + 2x + 7 = 0$

$$x^2 + 2x = -7 \qquad \text{Subtract 7.}$$

$$x^2 + 2x + 1 = -7 + 1 \qquad \left[\tfrac{1}{2}(2)\right]^2 = 1\text{; add 1 to each side.}$$

$$(x + 1)^2 = -6 \qquad \text{Factor on the left; add on the right.}$$

$$x + 1 = \pm i\sqrt{6} \qquad \text{Square root property}$$

$$x = -1 \pm i\sqrt{6} \qquad \text{Subtract 1.}$$

The solution set is $\{-1 + i\sqrt{6}, -1 - i\sqrt{6}\}$.

Work Problem 10 at the Side.

NOTE

We will use completing the square in **Section 10.6** when we graph quadratic equations and in **Section 12.2** when we work with circles.

9 Solve each equation by completing the square.

(a) $2r^2 - 4r + 1 = 0$

(b) $3z^2 - 6z - 2 = 0$

(c) $8x^2 - 4x - 2 = 0$

10 Solve each equation.

(a) $x^2 = -17$

(b) $(k + 5)^2 = -100$

(c) $5t^2 - 15t + 12 = 0$

ANSWERS

9. (a) $\left\{\frac{2 + \sqrt{2}}{2}, \frac{2 - \sqrt{2}}{2}\right\}$

(b) $\left\{\frac{3 + \sqrt{15}}{3}, \frac{3 - \sqrt{15}}{3}\right\}$

(c) $\left\{\frac{1 + \sqrt{5}}{4}, \frac{1 - \sqrt{5}}{4}\right\}$

10. (a) $\{i\sqrt{17}, -i\sqrt{17}\}$

(b) $\{-5 + 10i, -5 - 10i\}$

(c) $\left\{\frac{15 + i\sqrt{15}}{10}, \frac{15 - i\sqrt{15}}{10}\right\}$

10.2 The Quadratic Formula

OBJECTIVES

1. Derive the quadratic formula.
2. Solve quadratic equations using the quadratic formula.
3. Use the discriminant to determine the number and type of solutions.

The examples in the previous section showed that any quadratic equation can be solved by completing the square; however, completing the square can be tedious and time consuming. In this section, we complete the square to solve the general quadratic equation

$$ax^2 + bx + c = 0,$$

where a, b, and c are complex numbers and $a \neq 0$. The solution of this general equation gives a formula for finding the solution of any specific quadratic equation.

OBJECTIVE 1 Derive the quadratic formula. To solve $ax^2 + bx + c = 0$ by completing the square (assuming $a > 0$), we follow the steps given in **Section 10.1.**

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{Divide by } a. \text{ (Step 1)}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \qquad \text{Subtract } \tfrac{c}{a}. \text{ (Step 2)}$$

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} \qquad \text{(Step 3)}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \text{Add } \frac{b^2}{4a^2} \text{ to each side. (Step 4)}$$

Write the left side as a perfect square, and rearrange the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-c}{a} \qquad \text{(Step 5)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} + \frac{-4ac}{4a^2} \qquad \text{Write with a common denominator.}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \text{Add fractions.}$$

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad \text{Square root property (Step 6)}$$

Since

$$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} = \frac{\sqrt{b^2 - 4ac}}{2a},$$

the right sides of these equations can be expressed as

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

If $a < 0$, the same two solutions are obtained. The result is the **quadratic formula,** which is abbreviated as shown on the next page.

1 Identify the values of a, b, and c. (*Hint:* If necessary, first write the equation in standard form with 0 on the right side.) *Do not actually solve.*

(a) $-3q^2 + 9q - 4 = 0$

(b) $3x^2 = 6x + 2$

2 Solve $4x^2 - 11x - 3 = 0$ using the quadratic formula.

Answers
1. **(a)** -3; 9; -4 **(b)** 3; -6; -2
2. $\left\{-\frac{1}{4}, 3\right\}$

Quadratic Formula

The solutions of $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Audio

CAUTION
In the quadratic formula, ***the square root is added to or subtracted from the value of $-b$ BEFORE dividing by $2a$.***

OBJECTIVE 2 Solve quadratic equations using the quadratic formula. To use the quadratic formula, first write the given equation in standard form $ax^2 + bx + c = 0$; then identify the values of a, b, and c and substitute them into the quadratic formula, as shown in the next examples.

Work Problem 1 at the Side.

EXAMPLE 1 Using the Quadratic Formula (Rational Solutions)

You Try It

Solve $6x^2 - 5x - 4 = 0$.

Here a, the coefficient of the second-degree term, is 6, while b, the coefficient of the first-degree term, is -5, and the constant c is -4. Substitute these values into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)} \qquad a = 6,\ b = -5,\ c = -4$$

$$x = \frac{5 \pm \sqrt{25 + 96}}{12}$$

$$x = \frac{5 \pm \sqrt{121}}{12}$$

$$x = \frac{5 \pm 11}{12}$$

This last statement leads to two solutions, one from $+$ and one from $-$.

$$x = \frac{5 + 11}{12} = \frac{16}{12} = \frac{4}{3} \quad \text{or} \quad x = \frac{5 - 11}{12} = \frac{-6}{12} = -\frac{1}{2}$$

Check each solution in the original equation. The solution set is $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$.

Work Problem 2 at the Side.

We could have used factoring to solve the equation in Example 1.

$$6x^2 - 5x - 4 = 0$$

$$(3x - 4)(2x + 1) = 0 \qquad \text{Factor.}$$

$$3x - 4 = 0 \quad \text{or} \quad 2x + 1 = 0 \qquad \text{Zero-factor property}$$

$$3x = 4 \quad \text{or} \quad 2x = -1 \qquad \text{Solve each equation.}$$

$$x = \frac{4}{3} \quad \text{or} \quad x = -\frac{1}{2} \qquad \text{Same solutions as in Example 1}$$

When solving quadratic equations, it is a good idea to try factoring first. If the equation cannot be factored or if factoring is difficult, then use the quadratic formula. Later in this section, we will show a way to determine whether factoring can be used to solve a quadratic equation.

EXAMPLE 2 Using the Quadratic Formula (Irrational Solutions)

Solve $4r^2 = 8r - 1$.

Write the equation in standard form as $4r^2 - 8r + 1 = 0$.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic formula}$$

$$r = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} \qquad a = 4, b = -8, c = 1$$

$$= \frac{8 \pm \sqrt{64 - 16}}{8}$$

$$= \frac{8 \pm \sqrt{48}}{8}$$

$$= \frac{8 \pm 4\sqrt{3}}{8} \qquad \sqrt{48} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$$

$$= \frac{4(2 \pm \sqrt{3})}{4(2)} \qquad \text{Factor.}$$

$$= \frac{2 \pm \sqrt{3}}{2} \qquad \text{Lowest terms}$$

The solution set is $\left\{\frac{2 + \sqrt{3}}{2}, \frac{2 - \sqrt{3}}{2}\right\}$.

CAUTION

1. ***Every quadratic equation must be written in standard form*** $ax^2 + bx + c = 0$ ***before we begin to solve it,*** whether we use factoring or the quadratic formula.
2. ***When writing solutions in lowest terms, be sure to factor first; then divide out the common factor,*** as shown in the last two steps in Example 2.

Work Problem 3 at the Side.

EXAMPLE 3 Using the Quadratic Formula (Nonreal Complex Solutions)

Solve $(9q + 3)(q - 1) = -8$.

To write this equation in standard form, we first multiply and collect all nonzero terms on the left.

$$(9q + 3)(q - 1) = -8$$

$$9q^2 - 6q - 3 = -8$$

$$9q^2 - 6q + 5 = 0 \qquad \text{Standard form}$$

Continued on Next Page

3 Solve each equation using the quadratic formula.

(a) $6x^2 + 4x - 1 = 0$

(b) $2k^2 + 19 = 14k$

ANSWERS

3. (a) $\left\{\frac{-2 + \sqrt{10}}{6}, \frac{-2 - \sqrt{10}}{6}\right\}$

(b) $\left\{\frac{7 + \sqrt{11}}{2}, \frac{7 - \sqrt{11}}{2}\right\}$

4 Solve each equation using the quadratic formula.

(a) $x^2 + x + 1 = 0$

(b) $(z + 2)(z - 6) = -17$

From the equation $9q^2 - 6q + 5 = 0$, we identify $a = 9$, $b = -6$, and $c = 5$.

$$q = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(9)(5)}}{2(9)} \qquad \text{Substitute in the quadratic formula.}$$

$$= \frac{6 \pm \sqrt{-144}}{18}$$

$$= \frac{6 \pm 12i}{18} \qquad \sqrt{-144} = 12i$$

$$= \frac{6(1 \pm 2i)}{6(3)} \qquad \text{Factor.}$$

$$= \frac{1 \pm 2i}{3} \qquad \text{Lowest terms}$$

The solution set is $\left\{\frac{1 + 2i}{3}, \frac{1 - 2i}{3}\right\}$.

NOTE

We could have written the solutions in Example 3 in the form $a + bi$, the standard form for complex numbers, as follows:

$$\frac{1 \pm 2i}{3} = \frac{1}{3} \pm \frac{2}{3}i. \qquad \text{Standard form}$$

Work Problem 4 at the Side.

OBJECTIVE 3 Use the discriminant to determine the number and type of solutions. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad \leftarrow \text{Discriminant}$$

If a, b, and c are integers, the type of solutions of a quadratic equation—that is, rational, irrational, or nonreal complex—is determined by the expression under the radical sign, $b^2 - 4ac$. Because it distinguishes among the three types of solutions, $b^2 - 4ac$ is called the *discriminant*. By calculating the discriminant before solving a quadratic equation, we can predict whether the solutions will be rational numbers, irrational numbers, or nonreal complex numbers. (This can be useful in an applied problem, for example, where irrational or nonreal complex solutions are not acceptable.)

Discriminant

The **discriminant** of $ax^2 + bx + c = 0$ is $\boldsymbol{b^2 - 4ac}$. If a, b, and c are integers, then the number and type of solutions are determined as follows.

Discriminant	Number and Type of Solutions
Positive, and the square of an integer	Two rational solutions
Positive, but not the square of an integer	Two irrational solutions
Zero	One rational solution
Negative	Two nonreal complex solutions

Answers

4. (a) $\left\{\frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\right\}$
(b) $\{2 + i, 2 - i\}$

Calculating the discriminant can also help you decide whether to solve a quadratic equation by factoring or by using the quadratic formula. ***If the discriminant is a perfect square (including 0), then the equation can be solved by factoring. Otherwise, the quadratic formula should be used.***

You Try It

Video

EXAMPLE 4 Using the Discriminant

Find the discriminant. Use it to predict the number and type of solutions for each equation. Tell whether the equation can be solved by factoring or whether the quadratic formula should be used.

(a) $6x^2 - x - 15 = 0$

We find the discriminant by evaluating $b^2 - 4ac$.

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(6)(-15) \qquad a = 6, b = -1, c = -15 \\ &= 1 + 360 \\ &= 361 \end{aligned}$$

A calculator shows that $361 = 19^2$, a perfect square. Since a, b, and c are integers and the discriminant is a perfect square, there will be two rational solutions and the equation can be solved by factoring.

(b) $3m^2 - 4m = 5$

Write the equation in standard form as $3m^2 - 4m - 5 = 0$ to find $a = 3$, $b = -4$, and $c = -5$.

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(3)(-5) \\ &= 16 + 60 \\ &= 76 \end{aligned}$$

Because 76 is positive but not the square of an integer and a, b, and c are integers, the equation will have two irrational solutions and is best solved using the quadratic formula.

(c) $4x^2 + x + 1 = 0$

Since $a = 4$, $b = 1$, and $c = 1$, the discriminant is

$$1^2 - 4(4)(1) = -15.$$

Since the discriminant is negative and a, b, and c are integers, this quadratic equation will have two nonreal complex solutions. The quadratic formula should be used to solve it.

(d) $4t^2 + 9 = 12t$

Write the equation as $4t^2 - 12t + 9 = 0$ to find $a = 4$, $b = -12$, and $c = 9$. The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-12)^2 - 4(4)(9) \\ &= 144 - 144 \\ &= 0. \end{aligned}$$

Because the discriminant is 0, the quantity under the radical in the quadratic formula is 0, and there is only one rational solution. Again, the equation can be solved by factoring.

Work Problem 5 at the Side.

5 Find the discriminant. Use it to predict the number and type of solutions for each equation.

(a) $2x^2 + 3x = 4$

(b) $2x^2 + 3x + 4 = 0$

(c) $x^2 + 20x + 100 = 0$

(d) $15k^2 + 11k = 14$

(e) Which of the equations in parts (a)–(d) can be solved by factoring?

ANSWERS

5. (a) 41; two; irrational
(b) -23; two; nonreal complex
(c) 0; one; rational
(d) 961; two; rational **(e)** (c) and (d)

10.3 Equations Quadratic in Form

OBJECTIVE 1 Solve an equation with fractions by writing it in quadratic form. A variety of nonquadratic equations can be written in the form of a quadratic equation and solved by using one of the methods from **Sections 10.1 and 10.2.**

Video

EXAMPLE 1 Solving an Equation with Fractions That Leads to a Quadratic Equation

Solve $\frac{1}{x} + \frac{1}{x-1} = \frac{7}{12}$.

Clear fractions by multiplying each term by the least common denominator, $12x(x-1)$. (Note that the domain must be restricted to $x \neq 0$ and $x \neq 1$.)

$$12x(x-1)\frac{1}{x} + 12x(x-1)\frac{1}{x-1} = 12x(x-1)\frac{7}{12}$$

$$12(x-1) + 12x = 7x(x-1)$$

$$12x - 12 + 12x = 7x^2 - 7x \quad \text{Distributive property}$$

$$24x - 12 = 7x^2 - 7x \quad \text{Combine terms.}$$

Combine and rearrange terms so that the quadratic equation is in standard form. Then factor to solve the resulting equation.

$$7x^2 - 31x + 12 = 0 \quad \text{Standard form}$$

$$(7x-3)(x-4) = 0 \quad \text{Factor.}$$

$$7x - 3 = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{Zero-factor property}$$

$$x = \frac{3}{7} \quad \text{or} \quad x = 4 \quad \text{Solve each equation.}$$

Check by substituting these solutions in the original equation. The solution set is $\{\frac{3}{7}, 4\}$.

Work Problem 1 at the Side.

OBJECTIVE 2 Use quadratic equations to solve applied problems. In **Sections 2.4 and 8.5** we solved distance-rate-time (or motion) problems that led to linear equations or rational equations. Now we can extend that work to motion problems that lead to quadratic equations. We continue to use the six-step problem-solving method from **Section 2.3.**

Video

EXAMPLE 2 Solving a Motion Problem

A riverboat for tourists averages 12 mph in still water. It takes the boat 1 hr, 4 min to go 6 mi upstream and return. Find the speed of the current.

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** Let x = the speed of the current. The current slows down the boat when it is going upstream, so the rate (or speed) upstream is the speed of the boat in still water less the speed of the current, or $12 - x$. See Figure 1 on the next page.

Continued on Next Page

OBJECTIVES

1. Solve an equation with fractions by writing it in quadratic form.
2. Use quadratic equations to solve applied problems.
3. Solve an equation with radicals by writing it in quadratic form.
4. Solve an equation that is quadratic in form by substitution.

1 Solve each equation. Check your solutions.

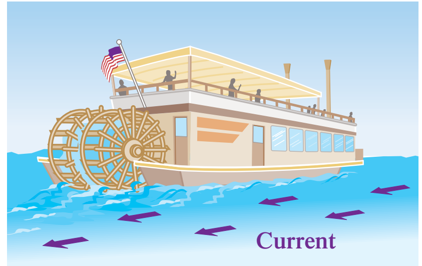
(a) $\frac{5}{m} + \frac{12}{m^2} = 2$

(b) $\frac{2}{x} + \frac{1}{x-2} = \frac{5}{3}$

(c) $\frac{4}{m-1} + 9 = -\frac{7}{m}$

ANSWERS

1. (a) $\left\{-\frac{3}{2}, 4\right\}$ (b) $\left\{\frac{4}{5}, 3\right\}$ (c) $\left\{\frac{7}{9}, -1\right\}$



Riverboat traveling *upstream*—the current slows it down.

Figure 1

Similarly, the current speeds up the boat as it travels downstream, so its speed downstream is $12 + x$. Thus,

$$12 - x = \text{the rate upstream;}$$
$$12 + x = \text{the rate downstream.}$$

This information can be used to complete a table. We use the distance formula, $d = rt$, solved for time t, $t = \frac{d}{r}$, to write expressions for t.

	d	r	t
Upstream	6	$12 - x$	$\frac{6}{12 - x}$
Downstream	6	$12 + x$	$\frac{6}{12 + x}$

Times in hours

Step 3 **Write an equation.** The total time, 1 hr and 4 min, can be written as

$$1 + \frac{4}{60} = 1 + \frac{1}{15} = \frac{16}{15} \text{ hr.}$$

Because the time upstream plus the time downstream equals $\frac{16}{15}$ hr,

$$\begin{array}{ccccc} \text{Time upstream} & + & \text{Time downstream} & = & \text{Total time} \\ \downarrow & & \downarrow & & \downarrow \\ \dfrac{6}{12 - x} & + & \dfrac{6}{12 + x} & = & \dfrac{16}{15}. \end{array}$$

Step 4 **Solve** the equation. Multiply each side by $15(12 - x)(12 + x)$, the LCD, and solve the resulting quadratic equation.

$$15(12 + x)6 + 15(12 - x)6 = 16(12 - x)(12 + x)$$
$$90(12 + x) + 90(12 - x) = 16(144 - x^2)$$
$$1080 + 90x + 1080 - 90x = 2304 - 16x^2 \quad \text{Distributive property}$$
$$2160 = 2304 - 16x^2 \quad \text{Combine terms.}$$
$$16x^2 = 144$$
$$x^2 = 9 \quad \text{Divide by 16.}$$
$$x = 3 \quad \text{or} \quad x = -3 \quad \text{Square root property}$$

Step 5 **State the answer.** The speed of the current cannot be -3, so the answer is 3 mph.

Step 6 **Check** that this value satisfies the original problem.

CAUTION

As shown in Example 2, when a quadratic equation is used to solve an applied problem, sometimes only *one* answer satisfies the application. ***Always check each answer in the words of the original problem.***

Work Problem 2 at the Side.

In **Section 8.5** we solved problems about work rates. Recall that a person's work rate is $\frac{1}{t}$ part of the job per hour, where t is the time in hours required to do the complete job. Thus, the part of the job the person will do in x hours is $\frac{1}{t}x$.

EXAMPLE 3 Solving a Work Problem

It takes two carpet layers 4 hr to carpet a room. If each worked alone, one of them could do the job in 1 hr less time than the other. How long would it take each carpet layer to complete the job alone?

Step 1 **Read** the problem again. There will be two answers.

Step 2 **Assign a variable.** Let x represent the number of hours for the slower carpet layer to complete the job alone. Then the faster carpet layer could do the entire job in $(x - 1)$ hours. The slower person's rate is $\frac{1}{x}$, and the faster person's rate is $\frac{1}{x-1}$. Together, they can do the job in 4 hr. Complete a table as shown.

	Rate	Time Working Together	Fractional Part of the Job Done
Slower Worker	$\frac{1}{x}$	4	$\frac{1}{x}(4)$
Faster Worker	$\frac{1}{x-1}$	4	$\frac{1}{x-1}(4)$

Sum is 1 whole job.

Step 3 **Write an equation.** The sum of the fractional parts done by the workers should equal 1 (the whole job).

Part done by slower worker + part done by faster worker = 1 whole job.

$$\frac{4}{x} + \frac{4}{x-1} = 1$$

Continued on Next Page

2 Solve each problem.

(a) In 4 hr, Kerrie can go 15 mi upriver and come back. The speed of the current is 5 mph. Complete this table.

	d	r	t
Up			
Down			

(b) Find the speed of the boat from part (a) in still water.

(c) In $1\frac{3}{4}$ hr, Ken rows his boat 5 mi upriver and comes back. The speed of the current is 3 mph. How fast does Ken row?

Answers

2. (a) row 1: 15; $x - 5$; $\frac{15}{x-5}$

row 2: 15; $x + 5$; $\frac{15}{x+5}$

(b) 10 mph **(c)** 7 mph

3 Solve each problem. Round answers to the nearest tenth.

(a) Carlos can complete a certain lab test in 2 hr less time than Jaime can. If they can finish the job together in 2 hr, how long would it take each of them working alone?

	Rate	Time Working Together	Fractional Part of the Job Done
Carlos			
Jaime			

(b) Two chefs are preparing a banquet. One chef could prepare the banquet in 2 hr less time than the other. Together, they complete the job in 5 hr. How long would it take the faster chef working alone?

Step 4 **Solve** the equation from Step 3.

$$\frac{4}{x} + \frac{4}{x-1} = 1$$

$$x(x-1)\left(\frac{4}{x} + \frac{4}{x-1}\right) = x(x-1)(1) \qquad \text{Multiply by the LCD.}$$

$$4(x-1) + 4x = x(x-1) \qquad \text{Distributive property}$$

$$4x - 4 + 4x = x^2 - x \qquad \text{Distributive property}$$

$$x^2 - 9x + 4 = 0 \qquad \text{Standard form}$$

This equation cannot be solved by factoring, so use the quadratic formula.

$$x = \frac{9 \pm \sqrt{81 - 16}}{2} = \frac{9 \pm \sqrt{65}}{2} \qquad a = 1, b = -9, c = 4$$

$$x = \frac{9 + \sqrt{65}}{2} \approx 8.5 \quad \text{or} \quad x = \frac{9 - \sqrt{65}}{2} \approx .5 \qquad \text{Use a calculator.}$$

Step 5 **State the answer.** Only the solution 8.5 makes sense in the original problem. (Why?) Thus, the slower worker can do the job in about 8.5 hr and the faster in about $8.5 - 1 = 7.5$ hr.

Step 6 **Check** that these results satisfy the original problem.

Work Problem 3 at the Side.

OBJECTIVE 3 Solve an equation with radicals by writing it in quadratic form.

EXAMPLE 4 Solving Radical Equations That Lead to Quadratic Equations

Solve each equation.

(a) $k = \sqrt{6k - 8}$

This equation is not quadratic. However, squaring both sides of the equation gives a quadratic equation that can be solved by factoring.

$$k^2 = 6k - 8 \qquad \text{Square both sides.}$$

$$k^2 - 6k + 8 = 0 \qquad \text{Standard form}$$

$$(k-4)(k-2) = 0 \qquad \text{Factor.}$$

$$k - 4 = 0 \quad \text{or} \quad k - 2 = 0 \qquad \text{Zero-factor property}$$

$$k = 4 \quad \text{or} \quad k = 2 \qquad \text{Potential solutions}$$

Recall from **Section 9.6** that squaring both sides of a radical equation can introduce extraneous solutions that do not satisfy the original equation. ***All potential solutions must be checked in the original (not the squared) equation.***

Check: If $k = 4$, then

$$k = \sqrt{6k - 8}$$
$$4 = \sqrt{6(4) - 8} \quad ?$$
$$4 = \sqrt{16} \quad ?$$
$$4 = 4. \quad \text{True}$$

If $k = 2$, then

$$k = \sqrt{6k - 8}$$
$$2 = \sqrt{6(2) - 8} \quad ?$$
$$2 = \sqrt{4} \quad ?$$
$$2 = 2. \quad \text{True}$$

Both solutions check, so the solution set is $\{2, 4\}$.

Continued on Next Page

Answers

3. **(a)** Jaime: 5.2 hr; Carlos: 3.2 hr **(b)** 9.1 hr

(b) $x + \sqrt{x} = 6$

$$\begin{aligned} \sqrt{x} &= 6 - x && \text{Isolate the radical on one side.} \\ x &= 36 - 12x + x^2 && \text{Square both sides.} \\ 0 &= x^2 - 13x + 36 && \text{Standard form} \\ 0 &= (x-4)(x-9) && \text{Factor.} \\ x - 4 = 0 \quad &\text{or} \quad x - 9 = 0 && \text{Zero-factor property} \\ x = 4 \quad &\text{or} \quad x = 9 && \text{Potential solutions} \end{aligned}$$

Check both potential solutions in the *original* equation.

If $x = 4$, then

$$\begin{aligned} x + \sqrt{x} &= 6 \\ 4 + \sqrt{4} &= 6 \quad ? \\ 6 &= 6. \quad \text{True} \end{aligned}$$

If $x = 9$, then

$$\begin{aligned} x + \sqrt{x} &= 6 \\ 9 + \sqrt{9} &= 6 \quad ? \\ 12 &= 6. \quad \text{False} \end{aligned}$$

Only the solution 4 checks, so the solution set is $\{4\}$.

Work Problem 4 at the Side.

OBJECTIVE 4 Solve an equation that is quadratic in form by substitution. A nonquadratic equation that can be written in the form

$$au^2 + bu + c = 0,$$

for $a \neq 0$ and an algebraic expression u, is called **quadratic in form.**

EXAMPLE 5 Solving Equations That Are Quadratic in Form

You Try It

Video

Solve each equation.

(a) $x^4 - 13x^2 + 36 = 0$

Because $x^4 = (x^2)^2$, we can write this equation in quadratic form with $u = x^2$ and $u^2 = x^4$. (Instead of u, any letter other than x could be used.)

$$\begin{aligned} x^4 - 13x^2 + 36 &= 0 \\ (x^2)^2 - 13x^2 + 36 &= 0 && x^4 = (x^2)^2 \\ u^2 - 13u + 36 &= 0 && \text{Let } u = x^2. \\ (u-4)(u-9) &= 0 && \text{Factor.} \\ u - 4 = 0 \quad \text{or} \quad u - 9 &= 0 && \text{Zero-factor property} \\ u = 4 \quad \text{or} \quad u &= 9 && \text{Solve.} \end{aligned}$$

To find x, we substitute x^2 for u.

$$\begin{aligned} x^2 = 4 \quad &\text{or} \quad x^2 = 9 \\ x = \pm 2 \quad &\text{or} \quad x = \pm 3 && \text{Square root property} \end{aligned}$$

The equation $x^4 - 13x^2 + 36 = 0$, a fourth-degree equation, has four solutions.* The solution set is $\{-3, -2, 2, 3\}$. Check by substitution.

(b)

$$\begin{aligned} 4x^4 + 1 &= 5x^2 \\ 4(x^2)^2 + 1 &= 5x^2 && x^4 = (x^2)^2 \\ 4u^2 + 1 &= 5u && \text{Let } u = x^2. \end{aligned}$$

Continued on Next Page

*In general, an equation in which an nth-degree polynomial equals 0 has n solutions, although some of them may be repeated.

4 Solve each equation. Check your solutions.

(a) $x = \sqrt{7x - 10}$

(b) $2x = \sqrt{x} + 1$

ANSWERS

4. **(a)** $\{2, 5\}$ **(b)** $\{1\}$

$$
\begin{aligned}
4u^2 - 5u + 1 &= 0 && \text{Standard form} \\
(4u - 1)(u - 1) &= 0 && \text{Factor.} \\
4u - 1 = 0 \quad \text{or} \quad u - 1 &= 0 && \text{Zero-factor property} \\
u = \frac{1}{4} \quad \text{or} \quad u &= 1 && \text{Solve.} \\
x^2 = \frac{1}{4} \quad \text{or} \quad x^2 &= 1 && \text{Substitute } x^2 \text{ for } u. \\
x = \pm\frac{1}{2} \quad \text{or} \quad x &= \pm 1 && \text{Square root property}
\end{aligned}
$$

Check that the solution set is $\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$.

(c) $x^4 = 6x^2 - 3$

First write the equation as

$$x^4 - 6x^2 + 3 = 0 \qquad \text{or} \qquad (x^2)^2 - 6x^2 + 3 = 0,$$

which is quadratic in form with $u = x^2$. Substitute u for x^2 and u^2 for x^4 to get

$$u^2 - 6u + 3 = 0.$$

Since this equation cannot be solved by factoring, use the quadratic formula.

$$
\begin{aligned}
u &= \frac{6 \pm \sqrt{36 - 12}}{2} && a = 1, b = -6, c = 3 \\
u &= \frac{6 \pm \sqrt{24}}{2} \\
u &= \frac{6 \pm 2\sqrt{6}}{2} && \sqrt{24} = \sqrt{4} \cdot \sqrt{6} = 2\sqrt{6} \\
u &= \frac{2(3 \pm \sqrt{6})}{2} && \text{Factor.} \\
u &= 3 \pm \sqrt{6} && \text{Lowest terms}
\end{aligned}
$$

$$
\begin{aligned}
x^2 = 3 + \sqrt{6} \quad &\text{or} \quad x^2 = 3 - \sqrt{6} && \text{Substitute } x^2 \text{ for } u. \\
x = \pm\sqrt{3 + \sqrt{6}} \quad &\text{or} \quad x = \pm\sqrt{3 - \sqrt{6}} && \text{Square root property}
\end{aligned}
$$

The solution set contains four numbers:

$$\left\{\sqrt{3 + \sqrt{6}}, -\sqrt{3 + \sqrt{6}}, \sqrt{3 - \sqrt{6}}, -\sqrt{3 - \sqrt{6}}\right\}.$$

NOTE

Some students prefer to solve equations like those in Examples 5(a) and (b) by factoring directly. For example,

$$
\begin{aligned}
x^4 - 13x^2 + 36 &= 0 && \text{Example 5(a) equation} \\
(x^2 - 9)(x^2 - 4) &= 0 && \text{Factor.} \\
(x + 3)(x - 3)(x + 2)(x - 2) &= 0. && \text{Factor again.}
\end{aligned}
$$

Using the zero-factor property gives the same solutions obtained in Example 5(a). Equations that cannot be solved by factoring (as in Example 5(c)) must be solved by substitution and the quadratic formula.

Work Problem 5 at the Side.

5 Solve each equation. Check your solutions.

(a) $m^4 - 10m^2 + 9 = 0$

(b) $9k^4 - 37k^2 + 4 = 0$

(c) $x^4 - 4x^2 = -2$

Answers

5. (a) $\{-3, -1, 1, 3\}$ (b) $\left\{-2, -\frac{1}{3}, \frac{1}{3}, 2\right\}$
(c) $\{\sqrt{2 + \sqrt{2}}, -\sqrt{2 + \sqrt{2}}, \sqrt{2 - \sqrt{2}}, -\sqrt{2 - \sqrt{2}}\}$

EXAMPLE 6 Solving Equations That Are Quadratic in Form

Solve each equation.

(a) $2(4m-3)^2 + 7(4m-3) + 5 = 0$

Because of the repeated quantity $4m-3$, this equation is quadratic in form with $u = 4m - 3$.

$$\begin{aligned}
2(\mathbf{4m-3})^2 + 7(\mathbf{4m-3}) + 5 &= 0 \\
2\mathbf{u}^2 + 7\mathbf{u} + 5 &= 0 && \text{Let } 4m-3 = u. \\
(2u+5)(u+1) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
2u + 5 &= 0 &\quad \text{or} \quad u + 1 &= 0 && \text{Zero-factor property} \\
\mathbf{u} &= -\frac{5}{2} &\quad \text{or} \quad \mathbf{u} &= -1 \\
\mathbf{4m-3} &= -\frac{5}{2} &\quad \text{or} \quad \mathbf{4m-3} &= -1 && \text{Substitute } 4m-3 \text{ for } u. \\
4m &= \frac{1}{2} &\quad \text{or} \quad 4m &= 2 && \text{Solve for } m. \\
m &= \frac{1}{8} &\quad \text{or} \quad m &= \frac{1}{2}
\end{aligned}$$

Check that the solution set of the original equation is $\{\frac{1}{8}, \frac{1}{2}\}$.

(b) $2a^{2/3} - 11a^{1/3} + 12 = 0$

Let $a^{1/3} = u$; then $a^{2/3} = (a^{1/3})^2 = u^2$. Substitute into the given equation.

$$\begin{aligned}
2u^2 - 11u + 12 &= 0 && \text{Let } a^{1/3} = u;\ a^{2/3} = u^2. \\
(2u-3)(u-4) &= 0 && \text{Factor.}
\end{aligned}$$

$$\begin{aligned}
2u - 3 &= 0 &\quad \text{or} \quad u - 4 &= 0 && \text{Zero-factor property} \\
\mathbf{u} &= \frac{3}{2} &\quad \text{or} \quad \mathbf{u} &= 4 \\
\mathbf{a^{1/3}} &= \frac{3}{2} &\quad \text{or} \quad \mathbf{a^{1/3}} &= 4 && u = a^{1/3} \\
(a^{1/3})^3 &= \left(\frac{3}{2}\right)^3 &\quad \text{or} \quad (a^{1/3})^3 &= 4^3 && \text{Cube each side.} \\
a &= \frac{27}{8} &\quad \text{or} \quad a &= 64
\end{aligned}$$

Check that the solution set is $\{\frac{27}{8}, 64\}$.

CAUTION

A common error when solving problems like those in Examples 5 and 6 is to stop too soon. ***Once you have solved for u, remember to substitute and solve for the values of the original variable.***

Work Problem 6 at the Side.

6 Solve each equation. Check your solutions.

(a) $5(r+3)^2 + 9(r+3) = 2$

(b) $4m^{2/3} = 3m^{1/3} + 1$

ANSWERS

6. **(a)** $\left\{-5, -\frac{14}{5}\right\}$ **(b)** $\left\{-\frac{1}{64}, 1\right\}$

10.4 Formulas and Further Applications

OBJECTIVES

1. Solve formulas for variables involving squares and square roots.
2. Solve applied problems using the Pythagorean formula.
3. Solve applied problems using area formulas.
4. Solve applied problems using quadratic functions as models.

OBJECTIVE 1 Solve formulas for variables involving squares and square roots. The methods presented earlier in this chapter and the previous one can be used to solve formulas with squares and square roots.

You Try It

Video

EXAMPLE 1 Solving for Variables Involving Squares or Square Roots

Solve each formula for the given variable.

