3

John G. Talbot
Sept. 1878.

Address

TO

THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE,

DELIVERED AT DUBLIN,

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BY

WILLIAM SPOTTISWOODE, M.A., F.R.S., LL.D., D.C.L.,

President.

ADDRESS.

On looking back at the long array of distinguished men who both in this and in the sister countries have filled the chair of the British Association; on considering also the increased pains which have been bestowed upon, and the increased importance attaching to, the Presidential Address; it may well happen when, as on this occasion, your choice has fallen upon one outside the sphere of professional Science, that your nominee should feel unusual diffidence in accepting the post. Two considerations have however in my own case outweighed all reasons for hesitation: First, the uniform kindness which I received at the hands of the Association throughout the eight years during which I had the honour of holding another office; and secondly the conviction that the same goodwill which was accorded to your Treasurer would be extended to your President.

Introductory.

These considerations have led me to arrange my observations under two heads, viz., I propose first to offer some remarks upon the purposes and prospects of the Association with which, through your suffrages, I have been so long and so agreeably connected; and secondly to indulge in a few reflections, not indeed upon the details or technical progress, but upon the external aspects and tendencies of the Science which on this occasion I have the honour to represent. The former of these subjects is perhaps trite; but as an old man is allowed to become garrulous on his own hobby, so an old officer may be pardoned for lingering about a favourite theme. And although the latter may appear somewhat unpromising, I have decided to make it one of the topics of my discourse, from the consideration that the holder of this office will generally do better by giving utterance to what has already become part of his own thought, than by gathering matter outside of its habitual range for the special occasion. For, as it seems to me, the interest (if any) of an address consists, not so much in the multitude of things therein brought forward, as in the individuality of the mode in which they are treated.

British Association, past history.

The British Association has already entered its fifth decade. It has held its meetings, this the 48th, in 28 different towns. In six cities of note, viz., York, Bristol, Newcastle-on-Tyne, Plymouth, Manchester, and Belfast, its curve of progress may be said to have a node, or point through which it has twice passed; in the five Universities of Oxford, Cambridge, Dublin, Edinburgh, and Glasgow, and in the two great commercial centres, Liverpool and Birmingham, it may similarly be said to have a triple point, or one through which it has three times passed. Of our 46 Presidents more than half (26, in fact) have passed away; while the remainder hold important posts in Science, and in the Public Service, or in other avocations not less honourable in themselves, nor less useful to the commonwealth. And whether it be due to the salubrity of the climate or to the calm and dispassionate spirit in which Science is pursued by its votaries here, I do not pretend to say; but it is a fact that the earliest of our ex-Presidents still living, himself one of the original members of the Association, is a native of and resident in this country.

At both of our former meetings held in Dublin, in 1835 and 1857 respectively, while greatly indebted to the liberal hospitality of the citizens at large, we were, as we now are, under especial obligations to the authorities of Trinity College for placing at our disposal buildings, not only unusually spacious and convenient in themselves, but full of reminiscences calculated to awake the scientific sympathies of all who may be gathered in them. At both of those former Dublin meetings the venerable name of Lloyd figured at our head; and if long established custom had not seemed to preclude it, I could on many accounts have wished that we had met for a third time under the same name. And although other distinguished men, such as Dr. Robinson, Professors Stokes, Tyndall, and Andrews, are similarly disqualified by having already passed the Presidential chair, while others again, such as Sir W. R. Hamilton, Dr. M'Cullagh, and Professor Jukes, are permanently lost to our ranks; still we should not have had far to seek, had we looked for a President in this fertile island itself. But as every one connected with the place of meeting partakes of the character of host towards ourselves as guests, it has been thought by our oldest and most experienced members that we should better respond to an invitation by bringing with us a President to speak as our representative than by seeking one on the spot; and we may always hope on subsequent occasions that some of our present hosts may respond to a similar call.

But leaving our past history, which will form a theme more appropriate to our jubilee meeting in 1881 at the ancient city of York, I will ask your attention to a few particulars of our actual operations.

Its relation to other Societies.

Time was when the Royal Societies of London and Edinburgh and the Royal Irish Academy were the only representative bodies of British Science and the only receptacles of memoirs relating thereto. But latterly, the division of labour, so general in industrial life, has operated in giving rise to special Societies, such as the Astronomical, the Linnæan, the Chemical, the Geological, the Geographical, the Statistical, the Mathematical, the Physical, and many others. To both the earlier or more general, and the later or more special societies alike, the British Association shows resemblance and affinity. We are general in our comprehensiveness; we are special in our sectional arrangement; and in this respect we offer not only a counterpart, but to some extent a counterpoise, to the general tendency to subdivision in Science. Further still, while maintaining in their integrity all the elements of a strictly scientific body, we also include, in our character of a microcosm, and under our more social aspect, a certain freedom of treatment, and interaction of our various branches, which is scarcely possible among separate and independent societies.

The general business of our meetings consists first, in receiving and discussing communications upon scientific subjects at the various sections into which our body is divided, with discussions thereon; secondly, in distributing, under the advice of our Committee of Recommendations, the funds arising from the subscriptions of members and associates; and thirdly, in electing a Council upon whom devolves the conduct of our affairs until the next meeting.

The communications to the sections are of two kinds, viz., papers from individuals, and reports from Committees.

Papers for the Sections, &c.

As to the subject matter of the papers, nothing which falls within the range of Natural Knowledge, as partitioned among our sections, can be considered foreign to the purposes of the Association; and even many applications of Science, when viewed in reference to their scientific basis, may properly find a place in our proceedings. So numerous, however, are the topics herein comprised, so easy the transition beyond these limits, that it has been thought necessary to confine ourselves within this range, lest the introduction of other matters, however interesting to individual members, should lead to the sacrifice of more important subjects. As to the form of the communications, while it is quite true that every scientific conclusion should be based upon substantial evidence, every theory complete before being submitted for final adoption, it is not the less desirable that even tentative conclusions and hypothetical principles when supported by sufficient primâ facie evidence, and enunciated in such a manner as to be clearly apprehended, should find room for discussion at our sectional meetings. Considering, however, our limitations of time, and the varied nature of our audience, it would seem not inappropriate to suspend, mentally if not materially, over the doors of our section rooms, the Frenchman's dictum, that no scientific theory "can be "considered complete until it is so clear that it can be "explained to the first man you meet in the street."

Special reports.

Among the communications to the sections, undoubtedly the most important as a rule, are the Reports; that is to say, documents issuing from specially appointed Committees, some of which have been recipients of the grants mentioned above. These reports are in the main of two kinds, first, accounts of observations carried on for a series of years, and intended as records of information on the special subjects; such for instance have been those made by the Kew Committee, by the Committees on Luminous Meteors, on British Rainfall, on the Speed of Steamships, on Underground Temperature, on the Exploration of certain Geological Caverns, &c. These investigations, frequently originating in the energy and special qualifications of an individual, but conducted under the control of a Committee, have in many cases been continued from year to year, until either the object has been fully attained, or the matter has passed into the hands of other bodies, which have thus been led to recognize an inquiry into these subjects as part and parcel of their appropriate functions. The second class is one which is perhaps even more peculiar to the Association; viz., the Reports on the progress and present state of some main topics of Science. Among these may be instanced the early Reports on Astronomy, on Optics, on the Progress of Analysis; and later, those on Electrical Resistance and on Tides; that of Prof. G. G. Stokes on Double Refraction; that of Prof. H. J. Smith on the Theory of Numbers; that of Mr. Russell on Hyperelliptic Transcendents; and others. On this head Professor Carey Foster, in his address to the Mathematical and Physical Section at our meeting last year, made some excellent recommendations, to which, however, I need not at present more particularly refer, as the result of them will be duly laid before the section in the form of the report from a Committee to whom they were referred. It will be sufficient here to add that the wide extension of the Sciences in almost every branch, and the consequent specialization of the studies of each individual, have rendered the need for such reports more than ever pressing; and if the course of true Science should still run smooth it is probable that the need will increase rather than diminish.

If time and space had permitted, I should have further particularised the Committees, occasionally appointed, on subjects connected with Education. But I must leave this theme for some future President, and content myself with pointing out that the British Association alone among scientific societies concerns itself directly with these questions, and is open to appeals for counsel and support from the great teaching body of the country.

Grants.

One of the principal methods by which this Association materially promotes the advancement of Science, and consequently one of its most important functions, consists in grants of money from its own income in aid of special scientific researches. The total amount so laid out during the 47 years of our existence has been no less than £44,000; and the average during the last ten years has been £1,450 per annum. These sums have not only been in the main wisely voted and usefully expended; but they have been themselves productive of much additional voluntary expenditure of both time and money on the part of those to whom the grants have been entrusted. The results have come back to the Association in the form of papers and reports, many of which have been printed in our volumes. By this appropriation of a large portion of its funds, the Association has to some extent anticipated, nay even it may have partly inspired the ideas, now so much discussed, of the Endowment of Research. And whether the aspirations of those who advocate such endowment be ever fully realised or not, there can, I think, be no doubt whatever that the Association in the matter of these grants has afforded a most powerful stimulus to original research and discovery.

Regarded from another point of view these grants, together with others to be hereafter mentioned, present a strong similarity to that useful institution, the Professoriate Extraordinary of Germany, to which there are no foundations exactly corresponding in this country. For, beside their more direct educational purpose, these Professorships are intended, like our own grants, to afford to special individuals an opportunity of following out the special work for which they have previously proved themselves competent. And in this respect the British Association may be regarded as supplying, to the extent of its means, an elasticity which is wanting in our own Universities.

Other Funds.

Besides the funds which through your support are at the disposal of the British Association there are, as is well known to many here present, other funds of more or less similar character at the disposal, or subject to the recommendations, of the Royal Society. There is the Donation Fund, the property of the Society; the Government Grant of £1,000 per annum, administered by the Society; and the Government Fund of £4,000 per annum (an experiment for five years) to be distributed by the Science and Art Department, both for research itself, and for the support of those engaged thereon, at the recommendation of a Committee consisting mainly of Fellows of the Royal Society. To these might be added other funds in the hands of different Scientific Societies.

