Application Demonstration Maplesoft

THE SOLUTION OF A VARIABLE BOUNDARY PROBLEM

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NOTE:

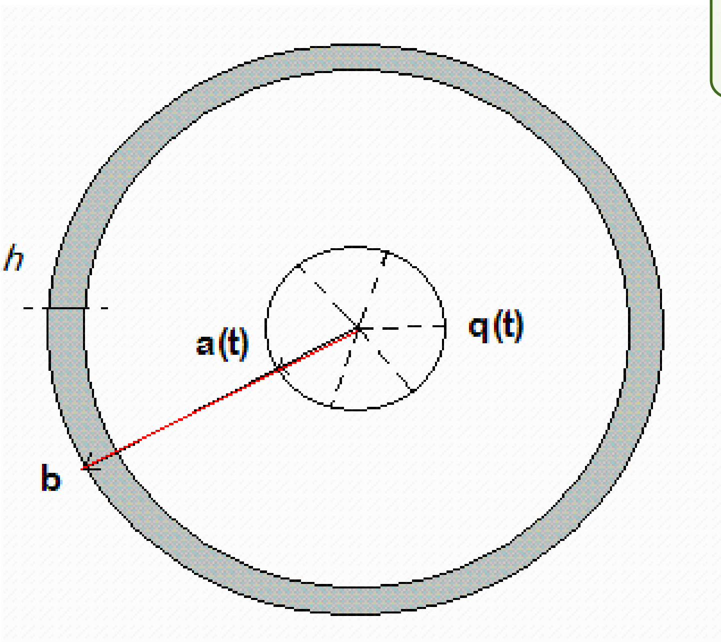
This worksheet demonstrates Maple's capabilities in researching the numerical and graphical solution of the variable boundary problem of a thick-walled cylinder of material enclosed in a thin metallic shell .

Abstract :

The relations between stress and strain in linear viscoelastic theory are discussed from the viewpoint of application to problems of stress analysis. This consideration includes some important differences from the estimation of linear viscoelastic laws for the representation of material properties , and the integral operators expressing the creep function or relaxation function can be applied . By using of the differential operator for the relation between stress and strain it is usually most convenient to solve some problems which have the variable boundary .

Problem Definition

We consider a thick-walled cylinder of material enclosed in a thin metallic shell . (Fig. 1)



(Fig 1) .

The outer radius of cylinder : *b* ; the thickness of the metallic shell : *h* .

The inner radius a is assumed by : *a(t)* , satisfied the condition $\frac{da}{dt} > 0$. The varying pressure influences upon the inner surface is a given function q(t) .
Notice that the tube and shell are under the plain strain conditions and the material is incompressible.

By setting σ_r : the radical stress , σ_θ : the circular stress ,
$u(r,t)$: the radical displacement ,
$e_r = \frac{du}{dr}$, $e_\theta = \frac{u}{r}$ are the deformation respectively .

The diferential equation of equilibrium:

Bywhat we introduce in (I) it follows that the diferential equation of equilibrium can be written consequently :

$$\frac{d\sigma}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (1)$$

In this case we have the law of viscoelasticity relation :

$$\sigma_\theta - \sigma_r = 2\widetilde{G}(e_\theta - e_r) \qquad (2)$$

At the inner boundary : r = a , σ_r = -q(t)
The circular deformation of the cylinder of material obtained by integrating :

$$e_\theta(b) = -\frac{b}{h}\frac{1-\upsilon^2}{E}\sigma_r(b) \qquad (3)$$

Here *E* and υ are the elastic constants of the shell material .

The second boundary condition : e = e_θ (b)
Because of incompressibility of material and zero axial strain :

$$e_\theta + e_r = \frac{du}{dr} + \frac{u}{r} = 0$$, it follows that : $u(r,t) = \frac{k(t)}{r}$, $e_r = -e_\theta = -\frac{k(t)}{r^2}$ (4.29)

From (2) we obtain :
$$\sigma_\theta - \sigma_r = 4\tilde{G}\frac{k(t)}{r^2} \tag{4}$$

By substituting (4) to (1) , the deduction can be rewritten :
$$\frac{d\sigma}{dr} - 4\tilde{G}\frac{k(t)}{r^3} = 0 \tag{5}$$

Integrating (5) gives :
$$\sigma(r,t) = -\frac{2\tilde{G}k(t)}{r^2} + k_1(t) \tag{6}$$

By using the condition : *r = a* , σ_r = *- q (t)*

then
$$-q(t) = -\frac{2\tilde{G}k(t)}{a^2(t)} + k_1(t)$$
and
$$k_1(t) = \frac{2\tilde{G}k(t)}{a^2(t)} - q(t) \tag{7}$$

The relation (6) becomes :
$$\sigma_r(r,t) = 2\left[\frac{1}{a^2(t)} - \frac{1}{r^2}\right]\tilde{G}k(t) - q(t) \tag{8}$$

Consider the second boundary condition : $e = -\frac{b}{h}\frac{1-\upsilon^2}{E}\sigma_r(b)$, then we arrive at the integral equation for k(t) :
$$\frac{1}{c}\frac{k(t)}{b^2} = 2\left[\frac{1}{a^2(t)} - \frac{1}{b^2}\right]\tilde{G}k(t) - q(t) \tag{9}$$

inwhich we set up :
$$\frac{1}{c}\frac{k(t)}{b^2} = 2\left[\frac{1}{a^2(t)} - \frac{1}{b^2}\right]\tilde{G}k(t) - q(t) \tag{10}$$

. The equation (9) can be considered as especial case of the general formation :
$$K(t,\tau) = g(t)\phi(t-\tau) \tag{11}$$

The authors of [2] have considered the simplest case of a viscoelastic body expressed by formular :
$$2\tilde{G} = \alpha\frac{\partial}{\partial t} + \beta \tag{12}$$

Here α , β are the given constants defined by experiments on material . The equation (9) now becomes differential for *k(t)* .
$$\alpha\left[\frac{1}{a^2(t)} - \frac{1}{b^2}\right]\frac{d}{dt}k(t) + \left[\beta\left(\frac{1}{a^2(t)} - \frac{1}{b^2}\right) - \frac{1}{b^2c}\right]k(t) = q(t) \tag{13}$$

From (13) , we find out the solution k(t) correspondently with choosing the functions *a(t) and q(t)* . For examples , some authors have taken :

$$a(t) = \frac{a_0}{\sqrt{1-\delta t}} \qquad (14)$$

Here q = const . [δ = E in case of elasticity , δ = 3 η in case of Newton's fluid] . By integrating (13) we obtain the solution of the viscoelasticity problem . In generally the differential equation for *k(t)* would be established in symbolic form by using the operator :

$$\tilde{G} = \frac{B(I_o^*)}{A(I_0^*)} \qquad (15)$$

Numerical and graphical solution:

Here we have used the parameters :

$t_0 := 6$ $\quad c := 1.2$ $\quad \delta := 0.00005$ $\quad s := 1.250000000$ $\quad h := 0.250000000$

$\alpha := 0.5$ $\quad b := 1.5$ $\quad a_0 := 1$ $\quad \beta := 2$ $\quad q_0 := 1$ $\quad k(0) = 0.1$;

(13) canbe rewitten as the following :

$$eq := \alpha \left(\frac{1-\delta t}{a_0^2} - \frac{1}{b^2} \right) \left(\frac{d}{dt} k(t) \right) + \left(\beta \left(\frac{1-\delta t}{a_0^2} - \frac{1}{b^2} \right) - \frac{1}{b^2 c} \right) k(t) = q_0$$

The analytic solution :

$$k(t) = \frac{\left(\int \frac{a_0^2 q_0 (-b^2 + b^2 \delta t + a_0^2)^{\left(\frac{c \alpha b^2 \delta + a_0^2}{c \alpha b^2 \delta} \right)}}{\alpha \, b^2 \, e^{\left(\frac{\beta t}{\alpha} \right)}} dt + \_C1 \right) e^{\left(-\frac{\beta t}{\alpha} \right)}}{(-b^2 + b^2 \delta t + a_0^2)^{\left(-\frac{a_0^2}{c \alpha b^2 \delta} \right)}}$$

the expression of function k(t) :

$$k(t) = \frac{\left(\int -4.50 \, (-1.25 + 0.0112 \, t)^{147} \, e^{(4.00 \, t)} \, dt - \_C1 \right) e^{(-4.00 \, t)}}{(-1.25 + 0.0112 \, t)^{148}}$$

from the given condition $k(0) = 0.1$ we obtain :

$$k0 := 0.4542742027 \; 10^{-14} \, (\_C1 + 0.2949026378 \; 10^{15} \, e^{0.} - 0.2949026378 \; 10^{15}) \, e^{0.}$$

$$edk := 0.4542742027 \; 10^{-14} \, \_C1 = 0.1$$

$$\_C1 = 0.22013 \; 10^{14}$$

' NUMERICAL AND GRAPHICAL SOLUTION "

' OUTPUT DATA '