(a) $w = \dfrac{kFr}{v^2}$ for v

$$w = \frac{kFr}{v^2} \quad \text{Isolate } v \text{ on one side.}$$

$$v^2w = kFr \quad \text{Multiply by } v^2.$$

$$v^2 = \frac{kFr}{w} \quad \text{Divide by } w.$$

$$v = \pm\sqrt{\frac{kFr}{w}} \quad \text{Square root property}$$

$$v = \frac{\pm\sqrt{kFr}}{\sqrt{w}} \cdot \frac{\sqrt{w}}{\sqrt{w}} = \frac{\pm\sqrt{kFrw}}{w} \quad \text{Rationalize the denominator.}$$

(b) $d = \sqrt{\dfrac{4A}{\pi}}$ for A

$$d = \sqrt{\frac{4A}{\pi}} \quad \text{Isolate } A \text{ on one side.}$$

$$d^2 = \frac{4A}{\pi} \quad \text{Square both sides.}$$

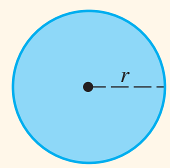
$$\pi d^2 = 4A \quad \text{Multiply by } \pi.$$

$$\frac{\pi d^2}{4} = A \quad \text{Divide by 4.}$$

Work Problem 1 at the Side.

1 Solve each formula for the given variable.

(a) $A = \pi r^2$ for r



(b) $s = 30\sqrt{\dfrac{a}{p}}$ for a

NOTE

In many formulas like $v = \frac{\pm\sqrt{kFrw}}{w}$ in Example 1(a), we choose the positive value. In our work here, we will include both positive and negative values.

You Try It

EXAMPLE 2 Solving for a Squared Variable

Solve $s = 2t^2 + kt$ for t.

Since the equation has terms with t^2 and t, write it in standard form $ax^2 + bx + c = 0$, with t as the variable instead of x.

$$s = 2t^2 + kt$$

$$0 = 2t^2 + kt - s \quad \text{Subtract } s.$$

Continued on Next Page

ANSWERS

1. (a) $r = \dfrac{\pm\sqrt{A\pi}}{\pi}$ **(b)** $a = \dfrac{ps^2}{900}$

$$2t^2 + kt - s = 0 \qquad \text{Standard form}$$

Now use the quadratic formula with $a = 2$, $b = k$, and $c = -s$.

$$t = \frac{-k \pm \sqrt{k^2 - 4(2)(-s)}}{2(2)} \qquad \text{Solve for } t.$$

$$t = \frac{-k \pm \sqrt{k^2 + 8s}}{4}$$

The solutions are $t = \dfrac{-k + \sqrt{k^2 + 8s}}{4}$ and $t = \dfrac{-k - \sqrt{k^2 + 8s}}{4}$.

Work Problem 2 at the Side.

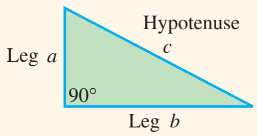
2 Solve $2t^2 - 5t + k = 0$ for t.

OBJECTIVE 2 Solve applied problems using the Pythagorean formula.

The Pythagorean formula

$$a^2 + b^2 = c^2,$$

illustrated by the figure in the margin, was introduced in **Section 9.3** and is used to solve applications involving right triangles. Such problems often require solving quadratic equations.



$a^2 + b^2 = c^2$
Pythagorean Formula

EXAMPLE 3 Using the Pythagorean Formula

Two cars left an intersection at the same time, one heading due north, the other due west. Some time later, they were exactly 100 mi apart. The car headed north had gone 20 mi farther than the car headed west. How far had each car traveled?

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** Let x be the distance traveled by the car headed west. Then $x + 20$ is the distance traveled by the car headed north. See Figure 2. The cars are 100 mi apart, so the hypotenuse of the right triangle equals 100.

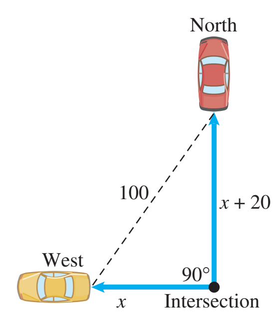


Figure 2

Step 3 **Write an equation.** Use the Pythagorean formula.

$$a^2 + b^2 = c^2$$
$$x^2 + (x + 20)^2 = 100^2$$

Step 4 **Solve.**

$$\begin{aligned}
x^2 + x^2 + 40x + 400 &= 10{,}000 && \text{Square the binomial.}\\
2x^2 + 40x - 9600 &= 0 && \text{Standard form}\\
x^2 + 20x - 4800 &= 0 && \text{Divide by 2.}\\
(x + 80)(x - 60) &= 0 && \text{Factor.}\\
x + 80 = 0 \quad \text{or} \quad x - 60 &= 0 && \text{Zero-factor property}\\
x = -80 \quad \text{or} \quad x &= 60
\end{aligned}$$

Step 5 **State the answer.** Since distance cannot be negative, discard the negative solution. The required distances are 60 mi and $60 + 20 = 80$ mi.

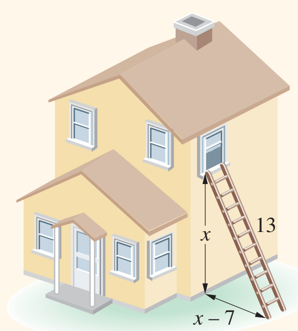
Step 6 **Check.** Since $60^2 + 80^2 = 100^2$, the answers are correct.

Work Problem 3 at the Side.

Video
You Try It

3 Solve the problem.

A 13-ft ladder is leaning against a house. The distance from the bottom of the ladder to the house is 7 ft less than the distance from the top of the ladder to the ground. How far is the bottom of the ladder from the house?



Answers

2. $t = \dfrac{5 + \sqrt{25 - 8k}}{4}$, $t = \dfrac{5 - \sqrt{25 - 8k}}{4}$
3. 5 ft

OBJECTIVE 3 **Solve applied problems using area formulas.**

EXAMPLE 4 Solving an Area Problem

A rectangular reflecting pool in a park is 20 ft wide and 30 ft long. The park gardener wants to plant a strip of grass of uniform width around the edge of the pool. She has enough seed to cover 336 ft^2. How wide will the strip be?

Step 1 **Read** the problem carefully.

Step 2 **Assign a variable.** The pool is shown in Figure 3. If x represents the unknown width of the grass strip, the width of the large rectangle is given by $20 + 2x$ (the width of the pool plus two grass strips), and the length is given by $30 + 2x$.

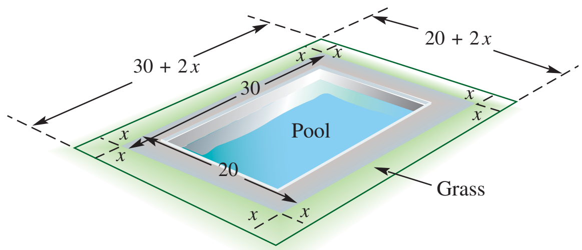


Figure 3

Step 3 **Write an equation.** The area of the large rectangle is given by the product of its length and width, $(30 + 2x)(20 + 2x)$. The area of the pool is $30 \cdot 20 = 600$ ft^2. The area of the large rectangle, minus the area of the pool, should equal the area of the grass strip. Since the area of the grass strip is to be 336 ft^2, the equation is

$$\underbrace{(30 + 2x)(20 + 2x)}_{\text{Area of rectangle}} - \underbrace{600}_{\text{area of pool}} = \underbrace{336.}_{\text{area of grass.}}$$

Step 4 **Solve.**

$$\begin{aligned}
600 + 100x + 4x^2 - 600 &= 336 && \text{Multiply.}\\
4x^2 + 100x - 336 &= 0 && \text{Standard form}\\
x^2 + 25x - 84 &= 0 && \text{Divide by 4.}\\
(x + 28)(x - 3) &= 0 && \text{Factor.}\\
x = -28 \quad \text{or} \quad x &= 3 && \text{Zero-factor property}
\end{aligned}$$

Step 5 **State the answer.** The width cannot be -28 ft, so the grass strip should be 3 ft wide.

Step 6 **Check.** If $x = 3$, then the area of the large rectangle is

$$(30 + 2 \cdot 3)(20 + 2 \cdot 3) = 36 \cdot 26 = 936 \text{ ft}^2. \qquad \text{Area of pool and strip}$$

The area of the pool is $30 \cdot 20 = 600$ ft^2. So, the area of the grass strip is $936 - 600 = 336$ ft^2, as required. The answer is correct.

Work Problem 4 at the Side.

OBJECTIVE 4 **Solve applied problems using quadratic functions as models.** Some applied problems can be modeled by *quadratic functions,* which can be written in the form

$$f(x) = ax^2 + bx + c,$$

for real numbers a, b, and c, with $a \neq 0$.

4 Solve the problem.

Suppose the pool in Example 4 is 20 ft by 40 ft and there is enough seed to cover 700 ft^2. How wide should the grass strip be?

ANSWERS

4. 5 ft

5 Solve the problem.

A ball is propelled vertically upward from the ground. Its distance in feet from the ground at t seconds is

$$s(t) = -16t^2 + 64t.$$

At what times will the ball be 32 ft from the ground? Use a calculator and round answers to the nearest tenth. (*Hint:* There are two answers.)

6 Use a calculator to evaluate

$$\frac{28.6 \pm \sqrt{(-28.6)^2 - 4(3.37)(-17)}}{2(3.37)}$$

for both solutions. Round to the nearest tenth. Which solution is valid for this problem?

EXAMPLE 5 Solving an Applied Problem Using a Quadratic Function

If an object is propelled upward from the top of a 144-ft building at 112 ft per sec, its position (in feet above the ground) is given by

$$s(t) = -16t^2 + 112t + 144,$$

where t is time in seconds after it was propelled. When does it hit the ground?

When the object hits the ground, its distance above the ground is 0. We must find the value of t that makes $s(t) = 0$.

$$0 = -16t^2 + 112t + 144 \qquad \text{Let } s(t) = 0.$$
$$0 = t^2 - 7t - 9 \qquad \text{Divide by } -16.$$
$$t = \frac{7 \pm \sqrt{49 + 36}}{2} = \frac{7 \pm \sqrt{85}}{2} \approx \frac{7 \pm 9.2}{2} \qquad \text{Use the quadratic formula and a calculator.}$$

The solutions are $t \approx 8.1$ or $t \approx -1.1$. Since time cannot be negative, discard the negative solution. The object will hit the ground about 8.1 sec after it is propelled.

Work Problem 5 at the Side.

EXAMPLE 6 Using a Quadratic Function to Model Company Bankruptcy Filings

The number of companies filing for bankruptcy was high in the early 1990s due to an economic recession. The number then declined during the middle 1990s, and in recent years has increased again. The quadratic function defined by

$$f(x) = 3.37x^2 - 28.6x + 133$$

approximates the number of company bankruptcy filings during the years 1990 through 2001, where x is the number of years that have elapsed since 1990. (*Source:* www.BankruptcyData.com)

(a) Use the model to approximate the number of company bankruptcy filings in 1995.

For 1995, $x = 5$, so find $f(5)$.

$$f(5) = 3.37(5)^2 - 28.6(5) + 133 \qquad \text{Let } x = 5.$$
$$= 74.25$$

There were about 74 company bankruptcy filings in 1995.

(b) In what year did company bankruptcy filings reach 150?

Find the value of x that makes $f(x) = 150$.

$$f(x) = 3.37x^2 - 28.6x + 133 \qquad \text{Let } f(x) = 150.$$
$$150 = 3.37x^2 - 28.6x + 133 \qquad \text{Standard form}$$
$$0 = 3.37x^2 - 28.6x - 17$$

Now use $a = 3.37$, $b = -28.6$, and $c = -17$ in the quadratic formula.

Work Problem 6 at the Side.

The positive solution is $x \approx 9$, so company bankruptcy filings reached 150 in $1990 + 9 = 1999$. (Reject the negative solution since the model is not valid for negative values of x.)

Answers

5. at .6 sec and at 3.4 sec
6. 9.0, −.6; 9.0

10.5 Graphs of Quadratic Functions

OBJECTIVES

1. **Graph a quadratic function.**
2. **Graph parabolas with horizontal and vertical shifts.**
3. **Predict the shape and direction of a parabola from the coefficient of x^2.**
4. **Find a quadratic function to model data.**

OBJECTIVE 1 Graph a quadratic function. Figure 4 gives a graph of the simplest *quadratic function,* defined by $y = x^2$.

x	y
-2	4
-1	1
0	0
1	1
2	4

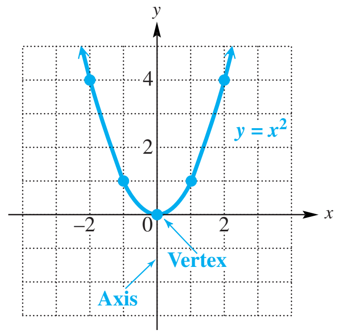


Figure 4

As mentioned in **Section 6.3,** this graph is called a **parabola.** The point (0, 0), the lowest point on the curve, is the **vertex** of this parabola. The vertical line through the vertex is the **axis** of the parabola, here $x = 0$. A parabola is **symmetric about its axis;** that is, if the graph were folded along the axis, the two portions of the curve would coincide. As Figure 4 suggests, x can be any real number, so the domain of the function defined by $y = x^2$ is $(-\infty, \infty)$. Since y is always nonnegative, the range is $[0, \infty)$.

In **Section 10.4,** we solved applications modeled by quadratic functions. We now consider graphs of general quadratic functions as defined here.

Audio

Quadratic Function

A function that can be written in the form

$$f(x) = ax^2 + bx + c$$

for real numbers a, b, and c, with $a \neq 0$, is a **quadratic function.**

The graph of any quadratic function is a parabola with a vertical axis.

NOTE

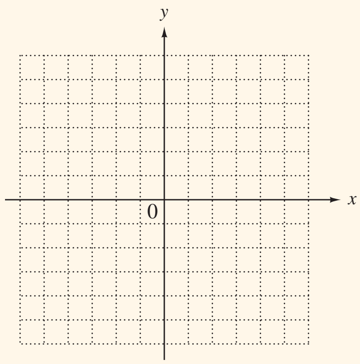
We use the variable y and function notation $f(x)$ interchangeably. Although we use the letter f most often to name quadratic functions, other letters can be used. We use the capital letter F to distinguish between different parabolas graphed on the same coordinate axes.

Parabolas, which are a type of *conic section* **(Chapter 12),** have many applications. Cross sections of satellite dishes and automobile headlights form parabolas, as do the cables that support suspension bridges.

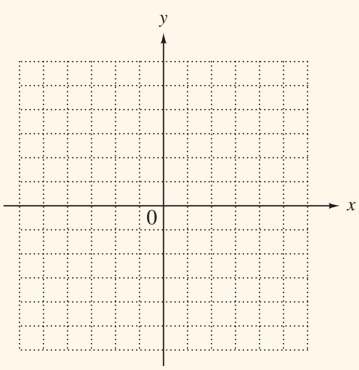
OBJECTIVE 2 Graph parabolas with horizontal and vertical shifts. Parabolas need not have their vertices at the origin, as does the graph of $f(x) = x^2$. For example, to graph a parabola of the form $F(x) = x^2 + k$, start by selecting sample values of x like those that were used to graph $f(x) = x^2$. The corresponding values of $F(x)$ in $F(x) = x^2 + k$ differ by k from those of $f(x) = x^2$. For this reason, the graph of $F(x) = x^2 + k$ is *shifted,* or *translated,* k units vertically compared with that of $f(x) = x^2$.

1 Graph each parabola. Give the vertex, domain, and range.

(a) $f(x) = x^2 + 3$



(b) $f(x) = x^2 - 1$



EXAMPLE 1 Graphing a Parabola with a Vertical Shift

Graph $F(x) = x^2 - 2$.

This graph has the same shape as that of $f(x) = x^2$, but since k here is -2, the graph is shifted 2 units down, with vertex $(0, -2)$. Every function value is 2 less than the corresponding function value of $f(x) = x^2$. Plotting points on both sides of the vertex gives the graph in Figure 5.

Notice that since the parabola is symmetric about its axis $x = 0$, the plotted points are "mirror images" of each other. Since x can be any real number, the domain is still $(-\infty, \infty)$; the value of y (or $F(x)$) is always greater than or equal to -2, so the range is $[-2, \infty)$. The graph of $f(x) = x^2$ is shown for comparison.

x	$f(x) = x^2$	$F(x) = x^2 - 2$
-2	4	2
-1	1	-1
0	0	-2
1	1	-1
2	4	2

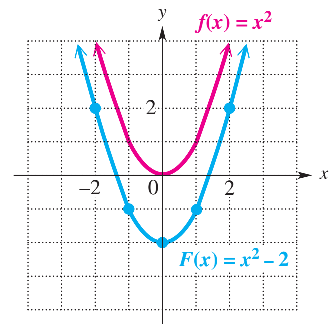


Figure 5

Vertical Shift

The graph of $\boldsymbol{F(x) = x^2 + k}$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted vertically: k units up if $k > 0$, and $|k|$ units down if $k < 0$. The vertex is $(0, k)$.

Work Problem 1 at the Side.

The graph of $F(x) = (x - h)^2$ is also a parabola with the same shape as that of $f(x) = x^2$. Because $(x - h)^2 \geq 0$ for all x, the vertex of $F(x) = (x - h)^2$ is the lowest point on the parabola. The lowest point occurs here when $F(x)$ is 0. To get $F(x)$ equal to 0, let $x = h$ so the vertex of $F(x) = (x - h)^2$ is $(h, 0)$. Based on this, the graph of $F(x) = (x - h)^2$ is shifted h units horizontally compared with that of $f(x) = x^2$.

EXAMPLE 2 Graphing a Parabola with a Horizontal Shift

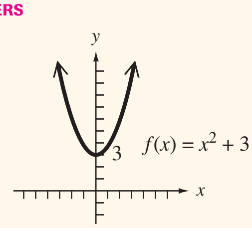
Graph $F(x) = (x - 2)^2$.

When $x = 2$, then $F(x) = 0$, giving the vertex $(2, 0)$. The graph of $F(x) = (x - 2)^2$ has the same shape as that of $f(x) = x^2$ but is shifted 2 units to the right. Plotting several points on one side of the vertex and using symmetry about the axis $x = 2$ to find corresponding points on the other side of the vertex gives the graph in Figure 6. Again, the domain is $(-\infty, \infty)$; the range is $[0, \infty)$.

Continued on Next Page

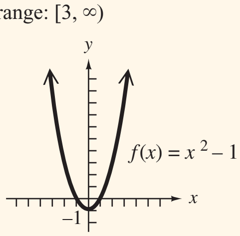
ANSWERS

1. (a)



vertex: $(0, 3)$; domain: $(-\infty, \infty)$; range: $[3, \infty)$

(b)



vertex: $(0, -1)$; domain: $(-\infty, \infty)$; range: $[-1, \infty)$

x	$F(x) = (x - 2)^2$
0	4
1	1
2	0
3	1
4	4

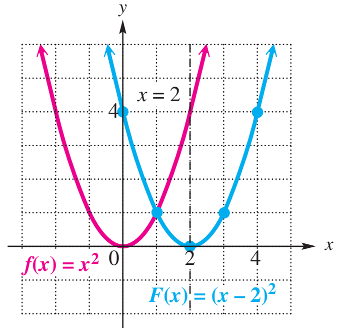


Figure 6

Horizontal Shift

The graph of $\boldsymbol{F(x) = (x - h)^2}$ is a parabola with the same shape as the graph of $f(x) = x^2$. The parabola is shifted h units horizontally: h units to the right if $h > 0$, and $|h|$ units to the left if $h < 0$. The vertex is $(h, 0)$.

CAUTION

Errors frequently occur when horizontal shifts are involved. To determine the direction and magnitude of a horizontal shift, find the value that would cause the expression $x - h$ to equal 0. For example, the graph of $F(x) = (x - 5)^2$ would be shifted 5 units to the *right,* because $+5$ would cause $x - 5$ to equal 0. On the other hand, the graph of $F(x) = (x + 5)^2$ would be shifted 5 units to the *left,* because -5 would cause $x + 5$ to equal 0.

Work Problem 2 at the Side.

A parabola can have both horizontal and vertical shifts.

EXAMPLE 3 Graphing a Parabola with Horizontal and Vertical Shifts

Graph $F(x) = (x + 3)^2 - 2$.

This graph has the same shape as that of $f(x) = x^2$, but is shifted 3 units to the left (since $x + 3 = 0$ if $x = -3$) and 2 units down (because of the -2). As shown in Figure 7, the vertex is $(-3, -2)$, with axis $x = -3$. This function has domain $(-\infty, \infty)$ and range $[-2, \infty)$.

x	$F(x)$
-5	2
-4	-1
-3	-2
-2	-1
-1	2

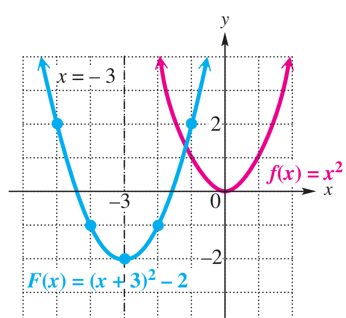
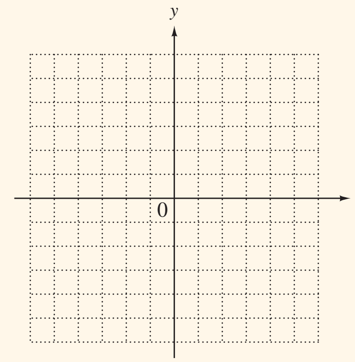


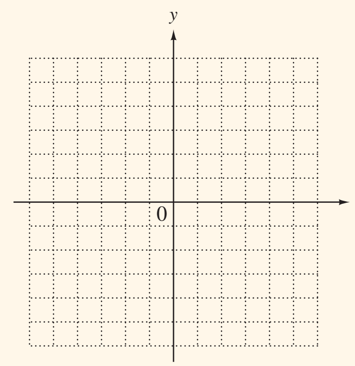
Figure 7

2 Graph each parabola. Give the vertex, axis, domain, and range.

(a) $f(x) = (x - 3)^2$

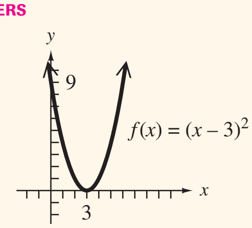


(b) $f(x) = (x + 2)^2$



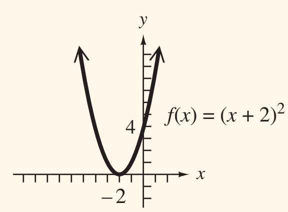
ANSWERS

2. (a)



vertex: $(3, 0)$; axis: $x = 3$; domain: $(-\infty, \infty)$; range: $[0, \infty)$

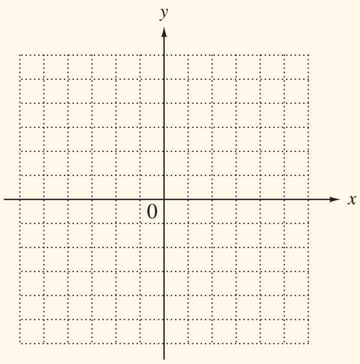
(b)



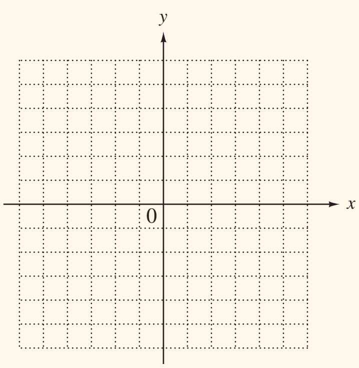
vertex: $(-2, 0)$; axis: $x = -2$; domain: $(-\infty, \infty)$; range: $[0, \infty)$

3 Graph each parabola. Give the vertex, axis, domain, and range.

(a) $f(x) = (x + 2)^2 - 1$



(b) $f(x) = (x - 2)^2 + 5$



The characteristics of the graph of a parabola of the form $F(x) = (x - h)^2 + k$ are summarized as follows.

Vertex and Axis of a Parabola

The graph of $\mathbf{F(x) = (x - h)^2 + k}$ is a parabola with the same shape as the graph of $f(x) = x^2$ with vertex (h, k). The axis is the vertical line $x = h$.

Work Problem 3 at the Side.

OBJECTIVE 3 **Predict the shape and direction of a parabola from the coefficient of x^2.** Not all parabolas open up, and not all parabolas have the same shape as the graph of $f(x) = x^2$.

EXAMPLE 4 Graphing a Parabola That Opens Down

Graph $f(x) = -\dfrac{1}{2}x^2$.

This parabola is shown in Figure 8. The coefficient $-\frac{1}{2}$ affects the shape of the graph; the $\frac{1}{2}$ makes the parabola wider (since the values of $\frac{1}{2}x^2$ increase more slowly than those of x^2), and the negative sign makes the parabola open down. The graph is not shifted in any direction; the vertex is still $(0, 0)$. Unlike the parabolas graphed in Examples 1–3, the vertex here has the *greatest* function value of any point on the graph. The domain is $(-\infty, \infty)$; the range is $(-\infty, 0]$.

x	$f(x)$
-2	-2
-1	$-\frac{1}{2}$
0	0
1	$-\frac{1}{2}$
2	-2

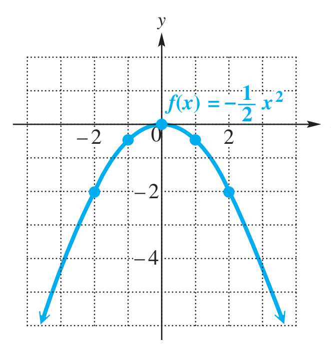


Figure 8

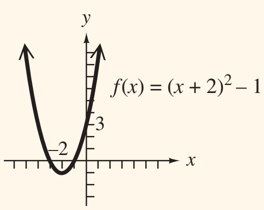
Some general principles concerning the graph of $F(x) = a(x - h)^2 + k$ are summarized as follows.

General Principles

1. The graph of the quadratic function defined by
$$F(x) = a(x - h)^2 + k, \quad a \neq 0,$$
is a parabola with vertex (h, k) and the vertical line $x = h$ as axis.
2. The graph opens up if a is positive and down if a is negative.
3. The graph is wider than that of $f(x) = x^2$ if $0 < |a| < 1$. The graph is narrower than that of $f(x) = x^2$ if $|a| > 1$.

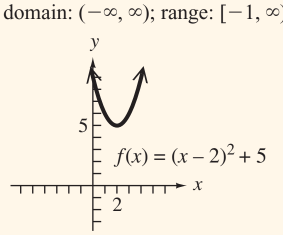
ANSWERS

3. (a)



vertex: $(-2, -1)$; axis: $x = -2$; domain: $(-\infty, \infty)$; range: $[-1, \infty)$

(b)



vertex: $(2, 5)$; axis: $x = 2$; domain: $(-\infty, \infty)$; range: $[5, \infty)$

Work Problems 4 and 5 at the Side.

EXAMPLE 5 Using the General Principles to Graph a Parabola

Graph $F(x) = -2(x + 3)^2 + 4$.

The parabola opens down (because $a < 0$), and is narrower than the graph of $f(x) = x^2$, since $|-2| = 2 > 1$, causing values of $F(x)$ to decrease more quickly than those of $f(x) = -x^2$. This parabola has vertex $(-3, 4)$ as shown in Figure 9. To complete the graph, we plotted the ordered pairs $(-4, 2)$ and, by symmetry, $(-2, 2)$. Symmetry can be used to find additional ordered pairs that satisfy the equation, if desired.

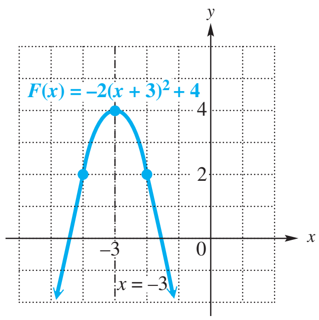


Figure 9

Work Problem 6 at the Side.

OBJECTIVE 4 Find a quadratic function to model data.

EXAMPLE 6 Finding a Quadratic Function to Model the Rise in Multiple Births

The number of higher-order multiple births in the United States is rising. Let x represent the number of years since 1970 and y represent the rate of higher-order multiples born per 100,000 births since 1971. The data are shown in the following table. Find a quadratic function that models the data.

U.S. HIGHER-ORDER MULTIPLE BIRTHS

Year	x	y
1971	1	29.1
1976	6	35.0
1981	11	40.0
1986	16	47.0
1991	21	100.0
1996	26	152.6
2001	31	185.6

Source: National Center for Health Statistics.

A scatter diagram of the ordered pairs (x, y) is shown in Figure 10 on the next page. The general shape suggested by the scatter diagram indicates that a parabola should approximate these points, as shown by the dashed curve in Figure 11. The equation for such a parabola would have a positive coefficient for x^2 since the graph opens up.

Continued on Next Page

4 Decide whether each parabola opens up or down.

(a) $f(x) = -\frac{2}{3}x^2$

(b) $f(x) = \frac{3}{4}x^2 + 1$

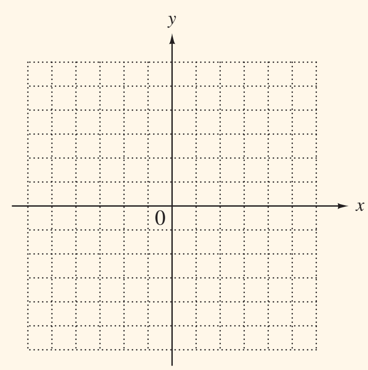
(c) $f(x) = -2x^2 - 3$

(d) $f(x) = 3x^2 + 2$

5 Decide whether each parabola in Problem 4 is wider or narrower than the graph of $f(x) = x^2$.

6 Graph

$f(x) = \frac{1}{2}(x - 2)^2 + 1$.

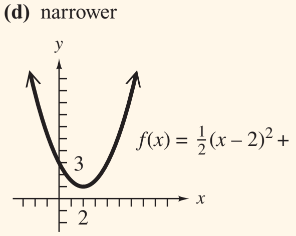


ANSWERS

4. **(a)** down **(b)** up **(c)** down **(d)** up

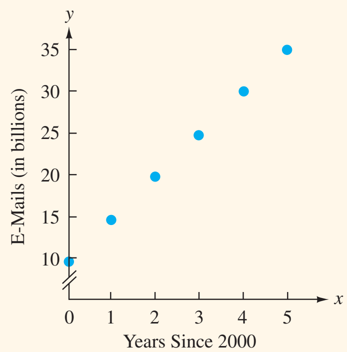
5. **(a)** wider **(b)** wider **(c)** narrower **(d)** narrower

6.



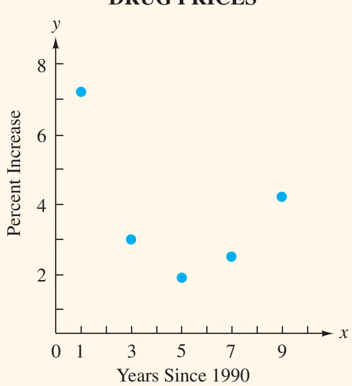
7 Tell whether a linear or quadratic function would be a more appropriate model for each set of graphed data. If linear, tell whether the slope should be positive or negative. If quadratic, tell whether the coefficient a of x^2 should be positive or negative.

(a) AVERAGE DAILY E-MAIL VOLUME



Source: General Accounting Office.

(b) INCREASES IN WHOLESALE DRUG PRICES



Source: *IMS Health*, Retail and Provider Perspective.

8 Using the points (1, 29.1), (6, 35), and (26, 152.6), find another quadratic model for the data on higher-order multiple births in Example 6.

ANSWERS

7. **(a)** linear; positive **(b)** quadratic; positive

8. $y = .188x^2 - .136x + 29.05$

U.S. HIGHER-ORDER MULTIPLE BIRTHS

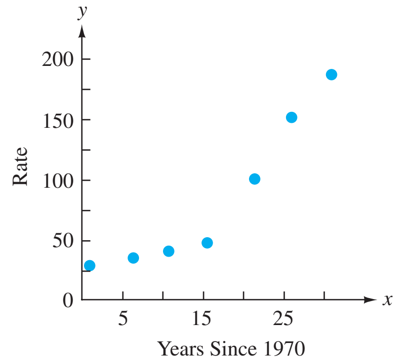


Figure 10

U.S. HIGHER-ORDER MULTIPLE BIRTHS

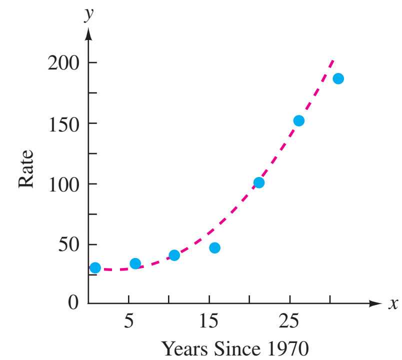


Figure 11

To find a quadratic function of the form

$$y = ax^2 + bx + c$$

that models, or *fits*, these data, we choose three representative ordered pairs and use them to write a system of three equations. Using (1, 29.1), (11, 40), and (21, 100), we substitute the x- and y-values from the ordered pairs into the quadratic form $y = ax^2 + bx + c$ to get the three equations

$$\begin{aligned} a(1)^2 + b(1) + c &= 29.1 && \text{or} & a + b + c &= 29.1 && (1)\\ a(11)^2 + b(11) + c &= 40 && \text{or} & 121a + 11b + c &= 40 && (2)\\ a(21)^2 + b(21) + c &= 100 && \text{or} & 441a + 21b + c &= 100. && (3) \end{aligned}$$

We can find the values of a, b, and c by solving this system of three equations in three variables using the methods of **Section 5.2.** Multiplying equation (1) by -1 and adding the result to equation (2) gives

$$120a + 10b = 10.9. \qquad (4)$$

Multiplying equation (2) by -1 and adding the result to equation (3) gives

$$320a + 10b = 60. \qquad (5)$$

We can eliminate b from this system of equations in two variables by multiplying equation (4) by -1 and adding the result to equation (5) to get

$$\begin{aligned} 200a &= 49.1 \\ a &= .2455. && \text{Use a calculator.} \end{aligned}$$

We substitute .2455 for a in equation (4) or (5) to find that $b = -1.856$. Substituting the values of a and b into equation (1) gives $c = 30.7105$. Using these values of a, b, and c, our model is defined by

$$y = .2455x^2 - 1.856x + 30.7105.$$

Work Problems 7 and 8 at the Side.

