UNIVERSITY OF TORONTO
3 1761 01206408 5

Presented to
The Library
of the
University of Toronto
by
Professor John Satterly
Department of Physics
University of Toronto

Presented to

John Satterly.

A prize

from the

University Correspondence College,

for the 13th place in Honours

at the January Matriculation Exam

of the University of London. 1899

Digitized by the Internet Archive
in 2008 with funding from
Microsoft Corporation

http://www.archive.org/details/advancedmechanic01briguoft

J Satterly

The Organized Science Series.

General Editor: WILLIAM BRIGGS, M.A., LL.B., F.C.S., F.R.A.S.

ADVANCED MECHANICS.

VOL. I.—DYNAMICS.

The Organized Science Series.

ADVANCED MECHANICS.

VOL. I.—DYNAMICS.

BEING "THE TUTORIAL DYNAMICS" TOGETHER WITH THE QUESTIONS IN DYNAMICS OF THE LAST THIRTEEN YEARS SET AT THE ADVANCED EXAMINATION OF THE SCIENCE AND ART DEPARTMENT.

BY

WILLIAM BRIGGS, M.A., F.C.S., F.R.A.S.,

GENERAL EDITOR OF THE UNIVERSITY TUTORIAL SERIES,
PRINCIPAL OF UNIVERSITY CORRESPONDENCE COLLEGE,

AND

G. H. BRYAN, Sc.D., F.R.S.,

SMITH'S PRIZEMAN, LATE FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE.

LONDON: W. B. CLIVE,
UNIVERSITY CORRESPONDENCE COLLEGE PRESS.
WAREHOUSE: 13 BOOKSELLERS ROW, STRAND, W.C.

QA
805
B75
1898
v. 1

LIBRARY
JAN 20 1964
UNIVERSITY OF TORONTO

876719

PREFACE.

In the following pages an attempt has been made to deal with the Dynamics necessary for the Science and Art Second Stage Examination in Theoretical Mechanics (Solids). In order, as far as possible, to separate the principles of Dynamics proper from those applications of Geometry and Trigonometry required to solve the more elaborate problems in the subject, the first ten chapters deal exclusively with Motion in a straight line, the Parallelogram Law being introduced in Chapter XI. This course was adopted in the first instance for two reasons. In the first place it must have been the experience of most teachers that those students who introduce higher analytic methods in a subject like Dynamics at too early a stage are apt to mistake their knowledge of such methods for a knowledge of Dynamics, and to overlook such matters of fundamental importance as relate to units and the like;—indeed we have seen students who could apply the Differential Calculus to problems on Dynamics but who were quite incapable of interpreting their results and became hopelessly mixed between foot-pounds, poundals, ergs and dynes. In the second place, teaching experience soon convinced us that the Parallelogram of Velocities is a far harder proposition to understand than is frequently supposed. The notion of a point possessing simultaneously *two* velocities or accelerations is of course absurd, and we therefore introduced the notion of

Relative Velocities at an early stage in the discussion of the laws of Composition and Resolution. It is interesting to note that since these chapters were written, a writer in one of the Mathematical journals has advocated exactly this mode of treatment. The latter chapters deal with motion down rough and smooth inclined planes, chords of quickest descent, the parabolic path of a projectile, circular and harmonic motion, small oscillations of a pendulum, impact of elastic spheres and the elements of rigid Dynamics. The last named subject has been treated perhaps rather more fully than would be necessary for mere examination purposes, owing to the want, felt by many students, of a preliminary insight into the principles of Rigid Dynamics treated without the use of the Calculus.

For exercises, every chapter but one will be found to be followed by a set of examples, and ten examination papers are also given. Moreover the dynamical questions from the Science and Art Examinations from 1885 to 1897 are given at the end of the book.

We think it desirable to caution readers against relying too much on the "Summaries of Results" to help them through examinations. If the bookwork has been thoroughly mastered they may prove of assistance in remembering some of the more important formulae, but knowledge of these formulae *alone* is practically of no value.

The parts peculiar to the syllabus for the Advanced Stage of the Science and Art Department were written for this edition by Mr. A. G. Cracknell; these we can unhesitatingly submit to the reader with every confidence.

CONTENTS.

EXAMPLES at the end of each Chapter except Chapter V.

INTRODUCTION.

UNITS.

1. **Mechanics defined.—Branches of Mechanics.**—The name **Mechanics** was originally used to designate the science of making machines. It is now, however, very generally applied to the whole theory which deals with motion and with bodies acted on by forces.

The subject Mechanics* is generally divided into two parts—

(1) **Dynamics,** which treats of moving bodies;

(2) **Statics,** which treats of bodies kept at rest under the action of forces.

2. By **force** is meant "any cause which changes or tends to change a body's state of rest or motion." In other words, whatever is capable of setting things in motion or stopping them when they are in motion, or altering the way in which they are moving, is called "force."

Noting that force is defined by means of motion, it is necessary, before considering the properties of forces, to consider the properties of motion itself. This branch of the subject is called **Kinematics.**

We then investigate the properties of force as deduced from the properties of motion; this branch is called **Kinetics.**

---

* There is a little diversity of opinion as to the use of the names Mechanics and Dynamics. Some writers include Statics in Dynamics, thus using the name Dynamics for what we have called Mechanics.

Lastly, in **Statics,** we treat of certain properties of forces which do not involve any consideration of motion.

It is thus evident that very little can be said about force until Kinematics has been dealt with; and for this reason we shall not treat of the measurement of force till Chapter VI.

3. **Origin and uses of Mechanics.**—Mechanics is one of the oldest sciences, for its study originated with the first attempts to make contrivances for raising weights. But it is only within the last three centuries that a simple and consistent theory of the relations of force to matter and motion has been developed. The laws of motion were first discovered by **Galileo** (about the year 1600) from a series of experiments on falling bodies dropped from the top of the leaning tower at Pisa. They were afterwards re-stated by **Newton** in his *Principia* (1687) in the form known as **Newton's Three Laws of Motion,** and as such they are now universally accepted as the basis of Mechanics. Since the time of Newton, no material change has been made in these laws, and, though different writers have modified the wording of them to suit their own particular views, the general principles have never been altered; for every experiment by which the truth of the laws has been tested has added evidence in their favour.

In this book we shall chiefly consider how the laws of motion may be applied to determine the behaviour of given bodies under given forces, not experimentally, but by calculation alone. Such applications of Mechanics are of the greatest practical use; without their aid none of the triumphs of modern engineering skill would have been feasible. In building a bridge, for example, it is of the utmost importance that we should be able to calculate beforehand exactly what pressures have to be sustained by the different parts; otherwise we could not be sure of the safety of the structure.

Nor is this the only use of Mechanics. Newton has applied his laws of motion to the solar system, and his investigations, supplemented by those of later astronomers, have shown that these laws are capable of accounting very simply for all the apparently complicated movements of the heavenly bodies. Moreover, the principles of Mechanics enter prominently into every branch of physical science, such as Heat, Sound, Electricity, and Light.

4. Three fundamental quantities to be measured.— In Mechanics we have to deal with three fundamental notions, namely, **space, time,** and **matter.** It would be difficult, if not impossible, to give an exact *definition* of either of these notions, but they are so familiar to us that this is hardly necessary. It is much more important to show how they can be measured, for in all applications of Mechanics exact measurements of all the quantities with which we are dealing are of the utmost importance.

It is easy enough to measure lengths with a foot rule or a tape, we thus obtain a measure of **space.**

A good watch or clock affords the means of measuring **time,** and it should be observed that in Mechanics we are chiefly concerned with measuring *intervals* of time.

Thus, in speaking of a "time 3 hours," or a "time t," we shall in general mean an *interval* 3 hours long, or an *interval* whose measure is t units of time, and shall not be referring to the *instant* when a clock indicates 3 o'clock or the instant when its indication is denoted by t.

But it is more difficult to specify how quantities of **matter** are to be measured, and before we can do so we must clear the way by the following definitions:—

5. Mass.—DEFINITIONS.—Quantity of matter is called **mass.**

Any limited quantity of matter is called **a body.**

Thus a stone, a piece of earth, wood, or metal, a drop of water, the whole of the Earth's globe, the Sun, and the other "celestial bodies," are all *bodies*.

A particle is a body whose size is so small that it may be regarded as a quantity of matter or mass collected at a single point.

A particle can only exist in theory, but it is often convenient to treat bodies as particles by imagining their mass to be concentrated at a single point.

Mass is characterized by the following properties :—

(1) *The masses of different portions of the same substance under the same conditions are proportional to the spaces they occupy.*

(2) *The mass of the same body is always the same, and is not altered by changing the size of the body either by compressing or heating it or otherwise.*

6. **Measurement of mass.**—If we were to take the *size* or *bulk* occupied by a body as a measure of the quantity of matter contained in it, the second of these conditions would not be satisfied.

Thus, we should have no hesitation in saying that two gallons of water contain twice as much matter as one gallon. But it seems unreasonable to suppose that a lump of lead represents the same quantity of matter as the air which would fill the same space.

Moreover, we can compress air so as to make the same quantity of air occupy a smaller bulk, and, on the other hand, we may convert water into steam by boiling, and it then occupies a far greater bulk than before. But this cannot alter the total quantity of material. Hence the quantity of matter in a body cannot be measured by its volume or bulk.

The usual way of estimating the quantity of matter in a body is by **weighing** it, *i.e.*, placing it in a pair of scales, and balancing it with suitable pieces of metal called "weights" placed in the opposite scale-pan. In the course of the present book it will be shown that *what is commonly called the "weight" of a body gives a correct measure of its mass.*

It would be impossible to use a pair of scales to weigh large quantities of matter, such as a mountain, the Earth, the Sun, or the Moon. Hence to speak, as many writers do, of the "weight of the Earth," is misleading.* The word "mass" is not liable to be misinterpreted, and is always used in books on Mechanics to denote "quantity of matter."

7. **Units.**—To measure any quantity of length, time, or mass, or any other such thing, we must first fix on some definite quantity of the same kind, and call this our **unit**

---

* If, however, "weight" were defined by means of the *universal gravitation* which exists between all bodies, it would be correct to speak of the "weight of the Earth."

of measurement. Having selected this unit, any other quantity will be measured by the *number* of units it contains.

The measurement of quantities in terms of some unit is familiar in every-day life, but the use of the word "unit" in this connection is not so familiar. A few illustrations will make the matter clear. If we speak of a sum of money as (say) five pounds, we imply that, taking a pound as the unit of money, the number of such units in the sum is 5. Similarly, in speaking of six yards of calico, the unit of length is a yard, and the number of such units is 6. And by ten pounds of sugar we mean that, if the unit of *mass* is a pound, the number of such units in the specified quantity of sugar is 10. Notice that the measure 10 pounds specifies the *mass* of the sugar.

The unit of measurement must always be something of the same kind as the quantity to be measured. For measuring a length, we must take some *length* for the unit; for measuring a quantity of matter, we must take some *mass* as our unit. The choice of a unit is, to a certain extent, arbitrary. Certain definite units are very generally adopted, and to these different *names* have been given.

The number which measures any definite quantity depends on what unit is taken. Thus, 24 pence and 2 shillings represent the same sum of money; when a penny is taken as the unit, the number measuring it is 24, and when a shilling is taken as the unit, the same sum is measured by 2. On the other hand, 2 shillings is not the same as 2 pence. Hence, in specifying a definite quantity of anything (*e.g.*, 2 *shillings*), we must give *two* data:—

(1) **The *name* of the unit chosen** (in this case *shillings*).

(2) **The *number* of units in the quantity measured** (in this case 2).

If we left out the word "shillings" and said simply "2," we should leave it quite vague whether we meant 2 shillings, 2 pence, or 2 pounds.

By **change of units** is meant the same thing as "reduction" in Arithmetic. When we reduce from yards to feet, we are given that a length contains (say) 2 yards, and we have to find its measure in feet (viz. 6). This process we shall call *changing the unit of length from a yard to a foot.*

8. The foot-pound-second system; or English system.—The most convenient **unit of length** in common use in England is the **foot** (ft.). A foot is one-third of a yard, the **yard** being defined as the distance between two marks on a certain bar of bronze kept at the offices of the Exchequer in London at a temperature of 62° Fahr.

Smaller lengths may be measured in *inches*, or twelfths of a foot; larger lengths in *miles* (mile = 5280 ft.); but in Mechanics it is better, as a general rule, to measure lengths in feet.

For measuring *areas*, such as the size of a plot of ground, we may take as unit a *square foot*, or the square whose length and breadth are each a foot; while for measuring *volumes*, as, for example, measuring the capacity of a tank, or a volume of water, the unit will be a *cubic foot*, or the capacity of a cube whose length, breadth, and depth are each one foot.

The **unit of time** is the **mean solar second,** the duration of which is derived from the average length of the solar day (1 day = $24 \times 60 \times 60$ seconds). We may, of course, measure long intervals of time in *minutes*, *hours*, or *days*, but for the sake of uniformity it is usually better to use the *second* in Mechanics.

The English **unit of mass** is the **pound avoirdupois** (lb.), and is the mass of a piece of platinum which is preserved in the Exchequer offices.* The mass of any other body is one pound if that body will balance the standard mass when placed in a pair of scales. In this way the standard pound is easily copied, and the mass of a body of moderate size can then be measured in pounds by finding how many pound masses are required to balance it in a pair of scales.

Although the pound is the most convenient unit of mass for general use in Mechanics, its multiple the *ton* (= 2240 lbs.) is often used to measure large masses, and its sub-multiple the *ounce* ($= \frac{1}{16}$ lb.) to measure small masses.

A cubic foot of water contains 1000 *ounces.*

The system of units based on taking the foot, pound, and second, as units of length, mass, and time, respectively, will be spoken of as the **foot-pound-second** or **F. P. S.** system.

---

* In the *Weights and Measures Act*, the pound is defined as the legal standard of weight, because the term "weight" is commonly used to denote what measures "mass," and masses are commonly compared by "weighing" them.

9. The Metric and C. G. S. systems.—The system of weights and measures in common use in France and certain other countries is called the **metric system.**

The **metric unit of length** is the **metre.** It was originally defined as the ten-millionth part of the length of a quadrant of the Earth's circumference measured from the North Pole to the Equator. Thus the whole circumference of the Earth is 40,000,000 or 4×10^7 metres.

The submultiple and multiple units of length are formed by repeatedly dividing or multiplying the metre by 10, as follows, the most important being printed in dark type:—

A metre	=	1000 **millimetres** (mm.).
,,	=	100 **centimetres** (cm.).
,,	=	10 *decimetres.*
10 metres	=	1 *decametre.*
100 ,,	=	1 *hectometre.*
1000 ,,	=	1 **kilometre** (km.).
10,000 ,,	=	1 *myriametre.*

For scientific purposes the unit of length generally adopted is the **centimetre**, or hundredth of a metre.

The unit of mass is the **gramme**, or **gram** (gm.), and was originally defined as the mass of a cubic centimetre of distilled water at the temperature 4° Centigrade.

Thus, if a small cubical box be made, having its length, breadth, and depth (inside measurement) each one centimetre, and if this box be filled with water at the right temperature, previously distilled to render it pure, the mass of this quantity of water is a gramme.*

* Since the introduction of the Metric System, the Earth's circumference and the weight of a cubic centimetre of water have been more accurately determined. But the original standard metre and gramme have been retained; hence the Earth's circumference is not *exactly* 40,000,000 metres, nor is the mass of a cubic centimetre of water *exactly* one gramme. The difference is, however usually neglected.

The submultiple and multiple units derived from the gramme by dividing or multiplying by ten are indicated by the same prefixes as in the case of the metre; thus:—

A gramme	=	1000 **milligrammes** (mgr.).
„	=	100 *centigrammes.*
„	=	10 *decigrammes.*
10 grammes	=	1 *decagramme.*
100 „	=	1 *hectogramme.*
1000 „	=	1 **kilogramme** (kilog. or kgr.).
10,000 „	=	1 *myriagramme.*

The units of time are the same in France as in England. The system of units based on the centimetre as unit of length, the gramme as unit of mass, and the second as unit of time, is called the **centimetre-gramme-second** system, or the **C. G. S.** system, and is used extensively in all countries for mechanical, physical, and electrical measurements.

10. **Advantages of the Metric System.**—From the above description it will be seen that the metric system possesses the following advantages:—

(i.) Each unit is exactly ten times the next smaller unit of the same kind, and therefore in changing the unit there is not the tedious multiplication or division required to reduce from one unit to another in the English system —*e.g.*, from feet to inches or from ounces to pounds.

(ii.) The units of length, volume, and mass are conveniently related. Thus we can write down at once the volume of a quantity of water in cubic centimetres if we know its mass in grammes, and *vice versâ.*

TABLES.

1. Metric Units of Length.

1 centimetre	=	0·3937079 inches.
1 metre	=	39·37079 „
	=	3·2808991 feet.
1 kilometre	=	3280·8991 „
	=	1093·6330 yards
	=	0·6213 miles.

2. Metric Units of Mass.

1 milligramme	=	·0154323488 grains.
1 gramme	=	15·4323488 „
	=	·0353739 oz.
1 kilogramme	=	2·20462 lbs.

3. Velocities.

Velocity of sound in air	=	1,120 feet per second.
„ light	=	186,330 miles per second.
	=	299,860 kilometres per sec.
„ Martini-Henri rifle bullet	=	1,330 feet per second.

4. Intensity of Gravity.

(The numbers represent, in feet and centimetres, *twice* the distance dropped by a falling body during the first second of its motion, at different places at the sea-level.)

Place.	*Ft. per sec. per sec.*	*Cm. per sec. per sec.*
The Equator	32·091	978·10
London	32·191	981·17
Edinburgh	32·203	981·54
The North Pole	32·255	983·11

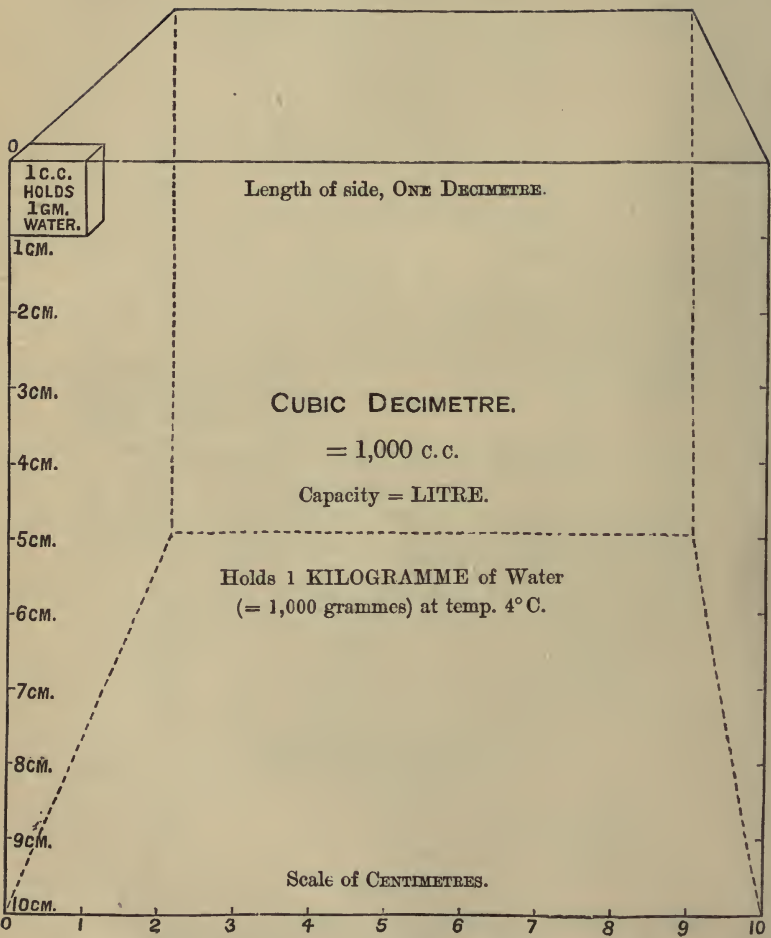
---

5. Densities (approximate).

	Mass of cubic foot in oz.	*Mass of cubic cm. in gms.*
Water	1000	1·0
Atmospheric air	1	·001
Mercury	13568	13·568

6. Power.

One horse power = 550 foot-pounds per second
= 7,460,000,000 ergs per second (roughly).


0
1 c.c.
HOLDS
1 GM.
WATER.
1 CM.
2 CM.
3 CM.
4 CM.
5 CM.
6 CM.
7 CM.
8 CM.
9 CM.
10 CM.
Length of side, One Decimetre.
Cubic Decimetre.
= 1,000 c.c.
Capacity = LITRE.
Holds 1 KILOGRAMME of Water
(= 1,000 grammes) at temp. 4° C.
Scale of Centimetres.
0 1 2 3 4 5 6 7 8 9 10

11. Diagram of the Metric System.—Useful facts.—The opposite diagram represents a cube whose side is one decimetre, the lengths on its front face being drawn to scale. The large cube would hold a kilogramme of water, while the small cube at the left-hand top corner would hold a gramme of water. For, the sides of the two cubes being in the proportion of 10 : 1, their volumes are as $10^3 : 1$, or 1000 : 1.

The following *rough* relations connecting the metric system with other measures will also be found useful for reference. More exact relations are given on page 9.

25 millimetres	= 1 inch.
30 centimetres	= 1 foot.
981 centimetres	= 32·2 feet (double the height dropped by a falling body in one second).
1 decimetre	= 4 inches.
1 metre	= 3 feet $3\frac{3}{8}$ inches.
8 kilometres	= 5 miles.
65 milligrammes	= 1 grain.
$28\frac{1}{3}$ grammes	= 1 ounce.
453 grammes	= 1 pound.
1 kilogramme	= 2 lbs. $3\frac{1}{4}$ oz.
10 kilogrammes	= 22 lbs.
1000 kilogrammes (the French "tonne")	= 1 ton.

The diameter of a halfpenny	= 1 inch.
,, ,, ,, penny	= 3 centimetres.
The mass of a penny	= $\frac{1}{3}$ ounce.
,, ,, sovereign	= 8 grammes.
1 metre	**= 39·3708 inches.**
1 kilogramme	**= 2·204 lbs.**

DYNAMICS.

CHAPTER I.

VELOCITY.

12. By **Kinematics** is meant the study of motion as motion only. Considerations of what is moving or what produces the motion do not enter this branch of the subject.

When a body continues to occupy the same position for any length of time, it is said to be at **rest.** When its position varies, it is said to be in **motion.***

DEFINITION.—**Velocity** is *rate of change of position.*

When a body is *continually* changing its position, the distance it moves depends on the length of time that it is in motion. But if several bodies are moving for the same length of time, the fastest one is that which gets over the greatest distance in the time.

Thus if one railway train travels 60 miles in an hour and another only goes 30 miles in an hour, we say that the former travels twice as fast as the latter.

* Although we shall always speak of the velocity of a *body*, yet the idea of velocity is not always necessarily associated with bodies. Thus *sound* travels with a certain velocity (about 1120 feet per second). It would be more general to speak of the velocity of a "*moving point*," but motion is far more easy to realize when it is some body that moves.

This shows that two velocities may be compared by comparing the distances traversed in the same interval of time.

We know from common experience that a moving body may continue to travel at the same rate for a considerable time, or it may move faster at certain times than at others, and in considering how velocity is to be measured it is necessary to examine whether the motion continues at the same rate; if it does we say that the velocity is *uniform.*

If, for example, a railway train is observed to travel for a number of successive miles, taking exactly one minute over each mile, we might naturally infer that the train is moving with a uniform velocity of one mile in a minute. We should however have a better test of whether the motion is really uniform or not if we could observe whether the distances travelled in each second of time were equal.

But if there were a number of stopping stations at equal distances of 40 miles apart, and a train were to go from each station to the next in an hour, we could not assert that the train was moving with a *uniform* velocity of 40 miles an hour. For if were to measure the distances passed over in smaller intervals of time, say, in each minute, we should find them to be far from equal; the train would be going much faster when midway between two stations than just before stopping at a station or just after starting.

We are now in a position to give the following definitions :—

13. Uniform and variable velocity.—DEFINITIONS.—The velocity of a moving point or body is said to be **uniform** when the distances which it traverses in equal intervals of time are equal, however short these equal intervals may be.

In other cases the velocity is said to be **variable.**

When velocity is uniform, it is measured by the distance traversed in a unit of time.

The word "per" is used in speaking of a rate. Thus we may restate the above definition of the measure of velocity in the following form, which, as we shall show later on, is also applicable to variable velocities :—

Velocity is measured by the distance traversed **per** *unit time.*

The **unit of velocity** is the velocity of a body which moves over a unit of length per unit of time.

The **F. P. S. unit of velocity** is a velocity of **one foot per second.**

The **C. G. S. unit of velocity** is a velocity of **one centimetre per second.**

14. To find the distance traversed in any interval of time by a body moving uniformly.

Let v be the velocity of the body; then, by definition, v is the distance traversed in each successive unit of time.

So, in 2 units of time the total distance traversed is $2v$,
in 3 units of time it is $3v$, and so on;
and in t units of time it is tv.

Hence, if s denote the distance traversed in the interval of time whose measure is t, we have

$$s = vt \quad \text{..........................} \quad \mathbf{(1)},$$

or **distance traversed = (velocity) × (time).**

Examples.—(1) If the velocity is 88 feet per second, the distance traversed in 25 seconds $= 88 \times 25 = 2200$ feet.

(2) If the velocity is 500 centimetres per second, the distance traversed in a minute (60 seconds) is 60×500 or 30,000 centimetres.

(3) To find the number of miles travelled in five minutes with a velocity of 88 feet per second. We cannot put $v = 88$ and $t = 5$ and say "$s = vt = 88 \times 5$," for the velocity 88 is measured in feet per *second* and the time 5 is measured in *minutes*. The formula $s = vt$ is not true unless everything is reduced to one system of units. If we use the foot-second system we must take the time t not as 5 *minutes* but as 300 *seconds*. We then have

$$\text{distance traversed} = 88 \times 300 = 26{,}400 \textit{ feet},$$

because we have taken a *foot* as our unit of length. Reducing this to miles, we find distance traversed = 5 miles.

(4) To find the number of metres described in an hour if the velocity be one centimetre per second.

Since a centimetre and a second are the units employed in defining the velocity, we must reduce the time (1 h.) to seconds. We then have $t = 3600$ sec., $v = 1$ cm. per sec.;

$$\therefore \quad s = 1 \times 3600 \text{ centimetres}$$
$$= 36 \text{ metres}.$$

15. Observations.—Examples (2) and (4) should be carefully studied, as they illustrate the following important points.

In working problems in Mechanics, it is necessary to begin by fixing on some system of units and reducing everything to these units. The answer must always be found in terms of these units in the first place, but it may afterwards be reduced to any other units if required. (Thus in Ex. 2 we reduced the final answer from feet to miles.)

In stating the answer the unit adopted must be mentioned, otherwise the answer might mean *anything* (*vide* § 7).

It should also be borne in mind that algebraical formulæ, as $s = vt$, are only convenient abbreviations of facts, and for this reason they should generally be remembered in words as well as in symbols, and where formulæ are employed the full meaning of the symbols should be distinctly kept in mind.

16. From (1) we have by division

$$v = \frac{s}{t};$$

hence the *velocity of a body may be found by dividing the distance traversed by the time taken in traversing it.*

Examples.—A cyclist rides from one milestone to the next in $4\frac{1}{2}$ minutes. To find his velocity in feet per second.

The distance traversed is one mile or 5280 feet, and the time taken is $4\frac{1}{2} \times 60$ or 270 seconds; therefore in one second the distance traversed in feet $= 5280 \div 270 = 19\cdot\dot{5}$;

$\therefore$ required velocity $= 19\cdot\dot{5}$ feet per second.

17. **Change of units.** — When a given velocity is expressed in terms of any given units of length and time, the same velocity may be referred to any other system of units by using the method illustrated in the following examples:—

Examples.—(1) To express a velocity of (*a*) one mile per hour, (*b*) 60 miles per hour, in feet per second.

(*a*) A mile contains 5280 feet and an hour contains 3600 seconds. Hence with velocity of one mile per hour

in 3600 seconds the distance traversed is 5280 feet;

$\therefore$ in 1 second " " " $\frac{5280}{3600}$ feet.

Therefore the velocity is represented in feet per second by $\frac{5280}{3600}$, *i.e.* $\frac{22}{15}$.

(*b*) A velocity of 60 miles an hour is 60 times as great, and it is therefore represented in feet per second by $\frac{22}{15} \times 60$ or 88.

$$n = n'\left(\frac{L_1}{L}\right)^x\left(\frac{M_1}{M}\right)^y\left(\frac{T_1}{T}\right)^z.$$

(2) When a foot and a second are the units of length and time, the measure of a certain velocity is 27. What is its measure when a yard and a minute are the units?

With a velocity of 27 ft. per sec., 27 ft. are passed over in a second, and, therefore, 27×60 ft. are passed over in a minute;
i.e., 27×20 yards are passed over in a minute;

$\therefore$ a velocity of 27 ft. per sec. = a velocity of 540 yards per minute;

$\therefore$ the measure of the velocity is 540 when estimated in terms of the new units.

OBSERVATION. — The student will find it useful to remember the relation

60 miles an hour = 88 feet per second ... (2).

18. **Positive and negative velocities.** — Where we are dealing with a number of motions in a straight line, some of which motions are in the reverse direction to others, it is convenient to regard velocities in one direction as positive and velocities in the opposite direction as negative, the measures of the latter velocities being negative quantities. A similar convention is also made with reference to the distance traversed, which is considered positive if a body has moved in one direction, and negative if it has moved in the reverse direction, the positive direction being the direction in which it would move with the positive velocity.

With these conventions the equation $s = vt$ always holds true.

The **velocity** of a body is always to be taken as defining both the rate at which it is travelling and the direction in which it is going.

The term **speed** is, however, often used to denote rate of motion considered without reference to direction.

Thus if we take the positive direction to be from left to right, the velocity of a body moving from left to right at the speed of 3 feet per second will be represented by 3, but the velocity of a body moving from right to left at the same speed will be represented by -3.

Again, if the body has moved 5 feet from left to right the distance traversed will be represented by 5, if it has moved 2 feet to the left the distance traversed will be represented by -2.

Example.—A balloon ascends with a velocity of 20 feet per second for half a minute, it then ascends with a velocity of 50 feet per second for one minute, it then descends at the rate of 10 feet per second for 20 seconds, and at the rate of 15 feet per second for 50 seconds. To find its final height above the ground.

If we regard the velocity of the balloon as positive when it is ascending, the velocity will be negative when descending. Hence the velocities during the four intervals, in feet per second, are represented algebraically by

$$+20, \quad +50, \quad -10, \quad -15 \text{ respectively.}$$

Also the intervals of time are

$$30, \quad 60, \quad 20, \quad 50 \text{ seconds respectively.}$$

Therefore the distances through which the balloon *rises* are represented algebraically by

$$20 \times 30, \quad 50 \times 60, \quad -10 \times 20, \quad -15 \times 50 \text{ feet respectively.}$$

The whole height to which the balloon has risen is the algebraical sum of the heights risen by the balloon in the several intervals (each taken with its proper sign), and is therefore

$$= 600 + 3000 - 200 - 750 = 2650 \text{ feet.}$$

Observation.—In this example the *minus* sign before the velocity may be simply regarded as a convenient way of representing the fact that "the height of the balloon is becoming less." Of course we could dispense with the use of signs by stating the same fact in words, and distinguishing the various distances as the heights "risen" and "fallen" respectively. This would not make the work much more laborious in the above example, but in more complicated problems the use of signs to denote directions greatly simplifies the formulæ and calculations.

19. **Representation of direction by the order of letters.**—In future, when we speak of "the straight line AB," we shall imply that the line is drawn *from* A *to* B, not from B to A. If we use the signs $+$ and $-$ to denote directions, as in § 18, the distance AB is considered positive if we have to go in the positive direction to get from A to B, negative if we have to go in the reverse direction. If we interchange the order of the letters, and write the distance as BA, we imply that it is measured *from* B *to* A, *i.e.*, in the reverse direction to what it was before. For this reason, BA is to be considered equal and opposite to AB, or, as we may express it,

$$BA = -AB, \text{ or } BA + AB = 0.$$

This relation may be taken as another way of stating that if we go a certain distance in one direction and then go an equal distance in the reverse direction, we get back to where we started from.

20. **Relative velocity.**—DEFINITION.—By the velocity of one body **relative** to another is meant the rate at which the first body is changing its position with respect to the second.

The meaning and importance of relative velocity will be best understood from the following simple illustrations:—

(1) Suppose that a man on board a large steamer is walking along the deck. We naturally say that he is in motion, because he is walking. But this motion along the deck is only a *relative motion*, and is not the true motion which he possesses, for he is also at the same time being carried forward by the motion of the steamer. And if the man remains standing in the same part of the deck we know that he is not really at rest, but that he is moving with the steamer.

(2) Next, suppose that the steamer overtakes a small boat out at sea, and after passing it leaves it behind. To a passenger on board the steamer the small boat presents the appearance of moving swiftly past the steamer from the bows towards the stern. But this appearance of motion is really produced by the steamer itself passing the boat in the opposite direction.

Moreover the boat may itself also be moving in the same direction as the steamer, but if the steamer is going more quickly the boat will fall behind and a passenger on board will think it is going in the opposite direction. All that he can observe is the *relative motion* of the boat and steamer with respect to one another. Unless there is land in sight he has no other object with which to compare this motion and find out what part of the motion belongs to the ship and what part to the boat.

(3) Suppose two railway trains, A and B, are drawn up side by side. A passenger in A only sees the carriages of the train B begin to move past the windows of his own carriage. From this he concludes that one of the trains is in motion, but he will find it impossible to tell whether (i.) his own train A is moving in one direction, or (ii.) the other train B is moving in the opposite direction, or (iii.) both trains are moving in opposite directions, or (iv.) both are moving in the same direction but one is moving faster than the other, so that they do not keep together. All he can say by observing B is that the two trains A, B have a *relative motion*.

If, however, he looks out at the station on the other side, he will see whether his position is changing with reference to the station and other surrounding objects. If so, he concludes that his train is moving. Even in this case the same appearance would be presented if his train were to remain at rest and the station were to begin moving backwards, as the telegraph poles by the line often seem to do. It is only from the results of previous experience that he is able to assert with certainty that his train and not the station is moving.

(4) If a fast train overtakes and passes a slow train, a passenger in the former will obtain the impression that the latter is going backwards, because he is going faster and leaves it behind. This apparent backward velocity is the *relative velocity* of the slow train with respect to the fast. The slow train is really moving forwards all the time, but as it is not going fast enough to keep up with the fast one, it appears to go in the opposite direction to the latter.

These illustrations show that our ideas of motion are purely relative. We can only fix the position of a body by comparing it with other bodies. If a body A is gradually changing its position with reference to another body B, we can, by observing this change of position, find the velocity of A relative to B. But the body B may itself be moving, and in that case the actual velocity of A will not be the same as its velocity relative to B.

Again, if we observe that A always remains in the same relative position with respect to B, we cannot say that the two bodies A and B are both at rest; for if they are moving together with the same velocity they will still continue to retain the same relative positions, and will therefore have the appearance of being at rest relatively to each other.

Unless, then, we are given some point which may be regarded as fixed, we can only regard velocities as relative.

21. **Application to the Earth.**—We are accustomed to consider the Earth as fixed, because we see on it trees, houses, hills, and other objects which appear to retain the same relative positions always. And so in measuring velocities we naturally refer them to the Earth. We observe that the Sun, Moon, and stars rise in the east, and set in the west. At first, we should naturally say the stars are moving, and that we are at rest. But when we watch the stars for a long time, their relative configurations never appear to change; hence, if they are moving, they must be moving together.

Now it is much easier to believe that a single body such as the Earth, which is only 8000 miles in diameter, should be moving as a whole than that the stars, which are separate and distinct bodies, enormously larger than the Earth, and at distances of many billions of miles apart, should be all revolving together about the Earth once in a day, so as to always remain in the same configurations. This and other reasons force upon us the fact that it is the Earth which rotates once a day, and not the stars that move. Further, we are taught that the Earth travels round the Sun, describing roughly a circle of radius 92 million miles in the course of the year, and flying through space at the rate of about $18\frac{1}{2}$ miles a second.

In most cases we do not have to take account of these motions of the Earth. The relative motions can be worked out in just the same way as if the Earth were fixed. In fact, most of our ideas of motion are based on experiments made with moving bodies on the Earth, and they really refer quite as much to relative motion as to actual motion.

22. **Properties of relative velocity.**—From the arguments and examples of the last paragraphs, the following properties will be evident.

When two bodies, A and B, are moving in any manner, the velocity of B *relative to* A is the velocity with which B would appear to move if the observer were moving with the body A.

If the bodies are moving in the same straight line, the rate at which the faster body overtakes and passes the slower one is the relative velocity of the former with respect to the latter.

If A and B are the bodies in this case, the relative velocity of B with respect to A is the rate at which the distance AB (measured from A towards B) increases, and is measured by the increase in the distance AB in a unit of time.

Examples.—(1) To find the relative velocity of two trains, A and B, both travelling in the positive direction at the rates of 30 and 50 miles an hour respectively.

In one minute the train A has travelled $\frac{1}{2}$ mile and B has travelled $\frac{5}{6}$ mile. Hence B gains $(\frac{5}{6}-\frac{1}{2})$ mile $= \frac{1}{3}$ mile on A in every minute. Hence the distance that B overtakes and passes A is the same as if A stood still and B moved forward at the rate of $\frac{1}{3}$ of a mile a minute, or 20 miles an hour, and this is therefore the velocity of B relative to A

But the train A falls behind B at the same rate, namely $\frac{1}{3}$ of a mile a minute. Hence the relative motion is the same as if the train B

stood still and the train A moved backward at the rate of 20 miles an hour. Therefore the velocity of A relative to B will be -20 miles an hour.

(2) Suppose the trains A, B are travelling in opposite directions at the rates of 30 and 50 miles an hour.

If the velocity of B be called $+50$ miles per hour or $+\frac{5}{6}$ miles per minute, that of A will be -30 miles an hour or $-\frac{1}{2}$ mile per minute. When the trains have passed each other it is clear that *both* velocities will tend to increase the distance between them. In one minute from the time they pass, they will evidently have separated a distance of $\frac{5}{6}+\frac{1}{2}$ miles or $\frac{4}{3}$ miles. B will then have got $\frac{4}{3}$ miles to the positive side of A, and A will have got $\frac{4}{3}$ to the negative side of B.

Hence the relative velocity of B is $+\frac{4}{3}$ miles per minute or $+80$ miles per hour, and that of A is $-\frac{4}{3}$ miles per minute or -80 miles per hour.

23. To find the relative velocity of two bodies moving with given velocities in the same straight line, we have the following rules:—

When two bodies are moving along the same straight line, their relative velocity is the difference of their actual velocities.

The velocity of one body relative to another is equal and opposite to that of the latter relative to the former.

To prove the above properties generally, let A, B be the positions of the moving bodies at any instant, A', B' their positions after a unit of time has elapsed, and let O be any point from which distances are measured. Since the distance between the bodies increases from AB to $A'B'$ in a unit of time, therefore the velocity of B relative to A is measured by the increase $A'B'-AB$. Also, AA', BB' measure the velocities of A and B respectively.

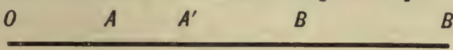


Fig. 2.

Now, $$A'B' = OB' - OA',$$
$$AB = OB - OA;$$

$\therefore$ subtracting, $$(A'B' - AB) = BB' - AA';$$

i.e. **(vel. of B rel. to A) = (vel. of B) − (vel. of A).**

Similarly, if due attention is paid to sign,

(vel. of A relative to B) = (vel. of A) − (vel. of B);

$\therefore$ **(vel. of A rel. to B) + (vel. of B rel. to A) = 0.**

24. Composition of velocities in one straight line.—From the consideration of relative velocities we naturally pass on to cases where the velocity of a body is due to a number of independent relative motions. We may take the following as illustrations of such motions:—

Examples.—(1) A river is flowing at the rate of 1 mile an hour, and a man can row a boat through the water at 4 miles an hour. To find his rate of progress (i.) down stream, (ii.) up stream.

Let $B'AB$ be the direction of the river, O the man's starting-point. Then in one hour the water that was at O will have flowed to a point A one mile from O, and if the man had allowed his boat to drift it would have reached A.

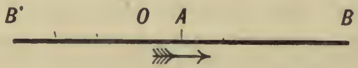


Fig. 3.

But the man has pulled his boat 4 miles through the water. Hence, if he is pulling down stream, his action in rowing during the hour will have taken the boat to a point B four miles below A.

The whole space OB is $= 4 + 1 = 5$ miles; hence the man's rate of progress down stream $= 5$ miles an hour.

But if the man pulls up stream, the action of his oars during the hour will take him 4 miles through the water to a point B' four miles above A.

In this case the whole space OB' (measured up stream) $= 4 - 1 = 3$ miles; hence the man's rate of progress up stream $= 3$ miles an hour.

(2) A steamer is travelling at 20 feet per second, and a man paces the deck at the rate of 4 feet per second. To find how far the man has actually moved in 10 seconds, and his actual velocity when he is going (i.) towards the bows, (ii.) towards the stern.

Let A represent his position on the deck at the beginning of the interval of 10 sec. At the end of the interval the ship has advanced 200 feet, and the part of the deck where he originally stood has moved to a point B, 200 feet in front of A.

But the man has walked over 40 feet of the deck, and has therefore got to a point C, 40 feet from B. Hence the actual distance the man has gone is from A to C.

(i.) When the man is going towards the bow (Fig. 4), we have

$$AC = AB + BC = 200 + 40 = 240 \text{ feet};$$

and, since the man moves over this distance in 10 sec., his velocity is 24 feet per sec.

Fig. 4.

Fig. 5.

(ii.) When the man is going astern (Fig. 5), we have

$$AC = AB - CB = 200 - 40 = 160 \text{ feet};$$

and the man's velocity is therefore 16 feet per second.

25. Component and resultant velocities.—DEFINITION.—If the different parts of a moving system have certain relative velocities which determine the motion of any body in the system, these relative velocities are called the **component** velocities of the body, and its actual velocity is called its **resultant** velocity.

Thus, in the first example of § 24, the rate of flow of the stream and the rate at which the man rowed are called the *components* of the velocity of the boat. The actual rate of progress is called the *resultant* velocity of the boat. In the second example, the velocity of the boat and the man's rate of walking are the man's *component* velocities, and his actual rate of moving through space is his *resultant* velocity.

The process of finding the resultant velocity from the components is called **compounding** velocities.

To compound several velocities in the same straight line we add them together.

For, if A, B, C are any bodies moving in the same straight line (Fig. 4),

$$\text{vel. of } B - \text{vel. of } A = \text{vel. of } B \text{ rel. to } A,$$

$$\text{vel. of } C - \text{vel. of } B = \text{vel. of } C \text{ rel. to } B;$$

$$\therefore \quad \text{vel. of } C = (\text{vel. of } A) + (\text{vel. of } B \text{ rel. to } A) + (\text{vel. of } C \text{ rel. to } B);$$

or, resultant vel. = sum of component vels.

This is always true for motions in one straight line, provided that their directions are denoted by the signs + and − in the manner explained in § 18.

26. **Variable velocity** may be measured in two ways—

(i.) By the **average velocity** in any given interval of time;

(ii.) By the velocity **at** any given instant.

Both these ways are commonly used in speaking of the speed of railway trains, steamers, &c. If a train travels from one station to another 20 miles distant in half-an-hour, we say that the **average velocity** of the train is 40 miles **per** hour. If, on the other hand, the train is observed at any part of the journey to go a mile a minute, we say that the velocity **at** that particular time is 60 miles **per** hour. The expression 60 miles **per** hour does not mean that the train actually goes 60 miles in any particular hour, but that it would go 60 miles in an hour if it were kept going at the same rate all the time.*

27. Average velocity.—DEFINITION.—The **average velocity** of a moving body in any given interval of time is the velocity with which a body would have to move *uniformly* in order to traverse the same distance in the same time.

If a body traverses a distance s in time t with *uniform* velocity v, then v is the distance passed over in a unit of time, and by §§ 14, 16, $s = vt$,

$$v = \frac{s}{t}.$$

If, however, the velocity is variable, the fraction s/t does not represent the *actual* velocity but the *average* velocity during the interval. Therefore

$$\text{average velocity} = \frac{\text{distance traversed}}{\text{time}},$$

or **distance traversed = (av. vel.) × (time).**

Example.—If 35 feet is traversed in $2\frac{1}{2}$ seconds, the average velocity is $35/2\frac{1}{2}$ or 14 feet **per** second. If the motion is uniform, the distance actually traversed in each second is 14 feet. If the velocity is variable,

* The use of the word "**per**" to denote a rate is familiar. We may invest a sum of money at 5 **per** cent. **per** annum, and by using the word "per" we avoid implying that the sum amounts to £100 or remains invested for a year.

this is not the case, and it is only in *one particular* interval of $2\frac{1}{2}$ seconds that the body traverses 35 feet. In the next $2\frac{1}{2}$ seconds it may traverse say 40 feet or 30 feet, and the average velocity will then be different.

28. Velocity at any instant. — DEFINITION. — The velocity of a body **at any instant** is measured by the rate **per** unit time at which distance is being traversed by the body *in the immediate neighbourhood of that instant.*

It will be noticed that a body can move over no distance in *no* time, so that we could not find its velocity by observing its position *at* one single instant. To find its rate of motion, we must observe the distance traversed during some interval of time near the given instant, even though we may make this interval as short as we like Hence the term velocity **at** any instant must be regarded as a convenient abbreviation for **average velocity during a very small interval** of time including the given instant.

By taking the interval very small, the velocity *has no time to alter in it*, and *the interval we consider must be so small that the rate of motion cannot change at all.*

Thus the speed of a railway train might vary considerably in 5 minutes, but in, say, so short an interval as a tenth of a second there would not be time for the rate of motion to alter appreciably.

SUMMARY OF RESULTS.

For motion with uniform velocity v, or variable velocity when v is the average velocity in the interval t,

$$s = vt \qquad \ldots\ldots (1);$$

$$60 \text{ miles an hour} = 88 \text{ feet per second} \qquad \ldots\ldots (2).$$

EXAMPLES I.

1. Find the measures of the following velocities, a foot and a second being the units of length and time :—

(i.) Sixty miles per hour ;

(ii.) Thirty yards per minute ;

(iii.) Four hundred feet in half-an-hour.

2. Find the measures of the following velocities, a yard and a minute being the units :—

(i.) Sixty feet per second;
(ii.) Sixty miles per hour;
(iii.) Thirty miles per half-hour.

3. A body has uniform velocity 16 feet per second; how far will it go in a minute?

4. A body moves with uniform velocity, whose measure is 180 if a yard be the unit of length and a minute the unit of time. How far will it go in 3 seconds?

5. How far will an express train, travelling uniformly at a rate of forty-five miles an hour, go in 6 seconds?

6. If the unit of time be a minute and the unit of length be a yard, what will be the measure of the velocity of a body which describes, at a uniform rate, 14 miles in 3 hours?

7. A mile race was run in 4 mins. 35 secs. What was the winner's average velocity in feet per second?

8. A train 215 yards long, going at the rate of 55 miles an hour, takes 10 seconds in passing another train going in the opposite direction at the rate of 35 miles an hour. What is the length of the second train?

9. A train going at the rate of 45 miles an hour takes half a minute in passing another train 230 yards long going in the same direction at the rate of 15 miles an hour. What is the length of the first train?

10. If u be the measure of a velocity in foot-second units, what is its measure in yard-minute units?

11. What is the measure of the centimetre-second unit of velocity (i.) in metres per minute, (ii.) in kilometres per hour, (iii.) when a quadrant of the Earth's circumference and a year are units of length and time?

12. A train travels 45 kilometres in an hour. What is its velocity (i.) in centimetres per second, (ii.) in metres per minute; and how many days would it take to travel over a distance equal to the Earth's circumference?

CHAPTER II.

ACCELERATION.

29. The velocities of which we shall treat in this chapter will be variable velocities, and we shall always suppose them to be measured by the velocities **at** different instants of time as defined in § 28.

DEFINITION.—**Acceleration** is *rate of change of velocity.*

When the velocity of a body is changing, its motion is said to be **accelerated.**

Thus, when a railway train has just started and is getting up speed its motion is accelerated.

30. Uniform and variable acceleration.—DEFINITIONS.—Acceleration is said to be **uniform** when the velocity always increases by equal amounts in equal intervals of time. In other cases acceleration is said to be **variable.**

Uniform acceleration is measured by the amount by which the velocity increases per unit of time.

31. Units of acceleration.—DEFINITION.—The **unit of acceleration** is the acceleration of a body which moves so that the measure of its velocity increases by unity in a unit of time.

In the F. P. S. system of units, where the unit of velocity is a velocity of one foot per second, the unit of acceleration is the acceleration which in one second increases the velocity by one foot per second, and this may be called an acceleration of **"one foot per second per second."**

Similarly, **the C. G. S. unit of acceleration** is an acceleration of **"one centimetre per second per second."**

OBSERVATION.—The words "per second" must be repeated because the unit of time is involved twice, firstly in measuring the velocity or change of velocity, and secondly in measuring the interval in which this change of velocity takes place.

Examples.—(1) If a body is moving at the rate of 5 feet per second at any instant, and its velocity one second later is 7 feet per second, the increase of velocity in one second is 2 feet per second, and therefore the acceleration is 2 feet per second per second.

(2) If in one second the velocity changes from 10 feet per second to 8 feet per second, the increase of velocity is $= 8-10 = -2$ feet per second, and the acceleration is -2 feet per second per second.

(3) Similarly, if the velocities at intervals of one second are 53, 59, 65, ... centimetres per second, the acceleration is 6 centimetres per second per second.

32. Having given the acceleration (supposed uniform), to find the velocity at any given instant.

Let f be the given acceleration, and let it be required to find the velocity acquired after t units of time have elapsed.

(i.) Suppose that the moving body starts from rest. Then, since the acceleration $= f$,

the velocity acquired in 1 unit of time $= f$.

In the next unit of time the velocity increases by f;

$\therefore$ the velocity acquired in 2 units of time $= 2f$,

similarly, the velocity acquired in 3 units of time $= 3f$,

and the velocity acquired in t units of time $= ft$.

Hence, if v denote the required velocity,

$$v = ft \text{ .... } (1).$$

(ii.) Suppose that the body starts with initial velocity u.

Then, as before, the amount by which the velocity *increases* in the interval $t = ft$.

But the velocity at the beginning of the interval $= u$;

$\therefore$ the velocity at the end of the interval $= u + ft$;

$\therefore$ in this case $v = u + ft$ .... **(2)**,

or $v - u = ft$.

In words,

(increase of velocity) = (acceleration) × (time).

OBSERVATION.—As in § 15, it should be noticed that the above formula only holds good provided that all the quantities are expressed in terms of the same units of length and time.

Examples.—(1) A train acquires a velocity of 60 miles an hour in two minutes. To find its acceleration in F.P.S. units.

In 2 min. (= 120 secs.) the velocity increases by 60 miles per hour = 88 feet per second;

$\therefore$ in one second the velocity increases $\frac{88}{120}$ feet per second.
Therefore the given acceleration is $\frac{88}{120}$ or ·73 feet per sec. per sec.

(2) If the acceleration is 32 feet per second per second, and the body starts with the velocity 100 feet per second, the velocity after ten seconds $= 100 + 32 \times 10 = 420$ feet per second.

33. **Change of units.**—When an acceleration is expressed in terms of one system of units, we may reduce it to any other system of units, by adopting the method illustrated in the following examples :—

Example.—(1) To express an acceleration of 32 feet per second per second in yards per minute per minute.

Here we are given that the increase of velocity in one second is 32 feet per second. In order to change to the new units we must

(1) find the increase of velocity in one minute;

(2) express this increase of velocity in yards per minute.

We accordingly proceed as follows :—

In 1 sec. total increase of velocity = 32 feet per second;

∴ in 1 minute (60 secs.) the total increase of velocity

$$= 32 \times 60 = 1920 \text{ feet per second}$$
$$= 1920 \times 60 \text{ feet per minute}$$
$$= \frac{1920 \times 60}{3} \text{ yards per minute}$$
$$= 38400 \text{ yards per minute.}$$

Therefore the given acceleration is 38400 yards per minute per minute.

OBSERVATION.—The change of the time unit from a second to a minute is repeated **twice**. There would be nothing absolutely wrong or illogical in speaking of the acceleration as "an acceleration of 1920 feet per *second* per *minute*," but such a hybrid representation, involving two different time units, would be confusing.

Example.—(2) A body is moving with an acceleration of 54000 miles per hour per hour. Express this in feet per second per second.

A velocity of 54000 miles per hr. = a vel. of $\frac{54000 \times 5280}{60 \times 60}$ ft. per sec.

= a vel. of 1800×44 ft. per sec.

This velocity is gained every hour;

∴ the gain per second is $\frac{1800 \times 44}{60 \times 60}$ ft. per sec. = 22 ft. per sec.;

therefore an acceleration of 54000 miles per hour per hour

= an acceleration of 22 ft. per sec. per sec.

34. **Positive and negative accelerations.**—In § 18 we explained how the velocities of bodies moving in opposite directions are distinguished by prefixing the signs + and − to the numbers which measure them.

Now accelerations are measured by the velocities added per unit time; hence an acceleration must be considered positive if this added velocity is positive, and negative if the added velocity is negative, and this will depend on the direction in which the change of motion is taking place. *When the velocity of a body is uniform the acceleration is zero*, for no increase takes place in the velocity.

Examples.—(1) If the velocities at intervals of a second are −9, −6, −3, 0, 3, 6, ... feet per second respectively, the acceleration is uniform and is +3 feet per sec. per sec., for each velocity is obtained by adding 3 to the previous velocity. It will be noticed that the *speed diminishes* when the velocity is *negative* and *increases* when the velocity is *positive*.

(2) If the velocities at successive seconds are 7, 4, 1, -2, -5, ..., the acceleration is -3; here the *speed* diminishes as long as the velocity is positive, but increases again when the velocity has changed sign and become negative. The effect is therefore the reverse of that in Ex. 1.

35. **Retardation.**—When the **speed** of a body is decreasing, the motion is said to be **retarded.** In the above examples it will be found that the motion is always retarded when the acceleration is in the opposite direction (of opposite sign) to the velocity, and we can easily see that this property is perfectly general.

Thus the acceleration of a body is *positive*—

(i.) If the velocity is positive and the speed is increasing;

(ii.) If the velocity is negative and the speed is decreasing.

Similarly, the acceleration of a body is *negative*—

(i.) If the velocity is positive and the speed decreasing;

(ii.) If the velocity is negative and the speed increasing.

36. Observation. — When signs are used to denote direction, *no alteration should on any account be made in the form of the fundamental equations of motion.* Thus (2) § 32 is *always* to be written $v = u + ft$, *never* $v = u - ft$, even if the motion is retarded. If a body moving in the positive direction is being retarded, the number f measuring the retardation is to be regarded as a negative quantity.

Examples.—(1) A body starts with velocity 144 feet per second, and is subject to a retardation of 32 feet per second per second. To find its velocity after 5 seconds.

Here $u = 144$, $f = -32$, $t = 5$; whence, substituting in the formula $v = u + ft$, we have

$$v = 144 + (-32) \times 5 = 144 - 160 = -16.$$

Hence the body is moving with speed 16 feet per second in the *opposite* direction to that in which it started.

(2) If a railway train moving at 60 miles an hour (88 feet per second) is brought to rest in one minute (60 secs.), and we consider the original velocity positive, the acceleration $= (v - u) \div t$

$$= (0 - 88) \div 60 = -88/60 = -22/15 \text{ feet per sec. per sec.},$$

and is negative. If this acceleration were continued for another minute, the train would acquire a velocity of -88 feet per second, that is its original velocity would be exactly reversed.

37. **Relative acceleration.** — *Definition. — The acceleration of any body B relative to another body A is the rate of increase of the velocity of B relative to A. It is therefore measured by the amount by which this relative velocity increases per unit time, and is subject to the usual conventions as regards algebraic signs.

Since always vel. of B rel. to A = vel. of B − vel. of A, the same relation connects the amounts by which these velocities increase in any interval, say in a unit of time;

∴ **accel. of B rel. to A = accel. of B − accel. of A.**

In like manner we may deduce from § 23 that

accel. of B rel. to A + accel. of A rel. to B = 0;

and other properties of relative velocities may be extended to relative accelerations in the same way.

For example, accelerations in the same straight line may be compounded by adding them together.

38. **Variable acceleration** is measured in a very analogous way to variable velocity. It may be measured either

(i.) By the average acceleration in any given interval;

(ii.) By the acceleration at any given instant.

39. Definition.—The **average acceleration** of a body in any given interval of time is measured by the amount by which the velocity would increase **per** unit time, if the body were to be uniformly accelerated during the interval, and to have the initial and final velocity.

Hence, since for uniform acceleration

$$v - u = ft, \quad \text{or} \quad f = \frac{v-u}{t},$$

it follows that $(v-u) \div t$ represents the **average acceleration** of any body whose velocity changes from u to v in time t.

* Observation.—The reader is urged to apply the following principles to some simple illustrative examples, *e.g.* to find the relative acceleration of two railway trains that are starting or being retarded, and are travelling in the same or opposite directions.

40. Definition.—**The acceleration at a given instant of time** is measured by the rate **per** unit time at which the velocity is increasing in the *immediate neighbourhood of the given instant*, or the **average acceleration in a small interval of time,** including the given instant.

As in the case of velocity, the acceleration at an instant could only be estimated by observing the increase of velocity in a small interval of time, including the given instant, and if the interval be taken sufficiently small, the acceleration will not have time to change appreciably in it.

Summary of Results.

For motion from rest, $v = ft$ .... (1)

For motion with initial velocity u,

$$v = u + ft \quad \text{.....................} \quad (2),$$

where the acceleration f is either uniform or is the average acceleration in the interval of time t.

EXAMPLES II.

1. How is the measure of an acceleration changed if

(i.) the unit of space be changed from a foot to a yard,

(ii.) the unit of time be changed from a second to a minute?

2. If the measure of the acceleration due to gravity be 32 when a foot and a second are taken as units, what will it be when the units of length and time are

(i.) an inch and a second,

(ii.) a yard and a minute,

(iii.) a mile and an hour?

3. Taking 72 as the measure of an acceleration when a yard and a minute are the units of length and time, find its measure when a furlong and an hour are the units.

4. A body uniformly accelerated is found to be moving, at the end of 8 seconds, with a velocity that would carry it through 30 miles in the next 10 minutes. Find its acceleration.

5. A body is moving with an acceleration of 1000 yards per minute per week. What is the measure of this acceleration when an inch and an hour are the units of space and time?

6. A body uniformly accelerated is found to be moving at the end of 8 seconds with a velocity which would carry it through 60 miles in the next hour. Find the acceleration.

7. If the measure of a uniform acceleration be 60 referred to a mile and a minute as units of space and time, what will be its measure when the units of space and time are a foot and a second respectively?

8. If V and f are the measures of a velocity and an acceleration when a foot and a second are the units of length and time, find their measures when a yard and a minute are the units of length and time.

9. Express the C.G.S. unit of acceleration, and the acceleration of gravity (980 cm. per sec. per sec.),

(i.) in metres per minute per minute,

(ii.) in terms of a kilometre and an hour as units.

10. A body is moving with a velocity sufficient to carry it through a distance equal to the Earth's circumference in 24 hours. Twelve hours later it is moving with an equal velocity in the opposite direction. Find the acceleration in C.G.S. units, supposing it uniform.

EXAMINATION PAPER I.

1. Explain what is meant by the *acceleration* of a point moving in a straight line.

2. How are velocity and acceleration measured (i.) when uniform, (ii.) when variable?

3. Give an account of the French system of weights and measures, and state accurately the connexion between the units of length, capacity, and mass.

4. Express a velocity of a mile an hour in terms of a velocity of 10 feet a second as unit.

5. The radius of a circle is half a mile; a horse runs round the circumference five times per hour. What is his velocity in feet per second?

6. The acceleration of a body is 13 feet per second per second. What is its measure in C.G.S. units?

7. If a mile per minute be the unit of velocity, and 32 feet per second per second that of acceleration, find the units of space and time.

8. The velocity of a train is known to have been diminishing uniformly; at 1 o'clock its velocity was 40 miles an hour; at 10 minutes past 1 its velocity was 10 miles an hour. What was its velocity 7 minutes past 1, and when did it come to rest?

CHAPTER III.

UNIFORMLY ACCELERATED MOTION.

41. **Preliminary observations.** — In Chapter I. we have shown that, for motion with *uniform velocity* v,

$$s = vt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

and in Chapter II. we have shown that, for motion *under uniform acceleration* f, the velocity *at any instant* is given by

$$v = u + ft \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

We shall now find expressions for the distance traversed in any time-interval t by a body moving with uniform acceleration.

We cannot do this by eliminating v from (1) and (2). For in uniformly accelerated motion the velocity is variable, and (1) does not hold good.

Examples.—(1) If a railway train starts from rest with uniform acceleration, and at the end of one minute it has acquired a velocity of a mile a minute, the train has *not travelled a mile in that minute.* For to do so it would have to go at full speed the whole time, but in reality the train never acquires this speed till the end of the minute; at the beginning of the minute it is not moving at all.

At the middle of the interval, or half-a-minute from starting, the velocity is $\frac{1}{2}$ a mile per minute, and in each second it increases by $\frac{1}{60}$ of a mile per minute. One second before the middle of the interval the velocity is $\frac{1}{60}$ of a mile per minute less than at the middle, and one second after the middle it is $\frac{1}{60}$ of a mile per minute greater. And generally, the velocity t seconds before the middle of the interval is as much below $\frac{1}{2}$ a mile per minute as the velocity t seconds after is above. Hence we are led to assume that the average velocity is $\frac{1}{2}$ a mile per minute, and that the distance traversed in the minute is $\frac{1}{2}$ a mile. The distance traversed in the first half-minute is, of course, less than $\frac{1}{2}$ of half a mile, and the distance traversed in the second half-minute is more than $\frac{1}{2}$ of half-a-mile by an equal amount; but the two together make up exactly one half-a-mile.

(2) A body moves from rest with an acceleration 10 feet per second; to find the distance traversed in 10 seconds, by making the supposition that the velocity is uniform (i.) during each second, (ii.) during each tenth of a second.

(i.) The velocities after

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 seconds

are 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 feet per second.

First suppose the velocity during each second to remain the same as at the beginning of that second. Then the distances traversed over during the several seconds are

0, 10, 20, 30, 40, 50, 60, 70, 80, 90 feet respectively.

Hence distance traversed in 1st and 10th seconds $= 0 + 90 = 90$ ft.
,, ,, 2nd ,, 9th ,, $= 10 + 80 = 90$ ft.
,, ,, 3rd ,, 8th ,, $= 20 + 70 = 90$ ft.
,, ,, 4th ,, 7th ,, $= 30 + 60 = 90$ ft.
,, ,, 5th ,, 6th ,, $= 40 + 50 = 90$ ft.

and the whole distance traversed is therefore $= 90 \times 5 = 450$ ft.

Next suppose the velocity during each second is the same as at the end of that second. Then the distances traversed during the several seconds are, respectively,

10, 20, 30, 40, 50, 60, 70, 80, 90, 100 feet;

and hence the distance traversed exceeds the distance previously traversed by the 100 feet passed over in the last second,

$\therefore$ the whole distance traversed is now $450 + 100 = 550$ feet.

Now the velocity at any intermediate time in any second is greater than at the beginning and less than at the end of that second. Hence the first result makes the distance traversed too small, and the second makes it too large. If we take the mean of the two results, we find

$$\text{distance traversed} = \tfrac{1}{2}(450 + 550) = 500 \text{ feet.}$$

The distance traversed is therefore the same as if the velocity were uniform and equal to 50 feet per second during the whole of the 10 seconds. Hence the average velocity is 50 feet per second, and is therefore half the final velocity, as in Example (1).

(ii.) Take tenths of a second. Then, supposing the velocity for the whole tenth to be the same as at the end of the tenth, the distance is the sum of 100 terms of the series $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots 9\frac{9}{10}, 10$. Combining the first and last term, the second and last but one, and so on, we have 50 terms each $= 10\frac{1}{10}$, and the distance $= 50 \times 10\frac{1}{10} = 505$ feet.

Next, supposing the velocity throughout the tenth of the second to be the same as that at the beginning of that tenth, the whole distance traversed will be the sum of 100 terms of the series $0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \ldots$ $\ldots 9\frac{9}{10}$. This is the same as before, but without the last term 10; hence the distance traversed is now $505 - 10$, or 495 feet.

The mean of the two values $= \frac{1}{2}(505 + 495) = 500$ feet as before.

Observation.—We have not actually proved that the distance traversed is 500 feet. We have, in fact, only shown that it lies (i.) between 450 feet and 550 feet; (ii.) between 495 and 505 feet. By dividing each second into hundredths, we should find closer limits. Thus, to find accurately the distance traversed under variable velocity, we must divide the time into a very large number of *very small* intervals, and add together the distances traversed in these intervals.

42. In uniformly accelerated motion, the average velocity in any interval of time is the arithmetic mean (*i.e.*, half the sum) of the velocities at the beginning and end of the interval.

Let f be the uniform acceleration,

t the number of units of time in the interval,

u the velocity at the beginning of the interval,

v the velocity at the end.

Divide the time t into a number of smaller intervals, each of length i, so that, if n denote the number of such intervals, we have $t = ni$,

and, therefore, $i = t/n$.

By making the number n very large, the intervals i will be very small. Let them be so small that *the velocity has not time to change appreciably during a single interval* i (§ 28). Then the velocity during any one interval may be treated as uniform, and we have

velocity	at beginning	of	1st	interval	$= u$,
,,	,,	,,	2nd	,,	$= u+fi$,
,,	,,	,,	3rd	,,	$= u+2fi$,
,,	,,	,,	4th	,,	$= u+(4-1)fi$,
	&c.,				&c.,
,,	,,	,,	$(m+1)$th	,,	$= u+mfi$.

Also velocity	at end of	last interval	$= v$,
,,	,,	last but 1	$= v-fi$,
,,	,,	last but 2	$= v-2fi$,
,,	,,	last but m	$= v-mfi$.

The distance traversed in any interval is found by multiplying the corresponding velocity by i. Now take the small intervals in pairs, and combine *the first interval with the last, the second with the last but one,* and so on, the $(m+1)$th interval being combined with the last but m.

Then sum of distances traversed
in first and last intervals $= (u+v)\,i$,
in 2nd and last but 1 $= (u+fi+v-fi)\,i = (u+v)\,i$,
in 3rd and last but 2 $= (u+2fi+v-2fi)\,i = (u+v)\,i$,
in $(m+1)$th and last but $m = (u+mfi+v-mfi)\,i$
$= (u+v)\,i$.

Therefore the distance traversed in each pair of intervals i is $(u+v)\,i$, and is the same as if the velocity in the pair were $\frac{1}{2}(u+v)$. And, since this is true of every pair, the whole distance traversed is the same as if the velocity were $\frac{1}{2}(u+v)$ throughout the whole of the time t.

Therefore distance traversed $= \frac{1}{2}(u+v)\,t$

and average velocity $= \frac{1}{2}(u+v)$,

as was to be proved.

Thus the distance traversed under uniform acceleration is the product of the time into half the sum of the initial and final velocities.

43. Observations.—We have supposed that, in the first, second, ... intervals, the body retains its initial velocity; and that, in the last, last but one, &c., the body moves with its final velocity. If we had supposed that in each interval the velocity retained its initial value, the velocities in the last, last but one, &c., would have been less than before by fi, and we should have found the average velocity to be $\frac{1}{2}(u+v-fi)$. If, on the other hand, the velocity were taken the same as at the end of each interval, the velocities in the first, second, &c., intervals would have been greater than before by fi, and we should have found the average velocity to be $\frac{1}{2}(u+v+fi)$.

But fi is the change of velocity in a single interval i, and we have supposed i so small that this change is practically zero. Hence, we put $fi = 0$, and we find the average velocity $= \frac{1}{2}(u+v)$, as before.

The velocities u, $u+fi$, $u+2fi$, ... in § 42 form an arithmetical progression, and so do the corresponding distances traversed in the intervals i. The formula given in Algebra for the sum of such a progression might therefore be used to find the distance traversed in the whole time t. The student may do this as an instructive exercise. But the proof of the formula—by combining the first term with the last, and so on—is identical with that used above.

44. Uniformly accelerated motion from rest.— From § 42 we see that: *When a body starts from rest with uniform acceleration, the average velocity is half the final velocity.*

The formulæ for the distance traversed in uniformly accelerated motion, which we shall now deduce, are very important.*

Let f be the uniform acceleration,

t the time, measured from the instant of rest,

v the velocity acquired,

s the distance traversed in the time t.

From the definition of average velocity (§ 27), we have

(distance traversed) = **(average velocity)** × **(time)**;

and average velocity $= \frac{1}{2}v$;

$$\therefore \quad s = \tfrac{1}{2}vt \quad \ldots\ldots\ldots\ldots \quad (1).$$

Also by § 32, remembering that the initial velocity is zero,

$$v = ft \quad \ldots\ldots\ldots\ldots \quad (2);$$

$$\therefore \quad s = \tfrac{1}{2}ft \times t,$$

or

$$s = \tfrac{1}{2}ft^2 \quad \ldots\ldots\ldots\ldots \quad (3).$$

Eliminating t from (1) and (2), we have

$$fs = \tfrac{1}{2}v^2,$$

or

$$v^2 = 2fs \quad \ldots\ldots\ldots\ldots \quad (4).$$

Formulæ (1), (2), (3), (4) are sufficient to work out any problem relating to uniformly accelerated motion from rest.

Example.—A train, starting from rest, acquires a velocity of 48 miles an hour in $2\frac{1}{2}$ minutes; to find the distance run in that time.

