AERONAUTICS

A CLASS TEXT

BY

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PREFACE

FOR several years I have been giving, at the Massachusetts Institute of Technology, courses of lectures on those portions of dynamics, both rigid and fluid, which are fundamental in aeronautical engineering. The more elementary parts of these courses, covering about ninety out of one hundred fifty lectures, are found in this book. Although it has been customary to teach the two subjects of rigid and of fluid dynamics in parallel or in rapid alternation, so that they are both developed as needed for each other and for the accompanying courses on airplane and airship design, it has seemed better in making a presentation in book form to separate them. The student should have completed Chaps. IX–XII of the fluid mechanics before undertaking the latter part of Chap. VI.

A number of topics which might well be included in a work on aeronautics have been omitted from the book, as they are from my lectures, because they can be taken up so much better in the parallel courses on design. In the preparation of the selected material I have had constantly in mind my own experience and needs relative to effective classroom instruction, particularly in the matter of lists of exercises. Although my students are supposed to have completed thorough courses in calculus, including the elements of differential equations, and in theoretical and applied mechanics, it has seemed better to assume too little, rather than too much, as retained in usable form. I hope, therefore, that with the present interest in aeronautics in particular, and in applied mathematics in general, this work may prove stimulating to other than technical students of aeronautical engineering.

Nobody can issue a book on aeronautics at this time without lamenting the fact that much, if not most, of the progress in theory which has been made during the war, particularly in England, has not yet been released for publication. To wait, however, until its release and subsequent digestion would mean a long delay. Indeed from one viewpoint no time is more appropriate for the printing of these elementary, introductory, and orienting lectures than just now when there impends a deluge of material for advanced study.

I desire to express my appreciation of the way Professor C. H. Peabody, in charge of the work in Aeronautical Engineering, has in every way encouraged and supported me. I am under the deepest personal and technical indebtedness to Dr. J. C. Hunsaker, U.S.N., with whom I was in close collaboration for three years, and upon whose published work I have had permission freely to draw for parts of Chap. VI and for most of Chap. VIII. Could I have consulted with him these last few years as I did earlier, this book would have been much improved.

EDWIN BIDWELL WILSON

July, 1919

CONTENTS

INTRODUCTION

RIGID MECHANICS

FLUID MECHANICS

AERONAUTICS

AERONAUTICS

INTRODUCTION

CHAPTER I

MATHEMATICAL PRELIMINARIES

1. Definitions. The term "aircraft" denotes any form of craft designed for the navigation of the air — airplanes, balloons, dirigibles, helicopters, kites, kite balloons, ornithopters, gliders, etc.

The term "airplane" denotes a form of aircraft heavier than air, which has wing surfaces for sustentation, with stabilizing surfaces, rudders for steering, and power plant for propulsion through the air. The landing gear may be suited for either land or water use.

The term "dirigible" denotes a form of balloon the outer envelope of which is elongated in shape, provided with a propelling system, car, rudders, and stabilizing surfaces.

2. Methods of Attack. The airplane and dirigible, except for their control and propelling surfaces, are essentially rigid bodies moving in a fluid medium, the air. It is not practicable, however, to treat the motion of the body and the medium simultaneously by the method developed by Thomson and Tait, and found in standard advanced treatises on hydromechanics. The moving object is, therefore, treated as a rigid or nearly rigid body acted upon by certain forces; namely, the propeller thrust, urging it forward, and the air pressures, urging it backward, and, in the case of the airplane, sustaining it against its weight W; and the theory of aeronautics is, consequently, divided into two parts: (1) the dynamics of flight, and (2) fluid dynamics.

In the dynamics of flight the motion of the craft is investigated with the aid of empirical laws for air pressures. In fluid dynamics those empirical laws are determined for the most part by experiments, but to some extent by general theoretical considerations involving the elements of hydromechanics. The two parts of the subject have been separated in preparing this work; they have been taught separately, but in parallel.

3. Engineering Units. As in all engineering practice, the forces are expressed in pounds or kilograms, according as the English or metric system is used. Work is expressed in foot-pounds or kilogram-meters, and power either in foot-pounds or kilogram-meters per second, or as horse power, whether English or French.

The pound-mass or kilogram-mass is a definite quantity of matter obtained by weighing on an arm-balance against the standard pound or kilogram, or replicas thereof; they do not depend upon the location in which the weighing is carried out.

The pound-force or kilogram-force are defined as the force of the earth's attraction upon the pound-mass or kilogram-mass. Owing to the variation in the acceleration of g from point to point, the units of force thus defined will vary from point to point unless it is agreed in advance that the units of force shall be the attraction of the earth upon the unit of mass at a standard place. Thus, it is necessary logically to separate local units and standard units of force, the latter being the value of the local pound where $g = 32.174$ feet per second per second, or, in the French system, of the local kilogram where $g = 980.665$ centimeters per second per second. As, however, the variation of g from its standard value is only about $\frac{1}{4}\%$ at most, it is customary in engineering practice when dealing with machines in which the accuracy of measurement and performance is no greater than that of aircraft to disregard any difference in local and standard units of force.

In some engineering work the unit of mass is taken as the slug (English or metric), which is defined as g pounds or g kilograms, and which, therefore, is about 32 pounds or 9.81 kilograms. If the slug is used, the same differentiation between local and standard slugs must logically be made; but again, the variation of the local from the standard value is so small as ordinarily to be neglected. As a matter of fact it is unnecessary to use the artificial unit slug with the attendant artificialities of density in slugs per unit volume, etc.; and, unless specified to the contrary, the units of mass will be the pound and kilogram.

The unit of time throughout the work is the second; the unit of distance, the foot or the meter; the units of velocity are the foot per second and meter per second, the mile per hour and the kilometer per hour. It would be possible to deal only with a single unit of velocity in each of the systems, English and the metric, but for practical reasons such a restriction seems unadvisable.

4. Change of Units. These matters of units and the matter of dimensions of physical quantities will be treated in detail later, but it should be pointed out now that in order to change the value of some quantity expressed in a certain set of units to the value which the same quantity has when expressed in another set of units it is only necessary to write the numerical value of the quantity followed by the symbols for the units, and substitute for those symbols their value in terms of the new units.

Example 1. Change the velocity of five miles per hour (mi/hr) to feet per second (ft/sec).

$$5\frac{\text{mi}}{\text{hr}} = 5\frac{5280 \text{ ft}}{3600 \text{ sec}} = 7\tfrac{1}{3} \text{ ft/sec}$$

The following table contains necessary or convenient data for changing from one system of units to another:

TABLE OF DATA

Metric		*English*	
1 centimeter (cm)	= 0.3937 in,	1 inch (in)	= 2.540 cm
1 meter (m)	= 3.2808 ft,	1 foot (ft)	= 0.3048 m
1 kilometer (km)	= 0.62137 mi,	1 mile (mi)	= 1.6093 km
36 km/hr	= 10 m/sec,	30 mi/hr	= 44 ft/sec
1 square centimeter (cm^2)	= 0.1550 in^2,	1 square inch (in^2)	= 6.452 cm^2
1 square meter (m^2)	= 10.764 ft^2,	1 square foot (ft^2)	= 0.0929 m^2
1 cubic meter (m^3)	= 35.315 ft^3,	1 cubic foot (ft^3)	= 0.028317 m^3
1 kilogram (kg)	= 2.2046 lb,	1 pound (lb)	= 0.4536 kg
1 kilogram-meter (kg.m)	= 7.233 ft.lb,	1 foot-pound (ft.lb)	= 0.13826 kg.m
gravity g cm/sec²	= 981 cm/sec²,	gravity g ft/sec²	= 32.2 ft/sec²
1 calorie (Cal)	= 3091 ft.lb,	1 foot-pound (ft.lb)	= 0.0003235 Cal
1 force de cheval	= 75 kg.m/sec,	1 horse power (H.P.)	= 550 ft.lb/sec
1 kg/cm²	= 14.22 lb/in²,	1 lb/in²	= 0.0703 kg/cm²
1 atmosphere	= 1.033 kg/cm²,	1 atmosphere	= 14.70 lb/in²

(An atmosphere "atmo" is 760 mm or 29.92 in. of mercury.)

standard $g = g_0$	= 980,665 cm/sec²,	standard $g = g_0$	= 32.174 ft/sec²

standard air, dry, density ρ = 0.07608 lb/ft³, temperature T = 62° F, pressure p = 29.921 in.Hg

5. Errors and Approximations. Throughout the work in aeronautics it is necessary to make approximations. An extremely important formula in this connection is the binomial expansion:

$$(1 \pm x)^n = 1 \pm nx + \tfrac{1}{2}n(n-1)x^2 \pm \ldots$$

If x is extremely small and n not too large, the result may be written:

$$(1 \pm x)^n = 1 \pm nx.$$

This shows that an error of x on 1 becomes an error of nx on 1 when a number is raised to the nth power. For example, if a velocity is known within an error of 2%, and if a force depends on the square of that velocity, the force is known only within an error of 4%. In like manner, if a length is known only to 2%, the volume of the cube with that length for its edge is known only to 6%. On the other hand, if an expression contains a square root ($\frac{1}{2}$ power) of a symbol whose value is known to within 2%, the value of the expression itself is accurate to within 1%.

When an expression involves a number of symbols, x, y, z, etc., as in the case $f(x,y,z\ldots)$, the change in the expression produced by slight changes in x, y, and z may be calculated approximately by the formula for the total differential of f, namely,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \ldots,$$

where dx, dy, dz, . . . represent the slight changes in x, y, z, etc.

In practical applications, dx, dy, dz, etc., are frequently not precisely known variations in x, y, z, etc., but merely possible errors in the measurement of those quantities. The possible error made in the calculation of f is estimated as the numerical sum of the terms in the expression for df, disregarding the significance that the derivatives may actually have.

Example 2. The time of oscillation of a pendulum (meaning by "oscillation" the swing from one extreme to the other) is

$$T = \pi\sqrt{l/g}.$$

To find the possible error in the time if l is three feet with a possible error of one-half inch, and g is 32.17 with a possible error of .01:

$$dT = \tfrac{1}{2}\pi\,(l^{-\frac{1}{2}}g^{-\frac{1}{2}}dl - l^{\frac{1}{2}}g^{-\frac{3}{2}}dg).$$

Substitute: $l = 3$, $dl = 1/24$, $dg = .01$ and add the numerical results, disregarding the negative sign. The result is: $dT = .0068$.

It often happens that the expression $f(x, y, z, \text{etc.})$ is in reality nothing but a product of different powers of x, y, z, etc. It is then particularly easy to estimate the relative or the percentage error in f in terms of the relative or the percentage errors in x, y, z, etc.; for if $f = A(x^p y^q z^r)$, then

$$\log f = \log A + p \log x + q \log y + r \log z,$$
$$df/f = p(dx/x) + q(dy/y) + r(dz/z).$$

Hence the relative or the percentage error in f is the sum of the relative or the percentage errors in each of the variables, each multiplied by the power to which that variable enters into the expression f.

Example 3. Find the possible relative or percentage error in the time T of example 2.

$$dT/T = \tfrac{1}{2}dl/l - \tfrac{1}{2}dg/g$$

The possible error in l is $\frac{1}{2}$ inch in 3 feet, which makes the relative error $dl/l = 1/72$, and its contribution to the relative error in T is 1/144. The relative error in g, namely dg/g, is only 1/3217, and its contribution to dT/T is only 1/6434, which is entirely negligible compared with 1/144. It follows, then, that the percentage error in T may be taken as about 2/3%.

A formula more general than that for the first differential is

$$\Delta f = df + \tfrac{1}{2}d^2 f + \tfrac{1}{6}d^3 f + \ldots,$$

which is Taylor's expansion, and is more frequently written (when there is only one independent variable involved) in the form

$$\Delta f = f'\Delta x + \tfrac{1}{2}f''\Delta x^2 + \tfrac{1}{6}f'''\Delta x^3 + \ldots,$$

where the successive derivatives are computed for that value of x from which Δx is measured. In particular if x is measured from O, then $\Delta x = x$ and Maclaurin's expansion is found.

6. Laws of Motion. The fundamental principles of mechanics which will be assumed as known are as follows:

1. Fundamental theorem of statics. When a rigid body is at rest, or in motion in a straight line with constant velocity and without rotation, the resultant force and the resultant couple acting on the body are each equal to zero. This means that the algebraic sum of the component forces along any direction vanishes, and that the algebraic sum of the moments of the forces about any axis also vanishes.

2. When a force acts upon a moving particle the rate of change of momentum of the particle is proportional to the force, and takes

place in the direction of the force. This means that the rate of change of a component momentum along any fixed direction is proportional to the component force along that direction.

3. Action and reaction are equal in magnitude and opposite in direction.

If W be the mass of a body, u the velocity in some fixed direction, and X the component force acting in that direction, the analytical statement of the second principle is:

$$\frac{d}{dt} Wu = kX \quad \text{or} \quad W\frac{du}{dt} = kX.$$

The value of the factor of proportionality, k, depends upon the units employed in measuring mass, velocity, and force. If the equation be applied to a body falling freely in vacuo with a mass measured as W pounds, the force will be W local pounds, and the acceleration, du/dt, will be the local value of g. Hence, k must be equal to g, and the equation becomes:

$$W\frac{du}{dt} = gX.$$

If it be desired to use standard pounds of force and a standard value of g, say g_o, the equation would be:

$$W\frac{du}{dt} = g_o X_o,$$

where X_o is the force in standard pounds.

Many persons desire, for algebraic simplicity, to be rid of the multiplier g. This may be accomplished by taking a unit of force $1/g$th of the pound, or the unit of mass as g pounds. Thus, the equation becomes:

$$\frac{W}{g}\frac{du}{dt} = W'\frac{du}{dt} = X \qquad \text{or} \qquad W\frac{du}{dt} = gX = X'$$

where, in the first equation, W' is the mass in slugs, and in the second equation X' is the force in poundals. In neither case does the analytical simplicity make up for the awkwardness of measuring mass or force as the case may be in units that are not given in everyday engineering experience.

A similar treatment may be given in the case of the metric system, when the kilogram mass and kilogram force, or the metric slug and the kilogram force, or the kilogram mass and the metric unit of force, which is $1/g$th of a kilogram, is used. The metric C. G. S. system of units is so defined that the multiplier k or g does not occur in the equation. The unit of mass is the gram and the unit of force the dyne, and the unit of acceleration the centimeter per second per second. The C. G. S. system is in common use in physics, but is not used in engineering in France.

The fundamental equations to be used in what follows are as follows:

$$W \frac{du}{dt} = gX, \qquad g = 32.17 \text{ (more or less)}$$

in the English system, where mass and force are measured in pounds, and acceleration in feet per second per second;

$$W \frac{du}{dt} = gX, \qquad g = 9.81 \text{ (more or less)}$$

in the metric system, where W and X are measured in kilograms, and acceleration in meters per second per second.

EXERCISES

Make the following changes in units:

1. 80 mi/hr to m/sec
2. 9.8 m/sec^2 to mi/hr.sec
3. 1 atmo to lb/ft^2
4. 0.08 lb/ft^3 to kg/m^3
5. 2000 lb/ft^2 to atmo
6. 1 atmo to kg/m^2
7. 10000 kg/m^2 to atmo
8. 1 ton.in to kg.m

9. If 13.6 be the specific gravity of mercury, what is its density in lb/ft^3 and kg/m^3?

10. A cylinder 10 in long and 5 in across is exhausted to vacuum. How many ft.lb and kg.m are done if the piston moves down the cylinder under atmospheric pressure?

11. Give approximate values for $\sqrt{1.02}$, $\sqrt[3]{0.97}$, $(1.005)^4$, $(0.985)^3$, $1/(1.033)$, $1/\sqrt{0.99}$, $1/(0.98)^2$.

12. If $p = \rho\, v^2/2g$, what relative error in p may arise from an error of 1% in ρ and 2% in v? *Ans.* 5%.

13. The inside radius of a small tube is measured by filling the tube with mercury, measuring the length of the column of mercury and weighing the mercury. What percentage error may arise in the determination of the radius through a 2% error in both length and weight? *Ans.* 2%.

14. The range of a projectile in vacuo is $R = V^2/g \sin 2i$. If $V = 100$ ft/sec and $i = 30°$, what error in the range may come from an error of 1 ft/sec in V and of 1 min of arc in i?

15. Find the percentage error in R in Ex. 14 independently of the actual error.

16. If the density and dimensions of a rigid body may be measured to $\frac{1}{2}$%, what may be the error in the calculated moment of inertia? Suppose the mass

and dimensions are known to $\frac{1}{2}\%$, what is then the possible error in the moment of inertia? *Ans.* 3%; 1.5%.

17. If $x < 25°$ show that the substitution of x in radius for $\sin x$ introduces an error of less than 4%. For how large values of x may $\cos x = 1 - x^2/2$ be used, with an error less than 4%? How about the approximation $\cos x = 1$ when x is small?

CHAPTER II

THE PRESSURE ON A PLANE

7. Normal planes. Various attempts have been made to obtain theoretically by the applications of the principles of mechanics an expression for the pressure upon a plane of area S situated in a stream of which the velocity is U and the density ρ. The fundamental equation of mechanics states that Pg (the force) is equal to the rate of change of momentum. If, then, the rate of change of momentum can be calculated, the force will be known as

$$P = 1/g \times \text{rate of change of momentum.}$$

An early calculation known to Newton was based on the consideration of the fluid as made up of particles impinging on the plate. Consider that a stream of cross section S has its forward momentum destroyed at the plate. The mass of the stream which falls upon the plate is ρSU per unit of time, and the velocity is U. The total momentum destroyed per unit time is therefore, ρSU^2, and the pressure is

$$P = \rho SU^2/g.$$

Now for air, $\rho = 0.08$ lbs/ft^3 (nearly), and as $g = 32$, the pressure would be

$$P = .0025\rho SU^2.$$

The experimental coefficient is (for square planes) about .0015 instead of .0025. The argument, therefore, is only qualitatively correct; but so far as it goes it does show that the pressure varies with the surface and with the square of the velocity; and both of these laws of variation are verified experimentally within reasonable limits. The reason that the coefficient comes out too high is because the plate does not actually stop a column of fluid of cross section equal to S. The fluid can, so to speak, evade the plate by flowing around it, and the effective column of fluid, whose momentum is destroyed, has a cross section only 60% of S. When the plate is a long rectangle, instead of a square, the fluid can only evade the plate on two sides, instead of four, and it is to be expected that the coefficient will be larger than .0015 but smaller than .0025, — and this is the fact.

8. Inclined Planes. In case the plate is inclined at an angle i to the general direction of the stream, the velocity of flow U may be resolved into a component $U \sin i$ perpendicular to the plate, and a component $U \cos i$ along the plate. The plate may be considered as stopping the momentum of the flow perpendicular to the plate while allowing the momentum of the flow along the plate to proceed unaltered. The amount of liquid which reaches the plate per unit of time may be taken as $\rho US \sin i$, because $S \sin i$ is the normal area which the plate exposes to the direction of the stream. The velocity being $U \sin i$, the rate of destruction of momentum is $\rho SU^2 \sin^2 i$, the product of mass and velocity, and gives the formula

$$Pg = \rho SU^2 \sin^2 i,$$

which is known as the "law of the sine-square," and is attributed to Newton. Experiments fail to verify this law. As the momentum destroyed is momentum perpendicular to the plate the pressure should be perpendicular to the plate, and this direction of pressure is verified except for a slight tangential drag due to friction.

A second argument attributed to Euler is as follows: The pressure on the plate must be (apart from friction) normal to the plate, but it should depend upon the apparent normal area which the plate exposes to the stream, which is $S \sin i$. The amount of liquid which reaches the plate per second is $\rho US \sin i$, and the momentum given up is $\rho SU^2 \sin i$, the product of the mass by the velocity. The pressure on this basis is

$$Pg = \rho SU^2 \sin i.$$

This is known as the "law of the sine," and is reasonably well verified in experiment so far as the direct variation with the sine of the angle is concerned, although the coefficient is not numerically equal to ρ/g.

Both arguments for a pressure formula are crude, and chiefly only of historical interest. It is, as a matter of fact, impossible to obtain a theoretical formula without a considerable knowledge of theoretical hydromechanics. In 1876 Lord Rayleigh obtained the result (for infinite strips)

$$P = \frac{\pi \sin i}{4 + \pi \sin i} \times \rho SU^2/g.$$

When the angle i is 90° the value becomes for air ($\rho = .08$)

$$P_{90} = .0011SU^2,$$

which is about half the experimental value .0021. When the angle i is small, $\sin i$ may be replaced by i in radians and the formula becomes

$$P = \pi\rho SU^2 i/4g = .002SU^2 i = .00003SU^2 i^\circ,$$

which is in poor agreement with the experimental facts. At the same time Lord Rayleigh obtained the formula

$$\frac{x}{b} = \frac{3}{4}\,\frac{\cos i}{4 + \pi \sin i} = .187 - .147i \text{ (approx.)}$$

for the distance of the center of pressure from the forward or leading edge of the plane, if b is the breadth of the plate.

Lanchester has given a calculation for the pressure upon a plane at the small angles in which the fact that the pressure must be due to a change in the momentum of the fluid is given a fundamental place. Let it be assumed that l is the length and b the breadth of the plane which will be taken as slightly curved, so that the air may follow it without discontinuity. Let it further be assumed that the particles of the fluid trace lines more or less parallel to the plate, leaving the rear of the plate at an angle of depression i. It is further assumed that the region of the stream above and below the plate which is affected by the presence of the plate has a total vertical dimension h. The volume of air which flows by per unit time, and is affected by the plate, is hlU. When the stream issues with velocity U from the trailing edge there is a downward component of velocity equal to $U \sin i$. Therefore, the downward momentum generated per unit time is the product of the mass ρhlU, which is affected, and the downward velocity $U \sin i$. The plate, therefore, by its pressure on the fluid has produced a rate of change of momentum in the fluid equal to $\rho hlU^2 \sin i$; and by the law of action and reaction the fluid must press against the plate with the pressure

$$Pg = \rho hlU^2 \sin i.$$

From experiments by Langley on the interference of superposed planes (biplane combinations) Lanchester inferred that the stratum of air affected by the plate was equivalent to the breadth of the plate both above and below, so that $h = 2b$. Substitute this value and the pressure on the plate is

$$Pg = 2\rho SU^2 \sin i,$$

since $S = bl$. Or $P = .005i = .00009i^\circ$, which is about right when l/b is about 6, but takes no account of the aspect ratio l/b.

The theory is capable of extension to the curved aerofoil. It is a matter of experience that in the neighborhood of the leading edge the air stream is rising, whereas in the neighborhood of the trailing edge it is falling. Let j be the angle at which the air is rising. Then the action of the aerofoil is to convert upward moving momentum into downward moving momentum, and the reaction of the fluid on the plate is

$$P = 2\rho SU^2 (\sin i + \sin j)/g.$$

The factor 2 might be replaced by a constant multiplier k if it should be found that a total vertical depth of $2b$ were not a just value for the stratum of air affected.

Nobody recognizes better than Lanchester himself that this argument compounded of equal amounts of appeal to experiment and to dynamics is justified mainly by the fact that it gives the desired result and is suggestive.

9. Experimental Results. The actual experimental results for the pressure, P_i, on a plane inclined at an angle i to a stream, are expressed in terms of the pressure P_{90} on an equal area normal to the stream. For air of standard density the values of P_{90} for square planes are

$P_{90} = .0015SU^2$, P in lbs, U in ft/sec, S in sq.ft,
$P_{90} = .0032SV^2$, P in lbs, V in mi/hr, S in sq.ft,
$P_{90} = .0060SV^2$, P in kg, V in km/hr, S in sq.m,
$P_{90} = .078SU^2$, P in kg, U in m/sec, S in sq.m.

These values are for square planes of large size; that is, of area upwards of 12 square feet (a trifle over 1 m²). For smaller areas a correction factor c must be applied, so that the pressure is not P but cP. The values of this factor are given in the accompanying table.

Correction Factor for Planes of less than 12 ft²

S	0.10	0.50	0.75	1.00	3.00	5.00	8.00	12.00
c	0.86	0.88	0.89	0.896	0.93	0.97	0.98	1.00

The pressure also varies with the aspect ratio, which is the ratio $r = l/b$ of length to breadth, that is, $P = c'P$ where c' is another correction factor dependent on the aspect ratio r.

Correction Factor for Aspect Ratio r

r	1	1.5	3	6	10	14.6	30	50
c'	1.00	1.04	1.07	1.10	1.145	1.25	1.40	1.47

The two tables show that as a normal plane is smaller the pressure is relatively less, and as the plane is more elongated across the wind the pressure is relatively greater.

For planes inclined at an angle i° to the direction of the stream, the pressure is decidedly irregular as a function of i, but for small values of i may be represented by the simple expression $P_i = kP_{90}i^\circ$, where k varies with the aspect ratio r and where the range of values of i for which the simple expression holds also varies with r.

Values of Factor k in $P_i = kP_{90}i^\circ$

Aspect Ratio	1	2	3	6	9
Value of k	0.036	0.043	0.050	0.061	0.075
Range of i	to 35°	to 20°	to 15°	to 12°	to 9°

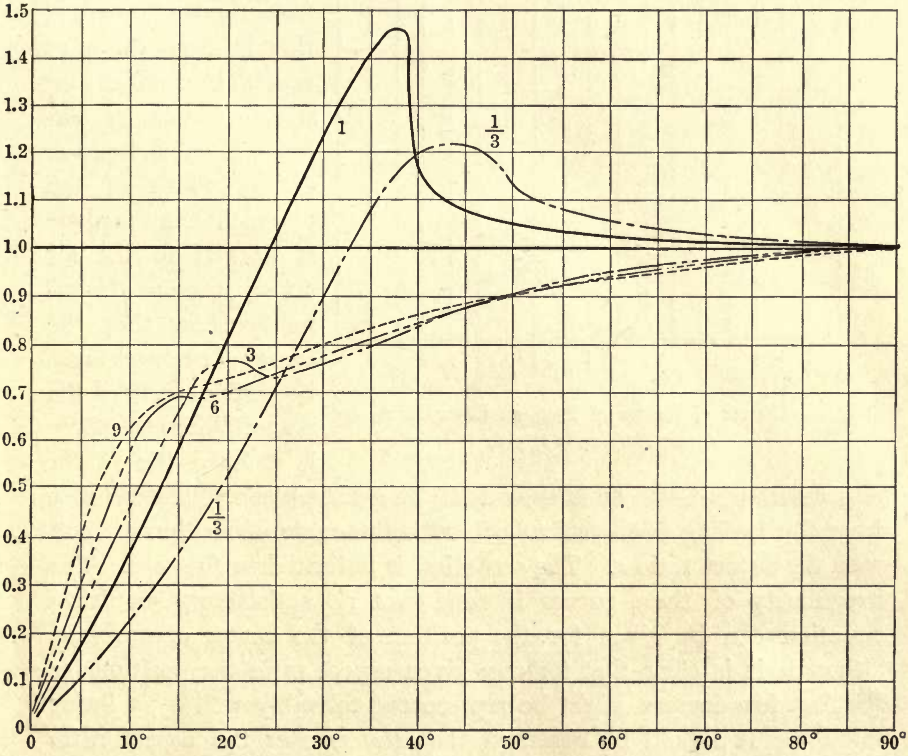

Fig. 1. Curves of the Ratio P_i/P_{90} (after Eiffel) for Aspect Ratios, $\frac{1}{3}$, 1, 3, 6, 9.

The relation between aspect ratio r and the constant k may be expressed approximately by the linear relation

$$k = .032 + .005r,$$

so that the pressure takes the form

$$P_i = P_{90}(.032 + .005r)i^{\circ},$$

where i is in degrees. The relation, however, is nowhere near valid except for small values of the angle i. The actual curves for the ratio P_i/P_{90} are given in figure 1. The irregularity of these curves will serve as an indication of how unsatisfactory in aerodynamics empirical formulas must necessarily be.

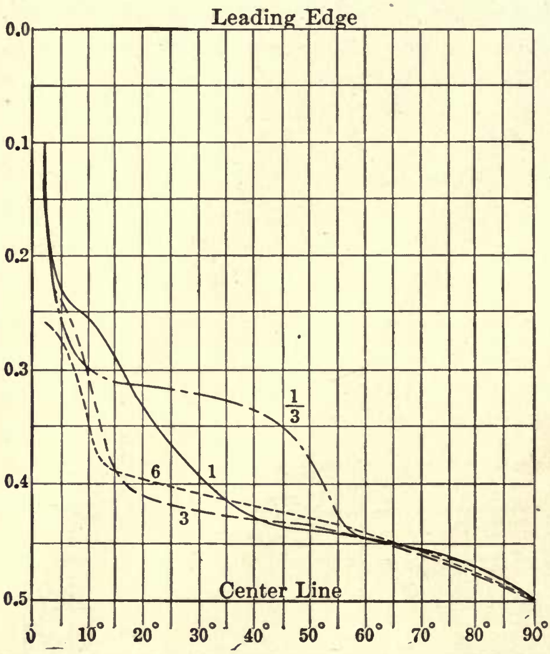

Fig. 2. Curves of Center of Pressure (after Eiffel) for Aspect Ratios, ⅓, 1, 3, 6.

The position of the center of pressure on the plane is measured from the leading edge; that is, from the edge which is further up stream; and is expressed as a fraction of the width of the plane. When the plane is normal to the air stream, reasons of symmetry show that the center of pressure must be at the center of the plane. When the plane is inclined the center of pressure must still lie in the middle line of the plane. The distance from the leading edge varies both with the angle of incidence i and with the aspect ratio r. The variation is indicated in figure 2. The irregularity of these curves is such that no satisfactory empirical equation can be given for the position of the center of pressure, although it is clear that for any given aspect ratio the position for the first few degrees might be represented tolerably well by a linear relation. It should be observed that the greater the aspect ratio the further the center of pressure lies from the leading edge.

The irregularities of the functions which arise in expressing various important properties of pressures in fluid motion are such that no satisfactory insight into the behavior of fluids can be obtained without a considerable study of theoretical fluid dynamics.

10. Factors of Proportionality. Not only is it necessary to change a physical quantity from one set of units to another; it is also important to change a factor of proportionality; for instance, if $P = .0015SU^2$ be the formula for the pressure on a normal plane in pounds when velocity is in feet per second, it may be required to find the formula when the velocity is in miles per hour. It is not possible here to change feet per second formally as indicated in Art. 4 to miles per hour. It is necessary that the value of $.0015SU^2$ when U is expressed in feet per second should be the same as the value of kSV^2 when V is expressed in the equivalent number of miles per hour. Now, if U is 1 ft/sec, $V = \frac{2}{3}$ mi/hr; and the equation

$$.0015S(1)^2 = kS(\tfrac{2}{3})^2$$

must hold. Hence

$$k = \tfrac{9}{4}(.0015) = .0034.$$

(The value $\frac{2}{3}$ is about 3% in error, and its square about 6%. If the 6% be deducted from 34 the result is 32, which checks with the formula given.) The general inference is that in converting the value of a factor of proportionality from one set of units to another it is necessary to multiply by the reciprocals of the ratios of the units, and not to substitute directly the ratios.

This inference may be demonstrated in detail as follows:

Take for illustration the formula

$$P = .0015SU^2.$$

The quantity .0015 is physically of the sort P/SU^2, as may be seen from the equation

$$.0015 = P/SU^2.$$

It is lb divided by ft², and by (ft/sec)². If the quantity be written with the symbols designating its quality the result is

$$\frac{.0015 \text{ lb}}{\text{ft}^2(\text{ft/sec})^2}.$$

Now if lb be changed to kg or to any other unit and ft to m or other units and ft/sec to any other units, by the formal rule of substituting values as in Art. 4, the result will be the correct value for the new factor of proportionality. For example:

$$\frac{.0015 \text{ lb}}{\text{ft}^2(\text{ft/sec})^2} = \frac{.0015(.45\text{kg})}{(.3 \text{ m})^2(.3 \text{ m/sec})^2} = .083.$$

The result is that in the metric system the formula for the pressure should be

$$P = .083SU^2.$$

(The value given is .078 instead of .083. The discrepancy is due to the use of .3m = ft, which is in error by $1\frac{1}{2}\%$. As this factor occurs in the denominator 4 times, the result should be in error about 6% of .083, *i.e.*, about .005, and subtracting this the result is .078, which checks. This example is a good illustration of the care that must be used to avoid a small percentage error in the value of a conversion factor such as .3 when that factor enters a large number of times into the expression to be calculated.)

11. A Moving Plane. It has been assumed that the plane is at rest at an inclination i to the direction of the air stream which is moving with a velocity U. The same pressure P would arise if the plane traveled with velocity U in still air. It is only the relative velocity which counts. In case it is convenient to consider both the air and the plane as in motion, the pressure P must be calculated by using for U the relative velocity of the plane and air and for i the angle between the plane and the direction of the relative wind. If U be the velocity of the air and u that of the plane, the relative velocity is $U - u$ provided both U and u be drawn as vectors and the difference $U - u$ be a vector difference. If, for example, a plane which makes the angle i with the wind velocity is itself moving perpendicular to the wind with velocity u, the vector diagram (Fig. 3) shows that the numerical value of the relative velocity is $(U^2 + u^2)^{\frac{1}{2}}$ and the actual angle between the plane and relative wind is not i but $\phi = i - \tan^{-1}(u/U)$ in radians or $i^\circ - \tan^{-1}(u/U)$ in degrees. The pressure is, therefore,

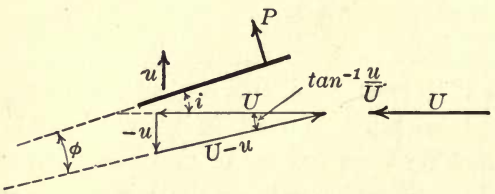

Fig. 3. Pressure on Plane Moving across Wind.

$$P = .0015S(U^2 + u^2)(.032 + .005r)(i^\circ - \tan^{-1}u/U).$$

This result may be used to obtain a preliminary idea of the turning of a windmill or of a propeller. Suppose two or more surfaces

S are inclined at an angle i to a wind stream and are centered on arms of length b from the axis of rotation about which the mill is rotating with angular velocity ω. The velocity of the vanes is across the wind and $u = \omega l$. With radian measure of angle the pressure on each vane is

$$P = kS(U^2 + \omega^2 l^2)[i - \tan^{-1}(\omega l/U)],$$

where

$$k = .0015(.032 + .005r) \times 57.3.$$

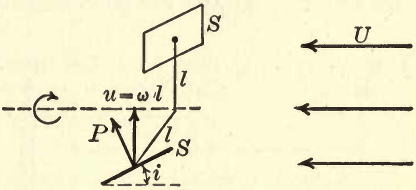

Fig. 4. Pressure and Torque on Moving Vane.

The thrust down the axis is $T = P \sin i$ and perpendicular to the axis it is $P \cos i$; the torque about the axis is $Pl \cos i$ due to each vane. The power absorbed by the mill is the torque multiplied by the angular velocity or

$$\Pi = nkS(U^2 + \omega^2 l^2)[i - \tan^{-1}(\omega l/U)]\omega l \cos i$$

if n be the number of vanes. This has a definite maximum as a function of ω if i be considered as known. Moreover ω comes in only in the form $\omega l/U$; for

$$\Pi = nkSU^3\left[1 + \left(\frac{\omega l}{U}\right)^2\right]\left[i - \tan^{-1}\frac{\omega l}{U}\right]\frac{\omega l}{U}\cos i.$$

These formulas are good only if $\phi = i - \tan^{-1}(\omega l/U)$ is a small angle so that P may be assumed to vary with the angle, and if the vanes do not interfere with each other's action. Moreover the tangential drag has been neglected, the pressure being assumed to be strictly normal to the vanes, whereas at high velocities and very small angles the tangential drag is not negligible.

In the calculation of Π it was assumed that the area S is centered at a distance l from the axis; if the area were distributed between $l - s$ and $l + s$ (with $S = 2sb$) the inner parts of the vane would have smaller total velocities and larger angles between the relative wind and the plane. The evaluation of P, T, Π would involve an integration. Another problem will, however, be treated to illustrate this principle. Suppose a plate $2s$ by b mounted on an arm of length l and rotated with angular velocity ω in a plane in still air, the plate being inclined at an angle i to the plane of rotation. Let y be the distance from the center to any strip dy by b of the plate; the total distance from the axis is $l + y$. The velocity of the center is $U = \omega l$; that of the strip is $U = \omega(l + y)$. The angle is constantly i. Hence

$$P = \int_{-s}^{+s} ki\omega^2(l + y)^2 b\, dy = kiS\omega^2 l^2\left(1 + \frac{s^2}{3l^2}\right).$$

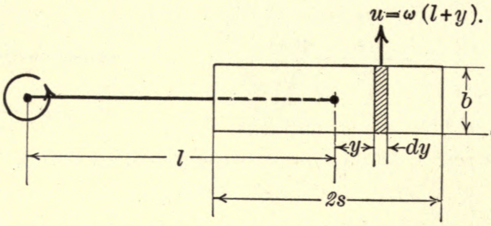

Fig. 5. Pressure on Vane Rotating in Still Air (Vane Distributed).

The pressure is somewhat greater than $kiS\omega^2 l^2$ which would have been the value on the assumption that S were centered at the distance l. Moreover the pressure is no longer centered at the distance l from the axis. Indeed the moment of the pressure about the central line of the plate at distance l is

$$M = \int_{-s}^{s} ki\omega^2(l + y)^2 yb\, dy = kiS\omega^2 l^2\left(\frac{2s^2}{3l}\right)$$

and hence the center of pressure has moved out radially by the distance

$$\Delta y = s\left(\frac{2}{3}\frac{s}{l}\right)\Big/\left(1 + \frac{s^2}{3l^2}\right) = \frac{2s^2}{3l} \text{ (approx.)}.$$

EXERCISES

1. Six hundred 30-gram bullets moving with a velocity of 1000 m/sec hit a target per minute. Find pressure in kg and lb. *Ans.* 30.6 kg; 67.3 lb.

2. A centrifugal gun is alleged to throw 33,000 shots per minute with the velocity of 1000 ft/sec. If the shots weigh one ounce, what is the back pressure on the gun? *Ans.* 1070 lb.

3. Prove that if the particles of fluid be reflected by the plane as though the impact were perfectly elastic the pressure would be

$$Pg = 2\rho U^2 S \sin^2 i.$$

(*Note:* This method of calculating pressure is useful in the kinetic theory of gases where molecules are moving in every direction, but gives the wrong result for pressures on aerofoils.)

4. If the pressure on a large square plate normal to a stream is $p = 15$ lb/ft² what is the pressure on a plate of 3 sq.ft? What must be the aspect ratio of a plate of 3 sq.ft if the pressure is 15 lb/ft²? *Ans.* 42; $r = 4$.

5. The aspect ratio is 6, the area 8 sq.ft, the velocity 60 mi/hr, and inclination 7°. Find total pressure and its components along and across the stream. *Ans.* 38.5 lb; 38.3 lb; 4.7 lb.

6. Make the change of units to verify $P_{90} = .078\ SU^2$ from $P_{90} = .0060SV^2$.

7. Make the change of units necessary to verify the formula: $P_{90} = .0015SU^2$ (P in lb, U in ft/sec) from $P_{90} = .0060SV^2$ (metric).

8. Given $W\ du/dt = 32.2F$ in the English system. Make the change of units to find the numerical multiplier if W is in kg, u in m/sec, F in kg. The answer should be about 9.81.

9. How much force is required to hold a plate of 10 ft² normal to a 20-mile wind? How much additional force if the plate is being moved 3 mi/hr into the wind? Can a 150-lb man hold the plate if his coefficient of friction is 0.1? At what angle must he stand?

10. If an automobile with top up exposes 6 ft² more of effective normal surface to the wind and if the automobile is driving 25 mi/hr into a 25-mile wind, how much is the additional force due to the top and how much additional power is required?

11. A rectangle is held at an angle of 6° to a 20-mile breeze. How fast must it be moved across the wind (*a*) to halve the pressure, (*b*) to double the pressure, (*c*) to reverse the pressure?

12. A skater whose coefficient of friction is $\mu = 0.03$ and whose weight is 150 lb exposes (with sail) an area of 12 ft² normal to a 15-mile wind before which he runs. Find his speed if $r = 3$. *Ans.* $4\frac{1}{2}$ mi/hr.

13. A skater has a sail of 20 ft², $r = 3$, and a following wind of 20 mi/hr. At what angle i must he set his sail so that there is equilibrium between his friction and the forward thrust on the sail? Take $W = 150$ lb, $\mu = 0.03$. What is the cross-wind thrust?

14. How fast will the skater in Ex. 13 move if $i = 9°$?

15. A plane 50 by 8 ft moves in a circle of 200-ft radius with a velocity of 100 ft/sec and an angle of $i = 8°$. Find the pressure if the surface is assumed centered, find the actual excess pressure, the moment about the central line, and the distance of the center of pressure from the line.

CHAPTER III

THE SKELETON AIRPLANE

12. Single Planes at Different Angles. Before proceeding to the discussion of actual airplanes with cambered wings the simpler analysis for the case of a skeleton airplane with a flat wing will be taken up.

The airplane will be considered as reduced, so far as its mass is concerned, to a single point, which will be called the center of gravity (C. G.). Support is furnished by a single flat surface of area S, corresponding to the main wings. (See figure 6.) The external applied forces are three: namely, the weight W, acting down, the propeller thrust T, acting forward, and assumed for the present to pass through the center of gravity, and the air pressure P, acting normal to the wing.

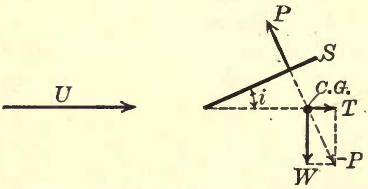

Fig. 6. Horizontal Flight, Propeller Thrust Horizontal.

Let it be assumed that the machine is in horizontal flight, and that the wing makes an angle i with the horizon, so that the wind pressure P, being normal to the surface by assumption, makes an angle i with the vertical.

The conditions of equilibrium for uniform motion are

$$T = P \sin i, \qquad W = P \cos i \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

If U be the velocity of flight, the air pressures may be represented approximately for small values of the angle i by the equation

$$P = kU^2Si, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

the pressure varying as the square of the velocity, and as the angle of inclination of the plane. The value of k depends on the units in which P, U, S, i are measured. Numerical values may be found in Chap. II.

The work done by the force T is the product of the force by the distance of horizontal motion; and the power or rate of doing work is, therefore, the product of T by the horizontal velocity U. Hence

$$\Pi = TU = kU^3Si^2, \quad \ldots \ldots \ldots \quad (3)$$

where the symbol Π represents the power, and where $\sin i$ has been set equal to i since i is assumed to be small and to be measured in radians.

If the weight and surface be regarded as given, the velocity and the angle i which the wing should make with the propeller shaft or the horizontal are connected by the equation

$$W = kU^2Si \quad \ldots \ldots \ldots \quad (4)$$

inasmuch as for small angles $\cos i$ may be set equal to 1. It is possible to eliminate the variable i between equations (3) and (4), and obtain a relation between the velocity of flight and the power expended, namely:

$$\Pi = W^2/kSU \quad \ldots \ldots \ldots \quad (5)$$

In this expression the value of k, suitable for radian measure of angle in (2), must be used.

This shows what might appear at first sight to be a paradoxical state of affairs, namely, that the greater the speed of flight the less the power required for the propulsion of the airplane, it being assumed that the design is so made as to give the correct value of i. The result is not paradoxical because as the machine is supposed to fly faster the angle i must be decreased in the design, so that we have to deal not with the faster flight of an actual machine, but with the faster flight of a series of machines designed to fly at a series of speeds. As the pressure varies with i and the backward component pressure varies with i^2 the diminution of i cuts down the force very rapidly — indeed so rapidly that, as shown above, the power required is actually diminished. It may be pointed out that one of the most serious difficulties in the early days of the airplane was to get the airplane started. In order for it to fly at all it had to have a considerable forward velocity. The situation was such that if the machine could get a fast enough start it had sufficient power to continue; but it did not have power enough to start.

13. Single Plane with Fixed Design. Suppose that instead of considering a series of airplanes with different angles i between the supporting plane and the direction of the propeller thrust we treat the steady flight of the same machine at different speeds. Let i be the fixed angle between the propeller thrust and the plane as before, and j the angle which the propeller thrust makes with the horizontal, measured positively when the direction of the thrust is above the horizontal (Fig. 7).

The angle between the plane and the direction of motion, which is horizontal, is then $i + j$, and

$$P = kSU^2(i + j) \quad \ldots\ldots\ldots\ldots \quad (6)$$

Horizontal and vertical components are

$$T \cos j = P \sin (i + j), \qquad W = T \sin j + P \cos (i + j). \quad (7)$$

As the angles i and j will be considered small, set $\sin j = j$, $\sin (i + j) = i + j$, and $\cos j = \cos (i + j) = 1$. The value of T is small compared with W because it contains $i + j$ squared. The value of $T \sin j$ or Tj is still smaller compared with W, and the equations may be written:

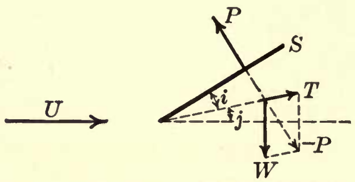

Fig. 7. Horizontal Flight, Propeller Thrust Inclined

$$T = kSU^2(i + j)^2,$$
$$W = kSU^2(i + j) \quad \ldots \quad (8)$$

If W, S and i be considered given, the second equation shows that the velocity of flight varies inversely as the square root of $i + j$. If, for instance, $i = 4°$ and $j = 0°$ for horizontal flight at the velocity U, j must be $12°$ for horizontal flight at the velocity $\frac{1}{2}U$. As the formula for the pressure which states that the pressure varies directly as the angle between the plane and the direction of flight is only reasonably correct up to angles of $14°$ to $18°$ for planes of ordinary shape or aspect ratio (see Art. 9), the argument which is given is intended only to show the very rapid increase in j which must take place as the speed diminishes. The propeller thrust may be expressed in terms of U by eliminating the variable j between the equations (8), so that

$$T = \frac{W^2}{kSU^2}, \qquad \Pi = TU = \frac{W^2}{kSU} \quad \ldots\ldots \quad (9)$$

The propeller thrust, therefore, increases as the speed diminishes; the power is again seen to increase as the speed decreases, and to decrease as the speed increases. Indeed, to the order of approximation used, the result is identical with (5).

14. Plane with Parasite Resistance. The next approximation to reality in an airplane will be made by assuming that, in addition to the wing S, there is an opposing surface S' normal to the wind. It is customary to treat the resistance of the body of the airplane, that is, the parasite resistance, as that due to a certain amount of surface normal to the direction of flight. The forces acting are, then: T, P, W, and $P' = k'S'U^2$. The conditions of equilibrium are (assuming $j = 0$)

$$T = P \sin i + k'S'U^2, \quad W = P \cos i \quad . \quad . \quad . \quad . \quad (10)$$

Let $\cos i = 1$, $\sin i = i$, and substitute for P. The result is

$$T = kSU^2i^2 + k'S'U^2, \quad W = kSU^2i \quad . \quad . \quad . \quad . \quad . \quad (11)$$

The elimination of i gives for the power

$$\Pi = TU = \frac{W^2}{kSU} + k'S'U^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The first term W^2/kSU corresponds to the power consumed by wing resistance alone, and, as before, diminishes as the speed increases, owing to the diminution of the angle between the plane and the direction of motion. The second term represents the power consumed by the parasite resistance, and increases as the speed increases. There must then be a particular speed, U, for which the power required is a minimum. The calculation for determining this speed and the power then required is as follows:

$$\frac{d\Pi}{dU} = -\frac{W^2}{kSU^2} + 3k'S'U^2 = 0,$$

$$U = \frac{W^{\frac{1}{2}}}{(3kk'SS')^{\frac{1}{4}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\Pi = \frac{4W^{\frac{3}{2}}}{3^{\frac{3}{4}}} \frac{(k'S')^{\frac{1}{4}}}{(kS)^{\frac{3}{4}}} = 4\frac{WU}{\sqrt{3}}\left(\frac{k'S'}{kS}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The value of k suited to radian measure of i must be used; k' does not involve angle at all because S' is the normal surface representative of the parasite resistance.

The last equation shows that for the same machine (S'/S unchanged) the weight carried per horse power varies inversely as the speed, when the speed is the most economical; and that for the same machine (SS' = const.) the most economical speed varies as the square root of the weight. Hence, if the machine be loaded with extra weight the speed increases, and the weight carried per horse power decreases.

If two machines were similar in design in all particulars the surfaces would vary as the square of the linear dimensions (length over all or length of wing), and the weight, depending on the volume, would vary with the cube of the linear dimensions. The most economical velocity would, therefore, vary with the square root of the linear dimensions, and the horse power required with the 7/2-power of the dimensions. This very rapid increase of power required would indicate that the linear dimensions of the machine could not be indefinitely extended unless changes in design could be made which would keep the weight from mounting in proportion to the cube of the dimensions. One of the directions in which weight has been progressively saved in the last decade is by the construction of engines more powerful in proportion to their weight, so that at present a good engine may develop about $\frac{1}{2}$ H.P. per pound weight, whereas in early days only $\frac{1}{5}$ H.P. or less per pound was developed.

The term "most economical speed" as used above refers only to the minimum consumption of power in flight, and not at all to the determination of the speed which would be most economical commercially where the important element is the amount of *useful* weight, that is, of weight above that of the machine and its equipment. Moreover, the power is here calculated as though due to an external applied force T.

Example. Let $W = 2000$, $S = 500$, aspect ratio = 6, so that $P_i = 0.06P_{90}i$. Find the angle i for horizontal flight if $\Pi = 30$ H.P. Find the angle j for flight at 70 ft/sec and 40 ft/sec. Find the additional H.P. required if the machine has the equivalent of 6 ft of parasite surface normal to the wind, and the most economical speed.

Take $P_{90} = .0015SU^2$. Then

$$W = 2000 = P = .00009 \times 500 \times U^2 i,$$
$$T = \Pi/U = 165{,}000/U = P \sin i = .00009 \times 500 \times U^2 \times i^2/57.3.$$

For these equations i is in degrees, and $\sin i$ has been replaced by $i^\circ/57.3$. The equations may be written

$$U^2 i = 44{,}400, \quad U^3 i^2 = 21{,}000{,}000,$$
$$Ui = 470, \quad U = 95 \text{ ft/sec},$$

and hence $i = 5^\circ$. Next,

$$W = 2000 = .045 \times (70)^2(i + j),$$
$$(i + j) : 5^\circ = (95)^2 : (70)^2 = 1.8,$$

from which $j = 4°$. At $U = 40$ ft/sec, $i + j = 28°$ and $j = 23°$. At such values of $i + j$, however, the empirical expression for P is far from right for aspect ratio 6, and the approximations for sine and cosine are no longer very good. Hence it is by no means certain that the machine could fly at all in this attitude, and it is probable from general considerations that it could not.

At $U = 95$ ft/sec the power absorbed by 6 ft of parasite surface is

$$\Pi = .0015 \times 6 \times (95)^3 = .009 \times 860{,}000 = 7740 \text{ ft. lbs/sec.}$$

The additional horse power is 14.

The most economical speed of operation is determined from formula (13), where k must be the value appropriate to radian measure of angle. The value of k above is .00009 when i is in degrees, or .0051 when i is in radians. Hence

$$U = \frac{\sqrt{2000}}{\sqrt[4]{3 \times .0051 \times .0015 \times 500 \times 6}} = \frac{45}{.51} = 88 \text{ ft/sec.}$$

15. Inclined Flight. — Consider next the case of the airplane which climbs at an angle Θ with the propeller thrust at an angle j to the line of flight, and with no parasite resistance. The conditions for uniform velocity are (Fig. 8)

$$T = W \sin(\Theta + j) + P \sin i, \quad W \cos(\Theta + j) = P \cos i. \qquad (15)$$

where the resolution of forces has been made along and perpendicular to the direction of the propeller thrust. With the customary approximations for the trigonometric functions of the small angles i and j,

$$T = W(\sin\Theta + j\cos\Theta) + kSU^2(i^2 + ij),$$

$$W(\cos\Theta - j\sin\Theta) = kSU^2(i + j).$$

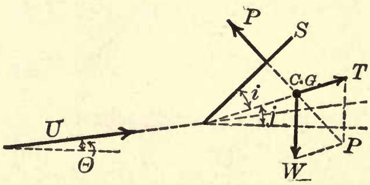

Fig. 8. Inclined Flight.

The equations may be further reduced if Θ is small as it generally is when positive; large negative values are of course of frequent occurrence, and for such values T may be zero because the machine may descend with the engine cut off.

If j be eliminated from the expression for the power Π and if $j \sin\Theta$ be neglected relative to $\cos\Theta$, and j^2 relative to 1,

$$\Pi = UW\sin\Theta + \frac{W^2\cos^2\Theta}{kSU}.$$

The first term is the power consumed in raising the machine and the second that in overcoming wing resistance. The power is a minimum when $U^2 = W\cos^2\Theta \div kS\sin\Theta$ for any given Θ. This gives

$$\Pi = \frac{2W^{\frac{3}{2}}\sin^{\frac{1}{2}}\Theta\cos\Theta}{(kS)^{\frac{1}{2}}}, \text{ and } \tan\Theta = i + j.$$

Since $i + j$ is small, by the assumed approximations, $\Theta = i + j$ is small, and the results hold only for small angles of climb. For these angles, the line of flight bisects the angle between the plane and the horizontal.

The velocity of climb is $v = U\sin\Theta$ and hence the velocity of climb when the power is least is found from

$$v^2 = U^2\sin^2\Theta = W\cos^2\Theta\sin\Theta/kS.$$

This is a maximum when $\Theta = \sin^{-1} .577 = 35.2^\circ$. For such large values of Θ the analysis is, however, not valid because $i + j = \tan\Theta$ is no longer small.

The equations for uniform motion with parasite are

$$T\cos j = W\sin\Theta + P\sin(i+j) + k'S'U^2,$$
$$W\cos\Theta = P\cos(i+j) + T\sin j,$$

if the resolution be made along and perpendicular to the path. With the usual approximations these equations become

$$T = W\sin\Theta + kSU^2(i+j)^2 + k'S'U^2,$$
$$W\cos\Theta = kSU^2(i+j),$$
$$\Pi = WU\sin\Theta + kSU^3(i+j)^2 + k'S'U^3$$
$$= WU\sin\Theta + \frac{W^2\cos^2\Theta}{kSU} + k'S'U^3.$$

The power is least when

$$\frac{d\Pi}{dU} = W\sin\Theta - \frac{W^2\cos^2\Theta}{kSU^2} + 3k'S'U^2 = 0,$$

$$U^2 = \frac{W\cos\Theta}{6k'S'}\left(\sqrt{\tan^2\Theta + 12k'S'/kS} - \tan\Theta\right)$$

for any given value of Θ. The value of $i + j$ is

$$i + j = \frac{6k'S'}{kS}\,\frac{1}{\sqrt{\tan^2\Theta + 12k'S'/kS} - \tan\Theta}.$$

If $k' = .0015$ and $k = .0051$ for radian measure, $S' = 8$, $S = 400$, and $W = 2000$, the results are

$$U^2 = \frac{2000 \cos \Theta}{.072}(\sqrt{\tan^2 \Theta + .0706} - \tan \Theta),$$

$$i + j = \frac{.0353}{\sqrt{\tan^2 \Theta + .0706} - \tan \Theta}.$$

It is only for values of Θ less than 12° that $i + j$ is small enough so that the approximations may be considered fair.

16. Circular Flight. Let the machine be turning uniformly in a horizontal circular path of radius R. Let the angle between the wing and the direction of flight be i, and let the plane be banked up at an angle ϕ. If the velocity of flight be U and if the area S be treated as centered at the distance R from the axis of revolution, the pressure on the plane is

$P = kSU^2 i$ normal to the plane.

Let this be resolved first into two components, one down the wind equal to $P \sin i$, one in the plane through the axis of revolution equal to $P \cos i$. The first component equilibrates the propeller thrust T supposed horizontal. The second must sustain the weight W and also furnish the centripetal force WU^2/Rg necessary to keep the machine in its circular path. Thus $P \cos i$ must be resolved into two components (Fig. 9)

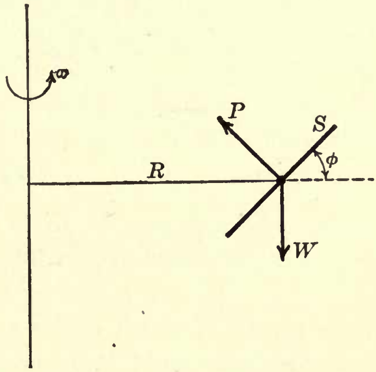

Fig. 9. Circular Flight.

$$P \cos i \cos \phi = W, \quad P \cos i \sin \phi = WU^2/Rg \; . \; . \; . \quad (16)$$

Let $\cos i = 1$. The relations are

$$kSU^2 i \cos \phi = W, \quad kSi \sin \phi = W/Rg \; . \; . \; . \; . \; . \quad (17)$$

Hence,

$$\tan \phi = U^2/Rg \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (18)$$

The banking angle must satisfy this relation; the tangent of the angle varies inversely as the radius of the circular path and directly as the square of the linear velocity. If $U = 100$, $\tan \phi$ will be 1 and $\phi = 45^\circ$ for a radius $R = 310$ ft.