NOTE
If we had chosen three different ordered pairs of data in Example 6, a slightly different model would have resulted.

Calculator Tip The *quadratic regression* feature on a graphing calculator can be used to generate a quadratic model that fits given data. See your owner's manual for details on how to do this.

10.6 More about Parabolas; Applications

OBJECTIVE 1 Find the vertex of a vertical parabola. When the equation of a parabola is given in the form $f(x) = ax^2 + bx + c$, we need to locate the vertex in order to sketch an accurate graph. There are two ways to do this:

1. Complete the square, as shown in Examples 1 and 2, or
2. Use a formula derived by completing the square, as shown in Example 3.

Video

EXAMPLE 1 Completing the Square to Find the Vertex

Find the vertex of the graph of $f(x) = x^2 - 4x + 5$.

To find the vertex, we need to write the expression $x^2 - 4x + 5$ in the form $(x - h)^2 + k$. We do this by completing the square on $x^2 - 4x$, as in **Section 10.1.** The process is a little different here because we want to keep $f(x)$ alone on one side of the equation. Instead of adding the appropriate number to each side, we *add and subtract* it on the right. This is equivalent to adding 0.

$$\begin{aligned} f(x) &= x^2 - 4x + 5 \\ &= (x^2 - 4x \quad\;) + 5 && \text{Group the variable terms.} \\ & \qquad \left[\tfrac{1}{2}(-4)\right]^2 = (-2)^2 = 4 \\ &= (x^2 - 4x + 4 - 4) + 5 && \text{Add and subtract 4.} \\ &= (x^2 - 4x + 4) - 4 + 5 && \text{Bring } -4 \text{ outside the parentheses.} \\ f(x) &= (x - 2)^2 + 1 && \text{Factor; combine terms.} \end{aligned}$$

The vertex of this parabola is (2, 1).

Work Problem 1 at the Side.

EXAMPLE 2 Completing the Square to Find the Vertex When $a \neq 1$

Find the vertex of the graph of $f(x) = -3x^2 + 6x - 1$.

We must complete the square on $-3x^2 + 6x$. Because the x^2-term has a coefficient other than 1, we factor that coefficient out of the first two terms and then proceed as in Example 1.

$$\begin{aligned} f(x) &= -3x^2 + 6x - 1 \\ &= -3(x^2 - 2x) - 1 && \text{Factor out } -3. \\ & \qquad \left[\tfrac{1}{2}(-2)\right]^2 = (-1)^2 = 1 \\ &= -3(x^2 - 2x + 1 - 1) - 1 && \text{Add and subtract 1.} \end{aligned}$$

Bring -1 outside the parentheses; be sure to multiply it by -3.

$$\begin{aligned} &= -3(x^2 - 2x + 1) + (-3)(-1) - 1 && \text{Distributive property} \\ &= -3(x^2 - 2x + 1) + 3 - 1 \\ f(x) &= -3(x - 1)^2 + 2 && \text{Factor; combine terms.} \end{aligned}$$

The vertex is (1, 2).

Work Problem 2 at the Side.

OBJECTIVES

1. Find the vertex of a vertical parabola.
2. Graph a quadratic function.
3. Use the discriminant to find the number of x-intercepts of a vertical parabola.
4. Use quadratic functions to solve problems involving maximum or minimum value.
5. Graph horizontal parabolas.

1 Find the vertex of the graph of each quadratic function.

(a) $f(x) = x^2 - 6x + 7$

(b) $f(x) = x^2 + 4x - 9$

2 Find the vertex of the graph of each quadratic function.

(a) $f(x) = 2x^2 - 4x + 1$

(b) $f(x) = -\frac{1}{2}x^2 + 2x - 3$

ANSWERS

1. (a) $(3, -2)$ (b) $(-2, -13)$
2. (a) $(1, -1)$ (b) $(2, -1)$

3 Use the formula to find the vertex of the graph of each quadratic function.

(a) $f(x) = -2x^2 + 3x - 1$

(b) $f(x) = 4x^2 - x + 5$

To derive a formula for the vertex of the graph of the quadratic function defined by $f(x) = ax^2 + bx + c$, complete the square.

$$f(x) = ax^2 + bx + c \quad (a \neq 0) \qquad \text{Standard form}$$

$$= a\left(x^2 + \frac{b}{a}x\right) + c \qquad \text{Factor } a \text{ from the first two terms.}$$

$$\left[\frac{1}{2}\left(\frac{b}{a}\right)\right]^2 = \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \qquad \text{Add and subtract } \frac{b^2}{4a^2}.$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c \qquad \text{Distributive property}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \qquad \text{Factor; combine terms.}$$

$$f(x) = a\left[x - \underbrace{\left(\frac{-b}{2a}\right)}_{h}\right]^2 + \underbrace{\frac{4ac - b^2}{4a}}_{k} \qquad f(x) = (x - h)^2 + k$$

Thus, the vertex (h, k) can be expressed in terms of a, b, and c. It is not necessary to remember the expression for k, since it can be found by replacing x with $\frac{-b}{2a}$. Using function notation, if $y = f(x)$, then the y-value of the vertex is $f\left(\frac{-b}{2a}\right)$.

Vertex Formula

The graph of the quadratic function defined by $f(x) = ax^2 + bx + c$ has vertex

$$\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right),$$

and the axis of the parabola is the line

$$x = \frac{-b}{2a}.$$

EXAMPLE 3 Using the Formula to Find the Vertex

Use the vertex formula to find the vertex of the graph of $f(x) = x^2 - x - 6$.

For this function, $a = 1$, $b = -1$, and $c = -6$. The x-coordinate of the vertex of the parabola is given by

$$\frac{-b}{2a} = \frac{-(-1)}{2(1)} = \frac{1}{2}.$$

The y-coordinate is $f\left(\frac{-b}{2a}\right) = f\left(\frac{1}{2}\right)$.

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 6 = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4}$$

The vertex is $\left(\frac{1}{2}, -\frac{25}{4}\right)$.

Work Problem 3 at the Side.

ANSWERS

3. (a) $\left(\frac{3}{4}, \frac{1}{8}\right)$ (b) $\left(\frac{1}{8}, \frac{79}{16}\right)$

OBJECTIVE 2 **Graph a quadratic function.** We give a general approach for graphing any quadratic function here.

Graphing a Quadratic Function f

Step 1 **Determine whether the graph opens up or down.** If $a > 0$, the parabola opens up; if $a < 0$, it opens down.

Step 2 **Find the vertex.** Use either the vertex formula or completing the square.

Step 3 **Find any intercepts.** To find the x-intercepts (if any), solve $f(x) = 0$. To find the y-intercept, evaluate $f(0)$.

Step 4 **Complete the graph.** Plot the points found so far. Find and plot additional points as needed, using symmetry about the axis.

Audio

YouTryIt

Video

EXAMPLE 4 Using the Steps to Graph a Quadratic Function

Graph the quadratic function defined by $f(x) = x^2 - x - 6$.

Step 1 From the equation, $a = 1$, so the graph of the function opens up.

Step 2 The vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, was found in Example 3 by substituting the values $a = 1$, $b = -1$, and $c = -6$ in the vertex formula.

Step 3 Now find any intercepts. Since the vertex, $\left(\frac{1}{2}, -\frac{25}{4}\right)$, is in quadrant IV and the graph opens up, there will be two x-intercepts. To find them, let $f(x) = 0$ and solve the equation.

$$\begin{aligned} f(x) &= x^2 - x - 6 & & \\ 0 &= x^2 - x - 6 & & \text{Let } f(x) = 0. \\ 0 &= (x-3)(x+2) & & \text{Factor.} \\ x - 3 = 0 \quad &\text{or} \quad x + 2 = 0 & & \text{Zero-factor property} \\ x = 3 \quad &\text{or} \quad x = -2 & & \end{aligned}$$

The x-intercepts are $(3, 0)$ and $(-2, 0)$. Find the y-intercept.

$$\begin{aligned} f(x) &= x^2 - x - 6 & & \\ f(0) &= 0^2 - 0 - 6 & & \text{Let } x = 0. \\ f(0) &= -6 & & \end{aligned}$$

The y-intercept is $(0, -6)$.

Step 4 Plot the points found so far and additional points as needed using symmetry about the axis $x = \frac{1}{2}$. The graph is shown in Figure 12. The domain is $(-\infty, \infty)$, and the range is $\left[-\frac{25}{4}, \infty\right)$.

x	y
-2	0
-1	-4
0	-6
$\frac{1}{2}$	$-\frac{25}{4}$
2	-4
3	0

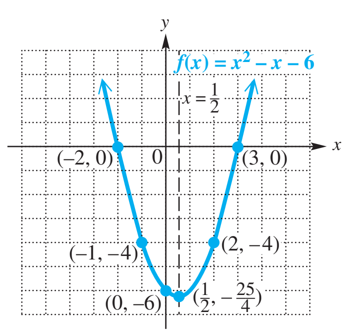


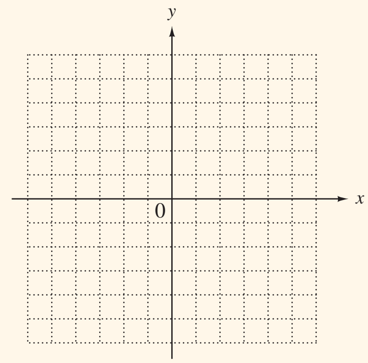
Figure 12

Work Problem 4 at the Side.

4 Graph the quadratic function defined by

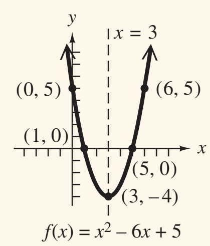
$$f(x) = x^2 - 6x + 5.$$

Give the axis, domain, and range.



ANSWERS

4.



axis: $x = 3$; domain: $(-\infty, \infty)$; range: $[-4, \infty)$

5 Use the discriminant to determine the number of x-intercepts of the graph of each quadratic function.

(a) $f(x) = 4x^2 - 20x + 25$

(b) $f(x) = 2x^2 + 3x + 5$

(c) $f(x) = -3x^2 - x + 2$

OBJECTIVE 3 **Use the discriminant to find the number of x-intercepts of a vertical parabola.** Recall from **Section 10.2** that the expression $b^2 - 4ac$ is called the discriminant of the quadratic *equation* $ax^2 + bx + c = 0$ and that we can use it to determine the number of real solutions of a quadratic equation. In a similar way, we can use the discriminant of a quadratic *function* to determine the number of x-intercepts of its graph. See Figure 13. If the discriminant is positive, the parabola will have two x-intercepts. If the discriminant is 0, there will be only one x-intercept, and it will be the vertex of the parabola. If the discriminant is negative, the graph will have no x-intercepts.

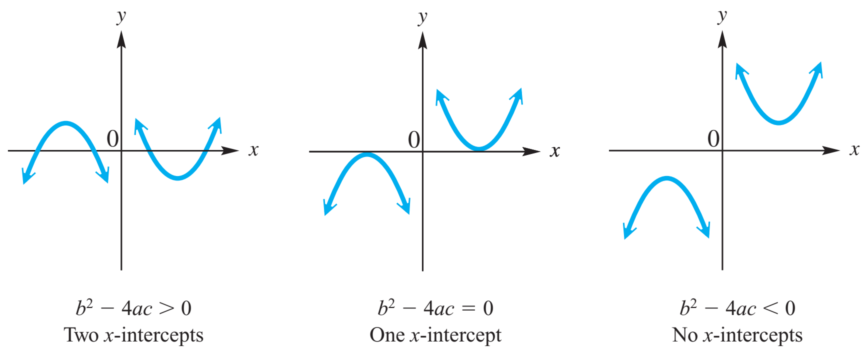


Figure 13

EXAMPLE 5 **Using the Discriminant to Determine the Number of x-Intercepts**

Use the discriminant to determine the number of x-intercepts of the graph of each quadratic function.

(a) $f(x) = 2x^2 + 3x - 5$

The discriminant is $b^2 - 4ac$. Here $a = 2$, $b = 3$, and $c = -5$, so

$$b^2 - 4ac = 9 - 4(2)(-5) = \mathbf{49}.$$

Since the discriminant is positive, the parabola has two x-intercepts.

(b) $f(x) = -3x^2 - 1$

In this equation, $a = -3$, $b = 0$, and $c = -1$. The discriminant is

$$b^2 - 4ac = 0 - 4(-3)(-1) = \mathbf{-12}.$$

The discriminant is negative, so the graph has no x-intercepts.

(c) $f(x) = 9x^2 + 6x + 1$

Here, $a = 9$, $b = 6$, and $c = 1$. The discriminant is

$$b^2 - 4ac = 36 - 4(9)(1) = \mathbf{0}.$$

The parabola has only one x-intercept (its vertex) because the value of the discriminant is 0.

Work Problem 5 at the Side.

OBJECTIVE 4 **Use quadratic functions to solve problems involving maximum or minimum value.** The vertex of a parabola is either the highest or the lowest point on the parabola. The y-value of the vertex gives the maximum or minimum value of y, while the x-value tells where that maximum or minimum occurs.

Answers

5. **(a)** discriminant is 0; one x-intercept **(b)** discriminant is -31; no x-intercepts **(c)** discriminant is 25; two x-intercepts

PROBLEM-SOLVING HINT

In many applied problems we must find the largest or smallest value of some quantity. When we can express that quantity as a quadratic function, the value of k in the vertex (h, k) gives that optimum value.

You Try It

Animation

EXAMPLE 6 Finding the Maximum Area of a Rectangular Region

A farmer has 120 ft of fencing to enclose a rectangular area next to a building. See Figure 14. Find the maximum area he can enclose.

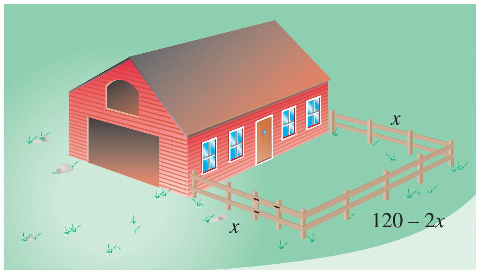


Figure 14

Let x represent the width of the rectangle. Since he has 120 ft of fencing,

$$x + x + \text{length} = 120 \quad \text{Sum of the sides is 120 ft.}$$

$$2x + \text{length} = 120 \quad \text{Combine terms.}$$

$$\text{length} = 120 - 2x. \quad \text{Subtract } 2x.$$

The area $A(x)$ is given by the product of the width and length, so

$$A(x) = x(120 - 2x)$$
$$= 120x - 2x^2.$$

To determine the maximum area, find the vertex of the parabola given by $A(x) = 120x - 2x^2$ using the vertex formula. Writing the equation in standard form as $A(x) = -2x^2 + 120x$ gives $a = -2$, $b = 120$, and $c = 0$, so

$$h = \frac{-b}{2a} = \frac{-120}{2(-2)} = \frac{-120}{-4} = 30;$$

$$A(30) = -2(30)^2 + 120(30) = -2(900) + 3600 = 1800.$$

The graph is a parabola that opens down, and its vertex is (30, 1800). Thus, the maximum area will be 1800 ft^2. This area will occur if x, the width of the rectangle, is 30 ft.

CAUTION

Be careful when interpreting the meanings of the coordinates of the vertex. The first coordinate, x, gives the value for which the ***function value*** is a maximum or a minimum. Be sure to read the problem carefully to determine whether you are asked to find the value of the independent variable, the function value, or both.

Work Problem 6 at the Side.

6 Solve Example 6 if the farmer has only 100 ft of fencing.

ANSWERS

6. The rectangle should be 25 ft by 50 ft with a maximum area of 1250 ft^2.

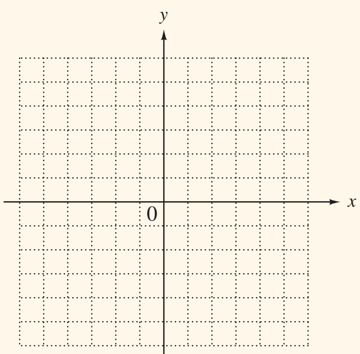
7 Solve the problem.

A toy rocket is launched from the ground so that its distance in feet above the ground after t seconds is

$$s(t) = -16t^2 + 208t.$$

Find the maximum height it reaches and the number of seconds it takes to reach that height.

8 Graph $x = (y + 1)^2 - 4$. Give the vertex, axis, domain, and range.



EXAMPLE 7 Finding the Maximum Height Attained by a Projectile

If air resistance is neglected, a projectile on Earth shot straight upward with an initial velocity of 40 m per sec will be at a height s in meters given by

$$s(t) = -4.9t^2 + 40t,$$

where t is the number of seconds elapsed after projection. After how many seconds will it reach its maximum height, and what is this maximum height?

For this function, $a = -4.9$, $b = 40$, and $c = 0$. Use the vertex formula.

$$h = \frac{-b}{2a} = \frac{-40}{2(-4.9)} \approx 4.1 \qquad \text{Use a calculator.}$$

Thus, the maximum height is attained at 4.1 sec. To find this maximum height, calculate $s(4.1)$.

$$s(4.1) = -4.9(4.1)^2 + 40(4.1) \approx 81.6 \qquad \text{Use a calculator.}$$

The projectile will attain a maximum height of approximately 81.6 m.

Work Problem 7 at the Side.

OBJECTIVE 5 Graph horizontal parabolas. If x and y are interchanged in the equation $y = ax^2 + bx + c$, the equation becomes $x = ay^2 + by + c$. Because of the interchange of the roles of x and y, these parabolas are horizontal (with horizontal lines as axes).

Graph of a Horizontal Parabola

The graph of

$$x = ay^2 + by + c \quad \text{or} \quad x = a(y - k)^2 + h$$

is a parabola with vertex (h, k) and the horizontal line $y = k$ as axis. The graph opens to the right if $a > 0$ and to the left if $a < 0$.

EXAMPLE 8 Graphing a Horizontal Parabola

Graph $x = (y - 2)^2 - 3$.

This graph has its vertex at $(-3, 2)$, since the roles of x and y are reversed. It opens to the right, the positive x-direction, and has the same shape as $y = x^2$. Plotting a few additional points gives the graph shown in Figure 15. Note that the graph is symmetric about its axis, $y = 2$. The domain is $[-3, \infty)$, and the range is $(-\infty, \infty)$.

x	y
-3	2
-2	3
-2	1
1	4
1	0

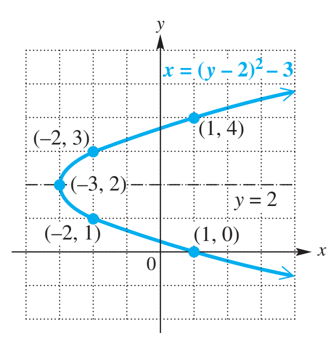


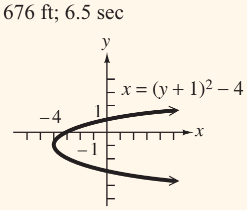
Figure 15

Work Problem 8 at the Side.

ANSWERS

7. 676 ft; 6.5 sec

8.



vertex: $(-4, -1)$; axis: $y = -1$; domain: $[-4, \infty)$; range: $(-\infty, \infty)$

When a quadratic equation is given in the form $x = ay^2 + by + c$, completing the square on y will allow us to find the vertex.

EXAMPLE 9 Completing the Square to Graph a Horizontal Parabola

Video

Graph $x = -2y^2 + 4y - 3$. Give the domain and range of the relation.

$$\begin{aligned} x &= -2y^2 + 4y - 3 \\ &= -2(y^2 - 2y) - 3 && \text{Factor out } -2. \\ &= -2(y^2 - 2y + 1 - 1) - 3 && \text{Complete the square; add and subtract 1.} \\ &= -2(y^2 - 2y + 1) + (-2)(-1) - 3 && \text{Distributive property} \\ x &= -2(y - 1)^2 - 1 && \text{Factor; simplify.} \end{aligned}$$

Because of the negative coefficient (-2), the graph opens to the left (the negative x-direction) and is narrower than the graph of $y = x^2$. As shown in Figure 16, the vertex is $(-1, 1)$. The domain is $(-\infty, -1]$, and the range is $(-\infty, \infty)$.

x	y
-3	2
-3	0
-1	1

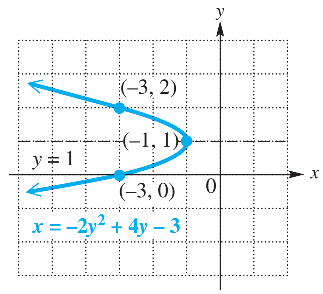


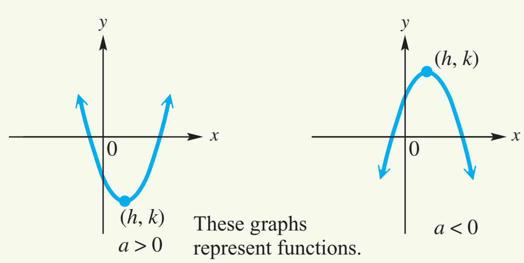
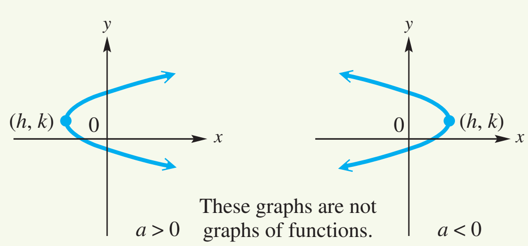
Figure 16

Work Problems 9 and 10 at the Side.

In summary, the graphs of parabolas studied in **Sections 10.5 and 10.6** fall into the following categories.

Audio

GRAPHS OF PARABOLAS

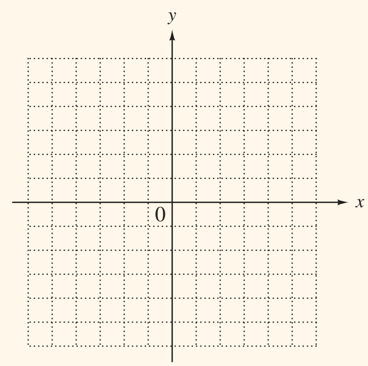
Equation	Graph
$y = ax^2 + bx + c$ $y = a(x - h)^2 + k$	
$x = ay^2 + by + c$ $x = a(y - k)^2 + h$	

9 Find the vertex of each parabola. Tell whether the graph opens to the right or to the left. Give the domain and range.

(a) $x = 2y^2 - 6y + 5$

(b) $x = -y^2 + 2y + 5$

10 Graph $x = -y^2 + 2y + 5$. Give the vertex, axis, domain, and range.

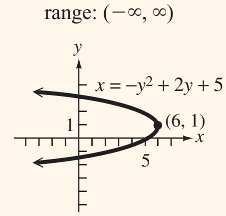


ANSWERS

9. (a) $\left(\frac{1}{2}, \frac{3}{2}\right)$; right; domain: $\left[\frac{1}{2}, \infty\right)$; range: $(-\infty, \infty)$

(b) $(6, 1)$; left; domain: $(-\infty, 6]$; range: $(-\infty, \infty)$

10.



vertex: $(6, 1)$; axis: $y = 1$; domain: $(-\infty, 6]$; range: $(-\infty, \infty)$

11 **(a)** Tell whether each of the following equations has a vertical or horizontal parabola as its graph.

A. $y = -x^2 + 20x + 80$

B. $x = 2y^2 + 6y + 5$

C. $x + 1 = (y + 2)^2$

D. $f(x) = (x - 4)^2$

(b) Which of the equations in part (a) represent functions?

ANSWERS

11. (a) A, D are vertical parabolas; B, C are horizontal parabolas.

(b) A, D

CAUTION

Only quadratic equations solved for y (whose graphs are vertical parabolas) are examples of functions. The horizontal parabolas in Examples 8 and 9 are ***not*** graphs of functions, because they do not satisfy the vertical line test. Furthermore, the vertex formula given earlier does not apply to parabolas with horizontal axes.

Work Problem 11 at the Side.

10.7 Quadratic and Rational Inequalities

OBJECTIVES

1. Solve quadratic inequalities.
2. Solve polynomial inequalities of degree 3 or more.
3. Solve rational inequalities.

We combine methods of solving linear inequalities and methods of solving quadratic equations to solve *quadratic inequalities*.

Audio

Quadratic Inequality

A **quadratic inequality** can be written in the form

$$ax^2 + bx + c < 0 \quad \text{or} \quad ax^2 + bx + c > 0,$$

where a, b, and c are real numbers, with $a \neq 0$.

As before, $<$ and $>$ may be replaced with $\leq$ and $\geq$.

OBJECTIVE 1 Solve quadratic inequalities. One method for solving a quadratic inequality is by graphing the related quadratic function.

EXAMPLE 1 Solving Quadratic Inequalities by Graphing

Solve each inequality.

(a) $x^2 - x - 12 > 0$

To solve the inequality, we graph the related quadratic function defined by $f(x) = x^2 - x - 12$. We are particularly interested in the x-intercepts, which are found as in **Section 10.6** by letting $f(x) = 0$ and solving the quadratic equation

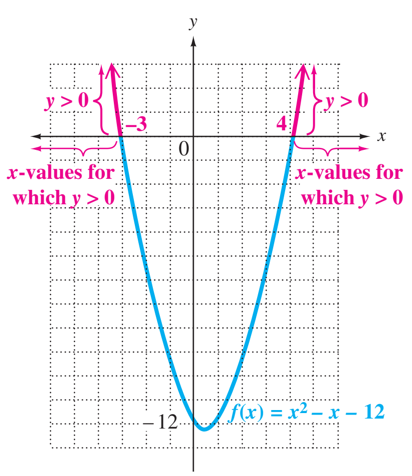
$$x^2 - x - 12 = 0.$$

$$(x - 4)(x + 3) = 0 \qquad \text{Factor.}$$

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-factor property}$$

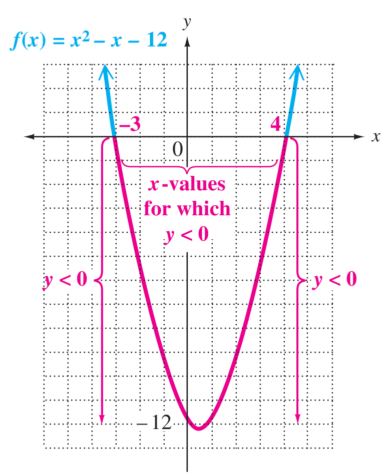
$$x = 4 \quad \text{or} \quad x = -3$$

Thus, the x-intercepts are $(4, 0)$ and $(-3, 0)$. The graph, which opens up since the coefficient of x^2 is positive, is shown in Figure 17(a). Notice from this graph that x-values less than -3 or greater than 4 result in y-values *greater than* 0. Therefore, the solution set of $x^2 - x - 12 > 0$, written in interval notation, is $(-\infty, -3) \cup (4, \infty)$.



The graph is *above* the x-axis for $(-\infty, -3) \cup (4, \infty)$.

(a)



The graph is *below* the x-axis for $(-3, 4)$.

(b)

Figure 17

Continued on Next Page

(b) $x^2 - x - 12 < 0$

Here we want values of y that are *less than* 0. Referring to Figure 17(b) on the previous page, we notice from the graph that x-values between -3 and 4 result in y-values less than 0. Therefore, the solution set of the inequality $x^2 - x - 12 < 0$, written in interval notation, is $(-3, 4)$.

NOTE

If the inequalities in Example 1 had used $\geq$ and $\leq$, the solution sets would have included the x-values of the intercepts and been written in interval notation as $(-\infty, -3] \cup [4, \infty)$ for Example 1(a) and $[-3, 4]$ for Example 1(b).

Work Problems 1 and 2 at the Side.

In Example 1, we used graphing to divide the x-axis into intervals. Then using the graphs in Figure 17, we determined which x-values resulted in y-values that were either greater than or less than 0. Another method for solving a quadratic inequality uses these basic ideas without actually graphing the related quadratic function.

EXAMPLE 2 Solving a Quadratic Inequality Using Test Numbers

You Try It

Video

Solve $x^2 - x - 12 > 0$.

First solve the quadratic equation $x^2 - x - 12 = 0$ by factoring, as in Example 1(a).

$$(x - 4)(x + 3) = 0$$

$$x - 4 = 0 \quad \text{or} \quad x + 3 = 0$$

$$x = 4 \quad \text{or} \quad x = -3$$

The numbers 4 and -3 divide the number line into the three intervals shown in Figure 18. ***Be careful to put the lesser number on the left.*** (Notice the similarity between Figure 18 and the x-axis with intercepts $(-3, 0)$ and $(4, 0)$ in Figure 17(a).)

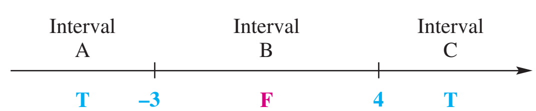


Figure 18

The numbers 4 and -3 are the only numbers that make the expression $x^2 - x - 12$ equal to 0. All other numbers make the expression either positive or negative. The sign of the expression can change from positive to negative or from negative to positive only at a number that makes it 0. Therefore, if one number in an interval satisfies the inequality, then all the numbers in that interval will satisfy the inequality.

To see if the numbers in Interval A satisfy the inequality, choose any number from Interval A in Figure 18 (that is, any number less than -3). Substitute this test number for x in the original inequality $x^2 - x - 12 > 0$. If the result is *true,* then all numbers in Interval A satisfy the inequality.

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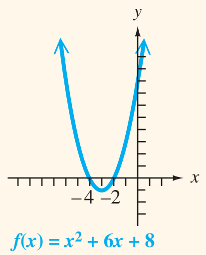
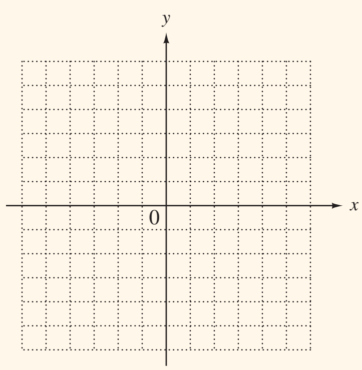
1 Use the graph to solve each quadratic inequality.



(a) $x^2 + 6x + 8 > 0$

(b) $x^2 + 6x + 8 < 0$

2 Graph $f(x) = x^2 + 3x - 4$ and use the graph to solve each quadratic inequality.

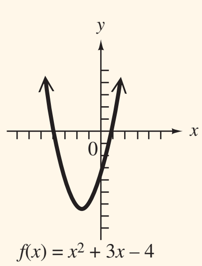


(a) $x^2 + 3x - 4 \geq 0$

(b) $x^2 + 3x - 4 \leq 0$

ANSWERS

1. **(a)** $(-\infty, -4) \cup (-2, \infty)$ **(b)** $(-4, -2)$
2. **(a)** $(-\infty, -4] \cup [1, \infty)$ **(b)** $[-4, 1]$



We choose -5 from Interval A. Substitute -5 for x.

$$x^2 - x - 12 > 0 \quad \text{Original inequality}$$
$$(-5)^2 - (-5) - 12 > 0 \quad ?$$
$$25 + 5 - 12 > 0 \quad ?$$
$$18 > 0 \quad \text{True}$$

Because -5 from Interval A satisfies the inequality, all numbers from Interval A are solutions.

Try 0 from Interval B. If $x = 0$, then

$$0^2 - 0 - 12 > 0 \quad ?$$
$$-12 > 0. \quad \text{False}$$

The numbers in Interval B are *not* solutions.

Work Problem 3 at the Side.

In Problem 3 at the side, the test number 5 satisfies the inequality, so the numbers in Interval C are also solutions.

Based on these results (shown by the colored letters in Figure 18), the solution set includes the numbers in Intervals A and C, as shown on the graph in Figure 19. The solution set is written in interval notation as

$$(-\infty, -3) \cup (4, \infty).$$

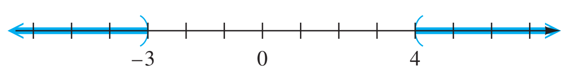


Figure 19

This agrees with the solution set we found by graphing the related quadratic function in Example 1(a).

In summary, a quadratic inequality is solved by following these steps.

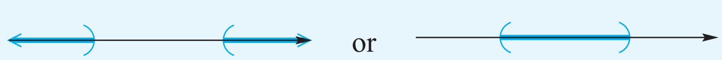
Audio

Solving a Quadratic Inequality

Step 1 **Write the inequality as an equation and solve it.**

Step 2 **Use the solutions from Step 1 to determine intervals.** Graph the numbers found in Step 1 on a number line. These numbers divide the number line into intervals.

Step 3 **Find the intervals that satisfy the inequality.** Substitute a test number from each interval into the original inequality to determine the intervals that satisfy the inequality. All numbers in those intervals are in the solution set. A graph of the solution set will usually look like one of these. (Square brackets might be used instead of parentheses.)



Step 4 **Consider the endpoints separately.** The numbers from Step 1 are included in the solution set if the inequality is $\leq$ or $\geq$; they are not included if it is $<$ or $>$.

Work Problem 4 at the Side.

3 Does the number 5 from Interval C satisfy $x^2 - x - 12 > 0$?

4 Solve each inequality, and graph the solution set.

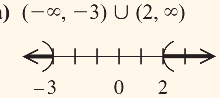
(a) $x^2 + x - 6 > 0$

(b) $3m^2 - 13m - 10 \leq 0$

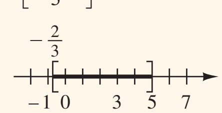
Answers

3. yes

4. (a) $(-\infty, -3) \cup (2, \infty)$



(b) $\left[-\frac{2}{3}, 5\right]$



5 Solve each inequality.

(a) $(3k - 2)^2 > -2$

(b) $(5z + 3)^2 < -3$

6 Solve each inequality, and graph the solution set.

(a) $(x - 3)(x + 2)(x + 1) > 0$

(b) $(k - 5)(k + 1)(k - 3) \leq 0$

Special cases of quadratic inequalities may occur, as in the next example.

EXAMPLE 3 Solving Special Cases

Solve $(2t - 3)^2 > -1$. Then solve $(2t - 3)^2 < -1$.