Here the initial velocity is zero, and the final velocity is $\frac{4}{5}$ mile per minute. Therefore the average velocity is $\frac{2}{5}$ mile per minute, and the distance run in the $2\frac{1}{2}$ minutes

$$= \tfrac{2}{5} \times 2\tfrac{1}{2} = 1 \text{ mile}.$$

* The student should not leave these until he is quite familiar with them, as a very large portion of the subject is dependent on them.

45. Uniformly accelerated motion with an initial velocity.—We have proved, in § 42, that

$$s = \tfrac{1}{2}(v+u)\,t \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and, in § 32, that $v-u=ft$, or $v=u+ft$ ....(6);

and from these two equations we may find a relation between any four of the quantities u, v, f, t, s. Thus, eliminating v, we have

$$s = \tfrac{1}{2}(u+u+ft)\,t,$$

or

$$s = ut+\tfrac{1}{2}ft^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7);$$

giving the distance traversed in terms of the time, the *initial* velocity, and the given acceleration.

Similarly, by eliminating u, we find

$$s = vt-\tfrac{1}{2}ft^2;$$

giving the distance traversed in terms of the *final* velocity and acceleration. This formula is not often used.

Lastly, eliminating t by multiplying (5) and (6) across, we have

$$(v-u)\times\tfrac{1}{2}(v+u)=fs, \quad\text{or}\quad \tfrac{1}{2}(v^2-u^2)=fs,$$

or $\quad v^2-u^2=2fs, \quad$ or $\quad v^2=u^2+2fs$ ... (8),

a relation between the distance traversed and the initial and final velocities.

46. **The average velocity in any interval is equal to the velocity at the middle of that interval.**

For we have seen that average velocity

$$\begin{aligned} &= \tfrac{1}{2}(u+v) = \tfrac{1}{2}(u+u+ft) \\ &= u+\tfrac{1}{2}ft = u+f(\tfrac{1}{2}t) \\ &= \text{velocity at time } \tfrac{1}{2}t \\ &= \text{vel. at middle of interval.} \end{aligned}$$

47. Observations.—To remember (7), it may be noticed that the expression for s, the distance traversed, consists of two terms. The term ut represents the distance traversed in time t with uniform velocity u, and the term $\frac{1}{2}ft^2$ represents the distance traversed under uniform acceleration f with no initial velocity. Hence the whole distance traversed is found by adding together the part due to the initial velocity and that due to the acceleration.

The equation (8) differs from the corresponding equation (4), (viz., $v^2 = 2fs$), in having the square of the initial velocity (u^2) added to its right-hand side.

In working numerical examples, it is always better to deduce the results from first principles, rather than to have recourse to formulæ. For this reason it is often more convenient to make use of the property of § 42, and to find the distance traversed by first determining the *average velocity*. If formulæ are used, great care must be taken not to lose sight of their full meaning, otherwise mistakes will inevitably occur in interpreting them.

48. **The signs of the letters should never be changed** in the formulæ, even when we are dealing with a retardation. In such a case, the *value* of f is a *negative* quantity, but the *formula* (7) must still be written $s = ut + \frac{1}{2}ft^2$, and not $s = ut - \frac{1}{2}ft^2$.

The following examples will show how the formula is to be applied to retarded motion :—

Examples.—(1) If a steamer starts from rest with an acceleration of 100 yards per minute per minute, it will at the end of five minutes have attained

$$\begin{array}{ccc} \text{a velocity} & \text{equal to} & 100 \times 5 \text{ yards per minute.} \\ (v) & = & (f) \times (t) \end{array}$$

(2) Now suppose that, when the steamer is going at the rate of 500 yards per minute, the engines are reversed, so as to produce a *backward* acceleration of 100 yards per minute per minute, and let it be required to find out how far the steamer will go in 3 minutes.

We must now put $f = -100$; and therefore

our formula $s = ut + \frac{1}{2}ft^2$

gives us $s = 500 \times 3 + \frac{1}{2}(-100) \times 3^2$;

i.e., $s = 1500 - 450$,

or distance traversed $= 1050$ yards.

A result which we might have arrived at as follows :

Supposing there had been no retardation, the steamer, moving with a velocity of 500 yards a minute, would have gone 1500 yards in 3 minutes. Again, if the steamer started from rest with a backward acceleration of 100 yards per minute per minute, it would have, at the end of 3 seconds, a velocity of 300 yards per minute. Hence its average velocity would be $\frac{300}{2}$ yards per second backwards, which gives us 150 yards × 3 = 450 yards as the distance it would move backwards in that time. Hence we may say that the initial velocity carries the steamer 1500 yards forwards, while the action of the reversed engines carries it backwards through 450 yards. Therefore on the whole the steamer has moved forward a distance of 1500 – 450 or 1050 yards, as before.

(3) If a train, when going at 50 miles an hour, can be pulled up in 48 seconds, find at what point the brakes must be applied.

When the train is being pulled up, the initial velocity is $\frac{5}{6}$ mile per minute, and the final velocity is zero; hence the average velocity is $\frac{5}{12}$ miles per minute. Also the time taken in pulling up equals $\frac{4}{5}$ of a minute.

Therefore the distance run when the brakes are on

$$= \tfrac{4}{5} \times \tfrac{5}{12} \text{ mile} = \tfrac{1}{3} \text{ of a mile.}$$

Hence the brakes must be applied when the train is $\frac{1}{3}$ of a mile from the station.

49. To find the distance traversed in the *n*th second of a body's motion.

With the usual notation, taking the second as the unit of time,

$$\text{velocity at end of } n-1 \text{ seconds} = u+f(n-1),$$

$$\text{velocity at end of } n \text{ seconds} = u+fn;$$

$\therefore$ average velocity during nth second

$$= \tfrac{1}{2}\left\{u+f(n-1)+u+fn\right\} = u+\tfrac{1}{2}(2n-1)f;$$

and, since the measure of a second is unity, the distance traversed in the nth second

$$= \left\{u+\tfrac{1}{2}(2n-1)f\right\} \times 1 = u+\tfrac{1}{2}(2n-1)f.$$

It is better, however, to remember the method by which this formula is obtained, and not the formula itself.

50. **To find the acceleration of a moving body by observation,** it is only necessary to observe the distances traversed in two successive seconds of the motion.

The distance traversed in the first second measures the average velocity per second, and equals the velocity at the middle of that second. Similarly, the distance traversed in the next second measures the velocity at the middle of that second. But from the middle of one second to the middle of the next is exactly one second. Hence the difference gives the increase of velocity in one second, and this measures the acceleration.

Examples.—(1) If the distances traversed in two successive seconds are 10 feet and 42 feet, to find the acceleration, supposing it uniform.

Here av. vel. in 1st sec. = vel. at mid. of 1st sec. = 10 ft. per sec.;
" " 2nd sec. = " " 2nd sec. = 42 ft. per sec.

Therefore, from middle of 1st to middle of 2nd second of time, velocity increases from 10 to 42 feet per second;

$\therefore$ increase of velocity in 1 sec. = 32 ft. per sec.;

$\therefore$ acceleration = 32 ft. per sec. per sec.

(2) If the distances traversed in three successive seconds are 5, 10, 14 feet, respectively, to show that the acceleration is not uniform.

The average velocities in the three seconds are 5, 10, 14 feet per second, respectively.

The increase in average velocity between the 1st and 2nd seconds of time $= 10-5 = 5$ ft. per sec.

The increase in average velocity between the 2nd and 3rd seconds of time $= 14-10 = 4$ ft. per sec.

If the acceleration were uniform, the average velocities would be the velocities at the middles of the respective seconds, and the two increases of velocity would be equal. But this is not the case. Therefore the acceleration is variable. It is evident that the acceleration is decreasing.

(3) A body traverses altogether 66 feet in the fifth, sixth, and seventh seconds of its motion from rest under uniform acceleration. To find the value of this acceleration.

The average velocity in the three seconds

$$= \tfrac{66}{3} = 22 \text{ ft. per sec.}$$

This is the velocity in the middle of the interval; *i.e.*, $5\frac{1}{2}$ seconds after starting;

$$\therefore \text{ the acceleration} = \frac{22}{5\frac{1}{2}} = 4 \text{ ft. per sec. per sec.}$$

It will be noticed that *two* observations of the position of a body are required to find its *velocity*, *three* to find its *acceleration*, and *four* to test whether this acceleration is *uniform*.

51.* **Graphic representation of variable velocity.**—We shall now show how motion with variable velocity can be fully represented by drawing a curve which serves as a sort of map or diagram of the velocity (Fig. 6).

Take a straight line OX (which we will suppose horizontal), and, having selected any point O on it, measure a length OM, such that the number of units of length in OM is equal to the number of units of time (say seconds) that have elapsed since the beginning of the motion. Then the point M will represent a certain instant of time; thus, if OM contains t units of length, the point M will represent the time t. The points a, b, c, distant respectively 1, 2, 3 units of length from O, will represent the times 1, 2, 3 seconds after the beginning of the motion, respectively.

* The rest of this chapter may be omitted on first reading, as it only contains alternative methods. The graphic representation of uniform motion is, however, simple and instructive.

Through M draw a line MP perpendicular to OM, and let the number of units of length in MP be equal to the number of units of velocity in the velocity of the moving point at the instant represented by M. Let similar perpendiculars be erected at every point on OX, so that (for example) aA, bB, cC, ... are to be taken proportional to the velocities at the times 1, 2, 3, ... seconds, respectively. Then the extremities of these perpendiculars will all be found to lie along a certain straight or curved line $ABCP$. This line may be called the **velocity curve** of the motion.

For negative velocities, we draw the perpendicular downwards instead of upwards, so that the velocity curve is below instead of above OX. Every horizontal length such as OM is called an **abscissa**, and every perpendicular MP is called an **ordinate.**

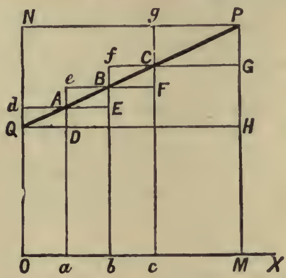


Fig. 6.

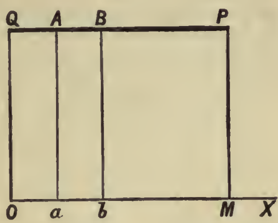


Fig. 7.

52. **When the motion is uniform,** the velocity curve is a straight line parallel to OX; for, if the velocity is u, all the "ordinates," such as MP, are u units long, and therefore the points on the velocity curve are at the same distance from OX. In this case, if t be the time OM (Fig. 7), we have (§ 14)

$$\text{distance traversed} = ut = MP \times OM = \text{area of rectangle } OP.$$

We shall now extend this result to variable velocities by showing that—

53. **The distance traversed in any interval of time is represented by means of the area contained by the velocity curve and the two bounding ordinates.**

Let OM represent the given interval, QP the velocity curve; then it is required to show that the area $OQPM$ measures the distance traversed.

Divide OM into any number of intervals at the points a, b, c, and draw the ordinates aA, bB, cC to meet the velocity curve in A, B, C, so that OQ, aA, bB, cC, MP represent the velocities at the instants of time represented by O, a, b, c, M.

If the velocity during each of the intervals Oa, ab, bc, cM were uniform and equal to the actual velocity at the *beginning* of that interval, the velocity curve would consist of the straight lines QD, AE, BF, CG parallel to OX. The distance traversed in the intervals would be $QY.Oa$, $aA.ab$, $bB.bc$, &c., and would be represented by the measures of the areas of the rectangles Qa, Ab, Bc, &c., and the whole distance traversed would be represented by the sum of the measures of these rectangles; that is, by the area of the inscribed figure $OQDAEBFCGMO$.

In like manner, if the velocity throughout each interval were equal to the actual velocity at its end, the velocity curve would consist of the lines dA, eB, fC ..., and the distance traversed would be represented by the area of the circumscribing figure $OdAeBfCgPMO$.

Now the distance actually traversed is intermediate between the distances described on the two above suppositions; it is, therefore, represented by an area intermediate between those of the inscribed and circumscribing figures. Now the area of the curve $OQPMO$ is intermediate between the areas of its inscribed and circumscribing figures, and is the only area which *always* possesses this property, however small the subdivisions O, a, ab, bc, ... Therefore the actual distance traversed must be measured by the area $OQPMO$.

If the velocity is negative, so that the curve descends below the horizontal line OX, the area of this portion is to be considered negative.

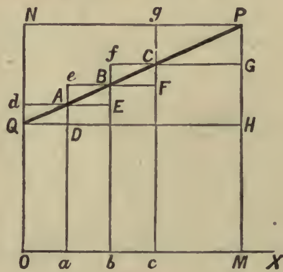


Fig. 8.

54. **To prove graphically the formula for uniformly accelerated motion,** $s = ut + \frac{1}{2}ft^2$.

We shall first show that the velocity curve under uniform acceleration is a straight line.

Let OQ denote the initial velocity u, and on OX measure any number of equal lengths Oa, ab, bc,..., representing equal intervals of time. Through a, b, c draw aA, bB, cC, representing the velocities of the instants a, b, c respectively. Draw QDH, AE, BF, ... parallel to OX, and complete the construction as in the figure.

Then DA, EB, FC represent the total increases of velocity in the equal intervals Oa, ab, bc. But, since the acceleration is uniform, these increases are equal; that is, $DA = EB = FC$. Therefore the triangles QDA, AEB, BFC are equal in all respects, and therefore

$$\angle AQD = \angle BAE = \angle CBF = \ldots.$$

Hence it may be readily seen that the points Q, A, B, C lie in a straight line, and therefore the velocity curve $QABCP$ is a straight line.

Now, if $OM = t$, MP represents the final velocity $u + ft$. Also, $MH = OQ = u$, and therefore $HP = ft$; and we have

$$\begin{aligned}\text{distance traversed} &= \text{area } OMPQ = \text{rect. } OMHQ + \Delta QHP \\ &= \text{rect. } OMHQ + \tfrac{1}{2}\text{ rect. } QHPN \\ &= OQ \cdot OM + \tfrac{1}{2} HP \cdot QM \\ &= u \cdot t + \tfrac{1}{2} ft \cdot t = ut + \tfrac{1}{2} ft^2;\end{aligned}$$

as was to be proved.

SUMMARY OF RESULTS.

When a body starts from rest,	When a body starts with initial velocity u,
Average vel. $= \frac{1}{2}$ final vel.;	Av. vel. $= \frac{1}{2}(u+v)$;
whence $s = \frac{1}{2}vt$......(1);	whence $s = \frac{1}{2}(u+v)t$...(5);
$v = ft$ ......(2);	$v = u + ft$ ......(6);
$s = \frac{1}{2}ft^2$ ...(3);	$s = ut + \frac{1}{2}ft^2$ ...(7);
$v^2 = 2fs$...... (4).	$v^2 = u^2 + 2fs$ ...(8);

where u is the initial velocity,
v is the final velocity,
f is the acceleration,
t is the time of motion,
s is the final distance from the starting point.

In each investigation we must assume some direction as positive.

Suppose, for instance, that we are dealing with a train on a line running north and south, and that at the beginning of the interval considered the train is at a certain station A. Then, if we have reason to believe that at the end of the interval the train will be north of A we may assume the direction from south to north as positive. If we obtain a negative value for the distance, it indicates that the ultimate position of the train is south of A.

EXAMPLES III.

1. What is meant by the statement that the acceleration of a particle is 32 foot-second units? With this acceleration, how far will the particle move in 10 seconds, and what will be its velocity at the end of that time?

2. A slip carriage is detached from a train and brought to rest under the action of a uniform retarding force, the train meanwhile proceeding with uniform velocity. Prove that, when the carriage stops, the distance of the train in front of it is equal to the distance through which the slip carriage has travelled from the instant of being detached.

3. A body starts from rest and moves with uniform acceleration 18 (foot-seconds). Find the time required by it to traverse the first, second, and third foot respectively.

4. If a body moving with uniform acceleration pass over 250 feet while its velocity increases from 40 to 60 feet per second, find the acceleration and the time of motion.

5. A particle is observed to describe 7 feet in 3 seconds, and 13 feet in the next 3 seconds. Find its acceleration.

6. A body moves over 30 feet during the 5th second and 42 feet during the 7th second of its motion. Find the whole space passed over in 10 seconds.

7. A body moving with uniform acceleration has the velocities u, v at two given points; given s, the distance between the points, find the time of describing it.

8. A body passes over a kilometre in 10 seconds under a uniform acceleration of 1000 C.G.S. units. Find its velocities at the beginning, middle, and end of the interval of time.

9. Show that the distance described from rest under a constant acceleration in the (p^2-p+1)th second is equal to the sum of the distances described in the first $(p-1)$ seconds and in the first p seconds.

10. A body A is moving with uniform velocity u in a straight line, and another body B moves from rest with given acceleration f along the same straight line, being initially at a distance a behind A. Find when and where B will overtake A.

CHAPTER IV.

GRAVITY.—MOTION OF BODIES FALLING VERTICALLY.

55. The principles proved in the foregoing sections are well illustrated by their applications to the motion of bodies falling to the ground.

The acceleration due to gravity is the same for all bodies. If we allow a coin and a sheet of paper or feather to drop freely from rest, both will be accelerated downwards, but the coin will reach the ground quicker than the paper; while a balloon will rise in the air instead of falling. From this it might be supposed that different bodies are differently accelerated by the action of gravity, the coin being more accelerated than the paper. But we must not forget that air itself has weight, and, moreover, a body falling through the air has to set in motion the particles of air which it displaces in its descent. Hence a light body of large size has to displace more air, and therefore encounters more resistance than a body of smaller size, but of the same weight. If the sheet of paper be rolled up into a ball, it will fall much quicker than before, because it offers less surface to the air and therefore has less air to push out of its way as it descends.

But if different bodies be allowed to fall in a tall jar which has been exhausted of air by means of an air-pump, they will all reach the bottom at the same instant, thus showing that all bodies are equally accelerated by gravity.

56. The same thing can be shown more simply without an air-pump by the following experiments, which should be performed by the student before proceeding further.

Experiment I.—Take a penny or other large coin and cut a round disc of paper slightly smaller than the coin. Lay the paper on the top of the coin, and carefully let the latter drop. Although the paper is uppermost, it will remain on the top of the coin, and both will fall together.

Here the coin, by going in front of the paper, overcomes the resistance of the air, which would otherwise retard the motion of the disc.

Experiment II.—Take a small tin canister without the lid (*e.g.*, a cocoa-tin), and in it place various objects, such as a coin, a feather, a piece of thin tissue paper, &c. Drop the canister from a height. All the objects will remain inside and will reach the ground together, showing that all are equally acted on by gravity.

Experiment III.—If a stone be allowed to drop from a height of 4 feet, it will reach the ground in half a second. If it be allowed to fall through 16 feet, it will take 1 second. If dropped through 64 feet, it will take 2 seconds.

Now if f be the acceleration, the formula $s = \frac{1}{2}ft^2$, taken in conjunction with these observations, gives

$$4 = \tfrac{1}{2}f \cdot (\tfrac{1}{2})^2, \quad 16 = \tfrac{1}{2}f \cdot 1^2, \quad 64 = \tfrac{1}{2}f \cdot 2^2,$$

whence $$f = 32.$$

Hence we conclude that a falling body descends with a uniform acceleration of about 32 *feet per second per second.*

That this acceleration is uniform is proved by the fact that each observation gives the same value for f.

Observation.—This experiment is, of course, only a very rough one, because it is very difficult to estimate times with sufficient accuracy.

57. The Intensity of Gravity.—The above and other experiments show that the acceleration of an unresisted falling body is uniform, and, since it is the same for different bodies, its magnitude at any place must be constant. *This constant acceleration is called the* **intensity of gravity,** *and is invariably denoted by the letter g.* Its value is not quite the same in different parts of the Earth. It is least at the Equator, where it amounts to only 32·091 F.P.S., or 978·10 C.G.S. units; and it is greatest at the North and South Poles, where it is estimated to be 33·255 F.P.S., or 983·11 C.G.S. units. It also depends on the altitude; it is greatest at the sea level, and diminishes slightly when we go *either* up to the top of a high mountain *or* down a deep mine. At London, at the sea level,

	F.P.S. units.		C.G.S. units.
$g =$	32·191 ft. per sec. per sec.	$=$	981·17 cm. per sec. per sec.

[N.B.—The above numbers are not to be committed to memory.]

For rough purposes it is usual to take

g = 32 feet per second per second.................... **(1),**

g = 981 centimetres per second per second ...... **(2).**

These numbers must be remembered, as they are constantly required. The more accurate value, $g = 32{\cdot}2$ ft. per sec. per sec., should also be remembered, although it is less often used.

The **vertical** at any place may be defined as the direction in which a body falling freely at that place is accelerated by gravity.

Observation.—It must be carefully borne in mind that **g is an acceleration, not a velocity.** For a body falls to the ground with uniform acceleration but with ever increasing velocity.

58. Motion from rest under gravity.—If we neglect the resistance of the air, a stone or other body dropped from a height will fall freely with a uniform acceleration g, or 32 ft. per sec. per sec. The distance fallen s and acquired velocity v at the end of t secs. will be given in feet and feet per sec. respectively by the formulæ

$$v = gt = 32t, \quad s = \tfrac{1}{2}gt^2 = 16t^2,$$

obtained by putting $f = g = 32$ in § 44.

The accompanying diagram serves to illustrate the motion. The round dots on the vertical line show the relative positions of the body at intervals of one second, each of the smaller divisions being supposed to represent 16 feet. The velocities at each second are also stated on the diagram.

Thus in one second a falling body will acquire a velocity of 32 feet per second. It will *not* have fallen through 32 feet, but only through 16 feet, because it started with no velocity at all; whereas to have gone 32 feet it would have had to have fallen with the full velocity of 32 feet per second during the whole time. By § 42, the average velocity during the second is *half* of 32, or 16 feet per second, and therefore the distance fallen is 16 feet.

If the diagram be held with the line horizontal it will represent the motion of a body moving from rest in a horizontal line with acceleration f, if the smaller divisions be taken to represent each $\frac{1}{2}f$ units of length. The particulars of the motion are given on the left.

When a falling body is small and heavy, such as a stone or a bullet, its motion will only be slightly affected by the resistance of the air, so that the results here obtained will give a fairly accurate idea of the actual motion. But the motion of a body which is very light for its size—such as a feather or a balloon—will depend very largely on the effect of the surrounding air, and will be entirely different.

59. **Distance fallen in the *n*th second.**

It will be noticed that the distances traversed in the *individual* seconds are 1, 3, 5, 7, 9, 11 ... times 16 feet respectively; and we should infer that the distance fallen in the nth second is $\frac{1}{2}g\,(2n-1)$ or $16\,(2n-1)$ feet.

This may be shown as in § 49, or as follows:—

The distance fallen in the nth second is the difference of the total distances fallen in n and $n-1$ seconds respectively, and is therefore

$$= \tfrac{1}{2}g\,.\,n^2 - \tfrac{1}{2}g\,(n-1)^2$$
$$= \tfrac{1}{2}g\,\{n^2-(n-1)^2\} = \tfrac{1}{2}g\,(2n-1) \quad\quad (3).$$

Observations.—It is better not to remember (3), but to obtain it in one or other of the above methods when required.

We notice that in each second the stone falls 32 more feet than in the preceding second. This follows from the fact that in each second the velocity increases by 32 feet per second.

Fig. 9.

I. Uniformly Accelerated Motion.

Time $t =$	Dist. $s = \frac{1}{2}ft^2 =$	Veloc. $v = ft =$
0	0	0
1	$\frac{1}{2}f.1^2$	f
2	$\frac{1}{2}f.2^2$	$2f$
3	$\frac{1}{2}f.3^2$	$3f$
4	$\frac{1}{2}f.4^2$	$4f$
5	$\frac{1}{2}f.5^2$	$5f$

II. Motion from Rest under Gravity.

Acceleration g = 32 feet per second per second.

Time in seconds.	Distance fallen in feet.	Velocity acquired in feet per second.
t.	$s = \frac{1}{2}g \times t^2$.	$v = g \times t$.
0 secs.	$s = \frac{1}{2}32 \times 0^2 =$ 0 ft.	$v = 32 \times 0 =$ 0 ft. per sec.
1 sec.	$s = \frac{1}{2}32 \times 1^2 =$ 16 ft.	$v = 32 \times 1 =$ 32 ft. per sec.
2 secs.	$s = \frac{1}{2}32 \times 2^2 =$ 64 ft.	$v = 32 \times 2 =$ 64 ft. per sec.
3 secs.	$s = \frac{1}{2}32 \times 3^2 =$ 144 ft.	$v = 32 \times 3 =$ 96 ft. per sec.
4 secs.	$s = \frac{1}{2}32 \times 4^2 =$ 256 ft.	$v = 32 \times 4 =$ 128 ft. per sec.
5 secs.	$s = \frac{1}{2}32 \times 5^2 =$ 400 ft.	$v = 32 \times 5 =$ 160 ft per sec.

60. **If a stone is dropped from a given height h,** the time taken in falling and the velocity of striking the ground may easily be got by substituting h for s and g for f in formulæ $s = \frac{1}{2}ft^2$, $v^2 = 2fs$, which thus become $\frac{1}{2}gt^2 = h$, $v^2 = 2gh$.

whence $$t = \sqrt{\frac{2h}{g}}, \quad v = \sqrt{2gh};$$

Or, if h be measured in feet, and $g = 32$,

$$t = \tfrac{1}{4}\sqrt{h} \text{ seconds}, \quad v = 8\sqrt{h} \text{ ft. per sec.}$$

It is convenient, though not essential, to remember the formula $v^2 = 2gh$, but it is much better to be able to use the fundamental formulæ.

It should require but little thought to see that the height from which the stone is dropped is the distance it must traverse before it strikes the ground.

Examples.—(1) To find the depth of a well, when a stone takes $1\frac{1}{2}$ seconds to reach the bottom.

The distance is given by

$$s = \tfrac{1}{2}gt^2 = 16 \times (\tfrac{3}{2})^2 = 36 \text{ feet},$$

which is, therefore, the depth of the well.

(2) If a brick drops off the roof of a chimney 100 feet high, in what time will it strike the ground, and with what velocity?

Here we have to find t and v, and we have given $s = 100$, and we know $f (= g) = 32$.

Now t is connected with s and f by the equation $s = \frac{1}{2}ft^2$, and so by substitution we get $100 = 16 \times t^2$;

$$\therefore \quad t^2 = \frac{100}{16}, \text{ or } t = 2\tfrac{1}{2}.$$

Similarly, v is connected with f and s by the formula $v^2 = 2fs$;

$$\therefore \quad v^2 = 2 \times 32 \times 100;$$

$$\therefore \quad v = 8 \times 10 = 80.$$

Therefore the brick will strike the ground in $2\frac{1}{2}$ seconds, with a velocity of 80 feet per second.

61. **Bodies projected downwards.**— If a body be projected downwards with initial velocity u, we merely have to write g for f in the formulæ of § 45 to obtain the relations between the time t, final velocity v, and distance fallen s. We thus have

$$v = u + gt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{4}),$$

$$s = \tfrac{1}{2}(u + v)\,t = ut + \tfrac{1}{2}gt^2 \quad \ldots\ldots\ldots\ldots (\mathbf{5}),$$

$$v^2 = u^2 + 2gs \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{6})$$

Examples.—(1) To find the velocity of projection, if the body descends 2000 feet in 10 seconds.

Let the required velocity be u feet per second. Putting $t = 10$, $s = 2000$, $g = 32$, in the formula

$$s = ut + \tfrac{1}{2}gt^2,$$

we have

$$2000 = 10u + 1600;$$

$$\therefore \quad 10u = 400, \quad u = 40.$$

Hence the body must be projected with a velocity of 40 feet per second.

(2) If a body is projected downwards with an initial velocity of 20 feet per second, to find the time taken to describe 500 feet.

Let t seconds be the required time.

Putting $u = 20$, $s = 500$, $g = 32$, in the formula (5), we have

$$500 = 20t + 16t^2.$$

To find t we have to solve this as a quadratic* equation. We may write the equation

$$t^2 + \frac{5t}{4} = \frac{500}{16}.$$

Completing the square on the left-hand side, we have

$$t^2 + \frac{5t}{4} + \left(\frac{5}{8}\right)^2 = \frac{500}{16} + \frac{25}{64} = \frac{2025}{64};$$

$$\therefore \quad t + \frac{5}{8} = \pm \frac{45}{8};$$

$$\therefore \quad t = \frac{45-5}{8} = 5, \quad \text{or} \quad \frac{-45-5}{8} = -\frac{25}{4}.$$

But the time t cannot be negative. Therefore the required time is 5 seconds.

(3) A body is projected downwards with a velocity of 500 centimetres per second; to find (i.) the velocity acquired, and (ii.) the time elapsed, when it has fallen 50 centimetres.

Let the acquired velocity be v centimetres per second. Then, using the C.G.S. units, we have $s = 50$, $u = 500$, $g = 981$; whence the formula

$$v^2 = u^2 + 2gs$$

gives $v^2 = (500)^2 + 2 \,.\, 981 \,.\, 50 = 250000 + 98100 = 348100$;

$$\therefore \quad v = 590 \text{ centimetres per second.}$$

The increase of velocity during the interval is therefore 90 centimetres per second, and this increase $= 981t$, where $t =$ time taken in falling;

$$\therefore \quad \text{required time } t = \frac{90}{981} = \frac{10}{109} = \cdot 091 \text{ secs. (approximately).}$$

* This quadratic equation may be avoided by the double method adopted in the next example. The student should do this for practice, finding $v = 180$.

62. Bodies projected upwards.—When a body is projected with a given upward velocity u, it is usually convenient to take the *upward* direction as positive. With this convention, s will always represent the height of the body *above* the point of projection; v will be positive when the body is rising and negative when the body is falling. Since acceleration due to gravity takes place in a downward direction, we must substitute $-g$ for f in our formulæ, which now become

$$v = u - gt \quad \text{.............................}(7),$$

$$s = \tfrac{1}{2}(u+v)\,t = ut - \tfrac{1}{2}gt^2 \quad \text{.........}(8),$$

$$v^2 = u^2 - 2gs \quad \text{........................}(9).$$

Here we consider g to represent the acceleration due to gravity, without reference to sign, so that $g = 32$, and not $= -32$.

For the upward motion, u is positive, and v, which is equal to u at starting, becomes less and less; for the formula $v = u - gt$ shows that v decreases as t increases, until $gt = u$, when v becomes $= 0$. At this instant the body remains stationary for an instant and then begins its downward course, which (as we shall prove in § 66) occupies exactly the same time as the ascent.

When the body returns to the point of projection, s vanishes; and if the body goes on *below* the point of projection, s is negative.

63. To find the time during which the body rises.

Since the body ceases rising, and begins falling, when $v = 0$, it follows from the equation $v = u - gt$, that the instant of time is given by

$$0 = u - gt,$$

whence

$$t = \frac{u}{g} \quad \text{....................}(10);$$

that is, the body rises during the interval $\dfrac{u}{g}$.

64. To find the greatest height to which the body rises.

The height is greatest when the body just ceases rising. We must therefore put $f = -g$ and $t = \dfrac{u}{g}$ in

$$s = vt + \tfrac{1}{2}ft^2;$$

and we have, for the greatest height,

$$s = u \cdot \frac{u}{g} - \tfrac{1}{2}g\left(\frac{u}{g}\right)^2 = \frac{u^2}{g} - \tfrac{1}{2}\frac{u^2}{g} = \frac{u^2}{2g} \quad \ldots\textbf{(11)}.$$

[We might otherwise find the greatest height from the equation

$$v^2 = u^2 - 2gs$$

by writing it, first, in the form

$$2gs = u^2 - v^2,$$

and then $$s = \frac{u^2 - v^2}{2g}.$$

Hence s is greatest when $u^2 - v^2$ is greatest, that is, when $v = 0$, for then v^2 is least. Therefore

$$\text{greatest height } s = \frac{u^2}{2g}.]$$

Examples.—(1) A stone is thrown upwards with a velocity of 48 ft. per sec. To find the greatest height, and the time taken in reaching it.

When the stone is at its greatest height, its velocity is zero, and the time is therefore given by

$$0 = 48 - gt = 48 - 32t;$$

$$\therefore \quad t = \tfrac{48}{32} = 1\tfrac{1}{2} \text{ secs.}$$

The height is therefore given by

$$s = 48t - \tfrac{1}{2}gt^2 = 48 \cdot \tfrac{3}{2} - 16 \cdot (\tfrac{3}{2})^2 = 72 - 36 = 36;$$

$$\therefore \quad \text{greatest height} = 36 \text{ feet.}$$

(2) To find the greatest height attained by a body which is thrown vertically upwards with a velocity of 100 ft. per sec.

The velocity at any height s is given by

$$v^2 = u^2 - 2gs = 100^2 - 2 \cdot 32 \cdot s = 10000 - 64s.$$

But, when the height is greatest, $v = 0$, and therefore

$$10000 - 64s = 0;$$

$$\therefore \quad \text{greatest height } s = \frac{10000}{64} = 156\tfrac{1}{4} \text{ feet.}$$

65. To find the whole time of flight.—After reaching its greatest height the body will begin to fall; its height will then decrease, and when this height becomes zero the body will have returned to the point of projection. Hence the time of flight t is found by putting $f = -g$, and $s = 0$, in $s = ut + \frac{1}{2}ft^2$.

We therefore have $ut - \frac{1}{2}gt^2 = 0$;

$$\therefore \quad t(u - \tfrac{1}{2}gt) = 0;$$

whence either $t = 0$,

or $u = \frac{1}{2}gt$, *i.e.* $t = \frac{2u}{g}$.

The factor $t = 0$ only tells us what we started with, namely, that the body was at the point of projection at the time $t = 0$. Thus the time of flight must be given by the other factor, and we have

$$\text{time of flight } t = \frac{2u}{g} \quad \ldots\ldots\ldots\ldots\ldots(\mathbf{12}).$$

66. Observations.—Comparing (10) and (12), we see that the body rises during half the time of flight. It therefore falls during the other half; hence *the time taken in rising to the highest point is equal to the time taken in returning to the point of projection.*

More generally, the time taken by the body in rising after passing *any* given point is equal to the time taken in again falling to that point; for, as we are not concerned with the body's motion *before* it first reaches the given point, we may treat that point as the point of projection.

Hence the diagram of § 58 gives a record of the upward as well as the downward motion of such a body, for its positions 1, 2, 3, ... seconds respectively *before* reaching the highest point are the same as 1, 2, 3, ... seconds *after* reaching the highest point.

Moreover, the upward velocity at any height when rising is numerically equal to the downward velocity at the same height when falling.

67. To find the time taken to reach a given height.

First Method.—We may use the equation (8), viz.,

$$s = ut - \tfrac{1}{2}gt^2,$$

where s the given height, u the initial velocity, and g the intensity of gravity, are supposed known. We want to find the time t; accordingly we must regard the equation as a quadratic equation in which t is the unknown quantity, and solve it to find t.

Now a quadratic equation has in general two solutions, and these determine the two instants at which the body is at the given height, when it is rising and when falling respectively.

Second Method.—Instead of finding the time at once, we may find the velocity v from (9)

$$v^2 = u^2 - 2gs;$$

and we may then find the time t from (7)

$$v = u - gt \quad \text{or} \quad t = (u - v) \div g.$$

Examples.—(1) If the velocity of projection is 80 ft. per sec., the *time of flight* is given by the equation

$$0 = s = ut - \tfrac{1}{2}gt^2 = 80t - \tfrac{1}{2} \cdot 32 \cdot t^2.$$

Rejecting the factor $t = 0$, this gives

$$t = \tfrac{80}{16} = 5 \text{ secs.}$$

(2) A body is projected upwards with a velocity of 96 feet per second; to find when it will be at the heights 80, 144, 160 feet above, and 112 feet below, the point of projection, respectively.

We shall employ the second method; accordingly we have to find the velocity v from the equation

$$v^2 = u^2 - 2gs = 96^2 - 2 \cdot 32 \cdot s;$$

where the height $s = 80$, 144, 160, -112 feet, respectively.

(i.) At height 80 feet,

$$v^2 = 96^2 - 64 \times 80 = 96^2 - 8^2 \times 4^2 \times 5 = 32^2 \times (9 - 5) = 32^2 \times 4;$$

$$\therefore \; v = 64, \text{ or } -64,$$

according as the body is rising or falling.

Hence the corresponding times t are given by

$$t = \frac{96 - 64}{g} = \frac{32}{32} = 1 \text{ second} \quad \text{(rising)},$$

or

$$t = \frac{96 - (-64)}{g} = \frac{96 + 64}{g} = \frac{160}{32} = 5 \text{ seconds} \quad \text{(falling)}.$$

(ii.) At height 144 feet, we have

$$v^2 = 96^2 - 64 \times 144 = 96^2 - 8^2 \times 12^2 = 0,$$

showing that the body is at its greatest height.

Here there is only one instant at which this height is reached, namely, at the time

$$t = \frac{96 \pm 0}{g} = \frac{96}{32} = 3 \text{ seconds.}$$

(iii.) At height 160 feet we should have

$$v^2 = 96^2 - 64 \times 160 = 9216 - 10240 = -1024;$$

but this is impossible, for a square cannot be negative.

This means that the body *never rises so high* as 160 feet. In fact, we have just seen that the greatest height is only 144 feet.

(iv.) At height -112 feet we have

$$v^2 = 96^2 - 64 \times (-112) = 96^2 + 8^2 \times 4^2 \times 7 = 32^2 \times (9+7) = 32^2 \times 4^2,$$

$$\therefore \quad v = 128 \quad \text{or} \quad -128 \text{ feet per second.}$$

Taking $v = 128$ feet per second, we find

$$t = \frac{96 - 128}{g} = -\frac{32}{32} = -1;$$

and, taking $v = -128$, we find

$$t = \frac{96 + 128}{g} = \frac{224}{32} = 7 \text{ seconds.}$$

Since the given point is *below* the point of projection, the body cannot reach it until it has begun to fall, and the required time is given by the positive value, *viz.* 7 seconds.

The negative value, -1 seconds, may be interpreted as follows:— If, instead of being *projected* with velocity 96 feet per second, the body had been thrown upwards from below so as to pass through the point of projection with this velocity, it would have been at a depth 112 feet 1 second *before* reaching the point of projection.

[If we had used the first method, we should have found the same results, for in the respective cases the equation

$$s = ut - \tfrac{1}{2}gt^2 = 96t - 16t^2$$

would give (i.) $80 = 96t - 16t^2$, or $t^2 - 6t + 5 = 0$;

$\therefore \quad (t-5)(t-1) = 0; \qquad \therefore \quad t = 1$ or 5 seconds.

(ii.) $144 = 96t - 16t^2$, or $t^2 - 6t + 9 = 0$;

$\therefore \quad (t-3)^2 = 0; \qquad \therefore$ only value is $t = 3$ seconds.

(iii.) $160 = 96t - 16t^2$, or $t^2 - 6t + 10 = 0$;

$\therefore \quad (t-3)^2 = -1$, and solution is impossible.

(iv.) $-112 = 96t - 16t^2$, or $t^2 - 6t - 7 = 0$;

$\therefore \quad (t-7)(t+1) = 0; \qquad \therefore \quad t = -1$ or 7 seconds.

These results would have to be fully interpreted as before, for an algebraic answer to a problem in Mechanics is of no value unless its meaning is properly interpreted and explained.]

68. Relative motion of two falling bodies.—Since the acceleration of gravity is the same for all bodies, *the relative acceleration of two bodies under gravity* (being the difference of their actual accelerations) *is zero.*

Therefore their relative velocity is constant.

This principle is of great use in finding when and where two bodies projected in the same vertical line will meet, or in finding their distance apart at any given instant of time.

Examples.—(1) A stone is dropped from the top of a tower 100 feet high, and at the same instant another stone is projected from the foot with a velocity of 80 feet per second; find when and where they meet.

Initially the velocities of the two stones are 0 and 80; hence the lower one approaches the upper with relative velocity 80 feet per second. And, since both have the same acceleration (viz., that due to gravity), this relative velocity remains constant.*

But their original distance apart is 100 feet. Hence they will be together in $\frac{100}{80}$ seconds; that is, in $1\frac{1}{4}$ seconds.

In this time the upper stone will have fallen through a distance

$$s = \frac{1}{2} . 32 . (\tfrac{5}{4})^2 = 25 \text{ feet.}$$

Hence the stones meet 25 feet below the top, and 75 feet above the bottom of the tower.

(2) If a stone is thrown vertically upwards with a velocity of 64 feet per second, and another stone is thrown up with the same velocity one second later, to find when they will meet.

At the instant that the second stone is projected the velocity of the first $= 64 - 32 . 1 = 32$ feet per second,

and the height through which it has risen

$$= 64 . 1 - \tfrac{1}{2} . 32 . 1^2 = 64 - 16 = 48 \text{ feet.}$$

The second stone is now projected upwards with velocity 64 feet per second. Its velocity upwards, relative to the first stone, is therefore $= 64 - 32 = 32$ feet per second.

But, since both stones are equally accelerated by gravity, their relative velocity is constant, and therefore the second continues to approach the first at a uniform rate of 32 feet per second.

But they are initially 48 feet apart.

Therefore they will meet in $\frac{48}{32}$ or $1\frac{1}{2}$ seconds from the instant when the second stone was projected.

* It is advisable to *state* this principle in solving any numerical example, for a correct knowledge of the principles employed is of the greatest importance in Mechanics.

69. **Bodies dropped from a moving balloon.** — If bodies be let fall from the car of a balloon in motion, they do not start from actual rest but from rest *relative to the balloon. They therefore have initially the same velocity as the balloon.* The same thing is true when bodies are dropped from a lift or cage which is ascending or descending a mine, or indeed from any vehicle in motion, such as a railway carriage or steamer.

If a stone is dropped from a balloon whose motion is being accelerated, *the subsequent motion of the stone will depend only on the velocity and not on the acceleration* of the balloon at the instant when the stone was let go, for the subsequent acceleration of the stone will always be that due to gravity.

Examples.—(1) A stone is dropped from a balloon at a height of 400 feet above the ground, and it reaches the ground in 6 seconds. To find the velocity with which the balloon was rising.

Let the upward velocity of the balloon be u feet per second. Then the stone starts with an upward velocity u, and in 6 seconds it is at a distance 400 below the point of projection. Therefore from

$$s = ut + \tfrac{1}{2}ft^2,$$
$$-400 = u \,.\, 6 - \tfrac{1}{2} \,.\, 32 \,.\, 6^2;$$
$$\therefore \quad 6u = 576 - 400 = 176;$$
$$\therefore \quad u = 29\tfrac{1}{3} \text{ feet per second.}$$

Note that the minus sign is given to g and s in this problem because they are measured *downwards*, and the positive sign to u because it is upward velocity. If the answer had come out negative, it would have indicated a downward initial velocity of the balloon.

(2) If a balloon be moving with any velocity whatever, but without acceleration, and a stone dropped from it reaches the ground in 5 seconds, to show that at the instant when the stone touches the ground the balloon will be at a height of 400 feet.

Consider the motion of the stone relative to the balloon. The acceleration of the stone is g, or 32 feet per second per second, while that of the balloon is zero. Therefore the relative acceleration of the stone is $32 - 0$, or 32 downwards.

We do not know the velocity of the balloon, but we know that the stone starts from relative rest, so that its initial relative velocity is zero. Hence in 5 seconds the stone will have fallen through a space $\frac{1}{2} \,.\, 32 \,.\, 5^2$ or 400 feet relatively to the balloon, and it will therefore be 400 feet below the balloon. But at this instant the stone strikes the ground. Therefore the balloon is at a height of 400 feet.

Observation.—The argument of the last example shows that if a stone is dropped from a balloon that is moving uniformly the depth of the stone below the position which the balloon occupies at any time t will be $\frac{1}{2}gt^2$, and will be the same as if the balloon were at rest. If the motion of the balloon itself were accelerated, this would not be the case.

Summary of Results.

If g is the acceleration of gravity,

$$g = 32 \text{ feet per sec. per sec. } \qquad (1)$$

(32·2 more accurately);

$$g = 981 \text{ cm. per sec. per sec. } \qquad (2);$$

$$\text{distance fallen in the } n\text{th second} = \tfrac{1}{2}g\,(2n-1) \qquad (3).$$

For bodies projected downwards, taking the downward direction as positive,

$$v = u + gt \qquad (4);$$

$$s = \tfrac{1}{2}(u+v)\,t = ut + \tfrac{1}{2}gt^2 \qquad (5);$$

$$v^2 = u^2 + 2gs \qquad (6).$$

For bodies projected upwards it is usually more convenient to take the upward direction as positive, and then $f = -g$; hence

$$v = u - gt \qquad (7);$$

$$s = \tfrac{1}{2}(u+v)\,t = ut - \tfrac{1}{2}gt^2 \qquad (8);$$

$$v^2 = u^2 - 2gs \qquad (9).$$

For the time taken in rising,

$$t = \frac{u}{g} \qquad (10);$$

$$\text{the height risen} = \frac{u^2}{2g} \qquad (11);$$

$$\text{the time of flight } t = \frac{2u}{g} \qquad (12)$$

$$= \text{twice time taken in rising.}$$

The relative velocity of two falling bodies is constant.

In the following examples, the value of g is taken to be 32 ft. per sec. per sec., unless the more accurate value, 32·2 ft., is expressly mentioned.

EXAMPLES IV.

1. Find the distances traversed in feet, and the velocities acquired in feet per second, by a body falling from rest for (i.) 5 seconds, (ii.) half a minute, (iii.) 15 minutes, (iv.) $\frac{1}{10}$ second.

Obtain the corresponding results in centimetres and centimetres per second, taking $g = 980$.

2. Find the velocities acquired and the times taken in falling freely through (i.) 100 feet, (ii.) 300 yards, (iii.) 3 inches, (iv.) 1000 centimetres.

3. What would the acceleration of gravity become if the unit of space were one yard, and the unit of time the time of falling from rest down a yard?

4. A falling particle in the last second of its motion passes through 224 feet. Find the height from which it fell, the acceleration of gravity being 32.

5. A body falls freely through 400 feet from rest. With what velocity will it reach the ground?

6. If, instead of falling from rest, the body (of the last question) be projected downwards so as to reach the ground with twice the former velocity, find the velocity of projection.

7. A cricket ball thrown up is caught by the thrower in 7 seconds. Draw to scale a figure showing its position at the end of every entire second since its start.

8. A ball thrown up is caught by the thrower 9 seconds afterwards. How high did it go, and with what speed was it thrown? How far below its highest point was it 5 seconds after its start?

9. With what velocities must two stones be projected upwards so that they may rise to heights of 100 and 121 feet above the ground respectively?

10. A particle is projected vertically under gravity. Prove that it will be at half its greatest height after times whose ratio is $3 + 2\sqrt{2} : 1$.

11. A stone is thrown vertically upward with a velocity of 160 feet a second. How high will it rise, and how long will it be before it returns to your hand?

12. If you let another stone drop down a well at the instant the first is within 20 feet of your hand on its return journey, at what distance below your hand will the two bodies meet? (See Example 11.)

13. Prove that two particles projected simultaneously from the same point cannot afterwards collide, whatever be their initial velocities.

14. From the edge of a cliff two stones are thrown at the same time, one vertically downwards with a velocity of 30 feet per second, the other vertically upwards with the same velocity. The first stone reaches the ground in $7\frac{1}{2}$ seconds. How much longer will the other be in the air?

15. From a balloon, which is ascending with a velocity of 32 feet per second, a stone is let fall, and reaches the ground in 17 seconds. How high was the balloon when the stone was dropped?

16. A man stands on a platform which is ascending with a uniform acceleration of 6 feet per sec. per sec.; and, at the end of four seconds after the platform has begun to move, he drops a stone. Find the velocity of the stone after three more seconds.

17. A and B are two points in the same vertical line. From B, the lower of the two points, a heavy particle is projected vertically upwards with a velocity which will just carry it to A, and at the same time a heavy particle is dropped from A. Show that when the particles meet, their velocities will be equal and opposite, and the spaces passed over by the particles will be as 3 : 1.

18. A body is projected upwards from the bottom of a well, whose depth below the surface is $8g$ feet, with a velocity of $5g$ feet per second. Find the time in which the body, after reaching its greatest height, will return to the level of the surface of the earth again.

EXAMINATION PAPER II.

1. Investigate the formula $s = \frac{1}{2}ft^2$, and deduce a corresponding expression in the case where the particle has an initial velocity u.

2. Show that the space passed over in the nth unit of time by a body moving with uniform acceleration f is $\frac{1}{2}f(2n-1)$.

3. Explain a convenient method of representing geometrically the velocity of a body moving according to a fixed law, and the distance passed over by it.

4. Find the acceleration necessary to make a body move from rest through 5 feet in 2 seconds.

5. What is meant by the statement $g = 32$? What units are employed in this equality?

6. A body is dropped from the top of a tower 145 feet high, and strikes the ground with a velocity of 96·6 feet per second. Find the value of g.

7. Prove that the relative velocity of two bodies falling vertically downwards is constant.

8. A ball is thrown up with a velocity of 110 feet per second. When will it be moving down with a velocity of 66 feet per sec.?

9. A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{12}$ seconds after it is let fall. Find from these data the velocity of sound in air.

10. A ball is allowed to drop to the ground from a height h, and at the same instant another ball is thrown up with sufficient velocity to carry it to a height $4h$. Where and when will the two balls meet?

CHAPTER V.

NEWTON'S FIRST LAW—MASS AND MOMENTUM.

70. **Kinetics.** — In the first three chapters we have considered motion in a straight line from a purely *kinematical* point of view. In the fourth chapter, we have had to assume one experimental fact—namely, that all bodies *in vacuo* fall to the ground with the same constant acceleration.

In the present part we shall consider motion generally with reference to (1) what moves, and (2) what causes it to move. This portion of the subject is called *kinetics*, in contradistinction to kinematics. To avoid introducing geometrical complications, we shall at present only consider motion in a straight line, a restriction which will be removed in Part III.

71. **Newton's Three Laws of Motion.** — As has been mentioned in the Introduction, Newton's Axioms or Laws of Motion are accepted as the foundation on which the relations between matter, motion, and force are built up. These laws were stated by Newton as follows:—

Lex I.—*Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum nisi quatenus illud a viribus impressis cogitur statum suum mutare.*

LEX II.—*Mutationem motus proportionalem esse vi motrici impressæ & fieri secundum lineam rectam qua vis illa imprimitur.*

LEX III. — *Actioni contrariam semper & æqualem esse reactionem, sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.*

First Law. — That every body perseveres in its state of remaining at rest or of moving uniformly in a right line, except in so far as it is compelled by impressed forces to change its state.

Second Law.—That change of motion is proportional to the impressed motive force, and takes place along the right line in which that force is impressed.

Third Law.—That reaction is always opposite and equal to action, or that the actions of two bodies mutually on one another are always equal, and tend in opposite directions.

We now proceed to examine Newton's Laws in detail.

72. **The First Law** furnishes us with the following

DEFINITION.—**Force** *is that which tends to change the state of rest or uniform motion of a material body.* (§ 2.)

Force may manifest itself to our senses in various ways. If we *push* or *pull* a body, we exert a force on it; and if the body is acted on by no other force, we shall set it in motion. Again, if we lift a heavy body off the ground, we shall have the body exerting a force on our hand, owing to its *weight*; and when we let the body go, this weight causes it to begin falling. A *magnet* placed near a bar of iron exerts a force of attraction on the iron. All these forces are capable, under suitable circumstances, of setting in motion, or changing the motion of, the bodies on which they act.

73. Measurement of time.—The First Law also furnishes us with a theoretical means of comparing different intervals of time. For it asserts that a body under no impressed forces would continue to move with uniform velocity; hence, by § 14, the distances traversed by such a body in different intervals would be proportional to the times taken, and conversely intervals of time might theoretically be compared by comparing the distances so traversed.

74. Evidence in favour of Newton's First Law.—The fact that a body at rest would, if left to itself, remain at rest, will probably be regarded as an obvious truism. It is not, however, so obvious that a body, if left to itself, would continue to move for ever with uniform velocity in a straight line; for common experience affords us no examples of bodies moving in this manner. The reason is that it is practically impossible to isolate a body from the action of force.

We have, however, abundant evidence that the more nearly a body is isolated from the action of force, the more nearly will it continue to move uniformly in a straight line.

A stone, if projected along a sheet of smooth ice, will continue to skid along for a considerable distance, and will move in a straight line, and the smoother the ice the longer will it travel. If the ice were perfectly smooth, and there were no air to resist the motion, the stone would always continue to travel with uniform velocity. But no ice is perfectly smooth, for even with the smoothest ice there is a small amount of friction. This, together with the resistance of the air, produces a small force on the stone, which gradually stops it, changing its state from a state of motion to a state of rest. When the stone has come to rest, these resisting forces cease to exist, and hence the stone remains at rest.

It will be easier, however, to furnish illustrations showing that *no force is required to maintain a body in uniform motion in a straight line, but that force is required to reduce a body from motion to rest, or to change its motion, as well as to start a body from rest into motion.*

If a man stand upright in a railway carriage, then, so long as the motion of the train is uniform and in a straight line, he will not feel that he is being *pushed forward* in any way. But if the train suddenly stops, the man will fall forwards owing to his tendency to go on moving.

As an instance in which force is expended in changing motion, consider a stone whirled rapidly round and round at the end of a string. The stone describes a circle, not a straight line; hence Newton's First Law tells us that it must be acted on by some force. We shall readily find that such is the case. Unless we hold the end of the string firmly, and exert a considerable pull on it, the stone will fly right off. In fact, if it be whirled sufficiently rapidly, the force required to continually change its direction of motion may become great enough to break the string, and the stone will then fly off in a straight line.

A body *may* remain at rest when it is acted on by several forces, for although each force may have a tendency to set the body in motion, these tendencies may be in opposite directions; and, since the body cannot move in two different ways at the same time, it will remain at rest, if these opposing tendencies balance one another. Thus, forces may exist even where there is no change of motion, and we must regard force as characterized by its *tending* to change a body from its state of rest or uniform motion, rather than by its actually producing any such change.

Thus, if two teams of equal strength pull at opposite ends of a rope in a tug of war, the rope will not move, in spite of the great forces which the two teams exert on it. This is because the two forces tend to set the rope moving in opposite directions, and these tendencies counteract each other.

75. **Evidence derived from celestial phenomena.**—Newton's laws of motion are really *axioms*, for they cannot be proved by purely mathematical reasoning. They rest on evidence derived from countless experiments and observations; in Chapter IX., we shall describe certain experiments by which they may be verified. But the most conclusive evidence in their favour lies in the fact that in every case in which they have been adopted as the basis of calculations, the results derived have been in strict accordance with actual observation.

As an instance we may mention their applications to the motions of the Earth and planets about the Sun, and of the Moon about the Earth. The Earth rotates on its axis once in just under 24 hours, and this rotation causes a point on the Equator to move at the rate of nearly 1000 miles an hour. The Earth also revolves about the Sun once in every year, and its speed from this cause amounts to about 1000 miles a minute. Yet we do not feel any sensation of moving at these enormous speeds, as we should certainly do if mere motion implied the existence of even a very small force. And, by assuming the Second and Third Laws of Motion, as well as the First Law, Newton and other astronomers have shown that all the observed complicated motions of the Moon about the Earth, and of the Earth and planets about the Sun, are natural consequences of the mutual attraction that exists between the different portions of matter forming them, and that it is this same attraction which gives rise to gravity on our Earth, and to the tides produced by the Moon. Were it not for the truth of the laws of motion, it would be impossible to account so simply for the motions of the solar system.

76. **Inertia and Mass.**—Newton's First Law is sometimes called the **law of inertia.** It states that material

bodies are unable of their own accord to change their state of rest or motion.

We know that some bodies are much easier set in motion, or stopped when moving, than others.

It is comparatively easy to set a small cricket-ball rolling along the ground with considerable velocity, but to set a large cannon-ball rolling even slowly requires a considerable effort. And, while we can easily stop the cricket-ball when it is thrown towards us, we should find it impossible to stop a cannon-ball travelling at anything like the same speed.

This difference cannot be due to the difference of *weight* of the balls, for, as we do not *lift* them off the ground, we do not have to overcome their weight in either case. And if the cannon-ball were removed to the surface of the Moon, it could be lifted far more easily than off the Earth; but exactly the same effort as before would be required to start it rolling along the ground.

The efforts required to produce the same change of velocity in different bodies are proportional to the **masses** of the bodies.

Thus, if the masses of two bodies are 1 lb. and 2 lbs., respectively, and if both are set in motion with the same velocity, the effort exerted in starting the second is double that exerted on the first.

Mass has been defined in § 5 as "quantity of matter." The property in virtue of which more or less effort is required to change the velocity of a body, is sometimes called *inertia*, so that **mass** may be said to be a measure of inertia.

The properties of mass and force are so intimately connected together that it is impossible to consider them separately; accordingly a clearer idea of mass will be formed when the next three chapters have been read.

77. Momentum.—DEFINITION.—The **momentum** of a body is a quantity measured by the *product of its mass and its velocity.*

The momentum of a system of bodies is the sum of the momenta of its different parts. If the mass be doubled, the momentum, with the same velocity, will be doubled, and with double that velocity it will be quadrupled (Newton).

If m denotes the mass, and v the velocity of the body,

the momentum $= mv$ .... **(1)**.

The **unit of momentum** *is the momentum of a unit mass moving with unit velocity.*

In the foot-pound-second system, the unit of velocity is a velocity of one foot per second. Hence the F.P.S. unit of momentum is the momentum of a pound of any substance moving at the rate of one foot per second.

Examples.—(1) The momentum of a 500-pound cannon-ball, when fired with a velocity of 1,000 feet per second, is

$$= 500 \times 1{,}000 = 500{,}000 \text{ foot-pound-second units.}$$

(2) What momentum is produced when a mass of 20 lbs. falls through a distance of 81 feet?

Using the equation $\quad v^2 = 2gs,$

we have $\quad v^2 = 2 \times 32 \times 81;$

$$\therefore \quad v = 72 \text{ ft. per sec.};$$

$$\therefore \quad \text{momentum} = mv = 20 \times 72 = 1440 \text{ ft.-lb.-sec. units.}$$

The C.G.S. unit of momentum is the momentum of a mass of one gramme moving with a velocity of one centimetre per second.

Example.—If a cannon-ball of 10,000 grammes is discharged with a velocity of 50,000 centimetres per second, its momentum

$$= 10{,}000 \times 50{,}000 = 500{,}000{,}000 \text{ C.G.S. units.}$$

SUMMARY OF RESULTS.

Newton's First Law.—Every body will continue in its state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by impressed forces.

Momentum of mass, m moving with velocity v

$= mv$ .... (1).

CHAPTER VI.

NEWTON'S SECOND LAW.

78. The quantity of motion (*quantitas motus*) of a body or the "motion" in Newton's Second Law, was measured by the product of its mass and its velocity, and accordingly is what we now call momentum. We may, therefore, restate the law thus:

Change of **momentum** *is proportional to the impressed force, and takes place in the direction in which the force is impressed.*

79. Observations. — Newton comments on this law roughly as follows:—

If any force generates a certain momentum, double that force will generate double the momentum, and treble the force will generate treble the momentum. This will be the case whether the forces have been impressed simultaneously at a single instant (as occurs when a body is struck with a violent blow) or gradually and successively (as when the forces continue to act on the body for a certain length of time). If the body was originally in motion, the momentum produced by the force (since this momentum is in the direction of the force) must be added to that of the body if both are in the same direction, or subtracted if they are in opposite directions. Or, if the force is in a direction inclined to the direction of motion, the added momentum and the original momentum are compounded to determine the motion of the body.

Newton's First Law gave us a definition of force. His Second Law tells us how different forces are to be compared and measured. We are not concerned in this chapter with the cases in which the force is inclined to the direction of motion.

Examples.—(1) If a cricket ball is thrown with a velocity of 50 feet per second, the impressed force used in throwing it is twice as great as if *the same ball* were thrown with a velocity of 25 feet per second, for the impressed forces are proportional to the momenta produced, and are therefore in the ratio 2 : 1.

(2) If two railway trains, of masses 120 and 90 tons, are started together, and one of them acquires a speed of 60 miles an hour in the same time as the other acquires a speed of 40 miles an hour, the forces exerted by the engines of the two trains, being, by Newton's Second Law, proportional to the momenta, are in the ratio of

$m_1v_1 : m_2v_2$, or $120 \times 60 : 90 \times 40$, or $7200 : 3600$, or $2 : 1$.

80. **The two measures of the effect of force.**—All forces with which we are acquainted have to act for a greater or shorter length of *time* before they can change the momentum of the bodies to which they are applied. Newton's First and Second Laws show that as long as a body is acted on by force, its momentum must continually keep changing; but as soon as the force ceases to act, the body will move uniformly in a straight line in obedience to Law I., and its momentum will remain constant. Hence the effect of a force might be measured in two different ways, as follows :—

(i.) By the *total* change of momentum of the body on which it acts.

This measure is called the **impulse** of the force, and is the measure of force contemplated by Newton in his Second Law. Newton's "*impressed force*" means "*impulse.*"

(ii.) By the *rate of change of momentum*, *i.e.* the change of momentum per unit time which the force tends to produce.

This is the usual measure of a force. Except where the "impulse of a force" is spoken of, it will always be assumed that forces are measured by the intensity with which they are applied at each instant, and not by the accumulated effects of their action during the time which has elapsed previously.

When a force is applied for two seconds, its total effect in producing changes of momentum is evidently twice as great as if the force only acted for one second. In other words, the *impulse of a constant force is proportional to the time during which it acts.* If the force continues to act during an indefinite time (as exemplified by the *weight* of a body, which is continually tending to pull it to the ground), its impulse continues to increase indefinitely, and would, therefore, be a most inconvenient measure of the force for practical purposes.

81. Application of Newton's Second Law to the comparison and measurement of forces. — Writing Newton's Second Law in the following form—

The total change of momentum is proportional to the **impulse** *of the applied force—*

we may apply it to the measurement of forces as follows:—

Let any specified force be chosen as the unit of force, and let k denote the velocity this force would impart to a body of unit mass if applied to it for a unit of time (say one second).

Then, since the mass in this case is unity, therefore 1 unit of force acting for 1 second produces a momentum or change of momentum whose measure is k;

Therefore 2 units of force, acting for 1 second, produce a change of momentum whose measure is $2k$, and so on.

Thus P units of force, acting for 1 second, produce a change of momentum whose measure is Pk.

If the force P, instead of acting during 1 second, acts during 2 seconds, the change of momentum during either second is Pk; and therefore the whole change of momentum in 2 seconds is $2Pk$; and so on.

Thus, in t seconds, the whole change of momentum produced by the force P is tPk.

Now let the force P be applied to a body of mass m, and let the velocity of the body be changed from u to v in the time t.

Then the momentum is changed from mu to mv, and therefore the whole change of momentum is $mv-mu$.

Hence, by what has been shown above,

$$mv-mu = tPk;$$

$$\therefore \quad kPt = m(v-u) \quad \text{.................} \quad (1);$$

and

$$\therefore \quad kP = \frac{mv-mu}{t}$$

$$= m \times \frac{v-u}{t} \quad \text{.................} \quad (1\text{A}).$$

Equation (1) or (1A) determines the force P which must be applied to a body of given mass in order to change its velocity by a given amount in a given time.

It is to be observed that the value of k depends upon what force is chosen as the unit of force.

Example.—If the unit of force is that force which, when acting on 1 lb. for 1 second, imparts to it a velocity of 32 feet per second, to find the measure of the force required to impart a velocity of 60 miles an hour to a railway train of 100 tons in 2 minutes.

Let P be the measure of the required force, and take 1 foot, 1 lb., 1 second as units of length, time, and mass.

The velocity acquired by the train in 2 minutes = 88 feet per second, and the mass of the train = 224,000 lbs.

Therefore the momentum imparted by P in 2 minutes

$$= 88 \times 224{,}000 \text{ F.P.S. units.}$$

Now a force 1, acting for 1 second, produces 32 such units of momentum.

Therefore a force P, acting for 1 second, produces $32P$ units of momentum; and a force P, acting for 120 seconds (or 2 minutes), produces $32 \times 120P$ units of momentum.

This must be equal to the momentum just found;

$$\therefore \quad 32 \times 120P = 88 \times 224{,}000;$$

$$\therefore \quad P = \frac{88 \times 224{,}000}{32 \times 120} = 5133\tfrac{1}{3} \text{ units of force.}$$

82. **To show that the rate of change of momentum of a body is proportional to the applied force.**

If k is the change of momentum produced in 1 second by the unit of force, we have shown that the change of momentum produced *in each second* by the force P is Pk.

But the change of momentum produced in each second measures the *rate of change of momentum per second.*

$$\therefore \textbf{ rate of change of momentum} = kP.$$

and is proportional to P, the applied force.

OBSERVATION.—Some writers have re-worded the Second Law of Motion thus: "**Rate of** *change of momentum is proportional to the impressed force.*" According to this statement the impressed force would be measured in the ordinary way, and not by its impulse. It would be convenient in some respects to adopt this as the second law of motion; but if this were done, we could no longer regard it as *Newton's* Second Law.

83. **To show that the force acting on a body is proportional to the product of its mass and its acceleration.**

In motion under uniform acceleration f, we have

$$v-u=ft.$$

But, by (1), $$kPt = m(v-u);$$

$$\therefore \quad kPt = mft;$$

$$\therefore \quad kP = mf,$$

or $$P = mf \div k \quad \ldots\ldots\ldots\ldots\ldots \textbf{(2)}.$$

Therefore P is proportional to the product mf, as was to be shown.

COROLLARY 1. — *A constant force produces a uniform acceleration.*

For, evidently, as long as P remains constant, the acceleration f remains constant, and its value does not depend on the length of time during which the force acts.

COROLLARY 2.—When the acceleration is *variable*, the force producing by it is variable, but the force *at any instant* is still proportional to the product of the mass into the acceleration *at that instant*, so that the relation (2)

$$kP = mf$$

still affords a measure of the force *at each instant* of the motion.

[To prove this, it would only be necessary to consider the change of momentum produced in an interval of time t so small that the acceleration *had not time to alter* during the interval (*cf.* § 40)].

84. **To compare different forces.**—The relation (1) shows that different forces are proportional to the velocities which they would impart to the same mass or to equal masses in the same or equal intervals of time.

For, supposing the equal masses m to start from rest and to acquire the velocities v, v' in the interval t, under the forces P, P', respectively, putting $u = 0$ in equation (1), we have

$$mv = kPt;$$

similarly

$$mv' = kP't;$$

therefore

$$\frac{mv}{mv'} = \frac{kPt}{kP't},$$

or

$$\frac{v}{v'} = \frac{P}{P'},$$

as was to be proved.

85. **Generalization.**—More generally: Let m, m' be the masses of two particles, and let their velocities be changed from u, u' to v, v' in the intervals of time t, t' respectively; then if P, P' be the forces required to produce the changes of motion, we have, by (1),

$$kPt = m(v-u), \quad kP't' = m'(v'-u');$$

and

$$\therefore \quad \frac{Pt}{P't'} = \frac{m(v-u)}{m'(v'-u')}.$$

This relation may be regarded as an analytical statement of the Second Law of Motion.

86. Dynamical Units of Force.

The relations (1), (2) will always be true whatever be the unit of force adopted, provided that a suitable value be given to the quantity k. Thus the unit of force may be chosen to be the weight of a pound, or the weight of a ton, or any other force. We shall now show that the unit of force may be chosen so that the constant $k = 1$, and that the equations of motion will then be much simplified.

DEFINITION.—**The Dynamical or Absolute Unit of Force** *is that force which, when applied to a unit of mass for a unit of time imparts to it a unit velocity.*

The magnitude of this unit depends on what units we adopt for measuring mass, length, and time. The dynamical unit of force in the foot-pound-second system is called the **poundal;** and the dynamical C.G.S. unit of force is called the **dyne.**

87. DEFINITION.—*The* **poundal** *is that force which, when applied to a pound of matter for one second, imparts to it a velocity of one foot per second.*

The poundal is roughly equal to the weight of half-an-ounce.

DEFINITION.—*The* **dyne** *is that force which, when applied to a mass of one gramme during one second, imparts to it a velocity of one centimetre per second.*

The dyne is a very small force indeed, being only $\frac{1}{445048}$ of the weight of a pound. For this reason forces are often measured in *megadynes*, the **megadyne** being one million dynes. A megadyne is rather more than the weight of a kilogramme.

Example.—To express the *poundal* in *dynes.*

A poundal acting on a pound for one second imparts a velocity of 1 foot per second.

But a pound = 453·7 grammes,

and a foot = 30·48 centimetres;

therefore a poundal acting on 453·7 grammes for 1 second imparts a velocity of 30·48 centimetres per second.

Therefore momentum imparted by a poundal in 1 second

$$= 453{\cdot}7 \times 30{\cdot}48 = 13780 \text{ C.G.S. units.}$$

But a dyne imparts 1 C.G.S. unit of momentum in 1 second;

$$\text{a poundal} = 13780 \text{ dynes, roughly.}$$

88. Equations of motion in Dynamical Units.— When force is measured in dynamical units, the value of k in equations (1), (2) is equal to 1.

For k has been defined as the velocity produced by a unit force acting on a unit mass for a unit time, and, by the definition of § 86, this velocity is unity;

$$\therefore \quad k = 1.$$

[Or, since by the definition of the dynamical unit we have $P = 1$, when $m = 1$, $t = 1$, $u = 0$, $v = 1$, therefore, by (1),

$$k \cdot 1 \cdot 1 = 1(1 - 0), \text{ or } k = 1.]$$

Putting $k = 1$, equation (1) assumes the form

$$Pt = m(v - u) = mv - mu \text{ .... (3)};$$

or *Change of momentum is* **equal** *to the impulse of the force measured in dynamical units.*

Also, by § 82,

The rate of change of momentum is **equal** *to the force,*

and, by (2),

$$P = mf \text{ .... (4)},$$

or *The measure of the applied force is* **equal** *to the product of the measures of the mass and the acceleration.*

It will be noticed that the dynamical unit of force is the force which, when applied to a unit mass, causes it to move with unit acceleration.