The forward thrust and power used are

$$T = P \sin i = kSU^2 i^2, \quad \Pi = kSU^3 i^2,$$

$$\Pi = \frac{W^2 \sec^2 \phi}{kSU} = \frac{W^2}{kSU}\left(1 + \frac{U^4}{R^2 g^2}\right) \; . \; . \; . \; . \; . \; . \quad (19)$$

It follows that the effect of the turn is to consume additional power, varying as the cube of the velocity — just as when parasite resistance was added. Put in another way, it may be stated that for S and U given the angle i between the wing and direction of flight must be as though the weight were $W \sec \phi$ instead of W, and hence i must be larger — for a sharp turn with a steep angle of bank it must be considerably larger than for straight-away flight. If there is parasite surface S', the expression for the power is

$$\Pi = \frac{W^2}{kSU} + \frac{W^2U^3}{kSR^2g^2} + k'S'U^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the first turn is that due to wing resistance in horizontal flight, the second the added amount due to turning, and the third that due to parasite. The additional power required for turning is by no means inconsiderable compared with the parasite consumption of power; it is in the ratio

$$W^2 : kk'SS'R^2g'^2$$

In the previous example, $W = 2000$, $k = .0051$, $k' = .0015$, $S = 500$ $S' = 6$. With these figures the ratio is $(415/R^2)$ which is 1 for $R = 415$, and 4 for a radius of 208 ft. Hence if this machine had to make a turn of 280 ft radius at 95 ft/sec, the power taken by parasite resistance would be 14 H.P., that by turning would be 56 H.P., whereas that due to forward motion would be about 30 H.P. — a total of 100 H.P.

In Art. 11 it was shown that when a plane turns in a circle the pressure is slightly greater than for the straight path at the same velocity. The proof applied only to the motion of a plane when not banked for the turn, but similar considerations would show that for the banked plane there would be a slight excess of pressure. This has been neglected in the calculations of this article. It was also shown that the resultant pressure was displaced outward so as to give a moment about the central line of the plane. This, too, is true when the plane is banked in turning — with the result that the plane cannot turn without setting up a moment tending to spin the plane about the horizontal (or a line nearly horizontal) in the direction of flight, and thus over-bank it. Rudder control alone must, therefore, be unsatisfactory.

17. Plane and Elevator. In the discussion thus far it has been assumed that the machine could fly at any angle i of the design, or at any angle j for the propeller thrust, whereas, as a matter of fact, not only must the forces be in equilibrium in respect to components, but also in respect to moments. Suppose the machine to fly with the propeller thrust horizontal, and to be in equilibrium in regard to moments. The three forces W, P, T must then pass through a point, and this point is the center of gravity if it be assumed that the propeller thrust goes through the center of gravity.

To fly faster the machine must pitch down slightly so that the angle j is negative, and the angle between the plane and the wind direction is less than i. This causes the pressure vector to move forward toward the leading edge of the plane, because the smaller the angle of attack i the nearer is the center of pressure to the leading edge (see Fig. 2, Art. 9). Hence, if the machine could fly with $j = 0$, it cannot fly with j negative unless some moment is introduced to equilibrate the moment which arises from the motion of the center of pressure; and, in like manner, if j should be positive the center of pressure would retreat, and a moment would arise which must be balanced.

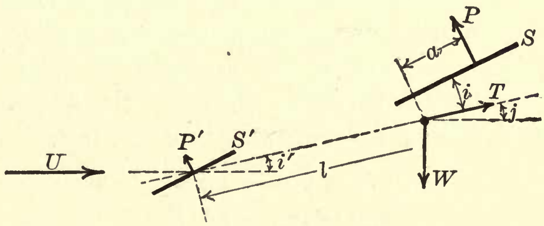

Fig. 10. Plane with Elevator.

Suppose, therefore, that at a distance l behind the center of gravity there is introduced a plane of area S' which may be tipped up and down. This plane is called the elevator, and is used for steering in a vertical plane. The size of the plane should be so adjusted relative to the main surface S that the pressure upon it can equilibrate the moments due to the motion of the center of pressure. The three conditions for equilibrium are (Fig. 10)

$$\begin{aligned} W &= kSU^2(i+j) + k'S'U^2i', \\ T &= kSU^2(i+j)^2 + k'S'U^2i'^2, \qquad (21) \\ M &= akSU^2(i+j) - lk'S'U^2i', \end{aligned}$$

where i' denotes the angle between the tail plane and the horizontal direction of flight (or better, from the "neutral" direction, if the down-wash from the wing be considered), where k' may differ from k because of the different aspect ratios of the tail plane or elevator from that of the main plane, and where a is the arm of the pressure P on the main plane with respect to the center of gravity. (The arm of the pressure P' on the tail plane is really $l \cos (i' - j)$ but may be taken as l since $i' - j$ is assumed to be small.) Stalling moments are positive, diving moments negative.

Suppose that when $j = 0$ the angle of the elevator i' is zero, so that the tail plane is "neutral" and $a = 0$. For flight with j at any value other than zero the tail plane cannot be neutral, nor can a be zero. The values for j which are reasonable for any flying attitudes are from $-i^\circ$ to such values that $i + j$ is some 15°; that is to say, a total range of 15° for the angle between the plane and the wind is about all that can be treated with any satisfaction. For this range of values the distance of the center of pressure (see Fig. 2, Art. 9), from the leading edge of the wing varies from 0.1 to 0.3 of the breadth b of the wing for square planes, from $0.2b$ to $0.35b$ for those of aspect ratio 3, and from $0.25b$ to $0.38b$ for those of aspect ratio 6. The variation is irregular, especially for angles under 5°. Between 5° and 15° one may write (by reading from the chart in Fig. 2),

$$x/b = .19 - .007i^\circ, \qquad \text{for aspect ratio} = 1,$$
$$x/b = .16 - .015i^\circ, \qquad \text{for aspect ratio} = 3,$$
$$x/b = .23 - .010i^\circ, \qquad \text{for aspect ratio} = 6,$$

where x is the distance of the center of pressure from the leading edge of the plane. This shows that the value of a is $a = -cbj^\circ$ where c is somewhere between .007 and .015, according to the aspect ratio of the elevator.

As S' is ordinarily small relative to S, a first approximation may be had by setting $W = kSU^2(i + j)$, and neglecting $k'S'U^2i'$. Substitute for U^2 in the expression for M. Then

$$M = W\left(a - \frac{lk'S'}{kS} \cdot \frac{i'}{i + j}\right).$$

If this is to vanish for any particular value of j, the elevator must be set at the angle

$$i' = \frac{akS}{lk'S'} \times (i + j).$$

Substitute for a its value $-cbj$. Then

$$i' = -cj\,(i + j)\,\frac{b}{l}\,\frac{S}{S'}\,\frac{k}{k'}. \qquad (22)$$

This shows that if $j = 10^\circ$ and $i + j = 15^\circ$, such as might be the case for slow flight, if $S/S' = 20$, which is a normal proportion in area between main wing and elevator, and if b/l is $\frac{1}{3}$, and if c be taken as 0.01, then $i' = -10^\circ$ if $k = k'$. The elevator would have to be turned up at the back edge. The pressure on the elevator could be calculated if the velocity of flight were known. If $U = 50$ ft/sec when $j = 10^\circ$, the pressure is about 60 lbs, though it will depend somewhat upon the aspect ratio of the elevator.

18. Plane and Stabilizer. Distinct from the question of the elevator is that of the stabilizer, which is a fixed plane of area S' at a distance l behind the center of gravity. If i' denotes the angle that this plane makes with the propeller thrust (not with direction of flight) the weight and moment equations are

$$W = kSU^2(i + j) + k'S'U^2(i' + j),$$
$$M = akSU^2(i + j) - lk'S'U^2(i' + j),$$

where a is the distance of the pressure vector forward of the center of gravity. In the normal attitude of flight j may be taken zero and a may be written a_0. The moment equation is

$$0 = a_0 kSU^2 i - lk'S'U^2 i'.$$

The center of gravity no longer lies in the line of pressure, but slightly behind it if i' is positive, that is if the stabilizer is turned down at the back, and in front of it if i' is negative. The moment developed for any other value of j becomes

$$M = W\left(a - \frac{lk'S'}{kS}\,\frac{i' + j}{i + j}\right).$$

Now it is $a - a_0$, the change in a, which equals $-cbj$. Hence

$$a = \frac{lk'S'}{kS} \times \frac{i'}{i} - cbj,$$

or

$$M = W\left(-cbj - \frac{lk'S'}{kS} \times \frac{i - i'}{i + j} \times \frac{j}{i}\right) \quad . \; . \; . \; . \; . \quad (23)$$

It is seen, therefore, that the introduction of the fixed tail plane (stabilizer) increases the moment $-Wcbj$ which must be taken care of with the elevator in uniform flight unless $i' > i$. Whether this means that a larger control (elevator) surface must be provided or that the elevator must be turned through a larger angle to regulate the flight of a machine with stabilizer, as compared with one without, cannot be surely inferred; for the surfaces of stabilizer and elevator are so near together that there is interference in their action.

Entirely distinct from the question of the moments which arise in uniform flight, where the condition $W = kSU^2(i+j)$ may be used, and where the problem is to set the elevator in such a way as to make $M = 0$, is the question of the moments which arise when the machine is slightly disturbed from uniform flight, as by a gust, or by an accidental use of the elevator whereby the machine is pitched to a slight angle without having time to adjust its velocity to the new angle of flight; so that the equation $W = kSU^2(i+j)$ cannot be used. Suppose the machine is moving uniformly with the velocity U, and that the angle $i+j$ is increased by the amount dj. The pressure on the main plane is increased by $kSU^2\,dj$, and the center of pressure moves back by the amount $cb\,dj$. The moment

$$akSU^2(i+j) \quad \text{becomes} \quad (a - cb\,dj)kSU^2(i+j+dj),$$

which is greater than the moment $a\,dSU^2(i+j)$ by the amount

$$akSU^2\,dj - cbkSU^2(i+j)dj.$$

The pressure on the stabilizer is increased from $lk'S'U^2(i'+j)$ to $lk'S'U^2(i'+j+dj)$, which produces a negative moment of the amount $k'S'U^2\,dj$. The total increase in the moment is, therefore,

$$U^2\,dj[kSr - cbkS(i+j) - lk'S' \times l].$$

In the condition of uniform flight

$$akSU^2(i+j) = lk'S'U^2(i'+j).$$

Substituting for a, the change in moment is

$$dM = U^2\,dj\left[lk'S'\left(\frac{i'-i}{i+j}\right) - cbkS(i+j)\right]. \quad . \quad . \quad . \qquad (24)$$

This moment dM will tend to restore the machine to its former attitude (pitch it down) if dM is negative, but will tend to pitch the machine up still further if dM is positive. The value of dM/dj is surely negative if i' is zero or negative, and the larger the negative value of i' the greater will be the moment tending to restore the machine to its previous attitude in uniform flight. The value of the restoring moment if $i' = 0$, $U = 95$ ft/sec, $S = 500$, $S' = 25$, $i = 5°$, $j = 4°$, $c = 0.01$, $k' = k = .00009$, $l = 27$, $b = 9$, is $dM = -630$ dj lbs. If $i' = -1°$ the result is increased by about 60 lbs to -690;

but if i' be 1°, the result is decreased by about 60 lbs to -570. If there were no stabilizer at all ($S' = 0$) the moment would be about 320 lbs. The considerable additional stabilizing effect due to the presence of the fixed tail plane is, therefore, apparent.

EXERCISES

1. Use the sine law $P = kSU^2 \sin i$ for the pressure. Write the values of T, W, II, for the airplane moving horizontally with the propeller thrust also horizontal, and eliminate i, thus determining the expression for the power without making the approximations $\sin i = i$ and $\cos i = 1$.

2. Show that on the hypothesis of Newton's sine-square law, $P = kSU^2 \sin^2 i$, the power required is independent of the velocity (the approximations $\cos i = 1$, $\sin i = i$ may be made). Compare with (5).

3. A machine spreads 400 sq.ft of a surface. The angle i is 6°. The weight of the machine is 1000 lbs. Use the formula $P_i = 0.06 P_{90} i^\circ$, with $P_{90} = 0.0015 SU^2$. Find the velocity of horizontal flight and the horse power required.

Ans. 68 ft/sec; 13 H.P.

4. With the data of the previous exercise find j, the angle of the propeller thrust with the horizontal if the speed is 30 mi/hr and if the speed is 70 mi/hr.

Ans. 8.3°; −3.4°.

5. Given parasite resistance equivalent to 4 ft² normal surface. Find the most economical speed in Ex. 3. *Ans.* 49 mi/hr.

6. In the case of horizontal flight at the most economical speed what proportion of the power is used in overcoming wind resistance, and what proportion in overcoming parasite resistance? What are these proportions if the speed is double the most economical speed?

7. Show that if a plane without parasite glides uniformly without power ($T = 0$) at the angle Θ, $i = -\Theta$ and U varies inversely as $i^{\frac{1}{2}}$. Comment on the conclusion that the plane glides faster on less inclined paths!

8. Let the angle Θ of glide be reckoned positive (instead of negative) below the horizontal. Assume a parasite surface S'. Show that for uniform velocity with power off ($T = 0$), the relation $i < \Theta$ holds and that

$$k'S'/kS = i \cos i\,(\tan\Theta - \tan i) = i \sin(\Theta - i)/\cos\Theta.$$

Plot the right-hand side as a function of i, the unknown, for $\Theta = 5^\circ, 10^\circ, 30^\circ, 60^\circ, 85^\circ$. Show that if $k'S'/kS$ is sufficiently small there are two values of i which satisfy the condition for uniform gliding. Show that for a given value of $k'S'/kS$ there is a minimum angle Θ of glide.

9. Solve the equation in Ex. 8 assuming i small and show that the minimum angle of glide is $\Theta = \tan^{-1} 2(k'S'/kS)^{\frac{1}{2}}$. Why must one of the roots be discarded when Θ is not small? Find a small value of i if Θ is large.

$$Ans.\ i = \tfrac{1}{2}\tan\Theta - \sqrt{\tfrac{1}{4}\tan^2\Theta - \frac{k'S'}{kS}};\quad i = \frac{k'S'}{kS}\cot\Theta.$$

10. Find the speed at which the glider descends if i is small.

11. Derive formulas for the increase of pressure and for the sidewise travel of the center of pressure out along the wing when the banking is ϕ. Show that the results reduce to those of Art. 11 if $\phi = 0$.

Ans. $\Delta P = Ps^2 \cos^2 \phi/3l^2$, $\Delta y = 2s^2 \cos \phi/3l$ (approx.).

12. If the data be as in Art. 16, in how small a radius may the plane turn at a velocity of 80 ft/sec without expending more than 80 H.P.?

13. A plane with $S = 400$ ft^2, $W = 1000$ lb, $r = 6$, can climb at $5\frac{3}{4}°$ (0.1 radian) with $U = 50$ ft/sec. In how small a circle may it turn at the same speed and what is the banking angle? What is the moment due to the sidewise motion of the resultant pressure?

14. Write equations for spiraling down (with engine off) on a helix with constant velocity, constant angle Θ of decline, constant banking angle ϕ, and constant angle i between the plane and the direction of flight. Compare the minimum gliding angle with that which holds when $\phi = 0$ to ascertain whether the spiral glide must be steeper than the straight glide.

15. From (22) calculate the value of i' if $j = -2°$, $i = 6°$, $b/l = \frac{1}{4}$, $S/S' = 20$, supposing the aspect ratio of the wing is 6 and of the elevator 3. Calculate the pressure on the elevator if $U = 100$ ft, and the moment M of the pressure.

16. Work out a formula similar to (22) from (21) on the assumption that for $j = 0$, the elevator is not neutral but has $i' = i'_0$ and the arm a is a_0.

17. Verify the formulas for x/b from the graphs in Fig. 2.

18. Given a tandem monoplane, *i.e.*, two planes of surfaces S, S' about equal, inclined at angles i, i' to the propeller thrust, a distance l apart. Assume no interference. Determine the position of the C. G. and the moment dM corresponding to a change from $j = 0$ to $j = dj$. Compare this with the result for a single surface.

RIGID MECHANICS

CHAPTER IV

MOTION IN A RESISTING MEDIUM

19. The Forces. As an object, such as an airplane, goes through the air there is a resistance or drag which, when the attitude of the machine is unchanged, is generally assumed to vary with the square of the velocity: $R = ku^2$. If the machine is falling vertically the weight W also acts. If the machine is running along the ground, as when alighting or departing, the acceleration of gravity does not act in the direction of the motion, but there is additional resistance of a frictional sort F which may be assumed in the first instance to be constant; and there may be, furthermore, the propeller thrust T acting forward. A more detailed analysis of the forces acting is given in Art. 21.

20. The Machine Stopping. To determine the motion of a machine when running along the ground to a stop, equation of motion is

$$W\,du/dt = -gF - gR, \quad \quad (1)$$

where the frictional force (in pounds) will be treated as constant, and where the air resistance (in pounds) will be taken as $R = ku^2$ proportional to the square of the velocity. The equation may also be written

$$W\frac{du}{dx}\frac{dx}{dt} = Wu\frac{du}{dx} = -gF - gR \quad \quad (2)$$

The first form (1) involves u and t, and its integration will determine the relation between the velocity and the time. The second form (2) involves u and x, and, when integrated, will give the relation between u and x.

To treat the first equation, we have

$$\int_{u_0}^{u}\frac{du}{F + ku^2} = -\int_0^t \frac{g\,dt}{W}.$$

Hence,

$$\left[\frac{1}{\sqrt{Fk}}\tan^{-1}\left(u\sqrt{\frac{k}{F}}\right)\right]_{u_0}^{u} = \frac{-gt}{W},$$

or

$$\tan^{-1}\left(u\sqrt{\frac{k}{F}}\right) = \tan^{-1}\left(u_0\sqrt{\frac{k}{F}}\right) - g\frac{\sqrt{Fk}}{W}t,$$

or

$$u = \frac{u_0 - \sqrt{F/k}\tan(gt\sqrt{Fk/W})}{1 + u_0\sqrt{k/F}\tan(gt\sqrt{Fk/W})} \quad . \ . \ . \ . \ . \ . \ . \quad (3)$$

The time the machine takes to stop is

$$T = \frac{W}{g\sqrt{Fk}}\tan^{-1}\left(u_0\sqrt{\frac{k}{F}}\right).$$

The value of k will generally be of the form $.0015S''$, where S'' is the total equivalent normal surface, and is

$$S'' = S' + (.032 + .005r)Si^\circ \sin i, \quad \sin i = i^\circ/57.3,$$

in the case of the skeleton airplane if the plane makes an angle of i° with the ground, if r is the aspect ratio, S the wing area, and S' the parasite equivalent normal area.

For the second equation the integration is

$$\int_{u_0}^{u}\frac{u\,du}{F + ku^2} = -\int_0^x\frac{g\,dx}{W},$$

or

$$\Big[\log(F + ku^2)\Big]_{u_0}^{u} = \frac{-2kgx}{W},$$

or

$$\log\frac{F + ku^2}{F + ku_0^2} = \frac{-2kgx}{W},$$

or

$$u^2 = \frac{F}{k}\left[\left(1 + \frac{ku_0^2}{F}\right)e^{\frac{-2kgx}{W}} - 1\right] \quad . \ . \ . \ . \ . \ . \ . \quad (4)$$

The value for the total distance run before stopping is

$$X = \frac{W}{2kg}\log\left(1 + \frac{ku_0^2}{F}\right).$$

If the air resistance is very small the total distance run reduces to $Wu_0^2/2gF$, as is usual in uniformly retarded motion, by virtue of the fact that $\log(1 + x) = x$ when x is small. The logarithm which enters in the expression for x is the natural logarithm to the base e, and must either be looked up in a table of natural logarithms, or must be converted from the base 10 by division by 0.4343, so that

$$X = \frac{W}{0.8686kg}\log_{10}\left(1 + \frac{ku_0^2}{F}\right).$$

The relation between x and t may be obtained by integrating either (3) or (4), or by elimination of u between these two expressions.

Example. Given a skeleton airplane with $W = 2000$ lb, $S = 500$ ft², $S' = 6$ ft², $r = 5$, landing with $i = 14.3°$. Find the landing speeds and the length of run if $F = \mu\, W = 0.2W$.

At landing the lift $P = 0.0015 \times .057 \times 500u_0^2 \times 14.3 = W = 2000$. Hence $u_0^2 = 3300$ and $u_0 = 57$ ft/sec. The equivalent normal surface is $S'' = 6 + .057 \times 500 \times 14.3 \times \frac{1}{4}$ or 108. Hence $k = .162$.

$$X = \frac{2000}{.8686 \times .162 \times 32.17} \log_{10}\left(1 + \frac{.162 \times 3300}{400}\right) = 162 \text{ ft.}$$

The run with the single drag of 400 lb would be 255 ft. The back pressure on the main plane has, therefore, materially reduced the run. The parasite surface $S' = 6$ has not much effect at these low speeds; the large surface $S = 500$ at the large angle $i = 14.3°$ introduces much more drag. Even if the machine were equipped with braking surfaces (*i.e.*, surfaces ordinarily held in the line of flight so as to offer no resistance, but capable of being thrown out perpendicular to the wind to give additional resistance on landing), the additional parasite thus obtainable would not be very effective unless the surfaces were large enough materially to increase the figure 108 for S''.

As a matter of fact, machines are landed into the wind where possible, and the motion depends on the wind velocity w. The equation is

$$W\, du/dt = -gF - kg(u + w)^2.$$

Let $v = u + w$. Then

$$W\, dv/dt = -gF - kgv^2.$$

The equation between v and t is the same as that between u and t above when at $t = 0$ the landing speed $v = v_0$ is the same relative air speed as u_0 regardless of the wind velocity w. The machine comes to rest when $u = 0$, that is, when $v = w$. The equation between velocity and distance is

$$W\frac{du}{dt} = W\frac{du}{dx}\,\frac{dx}{dt} = -gF - kg(u + w)^2.$$

Here dx/dt is the ground speed $u = v - w$. Therefore,

$$\frac{(v - w)dv}{F + kv^2} = \frac{-g\, dx}{W},$$

and the equation is not so simple as before. The integral is

$$\frac{1}{2k}\Big[\log (F + kv^2)\Big]_{v_0}^{v} - \frac{w}{\sqrt{Fk}}\Big[\tan^{-1}\Big(v\sqrt{\frac{k}{F}}\Big)\Big]_{v_0}^{v} = \frac{-gx}{W},$$

and the total distance of run is

$$X = \frac{W}{2kg}\log_e\frac{F+kv_0}{F+kw^2} - \frac{wW}{g\sqrt{Fk}}\Big(\tan^{-1} v_0\sqrt{\frac{k}{F}} - \tan^{-1} w\sqrt{\frac{k}{F}}\Big),$$

In the example worked above if $w = 30$ ft/sec the run is

$$X = \frac{2000}{.8686 \times .162 \times 32.17}\log_{10}\frac{400 + .162 \times 3300}{400 + .162 \times 900}$$

$$- \frac{30 \times 2000}{32.17\sqrt{.162 \times 400}}\Big(\tan^{-1} 57\sqrt{\frac{.162}{400}} - \tan^{-1} 30\sqrt{\frac{.162}{400}}\Big) = 22 \text{ ft.}$$

This is a great reduction over that found before, and is due to the fact that a large wind resistance is constantly operating.

21. Force Analysis. Such figures as have been obtained must be regarded as very rough approximations. To get any really accurate idea of the distance run a more careful analysis of the forces acting on the machine must be undertaken. These are: the weight W down at the C. G., the ground pressure N upward on the wheels of the landing gear, the rolling friction νN backward on the wheels, the pressure R up on the tail skid, the sliding friction μR backward on the skid, the drag D due to parasite, the wind pressure P on the main plane, and the pressure P' on the tail plane and elevator (which is sharply raised so as to bring the maximum pressure between the skid and ground and thus increase μR). If i is the angle between the main plane and the wind or ground, the components of P are $P \sin i$ back and $P \cos i$ up, approximately $Pi^\circ/57$ and P. The components of P' may be taken as L' (down) and D' back; they cannot well be represented in terms of the angles which the stabilizer and elevator make with the wind because of mutual interference. The equation of motion is (Fig. 11)

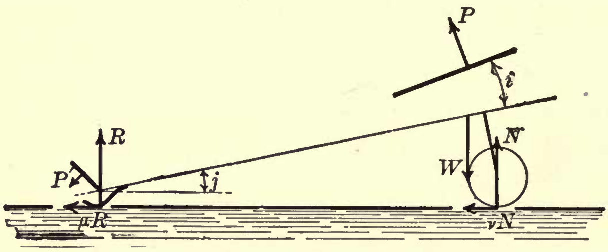

Fig. 11. Airplane Landing — Force Diagram.

$$\frac{W}{g}\frac{du}{dt} = -\mu R - D - D' - \nu N - Pi/57.$$

As there is no vertical motion, the vertical forces are in equilibrium.

$$0 = W - P - N - R + L' \quad (L' \text{ positive when downward}).$$

As there is no rotation about the C. G., movements about that point are zero. Let the distances of the skid below and behind the C. G. be h, l respectively. Let the distances of the center of pressure of P above and in front of the C. G. be b, a respectively, and the distance of N in front be a'. Then

$$0 = Pa + Na' + L'l + Pbi/57 - \nu Nh - \mu Rh - Rl - D'h,$$

neglecting any moment due to D. The two equilibrium equations suffice to determine the unknown reactions R and N. Probably μR is small relative to lR, and $D'h$ relative to $L'l$. Let $c = a + bi/57$ and $c' = a' - \nu h$. Then approximately

$$0 = cP + c'N + lL' - lR.$$

Hence

$$N = W - P - cP/l - c'N/l,$$

and the last two terms are surely small compared with either W or P. Except at the moment of landing when W and P are about equal, the conclusion $N = W - P$ seems justified. Also

$$R = L' + (c - c')P/l + c'W/l - c'L'/l - c'R/l.$$

The terms $c'R/l$ and $c'L'/l$ are negligible relative to R and L'; but as P and W are large compared with L' and R, it is not certain that $(c - c')P/l$ or $c'W/l$ are negligible even though the c's be much smaller than l.

In the equation of motion D and D' are small and will be omitted. Then

$$\begin{aligned} W\,du/g\,dt &= -\mu L' - \mu(c - c')P/l - \mu c'W/l - Pi/57 - \nu W + \nu P \\ &= -\mu L' - (\nu + \mu c'/l)W - (i/57 + \mu(c - c')/l)P. \end{aligned}$$

In this equation the term in W is constant, though small, and corresponds to F in the solution; the terms in L' and P both vary as u^2 and

$$ku^2 = \mu L' + (i/57 + \mu(c - c')/l)P$$

It is probable that $\mu(c - c')/l$ is not large relative to $i/57$ for most machines when in the landing position, but the term can hardly be considered as always negligible. In the example worked out it was assumed that $F = 0.2W$ which is probably a high value; on the other hand the resistance was taken as $Pi/57$ without the term $\mu L'$ and was probably somewhat small, for although L' is much less than P, μ is greater than $i/57$. The whole discussion shows how intricate the application of the most elementary mechanical principles may become when the airplane is in question.

22. The Vertical Dive. Consider next the case of free fall in a vertical direction, distance being measured vertically downward. The equation is then

$$W\,dv/dt = Wg - gkv^2 \qquad (5)$$

The acceleration becomes 0 when $W - kv^2 = 0$ or

$$V = (W/k)^{\frac{1}{2}}; \qquad (6)$$

for this speed, therefore, the motion is uniform. If the machine is falling with less than this speed V the acceleration of gravity prevails over the retardation, due to the wind, and the machine increases its speed. If, by any means, the machine could start downward at a velocity in excess of $V = (W/k)^{\frac{1}{2}}$ the resistance would prevail over the force due to gravity, and the machine would slow up. The critical velocity $V = (W/k)^{\frac{1}{2}}$ is called the *terminal velocity* of fall because it is the velocity which the machine approaches when it falls further and further irrespective of whether the initial downward velocity is greater or less than the terminal amount.

The equation of motion may be integrated either in the form

$$\int_{v_0}^{v} \frac{dv}{W - kv^2} = \int_0^t \frac{g\,dt}{W} \quad \text{or} \quad \int_{v_0}^{v} \frac{v\,dv}{W - kv^2} = \int_0^y \frac{g\,dy}{W},$$

where y is distance measured downward from the initial position. The value of the first integral is

$$\frac{1}{2}\sqrt{\frac{W}{k}}\left[\log \frac{\sqrt{\frac{W}{k}} + v}{\sqrt{\frac{W}{k}} - v}\right]_{v_0}^{v} = gt,$$

or

$$\log\left(\frac{V + v}{V - v}\;\frac{V - v_0}{V + v_0}\right) = \frac{2gt}{V}, \qquad \text{if } V = \sqrt{\frac{W}{k}}.$$

The solution for v in terms of t is as follows:

$$v = V\frac{(V + v_0) - (V - v_0)e^{-2gt/V}}{(V + v_0) + (V - v_0)e^{-2gt/V}} \qquad (7)$$

The time required to acquire a velocity v when falling from rest in the resisting medium is

$$T = \frac{V}{2g} \cdot \log \frac{V + v}{V - v} = \frac{V}{.8686g} \log_{10} \frac{V + v}{V - v}.$$

This equation shows that the time required to attain the terminal velocity is not finite but infinite, and the same conclusion holds when the fall is from any initial velocity v_0 whether greater or less than V. The equation (7) for v shows, however, that the velocity V is approached very rapidly as the time increases, owing to the presence of the term $e^{-2gt/V}$, which falls off exponentially; if $V = 320$ ft/sec, the term is $e^{-t/5}$ which is very small after 15 seconds.

To find the relation between the distance and the velocity, integrate. Then

$$\frac{-W}{2k}\left[\log\left(1 - \frac{kv^2}{W}\right)\right]_{v_0}^{v} = gy,$$

or

$$v^2 = \frac{W}{k}\left[1 - \left(1 - \frac{kv_0^2}{W}\right)e^{-2gky/W}\right] \quad . \; . \; . \; . \; . \; . \quad (8)$$

or, introducing the terminal velocity,

$$v^2 = V^2\left[1 - \left(1 - \frac{v_0^2}{V^2}\right)e^{-2gy/V^2}\right] . \; . \; . \; . \; . \; . \; . \quad (9)$$

Here again the expression e^{-2gy/V^2} is noteworthy because of the rapidity with which it falls off. If the terminal velocity be 200 ft/sec the expression is $e^{-y/625}$, and becomes equal to $1/e$ after a fall of 625 ft, to $1/e^2$ after a fall of 1250 ft, and so on decreasing in geometrical ratio. If the terminal velocity were 300 ft/sec, the exponential factor would be $e^{-y/1400}$, and would be reduced to $1/e$ only after a fall of 1400 ft, and $1/e^2$ only after 2800 ft.

As airplanes maneuver at great altitudes it is entirely possible to dive down a distance sufficiently great so that the terminal velocity will be nearly approached, particularly if the plane starts down at its normal flying speed. The above formulas have been deduced on the hypothesis that the machine is diving with the propeller cut off, for the propeller thrust T has been omitted from the equations. Experience seems to show that there is no very great addition to the terminal velocity if the engine is left running, because at the high values of the terminal velocity the propeller is exceedingly inefficient. However, if the propeller were left on, the machine would acquire its terminal velocity somewhat quicker, because during the first part of the dive the propeller would aid gravitation to a certain extent. As the propeller thrust at these speeds is not a large fraction of the weight in the case of most machines, the aid would not be very great.

23. Falling from a Great Height. The resistance of the air depends on the density of the air, and, indeed, varies directly with it, $R = ku^2 = c\rho u^2$. Therefore, at high altitudes, such as 20,000 ft, where the density is only about one-half what it is at the surface, as the terminal velocity varies inversely as $k^{\frac{1}{2}}$, that is inversely as $\rho^{\frac{1}{2}}$, the terminal velocity would be 40% greater than at sea level. In case a machine flying at 20,000 ft should be put into a dive, the velocity would increase towards a terminal velocity which exceeds that at the surface. It was remarked above, however, that if a flying body acquires a velocity in excess of the terminal velocity the body will slow down. It, therefore, is quite possible that a machine diving from a high level should acquire a velocity greater than it could maintain as it descended into denser air. The equation of motion when varying density is taken into account is

$$Wv\frac{dv}{dy} = Wg - c\rho(y)v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If $u = v^2$ be taken as the dependent variable, the equation is

$$\frac{du}{dy} + \frac{2c\rho(y)}{W}u = 2g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This is a linear equation of the type

$$\frac{dy}{dx} + P(x)y = Q(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

of which the solution in any standard book on differential equations is found to be

$$ye^{\int P\,dx} = \int e^{\int P\,dx}Q(x)dx + C \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In order to integrate this equation with tables of integrals it is necessary that P and Q should be such functions that

$$\int P\,dx \text{ and } \int e^{\int P\,dx}Q(x)\,dx,$$

should both be found in the tables of integrals (or should be reducible to forms there found).

It would, therefore, be necessary to fit to the observed curve of density on altitude (Art. 60) an empirical equation of such form that these integrals could be obtained (and this empirical equation might not be one which would be simplest for other purposes). One advantage of the use of empirical equations is that a given curve or table of data may be fitted reasonably well by a great variety of equations, and that the one chosen to represent the data may often be selected in such a manner as to simplify subsequent analytic work. Consider, for instance,

$$\rho = \frac{a}{y - b} \quad \ldots\ldots\ldots\ldots \quad (14)$$

This is convenient for integration because its integral is a logarithm and the exponential of a logarithm is a power of $y - b$, which in turn may be integrated because $Q(x)$ is a constant.

This equation contains two constants, a and b. The value of the expression increases as y increases (downward). Therefore, qualitatively, the formula appears to represent the change in density. It is true that the density by this formula would become infinite when $y = b$ but if the value determined for b were sufficiently large this would be no inconvenience. All that is necessary is that the empirical formula should represent the density between an altitude such as 20,000 or 25,000 ft and the earth.

The way to fit the curve to the observed data is to write

$$\rho y = b\rho - a \qquad \text{or} \qquad \rho y/\rho_0 = b\rho/\rho_0 - a/\rho_0,$$

to plot $z = \rho y/\rho_0$ against ρ/ρ_0, and fit in the best straight line that can be obtained. The constants b and a may then be read from the graph. The work is as follows: consider $y = 0$ at 24,000 ft altitude and take ρ/ρ_0 from the standard table:

Alt. =	24,000	20,000	16,000	12,000	8000	4000	0
ρ/ρ_0 =	.453	.524	.603	.690	.786	.889$_5$	1.000
y =	0	4000	8000	12,000	16,000	20,000	24,000
z =	0	2096	4824	8280	12,576	17,790	24,000
$\Delta\rho/\rho_0$ =	.071	.079	.087	.096	.103$_5$	.110$_5$	
Δz =	2096	2728	3456	4296	5224	6210	

The difference $\Delta(\rho/\rho_0)$ and Δz are taken because if the relation between z and ρ/ρ_0 is really linear the ratio of Δz to $\Delta(\rho/\rho_0)$ should be constant and equal to the slope of the line. As a matter of fact the ratio is not nearly constant and hence the relation cannot be well represented by a straight line, nor the relation between ρ and z by the form which was assumed because of its ease of integration. Nevertheless, the variables ρ/ρ_0 and z could be plotted and a straight line could be drawn as near as possible to all the points, and some sort of values of a and b could be determined from the graph. With these values of a and b the values of ρ/ρ_0 for different y's could be calculated from the assumed formula and compared with the actual known values of ρ/ρ_0 to see how bad or how good the agreement is. Another method will, however, be followed.

Consider the ground as $y = 0$ and the y axis as upward. This is more natural than to measure y down from some particular altitude, whenever an empirical equation must be obtained between density and altitude. The equation of motion is then

$$Wv\frac{dv}{dy} = \tfrac{1}{2}W\frac{d(v^2)}{dy} = -Wg + cg\rho(y)v^2 \quad . \quad . \quad . \quad . \quad (15)$$

and

$$\frac{du}{dy} - \frac{2cg\rho}{W}u = -2g, \qquad u = v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and

$$ue^{\frac{-2cg}{W}\int\rho\,dy} = -2g\int e^{\frac{-2cg}{W}\int\rho\,dy}dy + C \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Now $\rho\,dy$ is the amount of matter (air) between y and $y + dy$ in a column of unit cross section, and this is in pounds if ρ is in lb/ft³. Hence $\int\rho\,dy$ is the amount of air between the limits of integration in y. This amount is, however, proportional to the difference in barometric pressure between the levels, and thus a table of the pressure will give the value of $\int\rho dy$ — already integrated. As the integral appears in the exponent, it is desirable that it be represented empirically by a logarithm. Let $F(y)$ be the amount of air estimated in inches of mercury between $y = 0$ and $y = y$. Then, from the standard table (Art. 60), by subtracting the pressure at y from the pressure at $y = 0$, the table

$y = 0$	4000	8000	12,000	16,000	20,000	24,000
$F = 0$	4.07	7.68	10.85	13.62	16.02	18.09

may be obtained. Try

$$F(y) = b\log(1 + ay) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This has the desired logarithmic form; it vanishes at $y = 0$, and, since it has two constants, a and b may be passed through two points of the curve exactly, giving two equations

$$F_1 = b \log (1 + ay_1), \qquad F_2 = b \log (1 + ay_2),$$

for the determination of b and a. In particular

$$F_1 : F_2 = \log (1 + ay_1) : \log (1 + ay_2)$$

is an equation in a. It is equivalent to

$$(1 + ay_1) = (1 + ay_2)^{F_1/F_2},$$

and may easily be solved if F_1/F_2 is a small integer such as 2. Let $y_2 = 8000$, and $F_2 = 7.68$. Then $2F_2 = F_1 = 15.36$. The value of y_1 is just under 20,000. Try $y_1 = 19{,}000$. From the table $F(19{,}000) = 15.45$. This is quite near enough to 15.36, though by interpolation a nearer value might be had. The solution of

$$1 + 19{,}000a = (1 + 8000a)^2 = 1 + 16{,}000a + 64{,}000{,}000a^2$$

gives

$$a = 3/64{,}000, \qquad \text{whence} \qquad b = 24.1$$

is found by substitution. Hence

$$F(y) = 24.1 \log (1 + 3y/64{,}000) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

is the desired formula. If this be checked by substitution

$y = 0$	4000	8000	12,000	16,000	20,000	24,000
$F = 0$	4.05	7.68	10.8	13.5	15.9	18.1

The check is sufficiently good, though by trial a more accurate formula might be obtained.

$F(y)$ is the amount of air in inches of Hg; the amount in pounds is

$$13.6 \div 12 \times 62.5 \times F = 1706 \log (1 + 3y/64{,}000).$$

If V be the terminal velocity in air of density ρ_0, then $c\rho_0/W = 1/V^2$ and

$$\frac{2cg}{W}\int_0^y \rho\, dy = \frac{2g}{\rho_0 V^2}\int_0^y \rho\, dy = \frac{1{,}440{,}000}{V^2} \log\left(1 + \frac{3y}{64{,}000}\right).$$

The integral for the motion is

$$v^2\left(1 + \frac{3y}{64{,}000}\right)^{-\left(\frac{1200}{V}\right)^2} = -2g\int\left(1 + \frac{3y}{64{,}000}\right)^{-\left(\frac{1200}{V}\right)^2} dy + C, \quad (20)$$

or if $v = 0$ where $y = y_0$, then

$$v^2\left(1 + \frac{3y}{64{,}000}\right)^{-\left(\frac{1200}{V}\right)^2} = 2g\int_y^{y_0}\left(1 + \frac{3y}{64{,}000}\right)^{-\left(\frac{1200}{V}\right)^2} dy.$$

The value of v^2 is a maximum if $d(v^2)/dy = 0$; if the value of y_0 is sufficiently large, the maximum will occur when $y > 0$, *i.e.*, before the body reaches the earth. For example if $V = 600$ ft/sec,

$$v^2 = 2g\left(1 + \frac{3y}{64{,}000}\right)^4 \frac{64{,}000}{3 \times 3}\left[\left(1 + \frac{3y}{64{,}000}\right)^{-3} - \left(1 + \frac{3y_0}{64{,}000}\right)^{-3}\right].$$

If $y_0 = 16{,}000$ ft, the velocity when $y = 0$ is

$$v^2 = 2g \times \frac{64{,}000}{9}(1 - (1.75)^{-3}), \qquad v = 609.$$

The maximum velocity occurs when $y = 2175$ ft and is then 620.

The chief interest in this work, aside from the illustration that a body may actually attain a higher speed than V in falling from a great height, lies in its illustration of the methods of determining empirical equations, *i.e.*, analytical expressions to represent a table of data in a form suitable for subsequent analytical work. One leading way of fitting an expression to a table is to throw the expression into linear form, plot, and fit the best straight line; a second method is to fit the expression directly to several pairs of values of the table by solution of equations.

In connection with the variation of density it should be pointed out that the variation of gravity is too small to be of concern. According to Newton's law of gravitational attraction the accelerations due to the earth's mass must vary as the square of the distance from the center of the earth. If, then, g' be the acceleration of gravity at any height, h,

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} = \left(1 + \frac{h}{R}\right)^{-2} = 1 - \frac{2h}{R} + \frac{3h^2}{R^2} + \ldots,$$

where $R = 4000$ miles approximately.

As h is less than 5 mi, the variation in g will be less than one part in 400, as estimated from the first term in the series, and, consequently, too small to take into account when compared with the inevitable errors in determining a resistance coefficient such as k.

24. The Start. The third problem alluded to above was the get-away, where the machine runs along the ground under the action of its propeller. The equation for the motion is

$$\frac{W\,du}{dt} = Tg - Fg - kgu^2, \quad \ldots\ldots\ldots \quad (21)$$

where F is the frictional resistance and is small. The equation can be integrated only after some assumption is made as to the value of T. If T be assumed constant, the solution is analytically similar to that of the diving airplane just treated. As a matter of fact, T is by no means constant, but falls off rapidly with the increase of velocity. If it be assumed that T may be written in the form $T_0 + au + bu^2$, the integration may still be performed either to obtain the relation between u and t or between u and s, because

$$\frac{dx}{a + bx + cx^2} \quad \text{and} \quad \frac{x\,dx}{a + bx + cx^2}$$

may both be found in the tables. In the absence of experimental data for the determination of the change in T as a function of the velocity there is little use in performing the integration.

25. Curvilinear Motion. If the airplane is moving in a vertical plane in a curved path, as when pulling out of a dive, the equations of motion may be expressed either by writing the equations for horizontal and vertical motion or by resolving along and perpendicular to the path. This type of motion is also of importance in connection with the flying bomb, that is, in connection with any bomb where the air forces do not act tangentially to the path, but have a lift normal to the path in addition to the drift along it. The general problem is of a complexity too great to treat at this point. The equations of motion, however, relative to the path are of importance for estimating the centrifugal forces which arise in a sharp curvature of the path.

If the velocity be drawn in magnitude as well as in direction to scale the velocities at nearby points may be compared, and it is seen that the resultant change in velocity has a component perpendicular to the path, and one along it. The component along the path is simply the increase dv in velocity v regarded as a numerical quantity, that is, the increase in ds/dt. The component normal to the path is the product of the velocity by the change of inclination i of the path, namely, $v\,di$. Hence, the accelerations are: along the path, dv/dt; perpendicular to the path, $v\,di/dt$. The radius of curvature R is the reciprocal of the rate of turning di/ds of the tangent line with respect to the arc s, namely,

$$\frac{1}{R} = \frac{di}{ds} = \frac{di}{dt}\frac{dt}{ds} = \frac{1}{v}\frac{di}{dt}.$$

Consequently, the normal acceleration may be written

$$\text{normal acceleration} = \frac{v\,di}{dt} = \frac{v\,di}{ds}\frac{ds}{dt} = \frac{v^2}{R} \quad . \ . \ . \quad (22)$$

The tangential and normal forces acting on the center of gravity are, therefore,

$$\text{tangential force} = \frac{W}{g}\frac{dv}{dt}, \qquad \text{normal force} = \frac{Wv^2}{gR} \quad . \ . \quad (23)$$

When flying in a straight, horizontal path, the airplane wings must support the weight W of the airplane. When flying in a path of radius of curvature R the wings must support not only the weight W (or a component of W) but the additional normal force Wv^2/gR. Therefore, if v is sufficiently large or R sufficiently small to make v^2/gR a considerable multiple of the weight, a considerable extra strain will be put on the machine. For instance, if v = 200 ft/sec, and R = 200 ft, the additional force on the wings is about $6W$.

26. Forces on a Curved Wing. An interesting application of the formulas may be made to Lanchester's theory of the forces upon a curved aerofoil in terms of the angle of entry j and the trailing angle i (see Art. 8). It is assumed that the amount of air affected is a stratum above and below the wing of total thickness $2b$, where b is the breadth of the wing. The amount works out as $W = 2\rho lb^2 = 2\rho lS$, where l is the length of the wing, and S its area. If the air is drawn up at the leading edge of the wing so that the angle of entry is j, and issues behind the wing at an angle of depression i, the total angular change produced by the wing is $i + j$, and the curvature of the path is $(i + j)/b$, the radius of curvature being the reciprocal of this (Fig. 12).

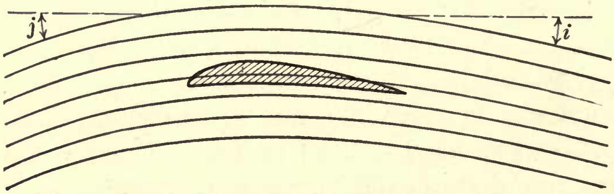

Fig. 12. Angle of Entry and Trailing Angle (Schematic).

If it be assumed that the acceleration and curvature in the air stream are uniform, or if the above average value is taken, the total force exerted should be

$$P = 2bSv^2\left(\frac{i+j}{gb}\right)\rho = \frac{2\rho}{g}Sv^2(i+j) \quad . \ . \ . \ . \ . \ . \quad (24)$$

As the angles are small, this pressure is for the most part exerted perpendicular to the wing direction, and may be taken as the lift on the wing, which must be equal to the weight of the machine.

If it be assumed that the resultant force is normal to the wing (supposed uniformly curved) at its middle point the pressure would make an angle, $(j - i)/2$, with the vertical, and the drag or component of the pressure backwards would be

$$\text{drag} = \frac{\rho}{g} S v^2 (i^2 - j^2); \quad \ldots \quad (25)$$

and would be negative, that is, the wing would be urged forward if j were greater than i, and if frictional resistance in the direction of the wind were neglected. As a matter of fact, the curvature of wings is not uniform, and when examined carefully the flow of air about the wing is not in parallel lines, so that no such obviously impossible result as having a negative drag can actually arise; but it is a fact that the distribution of pressures over the wing is such as to attribute to certain parts of the wing negative drag elements, and thereby to diminish the total drag on the wing (see Art. 67).

It has been assumed that the velocity of flow along the surface of the wing is everywhere constant. That is demanded by the equation of continuity of fluid motion if compressibility be neglected. The actual motion of the fluid and its resultant reaction upon the aerofoil are so complicated that no satisfactory theory can be constructed on any short general considerations such as those here advanced; and the modification of assumptions so as to bring as a final conclusion a proper value for the drag on an aerofoil is hardly worth while.

27. Bomb Trajectories. The two-dimensional motion of a rigid body in a resisting medium such as the air is of interest because of the problem of bomb dropping. Let the x-axis be horizontal, and the y-axis be vertically downward; and let i be the inclination of the trajectory to the x-axis measured positively downward. The forces acting are gravity and the resistance of the air. Two cases present themselves: (1) The body dropped is so compact that there is no planing action, and the resistance of the air is tangential to the path. (2) The body dropped is a species of flying bomb, where there is planing action, and a considerable lift perpendicular to the path.

The equations of motion for the first case are:

$$\frac{du}{dt} = -kgV^2 \cos i, \qquad \frac{dv}{dt} = g - kgV^2 \sin i \quad . \; . \; . \quad (26)$$

where V is the resultant velocity and where the resistance of the medium has been written as

$$R = kV^2/W \quad \text{(lbs)}. \; . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

Introduce s as the independent variable. Then

$$\frac{du}{ds} = -kgV \cos i = -kgu, \qquad \text{and} \qquad u = u_0 e^{-kgs} \; . \; . \quad (28)$$

The horizontal velocity, therefore, depends on the length of the curved trajectory. If the bomb moved in vacuo ($k = 0$), the trajectory would be a parabola. It is possible to make analytic approximations which give an estimate of the departure of the path from the parabola. To obtain the path it is necessary to eliminate the time from the equations of motion. When we write

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dx}{dt} = \frac{d}{dx}\left(\frac{dy}{dx}\frac{dx}{dt}\right)\frac{dx}{dt} = \frac{d^2y}{dx^2}u^2 + \frac{dy}{dx}\frac{du}{dt},$$

then

$$\frac{d^2y}{dx^2}u^2 + \frac{dy}{dx}\frac{du}{dt} = g - gkVv.$$

But

$$\frac{du}{dt} = -gkVu, \qquad \frac{du}{dt}\frac{dy}{dx} = -gkVv.$$

Hence, for the path in the resisting medium

$$\frac{d^2y}{dx^2} = \frac{g}{u^2}, \qquad u = u_0 e^{-kgs} \quad . \; . \; . \; . \; . \; . \; . \quad (29)$$

where $k = 1/U^2$ if U be the terminal velocity. Then

$$\frac{d^2y}{dx^2} = \frac{g}{u_0^2} e^{2gs/U^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

An approximation which is often made is to assume that the forward velocity u is always strictly equal to its initial value u_0. Under this assumption the path becomes strictly parabolic because the last equation becomes

$$\frac{d^2y}{dx^2} = \frac{g}{u_0^2},$$

a constant, and this parabola is the same parabola that the bomb would follow in vacuo. It is, therefore, only by the diminution of the forward velocity that the medium affects the path. If the terminal velocity U is about 900 ft/sec, the value of $2gs/U^2$ is nearly $s/12{,}500$, and hence $2gs/U^2$ is nearly equal to 1 until s is over 1000; even when $s = 2000$, the expression is not over 1.2. The departure from the parabola is, therefore, not great for a short fall.

Equation (30) is not apparently integrable. Several approximations may be made. It will be assumed that the initial direction of projection is horizontal, i.e., $v_0 = 0$, $dy/du = 0$. First, the strict parabolic path is

$$y = gx^2/2u_0^2 \quad \ldots\ldots\ldots \quad (31)$$

Next for motion near the vertex of the parabola (horizontal travel large compared with the drop), as may be the case in submarine bombing from seaplanes, s is nearly equal to x. The approximate equation is

$$\frac{d^2y}{dx^2} = \frac{g}{u_0^2}e^{2gx/U^2},$$

$$\frac{dy}{dx} = \frac{U^2}{2u_0^2}\left(e^{2gx/U^2} - 1\right)$$

$$y = \frac{U^4}{4gu_0^2}\left(e^{2gx/U^2} - 1\right) - \frac{U^2x}{2u_0^2} \quad \ldots\ldots \quad (32)$$

For small values of $2gx/U^2$ the exponential may be expanded

$$y = \frac{gx^2}{2u_0^2}\left[1 + \frac{2}{3}\frac{gx}{U^2}\right].$$

This approximate equation may be solved approximately for y by writing

$$x = \sqrt{\frac{2y}{g}}\,u_0\,(1 + \epsilon),$$

and substituting for x to find ϵ. Then, disregarding ϵ^2,

$$y = y(1 + 2\epsilon)\left[1 + \frac{2}{3}\frac{g}{U^2}\sqrt{\frac{2y}{g}}\,u_0(1 + \epsilon)\right],$$

or

$$1 = 1 + 2\epsilon + \frac{2}{3}\frac{g}{U^2}\sqrt{\frac{2y}{g}}\,u_0 + \text{small quantities},$$

and

$$\epsilon = -\frac{1}{3}\frac{g}{U^2}\sqrt{\frac{2y}{g}}\,u_0.$$

Hence,

$$x = \sqrt{\frac{2y}{g}}\,u_0 - \frac{2}{3}\frac{u_0^2}{U^2}y \quad \ldots\ldots\ldots \quad (33)$$

The correction term is small. For instance if $u_0 = 100$, $U = 900$, it is $y/120$. By the assumption that $s = x$, the expression (33) is no longer valid when y is not small relative to x. If $y = 100$, $x = 250$ and the correction is -0.8. If $y = 400$, $x = 500$ and the correction is -3.3. For larger values of x the assumption $s = x$ is far from true. The inference is that the correction is probably negligible in the cases where the assumption $s = x$ is reasonable.

The path (32) is parabolic in its general nature but drops more rapidly — it lies under the parabola. There is, however, a value of y for any value of x. As a matter of fact, the true path in a resisting medium approaches an asymptote, i.e., y becomes infinite when x is finite. This may be seen by another approximation. Always $s > y$ and

$$\frac{d^2y}{dx^2} = \frac{g}{u_0^2} e^{2gs/U^2} > \frac{g}{u_0^2} e^{2gy/U^2}.$$

The equation

$$\frac{d^2y}{dx^2} = \frac{g}{u_0^2} e^{2gy/U^2},$$

therefore, represents a curve which lies above the true path (gives too small a value of y) because the quantity integrated is always too small. Let $p = dy/dx$.

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} = \frac{g}{u_0^2} e^{2gy/U^2}$$

Then

$$p^2 = \frac{U^2}{u_0^2}\left(e^{2gy/U^2} - 1\right)$$

and

$$\frac{dy}{\sqrt{e^{2gy/U^2} - 1}} = \frac{e^{-gy/U^2}\,dy}{\sqrt{1 - e^{-2gy/U^2}}} = \frac{U}{u_0}\,dx.$$

Hence,

$$\cos^{-1}\left(e^{-gy/U^2}\right) = \frac{gx}{u_0U},$$

or

$$y = \frac{U^2}{g}\log\sec\frac{gx}{u_0U} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

The curve has an asymptote when $gx/u_0U = \pi/2$. The true trajectory, which lies under (34), must have an asymptote at least no further from the origin.

In a similar manner $s < x + y$ and

$$\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2} = \frac{g}{u_0^2} e^{2gz/U^2}, \qquad z = x + y,$$

is the differential equation of a curve which lies under the true trajectory. The integral is

$$\cos^{-1}\left(e^{-gz/U^2}\sqrt{1 - \frac{u_0^2}{U^2}}\right) - \cos^{-1}\sqrt{1 - \frac{u_0^2}{U^2}} = \frac{gx}{u_0 U}\sqrt{1 - \frac{u_0^2}{U^2}}$$

or

$$y + x = \frac{U^2}{g}\left[\log \sec\left(\frac{gx}{u_0 U}\sqrt{1 - \frac{u_0^2}{U^2}} + \sin^{-1}\frac{u_0}{U}\right) + \log\sqrt{1 - \frac{u_0^2}{U^2}}\right]. \quad (35)$$

If u_0 is small compared with U, the asymptote is at

$$x = \frac{u_0 U}{g}\left[\frac{\pi}{2} - \frac{u_0}{U}\right] = \frac{u_0 U}{g}\frac{\pi}{2} - \frac{u_0^2}{g},$$

instead of at $x = u_0 U\pi/2g$, and is nearer the origin. The true trajectory lies between the two, but its exact position requires an intricate series of approximations. The discussion of the trajectory of the flying bomb, where planing action must be considered, will not be undertaken.

EXERCISES

1. When landing the coefficient of mechanical friction is 0.2, the velocity is 40 mi/hr, the weight is 1500 lbs, and the air resistance is 700 lbs at 40 mi/hr. Find length and time of run.

2. In Exercise 1, suppose the landing made against a 15-mi wind. Find length and time of run.

3. A 1000-lb bomb has a terminal velocity of 900 ft/sec. Find k in $R = kv^2$. Find k if $R = kv^{1.87}$.

4. A 3000-lb machine has 9 sq ft parasite surface and a terminal velocity of 300 ft/sec. How much normal surface is equivalent to the wing in the altitude of free fall?