But although it must be admitted that the purposes of these various funds are not to be distinguished by any very simple line of demarcation, and that they may therefore occasionally appear to overlap one another, it may still, I think, be fairly maintained that this fact does not furnish any sufficient reason against their co-existence. There are many topics of research too minute in their range, too tentative in their present condition, to come fairly within the scope of the funds administered by the Royal Society. There are others, ample enough in their extent, and long enough in their necessary duration, to claim for their support a national grant, but which need to be actually set on foot or tried before they can fairly expect the recognition either of the public or of the Government. To these categories others might be added ; but the above-mentioned instances will perhaps suffice to show that even if larger and more permanent funds were devoted to the promotion of research than is the case at present, there would still be a field of activity open to the British Association as well as to other Scientific bodies which may have funds at their disposal.

On the general question it is not difficult to offer strong arguments in favour of permanent national Scientific Institutions ; nor is it difficult to picture to the mind an ideal future when Science and Art shall walk hand in hand together, led by a willing minister into the green pastures of the Endowment of Research. But while allowing this to be no impossible a future, we must still admit that there are other and less promising possibilities, which under existing circumstances cannot be altogether left out of our calculations. I am therefore on the whole inclined to think that, while not losing sight of larger schemes, the wisest policy, for the present at all events, and pending the experiment of the Government Fund, will be to confine our efforts to a careful selection of definite persons to carry out definite pieces of work; leaving to them the honour (or the onus if they so think it) of justifying from time to time a continuation of the confidence which the Government or other supporting body may have once placed in them.

Continuance of British Association.

Passing from the proceedings to other features and functions of our body, it should be remembered that the continued existence of the Association must depend largely upon the support which it receives from its members and associates. Stinted in the funds so arising, its scientific effectiveness would be materially impaired; and deprived of them, its existence would be precarious. The amount at our disposal in each year will naturally vary with the population, with the accessibility, and with other circumstances of the place of meeting; there will be financially, as well as scientifically, good years and bad years. But we have in our invested capital a sum sufficient to tide over all probable fluctuations, and even to carry us efficiently through several years of financial famine, if ever such should occur. This seems to me sufficient; and we have therefore, I think, no need to increase our reserve, beyond perhaps the moderate addition which a prudent treasurer will always try to secure, against expenditure which often increases and rarely diminishes.

But however important this material support may be to our existence and well being, it is by no means all that is required. There is another factor which enters into the product, namely, the personal scientific support of our best men. It is, I think, not too much to say, that without their presence our meetings would fail in their chief and most important element, and had best be discontinued altogether. We make, it must be admitted, a demand of sensible magnitude in calling upon men who have been actively engaged during a great portion of the year, at a season when they may fairly look for relaxation, to attend a busy meeting, and to contribute to its proceedings; but unless a fair quota at least of our veterans, and a good muster of our younger men, put in their appearance, our gatherings will be to little purpose. There was a period within my own recollection when it was uncertain whether the then younger members of our scientific growth would cast in their lot with us or not, and when the fate of the Association depended very much upon their decision. They decided in our favour; they have since become Presidents, Lecturers, and other functionaries of our body; with what result it is for you to judge.

Of the advantages which may possibly accrue to the locality in which our meetings are held, it is not for us to speak; but it is always a ground for sincere satisfaction to learn that our presence has been of any use in stimulating an interest, or in promoting local efforts, in the direction of Science.

The Council.

The functions of the British Association do not, however, terminate with the meeting itself. Beside the Special Committees already mentioned, there remains a very important body, elected by the General Committee, viz., the Council, which assembles at the office in London from time to time as occasion requires. To this body belongs the duty of proposing a President, of preparing for the approval of the General Committee the list of Vice-Presidents and sectional officers, the selection of evening lecturers, and other arrangements for the coming meeting.

At the present time another class of questions occupies a good deal of the attention of the Council. In the first generation of the Association, and during the period of unwritten, but not yet traditional, law, questions relating to our own organization or procedure either "settled themselves," or were wisely left to the discretionary powers of those who had taken part in our proceedings during the early years of our existence. These and other kindred subjects now require more careful formularisation and more deliberate sanction. And it is on the shoulders of the Council that the weight of these matters in general falls. These facts deserve especial mention on the present occasion, because one part of our business at the close of this meeting will be to bid farewell officially to one who has served us as Assistant Secretary so long and so assiduously that he has latterly become our main repertory of information, and our Mentor upon questions of precedent and procedure. The post hitherto held by Mr. Griffith (for it is to him that I allude) will doubtless be well filled by the able and energetic member who has been nominated in his place; but I doubt not that even he will be glad for some time to come to draw largely upon the knowledge and experience of his predecessor.

But, beside matters of internal arrangement and organization, the duties of the Council comprise a variety of scientific subjects referred to them by the General Committee, at the instance of the Committee of Recommendations, for deliberation and occasionally for action. With the increasing activity of our body in general, and more particularly with that of our various officers, these duties have of late years become more varied and onerous than formerly; nor is it to be wished that they should diminish in either variety or extent.

Once more, questions beyond our own constitution, and even beyond the scope of our own immediate action, such as education, legislation affecting either the promotion or the applications of science to industrial and social life, which have suggested themselves at our meetings, and received the preliminary sanction of our Committee of Recommendations, are frequently referred to our Council. These, and others which it is unnecessary to particularise, whether discussed in full Council or in Committees specially appointed by that body, render the duties of our councillors as onerous as they are important.

Its relations with Government.

While the Government has at all times, but in a more marked manner of late years, recognised the Royal Society of London, with representatives from the sister societies of Dublin and of Edinburgh, as the body to which it should look for counsel and advice upon scientific questions, it has still never shown itself indisposed to receive and entertain any well considered recommendation from the British Association. Two special causes have in all probability contributed largely to this result. First, the variety of elements comprised by the Association, on account of which its recommendations imply a more general concurrence of scientific opinion than those of any other scientific body. Secondly, the peculiar fact that our period of maximum activity coincides with that of minimum activity of other scientific bodies is often of the highest importance. At the very time when the other bodies are least able, we are most able, to give deliberate consideration, and formal sanction, to recommendations whether in the form of applications to Government or otherwise which may arise. In many of these, time is an element so essential, that it is not too much to say, that without the intervention of the British Association many opportunities for the advancement of Science, especially at the seasons in question, might have been lost. The Government has moreover formally recognised our scientific existence by appointing our President for the time being a member of the Government Fund Committee; and the public has added its testimony to our importance and utility by imposing upon our President and officers a variety of duties, among which are conspicuous those which arise out of its very liberal exercise of civic and other hospitality.

Presidential addresses.

Of the nature and functions of the Presidential address this is perhaps neither the time nor the place to speak; but if I might for a moment forget the purpose for which we are now assembled, I would take the opportunity of reminding those who have not attended many of our former meetings that our annual volumes contain a long series of addresses on the progress of Science, from a number of our most eminent men, to which there is perhaps no parallel elsewhere. These addresses are perhaps as remarkable for their variety in mode of treatment as for the value of their subject matter. Some of our Presidents, and especially those who officiated in the earlier days of our existence, have passed in review the various branches of Science, and have noted the progress made in each during the current year. But, as the various Sciences have demanded more and more special treatment on the part of those who seriously pursue them, so have the cases of individuals who can of their own knowledge give anything approaching to a general review become more and more rare. To this may be added the fact that although no year is so barren as to fail in affording sufficient crop for a strictly scientific budget, or for a detailed report of progress in research, yet one year is more fertile than another in growths of sufficient prominence to arrest the attention of the general public, and to supply topics suitable for the address. On these accounts apparently such a Presidential survey has ceased to be annual, and has dropped into an intermittence of longer period. Some Presidents have made a scientific principle, such as the Time-element in natural phenomena, or Continuity, or Natural Selection, the theme of their discourse, and have gathered illustrations from various branches of knowledge. Others again, taking their own special subject as a fundamental note, and thence modulating into other kindred keys, have borne testimony to the fact that no subject is so special as to be devoid of bearing or of influence on many others. Some have described the successive stages of even a single but important investigation; and while tracing the growth of that particular item, and of the ideas involved in it, have incidentally shown to the outer world what manner of business a serious investigation is. But there is happily no pattern or precedent which the President is bound to follow; both in range of subject-matter and in mode of treatment each has exercised his undoubted right of taking an independent line. And it can hardly be doubted that a judicious exercise of this freedom has contributed more than anything else to sustain the interest of a series of annual discourses extending now over nearly half a century.

The nature of the subjects which may fairly come within the scope of such a discourse has of late been much discussed; and the question is one upon which every one is of course entitled to form his own judgment; but lest there should be any misapprehension as to how far it concerns us in our corporate capacity, it will be well to remind my hearers that as, on the one hand, there is no discussion on the Presidential address, and the members as a body express no opinion upon it, so, on the other, the Association cannot fairly be considered as in any way committed to its tenour or conclusions. Whether this immunity from comment and reply be really on the whole so advantageous to the President as might be supposed need not here be discussed; but suffice it to say that the case of an audience assembled to listen without discussion finds a parallel elsewhere, and in the parallel case it is not always considered that the result is altogether either advantageous to the speaker or conducive to excellence in the discourse.

Their range of subjects.