This is at once evident from the equation $P = mf$. For $f = 1$ when $P = 1$ and $m = 1$.

The equation $P = mf$, together with the equations of uniformly accelerated motion of Chap. III., are sufficient to solve any problem relating to the motion of masses under the action of forces when these forces are measured in poundals or dynes.

Examples.—(1) A force of 3 poundals acts on a mass of 4 ounces. What is the acceleration produced?

In equation (4), substituting $P = 3$ poundals,

$$m = 4 \text{ oz.} = \tfrac{1}{4} \text{ lb.},$$

we have
$$3 = \tfrac{1}{4} f;$$
$$\therefore\ f = 12 \text{ ft. per sec. per sec.}$$

(2) How far can a force of 10 dynes move a kilogramme from rest in a minute?

Let f be the acceleration. Then the equation $P = mf$ or $10 = 1000f$ gives $f = \cdot 01$ cm. per sec. per sec.

And distance traversed from rest in one minute

$$= \tfrac{1}{2} f \,.\, t^2 = \tfrac{1}{2} \,.\, (\cdot 01) \times 60^2$$
$$= 18 \text{ centimetres.}$$

89. **Sudden changes of momentum. — Impulsive forces.**—Although all forces continue to act for a certain length of time, there are many forces which only act during a very short interval, and which nevertheless produce a considerable change of momentum in that interval. Such a force is called an **impulsive force** or **blow.**

As an illustration, consider the action called into play when a billiard ball is struck with a cue. The whole change of momentum takes place during the instant that the cue is in contact with the ball. The ball rebounds almost immediately, and as soon as contact ceases it begins to move uniformly along the table.

An impulsive force is measured by its *impulse* (*i.e.*, by the *first* method of § 80), and not like an ordinary force.

This mode of measurement is adopted for two reasons—

(i.) Because it is not often necessary to investigate the motion which takes place *during* the short interval while the impulsive force is acting. The total change in the motion is usually alone of importance.

(ii.) Because such forces do not usually remain constant during their time of action, and it would be very difficult to estimate their intensity at every instant of so short an interval.

It is convenient to take as the **dynamical unit of impulse** the impulse of a blow which produces a unit of momentum. Newton's Second Law asserts that a blow whose impulse is I will then produce a change of momentum of I units. If this blow be expended in changing the velocity of a mass m from u to v, we have, therefore,

$$I = m(v-u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5);$$

or, *Impulse = change of momentum.*

If we had chosen any other unit of impulse, we should have had

$$kI = m(v-u),$$

where the value of k would depend on the unit adopted.

Observation.—The dynamical unit of impulse in the foot-pound-second system is evidently equal to the impulse of a poundal acting during one second. Similarly the C.G.S. unit of impulse is the impulse of a dyne acting during one second.

Example.—If a cannon-ball of 50 lbs. is shot with a velocity of 1200 feet per second, the momentum produced is 50×1200 or 60,000 F.P.S. units, and therefore the impulse of the explosive force of the powder is 60,000 foot-pound-second units of impulse.

Summary of Results.

Newton's Second Law.—Change of momentum is proportional to the impulse of the force, and takes place in the direction in which that impulse is impressed.

For motion in a straight line, this gives

$$kPt = m(v-u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$kP = mf \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

When the forces are measured in **dynamical units,** $k = 1$, and the equations become

$$Pt = m(v-u) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$P = mf \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

For the change of motion due to a blow whose measure in dynamical units of impulse is I,

$$I = m(v-u) = \text{change of momentum} \quad \ldots \quad (5).$$

EXAMPLES V., VI.

1. What is the momentum acquired by (i.) a mass of 1 oz. after falling for 2 seconds, (ii.) a mass of 1 cwt. after falling through 1 foot, (iii.) a mass of 1 milligramme after falling through 1 metre?

2. Equal forces act on two bodies whose masses are M and m; at the end of a second the former is moving at the rate of 10 miles an hour, and the latter at the rate of 110 feet a second. Find the ratio of M to m. State the physical principle that justifies your answer.

3. A force P, acting on a body of weight (mass) 10 lbs., increases its velocity in every second by 7 feet a second; another force Q, acting on a body whose weight is 25 lbs., increases its velocity in every second by 9 feet per second Compare the forces.

4. If a constant force will pull a body through 10 feet in a second from rest, how far will it pull the body in a minute from rest? How fast will the body be moving at the end of the time?

5. A steam engine moves a train of mass 60 tons on a level road from rest, and acquires a speed of 5 miles an hour in 5 minutes. If the same engine move another train and give it a speed of 7 miles an hour in 10 minutes, find the mass of the second train. (The mass of the engine is included in that of the train, and the forces exerted by it are the same in both cases.)

6. If a force of 15 poundals act upon a mass of 13 pounds, what velocity will it generate in 8 seconds?

7. What force, acting for 6 seconds on a mass of 12 lbs., will change its velocity from 200 to 320 feet per second?

8. What force must be applied for one-tenth of a second to a mass of 10 tons in order to produce in it a velocity of 3,840 feet per minute. What would be the momentum of the mass so moving?

[N.B.—A numerical answer is meaningless unless the unit intended is also stated.]

9. A railway train whose mass is 100 tons, moving at the rate of a mile a minute, is brought to rest in 10 seconds by the action of a uniform force. Find how far the train runs during the time for which the force is applied. Also determine the force, stating the units employed.

10. A mass of 1 ton is moving at the rate of 60 miles an hour, and 1 minute later it is moving at the same rate, but in the reverse direction. What force (expressed in poundals) must have acted on the mass during the interval?

11. A certain force, acting on a mass of 11 lbs. for 5 seconds, gives it a velocity of 4 feet per second. Obtain (i.) the magnitude of the force, and (ii.) how long an equal force must act on a mass of 4 lbs. to move it through a distance of $27\frac{1}{2}$ feet from rest.

12. A cricket ball weighing 4 oz. is travelling horizontally at the rate of 45 miles an hour. What impulse must be given to it so that it may start back with a velocity of 35 miles an hour?

13. The velocity of a body is observed to increase by four miles per hour in every minute of its motion. Compare the force acting on it with the force of gravity. (Accel. of gravity = 32 ft. per sec. per sec.)

14. A mass of 1 kilogram starting from rest acquires a velocity of 1 metre per second in 1 second; another mass of 1 kilogram also starts from rest and acquires a velocity of 1 metre per second after moving a distance of 1 metre. Find the forces acting on the two masses, and state the *name* of the unit of force employed.

15. Two bodies initially at rest, whose masses are respectively 1 gramme and 100 grammes, move towards each other by virtue of an attractive force of 0·1 dyne. Find the velocity acquired by each in 3 seconds.

16. If 10 lbs., 1 yard, and 1 minute are the units of mass, length, and time, find the dynamical unit of force.

17. If the unit of mass is the mass of 12 lbs., and the units of length and time are 14 feet and 12 seconds respectively, find the measures of the mass, velocity, and momentum of a body which weighs 1 cwt. and is moving with a velocity of 35 feet per second.

CHAPTER VII.

NEWTON'S THIRD LAW.

90. Newton's Third Law may be stated thus:

To every action there is an equal and opposite reaction;

Or, **Action and reaction are always equal and opposite.**

Here **action** means the force which one body exerts on another, and the law states that the second body always exerts on the first an equal force in the opposite direction in the same straight line. This force is called the **reaction** of the second body on the first. In other words:

"*Whatever presses or pulls something else is pressed or pulled by it to the same amount.*"—(Newton.)

Observation.—The law is true whether the bodies are at rest or in motion, and whether they press against one another through being in contact, or act on each other at a distance (like a magnet acts on a bar of iron), *provided they act* **directly** *on one another*, *i.e.*, not through a third intermediate body, nor through a system of such bodies or machines.

91. **Statical illustrations of action and reaction.**

(1) "*If anyone presses a stone with his finger, his finger is also pressed by the stone.*"—(Newton.)

(2) "*If a horse draws a block of stone tied by a rope, the horse is, so to speak, drawn back equally towards the stone.*" —(Newton.)

Of course this reaction of the stone does not actually make the horse move backwards towards the stone, but only *tends* to do so; or, what is more correct, tends to prevent the horse from moving forward under the action exerted by his feet on the ground. If the rope were suddenly cut, and the horse continued to exert the same effort with his feet as before, he would start so quickly into motion that he would probably fall over forwards. As Newton puts it, the pull of the rope "impedes the progress of the one by the same amount that it promotes the progress of the other."

(3) If a ladder is allowed to lean against a wall, the ladder presses against the wall and the wall pushes with an equal force against the ladder. The action of the ladder tends to overturn the wall, and will actually overturn it if the masonry is weak and gives way. The reaction of the wall on the ladder prevents the ladder from falling over, as it would at once do if it were placed in the same position without such support.

92. **Thrust.**—DEFINITION.—When the action and reaction of two bodies tend to keep them apart from one another, or to prevent them from moving towards one another, they constitute a **thrust**, or a **push.**

The first illustration of § 91 affords an instance of a thrust. The finger exerts a thrust on the stone tending to push it away, and the stone exerts a thrust which prevents the finger from penetrating it.

93. **Pull.**—DEFINITION.—When the action and reaction of two bodies tend to keep them together or to prevent them from separating, they constitute a **pull,** or **tension.**

Thus in the second illustration of § 91 the horse exerts a pull on the stone, and the stone exerts an equal and opposite pull on the horse.

94. **Attraction and repulsion.**—DEFINITION.—When bodies act on one another *at a distance* (as a magnet acts on a bar of iron), the force between them is called an **attraction** if it tends to bring them together, or a **repulsion** if it tends to separate them.

Thus the Earth's *attraction* causes bodies to fall to the ground with the acceleration g (Chap. IV.).

95. **Friction.**—DEFINITION.—When the action and reaction between two bodies tend to prevent them from sliding one along the other, they constitute what is known as **friction.**

When a book rests on a table, and we try to push it along the table, we shall experience a certain resistance. This is due to the *friction* between the table and the book, which tends to prevent the book from slipping.

96. Applications of the Third Law to locomotion.

(1) The act of *walking* affords an excellent example of the equality of action and reaction, as well as of the properties of friction. In starting off to walk we press backwards on the ground with our feet, and the reaction of the ground gives us an equal and opposite impulse forwards, which sets us in motion.

This action and reaction are due to friction. If we try to walk across a smooth sheet of ice, we shall experience some difficulty, because only a very small amount of friction can be called into play between our feet and the ice.

(2) *Motion of a horse and cart.*—When a horse and cart are just starting into motion, the horse exerts a forward pull on the cart, and this pull sets the cart in motion.

It follows from Newton's Third Law that the cart exerts an equal and opposite backward pull on the horse. If this were the only force acting on the horse, the horse would move backwards towards the cart instead of forwards, and this we know is not the case.

But the *action* of the horse's feet in the act of walking presses backwards on the ground, and therefore the equal and opposite *reaction* of the ground (due to friction) tends to push the horse forwards. This reaction exceeds the backward drag of the cart by an amount sufficient to produce the acceleration with which the horse starts into motion.

Let the masses of the horse and cart be m and M respectively.

Let P be the pull between the horse and cart, and F the horizontal force of friction between the horse's feet and the ground, both expressed in dynamical units of force. Let f be the common acceleration of the horse and cart (both of which move together, of course).

The cart is acted on by the pull P drawing it forwards, and therefore, by § 88, $$P = Mf \quad \text{(i.).}$$

The horse is acted on by F pushing forwards and the reaction equal and opposite to P pulling backwards; since the latter acts in the negative direction, it is represented *algebraically* by $-P$. Hence the force instrumental in producing changes of motion is $F-P$.

Therefore $$F - P = mf \quad \text{(ii.).}$$

Adding (i.) and (ii.),

$$F = (M+m)f \quad \text{(iii.)}$$

$$= (\text{total mass of horse and cart}) \times (\text{acceleration}).$$

This shows that the change of motion in the horse and cart when considered together as a whole is that due to the force F acting on their combined mass, as we should expect.

Eliminating f from (i.) and (ii.) by cross multiplication, we have

$$M(F-P) - mP = 0;$$

$$\therefore \quad P = \frac{M}{M+m} F \quad \text{(iv.),}$$

giving P (the pull on the cart) in terms of F (the action of the horse's feet). In forming these equations, no account has been taken of friction or inertia of the cart-wheels, resistance of the air, &c.

(3) In a *railway engine* the action of the steam causes the driving wheels to press backwards on the rails, and the reaction of the rails not only sets the engine in motion, but also causes it to pull the train after it.

Here again action and reaction are due to friction, and, if the rails are greasy and the train heavy, the wheels will sometimes skid round instead of impelling the train forwards.

(4) *The propulsion of a bicycle* depends on exactly the same principle —the propelling force is the reaction of the ground, which is exactly equal and opposite to the action of the driving wheel produced by the rider pressing on the pedals.

97. Changes of momentum due to action and reaction.

In the last chapter we saw that the effects of forces might be measured either

by the total changes of momentum they produce (or tend to produce), *i.e.* their impulses; or

by the rates of change of momentum, *i.e.* their impulses per unit time.

Now action and reaction act during the same time, hence the Third Law asserts that the changes or tendencies to changes of momentum of two bodies due to their action and reaction are always equal and opposite. As Newton says:

"If one body impinges on another, and by its action changes the momentum of the latter in any way, the first body will in its turn undergo an equal change of momentum in the opposite direction due to the reaction of the second (because of the equality of the mutual pressure). These actions give rise to equal changes of *momentum*, not of *velocity*, provided that the bodies are not impeded by other forces. And since the changes of momentum are equal, the changes of velocity produced in opposite directions are inversely proportional to the masses of the bodies."

98. **The recoil of a gun** affords a good illustration of this property. The explosion of the powder inside the barrel exerts equal and opposite impulses on the shot and the gun, and causes them to move in opposite directions with equal momenta. Hence, if the speed of the shot be given, the speed of recoil can be found.

Examples.—(1) If a 700-lb. shot be fired from a 75-ton gun, with a speed of 1200 feet per second, to find the speed of recoil of the gun.

Here the momentum of the gun is equal and opposite to that of the shot.

Now, momentum of shot = 700×1200 foot-pound-second units, and
$\therefore$ momentum of gun is also = 840,000 F.P.S. units.

But mass of gun = $75 \times 2240 = 168000$ lbs.;

$$\therefore \text{ velocity of recoil} = \frac{\text{momentum}}{\text{mass}} = \frac{840000}{168000} = 5 \text{ feet per second.}$$

(2) If a 14-lb. shot leave the muzzle of a 2-ton gun with a *relative* speed of 540 feet per second, to find the speed of recoil.

Let v feet per second be the required speed of recoil; then, since the relative speed of the shot is 540, its actual speed in the direction opposite to that of recoil is $540 - v$. Also the masses of the shot and gun are 14 and 4580 lbs. respectively.

Therefore, since the momenta are equal and opposite,

$$14 \times (540 - v) = 4480 \times v;$$

$$\therefore \quad 540 - v - 320v = 0;$$

$$\therefore \quad v = \frac{540}{321} = \frac{180}{107} = 1{\cdot}682 \text{ feet per second approximately.}$$

OBSERVATION.—Note that the body of greater mass undergoes the smaller change of velocity, and *vice versâ*. Thus the speed of recoil of a large gun is very small compared with the speed of projection of the bullet.

99. **The propulsion of a rocket** depends on the same principle. As the contents of the rocket burn away, the products of combustion are projected with considerable velocity, and the downward impulses which project them continually give rise to equal and opposite upward impulses on the case of the rocket, causing it to rise in the air.

100. **Comparison of masses.**—The same principle suggests a simple means by which the masses of two bodies could theoretically be compared. Suppose the bodies A and B to have a small coiled-up spring placed between them, and let this spring be suddenly released, without jerking the bodies in any way. The spring will exert equal and opposite impulses on A and B, and therefore A and B will separate with equal and opposite momenta. Hence

$$(\text{mass of } A) \times (\text{speed of } A) = (\text{mass of } B) \times (\text{speed of } B);$$

$$\therefore \quad \frac{\text{mass of } A}{\text{mass of } B} = \frac{\text{speed of } B}{\text{speed of } A}.$$

Hence, by observing the speeds (or velocities in opposite directions) with which the bodies A, B move after leaving the spring, the ratio of the masses of A and B could be found.

If one of the masses, say B, be taken as the unit of mass, say a pound, the ratio of the speeds will give the mass of A in pounds, *i.e.*, in this case mass of A in pounds $=$ (speed of B)/(speed of A).

To compare the masses of two bodies it is not necessary to start them from rest by a spring placed between them. If they are allowed to collide with one another in any way, they will undergo equal and opposite changes of momentum, and therefore, as remarked by Newton, the changes of velocity will be inversely proportional to the masses.

Thus, if the velocities of two bodies (measured in the same direction) are changed from U, u to V, v by a collision between them, and if the masses of the bodies are M, m, then, since the changes of momentum are equal and opposite, $M(V-U) = -m(v-u)$,

or $$\frac{M}{m} = \frac{v-u}{U-V}.$$

OBSERVATION.—This relation may be written

$$MV + mv = MU + mu;$$

or, momentum after the action = momentum before the action.

101. Quantitative definition of mass. — We are, therefore, now in a position to give the following quantitative definition of mass :—

*The **mass** of any body is the measure of the quantity of matter in the body, defined by the law that the changes of velocity produced in two bodies by their mutual reactions are inversely proportional to their masses.*

102. The Principle of Conservation of Momentum. Since the changes of momenta are equal and opposite, the momentum of one body will increase by the same amount that the momentum of the other decreases. This will always be algebraically true, provided that we make the same conventions with regard to sign for momenta as for velocities (*i.e.* we regard the momenta of bodies moving in one direction as positive, and the momenta of bodies moving in the other direction as negative). The only effect of the action and reaction will be to transfer momentum from one body to the other, without altering the algebraic sum of their momenta. In other words :

The total momentum of a system of moving bodies in any direction is not altered by the mutual reactions of the several bodies.

This property is called the **Principle of Conservation of Momentum.** It holds good for any number of bodies. It is enunciated and proved by Newton as follows :—

"*The quantity of momentum formed by adding together the momenta of those bodies which are moving in one direction, and subtracting the momenta of those moving in the reverse direction, is not altered by the action of the bodies on one another.*"

"For, by Law III., action and reaction are equal and opposite, therefore by Law II. they produce equal changes of momentum in opposite directions. Hence, if the bodies are moving in the same direction, what is added to the momentum of the body in front is subtracted from the momentum of the body behind, so that the sum remains the same as before. If the bodies are moving in opposite directions, an equal amount of momentum will be subtracted from both, so that the difference of momentum of the parts moving in opposite directions will remain the same."

Examples.—(1) A ball of mass 3 lb., moving with velocity 2 feet per second, is struck by a ball of mass 1 lb., moving in the same direction, with a velocity of 10 feet per second. If after the blow the smaller ball comes to rest, find the subsequent velocity of the larger one.

Momentum of smaller ball before blow $= 1 \times 10 = 10$ units,

,, ,, ,, after ,, $= 0$;

$\therefore$ change of momentum of smaller ball $= 0 - 10 = -10$ units.

The change of momentum of the larger ball is equal and opposite; and therefore $= +10$ units.

But, before the blow, momentum of larger ball $= 3 \times 2 = 6$ units;

$\therefore$ after ,, ,, ,, ,, $= 6 + 10 = 16$ units;

and its mass $= 3$ lbs.;

$\therefore$ its velocity $= \frac{16}{3} = 5\frac{1}{3}$ ft. per sec.

(2) A billiard cue of mass 1 lb., moving with velocity $10\frac{1}{2}$ feet per second, strikes a ball of 5 oz. at rest. If immediately after the blow the cue and ball move with a common velocity, find this velocity.

Before the cue strikes the ball, we have 1 lb. moving with $10\frac{1}{2}$ F.P.S. units of velocity, and $\frac{5}{16}$ lb. without any velocity;

$\therefore$ total momentum $= 1 \times 10\frac{1}{2} + \frac{5}{16} \times 0 = 10\frac{1}{2}$ F.P.S. units.

After the blow the momentum is the same as before, but the whole mass, $1\frac{5}{16}$ lbs., is moving with a common velocity. Hence, if the required velocity in feet per second is v, we have

$$1\tfrac{5}{16} \cdot v = 10\tfrac{1}{2};$$

$$\therefore \quad \text{velocity } v = \tfrac{21}{2} \times \tfrac{16}{21} = 8 \text{ feet per second.}$$

Hence the ball and cue move off with a velocity of 8 feet per second.

(3) A goods truck of 6 tons, travelling at 3 miles an hour, collides with another truck at rest, and both move on together at 2 miles an hour. To find the mass of the second truck.

Taking a mile, an hour, and a ton, as units of length, time, and mass, the momentum of the first truck is decreased by the collision by $6 \times (3-2)$ or 6 units, and therefore 6 units of momentum are imparted to the second truck.

But the velocity acquired by the latter is 2 units;

$$\text{hence its mass} = \tfrac{6}{2} = 3 \text{ units of mass};$$

$$\therefore \quad \text{the mass of the second truck is 3 tons.}$$

Or, as we should more commonly express it, the second truck *weighs* 3 tons.

103. **Inelastic and elastic bodies.** — When two bodies (for instance, two balls) collide, they sometimes continue to remain together and move on with a common velocity. Such bodies are said to be *inelastic*. In other cases the bodies rebound, and separate after striking each other, and they are then said to be more or less *elastic*.

The laws which govern the rebound of elastic bodies have been determined experimentally, and will not be detailed here.

104. **External and internal forces.**—When we are dealing with the motion of a particular system of bodies, the actions and reactions between the different pairs of bodies are called **internal** forces of the system. Forces due to the action of bodies that do not belong to the system which we are considering are called **external forces** or **impressed forces** of the system.

By the Principle of Conservation of Momentum the total momentum of a system is unaltered by the internal forces of the system, but it may be altered by external forces. In that case an equal and opposite change is produced in the total momentum of the bodies (outside the system considered) by whose action the external forces are impressed.

105. **Reaction of motions relative to the Earth.**—Newton's Third Law shows that when a man jumps off the ground, he communicates to the Earth an amount of momentum equal and opposite to that of his own motion. But the mass of the Earth is so great—being about 6,067

million billion tons—that the *velocity* thus imparted to the Earth is absolutely imperceptible.

Moreover, the Earth yields slightly under the man, so that, instead of the motion getting transmitted to the whole Earth, only a slight vibration is produced in the Earth in his immediate neighbourhood. In the case of a man jumping, this vibration is imperceptible; but larger moving masses, such as traction engines and railway trains, as also sudden explosions, often shake the ground for a considerable distance.

Summary of Results.

Newton's Third Law. — Action and reaction are equal and opposite.

The Principle of Conservation of Momentum.—The total momentum of a system of bodies is not altered by their mutual reactions.

If the velocities of two masses M, m are changed from U, u to V, v by their mutual reactions,

$$\frac{M}{m} = -\frac{u-v}{U-V} = \frac{v-u}{U-V}, \quad \text{or} \quad MV+mv = MU+mu.$$

EXAMPLES VII.

1. Each of two bodies attracts the other with the same force. If allowed to move, show that in any given time they move over distances which are inversely proportional to their masses.

2. Enunciate Newton's Laws of Motion. A shot weighing 20 lbs. is fired from a gun weighing 5 tons, with a velocity of 1120 feet per second. Find the velocity with which the gun recoils.

3. A 10 lb. shot is fired from a gun weighing 1 ton, with a velocity of 1000 ft. per second. Find the velocity with which the gun recoils.

4. An 80-ton gun on a smooth horizontal plane projects a bolt of 5 cwt. horizontally with a velocity of 1200 feet per second. What is the velocity of recoil?

5. A gun weighing 5 tons is charged with a shot weighing 28 lbs. If the gun be free to move, with what velocity will it recoil when the ball leaves it with a velocity of 100 ft. per second? ✓

6. Just as a tramcar reaches a man standing by the tramway it has a velocity of $8\frac{1}{2}$ feet per second; the man takes hold of and mounts the car. What change of velocity takes place, the weights of the car and man being 1 ton and 10 stone respectively? ✓

7. A shell, moving with a velocity of 50 ft. per second, bursts into two parts, which weigh respectively 30 lbs. and 62 lbs. The velocity of the larger piece is increased to 80 feet per second. What is the velocity of the smaller?

8. A ball A, of weight 10 lbs., strikes a body B at rest, weighing 100 lbs., with a velocity of 100 ft. per second. Find the velocity of B, supposing A brought to rest by the impact. ✓

9. Three goods trucks, weighing respectively 5 tons, 7 tons, and 8 tons, are placed on the same line of rails. The first is made to impinge on the second with a velocity of 60 feet per second without rebounding. The first and second together impinge in the same way on the third. Find the final velocity.

10. Two wooden balls, weighing 12 oz. and 16 oz., are connected by a long coiled-up string. The smaller is projected with a velocity of 12 ft. per second. With what velocity must the larger be projected in order that both may come to rest when the string becomes tight? ✓

11. Two balls whose weights are 6 kilog. and 10 kilog., and whose velocities are 50 and 20 metres per second, approach in opposite directions, and, after impact, move on together. Find their common velocity.

12. Two pieces of magnetized iron, subject to their mutual attraction and to no other forces, start from rest. If their masses are 5 grammes and 2 grammes, and the acceleration of the larger one is 40 cm. per sec. per sec., find the acceleration of the smaller, and the force of attraction, expressed in dynes. ✓

EXAMINATION PAPER III.

1. State Newton's Second Law of Motion in forms applicable (i.) to finite forces, (ii.) to impulsive forces.

2. Explain the equation $P = Mf$.

3. State the Third Law of Motion, and explain clearly its application to the case of a horse starting a cart into motion.

4. A 30-ton mass is moving on smooth level rails at 20 miles an hour; what steady force can stop it (*a*) in half-a-minute, (*b*) in half-a-mile?

5. Find the acceleration produced, and the momentum acquired in one minute when a force of

(i.) 32 poundals acts on a mass of 1 cwt.;

(ii.) 5 dynes acts on a mass of 1 milligramme.

6. Define *momentum*. What is *inertia?* Is it a force?

7. A ball of mass 8 lbs. and velocity 60 ft. per second impinges directly on another ball of mass 45 lbs. and velocity 45 feet per second in the same direction. They move on together after impact. What is their common velocity?

8. When a man in a small boat moves forward, the boat begins to go backwards. Why is this? If the man weighs 12 stone, and the boat 18 stone, and the boat is 10 feet long, how far will the boat move back in the water when the man walks from one end to the other, if the resistance of the water be neglected?

9. What momentum is produced when a mass 10 lbs. falls for 7 seconds?

10. If the metre, the minute, and the kilogramme be taken as the units of length, time, and mass, compare the unit of force with the dyne.

CHAPTER VIII.

WEIGHT AND ITS RELATION TO MASS. GRAVITATION UNITS OF FORCE.

106. **Weight.**—DEFINITION.—*The* **weight** *of a body is the force with which it is attracted to the Earth.*

When we lift a body off the ground, we have to exert a certain force in order to overcome its weight. If the body rests on a table, it presses on the table with a force equal to its weight. If the body is unsupported, it will fall to the ground; hence Newton's First Law of Motion shows that some force must be acting on it. This force is the body's weight. We shall now show that

107. The weights of different bodies are proportional to their masses.

For in Chapter IV. we saw that all bodies fall to the ground with the same acceleration.

But, by § 83, the force acting on any body is proportional to the product of its mass and its acceleration.

In the case of a falling body the force is the weight of the body.

Therefore the weight is proportional to the mass, as was to be shown.

108. To express the weight of a given mass in dynamical units of force.

The acceleration of a falling body has been denoted by g, and it has been shown in Chapter IV. that

$$g = 32 \text{ ft. per sec. per sec.}$$
$$= 981 \text{ cm. per sec. per sec.}$$

This acceleration is produced by the weight of the body acting on its mass.

(i.) **In the foot-pound-second system** we have

$$g = 32.$$

Hence the weight of 1 lb. acting on the mass of 1 lb. produces an acceleration whose measure is 32.

But, by the definition of the poundal (§ 87),
a force of 1 poundal acting on a mass of 1 lb. produces a unit of acceleration, *i.e.* an acceleration whose measure is 1;

$\therefore$ **weight of a pound = 32 poundals** ...... (1).

Hence also,

a force of one poundal $= \frac{1}{32}$ weight of one pound

= weight of half-an-ounce.

[More accurately, the weight of a pound is 32·2 poundals.]

(ii.) **In the C.G.S. system,** we have

$$g = 981;$$

therefore the weight of 1 gramme acting on the mass of 1 gramme produces an acceleration of 981 C.G.S. units.

But, by the definition of the dyne (§ 87),
a force of 1 dyne acting on a mass of 1 gramme produces 1 C.G.S. unit of acceleration;

$\therefore$ **weight of a gramme = 981 dynes** ...... (2).

Hence also,

a force of 1 dyne $= \frac{1}{981}$ **weight of a gramme.**

(iii.) **Generally,** if W denote the weight of the mass m measured in dynamical units of force, then, in the equation

$$P = mf,$$

where $f = g$, $P = W$;

$$\therefore \quad W = mg \quad \text{..........................} \quad \mathbf{(3)},$$

or **weight** (in dynamical units) = (**mass**) $\times g$.

In this book, when the weight of a body is measured in dynamical units of force, we shall, for brevity and to avoid confusion, speak of it as the **"absolute weight"** of the body. Hence

absolute weight of mass $M = Mg$,

or in words,

(absolute weight) = (mass) × (accel. of gravity).

109. Gravitation unit of force.—The forces which occur most frequently in mechanical problems are those due to weight; moreover the weight of a given quantity of matter is a force which is easily reproduced as a standard of comparison, while a poundal or a dyne is a difficult unit to reproduce. For this reason it is convenient, both in engineering work and in all statical investigations (where the forces are due to weight, and no motion takes place), to measure force in terms of the weight of a definite quantity of matter.

Definition.—**The gravitation unit of force** *is the weight of the unit of mass.*

Where masses are measured in pounds, the gravitation unit of force is the weight of one pound.

If the gramme is taken as the unit of mass, the gravitation unit of force is the weight of one gramme.

The measure of a force in gravitation units is really the measure of the *mass* whose weight is equal to that force.

By **"a force of 1 lb."** is meant "*a force equal to the weight of* a pound."

Similarly, a force of 10 tons or "a force of 5 oz." denote forces equal to the weight of 10 tons or 5 oz., respectively. To avoid confusion, however, it is better always to add the word "weight" or its abbreviation "wt.," and we may therefore speak of the above forces as "1 *lb. wt.*," "10 *tons wt.*," "5 *oz. wt.*," respectively.

In like manner, by **forces of "1 gramme,"** "5 kilogrammes," or "31 milligrammes" are meant *forces equal to the weights of* 1 gramme, 5 kilogrammes, and 31 milligrammes, respectively, and these are more accurately spoken of as "1 *gm. wt.*," "5 *kilog. wt.*," "31 *mgr. wt.*," respectively.

Whatever be the system of units adopted, we always have, by § 108,

the gravitation unit of force = g absolute units,
the absolute unit of force = $1/g$ gravit. units.

110. **Equations of motion for gravitational units of force.**—In the first place it is important to observe that

When measured in *gravitation* units the force on a body is *not equal* to but only *proportional* to its {mass × acceleration, rate of change of momentum; and that change of momentum is *not equal* to but only *proportional* to the product of the force and the time.

In fact the relations proved in § 88 are based on the supposition that the unit of force is the dynamical unit.

In the general equation of motion

$$kP = mf$$

we must put $k = g$, for the case in which the force is measured in gravitation units. For if the force is the gravitation unit of force, *i.e.* is equal to the weight of unit mass, and it acts on a unit mass, the acceleration is g; hence if $P = 1$ and $m = 1$, then $f = g$.

Therefore, substituting in the above equation,

$$k \,.\, 1 = 1 \,.\, g.$$

Hence $k = g$, and therefore the equation of motion becomes $Pg = mf$..........................(4);
i.e. **(force in gravit. units) $\times g$ = (mass) $\times$ (accel.).**

Again, if the velocity changes from u to v in the interval of time t, the equation of momentum $kPt = m(v-u)$
becomes $gPt = m(v-u)$
or (change of momentum) = (force in gravit. units) $\times g \times$ (time).

Observations.—These equations might, of course, be applied to the solution of problems, and the work would probably be rather shorter than by either of the methods given below. But the work would not be so instructive, and confusion would be more likely to arise.

The student should spare no pains in becoming familiar with the dynamical and gravitational units of force, as well as the difference between "mass" and "absolute weight."

To understand these ideas fully may take some time, but the time will be well spent if this is done before proceeding further. And in working problems the only safeguard against confusion is **to specify at each step of the work the units in terms of which the different quantities** are measured—a caution which has already been given, but which applies with especial force to problems of the present class.

111. Application to problems.—In solving problems relating to the acceleration of masses under the action of forces where these forces are given or are required to be found in gravitation units, we may employ the formulæ of the last article, or adopt either of the following methods of solution. The first of the following methods is the safest method, and should be used whenever the problem presents any difficulty.

First method.—(1) *Reduce all the forces to dynamical units by multiplying their measures by the value of g (expressed in terms of the selected units of length and time).*

(2) *Work with the equations of motion, using dynamical units throughout; and, if any forces have to be calculated, obtain them in dynamical units.*

(3) *Finally, reduce the required forces to gravitation units by dividing by g.*

Examples.—(1) If a bucket of water, weighing 20 lbs., is pulled up from a well with an acceleration of 8 feet per second per second, to find, in lbs. weight, the force which must be applied to the rope.

Here the weight of the bucket

$$= 20 \text{ lbs. wt.} = 20 \times 32 = 640 \text{ poundals.}$$

The force applied to the rope must not only support the weight of the bucket, but must also produce an upward acceleration of 5 F.P.S. units.

Now the force required to support weight of bucket = 640 poundals, force required to produce an accel. 8 in 20 lbs. = 8×20 = 160 poundals.

$$\therefore \text{ total upward pull on bucket} = 640 + 160 \text{ poundals}$$
$$= 800 \text{ poundals} = 800/32 \text{ lbs. wt.}$$
$$= 25 \text{ lbs. wt.}$$

Therefore the rope must be pulled with a force equal to the weight of 25 pounds.

(2) A force equal to the weight of 5 lbs., acting on a body, produces an acceleration of 9600 yards per minute per minute. What is the mass of the body?

Here $P = \text{wt. of 5 lbs.} = 5 \times 32$ poundals,

$$f = 9600 \text{ yds. per min. per min.} = 8 \text{ ft. per sec. per sec.};$$
$$\therefore \quad 5 \times 32 = m \times 8;$$
$$\therefore \quad m = 20 \text{ lbs.}$$

112. **Second method.**—*We know that when any body is acted on by a force equal to its weight, it moves with acceleration g. Remembering that the force on a body is proportional to the product of its mass and its acceleration, the unitary method enables us to find a relation between the mass, the acceleration, and the impressed force measured in gravitation units.*

Examples.—(1) If a railway train of 120 tons is pulled by the engine with a force of 3 tons weight, to find how far it will have to travel to acquire a velocity of 60 miles an hour.

A force equal to the weight of the train, or 120 tons, would start it with an acceleration 32 feet per second per second.

Therefore a force of 3 tons weight produces an acceleration of $\frac{32}{40}$ or $\frac{4}{5}$ feet per second per second.

Let s be the required distance in feet. The acquired velocity = 88 feet per second, hence the formula for accelerated motion

$$v^2 = 2fs$$

gives $$88^2 = 2 \times \tfrac{4}{5} \times s,$$
whence $$s = 440 \times 11 \text{ ft.} = \tfrac{11}{12} \text{ of a mile.}$$

(2) If a 2-oz. bullet, travelling 1600 feet per second, penetrates 10 inches into a target, to find in lbs. the mean resistance of the target.

Let f be the acceleration of the bullet in feet per second per second: then, by the formula $v^2 - u^2 = 2fs$, we have
$$0^2 - 1600^2 = 2 \times \tfrac{10}{12} \times f;$$
whence $$f = -1600 \times 960 = -1536000.$$

Now a force of 2 oz. weight acting on the bullet (mass 2 oz.) would produce a retardation 32. Hence the force required to produce the given retardation
$$= 1{,}536{,}000 \times \tfrac{2}{32} \text{ oz. weight} = 96000 \text{ oz. weight}$$
$$= 6000 \text{ lbs. weight.}$$

113. **How mass is found by weighing.** — We are now in a position to explain why weighing a body in a pair of scales determines its mass.

A common balance consists of a beam or lever which can turn about its middle point, and at its ends are suspended the two scale-pans. The body to be weighed is placed in one scale-pan, and suitable weights are placed in the other. Now it will be shown in Statics that if the beam remains balanced in a horizontal position, the body and weights must press with equal forces on the two respective scale-pans. We thus infer that the *weight* of the body is equal to that of the weights employed.

But weight is proportional to mass.

Therefore also the *mass* of the body is equal to the mass of the weights used to balance it. If these weights are known multiples and sub-multiples of a pound, their amount is equal to the number of *lbs. wt.* in the *weight* of the body, or the number of *lbs.* in the *mass* of the body.

Thus **the weight of the body in pounds weight is numerically equal to its mass in pounds.**

Similarly, **the weight of a body in grammes weight is numerically equal to its mass in grammes.**

When, therefore, we say that **"a body weighs m lbs.,"** we imply that

(1) **The body is drawn towards the earth with a force m times as great as that which acts on a mass of 1 lb.;**

(2) **The MASS of the body is m lbs.;**

and we may draw the corresponding inferences when the weight of a body is given in grammes.

But this process of weighing **does not measure the absolute weight** of the body in poundals or dynes.

114. Difference between mass and weight. — Although the weight of a body may thus be measured by the same number as its mass, it is important to distinguish between the mass and the weight of a body. *Mass always represents a quantity of matter in the body, and does not depend on gravity; while weight always means the force with which a body is attracted to the ground.*

So long as we only have to compare the weight of one body with the weight of another body, and not to express the weight in dynamical units of force, the distinction between mass and weight is unimportant.

It does not matter, for example, whether we regard the weight of a packet of sugar or tea as measuring its relative heaviness *as compared with that of the pound weight* belonging to our scales or as measuring the quantity of material or mass in it.

But when weight is considered dynamically, with reference to its power of producing changes of momentum, and is measured in dynamical units of force, the distinction between mass and weight is at once apparent. In fact,

weight of mass $m = m$ times weight of unit mass
$= m$ gravitation units of force
$= mg$ dynamical units of force.

115. **Variations in the intensity of gravity.** — In Chapter IV., we stated that g, the intensity of gravity, is slightly different at different parts of the Earth, being greatest at the Poles and least at the Equator. Since the *absolute weight* of a mass m is mg, it follows that the absolute weight of a body is different in different places. From the results stated in § 56.

The **absolute weight of a pound mass** varies from **32·091 poundals** at the Equator to **32·255 poundals** at the Earth's North and South Poles.

The corresponding limits for the **absolute weight of a gramme** are **978·10** and **983·11 dynes,** respectively.

These variations do not affect the determination of masses by weighing with a balance, for *at any given place* equal masses have equal weights, whatever be the intensity of gravity.

Example.—Suppose that a pound of sugar is weighed out at the Equator, where $g = 32{\cdot}091$. The pound of sugar is attracted towards the Earth with a force of 32·091 poundals, but the pound weight used in weighing is also attracted with a force of 32·091 poundals; and, since these forces are equal, the two masses balance each other in the scales. If the same masses be taken to London, where $g = 32{\cdot}191$, the sugar will be attracted to the Earth with a force of 32·191 poundals, and the pound weight will also be attracted with a force of 32·191 poundals. Hence they will still balance each other in the scales.

116. **A spring balance** is often used for the purpose of weighing. One of the simplest forms is shown in Fig. 10. The scale-pan holding the goods to be weighed is suspended from a spiral spring. The spring is thus extended by the weight, and the greater the load the more is it extended. The required weight is indicated by a pointer, which moves up and down with the scale-pan, along a graduated scale at the side of the spring.

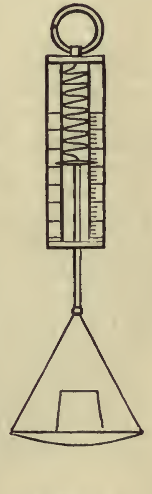


Fig. 10.

Unlike the common balance, the spring balance measures the *absolute weights* and not the *masses* of the bodies placed in the scale-pan. A force of one poundal will always extend the spring by an invariable amount, so that if the scale be graduated in poundals at one place, it will correctly measure forces in poundals at any other place.

Hence a spring balance really gives a constant measure of **force.**

But it does not give an accurate measure of **mass** unless it is used to weigh goods at the place for which it is graduated, as the following example will show :—

Example.—Suppose that it is graduated for weighing bodies in pounds at London, where $g = 32{\cdot}191$, and where a pound consequently weighs 32·191 poundals. Then the pointer will always indicate 1 lb. when the scale-pan is pulled down with a force of 32·191 *poundals*. At the Equator a pound only weighs 32·091 poundals, and therefore the weight of a pound mass does not pull the pointer quite down to the graduation marked "1 lb." To bring the pointer down to the 1 lb. reading, we should have to add an extra force of $\frac{1}{10}$ of a poundal, and this would require us to put in about $\frac{1}{20}$ of an ounce more into the scale-pan (since a poundal nearly equals $\frac{1}{2}$ oz. weight). Hence if a tradesman were to buy a spring balance in London, and to use it for weighing goods out at the Equator, he would have to add about $\frac{3}{10}$ per cent. ($\frac{1}{321}$ nearly) to the old cost price to find the cost under the new value of g.

[OBSERVATION.—*Practically*, such differences are too small to be detected except with the most sensitive spring balance.]

117. Apparent weight of a man in a moving lift.—When a man is ascending or descending in a lift with *uniform* velocity, the reaction of the floor of the lift is exactly equal to the man's weight. When, however, the lift is being *accelerated* upwards, the reaction of the floor must be greater than the man's weight, because it has not only to support his weight, but also has to give him an upward *acceleration*. And when the lift is being *accelerated* downwards, his weight must exceed the reaction of the floor on his feet by the amount necessary to impart to him the downward *acceleration* of the lift.

For let m be the mass of the man, and suppose the lift is moving with a downward acceleration f. Let R be the thrust on the floor of the lift, which is equal and opposite to the reaction of the floor on the man. We may call R the man's "apparent weight."

Suppose R measured in dynamical units. The forces acting on the man are his weight mg downwards and R upwards, giving on the whole a downward force $mg - R$. Remembering that

$$(\text{force}) = (\text{mass}) \times (\text{acceleration}),$$

we have, therefore, $mg - R = mf$;

$$\therefore \quad R = m(g-f)$$
$$= mg\left(1-\frac{f}{g}\right);$$

$\therefore$ thrust on floor = weight of man $\times \left(1-\frac{f}{g}\right)$...(i.).

Hence R, the man's thrust on the floor, is less than his actual weight by f/g of the latter, or, as we may express it, "*the man apparently loses f/g of his weight.*"

Similarly, if the lift is ascending with an *upward* acceleration f', $\quad R = mg\left(1+\frac{f'}{g}\right)$;

$\therefore$ thrust on floor = weight of man $\times \left(1+\frac{f'}{g}\right)$...(ii.),

or, "*the man will feel heavier by the fraction f'/g of his weight.*"

If the chain of the lift should break, it would descend with acceleration $f = g$, and (i.) shows that during the fall R would be $= 0$, or the man "would not feel his weight at all." In fact both man and lift would be falling freely.

Example.—If a man, weighing 12 stone, is descending a lift with acceleration 8 feet per second per second, the thrust of his feet on the floor will be $= 12$ stone wt. $\times (1 - \frac{8}{32}) =$ weight of 9 stone.

If he is ascending with the same acceleration, the thrust of his feet on the floor $= 12$ stone wt. $\times (1 + \frac{8}{32}) =$ weight of 15 stone.

118. **Observed effects when the lift is coming to rest.**—The changes in the man's apparent weight depend on the *acceleration* of the lift and not on its actual *velocity*, so that, when the lift is descending *uniformly* the man obtains the impression of being at rest, while the objects outside the lift appear to move upwards past him. When the downward motion is being retarded previous to stopping, this retardation is equivalent to an upward acceleration, and the man feels as if he were being *lifted up*. Similarly, when the lift is ascending uniformly, he receives the impression that the external objects are descending past him. When the lift begins to slacken speed before

coming to rest, the upward retardation is equivalent to a downward acceleration, and he feels as if the lift were beginning to fall from under him.*

SUMMARY OF RESULTS.

Weight is proportional to mass.

Gravitation unit of force = weight of unit mass.

Weight of mass of m lbs. $= mg$ poundals. $(g = 32 \cdot 2.)$...(1).

Weight of mass of m grammes $= mg$ dynes. $(g = 981.)$...(2).

$W = mg$ dynamical units ....(3).

EXAMPLES VIII.

1. Find the accelerations, the velocities acquired from rest in one minute, and the distances traversed in that minute, by the following given masses, when acted on by the given forces, namely:

(i.) Mass of 1 lb. under force of 2 oz. weight;
(ii.) Mass of 1 cwt. under force of 5 tons weight;
(iii.) Mass of 32 lbs. under force of 1 cwt. 1 qr.;
(iv.) Mass of 1 kilog. under force of 1 gram weight.

2. A certain force can just support a weight of 8 tons. How far would it move a mass of 16 tons in 1 minute if no other force acted on it?

3. A mass is acted on for 3 seconds by a uniform horizontal force that would just support 24 lbs. What momentum does it acquire?

4. If a force equal to the weight of 10 lbs. act on a mass of 10 lbs. for 10 seconds, what will be the momentum acquired?

5. A horizontal force, which would statically support 5 lbs., acted continuously for 3 seconds on a heavy body initially at rest on a smooth horizontal plane, and at the end of that time the body was moving with a velocity of 200 yards per minute. Determine (i.) the acceleration, and (ii.) the mass of the body.

---

* The student should take an early opportunity of verifying this by actual experience.

6. A ball, whose mass is 3 lbs., is falling at the rate of 100 feet per second. What force expressed in pounds weight will stop it (i.) in 2 seconds, (ii.) in 2 feet?

7. A railway train travels $\frac{1}{4}$ mile on a smooth level line, while its speed increases uniformly from 15 to 20 miles an hour. What proportion does the pull of the engine bear to the weight of the train?

8. Does the rope of a colliery-hoist have to bear most strain when the cage is at the top or at the bottom of the shaft? To eliminate the weight of the rope itself, consider only the portion immediately above the cage. Explain under what circumstances the stress may be greater than the weight of the cage attached to it.

9. 150 lbs. is drawn up the shaft of a coal-pit, and, starting from rest, acquires a velocity of 3 miles an hour in the first minute. Assuming that the acceleration is uniform, find how heavy the mass appears to one drawing it up.

10. In what time will a body fall from rest through 100 feet? If it be retarded in its fall by the tension of a string attached to it, so as to occupy 5 seconds in the fall, what is the pull of the string, the weight being supposed given.

11. If the dynamical measures of the mass and weight be the same, and the unit of length be 2 feet, find the unit of time.

12. A man whose weight is 160 lbs. is standing in a lift. With what force will he press on the bottom of the lift when it is (i.) ascending, (ii.) descending with uniform acceleration $\frac{1}{2}g$.

13. Ten pounds hangs by a string and is drawn up with an acceleration of 2 ft. per sec. per sec. Find the tension of the string.

14. A mass of 1000 kilogrammes is acted on for 1 hour by a force equal to the weight of a gramme. Find the distance traversed from rest. Find also (in centimetres per second) the initial velocity of projection in order that the mass might travel a kilometre in the hour.

15. A heavy vertical chain is drawn upwards by a given force of P lbs. weight, which exceeds its weight W. Find its acceleration and its tension at any assigned point. Show that the tension at its middle point is $\frac{1}{2}P$.

EXAMINATION PAPER IV.

1. Distinguish between *volume*, *mass*, and *weight*.

2. Find the relation between the units of mass and weight in order that W may be equal to Mg.

3. Explain how it is that the weight of a substance as determined by a pair of scales is the same anywhere, while it will vary if a spring balance be used.

4. A stone weighing two pounds falls for three seconds. What force will be required to stop it in two seconds?

5. If 1 lb. is the unit of mass, 1 ft. and 1 sec. being the units of space and time, what is the weight of the body whose mass is the unit of mass, and why?

6. A stone, after falling 2 seconds from rest, breaks a pane of glass, and in breaking it loses $\frac{1}{4}$ of its velocity. How far will it fall in the next second?

7. An ounce, a second, and an inch being taken as the units of mass, time, and length respectively, compare the (dynamical) unit of force with the weight of a pound.

8. A lift is descending and coming to rest with a uniform retardation of 4 F.P.S. units. A man in the lift weighs out a pound of tea with an ordinary balance and a pound of sugar with a spring balance. How many pounds of each does he really obtain?

9. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to a stop by a uniform force in the space of 18 feet. What is the tension in the rope while the stoppage is occurring?

10. If a force equal to the weight of one gramme pull a weight of a kilogramme along a smooth level surface, find the velocity when the body has moved one metre.

CHAPTER IX.

ATWOOD'S MACHINE—CONNECTED SYSTEMS.

119. The apparatus now to be described was invented by George Atwood, F.R.S., a Cambridge mathematician who published several works on Mechanics about the year 1784. It is now used for illustrating the laws of motion experimentally, and at one time was also employed to determine the intensity of gravity. For the latter purpose, however, it has been superseded by the pendulum, as observations of pendulum oscillations can be made with much greater accuracy. This method, however, does not depend on such elementary principles as those involved in Atwood's machine.

For this reason, consideration of the pendulum method is deferred till Chapters XVII. and XX., where the theories of the simple and compound pendulum are considered in § 244 and § 285 respectively.

We could not find g accurately by letting bodies fall down a shot-tower or down a mine, and timing them, because the velocity acquired in a few seconds would be so great that the motion could not be timed with sufficient exactness. A rough method has however been devised, in which a number of shot are released by an electric contrivance, and allowed to fall through a given height, in such a way that when one reaches the bottom the next is released. Thus the time taken by twenty to fall is twenty times the time taken by a single shot, and this interval can be easily timed by a stop-watch.

120. Atwood's Machine consists essentially of a light brass pulley (Figs. 14–16) fixed at a considerable height above the ground, over which passes a fine string supporting two weights P, Q attached to its ends. A *pulley* (Fig. 11) is a wheel with a groove cut round its rim to keep the string which it carries from slipping off. In Atwood's machine it is essential that this wheel should turn very freely, for which reason its shaft usually rolls

FIG. 11.

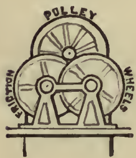


FIG. 12.

FIG. 13.

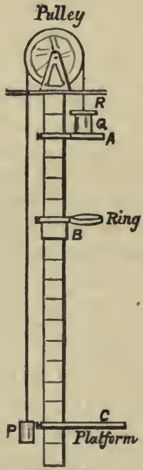


FIG. 14.

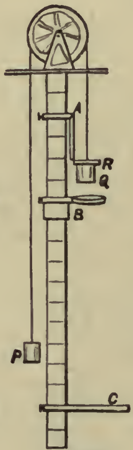


FIG. 15.

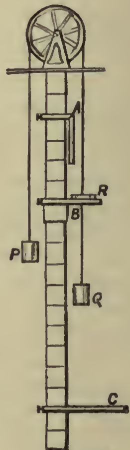


FIG. 16.

on sets of supporting wheels called "friction wheels" (Fig. 12), though any other arrangement which answered the same purpose might be used instead.

For measuring the heights of the weights in any position, a scale of inches or centimetres is attached to the pillar or wall on which the pulley is fixed, and for measuring time a clock is provided, whose pendulum ticks every second. In most experiments, the weights P

and Q are equal, and a small "rider" R, of the shape shown in Fig. 13, is placed on the top of Q, the string passing through the slot in R. A (Fig. 14) is a platform by which Q can be supported or released at will. B is a ring which is just large enough to let Q pass through, but which stops the weight R, and C is a fixed platform that will stop the weight Q. Both B and C can be fixed at any desired height, measured by the scale on the pillar.

When the weights Q, R are released, they are together heavier than P, so that they naturally begin to descend, at the same time pulling up the weight P (Fig. 15). When Q reaches the ring B, the weight R is detached, and the equal weights P, Q continue to move on alone (Fig. 16) until Q reaches the platform C, when it also stops. The times taken to fall to the ring B and then to the platform C can be reckoned by the clock, and the scale measures the depths fallen in these intervals.

121. In forming the equations of motion of the two weights in Atwood's machine, it is necessary to make use of the following facts :—

I. *The downward velocity of Q is equal to the upward velocity of P, and the downward acceleration of Q is equal to the upward acceleration of P.*

For, since the length of the string remains constant as one weight falls and the other rises,

$$\therefore \left.\begin{array}{c}\text{distance fallen by } Q\\ \text{per unit time}\end{array}\right\} = \left\{\begin{array}{c}\text{distance risen by } P\\ \text{per unit time};\end{array}\right.$$

i.e., downward velocity of Q = upward velocity of P.

Hence also

$$\left.\begin{array}{c}\text{increase of } Q\text{'s downward}\\ \text{velocity per unit time}\end{array}\right\} = \left\{\begin{array}{c}\text{increase of } P\text{'s upward}\\ \text{velocity per unit time};\end{array}\right.$$

i.e., downward accel. of Q = upward accel. of P.

II. *The tension of the string is the same throughout, so that its upward pulls on the weights P, Q are equal.*

This is only true provided there is no friction and the pulley and string have no appreciable mass (see §153 below); hence the investigations of this chapter would require modification when applied to an actual machine. The string pulls upwards on both weights; these pulls are therefore not opposite and are not the "action and reaction" of Newton's Third Law. In fact, the action is transmitted from one weight to the other through the string, and is not *direct* (§ 80).

Example.—Masses of 3 and 5 lbs. hang over a pulley, as in Atwood's machine. Find the tension of the string in lbs. weight, and the acceleration of either mass.

Let T be the tension of the string in poundals, and f the acceleration produced. The greater mass will move downwards, and the less upwards. Consider the motion of the 5-lb. mass. The forces acting on it are its own weight, $= 5 \times 32$ poundals, acting downwards, and the tension of the string acting upwards. The total downward force is therefore $160 - T$ poundals, and the mass moved is 5 lbs.;

$$\therefore \quad \frac{160-T}{5} = f \quad \text{.................(i.)}.$$

Now consider the motion of the 3-lb. mass. The forces acting on it are the tension acting upwards and its weight acting downwards. The total upward force is therefore $T-96$ poundals, and the mass moved is 3 lbs.

$$\therefore \quad \frac{T-96}{3} = f \quad \text{..........(ii.)}.$$

Hence, from (i.) and (ii.),

$$\frac{160-T}{5} = \frac{T-96}{3};$$

$$\therefore \quad 8T = 480 + 480 = 960 \text{ poundals};$$

$$\therefore \quad T = \frac{960}{8} \text{ poundals} = 3\tfrac{3}{4} \text{ lbs. weight.}$$

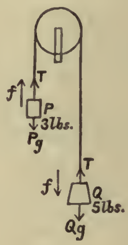


Fig. 17.

Substitute the value of T in poundals in either of the equations (i.) or (ii.). We thus obtain

$$f = 8 \text{ ft. per sec. per sec.}$$

Second Method.—If the value of T is not to be found, the following is the readiest method of calculating f:—

Resultant force producing motion

= weight of 5 lbs. − weight of 3 lbs. = weight of 2 lbs.

= 64 poundals.

Total mass moved is $(3+5)$ lbs. = 8 lbs.;

$$\therefore \quad f = \frac{\text{moving force}}{\text{mass moved}} = \frac{64}{8} = 8 \text{ ft. per sec. per sec.}$$

122. Two unequal masses P, Q* ($Q > P$), joined by a string passing over a light pulley, as in Atwood's machine, move under gravity (Fig. 17).

(i.) **To find their acceleration ;**

(ii.) **To find the pull in the string ;**

(iii.) **To find the force which the pulley has to support.**

(i.) The masses being P, Q, the absolute weights of the bodies are Pg and Qg.

Let the pull in the string be T dynamical units of force.

Let the downward acceleration of Q be f, then the upward acceleration of P is also f (§ 121, I.).

The forces acting on Q are therefore Qg downwards and T upwards, and their difference produces the downward acceleration f;

$$\therefore \quad Qf = Qg - T \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(i.)}.$$

Similarly, from considering the motion of P,

$$Pf = T - Pg \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \text{(ii.)}.$$

To find f we must eliminate T (as we do not know the value of T).

By addition, $(Q+P)f = (Q-P)g$;

$$\therefore \quad \text{required acceleration } f = \frac{Q-P}{Q+P}g \quad \ldots\ldots\ldots \quad \mathbf{(1)}.$$

We notice that the bodies move with *uniform* acceleration (for f is constant).

If the weight P is very small, $\frac{Q-P}{Q+P}g$ becomes nearly $\frac{Q}{Q}g$, $= g$, which is the acceleration of a free body.

---

* For convenience, we now use the letter Q for the total mass of the Q and R before mentioned.

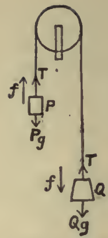


Fig. 18.

(ii.) To find T, the pull in the string, we must eliminate f from the equations

$$Qf = Qg - T$$

and $$Pf = T - Pg;$$

and we get

$$P(Qg - T) = Q(T - Pg),$$

or $$2PQg = (P+Q)T;$$

$$\therefore \quad T = \frac{2PQ}{P+Q} g \text{ dynamical units,}$$

pull in string $= \dfrac{2PQ}{P+Q}$ **gravitation units of force...(2).**

(iii.) The strings on either side of the pulley pull with a force T. Hence **the pulley has to support** altogether a force $2T$, equal to the weight of a mass

$$\frac{4PQ}{P+Q}.$$

Observation.—Notice that this force is not equal to the sum of the weights $P+Q$ unless $P = Q$.

123. Alternative method of finding the acceleration.

The weight of Q tends to pull down Q and to pull up P, while the weight of P has the opposite tendency. Hence the total force tending to accelerate Q downwards and P upwards is the difference of the weights, or $(Q-P)g$ dynamical units. The whole mass accelerated in this way is $Q+P$, and its acceleration is the required acceleration f. Hence the relation

$$\text{mass} \times \text{accel.} = \text{impressed force}$$

gives $$(Q+P)f = (Q-P)g;$$

giving, as before $$f = \frac{Q-P}{Q+P} g \text{ .... } (1).$$

Example.—Masses of 6 lbs. and 8 lbs. are hung over a pulley. In what time will they have moved over 7 feet?

Force causing motion

$$= (8-6) \text{ lbs. weight} = (8-6)\ 32 \text{ poundals.}$$

Mass moved $= 8+6$ lbs.

$$\therefore\ f = \frac{8-6}{8+6} \times 32 \text{ ft. per sec. per sec.}$$

$$= \frac{32}{7} \text{ ft. per sec. per sec.}$$

Let t be the required time; then

$$7 = \tfrac{1}{2} f t^2 = \frac{16}{7} t^2, \quad \therefore\ t = \sqrt{\frac{49}{16}} = 1\tfrac{3}{4} \text{ secs.}$$

124. **Experiments with Atwood's machine.**

The advantage of Atwood's machine is that by taking the weights at the ends of the string nearly equal, we can make the acceleration as small as we like, and the motion can then be investigated with great accuracy.

For, with the notation of § 122, the acceleration

$$f = \frac{Q-P}{Q+P} g,$$

and this is small if P and Q are nearly equal. If $Q = P$, the acceleration becomes zero, as the formula and simple experience show.

If the two masses are each equal to M, and a third mass m is placed on the top of one of them, we must write M for P and $M+m$ instead of Q in the expression we have found for f, and we have

$$f = \frac{m}{2M+m} g \text{ .... } (3),$$

which is small if m is small compared with M. This acceleration is the acceleration due to mg, the weight of m, acting on $2M+m$, the total mass of the three weights, as explained in § 123.

125. **To verify that bodies will continue to move uniformly when not acted on by force.**

Let the two weights P and Q be equal, and let a third weight R be placed on the top of Q. Let the ring B be fixed at any convenient distance below the platform A. As the weight R descends from A to B, it sets P and Q in motion. After R is detached by the ring, there is no force tending to change the motion of the system, because the equal weights P, Q tend to pull the system opposite ways. Hence, if Newton's first law be true, the velocity after leaving B ought to be uniform. To verify this, let the experiment be repeated with the platform C at different depths below B, and let the times taken by Q to traverse BC be observed. The times will always be found to be proportional to the depths traversed from B to C, showing that the velocity is uniform after leaving the ring. Thus, if the depth BC be doubled, the other circumstances being the same, the time taken in traversing it will be doubled.

126. **To verify that**

A constant weight acting on a constant mass produces a uniform acceleration, and that

(1) **If the system start from rest, $s \propto t^2$;** *

(2) **The acquired velocity $v \propto t$;**

(3) **The average velocity from rest $= \frac{1}{2}$ the final velocity.**

(1) Let the two equal weights P, Q and the third weight R be attached to the string as before. Let the experiment be performed several times with the ring B fixed at different depths below A, and let the times taken in falling from A to B be noted in each case. It will be found that this time is always proportional to the square root of the depth AB, or, what is the same thing, the distance AB is always proportional to the square of the time taken.

* The symbol $\propto$ denotes "varies as" or "is proportional to." Thus $s \propto t^2$ means that s is proportional to t^2.

Thus, suppose B is so adjusted that Q falls from A to B in one second. If the depth AB be increased four-fold, the time taken will be found to be 2 seconds; if AB be increased nine-fold, the time will be found to be 3 seconds; and so on.

The time taken can be measured by counting the ticks of the clock from the instant Q is released till the weight R is heard to strike the ring B.

(2) To measure the acquired velocity, fix the platform C at any convenient depth below B, and observe the time taken to traverse BC. Since the velocity is uniform after R is detached, the ratio of the distance BC to the time taken in traversing it gives the velocity of the system, which is therefore known. It only remains to show that for different positions of B this acquired velocity is proportional to the time taken to traverse AB, or (by the first part of the experiment) to the square root of the depth AB. With any number of experiments this relation will be found to hold good in every case.

Thus, if the time taken from A to B is 2 seconds, the velocity acquired will be double what it would be if B were raised to such a height that the time was 1 second.

(3) Arrange the platform and ring so that the depth BC is double the depth AB. Then on repeating the experiment it will be found that the time taken from A to B is equal to the time taken from B to C. Now the weight R is detached at B, and so BC is traversed with uniform velocity = the final velocity at B.

Hence the average velocity in traversing AB is half the final velocity at B. That is, a distance (BC) twice AB is traversed with a velocity equal to the final velocity at B in the same time that AB was traversed.

127. **To verify the relations between force, mass, and acceleration,** it is only necessary to vary the weights P, Q, and R, remembering that the mass moved is $2M+m$, the sum of the combined masses, and the moving force is the weight of m, or the difference of the

weights on opposite sides of the pulley. The accelerations in different cases may be compared by fixing the ring B at a constant depth s below A, and noting t the time taken to fall to the ring; if f is the acceleration, the equation

$$s = \tfrac{1}{2}ft^2$$

gives

$$f = \frac{2s}{t^2},$$

whence f is found.

The following different cases have to be considered:—

128. **To verify that the acceleration is proportional to the impressed force when the mass moved is kept constant,** we must vary the masses of P, Q, R in such a way as to keep the combined mass constant. This may be most easily accomplished by having the weights P, Q made up of a number of small weights (each equal, say, to R). By taking one of these weights from P, and another from Q, and attaching them both above the moveable weight R, we shall increase the weight of the latter without altering the total mass. When this is done, the acceleration of the system will always be found to be proportional to the total weight of R (as found by weighing in a pair of scales), and therefore to the impressed force.

129. **To verify that the force required to produce a given acceleration is proportional to the mass moved.**

If the weights P, Q, R are all doubled, it will be found that the system takes the same time as before to move through the distance AB, showing that the acceleration is unchanged. A similar result holds good if the three weights are all increased threefold or in any other proportion. Now the moving force is the weight of R; hence we infer that the acceleration is constant if the moving force and the mass moved are both increased in the same proportion, or the force is proportional to the mass.

130. To prove that the acceleration produced by a given force is inversely proportional to the mass moved.

Since the moving force is the weight of R, we must perform the experiment several times, using the same weight R and altering the equal weights P, Q each time. It will then be found that the acceleration is always proportional to $\frac{1}{2M+m}$, and is therefore inversely proportional to the mass moved.

131. To find g, the acceleration of gravity.

In Atwood's machine we have (§ 124)

$$f = \frac{m}{2M+m} g \quad \text{..................... (3).}$$

This acceleration may be measured, as in § 127, by observing the time required to fall a given height from rest. The masses M, m may be compared by weighing them in a pair of scales, and, knowing them, g may be found from (3), which gives

$$g = \frac{2M+m}{m} f.$$

132. To arrange the masses so that the system may move with an acceleration of one foot per second per second (taking $g = 32$).

Let us make the combined masses of P, Q, R equal to 1 lb. A force of 1 lb. weight acting on this mass would produce an acceleration of 32 feet per second per second; therefore the force required to produce an acceleration 1 is

$$\tfrac{1}{32} \text{ lb. wt.} = \tfrac{1}{2} \text{ oz. wt.};$$

$$\therefore \; m \text{ (the mass of } R) = \tfrac{1}{2} \text{ oz.}$$

Also $2M+m$ (the whole mass) $= 1$ lb. $= 16$ oz.;

$$\therefore \; 2M = 15\tfrac{1}{2} \text{ oz.,} \quad M = 7\tfrac{3}{4} \text{ oz.}$$

Hence the weights P, Q must each be $7\frac{3}{4}$ oz., and R must be half-an-ounce. Or P, Q, R may be taken to be any multiples whatever of $7\frac{3}{4}$ oz., $7\frac{3}{4}$ oz., and $\frac{1}{2}$ oz.

133. One weight drawn along a table by another.

A weight of m lbs., hanging freely by a string, draws a weight of M lbs. along a perfectly smooth table by means of a string passing over a small pulley at the edge of the table (Fig. 19). *To find the acceleration and the tension of the string in lbs. wt.*

Let T be the pull of the string in lbs. wt.; then its value in poundals $= Tg$. Also the weight of the hanging mass $= mg$ poundals, and that of the other mass is Mg poundals.

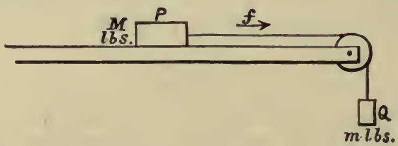


Fig. 19.

Hence, if f be the acceleration of the two masses in feet per second per second, we have, by considering the hanging mass,

$$m\,.f = mg - Tg\,;$$

and, by considering the mass on the table,

$$M\,.f = Tg.$$

Eliminating T, we have

$$(M+m)\,.f = mg\,;$$

$$\therefore \quad \text{acceleration } f = \frac{m}{M+m}\,g \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}.$$

Eliminating f, we have

$$mMg - MTg = mTg\,;$$

$$\therefore \quad \text{tension } T = \frac{Mm}{M+m} \text{ lbs. wt.} \quad \ldots\ldots\ldots \text{(5)}.$$

The result (4) also follows at once from the fact that the moving force is the m lbs. weight hanging freely and the whole mass moved is $m+M$ lbs.

Examples.—(1) A mass of 5 lbs., on a horizontal table, is connected by a string passing over an edge of the table with a mass of 3 lbs. hanging vertically. How far will the latter mass have fallen in one second?

Total mass moved $= (5 + 3)$ lbs. $=$ 8lbs.

Force producing motion $=$ weight of 3 lbs. $= 3 \times 32$ poundals;

$$\therefore \text{ acceleration produced } = \frac{3 \times 32}{8} \text{ ft. per sec. per sec.}$$

$$= 12 \text{ ft. per sec. per sec.}$$

$$\text{Distance fallen } = \tfrac{1}{2} ft^2 = \tfrac{1}{2} \times 12 \times 1^2 \text{ ft.} = 6 \text{ ft.}$$

(2) In the preceding example, what is the tension of the string?

Consider the mass on the table. The only force moving it is the tension T of the string;

$$\therefore \quad T = mf = 5 \times 12 \text{ poundals} = 60 \text{ poundals} = 1\tfrac{7}{8} \text{ lbs. weight.}$$

Summary of Results.

When two weights P, Q hang from the ends of a string passing over a smooth pulley,

$$\text{the acceleration } f = \frac{Q-P}{Q+P} g \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\text{the pull of the string, } T = \frac{2PQ}{Q+P} \text{ units of weight} \ldots (2);$$

the thrust on the pulley $= 2T$.

If two equal weights M are attached to the string, and a third weight m is placed on one of them,

$$f = \frac{m}{2M+m} g \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

If a weight M is drawn along a smooth table by a string carrying a weight m hanging over the edge,

$$\text{the acceleration } f = \frac{m}{m+M} g \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

$$\text{the pull of the string} = \frac{mM}{m+M} \text{ units of weight} \ldots (5).$$

In working numerical examples, it is advisable not to quote these formulæ, but to work from first principles.

EXAMPLES IX.

1. Two weights are attached to the ends of a string passing over a smooth pulley. Find the acceleration (stating the units employed), the tension in the string (in gravitation units), and the force which the pulley has to support, when the weights are:

(i.) 17 lbs. and 15 lbs.,	(ii.) 1 oz. and 15 oz.,
(iii.) 1 cwt. and 16 lbs.,	(iv.) 1 lb. and 14 oz.,
(v.) 5 lbs. and 4 lbs.,	(vi.) 20 lbs. and 4 lbs.,
(vii.) 490 grams and 491 grams,	(viii.) 1 kilog. and 90 grams.