5. The weight of a machine is 1800 lb, and the value of k in $R = kv^2$ is $k = .0325v^2$, v in mi/hr, at diving altitude. Find the terminal velocity.

6. A machine starts in a dive at a velocity of 120 mi/hr. The terminal velocity is 180 mi/hr. How far must the machine drop to require a velocity of 150 mi/hr, of 170 mi/hr? (Constant air density assumed.)

7. Find the time consumed in the drop in Ex. 6.

8. A machine with a terminal velocity 250 ft/sec dives 2000 ft with initial velocity 150 ft/sec. Find the velocity and the time required. (Constant air density.)

9. Suppose in Ex. 1 that the machine is equipped with "air brakes," i.e., with extra surfaces which may be set normal to the wind to add to parasite re-

sistance on landing. Let the extra surface be 20 sq ft. Calculate the shortening of the distance run in still air, and also against a 15-mi wind.

10. Suppose a bomb with terminal velocity 800 ft/sec is dropped from a height of 20,000; find the velocity on reaching the earth, the maximum velocity, and the level where the maximum is reached.

11. Show that the level of maximum velocity in a drop from a great height must be that at which the density of the air is such as to give a terminal velocity equal to that maximum, and thus check the result in the text and in Ex. 10.

12. The terminal velocity of a bomb is 800 ft/sec. The bomb is projected vertically upward with this velocity. How far will it rise and how long will it take? Assume ρ = const. Compare the results with the assumption of no air resistance.

13. Is the assumption ρ = const. justifiable in Ex. 12? Set up and integrate for rising motion, taking account of the variation of air density with level.

14. A 10-lb projectile projected upward with a velocity of 400 ft/sec is observed to rise 1000 ft. What can be inferred about the coefficient k in $R = kv^2$?

15. In Ex. 1 assume the machine lands (running up) on a slope of 1 in 10. Find the length of run.

16. Suppose a machine, traveling horizontally 100 mi/hr with the main plane at 7° to the direction of flight, could suddenly be changed to an altitude of 14°, what would be the retardation and the radius of curvature at the instant after the change?

17. A machine is diving at 225 ft/sec. With how short a radius could it be pulled out of the dive without putting more than $8W$ additional force on the structure? Suppose the path a circle. If the machine does not sensibly slow down, how long will it take to change from vertical dive to horizontal motion (neglect any effect of the weight)? Is this time short enough so that the assumption of constant velocity is justifiable?

18. On Lanchester's theory if the drag is one-fifth of the lift, what would be the relative magnitudes of the angle of entry and trailing angle?

19. Derive the equation $d^2y/dx^2 = g/u^2$ from the tangential and normal resolution of accelerations, and show that the result is independent of the law $R = f(v)$ of resistance. Check this analytically.

20. Assume that the resistance to each component velocity varies with the square of that component.

$$\frac{du}{dt} = -gku^2, \qquad \frac{dv}{dt} = g - gkv^2, \qquad k = 1/U^2.$$

Show that the path has no asymptote.

21. Given $U = 900$, $u_0 = 150$. Calculate y when $x = 1500$ by (31), (32), (34), (35).

CHAPTER V

HARMONIC MOTION

28. Physical Origin. Harmonic motion, damped, undamped, and forced is of constant use in the dynamics of airplanes, partly in connection with the motion of the airplane itself under certain flying conditions, partly because by means of experiments on the harmonic motion of models of airplanes certain aerodynamic properties of the models (and hence of the full-sized machine) may be determined which are necessary for setting up the general equations of motion of the airplan.

The fundamental applied force from which harmonic motion gets its name is a restitutive force proportional to the displacement from a fixed point. Restitutive forces of this simple type occur frequently on account of the wide application of Hooke's Law that in an elastic displacement the force is proportional to the displacement. Thus, if a particle is vibrating at the end of a helical spring, the force due to the spring obeys Hooke's Law when the vibration is longitudinal, that is in the line of the spring. Again, if a long wire or column be twisted the restoring moment is proportional to the angular twist. In this case moment and angle take the place of force and displacement, and moment of inertia takes the place of mass.

Harmonic motion also sometimes arises through the process of approximation; for example, if a body which is in equilibrium in a certain position under the action of certain forces be displaced a very small amount, the resultant force is ordinarily proportional to the displacement except for infinitesimals of higher order, because the force may ordinarily be expanded by Maclaurin's theorem in the form

$$F = ax + bx^2 + cx^3 \dots ,$$

and for sufficiently small displacements all terms except the first may be discarded.

29. Rectilinear Motion. If W be the mass, and if $-Ex$ be the force in pounds acting, the equation of motion is

$$W\frac{dv}{dt} = W\frac{v\,dv}{dx} = -Egx, \quad \ldots\ldots\ldots \quad (1)$$

and

$$\frac{W}{2g}(v^2 - v_0^2) = \frac{-E}{2}(x^2 - x_0^2) \quad \ldots\ldots\ldots \quad (2)$$

is the integral where x_0 and v_0 are the initial displacement and velocity.

When a body moves in a straight line under the action of a force which is a function of the distance from some point of the line the work done by the force is

$$\text{work} = \int F(x)\,dx,$$

and

$$\text{potential energy} = -\int F(x)\,dx \quad \ldots\ldots \quad (3)$$

is taken as the definition of potential energy. As the force is measured in pounds, the work or its negative the potential energy is measured in foot-pounds. The fundamental equation of motion is

$$W\frac{v\,dv}{dx} = F(x).$$

Hence

$$\frac{W}{2g}(v^2 - v_0^2) = \int_{x_0}^{x} F(x)\,dx \quad \ldots\ldots\ldots \quad (4)$$

This equation shows that the work done by the force is equal to the change in the quantity $Wv^2/2g$, which is defined as the kinetic energy, measured in foot-pounds or kilogram-meters. The work done is, therefore, equal to the change in the kinetic energy. Introducing the potential energy, V, the equation may be written

$$\frac{W}{2g}(v^2 - v_0^2) = -(V - V_0) \quad \ldots\ldots\ldots \quad (5)$$

or

$$\frac{W}{2g}v^2 + V = \frac{W}{2g}v_0^2 + V_0 = C \quad \ldots\ldots\ldots \quad (6)$$

The first equation states that the change in the kinetic energy is the negative of the change in the potential energy, and the second that the sum of the kinetic and potential energies is constant. This is the principle of conservation of energy in mechanics in a simple case. The equation (2) obtained in the special case of harmonic motion is the equation of energy. The kinetic energy is $Wv^2/2g$, as always, and potential energy is $Ex^2/2$, as is always the case when Hooke's Law applies, and the force is $-Ex$.

Suppose that the initial conditions are $x = 0$, $v = v_0$. Then from (2)

$$v = \sqrt{v_0^2 - \frac{Egx^2}{W}} \quad \text{or} \quad \frac{dx}{\sqrt{v_0^2 - gEx^2/W}} = dt.$$

On integrating and assuming that the time is zero when $x = 0$, the result is

$$\sqrt{\frac{W}{gE}} \sin^{-1} \sqrt{\frac{gE}{W}} \frac{x}{v_0} = t,$$

or

$$x = v_0 \sqrt{\frac{W}{gE}} \sin \sqrt{\frac{gE}{W}} t. \quad \quad (7)$$

The motion is periodic, and the complete period of oscillation is

$$T = 2\pi \sqrt{\frac{W}{gE}} \quad \quad (8)$$

By the complete period of oscillation is meant the time from the instant the particle starts to the time when it reaches that point again with the velocity in the same direction, — the round trip, so to speak. When discussing the oscillations of a pendulum the time of oscillation is ordinarily taken to be the time of beat from extreme to extreme, which is one-half the period of oscillation, as defined above.

Suppose a spring or elastic cord be hanging vertically, with a mass W attached to it, and that it be set in oscillation by depressing the mass a distance x_0, and then releasing it. Let x be measured from the position of the weight when at rest under the combined action of gravity and the tension in the spring. By Hooke's Law the tension is $F = Ed$, where d is the displacement. The displacement in the position of equilibrium must be such that $W = Ed$. The displacement in any position is $d + x$, and the force due to the spring is $F = E(d + x)$. The weight, however, is equal to Ed, and, consequently, the resultant force acting is simply Ex, and negative. The equation of motion is, therefore,

$$W \frac{v\,dv}{dx} = -Egx.$$

The energy equation, which is obtained by integrating, is

$$W \frac{v^2}{2g} = \frac{E}{2} (x_0^2 - x^2).$$

Hence
$$\frac{dx}{\sqrt{x_0^2 - x^2}} = \pm\sqrt{\frac{gE}{W}}\,dt.$$

The integral may be taken as
$$\cos^{-1}\frac{x}{x_0} = \sqrt{\frac{gE}{W}}\,t,$$
or
$$x = x_0 \cos\sqrt{\frac{gE}{W}}\,t \quad . \; . \; . \; . \; . \; . \; . \quad (9)$$

The motion is simple harmonic as before with period
$$T = 2\pi\sqrt{\frac{W}{gE}}.$$

It is important to recognize the period of oscillation in simple harmonic motion from the differential equation, or from the equation of energy. The differential equation may always be written
$$W\frac{dv}{dt} = -Egx.$$
The equation of energy
$$\frac{W}{2g}(v^2 - v_0^2) = \frac{-E}{2}(x^2 - x_0^2),$$

and the period being equal to $2\pi\sqrt{W/gE}$ is evidently 2π times the square root of the coefficient of either dv/dt or of v^2 in the two equations divided by the coefficient of x or of x^2.

30. Linear Differential Equations. The differential equation of harmonic motion is of the form
$$\frac{d^2x}{dt^2} + n^2x = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (10)$$

This is a linear equation with constant coefficients; linear because x occurs only to the first power, whether occurring by itself or in differentiated form, and with constant coefficients because n is constant. In such an equation if
$$x_1 = F(t) \qquad \text{and} \qquad x_2 = G(t)$$
are two different solutions, then
$$x = C_1x_1 + C_2x_2 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$
where C_1 and C_2 are any constants, will also be a solution. This solution contains two constants which correspond with constants of

integration which are obtained when the equation is actually integrated. It is often, however, possible to determine two solutions by inspection; for instance, in the above equation

$$x_1 = \sin nt \qquad \text{and} \qquad x_2 = \cos nt$$

are clearly solutions, as may be seen by substitution, and, hence, the general solution is

$$x = C_1 \sin nt + C_2 \cos nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It is possible to throw this trigonometric solution into a different form, as follows:

$$x = \sqrt{C_1^2 + C_2^2}\left(\frac{C_1}{\sqrt{C_1^2 + C_2^2}} \sin nt + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \cos nt\right). \quad (13)$$

Let

$$\frac{C_1}{\sqrt{C_1^2 + C_2^2}} = \cos \Gamma, \qquad \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \sin \Gamma \quad . \quad . \quad (14)$$

as is possible since the sum of the squares of these quantities is unity. Then

$$x = \sqrt{C_1^2 + C_2^2} \sin (nt + \Gamma) \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The coefficient $(C_2^2 + C_2^2)^{\frac{1}{2}}$ is called the amplitude of the motion, and represents the extreme value of x, whether positive or negative. The period is

$$T = 2\pi/n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and the displacement is zero when $t = -\Gamma/n$.

Still another method of solving the equation is to set

$$x = e^{mt},$$

and to substitute in the equation. On substituting, e^{mt} cancels out and leaves the equation

$$m^2 + n^2 = 0 \qquad \text{or} \qquad m = \pm in,$$

if $i = \sqrt{-1}$. This shows that two possible solutions of the equation are

$$x_1 = e^{int} \qquad \text{and} \qquad x_2 = e^{-int},$$

and, consequently,

$$x = C_1 e^{int} + C_2 e^{-int} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

is the general solution of the equation, but in imaginary form, since $i = \sqrt{-1}$.

Now

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4} + \ldots$$

and

$$e^{iy} = 1 + iy - \frac{y^2}{2} - \frac{iy^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \ldots$$

or

$$e^{iy} = \cos y + i \sin y \quad \ldots \ldots \quad (18)$$

In like manner

$$e^{-iy} = \cos y - i \sin y \quad \ldots \ldots \quad (19)$$

Hence, the solution is

$$x = (C_1 + C_2) \cos nt + i(C_1 - C_2) \sin nt.$$

And since C_1 and C_2 are undetermined constants, the solution may be written

$$x = K_1 \cos nt + K_2 \sin nt \quad \ldots \ldots \quad (20)$$

which agrees with that found before.

In solving linear equations with constant coefficients in physics it is customary to use this last method of substituting for the variable an exponential expression. This reduces the differential equation to an algebraic equation with a certain number of roots, and to each root corresponds a different exponential expression. If some of the roots are imaginary, it is necessary to transform the solution over into trigonometric form.

31. Rotatory Motion. Consider the motion of a mass rigidly attached to a vertical wire rotating about the axis of the wire, under the influence of the restoring torsional moment. The torsional moment is proportional to the angle of rotation, and may be, therefore, written $-E\theta$, where θ is the angle. The corresponding potential energy, that is, the energy stored in the wire when the angle is θ is $E\theta^2/2$ in foot-pounds. The kinetic energy in a rotating body is $I\omega^2/2g$, where

$$\omega = d\theta/dt$$

is the angular velocity, and I the moment of inertia. The energy equation is, therefore,

$$\frac{I}{2g}(\omega^2 - \omega_0^2) = \frac{-E}{2}(\theta^2 - \theta_0^2) \quad \ldots \ldots \quad (21)$$

The time of oscillation is

$$T = 2\pi \sqrt{\frac{gE}{I}} \quad \ldots \ldots \quad (22)$$

The differential equation for the rotating motion may be obtained from the energy equation by differentiating and dividing by $d\theta/dt$. It is

$$I\frac{d^2\theta}{dt^2} = -Eg\theta \quad \text{. . . . .} \quad (23)$$

The moment of inertia times the angular acceleration is equal to the moment of the force multiplied by g.

The equation of motion of the physical pendulum may be obtained either from the principles of energy or from the principle represented by the last equation. Let I be the moment of inertia of the pendulum about its point of support. Let c be the distance from the point of support to the center of gravity. Then

$$I\frac{d^2\theta}{dt^2} = -cgW\sin\theta = -cgW\theta, \quad \text{. . . . . .} \quad (24)$$

for the moment of the force is the arm $c\sin\theta$ multiplied by the force W and $\sin\theta = \theta$ very nearly. The period is

$$T = 2\pi\sqrt{\frac{cgW}{I}} \quad \text{. . . . .} \quad (25)$$

It is customary to write the moment of inertia as

$$I = Wk^2 \quad \text{. . . . .} \quad (26)$$

where k is called the radius of gyration. In the case of certain objects of simple form this method of regarding the moment of inertia as the product of the mass and the square of the radius of gyration is convenient. In many cases, however, it is necessary to write the equation in the form

$$k^2 = I/W \quad \text{. . . . .} \quad (27)$$

because the radius of gyration is actually determined from the ratio of the moment of inertia to the mass.

32. Moment of Inertia of an Airplane. The moment of inertia of a body about any axis is the vital physical constant of the body in respect to the motion of rotation about that axis. In the case of some simple bodies the moment of inertia may be calculated by the process of integration, as may be seen in any standard work on integral calculus; and the values of the moment of inertia or of the radius of gyration may be found for certain standard engineering shapes in engineering handbooks. In many cases, however, which arise in engineering, and in particular in the case of an irregular structure like the airplane, it may be desirable to determine the moment of inertia experimentally. The method of determining it is to suspend the airplane so that it shall oscillate about some axis and measure the time of oscillation. Three methods of suspension are worthy of consideration, because sometimes one and sometimes another is more convenient.

(1) The airplane is suspended as a simple pendulum, and allowed to rotate under the action of gravity about the axis of suspension. The distance of the center of gravity from the axis of suspension is measured, the value being c. Then

$$T = 2\pi\sqrt{\frac{cgW}{I}} \quad \text{or} \quad I = 4\pi^2\frac{cgW}{T^2} \quad . \quad . \quad . \quad . \quad (28)$$

This value of I is that of the moment of inertia about the axis of rotation. By a fundamental theorem of mechanics the moment of inertia about any axis is equal to the moment of inertia about the axis parallel to that axis passing through the center of gravity plus the mass times the square of the distance between the axes. Thus: If I_0 be the moment of inertia about an axis through the center of gravity,

$$I = I_0 + Wc^2 \quad \text{or} \quad I_0 = I - Wc^2 \quad . \quad . \quad . \quad . \quad (29)$$

Hence,

$$I_0 = \left(4\pi^2\frac{cg}{T^2} - c^2\right)W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

This formula is used to determine the moment of inertia about an axis through the center of gravity by measuring the time of oscillation of the airplane about some parallel axis. One difficulty with this formula is that it contains the difference of two quantities, and if these two quantities are nearly equal, each quantity must be determined to a considerable degree of accuracy in order that their difference may be known with reasonable accuracy. In the case where the mass of the body is greatly concentrated in the vicinity of the center of gravity so that the moment of inertia about an axis through the center of gravity is small the method is not particularly safe; but in the case of the airplane where there is a considerable dispersion of mass the difficulty is not serious.

(2) It is also possible to suspend the airplane on a piano wire so that it oscillates as a torsional pendulum. Then

$$T = 2\pi\sqrt{\frac{Eg}{I}} \quad \text{or} \quad I = 4\pi^2\frac{Eg}{T^2} \quad . \quad . \quad . \quad . \quad . \quad (31)$$

It is, therefore, only necessary in this suspension to measure the time of oscillation, provided the torsion modulus E of the wire when under a torsion equal to the weight of the airplane be known.

This torsional modulus E may have to be determined, which may be done in either of two ways; first, statically, second, kinetically. The airplane may be twisted through a measured angle θ and the moment about the center of gravity may be measured by determining the force required to hold the airplane at the angle θ; or there may be suspended from the wire a mass equal to that of the airplane, but of such simple form that its moment of inertia may be calculated, as, for instance, a mass in the form of a solid cylinder or a hollow ring, the latter being preferable. One timing experiment may then be used to determine the modulus E from a known value of I, and this value of E may be used to determine the moment of inertia of the airplane. If

$$I_1 = 4\pi^2 \frac{Eg}{T_1^2}, \qquad \text{then} \qquad E = \frac{T_1^2 I_1}{4\pi^2 g}, \quad \ldots \ldots \quad (32)$$

and

$$I = \frac{I_1 T_1^2}{T^2}, \quad \ldots \ldots \ldots \ldots \quad (33)$$

where I_1 is the known moment of inertia, and T_1 the time of oscillation of that body.

(3) A third method of suspension is the bifilar pendulum. Here the airplane is suspended by two wires a distance a apart, and of length l, the center of gravity of the airplane lying halfway between the vertical wires. If, now, the airplane be twisted about a vertical axis, it is raised by a small amount, and a certain quantity of potential energy is thus stored up. If the airplane be then released the potential energy is converted into kinetic, and an oscillatory rotation takes place about the vertical axis. If the angle of displacement is small and the length l of the wires is large compared with the distance between them, the up and down motion of the center of gravity is very small, the velocity of that motion is small, and its kinetic energy is negligible compared with the kinetic energy of the motion of rotation about the vertical axis. The energy equation is, therefore, approximately

$$\frac{I}{2g}\omega^2 + Wh = C, \quad \ldots \ldots \ldots \ldots \quad (34)$$

where h is the distance the center of gravity is raised, Wh being the potential energy. It is necessary to find the distance that the machine is raised when turned through an angle θ. To accomplish this take a set of axes in space with the origin halfway between the upper ends of the wires, the x-axis joining the upper ends, the z-axis drawn downward, and the y-axis perpendicular to both x and z. The coordinates of the upper end of one wire are

$$x = a/2, \qquad y = 0, \qquad z = 0.$$

The coordinates of the lower end of that wire in the equilibrium position are

$$x = a/2, \qquad y = 0, \qquad z = l.$$

If the airplane be turned through the angle θ the coordinates of the lower end are

$$x = \tfrac{1}{2}a\cos\theta, \qquad y = \tfrac{1}{2}a\sin\theta, \qquad z = z,$$

and the length of the wire may then be expressed as

$$l^2 = \tfrac{1}{4}a^2(1 - \cos\theta)^2 + \tfrac{1}{4}a^2\sin^2\theta + z^2.$$

Approximately, $1 - \cos\theta = \theta^2/2$, $\sin\theta = \theta$, and hence the first term is negligible compared to the second, and one may write

$$l^2 = \tfrac{1}{4}a^2\theta^2 + z^2.$$

The distance the center of gravity is raised is $l - z$. Now,

$$l^2 - z^2 = (l - z)(l + z) = a^2\theta^2/4,$$

and

$$l - z = a^2\theta^2/8l,$$

since $l + z$ is practically $2l$. The equation of motion is, therefore,

$$\frac{I}{2g}\left(\frac{d\theta}{dt}\right)^2 + \frac{Wa^2}{8l}\theta^2 = C \qquad (35)$$

Hence

$$T = \frac{4\pi}{a}\sqrt{\frac{lI}{Wg}}, \qquad \text{and} \qquad I = \frac{T^2Wa^2g}{16\pi^2 l} \qquad (36)$$

The moment of inertia can, therefore, be determined by measuring a, l, W, and T.

The time of oscillation T can be determined fairly accurately with a stop-watch if the pendulum be allowed to oscillate for a considerable number of beats, and the total time of oscillation be divided by the number of beats. The distance from the point of support to the center of gravity in the first method may be determined as accurately as the center of gravity can be placed in so complicated a structure as the airplane. The distance between the wires and the length of the wires in the third case (bifilar pendulum) can be determined with great accuracy. In the second and third cases the moment of inertia is really determined about the axis of oscillation, and it is therefore important that that axis should pass through the center of gravity, if the moment of inertia about an axis through the center of gravity is desired. However, a slight displacement of the center of gravity from the axis of rotation would increase the moment of inertia only by the small quantity Wc^2 where c was the displacement; and as the center of gravity is known pretty well, c could only be very small and the correction probably insignificant.

Example: Suppose that the radius of gyration of an airplane is 6 ft about some axis through the center of gravity. Find the time of oscillation when the machine is suspended as a simple pendulum to oscillate about a parallel axis 4 ft from the C. G. Find also the time of oscillation when the suspension is bifilar, with wires 8 ft long and 4 ft apart.

In the first case,

$$T = 2\pi\sqrt{I/cgW} = 2\pi\sqrt{k^2/cg} = 2\pi\sqrt{\frac{6^2 + 4^2}{4 \times 32.2}} = 3.9 \text{ sec.}$$

In the second case,

$$T = 2\pi\sqrt{4lI/Wa^2g} = 2\pi\sqrt{4lk^2/a^2g} = 2\pi\sqrt{\frac{4 \times 8 \times 6^2}{4^2 \times 32.2}} = 9.4 \text{ sec.}$$

In the case of the simple pendulum or bifilar pendulum it is important to keep the angle of oscillation small because as the angle increases the formula for the time becomes less accurate owing to the fact that the motion is not strictly simple harmonic. In the case of the torsional pendulum the angle of oscillation may be reasonably large without danger. Owing, however, to the presence of the wings and other surfaces upon the airplane, it is desirable in all cases to have the motion of oscillation very gentle so that the sweep of the surfaces through the air may not disturb the motion. The presence of these surfaces also will result in a damping of the harmonic motion, which, however, does not materially influence the period T, as will be shown.

Another caution to be observed in performing the experiment is to be sure that the motion is actually as intended. For instance, if in the suspension as a single pendulum suspending wires are used, it is possible that the pendulum may have a compound oscillation because the wires may not remain absolutely in line with the center of gravity; that is, there may be an oscillation of the wires combined with an oscillation of the machine about the ends of the wires. In the case of the torsion pendulum the machine unless carefully suspended may not oscillate strictly about the vertical axis coincident with the wire, but may have additional oscillations about other axes. In the case of a bifilar suspension there may be in addition to the oscillation about the vertical axis a certain oscillation about the horizontal axis connecting the two wires, or even an oscillation of the whole system in the plane of the two wires. A full discussion of the methods of determining the moments of inertia from observed times of oscillation would require a careful treatment of the errors of observation due in part to possible inaccuracies in the measured quantities, and in part to departures of the motion from the assumed simple harmonic form. These matters, however, will not be treated here.

33. Damped Harmonic Motion. In all oscillating systems there is likely to be a certain amount of friction, which manifests itself by a damping or decaying in the amplitude of the oscillation. This friction may vary with some power of the velocity, and will be assumed in the first instance to vary as the first power of the velocity. The equation of motion for a rectilinear oscillation may then be written

$$\frac{W\,d^2x}{dt^2} = -Egx - kg\frac{dx}{dt}, \qquad (37)$$

where $F = -Ex$ is the restoring force in pounds, and $R = -kv$ is the frictional resistance in pounds. The equation may be written

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0, \qquad (38)$$

$$a = gk/W, \qquad b = gE/W.$$

This equation is linear with constant coefficients, and may be solved by substituting

$$x = e^{mt}, \qquad dx/dt = me^{mt}, \qquad d^2x/dt^2 = m^2e^{mt}.$$

Then

$$m^2 + am + b = 0, \qquad m = -a/2 \pm \sqrt{a^2/4 - b}.$$

Three cases are possible, according as

$$b > a^2/4 \qquad \text{or} \qquad b = a^2/4 \qquad \text{or} \qquad b < a^2/4.$$

In the last two cases the exponential expressions are real, and the motion is non-oscillatory. These cases will not be treated.

In the first case the exponential expressions are imaginary, and the motion is oscillatory, namely,

$$x = e^{-at/2}(C_1 e^{i\sqrt{b-a^2/4}\,t} + C_2 e^{-i\sqrt{b-a^2/4}\,t})$$
$$= e^{-at/2}(A \cos \sqrt{b - a^2/4}\,t + B \sin \sqrt{b - a^2/4}\,t). \quad . \quad (39)$$

The term in the parenthesis is periodic, but the motion is damped or subject to decay on account of the multiplier, $e^{-at/2}$. The complete period of oscillation is

$$T = \frac{2\pi}{\sqrt{b - a^2/4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

The ratio of the displacement x at any time t, and at the time $t + T$ is $e^{aT/2}$ and $\delta = \log e^{aT/2} = aT/2$ is the logarithm of the ratio of the displacement at any time t to the displacement at the time $t + T$. This quantity δ is called the logarithmic decrement. As the time increases in arithmetic progression, the amplitudes die off in geometric progression. Thus, if $x_0, x_2, x_4 \ldots x_{2n}$ are the amplitudes of successive swings to the right,

$$x_2 = e^{\delta}x_0,\ x_4 = e^{\delta}x_2 = e^{2\delta}x_0,\ \ldots\ x_{2n} = e^{n\delta}x_0, \quad . \quad . \quad . \quad . \quad (41)$$

and if $x_1, x_3 \ldots$ are the amplitudes of the swings to the left,

$$x_1 = e^{\delta/2}x_0,\ x_3 = e^{\delta/2}x_1 = e^{3\delta/2}x_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

The logarithmic decrement

$$\delta = \log (x_0/x_2) = 2.303 \log_{10} (x_0/x_2) \quad . \quad . \quad . \quad . \quad . \quad (43)$$

is very nearly equal to 0 when x_2 is very nearly as large as x_0; that is, when the damping is very small. If, after n complete swings the motion is reduced to one-half amplitude,

$$\frac{x_0}{x_{2n}} = e^{anT/2} = e^{n\delta} = \tfrac{1}{2} \qquad \text{or} \qquad \delta = \frac{\log 2}{n} = \frac{.693}{n} \quad . \quad . \quad (44)$$

Now,

$$T = \frac{2\pi}{\sqrt{b - a^2/4}} = \frac{2\pi}{\sqrt{b - \delta^2/T^2}} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

since $aT/2 = \delta$. The solution for T is

$$T = \frac{2\pi}{\sqrt{b}}(1 + \delta^2/4\pi^2)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

This shows that if δ is a small quantity compared with 2π, the value of T is practically equal to $2\pi/\sqrt{b}$, which is precisely the value that T would have if there were no damping at all; that is, if $\delta = 0$, or $k = 0$. If the motion damps to half-amplitude after one complete oscillation, which is pretty severe damping, $\delta = 0.693$. Even in this case $\delta/2\pi$ is only about 1/10th, and $\delta^2/4\pi^2$ about 1/100th, so that no serious error is made by neglecting the effect of damping on the period. The approximate values for the period and for the logarithmic decrement when the decrement is small are

$$T = 2\pi/\sqrt{b}, \qquad \delta = \pi a/\sqrt{b} \quad \ldots \ldots \quad (47)$$

An experiment on oscillation with an observation of the time therefore determines b, and with an observation on the logarithmic decrement it determines also a, and consequently the damping coefficient k in $R = kv$. In making use of these formulas it is necessary to substitute for a and b their values in terms of k, g, E, W, as given under equation (38). In the case of rotatory motion the differential equation is of the form

$$I\frac{d^2\theta}{dt^2} = -Eg\theta - kg\frac{d\theta}{dt}, \quad \ldots \ldots \quad (48)$$

where θ is measured in radians, and the resistance is proportional to the angular velocity. This resistance is a resisting couple or moment just as the restoring term $-Eg\theta$ is a couple or moment. The expressions $kd\theta/dt$ and $E\theta$ are in ft.lbs.

34. Work and Energy. In case the damping is small, for instance, $\delta < 0.7$, as in the case where the motion damps to one-half amplitude (or less) after one complete oscillation, it is possible to estimate the damping approximately by means of the principle of work and energy. For the work done by the friction in foot-pounds is

$$\text{Work} = \int R\,dx = \int kv\,dx = \int kv^2\,dt \quad \ldots \quad (49)$$

Let it be assumed that the motion is essentially simple harmonic, with

$$x = x_0 \cos \sqrt{b}\,t.$$

The total work in a complete oscillation is, then,

$$\int_0^T kv^2\,dt = \int_0^{2\pi/\sqrt{b}} kx_0^2 b \sin^2 \sqrt{b}\,t\,dt = \pi k\sqrt{b}x_0^2 \quad \ldots \quad (50)$$

The initial energy in foot-pounds is $Ex_0^2/2$. The final energy is $Ex_2^2/2$, and the difference must be the work done. Hence,

$$\frac{Ex_0^2}{2} = \frac{Ex_2^2}{2} + \pi k\sqrt{b}x_0^2 \quad . \quad . \quad . \quad (51)$$

or

$$\left(\frac{x_0}{x_2}\right)^2 = \frac{1}{1 - 2\pi k\sqrt{b}/E} = 1 + 2\pi k\sqrt{b}/E,$$

since by hypothesis the damping is small. Extracting the root,

$$\frac{x_0}{x_2} = 1 + \pi k\sqrt{b}/E = e^{\delta} = 1 + \delta.$$

Hence,

$$\delta = \pi k\sqrt{b}/E = \pi a/\sqrt{b}.$$

The approximate method has, therefore, brought back the same value of δ as previously obtained.

The method of estimating the damping by the principle of work and energy is applicable to the case where the resistance varies as the square of the velocity. Suppose $R = kv^2$. Then the work done is

$$2\int_0^{T/2} kv^3\,dt = 2\int_0^{\pi/\sqrt{b}} kx_0^3 b^{3/2} \sin^3 \sqrt{b}\,t\,dt = 8kx_0^3 b/3 \quad . \quad . \quad (52)$$

The integration here has been taken for the half-swing, and the result has been doubled, a procedure which might have been used in the previous case. Then the equation of work and energy gives

$$\frac{Ex_0^2}{2} = \frac{Ex_2^2}{2} + \frac{8kx_0^3 b}{3} \quad . \quad . \quad . \quad (53)$$

or

$$\left(\frac{x_0}{x_2}\right)^2 = \frac{1}{1 - 16kx_0b/3E} = 1 + 16kx_0b/3E,$$

and

$$\frac{x_0}{x_2} = 1 + 8kx_0b/3E = e^{\delta} = 1 + \delta.$$

Hence

$$\delta = 8kx_0b/3E = 8kx_0g/3W \quad . \quad . \quad . \quad (54)$$

It is seen from this expression that the logarithmic decrement δ depends upon the initial amplitude x_0, and diminishes from oscillation to oscillation. It is only in case the damping is proportional to the velocity that the decrement, that is, the logarithm of successive amplitudes, is independent of the amplitude.

When observing an oscillatory motion which is damped, a test for the type of damping may be made by calculating

$$\log \frac{x_0}{x_2},\ \tfrac{1}{2} \log \frac{x_0}{x_4},\ \tfrac{1}{3} \log \frac{x_0}{x_6},\ \ldots, \qquad \log \frac{x_2}{x_4},\ \tfrac{1}{2} \log \frac{x_2}{x_6},\ \ldots.$$

If the motion is resisted by a force proportional to the velocity, these values of the decrement will be constant. If the values vary more than could be expected from the errors of observation, and if it were suspected that the resistance varied as the square of the velocity, it would be desirable to calculate the quantities

$$\frac{1}{x_0} \log \frac{x_0}{x_2},\ \frac{1}{x_2} \log \frac{x_2}{x_4},\ \frac{1}{x_4} \log \frac{x_4}{x_6},\ \ldots.$$

If the resistance did vary as the square of the velocity, these quantities should be constant except for the experimental errors. In general it cannot be expected that the resistance will vary either directly as the velocity or directly as the square of the velocity, but may vary with any power of the velocity between 1 and 2, or with a complicated function of the velocity. Fortunately, in most cases the resistance appears to vary very nearly with the velocity, and the plot of the different values of δ shows sensible constancy.

Example. The motion of the weather-vane is an illustration of damped harmonic motion. Consider the following idealized vane.

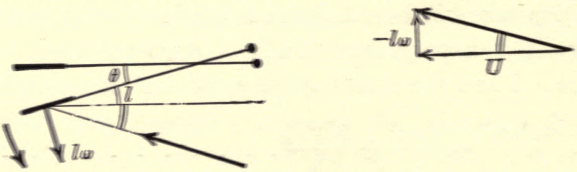

Fig. 13. Oscillation of a Weather-vane.

A surface S located (and considered as centered) at a distance l behind the axis of rotation. Let the wind velocity be U, and steady both in magnitude and in direction. Let I be the moment of inertia about the axis of rotation, and neglect the frictional couple on the axis. If the vane has an angular velocity ω about the axis, it has a velocity $l\omega$ across the wind. The total relative wind is $(U^2 + l^2\omega^2)^{\frac{1}{2}}$, which may be taken as simply U if $l\omega$ is small relative to U. The angle between the relative wind and the true wind is $\tan^{-1}(l\omega/U)$ or $l\omega/U$ in radians, and between the vane and the true wind is some value θ. The angle i between the vane and the relative wind is therefore (Fig. 13)

$$i^\circ = 57.3(\theta + l\omega/U),$$

and the normal pressure is $P = kSU^2i^\circ$, where k depends on the aspect ratio of the vane. The restoring moment is lP. Hence,

$$I\frac{d^2\theta}{dt^2} = -glkSU^2 \times 57{\cdot}3(\theta + l\omega/U)$$

with $\omega = d\theta/dt$. This equation is of the form

$$\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + b\theta = 0,$$

$$a = glkSU^2 \times 57{\cdot}3l/UI,$$

$$b = glkSU^2 \times 57{\cdot}3/I.$$

If l is numerically small compared with U, a is small compared with b, and the motion is damped harmonic.

The periodic time T varies inversely as the square root of b, i.e., inversely as the wind velocity U; the decrement δ varies as $a/b^{\frac{1}{2}}$ and is independent of U, i.e., the damping per oscillation is independent of the wind velocity, though the damping in time is faster the greater U. If vanes differing only in dimensions are considered, S varies as l^2 and I varies as l^5. The periodic time will therefore vary as l, and the decrement will not vary with l.

The simple weather-vane problem has a close relation to the more complex problem of the oscillations of the airplane in flight. For if the airplane takes up oscillatory motion in pitch it will whip the horizontal surfaces (particularly the elevator and stabilizer) across the wind and introduce a restoring moment proportional to the angular velocity, in addition to one proportional to the angular displacement, and the machine will execute a damped oscillation in pitch — though the calculation is no longer simple and the motion itself is complicated by up-and-down and fore-and-aft oscillatory departures from the steady horizontal motion.

35. Two Remarks. One remark with respect to the equation of rectilinear motion when there is a restoring force proportional to the displacement, and a resistance proportional to the square of the velocity should be made. The frictional force will always set itself against the motion. Now when $R = kv$, the expression for the force reverses its sign when the velocity reverses its sign, because v enters to the first power; and, consequently, the equation of motion may always be written as

$$W\frac{d^2x}{dt^2} = -Egx - kgv$$

with a similar expression where I replaces W, and θ replaces x in rotatory motion. But if $R = kv^2$, the expression for the resistance will not reverse its sign when the velocity reverses, and it is therefore necessary to deal with two different equations according as the oscillation is from right to left, that is, in the negative direction; or from left to right, that is, in the positive direction. For the first case,

$$W\frac{d^2x}{dt^2} = -Egx + kgv^2 \quad \dots \dots \dots \quad (55)$$

In the second,

$$W\frac{d^2x}{dt^2} = -Egx - kgv^2 \quad \dots \dots \dots \quad (56)$$

It is impossible to give a single analytical expression for both halves of the oscillation. It was for this reason that in calculating the work as above the calculation was made for the half-swing and then doubled.

The equation of motion from right to left may be integrated, for

$$\frac{d^2x}{dt^2} = \frac{v\,dv}{dx} = \frac{-Egx}{W} + \frac{kgv^2}{W}, \quad \dots \dots \dots \quad (57)$$

and if $u = v^2$, the equation becomes

$$\frac{du}{dx} - \frac{2gku}{W} = \frac{-2gEx}{W} \quad \dots \dots \dots \quad (58)$$

which is a linear equation of the type

$$\frac{dy}{dx} + P(x)y = Q(x), \quad \dots \dots \dots \quad (59)$$

of which the integral is

$$ye^{\int P\,dx} = \int Qe^{\int P\,dx}dx + C. \quad \dots \dots \quad (60)$$

The relation between $u = v^2$ and x may, therefore, be obtained. If x_0 be the initial amplitude, the relation between u and x is as follows:

$$ue^{-2gkx/W} = \frac{EW}{2gk^2}\left\{\frac{2gk}{E}\left(xe^{-2gkx/W} - x_0e^{-2gkx_0/W}\right) + e^{-2gkx/W} - e^{-2gkx_0/W}\right\}.$$

The position x_1, at which the swing stops upon the left, may be had by solving the equation obtained by setting $u = 0$, namely,

$$\xi_1 e^{-\xi_1} + e^{-\xi_1} = \xi_0 e^{-\xi_0} + e^{-\xi_0}, \qquad \text{if } \xi = 2gkx/W \quad \dots \quad (61)$$

This equation is transcendental, and can only be solved by approximate methods; either by expansion into series, or graphically. In most practical cases, however, the damping is so small that the approximate method of solution by means of the work and energy relation is sufficiently accurate.

A word must also be said about the mass of the spring or other physical body which gives the restoring force. Consider, for simplicity, the case in which a mass W is at the end of a spring. When the mass oscillates the whole spring also oscillates. The potential energy of the system, that is the work stored in the spring, is in any position $Ex^2/2$ if $x = 0$ in the position of equilibrium. The kinetic energy, however, is not merely $Wv^2/2g$, which is the energy of the moving mass, but must include the kinetic energy of the spring. Let it be assumed that the spring oscillates without surging; that is, that the motion of the spring is similar at all points, and that waves do not run up and down the spring. Let l be the natural length of the spring, ρ its mass per unit length, y the distance from the fixed end. Then the velocity u of the element of mass $dm = \rho\, dy$ is $u:v = y:l$, and the kinetic energy is, therefore,

$$\int_0^l \frac{v^2}{2g}\,\frac{y^2}{l^2}\,\rho\, dy = \frac{v^2}{6g}\,l\rho = \frac{w}{3}\,\frac{v^2}{2g}, \qquad (62)$$

where w is the mass of the whole spring. The total kinetic energy of the system is, therefore,

$$\frac{v^2}{2g}\left(W + \frac{w}{3}\right), \qquad (63)$$

which means that the effective total mass is $W + w/3$. In the case, therefore, in which the weight of the spring is any considerable portion of the weight of the vibrating mass, the time of oscillation will be not $T = 2\pi(W/Eg)^{\frac{1}{2}}$ but

$$T = 2\pi\sqrt{\frac{W + w/3}{Eg}} = 2\pi\sqrt{\frac{W}{Eg}}(1 + w/3W)^{\frac{1}{2}} \qquad (64)$$

36. Forced Harmonic Motion. In some problems in addition to the restoring force and the frictional force there is an external applied force, which is a function of the time, generally a simple sine or cosine function. The equation for the motion is

$$W\frac{d^2x}{dt^2} = -Egx - kg\frac{dx}{dt} + Cg\sin nt, \qquad (65)$$

where $C \sin nt$ is the applied force. Such a motion is called a forced harmonic motion. It arises in one method of obtaining certain of the aerodynamical coefficients of an airplane from experiments on models. For brevity the equation may be written

$$\frac{d^2x}{dt^2} + \frac{a\,dx}{dt} + bx = c \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

Now if any one solution I for this whole equation is known, the complete solution including two constants of integration may be obtained; for suppose

$$\frac{d^2I}{dt^2} + \frac{a\,dI}{dt} + bI = c \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Subtract this equation from the equation in x. The difference of two derivatives is the derivative of the difference. Hence

$$\frac{d^2(x - I)}{dt^2} + a\frac{d(x - I)}{dt} + b(x - I) = 0 \quad . \quad . \quad . \quad . \quad (68)$$

This equation, however, is the ordinary equation for unforced damped harmonic motion, and its solution, $x - I$, is known from the previous work.

To determine a particular solution the method of undetermined coefficients will be used. Let it be supposed that

$$I = A \cos nt + B \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

When this expression is substituted in the complete differential equation for I, the result is

$$(-An^2 + aBn + bA) \cos nt + (-Bn^2 - aAn + bB) \sin nt = c \sin nt.$$

This relation will be satisfied provided A and B satisfy the equations

$$-An^2 + aBn + bA = 0, \qquad -Bn^2 - aAn + bB = c.$$

The solution for A and B is as follows:

$$A = \frac{-anc}{(b - n^2)^2 + a^2n^2}, \qquad B = \frac{b - n^2}{(b - n^2)^2 + a^2n^2} \quad . \quad . \quad (70)$$

With these values of A and B the complete solution of the equation is

$$y = e^{-at/2}(C_1 \cos \sqrt{b - a^2/4}\, t + C_2 \sin \sqrt{b - a^2/4}\, t)$$
$$+ A \cos nt + B \sin nt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

The first terms represent a damped harmonic oscillation, which, after the lapse of a sufficient time, becomes as small as desired, and may be neglected; the second terms represent a permanent simple harmonic motion of which the amplitude is

$$\sqrt{A^2 + B^2} = \frac{c}{[(b - n^2)^2 + a^2n^2]^{\frac{1}{2}}} \quad . \; . \; . \; . \; . \quad (72)$$

Indeed, the value of I may be written

$$I = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \cos nt + \frac{B}{\sqrt{A^2 + B^2}} \sin nt\right)$$

$$= \sqrt{A^2 + B^2} \sin (nt - \Gamma) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (73)$$

where

$$\cos \Gamma = \frac{(b - n^2)c}{(b - n^2)^2 + a^2n^2}, \qquad \sin \Gamma = \frac{anc}{(b^2 - n^2)^2 + a^2n^2} \cdot \quad (74)$$

It is seen, therefore, that the forced oscillation represented by I is not in phase with the force, but lags behind it by the angle Γ/n, for the force vanishes when $t = 0$, but the oscillation I vanishes when $t = \Gamma/n$. The value of the tangent Γ is

$$\tan \Gamma = -\frac{A}{B} = \frac{an}{b - n^2} \quad . \; . \; . \; . \; . \; . \; . \quad (75)$$

After the natural damped oscillation has died out, and only the forced oscillation persists, the amplitude and lag angle of the forced oscillation depends upon the impressed frequency n. If a is small, and n is nearly equal to b, the amplitude has a small denominator, and is consequently large; that is, by tuning the impressed force to the frequency of the natural oscillation, a relatively small force may produce a considerable oscillation. This is the phenomena of resonance. In fact, the amplitude of I will be greatest when n is so selected that $(b - n^2)^2 + a^2n^2$ is smallest; that is, when

$$n^2 = b - \tfrac{1}{2}a^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (76)$$

For this value of n the amplitude is

$$\sqrt{A^2 + B^2} = \frac{c}{a\sqrt{b - a^2/4}} \cdot \quad . \; . \; . \; . \; . \; . \; . \quad (77)$$

and the angle of lag is

$$\Gamma = \tan^{-1} \frac{2\sqrt{b - a^2/2}}{a} = \tan^{-1} 2\sqrt{b}/a \quad . \; . \; . \; . \quad (78)$$

if the damping is small. The maximum amplitude of the motion, therefore, varies inversely as the damping coefficient a and the tangent of the lag angle also varies inversely as a. When the damping is small, not only is the amplitude large, but the lag angle is nearly 90°. When, however, the damping is large, the amplitude is relatively small, and the angle of lag is small.

Example. Suppose an ordinary clock pendulum beats seconds, weighs 10 lb (assumed concentrated in the bob), and damps to half-amplitude in 5 complete oscillations if not actuated by any driving force. How much driving force, resonant and perodic, is necessary to maintain an amplitude of 2.9° (1/20 radian) on each side of the vertical?

The equation of motion for the mass on its arc is (65) or (66) with $W = 10$, $E = W/l$, k unknown. As the pendulum beats seconds ($T = 2$), the length may be taken as $l = 3.3$ ft. Then $E = 3$. By (44) the decrement is $\delta = .14$. The resonant amplitude is by (77) with sufficient accuracy $c/ab^{\frac{1}{2}} = 1/20$. The values of a, b are by (47)

$$b = \frac{4\pi^2}{T^2} = \pi^2, \qquad a = \frac{\delta\sqrt{b}}{\pi} = .14, \qquad c = \frac{.14\pi}{20} = .022$$

By

$$c = \frac{gC}{W}, \qquad C = \frac{Wc}{g} = \frac{10(.022)}{32} = .007 \text{ lb.}$$

A periodic force of only a trifle over 0.1 oz maximum will keep up the motion. This force applied statically would produce a deflection in the 10-lb mass of only about .0007 radians instead of .05. Hence the resonant periodic force at maximum is only about 1/70 of the static force required for the same deflection. This is but a fair illustration of the relatively great effects which may come from a small resonant force when the damping is small.

EXERCISES

1. The force of a spring is 2 lb when stretched $\frac{1}{2}$ in. A mass of 5 lb is suspended by the spring. Find the time of oscillation.

2. A spiral spring 18 in long has suspended from it a mass of 8 oz. The periodic time is $1\frac{1}{2}$ sec. Find the elastic constant of the spring.

3. A cylindrical spar buoy stands vertically in the water. The diameter is 6 in, and the mass 200 lb. Find the time of oscillation if slightly depressed and released. *Ans.* $T = 4.46$ sec.

4. Write in exponential form and, when necessary, reduce to trigonometric equivalents, the solutions of

$$(a)\ \frac{d^2x}{dt^2} - 6\tfrac{1}{4}x = 0, \qquad (b)\ \frac{d^2x}{dt^2} + 2\tfrac{1}{4}x = 0.$$

5. The equation of energy is $v^2/64 + 9x^2 = C$. Find the periodic time. Find the maximum acceleration.

6. A couple of 3 ft.lb is necessary to twist a clamped rod through 19°. Find the time of oscillation if a circular disc of radius 6 in and weight 5 lbs is clamped to the free end of the rod with its plane perpendicular to the rod and its center in the rod.

7. An airplane weighing 1800 lb oscillates like a pendulum about an axis 3 ft from the C. G. The periodic time is $4\frac{1}{2}$ sec. Find the radius of gyration about a parallel axis through the C. G. How much error is made if there is an error of 0.1 sec in the time? If there is an error of 2 in in the measurement of the distance of the C. G. from the axis of rotation?

8. An airplane is on a bifilar suspension with wires 8 ft long, 4 ft apart. The time of oscillation is 10 sec. Find the radius of gyration. Estimate the error if the position of the C. G. is unknown within 2 in.

9. Taking into consideration the structure of the airplane discuss the probable advantages of the three types of suspension for determining the moments of inertia about the axes of pitch, of roll, and of yaw (see Fig. 26, Art. 75).

10. Given $\frac{d^2x}{dt^2} + 0.5\frac{dx}{dt} + 26x = 0$. Find T and δ accurately and approximately. *Ans.* $T = 1.2337, 1.2342$; $\delta = 0.30842, 0.30806$.

11. Given $\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + 5x = 0$. Find T and δ. Suppose $W = 10$ lb. What is the restitutive force per foot of extension? What is the resistance per ft/sec of velocity? *Ans.* $T = 2.82$, $\delta = 0.141$, $E = 1.55$, $k = 0.0311$.

12. Given $\delta = 0.2$, $T = 5$ sec. Write the differential equation. How many oscillations are needed for damping to half-amplitude?

13. You observe $\delta = 0.3$, $T = \frac{1}{2}$ sec. Give the values of E in $F = -Ex$ and of k in $R = kv$, if W is the mass. *Ans.* $E = 4.91W$, $k = .037\,W$.

14. A 10-lb mass is observed to oscillate 90 times per minute and to damp to half-amplitude in 15 sec. Find the differential equation. Find E in $F = -Ex$ and k in $R = kv$.

15. A 10-lb mass is acted upon by a restitutive force of 2 lb when the displacement is 1 ft. Find the period if the motion damps to 1/10 = amplitude after 300 oscillations. Find k in $R = kv$. *Ans.* $T = 2.47$, $k = .00193$.

16. A large square plate of area S and weight w lb per sq ft oscillates under a restitutive force of 1 lb per inch of displacement. If the resistance is $R = kv$, find the period and decrement.

17. A weather-vane consists of a 6-lb plate, one foot square, centered $1\frac{1}{2}$ ft behind the axis of rotation, counter-balanced by a weight 6 in forward of the axis. The wind is blowing 30 mi/hr. Calculate the period of oscillation and the ratio of two successive amplitudes.

18. If the pendulum of a clock, 10 in long, damps to half-amplitude in six complete oscillations, how large must be the amplitude of a periodic force (tuned to best resonance) to keep the clock running with a swing of $7.2° = \frac{1}{8}$ radian on each side of the vertical?

19. Find the time of oscillation of a system in bifilar suspension — wires 8 ft long, 4 ft apart — $W = 1800$ lb, radius of gyration 7 ft. *Ans.* 11 sec.

20. If in Ex. 19 the ratio of successive amplitudes $x_2 : x_0$ is 3 : 4, find the amplitude of the resonant periodic couple necessary to maintain an oscillation of 5° on each side of the equilibrium position. *Ans.* 14.2 ft.lb.

21. In Ex. 20, how much would the damping affect the period? How many ft.lb is the damping moment in the position of equilibrium and in the extreme position?

22. A child weighing 75 lb swings in a swing with 13-ft ropes. Find the period. If the motion damps to half-amplitude in 4 complete swings, what is the resonant periodic force necessary to maintain the amplitude of 10° on each side? What constant force applied for a distance of 6 in at the end of the swing will maintain the oscillation?

23. Calculate the work done or power delivered by the periodic force in a forced oscillation and compare with the energy absorbed or power consumed by the resisting force. What is the significance of the lag angle in this connection?

CHAPTER VI

MOTION IN TWO DIMENSIONS

37. Motion of the Center of Gravity. The motion of rotation about a fixed axis has been treated in the case of a rigid body by the use of the principle of work and energy. This is possible because that motion depends on only one independent variable, namely, angle; and the equation of work and energy furnishes one equation which is sufficient. When a rigid body moves in a plane, three independent variables are in general necessary to specify its position; two, such as (x, y) to determine the position of some point fixed in the body, and one, such as θ to determine the angle through which the body has turned. Ordinarily, except in the case of rotation about a fixed axis, the point (x, y) is taken as the center of gravity of the body. It is necessary to find the equations of motion of a body in the plane.

What the forces are which actually determine the rigid configuration of a rigid body we do not know. Rigidity can, however, be obtained by imagining that each particle of the body is connected to every other particle by a weightless rod of invariable length; for a body is rigid by definition when the distance between any two points remains unchanged. Moreover, any two systems of forces which maintain rigidity are necessarily equivalent in the statical sense; and, therefore, in deriving the equations of motion of a rigid body the internal forces of action and reaction between the particles may be assumed to be those which would arise from the system of inextensible rods. It is not necessary to imagine this system of rods at all if one postulates that the actions and reactions between the particles of the rigid body satisfy Newton's third law of motion; namely, that to each action there is an equal and opposite reaction; and that the line of the action and reaction for any two particles is the line connecting those particles. This law is verified in the imagined case of the rods, and it is only this law which is needed in the derivation of the equations. Moreover, if this law be assumed, a certain part of the derivation may be given without assuming the restriction of a rigid body; and this plan will be followed.

Let W be the mass of any particle in a system of particles. Let x, y be the coördinates of the particle; and let X, Y be the forces acting on the particle along the x and y directions. The equations of motion are, then,

$$W\frac{d^2x}{dt^2} = gX, \qquad W\frac{d^2y}{dt^2} = gY \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

Let these equations be added for all the particles in the system. Then,

$$\Sigma W\frac{d^2x}{dt^2} = g\Sigma X, \qquad \Sigma W\frac{d^2y}{dt^2} = g\Sigma Y \; . \; . \; . \; . \; . \; . \quad (2)$$

The center of gravity (x_c, y_c) of the body is determined by the equations

$$x_c = \frac{\Sigma Wx}{\Sigma W}, \qquad y_c = \frac{\Sigma Wy}{\Sigma W}, \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

or

$$\Sigma(Wx) = (\Sigma W)x_c, \qquad \Sigma(Wy) = (\Sigma W)y_c \quad . \; . \; . \; . \quad (4)$$

If these equations be differentiated once or twice the velocities and accelerations of the center of gravity are obtained from the equations

$$\Sigma W\frac{dx}{dt} = (\Sigma W)\frac{dx_c}{dt}, \qquad \Sigma W\frac{dy}{dt} = (\Sigma W)\frac{dy_c}{dt}, \; . \; . \; . \; . \quad (5)$$

$$\Sigma W\frac{d^2x}{dt^2} = (\Sigma W)\frac{d^2x_c}{dt^2}, \qquad \Sigma W\frac{d^2y}{dt^2} = (\Sigma W)\frac{d^2y_c}{dt^2} \quad . \; . \; . \quad (6)$$

The equations (2) may, therefore, be written

$$(\Sigma W)\frac{d^2x_c}{dt^2} = g\Sigma X, \qquad (\Sigma W)\frac{d^2y_c}{dt^2} = g\Sigma Y \; . \; . \; . \; . \; . \quad (7)$$

This shows that: The center of gravity (x_c, y_c) moves exactly as if all the mass (ΣW) were there concentrated, and all the force $(\Sigma X, \Sigma Y)$ were there applied.

Now, the forces $\Sigma X, \Sigma Y$ contain not only the external forces applied to the system, but also all the actions and reactions between individual particles. Since, however, these actions and reactions are equal and opposite in pairs, they will all cancel out from the sums $\Sigma X, \Sigma Y$; and, hence, the forces effective in moving the center of gravity are only the external forces, which may be denoted by X_e,

Y_e. The definitive equations of motion for the center of gravity are, therefore,

$$(\Sigma W)\frac{d^2x_c}{dt^2} = g\Sigma X_e, \qquad (\Sigma W)\frac{d^2y_c}{dt^2} = g\Sigma Y_e, \quad . \quad . \quad . \quad . \quad (8)$$

and the theorem is that: The center of gravity moves as though all the mass were there concentrated, and all the external forces there applied.