Because $(2t - 3)^2$ is never negative, it is always greater than -1. Thus, the solution set for $(2t - 3)^2 > -1$ is the set of all real numbers, $(-\infty, \infty)$. In the same way, there is no solution for $(2t - 3)^2 < -1$ and the solution set is $\emptyset$.

Work Problem 5 at the Side.

OBJECTIVE 2 Solve polynomial inequalities of degree 3 or more.

Higher-degree polynomial inequalities that can be factored are solved in the same way as quadratic inequalities.

EXAMPLE 4 Solving a Third-Degree Polynomial Inequality

Solve $(x - 1)(x + 2)(x - 4) \leq 0$.

This is a *cubic* (third-degree) inequality rather than a quadratic inequality, but it can be solved using the method shown in the box by extending the zero-factor property to more than two factors. Begin by setting the factored polynomial *equal* to 0 and solving the equation (Step 1).

$$(x - 1)(x + 2)(x - 4) = 0$$

$$x - 1 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 1 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 4$$

Locate the numbers -2, 1, and 4 on a number line, as in Figure 20, to determine the Intervals A, B, C, and D (Step 2).

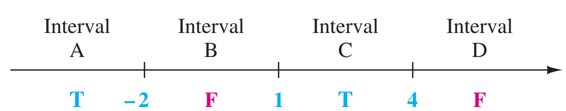


Figure 20

Substitute a test number from each interval in the *original* inequality to determine which intervals satisfy the inequality (Step 3). It is helpful to organize this information in a table.

Interval	Test Number	Test of Inequality	True or False?
A	-3	$-28 \leq 0$	T
B	0	$8 \leq 0$	F
C	2	$-8 \leq 0$	T
D	5	$28 \leq 0$	F

Verify the information given in the table and graphed in Figure 21. The numbers in Intervals A and C are in the solution set, which is written as

$$(-\infty, -2] \cup [1, 4].$$

The three endpoints are included since the inequality symbol is $\leq$ (Step 4).

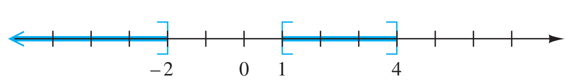


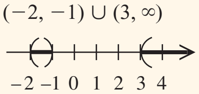
Figure 21

Work Problem 6 at the Side.

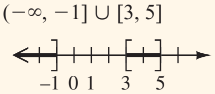
ANSWERS

5. (a) $(-\infty, \infty)$ **(b)** $\emptyset$

6. (a) $(-2, -1) \cup (3, \infty)$



(b) $(-\infty, -1] \cup [3, 5]$



OBJECTIVE 3 Solve rational inequalities. Inequalities that involve rational expressions, called **rational inequalities**, are solved similarly using the following steps.

Solving a Rational Inequality

Step 1 **Write the inequality** so that 0 is on one side and there is a single fraction on the other side.

Step 2 **Determine the numbers that make the numerator and denominator equal to 0.**

Step 3 **Divide a number line into intervals.** Use the numbers from Step 2.

Step 4 **Find the intervals that satisfy the inequality.** Test a number from each interval by substituting it into the *original* inequality.

Step 5 **Consider the endpoints separately.** Exclude any values that make the denominator 0.

EXAMPLE 5 Solving a Rational Inequality

Solve $\dfrac{-1}{p-3} > 1$.

Write the inequality so that 0 is on one side (Step 1).

$$\frac{-1}{p-3} - 1 > 0 \qquad \text{Subtract 1.}$$

$$\frac{-1}{p-3} - \frac{p-3}{p-3} > 0 \qquad \text{Use } p-3 \text{ as the common denominator.}$$

$$\frac{-1-p+3}{p-3} > 0 \qquad \text{Write the left side as a single fraction; be careful with signs in the numerator.}$$

$$\frac{-p+2}{p-3} > 0 \qquad \text{Combine terms in the numerator.}$$

The sign of the rational expression $\frac{-p+2}{p-3}$ will change from positive to negative or negative to positive only at those numbers that make the numerator or denominator 0. The number 2 makes the numerator 0, and 3 makes the denominator 0 (Step 2). These two numbers, 2 and 3, divide a number line into three intervals. See Figure 22 (Step 3).

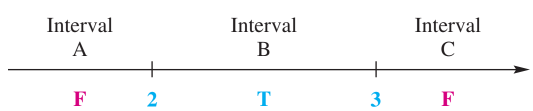


Figure 22

Testing a number from each interval in the *original* inequality, $\frac{-1}{p-3} > 1$, gives the results shown in the table (Step 4).

Interval	Test Number	Test of Inequality	True or False?
A	0	$\frac{1}{3} > 1$	F
B	2.5	$2 > 1$	T
C	4	$-1 > 1$	F

Continued on Next Page

7 Solve each inequality, and graph the solution set.

(a) $\dfrac{2}{x-4} < 3$

(b) $\dfrac{5}{z+1} > 4$

8 Solve $\dfrac{k+2}{k-1} \leq 5$, and graph the solution set.

The solution set of $\frac{-1}{p-3} > 1$ is the interval (2, 3). This interval does not include 3 since it would make the denominator of the original inequality 0; 2 is not included either since the inequality symbol is > (Step 5). A graph of the solution set is given in Figure 23.

Figure 23

Work Problem 7 at the Side.

CAUTION

When solving a rational inequality, ***any number that makes the denominator 0 must be excluded from the solution set.***

EXAMPLE 6 Solving a Rational Inequality

Solve $\dfrac{m-2}{m+2} \leq 2$.

Write the inequality so that 0 is on one side (Step 1).

$$\frac{m-2}{m+2} - 2 \leq 0 \qquad \text{Subtract 2.}$$

$$\frac{m-2}{m+2} - \frac{2(m+2)}{m+2} \leq 0 \qquad \text{Use } m+2 \text{ as the common denominator.}$$

$$\frac{m-2-2m-4}{m+2} \leq 0 \qquad \text{Write as a single fraction.}$$

$$\frac{-m-6}{m+2} \leq 0 \qquad \text{Combine terms in the numerator.}$$

The number -6 makes the numerator 0, and -2 makes the denominator 0 (Step 2). These two numbers determine three intervals (Step 3). Test one number from each interval (Step 4) to see that the solution set is the interval

$$(-\infty, -6] \cup (-2, \infty).$$

The number -6 satisfies the original inequality, but -2 cannot be used as a solution since it makes the denominator 0 (Step 5). A graph of the solution set is shown in Figure 24.

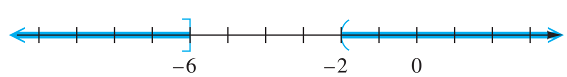
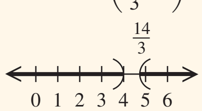


Figure 24

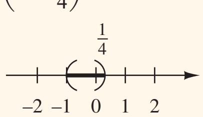
Work Problem 8 at the Side.

ANSWERS

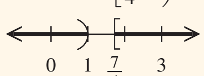
7. (a) $(-\infty, 4) \cup \left(\frac{14}{3}, \infty\right)$



(b) $\left(-1, \frac{1}{4}\right)$



8. $(-\infty, 1) \cup \left[\frac{7}{4}, \infty\right)$



Exponential and Logarithmic Functions

11

With the many advances made in electronics over the past decade, home theater is now a reality. The operating instructions for the Pioneer PD-F1009 compact disc player, a component in one author's system, includes a warning that loud noises can cause hearing damage, and a list of sound levels, measured in *decibels* and shown in the accompanying table, that can be dangerous under constant exposure.

In Exercise 39 of Section 11.5 we examine the meaning of decibel, which is based on *logarithmic functions,* one topic covered in this chapter.

Decibel Level	Example
90	Subway, motorcycle, truck traffic, lawn mower
100	Garbage truck, chain saw, pneumatic drill
120	Rock concert in front of speakers, thunderclap
140	Gunshot blast, jet plane
180	Rocket launching pad

Source: Deafness Research Foundation.

11.1 Inverse Functions

OBJECTIVES

1. Decide whether a function is one-to-one and, if it is, find its inverse.
2. Use the horizontal line test to determine whether a function is one-to-one.
3. Find the equation of the inverse of a function.
4. Graph f^{-1} from the graph of f.

In this chapter we will study two important types of functions, *exponential* and *logarithmic*. These functions are related in a special way: They are *inverses* of one another. We begin by discussing inverse functions in general.

Calculator Tip A calculator with the following keys will be essential in this chapter.

y^x, 10^x or LOG, e^x or $\ln x$

We will explain how these keys are used at appropriate places in the chapter.

OBJECTIVE 1 Decide whether a function is one-to-one and, if it is, find its inverse. Suppose we define the function

$$G = \{(-2, 2), (-1, 1), (0, 0), (1, 3), (2, 5)\}.$$

We can form another set of ordered pairs from G by interchanging the x- and y-values of each pair in G. Call this set F, with

$$F = \{(2, -2), (1, -1), (0, 0), (3, 1), (5, 2)\}.$$

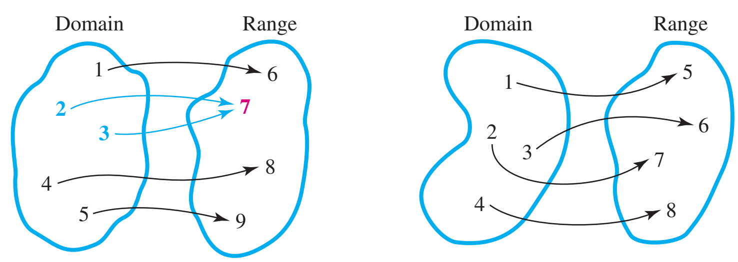
To show that these two sets are related, F is called the *inverse* of G. For a function f to have an inverse, f must be *one-to-one.*

One-to-One Function

In a **one-to-one function,** each x-value corresponds to only one y-value, and each y-value corresponds to just one x-value.

Audio

The function shown in Figure 1(a) is not one-to-one because the y-value 7 corresponds to *two* x-values, 2 and 3. That is, the ordered pairs (2, 7) and (3, 7) both appear in the function. The function in Figure 1(b) is one-to-one.



Not one-to-one
(a)

One-to-one
(b)

Figure 1

The *inverse* of any one-to-one function f is found by interchanging the components of the ordered pairs of f. The inverse of f is written $\boldsymbol{f^{-1}}$. Read f^{-1} as "the inverse of f" or "f-inverse."

CAUTION

The symbol $f^{-1}(x)$ does not represent $\dfrac{1}{f(x)}$.

The definition of the inverse of a function follows.

Audio

Inverse of a Function

The **inverse** of a one-to-one function f, written f^{-1}, is the set of all ordered pairs of the form (y, x), where (x, y) belongs to f. Since the inverse is formed by interchanging x and y, the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}.

For inverses f and f^{-1}, it follows that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

Video

EXAMPLE 1 Finding the Inverses of One-to-One Functions

Find the inverse of each function that is one-to-one.

(a) $F = \{(-2, 1), (-1, 0), (0, 1), (1, 2), (2, 2)\}$

Each x-value in F corresponds to just one y-value. However, the y-value 2 corresponds to two x-values, 1 and 2. Also, the y-value 1 corresponds to both -2 and 0. Because some y-values correspond to more than one x-value, F is not one-to-one and does not have an inverse.

(b) $G = \{(3, 1), (0, 2), (2, 3), (4, 0)\}$

Every x-value in G corresponds to only one y-value, and every y-value corresponds to only one x-value, so G is a one-to-one function. The inverse function is found by interchanging the x- and y-values in each ordered pair.

$$G^{-1} = \{(1, 3), (2, 0), (3, 2), (0, 4)\}$$

Notice how the domain and range of G become the range and domain, respectively, of G^{-1}.

(c) The U.S. Environmental Protection Agency has developed an indicator of air quality called the Pollutant Standard Index (PSI). If the PSI exceeds 100 on a particular day, that day is classified as unhealthy. The table shows the number of unhealthy days in Chicago for the years 1991–2002, based on new standards set in 1998.

Year	Number of Unhealthy Days	Year	Number of Unhealthy Days
1991	24	1997	10
1992	5	1998	12
1993	4	1999	19
1994	13	2000	2
1995	24	2001	22
1996	7	2002	21

Source: U.S. Environmental Protection Agency, Office of Air Quality Planning and Standards.

Let f be the function defined in the table, with the years forming the domain and the numbers of unhealthy days forming the range. Then f is not one-to-one, because in two different years (1991 and 1995), the number of unhealthy days was the same, 24.

Work Problem 1 at the Side.

1 Find the inverse of each function that is one-to-one.

(a) $\{(1, 2), (2, 4), (3, 3), (4, 5)\}$

(b) $\{(0, 3), (-1, 2), (1, 3)\}$

(c) A Norwegian physiologist has developed a rule for predicting running times based on the time to run 5 km (5K). An example for one runner is shown here. (*Source:* Stephen Seiler, Agder College, Kristiansand, Norway.)

Distance	Time
1.5K	4:22
3K	9:18
5K	16:00
10K	33:40

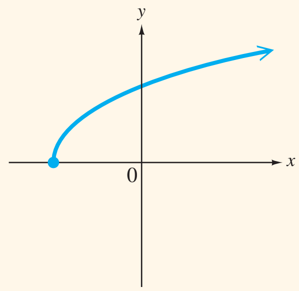
Answers

1. **(a)** $\{(2, 1), (4, 2), (3, 3), (5, 4)\}$
(b) not a one-to-one function
(c)

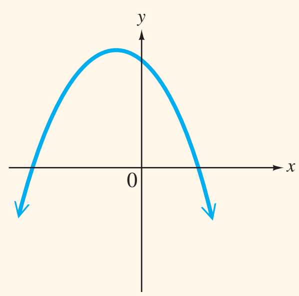
Time	Distance
4:22	1.5K
9:18	3K
16:00	5K
33:40	10K

2 Use the horizontal line test to determine whether each graph is the graph of a one-to-one function.

(a)



(b)



OBJECTIVE 2 **Use the horizontal line test to determine whether a function is one-to-one.** It may be difficult to decide whether a function is one-to-one just by looking at the equation that defines the function. However, by graphing the function and observing the graph, we can use the *horizontal line test* to tell whether the function is one-to-one.

Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

The horizontal line test follows from the definition of a one-to-one function. Any two points that lie on the same horizontal line have the same y-coordinate. No two ordered pairs that belong to a one-to-one function may have the same y-coordinate, and therefore no horizontal line will intersect the graph of a one-to-one function more than once.

EXAMPLE 2 Using the Horizontal Line Test

Use the horizontal line test to determine whether the graphs in Figures 2 and 3 are graphs of one-to-one functions.

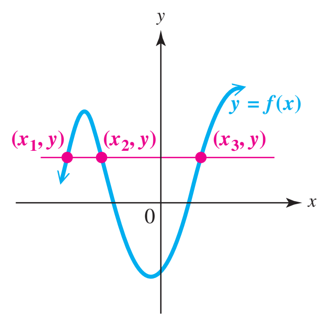


Figure 2

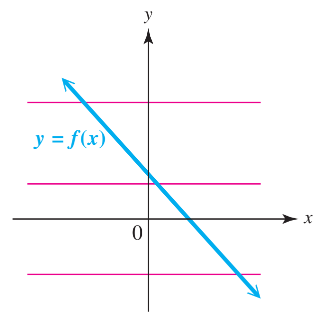


Figure 3

Because the horizontal line shown in Figure 2 intersects the graph in more than one point (actually three points), the function is not one-to-one.

Every horizontal line will intersect the graph in Figure 3 in exactly one point. This function is one-to-one.

Work Problem 2 at the Side.

OBJECTIVE 3 **Find the equation of the inverse of a function.** By definition, the inverse of a function is found by interchanging the x- and y-values of each of its ordered pairs. The equation of the inverse of a function defined by $y = f(x)$ is found in the same way.

Finding the Equation of the Inverse of $y = f(x)$

For a one-to-one function f defined by an equation $y = f(x)$, find the defining equation of the inverse as follows.

Step 1 Interchange x and y.

Step 2 Solve for y.

Step 3 Replace y with $f^{-1}(x)$.

ANSWERS

2. **(a)** one-to-one **(b)** not one-to-one

You Try It

Video

EXAMPLE 3 Finding Equations of Inverses

Decide whether each equation defines a one-to-one function. If so, find the equation of the inverse.

(a) $f(x) = 2x + 5$

The graph of $y = 2x + 5$ is a nonvertical line, so by the horizontal line test, f is a one-to-one function. To find the inverse, let $y = f(x)$ so that

$$y = 2x + 5$$
$$x = 2y + 5 \qquad \text{Interchange } x \text{ and } y. \text{ (Step 1)}$$
$$2y = x - 5 \qquad \text{Solve for } y. \text{ (Step 2)}$$
$$y = \frac{x-5}{2}$$
$$f^{-1}(x) = \frac{x-5}{2}. \qquad \text{Replace } y \text{ with } f^{-1}(x). \text{ (Step 3)}$$

Thus, f^{-1} is a linear function. In the function with $y = 2x + 5$, the value of y is found by starting with a value of x, multiplying by 2, and adding 5. The equation for the inverse has us *subtract* 5, and then *divide* by 2. This shows how an inverse is used to "undo" what a function does to the variable x.

(b) $y = x^2 + 2$

This equation has a vertical parabola as its graph, so some horizontal lines will intersect the graph at two points. For example, both $x = 3$ and $x = -3$ correspond to $y = 11$. Because of the x^2-term, there are many pairs of x-values that correspond to the same y-value. This means that the function defined by $y = x^2 + 2$ is not one-to-one and does not have an inverse.

If this is not noticed, following the steps for finding the equation of an inverse leads to

$$y = x^2 + 2$$
$$x = y^2 + 2 \qquad \text{Interchange } x \text{ and } y.$$
$$x - 2 = y^2 \qquad \text{Solve for } y.$$
$$\pm\sqrt{x-2} = y. \qquad \text{Square root property}$$

The last step shows that there are two y-values for each choice of $x > 2$, so the given function is not one-to-one and cannot have an inverse.

(c) $f(x) = (x - 2)^3$

Refer to **Section 6.3** to see from its graph that a cubing function like this is a one-to-one function.

$$y = (x-2)^3 \qquad \text{Replace } f(x) \text{ with } y.$$
$$x = (y-2)^3 \qquad \text{Interchange } x \text{ and } y.$$
$$\sqrt[3]{x} = \sqrt[3]{(y-2)^3} \qquad \text{Take the cube root on each side.}$$
$$\sqrt[3]{x} = y - 2$$
$$y = \sqrt[3]{x} + 2 \qquad \text{Solve for } y.$$
$$f^{-1}(x) = \sqrt[3]{x} + 2 \qquad \text{Replace } y \text{ with } f^{-1}(x).$$

Work Problem 3 at the Side.

3 Decide whether each equation defines a one-to-one function. If so, find the equation that defines the inverse.

(a) $f(x) = 3x - 4$

(b) $f(x) = x^3 + 1$

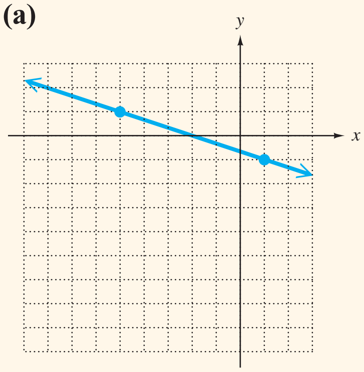
(c) $f(x) = (x - 3)^2$

ANSWERS

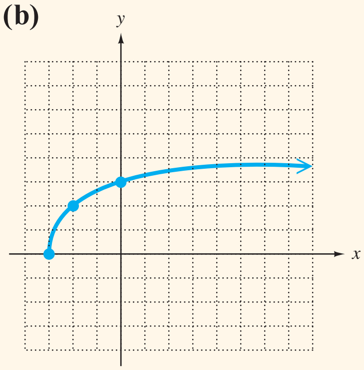
3. **(a)** one-to-one function; $f^{-1}(x) = \frac{x+4}{3}$
(b) one-to-one function; $f^{-1}(x) = \sqrt[3]{x-1}$
(c) not a one-to-one function

4 Use the given graphs to graph each inverse.

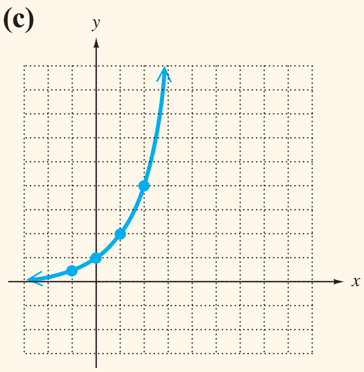
(a)



(b)



(c)



OBJECTIVE 4 Graph f^{-1} from the graph of f. One way to graph the inverse of a function f whose equation is known is to find some ordered pairs that belong to f, interchange x and y to get ordered pairs that belong to f^{-1}, plot those points, and sketch the graph of f^{-1} through the points. A simpler way is to select points on the graph of f and use symmetry to find corresponding points on the graph of f^{-1}.

For example, suppose the point (a, b) shown in Figure 4 belongs to a one-to-one function f. Then the point (b, a) belongs to f^{-1}. The line segment connecting (a, b) and (b, a) is perpendicular to, and cut in half by, the line $y = x$. The points (a, b) and (b, a) are "mirror images" of each other with respect to $y = x$. For this reason we can find the graph of f^{-1} from the graph of f by locating the mirror image of each point in f with respect to the line $y = x$.

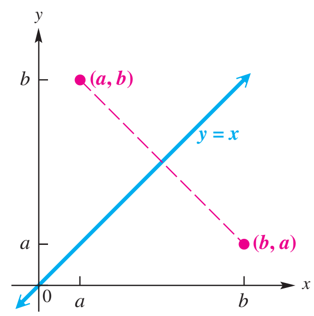


Figure 4

EXAMPLE 4 Graphing the Inverse

Graph the inverses of the functions shown in Figure 5.

In Figure 5 the graphs of two functions are shown in blue. Their inverses are shown in red. In each case, the graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$.

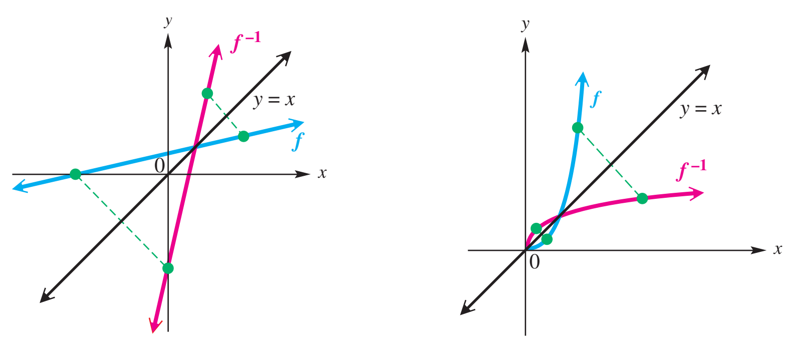


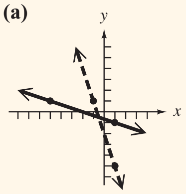
Figure 5

Work Problem 4 at the Side.

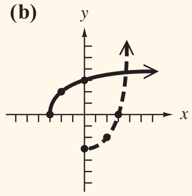
You Try It
Animation
Video

ANSWERS

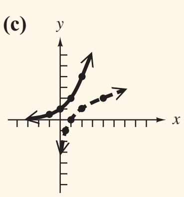
4. (a)



(b)



(c)



11.2 Exponential Functions

OBJECTIVES

1. Define exponential functions.
2. Graph exponential functions.
3. Solve exponential equations of the form $a^x = a^k$ for x.
4. Use exponential functions in applications involving growth or decay.

OBJECTIVE 1 Define exponential functions. In **Section 9.2** we showed how to evaluate 2^x for rational values of x. For example,

$$2^3 = 8, \qquad 2^{-1} = \frac{1}{2}, \qquad 2^{1/2} = \sqrt{2}, \qquad 2^{3/4} = \sqrt[4]{2^3} = \sqrt[4]{8}.$$

In more advanced courses it is shown that 2^x exists for all real number values of x, both rational and irrational. (Later in this chapter, we will see how to approximate the value of 2^x for irrational x.) The following definition of an exponential function assumes that a^x exists for all real numbers x.

Audio

Exponential Function

For $a > 0$, $a \neq 1$, and all real numbers x,

$$f(x) = a^x$$

defines the **exponential function with base a.**

NOTE

The two restrictions on a in the definition of an exponential function are important. The restriction that a must be positive is necessary so that the function can be defined for all real numbers x. For example, letting a be negative ($a = -2$, for instance) and letting $x = \frac{1}{2}$ would give the expression $(-2)^{1/2}$, which is not real. The other restriction, $a \neq 1$, is necessary because 1 raised to any power is equal to 1, and the function would then be the linear function defined by $f(x) = 1$.

OBJECTIVE 2 Graph exponential functions. We can graph an exponential function by finding several ordered pairs that belong to the function, plotting these points, and connecting them with a smooth curve.

You Try It

Video

EXAMPLE 1 Graphing an Exponential Function with $a > 1$

Graph $f(x) = 2^x$.

Choose some values of x, and find the corresponding values of $f(x)$.

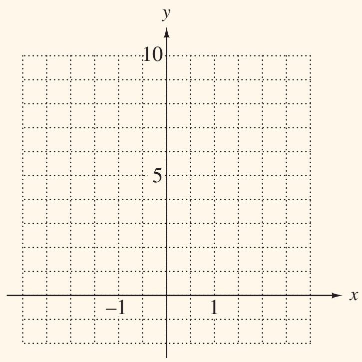
x	-3	-2	-1	0	1	2	3	4
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

Plotting these points and drawing a smooth curve through them gives the graph shown in Figure 6 on the next page. This graph is typical of the graphs of exponential functions of the form $F(x) = a^x$, where $a > 1$. The larger the value of a, the faster the graph rises. To see this, compare the graph of $F(x) = 5^x$ with the graph of $f(x) = 2^x$ in Figure 6.

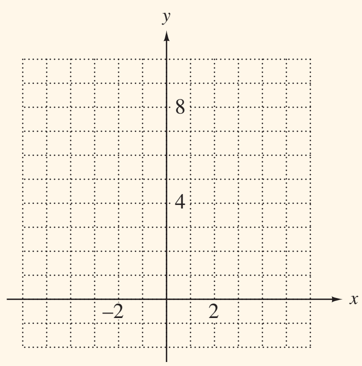
Continued on Next Page

1 Graph.

(a) $f(x) = 10^x$



(b) $g(x) = \left(\frac{1}{4}\right)^x$



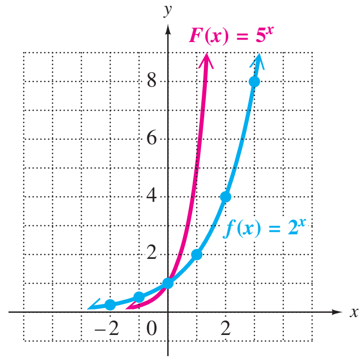


Figure 6

By the vertical line test, the graphs in Figure 6 represent functions. As these graphs suggest, the domain of an exponential function includes all real numbers. Because y is always positive, the range is $(0, \infty)$. Figure 6 also shows an important characteristic of exponential functions where $a > 1$: as x gets larger, y increases at a faster and faster rate.

CAUTION
Be sure to plot a sufficient number of points to see how rapidly the graph rises.

EXAMPLE 2 Graphing an Exponential Function with $a < 1$

Graph $g(x) = \left(\frac{1}{2}\right)^x$.

Again, find some points on the graph.

x	-3	-2	-1	0	1	2	3
$g(x) = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph, shown in Figure 7, is very similar to that of $f(x) = 2^x$ (Figure 6) with the same domain and range, except that here as x gets larger, *y decreases*. This graph is typical of the graph of a function of the form $F(x) = a^x$, where $0 < a < 1$.

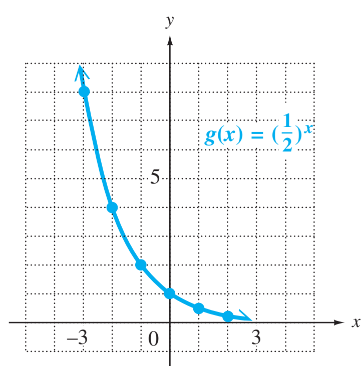


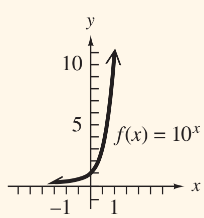
Figure 7

Work Problem 1 at the Side.

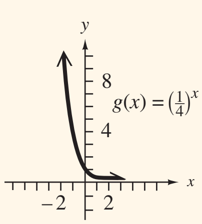
Based on Examples 1 and 2, we make the following generalizations about the graphs of exponential functions of the form $F(x) = a^x$.

ANSWERS

1. (a)



(b)



Characteristics of the Graph of $F(x) = a^x$

1. The graph contains the point (0, 1).
2. When $a > 1$, the graph *rises* from left to right. When $0 < a < 1$, the graph *falls* from left to right. In both cases, the graph goes from the second quadrant to the first.
3. The graph approaches the x-axis, but never touches it. (Recall from **Section 8.4** that such a line is called an *asymptote*.)
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

EXAMPLE 3 **Graphing a More Complicated Exponential Function**

Graph $f(x) = 3^{2x-4}$.

Find some ordered pairs.

$$\text{If } x = 0, \text{ then } y = 3^{2(0)-4} = 3^{-4} = \frac{1}{81}.$$

$$\text{If } x = 2, \text{ then } y = 3^{2(2)-4} = 3^{0} = 1.$$

These ordered pairs, $(0, \frac{1}{81})$ and (2, 1), along with the other ordered pairs shown in the table, lead to the graph in Figure 8. The graph is similar to the graph of $f(x) = 3^x$ except that it is shifted to the right and rises more rapidly.

x	y
0	$\frac{1}{81}$
1	$\frac{1}{9}$
2	1
3	9

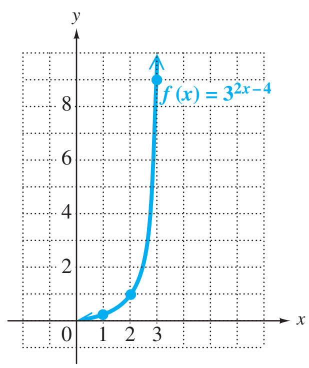


Figure 8

Work Problem 2 at the Side.

OBJECTIVE 3 **Solve exponential equations of the form $a^x = a^k$ for x.** Until this chapter, we have solved only equations that had the variable as a base, like $x^2 = 8$; all exponents have been constants. An **exponential equation** is an equation that has a variable in an exponent, such as

$$9^x = 27.$$

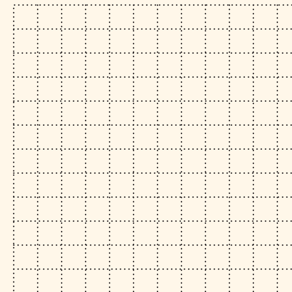
By the horizontal line test, the exponential function defined by $F(x) = a^x$ is a one-to-one function, so we can use the following property to solve many exponential equations.

Property for Solving an Exponential Equation

For $a > 0$ and $a \neq 1$, if $\quad a^x = a^y \quad$ then $\quad x = y.$

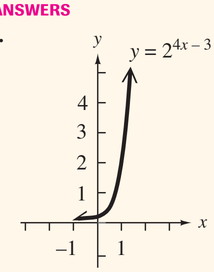
This property would not necessarily be true if $a = 1$.

2 Graph $y = 2^{4x-3}$.



ANSWERS

2.



To solve an exponential equation using this property, follow these steps.

Solving an Exponential Equation

Step 1 **Each side must have the same base.** If the two sides of the equation do not have the same base, express each as a power of the same base.

Step 2 **Simplify exponents,** if necessary, using the rules of exponents.

Step 3 **Set exponents equal** using the property given in this section.

Step 4 **Solve** the equation obtained in Step 3.

NOTE

These steps cannot be applied to an exponential equation like

$$3^x = 12$$

because Step 1 cannot easily be done. A method for solving such equations is given in **Section 11.6.**

EXAMPLE 4 Solving an Exponential Equation

Solve the equation $9^x = 27$.

We can use the property given in the box if both sides are written with the same base. Since $9 = 3^2$ and $27 = 3^3$,

$$9^x = 27$$
$$(3^2)^x = 3^3 \quad \text{Write with the same base. (Step 1)}$$
$$3^{2x} = 3^3 \quad \text{Power rule for exponents (Step 2)}$$
$$2x = 3 \quad \text{If } a^x = a^y, \text{ then } x = y. \text{ (Step 3)}$$
$$x = \frac{3}{2}. \quad \text{Solve for } x. \text{ (Step 4)}$$

Check that the solution set is $\left\{\frac{3}{2}\right\}$ by substituting $\frac{3}{2}$ for x in the original equation.

Work Problem 3 at the Side.

3 Solve each equation and check the solution.

(a) $25^x = 125$

(b) $4^x = 32$

(c) $81^p = 27$

EXAMPLE 5 Solving Exponential Equations

Solve each equation.

(a) $4^{3x-1} = 16^{x+2}$

Since $4 = 2^2$ and $16 = 2^4$,

$$(2^2)^{3x-1} = (2^4)^{x+2} \quad \text{Write with the same base.}$$
$$2^{6x-2} = 2^{4x+8} \quad \text{Power rule for exponents}$$
$$6x - 2 = 4x + 8 \quad \text{Set exponents equal.}$$
$$2x = 10 \quad \text{Subtract } 4x\text{; add 2.}$$
$$x = 5. \quad \text{Divide by 2.}$$

Verify that the solution set is $\{5\}$.

Continued on Next Page

ANSWERS

3. (a) $\left\{\frac{3}{2}\right\}$ (b) $\left\{\frac{5}{2}\right\}$ (c) $\left\{\frac{3}{4}\right\}$

(b) $6^x = \frac{1}{216}$

$6^x = \frac{1}{6^3}$ $\quad 216 = 6^3$

$6^x = 6^{-3}$ $\quad$ Write with the same base; $\frac{1}{6^3} = 6^{-3}$.