2. The pairs of weights of Example 1 are laid with one weight resting on a horizontal table and the other hanging from the edge of the table. Find the acceleration and the tension in the string in each of the cases, considering separately the two different arrangements when (*a*) the lighter, (*b*) the heavier weight rests on the table.

3. In what time will a weight of 37 lbs. draw another of 24 lbs. up through a height of 32 feet, and what velocity will each particle have at the end of that time?

4. Two weights of 5 lbs. and 7 lbs. respectively are fastened to the ends of a cord passing over a frictionless pulley supported by a hook. Show that when they are free to move, the pull on the hook is equal to $11\frac{2}{3}$ lbs. weight.

5. Two weights, 7 oz. and 9 oz., are attached to the ends of a string passing over the corner of a smooth table at the edge of a precipice, the larger weight being drawn along by the smaller, which descends vertically. After 3 seconds, the string is cut. How far will the 7 oz. weight have descended after another second?

6. A weight of 14 lbs. is moved from rest on a smooth horizontal table by a weight of 2 lbs., which hangs over the edge of the table and is connected with the large weight by means of a fine string passing over a small smooth pulley at the edge of the table. Find the tension of the string and the velocity of each weight at the end of 2 seconds.

7. A mass of 9 lbs. is attached to one end of a string, and masses of 7 and 4 lbs. to the other end, and the whole is hung up over a pulley. The system is allowed to move for 15 seconds, when the 4 lb. weight is cut away. How long will it be before the system comes instantaneously to rest?

8. A string passing over a smooth pulley has attached to it on one side (at different points) masses of 3 lbs. and 5 lbs., and on the other side masses of 4 lbs. and 6 lbs., the heavier mass on each side being lowest. Find (without using formulæ) the tension of each portion of the string, in lbs. weight.

9. A man weighing 12 stone and a sack weighing 10 stone are suspended over a smooth pulley by a rope whose weight may be neglected. Find their common acceleration.

10. If in the last example the man pulls himself up by the rope so as to diminish his downward acceleration by one half, find the upward acceleration of the sack in this case, and prove that the acceleration of the man upwards relative to the rope will be 3·2 (foot-second units).

11. If ACB be a string, C a pulley, and a weight of 5 lbs. be attached at A, a weight of 3 lbs. at B, and another of 3 lbs. between B and C, and if B be originally 11 feet above the ground, find the distance above B of the third weight in order that the latter may just reach the ground. Find also the time of motion. [$g = 32$.]

12. Two masses P, Q are attached to the ends of a string which passes over a smooth horizontal table and hangs over its two opposite edges. A third mass R is attached to the string near its middle point and rests on the table. Show that the system will move with acceleration $\frac{P-Q}{P+Q+R}g$, and find the tensions in the two portions of the string, expressing them in gravitation units.

EXAMINATION PAPER V.

1. Describe Atwood's machine, and explain how it is used to determine the acceleration of gravity.

2. Explain how to use Atwood's machine to show (i.) that a body acted on by a constant force moves with a uniform acceleration; (ii.) that the acceleration of a given mass is proportional to the force acting on it.

3. In Atwood's machine, one of the boxes is $\frac{1}{2}$ oz. heavier than the other. What must be the load of each in order that the overweighted box may fall through 1 ft. in the first second?

4. Describe an experiment to prove that the weight of half-an-ounce will produce in a mass of 1 lb. an acceleration of (approximately) 1 ft. per sec. per sec.

5. Describe experiments and observations tending to prove that the change caused by a given force is independent of the body's actual velocity.

6. Two scale-pans, each weighing 2 oz., are suspended by a weightless string over a smooth pulley. A mass of 10 oz. is placed in one and 4 oz. in the other. Find the tension of the string and the pressure on each scale-pan.

7. Two weights P and Q are connected by a fine thread passing over a smooth pulley. P descends through a distance h, when a part of P falls off, leaving only P_1, which is less than Q. How far will P_1 descend?

8. What must be the masses attached to the ends of the string of an Atwood's machine, and the mass of the rider, in order that the action of a force of 10,000 dynes upon a mass of 1000 grammes may be investigated?

9. Two unequal weights are connected by a string hanging over the edge of a smooth table. Show that the tension of the string is the same whichever weight is placed on the table.

10. A weight of 4 lbs., connected with another as in the previous question, falls 12 feet in the third second of motion. What is the mass of the latter weight?

CHAPTER X.

WORK, ENERGY, AND POWER.

134. **Work.**—*A force is said to* **do work** *when its point of application moves in the direction in which the force acts.*

When the point of application moves in a direction opposite to that of the force, **work** *is said to be* **done against** *the force.*

By the "*point of application*" of a force is meant the *particle* on which the force acts. When the force acts, not on a particle, but on a body of any size, the force may be supposed to be applied at some particular point of the body, and the "distance moved by the point of application" means the distance moved by the particle of the body at that point.

Examples of Work.—(1) An engine drawing a train does work, for the train moves in the direction in which the engine pulls. But when the train is being stopped by the brakes, the train does work against the brakes, because the resistance of the latter acts in the opposite direction to that in which the train is moving.

(2) If a heavy body falls to the ground, its weight does work. If we lift it up again, we must do work against its weight.

135. Definitions. — *The* **work done by a force** *is measured by the product of the force into the distance through which its point of application moves in the direction of the force.*

In the present chapter, we shall suppose the point of application to be moving in the same straight line as the force. If it is moving in the direction towards which the force tends, the work done will therefore be positive. If it is moving in the reverse direction, we may regard the distance traversed as negative (§ 18), so that the work done by the force is now a minus quantity.

Hence *work done against a force is the same thing as a negative quantity of work done by a force.*

Thus, when we do work against the weight of a body in lifting it off the ground, the weight of the body does a negative quantity of work.

136. Definition.—**The dynamical or absolute unit of work** *is the work done by the dynamical unit of force in moving its point of application through a distance of a unit of length*, whatever system of units be used.

The F.P.S. dynamical unit of work is the **foot-poundal,** and is the *work done by a force of one poundal in moving its point of application through one foot.*

The C.G.S. dynamical unit of work is called the **erg,** and is the *work done by a force of one dyne in moving its point of application through one centimetre.*

A million ergs is called a **megalerg.** It is the work done by a megadyne in moving through one centimetre. Owing to the smallness of the erg, work is often measured in megalergs.

Another Metric unit of work is the **joule,*** which contains ten million (10^7) ergs or 10 megalergs; this is, however, principally used in electrical measurements.

---

* Lately it has been the practice to give the names of eminent physicists to new constants which have been required by the advance of science; *e.g.*, watt, ampère, volt. Joule worked out the mechanical equivalent of heat.

Examples.—(1) To find the work done in moving 10 lbs. through a distance of 3 feet with an acceleration of 5 feet per second per second.

The force applied to the body

$$= \text{mass} \times \text{accel.} = 10 \times 5 \text{ poundals,}$$

and the work done

$$= \text{force} \times \text{distance traversed} = 50 \times 3 = 150 \textit{ foot-poundals.}$$

(2) The work done by a force of 980 dynes in moving through a distance of 10 centimetres is

$$980 \times 10, \quad \text{or} \quad 9800 \textit{ ergs.}$$

(3) To express the foot-poundal in ergs.

By § 87, a poundal contains 13,780 dynes, and a foot contains 30·48 centimetres. Hence, by definition, the foot-poundal is the work done by 13,780 dynes in moving through 30·48 centimetres, and

$$\therefore \quad \textit{a foot-poundal} = 13,780 \times 30{\cdot}48, \quad \text{or} \quad 420,000 \textit{ ergs.}$$

137. Definition.—**The gravitation unit of work** *is the work done in lifting the weight of a unit mass through a height equal to the unit of length.*

The English gravitation unit is the **foot-pound,** or *the work done in raising one pound of matter vertically through one foot.*

The C.G.S. unit is the **gramme-centimetre,** or *the work done in raising one gramme through a height of one centimetre.*

Owing to the smallness of the gramme-centimetre, another Metric gravitation unit, called the **kilogrammetre,** is generally used instead. This is *the work done in raising a kilogramme through one metre.*

Thus a kilogrammetre = work done in raising 1,000 grammes through 100 centimetres = 1000 × 100 or 100,000 or 10^5 gramme-centimetres.

Examples.—(1) To compare, and express in foot-pounds, the work done by a man weighing 10 stone in climbing a mountain 4,000 feet high; and the work done by the tide between low and high water in raising a ship of 500 tons through 20 feet.

The man raises a weight of 10 × 14 or 140 lbs. through a height of 4,000 feet; ∴ work done = 140 × 4,000 lbs. = 560,000 ft.-lbs.

The tide raises 500 × 2,240 or 1,120,000 lbs. through 20 feet;

∴ work done = 1,120,000 × 20 = 22,400,000 ft.-lbs.

These are in the ratio of 1 to 40.

(2) To compare the kilogrammetre with the foot-pound.

A kilogramme = $2\frac{1}{5}$ pounds, and a metre = $3\frac{3}{10}$ feet, roughly;

∴ a kilogrammetre = $2 \cdot 2 \times 3 \cdot 3 = 7 \cdot 26$ ft.-lbs., roughly.

138. **The gravitation unit of work is g times the corresponding dynamical unit.**

The weight of a pound is 32·2 poundals; hence the foot-pound, or work done in raising a pound weight through one foot, = 32·2 foot-poundals.

Again, a gramme weighs about 981 dynes. Hence the gramme-centimetre = work done by a lifting force of 981 dynes in moving through one centimetre = 981 ergs.

Generally, the weight of a unit mass is g dynamical units of force. Hence, in raising a weight of unit mass through a unit height, we have to apply a force of g dynamical units, and to move the point of application through a unit of length;

∴ work done $= g \times 1 = g$ dynamical units of work.

139. Energy.—DEFINITIONS.—By **energy** is meant *capacity for doing work.*

The **potential energy** of a body or system of bodies is *the amount of work which it is capable of performing in virtue of its position (or the positions of its parts).*

Examples.—(1) If a million tons of water are stored in a reservoir 500 feet above the sea level, the water may be said to have 500,000,000 *foot-tons of potential energy*, for if the water were allowed to run down to the sea it would be able to perform 500,000,000 foot-tons of work in its descent. By employing the water to drive a series of water wheels in its fall, this work may be utilized for driving machinery.

(2) If, in winding a clock, a weight of 8 lbs. is raised to a height of a yard from the bottom of the clock, its *potential energy* is then 24 *foot-pounds*, for in descending again it is able to perform 24 foot-pounds of work. This work is expended in driving the clock, and overcoming the friction of the machinery. When the weight has fallen one foot, its potential energy is only 16 foot-pounds, for it has only 2 more feet to fall, and it has already done 8 foot-pounds of work. When it has fallen another foot it has only one more foot to fall through, and its potential energy is only 8 foot-pounds.

140. **If a body of weight W is at a height h** above the ground, **its potential energy = Wh .... (1).**

For this is the amount of work its weight would do if the body fell to the ground.

If the weight W is expressed in dynamical units of force, the potential energy is Wh dynamical units of work. If W is the weight measured in gravitation units (so that W is numerically equal to the measure of the *mass* of the body), then Wh represents the number of gravitation units of work in the potential energy of the body.

Thus, if the mass of a body is M pounds, its weight $= M$ pounds weight $= Mg$ poundals, and its potential energy when at a height of h feet above the ground is $= Mh$ foot-*pounds* $= Mgh$ foot-*poundals* ($g = 32$, or $32 \cdot 2$).

Similarly, the potential energy of M grammes at a height of h centimetres $= Mh$ gramme-centimetres $= Mgh$ ergs ($g = 981$).

141. Definition.—**The kinetic energy** of a body is *its capacity for doing work in virtue of its motion.* It is measured by the amount of work that the body is capable of performing in coming to rest.

The following illustrations show that a moving body does actually possess energy.

Examples of kinetic energy.—

(1) A bullet when fired at a wooden target will penetrate a considerable distance into the wood, thereby doing work against the very great resistance to penetration offered by the target. Hence, before the bullet struck the target, it must have possessed kinetic energy, or capacity for doing work.

(2) A stone, when projected vertically upwards, will rise in the air, and thereby do work against gravity. Evidently, the capacity for doing work depends on the initial upward motion, and the kinetic energy is measured by the work done by the stone against gravity in coming to rest.

Thus, if a mass of 3 pounds is shot upwards with a velocity of 40 feet per second, it will rise to a height h, where (by $u^2 = 2gh$),

$$40^2 = 2 \, . \, 32 \, . \, h, \quad \text{or} \quad h = 25 \text{ ft.}$$

In rising through this height the body will do 3×25 or 75 foot-pounds of work. Hence the original kinetic energy must have been 75 foot-pounds, or 2400 foot-poundals.

142. To find an expression for the kinetic energy of a moving body.

Suppose a body of mass m to be moving with velocity u, and let us calculate the work it is capable of doing in coming to rest. If the velocity changes from u to 0 under the action of a force of P dynamical units, and if f denote the acceleration, s the space passed over, we have,

by § 46, $$(v^2-u^2 = 2fs),$$
$$0-u^2 = 2fs,$$
and, by § 88, $$P = mf.$$

Hence the work done *by* the force P, moving over a distance s,

$$= Ps = mf.s = 2fs \times \tfrac{1}{2}m = -u^2 \times \tfrac{1}{2}m = -\tfrac{1}{2}mu^2,$$

and the work done *by* the body *against* the force P is equal and opposite to this, and is therefore

$$= -Ps = -(-\tfrac{1}{2}mu^2) = \tfrac{1}{2}mu^2.$$

Therefore the body, in coming to rest, is capable of performing $\frac{1}{2}mu^2$ dynamical units of work, or

The kinetic energy of the body $= \frac{1}{2}mu^2$ .... **(2)**.

Or in words:

The kinetic energy of a body is half the product of its mass into the square of its speed.

Since the momentum of a moving body = mass × velocity, and its kinetic energy $= \frac{1}{2}$ (mass) × (velocity)^2,

∴ **kinetic energy = $\frac{1}{2}$ (momentum) × (velocity),**

an expression which is often useful. In all the above expressions the kinetic energy is supposed to be expressed in dynamical units of work (foot-*poundals* or ergs).

When expressed in foot-*pounds*, the kinetic energy

$$= mv^2 \div 2g.$$

Observations.—We notice that the work which a body can perform in coming to rest does not depend on *how* it is brought to rest. If it is retarded by a very great force it will stop after going a very short distance, while if it is retarded by a very small force it will run a considerable distance, but the *work done* will be the same in every case.

Many writers *define* the kinetic energy of a moving body as

$$\tfrac{1}{2}\,(\text{mass}) \times (\text{speed})^2,$$

and they *then* go on to *prove* that the kinetic energy is the work which the body will perform in coming to rest. This is, however, not so logical as the point of view that we have adopted, as it shows no reason why the factor $\frac{1}{2}$ is inserted. Still, candidates for an examination should be prepared to give either definition, and to deduce one definition from the other.

Example.—A cannon-ball, of weight 10 lbs., is fired horizontally, with a velocity of 1120 feet per second, from a gun, and the weight of the gun, with its carriage, is 5 tons. Find the kinetic energy of the gun immediately after the explosion, expressing it in foot-pounds.

The momentum of the cannon-ball is 10×1120 foot-pound-second units, and this is also the momentum of the gun (§ 98).

Now kinetic energy

$$= \tfrac{1}{2}Mv^2 = \frac{M^2v^2}{2M} = \frac{\text{square of momentum}}{\text{twice the mass}}.$$

Hence the kinetic energy of the gun is

$$\frac{(11200)^2}{2 \times 5 \times 2240} \text{ ft.-poundals.}$$

$$= \frac{(11200)^2}{22400 \times 32} \text{ ft.-lbs.}$$

$$= 175 \text{ ft.-lbs.}$$

143. In uniformly accelerated motion the increase of kinetic energy is always equal to the work done by the impressed forces.

We have, in motion under uniform acceleration f,

$$v^2 - u^2 = 2fs;$$

also, if P be the impressed force and m the mass moved,

$$P = mf.$$

$$\therefore \quad Ps = mfs = \tfrac{1}{2}m \times 2fs = \tfrac{1}{2}m(v^2 - u^2),$$

or

$$\boldsymbol{Ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2} \quad \ldots\ldots\ldots\ldots \textbf{(3)};$$

that is, **work done by P**

= (final kinetic energy) − (initial kinetic energy)

= **increase of kinetic energy.**

In particular, if the body start from rest, the whole kinetic energy acquired is equal to the work done by the impressed force.

Equation (3) is called the *Equation of Work.*

Alternative proof.—The same results may also be proved from first principles, as follows:—

If t be the time during which the force P acts, the total change of momentum is equal to the impulse of the force;

$$\therefore \quad Pt = m(v-u) \quad \text{(i.)}.$$

Also the motion is uniformly accelerated; therefore the average velocity is $\frac{1}{2}(v+u)$;

and therefore $s = \frac{1}{2}(v+u)t$ .... (ii.).

Multiplying (i.) and (ii.) together, and dividing throughout by t, we have, as before, $Ps = \frac{1}{2}m(v^2-u^2)$ .... (3).

144. **Comparison of the equations of momentum and work.**—The student should be careful to distinguish the property just proved from the property which forms the subject of Newton's Second Law [§ 88, equation (3)].

The Second Law states that

change of **momentum** = **impulse** of impressed force
= force × **time**;

and the Principle of Work states that

change of **kinetic energy** = **work** of impressed force
= force × **distance** traversed.

Examples.—(1) A stone weighing 3 lbs. falls through 7 ft. What is its kinetic energy, and what force will stop it in 2 ft.?

Kinetic energy of stone = work done by gravity
= weight × distance fallen
= 3 × 7 ft.-lbs. = 21 ft.-lbs.
= 21 × 32 ft.-poundals = 672 ft.-poundals.

Let P lbs. be the force required to stop it in 2 ft. Then we have P lbs. acting upwards and the weight 3 lbs. acting downwards. Hence the upward force retarding the motion of the stone is $P-3$ lbs. wt. When the stone is brought to rest,

work done *against* retarding force = kinetic energy *lost*;

$\therefore \quad (P-3) \times 2$ ft.-lbs. = 21 ft.-lbs.;

$\therefore$ required force $P = 13\frac{1}{2}$ lbs. wt.

(2) A stone weighing 8 oz. falls for 5 secs. What is its momentum, and what force will stop it in 3 secs.?

Weight of stone = $\frac{1}{2}$ lb. = $\frac{1}{2}g$ poundals.

Momentum acquired = impressed force × time

$= \frac{1}{2}g \times 5$ F.P.S. dynamical units

= 80 units (taking $g = 32$).

Let P lbs. be the force required to stop it in 3 secs. Then the actual retarding force $= (P - \frac{1}{2})$ lbs. wt. $= (P - \frac{1}{2})\, g$ poundals, and impulse of this force in 3 secs. = momentum destroyed;

$$\therefore \quad (P - \tfrac{1}{2})\, g \times 3 = \tfrac{1}{2}g \times 5;$$

$$\therefore \quad \text{required force } P = 1\tfrac{1}{3} \text{ lbs. wt.}$$

(3) To find the kinetic energy acquired by a kilogramme in falling through a metre.

The weight = 1000 gm. wt. = 1000 × 981 dynes,
and distance fallen = 100 centimetres.

Hence, by the Principle of Work,

acquired kinetic energy = work done by weight
= (force) × (distance)
= 981,000 × 100 (dynes, cm.)
= 98,100,000 *ergs* = 98·1 megalergs.

Observation.—Examples (1) and (3) show that when it is required to calculate the kinetic energy acquired by a body after moving through a given distance under a given force, it is not necessary to find the velocity and substitute in the expression $\frac{1}{2}mv^2$, for the acquired energy is simply the work done by the force.

(4) To find the velocity of an 8-lb. shot that will just penetrate an armour plate 10 inches think, the resistance being 84 tons.

Let u be the required velocity in ft. per sec. The resistance

$= 84 \times 2240$ lbs. wt. $= 84 \times 2240 \times 32$ poundals,

whence the Equation of Work

$$\tfrac{1}{2}m(v^2 - u^2) = P \times s$$

gives

$$\tfrac{1}{2} \cdot 8\,(0 - u^2) = -(84 \times 2240 \times 32) \times \tfrac{10}{12};$$

$$u^2 = 70 \times 2240 \times 8 = 1120 \times 1120;$$

$$\therefore \quad \text{required velocity} = 1120 \text{ ft. per sec.}$$

(5) If the velocity of the shot be doubled, what must the thickness of the plate be in order that the shot may only just penetrate it.

Let u' be the new velocity, s' the new thickness. Then the Equations of Work for the two cases give

$$\tfrac{1}{2}mu'^2 = -Ps',$$

$$\tfrac{1}{2}mu^2 = -Ps.$$

But $u' = 2u$; $\quad \therefore \quad u'^2 = 4u^2.$

Hence $\quad s' = 4s,$

or the thickness of the plate must be increased fourfold;

$$\therefore \quad \text{required thickness} = 40 \text{ inches.}$$

(6) A 1-oz. bullet is fired with a velocity of 1000 ft. per sec. Find the velocity with which a 2-oz. bullet could be fired from the same rifle with treble the charge of powder.

Since the explosive force always moves the bullet *through the same distance* (viz., the length of the barrel), the work done on the bullet is proportional to the charge of powder.

Hence the kinetic energy is trebled by trebling the charge, and, if m, m_1 be the masses, v, v_1 the velocities of the two bullets, we have

$$\tfrac{1}{2}\, m_1\, v_1^2 = 3 \times \tfrac{1}{2}\, m\, v^2,$$
$$\tfrac{1}{2} \cdot 2 \cdot v_1^2 = 3 \times \tfrac{1}{2} \cdot 1 \cdot v^2;$$
$$\therefore\ v_1^2 = \tfrac{3}{2} v^2;$$
$$\therefore \text{ required velocity } v_1 = v\sqrt{\tfrac{3}{2}} = 1000\sqrt{\tfrac{3}{2}} = \sqrt{(1{,}500{,}000)}$$
$$= 1225 \text{ ft. per sec. nearly.}$$

145. The following applications should be noticed.

If different forces act on equal masses *during the same time*, the forces are proportional to the velocities acquired.

For the times being the same, the impulses, and therefore the acquired momenta, are proportional to the forces.

But if the different forces move the bodies from rest *through equal distances*, the forces are proportional, *not* to the acquired velocities themselves, but to the *squares* of the acquired velocities.

For, since the forces move the bodies through equal distances, the works done by them are proportional to the forces; hence the forces are proportional to the kinetic energies of the two bodies.

The apparent discrepancy is easily accounted for. In the second case, the greater force moves the mass over the given distance more quickly, and therefore it acts for a shorter time than the lesser force. Hence the impulses, and therefore the momenta produced, are not proportional to the forces.

146. The Principle of Conservation of Energy.*— If a body is started in motion by any force, we see, from § 143, that the kinetic energy acquired is equal to the work done. If after a certain time another force acts on the body, the increase of kinetic energy is equal to the work done by the second force, and the total kinetic

* The Principle of Conservation of Energy, when it is first realized, is a most important revelation to every thinking being.

energy is therefore equal to the sum of the works done by the two forces. In like manner, if any number of forces act in succession on the body, the final kinetic energy is equal to the sum of the works done by the several forces. If therefore the body is again brought to rest, it will have done an amount of work equal to that done in setting the body in motion.

Thus we get as much work out of the body as was previously put into it.

This is a particular case of the **Principle of Conservation of Energy,** which may be briefly stated thus—

Energy can never be created nor destroyed, but can only be transformed from one form into another; or,

The total quantity of energy present in the universe always remains the same.

147. Observations.—The Principle of Conservation of Energy, like Newton's Laws of Motion, does not admit of a perfectly general proof, but is based on evidence derived from experiment. Energy may manifest itself in many other forms besides the ordinary mechanical (kinetic and potential) energy of moving bodies, and it is only when all these forms of energy are taken into account that the principle really holds good.

These forms of energy include energy of vibration which gives rise to sound, heat energy, radiant energy in the form of light, electrical energy, and chemical energy. The tendency of modern physical science is to regard all forms of energy as the kinetic and potential energies of the ultimate molecules of which matter is supposed to be built up. We cannot, of course, tell what these molecules are like or how they really move, for they are far too small to be seen with any microscope. All that we can do is to build up theories of them that will account for physical phenomena. By so doing physicists hope to represent all such phenomena by particular cases of the principles of dynamics.

The Principle of Conservation of Energy has, however, now been so thoroughly established upon accumulated evidence that if any result should be arrived at which appeared at variance with the principle, it would not be inferred that the principle was incorrect, but that energy had appeared in some form which had previously been overlooked.

The name **Mechanical Energy** is often applied to the two forms, kinetic and potential energy, to distinguish them from the other forms of energy mentioned above.

148. Particular cases of the principle.

Motion of a body projected under gravity.—If a mass m be projected vertically upwards with velocity u, we have, when the height above the ground is s,

$$v^2 = u^2 - 2gs;$$

$$\therefore \qquad \tfrac{1}{2}mv^2 + mgs = \tfrac{1}{2}mu^2.$$

But mgs is the potential energy (in dynamical units) at height s, and $\frac{1}{2}mv^2$ is the kinetic energy. Hence

kinetic energy + potential energy
= original kinetic energy.

Hence the total energy of the body always remains constant and equal to its original energy.

149. **Verification for Atwood's machine.**—Let P, Q be the total masses suspended from the ends of a string passing over a pulley, as in Atwood's machine, where $Q > P$. Then, if Q falls through a distance s, P rises through an equal distance s. The work done by the weight of Q is Qgs, and that done against P is Pgs; hence the potential energy of Q decreases by Qgs, and that of P increases by Pgs, so that the loss of potential energy

$$= (Q-P)\,gs.$$

If u is the initial velocity, and v the final velocity, the initial and final kinetic energies of the system are

$$\tfrac{1}{2}Qu^2 + \tfrac{1}{2}Pu^2 \quad \text{and} \quad \tfrac{1}{2}Qv^2 + \tfrac{1}{2}Pv^2;$$

hence gain of kinetic energy $= \frac{1}{2}(Q+P)(v^2-u^2)$.

The Principle of Conservation of Energy requires that

gain of kinetic energy = loss of potential energy,

or that $\quad \frac{1}{2}(Q+P)(v^2-u^2) = (Q-P)\,gs$,

or that
$$v^2 - u^2 = 2\,\frac{Q-P}{Q+P}\,gs \quad \ldots\ldots\ldots\ldots\ldots \text{(i.)}.$$

This relation **is** satisfied, for in uniformly accelerated motion $\quad v^2 - u^2 = 2fs$,
and we have seen in § 122 that

$$f = \frac{Q-P}{Q+P}\,g.$$

Whence (i.) follows immediately; hence the sum of the kinetic and potential energies is constant.

Similarly the principle may be verified for the case in which a body is drawn along a smooth horizontal table by a second body falling vertically (§ 133).

150. **Applications.**—Conversely, we may often determine the motion of a dynamical system by expressing in mathematical language the condition that the total mechanical energy is constant, or that the increase of kinetic energy is equal to the work done on the system.

This is really a most convenient way of finding the acceleration of the masses in Atwood's machine, especially if it is required to find the velocity acquired when these masses have moved through a given distance.

Examples.—(1) If a mass of 1 lb., hanging from the edge of a table, draws a mass of 8 lbs. along the table by means of a string, to find the velocity acquired in moving over 1 foot; and the acceleration.

Let the required velocity $= v$ ft. per sec.

Then the total kinetic energy $= \frac{1}{2}(1+8)v^2 = \frac{9}{2}v^2$ ft.-poundals; and the work done by the 1-lb. mass in falling

$$= 1 \text{ ft.-lb.} = 32 \text{ ft.-poundals.}$$

Therefore $\frac{9}{2}v^2 = 32, \quad \text{or} \quad v^2 = \frac{64}{9};$

whence $v = \frac{8}{3} = 2\frac{2}{3}$ ft. per sec.

Also the relation $v^2 = 2fs$

gives $\frac{64}{9} = 2f \cdot 1;$

whence the acceleration $f = \frac{32}{9}$ ft. per sec. per sec.

(2) A mass of 50 lbs. falls from a height of 50 feet, and penetrates 2 feet into loose sand. To find the resistance of the sand in pounds weight.

The kinetic energy acquired in falling is destroyed by the resistance of the sand. Hence the work done on the body by gravity is equal to the work done by the body against the resistance of the sand.

But the body falls altogether 52 feet,

$$\therefore \quad \text{work done by gravity} = 52 \times 50 = 2600 \text{ ft.-lbs.};$$

and, since the body moves 2 feet against the resistance,

$$\therefore \quad \text{resistance of sand} = \frac{2600}{2} = 1300 \text{ lbs. weight.}$$

[Notice in this example that we have not had to calculate the velocity of the body.]

*151. **Connection of the Principle of Energy with Newton's Third Law.—Conservative forces.**—Although, according to Newton's Third Law of Motion action and reaction are equal and opposite, it does not necessarily follow that the works done by them are equal and opposite. For two equal forces can only perform equal amounts of work provided that *their points of application move through equal distances.* Hence, if two bodies approach or separate from one another, there is a gain or loss of work done against their action and reaction.

In many cases this work will be restored if the bodies are brought back to their original position. When this is the case, the forces are said to be **conservative,** and the total energy, potential and kinetic, of the system is constant.

In other cases, however, work may be lost in altering the positions of two bodies, and may not be restored when the bodies are brought back to their original position. In such a case the forces are **non-conservative,** and there is a loss of mechanical (kinetic and potential) energy, which energy is transformed into heat or some other form of energy not usually considered in mechanical investigations.

In all cases where the forces between two bodies are of the nature of action and reaction, these forces are always equal and opposite, and therefore they both come into existence and both cease simultaneously, and therefore they act during the same time. Hence the changes of *momentum*, and not the changes of *energy*, are equal and opposite in such cases, as shown in Chapter VII.

Examples of non-conservative systems.—(1) If we push a book along a table, we do work *against the reaction* of the table on the book due to friction. But, since the table does not move, *no work is done by the action* of the book on the table. To bring the book back to its original place we *again* have to do work against the friction of the table. Hence there is a loss of work in both processes, and the work so lost is converted into heat.

(2) To find the loss of kinetic energy when a mass of 1 lb., moving with a velocity of 10 feet per second, strikes an equal mass of 1 lb., and both continue to move on together.

If v is the common velocity of the two masses after the blow, the constancy of momentum gives

momentum of 2 lbs. moving with vel. v = momentum of 1 lb. with vel. 10;

$$\therefore \quad 2v = 1 \times 10, \quad \text{or} \quad v = 5 \text{ ft. per sec.}$$

The kinetic energy of 1 lb. moving with a velocity of 10 feet per second

$$(= \tfrac{1}{2}mv^2) = \tfrac{1}{2} \cdot 1 \cdot 10^2 = 50 \text{ ft.-poundals.}$$

The kinetic energy of 2 lbs. moving with a velocity of 5 feet per second

$$= \tfrac{1}{2} \cdot 2 \cdot 5^2 = 25 \text{ ft.-poundals.}$$

Hence the loss of energy $= 50 - 25 = 25$ ft.-poundals $= \frac{25}{32}$ ft.-lbs.

*152. **Newton's "scholium" to the Third Law.**—The Principle of Conservation of Energy was first enunciated by Newton in a note or "scholium" on his Third Law, in a form the general purport of which may be stated as follows:—

"If action be measured by the rate at which a force works, and reaction be measured by the rate at which work is done against friction, gravity, and cohesion, together with the rate at which work is expended for producing kinetic energy, then action and reaction are equal and opposite."

This is to be regarded as the statement of an independent physical principle rather than as a necessary consequence of the Third Law of Motion, for the "action" and "reaction" in the above statement represent rates of working and not forces

153. **Tension of a string over a smooth pulley.**—*When a string passes over a pulley without friction, the tension is the same throughout, if the mass of the string and pulley be neglected.*

Let one end of the string be pulled with a force T, and suppose, if possible, that the pull at the other end is T', and is not equal to T. If a length s of the string is pulled over, the work done on the string by the force T is Ts, and the work done by the string at the other end is $T's$.

Their difference $(T-T')s$ is the mechanical energy communicated to the string and pulley. But there is no friction; hence the system is conservative, and this mechanical energy cannot be lost. Also the string and pulley have no mass; therefore they cannot acquire kinetic energy. Hence the communicated energy $(T-T')s$ must be zero, and therefore $T=T'$.

Therefore the pull is the same throughout the string (§ 121).

154. DEFINITION. — **Power** *is the* **rate of doing work.**—*The* **power** *of an "agent"* (*e.g.*, a steam engine, a horse, or whatever does work) *is measured by the amount of work the agent is capable of performing per unit of time.*

The F.P.S. dynamical unit of power is, of course, a rate of working of one foot-poundal per second. This unit is rarely used.

The C.G.S. dynamical unit of power is a power of one erg per second. This unit is too small for most purposes (being only about $\frac{1}{421400}$ of a foot-poundal per sec.), but from it is derived a larger unit called the **watt.** A watt is ten million (10^7) C.G.S. dynamical units of power, and is therefore = 10,000,000 ergs per second, or one joule per second. It is principally used in electrical engineering.

155. Horse-Power.—Gravitation Units of Power. —The power of a steam-engine is always measured in horse-power.

Definition.—*A* **horse-power** (h.-p.) *is a rate of working of* **550 foot-pounds per second**.......................... (4)

Hence 1 h.-p. = 33,000 foot-pounds per minute.

This unit of power was introduced by Watt, who estimated it as being the rate of working of a good horse, and it has been universally adopted by engineers as the unit of power. The power of an engine when expressed in horse-power is spoken of as the *horse-power of the engine.*

[Note that the horse-power is a gravitational unit of power.]

There is a corresponding gravitational unit in the Metric system, called the **force de cheval.** It is a power of 75 kilogrammetres per second.

When engineers speak of *an engine of so many horse-power* —say a 10 horse-power engine—they mean an engine which is capable, under favourable circumstances, of working at 10 horse-power—*i.e.*, performing 5500 foot-pounds per second. But such an engine might be worked more slowly and might be used to perform, say, only 4400 foot-pounds per second. It would then be said that the engine was working at $\frac{4}{5}$ of its full horse-power.

Examples.—(1) To find the horse-power of an engine which draws a railway train at 60 miles an hour against a resistance equal to the weight of 1 ton.

Here the engine moves 88 feet per second against a resistance of 2240 lbs. wt. Hence it performs

$$88 \times 2240 \text{ ft.-lbs. per sec.};$$

$$\therefore \text{ required horse-power of engine} = \frac{88 \times 2240}{550} = 358{\cdot}4.$$

(2) A steam pump raises 11 tons of water 15 feet high every minute. What is its horse-power?

$$\text{Work done per min.} = 11 \times 2240 \times 15 \text{ ft.-lbs.};$$

$\therefore$ work done per sec. $= 11 \times 2240 \times 15 \div 60$ ft.-lbs.
$= 11 \times 560$ ft.-lbs.

But one horse-power $= 550$ ft.-lbs. per sec.;

$$\therefore \text{ required horse-power} = \frac{11 \times 560}{550} = \frac{56}{5} = 11{\cdot}2.$$

(3) To express the horse-power in F.P.S. dynamical units.

A horse-power = rate of working of 550 ft.-lbs. per sec.
$= 550 \times 32$ ft.-poundals per sec.
$= 17600$ F.P.S. dynamical units.

156. **General expression for rate of working.**—If a body is moving with velocity v under the action of a force P, the distance traversed per unit of time by the body is v, and therefore the work done per unit of time is Pv; $\therefore$ **rate of working** $= Pv$

= (force) × (velocity of its point of application)...(5).

If P is expressed in poundals, this is rate of working in foot-*poundals* per second. To reduce to foot-*pounds* per second we should have to divide by g or 32, and to reduce to horse-power we should have further to divide by 550.

SUMMARY OF RESULTS.

Potential energy of weight W at height $h = Wh$......(1).

Kinetic energy of mass m moving with velocity v

$= \frac{1}{2}mv^2$ dynamical units of work .... (2)

$= \dfrac{mv^2}{2g}$ foot-pounds if m and v are in F.P.S. units.

The *equation of work* for a body moving in a straight line is $Ps = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$ .... (3),

or, work done = increase of kinetic energy.

This may be written

decrease of potential energy = increase of kinetic energy,

or, potential energy + kinetic energy = constant,

in accordance with Principle of Conservation of Energy.

A horse-power = 550 ft.-lbs. of work per second ... (4).

Rate of working = (force) × (vel. of pt. of application).. (5).

EXAMPLES X.

1. A lump of stone weighing 20 lbs. was dropped from a scaffolding, and, after falling freely through 35 feet, was brought to rest by penetrating $2\frac{1}{2}$ feet into mud. Assuming that the force of pressure of the mud on the stone was uniform, determine its magnitude.

2. A body weighing 4 lbs. falls 200 feet, and is then brought to rest by penetrating 2 feet into sand. What is the average resistance of the sand?

3. A cannon ball, whose mass is 60 lbs., falls through a vertical height of 400 feet. What is its energy? With what velocity must such a cannon ball be projected from a cannon to have initially an equal energy?

4. A body, whose weight is 3 lbs., is thrown vertically upwards with a velocity of 32 feet per second. What is its kinetic energy after (i.) $\frac{1}{2}$ second, (ii.) 1 second?

5. A shot is fired from a gun, which is fixed, with a certain charge of powder. If the quantity of powder be quadrupled, in what proportion will the velocity of the shot be increased?

6. Find the average force which will bring to rest, in 2 feet, an ounce bullet, moving at the rate of 1,500 feet per second. How long will it take to bring it to rest?

7. A stone, moving with a velocity of 15 feet per second, would just break through a pane of glass and come to rest. If the same stone be allowed to strike the pane with a velocity of 17 feet per second, what will be its velocity after passing through?

8. An inelastic mass of 13 lbs., moving along a smooth horizontal plane with a velocity of 87 feet per second, impinges directly on an inelastic mass of 16 lbs. at rest on the plane. What kinetic energy is lost? What has become of it?

9. Find, in miles per minute, the speed which would be maintained by an engine of 1 horse-power working against a resistance of 1 *poundal.*

10. How long will a man, whose weight is 11 stone, take in getting from the ground to the top of a steeple 400 feet high by means of ladders, if he exerts $\frac{4}{15}$ horse-power?

11. The resistance to the motion of a steam plough along level ground being supposed uniform and equal to the weight of $1\frac{1}{3}$ tons, and the horse-power of the engine employed 28, find the greatest uniform speed of the plough that can be maintained.

12. A steam crane of 6 horse-power raises a load to a height of 100 feet in 5 minutes. What is the greatest possible weight of the load?

13. In a railway train the resistance and friction of the rails is 1 lb. per ton. What is the horse-power of an engine which will maintain a speed on the level of 30 miles an hour in a train of 60 tons?

14. It has been calculated that a whale exerts 145 horse-power when swimming at 12 miles an hour. Find the resistance of the water in tons.

15. How many watts are there in a *force de cheval* (taking $g = 981$)?

16. Show that the rate at which work is done on a body is the product of its momentum and acceleration. What unit of power must be adopted in this case? Reduce the result to horse-power.

17. Having given that the unit of power is a million ergs per minute, that the unit of force is a thousand dynes, and the unit of time the tenth of a second, find what must be the units of mass and length.

18. Given that the Earth's radius is 4000 miles, that a cubic foot of water contains 1000 oz., that a quadrant of the Earth's Equator is 10^7 metres, that a cubic centimetre of water contains one gramme, find the ratios (i.) of a centimetre to a foot, (ii.) of a gramme to a pound, (iii.) of a dyne to a poundal, (iv.) of an erg to a foot-pound.

19. An ocean steamer does n knots when the engines indicate N horse-power. Find, in tons, the resistance of the steamer in her passage through the water. (A knot = 6086 feet per hour.)

EXAMINATION PAPER VI.

1. Define *work* and *horse-power*. How are they measured?

2. Distinguish between the *momentum* and the *energy* of a moving body.

3. How much work is done against gravity, by a man weighing 12 stone, in climbing a mountain a mile high?

4. A train of 200-ton mass is drawn by an engine of 120 horse-power. If the resistance is 4 lbs. to the ton, what is the velocity of the train?

5. A number of men can each do, on the average, 495,000 ft.-lbs. of work per day of eight hours. How many of such men are required to do work at the rate of 10 horse-power?

6. Find an expression for the whole amount of work done in raising several weights through different heights.

7. What is the horse-power of an engine which can project 10,000 lbs. of water per minute with a velocity of 80 feet per second, 20 per cent. of the whole work done being wasted by friction, &c.?

8. A bullet of mass 1 oz. leaves the muzzle of a gun 3 feet in length with a velocity of 1000 feet per second. Find the average pressure of the powder on the bullet.

9. A horse, drawing a cart along a level road at the rate of 2 miles an hour, performs 29,216 ft.-lbs. of work in 3 minutes. What pull in lbs. does the horse exert?

10. Enunciate and explain the Principle of Conservation of Energy.

CHAPTER XI.

COMPOSITION AND RESOLUTION OF VELOCITIES.

157. **Representation of uniform velocities by straight lines.**—We shall now deal with motions which are not all in one straight line; and in the first place we shall consider the properties of two or more motions which take place with uniform velocities in different straight lines.

In future, when we speak of a body as "**moving uniformly,**" we shall imply that it is moving *with uniform velocity in a straight line.*

In order to specify completely the velocity of a body, it necessary to state

(*a*) *How fast* it is moving;

(*b*) *In what direction* it is moving

The first of these two data is called the **speed** of the body, or the **magnitude** of its velocity; and, if the motion is uniform, it is measured by the distance traversed in a unit of time (Chap. I.).

The second is called the **direction** of the velocity, and is the direction of the straight line in which the body moves. It may be specified by referring it to certain fixed directions, such as the vertical and horizontal directions, the points of the compass, &c.

If then we draw the straight line which the body actually traverses in a unit of time, the length of this line will measure the speed, and its direction will indicate the direction of motion; hence the line will be sufficient to completely specify the velocity of the body. Such a line is said to **represent** the velocity in question.

Thus *uniform velocities may be represented by straight lines.*

Any equal and parallel straight line drawn anywhere would also represent the same velocity, since it would serve equally well to indicate the magnitude and direction.

The *sense* of the direction may be shown by an arrow drawn on or by the side of the line, or by the *order* of the letters used in naming the line. Thus AB represents a velocity which in unit time would carry a body from A to B; BA a velocity which in unit time would carry it from B to A (§ 19).

Example.—Two boats are sailing, one due east at 6 miles an hour, the other north-east at 7 miles an hour. To represent their velocities in a diagram.

Draw AB due east, and on it cut off AB, containing six units of length.

Draw AC, making an angle 45° with AB, and on it cut off AC, containing seven units of length.

Then, if a mile and an hour are the units of length and time, AB, AC represent completely the velocities of the two boats.

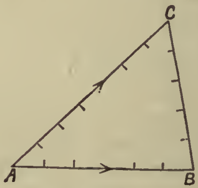


Fig. 20.

158. **Representation of variable velocities.**—When a body is not moving in a straight line, its velocity is variable, even if its speed remains constant.

Thus, if the body revolves in a circle so as to describe equal arcs of the circle in equal times, its velocity will be variable.

In dealing with variable velocity, it is usually necessary to specify it by the velocity **at** *any instant of time.* This is the velocity in a small interval of time, including the given instant, the interval being so short that *neither the speed nor the direction of motion* has time to change in it.

The velocity at any instant is *not* represented by the path *actually* traversed in a unit of time, but by the straight line which *would* be the path traversed if the velocity were to remain uniform from that instant onwards (as would be the case, by Newton's First Law, if the body were not acted on by any force). This line is a *tangent* to the curve along which the body actually moves. Thus, **velocities are always represented by straight lines,** never by arcs of curves.

159. **Relative velocity.** — As explained in §§ 20–22, the velocity of one body **relative** to another is the velocity with which the first body would *appear* to move if the person observing it were moving with the second body.

If two persons are travelling along parallel straight lines with the same velocity, each will always see the other at the same distance away, and in the same direction, and therefore they will be at rest relatively to one another. In other cases the change in position and direction of one as seen from the other determines their relative velocity.

In many cases the relative velocity may be found from first principles.

Examples.—(1) Two men start simultaneously to walk, one eastwards at 4 miles an hour, the other northwards at 3 miles an hour. To find their relative velocity and the direction in which they separate.

Let the men start from A. Then, in 1 hour the first man will have arrived at B, 4 miles east of A, and the second will have arrived at C, 3 miles north of A.

Since the two men started together, BC represents the distance the second man appears to have moved away in an hour, as observed by the first.

Therefore BC measures the relative velocity in miles per hour.

Now, by Euclid I. 47, since BAC is a right angle,

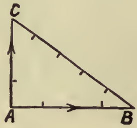


Fig. 21.

$$BC^2 = AB^2 + AC^2 = 4^2 + 3^2 = 16 + 9 = 25;$$
$$\therefore \quad BC = 5.$$

Hence the relative velocity is 5 miles per hour, in a direction parallel to BC.

(2) To determine (*a*) the direction taken by the smoke of a steamer, (*b*) the direction and velocity with which the wind appears to blow to a passenger on board.

[N.B.—The smoke is carried along with the wind.]

(*a*) Let AB represent the velocity of the wind, AC that of the steamer. Then the smoke will always be in a line through the steamer parallel to BC.

For, in a unit of time, the smoke which left the funnel at A will have been blown to B. Also the steamer will have gone from A to C, and smoke will be just leaving its funnel at C. Therefore the smoke will lie along BC. It is easy to see that, as the steamer moves on, the line of smoke always lies in the same direction, provided the velocities of the wind and steamer do not change.

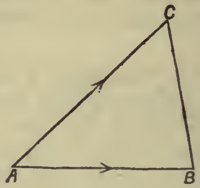


Fig. 22.

(*b*) In unit time the wind has blown the smoke from the steamer through the relative distance CB; therefore CB represents the relative velocity of the wind to a passenger on board, both in magnitude and direction.

(3) A carriage is travelling through a shower of rain, which is falling vertically with a velocity equal to that of the carriage. To show that, to a person in the carriage, the rain appears to fall at an angle of 45° with the vertical, and to find its apparent velocity.

Suppose that at any instant a raindrop appears to coincide with a speck on the carriage-window at A. Then, when the speck (with the carriage) has moved through a horizontal distance AB, the drop will have fallen through an equal vertical distance AC, and the relative positions of the speck and drop will be B, C. Therefore BC represents the direction in which the drop appears to move away from the speck, *i.e.* the apparent direction of the rain relative to the carriage.

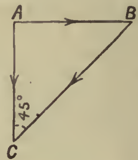


Fig. 23.

But ABC is a right-angled isosceles triangle, and therefore $ACB = 45°$.

Hence the direction of the rain appears to make an angle 45° with the vertical.

Also, $BC^2 = AB^2 + AC^2 = 2AC^2$; $\therefore$ $BC = AC\sqrt{2}$;

$\therefore$ apparent dist. traversed by drop $= \sqrt{2} \times$ (actual dist. traversed);

$\therefore$ apparent vel. of drop $= \sqrt{2} \times$ (actual vel. of rain).

160. **Having given the velocities of two bodies, to construct their relative velocity.**

Let the velocities of the two bodies be represented in magnitude and direction by the straight lines AB, AC, respectively, *both drawn* **from** *the point* A. Complete the triangle ABC.

Then BC represents the velocity of the second body relative to the first;

and CB (which is equal and opposite to BC) represents that of the first relative to the second.

If the bodies start together at A, then in a unit of time one will arrive at B and the other at C. The second will have separated from the first through the distance BC; therefore its relative velocity will be represented by BC,

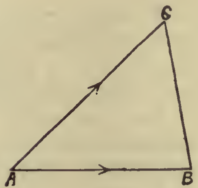


Fig. 24.

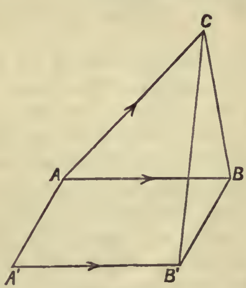


Fig. 25.

and the first will have separated from the second through the distance CB; therefore its relative velocity will be represented by CB.

If the bodies do not start from the same point, let them describe the straight lines $A'B'$ and AC, respectively, in unit time. Complete the parallelogram $ABB'A'$, and join CB.

Then AB or $A'B'$ represents the velocity of the first particle.

Also AA' is equal and parallel to BB'; therefore B occupies the same position relative to B' as A does relative to A'.

Hence the change per unit time in the relative positions of the bodies is the same as if the first body remained at B', and the second moved from B to C. Therefore BC represents the velocity of the second body relative to the first.

Example.—To find the relative velocity of the boats of § 157, Example. By careful measurement, BC (Fig. 20) = 5 units, approximately;

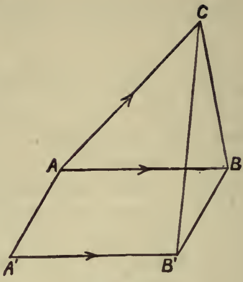


Fig. 26.

$\therefore$ required relative velocity = 5 miles an hour.

161. **Composition of velocities.** — A body cannot be in two places at the same time; therefore it cannot move in two different ways at the same time, and it cannot have two velocities at the same time. But it is often convenient to consider the motion of a body as made up or **compounded** of several independent velocities.

These velocities are called the **component** velocities of the body, and are in every case to be regarded as *relative velocities* on which the motion of the body depends.

The body's actual velocity is called its **resultant** velocity.

The process of determining the resultant velocity when the components are given is called **compounding** the several velocities.

Thus, the definitions of § 25 are perfectly general. But unless the motions are all in one straight line, the resultant velocity is not the algebraic sum of the components.

Thus, suppose a river is flowing, a steamer is being driven through the water by its engines, a man is walking across the deck of the steamer, and a fly is crawling up the man's hat. Then the component velocities of the fly are (*a*) the velocity of the water, (*b*) the velocity with which the steamer is driven *relative* to the water, (*c*) the velocity with which the man walks *relative* to the steamer, (*d*) the velocity with which the fly crawls *relative* to the man's hat. Each of these relative velocities affects the motion of the fly, but the actual or *resultant* velocity of the fly is different from any of them.

Examples.—(1) A ship is sailing at the rate of 12 feet per second, and a sailor climbs up the mast at the rate of $3\frac{1}{2}$ feet per second. To find the man's actual velocity.

Suppose the sailor originally at the foot of the mast at A. Then in one second the motion of the ship carries the foot of the mast from A to B, where $AB = 12$ feet. But the sailor has climbed up $3\frac{1}{2}$ feet, therefore he is at a point C, $3\frac{1}{2}$ feet above B, and AC is the distance actually traversed in one second.

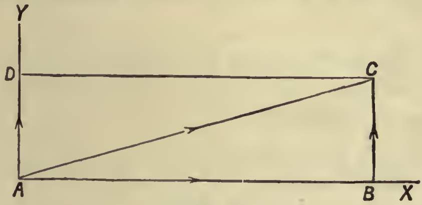


Fig. 27.

Now, since ABC is a right angle,

$$AC^2 = AB^2 + BC^2 = 12^2 + (\tfrac{7}{2})^2 = 144 + \tfrac{49}{4} = \tfrac{625}{4};$$

$$\therefore \quad AC = \tfrac{25}{2} = 12\tfrac{1}{2};$$

and therefore the sailor's actual velocity is $12\frac{1}{2}$ feet per second.

(2) A man rows a boat through the water at the rate of 3 miles an hour in a direction 60° east of north, in a current flowing southwards at the rate of $1\frac{1}{2}$ miles an hour. To show that the boat will travel due eastwards, and to find its rate of progress.

If a straw, dropped from the boat at A, were to drift with the current (supposed constant), it would in an hour reach a point B, $1\frac{1}{2}$ miles south of A.

But the man has rowed relatively to the water and straw through 3 miles in a direction 60° east of north.

Therefore the boat will have arrived at C, where $BC = 3$ miles, and $\angle ABC = 60°$. Complete the equilateral triangle BCD. Then $AB = 1\frac{1}{2}$ miles $= \frac{1}{2}BC = \frac{1}{2}DB$.

Therefore A is the middle point of BD, and AC is at right angles to AB.

Therefore the boat's course AC is due eastwards.

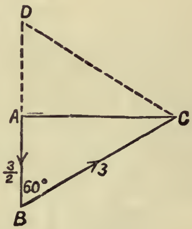


Fig. 28.

Also

$$AC^2 = BC^2 - AB^2 = 3^2 - (\tfrac{3}{2})^2 = 9 - \tfrac{9}{4} = \tfrac{27}{4};$$

$$\therefore \quad AC = \tfrac{3}{2}\sqrt{3} \text{ miles}.$$

Therefore the boat's actual velocity is $\frac{3}{2}\sqrt{3}$ miles an hour.

162. The Parallelogram of Velocities. — If two component velocities be represented in magnitude and direction by the two adjacent sides of a parallelogram drawn *from* a point, then their resultant velocity will be represented by the diagonal of the parallelogram drawn *from* that point.

Let AB, AD represent the two component velocities.

Then a body starting from A with velocity AB would, in unit time, arrive at B.

Let a second body start simultaneously from A with velocity relative to the first represented by AD; then the velocity of the latter body is the resultant of the two velocities AB, AD.

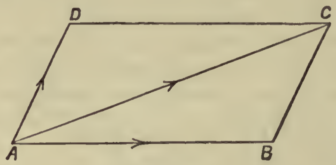


Fig. 29.

Since AD represents the relative velocity, therefore at the end of a unit time the bodies will have separated through a distance equal to AD, in a direction parallel to AD. But the first body is then at B. Therefore the second body is at a point C, such that BC is equal and parallel to AD.

$\therefore$ $ABCD$ is a parallelogram, and AC is its diagonal;

$\therefore$ in unit time the second body moves from A to C;

$\therefore$ AC represents its actual velocity;

$\therefore$ AC represents the resultant of the two velocities AB, AD, as was to be proved.

163. **If the two component velocities are uniform, the resultant velocity will also be uniform.**

Let AB be the distance which would be traversed in *any time t* by a body moving with one of the component velocities, AD the distance which would be traversed in *the same time t* by a body moving with the other component velocity (Fig. 29).

Then, as in the last paragraph, it may be shown that the distance which would be traversed in the time t by a body moving with the resultant of the two component velocities is represented by the diagonal AC of the parallelogram $ABCD$.

Since the component velocities are uniform,

∴ a body moving with either of these components would traverse equal distances in the same direction in equal intervals of time;

∴ AB, AD represent the distances which would be traversed in *every* interval of length t by bodies moving with the respective *component* velocities;

∴ AC represents the distance traversed in *every* interval of length t by a body moving with the *resultant* velocity;

∴ such a body will traverse equal distances in the same direction in equal intervals of time;

∴ the resultant velocity will be uniform.

164. Observation.—The student should be careful to distinguish between the constructions for the *relative* velocity of two bodies moving with given velocities, and the *resultant* of two given component velocities.

If the velocities of two bodies be represented by the sides BA, BC of a triangle, both drawn *from* B, the third side AC will represent their relative velocity.

Bnt if two velocities are represented by AB, BC, one drawn *towards* B and the other drawn *from* B, the third side AC will represent the resultant obtained by compounding the two velocities.

Now the velocity AB is equal and opposite to the velocity BA.

Hence *the relative velocity of two moving bodies is the same as the resultant velocity obtained by compounding the velocity of one body with a velocity equal and opposite to that of the other.*

165. DEFINITION.—The sides of a triangle or polygon are said to be **taken in order,** when of any two adjacent sides one is drawn *towards*, and the other *away from* their common angular point.

The phrase "taken in order" refers to the *sense* in which the sides are directed (p. 148). Thus, if we call the sides of a triangle *BC*, *CA*, *AB*, they are taken in order; but if we call them *CB*, *CA*, *BA*, they will not be taken in order.

In drawing a triangle or polygon without lifting the pencil off the paper, the sides will be described *taken in order*.

166. The Triangle of Velocities.—If a body have three component velocities which can be represented by the sides of a triangle taken in order, then the body will remain at rest.

Let the three component velocities be represented by *AB*, *BC*, *CA*.

Let a body start from *A*, with velocity represented by *AB*,

let a second body start from *A*, with velocity compounded of *AB*, *BC*; and

let a third body start from *A*, with velocity compounded of *AB*, *BC*, *CA*.

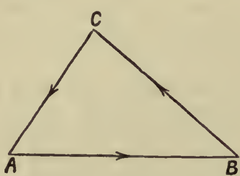


Fig. 30.

Then, at the end of a unit time, the first body will be at *B*,

the second body will have separated from the first through a relative distance *BC*, and will therefore have arrived at *C*;

and the third will have separated from the second through a relative distance *CA*, and will therefore be at *A*.

Hence the third body, which has the three velocities *AB*, *BC*, *CA*, remains at rest at *A*, as was to be proved.

The following is a generalization of the above proposition :—

167. The Polygon of Velocities.—If a body have any number of component velocities, which can be represented by the sides of a closed polygon taken in order, the body remains at rest.

Let *AB*, *BC*, *CD*, *DA* be the sides of the polygon representing the several component velocities.

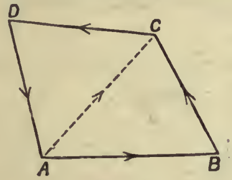


Fig. 31.

Let a number of bodies start simultaneously from *A*.
Let the first have a velocity *AB*,
  ,, second ,, ,, compounded of *AB* *BC*,
  ,, third ,, ,, ,, ,, *AB* .*BC* *CD*,
  ,, last ,, ,, ,, ,, *AB*, *BC*, *CD*, *DA*,
the number of bodies being equal to the number of sides.

Then, at the end of a unit of time,

the first has moved from *A* to *B*;

the second has separated through *BC* relative to the first, and is at *C* ;

the third has separated through *CD* relative to the second, and is at *D* ;

the last has separated through *DA* relative to the third, and is at *A*.

Hence the last body, whose motion is compounded of all the velocities, remains at rest at *A*.

168. To construct the resultant of any number of different component velocities.

Let the given velocities be represented by the straight lines AB, BC, CD, taken in order, forming all the sides but one of a polygon. Then, if the polygon be completed by drawing the remaining side *from* the first extremity A *to* the last extremity D, the line AD will represent the resultant velocity.

For in the course of the last proof it was shown that, if a body start from A with component velocities AB, BC, CD, it will in unit time arrive at D. Therefore AD represents the resultant velocity of the body.

169. To find the magnitude of the resultant of two velocities u, v in directions at right angles to one another.

Draw AB, AD at right angles, and let AB contain u, and AD contain v units of length.

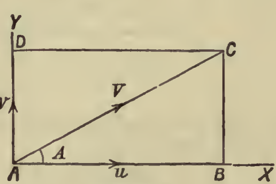


Fig. 32.

Then AB, AD represent the two velocities u, v.

Complete the parallelogram $ABCD$.

Then AC represents the resultant velocity.

Let $$AC = V.$$

By Euclid I. 47,

$$AC^2 = AB^2 + BC^2 = AB^2 + AD^2;$$

$$\therefore \quad V^2 = u^2 + v^2; \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)};$$

$\therefore$ **resultant velocity** $V = \sqrt{(u^2 + v^2)}$.

Examples.—(1) If the component velocities are 3 and 4 units respectively, $V^2 = 3^2 + 4^2 = 9 + 16 = 25 = 5^2$,
and the resultant velocity $V = 5$ units.

(2) If the component velocities are 5 and 12 units respectively,
$$V^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2,$$
and resultant velocity $V = 13$ units.

170. **To find the direction of the resultant of two given velocities u, v at right angles to one another.**

With the construction of the last paragraph, let
$$\angle BAC = A.$$
By the definition of the tangent of an angle,
$$\tan BAC = \frac{BC}{AB} = \frac{AD}{AB};$$
$$\therefore \quad \tan A = \frac{v}{u} \text{..........................} \quad \textbf{(2)}.$$

Knowing the tangent of A, the angle A may be found from a table of trigonometrical tangents, and the direction of the resultant is determined by this angle.

Cor.—The following particular cases are important:—

(i.) If $v = \frac{u}{\sqrt{3}}$, then $\tan A = \frac{1}{\sqrt{3}}$; and $\therefore A = 30°$.

(ii.) If $v = u$, then $\tan A = 1$; and $\therefore A = 45°$.

(iii.) If $v = u\sqrt{3}$, then $\tan A = \sqrt{3}$; and $\therefore A = 60°$.

If the tangent of the angle A has not either of these values, and a table of tangents is not at hand, the angle may be found approximately by drawing the diagram as true to scale as possible, and measuring the angle BAC with a protractor.

171. **Resolution of velocities.**—It may happen that we are given the resultant velocity in magnitude and direction, and that we have to find what are the component velocities along two given lines which have the given velocity for their resultant. This process is called **resolving** the given velocity into components in the given directions, and is the reverse of compounding velocities.

172. To resolve a given velocity into components in two different directions at right angles to one another.

Let AX, AY be the two given lines at right angles.

Let the given velocity be specified by its magnitude V and the angle A which its direction makes with AX.

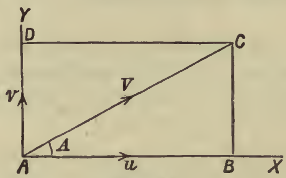


Fig. 33.

Let $\angle XAC = A$, and let $AC = V$.

Then AC represents the given velocity, and we have to find two velocities along AX, AY, whose resultant is AC.

Draw CB parallel to YA, and CD to XA.

Then $ABCD$ is a parallelogram, and therefore AC represents the resultant of the velocities represented by AB, AD. Therefore AB, AD represent the required components.

Let $AB = u$, $AD = v$. By Trigonometry,

$$\cos BAC = \frac{AB}{AC};$$

$$\therefore \quad AB = AC \cos BAC,$$

or

$$u = V \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

$$\sin BAC = \frac{BC}{AC} = \frac{AD}{AC};$$

$$\therefore \quad AD = AC \sin BAC,$$

or

$$v = V \sin A \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Therefore the required components are $V \cos A$ and $V \sin A$, respectively.

Cor.—The following cases are important:—

(i.) If $A = 30°$, then $u = \frac{1}{2}\sqrt{3} \,.\, V$, $v = \frac{1}{2}V$.

(ii.) If $A = 45°$, then $u = \sqrt{\frac{1}{2}} \,.\, V$, $v = \sqrt{\frac{1}{2}} \,.\, V$.

(iii.) If $A = 60°$, then $u = \frac{1}{2}V$, $v = \frac{1}{2}\sqrt{3} \,.\, V$.

173. Other properties of velocities.—In Chap. XIII. we shall show that forces may be compounded by the same rules as velocities. Hence all theorems relating to the composition of velocities will hold equally good for the composition of forces, and *vice versa*. In the earlier chapters of *The Tutorial Statics*, a number of other theorems about forces are proved, all of which are equally applicable to velocities. As, however, they are more often used in connection with forces than velocities, they are usually treated in Statics.

Summary of Results.

The Parallelogram of Velocities.— If two component velocities be represented by the two adjacent sides of a parallelogram drawn from a point, their resultant is represented by the diagonal of the parallelogram drawn from that point.

Triangle and Polygon of Velocities.—If a body have three or more component velocities which can be represented by the sides of a triangle or closed polygon taken in order, the body will remain at rest.

Resultant of two velocities u, v at right angles is given

in magnitude by $V^2 = u^2 + v^2$ .... (1),

and in direction by $\tan A = v \div u$..................... (2).

Components of a velocity V, along two lines inclined to its direction at angles A, $90° - A$, are given by

$u = V \cos A$ ...... (3), $v = V \sin A$ ...... (4).

EXAMPLES XI.

1. One body moves south uniformly at the rate of 9·8 inches per second, another east from the same point at the rate of 17·6 inches per second. Both started at the same time. How far will they be asunder in 3 minutes?

2. A body is approaching an observer with a velocity due east. In what direction will it appear to move if the observer is himself moving due north with an equal speed?

3. A ship is sailing north at the rate of 8 miles an hour through the sea, and a man walks at the rate of 7 feet per second straight across her level deck on a line drawn at right angles to her length. Draw a diagram (as well as you can to scale) by measuring (which one might) find the angle the man's resultant path makes with the north, and calculate his velocity with respect to the sea.

4. A fly crawls along a straight line ruled on a piece of paper, and the paper is made to slide along the table in a direction making an angle of 120° with this line, with a velocity equal to the fly's rate of crawling. Find the direction and rate at which the fly moves along the table.

5. A railway carriage is travelling at the rate of 60 feet per second, and a passenger rolls a ball across the floor of the carriage at the rate of 11 feet per second in a direction perpendicular to the line of motion of the train. Find the actual velocity with which the ball moves relative to the ground.

6. If a cannon ball is fired at 2000 yards range with a horizontal velocity of 1200 feet per second from a ship travelling 15 miles an hour, show that it strikes the water 110 feet in front of the point towards which the muzzle is pointed.

7. A ship is sailing north-east with a velocity of 10 miles an hour, and to a passenger on board the wind appears to blow from the north with a velocity of $10\sqrt{2}$ miles an hour. Find the true velocity of the wind.

8. A person on an express train moving 60 miles an hour wishes to hit a stationary object which is situated 100 yards off in a line through the marksman at right angles to the line of motion of the train. If his bullet moves 1200 feet per second, find out how much to one side of the object he should aim.

9. With what velocity must a man swim across a river 140 yards wide, flowing 2 miles an hour, so that he may not be carried further down the river than 40 yards?

10. A body has a velocity of 3 miles an hour to the north, and also a velocity of 5 miles an hour 30° south of east. It is brought to rest by the addition of a third velocity. Determine the magnitude of the additional velocity.

CHAPTER XII.

THE PARALLELOGRAM OF ACCELERATIONS. PROJECTILES.

174. **General definition of acceleration.**—When a body is moving in a straight line, its acceleration, if any, is in the line of motion, and may be defined as in Chap. II. In other cases we must define the acceleration as follows:—

DEFINITION.—**Acceleration** is measured by the *rate per unit time at which velocity is being acquired*, and the **direction** of the acceleration is the *direction of this acquired velocity*. The *velocity acquired* by a body in any interval of time is that velocity which must be compounded with the initial velocity in order to obtain the final velocity, *the composition being effected by the Parallelogram (or Triangle) of Velocities.*

From this definition it will be seen that changes in the direction of motion of a body involve acceleration, as well as changes in its actual speed. Unless a body is moving in a straight line, the direction of its acceleration will be found to be generally different from the direction of motion at any instant.

Example.—A body uniformly accelerated starts with a speed of 20 feet per second in a direction 30° west of south, and 10 seconds later it is moving with the same speed in a direction 30° east of south.

To find the acceleration and the velocity of the body 5 seconds after starting.

Draw AD due south. Make $\angle BAD = \angle DAC = 30°$, and take $AB = AC = 20$ units of length (Fig. 34). Then AB, AC represent the initial and final velocities of the body, and, by the Triangle of Velocities, BC represents the velocity which must be compounded with the former to obtain the latter. BC therefore represents the *change of velocity* in 10 seconds.

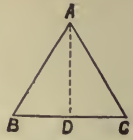


Fig. 34.

Since $AB = AC$ and $\angle BAC = 60°$, the triangle ABC is equilateral, and AD, the bisector of BAC, bisects the base BC at right angles. Thus $BC = AB = 20$, and BC points due east.

Therefore the velocity acquired in 10 secs. is 20 ft. per sec. in a direction due east, and therefore the body is subject to an eastward acceleration of 2 ft. per sec. per sec.

At 5 secs. from starting, the acquired velocity is half as great, and is represented by BD. Therefore the actual velocity is represented by AD. Since $\angle BAD = 30°$, therefore [$AD = AB \cos 30°$, or]

$$AD = AB\frac{\sqrt{3}}{2} = 20\frac{\sqrt{3}}{2} = 10\sqrt{3}.$$

Hence the velocity 5 secs. after starting is $10\sqrt{3}$ ft. per sec. due south.

175. **When two bodies have the same acceleration, their relative velocity is uniform.**

Let AB, AC represent the initial velocities of two bodies at any instant. Then BC represents their initial relative velocity (§ 160).

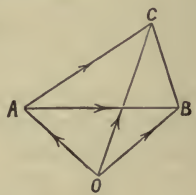


Fig. 35.

Let OA, drawn *towards* A, represent the velocity acquired by either body in *any* given interval of time, under the common acceleration.

Then the final velocities are obtained by compounding the velocity OA with AB and AC, respectively; and are, therefore, represented by OB, OC. Hence the final relative velocity is represented by BC, and is the same as the initial relative velocity. Therefore the relative velocity is constant.

COR. *If two bodies are projected in any directions and fall under gravity, their relative velocity will be uniform*, for the acceleration of gravity is the same for all bodies, and takes place in the vertical direction.

This property is of frequent use in investigating the motion of projectiles (§§ 184–188, below).

176. **Properties of velocities extended to accelerations.**—From the fact that an acceleration is a velocity acquired per unit time, it follows that, to most of the properties of velocities proved in the last chapter there correspond analogous properties of accelerations. These we shall now enumerate, in some cases without proof.

An acceleration may be represented by a straight line, for the velocity imparted *per* unit of time may be represented by a straight line (§ 157), and we may take this line to represent the acceleration.

Thus an acceleration of f ft. per sec. per sec. in any direction may be represented by drawing a line in that direction, and on it measuring a length representing f feet.

177. DEFINITION.—**The relative acceleration** of one body with respect to another may be measured by the relative velocity acquired *per* unit time, this acquired velocity being compounded with the original relative velocity.

It is also the acceleration with which the first body *would appear* to move, if observed by a person moving with the second body.

To find the relative acceleration of two bodies.—If the accelerations of two bodies be represented by AB, AC, the two sides of a triangle drawn from A, their relative acceleration will be represented by the third side BC.

For AB, AC represent the velocities acquired by the two bodies per unit time, and therefore, by § 160, BC represents the relative velocity acquired per unit time.