38. Motion about the Center of Gravity. The moments of the forces X, Y applied at the point (x, y), are about the origin $+xY$ and $-yX$, it being understood that a moment is positive when it tends to turn the x-axis into the y-axis. The moment equation about the origin for each particle is

$$W\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = g(xY - yX) \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

A "moment" may be defined for the momentum, of which the components are $W\,dx/dt$, $W\,dy/dt$, in exactly the same way that a moment is defined for a force with components X, Y; and the *moment of momentum* is thus defined as

$$\text{moment of momentum} = W\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \quad . \quad . \quad . \quad (10)$$

This is sometimes called the *angular momentum* instead of the moment of momentum. It depends not only on the velocity of the particle, but also on the position, because the arms of the momenta are x and y. The moment of the force multiplied by g is equal to the rate of change of the moment of momentum because

$$\frac{d}{dt}\left[W\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right] = W\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = g(xY - yX) \quad . \quad . \quad . \quad (11)$$

Let the moment equations for all the particles be added. The result is

$$\Sigma\frac{d}{dt}\left[W\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right] = g\Sigma(xY - yX) \quad . \quad . \quad . \quad . \quad (12)$$

The equation states that: The rate of change of the total angular momentum of the system about the origin is equal to g times the total moment of the forces. Here, again, the forces may be divided into the external applied forces and the internal actions and reactions which occur in equal and opposite pairs, and which have no moment about any point because the equal and opposite forces, being directed along the line joining the particles, produce no couple; that is, the arm of any action and the corresponding reaction are the same. In taking moments, therefore, only the external forces need be considered, and the equation becomes

$$\Sigma \frac{d}{dt}\left[W\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right] = g\Sigma(xY_e - yX_e). \quad . \quad . \quad . \quad (13)$$

The theorem is that: The rate of change of the total angular momentum of the system is g times the total moment of the external forces acting on it. The theorem and the formulas hold when moments are taken about the origin, regarded as a fixed point. The origin may be anywhere, but it must be fixed. The theorem and the formulas also hold when the point about which moments are taken is the center of gravity, although this point may be in motion; that is, it is true that

$$\Sigma \frac{d}{dt}\left[W\left((x - x_c)\frac{d(y - y_c)}{dt} - (y - y_c)\frac{d(x - x_c)}{dt}\right)\right]$$
$$= g\Sigma[(x - x_c)Y_e - (y - y_c)X_e].$$

The expression on the left may be written as

$$\Sigma \frac{d}{dt}\left[W\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\right] - \Sigma \frac{d}{dt}\left[W\left(x_c\frac{dy_c}{dt} - y_c\frac{dx_c}{dt}\right)\right]$$
$$-\Sigma \frac{d}{dt}\left[W\left((x - x_c)\frac{dy_c}{dt} - (y - y_c)\frac{dx_c}{dt}\right)\right]$$
$$-\Sigma \frac{d}{dt}\left[W\left(x_c\frac{d(y - y_c)}{dt} - y_c\frac{d(x - x_c)}{dt}\right)\right].$$

By (4) $\Sigma W(x - x_c) = 0$, $\Sigma W(y - y_c) = 0$. By (5) $\Sigma Wd(x - x_c)/dt = 0$ and $\Sigma Wd(y - y_c)/dt = 0$. Hence, the last two terms are zero as the summation Σ extends over $W, x, y, dx/dt, dy/dt$ and $x_c, y_c, dx_c/dt, dy_c/dt$ are constant with respect to it. By (13) the first term is $g\Sigma(xY_e - yX_e)$ and if moments be taken on (8), the second term becomes $-g\Sigma(x_cY_e - y_cX_e)$. The equation is, therefore, proved. Hence, the rate of change of the angular motion about the center of gravity is equal to g times the moment of the external forces about the center of gravity.

In the case of a rigid body, the angular momentum about any point is the product of the moment of inertia about that point by the angular velocity; for if r be the distance from the point to any particle of the body, and ω be the angular velocity, the velocity of the particle is $r\omega$, and is perpendicular to the radius, r. The momentum is $Wr\omega$, and the moment of the momentum is $r(Wr\omega) = Wr^2\omega$. Now, ω is the same for all particles, and on summing, the total angular momentum is

$$(\Sigma Wr^2)\omega = I\omega, \quad \dots\dots\dots\dots \quad (14)$$

if I be the moment of inertia.

The moment equation is, therefore,

$$\frac{d}{dt}\left(I\frac{d\theta}{dt}\right) = g\Sigma(xY_e - yX_e), \quad \dots\dots\dots \quad (15)$$

for any rigid body when moments are taken about any fixed point. The equation, as has just been proved, holds also for the center of gravity, though that point be moving; hence,

$$\frac{d}{dt}\left(I_c\frac{d\theta}{dt}\right) = g\Sigma[(x - x_c)\,Y_e - (y - y_c)X_e] \quad \dots\dots \quad (16)$$

In the rigid body I_c is constant, and the formula states that: The moment of inertia about the center of gravity multiplied by the angular acceleration is equal to the moment of the external forces about the center of gravity multiplied by g.

The moment equation (16), taken with the equations for the motion of the center of gravity (8), give three equations regulating the three independent variables x_c, y_c, and θ.

Now, let x, y represent henceforth the coördinates of the center of gravity of a rigid body, W the total mass (the weight), X, Y, the total components of the external forces, M the moment of those forces about the center of gravity, $I = Wk^2$, the moment of inertia about the center of gravity, and $q = \omega = d\theta/dt$, the angular velocity. The equations of motion are

$$W\frac{d^2x}{dt^2} = gX, \qquad W\frac{d^2y}{dt^2} = gY, \qquad Wk^2\frac{d^2\theta}{dt^2} = gM \quad \dots \quad (17)$$

The theorem on the center of gravity enables the first two equations to be replaced if desired by the corresponding equations for the tangential and normal resolution of forces along the path; namely,

$$\frac{W\,d^2s}{dt^2} = gT, \qquad \frac{WV^2}{R} = gN, \quad \dots\dots\dots \quad (18)$$

where V is the velocity in path, R the radius of curvature, T and N the tangential and normal components of external force.

39. Oscillations of the Airplane. The forces acting upon the airplane, being aerodynamic in nature and caused by the rush of the machine through the air, are so complicated that it is difficult and at present impossible to give general theoretical or empirical expressions for those forces in analytic form; and, consequently, the general motion of the airplane in a plane cannot be solved. The problem of coming out of a nose dive, or of looping the loop in a vertical plane are beyond our present powers to solve directly. In normal flight the airplane may be regarded as traveling with constant velocity in a straight line; but in actual flight the motion is never exactly uniform in a straight line; there is always more or less slight pitching, yawing, rolling, more or less slipping from one side to the other, or rising and falling, and more or less irregularity in the forward motion. One of the problems which can be solved is that of these slight variations from uniform motion. For the present it will be assumed that there is no yawing, sideslipping, or rolling, but that there may be slight variations in the forward velocity, slight velocities perpendicular to the general direction of motion, and a small amount of pitching.

The importance of the study of the small motions of an airplane about its normal motion of uniform flight in a straight line lies in the connection between these motions and the stability of the airplane. Owing to the unevenness of the structure of the air and to other accidental causes, a machine can never fly uniformly in a straight line, but can at best only approximate such motion. It is always being slightly disturbed so that small oscillations are set up. If these oscillations are damped, the effect of the accidental disturbances will diminish in time, and the machine, apart from subsequent disturbances, would settle back to its normal flying attitude. Such a machine is called dynamically stable. If, however, the small oscillations should have amplifying instead of damping effects, the machine would depart more and more from its normal flight, and would be called dynamically unstable. By the constant attention of the pilot a machine dynamically unstable may be flown, and if the instability is not too great, without any serious danger. But the machine which is dynamically stable does not require so constant an attention on the part of the pilot, and may, indeed, fly itself for long periods of time. Such very great disturbances in the air may occasionally be found as will throw a machine dynamically stable so far from its natural flying attitude that it will not return. These great disturbances come under the class of problems which are not yet satisfactorily solved. Much, however, is accomplished if the question of oscillation set up by small disturbances is settled.

The air forces will be separated from the force of gravity and the propeller thrust. For the purpose of making the notation conform to that adopted for general motion in space, the Z-axis will be taken vertical, and the X-axis horizontal in normal flight, the X-axis being drawn backward from the point of view of the pilot. The equations of motion are, then,

$$\frac{W}{g}\frac{du}{dt} = X - T,$$
$$\frac{W}{g}\frac{dw}{dt} = Z - W, \qquad (19)$$
$$\frac{Wk^2}{g}\frac{d^2\theta}{dt^2} = M - hT,$$

where u, w are the velocities along the X- and Z-axes, and u is very nearly equal to a constant quantity, U, which is itself negative; where X, Z, M are the air forces, and the moment of these forces; where k is the radius of gyration about the axis through the center of gravity perpendicular to the plane of the motion; where h is the arm of the propeller thrust, counted positive when the line of the thrust passes above the center of gravity. The conditions for uniform flight are

$$u = U,\ w = 0,\ q = d\theta/dt = 0,\ X = T,\ Z = W,\ M = hT. \qquad (20)$$

It is assumed here that the propeller thrust is horizontal. This would be impossible for flights at any speed other than a particular speed. If j be the angle which the propeller thrust makes with the horizontal, the first two equations would be

$$\left.\begin{aligned}\frac{W}{g}\frac{du}{dt} &= X - T\cos j,\\ \frac{W}{g}\frac{dw}{dt} &= Z - W + T\sin j.\end{aligned}\right\} \qquad (21)$$

Now, suppose that the motion is not uniform, but differs slightly from the uniform motion. The forces will differ slightly from those in the uniform motion. The air forces X, Z, M depend on the wind velocities relative to the machine, and on the attitude of the machine;

that is, upon the velocities u, w of the center of gravity of the machine, the angular velocity $q = d\theta/dt$ of the machine about its center of gravity, and upon the angle θ through which the machine has turned. It is assumed that w, q, θ are all small, and that u differs by a small quantity u' from its standard value U. Any force X may be written, therefore, as

$$X + dX = X + \frac{\partial X}{\partial u}u' + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial \theta}\theta + \frac{\partial X}{\partial q}q, \quad . \quad . \quad (22)$$

because since the standard values of u', w, θ, q are 0, and the values in any case are small, the quantities themselves may be written in place of their differentials in the formula for the total differential. A similar formula holds for Z and M. Moreover, when the motion of the machine varies, the action of the propeller, which depends upon the relative motion of the propeller and the air, will also probably vary, and, consequently, the propeller thrust T must be considered as expansible in the same form as X, Z, and M. However, the propeller thrust will be considered for simplicity to be constant, particularly as data on the variation of the propeller thrust are not yet available.

When the machine is in the general position, the equations are

$$\left.\begin{aligned} \frac{W}{g}\frac{d(U+u')}{dt} &= X + \frac{\partial X}{\partial U}u' + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial \theta}\theta + \frac{\partial x}{\partial q}q - T\cos(j+\theta),\\ \frac{W}{g}\frac{dw}{dt} &= Z + \frac{\partial Z}{\partial u}u' + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial \theta}\theta + \frac{\partial Z}{\partial q}q - W + T\sin(j+\theta),\\ \frac{Wk^2}{g}\frac{d^2\theta}{dt^2} &= M + \frac{\partial M}{\partial U}u' + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial \theta}\theta + \frac{\partial M}{\partial q}q - hT. \end{aligned}\right\} \quad (23)$$

As U is constant, and $X = T\cos j$, $Z = W - T\sin j$, and $M = hT$, the equations may be simplified; and, further, as θ is small, $\cos\theta = 1$, and $\sin\theta = \theta$ may be assumed. The equations for the small oscillations are, therefore,

$$\left.\begin{aligned} \frac{W}{g}\frac{du}{dt} &= \frac{\partial X}{\partial u}u' + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial q}q + \frac{\partial X}{\partial \theta}\theta + T\theta\sin j,\\ \frac{W}{g}\frac{dw}{dt} &= \frac{\partial Z}{\partial u}u' + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial \theta}\theta + T\theta\cos j,\\ \frac{Wk^2}{g}\frac{d^2\theta}{dt^2} &= \frac{\partial M}{\partial u}u' + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \theta}\theta. \end{aligned}\right\} \quad . \quad (24)$$

These equations, be it remembered, are obtained on the assumption that T is constant.

These equations in the three variables u', w, θ are linear, differential equations with constant coefficients, since all the derivatives are taken for the values

$$u = U, \qquad w = q = \theta = 0.$$

Before integrating the equations to determine the character of the oscillations it is advantageous to explain the meaning of each of the partial derivatives which occurs, and to describe a method of determining the value of that derivative for any particular machine.

(NOTE. The understanding of the work of the next article will be facilitated by a study of Chap. XII under Fluid Mechanics.)

40. Determination of the Coefficients. (*a*) The air forces on an airplane are supposed to vary with the square of the velocity. For example,

$$X = ku^2 = k(U + u')^2 = kU^2 + 2kUu' + ku'^2.$$

The last term may be neglected because it involves the square of the small quantity u', and is, therefore, an infinitesimal of the second order. The change in X is, therefore, $2kUu'$; and if this be divided by the change in u, namely, $u' = du$, the result is

$$\frac{\partial X}{\partial u} = 2kU = \frac{2kU^2}{U} = \frac{2X}{U} \quad \ldots \ldots \quad (25)$$

Thus, the first coefficient is determined as twice the X force for uniform velocity divided by that velocity. This is a negative quantity because U is negative. In the same way, Z and M vary with the square of the velocity, and

$$\frac{\partial Z}{\partial u} = \frac{2Z}{U} \quad \text{and} \quad \frac{\partial M}{\partial u} = \frac{2M}{U} \quad \ldots \ldots \quad (26)$$

Another way of writing these expressions is by introducing the values of X, Z and M in terms of T, W, and h; namely,

$$\frac{\partial X}{\partial u} = 2\frac{T\cos j}{U}, \quad \frac{\partial Z}{\partial u} = 2\frac{W - T\sin j}{U}, \quad \frac{\partial M}{\partial u} = 2\frac{hT}{U}. \quad (27)$$

The values of X, Z, M for any particular attitude of the machine may be obtained from the tabulated values of the lift, drag, and moment, as determined in wind tunnel experimentation; for it is precisely the values of X, Z, and M which are determined by the experiments on the model at the speed at which the experiment is run. It is simply necessary to determine for the attitude of the machine under investigation what is the speed U, or, if the speed U be given what is the attitude. This is found by comparing the lift with the weight. (See Chap. XII.) An actual calculation is carried through in Art. 43, step for step parallel with the theoretical discussion of this Article.

(*b*) The value θ represents a change in the attitude of the machine; and, consequently, the derivatives $\partial X/\partial\theta$, $\partial Z/\partial\theta$, $\partial M/\partial\theta$ are obtained by differentiating, or rather differencing the tabulated values for X, Z, M for the model after scaling the values up to those appropriate to the full-sized machine, running at the appropriate speed U.

(*c*) The values of $\partial X/\partial w$, $\partial Z/\partial w$, and $\partial M/\partial w$ are also obtained from the lift, drag, and moment tables for the model, but the calculation is less simple. A velocity w combined with the forward velocity U gives the machine a resultant velocity slightly inclined to the direction of U, the tangent of this angle being w/U in magnitude. This is the same as though the relative wind were less inclined to the aerofoil by the amount $\tan^{-1}(w/U)$, or, since w is small, by the

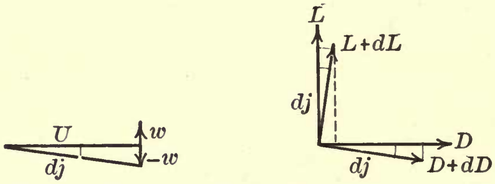

Fig. 14. Change of Forces with Cross-wind Motion.

amount w/U radians (see Art. 11). Hence, the velocity w is equivalent to the change in attitude of $57.3w/U$ degrees. The drag and lift tables give the forces along the wind and perpendicular to it. When the relative wind changes its direction, the lift L and drag D change theirs; this change in direction must be taken into account in calculating the change in X and Z. If the change be dj, the new values of X and Z are (Fig. 14)

$$\begin{aligned} X + dX &= (D + dD)\cos dj + (L + dL)\sin dj \\ &= D + dD + L\,dj, \\ Z - dZ &= (L + dL)\cos dj - (D + dD)\sin dj \\ &= L + dL - D\,dj, \quad \ldots\ldots\ldots\ldots \quad (28) \end{aligned}$$

if infinitesimals of higher order be neglected. As $X = D$, $Z = L$, and $dj = -w/U$, since U is negative, and the change, di, in attitude of the machine as entered in the drag and lift tables is $-57 \cdot 3\ dj$ in degrees, therefore,

$$dX = \left(\frac{57 \cdot 3}{U}\frac{dD}{di} - \frac{L}{U}\right)w, \qquad dZ = \left(\frac{57 \cdot 3}{U}\frac{dL}{D} + \frac{D}{U}\right)w, \quad . \quad (29)$$

and, hence, since $w = dw$,

$$\frac{\partial X}{\partial w} = \frac{57 \cdot 3}{U}\frac{dD}{di} - \frac{L}{U}, \qquad \frac{\partial Z}{\partial w} = \frac{57 \cdot 3}{U}\frac{dL}{di} + \frac{D}{U} \quad . \quad . \quad . \quad (30)$$

The coefficient dM/dw is obtained by differentiating, or, rather, by differencing the moment curve, using for the difference di the value w/U, the rate of change, or $57 \cdot 3w/U$ degrees. Hence,

$$\frac{\partial M}{\partial w} = +\frac{57 \cdot 3}{U}\frac{dM}{di} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(*d*) There remains to be determined the values of $\partial X/\partial q$, $\partial Z/\partial q$, $\partial M/\partial q$, which are the rates of change of the forces and moment with respect to angular velocity. If the machine in its normal attitude has a small angular velocity q the parts of the machine in front of the center of gravity have a small velocity in a general upward direction proportional to q, and to their distance from the C. G., whereas the parts behind the C. G. have a small downward velocity proportional to q and to their distance from the C. G. This means that for surfaces in front of the C. G. the effective angle of the relative wind is diminished, whereas for surfaces behind the center of gravity it is increased; but the stream lines around the body are so irregular, and their position so poorly known, that an accurate calculation of the effect of the rotation cannot be made from the lift, drag, and moment tables (cf. Art. 77). The change in M, for instance, is due partly to the down sweep of the tail, and partly to the travel of the center of pressure and the variation in the pressure on the wings. It is possible, however, to determine $\partial M/\partial q$ by observing the damping in a model oscillating about an axis located at the position of the center of gravity of the machine.

Let the model be suspended upon an axis and held in the wind in the tunnel in the position which the machine takes in flight, by springs which produce a restoring moment so that any slight derangement of the model tends to be corrected. If the model be then slightly displaced it will oscillate about the axis under the action of the restoring moment, due to the springs, and under the action of the wind forces upon the model. As the experiment on the model is to be conducted at different wind velocities in the tunnel, it is necessary to have the springs adjustable so that for all wind velocities the model may be held in the correct attitude toward the wind.

Let I_m be the moment of inertia of the model, and of any portion of the system by means of which the model is suspended, provided that portion oscillates with the model. The equation of motion for the oscillation is

$$I_m \frac{d^2\theta}{dt^2} = -g\left(c\theta + \frac{b\,d\theta}{dt}\right), \quad \quad (32)$$

where $c\theta$ is the restoring moment due to the action of the springs, and $b\,d\theta/dt$ is the frictional moment due partly to the action of the wind on the model and partly to the mechanical friction, which would be present even if the model oscillated in a vacuum. The solution for θ is of the type

$$\theta = Ce^{-\alpha t} \sin(\beta t + \Gamma),$$

where C and Γ are constants of the integration, and where

$$\alpha = \frac{gb}{2I_m}, \quad \beta = \sqrt{\frac{gc}{I_m} - \frac{g^2b^2}{4I_m^2}} \quad \quad (33)$$

The value of I_m is independent of the wind velocity in the tunnel. So is the value of c, the restoring moment due to the springs, provided Hooke's Law be assumed; for no matter how loosely or tightly the springs must be adjusted to keep the model in the proper attitude for any particular wind velocity in the tunnel, the restoring moment due to the springs for a slight displacement is always the same for the same displacement, if the force due to the spring is strictly proportional to the extension. On the other hand, b varies with the wind velocity from the small value of b_0, which is due to the mechanical friction alone, to a large and increasing value b as the wind velocity is increased. In all cases the damping must be sufficiently small so that a considerable number of oscillations may be executed without the model coming practically to rest, for otherwise it would be impossible to make accurate observations on the damping coefficient. It is, therefore, possible to use the approximate formulas

$$\beta = \sqrt{\frac{gc}{I_m}}, \quad T = \frac{2\pi}{\beta}, \quad \frac{gc}{I_m} = \frac{4\pi^2}{T^2}, \quad \quad (34)$$

where T is the time of a complete oscillation. These formulas will determine either c or I_m if the other is known.

The ratio of the amplitude of the oscillation at the start and after n complete oscillations is $r = e^{anT}$ and the logarithm of that ratio is

$$\log_e r = anT = \frac{ngbT}{2I_m} \quad \text{or} \quad b = \frac{2I_m \log_e r}{ngT} \quad . \ . \ (35)$$

Thus, by timing the oscillation and observing the damping the value of b may be obtained. This value contains three damping moments, namely, the mechanical damping, the damping due to the wind action upon other parts of the oscillating mechanism than the model itself, and, third, the damping due to the model. It is possible to measure the mechanical damping by allowing the model to oscillate in no wind. It is possible to determine the damping upon other parts of the apparatus by allowing the apparatus to oscillate without the model, and, thus, the net damping due to the model itself may be determined.

A rough calculation $\partial M/\partial q$ may be made by assuming that the damping moment is due mainly to the action of the tail. If the tail surface be S at a distance a behind the center of gravity, an angular velocity q will give the tail a velocity $w = -aq$. If the tail were originally neutral, the angle between the tail and the wing would become approximately w/U, and the normal pressure upon the tail would be

$$P = k\frac{W}{U}SU^2,$$

and the couple about the center of gravity would be aP. Hence, this couple would be

$$dM = ka\frac{W}{U}SU^2 = ka^2SUq, \qquad \frac{\partial M}{\partial q} = ka^2SU, \ . \ . \ . \ (36)$$

since $q = dq$. Here $k = .0015\ (.032 + .005r) \times 57.3$ for radian measure.

This formula shows that the coefficient $\partial M/\partial q$ varies directly with the speed, and directly as the fourth power of the linear dimension; and these laws are used to scale up the value $\partial M/\partial q$ from model experiments to the full-sized machine, the value for the model being multiplied by the fourth power of the ratio of the linear dimensions, and by the velocity ratio. Experiment shows, too, that the value of the coefficient does, in reality, vary almost exactly as the linear velocity within the range of velocities used in the wind tunnel. If the tail area be 30 sq. ft. placed 15 ft. behind the center of gravity, then for a velocity $U = -100$ ft/sec.,

$$\frac{\partial M}{\partial q} = .0015 \times \frac{57.3}{25} \times 225 \times 30 \times (-100) = -2320,$$

if the aspect ratio of the tail be taken as about $1\frac{1}{2}$.

There remain to be found $\partial X/\partial q$ and $\partial Z/\partial q$. The presumption is that $\partial X/\partial q$ is small, for a rotary velocity should not influence very greatly the value of the X force. The angle between the surfaces and the wind is diminished in front of the center of gravity, and increased behind it, and ought, more or less, to balance out. In fact, if the calculation just given for $\partial M/\partial q$ for the tail be applied, it will be seen that the variation in the X force is zero for a flat tail, because the pressure for a changed angle of the wind remains perpendicular to the tail. For the curved wing the change in the direction of the wind will introduce a slight change in the X force, but the wing surfaces are very near to the center of gravity, and the effect should be small. If the calculation be applied to determine $\partial Z/\partial q$, the result found is that

$$\frac{\partial Z}{\partial q} = -\frac{1}{a}\frac{\partial M}{\partial q} = 155.$$

As for the aerofoil itself, the change in the angle is small compared to that for the tail because the surface is so much nearer the center of gravity; and as the surface of the aerofoil itself is partly in front and partly behind the center of gravity, the chances are that even taking into account the large surface involved, the contribution to $\partial Z/\partial q$ is not great. Moroever, both $\partial X/\partial q$ and $\partial Z/\partial q$ occur, as will be seen, in the equation of motion in a way which renders them relatively unimportant for the motion.

41. Integration of the Equations. The linear equations (24) become

$$\left.\begin{aligned} \frac{du'}{dt} - a_1u' - a_2w - a_3\frac{d\theta}{dt} - a_4\theta &= 0, \\ -b_1u' + \frac{dw}{dt} - b_2w - b_3\frac{d\theta}{dt} - b_4\theta &= 0, \\ -c_1u' - c_2w + \frac{d^2\theta}{dt^2} - c_3\frac{d\theta}{dt} - c_4\theta &= 0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

where

$$a_1 = \frac{g}{W}\frac{\partial X}{\partial u}, \quad a_2 = \frac{g}{W}\frac{\partial X}{\partial w}, \quad a_3 = \frac{g}{W}\frac{\partial X}{\partial q}, \quad a_4 = \frac{g}{W}\left(\frac{\partial X}{\partial \theta} + T\sin j\right),$$

$$b_1 = \frac{g}{W}\frac{\partial Z}{\partial u}, \quad b_2 = \frac{g}{W}\frac{\partial Z}{\partial w}, \quad b_3 = \frac{g}{W}\frac{\partial Z}{\partial q}, \quad b_4 = \frac{g}{W}\left(\frac{\partial Z}{\partial \theta} + T\cos j\right),$$

$$c_1 = \frac{g}{Wk^2}\frac{\partial M}{\partial u}, \quad c_2 = \frac{g}{Wk^2}\frac{\partial M}{\partial w}, \quad c_3 = \frac{g}{Wk^2}\frac{\partial M}{\partial q}, \quad c_4 = \frac{g}{Wk^2}\frac{\partial M}{\partial \theta},$$

the coefficients a, b, c being introduced merely for brevity.

The set of equations in u', w, θ are simultaneous linear equations with constant coefficients. The method of solving these equations is to assume that each of the variables may be represented as a multiple of an exponential function; namely,

$$u' = C_1e^{\lambda t}, \quad w = C_2e^{\lambda t}, \quad \theta = C_3e^{\lambda t}.$$

Then

$$\frac{du'}{dt} = \lambda C_1e^{\lambda t}, \qquad \frac{dw}{dt} = \lambda C_2e^{\lambda t}, \qquad \frac{d\theta}{dt} = \lambda C_3e^{\lambda t}, \qquad \frac{d^2\theta}{dt^2} = \lambda^2C_3e^{\lambda t}.$$

If these assumed values of the variables and their derivatives be substituted in the equations, the exponential expression will cancel out all the way through, and leave the following three simultaneous linear algebraic equations homogeneous in the three unknowns C_1, C_2, C_3:

$$\left.\begin{aligned} (\lambda - a_1)C_1 - a_2C_2 - (a_3\lambda + a_4)C_3 &= 0, \\ -b_1C_1 + (\lambda - b_2)C_2 - (b_3\lambda + b_4)C_3 &= 0, \\ -c_1C_1 - c_2C_2 + (\lambda^2 - c_3\lambda - c_4)C_3 &= 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (38)$$

Now three homogeneous equations in three unknowns such as these cannot in general have a solution except the useless one

$$C_1 = C_2 = C_3 = 0,$$

for if the equations be divided through by one of the unknowns, say, C_3, they become a set of three linear equations, in the two unknowns C_1/C_3, C_2/C_3. Two such equations afford a solution for C_1/C_3 and C_2/C_3, and if the third equation is to be true, the result of this solution when substituted in that equation must check. In the notation of determinants the condition that the equations may have a solution is

$$\Delta = \begin{vmatrix} \lambda - a_1 & -a_2 & -a_3\lambda - a_4 \\ -b_1 & \lambda - b_2 & -b_3\lambda - b_4 \\ -c_1 & -c_2 & \lambda^2 - c_3\lambda - c_4 \end{vmatrix} = 0 \quad . \quad . \quad . \quad (39)$$

The rule for expanding a determinant is

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 + \beta_1\gamma_2\alpha_3 + \gamma_1\alpha_2\beta_3 - \gamma_1\beta_2\alpha_3 - \alpha_1\gamma_2\beta_3 - \beta_1\alpha_2\gamma_3, \quad . \; . \quad (40)$$

and when this is applied to the determinant involving λ the result is

$$\Delta = \lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0 \quad . \; . \; . \; . \quad (41)$$

where

$$B = -a_1 - b_2 - c_3,$$
$$C = -c_4 + a_1c_3 + b_2c_3 + a_1b_2 - c_2b_3 - b_1a_2 - c_1a_3,$$
$$D = a_1c_4 + b_2c_4 - c_2b_4 - c_1a_4 + a_1c_2b_3 + b_1a_2c_3 + c_1b_2a_3 - a_1b_2c_3 - b_1c_2a_3 - c_1a_2b_3,$$
$$E = a_1c_2b_4 - a_1b_2c_4 + b_1a_2c_4 - b_1c_2a_4 + c_1b_2a_4 - c_1a_2b_4.$$

This is an equation of the fourth degree — a biquadratic equation in λ, which will in general have four roots. To each root will correspond a possible type of motion for the airplane of the form $e^{\lambda t}$. If two roots of the equation are conjugate imaginaries, the two will be treated together, say as $\mu \pm \nu i$, and the corresponding type of motion will be of the form $e^{\mu t}(K_1 \cos \nu t + K_2 \sin \nu t)$. Such a pair of roots corresponds to an oscillation of the machine of total periodic time $T = 2\pi/\nu$, and with a time $t = \pm .693/\mu$ to amplify to double-amplitude or damp to half-amplitude.

42. Stability. If the machine when slightly disturbed is to return to its normal flying attitude, it is necessary that the real roots λ of the equation $\Delta = 0$ shall be negative, so that the exponential expression $e^{\lambda t}$ shall decrease with the time instead of increase; and it is further necessary that the real part μ of any pair of imaginary roots shall be negative so that the oscillation shall have a damping, not an amplifying, factor. Now, the solution of a biquadratic equation may be obtained by various approximations when the coefficients are known numerical quantities; but the solution is tedious. To determine whether or not there is stability, that is, whether or not the real roots and the real parts of imaginary roots are negative, it is sufficient to apply Routh's conditions, which are that all the coefficients in the biquadratic equation (41) and the expression

$$R = BCD - D^2 - B^2E \quad . \; . \; . \; . \; . \; . \; . \; . \quad (42)$$

shall be positive.

It is of great importance to distinguish between static and dynamic stability. An object at rest is stable if, when slightly displaced from its position of equilibrium, it tends to return to that position. The condition for static equilibrium is that for any imagined infinitesimal displacement of the body from its equilibrium position, there shall arise a restoring force that is negative. This kind of equilibrium is familiar. Dynamic stability has to do with bodies in motion, particularly in steady motion, and a body is said to possess dynamic stability if, when disturbed from a state of motion, it tends to return to that state. Thus the airplane in uniform horizontal flight is stable (dynamically) if the airplane, after being slightly disturbed in its velocity or angular velocity, tends to resume its condition of uniform horizontal flight with its original speed. The disturbed motion leads to differential equations which, as seen above, are linear with constant coefficients and of which the solutions are exponential expressions either real or imaginary. In case the exponential is real it either increases indefinitely with time or falls off indefinitely toward zero — the latter being required for stability. In case the exponential is imaginary it represents an oscillatory function which may amplify or damp — the latter being required for stability. The conditions for stability above are

$$B>0, \quad C>0, \quad D>0, \quad E>0, \quad R>0 \quad . \quad . \quad . \quad (43)$$

The application of this criterion requires the calculation of the aerodynamic coefficients in (38) and their substitution to find B, C, D, E, R.

In order, however, to tell how great is the damping (or amplifying) it is necessary actually to solve the biquadratic (42) by some approximate process. Now in the case of the airplane experience has shown that a fair literal approximation may be had. It is easy to surmise that the airplane should not experience nearly so great a resistance in moving slightly to and fro in its direction of motion as in moving up and down across its line of flight; for in the first case the surfaces are so largely parallel to the wind that the air may blow through, whereas in the second case the air is more seriously disturbed. Hence it might be inferred that there is one type of motion heavily damped, another lightly damped. Although such an argument proves nothing, it does give a valuable indication as to the method of attack in solving the biquadratic. For if two of the roots for λ are large

$\lambda^4 + B\lambda^3 + C\lambda^2$ should be large compared with $D\lambda + E$, and a trial equation for these roots would be

$$\lambda^4 + B\lambda^3 + C\lambda^2 = 0 \quad \text{or} \quad \lambda^2 + B\lambda + C = 0, \quad . \; . \quad (44)$$

with

$$\lambda = -\tfrac{1}{2}B \pm \sqrt{\tfrac{1}{4}B^2 - C} \quad . \; . \; . \; . \; . \; . \; . \quad (45)$$

On the other hand if two of the roots for λ are small, the higher powers of λ should be comparatively negligible and a trial equation for these is

$$C\lambda^2 + D\lambda + E = 0 \quad \text{with} \quad \lambda = -\frac{D}{2C} \pm \sqrt{\frac{D^2}{4C^2} - \frac{E}{C}} \quad . \; . \quad (46)$$

These assumptions amount to factoring the biquadratic into

$$(\lambda^2 + B\lambda + C)\left(\lambda^2 + \frac{D}{C}\lambda + \frac{E}{C}\right) = 0 \quad . \; . \; . \; . \; . \quad (47)$$

Experience seems to show that a closer factorization is given by

$$(\lambda^2 + B\lambda + C)\left[\lambda^2 + \left(\frac{D}{C} - \frac{BE}{C^2}\right)\lambda + \frac{E}{C}\right] = 0, \quad . \; . \quad (48)$$

and this form will be ordinarily taken as affording a sufficiently good approximation.

43. Illustrative Calculation for Stability. The case considered will be that of the "Clark" treated by Hunsaker in *Smithsonian Miscellaneous Collections*, vol. 62, No. 5, June 1916, pp. 1–78 in a discussion entitled Dynamical Stability of Aeroplanes. The machine has for weight and radius of gyration

$$W = 1600 \text{ lb}, \qquad k = 4.65 \text{ ft};$$

the model was $\frac{1}{26}$ size and tunnel tests were run at 30 mi/hr. The lift and drag in pounds in the model and the moments in pound-inches are as follows:

i	L	ΔL	D	ΔD	M	ΔM
−4	−.115		.128		.26	
		.227		−.020		−.10
−2	+.112		.108		.16	
		.128		−.004		−.04
−1	.240		.104		.12	
		.120		−.003		−.00
0	.360		.101		.12	
		.130		+.001		−.05

i	L	ΔL	D	ΔD	M	ΔM
+1	.490		.102		.07	
		.135		.003		−.08
2	.625		.105		−.01	
		.247		.010		−.11
4	.872		.115		−.12	
		.43		.038		−.16
8	1.30½		.153		−.28	
		.27		.060		−.13
12	1.57		.213		−.39	
		.07		.157		−.14
16	1.64		.370		−.53	

These data may be plotted and a fair curve may be drawn through the points — but that is not essential. The reliability of such data may be estimated by consulting the differences which should proceed in an orderly fashion when referred to the same difference of angle *i*. The differences ΔL are per degree as follows:

11, 13, 12, 13, 13½, 12, 11, 7, 2.

These form a satisfactory sequence. A greater regularity would have been indicated if the first 3 had been 11, 12, 13. When a curve is plotted and faired, the irregularities are somewhat smoothed out. The fairing process may equally well be applied arithmetically in the table. Only this caution must be observed: No more fairing is permissible than remains within the experimental errors. How much this is can only be determined by a careful "precision" discussion of the method and apparatus used in arriving at the final data. By making several runs for the same data the differences between the values found in the different runs will give an idea of the accidental errors in the work; the systematic errors could be checked in a similar way by comparing results found by different observers using different methods and different styles of apparatus. There is not yet available the material for a satisfactory comparison.

The differences for D are per degree

−10, −4, −3, +1, 3, 5, 10, 15, 39,

and form a satisfactory sequence. The large jump from 5 to 10 is not suspicious because the interval has been increased from 2 to 4 units in the table, where the differences are increasing. At degree intervals the differences might look like this:

−13, −7, −4, −3, 1, 3, 4, 6, 7, 9, 10, 12, 14, 15, 15, 16, 20, 30, 45, 62.

Different persons would fair the table differently just as different draughtsmen would fair the plotted curve differently. The data for M give per degree

$$-5,\ -4,\ 0,\ -5,\ -8,\ -5\tfrac{1}{2},\ -4,\ -3,\ -3\tfrac{1}{2}.$$

This is an unsatisfactory set of differences, particularly with respect to the 0 and the 8.

Consider another set of data for another model, that of the Curtis JN-2, obtained by Hunsaker and published in the *First Annual Report of the National Committee for Aeronautics* (Washington), 1916, pp. 23–51. For this machine

$$W = 1800 \text{ lb}, \qquad k = 5.83 \text{ ft}.$$

The model was $\frac{1}{24}$ size and turned tests were run at 30 mi/hr. The lift and drag are in lb, and moments in lb.in.

i	L	ΔL	D	ΔD	M	ΔM
$-4\frac{1}{2}$	$-.130$		.123		.21	
		.210		$-.018$		$-.03$
$-2\frac{1}{2}$	$+.080$		.105		.18	
		.220		$-.003$		$-.03$
$-\frac{1}{2}$	.300		.102		.15	
		.210		$+.003$		$-.05$
$+1\frac{1}{2}$	.510		.105		.10	
		.105		.005		$-.02$
$2\frac{1}{2}$	.615		.110		.08	
		.100		.005		$-.05$
$3\frac{1}{2}$	.715		.115		.03	
		.095		.007		$-.03$
$4\frac{1}{2}$	.810		.122		.0	
		.100		.008		$-.08$
$5\frac{1}{2}$	.910		.130		$-.08$	
		.18		.027		$-.10$
$7\frac{1}{2}$	1.09		.157		$-.18$	
		.28		.095		$-.22$
$11\frac{1}{2}$	1.37		.252		$-.40$	
		.11		.156		$-.20$
$15\frac{1}{2}$	1.48		.408		$-.60$	
		.01		.153		$-.16$
$19\frac{1}{2}$	1.49		.561		$-.76$	

The differences in L per degree are

$$10\tfrac{1}{2},\ 11,\ 11\tfrac{1}{2},\ 10\tfrac{1}{2},\ 10,\ 9\tfrac{1}{2},\ 10,\ 9,\ 7,\ 3,\ 0.$$

The series is satisfactory like the other L set. The differences for D per degree are

$$-9, -1\tfrac{1}{2}, +1\tfrac{1}{2}, 5, 5, 7, 8, 13\tfrac{1}{2}, 24, 39, 38.$$

These are tolerably satisfactory like those above for D. The set for M is

$$-1\tfrac{1}{2}, -1\tfrac{1}{2}, -2\tfrac{1}{2}, -2, -5, -3, -8, -5, -5, -5, -4,$$

and are, like the previous set, not so satisfactory as could be desired. Further the breaks do not come in similar places in the two cases; so that the inference must be either that different models exhibit queer irregularities in M in different parts of the range of observation or that the data for M in both cases have considerable inaccuracies in them. The latter inference is the more reasonable because the transformation from M_s, the moment about the spindle, to M involves the subtraction of numbers of nearly the same magnitude (see Art. 77).

(*a*) To calculate (25) and (26) for the "Clark" at $i = 0°$. The model is $\frac{1}{26}$ size. The speed V is found from

$$W = 1600 = .360 \times (26)^2 \times (V/30)^2, \qquad V = 76.9\,.$$

The value of U is, therefore, $U = -112\tfrac{1}{2}$. The percentage error in U will be half the percentage error in $L = .360$.

$$\frac{\partial X}{\partial u} = \frac{2X}{U} = 2(.101) \times (26)^2 \times (76.9/30)^2 \div (-112\tfrac{1}{2}) = -7.9,$$

$$\frac{\partial Z}{\partial u} = \frac{2W}{U} = \frac{2 \times 1600}{-112\frac{1}{2}} = -28.5, \qquad \frac{\partial M}{\partial u} = 0.$$

The assumption is made that the propeller thrust is horizontal so that it has no lifting effect; also that the moment is zero at $i = 0$ because of the use of the elevator and because the propeller thrust is assumed to pass through the C. G. so that $L = 0$. Notice that in calculating $\partial M/\partial u$, the tabulated value of M is not used. In order to fly at any given attitude the value of M, the moment of the air forces, has to be regulated not by a table but by the pilot with the elevator!

(*b*) To calculate the values of the derivatives with respect to θ. As θ is in radians and i is in degrees,

$$\frac{\partial X}{\partial \theta} = 57.3\,\frac{\partial X}{\partial i} = 57.3\,\frac{\Delta X}{\Delta i} = 57.3\,\frac{\Delta D}{\Delta i} \times (26)^2 \times \left(\frac{76.9}{30}\right)^2.$$

Now at $i = 0$, $\Delta D = -.001$ if the average of the adjacent differences $-.003$ and $+.001$ be taken for the value at $i = 0$. Hence

$$\frac{\partial X}{\partial \theta} = -255, \quad \text{and} \quad \frac{\partial Z}{\partial \theta} = 31{,}900,$$

$$\frac{\partial M}{\partial \theta} = \frac{57.3}{12} \frac{\Delta M}{\Delta i} \times (26)^3 \times \left(\frac{76.9}{30}\right)^2 = -17{,}000,$$

if ΔM be taken from the table as $-.03$. An average of .00 and .05 would be .025, but the adjacent differences are .04 and .08 and indeed if the whole range from -2° to $+2^\circ$ be taken, ΔM figures as over .04 per degree. It is very difficult to say what value should be taken for ΔM at $i = 0^\circ$; the value of $\partial M/\partial \theta$ estimated from the given table might be anywhere from $-10{,}000$ to $-25{,}000$. The values of these three derivatives may be obtained from the plotted curves of L, D, M by estimating the slope at $i = 0$. The slope, however, on a fair curve depends very much on the way it is faired and is just as liable to error as an arithmetical estimate from the tabulated values.

(*c*) Find next the derivatives with respect to w.

$$\frac{\partial X}{\partial w} = \left(\frac{57.3}{-112\frac{1}{2}} \frac{\Delta D}{\Delta i} + \frac{L}{112\frac{1}{2}}\right)(26)^2\left(\frac{76.9}{30}\right)^2.$$

At $i = 0$, $\Delta D/\Delta i = -.001$, $L = .360$. Hence

$$\frac{\partial X}{\partial w} = \frac{(26)^2}{112\frac{1}{2}}\left(\frac{76.9}{30}\right)^2\left(.360 + .057\right) = 16.5,$$

$$\frac{\partial Z}{\partial w} = \left(\frac{57.3}{-112\frac{1}{2}} \frac{\Delta L}{\Delta i} - \frac{D}{112\frac{1}{2}}\right)(26)^2\left(\frac{76.9}{30}\right)^2 = -287,$$

$$\frac{\partial M}{\partial w} = \frac{57.3}{-112\frac{1}{2}} \frac{\Delta M}{\Delta i} \frac{(26)^3}{12}\left(\frac{76.9}{30}\right)^2 = 151.$$

(*d*) Finally the derivatives by q must be found. The value of $\partial M/\partial q$ is had by an oscillation experiment. For the model by (36) the value of $b = -(\partial M/\partial q)_m$ may be calculated if the total time nT is observed in which the oscillation damps from a given amplitude to $1/r$ of that. In the experiments $r = 9$. The value of I_m itself was small compared with I_a, the moment of inertia of the apparatus to which the model was attached. Indeed

$$I_a = 1.18, \quad I_{a+m} = 1.26\tfrac{1}{2}, \quad I_m = .085.$$

Moreover, it is of course necessary to eliminate the mechanical friction so that only the wind friction remains for $\partial M/\partial q$. Let b, therefore, be divided into three parts, b_0 due to mechanical friction, b_a due to wind friction on the apparatus, and b_m due to the wind friction on the model, — then $b = b_0 + b_a + b_m$. The apparatus being set to oscillating in no wind the time $nT = 105$ seconds was required to damp to $\frac{1}{9}$. Hence

$$b_0 = \frac{2 \times 1.18 \times \log_e 9}{105 \times 32.2} = .00154.$$

Next the apparatus was oscillated in a 30 mi wind with $nT = 94$ secs observed. Hence

$$b_0 + b_a = \frac{2 \times 1.18 \times \log_e 9}{94 \times 32.2} = .00172.$$

Then the model was placed in the altitude $i = 0°$ and oscillated in the wind. The damping time nT was 17.5 secs. Hence

$$b_0 + b_a + b_m = \frac{2 \times 1.26\frac{1}{2} \times \log_e 9}{17.5 \times 32.2} = .00993.$$

Therefore, for the model

$$b_m = -\left(\frac{\partial M}{\partial q}\right)_m = -(.00993 - .00172) = -.0082.$$

It remains to scale this value up. Now (36) shows that $\partial M/\partial q$ varies as the fourth power of linear dimension ($a^2 S$) and as the velocity U. Hence, for the machine

$$\frac{\partial M}{\partial q} = \left(\frac{\partial M}{\partial q}\right)_m \times (26)^4 \times \left(\frac{76.9}{30}\right) = -9610.$$

The value of $\partial X/\partial q$ is small, that of $\partial Z/\partial q$ may be roughly estimated as around $-\frac{1}{20}$ of $\partial M/\partial q$ or say 500. The value will enter into the equation in such a way as not to make a large effect. This will be seen in the ensuing calculation, and the general discussion of large versus small quantities will be taken up in Chap. VIII.

(*e*) Come now the equations of motion (37). The coefficients are (since $g/W = 32/1600 = 1/50$, near enough):

$$\begin{array}{llll} a_1 = -.158, & a_2 = +.330, & a_3 = 0, & a_4 = -5.10. \\ b_1 = -.570, & b_2 = -5.74, & b_3 = 10, & b_4 = 646. \\ c_1 = 0, & c_2 = +.140, & c_3 = -8.90, & c_4 = -15.7. \end{array}$$

The determinant in λ is

$$\Delta = \begin{vmatrix} \lambda + .158 & -.330 & 5.10 \\ .570 & \lambda + 5.74 & -10\lambda - 646 \\ 0 & -.140 & \lambda^2 + 8.90\lambda + 15.7 \end{vmatrix} = 0.$$

Observe that $15.7 = .14 \times 112\frac{1}{2}$ and that $646 = 5.74 \times 112\frac{1}{2}$ and that $5.10 = .33 \times 112\frac{1}{2} - 32$, because always

$$\frac{\partial Z}{\partial \theta} + T = U\frac{\partial Z}{\partial w}, \qquad \frac{\partial M}{\partial \theta} = U\frac{\partial M}{\partial w}, \qquad \frac{\partial X}{\partial \theta} = U\frac{\partial X}{\partial w} - W. \quad (49)$$

It is a rule of determinants that the determinant is not altered if one column multiplied by any number is added to another column term for term. If the middle row of the above determinant be multiplied by $112\frac{1}{2}$ and added to the last column, then

$$\Delta = \begin{vmatrix} \lambda + .158 & -.330 & -32 \\ .570 & \lambda + 5.74 & 102\lambda \\ 0 & -.140 & \lambda^2 + 8.90\lambda \end{vmatrix} = 0.$$

This form of Δ is simpler to expand. Then

$$\begin{aligned} \Delta = (\lambda + .158)[(\lambda + 5.74)(\lambda^2 + 8.9\lambda) + .14 \times 102\lambda] \\ + .57[.14 \times 32 + .33(\lambda^2 + 8.9\lambda)] \\ = \lambda^4 + 14.5\lambda^3 + 67.9\lambda^2 + 12.0\lambda + 2.55 = 0. \end{aligned}$$

Hence,

$$B = 14.5, \qquad C = 67.9, \qquad D = 12.0, \qquad E = 2.55,$$

and

$$R = BCD - D^2 - B^2E > 0.$$

The machine is stable dynamically (as well as statically).

(*f*) Finally the roots λ must be formed to ascertain the periods of oscillation and the damping factors. The short oscillation, heavily damped, is obtained from

$$\lambda^2 + 14.5\lambda + 67.9 = 0, \qquad \lambda = -7.25 \pm 3.92i.$$

The periodic time is $T = 2\pi/3.92 = 1.6$ sec. The time to damp to half amplitude is $t = \log_e 2/7.25 = .095$ sec. The oscillation is indeed rapid and heavily damped; it would be only $(\frac{1}{2})^{16.8}$ or about 10^{-5} after one oscillation! The long weakly damped oscillation is obtained from (48) as

$$\lambda^2 + \left(\frac{12}{67.9} - \frac{14.5 \times 2.55}{(67.9)^2}\right)\lambda + \frac{2.55}{67.9} = 0.$$

This gives a period of about $T = 35$ sec with a damping in about $t = 8$ sec to half amplitude. The amplitude will be reduced to about $\frac{1}{20}$ in one complete oscillation. The machine is very stable.

NOTE. These results are given by Hunsaker in his Smithsonian paper above cited with slightly different numerical values. It cannot be expected that two calculations should agree more closely than 2 or 3%. The difficulties were pointed out in the discussion of the question of fairing tabulated or plotted values.

EXERCISES

1. Show that if two particles (or bodies) impinge, the momentum after impact must be equal to that before impact. Two bodies, of masses as 1 : 2 and with respective velocities as 4 : 3, moving in directions including an angle of 60° impinge. Find the direction and magnitude of the subsequent motion of their C. G.

2. Prove that if two particles are rigidly connected, any forces of action and reaction in the line joining them can do no work. (This result is really needed to justify the application of the principle of work and energy in the case of a rigid body.)

3. Two particles moving in the same line impinge. There is no loss of energy in the impact. Show that the relative velocity of the particles after impact must be equal and opposite to that before impact.

4. Discuss the problem of Ex. 3 when the particles are not moving in the same line.

5. A bar of mass W and length l is rotating about its C. G. (the axis being perpendicular to the bar) with angular velocity ω. What is the angular momentum? Find the numerical value if $W = 2$ lb, $l = 2$ ft, $\omega = 200$ R.P.M.

6. Solve Ex. 5 if the bar is rotating about one end.

7. A bar of length l and weight W is at rest. It is struck normally at one end by a particle of mass $2W$. The particle adheres to the rod after impact. Find the direction and magnitude of the velocity of the common C. G. after impact, and the angular velocity of the system about it.

8. A 600-lb, uniform disc of radius 4 ft is spinning about a fixed axis through its center with $\omega = 30$ R.P.M. A 100-lb boy is at the center. If he crawls out to the edge what will the angular velocity be?

9. A cube sliding on a horizontal plane strikes a small ridge which instantly stops the forward edge. Will the cube tip over or not? (Consider the ridge as an axis about which the angular momentum just before and just after impact must be the same.)

10. A cylindrical mass of fluid is circulating about an axis irrotationally (see Art. 80). The radius of the cylinder is a and the peripheral velocity is v. If the fluid congeals, find the angular velocity of the whole.

11. A flat disc, mass W, radius a, is moving with both translation and spin on a horizontal plane of which the coefficient of friction is μ. Find the magnitude and direction of the total friction, and the frictional torque. Write the equations of motion. Solve.

12. Prove that if the moment of inertia about the C. G. is I_c, that about a point (or axis perpendicular to the plane) at a distance d from the C. G. is $I_c + d^2W$.

13. A 100-lb disc of radius 2 ft has a force of 20 lb applied tangentially to the rim at one point. What accelerations, linear and angular, does the force set up?

14. A boy throws a hoop, $W = 1$ lb, $a = 1$ ft, with a forward velocity of 10 ft/sec, and a reverse spin of 2 revolutions per second, on to a horizontal plane of which the coefficient of friction is $\frac{1}{3}$. Describe qualitatively and find quantitatively the motion.

15. If an airplane weighs 1800 lb and has a radius of gyration equal to 6 ft, how great will be the angular acceleration set up by the elevator if a torque of 1500 ft lb is possible? How long will it take the machine to turn through 4° — supposing no other torques to act?

16. Calculate the derivatives with respect to u, w, θ for the "Clark" at $i = 3°$. Check by the equations (49).

17. Calculate the derivatives with respect to u, w, θ for the "Clark" at $i = 6°$. Check by equations (49).

18. Given as the results of oscillation experiments with $i = 6°$ that the time nT to damp to $\frac{1}{9}$ for the "Clark" in 30 mi wind is 20 secs. Find $(\partial M/\partial q)_m$, $\partial M/\partial q$ and estimate $\partial Z/\partial q$ as $\frac{1}{20}\ \partial M/\partial q$.

19. Given for $i = 12°$, $nT = 25$ sec, etc., as in Ex. 18.

20. Given that for the "Clark" at $i = 6°$ and $i = 12°$ the modified determinants Δ are respectively

$$\Delta_6 = \begin{vmatrix} \lambda + .12 & -.24 & -32 \\ 1.00 & \lambda + 2.9 & 60\lambda \\ 0 & -.106 & \lambda^2 + 4.4\lambda \end{vmatrix}, \quad \Delta_{12} = \begin{vmatrix} \lambda + .16 & 0 & -32 \\ 1.20 & \lambda + 1 & 51\lambda \\ 0 & -.066 & \lambda^2 + 2.8\lambda \end{vmatrix}$$

Calculate the coefficients and R from these determinants, and obtain the periodic and damping times.

21. Compare your results of Exs. 17, 18 with Δ_6 of Ex. 20 and make the calculation from your data.

22. Given that for the JN-2 model $I_a = 1.18$, $I_{a+m} = 1.34\frac{1}{2}$. Given that nT, the time to damp to $\frac{1}{9}$ is for apparatus in 30 mi wind 97 sec; for apparatus and model 16 sec when $i = 10$. Calculate $\partial M/\partial q$ for the whole machine. The value should be about -8400.

23. Make a stability calculation for the JN-2 at $i = 1°$.

CHAPTER VII

MOTION IN THREE DIMENSIONS

44. Angular Motion. For two-dimensional motion of a mass, the moments of inertia about the center of gravity, the position of the center of gravity, and the amount of turning about the center of gravity, the component forces, and the moments of the forces about the center of gravity entered into the equations of motion. In three dimensions the mass of the body, the position of the center of gravity, and the component forces will again enter a motion, and in a way entirely analogous to that found in the two-dimensional case, namely,

$$W\frac{d^2x_c}{dt^2} = gX, \qquad W\frac{d^2y_c}{dt^2} = gY, \qquad W\frac{d^2z_c}{dt^2} = gZ \quad . \; . \; . \quad (1)$$

To prove these equations the method used in the simpler case applies except for the necessity of carrying an additional equation.

The angular motion, however, is decidedly more complicated. In two dimensions rotation can take place only about a point, that is, about an axis perpendicular to the plane of motion. In three dimensions rotation may take place about various axes, and the composition of rotations, that is, the result of a rotation through a finite angle about one axis followed by the rotation through a finite angle about another axis is a complicated subject. Even the specification of the new positions of the axes in a body when rotated from a standard position to some other is itself complicated. All these matters will be omitted.

Angular velocity is a simpler quantity than angular displacement because if a body has simultaneous angular velocities about two axes intersecting at the point O, and if these angular velocities be represented as vector quantities by drawing along the axes of rotation a directed magnitude equal to the angular velocity, then the resultant motion of the body is an angular velocity determined both in regard to its magnitude and its axis of rotation by the law of composition of vectors, that is, by the parallelogram law. To prove this will require the values of the velocities of any point in space arising from an angular velocity about a particular axis.

Let p represent an angular velocity about the axis of x. Every point (x, y, z) then moves along a circle concentric with the x-axis, and the velocity of the point will have the components u, v, w as follows:

$$u = 0, \qquad v = -zp, \qquad w = yp \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In like manner, the velocities due to an angular velocity q about the y-axis must be

$$u = zq, \qquad v = 0, \qquad w = -xq \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the velocities due to the angular velocity r about the z-axis are

$$u = -yr, \qquad v = xr, \qquad w = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When all these angular velocities operate simultaneously, the resultant velocity of the point (x, y, z) will be the respective sums of these velocities, namely,

$$u = zq - yr, \qquad v = xr - zp, \qquad w = yp - xq \quad . \quad . \quad . \quad (5)$$

Now, if the law of vector composition of angular velocities is true, the velocities u, v, w here found must be those due to the angular velocity

$$\omega = \sqrt{p^2 + q^2 + r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

about a line whose direction cosines are respectively

$$\frac{p}{\omega} = \frac{p}{\sqrt{p^2 + q^2 + r^2}}, \quad \frac{q}{\omega} = \frac{q}{\sqrt{p^2 + q^2 + r^2}}, \quad \frac{r}{\omega} = \frac{r}{\sqrt{p^2 + q^2 + r^2}}. \quad (7)$$

That this is so may be proved geometrically. It is easier, however, to proceed differently. Suppose a second set of angular velocities p', q', r' be considered. The velocities due to these are by (5)

$$u' = zq' - yr', \qquad v' = xr' - 2p', \qquad w' = yp' - xq', \qquad (5')$$

the velocities due to the superposition of p, q, r and p', q', r' are then

$$u + u' = z(q + q') - y(r + r'), \text{ etc.}$$

and these are precisely the velocities due to angular velocities $p + p'$, $q + q'$, $r + r'$. Now if p, q, r be regarded as components of a vector, and p', q', r' as components of a second vector, the composition of the two angular velocities leads to a vector with components $p + p'$, $q + q'$, $r + r'$ when and only when the parallelogram law is obeyed. Hence the proof.