But, apart from this, the question of a limitation of range in the subject-matter for the Presidential address is not quite so simple as may at first sight appear. It must, in fact, be borne in mind that, while on the one hand knowledge is distinct from opinion, from feeling, and from all other modes of subjective impression; still the limits of knowledge are at all times expanding, and the boundaries of the known and the unknown are never rigid or permanently fixed. That which in time past or present has belonged to one category, may in time future belong to the other. Our ignorance consists partly in ignorance of actual facts, and partly also in ignorance of the possible range of ascertainable fact. If we could lay down beforehand precise limits of possible knowledge, the problem of Physical Science would be already half solved. But the question to which the scientific explorer has often to address himself is, not merely whether he is able to solve this or that problem, but whether he can so far unravel the tangled threads of the matter with which he has to deal as to weave them into a definite problem at all. He is not like a candidate at an examination with a precise set of questions placed before him; he must first himself act the part of the examiner and select questions from the repertory of Nature, and upon them found others, which in some sense are capable of definite solution. If his eye seem dim, he must look steadfastly and with hope into the misty vision, until the very clouds wreath themselves into definite forms. If his ear seem dull, he must listen patiently and with sympathetic trust to the intricate whisperings of Nature,—the goddess, as she has been called, of a hundred voices,—until here and there he can pick out a few simple notes to which his own powers can resound. If, then, at a moment when he finds himself placed on a pinnacle from which he is called upon to take a perspective survey of the range of Science, and to tell us what he can see from his vantage ground; if, at such a moment, after straining his gaze to the very verge of the horizon, and after describing the most distant of well defined objects, he should give utterance also to some of the subjective impressions which he is conscious of receiving from regions beyond; if he should depict possibilities which seem opening to his view; if he should explain why he thinks this a mere blind alley and that an open path; then the fault and the loss would be alike ours if we refused to listen calmly, and temperately to form our own judgment on what we hear; then assuredly it is we who would be committing the error of confounding matters of fact and matters of opinion, if we failed to discriminate between the various elements contained in such a discourse, and assumed that they had all been put on the same footing.

Presidential difficulties.

But to whatever decision we may each come on these controverted points, one thing appears clear from a retrospect of past experience; viz., that first or last, either at the outset in his choice of subject, or in the conclusions ultimately drawn therefrom, the President, according to his own account at least, finds himself on every occasion in a position of "exceptional, or more than usual difficulty." And your present representative, like his predecessors, feels himself this moment in a similar predicament. The reason which he now offers is, that the branch of Science which he represents is one whose lines of advance, viewed from a Mathematician's own point of view, offer so few points of contact with the ordinary experiences of life or modes of thought that any account of its actual progress which he might have attempted must have failed in the first requisite of an address, namely, that of being intelligible.

View of Mathematics here taken.

Now if this esoteric view had been the only aspect of the subject which he could present to his hearers, he might well have given up the attempt in despair. But although in its technical character Mathematical Science suffers the inconveniences, while it enjoys the dignity, of its Olympian position; still in a less formal garb, or in disguise, if you are pleased so to call it, it is found present at many an unexpected turn; and although some of us may never have learnt its special language, not a few have, all through our scientific life, and even in almost every accurate utterance, like Molière's well known character, been talking Mathematics without knowing it. It is, moreover, a fact not to be overlooked that the appearance of isolation, so conspicuous in Mathematics, appertains in a greater or less degree to all other Sciences, and perhaps also to all pursuits in life. In its highest flight each soars to a distance from its fellows. Each is pursued alone for its own sake, and without reference to its connection with, or its application to, any other subject. The pioneer and the advanced guard are of necessity separated from the main body; and in this respect Mathematics does not materially differ from its neighbours. And, therefore, as the solitariness of Mathematics has been a frequent theme of discourse, it may be not altogether unprofitable to dwell for a short time upon the other side of the question, and to inquire whether there be not points of contact in method or in subject-matter between Mathematics and the outer world which have been frequently overlooked; whether its lines do not in some cases run parallel to those of other occupations and purposes of life; and lastly, whether we may not hope for some change in the attitude too often assumed towards it by the representatives of other branches of knowledge and of mental activity.

In his Preface to the Principia, Newton gives expression to some general ideas which may well serve as the key note for all future utterances on the relation of Mathematics to Natural, including also therein what are commonly called Artificial, Phenomena.

"The ancients divided Mechanics into two parts, "Rational and Practical; and since artizans often work "inaccurately, it came to pass that Mechanics and "Geometry were distinguished in this way, that every- "thing accurate was referred to Geometry, and every- "thing inaccurate to Mechanics. But the inaccuracies "appertain to the artizan and not to the art, and "Geometry itself has its foundation in mechanical

Newton's preface.

"practice, and is in fact nothing else than that part of "Universal Mechanics which accurately lays down and "demonstrates the art of measuring." He next explains that rational Mechanics is the science of motion resulting from forces, and adds, "The whole difficulty of "Philosophy seems to me to lie in investigating the forces "of Nature from the phenomena of motion; and in "demonstrating that from these forces other phenomena "will ensue." Then, after stating the problems of which he has treated in the work itself, he says: "I "would that all other Natural Phenomena might similarly be deduced from mechanical principles. For "many things move me to suspect that everything "depends upon certain forces in virtue of which the "particles of bodies, through forces not yet understood, are either impelled together so as to cohere in "regular figures, or are repelled and recede from one "another."

Burrowes' remarks.

Newton's views, then, are clear: he regards Mathematics, not as a method independent of, though applicable to various subjects, but as itself the higher side or aspect of the subjects themselves; and it would be little more than a translation of his notions into other language, little more than a paraphrase of his own words, if we were to describe the mathematical as one aspect of the material world itself, apart from which all other aspects are but incomplete sketches, and, however accurate after their own kind, are still liable to the imperfections of the inaccurate artificer. Mr. Burrowes, in his Preface to the first volume of the Transactions of the Royal Irish Academy, has carried out the same argument, approaching it from the other side: "No one Science," he says, "is so little connected with "the rest as not to afford many principles whose use may

" extend considerably beyond the Science to which they " primarily belong, and no proposition is so purely theo- " retical as to be incapable of being applied to practical " purposes. There is no apparent connexion between " duration and the cycloidal arch, the properties of which " have furnished us with the best method of measuring " time; and he who has made himself master of the " nature and affections of the logarithmic curve has ad- " vanced considerably towards ascertaining the propor- " tionable density of the air at various distances from the " earth. The researches of the Mathematician are the " only sure ground on which we can reason from experi- " ments; and how far Experimental Science may assist " commercial interests is evinced by the success of manu- " factures in countries where the hand of the artificer " has taken its direction from the Philosopher. Every " manufacture is in reality but a chemical process, and " the machinery requisite for carrying it on but the " right application of certain propositions in rational " mechanics." So far your Academician. Every subject, therefore, whether in its usual acceptation scientific or otherwise, may have a mathematical aspect; as soon, in fact, as it becomes a matter of strict measurement, or of numerical statement, so soon does it enter upon a mathematical phase. This phase may, or it may not, be a prelude to another in which the laws of the subject are expressed in algebraical formulæ or represented by geometrical figures. But the real gist of the business does not always lie in the mode of expression; and the fascination of the formulæ or other mathematical paraphernalia may after all be little more than that of a theatrical transformation scene. The process of reducing to formulæ is really one of abstraction, the results of which are not always wholly on the side of gain; in fact, through the process itself the subject may lose in one respect even more than it gains in another. But long before such abstraction is completely attained, and even in cases where it is never attained at all, a subject may to all intents and purposes become mathematical. It is not so much elaborate calculations or abstruse processes which characterise this phase, as the principles of precision, of exactness, and of proportion. But these are principles with which no true knowledge can entirely dispense. If it be the general scientific spirit which at the outset moves upon the face of the waters, and out of the unknown depth brings forth light and living forms; it is no less the mathematical spirit which breathes the breath of life into what would otherwise have ever remained mere dry bones of fact, which reunites the scattered limbs and re-creates from them a new and organic whole.

And as a matter of fact, in the words used by Professor Jellett at our meeting at Belfast, viz., "Not "only are we applying our methods to many Sciences "already recognised as belonging to the legitimate "province of Mathematics, but we are learning to "apply the same instrument to Sciences hitherto wholly "or partially independent of its authority. Physical "Science is learning more and more every day to see in "the phenomena of Nature modifications of that one "phenomenon (namely, Motion) which is peculiarly "under the power of Mathematics." Echoes are these, far off and faint perhaps, but still true echoes, in answer to Newton's wish that all these phenomena may some day "be deduced from mechanical principles."

Mathematics, Literature, and Art.

If, turning from this aspect of the subject, it were my purpose to enumerate how the same tendency has evinced itself in the Arts, unconsciously it may be to the artists themselves, I might call as witnesses each one in turn with full reliance on the testimony which they would bear. And, having more special reference to Mathematics, I might confidently point to the accuracy of measurement, to the truth of curve, which according to modern investigation is the key to the perfection of classic art. I might triumphantly cite not only the architects of all ages, whose art so manifestly rests upon mathematical principles; but I might cite also the literary as well as the artistic remains of the great Artists of the Cinquecento, both Painters and Sculptors, in evidence of the Geometry and the Mechanics which, having been laid at the foundation, appear to have found their way upwards through the superstructure of their works. And in a less ambitious sphere, but nearer to ourselves in both time and place, I might point with satisfaction to the great school of English constructors of the 18th century in the domestic arts; and remind you that not only the engineer and the architect, but even the cabinet-makers, devoted half the space of their books to perspective and to the principles whereby solid figures may be delineated on paper, or what is now termed descriptive Geometry.

Nor perhaps would the Sciences which concern themselves with reasoning and speech, nor the kindred art of Music, nor even Literature itself, if thoroughly probed, offer fewer points of dependence upon the Science of which I am speaking. What, in fact, is Logic but that part of universal reasoning; Grammar but that part of Universal Speech: Harmony and Counterpoint but that part of Universal Music, "which accurately "lays down," and demonstrates (so far as demonstration is possible) precise methods appertaining to each of these Arts? And I might even appeal to the common consent which speaks of the mathematical as the pattern form of reasoning and model of a precise style.