178. **Component and resultant accelerations.**—If the velocity acquired by a body *per* unit of time be regarded as compounded of several independent velocities,

these may be defined as the **component** accelerations of the body.

The body's actual acceleration is called the **resultant** of the several component accelerations.

Component accelerations, like component velocities, are most easily realized by regarding them as the relative accelerations of a system of bodies on whose motion the resultant acceleration depends.

Thus, when a man is walking at a variable rate along the deck of a steamer which is starting into motion, the acceleration of the steamer and the man's acceleration relative to the steamer are the man's component accelerations.

179. **The Parallelogram of Accelerations.**—*If two component accelerations be represented by two adjacent sides of a parallelogram drawn from a point, their resultant acceleration shall be represented by the diagonal of the parallelogram drawn from the same point.*

For since the sides of the parallelogram represent the component accelerations, they represent the component velocities acquired by the moving body per unit time. By the Parallelogram of Velocities, therefore, the diagonal represents the resultant velocity acquired per unit time, due to the two components, and this is the resultant acceleration of the body.

Observation.—In the above proof we have assumed the Parallelogram of Velocities to hold good for the velocities communicated to the body. The following alternative proof shows how to take account of the initial velocity of the body.

180. **Alternative proof of the Parallelogram of Accelerations.**

Let the initial velocity of a body be represented by OA, and let the body be subject tothe two component accelerations represented by AB, AD. These accelerations are measured by the component velocities they impart in a unit of time; hence, in a unit of time (supposing the accelerations

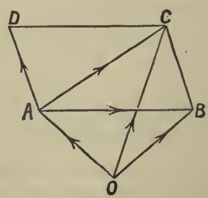


Fig. 36.

to remain uniform) a component velocity represented by AB will be acquired in virtue of the acceleration AB, and a component AD in virtue of the acceleration AD. The actual velocity at the end of a unit time is compounded of the three velocities OA, AB, AD.

Complete the parallelogram $ABCD$. Then BC represents the same velocity as AD, since they are equal and parallel. Hence the final velocity is the resultant of the velocities OA, AB, BC, and is represented by OC. But the initial velocity is represented by OA. Therefore the velocity acquired per unit time is represented by AC. Therefore AC represents the resultant acceleration, as was to be proved.

181. Triangle of Accelerations.—*If three accelerations be represented by the sides of a triangle taken in order then a body whose acceleration is compounded of the three will* ***either remain at rest or move uniformly in a straight line.****

Polygon of Accelerations.—*Generally, if a body have any number of component accelerations, represented by the sides of a closed polygon taken in order, the body* ***either remains at rest or moves uniformly in a straight line.****

For, in either case, the sides representing the accelerations also represent the component velocities imparted per unit of time. By the Triangle or Polygon of Velocities the resultant imparted velocity is zero. Hence no velocity is imparted to the body, and if it was originally at rest it remains at rest. If not, it continues to move uniformly onward with its initial velocity.

If a number of component accelerations be represented by *all the sides but one* of a polygon, their resultant will be represented by the remaining side required to complete the polygon, drawn *from* the extremity of the first side *to* that of the last.

* Notice the difference between these enunciations and those of the Triangle and Polygon of Velocities.

182. Composition of two accelerations at right angles.

If f_1, f_2 be the component accelerations in two directions at right angles, F the resultant acceleration, then

$$F^2 = f_1^2 + f_2^2.$$

Also the resultant acceleration makes with the direction of f an angle A, such that $\tan A = f_2/f_1$.

Resolution of a given acceleration in two directions at right angles.

Conversely, if we are given the resultant acceleration F, and we have to resolve it into two components in two given perpendicular directions, where the direction of F makes a given angle A with one of them, these components f_1, f_2 are given by

$$f_1 = F\cos A, \quad f_2 = F\sin A.$$

These results follow from the Parallelogram of Accelerations in exactly the same way as those of §§ 169–172 follow from the Parallelogram of Velocities.

183. **Projectiles.** — The properties of accelerations enable us to investigate the motion of a body thrown in any direction (not necessarily vertical) and falling under gravity. Such a body may be called a *projectile*. We shall always neglect the resistance of the air, and shall assume that the acceleration of gravity (g) is the same (both in magnitude and direction) at all points of the path.

184. **A body is thrown with a given velocity V in any given direction. To construct geometrically its position at any given instant of the motion.**

Let AP be the direction of projection.

On AP cut off

$AB = Vt =$ distance which would be traversed in time t, if the velocity were uniform and equal to V.

Draw AD vertically downwards, and make

$AD = \frac{1}{2}gt^2 =$ distance which would be traversed in time t by a body falling from rest at A.

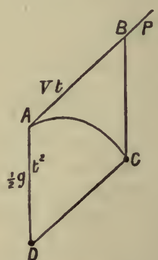


Fig. 37.

Complete the parallelogram $ABCD$. Then C represents the actual position of the body at the time t.

For suppose that, at the instant of projection, a second body is let fall freely from rest at A. Then at the time t this body will arrive at D. Also, since both bodies have the same acceleration (viz., that due to gravity), their relative velocity is uniform (§ 175), and equal to their initial relative velocity V. Hence, since both start together, their distance apart at time t is Vt in a direction parallel to AB. Hence the projectile must be at C, where DC is equal and parallel to AB, as was to be proved.

OBSERVATION.—Since $ABCD$ is a parallelogram, $\therefore BC = AD = \frac{1}{2}gt^2$, showing that at time t the projectile has been pulled down by gravity through a depth $\frac{1}{2}gt^2$ from the line of projection.

Hence, in firing at a target, the muzzle of the gun must be directed towards a point $\frac{1}{2}gt^2$ *above* the target, where t is the time taken by the bullet to reach the target.

185. Fig. 38 shows how this construction may be used to find the position of the body at every second of the motion. The points D_1, D_2, D_3 ... represent the positions of a body falling from rest after 1, 2, 3 ... seconds respectively. **They are therefore the points shown in the diagram on page 53.** On the direction of projection, we must take each of the divisions AB_1, B_1B_2, B_2B_3 ... to represent V units (supposing the unit of time to be one second). Completing the corresponding parallelograms, we find the points C_1, C_2, C_3 ... representing the positions of the projectile after 1, 2, 3 ... seconds respectively.

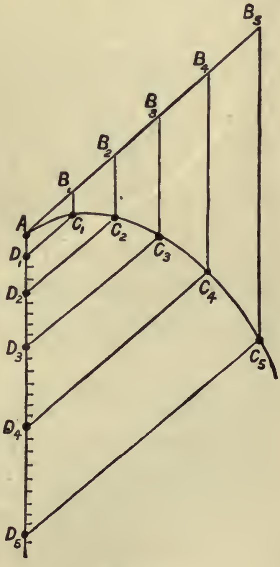


Fig. 38.

If the points A, C_1, C_2, C_3 ... be joined together by a curve, this curve, when well drawn, will represent the path described by the projectile. The curve must be drawn *touching* AB at A, for at the instant of projection, the direction of motion is *along* AB.

[The curve is called a *parabola*, and its properties are discussed in treatises on Conic Sections.]

186. **A body is projected with a velocity whose horizontal and vertical components are u and v. To find its position and velocity at any given instant.**

Let A be the point of projection. On the horizontal line through A measure off

$AB = ut =$ distance that would be traversed in time t with uniform horizontal velocity u.

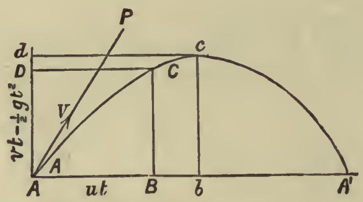


Fig. 39.

On the vertical, measure off

$AD = vt - \frac{1}{2}gt^2 =$ height at time t of a particle projected with vertical velocity v.

Complete the parallelogram $ABCD$. Then C represents the actual position of the projectile at time t.

Again, in the time t the acceleration g imparts a downward vertical velocity-component gt. This has to be compounded with the initial velocity-components u, v; hence, if u', v' are the horizontal and vertical velocity-components at time t,

$$u' = u, \quad v' = v - gt \quad \text{.................} \quad \textbf{(1).}$$

187. Greatest height and time of flight. — The vertical part of the motion is the same as that of a body D projected vertically upwards with initial velocity v.

[For if such a body is projected at the same instant as the projectile, the latter will separate from it with uniform horizontal relative velocity u.]

Therefore, by §§ 63–65,

The body ceases to rise when its vertical velocity $v-gt=0$; therefore time taken in rising $=\frac{v}{g}$ .... (**2**).

The greatest height (bc or Ad, Fig. 39)

$$=\frac{v^2}{2g} \text{ .... } (\mathbf{3}).$$

The time of flight on the horizontal plane AB is found by putting $0 =$ vertical height $= vt-\frac{1}{2}gt^2$, and is therefore

$$=\frac{2v}{g} = \text{twice time of rising ...... } (\mathbf{4}).$$

188. Range on a horizontal plane. — If t is the time of flight, the horizontal range (AA', Fig. 39) is ut, the distance traversed in a horizontal direction. Therefore, by (4),

$$\textit{horizontal range} = u \times \frac{2v}{g} = \frac{2uv}{g} \text{ .... } (\mathbf{5}).$$

Cor.—If the body is projected with velocity V at an inclination 45° to the horizon, then, by § 172, Cor.,

$$u=\frac{V}{\sqrt{2}},\quad v=\frac{V}{\sqrt{2}},\quad \text{and therefore } \textit{horizontal range} = \frac{V^2}{g} \text{...} (\mathbf{6}).$$

It can be shown that this is the greatest possible horizontal range for a body projected with velocity V. We observe that *the greatest range is double the height to which the body would ascend if projected vertically with velocity V*. See § 221.

$u=V\sin\alpha.\ v=V\cos\alpha.\quad \frac{2uv}{g} = \frac{V^2}{g}\sin 2\alpha$ ✓ ∴ maximum when $\alpha=\frac{\pi}{4}$.

Example. — If a bullet is fired at elevation 30° with velocity 1000 ft. per sec., then

horizontal velocity $u = 1000\cos 30° = 1000 \times \frac{1}{2}\sqrt{3} = 500\sqrt{3}$ ft. per sec.,

vertical velocity $v = 1000 \sin 30° = 1000 \times \frac{1}{2} = 500$ ft. per sec.;

hence time of flight $= \frac{2v}{g} = \frac{2\times 500}{32} = \frac{1000}{32} = 31\frac{1}{4}$ seconds,

and range on horizontal plane $= ut = 31\frac{1}{4} \times 500\sqrt{3} = 15625\sqrt{3}$ feet $= 27063$ feet (approximate) $= 9021$ yards $= 5\frac{1}{8}$ miles.

Summary of Results.

The Parallelogram of Accelerations.—(See page 166.)

The Triangle and Polygon of Accelerations.—(Page 167.)

For a body projected with horizontal and vertical velocity-components u, v, the horizontal and vertical distances traversed in time t are ut and $vt-\frac{1}{2}gt^2$.

The velocity-components at time t are

$$u' = u, \quad v' = v - gt \quad \text{.................} \quad (1)$$

$$\text{The time taken in rising} = \frac{v}{g} \quad \text{.........................} \quad (2).$$

$$\text{The greatest height} = \frac{v^2}{2g} \quad \text{.........................} \quad (3).$$

The time of flight on a horizontal plane

$$= \frac{2v}{g} \quad \text{.........................} \quad (4).$$

$$\text{The horizontal range} = \frac{2uv}{g} \quad \text{.......................} \quad (5).$$

The range is greatest when the elevation is 45°, and is

$$= \frac{V^2}{g} \quad \text{.........................} \quad (6).$$

EXAMPLES XII.

1. A body is initially moving eastward at the rate of 15 miles an hour, and 11 secs. later it is moving northward at the same rate. Find the direction and magnitude of the acceleration, supposed uniform, and the velocity $5\frac{1}{2}$ secs. after starting.

2. A particle moves uniformly along the sides of a regular hexagon. Calculate the change of velocity at each corner of the hexagon, and the magnitude of the blow required to cause this change.

3. A body is projected horizontally with a velocity of 32 ft. per sec., and falls under gravity. Represent in a diagram its velocities after 1, 2, 3 secs. respectively, and find their magnitudes.

4. Each of two projectiles is moving directly towards the other at a given instant. Show that they must ultimately meet.

5. A particle is projected in a horizontal direction with a velocity of 10 miles an hour, and at the same time falls under the action of gravity. Assuming that no other forces are acting, and taking $g = 32$ (feet, seconds), draw a figure representing the positions of the particle at the end of 1, $1\frac{1}{2}$, $2\frac{1}{4}$, and 3 secs.

6. A cannon ball is shot horizontally from the top of a tower 49 feet high, with a velocity of 2000 ft. per sec. Find at what distance from the tower the cannon ball will strike the ground.

7. A ball is thrown horizontally from a height of 100 ft., with a velocity of 60 ft. per sec. What is its velocity on reaching the ground?

8. A cannon ball of mass 7 lbs. is fired horizontally from a gun whose mass is 2 tons. The mouth of the cannon is 9 feet from the ground, and the ball strikes the ground $\frac{1}{8}$ of a mile off. What force will be required to bring the cannon to rest in 10 ft.?

9. A stone is thrown from the top of a tower with a velocity of 50 feet a second, in a direction making an angle of 30° with the horizon. Find the distance of the stone from the point of projection at the end of 5 seconds.

10. A stone is projected into the air with a velocity of 200 ft. per sec. in a direction inclined at 60° to a horizontal plane. With what velocity must another stone be projected vertically upwards so that the two stones may rise to the same height above the horizontal plane.

11. A body, thrown in a direction making an angle of 30° with the horizon, passes through a point $400\sqrt{3}$ feet horizontally from the point of projection and 76 feet above it. Find the velocity of projection.

12. The velocity of a projectile when at its greatest height is $\sqrt{\frac{2}{5}}$ of its velocity when at half its greatest height. Show that the angle of projection is 60°.

13. Two stones are simultaneously thrown from the top of a tower in any two directions at right angles, with velocities of 5 and 12 ft. per sec. respectively. Find their distance apart after 4 secs.

EXAMINATION PAPER VII.

1. Explain and illustrate the "Parallelogram of Velocities."

2. Show how to find the relative velocities of any set of bodies with regard to one of their number.

3. Find the direction in which a man must strike out across a river flowing half a mile an hour, if he swims at the rate of a mile an hour, and wishes to land at a point immediately opposite.

4. A person travelling eastward at the rate of 4 miles an hour observes that the wind seems to blow directly from the north; on doubling his speed, the wind appears to come from the north-east. Determine the direction and velocity of the wind.

5. Show that the highest point of the wheel of a carriage moves twice as fast as the carriage itself.

6. A ship is sailing due north with a velocity of 10 miles an hour. In what direction and with what velocity must a stone be thrown from its deck, that it may start in a north-westerly direction with a velocity $10\sqrt{2}$ miles an hour?

7. A particle is projected in any manner in a vertical plane. Show how to find its position at the end of a given time.

8. Explain a geometrical method of finding in direction and magnitude the velocity of a projectile at any instant, the initial circumstances being given.

9. A stone is projected horizontally from the top of a tower 100 ft. high with a velocity of 64 ft. per sec. What will be its distance from the foot of the tower when it strikes the ground?

10. Prove that the height to which a projectile ascends varies as the square of the velocity of projection.

CHAPTER XIII.

THE PARALLELOGRAM OF FORCES.

189. **Representation of forces by straight lines.**—Newton's Second Law tells us that force, like velocity, has direction as well as magnitude. For it asserts that change of momentum is proportional to the impulse of the force, *and takes place in the direction in which the force is impressed.* Hence the **magnitude** of a force is measured, as in Chapter VI., by the momentum per unit time which it imparts to the body on which it acts, and the **direction** of the force is the direction of this imparted momentum.

Or, what is equivalent, the magnitude of the force may be measured by the velocity it would impart to a unit mass in unit time, and its direction is the direction of this velocity, or the direction in which the body would begin to move if it started from rest.

If, therefore, this velocity be represented by a straight line, this line will indicate both the magnitude and direction of the force, and it may therefore be said to represent the force.

Thus forces may be represented by straight lines.

190. **The Principle of the Physical Independence of Forces.**—When a body, instead of starting from rest, is initially moving in a direction different to that of the impressed force, the velocity which the force imparts to the body in any given interval of time must be compounded with the body's initial velocity in order to obtain its final velocity (see Newton's comment on Law II., in § 79).

A few simple illustrations will show this.

(1) Let A, D be the positions at any instant of two men seated in a railway carriage moving uniformly with velocity AB. If the man at A throws a ball so as to reach the other in one second, *he will project it in the direction AD, in just the same way as he would have done if the carriage had been at rest.* But, owing to the motion of the train, the two men will, in one second, be carried, say, to B, C, and the actual path of the ball in space will be the diagonal AC.

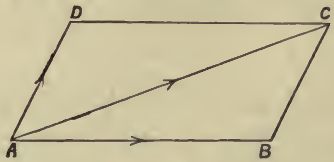


Fig. 40.

Hence the force exerted in throwing the ball merely imparts the *relative* velocity AD. But, before the ball was thrown, it had the same velocity AB as the carriage. Therefore the final velocity AC is obtained by compounding the initial velocity AB with the velocity AD due to the impressed force of projection.

(2) Again, when a stone is dropped, we say that its momentum is equal to the impulse of the force due to its weight. But, when the Earth's motion is taken into account, the momentum which we observe is only the momentum of the velocity *relative* to the Earth. The actual velocity of the stone in space is compounded of this relative velocity and the velocity of the Earth. But, before it dropped, the stone had the same velocity as the Earth. Therefore its final velocity is compounded of its initial velocity and the velocity due to the impulse of its weight, the latter component being given by Newton's Second Law.

This property may be stated more generally thus: *The velocity-component which any given force imparts to a body in any given time is independent of any other velocity-components which the body may possess or acquire.*

This is called the **Principle of the Physical Independence of Forces.**

Employing the definition of acceleration of § 174, it hence follows that the relation

$$P = mf,$$

or force = mass × acceleration,

holds good in every case of motion under force.

191. Composition of forces acting on a particle.—*A force cannot act on nothing;* it must be applied to some definite particle or body whose velocity it changes or tends to change, and the change of velocity will depend on the mass moved. Hence, to completely define a force, it is necessary to specify on what particle the force acts; *i.e.*, to specify its *point of application* (§ 134).

When, therefore, a force is represented by a straight line, this line must be drawn from its point of application. An equal and parallel straight line will represent a force of the same magnitude and direction, but with a different point of application, and therefore not the *same* force.

A body may be acted on by two or more independent forces at the same time.

We have abundant experience of this.

If two or more men pull a block of stone by means of separate ropes attached to it, the forces which they exert are *entirely independent.* Yet they all tend to set the stone in motion.

If we lift a body off the ground, the body is acted on simultaneously by two *entirely distinct* forces, namely, its weight and the lifting force exerted by our hand.

When two or more forces act simultaneously on the same **particle,** each force tends to impart a certain acceleration in the direction in which it is applied. But a particle cannot actually move in two different ways at the same time; it must move with a certain definite acceleration in some direction. Such an acceleration could always be produced by a single force of suitable magnitude applied to the particle in that direction. This force is called the **resultant** of the original system of forces. Hence we have the following

Definition.—The **resultant** of two or more forces is *that force which would produce the same acceleration that is produced by the several forces acting simultaneously.*

Any forces which have a given force for their resultant are called **components** of the given force.

Further, we see that *any number of forces must have a single resultant, provided that they all act on the same particle.*

192. The Parallelogram of Forces.—If two forces, acting simultaneously on the same particle, be represented by two adjacent sides of a parallelogram drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point.*

Let the two forces P, Q be represented by the sides AB, AD of the parallelogram $ABCD$. These lines represent

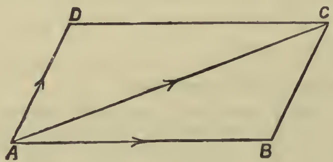


Fig. 40.

the velocities which P, Q, respectively, acting separately, would impart to a unit mass in a unit time (or to m units of mass in m units of time). When the two forces act on the same particle during the same time, the velocity-component imparted by either force is independent of the velocity-component imparted by the other (§ 190). Therefore the actual velocity acquired is found by compounding the velocities AB, AD by the Parallelogram of Velocities, and is therefore represented by the diagonal AC. Hence the change of momentum is the same as would be produced in the same time by a single force represented by AC; therefore the diagonal AC represents the resultant of the two given forces, as was to be proved.

193. Equilibrium.—DEFINITION.—A system of forces is said to **balance**, or to be in **equilibrium,** *when the forces, acting simultaneously, produce no change in the state of rest or uniform motion of the body or bodies to which they are applied.*

Thus, when several forces in equilibrium are applied to a body at rest, the body will remain at rest. If the forces are applied to a body in motion, the body will continue to move uniformly in a straight line as long as the forces balance. In each case the *acceleration* of the body is zero; therefore the resultant of the forces is zero.

194. **Deductions from the Parallelogram of Forces.**—The following properties of forces acting on a particle are analogous to those of velocities and accelerations (§§ 165–172 and 181, 182). As they will be considered more fully in treating of Statics, we shall now merely state them without proof.

Triangle of Forces.—If three forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle taken in order, they will be in equilibrium.

Polygon of Forces.—If any number of forces acting on the same particle can be represented in magnitude and direction by the sides of a closed polygon taken in order, they will be in equilibrium.

Composition of two forces at right angles.—If X and Y denote two forces acting at right angles on a particle, the magnitude of their resultant R is given by

$$R^2 = X^2 + Y^2.$$

Also, if this resultant makes an angle A with the force X, then

$$\tan A = \frac{Y}{X}.$$

Resolution of a force in two directions at right angles.—Conversely, if we have to resolve a force P into two components along two straight lines at right angles, and if A is the angle the force makes with one of these lines, the components (X, Y) are given by

$$X = P \cos A,$$

$$Y = P \sin A.$$

To find the resultant of two forces P, Q whose directions include a given angle A, we replace P by its components X, Y along and perpendicular to the direction of Q. Then P and Q are together equivalent to $Q+X$ and Y in these directions; hence R, the resultant, is given in magnitude by

$$R^2 = (Q+X)^2 + Y^2 = Q^2 + 2QX + X^2 + Y^2.$$

But $$X = P\cos A \quad \text{and} \quad X^2 + Y^2 = P^2;$$

$$\therefore \quad R^2 = Q^2 + 2QP\cos A + P^2,$$

a well-known formula.

194A. The following is an instructive illustration of the laws for the composition and resolution of velocities.

Example.—The sail of an ice-yacht* is set at an angle α to the keel. The wind is blowing at right angles to the keel with velocity V. Required to find the greatest possible velocity of the ice-yacht, supposing there be *no* resistance to motion along its keel.

Let v be the velocity. So long as there is any wind pressure on the sail, the speed of the yacht must be increasing, for there is no resistance to its motion. The speed will therefore continue to increase till the yacht is moving at such a rate that there is no wind pressure on the sail.

This will be the case when the velocity of the wind relative to the sail is along the surface of the sail, *i.e.*, when the resolved velocities

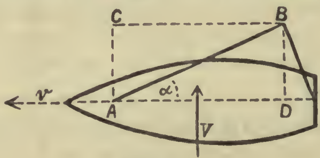


Fig. 41.

of the wind and of the sail, *at right angles to the sail*, are equal.

Whence $$V\cos\alpha = v\sin\alpha; \qquad \text{(Fig. 41)}$$

$$\therefore \quad v = V\cot\alpha.$$

Otherwise thus:—

Let AB be the sail. Draw AC and BC respectively perpendicular and parallel to the keel. Then, if a particle of air with velocity V moves from A to C in the same time that B moves from B to C, *the particle of air will just slide along the sail without pressing on it.*

But B moves from B to C in time BC/v, and a particle of air moves from A to C in time AC/V.

Hence $$\frac{BC}{v} = \frac{AC}{V}; \qquad \therefore \quad v = \frac{BC}{AC}\cdot V = V\cot\alpha.$$

COR.—If $\alpha = 45°$, $v = V$; if $\alpha < 45°$, $v > V$; and v increases as α decreases.

* An ice-yacht is a vessel used in America for sailing on frozen rivers. It rests on the ice on blades parallel to the keel, and runs along like a sleigh.

SUMMARY OF RESULTS.

The Parallelogram of Forces.—If two forces, acting simultaneously on the same particle, be represented by two adjacent sides of a parallelogram drawn from their point of application, they shall be equivalent to a single resultant force represented by the diagonal of the parallelogram drawn from that point.

For other results, see § 194.

EXAMPLES XIII.

[Further examples on Composition and Resolution of Forces will be given in *Statics*, Chaps. I., II. The following are miscellaneous examples.]

1. Find the resultants of the following pairs of forces acting at right angles to one another:—

(i.) 7 lbs. and 24 lbs.; (ii.) 8 oz. and 15 oz.;
(iii.) 20 cwt. and 21 cwt.; (iv.) 24 and 55 grammes weight.

2. Two forces of 12 lbs. and 13 lbs. act on a particle; what are the greatest and least values of their resultant?

3. A bullet is let fall from the mast-head, 30 feet above the deck, of a ship steaming at 20 miles an hour. Find how far the ship will have advanced before the bullet strikes the deck.

4. Describe, with diagrams, the apparent path of the bullet (of the last question) to an observer on board the ship, and its actual path in space.

5. A body, weighing 1 lb., is allowed to fall from rest under gravity, and is at the same time pulled aside by a horizontal force. If the body describes a straight line inclined to the horizon at an angle of 30°, what is the magnitude of the force, and what is the acceleration of the body?

6. Find the maximum velocity of an ice-yacht sailing at right angles to a wind of 10 miles an hour if the angle between the sail and the keel is $\sin^{-1}\frac{3}{5}$ (neglecting all resistances to motion along the keel).

7. Find the angle between the sail and the keel of the ice-yacht if, when the wind is perpendicular to the keel, the maximum velocity of the yacht is four times the velocity of the wind.

8. An ice-yacht is sailing at right angles to the wind at the rate of 20 miles an hour. Its sail makes with the keel an angle of 30°. Find the least possible velocity of the wind.

9. Find the greatest possible velocity of an ice-yacht sailing due N. under a N.W. wind of 20 miles an hour if the sail makes an angle of 30° with the keel.

10. Two particles A and B are moving along lines which meet at right angles at O. One is approaching, and the other receding from, O. If at each moment their velocities are inversely in the ratio of their distances from O, prove that the distance between them is constant.

11. AOB is a right angle; $AO = 20$ ft. Two particles start at the same instant with equal velocities, one from A toward O, the other from O toward B. What is their least distance apart?

CHAPTER XIV.

MOTION DOWN INCLINED PLANES.

195. Definitions.—**An inclined plane** may be exemplified by a plank tilted up at one end, so that bodies can slide down it, or by a road or railway running down hill at a uniform slope. It will, however, be convenient to take an inclined plane as the slanting face $ACC'A'$ of a block of material whose vertical face ABC is a right-angled triangle. The hypotenuse AC is called the **length** of the plane, AB is the **base** and is horizontal, the perpendicular BC is the **height** of the plane, and the angle BAC measures its **inclination** to the horizon.

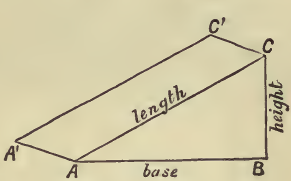


Fig. 42.

The plane is said to be at an inclination of **"1 in *n*,"** when its height is *one n^{th}* of its length, that is

$$BC = \frac{AC}{n}.$$

In this case $$\sin BAC = \frac{BC}{AC} = \frac{1}{n};$$

so that $1 \div n$ is the *sine* of the angle of inclination, and there would be a rise or fall of 1 foot for every n feet traversed up or down the plane.

Thus, if the angle is 30°, $\sin 30° = \frac{1}{2}$, and the inclination is 1 in 2.

By a **perfectly smooth** plane or other surface, we mean one that is perfectly slippery or devoid of friction, so that bodies can slide along it freely and without resistance. When a body presses against any surface, the surface exerts a reaction, for otherwise the body would penetrate it; *but the reaction of a perfectly smooth surface is perpendicular to the surface.*

If we stand on a slippery sheet of thick ice, the reaction of the ice prevents our going through; but if we try to walk, we cannot get much foothold, because the ice exerts but little friction. If the ice were perfectly smooth, we could not walk on it at all.

196. **A heavy body slides from rest down a perfectly smooth inclined plane. To construct the position of the body at a given time t.**

Let the body start from rest at A. Draw AC vertically downwards, and cut off

$AC = \frac{1}{2}gt^2 =$ distance that would be fallen in time t by a body dropped from A.

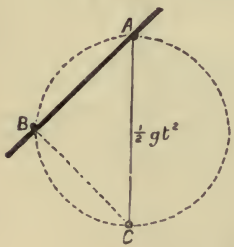


Fig. 43.

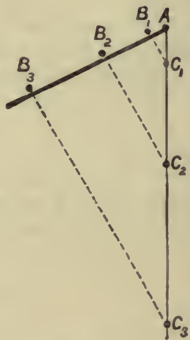


Fig. 44.

Drop CB perpendicular on the plane.

Then B represents the position of the body on the plane at the time t.

For let a second body be let fall from A at the instant that the first starts sliding down.

Then gravity tends to produce the same acceleration in both bodies, and therefore does not affect their relative motion.

The only other force is the reaction of the plane acting on the first body.

This reaction is perpendicular to the plane, and therefore constant in direction. Hence the relative velocity acquired by the bodies is perpendicular to the plane.

But both bodies start from rest together at A.

Therefore the line joining the bodies is always perpendicular to the plane, which proves the construction.

Cor. 1. Since the angle ABC is a right angle, B lies on a circle having AC as diameter (Euc. III. 31). Hence, *if any number of bodies start simultaneously from A, and slide down straight lines in the same vertical plane, their positions at any instant will all lie on a circle whose highest point is A.*

Cor. 2. Hence *the times taken to slide down different chords of a vertical circle, starting from the highest point of the circle, are all equal.*

197. Fig. 44 shows how the position of the body may be constructed at each second of the motion. The points C_1, C_2, C_3 are the positions of a freely falling body after 1, 2, 3 seconds, *aud these are given by the diagram on page* 53. Drawing perpendiculars on the plane, their feet B_1, B_2, B_3 represent the positions of a body sliding down the plane at the same instants.

198. To find the acceleration of a body sliding down a smooth incline of 1 in n.

Let the body be at C. Let its weight be represented by the vertical line Ca. Complete the parallelogram $Cabd$.

Then Cb and Cd represent the components of the weight along and perpendicular to the plane.

Since the body moves down the plane, the resultant force producing motion is down the

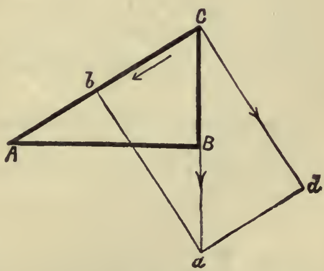


Fig. 45.

plane. Hence the reaction of the plane, acting perpendicular to it, must be equal and opposite to the component Cd, and the force producing motion is represented by Cb.

On the plane cut off

$$CA = Ca,$$

and draw AB horizontal. Then the right-angled triangles ABC, abc are equal in every respect;

$$\therefore \quad Cb = CB.$$

But, since the incline is 1 in n, the height CB is one nth of the length CA; therefore also

$$Cb = Ca \div n;$$

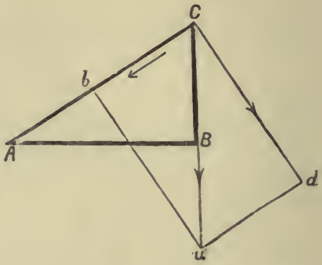


Fig. 46.

$\therefore$ resultant force producing motion = weight of body $\div n$.

But a force equal to the weight of the body would impart to it an acceleration g;

$\therefore$ **the acceleration down the plane** $= \frac{g}{n}$ ....**(1)**

$$= g \times \frac{\textbf{height of plane}}{\textbf{length of plane}}.$$

Cor. Let $\angle BAC = A$. Then $\sin A = \frac{BC}{AC} = \frac{Cb}{Ca}$;

$\therefore$ **acceleration** $= g \sin A$ .... **(2).**

In particular, if the inclination = 0°, 30°, 45°, 60°, 90°,

the acceleration down the plane $= 0, \ \frac{g}{2}, \ \frac{g}{\sqrt{2}}, \ \frac{g\sqrt{3}}{2}, \ g.$

199. Work.—When the direction of motion of a body is not in the same straight line with the force acting on it the work done must be defined as follows:—

DEFINITION.—Let a force P, constant in magnitude and direction, move its point of application from A to C. Draw CB perpendicular on the direction of P. Then the product of the force P into the distance AB measures the **work done** by the force, AB being considered positive or negative according as its direction is the same or opposite to that of the force.

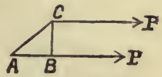


Fig. 47.

OBSERVATIONS. — *When the point of application moves perpendicular to the force, no work is done;* for, if AC is perpendicular to P, then B coincides with A, and AB vanishes.

If the point of application is moved first from A to B and then from B to C, the work done by P in the former displacement is $P \times AB$, and in the latter it is zero, because BC is perpendicular to P; therefore the whole work done is $P \times AB$, and is the same as if the point of application moved directly from A to C.

200. **Work on an inclined plane.—Work done by gravity.**—When a weight W slides down the inclined plane CA (Fig. 42 or 46), the work done by gravity is, by definition,

$$= W \times CB = W \times \text{vertical height descended};$$

and is the same as the work which would be done in falling vertically down the height of the plane.

Thus, *the work done by gravity on a body is always equal to the product of the weight of the body into the vertical height through which it descends,* whether the weight falls vertically or slides down an inclined plane.

Similarly, the work done *against* gravity in raising a body is the product of the weight into the vertical height through which it is raised.

When a body moves horizontally no work is done either by or against gravity.

Hence, in walking along a level road, no work is done against gravity, so that the fatigue felt after a walk is not entirely measured by the work done.

Examples.—(1) To find the work done against gravity by a horse in pulling a cart weighing 5 cwt. up a hill a mile long, at a slope of 1 in 40.

The vertical height risen

$$= \frac{1}{40} \text{ of a mile} = \frac{5280}{40} \text{ ft.} = 132 \text{ ft.},$$

and the weight raised $= 5 \times 112$ lbs. $= 560$ lbs.;

$\therefore$ work done $= 132 \times 560 = 73920$ ft.-lbs.

(2) To find the horse-power required to draw a train of 150 tons up an incline of 1 in 128 at 30 miles an hour, if the resistance due to friction is 10 lbs. per ton.

In one second the train moves 44 feet;

$$\therefore \text{ vertical height risen per sec.} = \frac{44}{128} \text{ ft.} = \frac{11}{32} \text{ ft.}$$

Also, weight of train $= 150 \times 2240$ lbs.;

$\therefore$ work done per sec. against gravity

$$= \frac{11}{32} \times 150 \times 2240 \text{ ft.-lbs.} = 115500 \text{ ft.-lbs.}$$

Also, total resistance due to friction $= 10 \times 150$ lbs. $= 1500$ lbs.;

$\therefore$ work done per sec. against resistance

$$= 1500 \times 44 \text{ ft.-lbs.} = 66000 \text{ ft.-lbs.};$$

$\therefore$ total work done per sec. $= 115500 + 66000$ ft.-lbs. $= 181500$ ft.-lbs.;

$$\therefore \text{ required horse-power} = \frac{181500}{550} = 330.$$

201. **To verify the principle of Conservation of Energy for motion down a smooth inclined plane.**

Let a mass m slide down an incline of 1 in n, starting with initial velocity u. By (1) § 198, the acceleration is $g \div n$. Hence, if v is the velocity after the body has gone a distance s, then, by (8) § 45,

$$v^2 - u^2 = 2\frac{g}{n}s.$$

$$\therefore \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mg \times \frac{s}{n}.$$

The left-hand side represents the increase of kinetic energy. Also mg is the weight of the body, and $s \div n$ is the vertical height through which it has fallen in moving a distance s on the plane; hence the right-hand side represents the work done by gravity.

Therefore increase of kinetic energy

= work done by gravity = decrease of potential energy.

Cor. If the body starts from rest, we have

$$v^2 = 2g\frac{s}{n} = 2g \times \text{height fallen.}$$

Hence, *if different bodies slide down inclined planes of the same height, they will all acquire the same speed on reaching the bottom.*

Examples.—(1) A body slides down a smooth plane whose height is one-third its base. To find the velocity acquired when it has travelled 12 feet.

Let the mass of the body be m lbs. In travelling 12 ft. it falls a vertical depth of $\frac{12}{3}$ ft. or 4 ft.;

$\therefore$ work done by gravity $= 4m$ ft.-lbs. $= 4mg$ ft.-poundals

$= 4m \times 32$ ft.-poundals.

This is equal to the kinetic energy. Hence, if v is the required velocity,

$$\tfrac{1}{2}mv^2 = 4m \times 32;$$

$$\therefore\ v^2 = 4 \times 2 \times 32 = 4 \times 64;$$

$$\therefore\ v = 2 \times 8 = 16 \text{ ft. per sec.}$$

(2) A weight of 3 lbs. draws a weight of 4 lbs. up an incline of 30° by means of a string passing over a pulley at the top of the plane and hanging vertically. To find the acceleration.

Let v be the velocity acquired when both weights have moved over s feet.

The 3-lb. wt. will have fallen vertically through s ft.;

$\therefore$ work done by 3-lb. wt. $= 3s$ ft.-lbs. $= 3gs$ ft.-poundals.

The 4-lb. weight will have moved s feet up the plane, and, since the incline is 1 in 2, it will have risen vertically through $\frac{1}{2}s$ feet;

$\therefore$ work done by 4-lb. wt. $= -4 \times \frac{1}{2}s = -2s$ ft.-lbs. $= -2gs$ ft.-poundals

The whole work done is equal to the kinetic energy;

$$\therefore\ \tfrac{1}{2}(4+3)v^2 = 3gs - 2gs = gs;$$

$$\therefore\ v^2 = \frac{2g}{7}s.$$

Comparing this with $v^2 = 2fs$, we have

required acceleration $f = \frac{1}{7}g$.

202. The problem of the previous example can also be solved by a somewhat different method without employing the Principle of Conservation of Energy; we shall now give an example of this method before applying it to a more general case.

Example.—(3) A mass of 14 lbs. hanging by a vertical string draws a mass of 10 lbs. up a smooth incline of 45°, the connecting string passing over a pulley at the summit. To find the acceleration and the tension of the string.

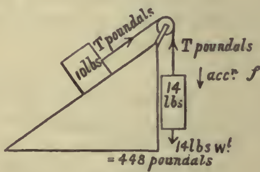


Fig. 48.

Let the acceleration be f ft./sec., and the tension T poundals.

Consider the motion of the 10-lb. mass.

If allowed to slide freely down the plane, its acceleration would be $g \sin 45°$ or $g\sqrt{\frac{1}{2}}$ down the plane.

Therefore the tension T must be sufficient to change the acceleration of the 10-lb. mass from $g \sin 45°$ downwards to f upwards, *i.e.*, it produces an acceleration-component $f + g \sin 45°$ *upwards*, and the property

$$\text{force} = \text{mass} \times \text{acceleration}$$

gives $$T = 10\,(f + g \sin 45°) \qquad \text{(i.)}.$$

Similarly, considering the 14-lb. mass, the tension T, acting upwards on it, changes its acceleration from g (the acceleration with which it would fall freely) to f, and therefore produces a downward acceleration-component, $f - g$, or, what is the same thing, an upward acceleration-component $g - f$. Therefore

$$T = 14\,(g - f) \qquad \text{(ii.)}$$

From (i.) and (ii.) we have

$$10\,(f + g \sin 45°) = 14\,(g - f);$$

$$\therefore \quad (10 + 14) f = (14 - 10 \sin^2 45°)\, g;$$

whence $$f = \frac{14 - 10 \sin 45°}{10 + 14}\, g = \left(\frac{7}{12} - \frac{5}{24}\sqrt{2}\right) g.$$

Again, by (ii.), $$T = 14g - 14\left(\tfrac{7}{12} - \tfrac{5}{24}\sqrt{2}\right) g;$$

$$\therefore \quad \text{tension} = \tfrac{35}{12}\,(2 + \sqrt{2})\, g \text{ poundals}$$
$$= \tfrac{35}{12}\,(2 + \sqrt{2}) \text{ lbs. wt.}$$

We now proceed to consider a further generalization of the preceding examples.

203. Motion of connected bodies on two inclined planes.—*A mass P is drawn up a smooth plane of inclination A by a mass Q sliding down a plane of inclination B, the two being connected by a string passing over a pulley at the common vertex of the planes. To find their acceleration and the tension of the string.*

Let $f =$ acceleration of mass P up AC

$= $ acceleration of mass Q down CB.

$T =$ tension of string (in dynamical units of force).

This tension acts upwards on either mass, *i.e.*, in the directions AC, BC on P, Q, respectively.

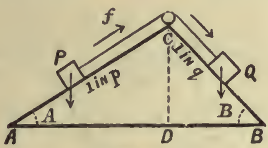


Fig. 49.

The tension T acting on the mass P changes its acceleration from $g \sin A$ downwards to f upwards, and therefore produces an upward acceleration-component $f+g\sin A$.

Hence $$T = P(f+g\sin A) \text{ .... (i.).}$$

Again, the tension T acting on the mass Q changes its acceleration from $g\sin B$ along CB to f in the same direction; it therefore produces an acceleration-component in the opposite direction of $g\sin B - f$. Hence

$$T = Q(g\sin B - f) \text{ .... (ii.).}$$

From (i.) and (ii.),

$$P(g\sin A + f) = Q(g\sin B - f);$$

$$\therefore \quad (P+Q)f = Qg\sin B - Pg\sin A;$$

$$\therefore \quad \text{required acceleration } f = \frac{Q\sin B - P\sin A}{Q+P}\,g \quad \dots \textbf{(3).}$$

To eliminate f, multiply (i.) by Q and (ii.) by P, and add;

$$\therefore \quad (P+Q)\,T = PQ(\sin A + \sin B)\,g;$$

$$\therefore \quad \text{tension } T = \frac{PQ}{P+Q}(\sin A + \sin B)\, g \quad \text{(dynamical units)}$$

$$= \frac{PQ}{P+Q}(\sin A + \sin B) \quad \text{(gravitation units)} \quad \ldots\ldots\ldots \textbf{(4)}.$$

[If formula (3) makes f *negative*, P pulls Q up.]

204. To find the line of quickest descent from a given point to a given straight line.

Let A be the given point, and BC the given straight line. (Fig. 50.)

Through A draw AD perpendicular to BC, and AE

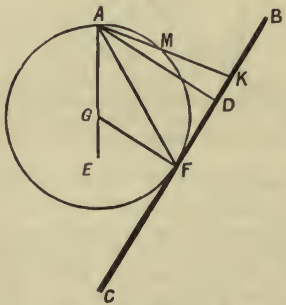


Fig. 50.

vertically down. Bisect $\angle DAE$ by AF, cutting BC in F. Then AF shall be the line of quickest descent; that is to say, a particle would slide more quickly down AF than down any other line joining A to BC.

Draw FG perpendicular to BC, cutting AE in G. Then, since FG and AD are parallel,

$$\therefore \quad \angle AFG = \angle DAF.$$

But $\qquad \angle DAF = \angle GAF. \qquad$ (Construction.)

$$\therefore \quad \angle GFA = \angle GAF;$$

$$\therefore \quad GA = GF.$$

Thus, if, with G as centre and GA as radius, we describe a circle, it will touch BC at F, and A will be its highest point.

Draw any other line AK to meet BC, cutting the circle in M. Then the time a particle would take to slide down $AF =$ time to slide down AM (§ 196, Cor. 2), and is therefore less than the time to slide down AK.

205. To find the line of quickest descent from a point to a circle.

Let A be the given point, BCF the given circle, and O its centre. (Fig. 51.)

Draw a radius OC, vertically down. Join AC, cutting the circumference in F; then AF shall be the line of quickest descent from A to the circle.

Join OF, and produce it to meet the vertical through A in E.

Then
$$\angle AFE = \angle OFC$$
$$= \angle OCF = \angle EAF \qquad \text{(by parallels).}$$

Hence $AE = EF$. Therefore, if, with centre E and radius EA, we describe a circle, it will touch BCF at F,

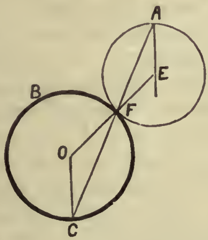


Fig. 51.

and A will be its highest point. Hence, as before, AF will be the line of quickest descent.

206. **To find the line of slowest descent from a point to a circle.**

Let *A* be the given point, and *BFC* the given circle (Fig. 52). Draw a radius *OC* vertically *up* from the centre. Join *AC*, and produce to

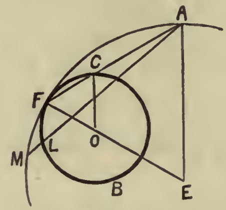


Fig. 52.

cut the circle again in *F*. Then *AF* shall be the line of slowest descent.

Join *FO*, and produce to meet the vertical through *A* at *E*.

Then $\angle EAF = \angle OCF$ (by parallels)

$= \angle CFO; \quad \therefore \; EA = EF.$

Hence, if, with centre *E* and radius *EA*, a circle be drawn, *A* will be its highest point, and it will touch the circle *BFC* *internally* at *F*.

Draw any other chord *ALM*. Then time to slide down *AF* = time to slide down *AM* > time to slide down *AL*; which proves the construction.

207. **Sliding Friction.**—If a body is sliding over a smooth surface, we know that the thrust between the body and the surface is always perpendicular to the surface. If, however, a body is sliding over a rough surface, the thrust is not perpendicular to the surface.

The reaction of the rough surface may be regarded as consisting of two forces: the one perpendicular to the surface, which is usually called the **normal reaction***; the other, along the surface, acting in the opposite direction to that in which the body is moving—this force is called the **friction.** The normal reaction prevents the body from penetrating into the surface; the friction is merely the resistance which the rough surface offers to sliding.

* The line drawn perpendicular to a surface at any point is called the *normal* at that point.

The magnitude of the friction depends partly on the roughness of the surface and of the body which is sliding over it, and partly on the magnitude of the normal reaction.

208. It is found by experiment that the friction between materials of given roughness is *always in constant ratio to the normal reaction.* This constant ratio is called the **Coefficient of Friction.** If we denote the coefficient of friction by μ, the friction by F, and the normal reaction by R, this experimental law is represented by the formula

$$\frac{F}{R} = \mu; \quad i.e., F = \mu R \quad \text{.................} \quad \textbf{(5)}.$$

It is important to notice that the value of μ does not depend on the velocity of sliding, nor on the magnitude of the area of the body which is in contact with the surface; but only on the roughness of the body and of the surface.

209. Let A represent a body sliding over a rough surface EH, with velocity u (Fig. 53); let AC and AB represent R and F. Then, completing the parallelogram $ABDC$, AD represents the **resultant reaction** of the surface.

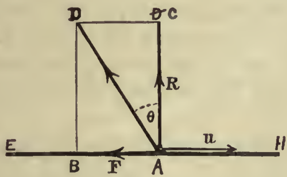


Fig. 53.

Let $\angle DAC = \theta$; then

$$\tan\theta = \frac{DC}{CA} = \frac{F}{R} = \mu \quad \text{.................} \textbf{(6)}.$$

Thus the *resultant reaction always makes with the normal an angle whose tangent is* μ; this angle is called the **angle of friction.**

210. Motion down a rough inclined plane.—Suppose a body D, of mass M, sliding down a rough plane AB inclined at an angle α to the horizon. (Fig. 54.)

Then the forces acting on the body are :—

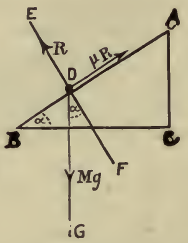


Fig. 54.

(i.) Its weight Mg, acting along DG; the normal reaction R, acting along DE; and

(ii.) The friction μR, acting along DA (*up* the plane since the body is sliding *down* it).

Also $\quad \angle FDG = 90° - \angle GDB = \angle ABC = \alpha.$

Thus Mg can be resolved into the two forces $Mg \cos \alpha$ along DF, and $Mg \sin \alpha$ along DB. (§ 194.)

Now, since the body has no acceleration perpendicular to the plane, ∴ the forces along EF must balance; *i.e.*,

$$R = Mg \cos \alpha.$$

Again, the resultant force down the plane is

$$Mg \sin \alpha - \mu R, \quad i.e., \ Mg \sin \alpha - \mu Mg \cos \alpha.$$

Thus, if f be the acceleration down the plane,

$$Mg \sin \alpha - \mu Mg \cos \alpha = Mf \text{.........[§ 88 (4)]};$$

whence $$\boldsymbol{f = g(\sin \alpha - \mu \cos \alpha)} \text{ .... } \mathbf{(7)}.$$

But, if θ be the angle of friction, $\mu = \tan \theta$;

$$\therefore \quad f = g(\sin \alpha - \cos \alpha \tan \theta)$$

$$= g \frac{\sin \alpha \cos \theta - \cos \alpha \sin \theta}{\cos \theta} = \boldsymbol{g \sin(\alpha - \theta) \sec \theta} \dots \mathbf{(8)}.$$

Cor. 1.—*If the plane be horizontal*, we must put $\alpha = 0$ in the above result; hence, attending to the negative sign of f, we see that a body sliding along a horizontal plane experiences a retardation μg. This retardation continues to act as long as the body is in motion; when it comes to rest, friction ceases.

This result may better be established independently from the equations $F = \mu R$, $R = mg$ and $mf = F$. Hence, $mf = \mu mg$, or $f = \mu g$, and, since the friction is in the direction opposite to that of motion, f is here a retardation, not an acceleration.

Cor. 2.—If $\theta = \alpha$, $f = g \sin(\alpha - \theta) \sec \theta = 0$; hence the body will remain at rest if it be initially at rest, or will descend with uniform velocity if it be initially projected down the plane. Thus, *if the inclination of the plane be equal to the angle of friction, the body will not begin to slide down the plane.*

If the angle of inclination be now diminished, we know from ordinary experience that the body will still remain in equilibrium; it is, however, important to understand that the equations used in this paragraph no longer hold, since they are based on the laws of Sliding Friction; these are not identical with the laws of Statical Friction, *i.e.*, of friction in cases of equilibrium; this subject will be fully discussed in Statics.

If, however, the particle be *projected down the plane* in this case, then, since $\alpha < \theta$, it follows that f will have a negative value, proving that the body will experience a retardation $g \sin(\theta - \alpha) \sec \theta$ till it comes to rest.

211. Motion of a body projected up a rough inclined plane.

This problem differs from that discussed in the last paragraph in one point only—the friction will now act *down* the plane since the body is moving *up;* hence, following the same line of reasoning, we shall find that the resultant force down the plane is now

$$Mg \sin \alpha + \mu Mg \cos \alpha;$$

whence

$$f = g(\sin \alpha + \mu \cos \alpha) = g \sin(\alpha + \theta) \sec \theta \quad \ldots \quad (9).$$

Hence, given the initial velocity up the plane, we could tell how far the body would go before this acceleration down the plane reduced it to rest.

When once the body was reduced to rest it would remain at rest if $\alpha = \theta$ or $< \theta$; but, if $\alpha > \theta$, the body would slide down again, and this part of its motion would be determined by the equations of the last paragraph.

212. **A heavy body slides from rest down a rough inclined plane. To construct the position of the body after a given time *t*.**

Let the body start from rest at A; draw AC vertically downwards, and cut off $AC = \frac{1}{2} gt^2 =$ distance that would be fallen in time t by a body dropped from A. (Fig. 55.)

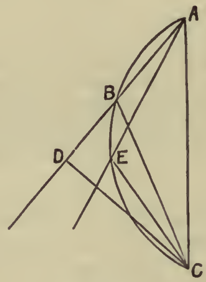


Fig. 55.

Draw CD perpendicular to the plane, and make angle $DCB = \theta$. Then B represents the position of the body after time t.

For let a second body be dropped from A at the instant when the first starts sliding down; then gravity tends to produce the same acceleration in both, and therefore does not affect their relative motion.

The only other force is the reaction of the plane acting on the first body.

Now the normal to the plane at any point is parallel to CD; and the reaction makes with the normal an angle θ. But angle $DCB = \theta$; therefore the reaction always acts in a direction parallel to CB.

Hence the relative velocity acquired by the bodies is parallel to CB. But both bodies start from rest at A.

Therefore the line joining the bodies is always parallel to CB. Thus when the one body is at C the other is at B.

COR.— $\angle ABC = \angle BDC + \angle BCD = 90^\circ + \theta$.

Draw a segment of a circle ABC, and draw any other chord AE; join CE.

Then $\angle CEA = \angle CBA = 90^\circ + \theta$;

∴ if a body slide down an equally rough plane in the position AE, it will be at E after t secs.

Hence, if a series of bodies start from A to slide down various equally rough chords of the segment on AC, which contains the angle $(90^\circ + \theta)$, they will all arrive at the arc at the same moment.

SUMMARY OF RESULTS.

If f is the acceleration down an incline of 1 in n,

$$f = \frac{g}{n} = g \times \frac{\text{height}}{\text{length}} \dots (1), \quad \text{or} \quad f = g \sin A \dots\dots\dots (2).$$

For two masses P, Q joined by string on two inclines, of angles A, B,

$$f = \frac{Q \sin B \sim P \sin A}{Q + P} g \dots\dots\dots (3),$$

$$\text{tension } T = (\sin A + \sin B) \frac{PQ}{P + Q} \dots\dots\dots (4).$$

To determine the magnitude of the friction when one body is sliding over another, $F = \mu R$ .... (5).

In the same case, if θ be the angle between the resultant reaction and the normal (*i.e.*, the angle of friction)

$$\tan \theta = \mu \dots\dots\dots (6).$$

Acceleration down a rough plane inclined to the horizon at an angle α greater than θ is

$$(g \sin \alpha - \mu \cos \alpha) \dots\dots\dots (7)$$

$$= g \sin (\alpha - \theta) \sec \theta \dots\dots\dots (8).$$

Retardation up a rough inclined plane is

$$g (\sin \alpha + \mu \cos \alpha) = g \sin (\alpha + \theta) \sec \theta \dots\dots (9).$$

EXAMPLES XIV.

1. Find the distances traversed in 1 sec., and the velocities acquired in that time, by particles sliding down planes of inclinations 30°, 45°, 60°.

2. A body, starting from rest on an inclined plane, describes 40 ft. in the third second; find the inclination of the plane.

3. A boy in a toboggan slides down a perfectly smooth hill, whose inclination is 1 in 20. At what rate will he be going (in miles per hour) when he has travelled 100 yds. from the start?

4. A body moves up an inclined plane, whose angle is 30°, starting with a velocity of 48 ft. per second. What is its velocity when it reaches a point 64 ft. from the starting-point?

5. The pull exerted by a rope which draws a carriage up an incline of 1 in 4, with an acceleration 2 (ft. per sec. per sec.), is $\frac{1}{2}$ ton. What is the weight of the carriage?

6. A mass of 52 lbs. lies on a plane inclined at an angle of 60° to the horizon. Find the work necessary to remove the mass 20 ft. up the plane.

7. An engine takes a train of 60 tons in all up an incline of 1 in 100 at a maximum speed of 30 miles per hour, and it can take a train of 150 tons on the level at the same speed. Find the fractional resistance of the road in lbs. per ton; and also the rate, in horse-power, at which the engine works when running at this speed.

8. Find the H.-P. of an engine which is taking a train of 200 tons up an incline of 1 in 224, at 30 miles an hour, assuming the resistance due to friction to be 20 lbs. per ton.

9. A body, weighing 187 lbs., is supported on an inclined plane, whose angle is 30°, by a horizontal force. Find the force and the work necessary to remove the body 20 ft. along the plane.

10. Three planes are inclined at angles of 30°, 45°, and 60°, respectively. Find the distance a body must slide down each plane in order to acquire a velocity of 10 cms. per second.

11. Find also, in ergs, the work required to move a mass of 1 gramme along each of the planes (of the last example) through a distance of 10 cms.

12. Two bodies, whose masses are P and Q, are connected by a fine stretched string; P hangs vertically, and Q is placed on a plane whose inclination to the horizon is 30°. Find the ratio of P to Q, if P descends from rest through a given space in (i.) twice, (ii.) four times the time in which it would fall freely through the same space.

13. A bullet, moving at the rate of 160 ft. per second, penetrates 7 ins. into a trunk of wood. With what velocity would another similar bullet, moving with the same velocity, emerge, after passing through a similar piece of wood, 3 ins. thick?

14. The side BC of a triangle ABC is vertical; show that, if the times of falling down the two sides BA, AC be equal, the triangle must be isosceles or right-angled.

15. A body is projected with velocity 20 ft. per second along a rough horizontal plane; it travels 25 ft. before it is brought to rest. Find the coefficient of friction.

16. Determine the acceleration of a body sliding down a rough plane whose coefficient of friction is $\frac{1}{6}\sqrt{3}$, if the inclination of the plane to the horizon is (i.) 30°, (ii.) 45°, (iii.) 60°.

17. A particle takes 2 secs. to slide down a rough plane inclined to the horizon at an angle of 60°. If the coefficient of friction is $\frac{1}{3}\sqrt{3}$, determine the length of the plane.

18. Find the velocity with which a body reaches the bottom of a rough plane, 48 yds. long, inclined at an angle of 30° to the horizon. ($\mu = \frac{1}{6}\sqrt{3}$.)

19. How far would a body travel before coming to rest if projected up a rough plane, inclined at an angle of 30° to the horizon, with initial velocity 40 ft. per sec. ($\mu = \frac{1}{2}\sqrt{3}$.)

20. How far would the body in the last question travel if projected down the plane with the same initial velocity?

21. A particle acquires a velocity of 8 ft. per sec. in sliding down a rough inclined plane whose base and height are both 2 ft. Find the coefficient of friction.

22. A body is projected up a rough plane inclined at an angle of 30° to the horizon; compare the times occupied in sliding up, and down again. ($\mu = \frac{1}{5}\sqrt{3}$.)

23. Two particles of equal mass are connected by a light smooth inextensible string, of length 6 ft. One is placed on a rough plane (inclination = 30°, $\mu = \frac{1}{6}\sqrt{3}$); the other is just hanging over the top of the plane. Find the acceleration of the system, the tension of the string, and the velocity with which the first particle reaches the top of the plane.

24. The inclination (α) of a rough plane to the horizon is less than its angle of friction (θ). Prove that the distances travelled by two bodies which are projected with equal velocities straight up and straight down the plane, respectively, are in the ratio

$$\sin(\theta-\alpha) : \sin(\theta+\alpha).$$

25. Find the line of quickest descent from a given line to a given point.

26. Find the line of quickest descent from a given circle to a given point.

27. Prove that the line of quickest descent from a given point to a given curve bisects the angle between the vertical and the normal to the curve at the point where it meets the curve.

28. Prove the same theorem for lines of slowest descent.

29. Find the line of quickest descent from a circle to a circle.

30. Find the line of quickest descent from a point within a given circle to the circumference.

31. The engine of a train of 200 tons exerts a steady hauling force of 3,000 lbs., and the frictional resistances to the motion of the train amount to 10 lbs. per ton. Find the times that the train would take to travel 5 miles, starting from rest, (i.) when the line is level, (ii.) when there is a down gradient of 1 in 150 for the first 2 miles and an up gradient of 1 in 450 for the remaining 3 miles.

32. A mass M lies on an inclined plane and is connected with another mass m by a thread, which passes over a smooth pulley at the top of the plane. When the plane is smooth, m is sufficiently great to pull M up the plane.

(i.) Show that, when the plane is smooth, the tension of the thread is constant, and find the velocity when the particles have moved over a given space from rest.

(ii.) If the plane is so rough as just to produce equilibrium, find the mass which must be added to m in order that M may be dragged up the plane at the same rate as in case (i.).

33. Two weights P and Q are connected by a string. P hangs vertically and draws Q up a plane of inclination α and coefficient of friction μ, the string passing over a pulley at the top of the plane. Prove that the acceleration is less than it would be if the plane were smooth by an amount

$$\frac{\mu Q \cos \alpha}{P+Q} g.$$

34. Find the line of quickest descent from a given circle to a given straight line without it.

35. Find the line of quickest descent from a given straight line without a given circle to the circle.

36. In Example 32, if the height of the plane be given, show that, if m pull up M in the shortest possible time, the inclination of the plane must be such that m is twice as great as it is for equilibrium.

37. A weight of 12 lbs., moving down the side of an isosceles triangle whose base is horizontal, draws a weight of 6 lbs. up the other side by means of a string passing over a pulley at the vertex. Determine the vertical angle of the triangle, that the tension of the string may be 4 lbs.

EXAMINATION PAPER VIII.

1. Give a dynamical proof of the proposition known as the Parallelogram of Forces.

2. Three equal forces P diverge from a point, the middle one being inclined at an angle of $60°$ to the others. Find the resultant of the three.

3. If a body, acted on by several forces, move in a straight line with uniform velocity, what conditions must the forces satisfy?

4. Find the acceleration down a smooth inclined plane.

5. Two bodies start from rest, one down a smooth inclined plane and the other falling freely. Prove that either body, as seen by a person moving with the other, appears to be moving from the observer in a straight line perpendicular to the plane with uniform acceleration.

6. A mass of 6 oz. slides down a smooth inclined plane, whose height is half its length, and draws another mass from rest over a distance of 3 feet in 5 seconds, along a horizontal table which is level with the top of the plane, the string passing over the top of the plane. Find the mass on the table.

7. Prove that, if a particle slide down a smooth inclined plane, the kinetic energy acquired is the same as if it had fallen vertically through an equal height.

8. A number of smooth rods meet in a point A, and rings placed on them slide down the rods, starting simultaneously from A. Prove that, after the time t, the rings are all on the surface of a sphere with radius $\frac{1}{4}gt^2$.

9. Find the retardation of a body projected down an inclined plane, supposing the tangent of the inclination to be less than the coefficient of friction.

10. A particle on a rough plane inclined at an angle α to the horizon is on the point of motion; if the plane were inclined at an angle β to the horizon, its acceleration would be doubled by making the plane smooth; prove that $\tan\beta = 2\tan\alpha$.

CHAPTER XV.

PARABOLIC MOTION.

213. **Properties of a parabola.**—In this chapter we shall consider more fully the motion of projectiles, to which reference was made in §§ 183–188 of Chap. XII. Before doing this it will be necessary to explain what is meant by a parabola, and to deduce one or two of the simpler properties of the curve.

Definition.—**A parabola** is the locus of a point which moves so that its distance from a given line is always equal to its distance from a given point.

The given line is called the **directrix** of the parabola; the given point is called its **focus.** The line through the focus perpendicular to the directrix is called its **axis.**

The curve is not a closed curve, but is of the shape represented in Fig. 56. (The branches AL and AK of the curve are both continued indefinitely.)

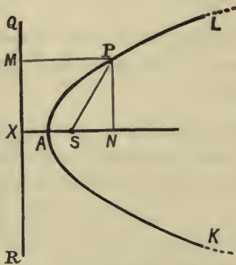


Fig. 56.

214. Let QR be the directrix, S the focus, SX the axis. Bisect SX at A. Then, since the distance of A from S = distance of A from QR, $\therefore$ A is one point on the locus. Let P be any other point, and PM the perpendicular on QR. Then, since P is on the locus, $SP = PM$.

Draw PN perpendicular to the axis.

Then $NP^2 = SP^2 - SN^2 = PM^2 - SN^2$
$= (AN + AX)^2 - (AN - AS)^2 = (AN + AS)^2 - (AN - AS)^2,$

i.e., $$NP^2 = 4AS \, . \, AN.$$

This is one of the fundamental properties of the parabola.

Definition.—The point A in which the curve cuts the axis is called the **vertex** of the parabola. The chord through S perpendicular to the axis is called the **latus rectum.** It is easily seen (from the definition of a parabola) that the latus rectum $= 2SX = 4AS$.

Cor.— $$\frac{NP^2}{AN} = 4AS = \text{a constant quantity.}$$

Hence, *conversely, if P move in such a way that $NP^2/AN = k$* (where k is any constant), *then the locus of P is a parabola, whose vertex is at A.* Also (since k corresponds to $4AS$) we see that *the distance of the focus from the vertex* $= k/4$.

215. **Horizontal and vertical motions of a projectile.**—Suppose a body projected with velocity U in a direction inclined to the horizon at an angle α. Then, resolving this velocity vertically and horizontally (§ 172), we have initial horizontal velocity $= U \cos \alpha$,

initial vertical velocity $= U \sin \alpha$.

Now let us consider the vertical and the horizontal motions separately. We have

initial horizontal velocity $= U \cos \alpha$,

horizontal acceleration $= 0$.

Thus the horizontal velocity at any time during the motion $= U \cos \alpha$ .... **(1)**,

and the horizontal space described in t secs.

$= Ut \cos \alpha$ .... (§ 14) **(2)**.

Again, initial vertical velocity $= U \sin \alpha$,

vertical acceleration $= -g$.

Hence, vertical velocity after t secs.

$= U \sin \alpha - gt$ .... (§ 32) **(3)**;

also, vertical space described in t secs.

$= Ut \sin \alpha - \frac{1}{2}gt^2$ (§ 45, equation 7) ...... **(4)**.

These four formulæ are most important, and they agree with those of § 186 on putting $u = U \cos \alpha$, $v = U \sin \alpha$.

216. DEFINITIONS.—Let a particle be projected from a point O, with velocity U, whose direction is inclined to the horizon at an angle α.

Let A be the highest point of its path, and let it fall to the ground again at C (Fig. 57). Draw AB perpendicular to OC.

Then OC is called the **horizontal range** of the projectile, and the time taken to reach OC is called the **time of flight.**

217. To find the time to the highest point—Let t_1 be the time taken in reaching A. Then the velocity of

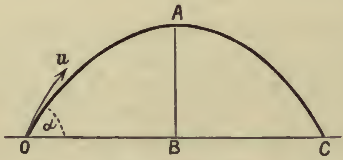


Fig. 57.

the particle at A is horizontal ; $\therefore$ vertical velocity after t_1 secs. is zero ; *i.e.*,

$$U \sin \alpha - g t_1 = 0, \qquad (\S\ 215)$$

whence $$t_1 = \frac{U \sin \alpha}{g} \quad \text{.......................} \quad \textbf{(5)}.$$

218. To find the greatest height to which the body rises.

$$AB = \text{vertical space described in } t_1 \text{ secs.}$$

$$= U t_1 \sin \alpha - \tfrac{1}{2} g t_1^2 = \frac{U^2 \sin^2 \alpha}{g} - \frac{1}{2} \frac{U^2 \sin^2 \alpha}{g}$$

$$= \frac{U^2 \sin^2 \alpha}{2g} \quad \text{....................} \quad \textbf{(6)}.$$

This result also follows from § 187 on putting $v = U \sin \alpha$.

219. To find the time of flight.—Let t_2 secs. be the time of flight. At C the particle is on the same horizontal level as at O; hence the vertical space described in t_2 secs. is zero, *i.e.*,

$$Ut_2 \sin \alpha - \tfrac{1}{2} g t_2^2 = 0;$$

whence

$$t_2 = \frac{2U \sin \alpha}{g} \quad \text{..................} \quad (7).$$

Cor.—Hence $t_2 = 2t_1$; that is, the projectile takes the same time from O to A as from A to C.

220. To find the horizontal range.

$$OC = \text{horizontal space described in } t_2 \text{ secs.}$$

$$= Ut_2 \cos \alpha = \frac{2U^2 \sin \alpha \cos \alpha}{g} = \frac{U^2 \sin 2\alpha}{g} \quad \text{...} \quad (8).$$

If u and v are the horizontal and vertical components of U, this expression assumes the form of (5) §188, viz., $2uv/g$.

Cor.— OB = horizontal space described in t_1 secs.

$$= Ut_1 \cos \alpha = \tfrac{1}{2} Ut_2 \cos \alpha = \tfrac{1}{2} OC.$$

Hence

$$OB = BC.$$

221. Greatest horizontal range.—*Given the initial velocity of a projectile, to find what angle of projection will give the greatest range.*

The range is $U^2 \sin 2\alpha / g$. This expression is greatest when $\alpha = 45°$: for then $\sin 2\alpha = 1$; in all other cases $\sin 2\alpha < 1$. Thus the greatest possible range, with initial velocity U, is

$$U^2/g \quad \text{......................} \quad (9).$$

This is *twice* the height ($U^2/2g$) to which the body would rise if projected *vertically* upwards with velocity U (see § 64).

When projected at an angle of 45°, so as to give the greatest horizontal range, the greatest height is $U^2 \sin^2 45°/2g$, or $U^2/4g$, and is one quarter of the range.

Cor.—Hence the greatest distance to which a cricket ball can be thrown is twice the greatest height to which it can be thrown up into the air.

222. The path of a projectile is a parabola, *and the highest point of the path is the vertex of the parabola.*

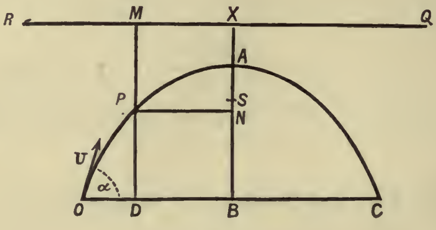


Fig. 58.

Let P be the position of the projectile after t secs.; and let O, A, B, and C represent the same points as in § 217. Draw PN and PD perpendicular to AB and OB respectively.

Then OD = horizontal space described in t secs.

$$= Ut\cos\alpha,$$

PD = vertical space described in t secs.

$$= Ut\sin\alpha - \tfrac{1}{2}gt^2;$$

$$\therefore\quad AN = AB - PD = \frac{U^2\sin^2\alpha}{2g} - Ut\sin\alpha + \tfrac{1}{2}gt^2$$

$$= \frac{U^2\sin^2\alpha - 2Ugt\sin\alpha + g^2t^2}{2g} = \frac{(U\sin\alpha - gt)^2}{2g}.$$

Also

$$PN = OB - OD = \tfrac{1}{2}OC - OD$$

$$= \frac{U^2\sin\alpha\cos\alpha}{g} - Ut\cos\alpha$$

$$= \frac{U\cos\alpha}{g}(U\sin\alpha - gt).$$

Hence

$$\frac{PN^2}{AN} = \frac{2U^2\cos^2\alpha}{g} = \text{a constant quantity};$$

for $2U^2\cos^2\alpha/g$ does not contain t, and therefore has the same value for every position of P.