45. Angular Momentum. If a body be rotating with angular velocities p, q, r about the three axes, the angular momentum of the body may be calculated as follows:

Let dW be an element of mass situated at x, y, z. The momentum in the x direction is $u\,dW$, representable as a directed quantity located at (x, y, z). The moment of this quantity about the x-axis is zero, but about the y-axis the angular momentum or moment of momentum is $zu\,dW$, whereas about the z-axis it is $-yu\,dW$. The velocity u of the mass dW, therefore, contributes the following angular momentum:

$$dh_1 = 0, \qquad dh_2 = zu\,dW, \qquad dh_3 = -yu\,dW, \quad . \; . \; . \quad (8)$$

where dh_1, dh_2, dh_3 are the elements of angular momentum or moment of momentum about the three axes. In like manner the velocities v and w contribute elements of angular momentum respectively as follows:

$$dh_1 = -zv\,dW, \qquad dh_2 = 0, \qquad dh_3 = xv\,dW, \quad . \quad (9)$$

$$dh_1 = yw\,dW, \qquad dh_2 = -xw\,dW, \qquad dh_3 = 0 \quad . \; . \; . \; . \quad (10)$$

The total angular momentum has, then, for its components

$$\left.\begin{aligned} dh_1 &= (yw - zv)dW, \; dh_2 = (zu - xw)dW, \\ dh_3 &= (xv - yu)dW, \end{aligned}\right\} \quad . \; . \; . \quad (11)$$

and if the values for u, v, and w be substituted from (5),

$$dh_1 = [(y^2 + z^2)p - zyq - xzr]dW,$$

with similar expressions of dh_2 and dh_3 obtained by advancing the letters.

The total value of h_1 is obtainable by integration as

$$h_1 = p\int(y^2 + z^2)\,dW - q\int xy\,dW - r\int xz\,dW \quad . \; . \quad (12)$$

with similar expressions for h_2 and h_3. The coefficient of p in h_1 is the moment of inertia of the body about the x-axis. The coefficients of q and r are quantities called products of inertia. The following notation is used for moments and products of inertia:

$$\left.\begin{aligned} A &= \int(y^2+z^2)\,dW, \; B = \int(z^2+x^2)\,dW, \; C = \int(x^2+y^2)\,dW, \\ D &= \int(yz)\,dW, \; E = \int(zx)\,dW, \; F = \int(xy)\,dW. \end{aligned}\right\} \quad (13)$$

Then,

$$h_1 = Ap - Fq - Er, \; h_2 = Bq - Dr - Fp, \; h_3 = Cr - Ep - Dq \quad (14)$$

That the angular momentum may be regarded as a vector with components h_1, h_2, h_3 and that angular momenta compound according to the parallelogram law follows from (14) and the composition of angular velocities just as the composition of angular velocities followed from (5) and the composition of velocities. Hence, although the moment of momentum or angular momentum of a plane rigid body is the moment of inertia I times the angular velocity ω namely, $h = I\omega$, the component moments of momentum for the rigid body in three dimensions contain not only three moments of inertia, but three products of inertia, and each component of angular momentum contains all three components of angular velocity. The angular momentum vector and the angular velocity vector are not in general in the same direction; that is, it is not in general true that the ratios $h_1 : h_2 : h_3$ are the same as the ratios $p : q : r$. In fact, the angle between the angular momentum and angular velocity, regarded as directed quantities, is (by a well-known formula of solid geometry)

$$\theta = \cos^{-1} \frac{ph_1 + qh_2 + rh_3}{\sqrt{p^2 + q^2 + r^2}\sqrt{h_1^2 + h_2^2 + h_3^2}} \quad . \quad . \quad . \quad . \quad (15)$$

46. Kinetic Energy of Rotation. The kinetic energy of rotation of the rigid body is

$$\text{K.E.} = \frac{1}{2g} \Sigma(u^2 + v^2 + w^2)\, dW, \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

when kinetic energy is measured as usual, in foot-pounds or similarly. If the values for u, v, w, be substituted from (5), the result is

$$\text{K.E.} = \frac{1}{2g}(Ap^2 + Bq^2 + Cr^2 - 2Dqr - 2Erp - 2Fpq) = \frac{1}{2g} I\omega^2, \quad (17)$$

for the kinetic energy is necessarily $1/2g$ times the product of the moment of inertia into the square of the angular velocity. The calculation of the kinetic energy of rotation in the plane is a simple problem involving the moment of inertia and the angular velocity; in space it involves the three moments of inertia, and the three products of inertia, and the component angular velocities.

From the formula for the kinetic energy it is easy to obtain the ellipsoid of inertia. The definition of the ellipsoid of inertia is that it is the locus obtained by laying off from the origin on each axis of rotation the reciprocal of the square root of the moment of inertia of the body about that axis. Now, the coördinates of the point x, y, z thus defined for each axis of rotation are

$$x = \frac{1}{\sqrt{I}} \frac{p}{\omega}, \quad y = \frac{1}{\sqrt{I}} \frac{q}{\omega}, \quad z = \frac{1}{\sqrt{I}} \frac{r}{\omega} \quad . \; . \; . \; . \quad (18)$$

The substitution of these values in the equation of kinetic energy (17) gives for the locus of (x, y, z) as

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = 1 \quad . \; . \; . \quad (19)$$

This is a quadric surface because it is an equation of the second degree. It is an ellipsoid (rather than a hyperboloid) because the radii being equal to the reciprocal of a moment of inertia must always be finite. This ellipsoid may be reduced to standard form by changing to a new set of axes directed along the principal axis of the ellipsoid. After this reduction the equation of the ellipsoid becomes merely

$$A'x'^2 + B'y'^2 + C'z'^2 = 1, \quad . \; . \; . \; . \; . \; . \; . \quad (20)$$

where A', B', C' are the moments of inertia about the principal axes of the ellipsoid of inertia. As the terms in $y'z'$, $z'x'$, and $x'y'$ do not occur in the reduced form, the products of inertia D', E', F' must all vanish when the axes of coördinates coincide with the axes of the ellipsoid of inertia.

The axes of the ellipsoid of inertia are called the principal axes of the rigid body. For these axes there are no products of inertia, and for these axes the relation between angular momentum and angular velocity is simply

$$h_1' = A'p', \quad h_2' = B'q', \quad h_3' = C'r', \quad . \; . \; . \; . \quad (21)$$

and the kinetic energy of rotation is

$$\text{K.E.} = 1/2g(A'p'^2 + B'q'^2 + C'r'^2) \quad . \; . \; . \; . \; . \quad (22)$$

The great simplifications for the angular momentum and for the kinetic energy which arise when principal axes are used make it desirable that whenever feasible those axes shall be used. It may be added that there is an even greater simplification in the dynamical equations of angular motion.

The reduction of an ellipsoid given in general form to its principal axes is a complicated piece of algebra. Fortunately, in most cases which arise in engineering, some, at least, of the principal axes can be foreseen from the symmetry of the figure.

If there is a plane of symmetry with respect to mass, that is to say, if there is a plane in the figure such that for every element of mass dW on one side of the plane, there is an equal element of mass dW on the opposite side of the plane (the line joining the two elements being perpendicular to the plane and bisected by the plane), the plane is one of mechanical symmetry. And if one axis, say the y-axis, is perpendicular to this plane, two of the products of inertia, namely, those that contain y,

$$D = \int yz\,dW, \qquad F = \int xy\,dW,$$

vanish; for each dW with coördinates (x, y, z) there is by hypothesis another dW, equal in magnitude, with coördinates $(x, -y, z)$. Hence, the terms in the integrals cancel in pairs, and

$$D = F = 0 \quad \text{. . . . .} \quad (23)$$

The ellipsoid of inertia then reduces to

$$Ax^2 + By^2 + Cz^2 - 2Ezx = 1 \quad \text{. . . . . .} \quad (24)$$

The y-axis is the principal axis, but the x and z axes are not unless $E = 0$. However, the algebraic problem of determining the two remaining principal axes now becomes merely that of finding the principal axes of the ellipse

$$Ax^2 + Cz^2 - 2Ezx = 1.$$

If a new set of axes x', z' be taken, inclined at an angle θ to the axes x and z, then

$$x' = x\cos\theta - z\sin\theta, \qquad z' = z\cos\theta + x\sin\theta,$$

and

$$E' = \int x'z'\,dW = \tfrac{1}{2}(A - C)\sin 2\theta + E\cos 2\theta \quad \text{. .} \quad (25)$$

For $E' = 0$, the result is

$$\tan 2\theta = 2E/(C - A) \quad \text{. . . . .} \quad (26)$$

The angle θ is, therefore, determined. The new values of the moments of inertia may also be calculated as

$$\left.\begin{aligned} A' &= \tfrac{1}{2}(A + C) + \tfrac{1}{2}(C - A)\cos 2\theta - E\sin 2\theta, \\ C' &= \tfrac{1}{2}(A + C) + \tfrac{1}{2}(A - C)\cos 2\theta + E\sin 2\theta. \end{aligned}\right\} \quad \text{. .} \quad (27)$$

The values of A' and C' can, therefore, be obtained from the expression for $\tan 2\theta$.

47. Equations of Rotatory Motion. It is next necessary to prove that the rate of change of angular momentum is equal to g times the moment of the forces. As in the case of the plane, this theorem is true first, when the point about which moments are taken is any fixed point in space, and, second, when moments are taken about the moving center of gravity. The proof, moreover, is identical with that given in the case of the plane, except that another variable must be carried throughout the demonstration, and that three equations must be carried in place of one. There is no need of giving the demonstration again, because the proof given before holds identically for the case of the component angular momentum and moment about the z-axis, whereas the proof for the components about the x and y-axes is obtainable by merely permuting the letters. The result is

$$\frac{dh_1}{dt} = \frac{d}{dt}(Ap - Fq - Er) = g(yZ - zY), \quad . \quad . \quad . \quad . \quad (28)$$

if Z, Y are the components of the external applied forces. The total moments about the three axes are denoted by L, M and N respectively. Hence,

$$\frac{d}{dt}(Ap - Fq - Er) = gL, \quad \frac{d}{dt}(Bq - Dr - Fp) = gM, \text{ etc.} \quad (29)$$

If the axes x, y, z through the center of gravity are fixed in direction in space, and the body moves relative to them, the moments and products of inertia are variable as the body moves, and the equations of motion must be obtained by differentiating not only p, q, r, but also A, F, E, etc. It is ordinarily very inconvenient to calculate moments and products of inertia with respect to fixed axes relative to which the body moves, and further inconvenient to calculate their rates of change. It is not only inconvenient, but exceedingly complicated. It is, therefore, customary to use axes moving in space, but fixed in the body. When this is done, the moments and products of inertia are constant, and may be calculated once for all. On the other hand, the expressions for the rate of change of the quantity referred to moving axes must be developed for axes in space. These expressions for rates of change are themselves somewhat complicated, but the final result is ordinarily simpler than would be found by regarding the axes as fixed in direction.

48. Moving Axes. Consider first the case of plane motion. Let V be a vector quantity with components u, v along x, y. Let the axes x, y be themselves rotating with an angular velocity r so that they turn through the angle $r\,dt$ in the time dt. It is required to determine the components of the rate of change dV/dt of the vector V along the axes. Draw the vector V and its new position $V + dV$. Now OM is the value of u, the component of V along x; and OM' is the value of $u + du$, the component of $V + dV$ along x'. Then

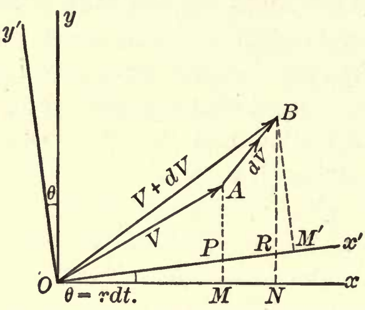

Fig. 15. Rate of Change of Vector, Moving Axes.

$$du = OM' - OM = OM' - OP$$
$$= PR + RM' = MN + RM'.$$

In writing this equation, $OM = OP\cos\theta = OP$ and $MN = PR\cos\theta = PR$ have been used since $\cos\theta = 1$, except for infinitesimals of the second order. Now $MN = \delta u$ is the component of dV along x and $\delta u/dt$ is the component of the rate of V. On the other hand $\delta u = MN$ is not the change in u because the rotation of the axes has brought in the additional part $RM' = BR\sin\theta = BR\,\theta$. Hence

$$\delta u = du - BRr\,dt, \qquad \frac{\delta u}{dt} = \frac{du}{dt} - BR\,r.$$

Now let dt approach zero. Then BR approaches $AM = v$, component of V perpendicular to x. Hence

$$\frac{\delta u}{dt} = \frac{du}{dt} - vr \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

The component along x of the rate of change of V is the rate of change of the component of V along x diminished by the product of the angular velocity r by the component of V perpendicular to x. In like manner it may be shown that the component along y of the rate of change of V is the rate of change of the component of V along y diminished by the product of the angular velocity r by the component of V perpendicular to y. This component is not u but $-u$ because it is $-x$ which bears to y the relation which y bears to x. Hence

$$\frac{\delta v}{dt} = \frac{dv}{dt} + ur \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

The need for these formulas lies in the fact that the force X along x (multiplied by g) is the mass W times the acceleration along x and

du/dt no longer gives the acceleration. In fact for the case of the plane, the equations of motion relative to rotating axes are

$$W\frac{\delta u}{dt} = W\left(\frac{du}{dt} - vr\right) = gX, \qquad W\frac{\delta u}{dt} = W\left(\frac{dv}{dt} + ur\right) = gY \quad (32)$$

Next proceed to the case of axes rotating with angular velocities p, q, r about a point O in space. If u, v, w are the components of a vector V along the moving axes x, y and z, the following are the formulas which must be demonstrated:

Component rate of change of vector V along $x = du/dt - vr + wq,$
" " " " " " " " $y = dv/dt - wp + ur,$ (33)
" " " " " " " " $z = dw/dt - uq + vp.$

It is sufficient to demonstrate any one of these formulas because the others are obviously obtainable merely by permuting the letters. A comparison of the rate of change along x with the expression obtained (30) for the plane shows that in addition to the term $-vr$ there is the term wq; whereas in addition to the term ur for the change along y there is the term $-wp$.

The proof of the formulas may be carried out as follows: The term du/dt is evidently due to the change of the component of V along x in the body, whereas the terms $-vr$ and wq are due to the change of the direction of the axis of x. This change is brought about by the infinitesimal rotation $r\,dt$ about the z-axis, and an infinitesimal rotation $q\,dt$ about the y-axis. The infinitesimal rotation $r\,dt$ will introduce the term $-vr$ in the component acceleration, just as in the demonstration for the plane. Both the demonstration and the figure there used may be repeated identically. There will also be a contribution $+ur$ to the component acceleration along y. The infinitesimal rotation $q\,dt$ about the y-axis may be treated in exactly the same manner, and it is then seen that the contribution to the acceleration along x is $+wq$ (not $-wq$), and to the acceleration along z is $-uq$.

49. Equations of Motion. When the formulas (33) are applied to the motion of the center of gravity itself, with u, v, w interpreted as the component velocities along the axes, the equations of motion for the center of gravity referred to moving axes become

$$W\left(\frac{du}{dt} - vr + wq\right) = gX, \text{ etc.} \quad \ldots\ldots\ldots (34)$$

When the same formulas are applied to the rate of change of angular momentum, the result is

$$\frac{dh_1}{dt} - h_2 r + h_3 q = gL, \quad \ldots\ldots\ldots \quad (35)$$

with similar expressions for the change in h_2 and h_3. If the values of h_1, h_2 and h_3 be substituted from (14), the result is

$$A(dp/dt) - F(dq/dt) - E(dr/dt) + (C - B)qr + D(r^2 - q^2) + Fpqr - Epq = gL,$$

with similar expressions obtained by permuting the letters.

These expressions are very much simplified when the body is referred to its principal axis. In that case,

$$D = E = F = 0,$$

and the equations become

$$\left.\begin{aligned} A(dp/dt) + (C - B)qr &= gL, \\ B(dq/dt) + (A - C)rp &= gM, \\ C(dr/dt) + (B - A)pq &= gN. \end{aligned}\right\} \quad \ldots\ldots \quad (36)$$

This set of equations is known as Euler's equations.

The airplane is itself a body which has a plane of symmetry. Perhaps the symmetry is not quite perfect. For instance, the two sides of the wing may not be of absolutely the same design. There may be some "washout," designed in part, at least, to compensate for the torque of the propeller. There may not be an absolutely perfect distribution of the masses of different parts of the machine relative to the plane of symmetry. Nevertheless, for practical purposes the central xz-plane is sufficiently near a plane of symmetry to be treated as such. As the y-axis is taken perpendicular to this plane, and the x and z axes in it, the products of inertia D and F vanish, and for the airplane one may write

$$h_1 = Ap - Er, \qquad h_2 = Bq, \qquad h_3 = Cr - Ep, \quad \ldots \quad (37)$$

and the equations of motion may be written as

$$\left.\begin{aligned} A(dp/dt) - E(dr/dt) + (C - B)qr - Epq &= gL, \\ B(dq/dt) + (A - C)rp + E(p^2 - r^2) &= gM, \\ C(dr/dt) - E(dp/dt) + (B - A)pq + Erq &= gN. \end{aligned}\right\} \quad \ldots \quad (38)$$

These equations will be used later to discuss stability.

50. Steady Motion. The motion of a body is said to be steady when the derivatives with respect to the time vanish in the equations of motion referred to moving axes. For a person who moves with the body, steady motion is no motion at all. When a top spins in the vertical position with constant angular velocity, the motion of the top is steady. When the top spins about an inclined axis, and the axis rotates uniformly about the vertical, the motion is also steady. If the body is referred to its principal axes, the condition for steady motion about the center of gravity is obtained from (36) by setting the derivatives equal to zero. Then

$$(C - B)qr = gL, \quad (A - C)rp = gM, \quad (B - A)pq = gN\,. \tag{39}$$

are the conditions connecting the angular velocities about the axes, the moments of inertia about the axes, and the moments of the forces.

It is necessary to note that the moments of the forces are not proportional to the component angular velocities. In fact, if the moment of the forces were proportional to the angular velocities, so that the resultant moment were in the same direction as the resultant angular velocity, the conditions to be satisfied would be

$$L : M : N = p : q : r.$$

As a matter of fact, the torque, that is, the resultant moment, is as a directed quantity perpendicular to the axis of spin in steady motion; that is, the torque is in a plane passing through the axis of spin. The condition that the vector whose components are L, M, N, be perpendicular to the vector whose components are p, q, r is $Lp + Mq + Nr = 0$. If the values for L, M and N be substituted from (39), it is seen at once that the condition for perpendicularity is satisfied, and the resultant torque is in a plane passing through the axis of rotation. Again, if the angular velocity be ω, and the direction cosines of its direction relative to the axes be l, m, n, so that

$$p = ls, \qquad q = ms, \qquad r = ns,$$

the components of the torque necessary to maintain this steady angular velocity ω about this direction are

$$gL = (C - B)mn\omega^2, \; gM = (A - C)ln\omega^2, \; gN = (B - A)lm\omega^2\,. \tag{40}$$

Whenever a rigid body is balanced upon an axis passing through the center of gravity, the body is necessarily in static equilibrium, with no tendency to rotate about the axis. Whenever that same body is rotating with an angular velocity ω about an axis passing through the center of gravity, there is in general a torque passing through that axis, and tending to deviate the axis into a new position. The amount of that torque is given by its components from (40). If the axis of rotation happens to be a principal axis, the angular velocity ω becomes either p or q or r, the other two components being zero, because p, q, r are by definition the component angular velocities along the principal axis. Moreover, l or m or n is equal to unity, and the other two vanish. It, therefore, appears from (40) that whenever the body rotates about a principal axis through the center of gravity, there is no torque acting upon the axis and tending to deviate it. A body is said to be in dynamical balance for rotation about an axis when there is no tendency for the axis to change direction under the action of the rotation; that is, when the axis is a principal axis. In the case of rapidly rotating shafts, armatures, and the like, in engineering structures, it is very important, if possible, to have the axis of rotation a principal axis passing through the center of gravity of the rotating body; for only in that case will the resultant torque on the bearings be zero. In case the body is out of dynamical balance, that is, in case the axis of rotation be not a principal axis, there is a torque upon the axis, and consequent excessive pressure upon the bearings, with much wear and tear.

51. Angular Momentum and Torque. The relations between angular momentum and torque, or moment of force, can best be explained with the aid of vectorial nomenclature.

First, consider the motion of a particle. Let Wv be the momentum drawn as a vector. Now, if the force F acts for the time dt, the contribution to the momentum is $F\,dt$, both in magnitude and in direction. If the force acts in the direction along which the particle is moving, then the magnitude of the momentum is increased; but if the force acts in a direction perpendicular to that in which the particle is moving, then the momentum is changed in direction without being altered in amount. However, the rate at which the momentum vector turns is $\omega = gF/Wv$. The same treatment may be accorded the velocity and acceleration that has been given to the momentum and the force. An acceleration dv/dt which acts in the direction of a velocity v produces an increase in the velocity without change of direction; but an acceleration which acts perpendicular to a velocity turns the velocity vector with an angular velocity determined by the relation $v\omega = dv/dt$.

Passing to angular velocity and angular acceleration, the facts are as follows: An angular acceleration, $d\omega/dt$, which takes place about the axis of the spin ω, increases the magnitude of ω. If, however, the angular acceleration generates angular velocity about an axis perpendicular to the axis of the spin ω the effect of the angular acceleration is to rotate the axis of spin, with an angular velocity Ω, which is the quotient of the angular acceleration and the angular velocity ω. That is, $\omega\Omega = d\omega/dt$. But it is not angular velocity which is of fundamental importance in the mechanics of a rigid body. The dynamical concept that must be used is angular momentum, which is not necessarily in the same direction as the angular velocity. The angular momentum has three components, and is regarded as a directed magnitude. The torque, or moment of the forces, has three components, and is also regarded as a directed magnitude. If, then, the torque is in the direction of the angular momentum, the effect of the torque is to increase the magnitude of the angular momentum; whereas if the torque is perpendicular to the angular momentum, the effect of the torque is to rotate the angular momentum vector in the plane determined by that vector, and the torque vector; and the angular velocity of rotation Ω is the quotient of g times the torque T by the angular momentum h. That is, $\Omega = gT/h$. It is also possible to regard the torque as a couple in the plane instead of as a vector perpendicular to that plane. The couple in a plane generates angular momentum perpendicular to that plane, and if the angular momentum is itself perpendicular to the plane, the effect is merely to increase the magnitude, whereas if the angular momentum lies in the plane, the effect is merely to rotate the angular momentum.

For example, consider the effect on the motion of an airplane of the angular momentum of the rotating parts, which consist, first of the propeller, and second of certain parts of the engine. In the case of a rotating engine, like the Gnome, the contribution of the rotation of the engine itself to the angular momentum is considerable. Imagine that the rotation appears clockwise from in front of the propeller, counterclockwise from behind. The angular momentum due to the rotating parts will, then, be drawn as a vector forward, because clockwise rotation is positive. If the pilot by operating the elevator tries to change from uniform horizontal motion to a climb, the angular momentum vector for the rotating parts must tip up. This means that a torque must act in the horizontal plane, and in the absence of an external force producing this torque, the machine will yaw off under the reaction set up by the rotating parts. In a similar manner, if the pilot tries to execute a turning (yawing) motion, the angular momentum vector turns to the right, which requires the operation of a torque in the vertical plane; and in the absence of such a torque, the machine will pitch or stall under the reaction of the motion due to the rotating parts. The torque required is the product of the angular momentum h by the angular velocity Ω of turning divided by g. That is $T = h\Omega/g$.

EXERCISES

1. Component angular velocities are 1, 3, 5. Find the resultant and the angles it makes with the axes.

2. A brick $2 \times 4 \times 8$ inches is spinning 240 R.P.M. about its main diagonal. Find the component angular velocities about the three edges.

3. A body is rotated 90° about a given horizontal axis and then 90° about the vertical. It is also first rotated 90° about the vertical and then 90° about that same horizontal axis. Show the difference in position illustrating that large angular displacements do not compound vectorially. If they did compound, what would the position be?

4. A body is rotated $\frac{1}{3}°$, $\frac{1}{2}°$, $\frac{2}{3}°$ about three perpendicular axes. Assuming that these rotations can be treated as infinitesimal find the resultant axis of rotation and the amount of rotation.

5. In Ex. 2 find the component velocities of each corner.

6. In Ex. 2 assume a density of 150 lb/ft^3 and calculate the moments of inertia about the C. G. about axes parallel to the edges.

7. In Ex. 6 calculate the products of inertia D, E, F if the origin is at one corner.

8. A thin rectangular plate $a \times b$ ft has two squares $c \times c$ nicked out of two diagonally opposite corners. Calculate the moments and products of inertia about axes through the C. G. parallel to the edges of the plate (one perpendicular to the plate).

9. The plate of Ex. 8 is spinning about an axis in the plate parallel to one edge. Find the component angular momenta, and the total angular momentum in magnitude and direction.

10. Find the kinetic energy in Ex. 9 and check (17).

11. If I_0 is the moment of inertia about an axis through the C. G., show that $I_0 + Wd^2$ is that about a parallel axis at a distance d.

12. If A, B, C, D, E, F are moments and products of inertia about axes through the C. G., find the corresponding quantities about parallel axes through (a, b, c).

13. Find the principal axes in Ex. 8, and the moments of inertia about them.

14. Write out the proof of (31) from a figure.

15. Suppose in Ex. 2 that the density is 150 lb/ft^3 and that the spin is dying out at the rate of 24 R.P.M. per second. Find the torques about the principal axes.

16. The plate of Ex. 8 is spinning steadily about the other diagonal. Find the torque.

17. If the rotating parts of an airplane have $I = 200$ and $\omega = 1200$ R.P.M., the axis of rotation being supposed to be along X (negative), what gyroscopic torque will be experienced if the machine pitches down 1° in 1 sec? Give magnitude, and tendency on the machine.

18. Data as in Ex. 17 with the mass of the machine 1600 and its radii of gyration about X, Y, Z equal to 5, 6, 7. Find the angular acceleration set up by the torque, and the angular velocity generated in 1 sec.

19. Suppose the airplane yaws 3° in 2 sec. Solve as in Ex. 17 and 18.

20. An airplane weighs 1600 and has radii of gyration of 5, 6, 7 about axes supposed principal. Find moments and products of inertia about new axes obtained by rotating the original axes about Y through 12° from X toward Z.

21. An airplane weighs 1800 and has radii of gyration 6, 6, $8\frac{1}{2}$ about axes supposed principal. Find moments and products of inertia about new axes obtained by rotating these through 15° from X to Z.

CHAPTER VIII

STABILITY OF THE AIRPLANE

52. The Form of the Equations. When an airplane moves in three dimensions the equations of motion referred to moving axes through the origin of the center of gravity with the y-axis perpendicular to the plane of symmetry, and with the x- and z-axes in the plane of symmetry, are as follows:

$$\left.\begin{aligned} W(du/dt - vr + wq) &= gX, & dh_1/dt - h_2r + h_3q &= gL, \\ W(dv/dt - wp + ur) &= gY, & dh_2/dt - h_3p + h_1r &= gM, \\ W(dw/dt - uq + vp) &= gZ, & dh_3/dt - h_1q + h_2p &= gN, \end{aligned}\right\} \quad (1)$$

with

$$h_1 = Ap - Er, \qquad h_2 = Bq, \qquad h_3 = Cr - Ep. \quad . \quad . \quad . \quad (2)$$

The conditions for steady motion in the direction of the axis of x with velocity $-U$ are

$$X = 0, \quad Y = 0, \quad Z = 0, \quad L = 0, \quad M = 0, \quad N = 0. \quad . \quad . \quad (3)$$

The forces and moments are not exclusively those due to the air forces. In the xz-plane there is acting a propeller thrust and the weight of the machine. Moreover, there is about the x-axis the constant action of the propeller torque, which is balanced or nearly balanced by the "wash-out" on the wing or is to be regarded as small compared with other moments. The moment L may, therefore, still be regarded as 0, at least in the preliminary discussion of stability. The forces X and Z and moment M, however, must be separated into the aerodynamical forces and the thrust and weight, namely,

$$X = X' - T, \qquad Z = Z' - W, \qquad M = M' - hT, \quad . \quad . \quad (4)$$

if the normal attitude is horizontal with propeller thrust horizontal and passing above the C. G. by the distance h. The conditions for steady motion are

$$u = U, \qquad v = w = q = r = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and for motions near the steady motion, which are the only ones discussed for stability, u is nearly equal to U, and $v, w, p, q,$ and r are all nearly equal to zero. The equations may be simplified by discarding the products of two small quantities. They then become

$$\left.\begin{aligned} W\frac{du}{dt} &= gX, & A\frac{dp}{dt} - E\frac{dr}{dt} &= gL, \\ W\left(\frac{dv}{dt} + Ur\right) &= gY, & B\frac{dq}{dt} &= gM, \\ W\left(\frac{dw}{dt} - Uq\right) &= gZ, & C\frac{dr}{dt} - E\frac{dp}{dt} &= gN. \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

The forces X, Y, Z and moments L, M, N are infinitesimal. The quantity u remaining in the first equation may also be taken as the infinitesimal departure of u from the constant value of U. It is necessary to calculate the infinitesimal values of the forces and moments. So far as the aerodynamic part of the forces is concerned, it may be assumed that the value depends wholly on the velocity and angular velocity of the machine relative to the air, that is (apart from the velocity U) upon u, v, w, p, q, r. The aerodynamic x-force may be written

$$X \doteq X_0 + \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial v}v + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial p}p + \frac{\partial X}{\partial q}q + \frac{\partial X}{\partial r}r, \quad . \quad (7)$$

where the partial derivatives are taken for the value appropriate to the steady state, and where X_0 is equilibrated by the thrust T.

(NOTE. In every equation beyond (6) the symbols X, Z, M will denote the aerodynamic forces and the effects of thrust and weight will be separately allowed for; moreover u will be the infinitesimal change in forward velocity.)

In like manner, the forces Y, Z and the moments $L, M,$ and N may be expanded to a first order of approximation into a linear expression in the small velocities, u, v, w, p, q, r. There arise, therefore, for the determination of the new values of the forces and moments 36 partial derivatives.

Of these 36 derivatives, however, 18 may be shown to vanish by using the fact that the airplane is a symmetrical structure. Those which vanish are the derivatives of $X, Z,$ and M with respect to $v, p,$ and r, and the derivatives of $Y, L,$ and N with respect to $u, w,$ and q. These may be written in the symbolic form

$$\frac{\partial(X, Z, M)}{\partial(v, p, r)} = 0 \quad \text{and} \quad \frac{\partial(Y, L, N)}{\partial(u, w, q)} = 0. \quad . \quad . \quad . \quad . \quad (8)$$

To prove, for instance, that $\partial X/\partial v = 0$ the argument is as follows: The change in X due to the velocity v is $(\partial X/\partial v)v$. The derivative is a constant, and the force due to v changes its sign when v changes sign, provided the derivative does not vanish; but owing to the symmetry of the structure, if a side slip v to the right produces a force along x, a side slip v to the left must produce the same, not the opposite, force along x. Therefore, $\partial X/\partial v$ must vanish. In the same way, the pitching moment about the y-axis due to an angular velocity p of roll is $(\partial M/\partial p)p$, and must change its sign with p. Now, as the structure is symmetrical, if a positive angular velocity of roll about the x-axis pitches the machine over, a negative angular velocity must also pitch it over in the same sense; so that the sign cannot change, and, hence, $\partial M/\partial p = 0$. The argument goes in exactly the same way for all the other 16 derivatives which have been stated to vanish.

It is now necessary to calculate the changes in the equation due to the propeller thrust and gravity when the axes are changed in direction by infinitesimal amounts. An infinitesimal angle of pitch is designated by θ, and measured positively when the machine tends to climb. An infinitesimal angle of yaw is denoted by ψ, and is measured positively when the machine yaws from left to right as viewed by the pilot. The infinitesimal angle ϕ of roll is positive when the right-hand side of the machine tips down, and the left hand up. The variation of the propeller thrust T with velocity and angular velocity is assumed to be 0; that is, the propeller thrust is taken to be constant. If the thrust were considered as changing an equation

$$T = T_0 + \frac{\partial T}{\partial u}u + \frac{\partial T}{\partial v}v + \frac{\partial T}{\partial w}w + \frac{\partial T}{\partial p}p + \frac{\partial T}{\partial q}q + \frac{\partial T}{\partial r}r,$$

analogous to (7) would have to be used, with $X_0 = T_0$ as the equilibrium condition. A yaw of the machine does not change the direction of action of the force of gravity relative to the axes; but a pitch θ or a roll ϕ does introduce changed components of gravity along the axes because of the variation in direction of the axes. The roll ϕ introduces a component $W\phi$, tending to make the machine slip in the y direction in the negative sense. A pitch θ introduces the component $W\theta$, tending to make the machine slip back in the x direction. The Z component of the weight is not changed except for infinitesimals of higher order. The equations of motion may then be written

$$\left.\begin{aligned}\frac{W}{g}\frac{du}{dt} &= \frac{\partial X}{\partial u}u + \frac{\partial X}{\partial w}w + \frac{\partial X}{\partial q}q + W\theta,\\ \frac{W}{g}\left(\frac{dw}{dt} - Uq\right) &= \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial q}q,\\ \frac{B}{g}\frac{dq}{dt} &= \frac{\partial M}{\partial u}u + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial q}q,\end{aligned}\right\} \quad . \ . \ . \ . \quad (9)$$

$$\left.\begin{aligned}\frac{W}{g}\left(\frac{dv}{dt} + Ur\right) &= \frac{\partial Y}{\partial v}v + \frac{\partial Y}{\partial p}p + \frac{\partial Y}{\partial r}r - W\phi,\\ \frac{A}{g}\frac{dp}{dt} - \frac{E}{g}\frac{dr}{dt} &= \frac{\partial L}{\partial v}v + \frac{\partial L}{\partial p}p + \frac{\partial L}{\partial r}r,\\ \frac{C}{g}\frac{dr}{dt} - \frac{E}{g}\frac{dp}{dt} &= \frac{\partial N}{\partial v}v + \frac{\partial N}{\partial p}p + \frac{\partial N}{\partial r}r,\end{aligned}\right\} \quad . \ . \ . \ . \quad (10)$$

with $q = d\theta/dt$ and $r = d\phi/dt$, because infinitesimal angles θ, ϕ, ψ compound like angular velocities so that infinitesimal rotations θ, ϕ, ψ are geometrically independent.

The first three equations contain only the variables u, w, and θ. The second three equations contain only the variables v, ϕ, and r. The first set of equations determine the so-called longitudinal motion. They are equivalent to, and, indeed, identical with (except for rotation and the difference between moving and fixed axes) the equations which have already been obtained (Art. 41) for the discussion of the motion in two dimensions. The theory of the longitudinal stability may, therefore, be considered as already largely completed. inasmuch as the work of this chapter has shown that for infinitesimal displacements the longitudinal and lateral motions are independent. The second set of three equations determines the so-called lateral motion; that is, the interrelation of side slip, roll, and yaw. The equations are linear with constant coefficients, and the method of treatment, so far as the analysis is concerned, does not differ materially from that given for longitudinal stability. In the details, however, the discussion is considerably different.

53. The Determination of the Coefficients. It has been stated that equations (9) are equivalent to (24) of Art. 39. The demonstration of this fact will throw light on the meaning of the coefficients and the significance of moving as contrasted with fixed axes. Compare first the equations

$$\frac{Wk^2}{g}\frac{d^2\theta}{dt^2} = \frac{\partial M}{\partial u}u' + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial q}q + \frac{\partial M}{\partial \theta}\theta, \quad . \ . \ . \quad (11)$$

and

$$\frac{B}{g}\frac{dq}{dt} = \frac{\partial M}{\partial u}u + \frac{\partial M}{\partial w}w + \frac{\partial M}{\partial q}q. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

The values of Wk^2 and B are the same — the moment of inertia about y. The meaning of u' and u are the same — the variation of the forward velocity from U — except for this difference: In (11), u' represents velocity along a fixed horizontal axis; in (12), u is velocity along a moving forward (or better, backward) axis. In like manner the two w's mean respectively small vertical velocity and small normal velocity. Now as U is large, a small value of θ will give a small, but not negligible, component velocity across the path. In fact if w is the velocity along z, the vertical velocity w' is (U being itself negative)

$$w' = w\cos\theta - U\sin\theta = w - U\theta. \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Next, in (11) the value of

$$\frac{\partial M}{\partial w} = +\frac{57{\cdot}3}{U}\frac{dM}{di} \qquad \text{and} \qquad \frac{\partial M}{\partial \theta} = 57{\cdot}3\,\frac{dM}{di},$$

and are obtained by differencing in the table of M or i as shown in Art. 39. Hence, if w' denote vertical velocity, two terms of (11) are

$$\frac{\partial M}{\partial w}w' + \frac{\partial M}{\partial \theta}\theta = +\frac{57{\cdot}3}{U}\frac{dM}{di}(w - U\theta) + 57{\cdot}3\,\frac{dM}{di}\theta = \frac{57{\cdot}3}{U}\frac{dM}{di}w.$$

The value of $\partial M/\partial w$ in (12) is also $57{\cdot}3dM/U\,di$ obtained by differencing in the table. For all these derivatives in both sets of equations are constant quantities calculated by altering a single one of the variables from its zero value, and whether moving or fixed axes be used the steady motion is the same and represented by the same values (zero) of the variables. Consider next another pair of equations, say,

$$\frac{W}{g}\frac{dw'}{dt} = \frac{\partial Z}{\partial u}u' + \frac{\partial Z}{\partial w}w' + \frac{\partial Z}{\partial q}q + \frac{\partial Z}{\partial \theta}\theta + T\theta\cos j, \quad . \qquad (14)$$

$$\frac{W}{g}\left(\frac{dw}{dt} - Uq\right) = \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial q}q, \quad . \quad . \quad . \quad . \qquad (15)$$

where in the first equation w has been written w' and denotes a true vertical component instead of one along the moving z-axis. Now by (31) of Art. 39 and (13) above

$$\frac{\partial Z}{\partial w}w' + \frac{\partial Z}{\partial \theta}\theta = \left(\frac{57.3}{U}\frac{dL}{di} + \frac{D}{U}\right)(w - U\theta) + 57.3\frac{dL}{di}\theta$$

$$= \left(\frac{57.3}{U}\frac{dL}{di} + \frac{D}{U}\right)w - D\theta.$$

But $D = T\cos j$ and hence the right-hand sides of (14) and (15) become identical. As for the left-hand sides dw'/dt represents the vertical acceleration, whereas $(dw/dt - Uq)$ represents the acceleration along z which though not quite vertical is practically so — and second-order infinitesimals have been systematically neglected. The discussion of the third pair of equations is left as an exercise.

With regard then to the coefficients in (9) the former results hold:

$$\frac{\partial X}{\partial u} = \frac{2X}{U} = \frac{2D}{U}, \qquad \frac{\partial Z}{\partial u} = \frac{2Z}{U} = \frac{2L}{U}, \qquad \frac{\partial M}{\partial U} = \frac{2M}{U},$$

$$\frac{\partial X}{\partial w} = \frac{57.3}{U}\frac{dD}{di} - \frac{L}{U}, \qquad \frac{\partial Z}{\partial w} = \frac{57.3}{U}\frac{dL}{di} + \frac{D}{U}, \qquad \frac{\partial M}{\partial w} = \frac{57.3}{U}\frac{dM}{di},$$

where L denotes the lift (not the rolling moment L). These six are obtained from the lift, drag, and moment tables, the first three without, the second three with differencing. The values of the derivatives by q are obtained so far as $\partial M/\partial q$ is concerned by an oscillation experiment, and so far as $\partial X/\partial q$ and $\partial Z/\partial q$ are concerned by simply neglecting them, partly because they are small, partly because they enter the equations of motion coupled with large quantities (or the value of $\partial Z/\partial q$ may be estimated as $\partial M/\partial q \div l$, if l be the arm of the tail).

For the coefficients in (10) the procedure is as follows:

(*a*) If the model is placed in the wind tunnel with a slight yaw, the side slipping force upon the model may be measured. Now, a slight yaw in the set-up in the wind tunnel is equivalent to a small velocity of side slip, the relation between the angle of yaw and the equivalent side slip in the wind being

$$\tan\psi = -\frac{v}{U} \qquad \text{or} \qquad \psi^\circ = -\frac{57.3v}{U}. \quad . \quad . \quad . \quad . \quad (16)$$

If, therefore, the side force Y or the moments L or N be measured for different angles of ψ, the derivative $\partial Y/\partial v$ may be obtained as the slope of the curve of Y on ψ when $\psi = 0$, multiplied by 57.3, divided by U, and changed in sign; that is,

$$\frac{\partial Y}{\partial v} = -\frac{\partial Y}{\partial \psi}\frac{57\cdot3}{U}, \quad \frac{\partial L}{\partial v} = -\frac{\partial L}{\partial \psi}\frac{57\cdot3}{U}, \quad \frac{\partial N}{\partial v} = -\frac{\partial N}{\partial \psi}\frac{57\cdot3}{U}. \quad (17)$$

The rules for scaling up forces and moments from the model to the full-sized machine are as before, namely: forces are scaled up according to the square of the velocity and the square of the linear dimension, moments according to the square of the velocity and the cube of the linear dimension. The results may be scaled up before calculating the derivative, inasmuch as the value of the derivatives themselves for the model are not important.

The derivatives $\partial L/\partial p$ and $\partial N/\partial r$ which are respectively the rates of change of rolling moment with respect to angular velocity of roll, and of yawing moment with respect to angular velocity of yaw, may be determined by allowing the model to oscillate in roll, or to oscillate in yaw; and by observing the damping. With respect to this experimental method nothing need be added to what was given in determining the rate of change of pitching moment with respect to angular velocity of pitch. It is merely necessary to rig the model so that the oscillation takes place about the proper axis. The result is scaled up to the full-sized machine by multiplying by the first power of the velocity ratio and by the fourth power of the linear dimension ratio.

(*b*) It is possible to give a fairly satisfactory calculation for $\partial L/\partial r$. This is the rate of change of rolling moment with respect to angular velocity in yaw. Consider the machine to be in its position of steady flight, but to be animated with an angular velocity of yaw. This means that any part of the wing with coördinate y is being carried forward with the velocity yr (*i.e.*, backward with velocity $-yr$) superposed upon the velocity U. The resultant velocity is, therefore, $U - yr$. Let b be the breadth or cord of one plane, and s the total span. The change in rolling moment $(\partial L/\partial r)r$ is produced by the excess of lift on the left wing due to the increased relative velocity $U - yr$, and the defect of lift on the right wing, due to the diminished relative velocity. Consider a strip across the wing of breadth dy and area $b\,dy$. The change of lift on this strip should be to the change of lift on the whole wing in proportion to the breadth dy of the strip, and the length s of the wing; but for the whole wing the change in lift with forward velocity u is

$$\frac{\partial Z}{\partial u}u = \frac{2Z}{U}u; \qquad \text{hence} \qquad \frac{dy}{s}\frac{2Z}{U} \qquad \text{and} \qquad \frac{y\,dy}{s}\frac{2Z}{U}u$$

are respectively the proportional change in the lift for the strip dy and in the moment of the change in the lift, which is the contribution of this strip to the rolling moment. Let the change in velocity u be set equal to $-yr$, and integrate over the entire wing from $y = -s/2$ to $y = +s/2$. Then

$$\frac{\partial L}{\partial r} r = \int_{-s/2}^{s/2} - \frac{2Z}{Us} ry^2 \, dy = - \frac{2Z}{Us} \frac{r}{3} \frac{2s^3}{8}$$

and

$$\frac{\partial L}{\partial r} = - \frac{Zs^2}{6U} = - \frac{Ws^2}{6U} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The derivative may, therefore, be calculated from the weight, the span, and the velocity. The calculation shows that the value of this derivative varies inversely with the speed U for a machine of given weight.

The rate of change of yawing moment due to rolling, namely, $\partial N/\partial p$ may be calculated in a similar manner. The machine is considered to be in the standard position, but to have superposed upon the velocity U an angular velocity p of roll. This means that a strip of breadth dy on the left-hand wing is being raised with a velocity $w = yp$. This will alter both the X and Z force on the strip, and it is the change in the X force which produces the yawing moment. The change in the X force with a velocity w is $(\partial X/\partial w)w$ for the entire wing, and for the strip of breadth dy the proportional change and its moment should be

$$\frac{\partial X}{\partial w} \frac{dy}{s} w \quad \text{and} \quad - \frac{\partial X}{\partial w} \frac{dy}{s} y^2 p.$$

Hence,

$$\frac{\partial N}{\partial p} p = - \int_{-s/2}^{s/2} \frac{\partial X}{\partial w} y^2 p \frac{dy}{s} \quad \text{and} \quad \frac{\partial N}{\partial p} = - \frac{\partial X}{\partial w} \frac{s^2}{12}. \quad (19)$$

Thus, the value of $\partial N/\partial p$ may be obtained from that of $\partial X/\partial w$ and the span of the wings.

In both these calculations only the effect of the wings themselves has been taken into consideration. It is not probable that the contribution of other surfaces to $\partial N/\partial p$ would be great, because those surfaces are relatively small and relatively near the axis of rotation. For $\partial L/\partial r$ the wings again seem to contribute practically all the effect, because an angular velocity of yaw would apparently produce from surfaces other than the wings very little rolling moment.

The arguments just given for the calculation are applicable also to calculate $\partial L/\partial p$ and $\partial N/\partial r$. The change in the rolling moment due to rolling $(\partial L/\partial p)\, p$ would arise from the changed value of the lift, just as $(\partial N/\partial p)\, p$ arose from the changed form of the drag. The result should, therefore, be so far as the wings are concerned,

$$\frac{\partial L}{\partial p} = \frac{\partial Z}{\partial w} \frac{s^2}{12}. \qquad (20)$$

It does not seem as though the surfaces other than the wings would contribute much to this effect. In like manner $\partial N/\partial r$ would so far as the wings are concerned be

$$\frac{\partial N}{\partial r} = \frac{\partial X}{\partial u} \frac{s^2}{12} \qquad (21)$$

In this case, however, it must be remembered that the angular velocity of yaw produces a sidewise velocity of the whole body, and of the vertical surfaces in the rear, so that the value of $\partial N/\partial r$, as calculated from the wings alone, might well be considered too small. As a matter of fact, a comparison between the calculated and observed values for these two aerodynamic coefficients shows that the argument is not as good as might be hoped.

This lack of agreement throws some suspicion on the accuracy of the values for $\partial L/\partial r$ and $\partial N/\partial p$ as calculated; and an experimental method of procedure is, therefore, advisable. The following method, dependent upon the theory of forced oscillations and resonance (see Art. 36) may be devised for the experimental determination of these quantities. The method may also be applied to the determination of other aerodynamic coefficients.

(*c*) Suppose the model mounted in the wind tunnel with the proper attitude for a particular speed of flight. Let it be free to oscillate about the x-axis under the action of a restoring moment due to springs; and let it be forced to take a periodic yawing motion about the z-axis. This may be accomplished by means of a suitably devised driving mechanism. The differential equation for the motion of the model will be of the form

$$I \frac{d^2\phi}{dt^2} + \frac{\mu\, d\phi}{dt} + \nu\phi = gL = gC\psi_0 \sin qt, \qquad (22)$$

where ψ_0 is the amplitude of the yawing motion and q is a number determining the frequency. The moment, $-\mu\, d\phi/dt$ is the frictional moment resisting oscillation about the x-axis, and is related to $\partial L/\partial p$. The restoring moment $-\nu\phi$ is due to the action of the springs. The moment $C \sin qt$ is the rolling moment due to the forced oscillation. in yaw, and is related to $\partial L/\partial r$. Let $\nu = n^2 I$. The solution of the equation for the forced oscillation will contain a complementary function and a particular integral. After a certain time, the complementary function will disappear, owing to the damping, which for a rolling motion is reasonably large, so that the time taken for the disappearance of the complementary function is not great. There would, then, remain merely the particular solution

$$\phi = \frac{gC\psi_0 \sin(qt - \beta)}{[(n^2 - q^2)^2 I^2 + \mu^2 q^2]^{\frac{1}{2}}}, \quad \quad (23)$$

where

$$\sin \beta = \frac{\mu q}{[(n^2 - q^2)^2 I^2 + \mu^2 q^2]^{\frac{1}{2}}}, \qquad \cos \beta = \frac{(n^2 - q^2) I}{[(n^2 - q^2)^2 I^2 + \mu^2 q^2]^{\frac{1}{2}}}.$$

Now, let the frequency, q, of the yawing motion be adjusted so as to produce the maximum periodic displacement ϕ corresponding to a given angular amplitude ψ_0 of yaw. If, then, ϕ_0 be the resonant amplitude of ϕ, the relation holds

$$\phi_0 = \frac{gC\psi_0}{\mu n \sqrt{1 - \mu^2/4n^2 I^2}} \quad \text{or} \quad C = \frac{\phi_0}{g\psi_0} \mu n \sqrt{1 - \frac{\mu^2}{4n^2 I^2}}. \quad (24)$$

The rolling moment due to an angular velocity of yaw is

$$(\partial L/\partial r) r = C\psi_0 \sin qt, \quad \quad (25)$$

where $\partial L/\partial r$ is a constant and r is a periodic variable. If the amplitude of the yawing angle be ψ_0, the amplitude of the yawing angular velocity is $\psi_0 q$. Hence, if amplitudes be taken in the equation (25), the result is that for the model

$$\frac{\partial L}{\partial r} = \frac{C}{q} = \frac{\phi_0}{\psi_0} \mu n \frac{\sqrt{1 - \frac{\mu^2}{4n^2 I^2}}}{\sqrt{n^2 - \frac{\mu^2}{2I^2}}}, \quad \quad (26)$$

or

$$\frac{\partial L}{\partial r} = \frac{\phi_0}{\psi_0} \mu \left(1 + \frac{\mu^2}{8n^2 I^2}\right), \quad \quad (27)$$

approximately.

The numerical value of $\partial L/\partial r$ for the model is, therefore, calculable from the ratio of the observed resonant amplitude ϕ_0 of roll to the forced amplitude ψ_0 of yaw and the frictional moment μ, which, for the model, is $\mu = g\partial L/\partial p$, apart from whatever mechanical friction there may be in the mechanism. The sign of $\partial L/\partial r$ is not determined by the experiment if only the amplitudes are observed. It may be determined from an observation on the phase difference between the forced oscillation and the forcing oscillation, or it may be considered as sufficiently well known from the calculation given above. The value of $\partial L/\partial r$ for the machine is obtained by scaling up the value for the model by multiplying by the velocity ratio and the fourth power of the ratio of the linear dimensions as in the case of other derivatives of moments by angular velocities.

In like manner, the value of $\partial N/\partial p$ may be obtained by leaving the model free to oscillate about the z-axis while forcing an oscillation in roll. The value of $\partial Z/\partial q$ could be obtained by forcing a pitching motion, and leaving the model free to oscillate in the vertical direction. The equation for that motion would be

$$W\frac{d^2z}{dt^2} + \mu\frac{dz}{dt} + \nu z = gZ = gC\theta_0 \sin qt,$$

and when adjusted to resonance, the value of $\partial Z/\partial q$ would be (with $\nu = n^2I$)

$$\partial Z/\partial q = \frac{Z_0}{\theta_0}\mu\left(1 + \frac{\mu^2}{8n^2W^2}\right),$$

approximately, where μ is $g\,\partial Z/\partial w$ except for mechanical friction. In a similar way $\partial X/\partial q$ could be measured by observing the oscillation in the X direction when an oscillation in pitch was forced.

(*d*) It is not necessary that the oscillations should take place about the center of gravity of the machine; but if they do not, the forces and moments introduced into the equations for the forced oscillation will have other terms in them than the simple ones here given. It is not even necessary that the motion should be a forced motion. Consider, for example, the oscillation as a pendulum of a model, held in the proper attitude of flight when in the position of equilibrium, about a point at a distance l below the axis of rotation. When the model on the end of its spindle of length l measured from the center of rotation to the center of gravity oscillates under the action of the wind, and of a restoring moment produced either by springs or by a counter-weight, the equation of motion will be

$$\frac{1}{g}(I + Wl^2 + I')\frac{d^2\theta}{dt^2} = -\nu\theta - \frac{\partial M}{\partial q}\frac{d\theta}{dt} - \frac{\partial X}{\partial u}u,$$

where I is the moment of inertia of the model about its point of attachment and I' is that of the suspension; for in this case there is also motion of the center of gravity backward and forward, which contributes the final term (the motion of the center of gravity in the normal direction being negligible if the oscillation be sufficiently small). But $u = lq = l\,d\theta/dt$, and hence

$$\frac{1}{g}\,(I + Wl^2 + I')\,\frac{d^2\theta}{dt^2} = -\,\nu\theta - \left(\frac{\partial M}{\partial q} + l\,\frac{\partial X}{\partial u}\right)\frac{d\theta}{dt} \quad \ldots \quad (28)$$

Therefore, the solution of this equation in θ would give for the model except for mechanical friction the two quantities $\partial M/\partial q$ and $\partial X/\partial u$ in combination. By making the experiment with two different values of l, for example, with $l = 0$ in one case, it would be possible to determine not only $\partial M/\partial q$ but $\partial X/\partial u$, without using the method of forced oscillations. In like manner, if the model were oscillating about a center at a distance l to the rear of the center of gravity, the differential equation obtained would involve $\partial M/\partial q$ and $\partial Z/\partial w$.

The rule for scaling up a quantity like $\partial X/\partial u$ or $\partial Z/\partial w$ calls for the first power of the velocity ratio and the square of the linear dimension, whereas the rule for $\partial M/\partial q$ calls for the first power of the velocity ratio and the fourth power of the linear dimension. It, therefore, follows that if the experiments were made with a $\frac{1}{24}$ sized model the value of

$$\frac{\partial X/\partial u \text{ for model}}{\partial M/\partial q \text{ for model}} = 576\,\frac{\partial X/\partial u \text{ for machine}}{\partial M/\partial q \text{ for machine}}.$$

Now, for the machine $\partial X/\partial u$ is very small, whereas $\partial M/\partial q$ is large, and if it were necessary to perform the oscillating experiment upon the full-sized machine it is probable that $\partial X/\partial u$ occurring in the same equation with $\partial M/\partial q$ would escape detection; but when the experiment is performed on the model the values of $\partial X/\partial u$ and $\partial M/\partial q$ are not so dissimilar that one would entirely mask the other (owing to the different scale ratio 1 : 576).

(*e*) There have now been treated the following lateral coefficients:

$$\frac{\partial(Y, L, N)}{\partial v}, \quad \frac{\partial(L, N)}{\partial p}, \quad \frac{\partial(L, N)}{\partial r}.$$

This leaves only two, $\partial Y/\partial p$ and $\partial Y/\partial r$. The value of $(\partial Y/\partial p)p$ is the variation of side force due to angular velocity of roll. This may arise from fin surface, *i.e.*, vertical surface parallel to the axis of the machine, or its equivalent. For example, if there is a surface of area S at a distance l above the axis of rotation, the angular velocity p will give the surface a lateral velocity lp, causing an angle lp/U with the relative wind and a cross pressure

$$Y = -.0015\ (.032 + .005r)SU^2 \times 57.3lp/U,$$

where r is aspect ratio. Hence,

$$\frac{\partial Y}{\partial p} = \frac{Y}{p} = -.0015\ (.032 + .005r)SlU \times 57.3. \quad . \quad . \quad . \qquad (29)$$

This derivative varies with the cube of the linear dimension and with the speed. If the wings have a dihedral angle so that they are not horizontal in normal flight but are either raised or lowered at the tips, the presence of the dihedral is equivalent to a certain amount of fin surface. For if the machine has an angular velocity p so that the right wing is moving down, and the left up, the pressure on the right is increased and on the left is decreased (owing to the change in direction of the relative wind). The increase and decrease are of equal amounts, and for wings without dihedral would produce no effect on Y, but with dihedral the pressures, being normal to the wing, have components toward the plane of symmetry as well as parallel to it, and one of these components is lengthened, whereas the other is shortened, thus setting up a side thrust $(\partial Y/\partial p)p$ equivalent to a fin surface.

Consider finally $(\partial Y/\partial r)r$, the variation of side thrust due to yawing, or rather to angular velocity of yaw. The value of this depends on the balance of vertical or fin surface fore and aft of the C. G., just as $(\partial Y/\partial p)p$ depended on the distribution above and below. It is a sort of weather-vane effect. So is the value of $(\partial N/\partial r)r$ — but with this difference: the yawing moment coefficient $\partial N/\partial r$ is increased by the addition of fin surface whether fore or aft of the C. G., whereas the side thrust coefficient is increased by fin surface fore and diminished by fin surface aft. In the ideal case of a fin surface of area S a distance l behind the C. G., the angular velocity r yields a linear velocity lr which sets up an angle lr/U with the relative wind and hence a force

$$Y = .0015(.032 + .005r)SU^2 \times 57.3lr/U,$$

and

$$\frac{\partial Y}{\partial r} = .0015(.032 + .005r)SlU \times 57.3 \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

For a positive yaw this is negative with U, *i.e.*, the thrust is toward the right.