$N := 10$, $M := 6$, $d := 1$

' NUMERICAL AND GRAPHICAL SOLUTION "

' OUTPUT DATA '

$N = 10$

$M = 6$

$d = 1$

$r := 0.125$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00080	0.00640
1.0	1.27172	0.01017	0.08139
2.0	1.37028	0.01096	0.08770
3.0	1.39498	0.01116	0.08928
4.0	1.41500	0.01132	0.09056
5.0	1.43504	0.01148	0.09184
6.0	1.45493	0.01164	0.09312

$r := 0.250$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00040	0.00160
1.0	1.27172	0.00509	0.02035
2.0	1.37028	0.00548	0.02192
3.0	1.39498	0.00558	0.02232
4.0	1.41500	0.00566	0.02264
5.0	1.43504	0.00574	0.02296
6.0	1.45493	0.00582	0.02328

$r := 0.375$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00027	0.00071
1.0	1.27172	0.00339	0.00904
2.0	1.37028	0.00365	0.00974
3.0	1.39498	0.00372	0.00992
4.0	1.41500	0.00377	0.01006
5.0	1.43504	0.00383	0.01020
6.0	1.45493	0.00388	0.01035

$r := 0.500$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00020	0.00040
1.0	1.27172	0.00254	0.00509
2.0	1.37028	0.00274	0.00548
3.0	1.39498	0.00279	0.00558
4.0	1.41500	0.00283	0.00566

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00020	0.00040
1.0	1.27172	0.00254	0.00509
2.0	1.37028	0.00274	0.00548
3.0	1.39498	0.00279	0.00558
4.0	1.41500	0.00283	0.00566
5.0	1.43504	0.00287	0.00574
6.0	1.45493	0.00291	0.00582

$r := 0.625$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00013	0.00018
1.0	1.27172	0.00170	0.00226
2.0	1.37028	0.00183	0.00244
3.0	1.39498	0.00186	0.00248
4.0	1.41500	0.00189	0.00252
5.0	1.43504	0.00191	0.00255
6.0	1.45493	0.00194	0.00259

$r := 0.875$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00011	0.00013
1.0	1.27172	0.00145	0.00166
2.0	1.37028	0.00157	0.00179
3.0	1.39498	0.00159	0.00182
4.0	1.41500	0.00162	0.00185
5.0	1.43504	0.00164	0.00187
6.0	1.45493	0.00166	0.00190

$r := 1.00$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00010	0.00010
1.0	1.27172	0.00127	0.00127
2.0	1.37028	0.00137	0.00137
3.0	1.39498	0.00139	0.00139
4.0	1.41500	0.00141	0.00141
5.0	1.43504	0.00144	0.00144
6.0	1.45493	0.00145	0.00145

$r := 1.12$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00009	0.00008
1.0	1.27172	0.00114	0.00101
2.0	1.37028	0.00122	0.00109
3.0	1.39498	0.00125	0.00111
4.0	1.41500	0.00126	0.00113
5.0	1.43504	0.00128	0.00114
6.0	1.45493	0.00130	0.00116

$r := 1.25$

t	k(t)	u(r,t)	e[theta]
0.0	0.10000	0.00008	0.00006
1.0	1.27172	0.00102	0.00081
2.0	1.37028	0.00110	0.00088
3.0	1.39498	0.00112	0.00089
4.0	1.41500	0.00113	0.00091
5.0	1.43504	0.00115	0.00092
6.0	1.45493	0.00116	0.00093

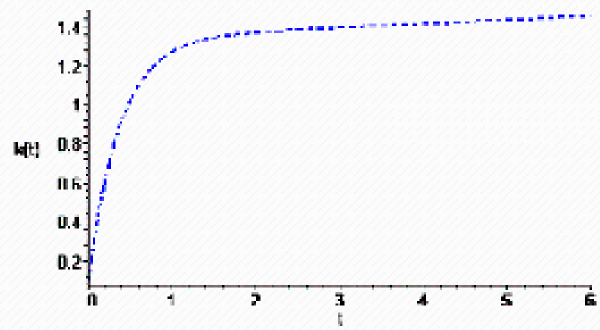


Fig.2 the diagram of $k(t)$

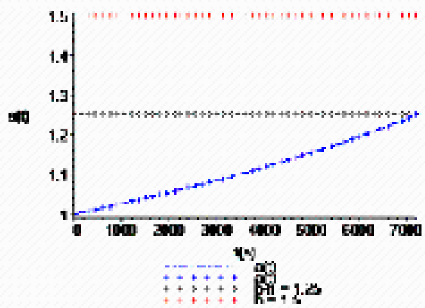


Fig.3 the diagram of $a(t)$

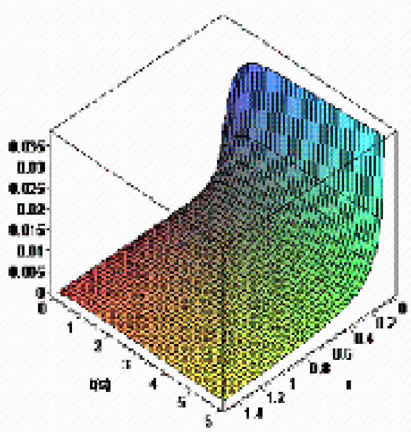


Fig.4 3D-graphical solution of $u(r,t)$

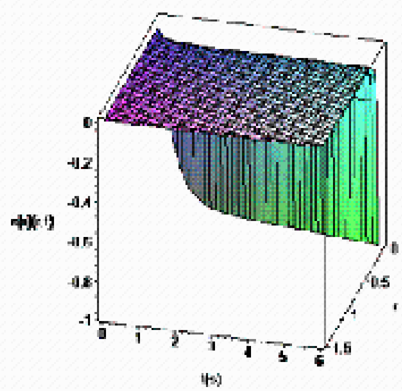


Fig.5 . 3D-graphical solution of $e_r(r,t)$

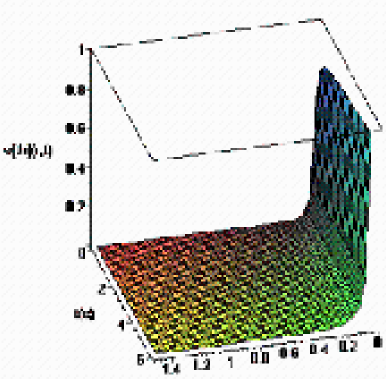


Fig.6 3D-graphical solution of $e_\theta(r,t)$

```
> restart;
with(DEtools):
a(t)=a[0]/sqrt(1-delta*t);
eq:=alpha*(1/a(t)^2-1/b^2)*diff(k(t),t)+(beta*(1/a(t)^2-1/b^2)-
1/(b^2*c))*k(t)=q(t);
eq:=subs(a(t)=a[0]/sqrt(1-delta*t),eq);
eq:=subs(q(t)=q[0],eq);
nok:=dsolve(eq,k(t));
```

$$a(t) = \frac{a_0}{\sqrt{1-\delta\, t}}$$

$$eq := \alpha \left(\frac{1}{a(t)^2} - \frac{1}{b^2}\right)\left(\frac{d}{dt} k(t)\right) + \left(\beta\left(\frac{1}{a(t)^2} - \frac{1}{b^2}\right) - \frac{1}{b^2\, c}\right) k(t) = q(t)$$

$$eq := \alpha \left(\frac{1-\delta\, t}{a_0^2} - \frac{1}{b^2}\right)\left(\frac{d}{dt} k(t)\right) + \left(\beta\left(\frac{1-\delta\, t}{a_0^2} - \frac{1}{b^2}\right) - \frac{1}{b^2\, c}\right) k(t) = q(t)$$

$$eq := \alpha \left(\frac{1-\delta\, t}{a_0^2} - \frac{1}{b^2}\right)\left(\frac{d}{dt} k(t)\right) + \left(\beta\left(\frac{1-\delta\, t}{a_0^2} - \frac{1}{b^2}\right) - \frac{1}{b^2\, c}\right) k(t) = q_0$$

$$nok := k(t) = \left(\int -\frac{q_0\, a_0^2\, b^2 \left(-b^2 + b^2 \delta\, t + a_0^2\right)^{\left(-\frac{c\,\alpha\, b^2\,\delta - a_0^2}{c\,\alpha\, b^2\,\delta}\right)} e^{\left(\frac{\beta\, t}{\alpha}\right)}}{\alpha}\, dt + \_C1 \right) e^{\left(-\frac{\beta\, t}{\alpha}\right)} \left(-b^2 + b^2\,\delta\, t + a_0^2\right)^{\left(-\frac{a_0^2}{c\,\alpha\, b^2\,\delta}\right)}$$

```
> restart;
with(DEtools):
cof := 1000:
eq := alpha*(1/a(t)^2-1/b^2)*(diff(k(t), t))+(beta*(1/a(t)^2-1/b^2)-
1/(b^2*c))*k(t) = q(t);eq := subs(a(t) = a[0]/sqrt(1-delta*t), eq);
eq := subs(q(t) = q[0], eq);
nok := dsolve(eq, k(t));
nok := evalf(subs(c = 1.2, delta = 0.5e-2, alpha = .5, b = 1.5, a[0] = 1,
beta = 2, q[0] = 1, nok), 3);
u := k(t)/(r*cof);
```

$$cof := 1000$$

$$eq := \alpha \left(\frac{1}{a(t)^2} - \frac{1}{b^2}\right) \left(\frac{d}{dt} k(t)\right) + \left(\beta \left(\frac{1}{a(t)^2} - \frac{1}{b^2}\right) - \frac{1}{b^2\, c}\right) k(t) = q(t)$$

$$eq := \alpha \left(\frac{1 - \delta\, t}{a_0^2} - \frac{1}{b^2}\right) \left(\frac{d}{dt} k(t)\right) + \left(\beta \left(\frac{1 - \delta\, t}{a_0^2} - \frac{1}{b^2}\right) - \frac{1}{b^2\, c}\right) k(t) = q(t)$$