$x = -3$ $\quad$ Set exponents equal.

Verify that the solution set is $\{-3\}$.

(c) $\left(\frac{2}{3}\right)^x = \frac{9}{4}$

$\left(\frac{2}{3}\right)^x = \left(\frac{4}{9}\right)^{-1}$ $\quad \frac{9}{4} = \left(\frac{4}{9}\right)^{-1}$

$\left(\frac{2}{3}\right)^x = \left[\left(\frac{2}{3}\right)^2\right]^{-1}$ $\quad$ Write with the same base.

$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^{-2}$ $\quad$ Power rule for exponents

$x = -2$ $\quad$ Set exponents equal.

Check that the solution set is $\{-2\}$.

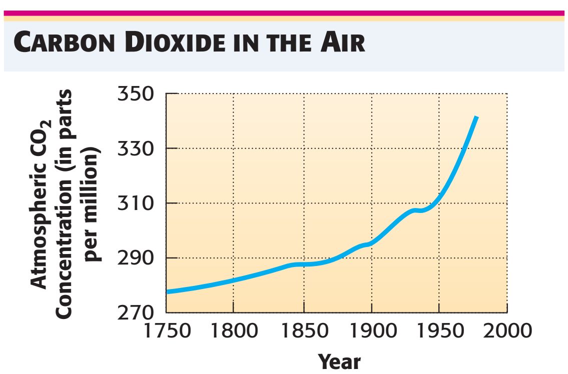
Work Problem 4 at the Side.

OBJECTIVE 4 Use exponential functions in applications involving growth or decay.

Video

EXAMPLE 6 Solving an Application Involving Exponential Growth

One result of the rapidly increasing world population is an increase of carbon dioxide in the air, which scientists believe may be contributing to global warming. Both population and carbon dioxide in the air are increasing exponentially. This means that the growth rate is continually increasing. The graph in Figure 9 shows the concentration of carbon dioxide (in parts per million) in the air.



Source: *Sacramento Bee,* Monday, September 13, 1993.

Figure 9

Continued on Next Page

4 Solve each equation and check the solution.

(a) $25^{x-2} = 125^x$

(b) $4^x = \frac{1}{32}$

(c) $\left(\frac{3}{4}\right)^x = \frac{16}{9}$

ANSWERS

4. (a) $\{-4\}$ **(b)** $\left\{-\frac{5}{2}\right\}$ **(c)** $\{-2\}$

5 Solve each problem.

(a) Use the function in Example 6 to approximate the carbon dioxide concentration in 1925.

(b) Use the function in Example 7 to find the pressure at 8000 m.

The data are approximated by the function defined by

$$f(x) = 278(1.00084)^x,$$

where x is the number of years since 1750. Use this function and a calculator to approximate the concentration of carbon dioxide in parts per million for each year.

(a) 1900
Since x represents the number of years since 1750, in this case $x = 1900 - 1750 = 150$. Thus, evaluate $f(150)$.

$$f(150) = 278(1.00084)^{150} \quad \text{Let } x = 150.$$
$$\approx 315 \text{ parts per million} \quad \text{Use a calculator.}$$

(b) 1950
Use $x = 1950 - 1750 = 200$.

$$f(200) = 278(1.00084)^{200}$$
$$\approx 329 \text{ parts per million}$$

EXAMPLE 7 Applying an Exponential Decay Function

The atmospheric pressure (in millibars) at a given altitude x, in meters, can be approximated by the function defined by

$$f(x) = 1038(1.000134)^{-x},$$

for values of x between 0 and 10,000. Because the base is greater than 1 and the coefficient of x in the exponent is negative, the function values decrease as x increases. This means that as the altitude increases, the atmospheric pressure decreases. (*Source:* Miller, A. and J. Thompson, *Elements of Meteorology,* Fourth Edition, Charles E. Merrill Publishing Company, 1993.)

(a) According to this function, what is the pressure at ground level?
At ground level, $x = 0$, so

$$f(0) = 1038(1.000134)^{-0} = 1038(1) = 1038.$$

The pressure is 1038 millibars.

(b) What is the pressure at 5000 m?
Use a calculator to find $f(5000)$.

$$f(5000) = 1038(1.000134)^{-5000} \approx 531$$

The pressure is approximately 531 millibars.

Work Problem 5 at the Side.

ANSWERS
5. **(a)** 322 parts per million
(b) approximately 355 millibars

11.3 Logarithmic Functions

OBJECTIVES

1. Define a logarithm.
2. Convert between exponential and logarithmic forms.
3. Solve logarithmic equations of the form $\log_a b = k$ for a, b, or k.
4. Define and graph logarithmic functions.
5. Use logarithmic functions in applications of growth or decay.

The graph of $y = 2^x$ is the curve shown in blue in Figure 10. Because $y = 2^x$ defines a one-to-one function, it has an inverse. Interchanging x and y gives

$$x = 2^y, \quad \text{the inverse of} \quad y = 2^x.$$

As we saw in **Section 11.1,** the graph of the inverse is found by reflecting the graph of $y = 2^x$ about the line $y = x$. The graph of $x = 2^y$ is shown as a red curve in Figure 10.

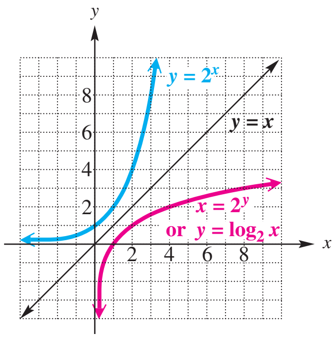


Figure 10

OBJECTIVE 1 Define a logarithm. We cannot solve the equation $x = 2^y$ for the dependent variable y with the methods presented up to now. The following definition is used to solve $x = 2^y$ for y.

Logarithm

For all positive numbers a, with $a \neq 1$, and all positive numbers x,

$$y = \log_a x \quad \text{means the same as} \quad x = a^y.$$

This key statement should be memorized. The abbreviation **log** is used for **logarithm.** Read $\log_a x$ as "the logarithm of x to the base a." To remember the location of the base and the exponent in each form, refer to the following diagrams.

Exponent ↓	Exponent ↓
Logarithmic form: $y = \log_a x$	Exponential form: $x = a^y$
↑ Base	↑ Base

In working with logarithmic form and exponential form, remember the following.

Audio

Meaning of $\log_a x$

A logarithm is an exponent. The expression $\log_a x$ represents the exponent to which the base a must be raised to obtain x.

1 Complete the table.

Exponential Form	Logarithmic Form
$2^5 = 32$	
$100^{1/2} = 10$	
	$\log_8 4 = \frac{2}{3}$
	$\log_6 \frac{1}{1296} = -4$

OBJECTIVE 2 Convert between exponential and logarithmic forms. We can use the definition of logarithm to write exponential statements in logarithmic form and logarithmic statements in exponential form. The following table shows several pairs of equivalent statements.

Exponential Form	Logarithmic Form
$3^2 = 9$	$\log_3 9 = 2$
$\left(\frac{1}{5}\right)^{-2} = 25$	$\log_{1/5} 25 = -2$
$10^5 = 100{,}000$	$\log_{10} 100{,}000 = 5$
$4^{-3} = \frac{1}{64}$	$\log_4 \frac{1}{64} = -3$

Work Problem 1 at the Side.

OBJECTIVE 3 Solve logarithmic equations of the form $\log_a b = k$ for a, b, or k. A **logarithmic equation** is an equation with a logarithm in at least one term. We solve logarithmic equations of the form $\log_a b = k$ for any of the three variables by first writing the equation in exponential form.

Video Animation You Try It

EXAMPLE 1 Solving Logarithmic Equations

Solve each equation.

(a) $\log_4 x = -2$

By the definition of logarithm, $\log_4 x = -2$ is equivalent to $x = 4^{-2}$. Solve this exponential equation.

$$x = 4^{-2} = \frac{1}{16}$$

The solution set is $\left\{\frac{1}{16}\right\}$.

(b) $\log_{1/2} (3x + 1) = 2$

$$3x + 1 = \left(\frac{1}{2}\right)^2 \quad \text{Write in exponential form.}$$

$$3x + 1 = \frac{1}{4}$$

$$12x + 4 = 1 \quad \text{Multiply by 4.}$$

$$12x = -3 \quad \text{Subtract 4.}$$

$$x = -\frac{1}{4} \quad \text{Divide by 12; lowest terms}$$

The solution set is $\left\{-\frac{1}{4}\right\}$.

(c) $\log_x 3 = 2$

$$x^2 = 3 \quad \text{Write in exponential form.}$$

$$x = \sqrt{3} \quad \text{Take the } \textit{principal} \text{ square root.}$$

Notice that only the principal square root satisfies the equation, since the base must be a positive number. The solution set is $\{\sqrt{3}\}$.

Continued on Next Page

ANSWERS

1. $\log_2 32 = 5$; $\log_{100} 10 = \frac{1}{2}$; $8^{2/3} = 4$; $6^{-4} = \frac{1}{1296}$

(d) $\log_{49}\sqrt[3]{7} = x$

$$49^x = \sqrt[3]{7} \quad \text{Write in exponential form.}$$
$$(7^2)^x = 7^{1/3} \quad \text{Write with the same base.}$$
$$7^{2x} = 7^{1/3} \quad \text{Power rule for exponents.}$$
$$2x = \frac{1}{3} \quad \text{Set exponents equal.}$$
$$x = \frac{1}{6} \quad \text{Divide by 2.}$$

The solution set is $\{\frac{1}{6}\}$.

Work Problem 2 at the Side.

For any real number b, we know that $b^1 = b$ and $b^0 = 1$. Writing these two statements in logarithmic form gives the following two properties of logarithms.

Audio

Properties of Logarithms

For any positive real number b, with $b \neq 1$,

$$\boldsymbol{\log_b b = 1} \quad \text{and} \quad \boldsymbol{\log_b 1 = 0.}$$

Video

EXAMPLE 2 Using Properties of Logarithms

Use the preceding two properties of logarithms to evaluate each logarithm.

(a) $\log_7 7 = 1$ **(b)** $\log_{\sqrt{2}}\sqrt{2} = 1$

(c) $\log_9 1 = 0$ **(d)** $\log_{.2} 1 = 0$

Work Problem 3 at the Side.

OBJECTIVE 4 Define and graph logarithmic functions. Now we define the logarithmic function with base a.

Audio

Logarithmic Function

If a and x are positive numbers, with $a \neq 1$, then

$$\boldsymbol{G(x) = \log_a x}$$

defines the **logarithmic function with base *a*.**

To graph a logarithmic function, it is helpful to write it in exponential form first. Then plot selected ordered pairs to determine the graph.

You Try It

Video

EXAMPLE 3 Graphing a Logarithmic Function

Graph $y = \log_{1/2} x$.

By writing $y = \log_{1/2} x$ in exponential form as $x = (\frac{1}{2})^y$, we can identify ordered pairs that satisfy the equation. Here it is easier to choose values for y and find the corresponding values of x. See the table of ordered pairs on the next page.

Continued on Next Page

2 Solve each equation.

(a) $\log_3 27 = x$

(b) $\log_5 p = 2$

(c) $\log_m \frac{1}{16} = -4$

(d) $\log_x 12 = 3$

3 Evaluate each logarithm.

(a) $\log_{2/5} \frac{2}{5}$

(b) $\log_\pi \pi$

(c) $\log_{.4} 1$

(d) $\log_6 1$

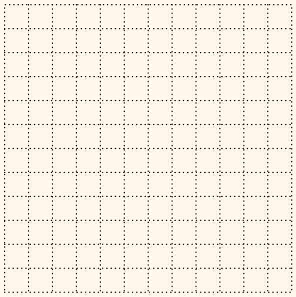
Answers

2. (a) $\{3\}$ **(b)** $\{25\}$ **(c)** $\{2\}$ **(d)** $\{\sqrt[3]{12}\}$

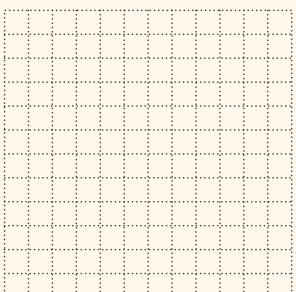
3. (a) 1 **(b)** 1 **(c)** 0 **(d)** 0

4 Graph.

(a) $y = \log_3 x$



(b) $y = \log_{1/10} x$



x	y
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2

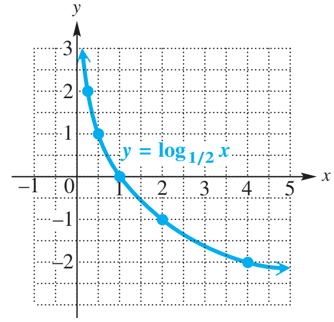


Figure 11

Plotting these points (be careful to get the values of x and y in the right order) and connecting them with a smooth curve gives the graph in Figure 11. This graph is typical of logarithmic functions with $0 < a < 1$. The graph of $x = 2^y$ in Figure 10, which is equivalent to $y = \log_2 x$, is typical of graphs of logarithmic functions with base $a > 1$.

Work Problem 4 at the Side.

Based on the graphs of the functions defined by $y = \log_2 x$ in Figure 10 and $y = \log_{1/2} x$ in Figure 11, we make the following generalizations about the graphs of logarithmic functions of the form $G(x) = \log_a x$.

Characteristics of the Graph of $G(x) = \log_a x$

1. The graph contains the point $(1, 0)$.
2. When $a > 1$, the graph *rises* from left to right, from the fourth quadrant to the first. When $0 < a < 1$, the graph *falls* from left to right, from the first quadrant to the fourth.
3. The graph approaches the y-axis, but never touches it. (The y-axis is an asymptote.)
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

Compare these characteristics to the analogous ones for exponential functions in **Section 11.2.**

OBJECTIVE 5 Use logarithmic functions in applications of growth or decay. Logarithmic functions, like exponential functions, can be applied to growth or decay of real-world phenomena.

EXAMPLE 4 Solving an Application of a Logarithmic Function

The function defined by

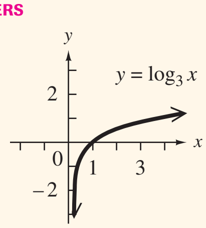
$$f(x) = 27 + 1.105 \log_{10}(x + 1)$$

approximates the barometric pressure in inches of mercury at a distance of x miles from the eye of a typical hurricane. (*Source:* Miller, A. and R. Anthes, *Meteorology,* Fifth Edition, Charles E. Merrill Publishing Company, 1985.)

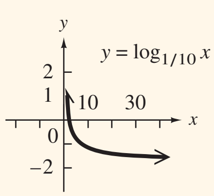
Continued on Next Page

ANSWERS

4. (a)



(b)



You Try It

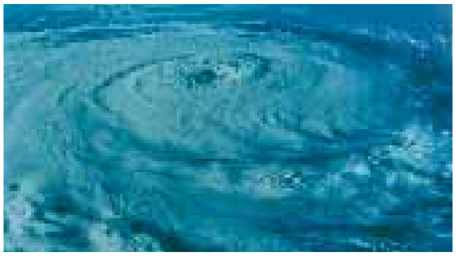
Animation

Video

(a) Approximate the pressure 9 mi from the eye of the hurricane.
Let $x = 9$, and find $f(9)$.

$$\begin{aligned} f(9) &= 27 + 1.105 \log_{10}(9 + 1) && \text{Let } x = 9. \\ &= 27 + 1.105 \log_{10} 10 && \text{Add inside parentheses.} \\ &= 27 + 1.105(1) && \log_{10} 10 = 1 \\ &= 28.105 && \text{Add.} \end{aligned}$$

The pressure 9 mi from the eye of the hurricane is 28.105 in.



(b) Approximate the pressure 99 mi from the eye of the hurricane.

$$\begin{aligned} f(99) &= 27 + 1.105 \log_{10}(99 + 1) && \text{Let } x = 99. \\ &= 27 + 1.105 \log_{10} 100 && \text{Add inside parentheses.} \\ &= 27 + 1.105(2) && \log_{10} 100 = 2 \\ &= 29.21 \end{aligned}$$

The pressure 99 mi from the eye of the hurricane is 29.21 in.

Work Problem 5 at the Side.

5 Solve the problem.

A population of mites in a laboratory is growing according to the function defined by

$$P(t) = 80 \log_{10}(t + 10),$$

where t is the number of days after a study is begun.

(a) Find the number of mites at the beginning of the study.

(b) Find the number present after 90 days.

(c) Find the number present after 990 days.

ANSWERS

5. (a) 80 **(b)** 160 **(c)** 240

11.4 Properties of Logarithms

Logarithms have been used as an aid to numerical calculation for several hundred years. Today the widespread use of calculators has made the use of logarithms for calculation obsolete. However, logarithms are still very important in applications and in further work in mathematics.

OBJECTIVES

1. **Use the product rule for logarithms.**
2. **Use the quotient rule for logarithms.**
3. **Use the power rule for logarithms.**
4. **Use properties to write alternative forms of logarithmic expressions.**

OBJECTIVE 1 Use the product rule for logarithms. One way in which logarithms simplify problems is by changing a problem of multiplication into one of addition. We know that $\log_2 4 = 2$, $\log_2 8 = 3$, and $\log_2 32 = 5$. Since $2 + 3 = 5$,

$$\log_2 32 = \log_2 4 + \log_2 8$$
$$\log_2(4 \cdot 8) = \log_2 4 + \log_2 8.$$

This is true in general.

Product Rule for Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, then

$$\log_b xy = \log_b x + \log_b y.$$

In words, the logarithm of a product is the sum of the logarithms of the factors.

NOTE

The word statement of the product rule can be restated by replacing "logarithm" with "exponent." The rule then becomes the familiar rule for multiplying exponential expressions: The *exponent* of a product is equal to the sum of the *exponents* of the factors.

To prove this rule, let $m = \log_b x$ and $n = \log_b y$, and recall that

$$\log_b x = m \quad \text{means} \quad b^m = x.$$
$$\log_b y = n \quad \text{means} \quad b^n = y.$$

Now consider the product xy.

$xy = b^m \cdot b^n$	Substitute.
$xy = b^{m+n}$	Product rule for exponents
$\log_b xy = m + n$	Convert to logarithmic form.
$\log_b xy = \log_b x + \log_b y$	Substitute.

The last statement is the result we wished to prove.

EXAMPLE 1 Using the Product Rule

Use the product rule to rewrite each expression. Assume $x > 0$.

(a) $\log_5(6 \cdot 9)$

By the product rule,

$$\log_5(6 \cdot 9) = \log_5 6 + \log_5 9.$$

(b) $\log_7 8 + \log_7 12 = \log_7(8 \cdot 12) = \log_7 96$

Continued on Next Page

(c) $\log_3 (3x) = \log_3 3 + \log_3 x$

$= 1 + \log_3 x \qquad \log_3 3 = 1$

(d) $\log_4 x^3 = \log_4 (x \cdot x \cdot x) \qquad x^3 = x \cdot x \cdot x$

$= \log_4 x + \log_4 x + \log_4 x \qquad$ Product rule

$= 3 \log_4 x$

Work Problem 1 at the Side.

OBJECTIVE 2 **Use the quotient rule for logarithms.** The rule for division is similar to the rule for multiplication.

Quotient Rule for Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, then

$$\log_b \frac{x}{y} = \log_b x - \log_b y.$$

In words, the logarithm of a quotient is the difference between the logarithm of the numerator and the logarithm of the denominator.

The proof of this rule is very similar to the proof of the product rule.

EXAMPLE 2 Using the Quotient Rule

Use the quotient rule to rewrite each logarithm.

(a) $\log_4 \frac{7}{9} = \log_4 7 - \log_4 9$

(b) $\log_5 6 - \log_5 x = \log_5 \frac{6}{x}, \quad x > 0$

(c) $\log_3 \frac{27}{5} = \log_3 27 - \log_3 5$

$= 3 - \log_3 5 \qquad \log_3 27 = 3$

CAUTION

There is no property of logarithms to rewrite the logarithm of a sum or difference. For example, we ***cannot*** write $\log_b (x + y)$ in terms of $\log_b x$ and $\log_b y$. Also,

$$\log_b \frac{x}{y} \neq \frac{\log_b x}{\log_b y}.$$

Work Problem 2 at the Side.

1 Use the product rule to rewrite each expression.

(a) $\log_6 (5 \cdot 8)$

(b) $\log_4 3 + \log_4 7$

(c) $\log_8 8k, \quad k > 0$

(d) $\log_5 m^2, \quad m > 0$

2 Use the quotient rule to rewrite each expression.

(a) $\log_7 \frac{9}{4}$

(b) $\log_3 p - \log_3 q,$
$p > 0, \quad q > 0$

(c) $\log_4 \frac{3}{16}$

ANSWERS

1. **(a)** $\log_6 5 + \log_6 8$ **(b)** $\log_4 21$
(c) $1 + \log_8 k$ **(d)** $2 \log_5 m$

2. **(a)** $\log_7 9 - \log_7 4$ **(b)** $\log_3 \frac{p}{q}$
(c) $\log_4 3 - 2$

OBJECTIVE 3 Use the power rule for logarithms. The next rule gives a method for evaluating powers and roots such as

$$2^{\sqrt{2}}, \quad (\sqrt{2})^{3/4}, \quad (.032)^{5/8}, \quad \text{and} \quad \sqrt[5]{12}.$$

This rule makes it possible to find approximations for numbers that could not be evaluated before. By the product rule for logarithms,

$$\begin{aligned} \log_5 2^3 &= \log_5(2 \cdot 2 \cdot 2) \\ &= \log_5 2 + \log_5 2 + \log_5 2 \\ &= 3 \log_5 2. \end{aligned}$$

Also,

$$\begin{aligned} \log_2 7^4 &= \log_2(7 \cdot 7 \cdot 7 \cdot 7) \\ &= \log_2 7 + \log_2 7 + \log_2 7 + \log_2 7 \\ &= 4 \log_2 7. \end{aligned}$$

Furthermore, we saw in Example 1(d) that $\log_4 x^3 = 3 \log_4 x$. These examples suggest the following rule.

Power Rule for Logarithms

If x and b are positive real numbers, where $b \neq 1$, and if r is any real number, then

$$\boldsymbol{\log_b x^r = r \log_b x.}$$

In words, the logarithm of a number to a power equals the exponent times the logarithm of the number.

As examples of this result,

$$\log_b m^5 = 5 \log_b m \quad \text{and} \quad \log_3 5^4 = 4 \log_3 5.$$

To prove the power rule, let

$$\begin{aligned} \log_b x &= m. & & \\ b^m &= x & & \text{Convert to exponential form.} \\ (b^m)^r &= x^r & & \text{Raise to the power } r. \\ b^{mr} &= x^r & & \text{Power rule for exponents} \\ \log_b x^r &= mr & & \text{Convert to logarithmic form.} \\ \log_b x^r &= rm & & \text{Commutative property} \\ \log_b x^r &= r \log_b x & & m = \log_b x \end{aligned}$$

This is the statement to be proved.

As a special case of the power rule, let $r = \frac{1}{p}$, so

$$\log_b \sqrt[p]{x} = \log_b x^{1/p} = \frac{1}{p} \log_b x.$$

For example, using this result, with $x > 0$,

$$\log_b \sqrt[5]{x} = \log_b x^{1/5} = \frac{1}{5} \log_b x \quad \text{and} \quad \log_b \sqrt[3]{x^4} = \log_b x^{4/3} = \frac{4}{3} \log_b x.$$

Another special case is

$$\log_b \frac{1}{x} = \log_b x^{-1} = -\log_b x.$$

3 Use the power rule to rewrite each logarithm. Assume $a > 0, b > 0, x > 0, a \neq 1$, and $b \neq 1$.

(a) $\log_3 5^2$

(b) $\log_a x^4$

(c) $\log_b \sqrt{8}$

(d) $\log_2 \sqrt[3]{2}$

4 Find the value of each logarithmic expression.

(a) $\log_{10} 10^3$

(b) $\log_2 8$

(c) $5^{\log_5 3}$

Answers

3. **(a)** $2 \log_3 5$ **(b)** $4 \log_a x$ **(c)** $\frac{1}{2}\log_b 8$ **(d)** $\frac{1}{3}$

4. **(a)** 3 **(b)** 3 **(c)** 3

NOTE
For a review of rational exponents, refer to **Section 9.2.**

EXAMPLE 3 Using the Power Rule

Use the power rule to rewrite each logarithm. Assume $b > 0, x > 0$, and $b \neq 1$.

(a) $\log_5 4^2 = 2 \log_5 4$

(b) $\log_b x^5 = 5 \log_b x$

(c) $\log_b \sqrt{7}$

When using the power rule with logarithms of expressions involving radicals, begin by rewriting the radical expression with a rational exponent.

$$\log_b \sqrt{7} = \log_b 7^{1/2} \qquad \sqrt{x} = x^{1/2}$$
$$= \frac{1}{2} \log_b 7 \qquad \text{Power rule}$$

(d) $\log_2 \sqrt[5]{x^2} = \log_2 x^{2/5} \qquad \sqrt[5]{x^2} = x^{2/5}$

$$= \frac{2}{5} \log_2 x \qquad \text{Power rule}$$

Work Problem 3 at the Side.

Two special properties involving both exponential and logarithmic expressions come directly from the fact that logarithmic and exponential functions are inverses of each other.

Special Properties
If $b > 0$ and $b \neq 1$, then

$$b^{\log_b x} = x, \; x > 0 \quad \text{and} \quad \log_b b^x = x.$$

To prove the first statement, let

$$y = \log_b x.$$
$$b^y = x \qquad \text{Convert to exponential form.}$$
$$b^{\log_b x} = x \qquad \text{Replace } y \text{ with } \log_b x.$$

The proof of the second statement is similar.

EXAMPLE 4 Using the Special Properties

Find the value of each logarithmic expression.

(a) $\log_5 5^4$
Since $\log_b b^x = x$,

$$\log_5 5^4 = 4.$$

(b) $\log_3 9 = \log_3 3^2 = 2$

(c) $4^{\log_4 10} = 10$

Work Problem 4 at the Side.

Here is a summary of the properties of logarithms.

Audio

Properties of Logarithms

If x, y, and b are positive real numbers, where $b \neq 1$, and r is any real number, then

Product Rule	$\log_b xy = \log_b x + \log_b y$
Quotient Rule	$\log_b \dfrac{x}{y} = \log_b x - \log_b y$
Power Rule	$\log_b x^r = r \log_b x$
Special Properties	$b^{\log_b x} = x$ and $\log_b b^x = x.$

OBJECTIVE 4 Use properties to write alternative forms of logarithmic expressions. Applying the properties of logarithms is important for solving equations with logarithms and in calculus.

Video

EXAMPLE 5 Writing Logarithms in Alternative Forms

Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers.

(a) $\log_4 4x^3 = \log_4 4 + \log_4 x^3$ Product rule

$= 1 + 3 \log_4 x$ $\log_4 4 = 1$; power rule

(b) $\log_7 \sqrt{\dfrac{m}{n}} = \log_7 \left(\dfrac{m}{n}\right)^{1/2}$

$= \dfrac{1}{2} \log_7 \dfrac{m}{n}$ Power rule

$= \dfrac{1}{2}(\log_7 m - \log_7 n)$ Quotient rule

(c) $\log_5 \dfrac{a^2}{bc} = \log_5 a^2 - \log_5 bc$ Quotient rule

$= 2 \log_5 a - \log_5 bc$ Power rule

$= 2 \log_5 a - (\log_5 b + \log_5 c)$ Product rule

$= 2 \log_5 a - \log_5 b - \log_5 c$ Distributive property

Notice the careful use of parentheses in the third step. Since we are subtracting the logarithm of a product and rewriting it as a sum of two terms, we must place parentheses around the sum.

(d) $4 \log_b m - \log_b n = \log_b m^4 - \log_b n$ Power rule

$= \log_b \dfrac{m^4}{n}$ Quotient rule

Continued on Next Page

5 Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers.

(a) $\log_6 36m^5$

(b) $\log_2 \sqrt{9z}$

(c) $\log_q \dfrac{8r^2}{m-1}, m > 1, q \neq 1$

(d) $2 \log_a x + 3 \log_a y, a \neq 1$

(e) $\log_4(3x + y)$

Answers

5. (a) $2 + 5 \log_6 m$ **(b)** $\log_2 3 + \frac{1}{2}\log_2 z$
(c) $\log_q 8 + 2 \log_q r - \log_q(m-1)$
(d) $\log_a x^2y^3$ **(e)** cannot be rewritten

(e) $\log_b(x+1) + \log_b(2x-1) - \dfrac{2}{3}\log_b x$

$$= \log_b(x+1) + \log_b(2x-1) - \log_b x^{2/3} \quad \text{Power rule}$$

$$= \log_b \frac{(x+1)(2x-1)}{x^{2/3}} \quad \text{Product and quotient rules}$$

$$= \log_b \frac{2x^2 + x - 1}{x^{2/3}}$$

(f) $\log_8 (2p + 3r)$ cannot be rewritten using the properties of logarithms. There is no property of logarithms to rewrite the logarithm of a sum.

Work Problem 5 at the Side.

11.5 Common and Natural Logarithms

As mentioned earlier, logarithms are important in many applications of mathematics to everyday problems, particularly in biology, engineering, economics, and social science. In this section we find numerical approximations for logarithms. Traditionally, base 10 logarithms were used most often because our number system is base 10. Logarithms to base 10 are called **common logarithms,** and $\log_{10} x$ is abbreviated as simply **log *x*,** where the base is understood to be 10.

OBJECTIVES

1. Evaluate common logarithms using a calculator.
2. Use common logarithms in applications.
3. Evaluate natural logarithms using a calculator.
4. Use natural logarithms in applications.

OBJECTIVE 1 Evaluate common logarithms using a calculator. We use calculators to evaluate common logarithms. In the next example we give the results of evaluating some common logarithms using a calculator with a LOG key. (This may be a second function key on some calculators.) For simple scientific calculators, just enter the number, then press the LOG key. For graphing calculators, these steps are reversed. We will give all approximations for logarithms to four decimal places.

You Try It

Video

EXAMPLE 1 Evaluating Common Logarithms

Evaluate each logarithm using a calculator.

(a) $\log 327.1 \approx 2.5147$ **(b)** $\log 437{,}000 \approx 5.6405$

(c) $\log .0615 \approx -1.2111$

Notice that $\log .0615 \approx -1.2111$, a negative result. ***The common logarithm of a number between 0 and 1 is always negative*** because the logarithm is the exponent on 10 that produces the number. For example,

$$10^{-1.2111} \approx .0615.$$

If the exponent (the logarithm) were positive, the result would be greater than 1 because $10^0 = 1$. See Figure 12.

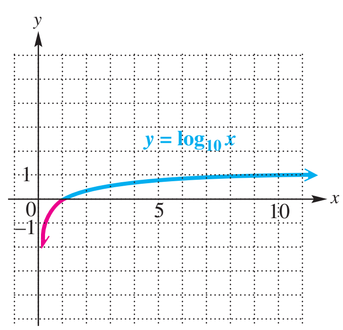


Figure 12

Work Problem 1 at the Side.

1 Evaluate each logarithm to four decimal places using a calculator.

(a) $\log 41{,}600$

(b) $\log 43.5$

(c) $\log .442$

OBJECTIVE 2 Use common logarithms in applications. In chemistry, pH is a measure of the acidity or alkalinity of a solution; water, for example, has pH 7. In general, acids have pH numbers less than 7, and alkaline solutions have pH values greater than 7. The **pH** of a solution is defined as

Audio

$$\text{pH} = -\log [H_3O^+],$$

where $[H_3O^+]$ is the hydronium ion concentration in moles per liter. It is customary to round pH values to the nearest tenth.

ANSWERS

1. (a) 4.6191 **(b)** 1.6385 **(c)** −.3546

2 Solve the problem.

Find the pH of water with a hydronium ion concentration of 1.2×10^{-3}. If this water had been taken from a wetland, is the wetland a rich fen, a poor fen, or a bog?

3 Find the hydronium ion concentrations of solutions with the following pH values.

(a) 4.6

(b) 7.5

Answers

2. 2.9; bog

3. **(a)** 2.5×10^{-5} **(b)** 3.2×10^{-8}

EXAMPLE 2 Using pH in an Application

Wetlands are classified as *bogs, fens, marshes,* and *swamps*. These classifications are based on pH values. A pH value between 6.0 and 7.5, such as that of Summerby Swamp in Michigan's Hiawatha National Forest, indicates that the wetland is a "rich fen." When the pH is between 4.0 and 6.0, the wetland is a "poor fen," and if the pH falls to 3.0 or less, it is a "bog." (*Source:* Mohlenbrock, R., "Summerby Swamp, Michigan," *Natural History,* March 1994.)

Suppose that the hydronium ion concentration of a sample of water from a wetland is 6.3×10^{-3}. How would this wetland be classified?

Use the definition of pH.

$$\begin{aligned} \text{pH} &= -\log(6.3 \times 10^{-3}) \\ &= -(\log 6.3 + \log 10^{-3}) && \text{Product rule} \\ &= -[.7993 - 3(1)] && \text{Use a calculator.} \\ &= -.7993 + 3 \\ &\approx 2.2 \end{aligned}$$

Since the pH is less than 3.0, the wetland is a bog.

Work Problem 2 at the Side.

EXAMPLE 3 Finding Hydronium Ion Concentration

Find the hydronium ion concentration of drinking water with pH 6.5.

$$\begin{aligned} \textbf{pH} &= -\log[H_3O^+] \\ \textbf{6.5} &= -\log[H_3O^+] && \text{Let pH} = 6.5. \\ \log[H_3O^+] &= -6.5 && \text{Multiply by } -1. \end{aligned}$$

Solve for $[H_3O^+]$ by writing the equation in exponential form, remembering that the base is 10.

$$\begin{aligned} [H_3O^+] &= 10^{-6.5} \\ [H_3O^+] &\approx 3.2 \times 10^{-7} && \text{Use a calculator.} \end{aligned}$$

Work Problem 3 at the Side.

OBJECTIVE 3 Evaluate natural logarithms using a calculator. The most important logarithms used in applications are **natural logarithms,** which have as base the number e. The number e is a fundamental number in our universe. For this reason e, like π, is called a *universal constant*. The letter e is used to honor Leonhard Euler, who published extensive results on the number in 1748. Since it is an irrational number, its decimal expansion never terminates and never repeats.