Hence from § 214, Cor., it follows that *the locus of P is* *a parabola; that A is its vertex, and AB its axis; and that the distance of the focus from A is*

$$\tfrac{1}{4}\frac{2U^2\cos^2\alpha}{g} = \frac{U^2\cos^2\alpha}{2g}.$$

Thus, if from AB we cut off

$$AS = \frac{U^2\cos^2\alpha}{2g} \quad\text{.................. } \mathbf{(10)},$$

S will be the focus of the parabola.

223. The height of the directrix above the point of projection is $U^2/2g$.

To construct the directrix, produce SA to X, making $SA = AX$; and through X draw QR perpendicular to SX. Then QR is the directrix. (*Cf.* § 214.)

Hence the height of the directrix above the point of projection $= XB = XA + AB$

$$= \frac{U^2\cos^2\alpha}{2g} + \frac{U^2\sin^2\alpha}{2g} = \frac{U^2}{2g} \quad\text{......... } \mathbf{(11)}.$$

COR. 1.—If the body were projected vertically, it would rise to a height $U^2/2g$. Hence *the height of the directrix above the ground is the height to which the body would have risen had it been projected vertically.*

COR. 2.—If several bodies were projected from the same point with the same velocity U, but in different directions, the directrix of each path would be at a height $U^2/2g$ above the point of projection; *i.e.*, *the parabolas which the several particles described would have a common directrix.*

224. The velocity of the projectile at any point of its path *is of the same magnitude as if it had fallen to that point from the directrix.*

Let P be the position of the projectile after t secs. Draw PM perpendicular to the directrix (Fig. 58). Let V be the velocity at P. Then V is the resultant of a horizontal velocity $U\cos\alpha$, and a vertical velocity

$$U\sin\alpha - gt;$$

$$\therefore \quad V^2 = U^2\cos^2\alpha + (U\sin\alpha - gt)^2$$
$$= U^2 - 2Ugt\sin\alpha + g^2t^2.$$

Had the particle been dropped from M, its velocity at P would be equal (*cf.* § 60) to

$$\begin{aligned}\sqrt{(2g \cdot MP)} &= \sqrt{\{2g\,(BX-PD)\}} \\ &= \sqrt{\{2g\left[U^2/2g-(Ut\sin\alpha-\tfrac{1}{2}gt^2)\right]\}} \\ &= \sqrt{(U^2-2Ugt\sin\alpha+g^2t^2)} = V,\end{aligned}$$

which proves the proposition.

*225. **The path of a projectile is a parabola.** (*Alternative proof.*)

Let A be the highest point to which the projectile rises, and let u be the velocity at A. Then the direction of the velocity u is horizontal. Hence, since the horizontal velocity is uniform, u is the horizontal component of the velocity at every point of the path.

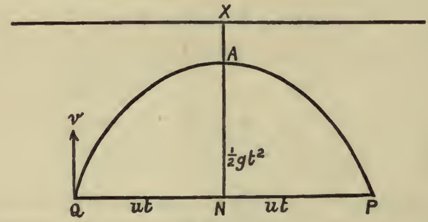


Fig. 59.

Firstly, let P be the position of the particle t secs. *after* reaching the highest point. Drop PN perpendicular on the vertical through A.

Then

NP = horizontal distance described in t secs. $= ut$ ... (i.),

AN = vertical distance fallen in t secs. $= \frac{1}{2}gt^2$ .... (ii.)

(remembering that the vertical velocity at A is zero).

Eliminating t, we have

$$\therefore \quad \frac{NP^2}{AN} = \frac{u^2t^2}{\frac{1}{2}gt^2} = \frac{2u^2}{g}, \quad \text{or} \quad NP^2 = \frac{2u^2}{g}AN.$$

Hence the locus of P in the downward motion from A is part of a parabola whose latus rectum is $u^2/2g$.

Secondly, let Q be the position of the projectile t secs. *before* reaching A. Draw the perpendicular QN, and let v be the vertical component of the velocity at Q.

Then, in t secs. after leaving Q, the projectile will be at A, and its vertical velocity will then be zero.

$\therefore$ QN = horizontal distance described in t secs. $= ut$...(i.),

AN = vertical distance risen in t secs. $= vt - \frac{1}{2}gt^2$ (ii.).

Also $v - gt$ = vertical velocity at $A = 0$ .... (iii.).

Hence, $v = gt$, and, by substituting for v in (ii.), we have

$$AN = \tfrac{1}{2}gt^2 \text{ .... (iv.).}$$

Eliminating t from (i.) and (iv.), we have, as before,

$$QN^2 = \frac{2u^2}{g} AN.$$

Hence the locus of Q in the upward motion to A is part of the same parabola as before.

$\therefore$ the complete path of the projectile is a parabola.

*226. **To find the velocity at any point of the path in terms of the depth of the point below the directrix.** (*Alternative Proof.*)

(i.) Let the vertical through A meet the directrix in X. Then we know that, if u is the horizontal velocity,

$$AX = \frac{1}{4}(\text{latus rectum}) = \frac{1}{4}\frac{2u^2}{g} = \frac{u^2}{2g};$$

$$\therefore \quad u^2 = 2g \,.\, XA,$$

or $$\text{velocity at } A = \sqrt{(2g \,.\, XA)}.$$

(ii.) If v is the vertical velocity at a point P after passing A, v is the velocity due to falling through the vertical height AN, and therefore $v^2 = 2g \,.\, AN$.

Hence the square of the resultant velocity at P

$$= u^2 + v^2 = 2g \,.\, XA + 2g\, AN = 2g \,.\, XN,$$

or $$\text{velocity at } P = \sqrt{(2g \,.\, XN)}.$$

(ii.) Similarly, if v is the vertical velocity at Q, v is the velocity required to carry the projectile to a height NA, and therefore $v^2 = 2g \cdot AN$, giving as before

$$\text{velocity at } Q = \sqrt{(2g \cdot XN)}.$$

Hence in every case the velocity at any point is that which would be acquired in falling vertically from the directrix to that point.

Examples.—(1) A ball is thrown with velocity 100 ft. per sec. in a direction inclined at an angle of 60° to the horizon. Find where it will strike a cliff distant 150 ft. from the point of projection.

Suppose the ball to start from O, and to strike the cliff QR at a point P after t secs. Let $PQ = x$. (Fig. 60.)

Then horizontal space described in t secs. = 150 ft.; also vertical space described in t secs. $= x$.

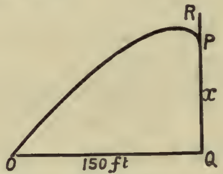


Fig. 60.

Hence $Ut \cos\alpha = 150$, $Ut \sin\alpha - \frac{1}{2}gt^2 = x$. Substituting the known values of U and α, we have

$$U\cos\alpha = 100 \cos 60° = 50,$$

$$U\sin\alpha = 100 \sin 60° = 100 \times \tfrac{1}{2}\sqrt{3} = 50\sqrt{3};$$

$$\therefore\ 50t = 150,\quad 50t\sqrt{3} - 16t^2 = x;$$

whence $$t = 3,\quad x = 150\sqrt{3} - 144 = 115{\cdot}8.$$

(2) Two seconds after its projection a projectile is travelling in a direction inclined at 30° to the horizon; after one more second it is travelling horizontally. Determine the magnitude and direction of its initial velocity.

Let U and α have their usual meanings. The velocity after 2 secs. is the resultant of a vertical velocity $U\sin\alpha - 2g$ and a horizontal velocity $U\cos\alpha$. The direction of the resultant velocity makes an angle 30° with the horizon. Hence, by § 170,

$$\frac{U\sin\alpha - 2g}{U\cos\alpha} = \tan 30°$$

$$= \frac{1}{\sqrt{3}} \quad \ldots\ldots\ldots\ldots\ldots \quad (a).$$

Again, after 3 secs. the vertical velocity is zero; *i.e.*, $U \sin\alpha - 3g = 0$; whence $U \sin\alpha = 3g$.

Substituting this value for $U\sin\alpha$ in (a), we obtain $U\cos\alpha = g\sqrt{3}$.

Thus $$U^2 = U^2\cos^2\alpha + U^2\sin^2\alpha = 9g^2 + 3g^2;$$

whence $$U = 2g\sqrt{3}.$$

Also $$\tan\alpha = \frac{U\sin\alpha}{U\cos\alpha} = \sqrt{3}; \quad \text{whence} \quad \alpha = 60°.$$

227. **Range on an inclined plane.**—*A body is projected with velocity u, at an inclination α to the horizon, from the foot of a plane inclined at an angle β to the horizon. Determine its range up the plane.*

Let O be the point of projection, OC the range up the plane (Fig. 61). Through C draw the vertical CD, meeting the horizontal through O at D. Let $OC = x$, and let the time of flight be t.

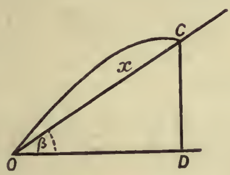


Fig. 61.

Then $$CD = x\sin\beta, \quad OD = x\cos\beta.$$

Also, horizontal space described in t secs. $= OD$,

i.e., $$ut\cos\alpha = x\cos\beta \quad \text{.......} \quad (a).$$

Again, vertical space described in t secs. $= CD$,

i.e., $$ut\sin\alpha - \tfrac{1}{2}gt^2 = x\sin\beta \quad \text{.......} \quad (b).$$

Find t from (a), substitute in (b), and solve the resulting equation for x. We then obtain

$$x = 2u^2\cos\alpha\,\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{g\cos^2\beta}$$

$$= u^2\,\frac{2\cos\alpha\,.\,\sin(\alpha-\beta)}{g\cos^2\beta} = u^2\,\frac{\sin(2\alpha-\beta) - \sin\beta}{g\cos^2\beta}.$$

Cor.—To find what value of α will make this expression a maximum:—$\sin(2\alpha-\beta)$ must have its maximum value; hence

$$2\alpha - \beta = 90°; \; i.e., \; \alpha = \tfrac{1}{2}(\beta + 90°).$$

It is easy to see that in this case the direction of projection bisects the angle between the inclined plane and the vertical.

SUMMARY OF RESULTS.

At t secs. after the projection of a projectile,

horizontal velocity $= U\cos\alpha$ .... (1),

horizontal space travelled $= tU\cos\alpha$ .... (2),

vertical velocity $= U\sin\alpha - gt$ ...... (3),

vertical space travelled $= tU\sin\alpha - \frac{1}{2}gt^2$... (4).

Time to highest point $= U\sin\alpha/g$......... (5).

Greatest height $= U^2\sin^2\alpha/2g$ ... (6).

Time of flight $= 2U\sin\alpha/g$ ...... (7).

Range $= U^2\sin 2\alpha/g$...... (8).

Greatest range $= U^2/g$ .... (9).

Distance of focus below highest point

$= U^2\cos^2\alpha/2g$ ... (10).

Height of directrix $= U^2/2g$ .... (11).

Velocity at any point equals that due to falling from the directrix to that point. (§§ 224, 226.)

EXAMPLES XV.

NOTE.—**tan**$^{-1}\frac{5}{12}$ denotes *the angle whose tangent is* $\frac{5}{12}$; similarly, **cos**$^{-1}\frac{3}{5}$ denotes *the angle whose cosine is* $\frac{3}{5}$, and so on.

1. Find the range, greatest height, and time of flight, of projectiles thrown with the following velocities, in directions inclined to the horizon at the following angles:—(i.) 640 ft./sec., 30°; (ii.) 100 ft./sec., 45°; (iii.) 1600 ft./sec., 60°; (iv.) 416 ft./sec., $\tan^{-1}\frac{5}{12}$; (v.) 800 ft./sec., $\cos^{-1}\frac{3}{5}$.

2. A particle is projected at an angle of 30°, with velocity 192 ft./sec. When will it be at a height of 80 ft. above the ground, and what will be the distance from the point of projection at that instant?

3. A body is projected from the ground with velocity 160 ft./sec. at an angle of 60°. At what height will it strike a cliff distant 120 ft. from the point of projection?

4. Find the velocity with which a stone must be thrown in order to strike horizontally the top of a cliff h ft. high, at a distance d ft. from the point of projection.

5. Two particles projected from different points in the same horizontal plane at the same instant meet in the air. Prove that their initial vertical velocities are equal.

6. A particle is projected from the ground straight up a smooth inclined plane. If it goes over the top of the plane, prove that it falls to the ground with a velocity equal to its initial velocity. (Use § 224.)

7. A series of particles are projected in a room from the same point with the same velocity in different directions. Prove that all of them which strike the ceiling strike it with the same velocity.

8. A body is projected from the ground in a direction inclined to the horizon at the angle 60°. Find velocity of projection, given that at a height of 96 ft. the direction of motion is 30° from the horizontal.

9. Find the smallest velocity with which a body may be projected in order to have a range of 100 ft. on a horizontal plane.

10. Given range = 100 ft., greatest height = 100 ft.; find the velocity of projection.

11. Given velocity of projection = 100 ft./sec., greatest height = 100 ft.; find the range.

12. Given velocity of projection = 50 ft./sec., range = 42 ft.; find the greatest height.

13. Express the velocity of projection in terms of the range and greatest height.

14. Determine the least velocity with which a ball can be thrown, so that its range up an incline of 30° should be 768 ft. (Use § 227, Corollary.)

15. Determine the least velocity with which a ball can be thrown to reach the top of a cliff 128 ft. high and $128\sqrt{3}$ ft. away from the thrower.

16. The direction of motion of a projectile at a certain instant is inclined at an angle α to the horizon; after t secs. it is inclined at an angle β. Prove that the horizontal component of the velocity of the projectile is $gt/(\tan\alpha - \tan\beta)$.

17. Given u the velocity of projection, and v the velocity at the highest point; determine the greatest height.

18. Find the latus rectum of the path of a projectile *in vacuo*, having given that at a certain instant the projectile is moving with velocity u in a direction inclined at an angle 45° to the horizon.

19. Smooth heavy particles are let fall simultaneously down chords of a vertical circle from its highest point. Show that they all reach the circumference again at the same instant and that their subsequent parabolic paths have the same directrix.

CHAPTER XVI.

CIRCULAR MOTION.

228. From our general definitions of acceleration it follows that, when the direction of motion of a body is constantly changing, it is subject to some acceleration even if its speed constantly remains the same. We shall now investigate this acceleration, considering particularly the case of a body moving in a circle. It will be convenient to regard this as the limiting case of that of a body moving along the sides of a polygon inscribed in the circle.

229. **Motion round the sides of a regular polygon.** —*A particle of mass m is moving with constant speed v round the perimeter of a regular polygon inscribed in a circle of radius r. To find the impulsive force on the particle at each angular point of the polygon.*

Let $ABCD$ ... be the polygon (Fig. 62), O the centre of the circle, t the time taken to describe each side. Then the length of each side is vt.

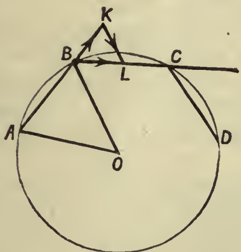


Fig. 62.

While the particle is moving with uniform velocity along AB, it follows from Newton's First Law that it is not under the action of any force. The sudden change in the direction of the velocity which occurs at B must be produced by an impulsive force. It is required to determine the magnitude and direction of its impulse.

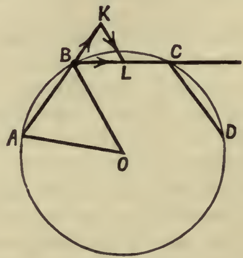


Fig. 63.

Join AO, BO; produce AB to a point K, such that BK contains v units of length. Draw KL parallel to BO, meeting BC in L.

Then, since the polygon is regular, BO bisects the angle ABC.

Also $\quad \angle BKL = \angle ABO$ (by parallels)

$$= \angle OBL = \angle BLK \text{ (by parallels)};$$

$$\therefore \quad BL = BK = v.$$

Thus BK and BL represent the velocities with which the particle travels along AB and BC respectively.

Now, the velocity of a particle after being subjected to an impulse is the resultant of the velocity of the particle before the impulse, and the velocity communicated by the impulse.

But, by the triangle of velocities, the velocity BL is the resultant of velocities BK and KL; hence KL *represents the velocity communicated to the particle by the impulse at* B; $\therefore$ the direction of the impulse at B is parallel to KL, *i.e.*, along BO; and its magnitude is $m \times KL$ (for it produces velocity KL in mass m).

Now, the triangles OAB and BKL are similar; for the four angles OAB, OBA, BKL, and BLK are all equal.

Thus $$\frac{KL}{BK}=\frac{AB}{AO}, \quad i.e., \quad \frac{KL}{v}=\frac{vt}{r};$$

whence $$KL=\frac{v^2t}{r}.$$

Thus the impulse at B is mv^2t/r toward the centre.

230. Motion in a circle with constant speed.—*A particle is travelling with constant speed v round the circumference of a circle; to determine its acceleration and the force which is acting upon it, and the time of revolution.*

Suppose that in the problem of the last paragraph we increase the number of sides indefinitely, keeping the values of v and r unaltered. Then t will become infinitely small, since the sides have become infinitely short. The perimeter of the polygon will now coincide with the circumference of the circle, and the series of small impulsive forces, each acting toward the centre, occurring in very rapid succession, will be equivalent to a continuous force always acting toward the centre.

This force produces an acceleration (say f) toward the centre, and the velocity ft produced by this acceleration in each time t must be the same as that produced at each corner of the polygon.

Thus $$ft=KL=\frac{v^2t}{r};$$

whence $$f=v^2/r \qquad \text{(1)}.$$

Thus *a particle describing a circle of radius r with velocity v has an acceleration v^2/r toward the centre.*

The *force* producing this acceleration in mass m must be a force mv^2/r toward the centre.

Again, let T be the time of revolution.

Then $$vT = \text{total length of path described}$$
$$= \text{circumference of circle} = 2\pi r;$$

$$\therefore \quad T=\frac{2\pi r}{v} \qquad \text{(2)}.$$

Note 1.—It is important to notice that the results of this paragraph and the preceding are based on the principle of the Physical Independence of Force (§ 190).

Cor.—Hence also $v = 2\pi r/T$,

and acceleration to centre $= \frac{4\pi^2}{T} r$ .... (**1**a).

231. **Centrifugal Force.**—It will be seen that if a particle is moving in a circle, there is an apparent tendency in the particle to leave the circular path, necessitating a force directed towards the centre to keep it on the circumference of the circle. To this force there has been applied the somewhat misleading name of **centripetal force.**

This force, whose magnitude is mv^2/r, must be due to the action of some other body. And, since action and reaction are equal and opposite, the particle exerts on the latter body a force mv^2/r *away* from the centre, and this is called **centrifugal force.**

In reality, the tendency of the particle at any point is (by Newton's First Law) to continue moving in the same straight line, that is, along the tangent to the circle at that point. This is spoken of as the **tendency to fly off at a tangent.** The force to the centre is necessary to make the particle deviate from the direction of the tangent, and continue to move along the curve.

Applications.—If a particle of mass m, not under gravity, is revolving with velocity v round a fixed point to which it is attached by an inextensible string of length r, the tension of the string is mv^2/r. Hence the particle exerts on the string an outward force mv^2/r, which is called the **centrifugal force.**

If a bead is travelling round a smooth circular wire, the same formula gives the reaction of the wire on the bead. The action of the bead on the wire is the centrifugal force in this case.

If a planet is describing a circular orbit round the Sun, the same formula gives the force with which the Sun attracts the planet.

Examples.—(1) A railway carriage of mass 1,000 lbs. is travelling with velocity 50 ft./sec. round a curve. If the radius of the curve be 1,000 ft., find the magnitude and direction of the resultant thrust on the rails.

The reaction of the rails must (i.) support the weight of the train; (ii.) supply the necessary force, mv^2/r poundals, toward the centre of the circle.

The first requires a vertical force of 1,000 lb. wt. The second requires a horizontal force toward the centre of the circle of

$$\frac{1000 \times 50^2}{1000} \text{ pdls.} = \frac{625}{8} \text{ lb. wt.}$$

The reaction of the rails is therefore the resultant of these two forces at right angles, $\sqrt{\{(1000)^2 + (\frac{625}{8})^2\}}$ lb. wt.

The direction of the resultant reaction is obviously inclined to the vertical at an angle whose tangent is

$$\tfrac{625}{8} \div 1000, \text{ or } \tfrac{5}{64}.$$

NOTE.—If the thrust between the carriage and the rails is not at right angles to the plane of the rails, there will be a tendency in the carriage either to wrench the rails from the sleepers, or more probably to run off the lines. To prevent this, in constructing a curved piece of railway, the outer rail is raised to a somewhat higher level than the inner, so that for a train travelling at average speed the plane of the rails is again perpendicular to the thrust between the carriage and the rails.

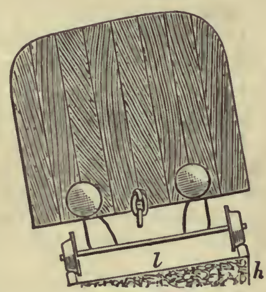


Fig. 64.

(2) If the rails are 4 ft. $8\frac{1}{2}$ ins. apart, to find how much the outer rail must be raised so that the carriage of Ex. 1 may press perpendicularly on the rails.

The plane of the rails must evidently make an angle of $\tan^{-1}\frac{5}{64}$ with the horizon. Calling this θ, the outer rail must be raised 4 ft. $8\frac{1}{2}$ ins. $\times \sin\theta$,

$$56\tfrac{1}{2} \times \frac{5}{\sqrt{(64^2+5^2)}} \text{ ins.} = 56\tfrac{1}{2} \times \tfrac{5}{64} \text{ ins. roughly, or nearly } 4\tfrac{1}{2} \text{ ins.}$$

232. **Angular velocity.**—When a particle P is moving in a plane, the line OP joining it to a fixed point O in that plane will, in general, revolve about the fixed point. The exception would be in the case where the particle P is moving either directly towards, or directly from, the fixed point O; the joining line will then obviously be stationary.

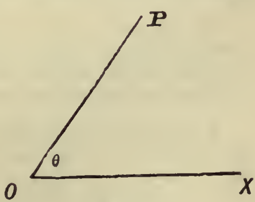


Fig. 65.

DEFINITION.—The **angular velocity** of a moving point about a fixed point is the *rate* (per unit time) *at which the line joining the two points is describing angles* about its fixed extremity.

If OX is a fixed line through O and P is moving in the plane XOP, the angular velocity of P is measured by the rate at which the angle XOP is increasing.

If the angles described in equal times be equal and in the same plane, however short these equal times be made, the angular velocity is said to be **uniform,** and it can be measured by the angle actually described in a unit of time.

In other cases the angular velocity is said to be **variable,** and must be measured in a similar way to variable velocity or variable acceleration by dividing the angle described in an infinitesimally short interval of time by the duration of that interval.

It follows from what has been said that the angular velocity of a moving particle about a fixed point will be zero if the direction of motion of the former passes through the latter.

233. **Unit of angular velocity.**—Angular velocities are usually measured in circular measure, *i.e.*, in **radians per second,** thus the **unit of angular velocity is the radian per second.**

In certain cases it may be convenient to measure angular velocity in *degrees per second* instead.

It is readily seen that, if θ be the angle described in time t by a particle revolving about a fixed point with uniform angular velocity w, then

$$\theta = wt \qquad \text{(3)}.$$

This gives $w = \theta/t$. If the angular velocity be variable, θ/t is the average angular velocity in any interval t; and, if this interval be made infinitely small, the average angular velocity becomes the angular velocity **at** the corresponding instant of time.

Again, in one complete revolution, four right angles are described, and the circular measure of four right angles is 2π (radians). Hence, if T be the time of revolution,

$$wT = 2\pi, \quad \text{or} \quad T = 2\pi/w.$$

234. **Theorem.**—*If a particle is describing a circle of radius r with constant speed v, and, if w be the angular velocity of the particle about the centre, then* $v = wr$.

Let O be the centre of the circle (Fig. 66), and let AB be the arc described in t secs. Then arc $AB = vt$ and

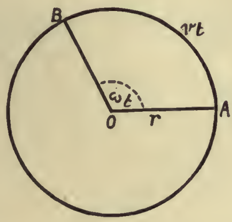


Fig. 66.

$\angle AOB = wt$ radians. But the arc AB divided by the radius gives the circular measure of the $\angle AOB$, *i.e.*,

$$vt/r = wt, \quad \text{or} \qquad \frac{v}{r} = w \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(4)}.$$

Cor.—The acceleration of the particle toward the centre

$$= \frac{v^2}{r} = w^2 r \ldots\ldots\ldots\ldots\ldots\ldots\ldots \textbf{(5)}.$$

Note.—If the speed in the circle is variable, the velocity and angular velocity *at any instant* are connected by the relation $v = wr$, as is evident, by making t infinitesimal.

235. **Motion round a circle with variable speed.**—If a particle is describing a circle with variable speed, and if v is its speed at any given instant, the particle at that instant will have an acceleration v^2/r toward the centre. If the speed along the curve is changing at that instant, it will also have an independent acceleration along the tangent to the curve.

This acceleration is measured by the rate of change of the speed with which the particle is moving.

236. **Angular velocity of a particle moving in a straight line.**—*A particle is moving with uniform velocity v along a given straight line; to determine its angular velocity at any instant about a given point.*

Let P be the position of the particle at the given instant, PQ the given straight line, O the given point (Fig. 67).

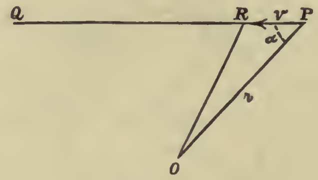


Fig. 67.

Let R be the position of the particle after time t, and w the *average* angular velocity of the particle about O during this time t. Let $\angle OPQ = \alpha$ and distance $OP = r$.

Then $PR = vt$; also $\angle POR =$ angle described in time t $=$ average angular velocity $\times$ time $= wt$.

But $$\frac{PR}{\sin POR} = \frac{RO}{\sin RPO}, \quad i.e., \quad \frac{vt}{\sin wt} = \frac{RO}{\sin \alpha}.$$

Now, suppose t to be indefinitely diminished, then the average angular velocity between P and R will become the *actual angular velocity at* P.

But, if t be indefinitely diminished,

the limit of $(\sin wt) \div wt = 1$ (by trigonometry);

hence $$\text{the limit of } \frac{vt}{\sin wt} = \frac{v}{w}.$$

Also $$OR = OP = r$$

(since PR is indefinitely small).

Hence $$\frac{v}{w} = \frac{r}{\sin \alpha}.$$

Whence $$w = \frac{v \sin \alpha}{r} \quad \text{.................. (6).}$$

Alternative proof.—Resolve the velocity v into its two components $v\cos\alpha$ along PO and $v\sin\alpha$ perpendicular to PO. The former produces no angular velocity about O (§ 232), and the latter gives rise to an angular velocity $w\sin\alpha \div r$ about O (§ 234). Hence $w = v\sin\alpha \div r$, as before.

Cor. 1.—The angular velocity about O will be a maximum when the particle is at the foot of the perpendicular from O to PQ. For then $\alpha = 90°$, and therefore $\sin\alpha$ is a maximum, and also r is a minimum.

Cor. 2.—Drop OM perpendicular on PQ.

Then $\sin\alpha = OM/OP$, and $\therefore$ $w = v\,.\,OM/OP^2$.

Hence the angular velocity varies inversely as the square of the distance from O.

237. **Motion in a smooth curve under gravity.**—*A body is sliding down a smooth wire or tube of any shape. To deduce its velocity at any point from the principle of Conservation of Energy.*

Suppose a bead of mass m to slide down a smooth wire of any shape from A to B. Let u and v be its velocities

Fig. 68.

at A and B respectively, and let AC $(=h)$ be the vertical height of A above B (Fig. 68).

Then, since no work is done in overcoming friction, the gain in kinetic energy in passing from A to B = loss in potential energy = work done by gravity;

$$\therefore \quad \tfrac{1}{2}m(v^2-u^2) = mgh;$$

whence $\quad v^2-u^2 = 2gh, \quad$ or $\quad v^2 = u^2+2gh \quad \ldots\ldots \quad (7).$

Cor.—The velocity at B is independent of the shape of the curve between A and B.

Note.—The investigation applies equally to a bead constrained to slide down a smooth wire, or to a small body allowed to slide, *without rolling*, in the interior of a smooth tube, or on the surface of any smooth body; but it is supposed in every case that no friction exists.

Example.—A bead of mass 1 oz. is free to move on a fixed smooth circular wire of radius 1 ft. whose plane is vertical. It starts from rest at one end of the horizontal diameter; find the thrust between the bead and the wire at the lowest point.

Let A be the starting point of the bead, B the lowest point of the wire, and O the centre (Fig. 69).

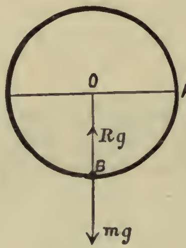


Fig. 69.

(a) To determine the velocity at B, apply the formula of the last paragraph, remembering $g = 32$.

Then $u = 0,\ h = OB = 1.$

Whence $v^2 = 2g \cdot OB = 2 \times 32 \times 1,$

giving $v = 8.$

(b) Let R lbs. weight be the reaction of the wire at B, acting along BO.

The only other force acting on the bead is its weight.

The acceleration of the bead is v^2/r, *i.e.*, $8^2/1$ or 64 ft./sec.^{2} along BO.

$\therefore$ resultant force along $BO = (Rg - mg)$ poundals.

Hence (from the formula $P = mf$)

$$Rg - mg = m \cdot 64; \quad \text{also} \quad m = \tfrac{1}{16};$$

whence $R = \frac{3}{16}$, or the thrust $= \frac{3}{16}$ lb. wt. $= 3$ oz. wt.

The same result could have been obtained as follows:—

The force of pressure of the bead on the wire at B is due (i.) to its weight, (ii.) to the centrifugal force.

The weight $= \frac{1}{16}$ lb. wt. $= 1$ oz. wt.
The centrifugal force $= mv^2/r$ pdls.
$= \frac{1}{8}$ lb. wt. $= 2$ oz. wt.
Hence, force of pressure $= 2 + 1 = 3$ oz. wt.

SUMMARY OF RESULTS.

If a particle is describing a circle with velocity v,
the acceleration towards the centre $= v^2/r$ .... (1),
the force towards the centre $= mv^2/r$, (§ 230)
and the time of a complete revolution $= 2\pi r/v$ ...... (2).

If a particle is describing a circle of radius r, with angular velocity w, the velocity $= wr$............ (4);
and the acceleration towards the centre $= w^2 r$ .... (5).

Angular velocity of any particle about a given point

$$= \frac{v \sin \alpha}{r} \text{ .... (6).}$$

Velocity of a bead sliding down a smooth wire or tube is given by $v^2 = u^2 + 2gh$ .... (7).

EXAMPLES XVI.

1. A body of mass 3 lbs. (not under gravity) is describing a circle round a point to which it is attached by a string 3 ft. long. If it makes 7 revolutions per second, find the tension of the string. [$\pi = \frac{22}{7}$.]

2. If the tension of the string in the last question is 18 lb. wt., find the number of revolutions per second.

3. An engine of mass 1 ton is travelling round a curve at the rate of 30 miles an hour. If the curve is an arc of a circle whose radius is 1210 ft., determine the horizontal thrust between the engine and the rails.

4. Two particles of equal mass are describing circles round fixed points to which they are attached by inextensible strings. Prove that, if the times of revolution are the same, the tensions of the strings are proportional to their lengths.

5. Two particles of mass $2m$ and m respectively are revolving round fixed points to which they are attached by strings of length l and $2l$ respectively. If the tensions in the strings are equal, compare the times of revolution of the particles.

6. A particle hanging, by a string 8 ft. long, from a fixed point is pulled aside till the string makes an angle of 60° with the vertical, and then let go. When the particle is passing through its equilibrium position, compare the tension of the string with the weight of the particle.

7. A smooth circular tube of radius 2 ft., whose plane is vertical, contains a particle of mass 1 oz. If the particle slide from rest at the highest point, find its velocity and the thrust between the particle and the curve at the lowest point.

8. In the last question, determine the thrust between the particle and the tube at the end of the horizontal diameter.

9. If the particle in Question 7 were projected from the lowest point with velocity 32 ft./sec., find its velocity and the pressure on the tube at the highest point.

10. Find the velocity with which the same particle must be projected from the lowest point in order that it should just rise to the end of the horizontal diameter.

11. A child weighing 5 stone is on a swing suspended by two cords. If the swing is describing an arc of 120° in each oscillation, find the tension of each cord when the swing is at the lowest point.

12. A light rod 3 ft. long is hinged at one extremity; at distances of 1, 2, and 3 ft., respectively from that extremity are attached masses of 3, 2, and 1 lb., respectively. Compare the tensions in each portion of the rod, if the 1 lb. mass is describing a circle at the rate of 10 ft./sec.

13. Determine the thrust between the particle and the tube in Question 7 when the radius through the particle is 30° below the horizontal diameter.

14. At what point of the motion of the particle in Question 7 will there be no thrust between the particle and the tube.

15. The attraction exerted by the Sun on any one of its planets varies directly as the mass of the planet, and inversely as the square of the planet's distance from the Sun. Assuming that all the orbits are circular, prove that the squares of the times of revolution of the planets vary as the cubes of the radii of the orbits.

16. A bicyclist is riding at the rate of 15 miles an hr. round a curve of radius 121 ft. Determine at what angle his machine must be inclined to the vertical.

17. Find the ratio of the centrifugal force on a body at the Equator (due to the Earth revolving on its axis once in 24 hrs.) to the weight of the body, taking the Earth's radius as 4,000 miles, and $g = 32{\cdot}2$.

18. A particle, suspended from a fixed point by a string of length r, hangs vertically. It is projected horizontally with velocity $\sqrt{(6gr)}$ and describes a circle in a vertical plane. Show that the tension of the string when the particle is at the end of a horizontal diameter is four times the tension when the particle is at the highest point.

19. A body of 5 lbs. weight is swung round in a horizontal circle of 4 ft. radius, making 40 revolutions per minute: find the force, in lbs. weight, with which it pulls outward.

CHAPTER XVII.

SIMPLE HARMONIC MOTION.

Simple and Conical Pendulums.

238. In various kinds of machinery, devices are used for converting to-and-fro motion in a straight line into rotatory motion; as instances, we may mention the common turning lathe or the crank of a steam engine. Hence, from circular motion we naturally pass on to considerations of to-and-fro rectilinear motion, the simplest form of which is known as *simple harmonic motion*. It will be necessary to preface the subject by establishing the following lemma:—

239. **Lemma.**—A particle P is describing a curved path CD, and a second particle Q is travelling along a straight line AB in such a manner that PQ is always perpendicular to AP. (Fig. 70.)

It is required to investigate the relation between the velocities and accelerations of the two particles.

Since PQ is always at right angles to AB, the velocity of P relative to Q is always at right angles to AB.

Also the velocity of Q is along AB.

Now, by definition of component velocities (§ 161), the velocity of P is compounded of the velocity of Q and the velocity of P relative to Q.

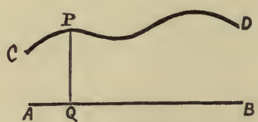


Fig. 70.

And these two components are at right angles; hence they are the resolved parts of the velocity of P along and perpendicular to AB*.

Hence *at any instant*

Q's *velocity* = P's *velocity resolved parallel to* AB.

Hence also the rate of change of Q's velocity at any instant = the rate of change of P's velocity parallel to AB.

i.e., at any instant,

Q's *acceleration* = P's *acceleration resolved parallel to* AB.

In short, Q's motion is simply the resolved part of P's motion parallel to AB, without P's motion perpendicular to AB.

240. Simple harmonic motion.

DEF.—Let ABC be a circle of radius r; let O be its centre, and AOC a diameter. Suppose a particle P to travel round the circumference with uniform angular velocity k, starting at C (Fig. 71).

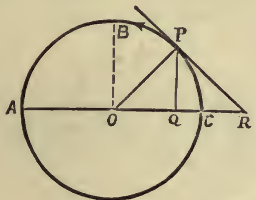


Fig. 71.

Also suppose a second particle Q to start from C at the same instant and to travel along the diameter CA in such a manner that the line joining P to Q is always perpendicular to AC. Then the motion of Q is called **simple harmonic motion.**

Obviously Q travels to and fro along AC; this is expressed by saying that Q **oscillates**; and the motion from C to A and back is called one **complete oscillation.**

* When a velocity or acceleration is resolved into two components, one along and the other perpendicular to a given line, the former is called the **resolved part** of the velocity or acceleration along the line. Thus, if a velocity V is in a direction making an angle A with a given line, its resolved part along the line is $V \cos A$ (from § 172).

The time occupied by one oscillation is the same as the time occupied by P in one revolution, and is therefore found by dividing the circumference by the velocity of P, *i.e.* by kr. Thus the time of one oscillation is

$$t = \frac{2\pi r}{kr} = \frac{2\pi}{k} \quad \text{.....................} \quad \mathbf{(1).}$$

241. In simple harmonic motion the acceleration varies as the distance from a fixed point.

By § 230, the acceleration of P is in the direction PO, and its magnitude

$$= \frac{(rk)^2}{r} = k^2 r\,;$$

$\therefore$ **acceleration of Q**

$$= P\text{'s acceleration resolved parallel to } CA \text{ (§ 235)}$$
$$= k^2 . OP \cos POQ = k^2 . OQ \quad \text{.....................} \quad \mathbf{(2).}$$

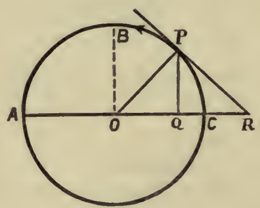


Fig. 72.

Thus *the acceleration of Q is always towards O and varies directly as the distance OQ.*

If we are given the initial velocity of a particle, and its acceleration at every point, the motion of the body is entirely determined.

Putting $k^2 = \mu$, we may state this result in a somewhat different form, thus:—

If a particle Q start from rest at C under an acceleration $\mu . OQ$, always towards O, the particle will oscillate to and fro along AC; and the time occupied in one oscillation will be $2\pi/\sqrt{\mu}$.

The motion of the particle will be *simple harmonic motion.*

DEFINITIONS.—The distance OC, or the greatest distance of the particle from the centre, is called the **amplitude** of the oscillation; and the time occupied in one oscillation is called its **period.**

COR. 1.—The period of oscillation is independent of r, *i.e.*, of its amplitude.

Such oscillations are called *isochronous*, which means that they are performed in the *same time* (whatever be their amplitude).

COR. 2.—If the mass of the particle is m, the force required to produce the acceleration $\mu . OQ$ is $m\mu . OQ$. Hence—*The motion of a particle under a force toward a given point, O, which varies directly as the distance is simple harmonic motion, and the oscillation is independent of the amplitude.*

242. **To determine the position and velocity of the particle after a given time t.**

(i.) Since P moves round the circle with velocity kr (Fig. 72),

$\therefore$ arc CP = distance travelled by P in time $t = ktr$.

$\therefore$ circular measure of angle POC

$$= \text{arc } PC/\text{radius} = ktr/r = kt.$$

Thus **distance of Q from starting point**

$$= CQ = CO - OQ = r - r\cos POQ$$

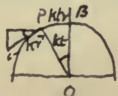
$$= r\,(1 - \cos kt) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \mathbf{(3).}$$

(ii.) Let the tangent at P meet OC produced in R; then **velocity of Q** = velocity of P resolved parallel to CA

$$= kr\cos PRO = kr\sin POR = kr\sin kt$$

$$= k\,.\,QP \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \mathbf{(4).}$$

If the motion be considered as starting from O, it is easily seen that P will start from B, and hence that the distance travelled in time t is given by $r\sin kt$; and the velocity by $kr\cos kt$.



243. **Simple harmonic motion along a curve.**—The same kind of motion would be possible along a curve. Thus, if a particle P (Fig. 73) is constrained to move along a curved path AOC, with an acceleration along the tangent to the curve directed towards O and of magnitude μ . (arc OP), the period of a complete oscillation will be independent of the amplitude, and will be $2\pi/\sqrt{\mu}$, &c.

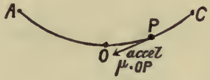


Fig. 73.

This naturally follows from considerations such as those given in § 235.

244. To find the period of a small oscillation of a simple pendulum.

DEFINITION.—A heavy particle suspended from a fixed point by a weightless inextensible string is called a **simple pendulum**.

Let P be the particle (Fig. 74), m its mass, l the length of the string, A the fixed point, AO the vertical position of the pendulum.

If the pendulum be pulled aside till the string is in the position AB, and then let go, the pendulum will swing to and fro, describing an arc of a circle BC, of which O is the middle point.

Let the tangent at P meet AO produced in Q. Draw PD vertically down, and PM perpendicular on AQ. Then the forces acting on P are the tension T along PA, and the weight mg.

Resolving along PQ, we have

$$\text{force along tangent} = mg \cos DPQ = mg \sin \theta$$
$$= mg \, . \, MP/AP.$$

This force, acting on the mass m, produces an acceleration $g \, . \, MP/l$ along the tangent, *i.e.*, along the direction of the arc at P.

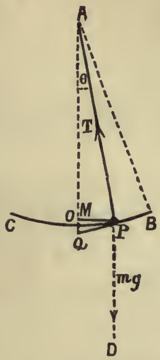


Fig. 74.

Now suppose the angle OAB subtended at A by the whole arc OB is small. Then, in any position of P, the perpendicular MP is approximately equal to the arc OP, and therefore the acceleration at P

$$= \frac{g}{l} \cdot \text{arc } OP = \frac{g}{l} \times \text{distance of } P \text{ from } O.$$

Hence, by § 243, the motion is a simple harmonic oscillation, whose period is

$$\frac{2\pi}{\sqrt{(g/l)}} = 2\pi \sqrt{\frac{l}{g}} \quad \text{.................... (5).}$$

245. Def.—A pendulum is said to **beat** when it swings from rest on one side of the vertical to rest on the other.

Hence each *oscillation* consists of two *beats*, backwards and forwards.

A clock usually ticks once in each beat of its pendulum.

A **seconds pendulum** is one which beats once a second; its period of oscillation is therefore *two seconds.*

246. **To find the length of the seconds pendulum,** we put $t = 2$ in the formula $t = 2\pi\sqrt{(l/g)}$. Then

$$\pi\sqrt{\frac{l}{g}} = 1; \quad \therefore \quad l = \frac{g}{\pi^2} \qquad (6).$$

and, if g be known, l can be found.

Conversely, knowing the length of the seconds pendulum at any place, the value of g can be found by the same formula.

More generally g may be found with great accuracy by observing the time of oscillation of a pendulum of known length. To do this a large number of oscillations are counted and the time they occupy observed; whence the time of one oscillation is found with a high degree of accuracy by division.

Thus the pendulum may be used to compare g at different places.

247. Conical Pendulum.—DEFINITION.—When a particle attached to a fixed point by a string, instead of swinging to and fro in a vertical plane, revolves in a horizontal circle, it is called a **conical pendulum.**

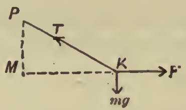


Fig. 75.

To find the time of revolution of a conical pendulum.

Let a particle K, of mass m, suspended from P by the string PK, revolve in a horizontal circle, whose centre M is vertically below P. Then the force mv^2/MK to the centre required to keep it in the circle is the resultant of the tension T along KP and the weight mg acting vertically downwards. Since the forces are parallel to the sides of the triangle PMK, therefore, by the triangle of forces,

$$\frac{mg}{PM} = \frac{T}{KP} = \frac{mv^2/KM}{KM};$$

$$\therefore \quad \frac{v^2}{KM^2} = \frac{g}{PM} \quad \text{or} \quad \frac{v}{KM} = \sqrt{\frac{g}{PM}}.$$

But, if t be the time of revolution,

$$t = \frac{2\pi . MK}{v}. \text{ Hence } \quad t = 2\pi\sqrt{\frac{PM}{g}} \quad \ldots\ldots \quad (7).$$

Hence the time of revolution depends only on the vertical depth of the particle below the point of support.

COR.—It follows that, if any number of particles be suspended from a fixed point and revolve in circles in the same periodic time, they will all lie in the same horizontal plane.

248. **Watt's Governor.**—The principal of the conical pendulum is well exemplified in the governor of a steam engine. Two balls are suspended like the particle of the last article and are driven round by the engine. If the speed of the engine increases, the time of revolution decreases, and therefore the depth (PM) of the balls below their attachment decreases, *i.e.*, the balls begin to rise. In doing so they are made by means of a suitable mechanism to act on the valves of the engine so as to shut off part of the steam. In this way the engine is prevented from working at too high a speed. This is particularly useful when the engine is sometimes employed to drive machinery and sometimes not; without such a governor the speed would become excessive when the engine was not working against any resistance.

SUMMARY OF RESULTS.

In *simple harmonic motion*

if the period of oscillation be $= 2\pi/k$ .... (1),

then the acceleration $= k^2 \times \text{distance}$ ...... (2),

the distance after time t, from extreme position

$= r(1-\cos kt)$ ...... (3),

and the velocity after time t $= kr \sin kt$ .... (4),

where kt is the *circular* or *radian measure* of the angle whose sine and cosine are taken.

Period of oscillation of *simple pendulum* of length l

$$= 2\pi\sqrt{\frac{l}{g}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

EXAMPLES XVII.

1. A particle, of mass 1 gramme, starts from rest at a distance 10 cms. from a fixed point O, towards which it is attracted with a force whose measure in dynes is 100 times the measure of the distance of the particle from O in centimetres. Determine after what time it will reach O; also its position and velocity after $\pi/40$ seconds.

2. A particle, of mass 1 gramme, is moving with simple harmonic motion. If its greatest velocity is 20π cm./sec., and the amplitude of the oscillation is 10 cm., find the period of oscillation and the force of attraction toward the centre when the particle is at its greatest distance.

3. In a simple harmonic motion, given the greatest velocity v, and the period of oscillation t, determine the amplitude.

4. Given the amplitude s, and the force of attraction on the particle when at its greatest distance from the centre of attraction P, determine the greatest velocity, if the mass of the particle is m.

5. Assuming $g = 32$, find the period of oscillation for a simple pendulum of length 1 yard.

6. What would be the measure of the acceleration of gravity in F.P.S. units if a simple pendulum, 39 in. long, made exactly one beat per second, in small oscillations?

7. What would be the measure of the acceleration of gravity in C.G.S. units, if the length of the "seconds pendulum" were exactly one metre?

8. The number of beats made in an hour by one pendulum is to the number made in half-an-hour by another as 13 : 6; and the length of the first pendulum is 1 ft. Determine the length of the second.

9. A seconds pendulum has its length slightly altered, and in consequence loses n seconds a day; find whether it has been lengthened or shortened, and by what fraction of its original length.

10. A particle is oscillating harmonically from A to B and back along a straight line AB whose middle point is O. If T is the period of oscillation, prove that the velocity at any point P is given by

$$v^2 = \frac{4\pi^2}{T^2}(OA^2 - OP^2).$$

11. A particle, suspended from a fixed point C by a string of length l, swings to and fro in a circular arc AOB, of which O is the lowest point. Prove, from the Principle of Conservation of Energy, that the velocity at any point P of the arc is given by

$$v^2 = g\,\frac{OA^2 - OP^2}{l},$$

where OA and OP are the *chords* joining O to A and P respectively. Hence deduce the time of a small oscillation (*cf.* Ex. 10), explaining why the result will not apply when the arc of oscillation is large.

12. A point moves uniformly with velocity u in a circle whose radius is a; prove that the projection of the moving point on a fixed diameter of the circle oscillates in periodic time $2\pi a/u$ as if it were a material particle under the influence of a force to the centre, producing acceleration u^2x/a^2, where x is the distance from the centre.

13. If gravity were 31·5 in feet and seconds, what would be the length of a pendulum performing complete vibrations in 2·5 seconds?

14. Find the acceleration of gravity (i.) in Paris, (ii.) at the Equator, having given that the lengths of the seconds' pendulums are 3·26 ft. and 3·251 ft. respectively.

15. Prove that the time of revolution in a conical pendulum of length l is ultimately $2\pi\sqrt{\frac{l}{g}}$ when the cone described by the pendulum is indefinitely small.

16. Prove that, if T is the time of revolution of the bob of a conical pendulum at the bottom of the shaft of a deep mine of depth l, the pendulum being suspended from the surface of the Earth, then the value of g at the bottom of the shaft is given by

$$g = \frac{4\pi^2 l}{T^2}\left(1 - \frac{l}{a}\right),$$

where a denotes the radius of the Earth.

17. In harmonic motion, show that, if the force to the centre be doubled, the period of oscillation will be altered roughly in the ratio of 5 to 7.

CHAPTER XVIII.

IMPACT OF SMOOTH SPHERES.

249. **Direct Impact.**—Definition.—When two bodies strike against or collide with one another they are said to **impinge** on one another, and the collision is termed an **impact**.

A sphere is said to impinge **directly** on a fixed plane, if the direction of its velocity, just before impact, is perpendicular to the plane.

A **direct impact** of two spheres is one in which the centres of the spheres, just before colliding, are moving along the same straight line.

From common experience (derived from the collision of billiard balls, the rebound of tennis balls, and the like), we know that after an impact the two impinging bodies usually rebound and separate. In cases of direct impact it is obvious (from considerations of symmetry) that the motion after impact is in the same straight line as before. The object of the present chapter is to investigate the *velocities* with which two bodies rebound after an impact between them.

The problem which we first require to solve may be stated thus :—

Given the masses of the two spheres (say M and m), *and their velocities before direct impact* (say U and u), *find their velocities after impact* (say V and v).

250. **The equation of momentum.**—We have already seen in Chapter VII. that, in the case of direct impact,

the *algebraic sum* of momenta after the impact

= the *algebraic sum* of momenta before;

the words "*algebraic sum*" being used to denote that the momentum of each body is to be reckoned positive or negative according as the body is moving in the positive or negative direction. From this principle we have also

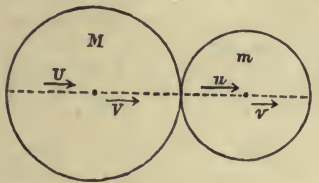


Fig. 76.

in the same chapter derived the equation

$$MV+mv=MU+mu \quad \text{..................} \quad \mathbf{(1)}.$$

This one equation is, however, insufficient to determine the two unknown quantities V and v. Moreover we know, as a matter of ordinary experience, that what happens after the collision does not depend only on the masses of the spheres and their velocities before impact, but also on their mutual elasticity.

251. Newton's Law of Impact.—The manner in which the elasticity affects the motion after impact can only be determined by experiment; and the result of Newton's investigations was to establish the following law:—

The ratio of the relative velocity after impact to the relative velocity before impact depends only on the materials of which the bodies are composed; *and is independent of the actual velocities or the masses of the bodies.*

For instance, if two spheres be made of glass, then the ratio of the relative velocity after impact to the relative velocity before impact is always approximately $\frac{15}{16}$.

This ratio is usually called the **coefficient of restitution** for the given materials; sometimes it is called the **coefficient of elasticity***; in formulæ it is usually represented by the letter e.

Hence the law is often quoted in the form—

velocity of separation $= e \times$ **velocity of approach.**

If e were equal to 1, the two bodies would separate at the same rate as they approached; they would then be called **perfectly elastic.**

If e were equal to 0, the two bodies would not separate at all; in which case they would be called **perfectly inelastic.**

In practice e is always found to be between 0 and 1, no known solids being either perfectly elastic or inelastic.

We will now express the law in mathematical language.

The relative velocity of M to m before impact is obviously $U-u$; and after impact it is $V-v$. Also these two relative velocities are in opposite directions and therefore of *opposite sign;* for the spheres are approaching each other before the impact, and receding apart after.

Thus $V-v$ and $e(U-u)$ are numerically equal, but of opposite sign.

$$\therefore \quad V-v=-e(U-u) \quad \text{.............. } \mathbf{(2)}.$$

This equation may be called the equation of restitution.

252. To determine the velocities of two given spheres after direct impact, we therefore write down the equation of momentum (with the notation of § 245)

$$MV+mv=MU+mu \text{ ....} \mathbf{(1)},$$

and the equation of restitution

$$V-v=-e(U-u) \text{ .... } \mathbf{(2)}.$$

If the value of e is known, equations (1) and (2), when

---

* Both names should be remembered, but the term "coefficient of restitution is the better, as the term "coefficient of elasticity" has several other meanings besides the present one.

solved as simultaneous equations, are sufficient to determine the two unknown quantities V and v.

It is important that the student should understand that the above formulæ include *both* cases of impact, viz., when the spheres before impact are moving in opposite directions, and when they are moving in the same direction. The argument is most easily followed if we assume all the velocities to be in the same direction, and take this as the positive direction. But it is equally true in all cases if we assign the correct signs to all the velocities when substituting in the formulæ, velocities in the direction opposite to the positive one being considered negative.

Example.—A sphere of weight 20 lbs., moving with velocity 10 ft. per sec., impinges directly on a second sphere of weight 5 lbs. moving in the opposite direction with velocity 30 ft. per sec. If the coefficient of elasticity is ·5, find the velocities after the impact.

Let the velocities after the impact be V and v. Then we must substitute the following values in equations (1) and (2):—

$$M = 20, \quad m = 5, \quad e = \tfrac{1}{2},$$
$$U = 10, \quad u = -30.$$

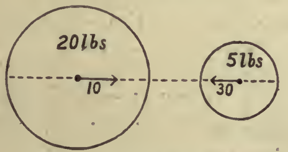


Fig. 77. Before impact.

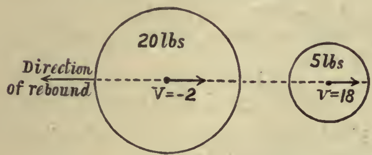


Fig. 78. After impact.

We thus obtain the equations

$$20V + 5v = 50,$$
$$V - v = -20.$$

Solving these, we find $V = -2$, $v = 18$.

Comparing the signs of U and u with the signs of V and v, we notice that the direction of motion of each sphere has in this case been reversed.

253. **Direct impact against a fixed plane.**—In this case the velocity of rebound will be e times the velocity of impact; where e is the coefficient of elasticity for the materials of which the sphere and plane are respectively composed.

254. Loss of kinetic energy in impact.—*To express the loss of kinetic energy in a direct impact between two spheres in terms of the masses and velocities before impact.*

The total kinetic energy of the two spheres after impact is

$$\tfrac{1}{2}MV^2 + \tfrac{1}{2}mv^2,$$

and their kinetic energy before impact is

$$\tfrac{1}{2}MU^2 + \tfrac{1}{2}mu^2.$$

Hence, if L denote the kinetic energy lost in impact,

$$L = \tfrac{1}{2}(MU^2 + mu^2) - \tfrac{1}{2}(MV^2 + mv^2) \quad \ldots\ldots\ldots(a),$$

where V and v are given by the equations

$$MV + mv = MU + mu \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$V - v = -e(U - u) \quad \ldots\ldots\ldots\ldots\ldots(2).$$

Hence all that now remains is to find the values of V and v by solving (1) and (2), and substitute them in (a), and simplify the resulting expression.

From here on the work is mere algebra.

The *best* plan would be for the student to work out the result as an exercise, proving that

$$L = \frac{1 - e^2}{2} \cdot \frac{Mm}{M + m}(U - u)^2.$$

If this should be found too difficult, the following will be found the *shortest way* of getting the result:—

Squaring (1) and (2), we have

$$M^2V^2 + 2MmVv + m^2v^2 = M^2U^2 + 2MmUu + m^2u^2,$$

$$V^2 - 2Vv + v^2 = e^2(U^2 - 2Uu + u^2).$$

Now L does not contain the produced Vv. To get rid of this we multiply the second of these equations by Mm, and add to the first, giving

$$(M^2 + Mm)V^2 + (m^2 + Mm)v^2$$
$$= M^2U^2 + 2MmUu + m^2u^2 + e^2Mm(U^2 - 2Uu + u^2).$$

The left-hand side is equal to

$$(M+m)(MV^2+mv^2)$$
$$= 2(M+m) \times \text{kinetic energy after impact.}$$

Hence, subtracting from both sides the expression

$$(M+m)(MU^2+mu^2),$$

we have

$$(M+m)\{(Mv^2+mv^2)-(Mu^2+mu^2)\}$$
$$= M^2U^2+2MmUu+m^2u^2-(M^2+mM)U^2-(m^2+Mm)u^2 + e^2Mm(U^2-2Uu+u^2),$$

or $(M+m)\times(-2L) = -Mm(U^2-2Uu+u^2)+e^2Mm(U^2-2Uu+u^2)$;

$$\therefore \quad 2(M+m)L = (1-e^2)Mm(U^2-2Uu+u^2),$$

or

$$L = \frac{1-e^2}{2}\frac{Mm}{M+m}(U-u)^2 \quad \text{.................. } \mathbf{(3)}.$$

255. This expression for the loss of kinetic energy, in terms of the masses and of the original velocities, gives us some very important results.

Firstly, e is never greater than 1, and therefore this expression can never be *negative*; that is to say, *the kinetic energy is* **never increased** *by the impact.*

If $e=1$, *the kinetic energy is unaltered; if* $e<1$, *it is diminished.*

In the latter case the lost kinetic energy mostly reappears in the form of heat energy, while part of it is expended in producing sound vibrations (as exemplified by the click of billiard balls). From the principle of Conservation of Energy we infer that this total energy which reappears in these forms is equal to the lost kinetic energy.

256. **To find the total impulse between the two spheres at the instant of collision.** Let I denote the impulse of the blow given by M to m; then, since the impulse is measured by the change of momentum produced in m,

$$\therefore \quad I = m(v-u).$$

Now, eliminating V from (1), (2) by multiplying (2) by M, and subtracting from (1), we have

$$(m+M)v = MU+mu+eM(U-u).$$

Subtracting $(m+M)u$ from both sides

$$(m+M)(v-u) = M(U-u)+eM(U-u);$$

$$\therefore \quad v-u = \frac{M(1+e)(U-u)}{M+m};$$

$$\therefore \quad I = m(v-u) = \frac{Mm(1+e)(U-u)}{M+m} \quad \text{.................. } \mathbf{(4)},$$

and the impulse of the blow which m gives to M is, of course, equal and opposite to this.

*257. Before we proceed to deduce further conclusions from the algebraical results of §§ 252, 256, we must analyse more closely the history of an impact.

The whole action between the balls takes place in an extremely short space of time, but this time can be divided into two distinct periods: firstly, the period of *compression*, during which each ball is approaching towards and *compressing* the other; secondly, the period of *restitution*, during which each ball is trying, more or less, to regain its original shape, and in so doing is *pushing the other away from it.** During the first period, the two centres are approaching one another; during the second, they are separating; and *at the instant of greatest compression their relative velocity is zero;* that is to say, they are travelling with a common velocity. But, if the balls were perfectly *inelastic*, there would be no period of restitution; the balls would make no attempt to regain their original shape, but would go on travelling together in their compressed state, with common velocity.

It follows from this that to find the common velocity at the instant of greatest compression we need only find what the ultimate velocity would be, if the balls were inelastic; and to find the measure of the impulse during the period of compression we need only find the total impulse, supposing the balls inelastic.

Now let I_1 and I_2 be the measures of the impulses during the periods of compression and restitution respectively; then, obviously,

$$I = I_1 + I_2.$$

Again, by the preceding argument, I_1 is found by putting

$$e = 0 \text{ in formula (4)},$$

$$i.e., \ I_1 = \frac{Mm\,(U-u)}{M+m};$$

also

$$\begin{aligned} I_2 &= I - I_1 \\ &= \frac{Mm\,(1+e)\,(U-u)}{M+m} - \frac{Mm\,(U-u)}{M+m} \\ &= \frac{e\,Mm\,(U-u)}{M+m}. \end{aligned}$$

Hence we see that $$I_2 = eI_1 \qquad \text{(5)}.$$

Again, by putting $e = 0$ in formula (3), we see that loss of kinetic energy during the period of compression is

$$\tfrac{1}{2}\,\frac{Mm\,(U-u)^2}{M+m};$$

* This is well exemplified by the case of a collision between two railway carriages fitted with spring buffers. Here the compression and subsequent expansion of the buffers can be very easily observed.

thus total loss of kinetic energy

$= (1 - e^2) \times$ loss during period of compression,

$=$ loss during compression $- e^2 \times$ loss during compression.

$\therefore$ during the period of restitution there must be a *gain* of kinetic energy equal to $e^2 \times$ loss during period of compression.

258. **Oblique impact.**—DEFINITION.—When the impact between two bodies is not direct it is said to be **oblique.**

Thus a sphere is said to impinge obliquely on a plane when its direction of motion before impact is not at right angles to the plane.

259. **Oblique impact of a sphere on a perfectly smooth plane.**—Let the velocities of the sphere before and after impact be u and v, and let their directions make angles α and β with the plane respectively. (Fig. 79.)

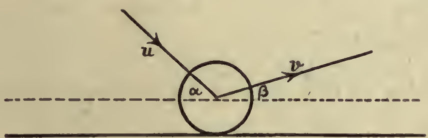


Fig. 79.

Given u and α, required to find v and β. Since no friction is supposed to act, the blow which the plane inflicts on the sphere is perpendicular to the plane, and therefore does not affect the resolved velocity of the sphere parallel to the plane.

The resolved velocity perpendicular to the plane is altered according to the same law as in the case of direct impact. Thus we have:—

(*a*) *resolved velocity parallel to the plane after impact* $=$ *resolved velocity parallel to the plane before impact;*

(*b*) *resolved velocity of separation perpendicular to the plane after impact* $= e \times$ *resolved velocity of approach perpendicular to the plane before impact*, these velocities being of course in reverse directions.

From these statements we derive the two equations—

$$v \cos \beta = u \cos \alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$v \sin \beta = eu \sin \alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (6).$$

Hence $v^2 = v^2 \cos^2 \beta + v^2 \sin^2 \beta = u^2 \cos^2 \alpha + e^2u^2 \sin^2 \alpha$,

and $$\tan \beta = \frac{v \sin \beta}{v \cos \beta} = \frac{eu \sin \alpha}{u \cos \alpha} = e \tan \alpha;$$

whence v and β are determined.

260. Oblique impact of smooth spheres.

If at the moment of impact two spheres are not both moving along the line joining their centres, the impact is said to be oblique.

Given the direction of the line of centres at impact and the magnitudes and directions of the velocities of the spheres before impact, to determine the velocities after impact.

Let ACB be the line through the two centres at the moment of impact (Fig. 80). Let U and u be the respective velocities of the spheres before impact, and let their

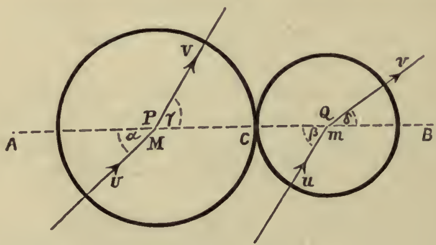


Fig. 80.

directions of motion make angles α and β with AB respectively.

Let V and v be the respective velocities after impact, and let their directions make angles γ and δ with AB respectively.

We are given M, m, U, u, α, β, and e; we are required to find V, v, γ, and δ.

Suppose all these velocities resolved along and perpendicular to AB; viz., U into $U\cos\alpha$ and $U\sin\alpha$; &c. ...

Now the spheres are smooth, and therefore the blow which m gives to M will be perpendicular to the common surface at C, *i.e.*, will be along the line CA. This blow will therefore *not* alter the resolved velocity of M perpendicular to AB, but only the resolved velocity along AB.

Thus (a) *resolved velocity of M perpendicular to AB after the impact = resolved velocity of M perpendicular to AB before the impact.*

In the same way we can show that—

(b) *Resolved velocity of m perpendicular to AB after the impact = resolved velocity of m perpendicular to AB before the impact.*

Again, exactly the same arguments and laws apply to the *resolved parts of the velocities* of the spheres *along AB* as in the case of direct impact (§§ 250, 251). We shall therefore obtain the two following results:—

(c) *The algebraic sum of momenta along AB after the impact = algebraic sum of momenta along AB before the impact.*

(d) *The difference of the velocities along AB after the impact = $-e\times$ difference of the velocities along AB before the impact.*

From these four statements we obtain the following four equations:—

from (a), $V\sin\gamma = U\sin\alpha$ .... **(7)**;

from (b), $v\sin\delta = u\sin\beta$ .... **(8)**;

from (c), $MV\cos\gamma + mv\cos\delta = MU\cos\alpha + mu\cos\beta$ .... **(9)**;

from (d), $V\cos\gamma - v\cos\delta = -e(U\cos\alpha - u\cos\beta)$ .... **(10)**.

These four equations are sufficient to determine the four unknown quantities, V, v, γ, and δ.

261. The method of solution requires careful attention.

Eliminate $v \cos \delta$ between equations (9) and (10), *i.e.*, multiply (10) by m, and add (9); from the resulting equation we obtain

$$V \cos \gamma = \frac{MU \cos \alpha + mu \cos \beta - me(U \cos \alpha - u \cos \beta)}{M+m}.$$

Using this last result with equation (7), we find

$$V^2 = V^2 \sin^2 \gamma + V^2 \cos^2 \gamma$$

$$= U^2 \sin^2 \alpha + \left\{ \frac{MU \cos \alpha + mu \cos \beta - me(U \cos \alpha - u \cos \beta)}{M+m} \right\}^2,$$

whence V is known.

Also, $$\cot \gamma = \frac{V \cos \gamma}{V \sin \gamma} = \frac{MU \cos \alpha + mu \cos \beta - me(U \cos \alpha - u \cos \beta)}{(M+m) U \sin \alpha},$$

whence γ is known.

In the same way, if we eliminate $V \cos \gamma$ between (9) and (10), we can determine $v \cos \delta$; and, taking this result in conjunction with equation (8), we can find v and δ.

The student should remember the statements (*a*), (*b*), (*c*), and (*d*), and should be able to write down at once the equations (7), (8), (9), (10), which are derived from them.

Example.—Two smooth spheres of mass 1 lb., each moving with velocity 20 ft. per sec. in directions at right angles to one another, impinge in such a way that the line joining their centres is the direction of motion of one of them. If $e = \frac{1}{2}$, determine the subsequent motion.

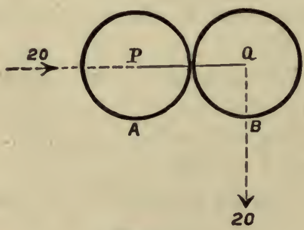


Fig. 81.

Using the accompanying figure (74), we have

velocity of A perpendicular to PQ after impact

= velocity of A perpendicular to PQ before impact = 0;

velocity of B perpendicular to PQ after impact

$$= \text{velocity of } B \text{ perpendicular to } PQ \text{ before impact} = 20.$$

[*Cf.* § 256 (a) and (b).]

Again, if u and v be the respective velocities in the direction PQ after impact, we have

$$u + v = 20, \qquad [Cf.\ \S\ 256\ (c)]$$

$$u - v = -\tfrac{1}{2}(20 - 0) = -10; \qquad [Cf.\ \S\ 256\ (d)]$$

whence $u = 5$, $v = 15$.

Thus the final velocity of A is 5 ft. per sec. along PQ; and the final velocity of B is the resultant of 15 ft. per sec. along PQ and 20 ft. per sec. perpendicular to PQ. The magnitude of this resultant

$$= \sqrt{(15^2 + 20^2)} = 5\sqrt{(3^2 + 4^2)} = 25 \text{ ft. per sec.}$$

262. **Pressure of a falling chain on an inelastic plane.**

Example.—A perfectly flexible chain is hanging from a point with its lower end just in contact with a horizontal plane. If it is allowed to fall, find the force of pressure on the plane at any time during the motion.

Let the mass of unit length of the chain be m.

The pressure on the plane after any time t is due to two causes:—(a) the weight of that portion of the chain which is already lying coiled up on the plane; (b) the continuous impact of fresh portions of the chain on the plane.

(a) After time t the chain has fallen a distance $\frac{1}{2}gt^2$; hence a length $\frac{1}{2}gt^2$ of the chain is lying coiled up on the plane. The mass contained in this length is $m \times \frac{1}{2}gt^2$, and the weight of it is therefore

$$m \times \tfrac{1}{2}gt^2 \times g \text{ poundals} = \tfrac{1}{2}mg^2t^2.$$

(b) Let P be the reaction of the plane due to the continuous impact, and let T be a very small interval of time during which the velocity of the chain may be considered as constant.

Then, since the chain has been falling for time t, its velocity is gt. Thus, during the interval T, a length gtT (and therefore a mass $mgtT$) falls on to the plane; thus, in time T, the force P reduces the velocity of a mass $mgtT$ from gt to 0.

Hence (§ 88, formula 3), we have

$$PT = mgtT(0 - gt);$$

whence $P = -mg^2t^2$ (the negative sign merely denoting that the reaction of the plane is in the opposite direction to the motion of the chain).

Combining (a) and (b), we obtain the total force of pressure on the plane, viz., $\frac{1}{2}mg^2t^2 + mg^2t^2 = \frac{3}{2}mg^2t^2$

= three times the weight of the portion lying coiled up on the plane.

SUMMARY OF RESULTS.