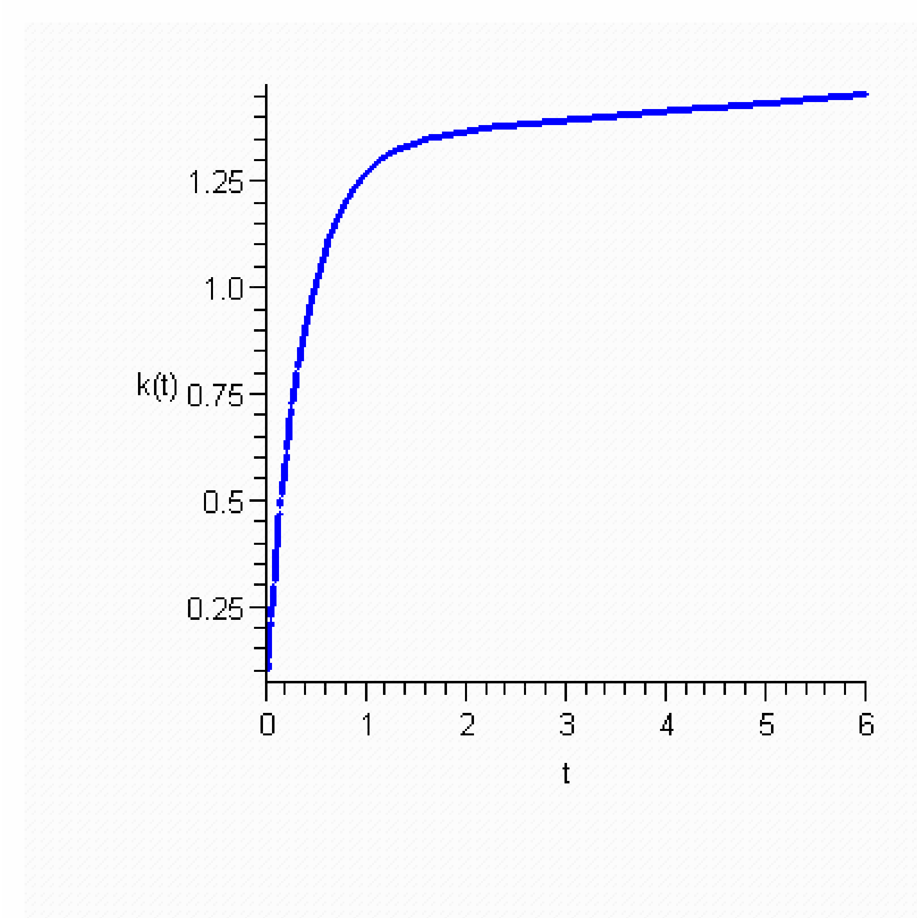
$$eq := \alpha \left(\frac{1 - \delta\, t}{a_0^2} - \frac{1}{b^2} \right) \left(\frac{\mathrm{d}}{\mathrm{d}t}\, k(t) \right) + \left(\beta \left(\frac{1 - \delta\, t}{a_0^2} - \frac{1}{b^2} \right) - \frac{1}{b^2\, c} \right) k(t) = q_0$$

$$nok := k(t) = \left(\int - \frac{b^2 \left(-b^2 + b^2\, \delta\, t + a_0^2 \right)^{\left(\frac{-c\, \alpha\, b^2\, \delta + a_0^2}{c\, \alpha\, b^2\, \delta} \right)} a_0^2\, q_0\, \mathrm{e}^{\left(\frac{\beta\, t}{\alpha} \right)}}{\alpha}\, \mathrm{d}t + \_C1 \right) \mathrm{e}^{\left(-\frac{\beta\, t}{\alpha} \right)} \left(-b^2 + b^2\, \delta\, t + a_0^2 \right)^{\left(-\frac{a_0^2}{c\, \alpha\, b^2\, \delta} \right)}$$

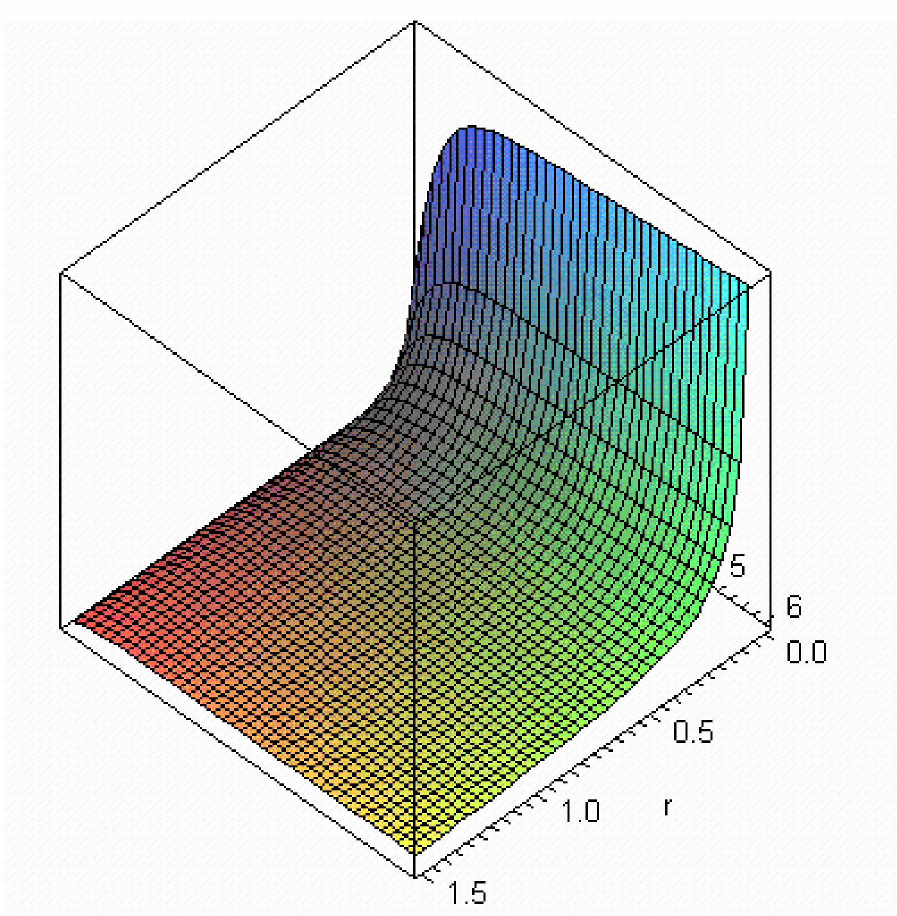
$$nok := k(t) = \frac{1}{(-1.25 + 0.0112\, t)^{148.}} \left(\left(\int - 4.50\, (-1.25 + 0.0112\, t)^{147.}\, \mathrm{e}^{(4.00\, t)}\, \mathrm{d}t + \_C1 \right) \mathrm{e}^{(-4.00\, t)} \right)$$

$$u := \frac{1}{1000} \frac{k(t)}{r}$$

```
> k(t):=(int(-4.50*(-1.25+.112e-
1*s)^147.*exp(4.00*s),s=0..t)+.22013e14)*exp(-4.00*t)/(-1.25+.112e-
1*t)^148.:
plot(k(t),t=0..6,labels=[t,"k(t)"],color=blue,style=line,linestyle=4,thickn
ess=2);
```