54. Longitudinal Stability. Although the longitudinal dynamical stability of the airplane was discussed with fixed axes, it will be taken up briefly here, with a revised notation, for the purpose of making a few additions. The longitudinal motion is in u, w, θ and the equations are (9). Let

$$\frac{g}{W}\frac{\partial X}{\partial u} = X_u, \qquad \frac{g}{W}\frac{\partial X}{\partial w} = X_w, \qquad \frac{g}{W}\frac{\partial M}{\partial q} = M_q, \text{ etc.}$$

Then if k_B^2 be written for the radius of gyration (squared) about the pitching axis y, the equations become

$$\left.\begin{aligned} du/dt &= X_u u + X_w w + X_q q + g\theta, \\ dw/dt &= Z_u u + Z_w w + (Z_q + U)q, \\ k_B^2 dq/dt &= M_u u + M_w w + M_q q. \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (31)$$

The substitution of expressions of the type $e^{\lambda t}$ gives the equation

$$\Delta_1(\lambda) = \begin{vmatrix} \lambda - X_u & -X_w & -(X_q\lambda + g) \\ -Z_u & \lambda - Z_w & -(Z_q + U)\lambda \\ -M_u & -M_w & k_B^2\lambda^2 - M_q\lambda \end{vmatrix} = 0. \; . \quad (32)$$

This is biquadratic of the form

$$\Delta_1(\lambda) = A_1\lambda^4 + B_1\lambda^3 + C_1\lambda^2 + D_1\lambda + E_1 = 0,$$

with

$$\begin{aligned} A_1 &= k_B^2, \qquad B_1 = -M_q + k_B^2(-Z_w - X_u), \\ C_1 &= k_B^2(X_u Z_w - X_w Z_u) - M_q(-Z_w - X_u) - (Z_q + U)M_w - M_u X_q, \\ D_1 &= -M_q(X_u Z_w - X_w Z_u) - (Z_q + U)(X_u M_w - M_u X_w) \\ &\quad - M_u g + (Z_w M_u - M_w Z_u)X_q, \\ E_1 &= +g(Z_w M_u - M_w Z_u). \end{aligned} \qquad (33)$$

Routh's discriminant, now that the coefficient of λ^4 is $k_B^2 = A_1$, is

$$R_1 = B_1 C_1 D_1 - A_1 D_1^2 - E_1 B_1^2. \; . \; . \; . \; . \; . \; . \quad (34)$$

The condition for stability is that all the coefficients and R be positive.

A typical set of values for the coefficients is as follows (JN-2):

$$\begin{aligned} &k_B^2 = 34, & &U = -115.5, & &W = 1800, & &g = 32.17, \\ &X_u = -.128, & &Z_u = -.557, & &X_w = +.162, & &Z_w = -3.95, \quad (35) \\ &M_w = +1.74, & &M_q = -150, & &M_u, X_q, Z_q \text{ all small.} \end{aligned}$$

Consider for example M_u. If the propeller thrust passes through the C. G., $M_u = 0$.

$$M_u = \frac{g}{W}\frac{\partial M}{\partial u} = \frac{g}{W}\frac{2M}{U} = \frac{g}{W}\frac{2hT}{U},$$

where h is the distance above the C. G. of the line of T. Suppose $T = 400$ lb, which is near the value of the drag in this case. Then

$$M_u = \frac{1}{56}\frac{2h \times 400}{-115} = -\frac{h}{8} \text{ (nearly)}.$$

If then h were as great as 2 ft (and it is not over a few inches in this machine), the value of M_u would be $\pm .25$ according as h were negative or positive. Would this be effectively a large or a small value? For that the general size of the coefficients (33) must be estimated. Now,

$$X_uZ_w - X_wZ_u = .128 \times 3.95 + .162 \times .557 = \text{roughly } 0.6,$$
$$X_uM_w - M_uX_w = -.128 \times 1.74 - M_u \times .162,$$
$$Z_wM_u - M_wZ_u = -3.95M_u + 1.74 \times .557.$$

The value of M_u does not affect A_1 or B_1; in C_1, which is the sum of three large terms with a total magnitude running into the hundreds, it appears only in the doubly small term $-M_uX_q$; in D_1 the initial term is about 90 and the second term so far as X_uM_w goes contributes about 20 more, whereas even if M_u were so large as 0.25 the term containing it would only contribute about ± 3. It is safe to assume, therefore, that M_u is without appreciable effect on any but E_1. Here, however, M_u is multiplied by the large quantity Z_w, whereas the other term is small (about 1.0). Hence a value of M_u approximating 0.25 would bring E_1 down toward the vanishing point and make toward instability. If M_u is to be regarded as "small" and to be put equal to 0, it must be decidedly smaller than 0.25, — as indeed it is for this machine — or M_u must be negative (the propeller must pass under the C. G. so that M is negative). In the latter case M_u increases E_1 which is favorable to stability unless the increase is so great as materially to reduce R_1.

The discussion for X_q and Z_q may be carried on similarly. As Z_q occurs only when combined with U, and then only in those terms of C_1 and D_1 which are added to other large terms, it appears that a value of 9 or 10 for Z_q could be considered "small." A rough guess at Z_q showed (Art. 39) that it should be of the magnitude of M_q divided by the length of the machine from C. G. to tail — some 15 ft. Hence Z_q is probably in the neighborhood of 10 and is "small." As for X_q, it occurs negligibly in C_1, and in D_1 appears with a coefficient of only about 1 whereas the rest of D_1 is around 100. Hence X_q would be "small" if not greater than 2 or 3. But the X force set up by an angular velocity of pitch can, judging from the configuration of the airplane, be at most only a small part of the Z force thus set up. Hence X_q appears surely as small as 2 or 3 and probably much smaller. The final conclusion is that X_q, Z_q, and M_u may be set equal to zero for the machine in question where h is small — whether M_u be negligible for a seaplane in which the engines are high is far from certain from the above analysis.

According to the approximation made before, the biquadratic may be factored into the two quadratics

$$\left(\lambda^2 + \frac{B_1}{A_1}\lambda + \frac{C_1}{A_1}\right)\left[\lambda^2 + \left(\frac{D_1}{C_1} - \frac{B_1E_1}{C_1^2}\right)\lambda + \frac{E_1}{C_1}\right] = 0, \quad . \quad (36)$$

and hence may be solved. The approximate periods and times of damping to half-amplitude may thus be found. If there is reason to doubt whether the approximation is good enough, the approximate value $\lambda = r$ found for one of the roots may be used as the basis for the calculation of a second approximation as follows: Suppose $x = r$ is near a root of $f(x) = 0$. The function $f(x)$ may be expanded in powers of $x - r$ by Taylor's formula.

$$f(x) = f(r) + (x - r)f'(r) + \tfrac{1}{2}(x - r)^2 f''(r) + \quad . \; . \; .$$

If the first two terms be taken as a sufficient approximation (since $x - r$ is small), the equation $f(x) = 0$ becomes linear and

$$x = r - \frac{f(r)}{f'(r)} = r - \frac{A_1r^4 + B_1r^3 + C_1r^2 + D_1r + E_1}{4A_1r^3 + 3B_1r^2 + 2C_1r + D_1} \quad . \; . \quad (37)$$

This will probably give a sufficient approximation. In the case, however, of a complex value of r such as $a + bi$, the calculation is laborious. The work may be shortened (especially in calculating the numerator) by using the equation

$$A_1r^2 + B_1r + C_1 = 0 \qquad \text{or} \qquad C_1r^2 + D_1r + E_1 = B_1E_1r/C_1,$$

according as r is a root of the first or second quadratic factor.

A rough estimate of the coefficients in the biquadratic is useful to indicate the main sources of stability in the design. Leaving aside the smaller terms as judged by (35),

$$A_1 = k_B^2, \quad B_1 = -M_q - k_B^2 Z_w, \quad E_1 = -gZ_u M_w,$$
$$C_1 = Z_w M_q - UM_w, \quad D_1 = -X_u(Z_w M_q - UM_w).$$

The short, quickly damped oscillation is so quickly vanishing as to be negligible (except possibly for the calculation of accelerations and stresses). The long, slowly damped oscillation is determined by

$$\lambda = -\frac{1}{2}\left(\frac{D_1}{C_1} - \frac{B_1 E_1}{C_1^2}\right) \pm i\sqrt{\frac{E_1}{C_1} - \left(\frac{D_1 C_1 - B_1 E_1}{C^2}\right)^2}.$$

For damped motion $C_1 D_1 > B_1 E_1$ or

$$-X_u(Z_w M_q - UM_w)^2 > gZ_u M_w(M_q + K_B^2 Z_w).$$

But $Z_u = 2g/U$ and $X_u : Z_u = X : Z = T : W$, where T is the propeller thrust or drag. Hence,

$$(Z_w M_q - UM_w)^2 > g\frac{W}{T}(-M_q - k_B^2 Z_w). \quad . \quad . \quad . \quad . \quad (38)$$

Now M_w is a measure of the statical stability; it is obtained as the slope of the moment curve, and may be varied widely merely by altering the setting of the stabilizer; a large M_w favors stability. M_q and Z_w enter squared on the left but only linearly on the right. Hence, increasing Z_w and M_q favors stability. Z_w will be larger as the wing loading is smaller so that a machine with ample wing area per unit mass should be more stable than one with high load. M_q is partly due to the side motion of the tail (in fact the part of M_q due to the tail is $-lM_w$ if l is the distance of the tail behind the C. G.) and that part of M_q may be increased by lengthening the body. The tail, however, is responsible for only a part of M_q and that part due to the wings would be increased (like Z_w) by lighter wing-loading. Inasmuch as the airfoil itself, by virtue of the contrary motion of the center of pressure, is unstable, the use of a wing or combination of wings with smaller travel of C. P. would be equivalent to additional stabilizer or to greater stability. Further, stability is increased by a low L/D or W/T and a small moment of inertia Wk_B^2. Increased speed is also clearly favorable to stability.

Each of these criteria has been stated independently on the basis of the inequality that must be satisfied for stability, *i.e.*, for damping, and the words "other things being equal" should everywhere be understood. When a given machine flies faster or slower, the attitude changes and with that the values of the coefficients.

But the changes are apparently conspiring to increase stability as the speed increases, because Z_w, M_q, U, M_w, are T and all increasing. The period should also be considered, as well as the damping, because a long period would make easier riding than a short period, *i.e.*, it is more comfortable to have

$$\frac{E_1}{C_1} - \frac{(D_1C_1 - B_1E_1)^2}{C_1^2} = \frac{-gZ_uM_w}{Z_wM_q - UM_w} - \frac{[-X_u(Z_wM_q - UM_w)^2 - gZ_uM_w(M_q + k_B^2Z_w)]^2}{(Z_wM_q - UM_w)^2}$$

small rather than large. The numerator of the last fraction is increasing as the stability, measured by the damping coefficient, increases. The denominator is also increasing, but apparently not so fast because $Z_wM_q - UM_w$ enters only squared instead of raised to the fourth power. Hence, the negative part is increasing — which is desired. The first fraction is positive and as $Z_u = 2g/U$ may be written as

$$\frac{-gZ_uM_w}{Z_wM_q - UM_w} = \frac{2g^2}{-U(Z_wM_q/M_w - U)}.$$

This part, therefore, decreases with Z_w, U, and M_q but increases with M_w. Hence for comfort the statical stability (M_w) should not be too large, and dynamical stability should be secured rather from Z_w, U, M_q. A machine, however, will not be executing oscillations unless it be disturbed. Hence, for comfort, it is important so to design the airplane that gusts do not disturb it; this puts further conditions on the aerodynamic coefficients and on the design. The matter will, however, not be pursued here.

Calculation shows that at low speed both the "Clark" and the "JN-2" become unstable. The change in the aerodynamic coefficients may be estimated as follows. Z_u which is $2g/U$ must increase as U decreases. As $X_u : Z_u = X : Z = D : L$, and as D/L increases at low speeds, $X_u : Z_u$ must increase, and so must X_u. But as D/L at low speed is considerably less than D/L at high speed, the value of X_u as U diminishes may first decrease, and increase only when the speed of least drag is approached; it cannot be affirmed that X_u is necessarily larger at a low speed than at high speed, but only that if the speed becomes low enough X_u increases. The whole range of X_u is ordinarily not great, though it runs up rapidly when the speed falls below that of minimum drag. The value of X_w diminishes and may even become negative; for X_w is calculated from

$$\frac{\partial X}{\partial w} = \frac{57.3}{U}\frac{dL}{di} + \frac{D}{U}, \qquad \frac{\partial Z}{\partial w} = \frac{57.3}{U}\frac{dD}{di} - \frac{L}{U},$$

and as i increases L approaches a maximum and dL/di tends to vanish, whereas D increases, so that evidently there must come a value of i, less than that of maximum lift, where X_w turns negative. The trend of Z_w is from a large negative value to a smaller negative value, though the formula does not make this so clear — and the minimum for Z_w may actually be passed. M_w obtained by differencing the moment curve and dividing by U increases at first, while the differences of M are nearly constant, but subsequently falls off as differences for M become smaller at large angles, whereas U changes only slowly at these attitudes. M_q falls off steadily as might be expected from its physical origin. Reference to the criterion of stability shows that every change is toward instability. It is advantageous to preserve stability down to as low as the landing speed, because irregularities in the air are great near the ground and amplifying oscillation of the airplane would make landing more hazardous.

These tables taken from Hunsaker's "Smithsonian" and "Nat. Adv. Comm." papers give data by which the above discussion may be checked.

Data for the "Clark," V = mi/hr, $W = 1600$, $k_B^2 = A_1 = 21.6$.

V	76.9	53.4	44.6	36.9
i	0	3	6	12
X_u	−.158	−.12	−.119	−.162
X_w	+.356	+.249	+.245	0
Z_u	−.57	−.823	−.985	−1.19
Z_w	−5.62	−3.77	−2.92	−1.0
M_w	+3.2	+3.99	+2.25	+1.41
M_q	−192	−123	−93.7	−60.5
B_1	317	207	159	85.1
C_1	1492	804	444	150
D_1	266	128	72.6	22.1
E_1	59.2	106	71.4	54
$10^{-6}R_1$	117	16.4	3.2	−.12
	stable	stable	stable	unstable
period T	34.7	17.6	15.8	10.6
time t	8.1	11.0	13.1	24.7

Data for Curtiss JN-2. V = mi/hr, W = 1800, $k_B^2 = 34$.

V	79	51.8	47	45.2	44.2	43.7
i	1	7	10	12	14	$15\frac{1}{2}$
X_u	−.128	−.121	−.151	−.189	−.223	−.276
X_w	+.162	+.113	−.075	−.236	−.132	−.292
Z_u	−.557	−.849	−.936	−.972	−.993	−1.01
Z_w	−3.95	−2.26	−1.46	−.736	−.553	−.673
M_w	+1.74	+2.45	+2.50	+2.15	+1.99	+2.02
M_q	−150	−113	−108	−106	−106	−106
B_1	289	194	165	$137\frac{1}{2}$	134	138
C_1	834	467	355	243	213	226
D_1	115	64.3	42.5	17.4	28	24.2
E_1	31	67	75.3	67.2	63.6	65.7
$10^{-6}R_1$	18	3.2	8.3	−.70	−.37	−.50
	stable	stable	stable	unstable	unstable	unstable
period T	34.3	16.7	13.7	12	...	11.6
time t	10.8	17.7	62.7	16	...	19.3

55. Lateral Stability. The equations for the lateral oscillations which involve roll, yaw, and side-slip are (10) which may be written

$$\left.\begin{aligned} dv/dt + Ur &= Y_v v + Y_p p + Y_r r - g\phi, \\ k_A^2 dp/dt - k_E^2 dr/dt &= L_v v + L_p p + L_r r, \\ k_C^2 dr/dt - k_E^2 dp/dt &= N_v v + N_p p + N_r r, \end{aligned}\right\} \quad \ldots \quad (39)$$

with k_A^2 and k_C^2 as the radii (squared) of gyration about x and z respectively, with $k_E^2 = E/W$, and with the coefficients on the right taken as the corresponding derivatives multiplied by g/W as in the former case. The variables in these equations are v, ϕ, r. It is necessary to use the angle of roll but not the angle of yaw. The substitution of $e^{\lambda t}$ leads to the equation

$$\Delta_2(\lambda) = \begin{vmatrix} \lambda - Y_v & -Y_p\lambda + g & -Y_r + U \\ -L_v & k_A^2\lambda^2 - L_p\lambda & -(k_E^2\lambda + L_r) \\ -N_v & -(k_E^2\lambda^2 + N_p\lambda) & k_C^2\lambda - N_r \end{vmatrix} = 0,$$

which again is biquadratic of the form

$$\Delta_2\lambda = A_2\lambda^4 + B_2\lambda^3 + C_2\lambda^2 + D_2\lambda + E_2,$$

with

$$A_2 = k_A^2 k_C^2 - k_E^4, \qquad E_2 = g(-N_r L_v + N_v L_r),$$
$$B_2 = -Y_v(k_A^2 k_C^2 - k_E^4) - N_r K_A^2 - L_p k_C^2 - k_E^2(L_r + N_p),$$

$$C_2 = N_rL_p - L_rN_p + k_C^2(Y_vL_p - L_vY_p) + k_A^2(N_vU - N_vY_r + Y_vN_r)$$
$$+ k_E^2(N_pY_v - Y_pN_v + Y_vL_r - L_vY_r + L_vU),$$
$$D_2 = -Y_v(N_rL_p - L_rN_p) + gk_C^2L_v + gk_E^2N_v - U(N_vL_p - L_vN_p)$$
$$+ Y_p(L_vN_r - N_vL_r) + Y_r(L_pN_v - N_pL_v).$$

The value of the product of inertia E being unknown it is assumed that $E = 0$ and that the forward and vertical axes are principal axes. This cannot be true for all values of i. But

$$B = \int(x^2 + z^2)dW, \qquad E = \int xz\,dW, \qquad B \pm 2E = \int(x \pm z)^2dW.$$

The last integral is necessarily positive and hence E must be numerically less than $B/2$. For the "Clark" $k_B^2 = 21.6$ and k_E^2 can only be about 10 at the most, whereas $k_A^2 = 27$ and $k_C^2 = 48.6$. For the JN-2, $k_B^2 = 34$ and k_E^2 is only about 16 at most, whereas $k_A^2 = 36.7$ and $k_C^2 = 70.6$. In both cases k_B^4 is small compared with $k_A^2k_C^2$ even at worst. As a matter of fact the third integral above can only vanish when the mass is all concentrated in a single plane $x = z$ or $x = -z$. This is far from the case in the airplane and hence it is probable that E is much less than $B/2$. The theory of change of axis (Art. 46) shows that if $E = 0$ when $i = 0$, E will be $\frac{1}{4}$ $(C - A)$ when the axes are turned through 15°. Hence if in the JN-2 the axes are principal at highest speed, k_E^2 will be about as high as 9 at lowest speed. It is probable that E may be neglected as assumed; but the applied mechanics of the airplane is indeed far from complete when, entirely apart from the aerodynamical coefficients, the dynamical characteristics are unknown. The experimental determination of the moments of inertia about three different axes through the C. G. in the x-z-plane would fix the position of the principal axes and give the moments of inertia about them.

To have an idea of the magnitude of the various coefficients the following data given for an old Blériot may be considered.

Data for Blériot (National Physical Laboratory, Teddington, Eng.)

$W = 1800$,	$i = 6°$,	$V = 65$,	$U = -95.4$,
$k_A = 5$,	$k_C = 6$,	$Y_p = 0$,	$Y_r = 0$,
$Y_v = -.108$,	$L_v = +.70$,	$N_v = -.44$	$L_p = -167$,
$N_p = +24$,	$L_r = +54$,	$N_r = -31$,	$A_2 = 900$,
$B_2 = 6780$,	$C_2 = 5580$,	$D_2 = 6640$,	$E_2 = -68$.

The machine is unstable because one of the coefficients, E_2, is negative; $R_2 = 21.5 \times 10^{10}$ and is positive. The biquadratic is

$$600\lambda^4 + 6780\lambda^3 + 5580\lambda^2 + 6640\lambda - 68 = 0.$$

The noteworthy feature is that the ratio E_2/D_2 is small, so very small as to indicate a small root about equal to $\lambda = -E_2/D_2 = .01$ and a typical solution $e^{.01t}$. This is not an oscillation but a straight amplifying term, doubling in 69 secs. A better approximation to the root may be found by (37), which is easy to apply when the root is real. A formula may also be developed by considering

$$C_2\lambda^2 + D_2\lambda + E_2 = 0, \qquad \lambda = -\frac{E_2}{D_2}(1 + \eta), \ \eta \text{ small},$$

$$C_2\frac{E_2^2}{D_2^2} + 2C_2\frac{E_2^2}{D_2^2}\eta - E_2 - E_2\eta + E_2 = 0,$$

$$\eta = \frac{C_2E_2/D_2^2}{1 - 2C_2E_2/D_2^2} = \frac{C_2E_2}{D_2^2}\left(1 + 2\frac{C_2E_2}{D_2^2}\right).$$

This shows that the percentage correction η to λ is small whenever C_2E_2/D_2^2 is small — in this case wholly negligible.

As the biquadratic has one real root, it must have another. Moreover physical reasons will show that there should be a large negative real root. For the frictional resistance to rolling is very great owing to the wing surface catching the air (L_p is large) and there is no large restoring moment opposing roll as there is in the case of pitching (M_w). The machine should act in roll like a pendulum in a very viscous fluid where the motion is non-oscillatory. A large root could be sought from the first two terms of the equation as $\lambda = -B_2/A_2 = -11.3$ corresponding to a heavily damped rolling motion $e^{-11.3t}$. If again the assumption be made that

$$A_2\lambda^2 + B_2\lambda + C_2 = 0, \qquad \lambda = -\frac{B_2}{A_2}(1 + \eta), \ \eta \text{ small},$$

$$A_2\frac{B_2^2}{A_2^2} + 2B_2\eta - \frac{B_2^2}{A_2} - \frac{B_2^2}{A_2}\eta + C_2 = 0,$$

$$\eta = \frac{C_2A_2}{B_2^2 - 2B_2A_2} = \frac{C_2A_2}{B_2^2}\left(1 + 2\frac{A_2}{B_2}\right).$$

The percentage correction η is in this case larger, some 8%. It will, therefore, be better to take as the value of λ

$$\lambda = -\frac{B_2}{A_2} + \frac{C_2}{B_2}; \qquad \text{or} \qquad \lambda + \frac{B_2^2 - A_2C_2}{B_2} = 0$$

as a factor of the biquadratic. Two roots or factors are, therefore, approximately known.

The other factor will represent a pair of complex roots; for obviously a machine may have an oscillation in yaw of the weather-vane type. One might try the intermediate terms $B_2\lambda^2 + C_2\lambda + D_2$ for that factor, and corrections to it could be made. The general practice has led to the use of the factor

$$\lambda^2 + \left(\frac{C_2}{B_2} - \frac{E_2}{D_2}\right)\lambda + \frac{B_2D_2}{B_2{}^2 - A_2C^2} = 0.$$

The correction term $-E_2/D_2$ is generally very small, but there are cases where it is not small. The product of all three factors gives the biquadratic approximately.

$$\left(\lambda + \frac{E_2}{D_2}\right)\left(\lambda + \frac{B_2}{A_2} - \frac{C_2}{B_2}\right)\left[\lambda^2 + \left(\frac{C_2}{B_2} - \frac{E_2}{D_2}\right)\lambda + \frac{B_2D_2}{B_2{}^2 - A_2C_2}\right] = 0.$$

NOTE. The aerodynamic coefficients have been taken as the corresponding derivatives multiplied by g and divided by W as was natural in the simplification of equations (38), Chap. VI. It would be possible to divide the moment equations by Wk^2 (with the appropriate k for each axis). Then the moment equations would become equations in angular acceleration, just as the force equations have become equations in acceleration. With this convention

$$M'_q = M_q/k^2_B, \quad M'_w = M_w/k^2_B, \quad M'_u = M_u/k^2_B,$$
$$L'_p = L_p/k^2_A, \quad L'_v = L_v/k^2_A, \quad L'_r = L_r/k^2_A,$$
$$N'_r = N_r/k^2_C, \quad N' = N_v/k^2_C, \quad N'_p = N_p/k^2_C.$$

This convention appears natural if the principal axes are the axes of reference so that $E = 0$, less natural when $E \neq 0$. Cowley and Levy in their *Aeronautics* give a set of coefficients with the k^2 divided out as:

$$X_u = -.14, \quad Z_u = -.08, \quad M_u = 0,$$
$$X_w = .19, \quad Z_w = -2.89, \quad M_w = .106,$$
$$X_q = \pm.5, \quad Z_q = 9.0, \quad M_q = -8.4,$$
$$Y_v = -.25, \quad L_v = .0332, \quad N_v = -.015,$$
$$Y_p = 1, \quad L_p = -8, \quad N_p = -.57,$$
$$Y_r = -3, \quad L_r = 2.6 \quad N_2 = -1.05.$$

To compare this set with that given above for the Blériot, it is merely necessary in the case of the Blériot to divide the L's by 25 and the N's by 36. Thus,

$$L_p = -6.7, \quad L_r = +2.2, \quad L_v = +.028,$$
$$N_r = -.86, \quad N_p = +.67, \quad N_v = -.012.$$

56. Stability Calculation. The details for longitudinal stability were well enough covered in Chap. VI. For lateral stability the procedure is as follows:

(*a*) To find the derivatives by v it is necessary to have the experimental data for Y, L, N when the machine is yawed. The following are Hunsaker's data on the "Clark." The values have been converted already to the full-sized machine, multiplied by g and divided by W.

	$i = 0°$			$i = 6°$			$i = 12°$		
ψ	Y	L	N	Y	L	N	Y	L	N
0	0	0	0	0	0	0	0	0	0
5	−2.06	25.9	−4.42	−.516	19.55	−2.04	−.45	8.65	−2.53
10	−4.31	40.2	−13.45	−1.124	34.1	−4.43	−.95	17.6	−5.95
15	−6.74	54.0	−22.1	−1.99	43.8	−8.54	−1.51	24.7	−9.35

Hence at $i = 0°$, $Y_v = -\partial Y/\partial\psi \times 57.3/U = +.401 \times 57.3/112\frac{1}{2} = -.205$. (The differences are −2.06 and −2.25 and increase .19 in 5°; it is probable that the difference at $\psi = 0$ should be something like −1.96. Yet if the value of Y is symmetric, the point $\psi = 0$ would be an inflection point and the difference may not be so far from −2.06 as backward interpolation would indicate.) Next $\Delta L = 25.9$ and $\Delta L/\Delta i = 5.18$. But here the difference is changing rapidly from 25.9 to 14.3. Probably an intermediate measurement, say at $\psi = 2\frac{1}{2}°$ should be made. It is unsafe either to plot ψ and fair the curve or to make an allowance for the curvature. However, one may guess $\Delta L = 30$ at $i = 0$ and $\Delta L/\Delta i = 6$. Then $L_v = 3.07$. In like manner $\Delta N = -4.42$ but increasing in 5° to 8.03. Choose $\Delta N = 3.00$, then $N_v = -.307$; whereas with $\Delta N = -4.42$, $N_v = .46$. Hunsaker has .449. Any result is liable to considerable error unless the data are given at closer intervals. However, if the machine is to yaw 5° in flight, the discussion for stability may perhaps be better given by using the values $\Delta Y/5$, $\Delta L/5$, $\Delta N/5$ which are average values from 0° to 5° than by using the actual values $dY/d\psi$, $dL/d\psi$, $dN/d\psi$ at $\psi = 0$.

(*b*) For $\partial L/\partial r$ use the formula (18) with the span $s = 40.2$ for the "Clark." Also to have L_r multiply by g and divide by W.

$$L_r = \frac{-g(40.2)^2}{-6 \times 112\frac{1}{2}} = \frac{32.17 \times 1616}{675} = +77.0.$$

For N_p use (19). Then

$$N_p = -X_w \frac{1616}{12} = -.356 \times 135 = -48.$$

To check L_p use (20)

$$L_p = Z_w \times 135 = -5.62 \times 135 = -759.$$

The value given by the oscillation of the model is -631. The check is not very good. To check N_r use (21).

$$N_r = X_u \times 135 = -.158 \times 135 = -21.3.$$

The oscillator gives -39.4. This cannot be considered a bad check because the calculation determines only the effect of the wings on N_r whereas the body and fin surface must have considerable effect. In the case of L_p the calculation gives for the wings more than the oscillator shows for the whole machine. As a matter of fact the far greater part of L_p must be due to the wings because of their large extent, distance from the axis of rotation, and motion about that axis. There is no need here to enter into the details of the calculation of L_p and N_r from the observed damping of the oscillation of the machine about the x- and z-axes — the work is exactly like that for M_q. The method of calculating L_r and N_p from observations on forced resonant oscillations will be omitted. The calculation of Y_p and Y_r both of which are "small" will not be given further than illustrated in principle by the formulas (29) and (30).

(*c*) The following data are given by Hunsaker for the "Clark" in three attitudes and the JN-2 in two, in his Smithsonian paper. I have, however, substituted my own calculation for N_p by (19) which differs from his.

"Clark": $W = 1600$, $k_A = 5.2$, $k_C = 6.975$, $s = 40.2$;
JN-2: $W = 1800$ $k_A = 6.06$, $k_C = 8.4$, $s = 36$.

	Clark Tractor			Curtiss JN-2	
V	76.9	44.6	36.9	78.9	43.6
i	0	6	12	1	$15\frac{1}{2}$
Y_v	$-.204$	$-.0878$	$-.106$	$-.248$	$-.09$
L_v	$+3.06$	$+3.44$	$+1.91$	$+.844$	$+2.7$
N_v	$-.449$	$-.351$	$-.53$	$-.894$	$-.45$
L_p	-631	-319	-224	-314	-78
N_p	-48	-33	0	-17.2	$+31.7$
L_r	$+77$	$+132.5$	$+160$	$+55.2$	$+101$
N_r	-39.4	-26.0	-38.9	-27.0	-30.4

(*d*) The remainder of the calculation requires merely first that these numerical values be substituted in the formulas for the coefficients using $k_E^2 = Y_p = Y_r = 0$, as "small" quantities; and second, that the values of the coefficients thus found be substituted in R_2 to ascertain whether all coefficients and R_2 be positive or whether one or more be negative. Thus stability is determined. To ascertain how great or small the stability may be the roots of the biquadratic must be calculated from the approximate factors given. In all this treatment of lateral stability, as of longitudinal, it has been assumed that only oscillations from horizontal flight are concerned. Calculations could also be made relative to uniform motion in an inclined line, motion at different levels (air-density different), gyroscopic effect of propeller, etc. These questions must be reserved for more advanced study.

57. Balance of the Airplane. The question of dynamical balance can only be settled when the principal axes are known. The matter of aerodynamical balance refers to the placing of surface fore and aft, above and below the center of gravity in such a way as to secure the desired flying properties. For longitudinal motion this was treated above. Most machines appear longitudinally stable except at very low speeds and the chief adjustments are in stabilizer, both in area and in angle, so as to get satisfactory values of M_w and M_q — the position of the C. G. being dependent on the stabilizer. For lateral motion the matter of balance may be presented thus: The preponderating terms in A_2, B_2, C_2, D_2, E_2 are

$$A_2 = k_A^2 k_B^2, \qquad E_2 = g(N_v L_r - L_v N_r),$$

$$B_2 = -Y_v k_A^2 k_B^2 - L_p k_C^2 - N_r k_A^2,$$

$$C_2 = (N_r L_p - L_r N_p) + Y_v L_p k_C^2 + k_A^2(N_v U + N_r Y_v),$$

$$D_2 = -Y_v(N_r L_p - N_p L_r) + U(N_p L_v - N_v L_p) + g k_C^2 L_v.$$

(*a*) "Rolling." Root $\lambda = -B_2/A_2 - C_2/B_2$. The main part is $-B_2/A_2$, and approximately

$$\lambda = -\frac{B_2}{A_2} = \frac{L_p}{k_A^2} + \frac{N_r}{k_C^2}.$$

The values of L_p and N_r are large and negative. The root λ is large and negative. Rolling dies out. Observe, however, that the value of λ depends both on the coefficients L_p, k_A^2 for roll and on the coefficients N_r, k_C^2 for yaw. It cannot be assumed that roll, yaw, and side-slip are independent. A "rolling" motion in reality is a combination of all three and the damping is on this combined motion. (Compare the physical theory of "normal" coördinates or "normal" modes of motion in coupled systems.) "Roll" is so strongly damped as to be negligible in the discussion of balance.

(*b*) "Spiral." Root $\lambda = -E_2/D_2$. The quantity D_2 is always large and positive. The criterion, therefore, as to whether the machine is stable or unstable is the sign of E_2, — stable if E_2 is positive, unstable if E_2 is negative, and in either case a small amount of stability or instability because E_2/D_2 is small. The condition for stability is

$$N_v L_r - L_v N_r > 0.$$

Now N_r, the (damping) moment due to yawing, is negative; it is a weather-vane effect and may be increased by lengthening the body or increasing the vertical fin surface. It is the amount of this surface and its distance from the C. G. which is important, rather than its position, whether fore or aft, in augmenting N_r negatively. As L_v is positive, any such increase in N_r favors stability. On the other hand N_v, the yawing moment due to side-slipping, depends on the balance of vertical fin fore and aft of the C. G. Equal amounts of surface added or subtracted at equal distances before and behind the C. G. would not alter N_v (other things being equal). From the data it is seen that N_v is negative, as might be expected from the configuration of the machine. Diminishing N_v favors stability, and this means evening up the fore and aft vertical surface. L_v, a rolling moment due to side-slip, depends on the balance of vertical surface above and below the C. G. Increasing L_v favors stability. This means the surface should rather be above than below the C. G. By (18) L_r depends on the span s — fixed for a given design except for major alterations.

As the numerical magnitudes of E_2 are small, it appears that relatively minor changes will make a shift from stability to instability or inversely. No serious changes would be needed. L_r increases as the speed decreases; this is unfavorable to stability at low speeds. N_r is reasonably constant relative to speed; in this respect it differs markedly from either L_p or M_q. But this difference is to be anticipated because as the angle i increases more and more of the larger wing surface is opposing rotation about the z-axis. For the "Clark" N_v increases (negatively) as the speed decreases, whereas for the

JN-2 the change is contrary. The result is that although at high speed the "Clark" is spirally stable, it becomes unstable for $i = 12°$; whereas the JN-2, unstable spirally at high speed, has become stable spirally at $i = 15\frac{1}{2}°$. The delicate nature of the balance for spiral stability could hardly be better illustrated. The change in L_v for the two machines is also contrary. The "Clark" has dihedral — the wing rises at $1°.63$. The effect of dihedral is to increase L_v, because the relative wind shifts to one side; but the effect is less marked for large values of i than for small.

Suppose the machine side-slips to the right (v negative). The negative N_v multiplied by the negative v causes a positive yawing moment, turning the machine to the right. The positive L_r banks it over positively as should be the case. If L_r is just right in magnitude, the bank will be right; if L_r is too large the machine will overbank, side-slip worse, yaw more, and so on into a spiral dive. N_r resists this tendency and so does L_v. This is the physical statement of the criterion for spiral stability — and from it the name "spiral" is justified.

(*c*) "Dutch Roll." Roots $\lambda^2 + \left(\frac{C_2}{B_2} - \frac{E_2}{D_2}\right)\lambda + \frac{B_2 D_2}{B_2{}^2 - A_2 C_2} = 0.$

This oscillation will be stable if $C_2 D_2 > B_2 E_2$. Ordinarily this condition is amply fulfilled. In cases where E_2 is negative (spiral instability) the condition is certainly fulfilled provided the other coefficients are all positive, as they are in all cases that have come to my attention. The "Dutch Roll" is ordinarily stable, damping 50% in one to six seconds. For the Curtiss JN-2 at $i = 15\frac{1}{2}°$, $B_2 = 6860$, $C_2 = 815$, $D_2 = 6670$, $E_2 = 1175$. Now

$$\frac{C_2}{B_2} - \frac{E_2}{D_2} = \frac{815}{6860} - \frac{1175}{6670} < 0.$$

Hence, this machine is unstable in this type of oscillation at high speed. The larger terms in the expression give

$$\frac{C_2}{B_2} - \frac{E_2}{D_2} = \frac{L_2}{k_C^2}\left(\frac{N_p}{L_p} - \frac{N_v}{L_v}\right) - Y_v - \frac{k_A^2 N_v}{k_C^2 L_p} U.$$

Hence large values (positive) of N_p are unfavorable. But N_p depends on X_w which changes from negative to positive for sufficiently large values of i. In the JN-2 this change comes around $i = 8\frac{1}{2}$. In the "Clark" it is just putting in its appearance at $i = 12$. The analysis of the changes necessary to avoid this type of instability could be pursued further on the basis of the equations that have been developed; but it should be remarked that the precision of the data are not such as to permit much confidence in any conclusions drawn from a too close scrutiny of numerical relationship. Much additional data is needed; much awaits release by European governments and by our own. The war has done much for aeronautics; but peace will bring an opportunity to digest the results and pursue the research.

EXERCISES

1. State the argument for the vanishing of $\partial Z/\partial p$ and $dL/\partial w$.

2. Show that for motion in a path inclined at an angle Θ the equations for the infinitesimal oscillation are the same as (9) and (10) except for these three

$$\frac{W}{g}\frac{du}{dt} = \frac{\partial x}{\partial u}u + \frac{\partial x}{\partial w}w + \frac{\partial x}{\partial q}q + W\theta\cos\Theta,$$

$$\frac{W}{g}\left(\frac{dw}{dt} - Uq\right) = \frac{\partial Z}{\partial u}u + \frac{\partial Z}{\partial w}w + \frac{\partial Z}{\partial q}q + W\theta\sin\Theta,$$

$$\frac{W}{g}\left(\frac{dv}{dt} + Ur\right) = \frac{\partial Y}{\partial v}v + \frac{\partial Y}{\partial p}p + \frac{\partial Y}{\partial r}r + W\psi\sin\Theta - W\phi\cos\Theta.$$

Treat θ, ϕ, ψ as infinitesimal so that they may add vectorially.

3. Prove equivalent the third pair of equations for the longitudinal motion as expressed in reference to moving and to fixed axes, *i.e.*, the X-equations.

4. Suppose the wings have a dihedral, *i.e.*, let each wing rise β degrees (generally not more than 1° to 3°) from the horizontal. Note that the lift L will be normal to the wing (assumed plane). Let the wing-chord make an angle of $i°$ to the horizontal. What is the angle between the wind and the wing? Treat i and β as small.

5. If a wing has dihedral β calculate the value of $\partial Y/\partial v$.

6. Prove (21), having regard to sign.

7. A surface of 1 sq.ft is placed in the x-z-plane, 2 ft above the C. G. and 15 ft back of it. What derivatives will this effect, and how much (V = 100 mi/hr).

8. Suppose the "Clark" model (Chap. VI) be attached to a pendulum 40 in long, the moment of inertia of the pendulum itself being 2 ft. lbs, and be placed in a 30-mi wind so that at the position of equilibrium $i = 0°$. Write the equation of oscillation for the pendulum near this position and determine the time to damp to $\frac{1}{9}$.

9. Using the tabulated values of the coefficients for the JN-2 at $i = 7°$ calculate the coefficients of the longitudinal biquadratics, find by (36) the values of λ for the long oscillation, and by (37) estimate the error in using (37). How much difference does this make in the times T and t?

10. If the statical stability (as estimated by M_w) for the "Clark" were 0 at $i = 0°$, all other coefficients remaining the same, would (38) indicate sta-

bility or instability? How small may M_w be at $i = 0°$ without loss of dynamical stability?

11. Take the data for the Blériot. Estimate the change in the values of A_2, B_2, C_2, D_2, E_2 if k_E^2 were as large as 6.

12. As in Ex. 11 estimate the size of Y_r (ordinarily assumed to vanish) which may be considered "small."

13. As in Ex. 11 estimate the size of Y_p which may be considered "small."

14. Using the Blériot data, find all the roots, the damping (or amplifying) times and the periods.

15. Make a calculation for longitudinal stability for the machine with data from Cowley and Levy. Find periods, etc.

16. As in Ex. 15, for lateral stability.

17. Calculate Y_v, L_v, N_v for the "Clark" at $i = 6°$ and compare with the tabulated values.

18. Calculate L_r, N_p, L_p, N_r for the "Clark" at $i = 6°$.

19. Oscillation data for L_p at $i = 0°$. $I_a = .0373g$, $I_{a+m} = .0399g$. Time to damp to $\frac{1}{9}$: apparatus, 78 sec; model and apparatus, 6 sec. Calculate L_p. Tunnel wind, 30 mi.

20. Oscillation data for N_r at $i = 0°$. $I_a = .0343g$, $I_{a+m} = .0396g$. Time: apparatus, 110 sec; model and apparatus, 57 sec. Test at 30 mi. Calculate N_r.

21. Determine damping times and periods for the lateral oscillations in each of these five cases: (a) "Clark" at $i = 0°$, (b) at $i = 6°$, (c) at $i = 12°$; (d) Curtiss JN-2 at $i = 1°$, (e) at $i = 15\frac{1}{2}°$.

22. Suppose the "Clark" flying at $i = 6°$ at an elevation where the air-density was 81% of that at the surface. Discuss longitudinal stability as thoroughly as possible and point out any additional experimental data that might be needed.

23. As in Ex. 22, for lateral stability.

24. Is a machine more or less stable on a hot than on a cold day? Barometer reading supposed the same.

FLUID MECHANICS

CHAPTER IX

MOTION ALONG A TUBE

58. The Variables Used. Suppose that a fluid is in motion along a tube which has a cross section S at a distance s from any point of the tube which may be chosen as origin. The motion of the fluid is characterized by two equations, one kinematic, the other dynamic. The kinematic equation expresses the fact of the indestructibility of matter; the dynamic equation gives the relation between the acceleration of the fluid at any point and the forces which act upon the fluid.

The term "fluid" itself is usually applied to either compressible or incompressible fluids. The term "liquid" is specific, and refers to an incompressible fluid. For every homogeneous fluid there is a so-called characteristic equation of the form $F(p, \rho, T) = 0$, connecting the pressure p in the fluid, the density ρ of the fluid, and the temperature T. In the case of the so-called permanent gases, such as air, the characteristic equation is

$$p = R\rho T, \quad \ldots\ldots \quad (1)$$

where T is the absolute temperature; that is, the temperature in Centrigrade degrees measured from $-273°$, or the temperature in Fahrenheit as measured from $-460°$. The value of the constant R depends upon the units used in measuring pressure, density, and temperature.

The effect of temperature is ordinarily disguised by the use of some assumption; for instance, the temperature is assumed to be invariable. Then,

$$p = k\rho, \quad \text{Boyle's Law} \quad \ldots\ldots \quad (2)$$

Or the motion of the fluid is supposed to be adiabatic, so that

$$p = k\rho^n, \quad \text{Adiabatic Law} \quad \ldots\ldots \quad (3)$$

where the value of n in air is about 1.4.

For a liquid the effect of pressure on density is negligible, and the effect of temperature is disregarded, so that the characteristic equation becomes ρ = const.

The variables used to characterize the motion of a fluid are the position s along the tube, and the time t. The velocity u along the tube, the pressure p in the fluid, and the density ρ depend on s and t. These variables are called the Eulerian variables. The motion of the individual particles of fluid is not followed. All that is stated is the state of motion of the fluid at different points and at different times. Lagrange introduced a method of studying fluids by following the motion of individual particles, as is done in the mechanics of a particle. His theory is more advanced than Euler's, and not necessary for the present work.

The dependent variables p, ρ, u, regarded as functions of the independent variables s and t, may have rates of change dependent either on a change in position or a change in time; and these rates are denoted by partial derivatives. Thus, $\partial p/\partial s$ represents the rate of change of pressure at any particular instant as one advances along the tube. On the other hand, $\partial p/\partial t$ represents the rate of change of pressure in time at any point of the tube, that is, pressures are compared at the same point but in different instants of time. The change in pressure when both position and time are changed is given by the formula for the total differential, namely

$$dp = \frac{\partial p}{\partial s} ds + \frac{\partial p}{\partial t} dt \quad . . . . (4)$$

Here ds and dt are wholly independent. A particular value of the differential is obtained if it be assumed that the displacement ds in space is that which actually occurs in the fluid in the time dt as the fluid is moving with the velocity u. This displacement is $ds = u\,dt$. Hence

$$dp = \left(\frac{\partial p}{\partial s} u + \frac{\partial p}{\partial t}\right) dt \quad . . . . (5)$$

is the infinitesimal change of pressure during the time dt if the motion of the fluid be followed. This value of the differential is called the fluid differential and, if it be divided by dt, the rate

$$\frac{dp}{dt} = \frac{\partial p}{\partial s} u + \frac{\partial p}{\partial t} \quad . . . . (6)$$

is the rate of change of pressure as one moves with the fluid.

The density ρ and the velocity u may be treated in exactly the same manner: $\partial u/\partial s$ is the rate of variation of the velocity in the stream at a particular instant, whereas $\partial u/\partial t$ is the rate of change of velocity in time at a particular point of the stream. The acceleration of the particles in the stream is the rate of change of the velocity when the motion of the particle is followed, and is

$$\frac{du}{dt} = \frac{\partial u}{\partial s} u + \frac{\partial u}{\partial t} \quad \ldots\ldots\ldots\ldots \quad (7)$$

that is, it is the fluid derivative of u.

59. Equations of Motion. Consider next a small volume of the fluid included between two nearby cross sections of the tube, a distance ds apart. Let dt be a short interval of time. The amount of fluid which flows across the cross section S in the time dt is the product of the density, the velocity, the time, and the cross section; namely, $\rho u S\, dt$. This product may be calculated for the cross section at s, and for the cross section at $s + ds$; and the difference between the values of the product $\rho u S\, dt$ at the two points represents the net outflow of the fluid in the time dt from the region between the two cross sections. The difference of the same function $\rho u S\, dt$ at s and $s + ds$ is simply

$$d(\rho u S)dt = \frac{\partial(\rho u S)}{\partial s} ds\, dt = \text{net outflow.}$$

The amount of matter in the infinitesimal region is the product of the density by the volume, namely, $\rho S\, ds$; and the amount of increase in the time dt is

$$d(\rho S)ds = \frac{\partial(\rho S)}{\partial t} dt\, ds = \text{amount of increase of fluid.}$$

By the principle of the indestructibility of matter the increase in the volume must be the negative of the outflow from the volume, or, if $ds\, dt$ be cancelled.

$$\frac{\partial(\rho u S)}{\partial s} = -\frac{\partial(\rho S)}{\partial t} \quad \ldots\ldots\ldots\ldots \quad (8)$$

This is the so-called "equation of continuity," the kinematic equation of fluid motion for the one dimensional case.

If the tube is of constant cross section, S may be cancelled out of the equation of continuity. If the density is constant, that is, if the fluid is a liquid, ρ may be cancelled out. The motion of the fluid is said to be steady when all partial derivatives with respect to the time are zero; that is, when the state of the fluid is at each point the same at all times, although the state may differ from point to point. For steady motion the equation of continuity may be integrated, and becomes merely

$$\rho u S = C \qquad \text{(const.)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The forces which act on a fluid are of three different types: first, the pressures p in the fluid, or between the fluid and its container; second, forces which act at a distance, such as the pull of gravity, or, if the fluid is rotating, the "centrifugal force"; and, third, the viscous actions and reactions of adjacent particles of fluid on one another.

It will be assumed for the present that the fluid is not viscous. This means that if one part of the fluid is moving relative to a neighboring part there is no tendency for the faster moving to accelerate the slower, or for the slower to retard the faster, the only action between the two being their common pressure acting normal to the surface separating the two particles. The forces which act at a distance upon a fluid are in reality accelerations, because the force upon a small element of mass, dW, of the fluid is proportional to dW. Thus, if gravity acts the force on dW is simply dW lb. The force equation is, therefore, taken as being in reality an acceleration equation, namely:

$$du/dt = \text{acceleration due to pressures} + \text{external acceleration.} \quad (10)$$

The effect of the fluid pressure in accelerating the element $\rho S ds$ of mass located between two cross sections of the tube is a composite of the pressure pS, acting forward on the rear face, of $(p + dp)(S + dS)$, acting back on the forward face, and of the forward component of the pressure p acting on the lateral surface of the element. The component in any direction of the force due to a pressure p acting upon a surface is the pressure multiplied by the projection of the surface on a plane normal to the direction. The projection of the lateral surface of the element of volume on the plane $S + dS$ is simply dS. The resultant pressure is, therefore (Fig. 16),

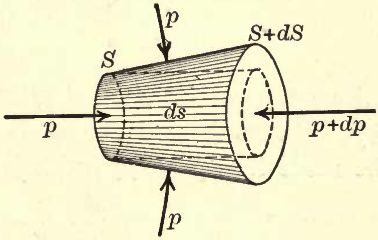

Fig. 16. Pressures on an Element of Fluid.

$$pS - (p + dp)(S + dS) + p\,dS = -S\,dp.$$

The acceleration due to this force $S\,dp$ is the quotient of the force (multiplied by g) by the mass $\rho S\,ds$. Hence,

$$\frac{du}{dt} = -\frac{g}{\rho}\frac{\partial p}{\partial s} + X, \quad \ldots \ldots \ldots \quad (11)$$

where X is an acceleration of the external applied forces. In the theory of fluid motion X is called a force, although it is really an acceleration. The equation just obtained is the dynamical equation of fluid motion.

An application of the kinematic equation is seen in the proof for the pressure on a plane (Art. 8) following Lanchester. It was there assumed that the fluid followed in lines essentially parallel to the plate, and, consequently, the fluid must emerge at the trailing edge with the same velocity with which it entered at the leading edge. If it were assumed that the lines of flow converged toward the trailing edge the velocity at the trailing edge would be greater than at the leading edge.

60. Hydrostatics. In case there is no motion in the fluid, the dynamical equation becomes

$$0 = -\frac{g}{\rho}\frac{\partial p}{\partial s} + X, \quad \ldots \ldots \ldots \quad (12)$$

which is the fundamental equation for hydrostatics in the simple case of one dimension. This equation may be applied to discuss the distribution of pressure with height in the atmosphere. Let $s = h$ be altitude above the earth's surface. The external acceleration is that due to gravity, namely, $X = -g$. The equation becomes

$$\frac{dp}{dh} = -\rho, \quad \ldots \ldots \ldots \quad (13)$$

where the partial sign of differentiation is not needed, because in hydrostatics there can be no change with the time. If the assumption be made that $p = k\rho$, which is Boyle's Law, the equation is

$$\frac{dp}{dh} = -\frac{p}{k}, \qquad k = \frac{p_0}{\rho_0}, \quad \ldots \ldots \ldots \quad (14)$$

where the value of k may be determined by comparing the pressure and density of the air at some known altitude such as the surface of the earth where $h = 0$. On integrating, the result is

$$\log p = -h/k + C \quad \ldots \ldots \ldots \quad (15)$$

or

$$p = Ce^{-h/k} \quad \ldots \ldots \ldots \quad (16)$$

The constant C may be determined by substituting $p = p_0$, $\rho = \rho_0$ when $h = 0$. The final result is

$$p = p_0 e^{-\rho_0 h/p_0} \qquad (17)$$

According to this formula, pressure in the atmosphere falls off indefinitely as the altitude increases indefinitely. The atmosphere does not come to an end. Moreover, the use of Boyle's law indicates that the temperature in the atmosphere is constant, whereas the temperature is known to fall off.

Another assumption that could be made is that the distribution of temperature in a vertical direction is such that air could move from one level to another without any loss of heat, that is, adiabatically. That is, we could assume

$$p = k\rho^n, \qquad k = p_0/\rho_0^n, \qquad (18)$$

The equation then is

$$\frac{dp}{dh} = -\left(\frac{p}{k}\right)^{\frac{1}{n}} = -\frac{\rho_0}{p_0^{\frac{1}{n}}} p^{\frac{1}{n}}. \qquad (19)$$

Integrating, this gives

$$\frac{n}{n-1} p^{\frac{n-1}{n}} = -\frac{\rho_0}{\sqrt[n]{p_0}}(h + C).$$

If $p = p_0$ when $h = 0$,

$$p^{\frac{n-1}{n}} = p_0^{\frac{n-1}{n}} - \frac{n-1}{n} \frac{\rho_0 h}{p_0^{\frac{1}{n}}}.$$

Or

$$p = p_0 \left(1 - \frac{n-1}{n} \frac{\rho_0 h}{p_0}\right)^{\frac{n}{n-1}}. \qquad (20)$$

According to this law the pressure becomes zero and the atmosphere ends at a height

$$h = \frac{n}{n-1}\left(\frac{p_0}{\rho_0}\right). \qquad (21)$$

For air $n = 1.4$, $\rho_0 = .08$, $p_0 = 2100$ lb/ft^2, all approximately. Hence approximately,

$$p = p_0\left[1 - \frac{h}{92,000}\right]^{3.5} \qquad (22)$$

The height of the atmosphere, therefore, is 92,000 ft, or about $17\frac{1}{2}$ miles. The absolute temperature in the atmosphere is obtained from $p = R\rho T$ as

$$T/T_0 = p/p_0 \cdot \rho_0/\rho = (1 - h/92,000). \qquad (23)$$

The temperature, therefore, falls off linearly from that at the surface to the absolute zero at 92,000 ft.

An empirical law connecting p and h and differing from (17) or (20) is determined in Art. 23 as

$$p = 29.92 - 24.1 \log_e (1 + 3h/64,000). \quad . \quad . \quad . \quad . \quad (24)$$

good between the 0 and 24,000-ft levels. This would make the height of the atmosphere 10 i, if mextrapolation were legitimate. Balloons reach levels between 20 and 25 mi and meteors become incandescent much higher up.

As a matter of fact, conditions in the atmosphere are not so simple as any of the laws (17), (22), or (24). The temperature may actually rise with the height for a short distance; that is, the surface temperature may be less than that somewhat higher up. Moreover, at a certain elevation the temperature ceases to fall off, and remains nearly constant, or even increases a little up to the greatest heights reached. The following tables give the actual record for one particular case and the so-called average or standard condition:

TABLE I

Ascension at Eccloo

Pressure, mm.	*Elevation, Meters*	*Temperature, Abs. °C.*
751	100	290
675	1,000	287
598	2,000	281
529	3,000	275
466	4,000	269
410	5,000	264
359	6,000	251
235	9,000	234
203	10,900	222
149	12,000	212
126	13,040	213
120	13,340	213
97	14,650	218
78	16,050	223
54	18,370	218
27	22,720	222
6	32,430	234

TABLE II

Standard Values

Elevation, feet	*Pressure, inches*	*Temperature, ° Centigrade*	ρ/ρ_0
0	29.92	15.00	1.0000
1,000	28.86	12.84	.9718
2,000	27.83	10.75	.9440
3,000	26.82	8.71	.9165
4,000	25.85	6.73	.8895
5,000	24.91	4.81	.8629
6,000	23.99	2.95	.8368
8,000	22.24	-0.60_5	.7859
10,000	20.60	-3.94	.7370
12,000	19.07	-7.05	.6901
14,000	17.63	-9.97	.6453
16,000	16.30	-12.68	.6026
18,000	15.05_5	15.21	.5621
20,000	13.90	-17.56	.5237
22,000	12.92_5	-19.74	.4874
24,000	11.83	-21.72	.4532
26,000	10.91	-23.64	.4210

It will be seen from the first table that the temperature falls off from 290° C. absolute at the surface of the earth to 212° at 12,000 meters (about $7\frac{1}{2}$ miles). The fall of 78° C. is equivalent to one of 140° F., the surface temperature being 53.5° F. The rate of change of temperature with altitude is not constant as required by the adiabatic law, but varies between about 1/20th and 1/12th of a Centigrade degree per 100 meters of ascent. Above 12,000 meters there is a slight rise of temperature, and at the greatest elevation reached the temperature has increased by 22° C. The inversion point, 12,000 meters, varies from season to season, and also very greatly with latitude. In the temperate zones the average height is between 10 and 11 km, and the temperature about 216° C. absolute; at the equator it is 17 km, and the temperature 182°. This altitude is above that of airplane flight. The average table shows a drop of one inch per thousand feet near the surface, shading off to a drop of only one-half inch per thousand at 4 to 5 miles, and a drop of 2° C. in temperature per thousand feet near the ground shading off to half that rate at 4 to 5 miles of elevation.