Taking, then, precision and exactness as the characteristics which distinguish the mathematical phase of a subject, we are naturally led to expect that the approach to such a phase will be indicated by increasing application of the principle of measurement, and by the importance which is attached to numerical results. And this very necessary condition for progress may, I think, be fairly described as one of the main features of scientific advance in the present day.

Measurements in Physics.

If it were my purpose, by descending into the arena of special sciences, to show how the most various investigations alike tend to issue in measurement, and to that extent to assume a mathematical phase, I should be embarrassed by the abundance of instances which might be adduced. I will therefore confine myself to a passing notice of a very few, selecting those which exemplify not only the general tendency, but also the special character of the measurements now particularly required, viz., that of minuteness, and the indirect method by which alone we can at present hope to approach them. An object having a diameter of an 80,000th of an inch is perhaps the smallest of which the microscope could give any well-defined representation; and it is improbable that one of 120,000th of an inch could be singly discerned with the highest powers at our command. But the solar beams and the electric light reveal to us the presence of bodies far smaller than these. And, in the absence of any means of observing them singly, Professor Tyndall has suggested a scale of these minute objects in terms of the lengths of luminiferous waves. To this he was led, not by any attempt at individual measurement, but by taking account of them in the aggregate, and observing the tints which they scatter laterally when clustered in the form of actinic clouds. The small bodies with which experimental Science has recently come into contact are not confined to gaseous molecules, but comprise also complete organisms; and the same philosopher has made a profound study of the momentous influence exerted by these minute organisms in the economy of life. And if, in view of their specific effects, whether deleterious or other, on human life, any qualitative classification, or quantitative estimate be ever possible, it seems that it must be effected by some such method as that indicated above.

Again, to enumerate a few more instances of the measurement of minute quantities, there are the average distances of molecules from one another in various gases and at various pressures; the length of their free path, or range open for their motion without coming into collision; there are movements causing the pressures and differences of pressure under which Mr. Crookes' radiometers execute their wonderful revolutions. There are the excursions of the air while transmitting notes of high pitch, which through the researches of Lord Rayleigh appear to be of a diminutiveness altogether unexpected. There are the molecular actions brought into play in the remarkable experiments by Dr. Ker, who has succeeded, where even Faraday failed, in effecting a visible rotation of the plane of polarisation of light in its passage through electrified dielectrics, and on its reflexion at the surface of a magnet. To take one more instance, which must be present to the minds of us all, there are the infinitesimal ripples of the vibrating plate in

Mr. Graham Bell's most marvellous invention. Of the nodes and ventral segments in the plate of the Telephone which actually converts sound into electricity and electricity into sound, we can at present form no conception. All that can now be said is that the most perfect specimens of Chladni's sand figures on a vibrating plate, or of Kundt's lycopodium heaps in a musical tube, or even Mr. Sedley Taylor's more delicate vortices in the films of the Phoneidoscope, are rough and sketchy compared with these. For notwithstanding the fact that in the movements of the Telephone-plate we have actually in our hand the solution of that old world problem the construction of a speaking machine; yet the characters in which that solution is expressed are too small for our powers of decipherment. In movements such as these we seem to lose sight of the distinction, or perhaps we have unconsciously passed the boundary between massive and molecular motion.

Through the Phonograph we have not only a transformation but a permanent and tangible record of the mechanism of speech. But the differences upon which articulation (apart from loudness, pitch, and quality) depends, appear from the experiments of Fleeming Jenkin and of others to be of microscopic size. The Microphone affords another instance of the unexpected value of minute variations,—in this case of electric currents; and it is remarkable that the gist of the instrument seems to lie in obtaining and perfecting that which electricians have hitherto most scrupulously avoided, viz., loose contact.

Once more, Mr. De La Rue has brought forward as one of the results derived from his stupendous battery of 10,000 cells, strong evidence for supposing that a voltaic discharge, even when apparently continuous, may still be an intermittent phenomenon; but all that is known of the period of such intermittence is, that it must recur at exceedingly short intervals. And in connexion with this subject, it may be added that, whatever be the ultimate explanation of the strange stratification which the voltaic discharge undergoes in rarefied gases, it is clear that the alternate disposition of light and darkness must be dependent on some periodic distribution in space or sequence in time which can at present be dealt with only in a very general way. In the exhausted column we have a vehicle for electricity not constant like an ordinary conductor, but itself modified by the passage of the discharge, and perhaps subject to laws differing materially from those which it obeys at atmospheric pressure. It may also be that some of the features accompanying stratification form a magnified image of phenomena belonging to disruptive discharges in general; and that consequently so far from expecting among the known facts of the latter any clue to an explanation of the former, we must hope ultimately to find in the former an elucidation of what is at present obscure in the latter. A prudent philosopher usually avoids hazarding any forecast of the practical application of a purely scientific research. But it would seem that the configuration of these striæ might some day prove a very delicate means of estimating low pressures, and perhaps also for effecting some electrical measurements.

Now, it is a curious fact that almost the only small quantities of which we have as yet any actual measurements are the wave lengths of light; and that all others, excepting so far as they can be deduced from these, await future determination. In the meantime, when unable to approach these small quantities individually, the method to which we are obliged to have recourse is, as indicated above, that of averages, whereby, disregarding the circumstances of each particular case, we calculate the average size, the average velocity, the average direction, &c. of a large number of instances. But although this method is based upon experience, and leads to results which may be accepted as substantially true; although it may be applicable to any finite interval of time, or over any finite area of space (that is, for all practical purposes of life) there is no evidence to show that it is so when the dimensions of interval or of area are indefinitely diminished. The truth is that the simplicity of nature which we at present grasp is really the result of infinite complexity; and that below the uniformity there underlies a diversity whose depths we have not yet probed, and whose secret places are still beyond our reach.

The present is not an occasion for multiplying illustrations, but I can hardly omit a passing allusion to one all important instance of the application of the statistical method. Without its aid social life, or the History of Life and Death, could not be conceived at all, or only in the most superficial manner. Without it we could never attain to any clear ideas of the condition of the Poor, we could never hope for any solid amelioration of their condition or prospects. Without its aid, sanitary measures, and even medicine, would be powerless. Without it, the politician, and the philanthropist, would alike be wandering over a trackless desert.

Mathematical methods and other subjects.

It is, however, not so much from the side of Science at large as from that of Mathematics itself, that I desire to speak. I wish from the latter point of view to indicate connexions between Mathematics and other subjects, to prove that hers is not after all such a far-off region, nor so undecipherable an alphabet, and to show that even at unlikely spots we may trace under-currents of thought which having issued from a common source fertilize alike the mathematical and the non-mathematical world.

Having this in view, I propose to make the subject of special remark some processes peculiar to modern Mathematics; and, partly with the object of incidentally removing some current misapprehensions, I have selected for examination three methods in respect of which Mathematicians are often thought to have exceeded all reasonable limits of speculation, and to have adopted for unknown purposes an unknown tongue. And it will be my endeavour to show not only that in these very cases our Science has not outstepped its own legitimate range, but that even Art and Literature have unconsciously employed methods similar in principle. The three methods in question are, first, that of Imaginary Quantities; secondly, that of Manifold Space; and thirdly, that of Geometry not according to Euclid.

First it is objected that, abandoning the more cautious methods of ancient Mathematicians, we have admitted into our formulæ quantities which by our own showing, and even in our own nomenclature, are imaginary or impossible; nay, more, that out of them we have formed a variety of new algebras to which there is no counterpart whatever in reality; but from which we claim to arrive at possible and certain results. Imaginaries.

On this head it is in Dublin, if anywhere, that I may be permitted to speak. For to the fertile imagination of the late Astronomer Royal for Ireland we are indebted for that marvellous calculus of Quaternions, which is only now beginning to be fully understood, and which has not yet received all the applications of which it is doubtless capable. And even although this calculus be not coextensive with another (the Ausdehnungslehre of Grassmann) which almost simultaneously germinated on the continent, nor with ideas more recently developed in America (Pierce's Linear Associative Algebras); yet it must always hold its position as an original discovery, and as a representative of one of the two great groups of generalised algebras, (viz., those the squares of whose units are respectively negative unity and zero) the common origin of which must still be marked on our intellectual map as an unknown region. Well do I recollect how in its early days we used to handle the method as a magician's page might try to wield his master's wand, trembling as it were between hope and fear, and hardly knowing whether to trust our own results until they had been submitted to the present and ever ready counsel of Sir W. R. Hamilton himself.

To fix our ideas, consider the measurement of a line, or the reckoning of time, or the performance of any mathematical operation. A line may be measured in one direction or in the opposite; time may be reckoned forward or backward; an operation may be performed or be reversed, it may be done or may be undone; and if having once reversed any of these processes we reverse it a second time, we shall find that we have come back to the original direction of measurement or reckoning, or to the original kind of operation.

Suppose, however, that at some stage of a calculation our formulæ indicate an alteration in the mode of measurement such that if the alteration be repeated, a condition of things, not the same as, but the reverse of the original, will be produced. Or suppose that, at a certain stage, our transformations indicate that time is to be reckoned in some manner different from future or past, but still in a way having definite algebraical connexion with time which is gone and time which is to come. It is clear that in actual experience there is no process to which such measurements correspond. Time has no meaning except as future or past; and the present is but the meeting point of the two. Or, once more, suppose that we are gravely told that all circles pass through the same two imaginary points at an infinite distance, and that every line drawn through one of these points is perpendicular to itself. On hearing the statement we shall probably whisper, with a smile or a sigh, that we hope it is not true, but that in any case it is a long way off, and perhaps, after all, it does not very much signify. If, however, we are not satisfied to dismiss the question on these terms, the Mathematician himself must admit that we have here reached a definite point of issue. Our Science must either give a rational account of the dilemma, or yield the position as no longer tenable.

Explanation of them.