The first few digits of the decimal value of e are 2.718281828. A calculator key e^x or the two keys INV and ln x are used to approximate powers of e. For example, a calculator gives

$$e^2 \approx 7.389056099, \quad e^3 \approx 20.08553692, \quad \text{and} \quad e^{.6} \approx 1.8221188.$$

Logarithms to base e are called natural logarithms because they occur in biology and the social sciences in natural situations that involve growth or decay. The base e logarithm of x is written **ln *x*** (read "el en x"). A graph of $y = \ln x$, the equation that defines the natural logarithmic function, is given in Figure 13.

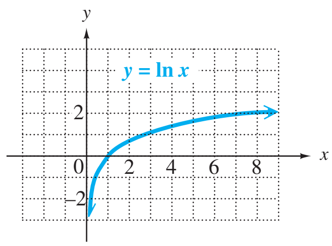


Figure 13

A calculator key labeled ln x is used to evaluate natural logarithms. If your calculator has an e^x key, but not a key labeled ln x, find natural logarithms by entering the number, pressing the INV key, and then pressing the e^x key. This works because $y = e^x$ defines the inverse function of $y = \ln x$ (or $y = \log_e x$).

EXAMPLE 4 Finding Natural Logarithms

Find each logarithm to four decimal places.

(a) $\ln .5841 \approx -.5377$

As with common logarithms, ***a number between 0 and 1 has a negative natural logarithm.***

(b) $\ln 192.7 \approx 5.2611$ **(c)** $\ln 10.84 \approx 2.3832$

Work Problem 4 at the Side.

OBJECTIVE 4 Use natural logarithms in applications. A common application of natural logarithmic functions is to express growth or decay of a quantity, as in the next example.

EXAMPLE 5 Applying Natural Logarithms

The altitude in meters that corresponds to an atmospheric pressure of x millibars is given by the logarithmic function defined by

$$f(x) = 51{,}600 - 7457 \ln x.$$

(*Source:* Miller, A. and J. Thompson, *Elements of Meteorology,* Fourth Edition, Charles E. Merrill Publishing Company, 1993.) Use this function to find the altitude when atmospheric pressure is 400 millibars.

Let $x = 400$ and substitute in the expression for $f(x)$.

$$\begin{aligned} f(400) &= 51{,}600 - 7457 \ln 400 \\ &\approx 6900 \quad \text{(to the nearest hundred)} \end{aligned}$$

Atmospheric pressure is 400 millibars at approximately 6900 m.

4 Find each logarithm to four decimal places.

(a) $\ln .01$

(b) $\ln 27$

(c) $\ln 529$

ANSWERS

4. (a) -4.6052 **(b)** 3.2958 **(c)** 6.2710

5 Use the logarithmic function in Example 5 to approximate the altitude at 700 millibars of pressure.

Calculator Tip In Example 5, the final answer was obtained using a calculator *without* rounding the intermediate values. In general, it is best to wait until the final step to round the answer; otherwise, a build-up of round-off error may cause the final answer to have an incorrect final decimal place digit.

Work Problem 5 at the Side.

ANSWERS

5. approximately 2700 m

11.6 Exponential and Logarithmic Equations; Further Applications

As mentioned earlier, exponential and logarithmic functions are important in many applications of mathematics. Using these functions in applications requires solving exponential and logarithmic equations. Some simple equations were solved in **Sections 11.2** and **11.3.** More general methods for solving these equations depend on the following properties.

OBJECTIVES

1. Solve equations involving variables in the exponents.
2. Solve equations involving logarithms.
3. Solve applications of compound interest.
4. Solve applications involving base e exponential growth and decay.
5. Use the change-of-base rule.

Audio

Properties for Solving Exponential and Logarithmic Equations

For all real numbers $b > 0$, $b \neq 1$, and any real numbers x and y:

1. If $x = y$, then $b^x = b^y$.
2. If $b^x = b^y$, then $x = y$.
3. If $x = y$, and $x > 0$, $y > 0$, then $\log_b x = \log_b y$.
4. If $x > 0$, $y > 0$, and $\log_b x = \log_b y$, then $x = y$.

We used Property 2 to solve exponential equations in **Section 11.2.**

OBJECTIVE 1 Solve equations involving variables in the exponents. The first examples illustrate a general method for solving exponential equations using Property 3.

Video

EXAMPLE 1 Solving an Exponential Equation

Solve $3^x = 12$.

$$\begin{aligned} 3^x &= 12 \\ \log 3^x &= \log 12 && \text{Property 3} \\ x \log 3 &= \log 12 && \text{Power rule} \\ x &= \frac{\log 12}{\log 3} && \text{Divide by } \log 3. \end{aligned}$$

This quotient is the exact solution. To find a decimal approximation for the solution, use a calculator.

$$x \approx 2.262$$

The solution set is $\{2.262\}$. Check that $3^{2.262} \approx 12$.

CAUTION

Be careful: $\dfrac{\log 12}{\log 3}$ is *not* equal to log 4. Note that $\log 4 \approx .6021$, but $\dfrac{\log 12}{\log 3} \approx 2.262$.

Work Problem 1 at the Side.

1 Solve each equation and give the decimal approximation to three places.

(a) $2^x = 9$

(b) $10^x = 4$

ANSWERS

1. **(a)** $\{3.170\}$ **(b)** $\{.602\}$

2 Solve $e^{-.01t} = .38$.

When an exponential equation has e as the base, it is appropriate to use base e logarithms.

Video

EXAMPLE 2 **Solving an Exponential Equation with Base e**

Solve $e^{.003x} = 40$.

Take base e logarithms on both sides.

$$\ln e^{.003x} = \ln 40$$

$$.003x \ln e = \ln 40 \qquad \text{Power rule}$$

$$.003x = \ln 40 \qquad \ln e = \ln e^1 = 1$$

$$x = \frac{\ln 40}{.003} \qquad \text{Divide by .003.}$$

$$x \approx 1230 \qquad \text{Use a calculator.}$$

The solution set is $\{1230\}$. Check that $e^{.003(1230)} \approx 40$.

Work Problem 2 at the Side.

Audio

General Method for Solving an Exponential Equation

Take logarithms to the same base on both sides and then use the power rule of logarithms or the special property $\log_b b^x = x$. (See Examples 1 and 2.)

As a special case, if both sides can be written as exponentials with the same base, do so, and set the exponents equal. (See **Section 11.2.**)

3 Solve $\log_3(x + 1)^5 = 3$. Give the exact solution.

OBJECTIVE 2 **Solve equations involving logarithms.** The properties of logarithms from **Section 11.4** are useful here, as is using the definition of a logarithm to change the equation to exponential form.

Video You Try It

EXAMPLE 3 **Solving a Logarithmic Equation**

Solve $\log_2(x + 5)^3 = 4$. Give the exact solution.

$$(x + 5)^3 = 2^4 \qquad \text{Convert to exponential form.}$$

$$(x + 5)^3 = 16$$

$$x + 5 = \sqrt[3]{16} \qquad \text{Take the cube root on each side.}$$

$$x = -5 + \sqrt[3]{16} \qquad \text{Add } -5.$$

$$x = -5 + 2\sqrt[3]{2} \qquad \text{Simplify the radical.}$$

Verify that the solution satisfies the equation, so the solution set is $\{-5 + 2\sqrt[3]{2}\}$.

CAUTION

Recall that the domain of $y = \log_b x$ is $(0, \infty)$. For this reason, ***it is always necessary to check that the solution of an equation with logarithms yields only logarithms of positive numbers in the original equation.***

Work Problem 3 at the Side.

ANSWERS

2. $\{96.8\}$
3. $\{-1 + \sqrt[5]{27}\}$

Video

EXAMPLE 4 Solving a Logarithmic Equation

Solve $\log_2(x+1) - \log_2 x = \log_2 7$.

$$\log_2(x+1) - \log_2 x = \log_2 7$$

$$\log_2 \frac{x+1}{x} = \log_2 7 \qquad \text{Quotient rule}$$

$$\frac{x+1}{x} = 7 \qquad \text{Property 4}$$

$$x + 1 = 7x \qquad \text{Multiply by } x.$$

$$\frac{1}{6} = x \qquad \text{Subtract } x\text{; divide by 6.}$$

Check this solution by substituting in the original equation. Here, both $x+1$ and x must be positive. If $x = \frac{1}{6}$, this condition is satisfied, so the solution set is $\left\{\frac{1}{6}\right\}$.

Work Problem 4 at the Side.

You Try It

Animation

EXAMPLE 5 Solving a Logarithmic Equation

Solve $\log x + \log(x - 21) = 2$.

For this equation, write the left side as a single logarithm. Then write in exponential form and solve the equation.

$$\log x + \log(x-21) = 2$$

$$\log x(x-21) = 2 \qquad \text{Product rule}$$

$$x(x-21) = 10^2 \qquad \log x = \log_{10} x\text{; Write in exponential form.}$$

$$x^2 - 21x = 100 \qquad \text{Distributive property; multiply.}$$

$$x^2 - 21x - 100 = 0 \qquad \text{Standard form}$$

$$(x-25)(x+4) = 0 \qquad \text{Factor.}$$

$$x - 25 = 0 \quad \text{or} \quad x + 4 = 0 \qquad \text{Zero-factor property}$$

$$x = 25 \quad \text{or} \quad x = -4 \qquad \text{Solve each equation.}$$

The value -4 must be rejected as a solution since it leads to the logarithm of at least one negative number in the original equation.

$$\log(-4) + \log(-4 - 21) = 2 \qquad \text{The left side is undefined.}$$

The only solution, therefore, is 25, and the solution set is $\{25\}$.

CAUTION
Do not reject a potential solution just because it is nonpositive. Reject any value that leads to the logarithm of a nonpositive number.

Work Problem 5 at the Side.

4 Solve
$\log_8(2x+5) + \log_8 3 = \log_8 33$.

5 Solve
$\log_3 2x - \log_3(3x+15) = -2$.

ANSWERS
4. $\{3\}$
5. $\{1\}$

6 Find the value of \$2000 deposited at 5% compounded annually for 10 yr.

Audio

In summary, we use the following steps to solve a logarithmic equation.

Solving a Logarithmic Equation

Step 1 **Transform the equation so that a single logarithm appears on one side.** Use the product rule or quotient rule of logarithms to do this.

Step 2 **(a)** **Use Property 4.** If $\log_b x = \log_b y$, then $x = y$. (See Example 4.)

(b) **Write the equation in exponential form.** If $\log_b x = k$, then $x = b^k$. (See Examples 3 and 5.)

OBJECTIVE 3 **Solve applications of compound interest.** So far in this book, problems involving applications of interest have been limited to simple interest using the formula $I = prt$. In most cases, interest paid or charged is **compound interest** (interest paid on both principal and interest). The formula for compound interest is an important application of exponential functions.

Audio

Compound Interest Formula (for a Finite Number of Periods)

If a principal of P dollars is deposited at an annual rate of interest r compounded (paid) n times per year, the account will contain

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

dollars after t years. (In this formula, r is expressed as a decimal.)

Video

EXAMPLE 6 **Solving a Compound Interest Problem for *A***

How much money will there be in an account at the end of 5 yr if \$1000 is deposited at 6% compounded quarterly? (Assume no withdrawals are made.)

Because interest is compounded quarterly, $n = 4$. The other values given in the problem are $P = 1000$, $r = .06$ (because $6\% = .06$), and $t = 5$. Substitute into the compound interest formula to get the value of A.

$$A = 1000\left(1 + \frac{.06}{4}\right)^{4 \cdot 5}$$

$$A = 1000(1.015)^{20}$$

Now use the y^x key on a calculator, and round the answer to the nearest cent.

$$A = 1346.86$$

The account will contain \$1346.86. (The actual amount of interest earned is \$1346.86 − \$1000 = \$346.86. Why?)

Work Problem 6 at the Side.

EXAMPLE 7 **Solving a Compound Interest Problem for t**

Suppose inflation is averaging 3% per year. How many years will it take for prices to double?

We want to find the number of years t for \$1 to grow to \$2 at a rate of 3% per year. In the compound interest formula, we let $A = 2$, $P = 1$, $r = .03$, and $n = 1$.

Continued on Next Page

ANSWERS

6. about \$3257.79

$$2 = 1\left(1 + \frac{.03}{1}\right)^{1t} \quad \text{Substitute in the compound interest formula.}$$

$$2 = (1.03)^t \quad \text{Simplify.}$$

$$\log 2 = \log(1.03)^t \quad \text{Property 3}$$

$$\log 2 = t \log 1.03 \quad \text{Power rule}$$

$$t = \frac{\log 2}{\log 1.03} \quad \text{Divide by log 1.03.}$$

$$t \approx 23.45 \quad \text{Use a calculator.}$$

Prices will double in about 23 yr. (This is called the **doubling time** of the money.) To check, verify that $1.03^{23.45} \approx 2$.

Work Problem 7 at the Side.

Interest can be compounded annually, semiannually, quarterly, daily, and so on. The number of compounding periods can get larger and larger. If the value of n is allowed to approach infinity, we have an example of **continuous compounding.** However, the compound interest formula above cannot be used for continuous compounding since there is no finite value for n. The formula for continuous compounding is an example of exponential growth involving the number e.

Continuous Compound Interest Formula

If a principal of P dollars is deposited at an annual rate of interest r compounded continuously for t years, the final amount on deposit is

$$A = Pe^{rt}.$$

EXAMPLE 8 Solving a Continuous Interest Problem

(a) In Example 6, we found that \$1000 invested for 5 yr at 6% interest compounded quarterly would grow to \$1346.86. How much would this same investment grow to if compounded continuously?

$$A = Pe^{rt} \quad \text{Continuous compound interest formula}$$

$$A = 1000e^{(.06)5} \quad \text{Let } P = 1000,\ r = .06, \text{ and } t = 5.$$

$$A \approx 1349.86 \quad \text{Use a calculator; round to two decimal places.}$$

The account will grow to \$1349.86.

(b) How long would it take for the initial investment amount to double?

We must find the value of t that will cause A to be $2(\$1000) = \$2000$.

$$A = Pe^{rt}$$

$$2000 = 1000e^{.06t} \quad \text{Let } A = 2P = 2000.$$

$$2 = e^{.06t} \quad \text{Divide by 1000.}$$

$$\ln 2 = .06t \quad \text{Take natural logarithms; } \ln e^k = k.$$

$$t = \frac{\ln 2}{.06} \quad \text{Divide by .06.}$$

$$t \approx 11.55 \quad \text{Use a calculator.}$$

It would take about 11.55 yr for the original investment to double.

Work Problem 8 at the Side.

7 Find the number of years it will take for \$500 to increase to \$750 in an account paying 4% interest compounded semiannually.

8 **(a)** How much will \$2500 grow to at 4% interest compounded continuously for 3 yr?

(b) How long would it take for the initial investment in part (a) to double?

ANSWERS

7. about 10.24 yr

8. **(a)** \$2818.74 **(b)** about 17.33 yr

9 Radioactive strontium decays according to the function defined by

$$y = y_0 e^{-.0239t},$$

where t is time in years.

(a) If an initial sample contains $y_0 = 12$ g of radioactive strontium, how many grams will be present after 35 yr?

(b) What is the half-life of radioactive strontium?

OBJECTIVE 4 **Solve applications involving base *e* exponential growth and decay.** You may have heard of the carbon-14 dating process used to determine the age of fossils. The method used is based on a base e exponential decay function.

EXAMPLE 9 Solving an Exponential Decay Application

Carbon-14 is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radioactive carbon-14 disintegrates according to the function defined by

$$y = y_0 e^{-.000121t},$$

where t is time in years, y is the amount of the sample at time t, and y_0 is the initial amount present at $t = 0$.

(a) If an initial sample contains $y_0 = 10$ g of carbon-14, how many grams will be present after 3000 yr?
Let $y_0 = 10$ and $t = 3000$ in the formula, and use a calculator.

$$y = 10e^{-.000121(3000)} \approx 6.96 \text{ g}$$

(b) How long would it take for the initial sample to decay to half of its original amount? (This is called the **half-life.**)
Let $y = \frac{1}{2}(10) = 5$, and solve for t.

$$\begin{aligned} 5 &= 10e^{-.000121t} && \text{Substitute.} \\ \frac{1}{2} &= e^{-.000121t} && \text{Divide by 10.} \\ \ln\frac{1}{2} &= -.000121t && \text{Take natural logarithms; } \ln e^k = k. \\ t &= \frac{\ln\frac{1}{2}}{-.000121} && \text{Divide by } -.000121. \\ t &\approx 5728 && \text{Use a calculator.} \end{aligned}$$

The half-life is just over 5700 yr.

Work Problem 9 at the Side.

OBJECTIVE 5 **Use the change-of-base rule.** In **Section 11.5** we used a calculator to approximate the values of common logarithms (base 10) or natural logarithms (base e). However, some applications involve logarithms to other bases. For example, for the years 1980–1996, the percentage of women who had a baby in the last year and returned to work is given by

$$f(x) = 38.83 + 4.208 \log_2 x,$$

for year x. (*Source:* U.S. Bureau of the Census.) To use this function, we need to find a base 2 logarithm. The following rule is used to convert logarithms from one base to another.

Change-of-Base Rule

If $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, and $x > 0$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

ANSWERS
9. (a) 5.20 g **(b)** 29 yr

NOTE
Any positive number other than 1 can be used for base b in the change-of-base rule, but usually the only practical bases are e and 10 because calculators give logarithms only for these two bases.

To derive the change-of-base rule, let $\log_a x = m$.

$$\begin{aligned}
\log_a x &= m & &\\
a^m &= x & &\text{Change to exponential form.}\\
\log_b(a^m) &= \log_b x & &\text{Property 3}\\
m \log_b a &= \log_b x & &\text{Power rule}\\
(\log_a x)(\log_b a) &= \log_b x & &\text{Substitute for } m.\\
\log_a x &= \frac{\log_b x}{\log_b a} & &\text{Divide by } \log_b a.
\end{aligned}$$

The last step gives the change-of-base rule.

Video

EXAMPLE 10 Using the Change-of-Base Rule

Find $\log_5 12$.

Use common logarithms and the change-of-base rule.

$$\begin{aligned}
\log_5 12 &= \frac{\log 12}{\log 5}\\
&\approx 1.5440 \qquad \text{Use a calculator.}
\end{aligned}$$

NOTE
Either common or natural logarithms can be used when applying the change-of-base rule. Verify that the same value is found in Example 10 if natural logarithms are used.

Work Problem 10 at the Side.

EXAMPLE 11 Using the Change-of-Base Rule in an Application

Use natural logarithms in the change-of-base rule and the function defined by

$$f(x) = 38.83 + 4.208 \log_2 x$$

(given earlier) to find the percent of women who returned to work after having a baby in 1995. In the function, $x = 0$ represents 1980.

Substitute $1995 - 1980 = 15$ for x.

$$\begin{aligned}
f(15) &= 38.83 + 4.208 \log_2 15 & &\\
&= 38.83 + 4.208\left(\frac{\ln 15}{\ln 2}\right) & &\text{Change-of-base rule}\\
&\approx 55.3\% & &\text{Use a calculator.}
\end{aligned}$$

This is very close to the actual value of 55%.

Work Problem 11 at the Side.

10 **(a)** Find $\log_3 17$ using common logarithms.

(b) Find $\log_3 17$ using natural logarithms.

11 In Example 11, what percent of women returned to work after having a baby in 1990?

ANSWERS

10. **(a)** 2.5789 **(b)** 2.5789

11. 52.8%; This is very close to the actual value of 53%.

Nonlinear Functions, Conic Sections, and Nonlinear Systems

12

In this chapter, we study a group of curves known as *conic sections*. One conic section, the *ellipse,* has a special reflecting property responsible for "whispering galleries." In a whispering gallery, a person whispering at a certain point in the room can be heard clearly at another point across the room.

The Old House Chamber of the U.S. Capitol, now called Statuary Hall, is a whispering gallery. History has it that John Quincy Adams, whose desk was positioned at exactly the right point beneath the ellipsoidal ceiling, often pretended to sleep there as he listened to political opponents whispering strategies across the room. (*Source: We, the People, The Story of the United States Capitol,* 1991.)

In Section 12.2, we investigate ellipses.

12.1 Additional Graphs of Functions; Composition

OBJECTIVES

1. Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations.
2. Recognize and graph step functions.
3. Find the composition of functions.

In earlier chapters we introduced the function defined by $f(x) = x^2$, sometimes called the **squaring function.** This is one of the most important elementary functions in algebra.

OBJECTIVE 1 Recognize the graphs of the elementary functions defined by $|x|$, $\frac{1}{x}$, and $\sqrt{x}$, and graph their translations. Another one of the elementary functions, defined by $f(x) = |x|$, is called the **absolute value function.** Its graph, along with a table of selected ordered pairs, is shown in Figure 1. Its domain is $(-\infty, \infty)$, and its range is $[0, \infty)$.

x	y
0	0
± 1	1
± 2	2
± 3	3

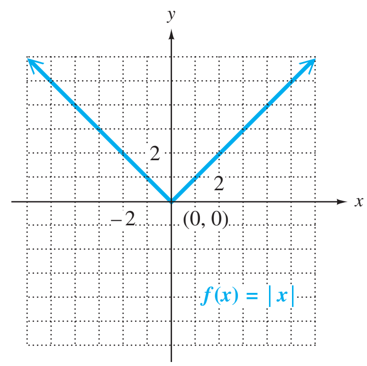


Figure 1

The **reciprocal function,** defined by $f(x) = \frac{1}{x}$, was introduced in **Section 8.4.** Its graph is shown in Figure 2, along with a table of selected ordered pairs. Notice that x can never equal 0 for this function, and as a result, as x gets closer and closer to 0, the graph approaches either ∞ or $-\infty$. Also, $\frac{1}{x}$ can never equal 0, and as x approaches ∞ or $-\infty$, $\frac{1}{x}$ approaches 0. The axes are called **asymptotes** for the function. (Asymptotes are studied in more detail in college algebra courses.) For the reciprocal function, the domain and the range are both $(-\infty, 0) \cup (0, \infty)$.

x	y
$\frac{1}{3}$	3
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$

x	y
$-\frac{1}{3}$	-3
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-3	$-\frac{1}{3}$

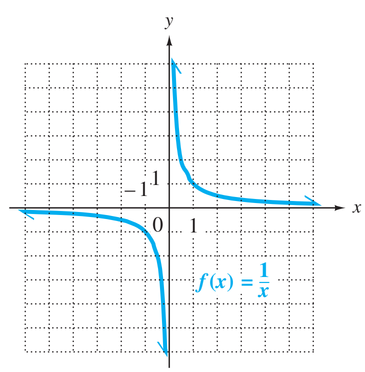


Figure 2

The **square root function,** defined by $f(x) = \sqrt{x}$, was introduced in **Section 9.1.** Its graph is shown in Figure 3 on the next page. Notice that since we restrict function values to be real numbers, x cannot take on negative values. Thus, the domain of the square root function is $[0, \infty)$. Because the principal square root is always nonnegative, the range is also $[0, \infty)$. A table of values is shown along with the graph.

x	y
0	0
1	1
4	2

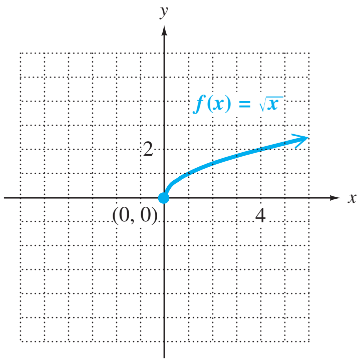


Figure 3

Just as the graph of $f(x) = x^2$ can be shifted, or translated, as we saw in **Section 10.5,** so can the graphs of these other elementary functions.

Video

EXAMPLE 1 Applying a Horizontal Shift

Graph $f(x) = |x - 2|$.

The graph of $y = (x - 2)^2$ is obtained by shifting the graph of $y = x^2$ two units to the right. In a similar manner, the graph of $f(x) = |x - 2|$ is found by shifting the graph of $y = |x|$ two units to the right, as shown in Figure 4. The table of ordered pairs accompanying the graph supports this, as can be seen by comparing it to the table with Figure 1. The domain of this function is $(-\infty, \infty)$, and its range is $[0, \infty)$.

x	y
0	2
1	1
2	0
3	1
4	2

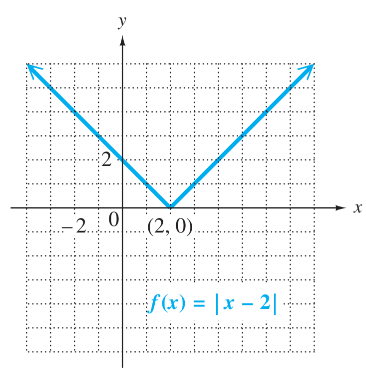


Figure 4

Work Problem 1 at the Side.

Video

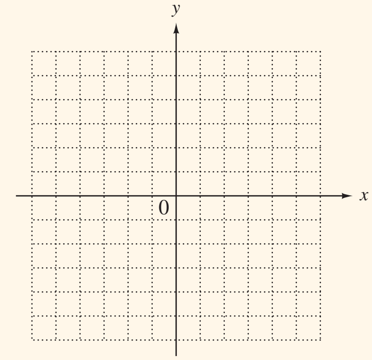
EXAMPLE 2 Applying a Vertical Shift

Graph $f(x) = \dfrac{1}{x} + 3$.

The graph of this function is found by shifting the graph of $y = \frac{1}{x}$ three units up. See Figure 5 on the next page. The domain is $(-\infty, 0) \cup (0, \infty)$, and the range is $(-\infty, 3) \cup (3, \infty)$.

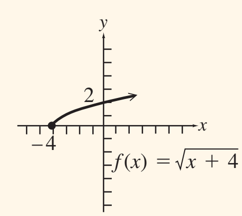
Continued on Next Page

1 Graph $f(x) = \sqrt{x + 4}$. Give the domain and range.



Answers

1.



$[-4, \infty)$; $[0, \infty)$

x	y
$\frac{1}{3}$	6
$\frac{1}{2}$	5
1	4
2	3.5

x	y
$-\frac{1}{3}$	0
$-\frac{1}{2}$	1
-1	2
-2	2.5

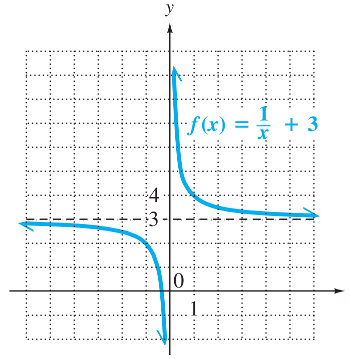


Figure 5

Work Problem 2 at the Side.

EXAMPLE 3 Applying Both Horizontal and Vertical Shifts

Graph $f(x) = \sqrt{x + 1} - 4$.

The graph of $y = (x + 1)^2 - 4$ is obtained by shifting the graph of $y = x^2$ one unit to the left and four units down. Following this pattern here, we shift the graph of $y = \sqrt{x}$ one unit to the left and four units down to get the graph of $f(x) = \sqrt{x + 1} - 4$. See Figure 6. The domain is $[-1, \infty)$, and the range is $[-4, \infty)$.

x	y
-1	-4
0	-3
3	-2

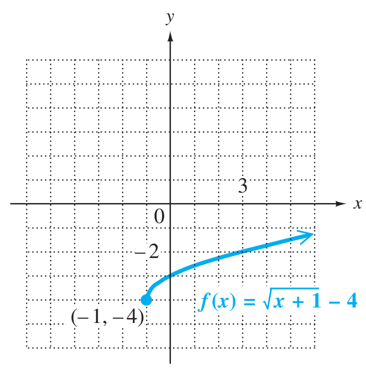


Figure 6

Work Problem 3 at the Side.

OBJECTIVE 2 Recognize and graph step functions. The **greatest integer function,** usually written $f(x) = [\![x]\!]$, is defined as follows:

$[\![x]\!]$ denotes the largest integer that is less than or equal to x.

For example,

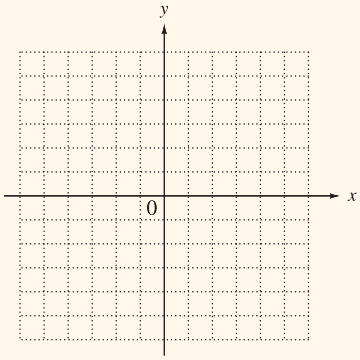
$$[\![8]\!] = 8, \quad [\![7.45]\!] = 7, \quad [\![\pi]\!] = 3, \quad [\![-1]\!] = -1, \quad [\![-2.6]\!] = -3,$$

and so on.

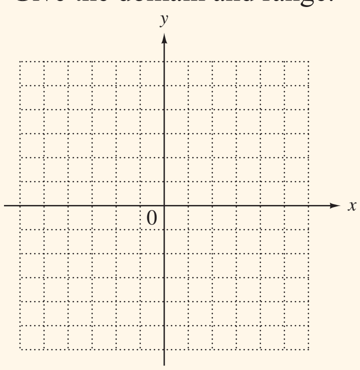
Work Problem 4 at the Side.

2 Graph $f(x) = \dfrac{1}{x} - 2$.

Give the domain and range.



3 Graph $f(x) = |x + 2| + 1$.
Give the domain and range.



4 Find each of the following.

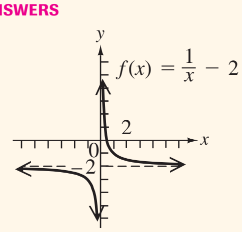
(a) $[\![18]\!]$ **(b)** $[\![8.7]\!]$

(c) $[\![-5]\!]$ **(d)** $[\![-6.9]\!]$

(e) $\left[\!\!\left[\frac{1}{2}\right]\!\!\right]$

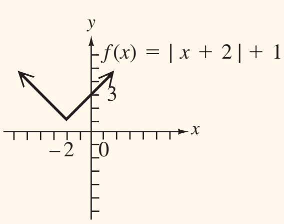
ANSWERS

2.



$(-\infty, 0) \cup (0, \infty)$; $(-\infty, -2) \cup (-2, \infty)$

3.



$(-\infty, \infty)$; $[1, \infty)$

4. (a) 18 **(b)** 8 **(c)** -5 **(d)** -7 **(e)** 0

EXAMPLE 4 Graphing the Greatest Integer Function

Graph $f(x) = [\![x]\!]$.

For $[\![x]\!]$,

$$\begin{array}{llll} \text{if} & -1 \le x < 0, & \text{then} & [\![x]\!] = -1; \\ \text{if} & 0 \le x < 1, & \text{then} & [\![x]\!] = 0; \\ \text{if} & 1 \le x < 2, & \text{then} & [\![x]\!] = 1, \end{array}$$

and so on. Thus, the graph, as shown in Figure 7, consists of a series of horizontal line segments. In each one, the left endpoint is included and the right endpoint is excluded. These segments continue infinitely following this pattern to the left and right. Since x can take any real number value, the domain is $(-\infty, \infty)$. The range is the set of integers $\{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$. The appearance of the graph is the reason that this function is called a **step function.**

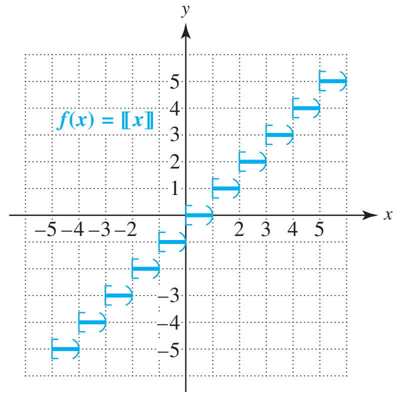


Figure 7

The graph of a step function also may be shifted. For example, the graph of $h(x) = [\![x - 2]\!]$ is the same as the graph of $f(x) = [\![x]\!]$ shifted two units to the right. Similarly, the graph of $g(x) = [\![x]\!] + 2$ is the graph of $f(x)$ shifted two units up.

Work Problem 5 at the Side.

You Try It

Video

EXAMPLE 5 Applying a Greatest Integer Function

An overnight delivery service charges \$25 for a package weighing up to 2 lb. For each additional pound or fraction of a pound there is an additional charge of \$3. Let $D(x)$ represent the cost to send a package weighing x pounds. Graph $D(x)$ for x in the interval $(0, 6]$.

For x in the interval $(0, 2]$, $\quad y = 25$.

For x in the interval $(2, 3]$, $\quad y = 25 + 3 = 28$.

For x in the interval $(3, 4]$, $\quad y = 28 + 3 = 31$, and so on.

The graph, which is that of a step function, is shown in Figure 8.

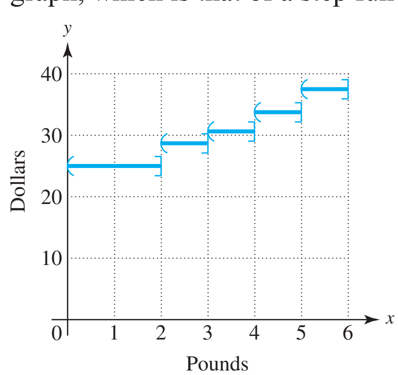
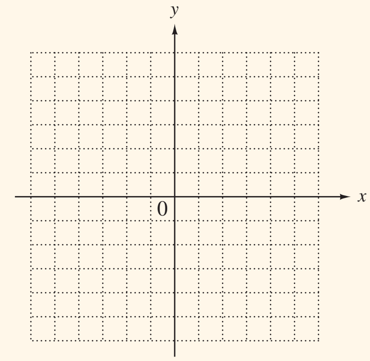


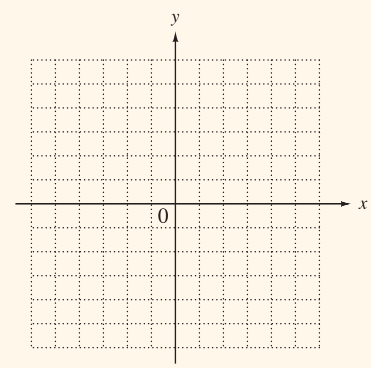
Figure 8

Work Problem 6 at the Side.