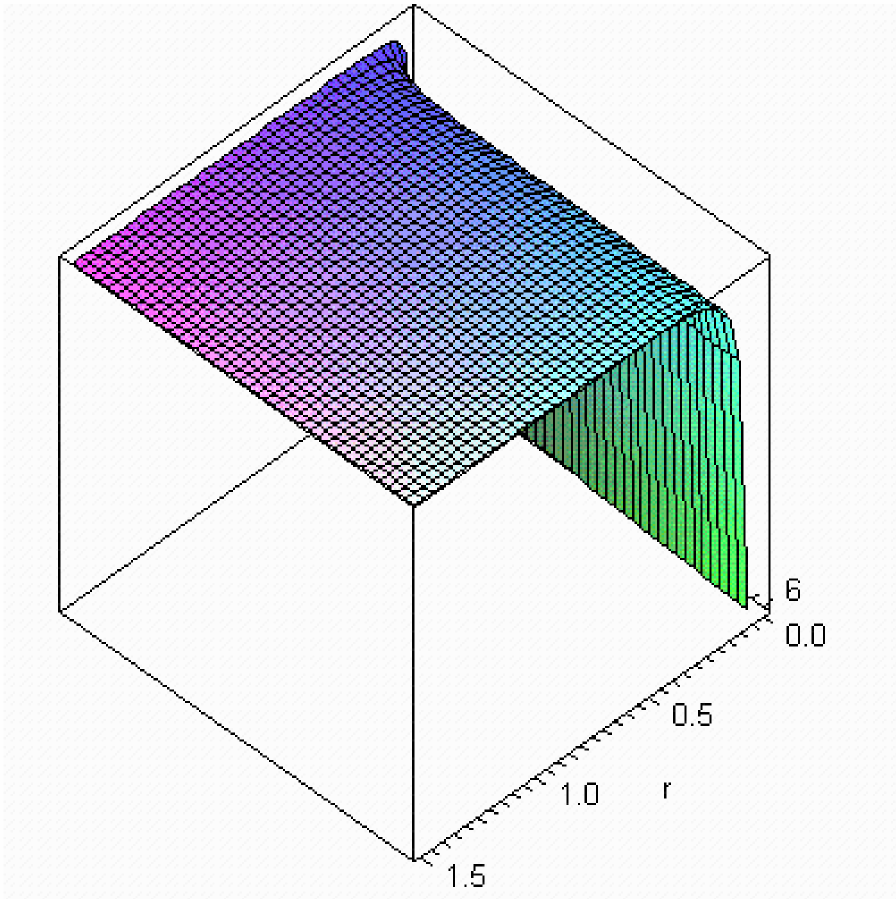

```
> plot3d(u, r = 0 .. 1.5, t = 0 .. 6, axes = boxed, labels = ["r", "t(s)",
"u(r,t)"], grid = [40, 40]);
```



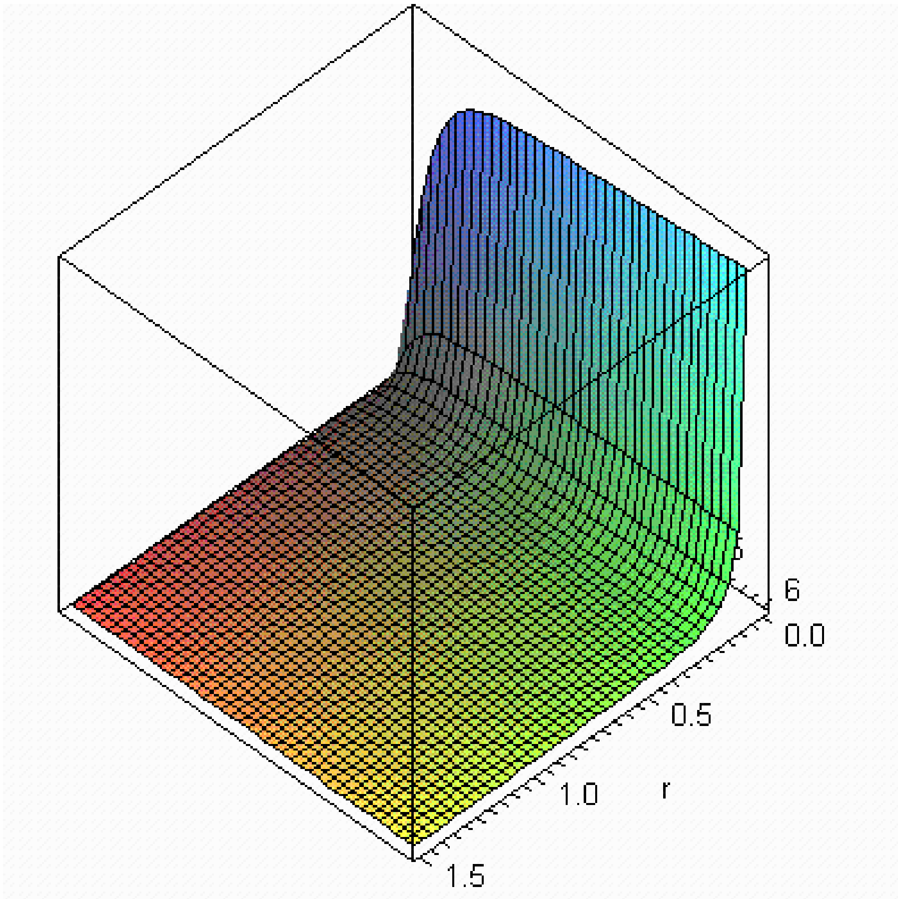

```
> epsilon[r] := -k(t)/r^2;
plot3d(epsilon[r]/cof, r = 0 .. 1.5, t = 0 .. 6, axes = boxed, labels =
["r", "t(s)", "e[r](r,t)"], grid = [40, 40]);
```

APPROVED

By CO HONG TRAN at 12:06 pm, Dec 02, 2007




```
> epsilon[theta] := k(t)/r^2;
plot3d(epsilon[theta]/cof, r = 0 .. 1.5, t = 0 .. 6, axes = boxed, labels =
["r", "t(s)", "e[th](r,t)"], grid = [40, 40]);
```




```
>
> restart;
k(t) = (Int(-4.50*(-1.25+0.112e-1*t)^147.*exp(4.00*t), t)+_C1)*exp(-4.00*t)/(-1.25+0.112e-1*t)^148.;
```

$$k(t) = \frac{1}{(-1.25 + 0.0112\,t)^{148.}} \left(\left(\int -4.50\,(-1.25 + 0.0112\,t)^{147.}\, e^{(4.00\,t)}\, dt + \_C1 \right) e^{(-4.00\,t)} \right)$$

```
> k(t):=(int(-4.50*(-1.25+.112e-1*tau)^147.*exp(4.00*tau),tau=0..t)+.22013e14)*exp(-4.00*t)/(-1.25+.112e-1*t)^148.:;k0:=subs(t=0,k(t));
```

$$k0 := 4.542742027\ 10^{-15}\ (-2.728896378\ 10^{14} + 2.949026378\ 10^{14}\ e^{0.})\ e^{0.}$$

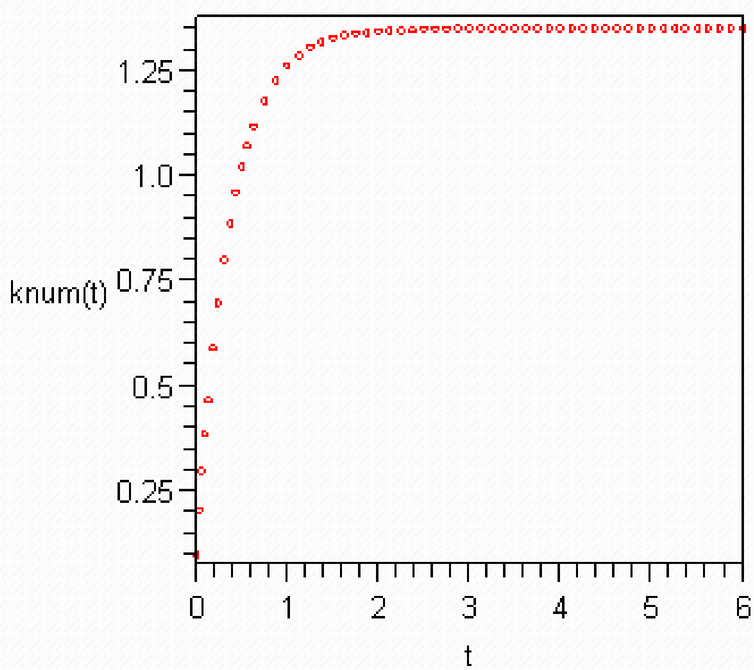
```
> edk := k0 = .1;
C1 = evalf(solve(edk, _C1), 5);
```