Special modes of explaining this anomalous state of things have occurred to Mathematicians. But, omitting details as unsuited to the present occasion, it will, I think, be sufficient to point out in general terms that a solution of the difficulty is to be found in the fact that the formulæ which give rise to these results are more comprehensive than the signification which has been given to them; and when we pass out of the condition of things first contemplated they cannot (as it is obvious they ought not) give us any results intelligible on that basis. But it does not therefore by any means follow that upon a more enlarged basis the formulæ are incapable of interpretation; on the contrary, the difficulty at which we have arrived indicates that there must be some more comprehensive statement of the problem which will include cases impossible in the more limited, but possible in the wider view of the subject.

A very simple instance will illustrate the matter. If from a point outside a circle we draw a straight line to touch the curve, the distance between the starting point and the point of contact has certain geometrical properties. If the starting point be shifted nearer and nearer to the circle the distance in question becomes shorter, and ultimately vanishes. But as soon as the point passes to the interior of the circle the notion of a tangent and distance to the point of contact cease to have any meaning; and the same anomalous condition of things prevails as long as the point remains in the interior. But if the point be shifted still further until it emerges on the other side, the tangent and its properties resume their reality; and are as intelligible as before. Now the process whereby we have passed from the possible to the impossible, and again repassed to the possible (namely the shifting of the starting point) is a perfectly continuous one, while the conditions of the problem as stated above have abruptly changed. If, however, we replace the idea of a line touching by that of a line cutting the circle, and the distance of the point of contact by the distances at which the line is intercepted by the curve, it will easily be seen that the latter includes the former as a limiting case, when the cutting line is turned about the starting point until it coincides with the tangent itself. And further, that the two intercepts have a perfectly distinct and intelligible meaning whether the point be outside or inside the area. The only difference is that in the first case the intercepts are measured in the same direction; in the latter in opposite directions.

The foregoing instance has shown one purpose which these imaginaries may serve, viz., as marks indicating a limit to a particular condition of things, to the application of a particular law, or pointing out a stage where a more comprehensive law is required. To attain to such a law we must, as in the instance of the circle and tangent, reconsider our statement of the problem; we must go back to the principle from which we set out, and ascertain whether it may not be modified or enlarged. And even if in any particular investigation, wherein imaginaries have occurred, the most comprehensive statement of the problem of which we are at present capable fails to give an actual representation of these quantities; if they must for the present be relegated to the category of imaginaries; it still does not follow that we may not at some future time find a law which will endow them with reality, nor that in the meantime we need hesitate to employ them, in accordance with the great principle of continuity, for bringing out correct results.

Illustration from Art

If, moreover, both in Geometry and in Algebra we occasionally make use of points or of quantities which from our present outlook have no real existence, which can neither be delineated in space of which we have experience, nor measured by scale as we count measurement; if these imaginaries, as they are termed, are called up by legitimate processes of our Science; if they serve the purpose not merely of suggesting ideas, but of actually conducting us to practical conclusions; if all this be true in abstract Science, I may perhaps be allowed to point out, at all events in illustration, that in Art unreal forms are frequently used for suggesting ideas, for conveying a meaning for which no others seem to be suitable or adequate. Are not forms unknown to Biology, situations incompatible with gravitation, positions which challenge not merely the stability but the very possibility of equilibrium,—are not these the very means to which the Artist often has recourse in order to convey his meaning and to fulfil his mission? Who that has ever revelled in the ornamentation of the Renaissance, in the extraordinary transitions from the animal to the vegetable, from faunic to floral forms, and from these again to almost purely geometric curves, who has not felt that these imaginaries have a claim to recognition very similar to that of their congeners in Mathematics? How is it that the grotesque paintings of the middle ages, the fantastic sculpture of remote nations, and even the rude art of the Prehistoric Past, still impress us, and have an interest over and above their antiquarian value; unless it be that they are symbols which, although hard of interpretation when taken alone, are yet capable from a more comprehensive point of view of leading us mentally to something beyond themselves, and to truths which, although reached through them, have a reality scarcely to be attributed to their outward forms?

and Literature.

Again, if we turn from art to letters, truth to nature and to fact is undoubtedly a characteristic of sterling literature; and yet in the delineation of outward nature itself, still more in that of feelings and affections, of the secret springs of character and motives of conduct, it frequently happens that the writer is driven to imagery, to an analogy, or even to a paradox, in order to give utterance to that of which there is no direct counterpart in recognized speech. And yet which of us cannot find a meaning for these literary figures, an inward response, to imaginative poetry, to social fiction, or even to those tales of giant and fairyland written, it is supposed, only for the nursery or schoolroom? But in order thus to reanimate these things with a meaning beyond that of the mere words, have we not to reconsider our first position, to enlarge the ideas with which we started; have we not to cast about for some thing which is common to the idea conveyed and to the subject actually described, and to seek for the sympathetic spring which underlies both; have we not, like the mathematician, to go back as it were to some first principles, or, as it is pleasanter to describe it, to become again as a little child?

Manifold space.

Passing to the second of the three methods, viz., that of manifold space, it may first be remarked that our whole experience of space is in three dimensions, viz., of that which has length, breadth, and thickness; and if for certain purposes we restrict our ideas to two dimensions as in plane geometry, or to one dimension as in the division of a straight line, we do this only by consciously and of deliberate purpose setting aside, but not annihilating, the remaining one or two dimensions. Negation, as Hegel has justly remarked, implies that which is negatived, or, as he expresses it, affirms the opposite. It is by abstraction from previous experience, by a limitation of its results, and not by any independent process, that we arrive at the idea of space whose dimensions are less than three.

It is doubtless on this account that problems in plane geometry which, although capable of solution on their own account, become much more intelligible, more easy of extension, if viewed in connexion with solid space, and as special cases of corresponding problems in solid geometry. So eminently is this the case, that the very language of the more general method often leads us almost intuitively to conclusions which from the more restricted point of view require long and laborious proof. Such a change in the base of operations has, in fact, been successfully made in geometry of two dimensions, and although we have not the same experimental data for the further steps, yet neither the modes of reasoning, nor the validity of its conclusions, are in any way affected by applying an analogous mental process to geometry of three dimensions; and by regarding figures in space of three dimensions as sections of figures in space of four, in the same way that figures in plano are sometimes considered as sections of figures in solid space. The addition of a fourth dimension to space, not only extends the actual properties of geometrical figures, but it also adds new properties which are often useful for the purposes of transformation or of proof. Thus it has recently been shown that in four dimensions a closed material shell could be turned inside out by simple flexure, without either stretching or tearing; and that in such a space it is impossible to tie a knot.

Again, the solution of problems in geometry is often effected by means of algebra; and as three measurements, or co-ordinates as they are called, determine the position of a point in space, so do three letters or measurable quantities serve for the same purpose in the language of algebra. Now, many algebraical problems involving three unknown or variable quantities admit of being generalized so as to give problems involving many such quantities. And as, on the one hand, to every algebraical problem involving unknown quantities or variables by ones, or by twos, or by threes, there corresponds a problem in geometry of one or of two or of three dimensions; so on the other it may be said that to every algebraical problem involving many variables there corresponds a problem in geometry of many dimensions.

There is, however, another aspect under which even ordinary space presents to us a four-fold, or indeed a mani-fold, character. In modern Physics, space is regarded not as a vacuum in which bodies are placed and forces have play, but rather as a plenum with which matter is coextensive. And from a physical point of view the properties of space are the properties of matter, or of the medium which fills it. Similarly from a mathematical point of view, space may be regarded as a locus in quo, as a plenum, filled with those elements of geometrical magnitude which we take as fundamental. These elements need not always be the same. For different purposes different elements may be chosen; and upon the degree of complexity of the subject of our choice will depend the internal structure or mani-foldness of space.

Thus, beginning with the simplest case, a point may have any singly infinite multitude of positions in a line, which gives a one-fold system of points in a line. The line may revolve in a plane about any one of its points, giving a two-fold system of points in a plane; and the plane may revolve about any one of the lines, giving a three-fold system of points in space.

Suppose, however, that we take a straight line as our element, and conceive space as filled with such lines. This will be the case if we take two planes, *e.g.* two parallel planes, and join every point in one with every point in the other. Now the points in a plane form a two-fold system, and it therefore follows that the system of lines is four-fold; in other words, space regarded as a plenum of lines is four-fold. The same result follows from the consideration that the lines in a plane, and the planes through a point, are each two-fold.

Again, if we take a sphere as our element we can through any point as a centre draw a singly infinite number of spheres, but the number of such centres is triply infinite; hence space as a plenum of spheres is four-fold. And generally, space as a plenum of surfaces has a mani-foldness equal to the number of constants required to determine the surface. Although it would be beyond our present purpose to attempt to pursue the subject further, it should not pass unnoticed that the identity in the four-fold character of space, as derived on the one hand from a system of straight lines, and on the other from a system of spheres, is intimately connected with the principles established by Sophus Lie in his researches on the correlation of these figures.

If we take a circle as our element we can around any point in a plane as a centre draw a singly infinite system of circles; but the number of such centres in a plane is doubly infinite; hence the circles in a plane form a three-fold system, and as the planes in space form a three-fold system, it follows that space as a plenum of circles is six-fold.

Again, if we take a circle as our element, we may regard it as a section either of a sphere, or of a right cone (given except in position) by a plane perpendicular to the axis. In the former case the position of the centre is three-fold; the directions of the plane, like that of a pencil of lines perpendicular thereto, two-fold; and the radius of the sphere one-fold; six-fold in all. In the latter case, the position of the vertex is three-fold; the direction of the axis two-fold; and the distance of the plane of section one-fold; six-fold in all, as before. Hence space as a plenum of circles is six-fold.