61. Bernoulli's Equation. For steady motion in one dimension the dynamical equation is

$$\frac{du}{dt} = u\frac{du}{ds} = \frac{-g}{\rho}\frac{dp}{ds} + X.$$

This may be integrated

$$\frac{u^2}{2} + \int \frac{g\,dp}{\rho} - \int X\,ds = C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

or

$$\frac{u^2 - u_0^2}{2g} + \int_{p_0}^{p} \frac{dp}{\rho} - \frac{1}{g}\int_{s_0}^{s} X\,ds = 0 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This is known as Bernoulli's equation, and has many applications in hydromechanics and in hydraulics.

Bernoulli's equation may be used to demonstrate Torricelli's Law. Suppose that a small hole is opened in a vertical cylinder of water at a depth h below the surface. As the area of the hole is small compared to that of the cylinder, the velocity of drop at the top of the surface of the liquid must be small compared with the velocity of efflux, and, hence, the square of the velocity exceedingly small. The external acceleration is $+g$ if distance be measured downward from the top of the liquid. Then

$$\frac{u^2 - 0}{2g} + \frac{p - p_0}{\rho} - h = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The pressure at the top of the liquid is the atmospheric pressure, and in the stream outside the vent is also the atmospheric pressure. The equation, therefore, gives

$$u^2 = 2gh \qquad \text{or} \qquad u = \sqrt{2gh},$$

which is Torricelli's Law.

In the demonstration it has been assumed that the fluid was frictionless, and, consequently, the value for u must be expected to be slightly too high.

(NOTE: It is entirely incorrect to assume that the amount of fluid discharged per unit time through the orifice is ρuS where S is the area of the orifice. The stream lines in the liquid converge toward the orifice on the inside, and continue to converge for some small distance outside, so that the cross section of the stream at its narrowest point where the stream lines are parallel is not S, but a fraction, about 0.62, of S.)

A second application of Bernoulli's formula is to the efflux of a gas from a large tank, in which the pressure at a distance from the orifice is p_0, and the velocity $u_0 = 0$. The effect of gravity in this case is negligible, owing to the small density of the gas. The theorem gives

$$\frac{u^2 - 0}{2g} + \int_{p_0}^{p} \frac{dp}{\rho} = 0, \qquad \text{where} \qquad \frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^n,$$

if the gas is supposed, as is natural, to follow the adiabatic law. For air $n = 1.4$, for steam in some conditions $n = 1.2$, and for all gases n lies between 1.67 and 1.00. On integration the equation gives for the velocity of efflux

$$u^2 = \frac{2gn}{n-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{n-1}{n}}\right] \quad \ldots\ldots \quad (28)$$

This formula must not be applied in cases where the velocity is exceedingly high, so high as to approach the velocity of sound, for at these high velocities whistling, throbbing, and other complicated phenomena set in.

(Bernoulli's equation for a liquid under gravity may be written $u^2/2g + p/\rho + h = C$. The height h represents the static "head" above a certain level. The term $u^2/2g$ is called the kinetic head, and p/ρ the pressure head. The formula states that the total head, kinetic plus pressure plus static, must be constant in steady motion.

The formula really is merely the statement of the conservation of energy for the unit mass of liquid.)

If the pressure p differs little from p_0 so that $p = p_0(1 - a)$, where a is small, the value found for u^2 may be expanded by the binomial theorem. Then

$$u^2 = \frac{2gn}{n-1}\frac{p_0}{\rho_0}\left[1 - \left(1 - \frac{n-1}{n}a - \frac{n-1}{2n^2}a^2 + \ldots\right)\right]$$

$$= 2g\frac{p_0}{\rho_0}\left(a + \frac{a^2}{2n}\right) \quad \ldots \ldots \quad (29)$$

In this form an estimate of the error in the determination of u^2 due to the neglect of the compressibility in cases where the change of pressure is not great may be made. For a liquid $n = \infty$, since ρ/ρ_0 must be constant and independent of p/p_0. For a liquid, then, $u^2 = 2gp_0a/\rho$, when the percentage change in pressure is $100a$ and negative. For a gas the formula contains also $a^2/2n$; the ratio of this to a is $a/2n$, and n lies between 1 and 1.66. The relative error in neglecting compressibility is, therefore, $a/2n$, which is never greater than $a/2$. If, then, the pressure difference observed is less than 1% of the initial pressure p_0, the calculated value of u^2 based on the assumption of incompressibility will differ from the true value by less than ½%, and the calculated value of u by less than ¼%. For almost all practical purposes in aeronautics the formula

$$u^2 = \frac{2g}{\rho_0}(p_0 - p) \quad \ldots \ldots \quad (30)$$

may be used. If the air has a velocity u_0 when $p = p_0$, the formula for the velocity is

$$u^2 = u_0^2 + \frac{2g}{\rho}(p_0 - p) \quad \ldots \ldots \quad (31)$$

62. The Pitot and Venturi Tubes. The Pitot tube is a device for measuring velocity in a stream of air. The tube is double, the inner tube being exposed directly to the flow of the air. The outer tube has on its lateral side perforations which transmit the pressure of the air in the moving stream. Now in both tubes the air is necessarily at rest. The center tube transmits the pressure of the air at the nozzle, where the stream is stopped, and the velocity is zero.

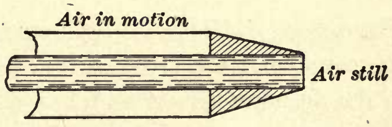

Fig. 17. Pitot Tube (Air coming from the Right).

The outer tube, through its lateral openings, transmits the pressure in the stream when in motion. By Bernoulli's proposition the pressure where the stream is in motion and the pressure where the stream has been brought to rest differ by an amount proportional to the velocity. The velocity in the stream, therefore, may be taken as

$$u^2 = \frac{2g(p_0 - p)}{\rho}, \qquad (32)$$

the density of the stream being considered constant.

In careful experimentation with the Pitot tube it is necessary to reduce to a standard temperature, pressure, and density. These standards are

$$p = 29.921 \text{ in. Hg}, \qquad T = 62^\circ \text{ F.}, \qquad \rho = .07608 \text{ lb/ft}^3.$$

Changes in the barometric pressure or temperature introduce changes in the density of the air, which may be calculated by Boyle's Law. Moreover, for really accurate work the humidity of the air must be taken into account.

The Venturi tube is a device designed for magnifying the effect obtained by the Pitot tube. In the Venturi tube there are two cones placed end to end with the shorter cone opening upstream. The narrowing of this cone from its cross section S_1 at the mouth to S_2 at the throat produces, on account of the equation of continuity, an increase in velocity from the value u_0 at the mouth to the value $u_1 = S_1u_0/S_2$ at the throat. If then the air pressures are taken off at the mouth and the throat from perforations in the tube,

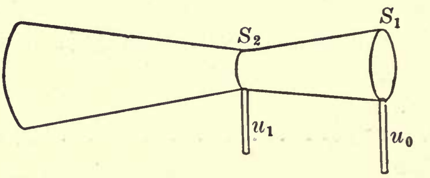

Fig 18. Venturi Tube (Air coming from the Right).

$$u_1^2 - u_0^2 = \frac{2g(p_0 - p_1)}{\rho} = u_0^2\left(\frac{S_1^2}{S_2^2} - 1\right). \qquad (33)$$

The pressure difference is magnified in the ratio approximately as the squares of the cross sections, and is, therefore, easier to measure. If the pressure should be taken off at the orifice from a tube pointing upstream instead of from a lateral perforation, the pressure difference and velocity would be connected by the relation

$$2g\,\frac{(p_0 - p_1)}{\rho} = \frac{u_0^2S_1^2}{S_2^2}. \qquad (34)$$

Example: Find the pressure difference in a Pitot tube if the stream is moving 60 miles an hour.

$$p_0 - p = 88^2(.07608)/2g = 9.15 \text{ lb/ft}^2.$$

This value is in pounds per square foot. Its equivalent in inches of mercury is 0.13, and in inches of water 1.76. This example illustrates the very small pressure difference (compared with the value of the atmosphere itself) which is due to a velocity of 60 miles an hour.

If the temperature of the air were not at 62° F., and the pressure not at 29.921, some value of ρ other than 0.07608 would have to be substituted in the formula. The value of the density varies inversely as the absolute temperature; the absolute Fahrenheit temperature corresponding to 62° is 522°. An increase in temperature to 92° F. would represent a decrease of about 6% in density, and a drop to 0° F. an increase of about 12% in the density. As the pressure difference in the Pitot tube varies directly with the density, changes of temperature such as are normal in temperate zones may, therefore, introduce considerable errors in the calculated pressure drop unless allowance could then be made. A variation of 1 inch in barometric pressure represents a variation of about 3% in the value of ρ, and, consequently, the extreme ranges in barometric pressures even at the surface of the earth are sufficiently great so that they should be taken into account.

EXERCISES

1. Given that for standard dry air $\rho = .07608$ lb/ft³, $T = 62°$ F., $p = 29.921$ in. Hg. Find R when $p = R\rho T$, where T is absolute F.

2. Given that for standard air $p = 29.921$ in. Hg when $T = 15°$ C. and $\rho = \rho_0$. Find ρ/ρ_0 when $T = -7°$ C. and $p = 19.06$ in. Hg.

3. Show that if liquid is diverging in a narrow cone the velocity must vary inversely as the square of the distance from the vertex.

4. If liquid diverges in a sector (region bounded by two parallel planes and by two intersecting planes), the velocity must vary inversely as the distance from the vertex.

5. Given that $p = p_0 e^{-\rho_0 h/p_0}$, $p_0 = 29.92$ in. Hg, $\rho_0 = .0761$ lb/ft³. Compute the pressures at altitudes of $h = 3000, 6000, 9000, 12{,}000, 15{,}000, 18{,}000, 21{,}000$ ft, and compare with the tabulated values.

6. Given that $p = p_0\left(1 - \frac{n-1}{n}\frac{\rho_0 h}{p_0}\right)^{\frac{n}{n-1}}$ with $n = 1.41$, $p_0 = 29.92$ in. Hg, $\rho_0 = .0761$ lb/ft³. Compute p for $h = 3000, 6000, 9000, 12{,}000, 15{,}000, 18{,}000, 21{,}000$ ft, and compare with the tabulated values. Find the temperatures at these levels if $T = 15°$ C. for $h = 0$.

7. The gage pressure in an air tank is 200 lb/ft^2 (in excess of atmospheric). How fast will the air flow out (neglecting viscous friction) through an orifice? Take $n = 1.41$ and use $u^2 = \frac{2gn}{n-1}\frac{p_0}{\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{n-1}{n}}\right]$. Is the velocity so high that the velocity of sound is approached?

8. The velocity in a stream (at a point at an instant) is 10 ft/sec and is observed to be increasing 2 ft/sec per foot and decreasing 5 ft/sec per second. What is the acceleration?

9. The acceleration in a stream (at a certain point and at a given instant) is 15 ft/sec^2, and is decreasing 5 ft/sec per second. If the velocity is 20 ft/sec. how fast is it increasing as one advances along the stream, and what is its value 1 in from the given point?

10. A tank 10 ft high and 2 ft in diameter has a hole of 2 in diameter from the bottom. Use Torricelli's law for the velocity of efflux and a contraction coefficient 0.62 to find a differential equation between the height h of the water at any time and the time t. Integrate and determine how long it takes for the tank to become half emptied.

11. It is desired to make an hour-glass, in the form of a surface of revolution with a hole in the bottom, so that if the glass be filled with water, the level of the water will descend equal amounts in equal times. Find the shape of the surface of revolution. Show that y must vary as x^4, if y is the height above the bottom and x is the radius of the glass at that height.

12. Find the pressure difference recorded in the Pitot for a velocity of 150 mi/hr in standard air. What would be the pressure difference on the 20,000 ft level?

13. A Venturi tube records a pressure difference 24 times as great as the Pitot; what is the ratio of the diameters of mouth and throat?

14. On a boat the (relative) wind comes from a direction 10° out from dead ahead. A Pitot held in the wind registers a pressure difference of 1 inch of water. Find the velocity of the boat and of the true wind, also the direction of the wind, if the velocity of the boat is 20 knots ($1\frac{1}{5}$ mi/hr).

15. A Pitot in a stream of water and one in a stream of air register the same pressure difference. Find the ratio of the velocities of the streams.

16. Suppose a small mass $dW = \rho S\,ds$ of fluid is transferred from a point where the pressure is p to one where it is $p + dp$. Show that the work done against the pressure is $dW\,dp/\rho$. Hence show that Bernoulli's formula $(u^2 - u_0^2)/2g + \int dp/\rho - 1/g \cdot \int X\,ds = 0$ is, when multiplied by dW, the statement that the gain in kinetic energy is equal to the work done by the forces.

17. A tube filled with liquid of specific gravity 0.8 is inclined at an angle of 5° to the horizontal and graduated in mm. When the ends of the tube are connected with the two pressure chambers of a Pitot tube, the liquid moves 170 mm. What is the velocity of the air stream?

18. In standard air the pressure difference registered in the Pitot tube is 20 lb/ft^2. What is the speed? What would be the speed if the air density were only half as much?

19. If liquid is rotating steadily with angular velocity ω about a vertical axis, show that the free surface must be a parabola.

CHAPTER X

PLANAR MOTION

63. The Variables. If a fluid moves in two dimensions, that is, if the flow is parallel to a plane and is the same in all planes parallel to that plane, the notations required will be as follows: p for the pressure in the fluid at any point (x, y), ρ for the density, u and v for velocities parallel to the axes of x and y, t for time. In this case there are three independent variables, x, y, t, the variables p, ρ, u, v being regarded as dependent upon them. The equations of motion are (1) the equation of continuity, which expresses the indestructibility of matter, and (2) the two dynamical equations which determine the accelerations of each particle of fluid along the axes.

The fluid derivative for 2-dimensional motion, that is, the rate of change of any quantity when both space and time change together as they do in the actual motion of the fluid, is

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \frac{u\partial(\ )}{\partial x} + \frac{v\partial(\ )}{\partial y}, \quad . \ . \ . \ . \ . \quad (1)$$

where any symbol may be put into the parenthesis. The proof of this expression is from the formula for the total differential,

$$d(\ ) = \frac{\partial(\ )}{\partial t}dt + \frac{\partial(\ )}{\partial x}dx + \frac{\partial(\ )}{\partial y}dy.$$

It is only necessary to divide by dt to obtain the time rate, and to put

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}$$

to express the fact that the changes in x and y are thus actually due to the fluid velocities u and v. In the case of the steady motion

$$\frac{\partial(\ )}{\partial t} = 0 \quad \text{and} \quad \frac{d(\ )}{dt} = \frac{u\partial(\ )}{\partial x} + \frac{v\partial(\ )}{\partial y}. \quad . \ . \ . \ . \quad (2)$$

64. The Equations of Motion. To determine the equation of continuity consider a small rectangle $dx\,dy$ parallel to the axis. The flow into this rectangle per unit time across the various edges is as follows: Flow in across left-hand edge = $\rho u\, dy$, flow out across right-hand edge = $(\rho u + d_x(\rho u)dx)\, dy$, flow in across bottom edge = $\rho v\, dx$, flow out across top edge = $(\rho v + d_y(\rho v)dy)dx$, where

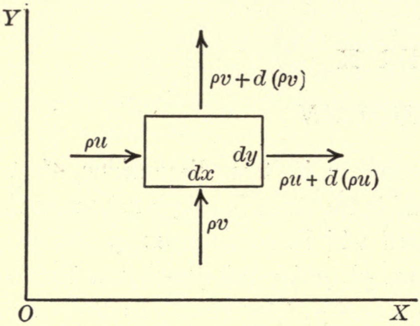

Fig. 19. Flow across Infinitesimal Rectangle.

$$d_x(\rho u) = \frac{\partial(\rho u)}{\partial x}dx,$$

$$d_y(\rho v) = \frac{\partial(\rho v)}{\partial y}dy.$$

Each rate of flow has been determined as the product of the density, the velocity in the direction of the flow, and length of the line across which the flow takes place. The net outflow from the rectangle is

$$\text{net outflow} = \left(\frac{\partial \rho u}{dx} + \frac{\partial \rho v}{dy}\right)dxdy = -\frac{\partial \rho}{\partial t}(dx\,dy), \quad . \quad . \quad . \quad (3)$$

for the net outflow must be equal to the rate of diminution of the amount of matter $\rho\, dxdy$ inside the rectangle. The equation of continuity, therefore, is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The derivation of the dynamical equations is simpler. The accelerations are those due to the fluid pressures and those due to the external accelerations. The pressure on the left-hand of the rectangle is $p\, dy$, and on the right-hand side is $(p + d_x p)dy$. The net pressure is backward, and equals

$$-d_x p\, dy = -\frac{\partial p}{\partial x}\partial x\, dy.$$

The acceleration produced is the product of the pressure by g divided by the mass $\rho\, dxdy$. Hence,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-g}{\rho}\frac{\partial p}{\partial x} + X,$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{dv}{\partial y} = \frac{-g}{\rho}\frac{\partial p}{\partial y} + Y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where X, Y are the external applied accelerations and where the second equation is obtained in a manner entirely similar to the first.

The equations for steady motion are by (2)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-g}{\rho}\frac{\partial p}{\partial x} + X, \qquad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-g}{\rho}\frac{\partial p}{\partial y} + Y. \quad (6)$$

65. Bernoulli's Formula. These equations cannot be integrated in general in any simple manner, but may be integrated in a number of cases where some simplifying assumption is made. In particular, they may be integrated to determine the relation between the velocity and pressure along a stream line in the steady motion (6). For a stream line

$$u : v : q = dx : dy : ds, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where q is the resultant velocity, namely, $(u^2 + v^2)^{\frac{1}{2}}$. Furthermore,

$$\frac{d(\ )}{ds} = \frac{\partial(\ )}{\partial x}\frac{dx}{ds} + \frac{\partial(\ )}{\partial y}\frac{dy}{ds} = \frac{\partial(\ )}{\partial x}\frac{u}{q} + \frac{\partial(\ )}{\partial y}\frac{v}{q}.$$

The equations, therefore, may be written

$$q\frac{du}{ds} = \frac{-g}{\rho}\frac{\partial p}{\partial x} + X, \qquad q\frac{dv}{ds} = \frac{-g}{\rho}\frac{\partial p}{\partial y} + Y.$$

Multiply by dx and dy respectively, and add, placing $q\,dx = u\,ds$, $q\,dy = v\,ds$. Then

$$u\,du + v\,dv = -g\,dp/\rho + X\,dx + Y\,dy,$$

and, integrating, the result is

$$\frac{u^2 + v^2}{2} = -g\int\frac{dp}{\rho} + \int\Big(X\,dx + Y\,dy\Big) + C. \quad . \quad . \quad . \quad (8)$$

or

$$\frac{q^2 - q_0^2}{2g} + \int\frac{dp}{\rho} - \frac{1}{g}\int\Big(X\,dx + Y\,dy\Big) = 0. \quad . \quad . \quad . \quad (9)$$

This equation is Bernoulli's formula, applicable along any stream line in steady motion. The terms in it are identical with those found for motion in one dimension, and should be so, because steady motion in two dimensions when a single stream line is considered may be regarded as motion in one dimension, namely in a tube along the stream line.

It is customary to interpret the different terms in Bernoulli's equation as "heads." This may be seen most clearly by supposing that the fluid is liquid, and that the acceleration acting is gravity. Then

$$\frac{q^2 - q_0^2}{2g} + \frac{p - p_0}{\rho} + h = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

if h is the difference in level between the initial end of the stream line and any point on it. The last term appears, then, as the "head" (of a certain number of feet or meters) of the fluid and may be called the static head, being simply a difference in level. The middle term is the pressure difference divided by the density, and may be called the pressure head. The first term depends on the square of the velocity, which enters into the kinetic energy in mechanics, and is consequently called the kinetic head or velocity head, and Bernoulli's theorem states that the total head is always zero along a stream line in steady motion. Or if the equation be written as

$$\frac{q^2}{2g} + \frac{p}{\rho} + h = C, \qquad (11)$$

the theorem is stated that the total head, kinetic, pressure, and static, is constant along a stream line. There is no contradiction in the two statements, because in one case the origin is not determined, whereas in the other case the various heads are all measured from a given origin.

66. Experimental Discussion. If the external applied accelerations, X, Y, are so small as to be negligible, as in the case of gases except in exceedingly large columns, or if the motion is such that the external accelerations do no work, as in the case when a liquid flows horizontally under the action of no force other than gravity, the equation of Bernoulli states that

$$\frac{q^2}{2g} + \frac{p}{\rho} = C,$$

which means that: *The velocity is larger where the pressure is smaller, and smaller where the pressure is larger.* This is to many a paradox, because there is an intuition that the pressure must be high where the velocity is great. The following illustrations will serve to show that the general intuition is wrong, and that Bernoulli's equation is right.

(1) What moves the fluid apart from its own inertia and in the absence of external force is the pressure difference in the fluid. If, therefore, the velocity is increasing down a stream line, there must be more pressure on the back side of each little element than on the forward side, which means that the pressure must be decreasing down the stream line. On the other hand, if the fluid is slowing up, that is, suffering a retardation, there must be a resultant pressure

difference acting back along the stream line, which means that the pressure in advance is greater than the pressure behind, and the pressure must be increasing down the stream line as the velocity decreases.

(2) Consider the motion of a ball pitched so as to describe a curve — an in or out curve — and neglect the drop of the ball under gravity. The ball is moving forward, and rotating about a vertical axis. The rule for the deflection of the ball to the right or left is that the ball follows its nose; that is, if the front of the ball is by the rotation moving to the right, the ball moves to the right. Now the motion of the air about a moving ball is not steady; the relative motion of the ball and the air is, however, the same as though the ball were at rest, except for its spinning, and the air were moving by the ball. In this case the motion of the air is steady. If, then, a ball be spinning on a vertical axis, and the air be moving past it with a certain velocity, the air is dragged by the friction of the ball against the air so that the velocity in the stream is greater on that side of the ball on which the rotation causes the surface to move in the same direction as the stream, and less on that side of the ball in which the rotation causes the surface to move opposite to the general direction of the stream. According to Bernoulli's principle there should be a pressure urging the ball sideways, and acting from the side where the air velocity is least (and pressure greatest) toward the side where the air velocity is greatest (and pressure least); this pressure is in the direction which urges a ball to follow its nose.

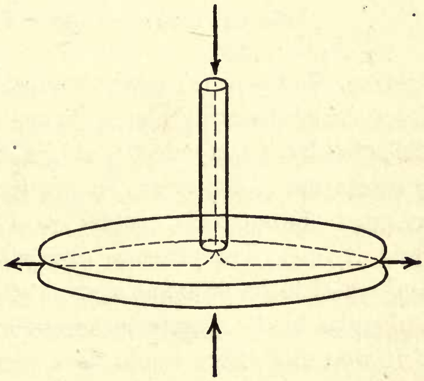
Fig. 20. Suction between Discs.

(3) If a tube penetrates a flat disk, and a light disk be held near the flat disk, and a stream of air be blown through the tube, the velocity of the stream between the two disks is greater than the velocity on the outside of the disk where the air is relatively at rest; therefore, the pressure between the two disks should be less than the pressure on the outside, and the lower disk should be drawn up toward

the other. And this is exactly what happens. One might think, intuitively, that by blowing hard enough through the tube in the upper disk he could blow off the lower disk, but, as a matter of fact, the harder he blows the more tightly the lower disk sticks to the upper one. The experiment is, therefore, in corroboration of Bernoulli's principle, and not at all in corroboration of the intuitive feeling.

(4) Suppose a long rectangular plate be held normal to a stream of fluid. The motion of the fluid in this case is to a certain extent discontinuous. The fluid is deflected by the plate, and does not close in behind the plate. The result is that there is a more or less "dead" wake behind the plate. It will be assumed that this dead wake is clearly defined, and separated from the moving fluid by a

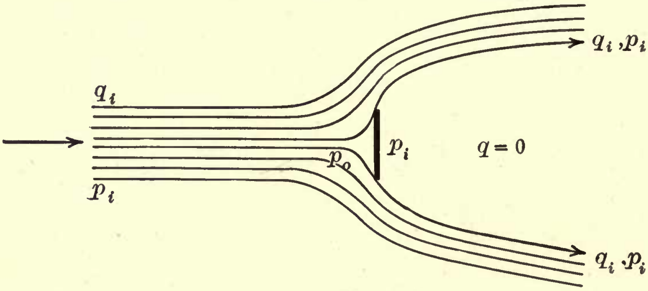

Fig. 21. Pressure on a Rectangle — Discontinuous Motion.

surface. Let q_i be the general velocity of the stream at great distances from the obstructing plate. When the fluid reaches the plate and is deflected by it the velocity at the center of the plate is zero, and the pressure must, therefore, be greater at the center of the plate than at great distances. Consider, next, the wake, which is a portion of fluid at rest. The pressure throughout this region must be constant, and equal to its value at a great distance. Now, the pressure in the wake and in the stream just outside the wake must be the same, for if it were not there would be a resultant pressure upon the surface, separating the wake from the moving fluid, urging the surface inward toward the wake, or outward from it; whereas in the steady state of motion which is assumed, the surface has taken a certain definite position of equilibrium, and is not urged in one direction or the other. The consequence is that the pressure along the surface of separation is everywhere the same. Applying Bernoulli's formula, it appears that the velocity in the fluid on the outside of the surface must also be everywhere the same.

At a great distance the velocity is q_i, and the pressure in the moving fluid and in the wake is the same, and equal to p_i. Returning now to the front face of the plate, where the fluid is at rest at the center, and moving at the edge with the velocity q_i, it may be inferred naturally that the velocity along the plate from the center toward the edge increases from zero to q_i and is, therefore, everywhere less than q_i. The average pressure over the front of the plate must, therefore, be greater than p_i, whereas the pressure on the back is equal to p_i; and there must be a force tending to move the plate down the stream, as is known to be the case. Bernoulli's principle is, therefore, again corroborated, although the introduction of the idea of a discontinuity in the motion has somewhat complicated the demonstration, because it has made necessary the application of Bernoulli's theorem both in the dead wake and in the moving fluid.

It will not do to apply Bernoulli's theorem simultaneously to the moving fluid and to the wake, because in that case the argument would be as follows: The wake is at rest. The fluid on the front of the plate is in motion, except at the central point. The pressure, therefore, on the back should be greater than on the front, and the plate should be urged up stream. A comparison of the experimental effect cited under (3) with the experimental effect cited under (4) reveals a considerable complexity in the results obtained by the application of Bernoulli's principle. It must be always borne in mind that Bernoulli's principle applies along a stream line. For the demonstration in (3) it is supposed that the stream lines close in behind the lower disk. For the demonstration in (4) it is assumed that there is a *discontinuity*, and that the stream lines do not close in. In any particular case the question of whether or not the stream lines do close in is of vital importance, and the complexity of nature is such that sometimes no conclusion can be reached apart from an appeal to experiment.

Furthermore, in any natural fluid there is always a certain amount of viscosity, or friction, so that a surface of discontinuity is not actually possible, and the wake is not actually dead. As a matter of fact, however, the wake is in much less rapid motion than the fluid outside, and such motion as there is is not at all stream line, but quite heterogeneous, consisting of a system of more or less well-defined eddies. Wherever there is a tendency to form a surface of discontinuity, there is also a tendency for the surface to break down into a series of eddies, so that whereas outside of the surface the motion is stream line, at the surface it is eddy motion, and inside the surface it may be entirely irregular. This tendency for a surface to break down may be illustrated by a flame experiment. If a gas flame be lighted and not turned up too high there is a general stream line motion in the flame, and a general stream line motion in the air outside the flame, with a surface of discontinuity between the two. If the flame be turned so high that the surface of discontinuity breaks down into a series of eddies, the flame itself will flicker and roar, indicating a disturbed motion in the flame, and near by it on the outside.

67. Pressure on Aerofoil. Bernoulli's theorem throws a good deal of light on the distribution of pressure between the under and

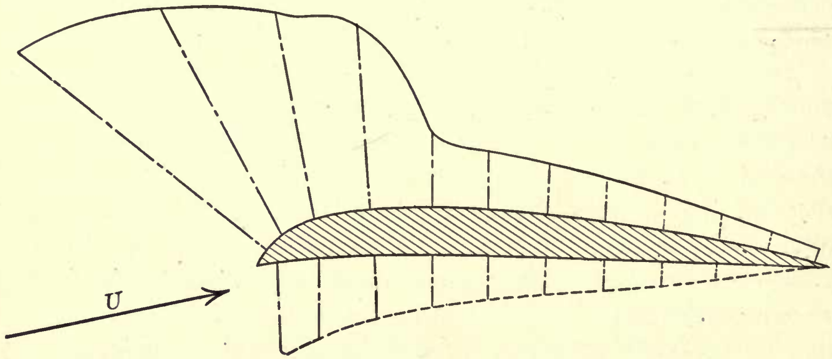

Fig. 22. Distribution and Pressure on Cross Section of Aerofoil.
Wind direction indicated by arrow U. Suction (above) and pressure (below) laid off along normals to the surface of the aerofoil.

upper surface of an aerofoil. Where the velocity of the air along the foil is checked the pressure will be increased, and where it is accelerated the pressure will be diminished. The pressure distribution over the under surface (excess pressure) and over the upper surface (pressure defect) of the aerofoil is indicated in the figure. The remarkable phenomenon is that it is the deficiency of pressure on the upper surface much more than the excess of pressure on the lower surface which sustains the wing. The deficiency of pressure will tend to suck off the fabric on the top surface of the wing, and accidents have actually happened in cases where the fabric was not sufficiently strong or not sufficiently well attached to withstand the suction.

The direction as well as magnitude of the pressure is worth study. As the pressure is normal to the surface certain areas of the wing tend to urge the wing forward and thus diminish the drag on the whole wing. The figure, which is purely diagrammatic, illustrates this. As the drag represents power required to maintained flight, whereas the lift represents sustentation afforded to support the machine, an increase of lift accompanied by no increase in drag would indicate an increase of efficiency in wing design; so would a decrease of drag with no decrease in lift. Even small changes in the shape of the wing may produce such relatively larger effects on lift or drag (including anti-drag) as materially to alter the flying properties of a wing. No hydrodynamic theory is adequate to the prediction of the best design of wing for a given purpose — appeal must be made to experiment with a great variety of shapes.

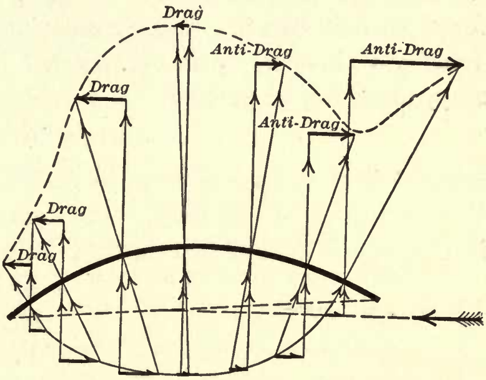

Fig. 23. Suction and Pressure on Idealized Aerofoil (Schematic).
Forces resolved in lift and drag or anti-drag.

Further theoretical discussion of two-dimensional motion of a perfect fluid, that is, of one without viscosity, will be postponed to a later chapter, and the phenomenon of viscosity will now be briefly discussed.

68. Viscous Fluids. When the fluid on one side of an imaginary surface in moving more rapidly than that on the other side, the internal friction or viscosity of the fluid sets up an action and its equal and opposite reaction. The faster moving fluid tends to drag with it the slower moving, and the slower moving tends to retard the faster moving. Consider the special case of motion between two parallel planes, where the fluid

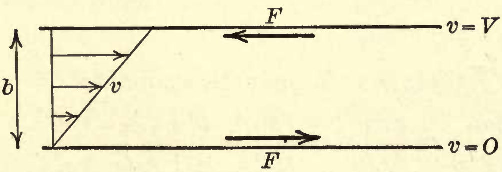

Fig. 24. Action and Reaction due to Viscosity.

is at rest in contact with the lower plane, and is in motion with the velocity V in contact with the upper plane (the plane itself may be supposed to be in motion with the velocity V). The upper plane experiences a tangential drag tending to retard it, and the lower one a tangential force tending to set it in motion. The amount of this force per unit area is found by experiment to vary directly with the velocity V, inversely with the distance between the two planes. It may, therefore, be written

$$F = \mu V/b \text{ lbs/sq.ft} \quad \dots \dots \dots \quad (12)$$

where b is the distance between planes. The coefficient μ is called the coefficient of viscosity. The motion between the two planes in case one is held at rest and the other moved at the velocity V is such that the velocity of motion increases linearly from zero to V. If the breadth b be very small, say db, and the difference of velocity V be very small, say dV, the force may be written

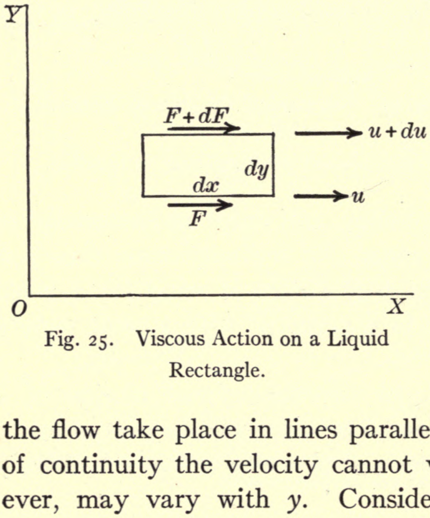

Fig. 25. Viscous Action on a Liquid Rectangle.

$$F = \mu dV/db. \quad . \quad . \quad (13)$$

Suppose, now, that a liquid is in motion between two parallel planes. Let the x-axis be parallel to the planes, and the y-axis be perpendicular; and let the flow take place in lines parallel to the x-axis. By the equation of continuity the velocity cannot vary with x. The velocity, however, may vary with y. Consider a small stratum of the fluid between y and $y + dy$. At the lower face the rate of change of velocity with y is $\partial u/\partial y$, and there must be a viscous action equal to

$$F = \mu \frac{\partial u}{\partial y}.$$

On the upper face, at $y + dy$, there is a viscous action equal to

$$F + d_yF = \mu \left(\frac{\partial u}{\partial y} + d_y \frac{\partial u}{\partial y}\right) = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} dy\right),$$

and there is, consequently, a differential effect of the amount

$$dF = \mu \frac{\partial^2 u}{\partial y^2} dy \quad \dots \dots \dots \quad (14)$$

per unit area, accelerating or retarding the material between y and $y + dy$. The amount of this material per unit surface is $\rho\, dy$, and, consequently, the acceleration is

$$g \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}.$$

The equation for the motion of the fluid is, therefore,

$$\frac{du}{dt} = -\frac{g}{\rho}\frac{\partial p}{\partial x} + \frac{\mu g}{\rho}\frac{\partial^2 u}{\partial y^2} + X, \qquad \frac{dv}{dt} = -\frac{g}{\rho}\frac{\partial p}{\partial y} + Y. \quad . \quad (15)$$

(*a*) Consider, now, the motion of a viscous fluid, between two planes which are at rest, under the action of no external force X, in the particular case when the motion is steady. There is no need here of using a partial derivative for u, because u varies only with x by virtue of the hypothesis introduced. If there is no force acting in the y direction $\partial p/\partial y$ must be zero, because by hypothesis there is no motion in the y direction, and, consequently, $\partial p/\partial x$ may be written dp/dx, because p does not depend on y or t. The dynamical equation reduces to

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 u}{dy^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Now, the second term varies with y only, but not with x, and the first term varies with x, only, but not with y. Consequently, both terms must be constant; for a function of y cannot be equal to a function of x unless each be a constant. Hence,

$$\frac{dp}{dx} = -C, \qquad \mu\frac{d^2u}{dy^2} = C-. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If a column of liquid between the two planes of total length l be under consideration, and if the pressure be p_0 at the beginning and p_1 at the end of this length, the pressure gradient or rate of change of pressure with x must be

$$\frac{dp}{dx} = \frac{p_1 - p_0}{l}, \qquad C = \frac{p_0 - p_1}{l}.$$

The integration of the equation for u gives

$$\mu \frac{du}{dy} = -Cy + C_1 \qquad \mu u = -\tfrac{1}{2}Cy^2 + C_1 y + C_2, \quad . \quad . \quad (18)$$

where C_1, C_2 are constants of integration. Let the origin for y be taken at the middle of the column of fluid, and let b be the whole distance between the planes. Now, owing to the viscosity, the fluid will stick to both planes, and, consequently u must be zero when $y = +b/2$ and when $y = -b/2$. Hence,

$$0 = -\tfrac{1}{8}Cb^2 \pm C_1 b + C_2,$$

and from these equations $C_1 = 0$, $C_2 = Cb^2/8$. Hence,

$$\mu u = \frac{1}{2}C\left(\frac{1}{4}b^2 - y^2\right) = \frac{p_0 - p_1}{2l}\left(\frac{1}{4}b^2 - y^2\right). \quad . \quad . \quad (19)$$

The velocity is greatest midway between the planes and falls off according to the parabolic law to zero at the planes.

The rate of flow of the fluid for any value of y is u, and the average rate of flow between the planes is

$$\bar{u} = \frac{1}{b}\int_{-b/2}^{b/2} u\,dy = \frac{(p_0 - p_1)b^2}{12l\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

when measured by the volume passed per unit time; it is ρ times this when measured by the weight. The central rate of flow at $y = 0$ is

$$\frac{(p_0 - p_1)b^2}{8l\mu} = \frac{3}{2}\bar{u},$$

which shows that the average velocity is two-thirds of the central velocity. The relation between the total volume of flow $F = \bar{u}b$, the pressure difference at the two ends of the column, the distance between the planes, and the coefficient μ of viscosity is

$$\mu = \frac{(p_0 - p_1)b^3}{12lF}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This formula affords an experimental procedure for determining μ. It is merely necessary to force a viscous fluid between two planes for a given length under known pressure difference, and measure the rate of flow F by volume per unit distance along the plane perpendicular to the direction of flow.

(*b*) Better than to force the fluid between two parallel planes is to force it down a cylindrical tube. The equations of motion for the fluid in a vertical tube may be obtained as follows: Let a be the radius of the tube, and r the distance from the axis to any point. Consider the cylindrical shell of fluid which lies between r and $r + dr$. At r the viscous drag is $\mu\, du/dr$, and is applied to a surface proportional to $2\pi r$. At $r + dr$ the force per unit surface is

$$\mu\left(\frac{du}{dr} + \frac{d}{dr}\frac{du}{dr}dr\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

applied to a surface proportional to $2\pi(r + dr)$. The differential effect of the drag is, therefore,

$$2\pi\mu\left[(r + dr)\left(\frac{du}{dr} + \frac{d^2u}{dr^2}dr\right) - r\frac{d^2u}{dr^2}\right] = 2\pi\mu\left(\frac{r\,d^2u}{dr^2} + \frac{du}{dr}\right)dr. \quad (23)$$

The amount of matter in the cylindrical shell is proportional to $2\pi r\,dr$, and hence the acceleration due to the frictional drag is

$$\frac{\mu g}{\rho r}\left(\frac{r\,d^2u}{dr^2} + \frac{du}{dr}\right) = \frac{\mu g}{\rho r}\frac{d}{dr}\frac{r\,du}{dr}. \quad (24)$$

The dynamical equation is for steady motion

$$0 = -\frac{g}{\rho}\frac{dp}{dx} + \frac{\mu g}{\rho r}\frac{d}{dr}\left(\frac{r\,du}{dr}\right) + X. \quad (25)$$

As before, dp/dx depends only on x, whereas u depends only on r, and, consequently, if X be zero, or if it be constant, the equation when written in the form

$$\frac{dp}{dx} = \frac{\mu}{r}\frac{d}{dr}\frac{r\,du}{dr} + \frac{\rho X}{g} \quad (26)$$

shows that dp/dx must be constant; that is,

$$\frac{dp}{dx} = -C = \frac{p_1 - p_0}{l},$$

and

$$\frac{\mu}{r}\frac{d}{dr}\frac{r\,du}{dr} + \frac{\rho X}{g} = -C = \frac{p_0 - p_1}{l}. \quad (27)$$

This equation will be integrated when $X = 0$. Then

$$\mu\frac{r\,du}{dr} = -\frac{1}{2}Cr^2 + C_1, \quad (28)$$

or

$$\mu\frac{du}{dr} = -\frac{Cr}{2} + \frac{C_1}{r}.$$

Then

$$\mu u = -\frac{Cr^2}{4} + C_1 \log r + C_2. \quad (29)$$

where C_1 and C_2 are constants of integration. Unless $C_1 = 0$, $u = \infty$ when $r = 0$, owing to the presence of $\log r$. As the velocity in the middle of the cylinder cannot be ∞, C_1 must be zero. As the velocity when $r = a$, the radius of the tube, must be zero, $C_2 = \frac{1}{4}Ca^2$, and

$$u = -\frac{C}{4\mu}(a^2 - r^2) = \frac{p_0 - p_1}{4\mu l}(a^2 - r^2). \quad (30)$$

This gives the law of variation of velocity with distance from the axis of the cylinder. To calculate the flow measured by volume,

$$F = \int_0^a 2\pi ru\, dr = \frac{(p_0 - p_1)a^4}{8l\mu} \quad \text{. . . . . . . (31)}$$

and the value of μ may be obtained as

$$\mu = \frac{(p_0 - p_1)a^4}{8lF} \quad \text{. . . . . (32)}$$

This is known as Poiseuille's formula, and is useful in obtaining the value of μ from experimental data. It is possible to draw capillary tubes of such small radius that the flow is both small and steady under the pressure difference $p_0 - p_1$, and, consequently, reasonably satisfactory values of μ may be obtained.

The definition above given for μ and used throughout the analysis is one in which the force is measured as is customary in engineering, in pounds or kilograms, whereas the definition of μ given in treatises on theoretical physics is one in which the force is measured in dynes, or, if the English system be used in "poundals." The value of μ here used is, therefore, the ordinary value divided by g.

The ratio of μ to ρ, namely $\nu = \mu/\rho$, is called the kinematic coefficient of viscosity; it is this ratio which enters into the expression for the acceleration, where μ enters into the expression for force. The following are the values of μg or νg for some substances:

Substance	Temperature	μg	νg	System
Air	15° C. (760 mm.)	....	.000159	English
"	0° C.	....	.000144	"
Water	5° C.	....	.0000164	"
"	10° C.	....	.0000141	"
"	15° C.	....	.0000123	"
"	20° C.	....	.0000108	"
"	17° C.	.01112	....	Metric
Mercury	70° C.	.01602	....	"
Air	15° C.	.000179	....	"

(NOTE: There is a rapid variation of μ or ν with the temperature. Different methods of determining the viscosity give different results to such an extent that the above figures are probably not accurate to as many places as written.)

69. Physical Observations. In the discussion of this chapter it has been pointed out that there is (1) continuous stream line flow, (2) discontinuous flow, as when a stream passes around an obstructing object, (3) continuous flow in which viscosity is taken into account, the first two types of flow being discussed on the basis of no viscosity. If, in experiments on viscosity, the fluid is passed too quickly through the tube, or if the tube is too large, the fluid will adhere, as required by the theory of viscosity, to the tube at the outside, but the flow will not be the simple flow found above, because if the velocity in the center is too great the flow becomes discontinuous; near the boundary it is as determined for simple viscous flow; near the center it is more like simple stream line flow without viscosity, and there is at a certain distance from the center a surface of discontinuity full of small eddies or rollers. Hydraulic engineers, therefore, cannot assume that when liquid flows in a pipe they are in the presence of either the simple viscous flow or of simple stream line flow without viscosity. If the velocity of flow down the middle of the pipe is increased too much the motion becomes still more complicated. Instead of having the region of simple viscous flow near the boundary and the region of steady stream line flow on the interior, separated by a surface of discontinuity made up of small rollers or eddies, the eddies break down and move in toward the center, disturbing the stream line motion; and there thus arises what is known as (4) turbulent flow. This is a fourth type, separate from the three enumerated above; and for the most part what is known about turbulent flow has been derived by experimentation, and is expressed by empirical formulas. In turbulent flow there is a general velocity of the stream, as in stream line flow in a nonviscous fluid, or steady flow in a viscous fluid; but the actual motion of the particles of fluid is not even approximately the same in velocity as the general motion of the stream. The velocities in the fluid vary rapidly from point to point, and hence the name "turbulent flow." This is the kind of motion found in the wake of a steamer.

When one observes the motion of the fluid around a steamer the following facts stand out:

(*a*) At reasonable distances from the steamer the flow is stream line, slow and steady relative to the steamer, apparently not much influenced by viscosity.

(*b*) Very close to the hull the water is dragged with the steamer.

The motion is slow and steady relative to the steamer, and apparently depends mainly on viscosity, in that the velocity increases more or less uniformly as one passes out from the hull into the fluid.

(*c*) There is a region not particularly wide in which the viscous motion goes over into the nonviscous, and in which numerous small eddies are observable, so that judged from the point of view of a person on deck the outside motion appears to roll on the inside motion through the intermediary of these eddies.

(*d*) At the stern there is a long, turbulent wake in which no velocity can be assigned to the particles of fluid that is anything like constant even over relatively small volumes.

It is clear, therefore, that the phenomena which arise when a body moves through a fluid are of great complexity; that only the simplest cases can be stated in simple mathematical equations and solved; and that to a considerable extent general physical arguments or empirical data must be used to represent the effects of fluid motion on a body moving in a fluid. These general physical principles will be taken up next. It will not do, however, to overlook the fact that when a body moves through a fluid there are both close to the body and particularly at a considerable distance simple types of fluid motion, and that the discussion of these simple types of motion is important because of the information it will give in respect to some of the main characteristics of the flow when a body moves in a fluid, and of the effects of that flow on the body. This matter will be taken up later.

EXERCISES

1. Show that for steady flow out radially from a point and equal along each radius, the fact of "continuity" is expressible as $\rho ur = \text{const.}$, if u is the radial velocity at the distance r. Hence infer that for steady radial flow of a liquid the velocity must vary inversely as the distance.

2. Show that for radial flow $\frac{\partial(\rho r)}{\partial t} + \frac{\partial(\rho r u)}{\partial r} = 0.$

3. Derive the equation of continuity in polar coördinates by examining the flow across the sides of the element $r d\theta dr$ of area. The radial and normal velocities are u, v.
Ans. $\frac{\partial(\rho r)}{\partial t} + \frac{\partial(\rho r u)}{\partial r} + \frac{\partial(\rho v)}{\partial \theta} = 0.$

4. Derive the dynamical equations of fluid motion in polar coördinates.

Ans. $\frac{du}{dt} - \frac{v^2}{r} = -\frac{g}{\rho}\frac{\partial p}{\partial r} + R, \quad \frac{1}{r}\frac{d}{dt}(rv) = -\frac{g}{\rho}\frac{\partial p}{r\partial \theta} + \Theta.$

5. Show that $\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial r} + v\frac{\partial(\ )}{r\partial\theta}$ in polar coördinates.

6. Suppose that the forces on the moving spinning ball may be calculated from the relative velocities and the sine law $P = kU^2 \sin i$. Assume that the velocity at the periphery is a composition of the forward and rotary motions, disregarding the actual complex disturbance of the stream. Is the deflecting force in the right direction?

7. Suppose that in the problem (*a*), Art. 68, the planes are vertical and that $X = g$ cannot be neglected. Find the law of variation of u with y and the total volume of flow (per second per unit breadth along the plate perpendicular to the flow).

8. If in problem (*b*), Art. 68, the liquid were forced between two concentric cylinders, find the law of variation of the velocity and the total flow. Show that if the inner radius is large and the outer radius only slightly larger, the results reduce to the flow between parallel planes.

9. Suppose that in problem (*b*), Art. 68, the tube is vertical and $X = g$ cannot be neglected. Find the law of flow and the total amount. Compare with Poiseuille's formula to exhibit the effect of the action of g.

10. A cylinder of radius a spins with angular velocity ω in a larger coaxial cylinder of radius b, the space between the cylinders being filled with a viscous fluid. Assume the motion to be steady and in circles concentric with the axis of the cylinders. (Neglect end effects, *i.e.*, assume both cylinders long.) Calculate the torque per unit length necessary to maintain the spinning. Plot as a function of b/a.

11. In the problem of the steady motion of a viscous liquid between parallel plates calculate the work consumed by the forces of viscosity and compare it with the work done by the pressures. (See Ex. 16, Chap. IX.)

12. Compare the tabulated values $\mu g = .000179$ (metric) and $\nu g = .000159$ (English) to see how well they check.

13. Two parallel planes are 1 in apart and the relative motion is 3 ft/sec. If the true viscous flow is maintained, calculate the force per square foot on the planes due to viscosity.

14. Water is forced through a tube of radius 1/2000 in at a mean velocity of 4 in/sec. What is the pressure gradient?

15. Show that the motion of a viscous liquid between two parallel plates is rotational and find the angular velocity in the fluid.

16. Show that the motion of a viscous liquid in a tube is rotational and find the angular velocity at any distance from the center.

17. In the flow of Ex. 14 calculate the angular velocity of the elements of the fluid near the boundary $r = a$ in R.P.M.

(NOTE. In Exs. 15–17 the angular velocity of rotation of the particles of the fluid may be calculated from the formula $\omega = \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]$. Motion is called rotational if $\omega \neq 0$, irrotational if $\omega = 0$. See Art. 80.)

CHAPTER XI

THEORY OF DIMENSIONS

70. Fundamental Units. The dimensions of a physical quantity have to do not with the magnitude of the quantity, but with its quality. From geometry the fundamental dimension of length, designated L, is taken, and certain derived dimensions, for example, area, which is the product of two lengths in the case of the rectangle, and which for any set of similar figures varies with the square of the corresponding linear dimensions, is said to have the dimensions L^2. This type of dimensionality is indicated in the abbreviations for the units of measure; for example, for length the abbreviation is ft, for area, ft^2. The dimensions of volume are L^3; and angle is denoted by L°, because the definition of angle in radian measure is arc divided by radius; and in general, since the angles in similar figures are equal, the measure of angle must be independent of the measure of length. A number which has no dimensions is called a pure number. Angle is, therefore, a pure number. This must not be taken to mean that angle has no other characteristics than those of pure number, but that whatever other characteristics it has are not dependent on the unit of length.

In kinematics time is introduced, and the dimensions of time are written T, and considered independent of linear dimensions. By combining time and space the dimensions of various kinematic quantities may be obtained obviously, from the definitions appropriate to those quantities. Thus, velocity being a quotient of a distance (finite or differential) by a time (finite or differential) has the dimensions L/T, which corresponds to the customary way of writing the unit of velocity, ft/sec, or m/sec. Acceleration, in like manner, has the dimensions L/T^2, whereas those of angular velocity and angular acceleration are $1/T$ and $1/T^2$, respectively, because angle has no dimensions. According to one of Kepler's laws, when a planet moves around the sun the radius vector drawn from the sun to the planet describes equal areas in equal times, and the rate at which the radius sweeps out area is called the areal velocity, to which the dimensions L^2/T are naturally assigned.

Another element in mechanics is mass, or quantity of matter, which is independent both of length and time measurements, and is assigned the dimensional symbol M. From this many units may be derived; for example, the density is the mass per unit volume, and its dimensions are M/L^3; and in the case of chains or steel rails, it may be convenient to define linear density as the mass per linear foot, and linear density would, therefore, have the dimensions M/L. In the equations of fluid motion there occurs a term corresponding to rate of change of density, $d\rho/dt$, and the dimensions of this quantity must be M/L^3T. In mechanics there occurs momentum of dimensions ML/T, moment of inertia of dimensions, ML^2 (because the definition of moment of inertia is the mass times the square of the distance), angular momentum or moment of momentum (which is a momentum times the distance) of dimensions ML^2/T, and work or energy, ML^2/T^2.

It is entirely possible that two different physical things should be numerically equal but of different dimensions; for example, the specific gravity of a substance is the ratio of the density of that substance to the density of water in a standard condition, and specific gravity is, therefore, a pure number without dimensions, because it is the ratio of two physically like quantities. In the metric system the density of standard water is unity; and, therefore, the specific gravity (a pure number) and the density (of dimensions M/L^3) are numerically equal for all substances. In the English system, however, the density of water is not 1, but about 62 lb/ft^3, so that specific gravity and density are not numerically equal.

Whenever an equation is written between physical quantities the terms of the two sides of the equality must be equal not only in magnitude but in quality, that is, in dimensions. A momentum cannot be equated to a mass or a time, but only to a momentum. Dissimilar things are never equal in physics, and cannot be added to one another, or subtracted from one another, though they may be multiplied as is the case in obtaining momentum from mass and velocity, or divided, as in the case of density. When, therefore a physical law is stated as an equation it is necessary that the quantities on the two sides of the equality should have the same dimensions. Thus, Newton's second law states that the mass times the acceleration (or the rate of change of momentum) is proportional to the force; or $Wa = kF$, if W denoted mass, a, acceleration, and F, force. From the dimensional point of view this means that

$$ML/T^2 = (k)\ (F),$$

where the parentheses denote "dimensions of"; that is to say the factor of proportionality k and force F must have such dimensions that together the product kF has the dimensions ML/T^2; and except as some physical characteristics of force are known which limit the dimensions assigned to F, any dimensions whatsoever may be assigned to it, provided that in the dynamical equation other dimensions be assigned to k which make the product kF of dimensions ML/T^2. It is customary in physics to assign to F in Newton's equation the dimensions ML/T^2, and to treat k as a pure number equal to 1 in the metric system, and equal to 32.174 in the English system where force is in standard pounds.

It would be equally possible to consider F to have the same dimensions as mass, namely, $(F) = M$, and to assign to k the dimensions of an acceleration, particularly as k is numerically equal to the acceleration of gravity, or to its standard value.

According to another law of Newton, the force of attraction between two masses varies as the product of the masses, and inversely as the square of the distance between them; or

$$F = c\frac{WW'}{R^2}, \qquad \text{and} \qquad (F) = (c)\frac{M^2}{L^2},$$

would then be the dimensional equation for F. If here the dimensions of c be taken as zero, that is, if it could be assumed that c is a pure number, which is the assumption made in the electrical theory in connection with Coulomb's Law, the dimensions of force would be M^2/L^2, and the dimensions of k in the dynamical equation would be L^3/MT^2. As a matter of fact, the dimensions of F are taken as ML/T^2, and the dimensions of c in the equation for gravitational attraction are, therefore, assigned as L^3/MT^2.

From the engineers' point of view, where force is regarded as perhaps more fundamental even than mass, it would be natural to take length, time and force as a system of quantities upon which to found the theory of dimensions; and if the multiplier in the dynamical equation be considered as having no dimensions, the dimensions of mass would then be

$$(M) = FT^2/L.$$

There are authors who use this system of dimensions, but in this text the customary physical system will be used. The symbol W will, however, always stand for mass, and the force due to a weight will be written Wg with g assigned the usual dimensions L/T^2 of an acceleration.

71. Use of Dimensions. There are three reasons why the theory of dimensions is important:

(1) It is often possible to detect careless errors in analytical work by checking through each and every equation to see whether it is dimensionally correct.

(2) It is often possible, as will be seen, to predict the quality of a formula for something that is unknown merely from a knowledge of the dimensions of the result, and the dimensions of the variables upon which the result depends.

(3) The theory of dimensions is constantly used in scaling up model experiments to full-sized machines.

As an illustration of (1), consider the motion of a body in a resisting medium determined by the equation

$$\frac{W\,dv}{dt} = Wg - kgv^2. \quad \text{(1)}$$

In dimensions this is written

$$M\frac{L}{T^2} = M\frac{L}{T^2} - (k)\frac{L}{T^2}\frac{L^2}{T^2}. \quad \text{(2)}$$

As all the terms in this equation must have the same dimensions,

$$(kg) = M/L \qquad \text{or} \qquad (k) = MT^2/L^2.$$

The terminal velocity (Art. 22) is

$$V = \sqrt{W/k}. \quad \text{(3)}$$

Now W has the dimensions M, and k the dimensions MT^2/L^2, and therefore,

$$(V) = \sqrt{M \div MT^2/L^2} = \sqrt{L^2/T^2} = L/T, \quad \text{(4)}$$

and the dimensions check.

As an illustration of (2), suppose it be required to determine the time of oscillation of a simple pendulum in vacuo. The time of oscillation of the pendulum can depend only on its length l and on the acceleration of gravity g at the location where the pendulum oscillates. The mass, which is supposed to be located at a point, does not enter into the time of oscillation because the forces will all be proportional to that mass. Suppose, then, that the time is regarded as a product of a certain power of l by a certain power of g, namely as $T = al^n g^m$, where a has no dimensions; that is, suppose that the whole physical dependence of T on l and g can be accounted for by the proper choice of n and m. The dimensional equation is

$$(T) = L^n(g)^m \quad \text{or} \quad T = L^n(L/T^2)^m = L^{n+m}T^{-2m}. \quad . \quad . \quad (5)$$

That the two sides may be the same, it is necessary that

$$n + m = 0, \qquad -2m = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

from which it follows that $m = -\frac{1}{2}$, $n = +\frac{1}{2}$, and

$$T = a(l/g)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The expression for the time of oscillation is, therefore, entirely determined except for the value of the pure number a.