In direct impact the velocities are found from the equations—

$$MV + mv = MU + mu \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (1),$$

$$V - v = -e(U - u) \quad \ldots\ldots\ldots\ldots\ldots \ (2).$$

Loss of kinetic energy in an impact

$$= \frac{1}{2}\frac{Mm(1-e^2)(U-u)^2}{M+m} \quad \ldots\ldots\ldots\ldots\ldots \ (3).$$

$$\textit{Measure of impulse} = \frac{Mm(1+e)(U-u)}{M+m} \quad \ldots\ldots\ldots\ldots \ (4),$$

$$I_2 = eI_1.$$

For *oblique impact against a fixed plane*

$$V\cos\beta = U\cos\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (5),$$

$$V\sin\beta = eU\sin\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (6).$$

Results of *oblique impact between two spheres* determined from equations—

$$V\sin\gamma = U\sin\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (7);$$

$$v\sin\delta = u\sin\beta \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (8);$$

$$MV\cos\gamma + mv\cos\delta = MU\cos\alpha + mu\cos\beta \ \ldots \ (9);$$

$$V\cos\gamma - v\cos\delta = -e(U\cos\alpha - u\cos\beta) \ (10).$$

EXAMPLES XVIII.

1. A sphere of mass 10 lbs., travelling with velocity 20 ft. per sec., overtakes a sphere of mass 20 lbs., travelling in the same direction with velocity 10 ft. per sec. If the coefficient of elasticity be ·5, find the velocities after impact.

2. If in the last question the spheres were moving in opposite directions, determine the subsequent motion.

3. Two spheres of masses 6 and 8 lbs. are moving directly towards one another, each with velocity 20 ft. per sec. After impact the velocity of the first sphere is reversed; find the coefficient of restitution and the subsequent motion of the second sphere.

4. A sphere impinges directly on another sphere at rest; the coefficient of restitution is e; the first sphere is reduced to rest by the impact; prove that the masses of the spheres are in the ratio $e : 1$.

5. A sphere impinges directly on another sphere at rest; the coefficient of restitution is e; the final velocity of the second sphere is equal to the initial velocity of the first; prove that the masses of the spheres are in the ratio $1 : e$.

6. After a direct impact two spheres are observed to be moving in the same direction with velocities 20 and 10 respectively; their masses are 1 and 3 lbs. respectively; the coefficient of restitution is known to be 1/3; find the velocities before impact.

7. The centres of two equal and perfectly elastic smooth spheres are moving with equal and opposite velocities along parallel lines. If the distance between the parallel lines is $\sqrt{2} \times$ the radius of either sphere, show that after impact the velocity of each sphere will be at right angles to its former direction.

8. A smooth sphere of mass 5 lbs., travelling with velocity 20 ft. per sec., impinges on another of mass 10 lbs., travelling in a direction at right angles to its own with velocity 10 ft. per sec. At the moment of impact the centre of the first is on the line of motion of the centre of the second. Determine the subsequent motion, if $e = \cdot 5$.

9. A body, dropped from a height h on to a horizontal plane, bounces up and down, and finally comes to rest. If the coefficient of restitution be e, prove that—

(*a*) The velocities at the beginnings of two consecutive rebounds are in the ratio $1 : e$.

(*b*) The times occupied by two consecutive rebounds are in the ratio $1 : e$.

(*c*) The distances travelled in two consecutive rebounds are in the ratio $1 : e^2$.

10. Using the results of Question 9, find the time before a ball is reduced to rest which falls from a height of 16 ft. and bounces on a horizontal plane, if the coefficient of elasticity is ·5. Find also the total space traversed.

11. Considering the earth as a sphere of infinite mass at rest, investigate the subsequent motion if an elastic ball of finite mass impinges directly on the earth, deducing the result of § 249.

12. A sphere of mass 1 lb., moving with velocity 27 ft. per sec., impinges directly on a sphere of mass 3 lbs. at rest. The second sphere then impinges directly on a plane, and afterwards impinges again on the first. Determine the final velocity of the first sphere, if the coefficient of restitution for each impact is 1/3.

13. A ball in a square courtyard with smooth walls and floor is projected along the ground in a direction parallel to one of the diagonals; prove that it will constantly be returning to the point from which it started.

14. A smooth sphere, moving with velocity u, impinges on an equal smooth sphere at rest; the impact is oblique and perfectly elastic. Prove that after impact the two spheres are moving at right angles.

15. A ball is dropped from the top of a tower 100 ft. high, and at the same instant another is thrown vertically up from the foot of the tower. The two balls impinge directly, and the first ball just reaches the top of the tower in the rebound. If the balls are perfectly elastic, find the velocity with which the second was thrown up.

16. Three imperfectly elastic particles, of masses m_1, m_2, m_3, respectively, and of the same material, are lying at rest in one straight line. If the first be projected toward the second with velocity u, and if the second impinging on the third imparts to it a velocity v, prove that

$$(m_1 + m_2)(m_2 + m_3)v = m_1 m_2 u (1 + e)^2.$$

17. A hose discharges every second 20 lbs. of water with velocity 40 ft. per sec. against an inelastic plane. Determine the force of pressure on the plane.

18. A Maxim gun is firing off 8 bullets a second; each bullet weighs 5 ozs., and is discharged with velocity 1,000 ft. per sec. Determine the force necessary to keep the gun from recoiling.

19. Determine the H.P. at which the gun in the last question is working.

20. Two perfectly elastic balls, equal in all respects, are in contact, and are impinged upon simultaneously by a third ball, in all respects equal to each of the former, moving with velocity u perpendicular to the line of centres of the two former. Find the velocity of the balls after the impact.

21. A series of 21 inelastic particles of equal mass are arranged at equal intervals in a straight line. The first particle is projected in the direction of the second, and strikes it in one minute. Find the total time before the last impact.

22. What would be the result in the last question if the masses were proportional to 1, 1, 2, 4, 8, &c.?

23. Two elastic spheres, equal in all respects, are moving towards each other with equal velocities, their centres being on two parallel lines whose distance apart is d_1 (less than d, the diameter of either sphere). Prove that after impact they will move away from each other with equal velocities, so that their centres are on two parallel lines whose distance apart d_2 is given by the equation

$$d_2^2 \left\{e^2 d^2 + (1-e^2)\, d_1^2\right\} = d^2 d_1^2.$$

24. A sphere moving with a velocity v impinges directly on another of twice its mass. Find the velocities after impact (i.) if the two spheres are inelastic, (ii.) if they are perfectly elastic.

25. An imperfectly elastic ball impinges upon a plane at an angle of 30°, and is deflected from it at an angle of 60° from the perpendicular. Find the coefficient of elasticity.

26. Two spheres, of masses m and n, moving in the same right line with velocities u and v, being supposed to interchange velocities by direct collision with each other; required their ratio of masses, and coefficient of elasticity.

27. An elastic ball, moving vertically under the action of gravity, being supposed to fall through a height h upon a horizontal plane; required its coefficient of elasticity in order that, after rebounding from the plane, it may ascend again to a height k above it.

EXAMINATION PAPER IX.

1. Prove fully that a particle projected horizontally describes a parabola whose vertex is the point of projection and whose axis is vertical.

2. A particle is projected from the foot of an inclined plane; its first impact with the plane is direct: prove that $3 \cos \alpha = \cos (\alpha - 2\beta)$, where α and β are the inclinations to the horizon of the initial direction of motion, and of the plane, respectively.

3. A particle describes a circle of radius r with uniform speed: show that its acceleration is directed towards the centre of the circle and is equal to v^2/r.

4. If a point P describes uniformly a circle, whose centre is O, show that M, the foot of the perpendicular from P on any diameter, moves with an acceleration which varies as OM, and find (i.) the period of the motion of M, and (ii.) its velocity when at a given distance from O.

5. Find the period of a small oscillation of a simple pendulum.

6. On a certain planet the length of the seconds pendulum is exactly 2 metres. How far will a body fall in 1 second on that planet?

7. An imperfectly elastic particle, moving with given velocity, impinges obliquely on a smooth fixed plane: find the magnitude and direction of the velocity after impact.

8. A particle impinges directly on a second particle of three times its own mass, initially at rest; the second particle then impinges directly on a fixed plane; the coefficient of restitution for both impacts is e: prove that there will always be a third impact unless $e = 1$.

9. A particle of mass M impinges directly on a particle of mass m initially at rest: determine the condition that exactly half the kinetic energy should be lost in the impact.

10. At any point within a solid sphere the attraction towards the centre varies directly as the distance from the centre. Supposing a small hole were bored straight through the earth, and a stone dropped down it, determine roughly with what velocity the stone would reach the centre ($g = 32$; diameter of earth $= 8,000$ miles), and after what time it would come up at the other side of the earth.

CHAPTER XIX.

MOMENTS OF INERTIA.

263. **Rigid Dynamics.**—In the preceding chapters we have considered only the motion of particles and bodies which move as a whole without rotation. That portion of dynamics which deals with the motion of rigid bodies when rotation takes place is called Rigid Dynamics.

By a **rigid body** is meant a body which always remains of the same size and shape however it is moved about. This implies that the line joining any two particles of a rigid body always remains of the same length, and, if three particles are joined together, it follows that the angles as well as the sides of the triangle so formed are constant in magnitude.

264. **Angular velocity of a rigid body.**

If a rigid lamina is revolving in its own plane about a fixed point in itself, it is clear that, since the lamina is rigid, all points on it describe equal angles about the fixed point, and hence we can measure its rate of revolving by measuring the angle described per unit time by any straight line in the lamina which passes through the fixed point.

In the same way, if a rigid body be revolving about a fixed axis, we measure its rate of revolving by measuring the angle described per unit time by any straight line in the solid which passes through *and is perpendicular** to

* A line not perpendicular to the axis would describe a *cone* instead of revolving through a *plane angle* in any given time, as does a line which is perpendicular.

the fixed axis. This angle measures the **angular velocity** of the lamina or solid body.

In investigating the rotation of rigid bodies under forces, certain quantities called *moments of inertia* are constantly occurring. We now proceed to define these and to show how they enter into the expressions for the kinetic energy of rigid bodies.

265. Moments of inertia.—DEFINITION.—If a series of particles of masses m_1, m_2, m_3, &c., are arranged at perpendicular distances r_1, r_2, r_3, &c., from a given line, then the **moment of inertia** of the system about this line is the quantity

$$m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \&c. \quad \ldots\ldots\ldots\ldots \quad \textbf{(1)}.$$

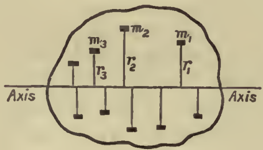


Fig. 82.

This expression may be conveniently represented by the notation $\Sigma(mr^2)$. The symbol Σ (the Greek capital *S*, called sigma) is used in Mathematics to denote *summation*. $\Sigma(mr^2)$ denotes the sum of all the terms formed on the model of the term mr^2.

In this connection the term **axis** is usually applied to the given line about which the moment of inertia is taken.

If, instead of a system of particles, we are dealing with a continuous rigid body, we may divide the body up into a very large number of parts, and make these parts so small that each may be regarded as a single particle. The value of $\Sigma(mr^2)$ for these particles will be defined to be the **moment of inertia of a body**.

266. Kinetic energy of a rotating body.

Suppose a body rotating about a fixed axis with angular velocity w; and let I be its moment of inertia about this axis. Let m_1, m_2, ...; r_1, r_2, ... have the same meanings as in § 265.

Fig. 83.

Then, by § 234, velocity of m_1 is wr_1; hence the kinetic energy of m_1 is $\frac{1}{2}m_1w^2r_1^2$. Thus, since w is the same for each of the masses, the kinetic energy of the body

$$= \Sigma\left(\tfrac{1}{2}mw^2r^2\right) = \tfrac{1}{2}m_1w^2r_1^2 + \tfrac{1}{2}m_2w^2r_2^2 + \tfrac{1}{2}m_3w^2r_3^2 + \dots$$
$$= \tfrac{1}{2}w^2\left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots\right) = \tfrac{1}{2}w^2\Sigma\left(mr^2\right)$$
$$= \tfrac{1}{2}Iw^2 \quad \text{.......... (2).}$$

Examples.—(1) Masses 1, 2, 3, 4 grammes are placed at the corners of a square $ABCD$ whose sides are 10 cm. long. To find their moments of inertia about straight lines through their centre of gravity parallel to AB and perpendicular to the plane of the square respectively.

Let x be the distance of the C.G. from DA, y its distance from AB. By a well-known formula in Statics, we have

$$x = \frac{1.0 + 2.10 + 3.10 + 4.0}{1 + 2 + 3 + 4} = 5 \text{ cm.},$$

$$y = \frac{1.0 + 2.0 + 3.10 + 4.10}{1 + 2 + 3 + 4} = 7 \text{ cm.}$$

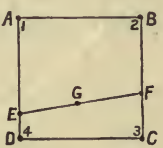


Fig. 84.

(i.) If an axis be taken through G parallel to AB, the points A, B are distant 7 cm. from it on one side and B, C are distant 3 cm. on the other side; if the former distance be called $+7$, the latter will be -3, and the moment of inertia will be

$$1.7^2 + 2.7^2 + 3.(-3)^2 + 4(-3)^2 = 49 + 98 + 27 + 36$$
$$= 210 \text{ gramme-centimetre units.}$$

Since moments of inertia are the result of multiplying masses by the squares of lengths, we write the above result thus:—210 gm.-cm.^{2}.

[Note also that it is not necessary to take account of the *signs* of the various distances, since the negative ones on being squared give a positive result.]

(ii.) When the axis is perpendicular to the plane of the paper $ABCD$, the lengths AG, BG, CG, DG are the perpendicular distances from it, and therefore, the moment of inertia

$$= 1.AG^2 + 2.BG^2 + 3.CG^2 + 4.DG^2.$$

By Geometry,

$$AG^2 = BG^2 = 5^2 + 7^2, \quad CG^2 = DG^2 = 5^2 + 3^2;$$

whence we find required moment of inertia = 460 gm.-cm.^{2}.

(2) When a foot and a pound are units the measure of a certain moment of inertia is 14. To find its measure when an inch and a hundredweight are units.

The moment of inertia is evidently equal to that of a single mass of 14 lbs. placed at a distance 1 ft. from the axis, *i.e.*, $\frac{14}{112}$ cwt. at a distance 12 in. from the axis. Hence, in the new units,

$$\text{the moment of inertia} = (\text{mass}) \times (\text{distance})^2$$
$$= \tfrac{14}{112} \times 12^2 = \tfrac{1}{8} \times 144 = 18 \text{ units.}$$

Otherwise thus:

$$14 \text{ lbs.-ft.}^2 = 14\,(\tfrac{1}{112}\text{ cwt.})\,(12\text{ ins.})^2 = 14 \times \tfrac{1}{112}\text{ cwt.} \times 12^2\text{ ins.}^2$$
$$= 14 \times \tfrac{1}{112} \times 12^2 \text{ cwt.-ins.}^2 = 18 \text{ cwt.-ins.}^2$$

267. Relation between the moments of inertia of a lamina.—*Given the moments of inertia of a lamina about two axes in its own plane at right angles to one another; required to determine the moment of inertia about the line through their point of intersection perpendicular to the plane of the lamina.*

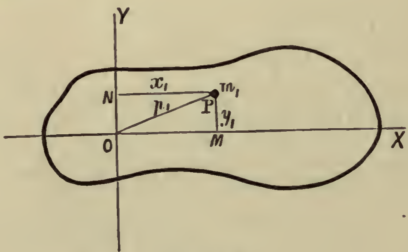


Fig. 85.

Let OX, OY be the two axes at right angles (Fig. 85). Let m_1, m_2, &c., be the masses of the various particles of which the lamina is composed; let y_1, y_2, &c. be their respective perpendicular distances from OX; x_1, x_2, &c., their respective perpendicular distances from OY; and

r_1, r_2, &c., their respective distances from O. Then $r_1^2 = x_1^2 + y_1^2$, &c. Also r_1, r_2, &c., are the respective perpendicular distances of the particles from the line through O perpendicular to the lamina. Let I_1, I_2, and I, be the moments of inertia about OX, OY and the new axis respectively. Then

$$\begin{aligned} I &= m_1 r_1^2 + m_2 r_2^2 + \&c. \\ &= m_1 (x_1^2 + y_1^2) + m_2 (x_2^2 + y_2^2) + \&c. \\ &= m_1 y_1^2 + m_2 y_2^2 + \&c. \\ &\quad + m_1 x_1^2 + m_2 x_2^2 + \&c. \end{aligned}$$

But

$$I_1 = m_1 y_1^2 + m_2 y_2^2 + \ldots \quad \text{and} \quad I_2 = m_1 x_1^2 + m_2 x_2^2 + \ldots;$$

$$\therefore \quad I = I_1 + I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \textbf{(3)}.$$

268. The principle of parallel axes.—*Given the moment of inertia of a lamina about any axis in its plane through its centre of gravity, required to find its moment of inertia about a parallel axis also in the plane of the lamina at a distance h from the original axis.*

Let AB be the axis through G, the centre of gravity of the lamina; and CD the parallel axis (Fig. 86). Let the

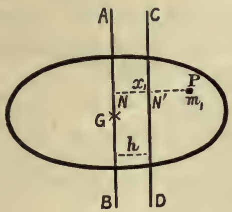


Fig. 86.

moments of inertia about AB and CD be I and I' respectively.

Let M be the total mass of the lamina; m_1, m_2, &c. the masses of the particles of which it is composed; x_1, x_2, &c., their distances from AB, the signs of x_1, x_2, &c. being reckoned positive or negative according as the particles are on the positive or negative side of AB.

Then $$I = \Sigma\,(mx^2).$$

Again, the distance of m from CD will be $x_1 - h$*; and similarly for m_2, &c.

Thus

$$I' = m_1\,(x_1 - h)^2 + m_2\,(x_2 - h)^2 + \&c.,$$

or, on squaring out and rearranging,

$$= m_1 x_1^2 + m_2 x_2^2 + \&c.$$
$$- 2h\,(m_1 x_1 + m_2 x_2 + \&c.)$$
$$+ h^2\,(m_1 + m_2 + \&c.)$$
$$= \Sigma\,(mx^2) - 2h\Sigma\,(mx) + Mh^2,$$

where

$M = \Sigma\,(m) =$ total mass of the lamina.

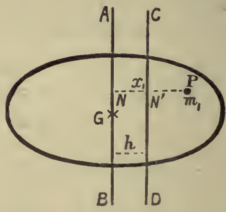


Fig. 87.

Now the distance of G from $AB = 0$; which gives, by a well-known proposition in Statics,

$$\frac{\Sigma\,(mx)}{\Sigma\,(m)} = 0;$$

hence $$\Sigma\,(mx) = 0.$$

Thus $$I' = \Sigma\,(mx^2) + Mh^2$$

$$\therefore\quad I' = I + Mh^2 \quad\text{.......................}\quad (4).$$

Now Mh^2 would be the moment of inertia about CD of a single particle of mass M at G.

Hence the moment of inertia of a lamina about any axis in its plane is equal to the moment of inertia about the parallel axis through the centre of gravity, together with the moment of inertia about the given axis of the whole mass collected at the centre of gravity.

This property is known as the **principle of parallel axes.**

The same principle is also true for a solid body about any axis, or a lamina about an axis not in its own plane; but the proof is more complicated.

* This is true, taking account of the sign of x_1, no matter on which side of AB m_1 may lie.

Example.—To deduce the Principle of Parallel Axes for a lamina when the axes are perpendicular to its plane.

Let the axes AB, CD cut the lamina in its centre of gravity G and in O. In the plane of the lamina, draw $Y'GY$, $H'OH$ perpendicular to GO, and produce GO to X. Let A, B be the moments of inertia of the

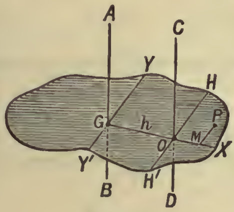


Fig. 88.

lamina about GX, GY, I and I' its moments of inertia about AB and CO respectively.

Then, by what has just been shown, if $GO = h$, the moment of inertia about $HOH = B + Mh^2$, and then, by § 267, we have

$$I = A + B, \quad I' = A + B + Mh^2; \quad \therefore \; I' = I + Mh^2.$$

269. The following well-known theorems in Algebra will be used in the ensuing paragraphs :—

$$(a) \quad 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2};$$

$$(b) \quad 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6};$$

$$(c) \quad 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

The sums in these formulæ may be written

$$\Sigma(n), \; \Sigma(n^2), \; \Sigma(n^3).$$

In what follows we shall in general suppose the bodies to be of *uniform density*, so that the masses of equal volumes (however small) in different parts of the same body are equal. In dealing with *laminæ*, or thin flat sheets of matter, we shall suppose them of uniform thickness and density, so that the masses of equal areas are equal.

270. To determine the moment of inertia of a rectangular lamina about one side.

Required the moment of inertia of the rectangle $ABCD$, about the side AD (Fig. 89).

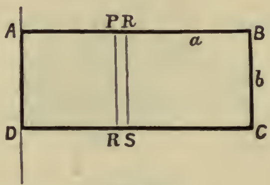


Fig. 89.

Let $AB = a$, $BC = b$, total mass $= M$, and let the lamina be divided into n equal strips by lines parallel to AD, where n is some very large number.

Then the breadth of each strip $= a/n$, and the mass of each strip $= M/n$.

Let $PQRS$ be the qth strip from AD; then $AQ = qa/n$. Also, since this strip is very narrow, all the particles which compose it are approximately at a distance ra/n from AD.

Hence the moment of inertia of this strip about AD

$$= \frac{M}{n} \times \left(\frac{qda}{n}\right)^2 = \frac{Ma^2}{n^3} q^2.$$

The moment of inertia of all the strips about AD will be obtained by giving to q in this expression the successive values 1, 2, 3, ..., n.

Thus the moment of inertia of the lamina about AD

$$= \text{sum of moments of inertia of the strips about } AD$$

$$= \text{the limit of } \frac{Ma^2}{n^3} . 1^2 + \frac{Ma^2}{n^3} . 2^2 + \frac{Ma^2}{n^3} . 3^2 + \ldots + \frac{Ma^2}{n^3} . n^2$$

$$= \frac{Ma^2}{n^3}(1^2 + 2^2 + 3^3 + \ldots + n^2) = \frac{Ma^2}{n^3} . \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{Ma^2}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right).$$

Now, if we make n infinite, this result will be no longer merely approximate, but exact; thus we obtain the value of the moment of inertia as $\frac{Ma^2}{3}$ .... (5).

Cor.—If I_0 be the moment of inertia of the rectangle about a line through the C.G. parallel to AD, then, by the principle of parallel axes, since the distance of the C.G. from AD is $\frac{1}{2}a$,

$$\frac{Ma^2}{3} = I_0 + M\left(\frac{a}{2}\right)^2;$$

$$\therefore \quad I_0 = \frac{Ma^2}{3} - \frac{Ma^2}{4} = \frac{Ma^2}{12} \text{ .... (6).}$$

271. **Radius of gyration.**—Suppose a mass M equal to that of the lamina in the last paragraph collected at a point at distance k from AD. Then the moment of inertia of this mass about AD is Mk^2. This will be equal to the actual moment of inertia of the lamina, if $k = a/\sqrt{3}$. $a/\sqrt{3}$ is then called the radius of gyration of the lamina about AD. Hence we have the following:—

Definition.—The **radius of gyration** of a body about a given axis is the distance from that axis at which a particle of equal mass to the body must be placed in order that it should have the same moment of inertia about that axis.

272. **To find the moment of inertia of a straight line** *about an axis through one extremity, perpendicular to the line.*

This may be deduced at once from § 270, for, if $b = 0$, the rectangle becomes a line; and the required moment of inertia is $Ma^2/3$ .... (7).

273. **The moment of inertia of a parallelogram** $ABCD$ about the side AB can be found by dividing its area into an infinitely large number of equal narrow strips by lines parallel to AB. If h is the perpendicular distance between AB and DC, and there are n strips, the distance of the qth strip from AB is qh/n, and its mass is M/n; whence, as in the case of a rectangle, we find that the moment of inertia about AB $= \frac{1}{3}Mh^2 = \frac{1}{3}M \cdot BC^2 \sin^2 ABC$.

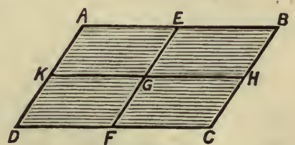


Fig. 90.

274. To find the moment of inertia of a triangle *about a line through either vertex parallel to the opposite side.*

Let ABC be a triangle of mass M. Draw AX parallel to, and AN perpendicular to, BC (Fig. 91); let

$$AN = p, \quad BC = a.$$

Divide AN into a large number (n) of equal parts.

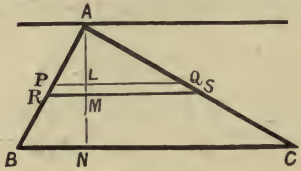


Fig. 91.

Let LM be the qth of these parts, counting from A. Divide the triangle into strips by lines through the points of division of AN, and let $PQRS$ be the strip cutting AN in LM. Then the area of the strip is practically equal to $LM \times SR$, since the strip is very narrow.

Now, by similar triangles,

$$SR : BC = AR : AC = AM : AN = q : n;$$

whence $SR = aq/n$. Also $LM = p/n$. Hence

$$\text{area of } PQRS = qpa/n^2.$$

Again,

$$\text{mass of } PQRS : \text{mass of } ABC = \text{area of } PQRS : \text{area of } ABC$$
$$= qpa/n^2 : pa/2;$$

whence $\qquad$ mass of $PQRS = 2qM/n^2$.

Also, distance of this mass from the axis AX

$$= AM = qp/n.$$

$\therefore$ the moment of inertia of $PQRS$ about the axis AX

$$= \frac{2qM}{n^2}\left(\frac{q}{n}\right)^2 p^2 = 2Mp^2\,\frac{q^3}{n^4}.$$

Hence, giving to q the successive values 1, 2, 3, ..., n, and summing, we see that the moment of inertia of the whole triangle about AX

$$= \text{the limit of } 2Mp^2 \times \frac{1^3 + 2^3 + \ldots + n^3}{n^4}$$

$$= 2Mp^2 \times \frac{n^2(n+1)^2}{4n^4} = 2Mp^2 \left\{ \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \right\}.$$

Putting $n = \infty$, we obtain

$$\text{moment of inertia} = \frac{Mp^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(8)}.$$

Cor.—The moment of inertia about the side BC can easily be deduced from the principle of parallel axes. Let I and I_0 be the moments of inertia about BC and about a parallel to BC through the C.G. Then, since the distances of the C.G. from BC and from AX are $\frac{1}{3}p$ and $\frac{2}{3}p$, respectively, we have, by (6),

$$\frac{Mp^2}{2} = I_0 + M(\tfrac{2}{3}p)^2 \quad \text{and} \quad I = I + M(\tfrac{1}{3}p)^2,$$

whence $$I_0 = \frac{Mp^2}{18} \quad \text{and} \quad I = \frac{Mp^2}{6} \quad \ldots\ldots\ldots\ldots \text{(9)}.$$

Example.—To find the moment of inertia of a rectangle about a diagonal.

Let $ABCD$ be the rectangle (Fig. 92); let $AB = a$, $BC = b$, and let the mass of the rectangle $= M$. Draw AN perpendicular to BD.

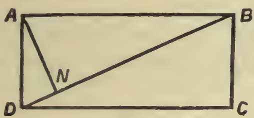


Fig. 92.

Then, moment of inertia of $ABCD$ about BD

= sum of moments of inertia about BD of $\triangle$s ABD, CBD

= twice moment of inertia of $\triangle ABD$

$$= 2\frac{M}{2} \cdot \frac{AN^2}{6}. \qquad (\S\ 274,\ \text{Cor.})$$

Also $$AN \cdot BD = 2 \text{ area of } \triangle ABD = ab;$$

$$\therefore\ AN = \frac{ab}{BD} = \frac{ab}{\sqrt{a^2+b^2}}.$$

Hence required moment of inertia

$$= \frac{M}{6} \cdot \frac{a^2b^2}{a^2+b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}.$$

275. The moment of inertia of a uniform circular wire *of radius r about an axis through its centre perpendicular to its plane* is obviously

$$= Mr^2 \quad \dots\dots\dots\dots \quad \textbf{(11)}.$$

For every particle is at the same distance r from the axis.

We can now determine the moment of inertia of a circular wire about a diameter.

Let this be I. Then if we take two diameters, at right angles, and apply the theorem of § 267, we see that the sum of the moments of inertia about these two diameters = moment of inertia about the axis through the centre perpendicular to the plane of the wire; *i.e.*, since the moments about the two diameters are equal,

$$2I = Mr^2;$$

whence

$$I = \frac{Mr^2}{2} \quad \dots\dots\dots\dots \quad \textbf{(12)}.$$

276. To find the moment of inertia of a circular lamina of radius *r about an axis through its centre perpendicular to its plane.*

Let OA be a radius of the lamina (Fig. 93). Suppose OA divided into a large number (n) of equal parts, and

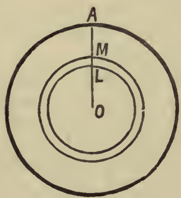


Fig. 93.

suppose the lamina divided into n concentric rings by means of circles drawn with centre O, through the points of division of OA.

Let LM be the pth division. Then the area of the ring corresponding to LM is practically the circumference of the ring × its breadth

$$= 2\pi OM \times ML = 2\pi \frac{pr}{n} \cdot \frac{r}{n}.$$

Again, mass of the ring : mass of the lamina = area of ring : area of lamina; whence mass of the ring

$$= \frac{2pM}{n^2}.$$

Hence moment of inertia of this ring about the given axis

$$= \frac{2pM}{n^2} \cdot \frac{p^2r^2}{n^2} = \frac{2Mr^2}{n^4} \cdot p^3.$$

Hence moment of inertia of the lamina

$$= \text{sum of moments of the rings}$$

$$= \text{limit of } \frac{2Mr^2}{n^4}(1^3 + 2^3 + \ldots + n^3)$$

$$= \frac{Mr^2}{2}\left(1 + \frac{2}{} + \frac{1}{n^2}\right). \qquad (\S\, 269)$$

Making n infinite, this reduces to

$$\frac{Mr^2}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(13)}.$$

By a similar argument to that of § 275, we see that the moment of inertia of a circular lamina about a diameter is given by

$$I = \frac{Mr^2}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(14)}.$$

Cor.—The moment of inertia of a solid cylinder of radius r about its axis is also equal to $\frac{1}{2}Mr^2$, because the cylinder can be split up into thin laminæ by planes perpendicular to its axis, and the moment of inertia of each lamina is equal to its mass multiplied into $\frac{1}{2}r^2$.

277. **Sphere, and Spherical Shell.** We give the following results without proof:—

The moment of inertia of a uniform thin *spherical surface* of radius a about a diameter $= \frac{2}{3}Ma^2$ .... **(15)**.

That of a *solid sphere* of radius a about a diameter

$= \frac{2}{5}Ma^2$ .... **(16)**.

Example.—To find the moment of inertia of a hollow sphere of radius a containing a concentric cavity of radius b.

Let d be the density of the matter in the hollow sphere.

The moment of inertia is clearly the difference between that of a solid sphere of radius a of density d, and that of the matter which would fill the cavity, *i.e.*, that of a solid sphere of radius b and of the same density d.

The masses of the solid spheres are $\frac{4}{3}\pi a^3 d$ and $\frac{4}{3}\pi b^3 d$; hence the mass of the hollow sphere is $\quad \frac{4}{3}\pi(a^3-b^3)d = M$, say.

The moments of inertia of the two spheres are therefore

$$\frac{4}{3}\pi a^3 d \times \frac{2}{5}a^2 \quad \text{and} \quad \frac{4}{3}\pi b^3 d \times \frac{2}{5}b^2. \qquad \text{(from above)}$$

Hence the moment of inertia of the hollow sphere

$$= \frac{4}{3}\pi a^3 d \times \frac{2}{5}a^2 - \frac{4}{5}\pi b^3 d \times \frac{2}{5}b^2 = \frac{4}{3}\pi d \times \frac{2}{5}(a^5-b^5)$$

$$= \frac{2}{5}M\frac{a^5-b^5}{a^3-b^3} \quad \textbf{(17).}$$

278. **Problem.**—*Given the moments of inertia of a lamina about two perpendicular axes in its plane, with respect to one at least of which it is symmetrical; required to find its moment of inertia about another axis in its plane through their point of intersection making an angle* α *with one of the given axes.*

Let m_1 be the mass of any particle at P. Let OX, OY be the two given axes (Fig. 94), and let PN and PM, the perpendiculars on OY and

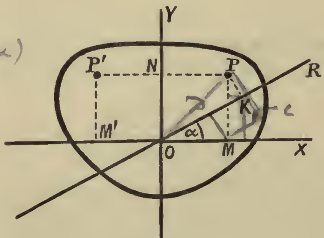


Fig. 94.

OX, be x_1 and y_1, respectively. Suppose the lamina symmetrical with respect to OY. Draw PK perpendicular to the new axis OR.

Then it can be easily shown that

$$PK = y_1 \cos\alpha - x_1 \sin\alpha.$$

Thus the moment of inertia of m_1 about OR is

$$m_1(y_1\cos\alpha - x_1\sin\alpha)^2.$$

Hence the moment of inertia of the lamina is

$$m_1(y_1\cos\alpha - x_1\sin\alpha)^2 + m_2(y_2\cos\alpha - x_2\sin\alpha)^2 + \&c.$$
$$= \cos^2\alpha\,(m_1y_1^2 + m_2y_2^2 + \&c.) + \sin^2\alpha\,(m_1x_1^2 + m_2x_2^2 + \&c.)$$
$$-2\sin\alpha\cos\alpha\,(m_1x_1y_1 + m_2x_2y_2 + \&c.).$$

Now, since the lamina is symmetrical to OY, there will be another particle of equal mass to m_1 (say m_r) at a point P' on PN produced to the opposite side of OY, such that $NP' = PN$. Hence

$$M'P' = MP \quad \text{and} \quad NP' = PN;$$

whence $$y_r = y_1 \quad \text{and} \quad x_r = -x_1.$$

Also $$m_r = m_1.$$

Hence $$m_rx_ry_r + m_1x_1y_1 = 0.$$

Since all the particles can be paired off in this way, it follows that

$$m_1x_1y_1 + m_2x_2y_2 + \&c. = 0.$$

Hence the moment of inertia about OR

$$= \cos^2\alpha\,(m_1y_1^2 + m_2y_2^2 + \&c.) + \sin^2\alpha\,(m_1x_1^2 + m_2x_2^2 + \&c.)$$
$$= I_1\cos^2\alpha + I_2\sin^2\alpha \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(18)},$$

where I_1 and I_2 are the given moments of inertia about OX and OY respectively.

*279. **If the body were not symmetrical about either** OX **or** OY, the equation for I would generally take the form

$$I = I_1\cos^2\alpha + I_2\sin^2\alpha - 2\sin\alpha\cos\alpha\,\Sigma(mxy).$$

$\Sigma(mxy)$ is called the *product of inertia* about the two axes OX, OY (9A).

If this product of inertia be known as well as I_1 and I_2, the moment of inertia of the lamina about any line through O in its plane can be found.

**Examples.*—(1) Given the mass of a lamina, the position of its centre of gravity, and the moments of inertia about any three lines in its plane, no two of which are parallel, to find the moment of inertia about any other line in the plane of the lamina.

First, let the three lines intersect in the C.G. of the lamina, and let them make angles α, β, γ with the axis of x. Then, if A, B, C are the moments of inertia about them, we have, with the notation of the above paragraph,

$$A = I_1\cos^2\alpha + I_2\sin^2\alpha - 2\Sigma(mxy)\sin\alpha\cos\alpha,$$
$$B = I_1\cos^2\beta + I_2\sin^2\beta - 2\Sigma(mxy)\sin\beta\cos\beta,$$
$$C = I_1\cos^2\gamma + I_2\sin^2\gamma - 2\Sigma(mxy)\sin\gamma\cos\gamma.$$

These three equations suffice to determine I_1, I_2, and $\Sigma(mxy)$.

The moment of inertia about any other line through G can now be found, and the moment of inertia about any line not through G can be deduced from the principle of parallel axes.

If the lines about which the moments of inertia are A, B, C do not pass through G, let them be at distances a, b, c from G. By the principle of parallel axes, the moments of inertia about three lines through G parallel to them are $A-Ma^2$, $B-Mb^2$, and $C-Mc^2$, respectively, and, these being known, we proceed as in the first case.

*(2) *The moment of inertia of a triangular lamina of mass M about any axis is the same as that of three equal particles of mass $\frac{1}{3}M$ placed at the middle points of its sides.*

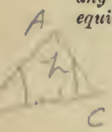
The three particles have the same mass and the same C.G. as the triangle; hence we only have to show that their moments of inertia about three straight lines in the plane of the triangle are the same.

Take these lines to be the sides of the triangle. If p is the distance of A from BC, the middle points of AB, AC, BC are distant $\frac{1}{2}p$, $\frac{1}{2}p$, and 0 from BC. Hence the moment of inertia of the particles about BC

$$= \tfrac{1}{3}M(\tfrac{1}{2}p)^2 + \tfrac{1}{3}M(\tfrac{1}{2}p)^2 + 0 = \tfrac{1}{6}Mp^2$$

= moment of inertia of triangle about BC. (§ 274, Cor.)

Similarly, the moments of inertia of the particles and triangle about CA and AB are the same, and therefore their moments of inertia about *any* straight line are the same. The particles are therefore said to be *equimomental* to the lamina.



Summary of Results.

The *moment of inertia* of a system of particles

$$= \Sigma mr^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Kinetic energy of a rotating body $= \frac{1}{2}Iw^2$ .... (2).

The moments of inertia of a lamina are connected by the relation $I = I_1 + I_2$ (§ 267) .... (3).

The Principle of Parallel Axes gives

$$I' = I + Mh^2 \quad (\S\,268) \quad \ldots\ldots\ldots \quad (4).$$

The moments of inertia of a symmetrical lamina about different directions give

$$I = I_1\cos^2 a + I_2 \sin^2 a \quad (\S\,278) \quad \ldots \quad (5).$$

Moment of inertia of

a *rectangle* about one edge $= \frac{Ma^2}{3}$ .... (6),

,, about parallel axis through centre $= \frac{Ma^2}{12}$ (7);

a *straight line* about one end $= \frac{Ma^2}{3}$ .... (8);

a *triangle* about line through vertex parallel to one side

$= \frac{Mp^2}{2}$ .... (9),

where p is the altitude,

,, about line through C.G. parallel to side

$= \frac{Mp^2}{18}$ .... (10),

,, about one side $= \frac{Mp^2}{6}$ .... (11);

a *rectangle* about a diagonal $= \frac{M}{6} \cdot \frac{a^2b^2}{a^2+b^2}$ .... (12);

a *circular wire* about an axis through its centre, perpendicular to its plane $= Mr^2$ .... (13),

,, about a diameter $= \frac{Mr^2}{2}$ .... (14);

a *circular lamina* about an axis through its centre perpendicular to its plane $= \frac{Mr^2}{2}$ ...... (15),

,, about a diameter $= \frac{Mr^2}{4}$ .... (16);

a *spherical surface* about a diameter $= \frac{2}{3}Ma^2$ .... (17);

a *solid sphere* about a diameter $= \frac{2}{5}Ma^2$ .... (18);

a *triangle* about any axis is same as that of masses $\frac{1}{3}M$ at middle points of sides .... (19).

EXAMPLES XIX.

1. ABC is an equilateral triangle, whose sides are 1 ft. long, D, E, F are the middle points of its sides. If masses of 1 lb. are placed at A, B, C, and masses of 2 lbs. at D, E, F, find the moment of inertia of the system of masses (i.) about BC, (ii.) about DE, (iii.) about a line through the centre of gravity parallel to BC.

2. In the last example find the moments of inertia of the system about axis drawn at right angles to the plane of the triangle (i.) through A, (ii.) through D, (iii.) through the centre of gravity of the triangle.

3. Taking a foot and a pound as units, the moment of inertia of a certain body is 54. What is its amount when a yard and an ounce are taken as units?

4. The measure of a certain moment of inertia is 260 when a centimetre and a gramme are units. What is its measure (i.) when a millimetre and a kilogramme are units, (ii.) when a metre and a milligramme are units, (iii.) when a metre and a kilogramme are units?

5. If equal masses m be placed at the corners of a regular hexagon whose side is a, find their moment of inertia (i.) about a side of the hexagon, (ii.) about a diagonal, (iii.) about an axis perpendicular to the plane of the hexagon through its centre, (iv.) about an axis perpendicular to the plane through an angular point. [The Principle of Parallel Axes is not to be assumed.]

6. Verify that the Principle of Parallel Axes holds good in connection with the first and second results of Example 5, as also in connection with the third and fourth.

7. A rectangle without mass, whose sides are 2 metres and 1 metre long, has masses of 1 kilogramme placed at the four corners, and masses of 5 kilogrammes at the middle points of the two longest sides. Taking a kilogramme and a metre as units, find the moments of inertia of the system of masses (i.) about the sides of the rectangle, (ii.) about its diagonals, (iii.) about parallels to the sides through its centre of gravity.

8. Deduce from § 270 the moment of inertia of a rectangle about an axis through its centre parallel to one side, not using § 268.

9. Hence determine its moment of inertia about an axis half-way between one side and its centre. (Use § 268.)

10. Deduce from § 273 the moment of inertia of a parallelogram about one side.

11. Determine the moment of inertia of a triangle about a median.

12. Find the moment of inertia of a circular wire about a tangent.

13. Find the moment of inertia of a circular lamina about a tangent.

14. Verify that the moment of inertia of a triangle about a lino through its C.G. parallel to either side is equal to the moment of inertia of three particles placed at the middle points of the sides, the mass of each particle being one-third that of the triangle.

15. From a circular lamina of radius a, a concentric aperture of radius b is cut out. Prove that the moment of inertia of the remainder about an axis through the centre perpendicular to the plane

$$= \tfrac{1}{2}M(a^2 + b^2).$$

16. A wire of mass M and length $3a$ is bent into the form of an equilateral triangle. Find its moment of inertia about an axis through its centre of gravity perpendicular to its plane.

17. A wire of mass M and length $4a$ is bent into the form of a square. Find its moment of inertia about a side of the square.

18. Find the moment of inertia of a thin cylindrical shell about its axis, the ends of the cylinder being made of material of the same thickness and density as the curved surface.

CHAPTER XX.

PRINCIPLE OF WORK FOR RIGID BODIES.

280. In the last chapter (§ 266) we proved that the kinetic energy of a rigid body rotating about a fixed axis with angular velocity w is $\frac{1}{2}Iw^2$, where the moment of inertia I can be calculated for bodies of certain given shapes. By assuming the Principle of Conservation of Energy, we are now able to investigate very easily the motion of certain rigid bodies when acted on by given forces.

It will be necessary to remember that, in the case of a body acted on by gravity, the work done by gravity in any change of position is the same as if the mass of the body were all collected at its centre of gravity, and is therefore equal to Mgh dynamical units of work, M being the mass of the body and h the vertical depth through which its centre of gravity has fallen.

We may introduce the subject by the following example:—

Example.—A rectangular lamina $ABCD$ is free to revolve about the edge AB, which is horizontal. If it be held horizontally and then let go, determine its angular velocity when passing through its vertical position; given $AB = 3$ ft., $BC = 4$ ft.

Let w be the required angular velocity; then the kinetic energy of the lamina

$$= \tfrac{1}{2}Iw^2 = \tfrac{1}{2}\frac{M\,.\,16}{3}w^2 = \frac{8Mw^2}{3}. \qquad (\S\ 270)$$

Also, loss of potential energy in falling from the horizontal to the vertical position = work done by gravity = weight × vertical distance through which the centre of gravity has fallen $= Mg \times 2$.

But *loss of potential energy = gain of kinetic energy;*

whence

$$2Mg = \frac{8Mw^2}{3};$$

$$\therefore \quad w = \sqrt{\frac{3g}{4}} = 2\sqrt{6} \text{ (radians per sec.)}.$$

281. To find the acceleration in Atwood's Machine when the inertia of the pulley is taken into account.

Let the two masses P, Q be connected by a string passing over a pulley. Let a be the radius of the pulley

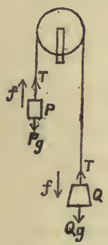


Fig. 95.

in the groove where the string passes, I the moment of inertia of the pulley about its axis.

Let v be the velocity of the two masses when the heavier, Q, has fallen, and the lighter, P, has risen, through a distance s.

Then, if w be the angular velocity of the pulley, it is clear that the velocity of the particles of the pulley in contact with the string is wa, and hence, if the string does not slip, $w \cdot a = v$, whence $v = wa$.

The total kinetic energy of the masses and pulley

$$= \tfrac{1}{2}Pv^2 + \tfrac{1}{2}Qv^2 + \tfrac{1}{2}Iw^2 = \tfrac{1}{2}v^2(P + Q + I/a^2).$$

Also the work done by gravity

$$= (Q - P)\,gs.$$

Equating these, we have, by the Principle of Work,

$$\tfrac{1}{2}v^2(P + Q + I/a^2) = (Q - P)\,gs;$$

$$\therefore \quad v^2 = 2\,\frac{Q - P}{Q + P + I/a^2}\,gs.$$

Comparing this with the equation for uniformly accelerated motion $v^2 = 2fs$, we have

$$\text{acceleration } f = \frac{Q - P}{Q + P + I/a^2}\,g.$$

If M is the mass and k the radius of gyration of the pulley, we have $$I = Mk^2,$$

and $$\therefore\quad f = \frac{Q-P}{Q+P+Mk^2/a^2}\,g \quad \text{...........} \ \mathbf{(1)}.$$

The acceleration is therefore the same as if the pulley were without mass, and masses equal to $\frac{1}{2}Mk^2/a^2$ were added to each of the two masses P and Q.

NOTE.—In the present case, the tensions in the *two parts of the string are not equal.* The difference between these tensions is the force producing motion of the pulley.

Example.—To determine the value of g by two observations made with Atwood's Machine, the moment of inertia of the pulley being unknown.

Let F be the observed acceleration when the mass P descending pulls the mass Q up. Let other masses p, q be substituted for P and Q, and let the new acceleration be observed to be f. Then

$$F = \frac{Q-P}{Q+P+I/a^2}\,g, \quad f = \frac{q-p}{q+p+I/a^2};$$

I/a^2 is unknown, and must therefore be eliminated. Writing the equations—

$$Q+P+I/a^2 = (Q-P)\,g/F,$$
$$q+p+Ia^2 = (q-p)\,g/f,$$

we have, by subtraction,

$$(Q+P)-(q+p) = \left\{\frac{Q-P}{F} - \frac{q-p}{f}\right\} g;$$

whence g may be found in terms of the known accelerations F, f, and the known masses P, Q, p, q.

282. **Instantaneous centre of rotation of a rolling wheel.**

If a wheel is rolling, without slipping, over any surface, its motion at any instant may be regarded as a rotation round that point which is in contact with the surface.

For suppose, first, that the wheel, instead of being circular, is in the shape of a regular polygon of a large number of sides. Then, as it rolls over the surface, the different angular points of the polygon come successively in contact with the surface; and, if there is no slipping, the wheel revolves round the one angular point till the next comes into contact with the surface. But, if we

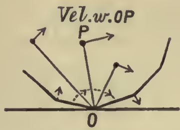


Fig. 96.

suppose the number of sides to be indefinitely increased, the polygon becomes a circle; whence the theorem follows.

For this reason the point of the wheel in contact with the surface is called the **instantaneous centre of rotation.**

283. **To determine the magnitude and direction of the velocity of any point on a rolling wheel** *of radius r, which is travelling with velocity v along a fixed plane.*

Let A be the centre of the wheel, O the point of contact with the plane, and P the given point on the wheel (not necessarily on the circumference). (Fig. 97.)

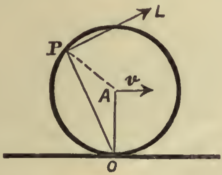


Fig. 97.

Let $OP = d$, and let w be the angular velocity with which the wheel is revolving.

Draw PL perpendicular to OP.

(i.) To determine w:—

Since the wheel is travelling with velocity v, the velocity of A is $= v$.

But, since O is the instantaneous centre of rotation, the velocity of A is $= w \,.\, OA = wr$; (§ 234)

$$\therefore \quad wr = v; \quad \text{whence} \quad w = v/r.$$

(ii.) To determine the velocity of P.

The magnitude of P's velocity is

$$w \,.\, OP = \frac{v}{r} \,.\, d.$$

and its direction is perpendicular to OP, *i.e.*, along PL.

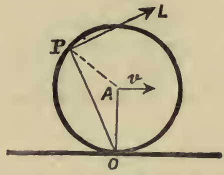


Fig. 97.

Alternative method.—Consider first the motion relative to the centre of the wheel. Since the wheel is turning with angular velocity w, the velocity of any point P on the wheel relative to A is $w \,.\, AP$.

(i.) Hence, since $AO = r$, the velocity of O relative to A is wr perpendicular to AO, and the velocity of A relative to O is equal and opposite:

$$\therefore \quad v = wr \quad \text{and} \quad w = v/r.$$

(ii.) The velocity of P is compounded of the velocity of P relative to A and the velocity of A.

The former is $w \,.\, AP$ perpendicular to AP, and the latter is $w \,.\, OA$ perpendicular to OA. These velocities are perpendicular to the sides AP, OA of the triangle AOP; hence, if the triangle AOP were turned through a right angle, it would become a Triangle of Velocities. Hence the resultant velocity of P is perpendicular to the third side OP, and its magnitude is $w \,.\, OP$.

284. To find the acceleration of a wheel rolling down a rough inclined plane.

[The plane is supposed to be sufficiently rough to prevent the point of contact of the wheel from slipping.]

Let M be the mass of the wheel, r its radius, k its radius of gyration about an axis through its centre perpendicular to the plane of the wheel, and let A be the angle of inclination of the plane.

Let v be the velocity of the centre of the wheel when it has moved through a distance s. Then, by § 282, the wheel is at this instant rotating about its point of contact with the plane, with angular velocity w equal to v/r; hence, by § 266, the kinetic energy of the wheel is $\frac{1}{2}Iw^2$ or Iv^2/r^2, where I is its moment of inertia about an axis *through the point of contact*, perpendicular to its plane.

The work done by gravity

$$= Mgs \sin A.$$

Equating these, we have

$$\tfrac{1}{2}Iv^2/r^2 = Mgs \sin A,$$

whence $$v^2 = 2\frac{Mr^2}{I} g \sin A \,.\, s.$$

Comparing this with $$v^2 = 2fs,$$

we have $$f = \frac{Mr^2}{I} g \sin A \text{ .... (i.).}$$

Now the moment of inertia about an axis through the centre perpendicular to the wheel is Mk^2.

Hence, by the Theorem of Parallel Axes,

$$I = Mk^2 + Mr^2.$$

Substituting this value in (i.), we have

$$f = \frac{r^2}{r^2+k^2} g \sin A \text{ .... (2).}$$

Hence *the acceleration of the wheel is to the acceleration of a body sliding down a smooth plane of the same inclination in the ratio of* r^2 *to* r^2+k^2.

COR.—For a *circular hoop*,

$$k = a \ (\S\, 275); \quad \therefore \quad f = \tfrac{1}{2}g \sin A.$$

For a *circular disc* or *cylinder*,

$$k^2 = \tfrac{1}{2}r^2; \qquad \therefore \quad f = \tfrac{2}{3}g \sin A.$$

For a *thin spherical shell*,

$$k^2 = \tfrac{2}{3}r^2; \qquad \therefore \quad f = \tfrac{3}{5}g \sin A.$$

For a *solid sphere*,

$$k^2 = \tfrac{2}{5}r^2; \qquad \therefore \quad f = \tfrac{5}{7}g \sin A.$$

By observing the acceleration of a sphere or cylinder rolling down an inclined plane, the value of g could be found.

Examples.—(1) To find the acceleration of a truck running down an incline on wheels.

Let M be the mass of the truck; $m_1, m_2, \ldots$ the masses of the wheels; $r_1, r_2, \ldots$ their radii; $k_1, k_2, \ldots$ their radii of gyration about the axes through their centres; $I_1, I_2, \ldots$ their moments of inertia about their points of contact with the ground. Let A be the inclination of the incline, and, if the truck does not start from rest, let the velocity change from u to v when the truck moves over a distance s.

Then the initial and final kinetic energies are, respectively,

$$\tfrac{1}{2}(Mu^2 + I_1u^2/r_1^2 + I_2u^2r_2^2 + \ldots),$$

and a similar expression with v written for u.

Remembering that $\quad I_1 = m_1(k_1^2 + r_1^2)$, &c.,

and equating the increase of kinetic energy to the work done, we have

$$\tfrac{1}{2}\left(M + m_1\frac{r_1^2 + k_1^2}{r_1^2} + m_2\frac{r_2^2 + k_2^2}{r_2^2} + \ldots\right)(v^2 - u^2)$$
$$= (M + m_1 + m_2 + \ldots)\,gs\sin A.$$

Comparing this equation with $\quad v^2 - u^2 = 2fs, \quad$ we have

$$\text{acceleration } f = \frac{M + \Sigma m}{M + \Sigma m + \Sigma(mk^2/r^2)}\,g\sin A,$$

where the symbol Σ indicates summation for the several wheels on which the truck stands.

(2) Two spheres of the same size and mass are exactly of the same outward appearance, but one is solid and the other contains a hollow cavity, the hollow one being made of matter of greater density so as to make the masses equal. To find which is the hollow sphere.

Allow the two spheres to roll down an inclined plane, starting simultaneously from rest. Then, from the formula

$$f = \frac{r^2}{r^2 + k^2}\,g\sin A,$$

we see that the sphere for which k^2 is the greatest will have the least acceleration.

Now the particles of the solid sphere are distributed uniformly throughout its volume, while those of the hollow sphere all lie outside the cavity. It is clear that the matter of the hollow sphere lies on the whole further from the centre than that of the solid one, and therefore further from any diameter, and hence the value of k^2 (which may be regarded as the mean square of the distance of the particles from the axis) is greater for the hollow sphere.

Hence the acceleration of the hollow sphere will be less than that of the solid one, and, by observing the two, they may be distinguished.

285. **Compound Pendulum.**—DEFINITION.—Any body which is capable of swinging to and fro about a fixed axis under the action of gravity may be called a **pendulum.** Such a pendulum is frequently referred to as a **compound** pendulum, to distinguish it from the simple pendulum consisting of a single suspended particle as considered in Chap. XVII.

Two pendulums which will oscillate through equal angles in equal times are said to be **equivalent.**

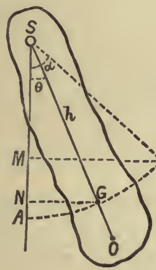


Fig. 98.

286. A body oscillates under gravity about a fixed horizontal axis. To prove that this pendulum is equivalent to a simple pendulum, and to find the length of the latter.

Let a body of mass M be suspended from a fixed horizontal axis through S, perpendicular to the plane of the diagram (Fig. 98). Let G be the C.G. of the body, so that in equilibrium SG is in the vertical position SZ.

Let the body be drawn aside from the equilibrium position through an angle α (so that its C.G. is brought to H), and then let go.

Then, if w is the angular velocity acquired when SG makes an angle θ with the vertical and I the moment of inertia about the fixed axis,

$$\text{the kinetic energy} = \tfrac{1}{2}Iw^2,$$

Fig. 98.

and the work done by gravity

$$= Mg \times \text{vertical distance of } G \text{ below } H$$
$$= Mg \times (SN - SM)$$
$$= Mgh\,(\cos\alpha - \cos\theta), \quad \text{where} \quad h = SG.$$

Equating these, we have

$$\tfrac{1}{2} Iw^2 = Mgh\,(\cos\alpha - \cos\theta)\,;$$

$$\therefore \quad w^2 = 2\frac{Mh}{I} g\,(\cos\alpha - \cos\theta) \;\ldots\ldots\ldots\ldots\ldots\; \text{(i.)}.$$

This equation gives the angular velocity after falling from the initial position through an angle $\alpha - \theta$.

Now compare the motion with that of a simple pendulum of mass m and length l, initially drawn aside through the same angle α and allowed to fall through the same angle $\alpha - \theta$. If w is the angular velocity acquired, the velocity of the mass is lw, and the equation of energy gives

$$\tfrac{1}{2} ml^2 w^2 = mgl\,(\cos\alpha - \cos\theta),$$

or

$$w^2 = \frac{2}{l} g\,(\cos\alpha - \cos\theta) \;\ldots\ldots\ldots\ldots\ldots\; \text{(ii.)}.$$

Comparing (i.) and (ii.), we see that the value of w will be the same in the two pendulums, provided that

$$\frac{1}{l} = \frac{Mh}{I} \quad \text{or} \quad l = \frac{I}{Mh}.$$

If l have this value, then, since the angular velocities of the simple and compound pendulums after describing equal angles are equal, it follows that the angular motions of the two pendulums are identical, so that if started together they will continue to swing together, describing the same angles in the same times.

Hence *the body is equivalent to a simple pendulum of length I/Mh.*

Now let k be the radius of gyration of the body about an axis through its centre of gravity parallel to the fixed axis. The moment of inertia about the new axis is therefore Mk^2, and hence, by the Theorem of Parallel Axes,

$$I = M(k^2+h^2).$$

Substituting, we see that **the length of the simple pendulum is given by**

$$l = \frac{k^2+h^2}{h} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(3)}.$$

Cor.—If a simple pendulum oscillates through a *small* angle, we know that its time of oscillation is $2\pi\sqrt{(l/g)}$. Hence **the time of a small oscillation of the body** is given by

$$T = 2\pi\sqrt{\frac{k^2+h^2}{hg}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(4)}.$$

287. **Centres of suspension and oscillation.**—The point S about which the body swings is called the **centre of suspension,** and, if on SG a length SO is marked off equal to l, the length of the simple equivalent pendulum, the point O is called the **centre of oscillation.**

From this we have the following :—

Definition.—The **centre of oscillation** of a compound pendulum is the point at which a single mass would have to be placed in order that it should oscillate in the same time as the original pendulum.

[Note that the time of oscillation is *not* the same as if the mass were collected at its centre of *gravity.*]

Examples.—(1) A rod of length $2a$ oscillates about a point distant h from its middle point. To find the length of the simple equivalent pendulum.

Here $$k^2 = \tfrac{1}{3}a^2,$$

and $$l = \frac{k^2+h^2}{h} = \frac{\frac{1}{3}a^2+h^2}{h} = \frac{a^2}{3h}+h.$$

(2) A pendulum consists of a sphere of mass M and radius a suspended from S by a rod of mass m and length b. To find the length of the simple equivalent pendulum.

The centre of the sphere is at a distance $a+b$ from S.

Moment of inertia of rod about S

$$=\tfrac{1}{3}mb^2.$$

Moment of inertia of sphere about centre

$$=\tfrac{2}{5}Ma^2.$$

Therefore moment of inertia of sphere about S

$$=\tfrac{2}{5}Ma^2+M(a+b)^2.$$

Therefore moment of inertia of whole pendulum

$$=\tfrac{1}{3}mb^2+M\left\{\tfrac{2}{5}a^2+(a+b)^2\right\}.$$

Distance SG of centre of gravity

$$=\frac{m\,.\,\tfrac{1}{2}b+M(a+b)}{m+M},$$

and total mass

$$=m+M.$$

Hence [by the formula $l=I(Mh)$] we have

$$l=\frac{\tfrac{1}{3}mb^2+M\left\{a^2+(a+b)^2\right\}}{(m+M)\dfrac{m\,.\,\tfrac{1}{2}b+M(a+b)}{m+M}}$$

$$=\frac{\tfrac{1}{3}mb^2+M\left\{\tfrac{2}{5}a^2+(a+b)^2\right\}}{\tfrac{1}{2}mb+M(a+b)}.$$

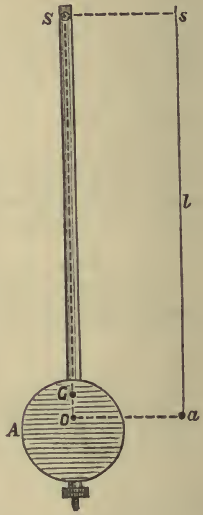


Fig. 99.

288. **The centres of suspension and oscillation are convertible.**

From (1) we have

$$SO=l=\frac{k^2+h^2}{h}=\frac{k^2}{h}+h=\frac{k^2}{SG}+SG;$$

$$\therefore\quad GO=SO-SG=\frac{k^2}{SG},$$

and $$GO\,.\,SG=k^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; \textbf{(5)}.$$

In this formula, the lengths GO, GS are interchangeable; thus we obtain

$$GS=\frac{k^2}{OG},\ \text{ and }\ OS=\frac{k^2}{OG}+OG.$$

Fig. 100.

It follows that, if the body were inverted and suspended from O, the centre of oscillation would be S; in other words, *the centres of suspension and oscillation are convertible.* We notice that, the length of the equivalent pendulum being equal to OS, whether the body be suspended from S or from O, the times of oscillation about O and S are equal.

COR.—On any straight line through G there are four points about which the times of oscillation will be equal. For, if we cut off $GS' = GS$ and $GO' = GO$, the time of oscillation about S' is clearly the same as about S (the length of the simple equivalent pendulum being

$$= \frac{k^2}{GS'} + GS' = \frac{k^2}{GS} + GS;$$

and that about O' is clearly the same as that about O. We notice that the four points are symmetrically situated in pairs on opposite sides of G, that

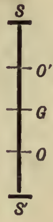


Fig. 101.

$$OS = O'S' = l,$$

and that $$GS \cdot GO = GS' \cdot GO' = k^2.$$

*289. **Captain Kater's Pendulum** is a loaded bar which can be suspended from either of two parallel axes formed by knife edges in the same plane as its C.G. and on opposite sides of it. By varying the positions of the loads on the bar or of the knife-edges (all of which are adjustable), the period of a small oscillation about either knife-edge may be varied.

To find the length of the seconds' pendulum at any place, and thus determine the intensity of gravity, the knife-edges and loads are so arranged (by repeated trials) that the period of a complete small oscillation about either knife-end is exactly 2 secs. (§ 245), but the centre of gravity is *not* midway between the knife-edges. Since the periods about the two centres of suspension are thus equal, we know that each is the centre of oscillation about the other; hence the distance between them is equal to the length of the simple equivalent pendulum, *i.e.*, the seconds' pendulum. By measuring the distance between the knife-edges, this length is found, and hence g is determined.

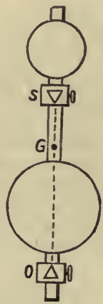


Fig. 102.

***290. Motion of any rigid body about a fixed axis.**

A rigid body is free to turn about a fixed axis AB, and is acted on by given forces. It is required to investigate the motion.

Let P be a small portion of the body, of mass m, small enough to be regarded as a particle, m its mass, r its distance PN from the axis AB.

Then, if w be its angular velocity, its actual velocity will be rw perpendicular to the plane ABP, and its acceleration will consist of two components—that along PN being of magnitude w^2r (since the particle is revolving in a circle of radius r), and the component perpendicular to the plane APB being

$$= \text{rate of change of the velocity } wr$$

$$= r \times \text{rate of change of } w$$

$$= rw',$$

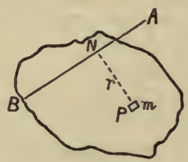


Fig. 103.

where w' represents the rate of change of the angular velocity w; so that w' may be called the *angular acceleration* of the particle.

It follows that the resultant forces producing motion of the particle are mw^2r along PN and mrw' perpendicular to the plane APB. Let these be called the "effective forces" on P. If, then, additional forces equal and opposite to these were applied to the particle, they would destroy its acceleration, and, if similar sets of forces were applied to the whole body, they would keep it in equilibrium.

Hence the sum of the moments about AB of the actual forces producing motion, minus that of the effective forces, is equal to zero.

Now the forces such as mw^2r have no moment about AB, and the moment of the force mrw'

$$= mrw' \times r = mr^2w'.$$

Hence sum of moments of impressed forces

$$= \Sigma mr^2w'$$

$$= w' \times \Sigma mr^2$$

(since w' is the same for all the particles)

$$= w'I,$$

where I is the moment of inertia about AB.

Hence, for angular motion about AB, the analogue of the equation of linear motion,

impressed force = mass × acceleration,

is impressed moment = moment of inertia × angular accel.

We can thus find the angular acceleration of a body made to revolve about a fixed axis by given forces.

*291. **Angular momentum.**—Suppose P is a particle of mass m moving with velocity V in any given direction PQ, and let AB be any axis not in the same plane as PQ. Resolve the velocity V into two rectangular components, one in and one perpendicular to the plane APB, and let these be u and v. Drop PN perpendicular on the axis AB, and let $NP = r$. Then the product mvr is called the **angular momentum** or **moment of momentum** of the particle

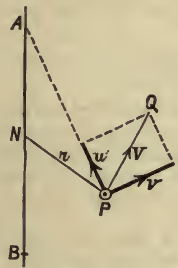


Fig. 104.

about AB. Hence the following:—

Definition.—The **angular momentum of a particle** about a given axis is thus the product of its distance from the axis into the resolved part of its momentum perpendicular to the plane through the particle and the axis.

The angular momentum is thus the moment of the momentum about the axis formed in the same way as the moment of a force.

Definition. — The **angular momentum of a system of particles or a rigid body** about a given axis is equal to the sum of the angular momenta of the particles of which it is composed.

In the case of a rigid body rotating about the fixed axis AB, with angular velocity w, the velocity of any particle P is $w.NP$

perpendicular to the plane APB; hence the angular momentum of the mass m at P $= mwr^2$, where $r = NP$.

Therefore the angular momentum of the body

$$= \Sigma mr^2 w = w \times \Sigma mr^2 = wI \quad \text{.....................} \quad (6),$$

where I is the moment of inertia about AB.

It follows from § 290 that the rate of change of the angular momentum about AB is equal to the moment of the impressed forces about AB.

*292. **General Equations of Motion of a Plane Body.**—When a plane lamina is moving in its plane under the action of forces in that plane, it is known from statical considerations that the forces can be replaced by a single force at the centre of gravity and a couple whose moment equals the sum of the moments of the forces about the centre of gravity. If P denotes the force, L the couple, M the mass of the body, and k its radius of gyration about its centre of gravity, then the acceleration of the centre of gravity takes place in the direction of P; and, just as in the case of a particle, it is given by the equation $P = Mf$ .... (1).

If w is the angular velocity of the body about its C.G., the rotational motion is determined by the equation

$$L = Mk^2 \times (\text{rate of change of } w) \quad \text{..................} \quad (2).$$

Equation (1) expresses the fact that the rate of change of the translational momentum is equal to the resultant of the applied forces, and equation (2) expresses the fact that the rate of change of the angular momentum about the C.G. is equal to the moment of the applied forces about that point.

It may also be shown that the rate of change of the angular momentum of any body or system of bodies about a *fixed* axis is equal to the moment of the impressed forces about that axis. If then a system is acted on by no forces beyond the mutual actions and reactions of its parts, the angular momentum of the system about any fixed axis remains constant; and the same is the case when the only forces acting on the body intersect or are parallel to the fixed axis. This property is called the **Principle of Conservation of Angular Momentum.**

SUMMARY OF RESULTS.

Acceleration in Atwood's Machine, taking account of inertia of pulley, is

$$\frac{Q-P}{Q+P+Mk^2/a^2}g \text{ .... } (1).$$

Instantaneous centre of a rolling wheel is at its point of contact with ground.

For acceleration of *wheel rolling down inclined plane*

$$f=\frac{r^2}{r^2+k^2}g\sin A \text{ .... } (2).$$

A *compound pendulum* is equivalent to a simple pendulum of length

$$l=\frac{k^2+h^2}{h} \text{ .... } (3).$$

The *centres of suspension and oscillation* are convertible, and

$$GO.SG=k^2 \text{ .... } (5).$$

Angular momentum of a body rotating about a fixed axis

$$=wI \text{ .... } (6).$$

EXAMPLES XX.

1. A circular wire, of radius 1 ft., is free to turn about a horizontal axis through a point in its rim, perpendicular to its plane. If it be held in such a position that the centre of the wire is on the same level as the axis, and then let go, determine its angular velocity when passing through the position of equilibrium.

2. Solve the corresponding problem, given a circular lamina instead of a circular wire.

3. Two masses, P and Q, are respectively attached to the rim and axle of a wheel and axle, whose radii are a and b respectively. If the moment of inertia of the wheel and axle is I, determine the velocity of the mass P after falling s ft.

4. A mass P is attached to the end of a light inextensible string, which is wound round the rim of a wheel of radius r, free to turn in its own vertical plane, about its centre. If the moment of inertia of the wheel is I, determine the velocity of the mass after falling s ft.

5. If a wheel is rolling on any surface, plane or curved, prove that at any given instant the direction of motion of all points on the rim passes through that point which is farthest from the point of contact.

6. A wheel is rolling on a horizontal plane. Compare the velocities of the highest point, the centre, and either extremity of the horizontal diameter.

7. A piece of mud is thrown off from the top of a cab-wheel. Prove that, when it falls to the ground, the distance between it and the point of the wheel then in contact with the ground will be equal to the distance moved by the wheel since it was thrown off.

8. A wheel is rolling along a horizontal road, and a piece of mud is thrown off from its hindmost point. Prove that it will just touch the wheel again as it falls.

9. A wheel is rolling on a horizontal plane. Find at any instant the locus of those points on the wheel whose direction of motion passes through a given point not vertically above the point of contact.

10. A railway carriage is moving at 30 miles an hour; its wheels are 3 ft. in diameter. Determine their angular velocity.

11. A top whose moment of inertia is I foot-pound units is spun by pulling a string of length l feet wound round its axis. Supposing a force of P lbs. weight exerted in pulling the string, what is the angular velocity with which the top spins?

12. Compare the times in which a sphere and a cylinder of the same radius roll down a rough inclined plane.

13. A fine piece of string is wound round a heavy solid sphere, one end being attached to the sphere, and the other end being fixed up. Prove that, if the sphere be allowed to fall so that the string uncoils, its acceleration will be $\frac{5}{7}g$.

14. Find the corresponding acceleration of a falling solid cylinder round which a string has been wound in the same way, supposing the axis of the cylinder to remain horizontal.