Similarly, if we take a conic as our element we may regard it as a section of a right cone (given except in position) by a plane. If the nature of the conic be defined, the plane of section will be inclined at a fixed angle to the axis; otherwise it will be free to take any inclination whatever. This being so, the position of the vertex will be three-fold; the direction of the axis two-fold; the distance of the plane of section from the vertex one-fold; and the direction of that plane, one-fold if the conic be defined, two-fold if it be not defined. Hence, space as a plenum of definite conics will be seven-fold, as a plenum of conics in general eight-fold. And so on for curves of higher degrees.

This is in fact the whole story and mystery of manifold space. It is not seriously regarded as a reality in the same sense as ordinary space; it is a mode of representation, or a method which, having served its purpose, vanishes from the scene. Like a rainbow, if we try to grasp it, it eludes our very touch; but, like a rainbow, it arises out of real conditions of known and tangible quantities, and if rightly apprehended it is a true and valuable expression of natural laws, and serves a definite purpose in the science of which it forms part.

Illustrations.

Again, if we seek a counterpart of this in common life, I might remind you that perspective in drawing is itself a method not altogether dissimilar to that of which I have been speaking; and that the third dimension of space, as represented in a picture, has its origin in the painter's mind, and is due to his skill, but has no real existence upon the canvass which is the groundwork of his art. Or again, turning to literature, when in legendary tales, or in works of fiction, things past and future are pictured as present, has not the poetic fancy brought time into correlation with the three dimensions of space, and brought all alike to a common focus? Or once more, when space already filled with material substances is mentally peopled with immaterial beings, may not the imagination be regarded as having added a new element to the capacity of space, a fourth dimension of which there is no evidence in experimental fact?

Non-Euclid geometry.

The third method proposed for special remark is that which has been termed Non-Euclidean Geometry; and the train of reasoning which has led to it may be described in general terms as follows: some of the properties of space which on account of their simplicity, theoretical as well as practical, have, in constructing the ordinary system of geometry, been considered as fundamental, are now seen to be particular cases of more general properties. Thus a plane surface, and a straight line, may be regarded as special instances of surfaces and lines whose curvature is everywhere uniform or constant. And it is perhaps not difficult to see that, when the special notions of flatness and straightness are abandoned, many properties of geometrical figures which we are in the habit of regarding as fundamental will undergo profound modification. Thus a plane may be considered as a special case of the sphere, viz., the limit to which a sphere approaches when its radius is increased without limit. But even this consideration trenches upon an elementary proposition relating to one of the simplest of geometrical figures. In plane triangles the interior angles are together equal to two right angles; but in triangles traced on the surface of a sphere this proposition does not hold good. To this, other instances might be added.

Further, these modifications may affect not only our ideas of particular geometrical figures, but the very axioms of the Science itself. Thus, the idea which, in fact, lies at the foundation of Euclid's method that a geometrical figure may be moved in space without change of size or alteration of form, entirely falls away, or becomes only approximate in a space wherein dimension and form are dependent upon position. For instance, if we consider merely the case of figures traced on a flattened globe like the earth's surface, or upon an eggshell, such figures cannot be made to slide upon the surface without change of form, as is the case with figures traced upon a plane or even upon a sphere. But, further still, these generalizations are not restricted to the case of figures traced upon a surface; they may apply also to solid figures in a space whose very configuration varies from point to point. We may, for instance, imagine a space in which our rule or scale of measurement varies as it extends, or as it moves about, in one direction or another; a space, in fact, whose geometric density is not uniformly distributed. Thus we might picture to ourselves such a space as a field having a more or less complicated distribution of temperature, and our scale as a rod instantaneously susceptible of expansion or contraction under the influence of heat; or we might suppose space to be even crystalline in its geometric formation, and our scale and measuring instruments to accept the structure of the locality in which they are applied. These ideas are doubtless difficult of apprehension, at all events at the outset; but Helmholtz has pointed out a very familiar phenomenon which may be regarded as a diagram of such a kind of space. The picture formed by reflexion from a plane mirror may be taken as a correct representation of ordinary space, in which, subject to the usual laws of perspective, every object appears in the same form and of the same dimensions whatever be its position. In like manner the picture formed by reflexion from a curved mirror may be regarded as the representation of a space wherein dimension and form are dependent upon position. Thus in an ordinary convex mirror objects appear smaller as they recede laterally from the centre of the picture; straight lines become curved; objects infinitely distant in front of the mirror appear at a distance only equal to the focal length behind. And by suitable modifications in the curvature of the mirror, representations could similarly be obtained of space of various configurations.

Its meaning and use.

The diversity in kind of these spaces is of course infinite; they vary with the mode in which we generalize our conceptions of ordinary space; but upon each as a basis it is possible to construct a consistent system of geometry, whose laws, as a matter of strict reasoning, have a validity and truth not inferior to those with which we are habitually familiar. Such systems having been actually constructed, the question has not unnaturally been asked, whether there is anything in nature or in the outer world to which they correspond; whether, admitting that for our limited experience ordinary geometry amply suffices, we may understand that for powers more extensive in range or more minute in definition some more general scheme would be requisite? Thus, for example, although the one may serve for the solar system, is it legitimate to suppose that it may fail to apply at distances reaching to the fixed stars, or to regions beyond? Or again, if our vision could discern the minute configuration of portions of space, which to our ordinary powers appear infinitesimally small, should we expect to find that all our usual Geometry is but a special case, sufficient indeed for daily use, but after all only a rough approximation to a truer although perhaps more complicated scheme? Traces of these questions are in fact to be found in the writings of some of our greatest and most original Mathematicians. Gauss, Riemann, and Helmholtz have thrown out suggestions radiating as it were in these various directions from a common centre; while Cayley, Sylvester, and Clifford in this country, Klein in Germany, Lobatcheffsky in Russia, Bolyai in Hungary, and Beltrami in Italy, with many others, have reflected similar ideas with all the modifications due to the chromatic dispersion of their individual minds. But to the main question the answer must be in the negative. And, to use the words of Newton, since "Geometry has its foundation in mechanical practice," the same must be the answer until our experience is different from what it now is. And yet, all this notwithstanding, the generalised conceptions of space are not without their practical utility. The principle of representing space of one kind by that of another, and figures belonging to one by their analogues in the other, is not only recognised as legitimate in pure mathematics, but has long ago found its application in cartography. In maps or charts, geographical positions, the contour of coasts, and other features, belonging in reality to the Earth's surface, are represented on the flat; and to each mode of representation, or projection as it is called, there corresponds a special correlation between the spheroid and the plane. To this might perhaps be added the method of descriptive geometry, and all similar processes in use by engineers, both military and civil.

It has often been asked whether modern research in the field of Pure Mathematics has not so completely outstripped its physical applications as to be practically useless; whether the analyst and the geometer might not now, and for a long time to come, fairly say, "hic artem remumque repono," and turn his attention to Mechanics and to Physics. That the Pure has outstripped the Applied is largely true; but that the former is on that account useless is far from true. Its utility often crops up at unexpected points; witness the aids to classification of physical quantities, furnished by the ideas (of Scalar and Vector) involved in the Calculus of Quaternions; or the advantages which have accrued to Physical Astronomy from Lagrange's Equations, and from Hamilton's Principle of Varying Action; on the value of Complex Quantities, and the properties of general Integrals, and of general theorems on integration for the Theories of Electricity and Magnetism. The utility of such researches can in no case be discounted, or even imagined beforehand; who, for instance, would have supposed that the Calculus of Forms or the Theory of Substitutions would have thrown much light upon ordinary equations; or that Abelian Functions and Hyperelliptic Transcendents would have told us anything about the properties of curves; or that the Calculus of Operations would have helped us in any way towards the Figure of the Earth. But upon such technical points I must not now dwell. If however, as I hope, it has been sufficiently shown that any of these more extended ideas enable us to combine together, and to deal with as one, properties and processes which from the ordinary point of view present marked distinctions, then they will have justified their own existence; and in using them we shall not have been walking in a vain shadow, nor disquieting our brains in vain.

Mathematical symmetry.

These extensions of mathematical ideas would however be overwhelming, if they were not compensated by some simplifications in the processes actually employed. Of these aids to calculation I will mention only two, viz., symmetry of form, and mechanical appliances; or, say, Mathematics as a Fine Art, and Mathematics as a

Handicraft. And first, as to symmetry of form. There are many passages of algebra in which long processes of calculation at the outset seem unavoidable. Results are often obtained in the first instance through a tangled maze of formulæ, where at best we can just make sure of our process step by step, without any general survey of the path which we have traversed, and still less of that which we have to pursue. But almost within our own generation a new method has been devised to clear this entanglement. More correctly speaking, the method is not new, for it is inherent in the processes of algebra itself, and instances of it, unnoticed perhaps or disregarded, are to be found cropping up throughout nearly all mathematical treatises. By Lagrange, and to some extent also by Gauss, among the older writers, the method of which I am speaking was recognized as a principle; but beside these perhaps no others can be named until a period within our own recollection. The method consists in symmetry of expression. In algebraical formulæ combinations of the quantities entering therein occur and recur; and by a suitable choice of these quantities the various combinations may be rendered symmetrical, and reduced to a few well known types. This having been done, and one such combination having been calculated, the remainder, together with many of their results, can often be written down at once, without further calculations, by simple permutations of the letters. Symmetrical expressions, moreover, save as much time and trouble in reading as in writing. Instead of wading laboriously through a series of expressions which, although successively dependent, bear no outward resemblance to one another, we may read off symmetrical formulæ, of almost any length, at a glance. A page of such formulæ becomes a picture: known forms are seen in definite groupings; their relative positions, or perspective as it may be called, their very light and shadow, convey their meaning almost as much through the artistic faculty as through any conscious ratiocinative process. Few principles have been more suggestive of extended ideas or of new views and relations than that of which I am now speaking. In order to pass from questions concerning plane figures to those which appertain to space, from conditions having few degrees of freedom to others which have many—in a word, from more restricted to less restricted problems—we have in many cases merely to add lines and columns to our array of letters or symbols already formed, and then read off pictorially the extended theorems.