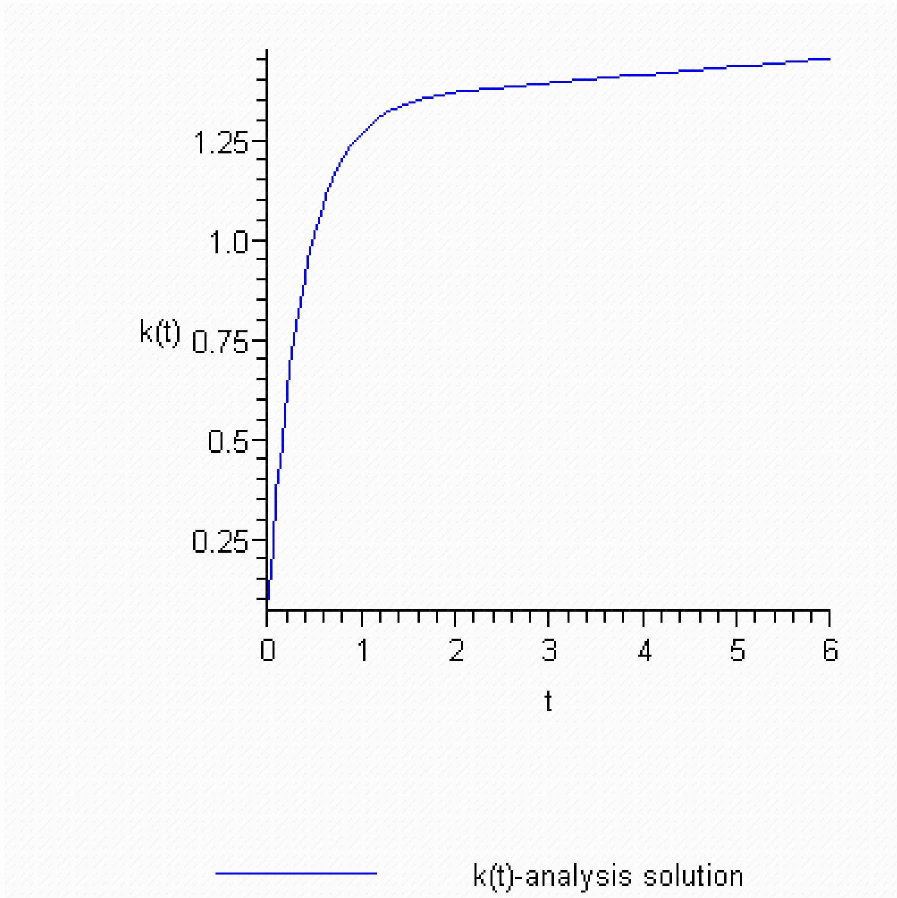
$$edk := 0.09999938024 = 0.1$$

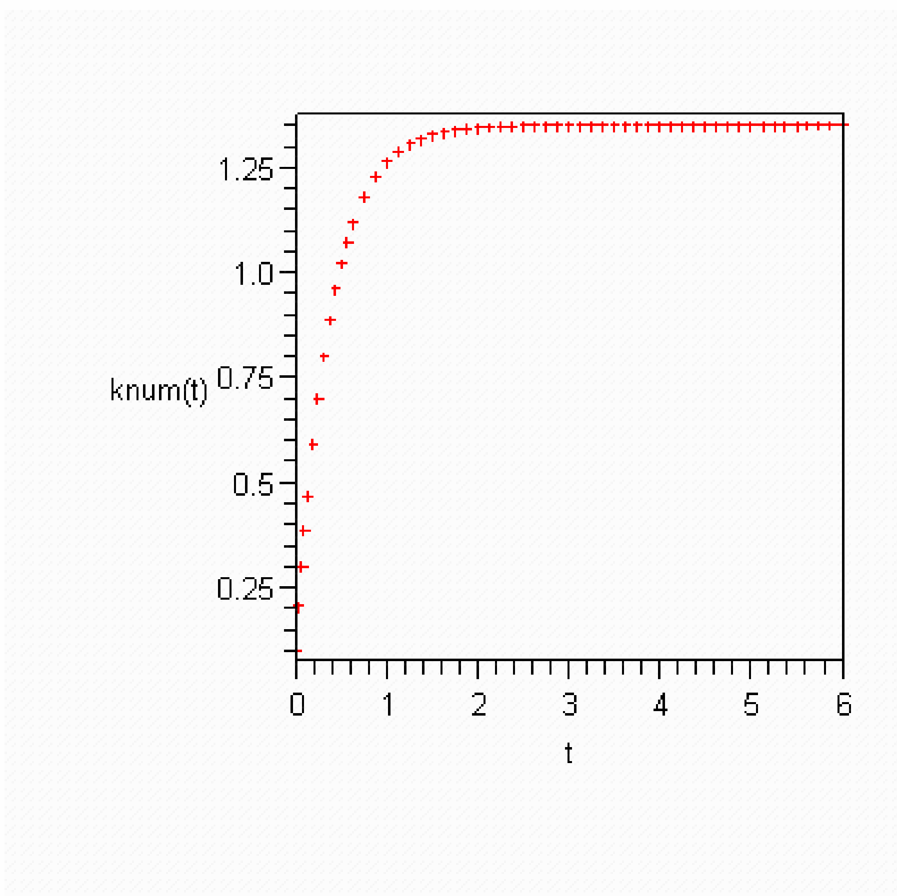
$$C1 = (\,)$$

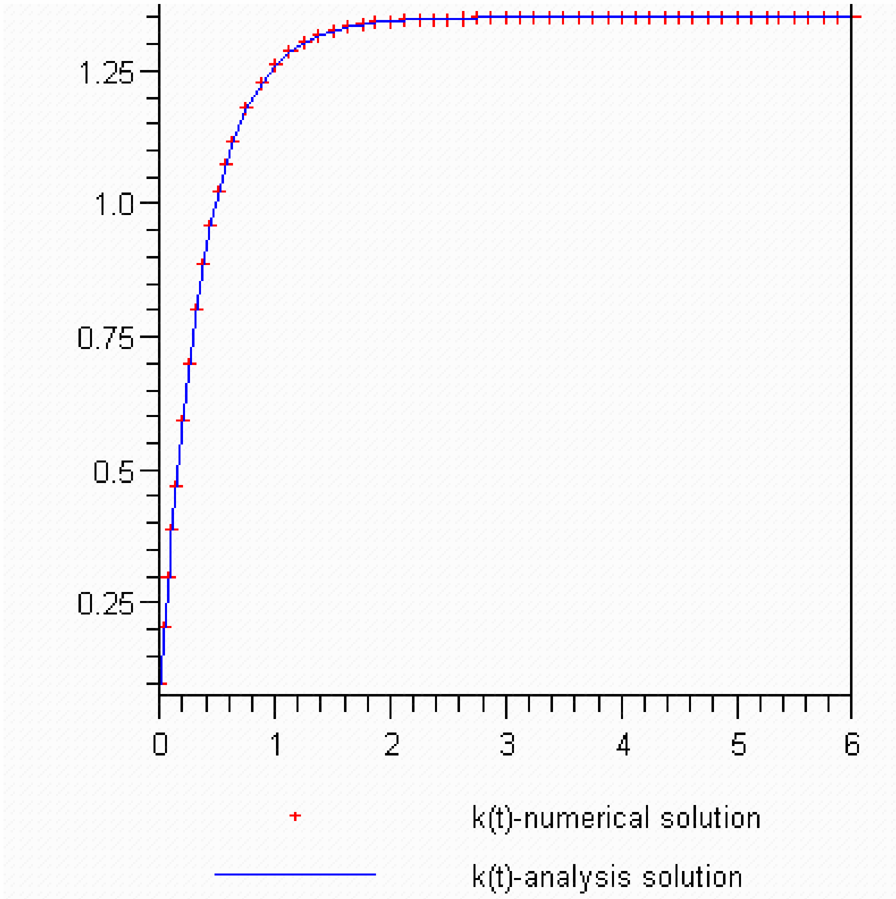
```
> u:=k(t)/r/10^3:
```

```
epsilon[theta]:=k(t)/r^2/10^3:
epsilon[rt]:=-k(t)/r^2/10^3:
print(" NUMERICAL AND GRAPHICAL SOLUTION ");
print("   OUTPUT   DATA  ");
N:=10;
M:=6;
d:=1;

> for m from 1 to N  do
r:=evalf(1.25*m/N,3);
u:=k(t)/r/10^3:
epsilon[theta]:=k(t)/r^2/10^3:
epsilon[rt]:=-k(t)/r^2/10^3:
printf("        t           k(t)            u(r,t)          e[theta]
\n\n");
for j from 0 to M do
printf("%10.1f    %10.5f       %10.5f       %10.5f       \n", d*j,
subs(t=d*j,k(t)), subs(t=d*j,u), subs(t=d*j,epsilon[theta]));
end do;
end do;
```