5 Graph $f(x) = [\![x + 1]\!]$. Give the domain and range.

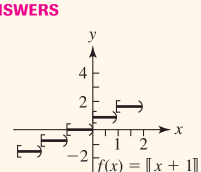


6 Assume that the post office charges 80¢ per oz (or fraction of an ounce) to mail a letter to Europe. Graph the ordered pairs (ounces, cost) for x in the interval $(0, 4]$.



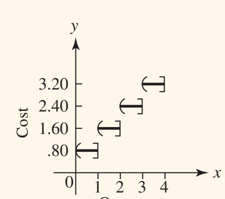
ANSWERS

5.



$(-\infty, \infty)$; $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

6.



OBJECTIVE 3 Find the composition of functions. The diagram in Figure 9 shows a function f that assigns to each element x of set X some element y of set Y. Suppose that a function g takes each element of set Y and assigns a value z of set Z. Using both f and g, then, an element x in X is assigned to an element z in Z. The result of this process is a new function h, which takes an element x in X and assigns an element z in Z.

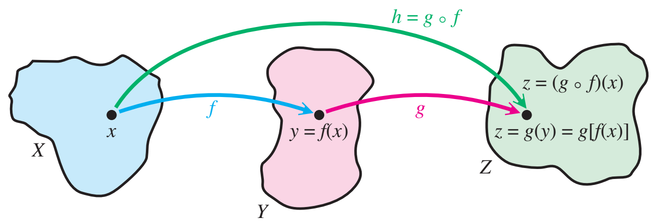


Figure 9

This function h is called the *composition* of functions g and f, written $g \circ f$, and is defined as follows.

Audio

Composition of Functions

If f and g are functions, then the **composite function,** or **composition,** of g and f is defined by

$$(g \circ f)(x) = g[f(x)]$$

for all x in the domain of f such that $f(x)$ is in the domain of g.

Read $g \circ f$ as "g of f."

As a real-life example of how composite functions occur, consider the following retail situation.

A $40 *pair of blue jeans is on sale for* 25% *off. If you purchase the jeans before noon, the retailer offers an additional* 10% *off. What is the final sale price of the blue jeans?*

You might be tempted to say that the jeans are 35% off and calculate $40(.35) = $14, giving a final sale price of $40 − $14 = $26 for the jeans. ***This is not correct.*** To find the final sale price, we must first find the price after taking 25% off, and then take an additional 10% off that price.

$40(.25) = $10, giving a sale price of $40 − $10 = $30. Take 25% off original price.

$30(.10) = $3, giving a ***final sale price*** of $30 − $3 = $27. Take additional 10% off.

This is the idea behind composition of functions.

As another example, suppose an oil well off the California coast is leaking, with the leak spreading oil in a circular layer over the surface. See Figure 10.

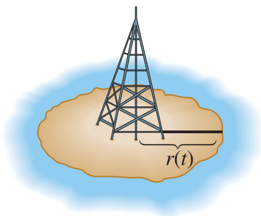


Figure 10

At any time t, in minutes, after the beginning of the leak, the radius of the circular oil slick is given by $r(t) = 5t$ feet. Since $A(r) = \pi r^2$ gives the area of a circle of radius r, the area can be expressed as a function of time by substituting $5t$ for r in $A(r) = \pi r^2$ to get

$$A(r) = \pi r^2$$
$$A[\mathbf{r(t)}] = \pi(\mathbf{5t})^2 = 25\pi t^2.$$

The function $A[r(t)]$ is a composite function of the functions A and r.

EXAMPLE 6 Evaluating a Composite Function

Let $f(x) = x^2$ and $g(x) = x + 3$. Find $(f \circ g)(4)$.

$$\begin{aligned} (f \circ g)(4) &= f[g(\mathbf{4})] && \text{Definition} \\ &= f(\mathbf{4} + 3) && \text{Use the rule for } g(x);\ g(4) = 4 + 3. \\ &= f(\mathbf{7}) && \text{Add.} \\ &= \mathbf{7}^2 && \text{Use the rule for } f(x);\ f(7) = 7^2. \\ &= 49 \end{aligned}$$

Notice in Example 6 that if we reverse the order of the functions, the composition of g and f is defined by $g[f(x)]$. Once again, letting $x = 4$, we have

$$\begin{aligned} (g \circ f)(\mathbf{4}) &= g[f(\mathbf{4})] && \text{Definition} \\ &= g(\mathbf{4}^2) && \text{Use the rule for } f(x);\ f(4) = 4^2. \\ &= g(\mathbf{16}) && \text{Square 4.} \\ &= \mathbf{16} + 3 && \text{Use the rule for } g(x);\ g(16) = 16 + 3. \\ &= 19. \end{aligned}$$

Here we see that $(f \circ g)(4) \neq (g \circ f)(4)$ because $49 \neq 19$. In general,

$$(f \circ g)(x) \neq (g \circ f)(x).$$

EXAMPLE 7 Finding Composite Functions

Let $f(x) = 4x - 1$ and $g(x) = x^2 + 5$. Find each of the following.

(a) $(f \circ g)(2)$

$$\begin{aligned} (f \circ g)(2) &= f[g(2)] & \\ &= f(2^2 + 5) & g(x) = x^2 + 5 \\ &= f(9) & \\ &= 4(9) - 1 & f(x) = 4x - 1 \\ &= 35 & \end{aligned}$$

(b) $(f \circ g)(x)$

Here, use $g(x)$ as the input for the function f.

$$\begin{aligned} (f \circ g)(x) &= f[g(x)] & \\ &= 4(g(x)) - 1 & \text{Use the rule for } f(x);\ f(x) = 4x - 1. \\ &= 4(x^2 + 5) - 1 & g(x) = x^2 + 5 \\ &= 4x^2 + 20 - 1 & \text{Distributive property} \\ &= 4x^2 + 19 & \text{Combine terms.} \end{aligned}$$

(c) Find $(f \circ g)(2)$ again, this time using the rule obtained in part (b).

$$\begin{aligned} (f \circ g)(x) &= 4x^2 + 19 & \text{From part (b)} \\ (f \circ g)(2) &= 4(2)^2 + 19 & \text{Let } x = 2. \\ &= 4(4) + 19 & \\ &= 16 + 19 & \\ &= 35 & \end{aligned}$$

The result, 35, is the same as the result in part (a).

(d) $(g \circ f)(x)$

Here, use $f(x)$ as the input for the function g.

$$\begin{aligned} (g \circ f)(x) &= g[f(x)] & \\ &= [f(x)]^2 + 5 & \text{Use the rule for } g(x);\ g(x) = x^2 + 5. \\ &= (4x - 1)^2 + 5 & f(x) = 4x - 1 \\ &= 16x^2 - 8x + 1 + 5 & (x - y)^2 = x^2 - 2xy + y^2 \\ &= 16x^2 - 8x + 6 & \text{Combine terms.} \end{aligned}$$

Compare this result to that in part (b). Again, $(f \circ g)(x) \neq (g \circ f)(x)$.

Work Problem 7 at the Side.

7 Let $f(x) = 3x + 6$ and $g(x) = x^3$. Find each of the following.

(a) $(f \circ g)(2)$

(b) $(g \circ f)(2)$

(c) $(f \circ g)(x)$

(d) $(g \circ f)(x)$

ANSWERS

7. **(a)** 30 **(b)** 1728 **(c)** $3x^3 + 6$ **(d)** $(3x + 6)^3$

12.2 The Circle and the Ellipse

When an infinite cone is intersected by a plane, the resulting figure is called a **conic section.** The parabola is one example of a conic section; circles, ellipses, and hyperbolas may also result. See Figure 11.

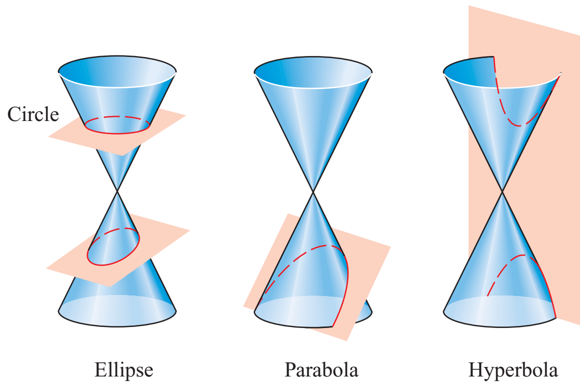


Figure 11

OBJECTIVES

1. Find the equation of a circle given the center and radius.
2. Determine the center and radius of a circle given its equation.
3. Recognize the equation of an ellipse.
4. Graph ellipses.

OBJECTIVE 1 Find an equation of a circle given the center and radius. A **circle** is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed point is called the **center,** and the fixed distance is called the **radius.** We use the distance formula from **Section 9.3** to find an equation of a circle.

EXAMPLE 1 Finding an Equation of a Circle and Graphing It

Animation

Find an equation of the circle with radius 3 and center at (0, 0), and graph it.

If the point (x, y) is on the circle, the distance from (x, y) to the center $(0, 0)$ is 3. By the distance formula,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d \qquad \text{Distance formula}$$

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 3$$

$$x^2 + y^2 = 9. \qquad \text{Square both sides.}$$

An equation of this circle is $x^2 + y^2 = 9$. The graph is shown in Figure 12.

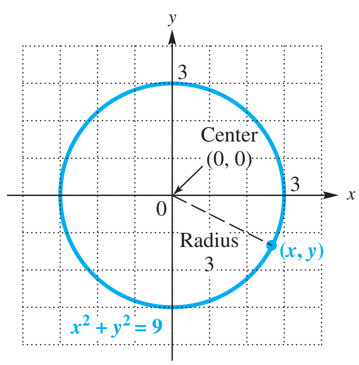
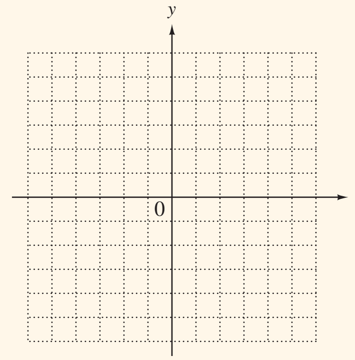


Figure 12

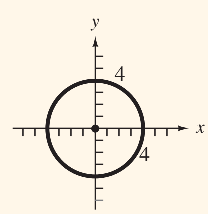
Work Problem 1 at the Side.

1 Find an equation of the circle with radius 4 and center (0, 0). Sketch its graph.

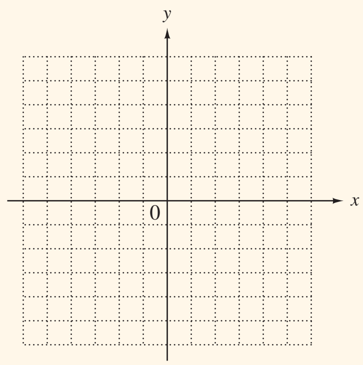


ANSWERS

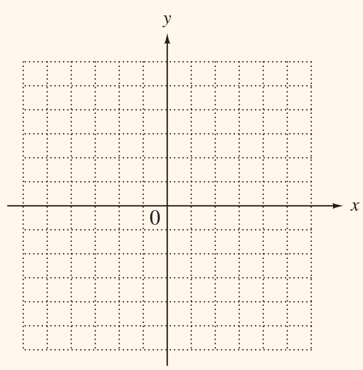
1. $x^2 + y^2 = 16$



2 **(a)** Find an equation of the circle with center at $(3, -2)$ and radius 4. Graph the circle.



(b) Use the center-radius form to determine the center and radius of $(x-5)^2 + (y+2)^2 = 9$, and then graph the circle.



A circle may not be centered at the origin, as seen in the next example.

EXAMPLE 2 Finding an Equation of a Circle and Graphing It

Find an equation of the circle with center at $(4, -3)$ and radius 5, and graph it.

Use the distance formula again.

$$\sqrt{(x - \mathbf{4})^2 + [y - (\mathbf{-3})]^2} = \mathbf{5}$$

$$(x-4)^2 + (y+3)^2 = 25 \qquad \text{Square both sides.}$$

To graph the circle, plot the center $(4, -3)$, then move 5 units right, left, up, and down from the center. Draw a smooth curve through these four points, sketching one quarter of the circle at a time. The graph of this circle is shown in Figure 13.

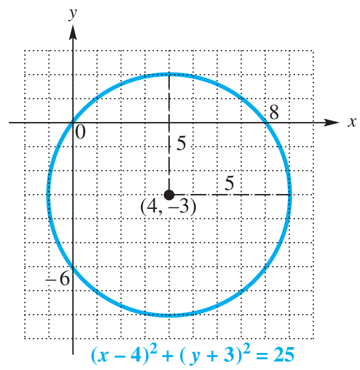


Figure 13

Examples 1 and 2 suggest the form of an equation of a circle with radius r and center at (h, k). If (x, y) is a point on the circle, then the distance from the center (h, k) to the point (x, y) is r. By the distance formula,

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Squaring both sides gives us the following **center-radius form** of the equation of a circle.

Equation of a Circle (Center-Radius Form)

An equation of a circle of radius r with center at (h, k) is

$$(x-h)^2 + (y-k)^2 = r^2.$$

EXAMPLE 3 Using the Center-Radius Form of the Equation of a Circle

Find an equation of the circle with center at $(-1, 2)$ and radius 4.

Use the center-radius form, with $h = -1$, $k = 2$, and $r = 4$.

$$(x-h)^2 + (y-k)^2 = r^2$$

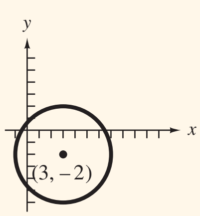
$$[x - (\mathbf{-1})]^2 + (y - \mathbf{2})^2 = \mathbf{4}^2$$

$$(x+1)^2 + (y-2)^2 = 16$$

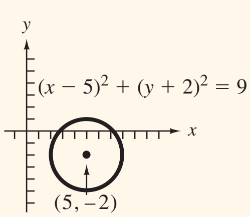
Work Problem 2 at the Side.

Answers

2. **(a)** $(x-3)^2 + (y+2)^2 = 16$



(b) center at $(5, -2)$; radius 3



OBJECTIVE 2 Determine the center and radius of a circle given its equation. In the equation found in Example 2, multiplying out $(x - 4)^2$ and $(y + 3)^2$ and then combining like terms gives

$$\begin{aligned} (x-4)^2 + (y+3)^2 &= 25 \\ x^2 - 8x + 16 + y^2 + 6y + 9 &= 25 \\ x^2 + y^2 - 8x + 6y &= 0. \end{aligned}$$

This general form suggests that an equation with both x^2- and y^2-terms with equal coefficients may represent a circle. The next example shows how to tell, by completing the square. This procedure was introduced in **Section 10.1.**

EXAMPLE 4 Completing the Square to Find the Center and Radius

Graph $x^2 + y^2 + 2x + 6y - 15 = 0$.

Since the equation has x^2- and y^2-terms with equal coefficients, its graph might be that of a circle. To find the center and radius, complete the squares on x and y.

$$x^2 + y^2 + 2x + 6y = 15 \qquad \text{Get the constant on the right.}$$

$$(x^2 + 2x \quad) + (y^2 + 6y \quad) = 15 \qquad \text{Rewrite in anticipation of completing the square.}$$

$$\left[\frac{1}{2}(2)\right]^2 = 1 \qquad \left[\frac{1}{2}(6)\right]^2 = 9 \qquad \text{Square half the coefficient of each middle term.}$$

$$(x^2 + 2x + 1) + (y^2 + 6y + 9) = 15 + 1 + 9 \qquad \text{Complete the squares on both } x \text{ and } y.$$

$$(x + 1)^2 + (y + 3)^2 = 25 \qquad \text{Factor on the left; add on the right.}$$

$$[x - (-1)]^2 + [y - (-3)]^2 = 5^2 \qquad \text{Center-radius form}$$

The last equation shows that the graph is a circle with center at $(-1, -3)$ and radius 5. The graph is shown in Figure 14.

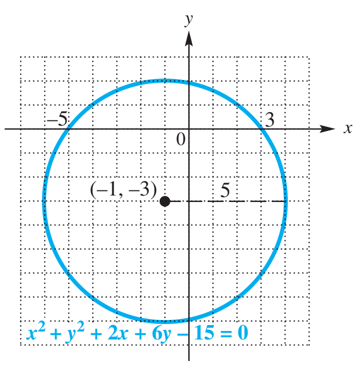


Figure 14

NOTE

If the procedure of Example 4 leads to an equation of the form $(x - h)^2 + (y - k)^2 = 0$, then the graph is the single point (h, k). If the constant on the right side is negative, then the equation has no graph.

Work Problem 3 at the Side.

3 Find the center and radius of the circle with equation

$x^2 + y^2 - 6x + 8y - 11 = 0.$

ANSWERS

3. center at $(3, -4)$; radius 6

OBJECTIVE 3 Recognize the equation of an ellipse. An **ellipse** is the set of all points in a plane the *sum* of whose distances from two fixed points is constant. These fixed points are called **foci** (singular: *focus*). Figure 15 shows an ellipse whose foci are $(c, 0)$ and $(-c, 0)$, with x-intercepts $(a, 0)$ and $(-a, 0)$ and y-intercepts $(0, b)$ and $(0, -b)$. It can be shown that $c^2 = a^2 - b^2$ for an ellipse of this type. The origin is the **center** of the ellipse.

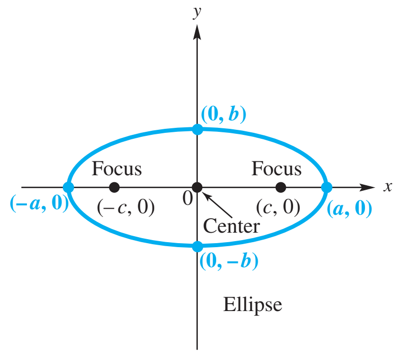


Figure 15

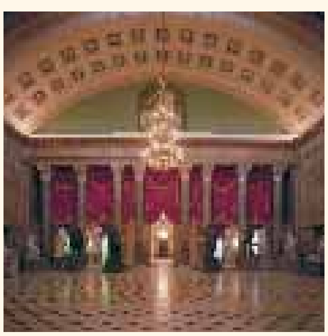
From the preceding definition, it can be shown by the distance formula that an ellipse has the following equation.

Equation of an Ellipse

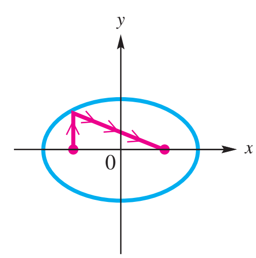
The ellipse whose x-intercepts are $(a, 0)$ and $(-a, 0)$ and whose y-intercepts are $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

NOTE
A circle is a special case of an ellipse, where $a^2 = b^2$.



When a ray of light or sound emanating from one focus of an ellipse bounces off the ellipse, it passes through the other focus. See the figure. As mentioned in the chapter introduction, this reflecting property is responsible for whispering galleries. John Quincy Adams was able to listen in on his opponents' conversations because his desk was positioned at one of the foci beneath the ellipsoidal ceiling and his opponents were located across the room at the other focus.



Reflecting property of an ellipse

The paths of Earth and other planets around the sun are approximately ellipses; the sun is at one focus and a point in space is at the other. The orbits of communication satellites and other space vehicles are also elliptical.

Elliptical bicycle gears are designed to respond to the legs' natural strengths and weaknesses. At the top and bottom of the powerstroke, where the legs have the least leverage, the gear offers little resistance, but as the gear rotates, the resistance increases. This allows the legs to apply more power where it is most naturally available. See Figure 16.

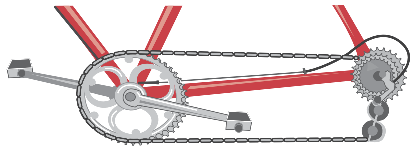


Figure 16

OBJECTIVE 4 Graph ellipses. To graph an ellipse centered at the origin, we plot the four intercepts and then sketch the ellipse through those points.

You Try It

Animation

Video

EXAMPLE 5 Graphing Ellipses

Graph each ellipse.

(a) $\dfrac{x^2}{49} + \dfrac{y^2}{36} = 1$

Here, $a^2 = 49$, so $a = 7$, and the x-intercepts for this ellipse are $(7, 0)$ and $(-7, 0)$. Similarly, $b^2 = 36$, so $b = 6$, and the y-intercepts are $(0, 6)$ and $(0, -6)$. Plotting the intercepts and sketching the ellipse through them gives the graph in Figure 17.

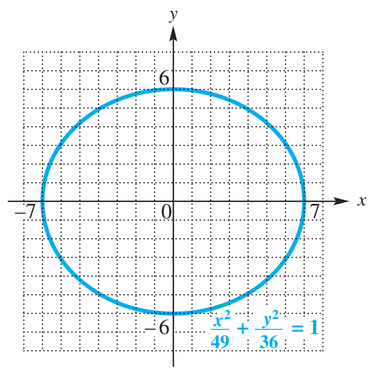


Figure 17

(b) $\dfrac{x^2}{36} + \dfrac{y^2}{121} = 1$

The x-intercepts for this ellipse are $(6, 0)$ and $(-6, 0)$, and the y-intercepts are $(0, 11)$ and $(0, -11)$. Join these intercepts with the smooth curve of an ellipse. The graph has been sketched in Figure 18.

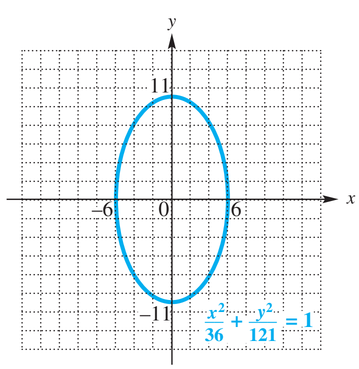
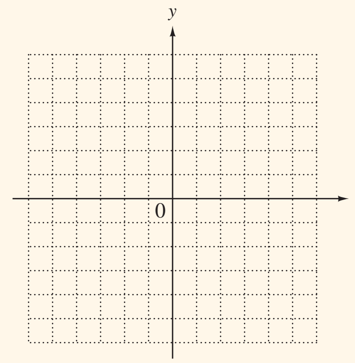


Figure 18

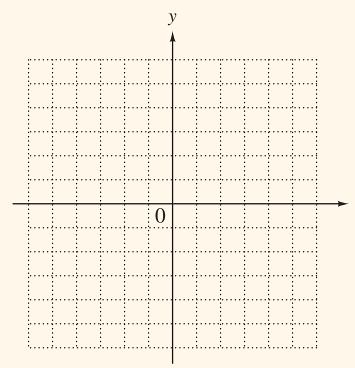
Work Problem 4 at the Side.

4 Graph each ellipse.

(a) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

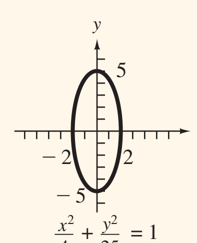


(b) $\dfrac{x^2}{64} + \dfrac{y^2}{49} = 1$

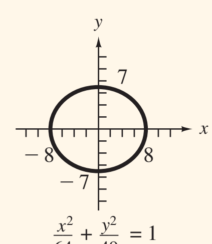


ANSWERS

4. (a)

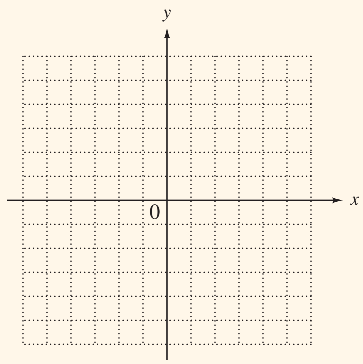


(b)



5 Graph

$$\frac{(x+4)^2}{16} + \frac{(y-1)^2}{36} = 1.$$



As with the graphs of parabolas and circles, the graph of an ellipse may be shifted horizontally and vertically, as in the next example.

EXAMPLE 6 Graphing an Ellipse Shifted Horizontally and Vertically

Graph $\dfrac{(x-2)^2}{25} + \dfrac{(y+3)^2}{49} = 1.$

Just as $(x-2)^2$ and $(y+3)^2$ would indicate that the center of a circle would be $(2, -3)$, so it is with this ellipse. Figure 19 shows that the graph goes through the four points $(2, 4)$, $(7, -3)$, $(2, -10)$, and $(-3, -3)$. The x-values of these points are found by adding $\pm a = \pm 5$ to 2, and the y-values come from adding $\pm b = \pm 7$ to -3.

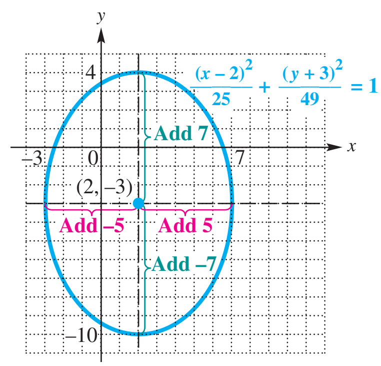


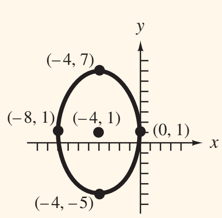
Figure 19

Work Problem 5 at the Side.

> **NOTE**
> ***The graphs in this section are not graphs of functions.*** The only conic section whose graph is a function is the vertical parabola with equation $f(x) = ax^2 + bx + c$.

ANSWERS

5.



$$\frac{(x+4)^2}{16} + \frac{(y-1)^2}{36} = 1$$

12.3 The Hyperbola and Other Functions Defined by Radicals

OBJECTIVES

1. Recognize the equation of a hyperbola.
2. Graph hyperbolas by using asymptotes.
3. Identify conic sections by their equations.
4. Graph certain square root functions.

OBJECTIVE 1 Recognize the equation of a hyperbola. A **hyperbola** is the set of all points in a plane such that the absolute value of the *difference* of the distances from two fixed points (called *foci*) is constant. Figure 20 shows a hyperbola. Using the distance formula and the definition above, we can show that this hyperbola has equation $\frac{x^2}{16} - \frac{y^2}{12} = 1$.

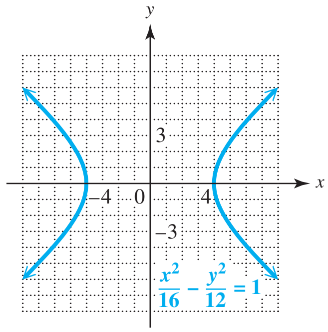


Figure 20

To graph hyperbolas centered at the origin, we need to find their intercepts. For the hyperbola in Figure 20, we proceed as follows.

***x*-Intercepts**

Let $y = 0$.

$$\frac{x^2}{16} - \frac{0^2}{12} = 1 \qquad \text{Let } y = 0.$$

$$\frac{x^2}{16} = 1$$

$$x^2 = 16 \qquad \text{Multiply by 16.}$$

$$x = \pm 4$$

The x-intercepts are $(4, 0)$ and $(-4, 0)$.

***y*-Intercepts**

Let $x = 0$.

$$\frac{0^2}{16} - \frac{y^2}{12} = 1 \qquad \text{Let } x = 0.$$

$$-\frac{y^2}{12} = 1$$

$$y^2 = -12 \qquad \text{Multiply by } -12.$$

Because there are no *real* solutions to $y^2 = -12$, the graph has no y-intercepts.

The graph of $\frac{x^2}{16} - \frac{y^2}{12} = 1$ in Figure 20 has no y-intercepts. On the other hand, the hyperbola in Figure 21 has no x-intercepts. Its equation is $\frac{y^2}{25} - \frac{x^2}{9} = 1$, with y-intercepts $(0, 5)$ and $(0, -5)$.

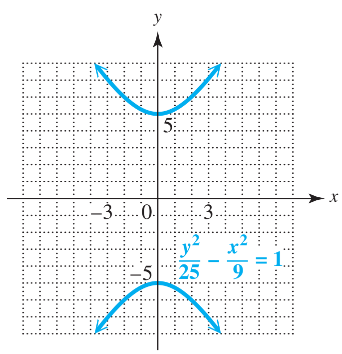


Figure 21

Equations of Hyperbolas

A hyperbola with x-intercepts $(a, 0)$ and $(-a, 0)$ has an equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and a hyperbola with y-intercepts $(0, b)$ and $(0, -b)$ has an equation of the form

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

OBJECTIVE 2 Graph hyperbolas by using asymptotes. The two branches of the graph of a hyperbola approach a pair of intersecting straight lines, which are its *asymptotes.* (See Figure 22 on the next page.) The asymptotes are useful for sketching the graph of the hyperbola.

Audio

Asymptotes of Hyperbolas

The extended diagonals of the rectangle with vertices (corners) at the points (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$ are the **asymptotes** of the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This rectangle is called the **fundamental rectangle.** Using the methods of **Chapter 4,** we could show that the equations of these asymptotes are

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

To graph hyperbolas, follow these steps.

Audio

Graphing a Hyperbola

Step 1 **Find the intercepts.** Locate the intercepts at $(a, 0)$ and $(-a, 0)$ if the x^2-term has a positive coefficient, or at $(0, b)$ and $(0, -b)$ if the y^2-term has a positive coefficient.

Step 2 **Find the fundamental rectangle.** Locate the vertices of the fundamental rectangle at (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$.

Step 3 **Sketch the asymptotes.** The extended diagonals of the rectangle are the asymptotes of the hyperbola, and they have equations $y = \pm \frac{b}{a}x$.

Step 4 **Draw the graph.** Sketch each branch of the hyperbola through an intercept and approaching (but not touching) the asymptotes.

EXAMPLE 1 Graphing a Horizontal Hyperbola

Graph $\dfrac{x^2}{16} - \dfrac{y^2}{25} = 1$.

Step 1 Here $a = 4$ and $b = 5$. The x-intercepts are $(4, 0)$ and $(-4, 0)$.

Step 2 The four points $(4, 5)$, $(-4, 5)$, $(-4, -5)$, and $(4, -5)$ are the vertices of the fundamental rectangle, as shown in Figure 22.

Steps 3 and 4 The equations of the asymptotes are $y = \pm \frac{5}{4}x$. The hyperbola approaches these lines as x and y get larger in absolute value.

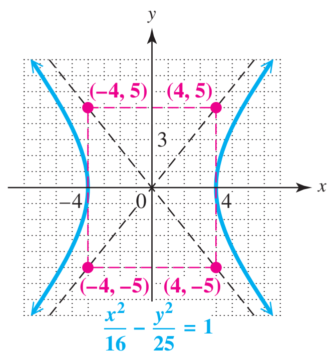


Figure 22

CAUTION
When sketching the graph of a hyperbola, be sure that the branches do not touch the asymptotes.

Work Problem 1 at the Side.

EXAMPLE 2 Graphing a Vertical Hyperbola

Graph $\dfrac{y^2}{49} - \dfrac{x^2}{16} = 1$.

This hyperbola has y-intercepts $(0, 7)$ and $(0, -7)$. The asymptotes are the extended diagonals of the rectangle with vertices at $(4, 7)$, $(-4, 7)$, $(-4, -7)$, and $(4, -7)$. Their equations are $y = \pm \frac{7}{4}x$. See Figure 23.

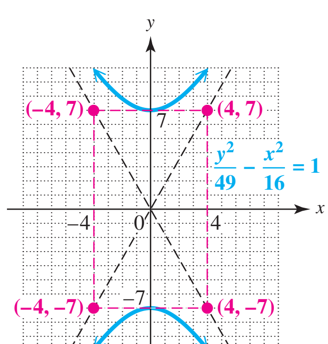
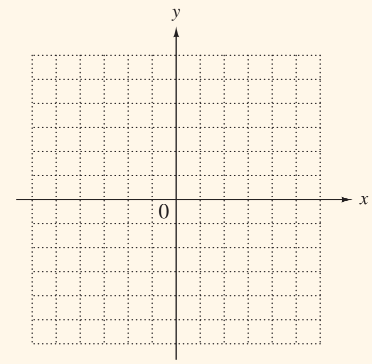


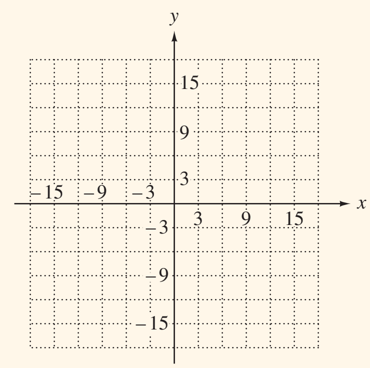
Figure 23

Work Problem 2 at the Side.

1 Graph $\dfrac{x^2}{4} - \dfrac{y^2}{25} = 1$.

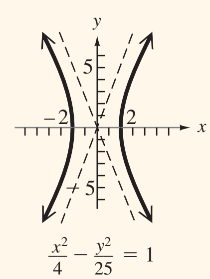


2 Graph $\dfrac{y^2}{81} - \dfrac{x^2}{64} = 1$.

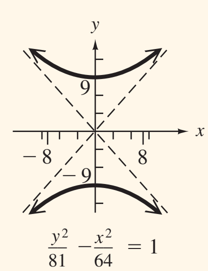


ANSWERS

1.

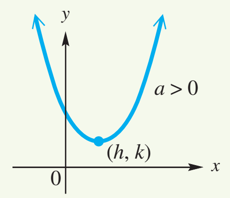
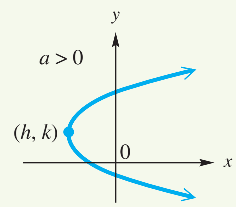
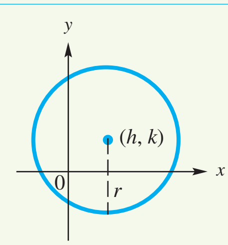
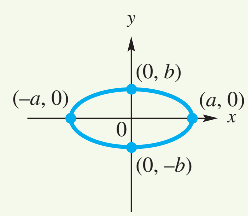
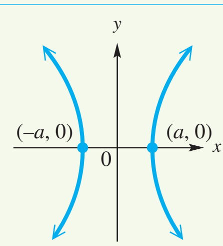
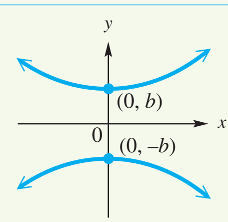


2.