15. One side of a uniform lamina, in the form of a square of side $2a$, is attached to a horizontal bar by two smooth hinges equidistant from the ends of the side. The lamina is held in a horizontal position and is allowed to fall *in vacuo* under the action of gravity. Determine its angular velocity when it is vertical, and find the length of a simple pendulum which will acquire the same angular velocity, if let fall in a similar manner.

16. A triangular lamina swings about one of its sides which is horizontal. Prove that the length of the simple equivalent pendulum is half the altitude of the triangle.

17. Find the lengths of the simple equivalent pendulums in the following cases:—

(i.) An isosceles triangle of base a and altitude h suspended from its vertex and swinging in its own plane.

(ii.) The same triangle swinging about an axis through its vertex parallel to the base.

(iii.) A square hung up by one corner and swinging in its own plane.

(iv.) A rod of length $2a$ placed at the bottom of a spherical bowl of radius r and oscillating about its position of equilibrium in a vertical plane.

18. The radius of gyration of a certain body about a fixed axis through its centre of gravity is k. Prove that, if the body be suspended about a parallel axis, the time of oscillation cannot be less than $2\pi\sqrt{(2k/g)}$.

19. P, Q, R are three points on the rod of a compound pendulum such that $PQ = QR$, and the time of oscillation (t) about each is the same. Prove that, if T is the minimum time of oscillation of the pendulum (see the last example), $3t^4 = 4T^4$.

20. A door consisting of a uniform rectangular lamina 7 ft. high and 3 ft. wide is hinged in such a way that when swinging to after being opened its centre of gravity falls 1 inch. Find the angular velocity acquired.

21. A long rod AB, hinged at A to a horizontal plane, rests on a smooth cylinder of radius a which is moving along the plane toward A with velocity v. Prove that the angular velocity of the rod is $2\sin^2\frac{\theta}{2}\cdot\frac{v}{a}$, where θ is the inclination of the rod to the plane.

22. In the last example, the masses of the rod and cylinder are m and M, and the length of the rod is l. Apply the principle of energy to find the greatest value of v in order that the rod may not be overturned; and, supposing v to be less than this value, find the inclination of the rod to the horizon when it and the cylinder come to rest.

23. Two masses P and Q are connected by a string passing over an Atwood pulley whose radius is a, mass M, and radius of gyration k. Find the acceleration of the masses, the angular acceleration of the pulley, and the tensions of the two portions of the string, by writing down the separate equations of motion of the masses and pulley (the latter expressing the fact that the rate of change of the angular momentum is equal to the difference of the moments of the tensions of the two parts of the string).

24. Find the *total* angular momentum about the axis of rotation of the *whole system* consisting of the masses and pulley of the previous example. Verify that the rate of change of this total angular momentum is equal to the difference of the moments of the weights of P and Q about the axis.

25. A lamina is moving in any manner in its plane. If x, y are the coordinates of any particle of mass m of the lamina, u, v its component velocities parallel to the axes of x and y, prove that the angular momentum of the lamina about an axis through the origin perpendicular to its plane is equal to $\Sigma m(vx-uy)$.

26. If on the lamina of the last question there be impressed an additional velocity whose components are U and V and are the same for every particle, prove that the angular momentum of the whole is increased by an amount

$$M(VX-UY)$$

where M is the whole mass of the lamina, and X, Y the coordinates of its centre of mass.

EXAMINATION PAPER XI.

1. Find the kinetic energy of a rigid body rotating about a fixed axis with angular velocity w.

2. Prove that the moment of inertia of a uniform plane lamina about an axis OC perpendicular to its plane at O is equal to the sum of its moments of inertia about any two perpendicular lines OA, OB in the plane at O.

3. Find the moment of inertia of a triangular lamina about a side by direct summation of the moments of inertia of the thin strips into which it may be divided by equidistant lines parallel to the sides.

4. Given that the moments of inertia of a lamina about two lines OA, OB in its plane are I_1 and I_2, and that the lamina is symmetrical about one of them, find the moment of inertia about any other axis through O in the same plane; and, if $I_1 = I_2$, prove that the new moment of inertia is equal to either of them.

5. Two masses, P and Q, are attached to the ends of a light rough inextensible string, which passes over a pulley of radius r. Given that the moment of inertia of the pulley about its axis is I, determine, by the Principle of Conservation of Energy, the velocity of P after it has fallen s ft. from rest.

6. Find the acceleration of a hoop rolling down a rough inclined plane.

7. Prove that the centres of suspension and oscillation of a compound pendulum are convertible, and that the time about each is the same.

8. A walking-stick, which may be regarded as a thin uniform rod 3 feet long, is stood vertically on end on a horizontal plane, and topples over from this position of unstable equilibrium. If the plane be rough enough to prevent slipping, find the angular velocity of the stick, and the velocity of its upper end when it strikes the plane (assume $g = 32$).

EXAMPLES XXI. (Miscellaneous.)

1. Two stations are $1\frac{1}{3}$ miles apart. A goods train starts from rest at one of them, moving with uniform acceleration 6 ins. per sec. per sec., until the steam is shut off and the brakes are applied so as to bring it to rest at the other station. If the brakes cause a retardation of 5 ft. per sec. per sec. in the train, at what distance from the second station must they be applied?

2. A bicyclist, riding at 20 miles an hour, passes a horseman, who immediately starts off in pursuit. The horse can gallop at 30 miles an hour, and he attains this speed with an acceleration of 11 ft. per sec. per sec. Find how far the cyclist has gone before the horseman overtakes him.

3. Two trains on the same line are approaching one another with velocities U and u, respectively. When there is a distance s between them, each is seen from the other. Prove that it is just possible to avoid a collision if $u^2F + U^2f = 2Ffs$, where F and f are the greatest retardations which the brakes can produce in the respective trains. [Omit the possibility of one train being able to move back before the other is brought to rest.]

4. An express train is overtaking a goods train on the same line; their velocities are U and u, respectively. When there is a distance s between them, each is seen from the other. Prove that it is just possible to avoid a collision if $(U-u)^2 = 2(F+f)s$, where F is the greatest retardation and f the greatest acceleration which can be produced in the two trains respectively.

5. A train weighs M tons, and the resistance of friction is p lbs. per ton. If the engine can exert a pull of P lbs., and the brake a resistance of R lbs., find the distances passed over in attaining a speed of v miles per hour from rest, and in slowing down from that speed to rest respectively.

6. A pile of mass 160 lbs. is driven into the ground by a weight of $\frac{1}{2}$ ton, which is repeatedly let fall on it from a height of 16 ft. If each blow drives the pile 1 ft. farther in, determine the average resistance of the ground.

7. A steam hammer of mass $\frac{1}{2}$ ton is hammering a red-hot plate of steel. At every stroke the hammer falls 10 ft.; the average force exerted by the steam pressure behind the hammer is $4\frac{1}{2}$ tons weight. If the hammer compresses the steel 3 ins. in one blow, determine the average pressure of the hammer on the steel.

8. A hammer-head of mass 14 oz., travelling with velocity 16 ft. per sec., strikes an inelastic nail of mass 2 oz., and drives it 2 ins. into a fixed block of wood. Determine in lbs. wt. the average resistance of the wood to the nail.

9. Suppose that in the last question the block of wood weighs 20 lbs., and is free to move along a smooth horizontal plane; also suppose the average resistance of the wood to the nail to be 50 lbs. wt. Determine how far the nail will penetrate; also find the final velocity of the block.

10. A mass m with initial velocity v penetrates into a mass M initially at rest. The average resistance to penetration is R. Prove that the mass m penetrates a distance $\frac{v^2Mm}{2R(M+m)}$, and that the final velocity of the system is $\frac{mv}{M+m}$.

11. A rifle bullet of mass 1 oz., travelling with velocity 1,000 ft. per sec., passes through a block of wood 1 ft. thick, weighing 50 lbs. If it leaves the wood with velocity 200 ft. per sec., determine the final velocity of the wood.

12. Given a smooth perfectly elastic plane and two points in space on the same side of it, determine geometrically in what direction a particle must be projected from the one point in order that it should pass through the other after impact with the plane. [Gravity *not* being taken into account.]

13. Given two smooth perfectly elastic planes and two points in space; find how to project a particle from one point in order that, after impact with each plane, it should pass through the other point. (See Ex. 12.)

14. A bullet of mass m travelling with velocity u passes through the centre of a sphere of wood of mass M and diameter d, which was

originally at rest on a smooth horizontal surface. If the bullet leave the sphere with velocity v, find (i.) the final velocity of the sphere; (ii.) the time occupied by the bullet in penetrating; (iii.) the average pressure between the bullet and the sphere.

15. A particle of elasticity e is projected in a direction inclined to the vertical, and bounces along a smooth horizontal plane. The range of one rebound is r; find the range of the next.

16. A heavy slab, whose under surface is rough, but the upper smooth, slides down a given inclined plane. Find the acceleration with which a small particle laid on its upper surface will move along the slab. [Given inclination of plane $= \alpha$, coefficient of friction $= \mu$.]

17. A heavy particle slides from the top of a smooth sphere of diameter 2 ft. Find at what point it will leave the sphere.

18. A flexible heavy string, length $2l$, is moving over a smooth fixed peg, the two unequal portions hanging vertically. Prove that at the instant when its middle point is at a distance x below the peg the acceleration of the motion is xg/l.

19. Determine the tension at the middle point of the string in the last question.

20. A weight W hangs by a string over a pulley. A monkey takes hold of the other end, and at an instant when W is at rest commences to climb. He climbs h ft. in t secs. without disturbing W. Determine his motion, and find his weight. If at the end of t secs. he cease to climb, how much farther will he ascend in the next t secs.?

21. Two monkeys of masses M and m start climbing, each with uniform acceleration, up the two ends of a rope which is hanging over a pulley. Find the relation between their accelerations if the rope always remains in exactly the same position.

22. A light wheel of radius 3 ins., which is rigidly connected with a light axle of radius 1 in., is free to revolve in a vertical plane. A string wound round the rim of the wheel carries a weight of 5 lbs., and a string wound round the axle in the opposite direction carries a weight of 9 lbs. If the system be left to itself, determine the acceleration of each weight.

23. A heavy uniform perfectly flexible string is placed over a perfectly smooth peg in such a way that a length L ft. hangs over one side, and l ft. over the other. The string is then left free to move. Prove that the velocity of the string just as it leaves the peg is $\sqrt{2Llg/(L+l)}$.

24. A weightless rod 30 cm. long is hinged at one end; masses 2, 4, and 6 grammes are attached to it at distances 10, 20, and 30 cm., respectively from the hinge. If it is held in a horizontal position, and then let go, find the angular velocity when the rod is vertical, and also the pull on the hinge.

25. Equal heavy particles are attached to the middle and end of a light rod. The other end is fixed. If the system be set rotating about the fixed end, prove that the tensions in the two parts of the rod are in the ratio 3 : 2.

26. If H be the greatest height of a projectile, R the horizontal range, and V the velocity of projection, prove that $R^2 = 16H(\frac{1}{2}V^2/g - H)$.

27. A 111-ton Armstrong gun, whose muzzle is 16 ft. above the ground, discharges a projectile horizontally to strike the ground 750 yds. off. Neglecting the resistance of the air and the friction of the gun, compare the work done by gravity with that done by the gunpowder on the projectile.

28. If two particles be projected from the same point in the same vertical plane with equal velocities u, in different directions, so as to have the same range R, find the difference between their times of flight.

29. A particle is placed on a rough horizontal plate ($\mu = \cdot\dot{6}$) at a distance of 9 ins. from a vertical axis about which the plate can turn. Find the greatest number of revolutions per minute the plate can make without causing the particle to slip upon it.

30. Two guns are pointed at each other, one upwards at the angle of elevation α, and the other downwards at the same angle of depression, the muzzles being 100 ft. apart. If the bullets leave the muzzles with velocities 2,200 and 1,800 ft. per sec., prove that they will meet, and find after what time.

31. Let $O, A_n, A_{n-1}, \ldots A_2, A_1, A$ be points arranged in this order on a straight line. A body starting from rest at A moves towards O under the influence of a uniform force $\frac{1}{2}\mu(OA + OA_1)$ until it reaches

A_1. Then the force becomes $\frac{1}{2}\mu(OA_1 + OA_2)$, and remains so until the body reaches A_2. Then the force becomes $\frac{1}{2}\mu(OA_2 + OA_3)$, and remains so until the body reaches A_3, and so on; and whilst the body is moving from A_n to O the force is uniform and equal to $\frac{1}{2}\mu(OA_n)$. Prove that the body will reach O with velocity $OA\sqrt{\mu}$.

Apply the result to determine the velocity with which a body moving in a straight line under the influence of a force, varying as the distance from a point on a straight line, will arrive at the centre of force.

32. O, B, A are three points arranged in this order on a straight line. A particle is projected from A towards B with velocity u, and reaches B with velocity v, being acted on during the motion by a force which produces uniform acceleration $\mu \div (OB \cdot OA)$ towards O.

Prove that $$\frac{1}{2}(v^2 - u^2) = \mu\left(\frac{1}{OB} - \frac{1}{OA}\right).$$

33. O, A_n, A_{n-1}, ... A_2, A_1, A are points arranged in this order on a straight line. A particle is projected from A towards A_n, with velocity u, and reaches A_n with velocity v, being acted on during the motion by a force of the following nature:—Whilst the particle moves from A to A_1, it produces uniform acceleration $\mu/(OA \cdot OA_1)$; whilst the particle moves from A_1 to A_2, it produces uniform acceleration $\mu/(OA_1 \cdot OA_2)$; whilst the particle moves from A_2 to A_3, it produces uniform acceleration $\mu/(OA_2 \cdot OA_3)$; and so on.

Prove that $$\frac{1}{2}(v^2 - u^2) = \mu\left(\frac{1}{OA_n} - \frac{1}{OA}\right).$$

34. In Example 33 show how, by keeping the points A_n, A fixed in position, and making n infinitely great, to prove the formula

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C;$$

where v is the velocity, and μ/r^2 the acceleration at the distance r from O, and C a constant.

35. A point moves in a plane so that its projection on the axis of x performs a harmonic vibration of period 1 second with an amplitude of 1 foot, whilst its projection on the axis of y (which is at right angles to the axis of x) performs a harmonic vibration of period 2 seconds with an amplitude of 1 foot. Find the equation of the path of the point, it being given that the point, whose coordinates, measured in feet, are 1, 0, is on the path. Draw the path.

RESULTS IN MENSURATION.

The following facts in Solid Geometry and Mensuration are assumed. The references given below are to the articles in Briggs and Edmondson's *Mensuration*, where the reader will find the properties in question fully proved. Proofs of them are also given in most elementary treatises on Solid Geometry. The *results* alone need be remembered:—

(1) **The area of a triangle**

$$= \frac{1}{2}\ (base) \times (altitude). \qquad (\S\ 45.)$$

(2) **The area of a trapezoid** (*i.e.* a quadrilateral with two sides parallel) $= (its\ height) \times (\frac{1}{2}\ sum\ of\ parallel\ sides)$. (§ 49.)

(3) **The length of the circumference of a circle** of radius r

$$= \pi \times (diameter)$$
$$= 2\pi r; \qquad (\S\ 57.)$$

where the Greek letter π ("pi") stands for a certain "incommensurable" number (that is, a number which cannot be expressed as an exact arithmetical fraction), whose value lies between 3·141592 and 3·141593. The following approximate values should be remembered and used, unless otherwise stated.

$$\pi = \frac{22}{7}, \quad \text{for all } rough \text{ calculations;}$$

$$\pi = 3\text{·}1416, \text{ more approximately.}$$

(4) **The area of the circle**

$$= \frac{1}{2}\ (radius) \times (circumference)$$
$$= \pi r^2. \qquad (\S\ 58.)$$

(5) **The volume of a pyramid**

$$= \frac{1}{3}\,(\textit{height}) \times (\textit{area of base})$$

$$= \frac{1}{3}\,hA, \qquad (\S\ 105.)$$

the height h being the perpendicular from the vertex on the plane of the base, and A the area of the base.

(6) **The area of the curved surface of a cylinder,** whose height is h and the radius of whose base is r,

$$= (\textit{height}) \times (\textit{circumference of base})$$

$$= 2\pi rh. \qquad (\S\ 115.)$$

(7) **The volume of the cylinder**

$$= (\textit{height}) \times (\textit{area of base})$$

$$= \pi r^2 h. \qquad (\S\ 116.)$$

(8) **The area of the curved surface of a right circular cone,** whose height is h and the radius of whose base is r,

$$= \frac{1}{2}\,(\textit{circumference of base}) \times (\textit{length of slant side})$$

$$= \pi r \sqrt{(h^2+r^2)}\,; \qquad (\S\ 117.)$$

a *slant side* being a line drawn from the vertex to a point in the circumference of the base.

(9) **The volume of the cone**

$$= \frac{1}{3}\,(\textit{vol. of cylinder of same base and height})$$

$$= \frac{1}{3}\,\pi r^2 h. \qquad (\S\ 118.)$$

(10) **The area of the surface of a sphere** of radius r

$$= 4 \textit{ times area of circle of same radius}$$

$$= 4\pi r^2. \qquad (\S\ 126.)$$

(11) **The volume of the sphere**

$$= \frac{1}{3}\,(\textit{radius}) \times (\textit{surface})$$

$$= \frac{4}{3}\,\pi r^3. \qquad (\S\S\ 127,\ 128.)$$

EXAMINATION QUESTIONS IN DYNAMICS

FROM THE

SCIENCE AND ART PAPERS.

(*ADVANCED STAGE.*)

1885.

1. (*a*) It is said that a horse can do about 13,200,000 ft.-lbs. of work in a day of 8 hours, walking at the rate of $2\frac{1}{2}$ miles per hour. What pull (in pounds) could such a horse exert continuously during the working day? (*b*) How many such horses would be required to do as much work as an engine of 10 horse-power, working day and night?

2. (*a*) A body in motion is observed to increase its velocity in every second by $5\frac{1}{3}$ ft. per sec.; how far would it move from rest in 12 secs.? If the body has a mass of 10 lbs., what is the numerical value of the force producing the motion, (*b*) in absolute units, (*c*) in gravitation units? ($g = 32$.)

3. A particle slides along a rough horizontal plane; find the retardation of its velocity.

   If the coefficient of friction between the particle and the plane is 0·05, and the velocity of the particle at a certain point 40 ft. a second, at what distance from that point will it come to rest, and after what time?

4. Two masses, P and Q, are connected by a fine thread passing over a perfectly smooth fixed horizontal cylinder; the mass of P is greater than that of Q; P is allowed to descend through a distance h, drawing up Q; at the instant the distance has been described, part of P falls off, leaving only P_1, the mass of which is less than that of Q; find how far P_1 will descend.

5. A particle, whose mass is 4, moving with a velocity 12, meets and impinges directly on a particle whose mass is 8 and velocity 4; the coefficient of restitution is 0·5; find, from first principles, their velocities at the end of the impact, and what part of their joint kinetic energy has disappeared in the impact.

1886.

1. Define a *foot-pound* of work, and a *horse-power*. A steam-crane, working with 3 horse-powers, is found to raise a weight of 10 tons to a height of 50 ft. in 20 minutes; what part of the work is done against friction? If the crane is kept at similar work for 8 hours, how many foot-pounds of the work are wasted on friction?

2. Find the position of a body at the end of a given time from the instant at which it is thrown with a given velocity in a given direction, the motion being supposed to take place *in vacuo*. A body is thrown in a direction making an angle of 30° with the horizon, and passes through a point whose horizontal distance from the point of projection $400\sqrt{3}$ ft., and vertical height above the point of projection 76 ft.; find the velocity of projection. $(g = 32.)$

3. A particle, whose mass is 10, moving with a velocity 5, meets and impinges directly on another particle whose mass is 20 and velocity 3; the coefficient of restitution is 0·125; find from first principles the velocities of the particles at the end of the impact.

State the dynamical principles employed in answering this question, and define the coefficient of restitution.

4. (*a*) A flywheel weighs 10,000 lbs., and is of such a size that the matter composing it may be treated as if concentrated on the circumference of a circle 12 ft. in radius; what is its kinetic energy when moving at the rate of 15 revolutions a minute? $(g = 32, \pi = 3{\cdot}1416.)$

(*b*) How many turns would it make before coming to rest, if the steam were cut off and it moved against a friction of 400 lbs. exerted on the circumference of an axle 1 ft. in diameter?

1887.

1. A shaft, 560 ft. deep and 5 ft. in diameter, is full of water; how many foot-pounds of work are required to empty it, and how long would it take an engine of $3\frac{1}{2}$ horse-power to do the work? (N.B.—Of course, it is to be assumed that there is no flow of water into the shaft. Take $\pi = 3\frac{1}{7}$.)

2. Find the ratio of the height to the length of a smooth inclined plane, down which when a particle slides the acceleration of its velocity is one-fifth of the acceleration of the velocity of a body falling freely under the action of gravity.

If such a particle has a mass of 12 lbs., find its velocity and its kinetic energy acquired in descending along 100 ft. of the length of the plane. $(g = 32.)$

3. A particle describes a circle with a constant velocity; show that the force acting upon it is always directed to the centre, and find the magnitude of the force in terms of the radius of the circle, and the mass and velocity of the particle.

4. (*a*) Define the *moment of inertia* of a system of particles. Find the moment of inertia of a rod of uniform density, with reference to an axis passing through one end at right angles to its length.

(*b*) A rod, 6 ft. long, weighing 12 lbs., revolves uniformly 30 times a minute about an axis at right angles to its length and passing through one end; find its kinetic energy.

1888.

1. Find the time in which a particle will slide down a chord drawn through the highest point of a circle whose plane is vertical.

Find the straight line down which a particle will slide in the shortest time from a given point to a given plane.

2. (*a*) The mass of a particle is m lbs., and its velocity v ft. a second; find the number of foot-pounds of work it can do against a resistance.

(*b*) A particle weighs 10 lbs., and moves at the rate of 1250 ft. a second; find the distance through which it could overcome a resistance of one million pounds.

3. If V is the velocity of a simple pendulum at its lowest point, show that at any time, t, after passing through the lowest point its velocity is $V \cos 2\pi t/T$, where T denotes the time of one complete oscillation.

4. Find the moment of inertia of a rectangular lamina about an edge.

A rectangular lamina, whose shorter edges are 4 ft. long, turns round one of its longer edges 50 times a minute; it weighs 441 lbs. Find its kinetic energy (*a*) in foot-poundals, (*b*) in foot-pounds.

1889.

1. Find the number of foot-pounds of work required to wind up a given chain which hangs by one end.

2. A particle describes the perimeter of a regular hexagon with a constant velocity of 100 ft. a second. Find the magnitude and direction of the velocity that must be communicated to it at the instant it reaches an angular point.

3. Investigate the time of a small oscillation of a simple pendulum.

A seconds pendulum has its length slightly altered, and, in consequence, loses n secs. a day. Find whether it has been lengthened or shortened, and by what fraction of its original length.

4. Find the moment of inertia of a circular lamina of uniform density, with reference to an axis through its centre at right angles to its plane.

If the mass of the lamina is 100 lbs., and its diameter $3\frac{1}{2}$ ft., and if it turns round the axis 120 times a minute, find its kinetic energy (*a*) in foot-poundals; (*b*) in foot-pounds. ($\pi = 3\frac{1}{7}$, $g = 32$.)

1890.

1. Find an expression for the whole amount of work done in raising several weights through different heights.

A uniform beam weighs 1,000 lbs., and is 20 ft. long. It hangs by one end, round which it can turn freely. How many foot-pounds of work must be done to raise it from its lowest to its highest position?

2. State the meaning of each letter in the formula $v^2 = V^2 + 2fs$, and prove the formula.

A particle, whose velocity undergoes a constant acceleration, starts from rest, and, after describing 50 ft., has a velocity of 20 ft. a second. Find the increase of its velocity per second, and the time in which it describes the distance of 50 ft.

3. A body is thrown obliquely *in vacuo*. Find expressions for its horizontal distance from, and its vertical height above, the point of projection after the lapse of a certain number of seconds.

4. A body impinges directly with a given velocity against a fixed plane. Given the coefficient of restitution, find the velocity of rebound

If the mass of a body is 10 lbs., the velocity of impact 20 ft. a second, and the coefficient of restitution 0·5, how many foot-poundals of energy disappear in the collision?

1891.

1. Show that the time in which a particle falls from rest down a chord drawn through the highest point of a vertical circle is constant.

Find the straight line of quickest descent from a point within a given vertical circle to the circumference.

2. Two particles, whose masses are P and Q, are connected by a thread which is placed on a smooth point. If P goes down and draws Q up, find the acceleration of its velocity.

Find the mass of P when the tension of the thread equals three times the weight of Q.

3. Two bodies of given masses, moving with given velocities, impinge on each other directly. Find their velocities after impact, the coefficient of restitution being known.

If one of the bodies is at rest, and its mass is indefinitely greater than that of the other, find the velocity of the second body after impact.

4. Define the *centre of percussion*, and find its position from first principles in the case of a rod of uniform density suspended freely by one end.

1892.

1. Define *angular velocity*. P is a point of a body turning uniformly round a fixed axis, and PN is a line drawn from P at right angles to the axis. If PN describes an angle of 375° in 3 secs., what is the angular velocity of the body, and, if PN is 6 ft. long, what is the linear velocity of P?

2. State Newton's Third Law of Motion, and give his illustration of it. How does it appear that, when one body impinges directly on another, their velocities undergo changes which take place in opposite directions, and are inversely as their masses?

3. If two circles touch each other (internally) at their highest point, and a straight line be drawn through this point, show that the time of falling from rest down the part of the straight line intercepted between their circumferences is constant.

4. Two particles of given masses are connected by an inextensible thread which passes over a smooth point. One of the particles is at rest on a table, while the other descends. Find their common velocity the instant after the thread is drawn straight.

The mass of the particle at rest is 5 lbs.; the mass of the descending particle is 3 lbs., and it falls through 10 ft. before the string is drawn straight. How much of the kinetic energy disappears when the thread is drawn straight?

5. If a particle slides down a rough inclined plane, find an expression for the acceleration of its velocity.

If the inclination of the plane is 30°, and the acceleration is at the rate of 12 ft. a second in each second, find the coefficient of friction. ($g = 32$.)

6. If a curve is drawn in such a way that the ordinates represent

the successive values of a variable force, and the corresponding abscissæ the distance through which it acts, show that the area represents the work done by the force.

7. A particle describes successively the the sides of an equilateral triangle with a constant velocity V. Find the magnitude and direction of the velocity v that must be communicated to it when it comes to an angular point.

Compare the results in this case with those that would be obtained in the case of a regular hexagon.

8. Define *moment of inertia*. Find the moment of inertia of a circular lamina about an axis at right angles to its plane and passing through its centre.

9. A particle, whose mass is 5 lbs., moves at the rate of 20 ft. a second. Express its kinetic energy in foot-poundals. If it moves over a distance of 30 ft. against a constant resistance R, and its velocity is thereby reduced to 15 ft. a second, fiind R in poundals.

10. A particle, whose mass is 10 lbs., is constrained to move in a horizontal circle by a string 5 ft. long fastened to a fixed point. If at any instant the tension of the string is 98 poundals, find the velocity of the particle, and its angular velocity about a fixed point.

1893.

1. Two points, P and Q, move with different velocities along the same line. What is the relative velocity of Q to P?

If Q is allowed to fall freely, and, 2 secs. after, P is allowed to fall freely from the same point, find the relative velocity of Q to P at any subsequent time.

2. What is the numerical value of the angular velocity of a body which turns uniformly round a fixed axis twenty-five times a minute?

ABC is a triangle with a right angle at C. It is turning with a given angular velocity round an axis through A at right angles to its plane. Find the magnitude and direction of the velocities of B and C; find also the relative velocity of B to C.

3. Define an *absolute unit of force* and a *poundal*.

It is sometimes said that a poundal is half an ounce or the weight of half an ounce. Point out (a) the inexactness of the statement, (b) why the statement, even if exact, would not be a definition of a poundal.

4. O, A, B are three points in order along a line Ox. AH is drawn at right angles to Ox. P is a force acting in the direction O to x, and its point of application moves in the direction O to x. The magnitude of P varies inversely as the distance of its point of application from O. Given that its magnitude at A is represented by AH, show how to draw a diagram of the work done while the point of application moves from A to B.

If $OA = 1$ ft., $OB = 9$ ft., $AH = 12$ lbs., draw the diagram of work, and from it determine, approximately, the number of foot-pounds of work done by the force.

5. Prove the formula $s = \frac{1}{2} ft^2$ for the distance described by a particle whose velocity undergoes constant acceleration.

AB is a given straight line, and P a point above it. Find the straight line down which a particle would fall from P to AB in the shortest time.

6. In the conical pendulum, given the mass of the bob, the length of the thread, and the angle of the cone, find the tension of the thread and the time of one revolution of the pendulum.

If the pendulum is 10 ft. long, the half angle of the cone 30°, and the mass of the bob 12 lbs., find the tension of the thread and the time of one revolution.

7. Define the *coefficient of restitution.*

A particle, whose mass is 10, moving with a velocity of 12, meets and impinges directly on a particle, whose mass is 8, moving with a velocity of 7·5. The coefficient of restitution is 0·8, Find, from first principles, the momentum gained by the one particle and lost by the other during the impact, and, hence, the momenta of the particles after impact.

8. Define *moment of inertia.*

If the moment of inertia of a body with reference to an axis passing through the centre of gravity is known, how can the moment of inertia be found with respect to a parallel axis?

9. Define *angular velocity.*

A particle, whose mass is 3 lbs., moves uniformly in a circle. It describes the circumference 42 times a minute. Find its angular velocity about the centre, and, if the radius is 14 ft., find its kinetic energy.

1894.

1. State the rule for the composition of two velocities.

Draw two lines AB, AC containing an acute angle. A particle is at A, moving with a given velocity V from A towards B. Give a construction for determining the velocity that must be impressed on it to make it move with a velocity $2V$ from A towards C.

2. Draw a circle with centre O, and two diameters AB, CD at right angles to each other, and let the direction A, C, B be contrary to that of the motion of the hands of a watch. Produce OA to P, making AP equal to OA. Let the radius of the circle be 20 ft. Suppose a particle to move along the circumference of the circle in the direction A, C, B at the rate of 12 ft. a second. Find its angular velocity with respect to O; find also its angular velocity with respect to P (i.) when it is at A, (ii.) when it is at C, (iii.) when it is at B.

3. State Newton's Three Laws of Motion. Give his illustrations of the Third Law of Motion. Give Newton's proof of the Parallelogram of Forces.

4. Define a *poundal.*

It is known that the acceleration due to the attraction of a planet on a body near its surface varies nearly as the mass of the planet directly, and the square of its radius inversely. It is found that the mass of Jupiter is about 370 times that of the earth, and that his radius is about 11 times that of the earth. With these data, and assuming that the acceleration due to gravity near the earth's surface is 32 in feet and seconds, find the force, in poundals, of Jupiter's attraction on a pound of matter near his surface.

5. A particle slides down a rough inclined plane. Find the acceleration of its velocity.

The angle of inclination of the plane is 30°, and the coefficient of friction is 0·5. If one body falls freely through a given distance, and another body slides down the plane along an equal distance, show that the time of the first body's motion will very nearly equal one fourth part of the time of the second body's motion. ($\sqrt{3} = 1\cdot73205$.)

6. A body is tied to a string and is whirled round in a vertical circle. Find the least velocity it can have at the lowest point if it is to describe the whole circle.

If the mass of the body is 12 lbs. and the length of the string 6 ft., find the tension of the string at the lowest point if the body just stays in the circle. In what units is your answer expressed? ($g = 32$.)

7. A is a particle at rest, and B, moving with a given velocity, impinges on it directly. Find, from first principles, the coefficient of restitution if B is brought to rest by the collision.

The masses of A and B are 12 lbs. and 4 lbs. respectively, and B's velocity is 10 ft. a second. Find how many foot-poundals of kinetic energy disappear in the collision if B is brought to rest.

8. The sides of a rectangle are a and b. Find the moment of inertia about the side a of a diagonal considered as a line of uniform density.

9. Define *angular velocity*.

Draw a straight line and mark on it four points, in order, A, B, C, D. Let AB, BC, CD be respectively 2 ft., 1 ft., 3 ft. Suppose a particle to be at C, moving at right angles to AD at the rate of 4 ft. a second. Find its angular velocity with reference to A, B, and D respectively.

In what respect does the angular velocity with reference to A differ from the angular velocity with reference to D?

1895.

1. Define *power* and *horse-power*.

An engine with its tender weighs 80 tons. It is moving uniformly at the rate of 20 miles an hour, against a resistance of 7 lbs. a ton. At what horse-power is it working?

If it drew after it a train of 12 carriages, each weighing 10 tons, at the rate of 40 miles an hour, against a resistance of 8 lbs. a ton, at what horse-power would it now be working?

2. Define a *poundal* (or British absolute unit of force).

If a force of 5 poundals acts on a mass of 10 lbs. in the direction of the motion, what velocity would it impart to the mass in 3 seconds?

3. A body is thrown upward from the top of a tower with a velocity of 48 ft. a second. Find where it will be at the end of 4 seconds. ($g = 32$.)

Write down the formula or formulæ by means of which you answer this question, and state what it means (or they mean).

4. Draw a square $ABCD$. A particle moves along AB with a velocity 10. It is made to move along BC with a velocity 20. Find the magnitude and direction of the velocity that must be impressed on it at B.

5. A body moves with a constant velocity in a given circle. State what is known as to the force which acts on the body.

Find a numerical result when the mass of the body is 10 lbs., and it moves at the rate of 900 ft. a minute in a circle 3 yds. in diameter.

6. A point (A) is moving with a given velocity (V) along a given line. Another point (B) is moving with a given velocity (v) along a given line intersecting the former. Show how to find the velocity of B relative to A.

7. A point moves along a line. Find its angular velocity with respect to an assigned point outside the line.

Given the line and the fixed point, and that the movable point has a constant velocity along the line, find the position of the movable point when its angular velocity about the fixed point is greatest. Find also its position when its angular velocity is one-fourth of that greatest value.

8. Draw ABC, an equilateral triangle, with the base AB horizontal and B downwards. Let a weight at C be tied by threads AC, BC to fixed points at A and B. If the thread BC is cut, show that the tension of AC is suddenly increased by one-half.

9. Given V, the velocity of projection of a projectile, and v, its velocity at its highest point, find the position of the highest point.

10. A bullet, moving at the rate of 1,100 ft. a second, passes through a thin plank, and comes out with a velocity of 1,000 ft. a second. If it then passes through another plank exactly like the former, with what velocity will it come out of this second plank?

11. Assuming that the earth turns once in 86,164 seconds, that the equatorial radius is 20,900,000 ft., and that the acceleration due to gravity at the equator is 32·1, find what part of the weight of a body is used up in keeping the body on the equator. (N.B.—Take $\pi^2 = 9{\cdot}87$.)

12. Describe the action that takes place between two smooth spheres when one of them impinges directly on the other.

A shot, whose mass is 1 lb., moving at the rate of 1,377 ft. a second, strikes a body, whose mass is 50 lbs., in such a way as to cause no rotation. It enters the body and stays in it. Find the velocity of the body after the impact.

1896.

1. Define the *angular velocity of a moving point* with respect to a fixed point. Under what circumstances will the angular velocity of the moving point be equal to its linear velocity divided by its distance?

Draw an equilateral triangle ABC, having each side 12 ft. long; a point moves along BC with a velocity of 10 ft. a second. When it is at C, what is its angular velocity with respect to A?

2. State Newton's Second Law of Motion. Explain briefly how the measure of force is derived from this law?

In the equation $P = mf$, in what units is P when the units of mass, distance, and time are a pound, a foot, and a second?

3. When two smooth bodies are pressed together, in what direction does the mutual action take place? If the bodies are rough, what other force may be called into play?

A particle of given weight is placed on an inclined plane and stays at rest: what is the magnitude of the friction called into play? Under what circumstances would the particle stay at rest if the inclination of the plane were increased?

4. A curve is drawn, and AN, NP are the abscissa and ordinate of any point (P) of the curve. If AN represents the distance through which a force has acted, and PN represents the magnitude of the force when it has acted through that distance, show that the area of the curve represents the work done by the force.

5. Two circles touch each other externally, and the point of contact (A) is in the same vertical line as the centres; from any point (P) of the upper circumference draw a straight line PAQ to meet the lower circumference in Q. If a particle is allowed to fall from P along PQ, show that the time it takes to reach Q is constant for all positions of P.

Also compare the times in which PA and AQ are described.

6. A particle slides down a rough inclined plane: find the acceleration of its velocity.

Under what circumstances would the velocity be retarded?

The angle of friction between a particle and a plane is 30°, and the angle of inclination is 15°. If the particle begins to slide down with a velocity of 100 ft. a second, how far will it slide before coming to rest?

7. A particle of given mass moves with a given velocity in a circle of given radius: state what is known as to the force which acts on the particle.

Prove the statement.

8. A particle (A) whose mass is 3 is tied by an inextensible thread to a particle (B) whose mass is 5, and B is placed on a smooth table, while A is allowed to fall from the edge; at first the thread is slack, but at the instant A has fallen 9 ft. the thread is drawn tight. With what velocity does B begin to move?

What part of the kinetic energy of the system disappears when the thread is drawn tight?

1897.

1. Define the angular velocity of a moving point with respect to a given fixed point. Under what circumstances will a moving point have no angular velocity with reference to a given fixed point?

AB is a given straight line, and P a given fixed point without it; a particle Q moves along AB with a given constant velocity; when Q is in any assigned position, find its angular velocity with respect to P.

2. Two particles move in a straight line, and are acted on by forces P and Q respectively; the mass of the one particle is m, and its velocity is increased by f in a certain time; the mass of the other is m_1, and its velocity is increased in the same time by f_1; show that it follows from Newton's Second Law that—

$$P : Q :: mf : m_1 f_1.$$

Explain how we can reduce from this proportion the number of absolute units of force in P.

3. A uniform rope hangs by one end, and carries a weight at the other; show how to draw a diagram to represent the work done in winding up the rope, and thereby lifting the weight.

4. A particle is projected in a given direction *in vacuo;* show that it describes a parabola.

A is the highest point, and S the focus of the path of a projectile; N is the point in which AS produced meets the horizontal plane through the point of projection; given that SN is twice AS, find the point of projection and the direction of projection.

5. A uniform and perfectly flexible thread (or chain) is placed on a smooth horizontal table so that one end just hangs over the edge, and consequently that end falls, dragging the rest of the thread after it. Find (*a*) the acceleration of the velocity at the instant when an assigned length is hanging; (*b*) the work that has been done by gravity up to that instant; (*c*) the velocity of the thread at that instant.

State the mechanical principles that justify your results.

6. A particle falls from a given height on to a smooth horizontal plane; the coefficient of restitution between the particle and the plane is given; find the height of the first, and also of the second rebound.

7. Draw AB inclined at a given angle to the vertical and BC horizontal in such a way that A is above the prolongation of CB. Let AB represent a thread fastened to a fixed point at A, and to a heavy particle at B; the particle is kept in position by a force pulling it along a thread BC; after a time the thread BC breaks; show that the weight of the particle is a mean proportional between the tension of AB before BC breaks and its tension immediately after BC breaks.

ANSWERS.

Examples I. (Pages 25, 26.)

1. (i.) 88 (ii.) $\frac{3}{2}$. (iii.) $\frac{2}{9}$. **2.** (i.) 1200. (ii.) 1760. (iii.) 1760
3. 960 ft. **4.** 9 yds. **5.** 396 ft. **6.** $136\frac{8}{9}$.
7. $19\frac{1}{5}$ ft. per sec. **8.** 225 yds. **9.** 210 yds. **10.** $20u$.
11. (i.) ·6. (ii.) ·036. (iii.) ·0315576.
12. (i.) 1250. (ii.) 750. (iii.) $37\frac{1}{27}$ days.

Examples II. (Pages 33, 34.)

1. (i.) Diminished to $\frac{1}{3}$ of its former value. (ii.) Increased to 3600 times its former value.
2. (i.) 384. (ii.) 38400. (iii.) $78545\frac{5}{11}$.
3. $1178\frac{2}{11}$. **4.** 33 ft. per sec. per sec. **5.** $12857\frac{1}{7}$.
6. 11 ft. per sec. per sec. **7.** 88. **8.** $20V$, $1200f$.
9. (i.) 36; 35280. (ii.) 129·6; 127008. **10.** $-2\frac{209}{1458}$ cm. per sec. per sec.

Examination Paper I. (Page 35.)

1. See § 29. **2.** See §§ 30, 38–40. **3.** See §§ 9–11.
4. $\frac{11}{75}$. **5.** $23\frac{1}{21}$ ft. per sec. **6.** 390.
7. 242 ft.; $2\frac{3}{4}$ secs. **8.** 19 miles an hour; $13\frac{1}{3}$ mins. past one.

Examples III. (Page 48.)

1. 1600 ft.; 320 ft. per sec. **3.** 1 sec., $(\sqrt{2}-1)$ sec., $(\sqrt{3}-\sqrt{2})$ sec.
4. 4 ft. per sec. per sec.; 5 secs. **5.** $\frac{2}{3}$ ft. per sec. per sec.
6. 330 ft. **7.** $2/(u+v)$. **8.** 50, 100, and 150 metres per sec., respectively.
10. Time $= (u+\sqrt{u^2+2af})/f$;
B has travelled over $(u^2+af+u\sqrt{u^2+2af})/f$.

Examples IV. (Pages 64, 65.)

1. (i.) 400 ft., or 12,250 cm.; 160 ft., or 4900 cm. per sec.
(ii.) 14,400 ft., or 441,000 cm.; 960 ft., or 29,400 cm. per sec.
(iii.) 12,960,000 ft. or 396,900,000 cm.; 28,800 ft. or 882,000 cm. per sec.
(iv.) ·16 ft., or 4·9 cm.; 3·2 ft. per sec., or 98 cm. per sec.
2. (i.) 80 ft. per sec.; $2\frac{1}{2}$ secs. (ii.) 240 ft. per sec.; $7\frac{1}{2}$ secs.
(iii.) 4 ft. per sec.; $\frac{1}{8}$ sec. (iv.) 1400 ft. per sec.; $1\frac{3}{7}$ secs.
3. 2. **4.** 900 ft. **5.** 160 ft. per sec. **6.** About 277 ft. per sec.
8. 324 ft.; 144 ft. per sec.; 4 ft. **9.** 80 ft. per sec.; 88 ft. per sec.
11. 400 ft.; 10 secs. **12.** $\frac{5}{19}$ ft. **14.** $1\frac{7}{8}$ secs.
15. 4080 ft. **16.** 72 ft. per sec. downwards. **18.** 3 secs. ? 8 sec

Examination Paper II. (Page 66.)

1. See §§ 44, 45. **2.** See § 49. **3.** See § 51.
4. $2\frac{1}{2}$ ft. per sec. per sec. **6.** 32·18 ft. per sec. per sec., nearly.
7. See § 68. **8.** $5\frac{1}{2}$ secs. **9.** 1200 ft. per sec.
10. $\frac{1}{16}\sqrt{h}$ secs.; $\frac{15}{16}h$ above the ground.

Examples V., VI. (Pages 83, 84.)

1. (i.) 4 F.P.S. units. (ii.) 896 F.P.S. units.
(iii.) $\frac{7}{50}\sqrt{10}$ C.G.S. units.
2. 15 : 2. **3.** 14 : 45. **4.** 12000 yds.; 1200 ft. per sec.
5. $85\frac{5}{7}$ tons. **6.** $9\frac{3}{13}$ ft. per sec. **7.** $7\frac{1}{2}$ lbs. weight.
8. 14,336,000 poundals, or 200 tons weight; 1,433,600 F.P.S. units of momentum, or 640 tons-ft. per sec.
9. 440 ft.; $27\frac{1}{2}$ tons weight. **10.** $6570\frac{2}{3}$ poundals.
11. (i.) $8\frac{4}{5}$ poundals. (ii.) 5 secs. **12.** $29\frac{1}{3}$ F.P.S. units.
13. 11 : 3600. **14.** 100,000 dynes; 50,000 dynes.
15. ·3 cm. per sec.; ·003 cm. per sec. **16.** $\frac{1}{120}$ poundal.
17. Mass $= 9\frac{1}{3}$, velocity $= 30$, momentum $= 280$.

Examples VII. (Pages 94, 95.)

2. See § 71; 2 ft. per sec. **3.** $4\frac{13}{28}$ ft. per sec. **4.** $3\frac{3}{4}$ ft. per sec.
5. $\frac{1}{4}$ ft. per sec. **6.** 8 ft. per sec. **7.** 12 ft. per sec.
8. 10 ft. per sec. **9.** 15 ft. per sec. **10.** 9 ft. per sec.
11. $6\frac{1}{4}$ metres per sec. **12.** 100 cm. per sec. per sec.; 200 dynes.

Examination Paper III. (Page 96.)

1. See §§ 78, 81, 82. **2.** See § 88. **3.** See §§ 90, 91, 96 (2).
4. (*a*) $\frac{11}{12}$ ton weight, or $65706\frac{2}{3}$ poundals.
(*b*) $\frac{11}{72}$ ton weight, or $10951\frac{1}{3}$ poundals.
5. (i.) $\frac{2}{7}$ ft. per sec. per sec.; 1920 F.P.S. units.
(ii.) 5000 cm. per sec. per sec.; 300 C.G.S. units.
6. See §§ 77, 76. **7.** $47\frac{14}{33}$ ft. per sec. **8.** 4 ft.
9. 2240 F.P.S. units. **10.** $27\frac{7}{9}$ dynes.

Examples VIII. (Pages 108, 109.)

1. (i.) 4 ft. per sec. per sec.; 240 ft. per sec.; 7200 ft.
(ii.) 3200 ft. per sec. per sec.; 64000 yds. per sec.; $1090\frac{10}{11}$ miles.
(iii.) 140 ft. per sec. per sec.; 2800 yds. per sec.; $47\frac{8}{11}$ miles.
(iv.) ·981 cm. per sec. per sec.; 58·86 cm. per sec.; 1765·8 cm.
2. 28800 ft. **3.** 2304 F.P.S. units. **4.** 3200 F.P.S. units.
5. $3\frac{1}{3}$ ft. per sec. per sec.; 48 lbs. **6.** (i.) $7\frac{11}{16}$, (ii.) $237\frac{3}{8}$ lbs. weight.
7. 77 : 17280. **8.** At the bottom. **9.** $150\frac{11}{32}$ lbs. weight.
10. $2\frac{1}{2}$ secs.; $\frac{3}{4}$ of the weight. **11.** $\frac{1}{4}$ sec.
12. (i.) 240 lbs. weight. (ii.) 80 lbs. weight. **13.** $10\frac{2}{3}$ lbs. weight.
14. 63·5688 metres; $26\cdot0119\dot{7}$ cm. per sec. per sec.
15. $(P-W)g/W$; nP, if n be the fraction of the chain below point considered.

Examination Paper IV. (Page 110.)

1. See §§ 5, 6, 106, 107, 114. **2.** See § 108. **3.** See §§ 113, 115, 116.
4. 5 lbs. weight. 3 lb **5.** See § 108 (i.) **6.** 64 ft. **7.** 1 : 6144.
8. 1 lb. of tea, $\frac{8}{9}$ lb. of sugar. **9.** $4\frac{115}{128}$ tons. **10.** 14 cm. per sec.

Examples IX. (Pages 124, 125.)

1. (i.) $15\frac{15}{16}$ lbs. weight; 2 ft. per sec. per sec.; $31\frac{7}{8}$ lbs. weight.
(ii.) $1\frac{7}{8}$ oz. weight; 28 ft. per sec. per sec.; $3\frac{3}{4}$ oz. weight.
(iii.) 28 lbs. weight; 24 ft. per sec. per sec.; 56 lbs. weight.
(iv.) $\frac{14}{15}$ lb. weight; $2\frac{2}{15}$ ft. per sec. per sec.; $1\frac{13}{15}$ lbs. weight.
(v.) $4\frac{4}{9}$ lbs. weight; $3\frac{5}{9}$ ft. per sec. per sec.; $8\frac{8}{9}$ lbs. weight.
(vi.) $6\frac{2}{3}$ lbs. weight; $21\frac{1}{3}$ ft. per sec. per sec.; $13\frac{1}{3}$ lbs. weight.
(vii.) $490\frac{490}{981}$ gm. weight; 1 cm. per sec. per sec.; $980\frac{980}{981}$ gm. weight.
(viii.) $165\frac{15}{109}$ gm. weight; 99 cm. per sec. per sec.; $330\frac{30}{109}$ gm. weight.

2. (i.) (*a*) 17, (*b*) 15 ft. per sec. per sec.; $7\frac{31}{32}$ lbs. weight.
(ii.) (*a*) 30, (*b*) 2 ft. per sec. per sec.; $\frac{15}{8}$ oz. weight.
(iii.) (*a*) 28, (*b*) 4 ft. per sec. per sec.; 14 lbs. weight.
(iv.) (*a*) $17\frac{1}{15}$, (*b*) $14\frac{14}{15}$ ft. per sec. per sec.; $\frac{7}{15}$ lbs. weight.
(v.) (*a*) $17\frac{7}{9}$, (*b*) $14\frac{2}{9}$ ft. per sec. per sec.; $2\frac{2}{9}$ lbs. weight.
(vi.) (*a*) $26\frac{2}{3}$, (*b*) $5\frac{1}{3}$ ft. per sec. per sec.; $3\frac{1}{3}$ lbs. weight.
(vii.) (*a*) 491, (*b*) 490 cm. per sec. per sec.: $245\frac{245}{981}$ grams weight.
(viii.) (*a*) 900, (*b*) 81 cm. per sec. per sec.; $82\frac{62}{109}$ grams weight.

3. $\sqrt{\frac{122}{13}}$ sec.; $32\sqrt{\frac{26}{61}}$ ft. per sec. **5.** 58 ft.

6. 8 ft. per sec.; $1\frac{3}{4}$ lbs. weight. **7.** 12 secs.

8. Tension between 3 lbs. mass and 5 lbs. mass = $5\frac{5}{9}$ lbs. weight; tension between 4 lbs. mass and 6 lbs. mass = $5\frac{1}{3}$ lbs. weight; tension of portion over pulley = $8\frac{8}{9}$ lbs. weight.

9. $2\frac{10}{11}$ ft. per sec. per sec. **10.** $4\frac{36}{55}$ ft. per sec. per sec. **11.** 4 ft.

12. Tension between P and $R = (2Q+R)\,P/(P+Q+R)$;
,, between Q and $R = (2P+R)\,Q/(P+Q+R)$.

Examination Paper V. (Page 126.)

1. See §§ 120, 131. **2.** See §§ 126, 128. **3.** $4\frac{1}{4}$ oz., and $3\frac{3}{4}$ oz.

4. See § 132. **5.** See § 126.

6. Tension = $\frac{1}{2}$ lb. weight; pressures on pans = $6\frac{2}{3}$ oz. weight and $5\frac{1}{3}$ oz. weight.

7. $\dfrac{(P-Q)(Q+P_1)}{(P+Q)(Q-P_1)}h$.

8. Each mass = $494\frac{886}{981}$ grams; rider = $10\frac{910}{981}$ grams.

9. See § 122. **10.** $22\frac{2}{3}$ lbs.

Examples X. (Pages 144, 145.)

1. 300 lbs. weight. **2.** 404 lbs. weight.

3. 24000 ft.-lbs.; 160 ft. per sec. **4.** (i.) 384 ft.-poundals; 0.

5. The velocity is doubled. **6.** $351\frac{9}{16}$ poundals; $\frac{1}{75}\sqrt{2}$ secs.

7. 8 ft. per sec. **8.** $848\frac{1}{4}$ ft.-lbs. **9.** 200 miles per min.

10. 7 min. **11.** 275 ft. per min. **12.** 9900 lbs.

13. $4\frac{4}{5}$. **14.** $4531\frac{1}{4}$ lbs. **15.** 735·75 watts.

16. 1 ft.-poundal per sec., or $\frac{1}{17600}$ horse-power.

17. Unit of mass = 6 grams; unit of length = $1\frac{2}{3}$ cm.

18. (i.) 1 cm. = ·033173 ft. (ii.) 1 gram = ·00228 lb.
(iii.) 1 dyne = ·0000757 poundal.
(iv.) 1 erg = ·000000785 ft.-poundal.

19. $12375N/85204n$ tons.

EXAMINATION PAPER VI. (PAGE 146.)

1. See §§ 134, 135, 155. **2.** See §§ 77, 139, 142.
3. 887040 ft.-lbs. **4.** $56\frac{1}{4}$ miles per hour. **5.** 320. **7.** $37\frac{29}{33}$.
8. 328 lbs. $8\frac{1}{3}$ oz. weight. **9.** $55\frac{1}{3}$ lbs. **10.** See §§ 146, 147.

EXAMPLES XI. (PAGES 161, 162.)

1. 302 ft. 2 ins., nearly. **2.** South-east. **3.** 13·66 ft. per sec.
4. In a line making 60° with the line on the paper, and with the same velocity as that with which it crawls along the paper.
5. 61 ft. per sec. **7.** 10 miles an hour from the north-west.
8. 22 ft. behind the object, measured parallel to the direction of the train's motion.
9. 7 miles an hour. **10.** $\sqrt{19}$ miles an hour.

EXAMPLES XII. (PAGES 172, 173.)

1. N.W. acc. of $\frac{15}{11}\sqrt{2}$ ft. per sec. per sec.; $\frac{15}{2}\sqrt{2}$ ft. per sec. N.E.
2. Change of velocity = velocity of particle;
momentum of blow = momentum of particle.
3. $32\sqrt{2}$ ft. per sec.; $32\sqrt{5}$ ft. per sec.; $32\sqrt{10}$ ft. per sec.
6. 3500 ft. **7.** 100 ft. per sec. **8.** 13·2 lbs. wt. **9.** 350 ft.
10. $100\sqrt{3}$ ft. per sec. **11.** 1600/9 ft. per sec. **12.** 60°. **13.** 52 ft.

EXAMINATION PAPER VII. (PAGE 174.)

1. See § 162. **2.** See § 160. **3.** 60° with the bank, up stream.
4. $4\sqrt{2}$ miles an hour from N.W. **6.** 10 miles an hour, W.
7. See § 184. **9.** 160 ft. **10.** See § 187.

EXAMPLES XIII. (PAGE 180.)

1. (i.) 25 lbs. (ii.) 17 oz. (iii.) 29 cwt. (iv.) 60 grams weight, nearly.
2. 25 lbs.; 1 lb. **3.** 40·16 ft., nearly.
5. $64/\sqrt{3}$ ft. per sec. per sec.; $1/\sqrt{3}$ lbs. weight. **6.** $13\cdot\dot{3}$ ft./sec.
7. $\tan^{-1}\cdot 25$. **8.** $20\sqrt{3}/3$ ft./sec. **9.** $10\sqrt{2}(\sqrt{3}-1)$ ft./sec.
11. $10\sqrt{2}$ ft.

EXAMPLES XIV. (PAGES 200–204.)

1. 8 ft., 16 ft. per sec.; $8\sqrt{2}$ ft., $16\sqrt{2}$ ft. per sec.; $8\sqrt{3}$ ft., $16\sqrt{3}$ ft. per sec.
2. 30°. **3.** 21·1 miles per hour. **4.** 16 ft. per sec. **5.** $1\frac{3}{5}$ tons.
6. $520\sqrt{3}$ ft.-lbs. **7.** $14\cdot8\dot{6}$ lbs. per ton; 179·2 H.P. **8.** 480.

9. 107·96 lbs. wt.; 1870 ft.-lbs. **10.** $\frac{100}{981}$ cm.; $\frac{50\sqrt{2}}{981}$ cm.; $\frac{100\sqrt{3}}{2943}$ cm.

11. 4905 ergs; $4905\sqrt{2}$ ergs; $4905\sqrt{3}$ ergs.

12. (i.) $P = Q$. (ii.) $P = \frac{3}{5}Q$. **13.** $\frac{480}{7}\sqrt{7}$ ft. per sec.

15. ·25. **16.** (i.) $\cdot 25g$; (ii.) $\cdot 503g$; (iii.) $\cdot 722g$. **17.** $\frac{1}{3}g\sqrt{3}$ ft.

18. 48 ft./sec. **19.** 20 ft. **20.** 100 ft. **21.** ·5.

22. 1 : 2. **23.** 4 ft./sec.^{2}, $\frac{7}{8}w$ lbs., $4\sqrt{3}$ ft./sec.

25. Through the given point draw a line vertically up, and a line perpendicular to the given line; bisect the angle between these two lines.

26. The line from the point to the circumference which, if produced, passes through the highest point of the circle.

29. The line which, if produced both ways, passes through the highest point of the first circle and the lowest point of the second.

30. The line which, if produced, passes through the highest point of the circle.

31. 860 secs. on level, 270 secs. down, 206 secs. up (to the nearest sec.).

34, 35. The line in either case bisects the angle between the vertical and a perpendicular to the given line, and, when produced, passes through the highest point of the circle in Ex. 34, the lowest point in Ex. 35. **37.** 30°.

Examination Paper VIII. (Page 205.)

1. See § 192. **2.** $2P$, acting along the direction of the middle one.
3. See § 193. **4.** See § 198. **6.** 24 lbs. 10 oz. **7.** See § 201.

Examples XV. (Pages 216–218.)

1. (i.) $6400\sqrt{3}$ ft.; 1600 ft.; 20 secs. (ii.) 312·5 ft.; 78·125 ft.; 4·419 secs. (iii.) $40000\sqrt{3}$ ft.: 30000 ft.; $50\sqrt{3}$ secs.; (iv.) 3840 ft.; 400 ft.; 10 secs. (v.) 19200 ft.; 6,400 ft.; 40 secs. **2.** After 1 or 5 secs.; $16\sqrt{133}$ ft., or $80\sqrt{109}$.

3. $12(10\sqrt{3}-3)$ ft. **4.** $4\sqrt{(4h+d^2/h)}$ ft./sec. **8.** 96 ft./sec.

9. $40\sqrt{2}$ ft./sec. **10.** $20\sqrt{17}$ ft./sec. **11.** 300 ft.

2. 36, or 3·0625 ft. **13.** $\sqrt{\{(16H+R^2/H)g/8\}}$ ft./sec.

14. 192 ft./sec. **15.** $64\sqrt{3}$ ft./sec. **17.** $\frac{u^2-v^2}{2g}$ ft. **18.** U^2/g.

Examples XVI. (Pages 229–231.)

1. 544·5 lbs. wt. **2.** $4/\pi$. **3.** ·05 ton wt. **5.** Equal.
6. 2 : 1. **7.** 16 ; 5 oz. wt. **8.** 2 oz. wt. **9.** 11 oz. wt.
10. $8\sqrt{2}$. **11.** 5 stone. **12.** 10 : 7 : 3. **13.** 3·5 oz. wt.
14. When the particle has fallen a vertical distance of 8 ins.
16. $\tan^{-1}\frac{1}{8}$. **17.** About $\frac{1}{289}$, or ·0034 approx. **19.** 10·97.

Examples XVII. (Pages 240, 241.)

1. $\pi/20$ sec. ; $5\sqrt{2}$ cm. from 0 ; $50\sqrt{2}$ cm. per sec.
2. 1 sec. ; $40\pi^2$ dynes. **3.** $vt/2\pi$. **4.** $\sqrt{(Ps/m)}$.
5. 1·92 secs. **6.** 32·08. **7.** 986·96. **8.** 169/144 ft.
9. Lengthened by $n/43200$ of original length, approximately.
13. 4·892 ft. **14.** (i.) 32·182. (ii.) 32·09.

Examples XVIII. (Pages 254–257.)

1. 10, 15 ft./sec.
2. 10, 5 ft./sec. direction of motion reversed in each sphere.
3. ·75 ; returns with velocity 10 ft./sec.
6. 20 ft./sec., 10 ft./sec. ; in opposite directions.
8. The 10-lb. mass proceeds in its original line of motion with velocity 5 ft./sec. ; the 5-lb. mass proceeds obliquely with velocity $10\sqrt{5}$ ft./sec. **10.** 3 secs., 26 ft. 8 ins.
12. 3. **15.** 80 ft./sec. **17.** 25 lbs. wt.
18. 125 oz. wt. **19.** 625/88 H.P. **21.** 3·5 hrs.
22. $(2^{20}-1)$ min. **24.** (i.) $\frac{1}{3}v$. (ii.) $-\frac{1}{3}v$ and $\frac{2}{3}v$.
25. $\frac{1}{3}$. **26.** $m = n$, $e = 1$. **27.** $\sqrt{(k/h)}$.

Examination Paper IX. (Page 258.)

6. π^2 metres. **9.** $M = m(1-2e^2)$. **10.** 17700 mls./hrs., 42·5 mins.

Examples XIX. (Pages 276, 277.)

1. (i.) $\frac{3}{2}$. (ii.) $\frac{3}{4}$. (iii.) $\frac{3}{4}$. **2.** (i.) $\frac{9}{2}$. (ii.) $\frac{9}{4}$. (iii.) $\frac{3}{2}$.
3. 96. **4.** (i.) 26. (ii.) 26. (iii.) ·000026.
5. (i.) $15ma^2/2$. (ii.) $3ma^2$. (iii.) $6ma^2$. (iv.) $12ma^2$.
7. (i.) 7 ; 18. (ii.) $3\frac{3}{5}$. (iii.) $3\frac{1}{2}$; 4.
8. $Ma^2/3$ ($2a$ = side of rectangle). **9.** $7Ma^2/12$.
10. $Mp^2/3$ (p = breadth of parallelogram).
11. $Mq^2/6$ (q = perpendicular from angular point on the median).
12. $3Mr^2/2$. **13.** $5Mr^2/4$. **16.** $Ma^2/6$. **17.** $5Ma^2/12$. **18.** Mr^2.

EXAMPLES XX. (PAGES 293–296.)

1. $4\sqrt{2}$ radians per sec. **2.** $\frac{8}{3}\sqrt{6}$ radians per sec.
3. $v^2 = 2ags(Pa - Qb)/(Pa^2 + Qb^2 + I)$. **4.** $v^2 = 2r^2gsP/(Pr^2 + I)$.
6. $2 : 1 : \sqrt{2}$.
9. A circle whose diameter is the line joining the given point to the point of contact.
10. 88/3 radians per sec. **11.** $\sqrt{\{2Pgl/I\}}$ radians per sec.
12. $\sqrt{(14/15)}$. **14.** $\frac{2}{3}g$. **15.** $\sqrt{(3g/2a)}$ radians per sec.; $4a/3$.
17. (i.) $\frac{1}{4}(3h + a^2/h)$. (ii.) $3h/4$. (iii.) $2\sqrt{2}a/3$, (a = side of square).
(iv.) $(3r^2 - 2a^2)/\sqrt{(r^2 - a^2)}$. **20.** $1\frac{1}{3}$ radians per sec.
22. $\sqrt{\left\{ mgl\,(1 - \sin\theta) \Big/ \left(M + \frac{4}{3}\frac{ml^2}{a^2}\sin^4\frac{\theta}{2} \right) \right\}}$;

$\sin^{-1}\sqrt{\left\{ \frac{v^2}{gl}\left(\frac{M}{m} + \frac{4}{3}\frac{l^2}{a^2}\sin^4\frac{\theta}{2} \right) + \sin\theta \right\}}$.

23. $2(P - Q)g/(2P + 2Q + M)$; $2(P - Q)g/a(2P + 2Q + M)$;
$P(4Q + M)g/(2P + 2Q + M)$; $Q(4P + M)g/(2P + 2Q + M)$.

EXAMINATION PAPER X. (PAGE 297.)

4. $I_1\cos^2\theta + I_2\sin^2\theta$. **5.** $v^2 = 2r^2gs(P - Q)/(Pr^2 + Qr^2 + I)$.
6. $\frac{1}{2}g\sin\alpha$. **8.** $4\sqrt{2}$ radians per sec.; $12\sqrt{2}$ ft. per sec.

EXAMPLES XXI.—MISCELLANEOUS. (PAGES 298–302.)

1. 640 ft. **2.** 176 ft.
5. $77Mv^2/5400(P - Mp)$ miles; $77Mv^2/5400(R + Mp)$ miles.
6. 53/7 tons wt. **7.** 205 tons wt. **8.** 294 oz. wt.
9. ·7 in.; 8 in. per sec. **11.** 1 ft. per sec.
12. From second point P drop PM perp. on plane; produce it to P_1, making $MP_1 = PM/e$, and project particle towards P_1.
14. $m(u - v)/M$; $2Ms/\{M(u + v) - m(u - v)\}$;
$m(u - v)\{M(u + v) - m(u - v)\}/2Ms$.
15. er. **16.** $\mu g\cos\alpha$. **17.** After falling a vertical distance of 4 ins.
19. $(l - x)/2l \times$ weight of string.
20. Uniform acceleration of $\frac{2h}{t^2}$ ft./sec.^{2}; $\frac{t^2g}{t^2g + 2h}W$; $\frac{3hgt^2}{2(h + gt^2)}$.
21. $M(g + F) = m(g + f)$. **22.** $g/3$; $g/9$.
24. $7\sqrt{14}/3$ radians per sec.; $33\dot{7}$ gr. wt. **27.** 256 : 1265625.
28. $2\sqrt{(u^2 - gR)}/g$. **29.** $160/\pi$. **30.** ·025 sec.
31. $-2u(\sqrt{3} - 1)/5$; $u(2 + \sqrt{3})/5$. **35.** $x = 2y^2 - 1$, or $y^2 = \frac{1}{2}(x + 1)$.

Science and Art Examination Questions. (Pages 305–315.)

1885.

1. (*a*) 125 lbs.; (*b*) 36. **2.** (*a*) 384 ft.; (*b*) $53\frac{1}{2}$ poundals; (*c*) $1\frac{2}{3}$ lbs.
3. (*a*) μg; (*b*) 500 ft., 25 secs. **4.** $(P-Q)(Q+P_1)h/(P+Q)(Q-P_1)$.
5. -4; 4; 256.

1886.

1. $\frac{4}{9}\frac{3}{9}$; 20,640,000 ft.-lbs. **2.** $177\frac{7}{9}$ ft. per sec.
3. -1; 0. **4.** (*a*) 55561·2 ft.-lbs.; (*b*) 44·1964 turns.

1887.

1. 192,500,000 ft.-lbs.; $1666\frac{2}{3}$ minutes.
2. 1 : 5; $16\sqrt{5}$ ft. per sec.; 7680 ft.-poundals.
4. (*a*) $\frac{1}{3}ml^2$, where m is the mass and l the length of the rod; (*b*) $72\pi^2$ ft.-poundals.

1888.

2. (*a*) $mv^2/2g$; (*b*) 0·244 ft.
4. (*a*) $9800\pi^2/3$ ft.-poundals; (*b*) $9800\pi^2/3g$ ft.-lbs.

1889.

1. $\frac{1}{2}Wl$ ft.-lbs. **2.** 100 ft. per sec. towards centre of hexagon.
3. Lengthened by $n/43200$ of its original length.
4. (*a*) 12,100 ft.-poundals; (*b*) 378·125 ft.-lbs.

1890.

1. 20,000 ft.-lbs. **2.** 4 ft. per sec.; 5 secs. **4.** 1500 ft.-poundals.

1891.

2. $P = 3Q$.

1892.

1. $\cdot 7\pi$ radians per sec.; $4\frac{1}{6}$ ft. per sec. **4.** 600 ft.-poundals.
5. $\sqrt{3}/12$. **7.** $2V$, V. **8.** $\frac{1}{2}Mv^2$. **9.** 1000; $14\frac{7}{12}$.
10. 7 ft. per sec.; 1·4 radians per sec.

1893.

1. 64 ft. per sec. **2.** $5\pi/6$ radians per sec. **4.** 26·366 ft.-lbs.
6. $8\sqrt{3}$ lbs.; 3·27 secs. **7.** 156; 156; -36; 96.
9. $4\frac{2}{3}$ radians per sec.; 5691·84 ft.-poundals.

1894.

2. $\frac{2}{3}$; $\frac{3}{3}$; $3/5\sqrt{5}$; $\frac{1}{5}$ radian per sec. **6.** $\sqrt{5gr}$; 72 lbs.
7. m/M; $133\frac{1}{3}$ ft.-poundals. **8.** $Mb^2/3$. **9.** $\frac{4}{3}$; 4; $-\frac{4}{3}$ radians per sec.

1895.

1. $29\frac{13}{15}$ H.P., $170\frac{2}{3}$ H.P. **2.** $1\frac{1}{2}$ ft. per sec. **3.** 64 ft. below top.
4. $10\sqrt{5}$. **5.** 250 poundals. **10.** $100\sqrt{79}$ ft./sec.
11. 0·003462315. **12.** 27 ft./sec.

1896.

1. (*a*) When the line of motion is at right angles to the line joining the moving point to the fixed point; (*b*) $\frac{5}{12}\sqrt{3}$ radians per sec.

3. The particle will remain at rest as long as the angle of inclination of the plane is less than the angle of friction.

5. $\sqrt{a} : \sqrt{b}$, where a and b are the radii of the circle.

6. 80 ft. **8.** 15 f.s.; 1800 ft.-poundals.

1897.

1. When the fixed point lies in the line of motion; ud/PQ^2, where u is the constant velocity, and d the length of the perpendicular from P to AB.

4. Point of projection is at P in horizontal plane through N, where $PN = 2\sqrt{3}AS$; angle of projection $= 60°$.

5. (*a*) gx/l; (*b*) $\frac{1}{2}mgx^2/l$; (*c*) $\sqrt{(gx^2/l)}$, where l and m are the length and mass of the thread and x is the length hanging over the edge.

6. e^2h, e^4h, where h is the given height and e the co. of restitution.

QA
805
B75
1898
v.1
Physical &
Applied Sci.

Briggs, William
Advanced mechanics

PLEASE DO NOT REMOVE
CARDS OR SLIPS FROM THIS POCKET

UNIVERSITY OF TORONTO LIBRARY