1.25
1.0
k(t)
0.75
0.5
0.25
0
1
2
3
4
5
6
t
k(t)-analysis solution

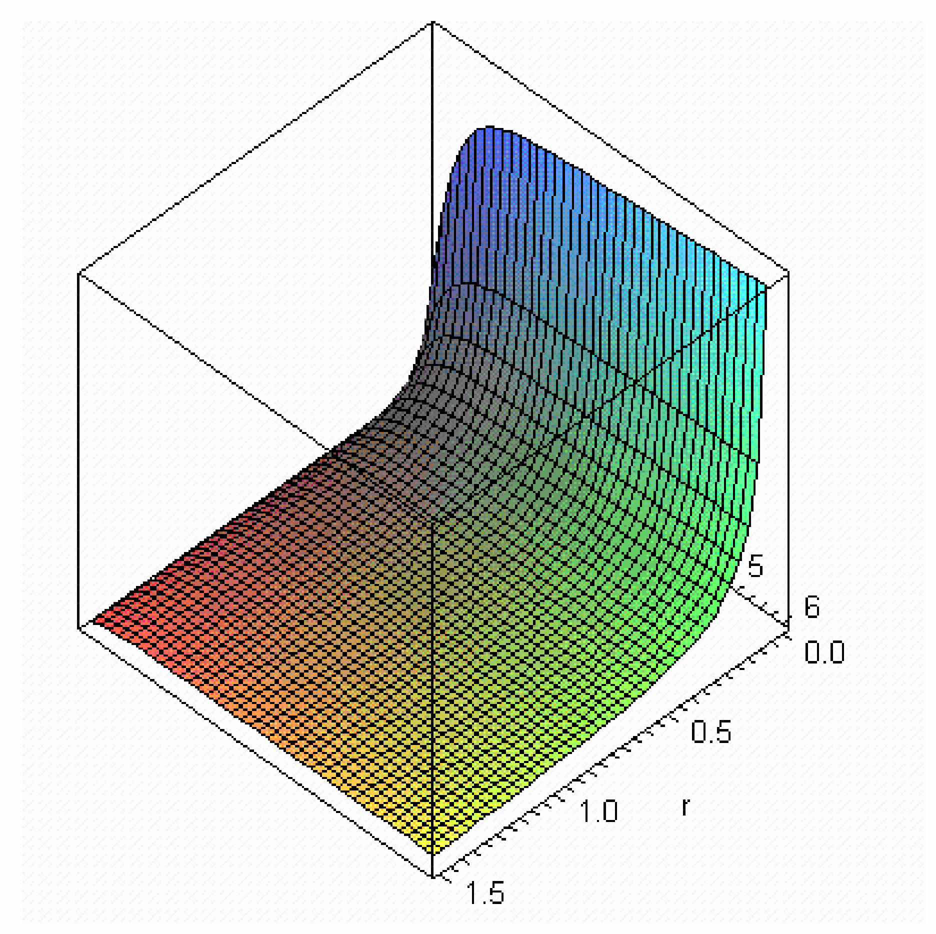

1.25
1.0
knum(t)
0.75
0.5
0.25
0
1
2
3
4
5
6
t

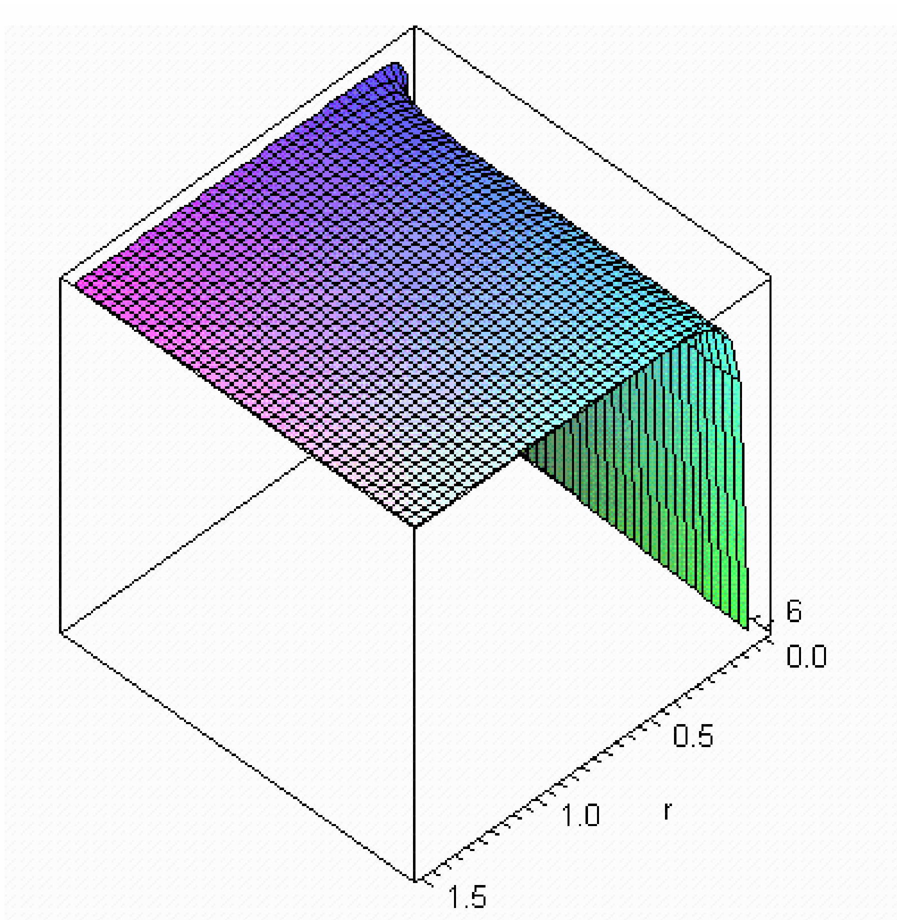


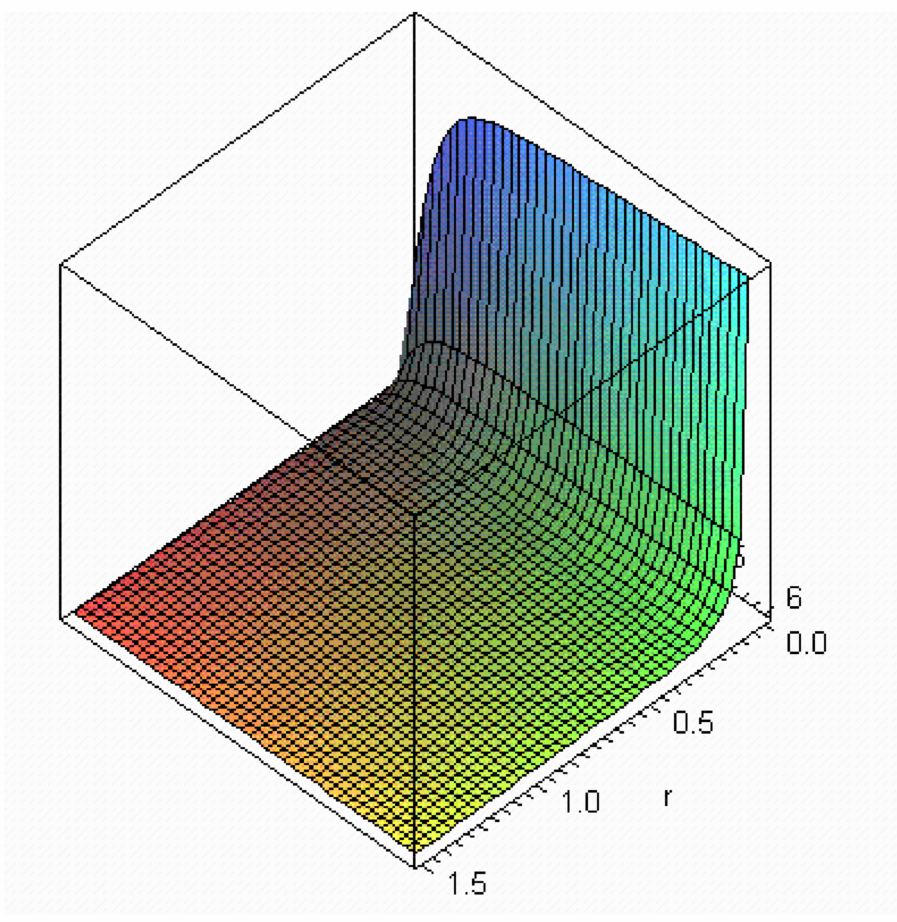
$$unum := \frac{knum}{r}$$

```
>
plot3d(u,r=0..1.5,t=0..6,axes=boxed,labels=["r","t(s)","u(r,t)"],grid=[40,40]);

cof:=10^3:
epsilon[r]:=-k(t)/r^2:

plot3d(epsilon[r]/cof,r=0..1.5,t=0..6,axes=boxed,labels=["r","t(s)","e[r](r,t)"],grid=[40,40]);
epsilon[theta]:=k(t)/r^2:

plot3d(epsilon[theta]/cof,r=0..1.5,t=0..6,axes=boxed,labels=["r","t(s)","e[th](r,t)"],grid=[40,40]);

with(plots):
N:=3;

for j from 0 to N do
a[0]:=1/(j+1):
plot([a[0]/sqrt(1-delta*t),a[0]/sqrt(1-delta*t),1.25,1.5],t=0..(1.95+j)*3600,color=[blue,blue,black,red],thickness=[1,2,2,2],style=[line,point,point,point],labels=["t(s)","a(t)"],symbol=[diamond,cross],linestyle=4,legend=[`a(t)`,`a(t)`,`b-h = 1.25`,`b = 1.5`]);
```

```
od;
```