Mechanical methods.

Next as to mechanical appliances. Mr. Babbage, when speaking of the difficulty of ensuring accuracy in the long numerical calculations of theoretical astronomy, remarked, that the science which in itself is the most accurate and certain of all had, through these difficulties, become inaccurate and uncertain in some of its results. And it was doubtless some such consideration as this, coupled with his dislike of employing skilled labour where unskilled would suffice, which led him to the invention of his calculating machines. The idea of substituting mechanical for intellectual power has not lain dormant; for beside the arithmetical machines whose name is legion (from Napier's Bones, Earl Stanhope's calculator, to Schultz's and Thomas's machines now in actual use) an invention has lately been designed for even a more difficult task. Prof. James Thomson has in fact recently constructed a machine which, by means of the mere friction of a disk, a cylinder, and a ball, is capable of effecting a variety of the complicated calculations which occur in the highest application of Mathematics to physical problems. By its aid it seems that an unskilled labourer may, in a given time, perform the work of ten skilled arithmeticians. The machine is applicable alike to the calculation of tidal, of magnetic, of meteorological, and perhaps also of all other periodic phenomena. It will solve differential equations of the second and perhaps of even higher orders. And through the same invention the problem of finding the free motions of any number of mutually attracting particles, unrestricted by any of the approximate suppositions required in the treatment of the Lunar and Planetary Theories, is reduced to the simple process of turning a handle.

When Faraday had completed the experimental part of a physical problem, and desired that it should thenceforward be treated mathematically, he used irreverently to say, "Hand it over to the calculators." But truth is ever stranger than fiction; and if he had lived until our day, he might with perfect propriety have said, "Hand it over to the machine."

Mathematics and observation.

Had time permitted, the foregoing topics would have led me to point out that the mathematician, although concerned only with abstractions, uses many of the same methods of research as are employed in other sciences, and in the arts, such as observation, experiment, induction, imagination. But this is the less necessary because the subject has been already handled very ably, although with greater brevity than might have been wished, by Professor Sylvester in his address to Section A. at our meeting at Exeter.

Origin of mathematical ideas.

In an exhaustive treatment of my subject there would still remain a question which in one sense lies at the bottom of all others, and which through almost all time has had an attraction for reflective minds, viz., what was the origin of mathematical ideas? Are they to be regarded as independent of, or dependent upon, experience? The question has been answered sometimes in one way and sometimes in another. But the absence of any satisfactory conclusion may after all be understood as implying that no answer is possible in the sense in which the question is put; or rather that there is no question at all in the matter, except as to the history of actual facts. And, even if we distinguish, as we certainly should, between the origin of ideas in the individual and their origin in a nation or mankind, we should still come to the same conclusion. If we take the case of the individual, all we can do is to give an account of our own experience; how we played with marbles and apples; how we learnt the multiplication table, fractions, and proportion; how we were afterwards amused to find that common things conformed to the rules of number; and later still how we came to see that the same laws applied to music and to mechanism, to astronomy, to chemistry, and to many other subjects. And then, on trying to analyse our own mental processes, we find that mathematical ideas have been imbibed in precisely the same way as all other ideas, viz., by learning, by experience, and by reflexion. The apparent difference in the mode of first apprehending them and in their ultimate cogency arises from the difference of the ideas themselves, from the preponderance of quantitative over qualitative considerations in Mathematics, from the notions of absolute equality and identity which they imply.

If we turn to the other question, How did the world at large acquire and improve its idea of number and of figures? How can we span the interval between the savage who counted only by the help of outward objects, to whom 15 was "half the hands and both the feet," and Newton or Laplace? The answer is the history of Mathematics and its successive developments, arithmetic, geometry, algebra, &c. The first and greatest step in all this was the transition from number in the concrete to number in the abstract. This was the beginning not only of Mathematics but of all abstract thought. The reason and mode of it was the same as in the individual. There was the same general influx of evidence, the same unsought for experimental proof, the same recognition of general laws running through all manner of purposes and relations of life. No wonder then if, under such circumstances, Mathematics, like some other subjects, and perhaps with better excuse, came after a time to be clothed with mysticism; nor that, even in modern times, they should have been placed upon an *à priori* basis, as in the philosophy of Kant. Number was so soon found to be a principle common to many branches of knowledge that it was readily assumed to be the key to all. It gave distinctness of expression, if not clearness of thought, to ideas which were floating in the untutored mind, and even suggested to it new conceptions. In "the one," "the all," "the many in one," (terms of purely arithmetic origin,) it gave the earliest utterance to men's first crude notions about God and the world. In "the equal," "the solid," "the straight," and "the crooked," which still survive as figures of speech among ourselves, it supplied a vocabulary for the moral notions of mankind, and quickened them by giving them the power of expression. In this lies the great and enduring interest in the fragments which remain to us of the Pythagorean philosophy.

Their survival,

and transition.

The consecutive processes of Mathematics led to the consecutive processes of Logic; but it was not until long after mankind had attained to abstract ideas that they attained to any clear notion of their connexion with one another. The leading ideas of Mathematics became the leading ideas of Logic. The "one" and the "many" passed into the "whole" and its "parts"; and thence into the "Universal" and the "Particular." The fallacies of Logic, such as the well known puzzle of Achilles and the tortoise, partake of the nature of both Sciences. And perhaps the conception of the Infinite and the Infinitesimal, as well as of Negation, may have been in early times transferred from Logic to Mathematics. But the connexion of our ideas of number is probably anterior to the connexion of any of our other ideas. And as a matter of fact, geometry and arithmetic had already made considerable progress when Aristotle invented the Syllogism.

General ideas.

General ideas there were, beside those of Mathematics—true flashes of genius which saw that there must be general laws to which the universe conforms, but which saw them only by occasional glimpses, and through the distortion of imperfect knowledge; and although the only records of them now remaining are the inadequate representations of later writers, yet we must still remember that to the existence of such ideas is due not only the conception but even the possibility of Physical Science. But these general ideas were too wide in their grasp, and in early days at least were connected to their subjects of application by links too shadowy, to be thoroughly apprehended by most minds; and so it came to pass that one form of such an idea was taken as its only form, one application of it as the idea itself; and Philosophy, unable to maintain itself at the level of ideas, fell back upon the abstractions of sense, and, by preference, upon those which were most ready to hand, namely those of Mathematics. Plato's ideas relapsed into a doctrine of numbers; Mathematics into Mysticism, into Neo-Platonism, and the like. And so, through many long ages, through good report and evil report, Mathematics have always held an unsought for sway. It has happened to this Science, as to many other subjects, that its warmest adherents have not always been its best friends. Mathematics have often been brought in to matters where their presence has been of doubtful utility. If they have given precision to literary style, that precision has sometimes been carried to excess, as in Spinoza and perhaps Descartes; if they have tended to clearness of expression in Philosophy, that very clearness has sometimes given an appearance of finality not always true; if they have contributed to definition in theology, that definiteness has often been fictitious, and has been attained at the cost of spiritual meaning. And, coming to recent times, although we may admire the ingenuity displayed in the logical machines of Earl Stanhope and of Stanley Jevons, in the Formal Logic of De Morgan, and in the calculus of Boole; although as mathematicians we may feel satisfaction that these feats (the possibility of which was clear à priori) have been actually accomplished; yet we must bear in mind that their application is really confined to cases where the subject matter is perfectly uniform in character, and that beyond this range they are liable to encumber rather than to assist thought.

Not unconnected with this intimate association of ideas and their expression is the fact that, which ever may have been cause, which ever effect, or whether both may not in turn have acted as cause and effect, the culminating age of classic art was contemporaneous with the first great development of mathematical science. In an earlier part of this discourse I have alluded to the importance of mathematical precision recognised in the technique of Art during the Cinquecento; and I have now time only to add that, on looking still further back it would seem that sculpture and painting, architecture and music, nay even poetry itself, received a new, if not their first true, impulse at the period when geometric form appeared fresh chiselled by the hand of the mathematician, and when the first ideas of harmony and proportion rang joyously together in the morning tide of art.

Relations of Science to Literature and Art.

Whether the views on which I have here insisted be in any way novel, or whether they be merely such as from habit or from inclination are usually kept out of sight, matters little. But whichever be the case, they may still furnish a solvent of that rigid aversion which both Literature and Art are too often inclined to maintain towards Science of all kinds. It is a very old story that, to know one another better, to dwell upon similarities rather than upon diversities, are the first stages towards a better understanding between two parties; but in few cases has it a truer application than in that here discussed. To recognise the common growth of scientific and other instincts until the time of harvest is not only conducive to a rich crop; but it is also a matter of prudence, lest in trying to root up weeds from among the wheat, we should at the same time root up that which is as valuable as wheat. When Pascal's father had shut the door of his son's study to Mathematics, and closeted him with Latin and Greek, he found on his return that the walls were teeming with formulæ and figures, the more congenial product of the boy's mind. Fortunately for the boy, and fortunately also for Science, the Mathematics were not torn up, but were suffered to grow together with other subjects. And all said and done, the lad was not the worse scholar or man of letters in the end. But, truth to tell, considering the severance which still subsists in education and during our early years between Literature and Science, we can hardly wonder if when thrown together in the afterwork of life, they should meet as strangers; or if the severe garb, the curious implements, and the strange wares of the latter should seem little attractive when contrasted with the light companionship of the former. The day is yet young, and in the early dawn many things look weird and fantastic which in fuller light prove to be familiar and useful. The outcomings of Science, which at one time have been deemed to be but stumbling blocks scattered in the way, may ultimately prove stepping stones which have been carefully laid to form a pathway over difficult places for the children of "sweetness and of light."

Concluding remarks.