OBJECTIVE 3 Identify conic sections by their equations. Rewriting a second-degree equation in one of the forms given for ellipses, hyperbolas, circles, or parabolas makes it possible to identify the graph of the equation.

SUMMARY OF CONIC SECTIONS

Equation	Graph	Description	Identification
$y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$	 Parabola	It opens up if $a > 0$, down if $a < 0$. The vertex is (h, k).	It has an x^2-term. y is not squared.
$x = ay^2 + by + c$ or $x = a(y - k)^2 + h$	 Parabola	It opens to the right if $a > 0$, to the left if $a < 0$. The vertex is (h, k).	It has a y^2-term. x is not squared.
$(x - h)^2 + (y - k)^2 = r^2$	 Circle	The center is (h, k), and the radius is r.	x^2- and y^2-terms have the same positive coefficient.
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	 Ellipse	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The y-intercepts are $(0, b)$ and $(0, -b)$.	x^2- and y^2-terms have different positive coefficients.
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	 Hyperbola	The x-intercepts are $(a, 0)$ and $(-a, 0)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	x^2 has a positive coefficient. y^2 has a negative coefficient.
$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$	 Hyperbola	The y-intercepts are $(0, b)$ and $(0, -b)$. The asymptotes are found from (a, b), $(a, -b)$, $(-a, -b)$, and $(-a, b)$.	y^2 has a positive coefficient. x^2 has a negative coefficient.

You Try It

Video

EXAMPLE 3 Identifying the Graphs of Equations

Identify the graph of each equation.

(a) $9x^2 = 108 + 12y^2$

Both variables are squared, so the graph is either an ellipse or a hyperbola. (This situation also occurs for a circle, which is a special case of the ellipse.) To see which one it is, rewrite the equation so that the x^2- and y^2-terms are on one side of the equation and 1 is on the other.

$$9x^2 - 12y^2 = 108 \qquad \text{Subtract } 12y^2.$$

$$\frac{x^2}{12} - \frac{y^2}{9} = 1 \qquad \text{Divide by 108.}$$

Because of the minus sign, the graph of this equation is a hyperbola.

(b) $x^2 = y - 3$

Only one of the two variables, x, is squared, so this is the vertical parabola $y = x^2 + 3$.

(c) $x^2 = 9 - y^2$

Write the variable terms on the same side of the equation.

$$x^2 + y^2 = 9 \qquad \text{Add } y^2.$$

The graph of this equation is a circle with center at the origin and radius 3.

Work Problem 3 at the Side.

3 Identify the graph of each equation.

(a) $3x^2 = 27 - 4y^2$

(b) $6x^2 = 100 + 2y^2$

(c) $3x^2 = 27 - 4y$

(d) $3x^2 = 27 - 3y^2$

OBJECTIVE 4 Graph certain square root functions. Recall from **Section 4.5** that no vertical line will intersect the graph of a function in more than one point. Thus, horizontal parabolas and all circles, ellipses, and hyperbolas are examples of graphs that do not satisfy the conditions of a function. However, by considering only a part of the graph of each of these we have the graph of a function, as seen in Figure 24.

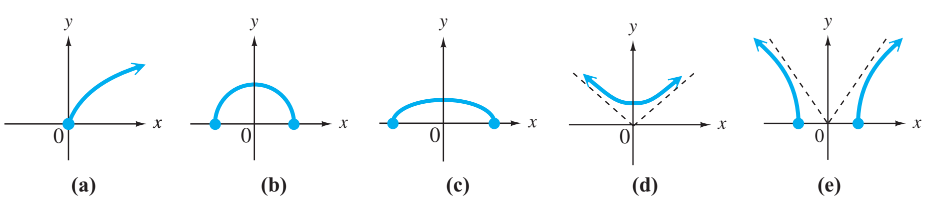


Figure 24

In parts (a)–(d) of Figure 24, the top portion of a conic section is shown (parabola, circle, ellipse, and hyperbola, respectively). In part (e), the top two portions of a hyperbola are shown. In each case, the graph is that of a function since the graph satisfies the conditions of the vertical line test.

In **Sections 9.1** and **12.1** we observed the square root function defined by $f(x) = \sqrt{x}$. To find equations for the types of graphs shown in Figure 24, we extend its definition.

Audio

Square Root Function

For an algebraic expression u, with $u \geq 0$, a function of the form

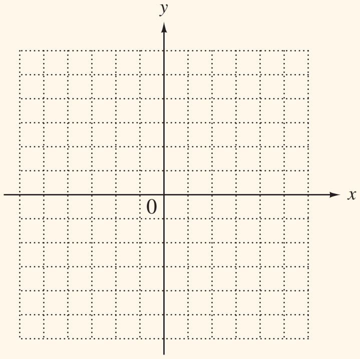
$$f(x) = \sqrt{u}$$

is called a **square root function.**

ANSWERS

3. **(a)** ellipse **(b)** hyperbola **(c)** parabola **(d)** circle

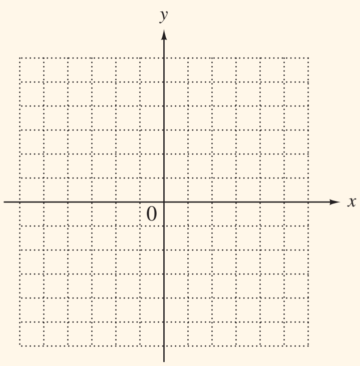
4 Graph $f(x) = \sqrt{36 - x^2}$. Give the domain and range.



5 Graph

$$\frac{y}{3} = -\sqrt{1 - \frac{x^2}{4}}.$$

Give the domain and range.



EXAMPLE 4 Graphing a Semicircle

Graph $f(x) = \sqrt{25 - x^2}$. Give the domain and range.

Replace $f(x)$ with y and square both sides to get the equation

$$y^2 = 25 - x^2, \quad \text{or} \quad x^2 + y^2 = 25.$$

This is the graph of a circle with center at $(0, 0)$ and radius 5. Since $f(x)$, or y, represents a principal square root in the original equation, $f(x)$ must be nonnegative. This restricts the graph to the upper half of the circle, as shown in Figure 25. Use the graph and the vertical line test to verify that it is indeed a function. The domain is $[-5, 5]$, and the range is $[0, 5]$.

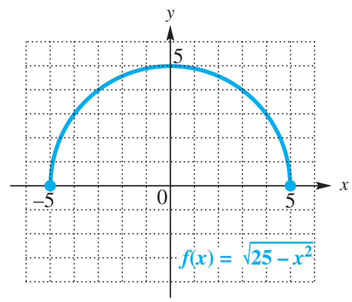


Figure 25

Work Problem 4 at the Side.

EXAMPLE 5 Graphing a Portion of an Ellipse

Graph $\frac{y}{6} = -\sqrt{1 - \frac{x^2}{16}}$. Give the domain and range.

Square both sides to get an equation whose form is known.

$$\frac{y^2}{36} = 1 - \frac{x^2}{16}$$

$$\frac{x^2}{16} + \frac{y^2}{36} = 1 \qquad \text{Add } \frac{x^2}{16}.$$

This is the equation of an ellipse with x-intercepts $(4, 0)$ and $(-4, 0)$ and y-intercepts $(0, 6)$ and $(0, -6)$. Since $\frac{y}{6}$ equals a negative square root in the original equation, y must be nonpositive, restricting the graph to the lower half of the ellipse, as shown in Figure 26. Verify that this is the graph of a function, using the vertical line test. The domain is $[-4, 4]$, and the range is $[-6, 0]$.

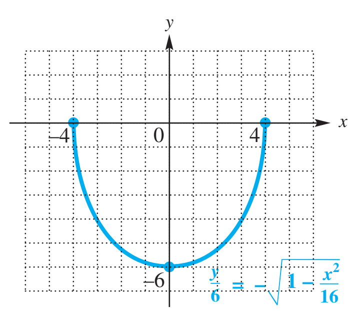
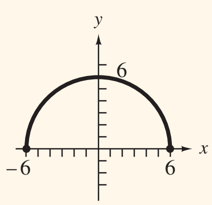


Figure 26

Work Problem 5 at the Side.

ANSWERS

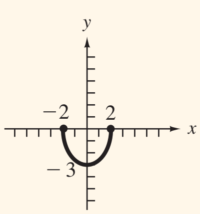
4.



$f(x) = \sqrt{36 - x^2}$

$[-6, 6]$; $[0, 6]$

5.



$\frac{y}{3} = -\sqrt{1 - \frac{x^2}{4}}$

$[-2, 2]$; $[-3, 0]$

12.4 Nonlinear Systems of Equations

OBJECTIVES

1. Solve a nonlinear system by substitution.
2. Use the elimination method to solve a system with two second-degree equations.
3. Solve a system that requires a combination of methods.

An equation in which some terms have more than one variable or a variable of degree 2 or greater is called a **nonlinear equation.** A **nonlinear system of equations** includes at least one nonlinear equation.

When solving a nonlinear system, it helps to visualize the types of graphs of the equations of the system to determine the possible number of points of intersection. For example, if a system includes two equations where the graph of one is a parabola and the graph of the other is a line, then there may be 0, 1, or 2 points of intersection, as illustrated in Figure 27.

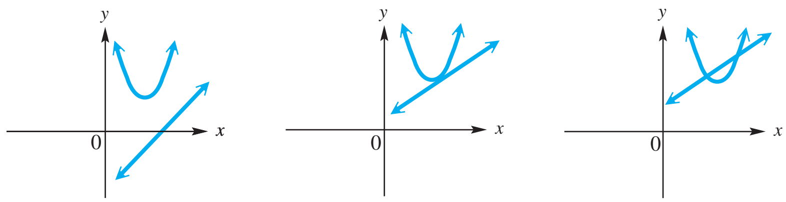


Figure 27

OBJECTIVE 1 Solve a nonlinear system by substitution. We solve nonlinear systems by the substitution method, the elimination method, or a combination of the two. The substitution method **(Section 5.1)** is usually appropriate when one of the equations is linear.

EXAMPLE 1 Solving a Nonlinear System by Substitution

Solve the system.

$$x^2 + y^2 = 9 \qquad (1)$$
$$2x - y = 3 \qquad (2)$$

The graph of (1) is a circle and the graph of (2) is a line. Visualizing the possible ways the graphs could intersect indicates that there may be 0, 1, or 2 points of intersection. See Figure 28. First solve the linear equation for one of the two variables, and then substitute the resulting expression into the nonlinear equation to obtain an equation in one variable.

$$2x - y = 3 \qquad (2)$$
$$y = 2x - 3 \qquad (3)$$

Substitute $2x - 3$ for y in equation (1).

$$x^2 + y^2 = 9 \qquad (1)$$
$$x^2 + (2x - 3)^2 = 9$$
$$x^2 + 4x^2 - 12x + 9 = 9$$
$$5x^2 - 12x = 0 \qquad \text{Subtract 9; combine terms.}$$
$$x(5x - 12) = 0 \qquad \text{Factor; GCF is } x.$$
$$x = 0 \quad \text{or} \quad x = \frac{12}{5} \qquad \text{Zero-factor property}$$

Let $x = 0$ in equation (3) to get $y = -3$. If $x = \frac{12}{5}$, then $y = \frac{9}{5}$. The solution set of the system is $\{(0, -3), (\frac{12}{5}, \frac{9}{5})\}$. The graph in Figure 29 on the next page confirms the two points of intersection.

Continued on Next Page

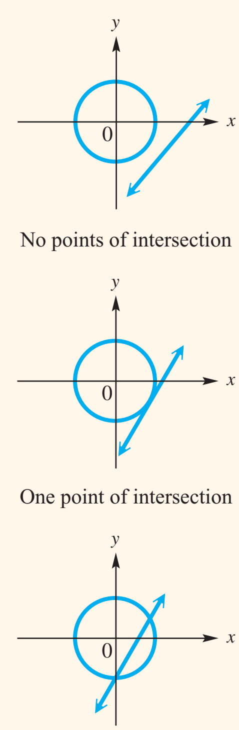


Figure 28

1 Solve each system.

(a) $x^2 + y^2 = 10$
$x = y + 2$

(b) $x^2 - 2y^2 = 8$
$y + x = 6$

2 Solve each system.

(a) $xy = 8$
$x + y = 6$

(b) $xy + 10 = 0$
$4x + 9y = -2$

Answers
1. (a) $\{(3, 1), (-1, -3)\}$
(b) $\{(4, 2), (20, -14)\}$
2. (a) $\{(4, 2), (2, 4)\}$
(b) $\left\{(-5, 2), \left(\frac{9}{2}, -\frac{20}{9}\right)\right\}$

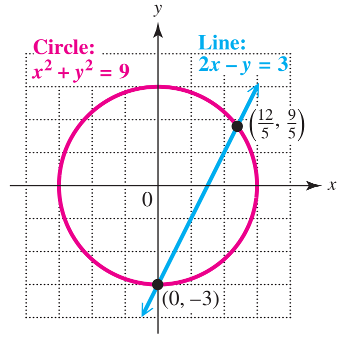


Figure 29

Work Problem 1 at the Side.

You Try It

EXAMPLE 2 Solving a Nonlinear System by Substitution

Solve the system.

$$6x - y = 5 \quad (1)$$
$$xy = 4 \quad (2)$$

The graph of (1) is a line. We have not specifically mentioned equations like (2); however, it can be shown by plotting points that its graph is a hyperbola. Visualizing a line and a hyperbola indicates that there may be 0, 1, or 2 points of intersection. Since neither equation has a squared term, we can solve either equation for one of the variables and then substitute the result into the other equation. Solving $xy = 4$ for x gives $x = \frac{4}{y}$. Substitute $\frac{4}{y}$ for x in equation (1).

$$6\left(\frac{4}{y}\right) - y = 5 \qquad \text{Let } x = \tfrac{4}{y}.$$
$$\frac{24}{y} - y = 5 \qquad \text{Multiply.}$$
$$24 - y^2 = 5y \qquad \text{Multiply by } y,\ y \neq 0.$$
$$0 = y^2 + 5y - 24 \qquad \text{Standard form}$$
$$0 = (y - 3)(y + 8) \qquad \text{Factor.}$$
$$y = 3 \quad \text{or} \quad y = -8 \qquad \text{Zero-factor property}$$

We substitute these results into $x = \frac{4}{y}$ to obtain the corresponding values of x.

$$\text{If } y = 3, \text{ then } x = \frac{4}{3}. \qquad \text{If } y = -8, \text{ then } x = -\frac{1}{2}.$$

The solution set of the system is $\left\{\left(\frac{4}{3}, 3\right), \left(-\frac{1}{2}, -8\right)\right\}$. See Figure 30.

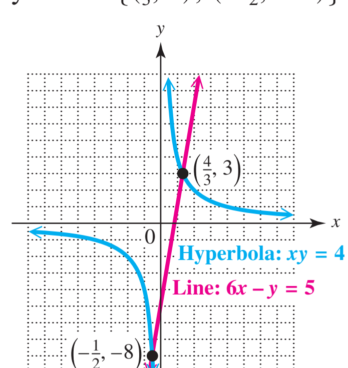


Figure 30

Work Problem 2 at the Side.

OBJECTIVE 2 Use the elimination method to solve a system with two second-degree equations. The elimination method **(Section 5.1)** is often used when both equations are second degree.

You Try It

Video

EXAMPLE 3 Solving a Nonlinear System by Elimination

Solve the system.

$$x^2 + y^2 = 9 \quad (1)$$
$$2x^2 - y^2 = -6 \quad (2)$$

The graph of (1) is a circle, while the graph of (2) is a hyperbola. By analyzing the possibilities we conclude that there may be 0, 1, 2, 3, or 4 points of intersection. Adding the two equations will eliminate y, leaving an equation that can be solved for x.

$$\begin{aligned} x^2 + y^2 &= 9 \\ 2x^2 - y^2 &= -6 \\ \hline 3x^2 \qquad &= 3 \\ x^2 &= 1 && \text{Divide by 3.} \\ x = 1 \quad \text{or} \quad x &= -1 && \text{Square root property} \end{aligned}$$

Each value of x gives corresponding values for y when substituted into one of the original equations. Using equation (1) is easier since the coefficients of the x^2- and y^2-terms are 1.

If $x = 1$, then

$$1^2 + y^2 = 9$$
$$y^2 = 8$$
$$y = \sqrt{8} \quad \text{or} \quad y = -\sqrt{8}$$
$$\mathbf{y = 2\sqrt{2} \quad or \quad y = -2\sqrt{2}.}$$

If $x = -1$, then

$$(-1)^2 + y^2 = 9$$
$$y^2 = 8$$
$$y = \sqrt{8} \quad \text{or} \quad y = -\sqrt{8}$$
$$\mathbf{y = 2\sqrt{2} \quad or \quad y = -2\sqrt{2}.}$$

The solution set is

$$\{(1, 2\sqrt{2}), (1, -2\sqrt{2}), (-1, 2\sqrt{2}), (-1, -2\sqrt{2})\}.$$

Figure 31 shows the four points of intersection.

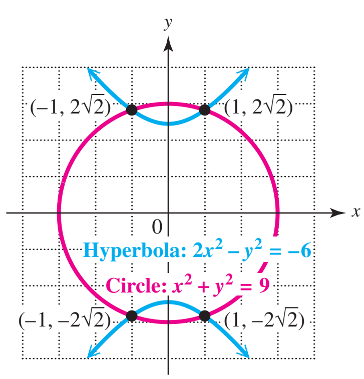


Figure 31

Work Problem 3 at the Side.

3 Solve each system.

(a) $x^2 + y^2 = 41$
$x^2 - y^2 = 9$

(b) $x^2 + 3y^2 = 40$
$4x^2 - y^2 = 4$

ANSWERS

3. (a) $\{(5, 4), (5, -4), (-5, 4), (-5, -4)\}$
(b) $\{(2, 2\sqrt{3}), (2, -2\sqrt{3}), (-2, 2\sqrt{3}), (-2, -2\sqrt{3})\}$

OBJECTIVE 3 Solve a system that requires a combination of methods. Solving a system of second-degree equations may require a combination of methods.

EXAMPLE 4 Solving a Nonlinear System by a Combination of Methods

Solve the system.

$$x^2 + 2xy - y^2 = 7 \quad (1)$$
$$x^2 - y^2 = 3 \quad (2)$$

While we have not graphed equations like (1), its graph is a hyperbola. The graph of (2) is also a hyperbola. Two hyperbolas may have 0, 1, 2, 3, or 4 points of intersection. We use the elimination method here in combination with the substitution method. We begin by eliminating the squared terms by multiplying each side of equation (2) by -1 and then adding the result to equation (1).

$$\begin{array}{rcl} x^2 + 2xy - y^2 &=& 7 \\ -x^2 \qquad\quad + y^2 &=& -3 \\ \hline 2xy \qquad &=& 4 \end{array}$$

Next, we solve $2xy = 4$ for y. (Either variable would do.)

$$2xy = 4$$
$$y = \frac{2}{x} \quad (3)$$

Now, we substitute $y = \frac{2}{x}$ into one of the original equations. It is easier to do this with equation (2).

$$\begin{aligned} x^2 - y^2 &= 3 && (2) \\ x^2 - \left(\frac{2}{x}\right)^2 &= 3 \\ x^2 - \frac{4}{x^2} &= 3 \\ x^4 - 4 &= 3x^2 && \text{Multiply by } x^2,\ x \neq 0. \\ x^4 - 3x^2 - 4 &= 0 && \text{Subtract } 3x^2. \\ (x^2 - 4)(x^2 + 1) &= 0 && \text{Factor.} \end{aligned}$$

$$x^2 - 4 = 0 \quad \text{or} \quad x^2 + 1 = 0$$
$$x^2 = 4 \quad \text{or} \quad x^2 = -1$$
$$x = 2 \quad \text{or} \quad x = -2 \qquad x = i \quad \text{or} \quad x = -i$$

Substituting these four values of x into equation (3) gives the corresponding values for y.

If $x = 2$, then $y = 1$. If $x = i$, then $y = -2i$.
If $x = -2$, then $y = -1$. If $x = -i$, then $y = 2i$.

Note that if we substitute the x-values we found into equation (1) or (2) instead of into equation (3), we get extraneous solutions. ***It is always wise to check all solutions in both of the given equations.*** There are four ordered pairs in the solution set, two with real values and two with nonreal complex values. The solution set is

$$\{(2, 1), (-2, -1), (i, -2i), (-i, 2i)\}.$$

Continued on Next Page

The graph of the system, shown in Figure 32, shows only the two real intersection points because the graph is in the real number plane. The two ordered pairs with nonreal complex components are solutions of the system, but do not appear on the graph.

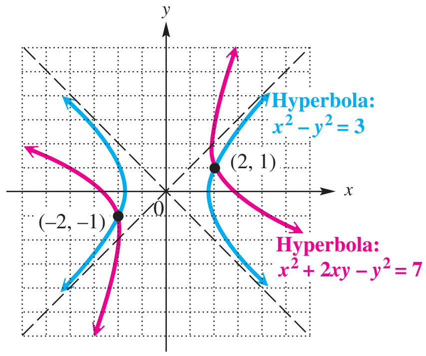


Figure 32

Work Problem 4 at the Side.

NOTE
In the examples of this section, we analyzed the possible number of points of intersection of the graphs in each system. However, in Examples 2 and 4, we worked with equations whose graphs had not been studied. Keep in mind that it is not absolutely essential to visualize the number of points of intersection in order to solve the system. Furthermore, as in Example 4, there are sometimes nonreal complex solutions to nonlinear systems that do not appear as points of intersection in the real plane. Visualizing the geometry of the graphs is only an aid to solving these systems.

4 Solve each system.

(a) $x^2 + xy + y^2 = 3$
$x^2 + y^2 = 5$

(b) $x^2 + 7xy - 2y^2 = -8$
$-2x^2 + 4y^2 = 16$

Answers

4. (a) $\{(1, -2), (-1, 2), (2, -1), (-2, 1)\}$
(b) $\{(0, 2), (0, -2), (2i\sqrt{2}, 0), (-2i\sqrt{2}, 0)\}$

12.5 Second-Degree Inequalities and Systems of Inequalities

OBJECTIVES

1. Graph second-degree inequalities.
2. Graph the solution set of a system of inequalities.

OBJECTIVE 1 Graph second-degree inequalities. The linear inequality $3x + 2y \leq 5$ is graphed by first graphing the boundary line $3x + 2y = 5$. A **second-degree inequality** is an inequality with at least one variable of degree 2 and no variable with degree greater than 2. An example is $x^2 + y^2 \leq 36$. Such inequalities are graphed in the same way. The boundary of the inequality $x^2 + y^2 \leq 36$ is the graph of the equation $x^2 + y^2 = 36$, a circle with radius 6 and center at the origin, as shown in Figure 33.

The graph of the inequality $x^2 + y^2 \leq 36$ will include either the points outside the circle or the points inside the circle, as well as the boundary. We decide which region to shade by substituting any test point not on the circle, such as (0, 0), into the original inequality. Since $0^2 + 0^2 \leq 36$ is a true statement, the original inequality includes the points inside the circle, the shaded region in Figure 33, and the boundary.

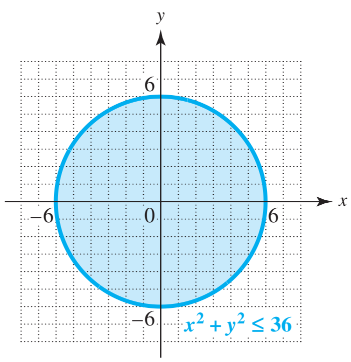


Figure 33

You Try It

Video

EXAMPLE 1 Graphing a Second-Degree Inequality

Graph $y < -2(x - 4)^2 - 3$.

The boundary, $y = -2(x - 4)^2 - 3$, is a parabola that opens down with vertex at $(4, -3)$. Using (0, 0) as a test point gives

$$0 < -2(0 - 4)^2 - 3 \quad ?$$
$$0 < -32 - 3 \quad ?$$
$$0 < -35. \quad \text{False}$$

Because the final inequality is a false statement, the points in the region containing (0, 0) do not satisfy the inequality. Figure 34 shows the final graph. The parabola is drawn as a dashed curve since the points of the parabola itself do not satisfy the inequality, and the region inside (or below) the parabola is shaded.

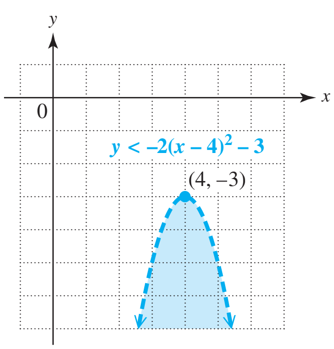
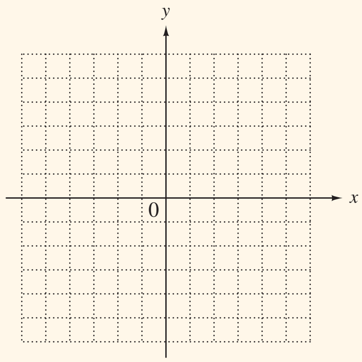
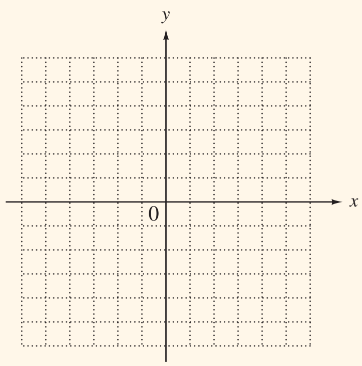


Figure 34

1 Graph $y \geq (x+1)^2 - 5$.



2 Graph $x^2 + 4y^2 > 36$.



NOTE

Since the substitution is easy, the origin is the test point of choice unless the graph actually passes through (0, 0).

Work Problem 1 at the Side.

EXAMPLE 2 Graphing a Second-Degree Inequality

Graph $16y^2 \leq 144 + 9x^2$.

First rewrite the inequality as follows.

$$16y^2 - 9x^2 \leq 144 \qquad \text{Subtract } 9x^2.$$

$$\frac{y^2}{9} - \frac{x^2}{16} \leq 1 \qquad \text{Divide by 144.}$$

This form shows that the boundary is the hyperbola given by

$$\frac{y^2}{9} - \frac{x^2}{16} = 1.$$

Since the graph is a vertical hyperbola, the desired region will be either the region between the branches or the regions above the top branch and below the bottom branch. Choose (0, 0) as a test point. Substituting into the original inequality leads to $0 \leq 144$, a true statement, so the region between the branches containing (0, 0) is shaded, as shown in Figure 35.

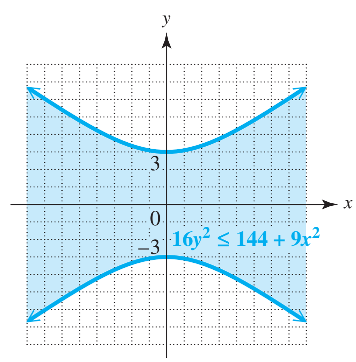


Figure 35

Work Problem 2 at the Side.

OBJECTIVE 2 Graph the solution set of a system of inequalities. If two or more inequalities are considered at the same time, we have a **system of inequalities.** To find the solution set of the system, we find the intersection of the graphs (solution sets) of the inequalities in the system.

EXAMPLE 3 Graphing a System of Two Inequalities

Graph the solution set of the system.

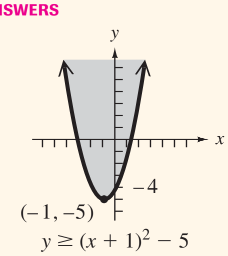
$$2x + 3y > 6$$
$$x^2 + y^2 < 16$$

Begin by graphing the solution set of $2x + 3y > 6$. The boundary line is the graph of $2x + 3y = 6$ and is a dashed line because of the symbol $>$. The test point (0, 0) leads to a false statement in the inequality $2x + 3y > 6$,

Continued on Next Page

ANSWERS

1.



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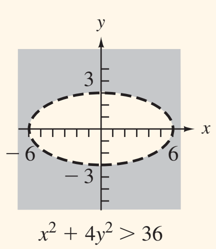


so shade the region above the line, as shown in Figure 36. The graph of $x^2 + y^2 < 16$ is the interior of a dashed circle centered at the origin with radius 4. This is shown in Figure 37.

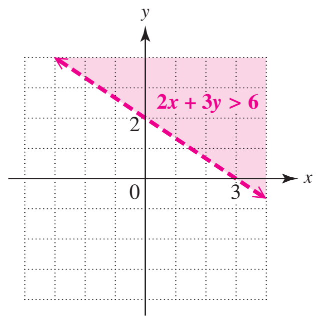


Figure 36

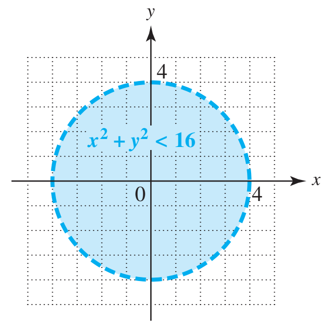


Figure 37

Finally, to show the graph of the solution set of the system, determine the intersection of the graphs of the two inequalities. The overlapping region in Figure 38 is the solution set.

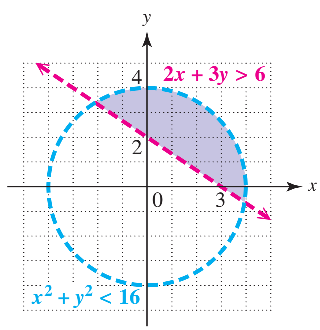


Figure 38

Work Problem 3 at the Side.

EXAMPLE 4 Graphing a Linear System with Three Inequalities

Animation

Video

Graph the solution set of the system.

$$x + y < 1$$
$$y \leq 2x + 3$$
$$y \geq -2$$

Graph each inequality separately, on the same axes. The graph of $x + y < 1$ consists of all points below the dashed line $x + y = 1$. The graph of $y \leq 2x + 3$ is the region that lies below the solid line $y = 2x + 3$. Finally, the graph of $y \geq -2$ is the region above the solid horizontal line $y = -2$. The graph of the system, the intersection of these three graphs, is the triangular region enclosed by the three boundary lines in Figure 39, including two of its boundaries.

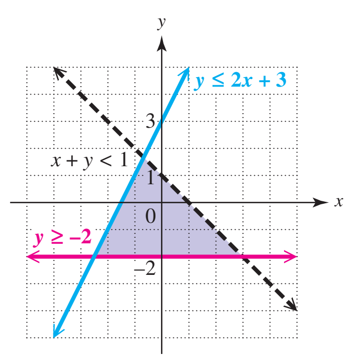
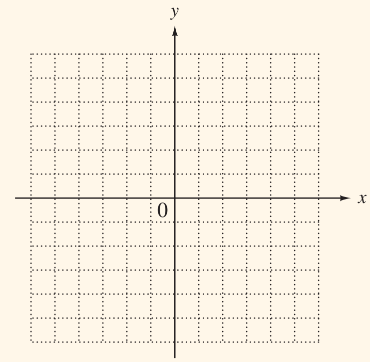


Figure 39

Work Problem 4 at the Side.

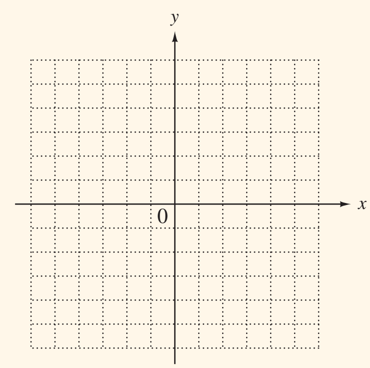
3 Graph the solution set of the system.

$$x^2 + y^2 \leq 25$$
$$x + y \leq 3$$



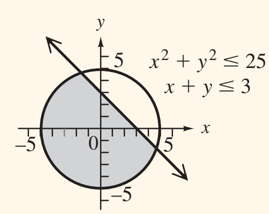
4 Graph the solution set of the system.

$$3x - 4y \geq 12$$
$$x + 3y \geq 6$$
$$y \leq 2$$

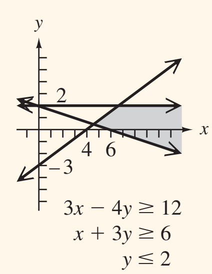


ANSWERS

3.

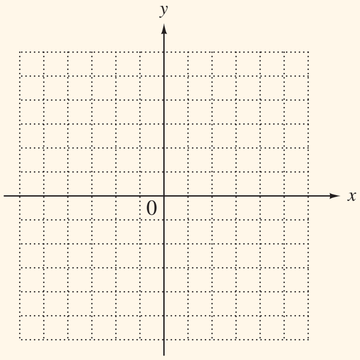


4.



5 Graph the solution set of the system.

$$y \geq x^2 + 1$$
$$\frac{x^2}{9} + \frac{y^2}{4} \geq 1$$
$$y \leq 5$$



EXAMPLE 5 Graphing a System with Three Inequalities

Graph the solution set of the system.

$$y \geq x^2 - 2x + 1$$
$$2x^2 + y^2 > 4$$
$$y < 4$$

The graph of $y = x^2 - 2x + 1$ is a parabola with vertex at (1, 0). Those points above (or in the interior of) the parabola satisfy the condition $y > x^2 - 2x + 1$. Thus, points on the parabola or in the interior are in the solution set of $y \geq x^2 - 2x + 1$.

The graph of the equation $2x^2 + y^2 = 4$ is an ellipse. We draw it as a dashed curve. To satisfy the inequality $2x^2 + y^2 > 4$, a point must lie outside the ellipse.

The graph of $y < 4$ includes all points below the dashed line $y = 4$. Finally, the graph of the system is the shaded region in Figure 40 that lies outside the ellipse, inside or on the boundary of the parabola, and below the line $y = 4$.

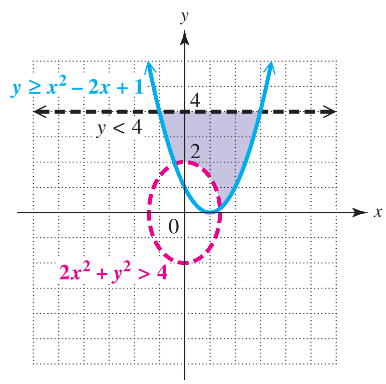


Figure 40

Work Problem 5 at the Side.

ANSWERS

5.