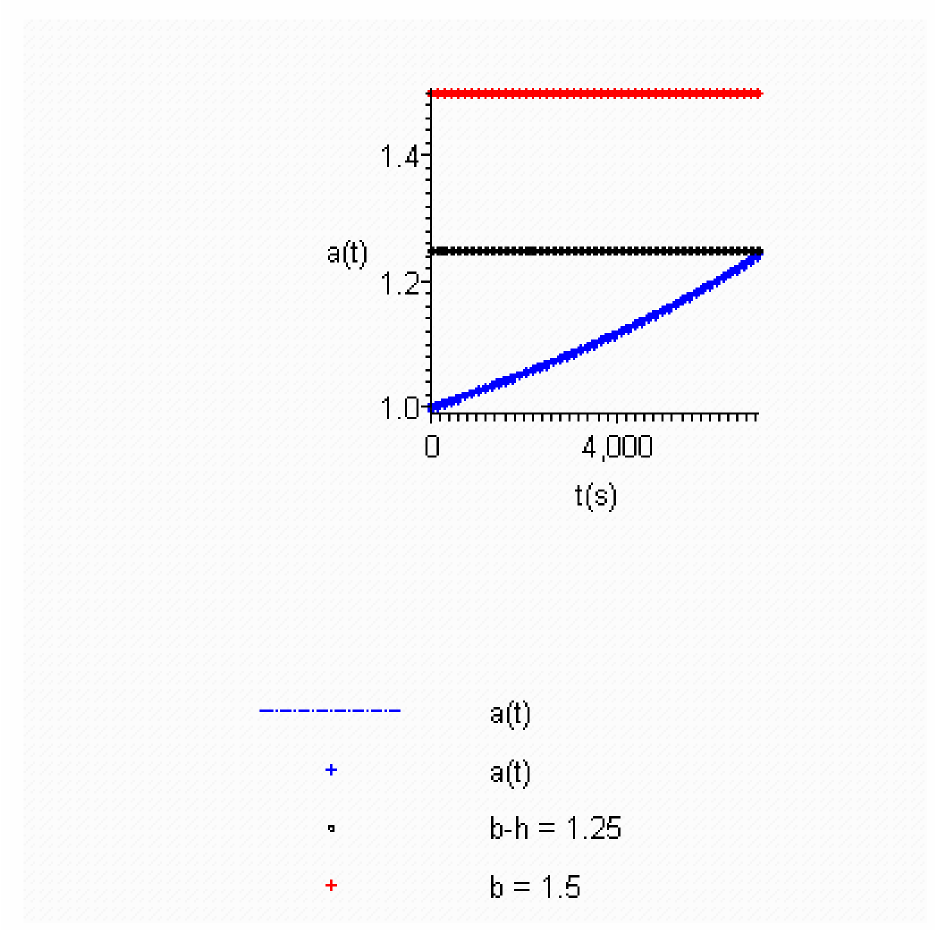



6
0.0
0.5
r
1.0
1.5

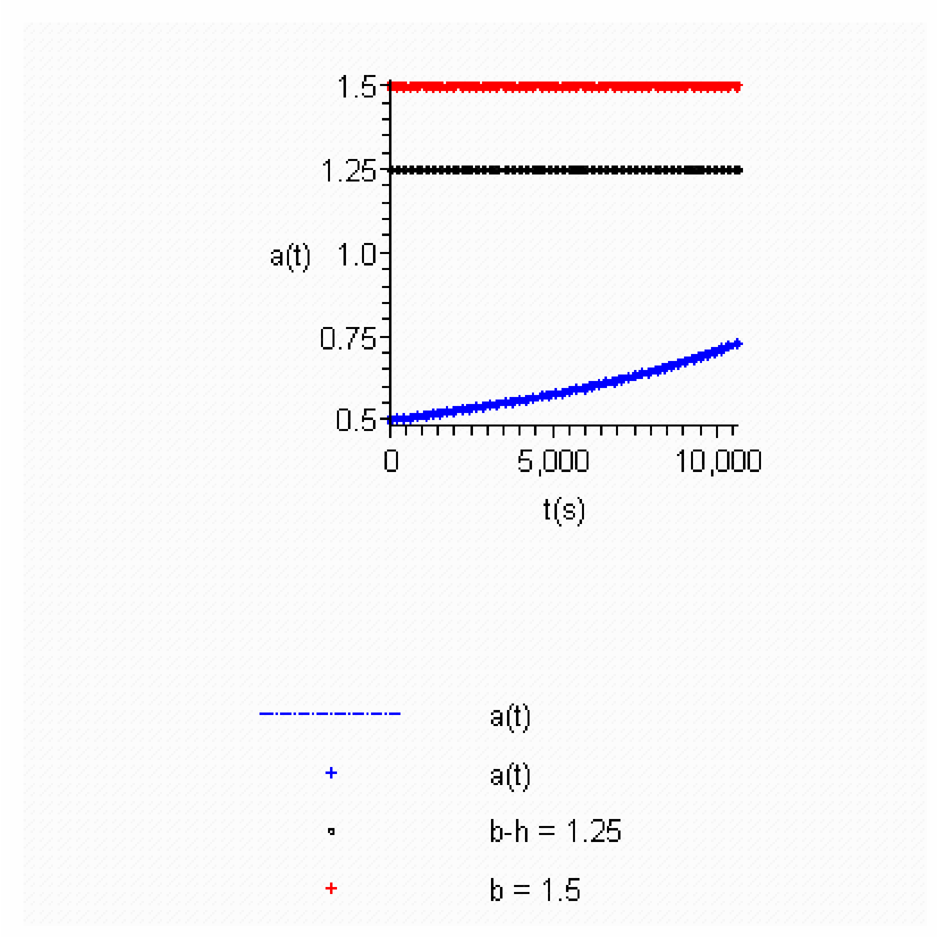

6
0.0
0.5
r
1.0
1.5

$N := 3$

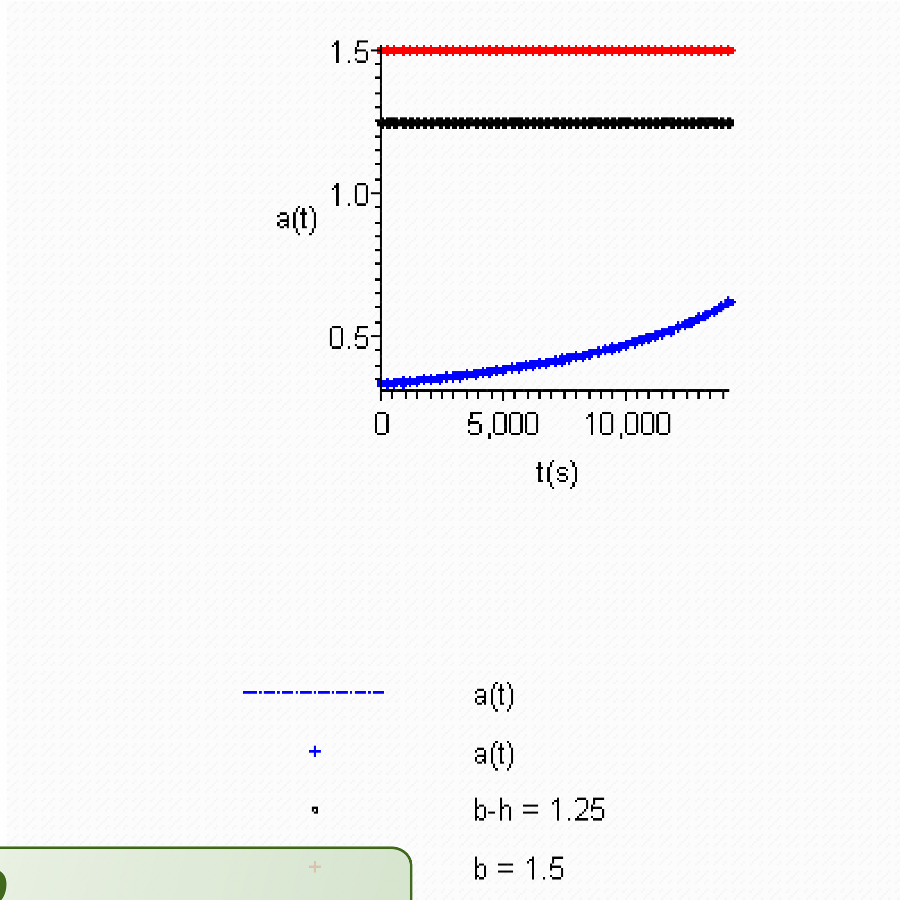
$a_0 := 1$



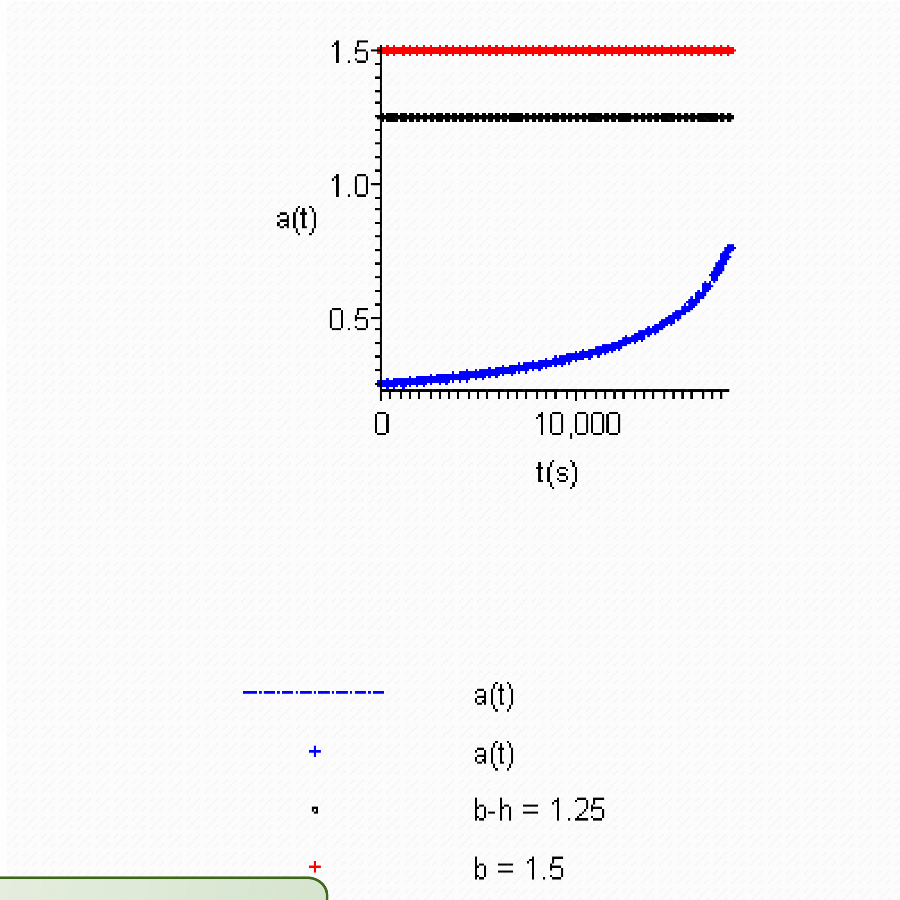
$a_0 := \frac{1}{2}$

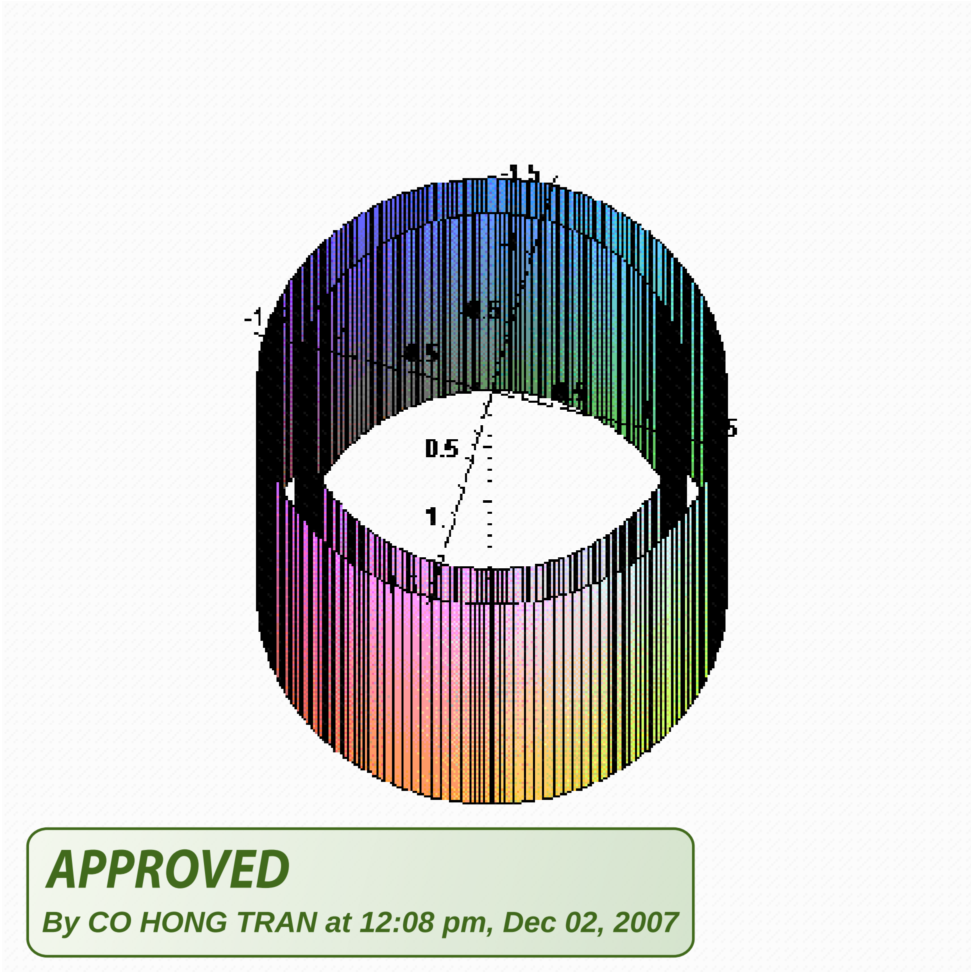


$$a_0 := \frac{1}{3}$$

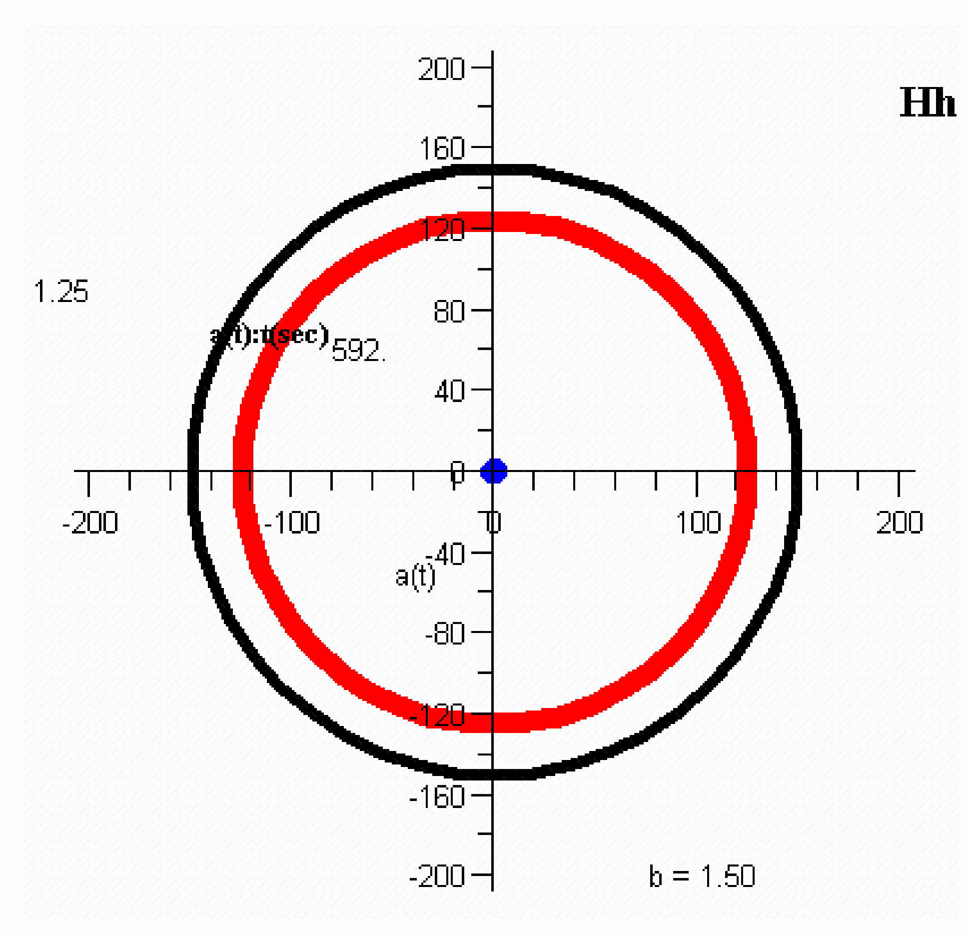


$$a_0 := \frac{1}{4}$$


1.5
1.0
a(t)
0.5
0
10,000
t(s)
a(t)
a(t)
b-h = 1.25
b = 1.5




```
>
> func_bankinhtrong(50, 5.2, 2, 7, Hh);
```



REFERENCES

[1] **Ю.Н. РАБОТНОВ ,** *Ползучесть элементов конструкций* , издательство наука , МОСКВА 1966
[2] **Lee E .H ., Radok J.R.M., Woodward W.B.,** *Stress Analysis for Linear Viscoelastic Materials , Trans. Soc. Rheol., 3, 41-59 (1959)*