The instances on which we have dwelt are only a few out of many in which Mathematics may be found ruling and governing a variety of subjects. It is as the supreme result of all experience, the framework in which all the varied manifestations of nature have been set, that our Science has laid claim to be the Arbiter of all knowledge. She does not indeed contribute elements of fact, which must be sought elsewhere; but she sifts and regulates them; she proclaims the laws to which they must conform if those elements are to issue in precise results. From the data of a problem she can infallibly extract all possible consequences, whether they be those first sought, or others not anticipated; but she can introduce nothing which was not latent in the original statement. Mathematics cannot tell us whether there be or be not limits to time or space; but to her they are both of indefinite extent, and this in a sense which neither affirms nor denies that they are either infinite or finite. Mathematics cannot tell us whether matter be continuous or discrete in its structure; but to her it is indifferent whether it be one or the other, and her conclusions are independent of either particular hypothesis. Mathematics can tell us nothing of the origin of matter, of its creation or its annihilation; she deals only with it in a state of existence; but within that state its modes of existence may vary from our most elementary conception to our most complex experience. Mathematics can tell us nothing beyond the problems which she specifically undertakes; she will carry them to their limit, but there she stops, and upon the great region beyond she is imperturbably silent.

Conterminous with space and coeval with time is the kingdom of Mathematics; within this range her dominion is supreme; otherwise than according to her order nothing can exist; in contradiction to her laws nothing takes place. On her mysterious scroll is to be found written for those who can read it that which has been, that which is, and that which is to come. Everything material which is the subject of knowledge has number, order, or position; and these are her first outlines for a sketch of the universe. If our more feeble hands cannot follow out the details, still her part has been drawn with an unerring pen, and her work cannot be gainsaid. So wide is the range of mathematical science, so indefinitely may it extend beyond our actual powers of manipulation, that at some moments we are inclined to fall down with even more than reverence before her majestic presence. But so strictly limited are her promises and powers, about so much that we might wish to know does she offer no information whatever, that at other moments we are fain to call her results but a vain thing, and to reject them as a stone when we had asked for bread. If one aspect of the subject encourages our hopes, so does the other tend to chasten our desires; and he is perhaps the wisest, and in the long run the happiest among his fellows, who has learnt not only Mathematics, but also the larger lesson which they indirectly teach, namely, to temper our aspirations to that which is possible, to moderate our desires to that which is attainable, to restrict our hopes to that of which accomplishment, if not immediately practicable, is at least distinctly within the range of conception. That which is at present beyond our ken may, at some period and in some manner as yet unknown to us, fall within our grasp; but our science teaches us, while ever yearning with Goethe for "Light, more light," to concentrate our attention upon that of which our powers are capable, and contentedly to leave for future experience the solution of problems to which we can at present say neither yea nor nay.

It is within the region thus indicated that knowledge in the true sense of the word is to be sought. Other modes of influence there are in society and in individual life, other forms of energy beside that of intellect. There is the potential energy of sympathy, the actual energy of work; there are the vicissitudes of life, the diversity of circumstance, health, and disease, and all the perplexing issues, whether for good or for evil, of impulse and of passion. But although the book of life cannot at present be read by the light of Science alone, nor the wayfarers be satisfied by the few loaves of knowledge now in our hands; yet it would be difficult to overstate the almost miraculous increase which may be produced by a liberal distribution of what we already have, and by a restriction of our cravings within the limits of possibility.

In proportion as method is better than impulse, deliberate purpose than erratic action, the clear glow of sunshine than irregular reflexion, and definite utterances than an uncertain sound; in proportion as knowledge is better than surmise, proof than opinion; in that proportion will the mathematician value a discrimination between the certain and the uncertain, and a just estimate of the issues which depend upon one motive power or the other. While on the one hand he accords to his neighbours full liberty to regard the unknown in whatever way they are led by the noblest powers that they possess; so on the other he claims an equal right to draw a clear line of demarcation between that which is a matter of knowledge, and that which is at all events something else, and to treat the one category as fairly claiming our assent, the other as open to further evidence. And yet, when he sees around him those whose aspirations are so fair, whose impulses so strong, whose receptive faculties so sensitive, as to give objective reality to what is often but a reflex from themselves, or a projected image of their own experience, he will be willing to admit that there are influences which he cannot as yet either fathom or measure, but whose operation he must recognize among the facts of our existence.

NOTES.

Page 11, line 28. It is worth while to compare the following passage from Plato's "Republic," Book vii. (Jowett's translation):

After plane geometry, we took solids in revolution instead of taking solids in themselves; whereas after the second dimension the third, which is concerned with cubes and dimensions of depth, ought to have been followed.

It is true, Socrates; but these subjects seem to be as yet hardly explored.

Why, yes, I said, and for two reasons; in the first place, no government patronises them, which leads to a want of energy in the study of them, and they are difficult; in the second place, students cannot learn them unless they have a teacher. But then a teacher is hardly to be found, and even if one could be found, as matters now stand the students of these subjects, who are very conceited, would not mind him; that, however, would be otherwise if the whole state patronised and honoured them, then they would listen, and there would be continuous and earnest search, and discoveries would be made; since even now, disregarded as they are by the world, and maimed of their fair proportions, and although none of their votaries can tell the use of them, still these studies force their way by their natural charm, and very likely they may emerge into light.

P. 22, l. 3. Compare with this the latter part of Plato's "Philebus," on knowledge and the handicraft arts; also Prof. Jowett's Introduction thereto.

P. 25, l. 15. See "Trattato della Pittura," by Leonardo da Vinci; also the Memoir on the MSS. of L. d. V., by Venturi, 1797.

P. 25, l. 24. "The Gentleman and Cabinet Maker's Director," by Thomas Chippendale, London, 1754.

"The Cabinet Maker and Upholsterer's Drawing Book," by Thomas Sheraton, London, 1793.

P. 26, l. 30. See Sorby's Address to the Microscopical Society, 1876.

P. 27, l. 5. Phil. Trans. of the Royal Society, 1870, p. 333; and 1876, p. 27.

Page 27, line 10. Phil. Trans. 1877, p. 149.

P. 27, l. 23. "On Attraction and Repulsion resulting from Radiation," Phil. Trans. 1874, p. 501; 1875, p. 519; 1876, p. 325.

P. 27. l. 26. Philosophical Magazine, April 1878.

P. 27, l. 28. Philosophical Magazine, 1875, Vol. ii., pp. 337, 446; 1877, Vol. i., p. 321; 1878, Vol. i., p. 161.

P. 28, l. 7. Poggendorff's Annalen, Tom. xxxv., p. 337.

P. 28. l. 9. Royal Society's Proceedings, 1878.

P. 28, l. 18. The Papers on the Telephone are too numerous to specify.

P. 28, l. 19. See various Papers in "Nature," and elsewhere, during the last twelve months.

P. 28, l. 25. Royal Society's Proceedings, May 9, 1878.

P. 28, l. 32. Phil. Trans., Vol. 169, pp. 55 and 155, and other Papers catalogued in the Appendix to Part II. of the Memoir.

P. 30, l. 4. See Maxwell "On Heat," chap. xxii.

P. 32, l. 1. Grunert's Archiv., Vol. vi., p. 337; also separate work, Berlin, 1862.

P. 32, l. 1. "Linear Associative Algebra," by Benjamin Peirce, Washington City, 1870.

P. 33, l. 1. Sir W. Thomson, "Cambridge Mathematical Journal," vol. iii., p. 174. Jevons' "Principles of Science," Vol. ii., p. 438.
But an explanation of the difficulty seems to me to be found in the fact that the problem, as stated, is one of the conduction of heat, and that the "impossibility" which attaches itself to the expression for the "time" merely means, that previous to a certain epoch the conditions which gave rise to the phenomena were not those of conduction, but those of some other action of heat. If, therefore, we desire to comprise the phenomena of the earlier as well as of the later period in one problem, we must find some more general statement; viz., that of physical conditions which at the critical epoch will issue in a case of conduction. I think that Prof. Clifford has somewhere given a similar explanation.

P. 38, l. 13. S. Newcomb "On Certain Transformations of Surfaces." American Journal of Mathematics," Vol. i., p. i.

P. 38, l. 14. Tait "On Knots." Transactions of the Royal Society of Edinburgh," Vol. xxviii., p. 145. Klein, "Mathematische Annalen," ix., p. 478.

Page 48, line 31. Royal Society's Proceedings," February 3rd, 1876, and May 9, 1878.

P. 53, l. 17. For example, in Herbart's "Psychologie."

P. 53, l. 19. A specimen will be found in the Moralia of Gregory the Great, Lib. I., c. xiv., of which I quote only the arithmetical part: "Quid in septenario numero, nisi summa perfectionis accipitur? Ut enim humanæ rationis causas de septenario numero taceamus, quæ afferunt, quòd idcirco perfectus sit, quia exprimo pari constat, et primo impari; ex primo, qui dividi potest, et primo, qui dividi non potest; certissimè scimus, quòd septenarium numerum Scriptura Sacra pro perfectione ponere consuevit. . . . . A septenario quippe numero in duodenarium surgitur. Nam septenarius suis in se partibus multiplicatus, ad duodenarium tenditur. . Sive enim quatuor per tria, sive per quatuor tria ducantur, septem in duodecim vertuntur. . . . . Jam superiùs dictum est, quòd in quinquagenario numero, qui septem hebdomadibus ac monade additâ impletur, requies designatur; denario autem numero summa perfectionis exprimetur."

P. 54, l. 2. Approximate dates B.C. of—

Sculptors, Painters, and Poets.		Mathematicians.	
Stesichorus,	600.	Thales,	600.
Pindar,	522–442.	Pythagoras,	550.
Æschylus,	500-450.	Anaxagoras,	500-450.
Sophocles,	495–400.	Hippocrates,	460.
Euripides,	480–400.		
Phidias,	488–432.		
Praxiteles,	450–400.	Theætetus,	440.
Zeuxis,	400.	Archytas,	400.
Apelles,	350.		
Scopas,	350.	Euclid,	323–283.

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