In the case of the time of oscillation of the pendulum, dynamical theory is sufficient to prove that $T = \pi(l/g)^{\frac{1}{2}}$, so that the value of a and the form of T are both determined theoretically. Suppose, however, that dynamical theory could not predict the formula for the time of oscillation, but that the formula must be obtained empirically by experiment. For the empirical determination it would be necessary to treat pendulums of all sorts of length to find the variation T with l, and in all possible gravitational fields to determine the dependence of T on g; and when the experiments had been made and the results tabulated or plotted, it would be necessary to fit empirical equations to the data. If, however, it is known in addition from the dimensional argument above that T is necessarily of the form $a(l/g)^{\frac{1}{2}}$, it is necessary only to perform a single experiment to determine from the observed values of T, l, and g the numerical value of a. This is a great saving.

(3) The use of the theory of dimensions in scaling up the results of model experiments to full-sized machines will be taken up later. At this point a determination of fundamental laws of fluid resistance by means of (2) will be discussed.

72. Fluid Resistance. To determine the pressure on a circular disc of radius r in a stream of incompressible fluid, moving normal to the disc with velocity U, and of density ρ, under the assumption that viscosity will be neglected. The pressure Pg (P in lb) is a force to which dimensions ML/T^2 have been assigned; and it can depend only on the radius r of the disc, the velocity U of the stream, and the density ρ; that is,

$$Pg = f(r, U, \rho) = ar^p U^q \rho^s, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

where it has been assumed, as always, that the function may be considered as a product of properly chosen powers of the variables entering the function, and a numerical multiplier a without dimensions. Then the dimensional equation is

$$LM/T^2 = L^p(L/T)^q(M/L^3)^s = L^{p+q-3s}T^{-q}M^s, \quad . \ . \ . \quad (9)$$

and the exponents p, q, s, must be determined so that

$$s = 1, \qquad -q = -2, \qquad p + q - 3s = 1. \quad . \ . \ . \quad (10)$$

Here are three equations in three unknowns, and the solution is

$$s = 1, \qquad q = 2, \qquad p = 2. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

Hence,

$$Pg = ar^2U^2\rho = a'SU^2\rho, \quad . \ . \ . \ . \ . \ . \ . \quad (12)$$

if S be the surface of the disc, which varies as r^2. It has, therefore been proved that the pressure upon the circular disc varies jointly with the surface exposed, the square of the velocity of the stream, and the density. (The pressure has been written in the form Pg instead simply of the form P, because the force inserted in any dynamical equation is inserted in the form Pg if P is in pounds or kilograms. It is advisable when forces are measured in weight units to keep the value of the force and the multiplier g together, in order to avoid any difficulties that might arise from a confusion in dimensions.)

Suppose the pressure is desired on a rectangle of dimensions l by b in a stream moving with velocity U, and of density ρ. Then

$$Pg = f(l, b, U, \rho) = al^pb^qU^r\rho^s, \quad . \ . \ . \ . \ . \ . \quad (13)$$

and

$$\frac{ML}{T^2} = L^pL^q\left(\frac{L}{T}\right)^r\left(\frac{M}{L^3}\right)^s. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (14)$$

The equations to be solved for establishing the identity of dimensions on the two sides of the equations are

$$p + q + r - 3s = 1, \qquad s = 1, \qquad -r = -2. \quad . \ . \ . \quad (15)$$

Here are three equations in four unknowns, and it is only possible to solve the equations for three of the unknowns in terms of one of the others. One method of solution gives

$$s = 1, \qquad q = 2 - p, \qquad r = 2. \quad \ldots \ldots \quad (16)$$

Hence,

$$Pg = al^p b^{2-p} U^2 \rho = aSU^2 \rho (l/b)^{p-1}, \quad \ldots \ldots \quad (17)$$

where $S = lb$ is the area. The formula for the pressure, therefore, contains the aspect ratio l/b to an unknown power. Moreover, the power may not only be unknown, but the constant a may depend on the power, and, finally, the pressure may be any combination of terms containing $SU^2\rho$ and some power of the aspect ratio and a constant depending on that power or on the aspect ratio itself, since the aspect ratio is a pure number. The value for the pressure may, therefore, be written

$$Pg = SU^2 \rho f(l/b), \quad \ldots \ldots \quad (18)$$

and although this form is apparently more general in that it contains an unknown function f of the aspect ratio, it is not in reality more general because of the possible variations in p and a in the previous formula.

The indetermination which has arisen in the previous problem is typical of that which arises in every problem to which the dimensional argument is applied when the number of variables exceeds three. There are only three independent dimensions in mechanics, M, L, T and, consequently, only three equations can be obtained from the dimensional argument. The final form of a formula determined by the dimensional argument can, therefore, be obtained qualitatively only in case three or fewer physical variables enter into the quantity which it is desired to determine. To express this another way, any physical quantity in mechanics may be represented dimensionally in terms of any three independent physical elements E_1, E_2, E_3 where by independent is meant that the three contain M, L, and T in such a form that any physical quantity with arbitrary dimensions may be determined as a product of certain powers of those three, or in other words that no one of the three may be expressed in the form $E_1 = aE_2^p E_3^q$.

In fluid mechanics the area S, the velocity U, and the density ρ are generally taken as the three fundamental physical quantities. The pressure P upon any body may be written as

$$Pg = SU^2 \rho f(x, y, z, \ldots). \quad \ldots \ldots \quad (19)$$

where x, y, z, are not merely the variables other than S, U^2, ρ which enter into the problem, but are combinations of those variables with S, U, and ρ in such a manner as to be free of dimensions. Thus, in the above case the function depended on l/b, a dimensionless quantity. It should be noted in general that if E_1, E_2, E_3 are three independent physical quantities any other quantity may be expressed dimensionally as $aE_1^pE_2^qE_3^r$.

The pressure on a rectangle inclined at an angle i to the stream would be of the form

$$Pg = SU^2\rho f(l/b, i), \quad \ldots \ldots \ldots \quad (20)$$

and would depend on two dimensionless variables, the aspect ratio and the inclination of the plane to the stream. As angle has no dimensions, it will not be possible in any way to determine the variation of the pressure with the angle by a dimensional argument, and for the same reason the variation of the aspect ratio cannot be determined. It remains necessary to fix by experiment the expression of the function f in terms of aspect ratio and angle of incidence. To find this is far simpler than to determine the dependence not only on these two variables but also on S, U, and ρ which would have been necessary if the dimensional argument had not first been given. It has been assumed in all the demonstrations above that the plane, whether circular or rectangular, had no thickness. If a plane of appreciable thickness were opposed to a stream not only the aspect ratio l/b would occur in the function f, but the ratio t/l of the thickness to the length, or, if more convenient, the ratio of t/b; or, if still more convenient, the ratio t^2/S.

73. Viscosity. The effect of viscosity on the pressure may be discussed by the dimensional argument. The form in which the coefficient of viscosity μ enters into the dynamical equation is seen from the equation, namely,

$$\frac{du}{dt} = -\frac{g}{\rho}\frac{\partial p}{\partial x} + \frac{\mu g}{\rho}\frac{d^2u}{dy^2} + X,$$

where every term in the equation is necessarily an acceleration, and where for convenience μg will be treated together just as Pg is. The dimensional equation is

$$\frac{L}{T^2} = \left[(\mu g) \div \frac{M}{L^3}\right] \times \left[\frac{L}{T} \div L^2\right] \quad \text{and} \quad (\mu g) = \frac{M}{LT}.$$

The dimensions of the coefficient of viscosity in the form μg are, therefore, mass divided by the product of distance and time; that is, coefficients of viscosity are measured in lb/(ft. sec.). The particular combination $\mu g/\rho$ has the dimensions L^2/T. The ratio $\mu/\rho = \nu$ is called the kinematic coefficient of viscosity, or coefficient of kinematic viscosity, because it depends only on space and time units, and not on mass.

To summarize: Dimensions of the coefficients of viscosity are

$$(\mu g) = M/LT, \qquad (\nu g) = L^2/T. \qquad (21)$$

(The presence of g accompanying the symbol P for pressure, μ for viscosity, and ν for kinematic viscosity is caused by the use of force in pounds or kilograms instead of force in "poundals" or "dynes," coupled with the desire to use the ordinary M,L,T system of dimensions, and to treat g as an acceleration instead of as a pure number so that the force which is the weight or earth-pull in a mass W shall be Wg. This is one of the awkwardnesses of using the engineers' system of force units without going over to the corresponding system of mass units, namely the English or metric slug (g pounds or g kilograms). It seems, however, that occasional occurrence of analytic awkwardness such as this is hardly sufficient to make it worth while to do away in engineering with those units of mass and force which are of nearly universal acceptance, and which are certainly far more natural to the student. In other words, occasional analytical artificiality or complication is preferable to constant engineering artificiality.)

(1) Suppose that it be required to determine the force upon an object in a moving fluid, when the viscosity is important, and the density relatively unimportant, as is the case when the fluid is moving very slowly, so that the dynamical effects are due to the stickiness, not to inertia. Then

$$Pg = f(l, U, \mu g) = al^p U^q (\mu g)^r, \qquad (22)$$

and

$$\frac{ML}{T^2} = L^p \left(\frac{L}{T}\right)^q \left(\frac{M}{LT}\right)^r.$$

The equations are as follows:

$$p + q - r = 1, \qquad r = 1, \qquad -q - r = -2.$$

From which $r = 1$, $q = 1$, $p = 1$ and

$$Pg = alU\mu g \qquad \text{or} \qquad P = lU\mu. \qquad (23)$$

It appears that in this case the pressure of the fluid on the immersed object varies directly with the linear dimension instead of with its square, directly with the velocity, instead of with its square, and directly with the coefficient of viscosity. This law of resistance is known as Stokes's Law, and is applicable only when the motion is very slow, as in the case of minute particles of dust suspended in the air.

(2) If the pressure be considered to depend on both the coefficient of viscosity and the density,

$$Pg = f(l, U, \rho, \mu g) = al^p U^q \rho^r (\mu g)^s, \quad . \quad . \quad . \quad . \quad (24)$$

or

$$\frac{ML}{T^2} = L^p \left(\frac{L}{T}\right)^q \left(\frac{M}{L^3}\right)^r \left(\frac{M}{LT}\right)^s$$

and

$$p + q - 3r - s = 1, \qquad r + s = 1, \qquad -q - s = -2.$$

Hence $r = 1 - s$, $q = 2 - s$, $p = 2 - s$ and

$$Pg = aSU^2\rho\left(\frac{\mu g}{lU}\right)^s = SU^2\rho f\left(\frac{\nu g}{lU}\right). \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In this solution s is not determined, and the general functional form may be given. As g is constant, within the accuracy generally postulated, the pressure P is a function of the quotient of the kinematic coefficient ν by the product of the stream velocity and the linear dimension of the object.

(3) Intermediate between (12), in which viscosity is neglected, and (23), in which inertia is neglected, comes a formula which is necessarily a special case of (25) and which is known as the law of skin friction. In Art. 69, it was pointed out that when a body moved through a liquid there adhered to the skin of the body a boundary layer in which viscosity was the ruling factor. Suppose the body at rest and the fluid moving by with velocity U. Let the boundary layer be of thickness w. Consider the body as a rectangle parallel to the direction of the motion. Let the length of the rectangle be l and the breadth be b. The wetted surface is $2lb$; the volume of the adhering layer is $2lbw$. The motion in the layer is slower than that outside, so that the "trapped" volume $2lbw$ represents a loss of momentum. The amount of fluid trapped per unit time is proportional to U (w being assumed constant) but does not depend on the length l in the direction of motion, and the loss of velocity

is proportional to U. Hence the force is proportional to $bw\rho UU$ or

$$F = kbw\rho U^2, \qquad \text{but} \qquad F = 2\mu lbU/w, \quad . \quad . \quad . \quad . \quad (26)$$

is by definition the viscous reaction. Hence,

$$kw\rho U = 2\mu l/w \quad \text{or} \quad w \propto (\mu l/\rho U)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

This is not unreasonable because the thickness of the boundary layer should physically increase with μ and decrease with ρ and U. Substitute (27) in (26). Then

$$F \propto \mu^{\frac{1}{2}} \rho^{\frac{1}{2}} bl^{\frac{1}{2}} U^{\frac{3}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

This is the law of skin friction or Allen's law. The force varies as the $\frac{3}{2}$ power of the velocity, as the $\frac{3}{4}$ power of the wetted surface, and as the square root of both viscosity and density jointly.

It is difficult to separate the skin friction from the total resistance in experiment. It might be expected that the total force acting on a body moving in a liquid would be of the form

$$F = a\mu S^{\frac{1}{2}} U + a' \mu^{\frac{1}{2}} \rho^{\frac{1}{2}} S^{\frac{3}{4}} U^{\frac{3}{2}} + a'' \rho S U^2. \quad . \quad . \quad . \quad . \quad (29)$$

The first term would be small, if at all detectable, because the straight viscous resistance is known to apply only when the velocity is very small so that the complicated disturbances accompanying turbulence do not appear. The main part of F should be found in the last two terms because as a matter of fact when a body moves in a liquid the phenomenon of the boundary layer and of the outer general streamline disturbance do clearly appear. A compromise could be made by writing

$$F = kS^p U^{2p}, \qquad \tfrac{3}{4} < p < 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

when μ and ρ are given constants. Zahm has made measurements indicating that $p = 0.93$ in his range of experiments. When an aeroplane wing moves through the air, the lift L sustains the weight and the drag D is equilibrated by the propeller thrust. It would not be unreasonable to expect that the drag, which is mainly tangential to the wing, might have in its make-up a much larger percentage of the skin friction and a much smaller percentage of the general inertia reaction than the lift which is mainly noımal to the wing. That is some such result as

$$D \propto S^p U^{2p}, \qquad L \propto S^q U^{2q}, \qquad \tfrac{3}{4} < p < q < 1 \quad . \quad . \quad . \quad (31)$$

might be expected.

74. Dynamical Similarity and Models. For geometrical similarity a single ratio, the ratio of lengths of corresponding lines is sufficient to determine one figure from another. For dynamical similarity three other ratios are necessary, namely, the ratio of the masses, the ratio of the forces, and the ratio of the times.

Let it be supposed that X, x, m, t, represent respectively a force, distance described, a mass, and the time of describing the distance in the case of a model of a machine; and that X', x', m', t', represent the corresponding quantities for the machine itself. Further let it be assumed that

$$X' = \xi X, \qquad x' = \lambda x, \qquad m' = \mu m, \qquad t' = \tau t, \quad . \; . \quad (32)$$

ξ, λ, μ, τ being four ratios of similitude. It is understood that all forces are in the ratio ξ as between model and machine, all masses in the ratio μ, etc. Velocities will be in the ratio λ/τ, accelerations in the ratio of λ/τ^2, and the fundamental equation of dynamics connecting mass, acceleration, and force requires that for similarity

$$\mu\lambda = \xi\tau^2. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

Hence, there is a relation between the four ratios of similitude, and only three of those ratios are in fact independent.

If a machine and model are geometrically similar and of the same material, if the masses are in corresponding proportion, and if the machine and model begin to move in similar fashions, they will continue to move in a similar manner, provided the applied forces are proportional to the mass multiplied by the linear dimensions and divided by the square of the time. If the velocity ratio be introduced as $\lambda/\tau = \eta$, this statement may be phrased as follows: The forces must be proportional to the mass multiplied by the square of the velocity, divided by the linear dimensions — as may be seen by eliminating τ between (33) and $\lambda/\tau = \eta$ to obtain $\xi = \mu\eta^2/\lambda$.

In many cases the weight of the machine is one of the fundamental forces to be considered; and the weight is proportional to the mass or $\xi = \mu$. Hence $\eta^2 = \lambda$ and the velocity of the model, when working similarly to the machine, must be proportional to the square root of the linear dimensions. When the velocities are proportional to the square root of the linear dimensions the model and machine will be said to be working at "corresponding" velocities. If, as is assumed here, the weight is an important force for the proper working of the machine, all other forces must be in the same ratio as the weights, that is, as the masses. Now, if the model and machine be made of the same material, the weights of corresponding parts assumed geometrically similar are as the cubes of the linear dimensions; and, hence, all the applied forces must vary as the cubes of the linear dimensions. In the case of a model of a gas engine the piston area is proportional to the square of the linear dimensions; and, consequently, the pressure in the cylinder must be proportional to the linear dimensions. The performance of a full-sized engine cannot be predicted from that of a model unless this proportionality between the fluid pressures in the cylinder be preserved.

(1) In addition to the external or applied forces there are developed in any machine or model certain internal actions and reactions; and on the hypothesis that the weights are important forces it is necessary that all the actions and reactions developed by the working of the model should also be proportional to the masses, that is, to the cube of the dimensions. For example, in the case of the airplane the forces which are developed by the motion of the machine or model are the friction resistances of the air. These are generally assumed to vary with the surface and with the square of the velocity. As the velocity varies as the square root of the linear dimensions, the air pressures will indeed vary with the cube of the linear dimensions at corresponding velocities; and, hence, the prediction of performance from the flying model to the machine is justified so far as the forces of air resistance are concerned, provided at any rate those forces do not in fact differ much from the law assumed and used throughout this work. This principle was used by Langley in the predicting of performance of a man-carrying machine from the performance of a one-quarter size model. In ordinary practice at the present time the dynamical performance of machines is calculated and not obtained from flying models.

In naval architecture the resistance of full-sized ships is investigated by measuring resistance on small models. If it be assumed that the model and the ship are of the same material, that is, that they have the same mean densities, the weights vary as the cube of the linear dimensions. Corresponding velocities vary as the square root of the linear dimensions; and the resistances vary approximately as the squares of the velocities multiplied by the areas immersed. The results are entirely similar to those in the case of the air ship or airplane. The forces vary as the cubes of the linear dimensions;

and thus if a 1/25th sized model is made and drawn through the water at the proper speed, which is 1/5th the speed of the full-sized ship, the forces upon it should be the $1/(25)^3$ of the forces upon the ship; and the resistance to motion of the ship may be taken as $(25)^3$ times the resistance measured on the model. This is not, in fact, the procedure used in aeronautics, for in wind tunnel experiments it is not assumed that the velocities are in the proper ratios. The forces are obtained by multiplying the observed force on the model by the ratio of the surfaces (the square of the ratio of linear dimensions), and by the square of the ratio of the velocities. As a matter of fact, if it were convenient to run the wind tunnel at the corresponding speed, which would be to the speed of the airplane in the ratio of the square root of the linear dimensions between model and airplane, the forces on the machine could be calculated directly by multiplying by the cube of the linear dimensions.

If instead of assuming that the forces vary with the square of the velocity, the general law of variation (25) be used which includes any possible effect of viscosity, then

$$P = \rho SU^2 \phi\left(\frac{\nu}{lU}\right), \qquad P' = \rho' S' U'^2 \phi\left(\frac{\nu'}{l'U'}\right) \quad . \; . \; . \quad (34)$$

may be taken as the forces on the machine and model respectively. Hence

$$P = P' \frac{\rho SU^2}{\rho' S' U^2} \frac{\phi\,(\nu/lU)}{\phi\,(\nu'/l'U')} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (35)$$

The force on the machine may then be calculated from that on the model taken with the respective values of $\rho, S, U, \rho', S', U'$, provided the arguments ν/lU and $\nu'/l'U'$ of the unknown function ϕ are the same. Hence, the test on the model should not be run at an arbitrary velocity U' but at one such that

$$U' : U = l/\nu : l'/\nu' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (36)$$

If the test be run in air, $\nu = \nu'$; and the wind velocity in the test should be to the velocity of flight inversely as the lengths of model and machine. That is, a test on a 1/24th size model should be run at 24 times the velocity of flight of the machine. This is entirely impracticable, for such velocities cannot be obtained; and would be valueless if practicable, for at high velocities the compression of the air cannot be neglected, but has to be regarded as a variable in addition to μ, ρ, S, U which were used in deducing (25).

(2) The theory of models may also be applied to statical structures. In recent years it has become customary to sand-test airplanes to determine the limits of their structural strength. In case it became necessary to sand-test exceedingly large machines the cost of the construction of the machine for sand-testing might be very high. The question would, therefore, arise whether one might not equally well construct an exactly similar model of the machine and sand-test that. The forces involved here are the weights; and to be able to sand-test the model with assurance that results could be carried over to the full-sized machine it would be necessary to carry out a test with weights of sand proportional to the cubes of the linear dimensions. It would also be necessary to show that in geometrically similar structures the forces developed by slight deformation were all proportional to the cubes of the linear dimensions. Consider, for example, a stay wire. The strength of the wire varies with the cross section; $F = kS\Delta l/l$, where k is a constant of the material. The extension, in similar figures, would vary with the linear dimensions. The total force developed would, therefore, vary only with the square of the dimensions. A similar argument would apply to compression members under slight compression. With bending moments the argument is different; for

$$EI\frac{d^2y}{dx^2} = \text{Moment}, \qquad (37)$$

is the equation, in which E is a constant of the material (Young's modulus). Now if forces vary as L^3, moments vary as L^4. But the moment of inertia I of the area of a section varies as L^4 and d^2y/dx^2 as $1/L$. Hence moments should vary as L^3 instead of as L^4, and similar structures loaded similarly in proportion to their weights would not suffer similar deflections.

The general discussion for sand-testing will give an indication of the method to be pursued. Let f be the load, s the strain, E Young's modulus, l the length, W the weight, A the cross section, and I the moment of the section. For different members these may all be different. The dimensions are, if force F be taken as the fundamental unit with length L as the other, as would be natural in statics,

$$(E) = F/L^2, \qquad (f) = F, \qquad (s) = L, \qquad (I) = L^4. \qquad (38)$$

There are two fundamental quantities and any quantity may, by the general theorem, be written as the product of any two raised to the proper powers multiplied by a function containing the others expressed in terms of them in such a way as to make the combination non-dimensional. Let l and E be taken as the two fundamental quantities. Then

$$(f) = (El^2), \quad (s) = (l), \quad (A) = (l^2), \quad (W) = F = (El^2), \quad (39)$$

and

$$f = El^2 \text{ func.} \left(\frac{s}{l}, \frac{A}{l^2}, \frac{W}{El^2}, \frac{I}{l^4}\right). \quad \ldots\ldots\ldots \quad (40)$$

The function has the four variables s, A, W, I in the desired combinations with E and l so as to make the ratios pure numbers. If the structures are similar geometrically, two of the numbers become 1 and the function may be written

$$f = El^2\phi(s/l,\ W/El^2). \quad \ldots\ldots\ldots \quad (41)$$

If the variables for the model are designated by accents,

$$\frac{f}{f'} = \frac{El^2}{E'l'^2}\,\frac{\phi(s/l,\ W/El^2)}{\phi(s'/l',\ W'/E'l'^2)}, \quad \ldots\ldots\ldots \quad (42)$$

or

$$\frac{f}{El^2} : \frac{f'}{E'l'^2} = \phi\left(\frac{s}{l}, \frac{W}{El^2}\right) : \phi\left(\frac{s'}{l'}, \frac{W'}{E'l'^2}\right). \quad \ldots\ldots \quad (43)$$

The value of f for the machine can be predicted from that of f' for the model, taken with the values of E, E', l, l', only when the values of the function ϕ are the same; and as the function is unknown, this means that the variables must be equal, namely,

$$\frac{s}{l} = \frac{s'}{l'}, \quad \frac{W}{El^2} = \frac{W'}{E'l'^2}.$$

If then the values of E and E' are the same, the weights must vary as l^2 which would be impossible unless two different materials could be found with the same value of E but with densities inversely proportional to the lengths of machine and model. The test of a statical structure could, however, be made if the weight was so small relative to the load (both for machine and model) that the variables W and W' could be omitted. Then with the strains similar the loads would be as $El^2 : E'l'^2$.

EXERCISES

1. Derive the formula for the time of oscillation of a pendulum without assuming that the time does not depend on W, *i.e.*, take $T = f(l, g, W)$. What information can be had as to the dependence of T on the angular amplitude of the motion?

2. Discuss the time of oscillation in simple harmonic motion of a mass W under a restitutive force $F = Ex$ (lb) when the amplitude is A.

3. The central acceleration of a particle of weight W moving uniformly with velocity v in a circle of radius R can clearly depend only on W, v, R. What information does this give as to the formula for the acceleration?

4. If the velocity of a sea-wave be assumed to be dependent only on the depth of the sea, the length λ of the wave, the density ρ of the water, and the value of g, what information is furnished by the theory of dimensions?

5. If the thrust T of a propeller depends only on its diameter D, the forward speed U, the number N of revolutions per minute, and the density ρ of the air, show that $T = \rho D^2U^2\phi(DN/U)$.

6. Discuss the torque Q of the propeller along the lines of Ex. 5. Show that $Q = \rho D^3U^2\phi(DN/U)$. What other features of the propeller might have to be considered?

7. Suppose the resistance of a fluid depends not only on S, ρ, U, but also on the hydrostatic pressure p_0 at a distance in the fluid. Prove $P = SU^2\rho f(p_0/\rho U^2)$.

8. If the thrust of a propeller depends on the quantities in Ex. 5 and also on the viscosity, what other variable must enter into ϕ in addition to DN/U?

9. Assume the central acceleration in a planetary orbit is $f = -\mu/r^2$ according to Newton's law. Let a be a dimension of the orbit. Find the dependence of the periodic time T on μ and a. If μ is proportional to the mass W of the central attracting body, how should the periodic time depend on W? What would be the earth's period if its orbit were the same size and the sun were twice as massive?

10. Give the argument leading to a formula like (28) for a body of total surface S wholly immersed in a fluid.

11. If Langley's quarter-size model used $\frac{1}{4}$ H.P. in flight, how much would the full-size machine use?

12. Show that according to the ordinary law for sliding friction, the forces of friction satisfy the condition for dynamical similarity. Is this true of rolling friction?

13. Experiments on the deflection of a 50-ft bridge weighing 100 tons when a 20-ton engine runs over it at 40 mi/hr are to be made on a model bridge 5 ft long weighing $6\frac{1}{4}$ lb. What should be the weight of the model engine? If the stiffness of the model bridge is such that its statical deflection under the model engine is $\frac{1}{10}$ the statical deflection of the bridge under the engine, what should be the velocity of the model for the test?

14. If experiments to find the resistance of air to a dirigible are to be made by measuring the resistance of a model in water, what should be the relation of the dimensions and velocities of dirigible and model. Is this practicable?

15. Two airplanes are exactly similar in the ratio 1:2. If the efficiency of the two engines is the same, what is the ratio of radii of action?

16. If the range R of a projectile (neglecting air resistance) depends only on the velocity and inclination of projection and on the value of g, find R.

17. Check for dimensional correctness a number of formulas in Chaps. IV, V, IX, X.

CHAPTER XII

THE FORCES ON AN AIRPLANE

75. Lift and Drag. The statical forces and moments on an airplane may be determined from model experiments in the wind tunnel. The model may be fixed in the desired attitude toward the wind, and the forces or moments on the model may be measured by the aerodynamical balance, and may be scaled up to the full-sized machine by the following equations, where F' denotes the force on the model, and F that on the machine, l the linear dimension, and U the velocity:

$$F = F'\frac{l^2U^2}{(l')^2(U')^2}, \qquad M = M'\frac{l^3U^2}{(l')^3(U')^2} \quad . \quad . \quad . \quad (1)$$

The forces are scaled up as the square of the linear dimensions and as the square of the velocities of the relative wind. The moments are scaled up as the cube of the linear dimensions and as the square of the velocity. This rule is dependent upon the ordinary assumption that the pressures vary with the surfaces and with the squares of the velocity, and are practically independent of the quantity lU/ν.

It is desirable to have a convention as to the position of axes of reference in the model or airplane. The origin is taken at the center of gravity of the airplane, and at its corresponding point in the model (which need not be the center of gravity in the model, because it is only with respect to the surfaces exposed to the wind that the model is true). The Y axis is fixed as the line perpendicular to the plane of symmetry of the machine, and extending from the center of gravity to the left of the pilot. The X and Z axes are in the plane of symmetry, and are so chosen that for flight in a horizontal line the X axis is horizontal and toward the rear, the Z axis vertical and upward. Now, for uniform flight at different speeds, different lines with the machine are horizontal; consequently, the X and Z axes by the above definition are not fixed in the airplane except when the speed of horizontal flight is given. In most machines the attitude may change through a range of something like 15° between the position taken at highest speed and the position at lowest speed. In case it is desirable to use the same X and Z axes over a wide range of speeds, it is necessary to specify what direction in the machine shall be taken as the X direction. (See Figure 26.)

As the X axis is drawn backward, the velocity of the machine when in uniform flight is a negative quantity. To a certain extent this is an artificial convention, but it is natural when viewed from the wind tunnel experiment, because the direction of the relative wind (the model being fixed) is along the X axis, and the velocity of the relative wind is positive. When it is a question of a machine in flight it is necessary to remember that it is not the velocity of the machine but the velocity of the relative wind which is positive. Angles and angular velocities will be considered positive when the rotation about any axis is clockwise; that is, a yaw is positive when the machine is turning from left to right from the point of view of the pilot, namely, when there is rotation about the Z axis which tends to carry the X axis into the Y axis. In the same way, rotation about the Y axis from Z to X is positive; that is, pitching down is negative, pitching up or stalling is positive. Finally, rolling about the X axis is positive when the right hand of the pilot drops, and the left hand rises; that is, the bank proper to a positive yaw is a positive bank. Forces are positive when directed in a positive direction along the axes; moments are positive when they tend to produce positive angular accelerations.

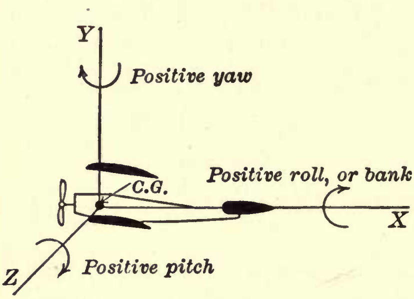

Fig. 26. Position of Axes Relative to Airplane.

The relations between different sets of rectangular axes in space are complicated in the general case, but simple when the angular displacements between the two positions of the axes are small. Consequently, it will be assumed that during the motion of the machine the axes are not much disturbed from their given positions, at least, in yaw and roll. The angular displacement in pitch if unaccompanied by yaw and roll need not be restricted to being infinitesimal.

If the model be placed in the wind tunnel with the Y axis horizontal and athwart the tunnel, the forces and moments produced will be a force down the wind; that is, essentially in the direction of the X axis; a force across the wind, that is, essentially in the direction of the Z axis; and a moment about the Y axis. The moments about the axes of X and Z and the forces around the Y axis will vanish, owing to the symmetry of the model with respect to the x–z plane. The force down the wind is called the drag, and the force cross the wind the lift, because in the standard position it is the force across the wind which sustains the machine against gravity.

The following data are for the lift and drag on a model of the Curtiss JN-2. The angle i is the angle of the wind-chord to the relative wind. The linear dimensions of the model were $\frac{1}{24}$ those of the full-sized machine.

Data on the Curtiss JN-2 (Hunsaker). Model $\frac{1}{24}$ size

i	L	ΔL	$\Delta^2 L$	D	ΔD	$\Delta^2 D$
−4	−0.08			.115		
		.22			−.011	
−2	+0.14		−.01	.104		+.009
		.21			−.002	
0	.35		.00	.102		.008
		.21			+.006	
2	.56		.00½	.108		.004
		.20½			.010	
4	.76½		.01	.118		.008
		.19½			.018	
6	.96		.01½	.136		.011
		.17			.029	
8	1.13		.02	.165		.016
		.15			.045	
10	1.28		.04	.210		.015
		.11			.060	
12	1.39		.05	.270		.018
		.06			.078	
14	1.45		.03	.348		.002
		.03			.080	
16	1.48		.01	.428		.000
		.02			.080	
18	1.50		.04	.508		−.007
		−.02			.073	
20	1.48			.581		
1	.45			.104		
7	1.05			.150		

Wind velocity in tunnel = 30 mi/hr. Lift and drag in pounds on model.

To examine the values one may either plot L and D against i, or one may consider the differences in the table. If the differences are constant, the curve is straight; if the differences of the differences, that is, the second differences are constant, the curve is a parabola. The differences in the lift L are exceedingly constant over a considerable range of angle, but beyond that neither the differences nor the second differences are constant. For the restricted range from -4° to about 8° a linear or straight line formula holds very well. It is difficult to obtain a satisfactory empirical formula to represent L over the whole range of the table, and one reason for this difficulty lies precisely in the extraordinarily good fit of the linear formula for the restrictive range.

The differences in the drag are not at all constant; but over a restricted range, such as from -4° to 8°, the second differences are reasonably constant; and for this reason a parabola will fit the curve very well over this range. For larger values of i, however, the curve first arises more steeply than this parabola, because the second differences are much larger; and then the curve straightens out, and toward the end of the table actually reverses its curvature.

In taking differences in a series of experimental values it must be borne in mind that the values themselves are liable to slight errors. In the case of the drag, if the figure .108 for $i = 2^\circ$ be changed by one unit to .107, the adjacent first differences are each changed by one unit to .005 and .011, and the corresponding second difference is changed by two units to .006. It can hardly be hoped that the experimental values are accurate to more than about 1%; and, consequently, the variation of two units in the second difference is not serious except when there is a progressive change of considerable magnitude. All in all, it is not considered at present worth while to develop empirical formulas for lift and drag on a particular model, because the empirical formula is either not nearly so accurate as the table, or is too complicated to be of much value — entirely apart from the difficulty of obtaining an accurate empirical formula, which, in itself, is a time-consuming process.

76. Performance Curves. Performance curves for the machine may be obtained from the data in the following manner: Suppose the machine to weigh 1800 lbs, and to fly at the angle $i = 2^\circ$. The lift on the model is 0.56 lb at 30 mi/hr. The lift on the machine is $24^2 = 576$ times as much at 30 mi/hr. Assuming that the lift varies as the square of the speed, the speed of the machine may be determined so that the total lift shall be 1800 lbs, namely,

$$1800 = 576 \times 0.56 \times (V/30)^2 \quad \text{or} \quad V = 71 \text{ mi/hr}.$$

The drag on the model at 2° is 0.108 lbs. The drag on the machine is 576 times as great, and is further multiplied by the square

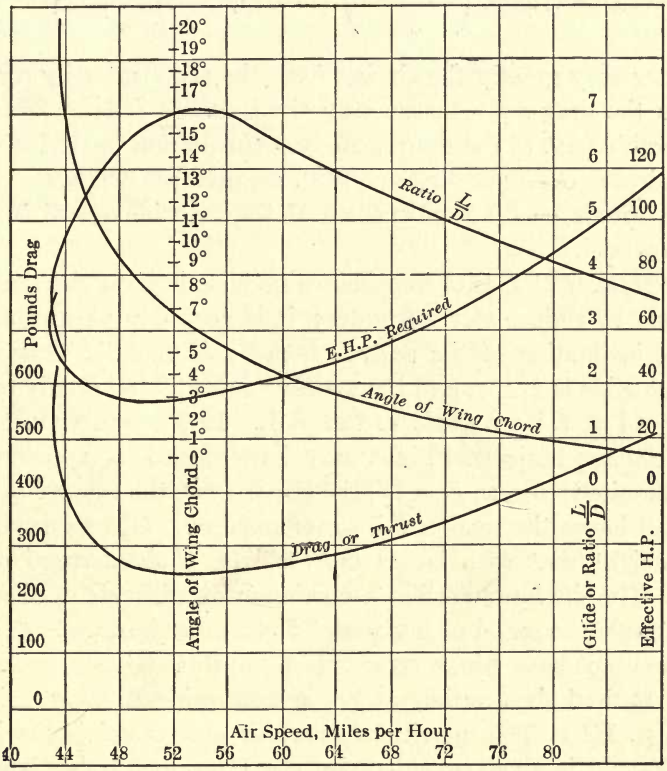

Fig. 27. Performance Curves for Curtiss JN-2 (after Hunsaker).

of the velocity-ratio, namely, 5.58; hence $D = 347$ lb. The propeller thrust must be equal to the drag, as the cosine of the small angle is practically 1. The horse power required is the product of the drag by the velocity divided by 375, and is in this case H.P. $= 347 \times 71/375 = 66$. The points in figure 27 have been determined by carrying through the calculations for different angles of assumed flight.

It is known that the forces on a body moving in a fluid do not vary strictly with the square of the velocity and with the square of the linear dimension (see Art. 73). It is, therefore, possible to take account of this variation if desired in calculating the drag and the horse power required. For instance, suppose that the lift varies as the square. The velocity is then 71 mi/hr as before; but let the drag vary as $(lV)^{1.9}$. Then the drag in the machine is

$$D = .108 \times \left(\frac{24 \times 71}{30}\right)^{1.9} = 233, \qquad \text{H.P.} = 44.$$

Note how very greatly the change from the exponent 2 by 0.1 to 1.9 changes the drag and consequently the H.P. It follows that if any considerable part of the drag is due to skin-friction instead of inertia in the air, the curves of drag and H.P. required are decidedly too high at high speeds, — the performance of the machine would be better than expected.

The ratio of D/L is of importance because it gives the tangent of the angle of glide under no power; it is generally plotted as L/D, and the inclination of the path is then "1 in L/D." Thus if L/D be 7, the glide is 1 in 7, and the machine comes down slowly with the wing-chord at a large angle to the path. In a rough way, too, the ratio L/D for horizontal flight may be regarded as a criterion for power required; for, as $L = W$, the larger L/D, the smaller the value of D, and hence the smaller the power required. Of two machines of equal weight that with larger L/D will for a given speed use less power. The lift itself reaches a maximum at an angle of about 18°; at this angle the speed of horizontal flight must be least. (The machine may not have power enough to fly in this attitude, because the power required rises rapidly at low speeds owing to wing resistance, see Chap. III.) This minimum speed (if attainable) is the landing speed V_0 and is about 43 mi/hr for the JN-2. (Whether landing at so slow a speed would be safe from the point of view of the control or stability of the machine is another matter; there is dynamical instability below 46 mi/hr.) If L_0 be the maximum lift, then $V : V_0 = L_0^{\frac{1}{2}} : L^{\frac{1}{2}}$ is a proportion which determines V.

The discussion of different aerofoils may be carried on like that for the whole machine — so may that for different bodies. In the design of the body the aim should be to diminish the drag by using a stream-line shape. The parasite resistance is a very serious factor in power consumption at high speeds. In the design of an aerofoil different features must be kept in mind according to the use for which the machine is intended. The discussion may be made with reference to the curve for L/D for the aerofoil. As most of the weight of the machine is carried on the wing, $L = W$, nearly, for horizontal flight, and large values of L/D correspond to small values of D. If a machine is to fly mainly at high speeds, the values of L/D should be large at high speeds, *i.e.*, for small angles of attack; but if the machine is to fly at lower speeds, it is for this range that L/D should be large, *i.e.*, for larger angles of attack. By varying the shape of the section of the aerofoil considerable differences in form of the L/D curve may be obtained. It is not possible here to go much into detail — that belongs rather to the course in design — but it is desirable to point out the main facts that should be borne in mind. Other things being equal, it would be better, for structural reasons, to have a thick wing rather than a thin one. Indeed if the wing without much loss of aerodynamic efficiency could be so thickened that it could be supported by internal spars instead of by external wires and struts, a considerable saving in parasite resistance might be effected.

In giving data on wings it is usual to give not the values of L and D but those of the coefficients K_x, K_y in the expressions

$$L = K_y SV^2, \qquad D = K_x SV^2.$$

The units of K_y and K_x are, therefore, lb/(ft.^{2}mi^2/hr^2). The ratios L/D and K_y/K_x are the same. In the following table the last column gives the distance x of the center of pressure from the leading edge of the aerofoil as the fraction $f = x/b$ of that distance to the width of the chord.

i	$1000K_y$	$1000K_x$	L/D	f
−4	−.123	+.164	−0.75	....
−2	+.520	.115	+4.52	.67
−1	.765	.108	7.11	.52½
0	.975	.103	9.44	.46
1	1.118	.100	11.8	.41½
2	1.38	.099½	13.9	.39
4	1.77	.111½	15.9	.35
5	1.98	.134	14.8	.33
8	2.56	.190	13.5	.30
12	3.31	.310	10.7	.27½
14	3.60	.354½	10.1½	.27½
16	3.61½	.443	8.15	.27½
18	3.47	.558	6.22	.30

Here the ratio L/D reaches nearly 16, which is not at all excessive for a wing alone (though the whole machine will usually not reach half so much). Moreover the ratio remains high over a considerable range of angle i. The position of the center of pressure has to be determined by measuring moments as explained in connection with the Vector Diagram of the next Art. But this fact is noteworthy: The center of pressure approaches the leading edge as the angle of attack is increased (until it is very large). This is characteristic of most curved aerofoils in distinction to what happens in the case of a flat plate where the C. P. retreats from the leading edge with increasing values of i. In discussing the stabilizer (Art. 18) it was seen that the effect of the stabilizer was to increase the restoring moment tending to bring the skeleton airplane back to its normal flying attitude when slightly disturbed. But the skeleton airplane is stable anyhow, because if the angle i be increased the C. P. retreats, giving rise to a differential diving moment, whereas if the angle i be decreased the C. P. advances toward the leading edge, giving rise to a differential stalling moment. With the curved wing, however, the motion of the C. P. is contrariwise, and an airplane with the wing only (without stabilizer) would not tend to return to its normal attitude but would depart further from it by virtue of the moment brought to play.

NOTE. In place of the lift and drag coefficients K_y, K_x above used, there may be introduced the coefficients K'_y, K'_x defined as

$$Lg = K'_y \rho S U^2, \qquad Dg = K'_x \rho S U^2,$$

and called the absolute lift and drag coefficients. The dimensions are

$$(K'_x) = \frac{(Dg)}{(\rho)(S)(V^2)} = \frac{ML/T^2}{M/L^3 \cdot L^2 \cdot L^2/T^2} = 1.$$

Thus K'_x and K'_y are free from dimensions, and hence "absolute." Indeed the values of K'_x, K'_y are the same in any two different but consistent set of units, *e.g.* ft, lb, ft/sec, $g = 32.2$ and m, kg, m/sec, $g = 9.81$; they will, however, not be the same if V is in mi/hr unless ρ is in lb/mi^3 and g in mi/hr^2. The maximum values of K'_y run as high as 0.70. The "absolute" coefficients are used by the National Physical Laboratory, England, but have never found favor with Eiffel or in this country.

77. The Vector Diagram. The knowledge of the lift and drag serves to determine the total resultant as the square root of the sum of their squares, namely,

$$R = (L^2 + D^2)^{\frac{1}{2}}.$$

The angle made by the resultant with the wind direction is $\tan^{-1}(L/D)$. It is necessary to observe carefully the fact that the direction of the resultant is specified relative to the wind, and not relative to a fixed direction in the model. As the wing-chord is used as a fundamental line of reference in the model, the angle between the resultant and the wing-chord may be found as $\tan^{-1}(L/D) + i$ measured from the rear of the chord. Thus, both the magnitude and the direction of the resultant relative to the model are known.

To determine the actual line of action of the resultant on the model (regarded as a rigid body) it is necessary to know the distance of the resultant from some fixed point; and this is determined by measuring the moment of the air forces on the model about an axis passing through the center of gravity of the model. The following table gives the results for the Curtiss JN-2.

Data on the Curtiss JN-2 (Hunsaker). Model $\frac{1}{24}$ size

i	L	D	R	M_s	M_s/R
$-4\frac{1}{2}$	$-.13$	.123	.179	-0.022	-0.12
$-2\frac{1}{2}$	$+.08$	.105	.132	.40	3.0
$-\frac{1}{2}$	.30	.102	.317	1.05	3.3
$1\frac{1}{2}$	.51	.105	.521	1.65	3.2
$3\frac{1}{2}$	$.71\frac{1}{2}$	.115	.724	2.21	3.0
$5\frac{1}{2}$	.91	.130	.92	2.71	$2.9\frac{1}{2}$
$7\frac{1}{2}$	1.09	.157	1.10	3.17	2.9
$9\frac{1}{2}$	1.24	.196	....	....	....
$11\frac{1}{2}$	1.37	.252	1.39	3.81	$2.7\frac{1}{2}$
$13\frac{1}{2}$	1.40	.330	....	....	....
$15\frac{1}{2}$	1.48	.408	1.53	4.00	2.6
$17\frac{1}{2}$	1.49	.482	....	....	....
$19\frac{1}{2}$	1.49	.561	1.59	3.95	2.5
$2\frac{1}{2}$	$.61\frac{1}{2}$	.110	....	1.93	....
$4\frac{1}{2}$	.81	.122	.82	2.48	....

Wind velocity in tunnel = 30 mi/hr. Lift and drag in pounds on the model; moments about the spindle in pound-inches; M_s/R is in inches on the model, not on the drawing (Fig. 28).

The spindle, or axis about which moments are taken, pierces the model through its center of gravity, which is not the point corresponding to the center of gravity of the machine. The advantage of having the spindle pass through the center of gravity of the model itself arises from the fact that the moment about this axis is the moment of the air forces alone; whereas the moment about any other axis would be due partly to the moment of the air forces, and partly to the weight of the model located at its center of gravity. If the moment M_s be divided by the resultant R, the quotient is the arm of the resultant; and the knowledge of this arm, taken with the direction of the resultant, makes it possible to locate on a drawing of the model the resultant of the air forces (Fig. 29.).

In order to fly at any particular attitude, that is, at any particular speed in a horizontal line, it is necessary to set the elevator at such

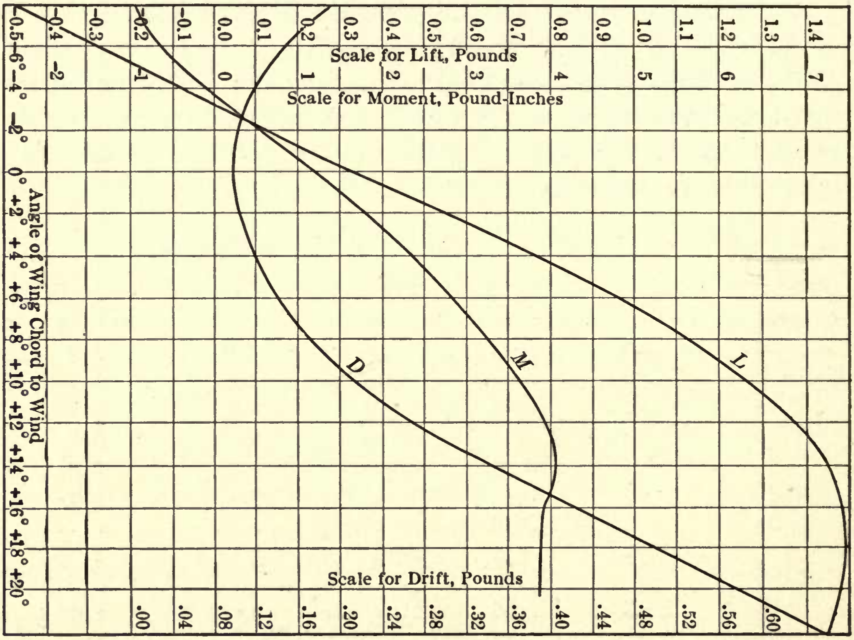

Fig. 28. Lift, Drag, and Moment Curves for Curtiss JN-2 Model (after Hunsaker).

an angle that the resultant force passes through the center of gravity, so that the condition for equilibrium in respect to rotation is satisfied. The resetting of the elevator changes in some small measure, but only in a small measure, both the lift and the drag. The major change is in the moment, and not in either the magnitude or direction of the resultant. The effect, therefore, on the vector diagram of a change in the elevator is to move the lines of action back and forth relative to the center of gravity.

If the machine is supposed to be flying with some given velocity at a certain attitude, and is accidentally tipped up or pitched down a little, a moment should be developed which will right the machine;

but when the attitude of the machine is changed, the result may be expressed by saying that instead of the force through the center of gravity, an adjacent force passing not quite through the center of

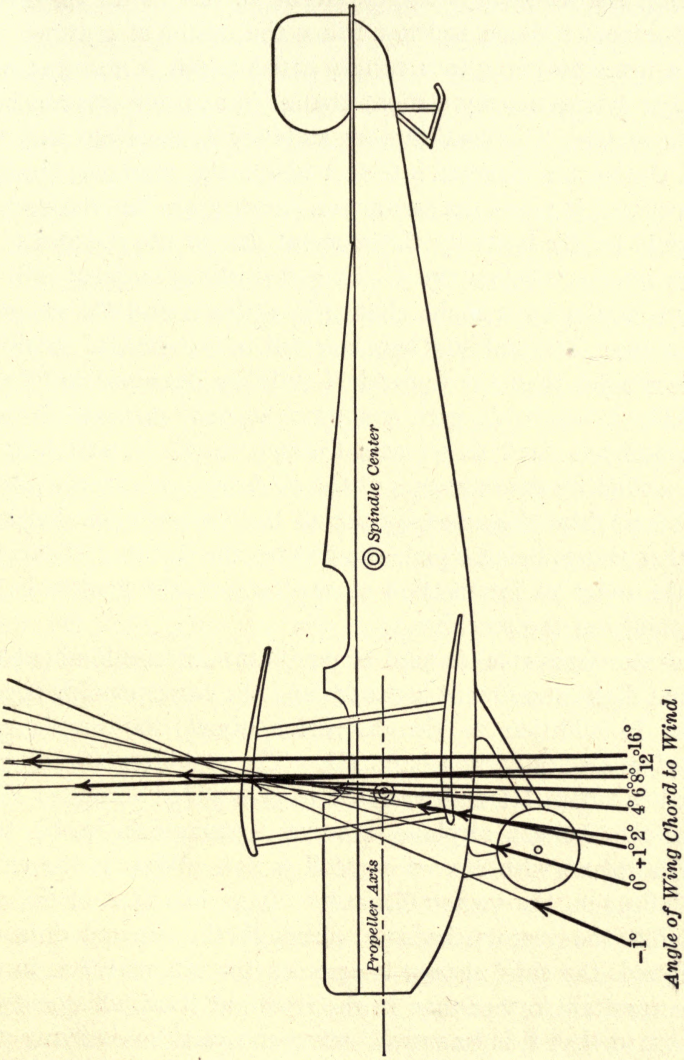

Fig. 29. Vector Diagram showing Position of Resultant Forces for Different Altitudes and their Magnitude for Constant Wind Velocity.

gravity is brought into action. If the machine is tipped up, the angle with the relative wind is increased, and a vector to the rear on the diagram comes into action. This has a tendency to pitch the machine down. If, however, the machine were depressed slightly, the new vector would be forward of the center of gravity, and would tend to restore the machine. The condition, therefore, that a restoring moment shall be brought into action by momentary changes of attitude in the machine is that adjacent vectors in the force diagram shall intersect above and not below the center of gravity.

When a machine flying in a certain attitude has a restoring moment brought into action by a slight change in attitude, the machine is said to be stable. The condition for stability is, therefore, that the vectors on the vector diagram intersect above the center of gravity. (Strictly speaking it is not the vectors on the diagram but the vectors as they would be displaced by the moment due to the elevator.) If the vectors intersect below the C. G. a disturbing moment will be brought into action by a slight change in attitude and the machine will be unstable. The stability here referred to is a statical stability, very much simpler than the dynamical stability discussed in Chaps. VI and VIII; it has to do with conditions of rest, as when the machine is considered fixed to an axis through its C. G. and held in balance in a wind by the moments of the air forces; it corresponds to the stability of ships discussed by use of the "metacentric height." Indeed, other things being equal, the greater the distance above the C. G. of the point of intersection of the vectors, the greater is the statical stability of the machine.

The elevator (movable) is used to equilibrate moments about the C. G. so that flight at different attitudes and the corresponding speeds is possible; in addition to the discussion above, the simple case was treated analytically in Chap. III. The stabilizer (fixed) was also treated in Chap. III and was seen to give added stability. The stability of the skeleton airplane, however, is ample in itself; that of the real airplane with curved aerofoil is not. What is the effect of the stabilizer on the vector diagram? It is, like that of the elevator, to shift the vectors, because, although the lift and drag are slightly altered, the chief change is again in the moment, *i.e.*, in the arm of the resultant rather than in the resultant itself. Suppose the stabilizer set so that it is "neutral" when the machine is flying in a certain attitude, say with $i = 4°$. Then the presence of the stabilizer has no effect upon the position of the resultant for $i = 4°$ in the diagram. If the machine stalls a bit, so that $i = 6°$, say, the stabilizer is blown upon from the bottom and produces a diving (negative) moment, which means that the vector for $i = 6°$ must be moved to the rear (as compared with its position in a machine without stabilizer); but if the machine noses over, say to $i = 2°$, the stabilizer is blown upon from above and there is a stalling (positive) moment and the vector for $i = 2°$ must be moved forward. The result is, therefore, to cause the vectors for 2° and 6° to intersect that for 4° further above the C. G., thus increasing the "metacentric height" and the statical stability for flight at $i = 4°$. Indeed with a large enough stabilizer far enough aft, it is clear that any instability due to the aerofoil could be overcome.

It has been assumed that the stabilizer was neutral. This can be the case only for one attitude, which may not be any attitude actually flown. If the stabilizer is a lifting surface at that speed of flight for which the design calls for a neutral elevator, the location of the C. G. relative to the wings must be rearward of where it should be if there were no stabilizer. If there is a negative lift, the C. G. must be further forward. Thus if the stabilizer be considered as set at various negative angles (turned up at the rear) relative to the wing-chord, the machine must be designed with its C. G. in various positions, and the greater the (negative) angle, the further forward must the C. G. go — for the larger angles means a relatively larger negative lift. These details were worked out for the simple case in Chap. III; the principles still apply, but the numerical details are different because the travel of the center of pressure on the aerofoil is contrary in direction and because the direction as well as the magnitude of the resultant force varies.

Suppose that with the elevator neutral and the stabilizer at a given angle $-i'$ with the wing-chord the machine balances with the C. G. in a certain position, corresponding to a definite attitude. The resultant through the C. G. is $L = W$. Now if the stabilizer be set at a greater negative angle $-i' - \Delta i'$ there is a stalling moment ΔM brought into play. This change in moment on the stabilizer will be the same for all attitudes of the machine so long as the angle i is not so greatly altered that the pressure on the stabilizer can no longer be regarded as a linear function of the angle between the stabilizer and the wind. Corresponding to this ΔM, the C. G. must be set forward by the amount $x = \Delta M/W$. The vectors in the vector diagram have their arms changed by the amount $\Delta M/R$. Now for the vectors originally ahead of the C. G., R is less than W (since R and L are nearly equal, as may be seen in the table for the range $i = -\frac{1}{2}$

to $i = 19\frac{1}{2}$) whereas for the vectors originally behind the C. G., R is greater than W. Hence, the former vectors are advanced more, the latter less than C. G., and there is a greater general dispersion of the new vectors about the new C. G. than of the old vectors about the old C. G. The result is that the new vectors intersect further above the C. G., and the statical stability has been increased by the change Δi in the setting of the stabilizer, — as in the simple case of Chap. III.

This result may be exemplified by figures from a test by Hunsaker on the same model ("Clark") with three different tail settings at $-2\frac{3}{4}^\circ$, -5°, -7° respectively to the wing-chord. The moments have been reduced to the estimated C. G. L and D in pounds, M in pound-inches about the C. G., V = 30 mi/hr. Model $\frac{1}{26}$.

	$i' = -2\frac{3}{4}^\circ$			$i' = -5^\circ$			$i' = -7^\circ$		
i	L	D	M	L	D	M	L	D	M
−2	.18	.10	.01	.11	.11	.16	.03	.11	.47
−1	.32	.10	.02	.24	.10	.12	.14	.10	.45
0	.45	.10	.02	.36	.10	.12	.30	.10	.29
+1	.57	.10	.03	.49	.10	.07	.44	.10	.14
+2	.70	.11	.04	.62	.10	−.01	.57	.10	.04
4	.93	.12	.04	.87	.11	−.12	.81	.11	−.16
8	1.32	.17	.02	1.30	.15	−.28	1.22	.14	−.48
12	1.60	.24	.09	1.57	.21	−.39	1.54	.22	−.88

The effect of the different tail setting is (1) to change the attitude of "no lift," as might be expected; (2) to leave the minimum drag the same; (3) to leave the maximum lift, not shown in the table, the same; (4) to change a practically constant and zero moment when $i' = -2\frac{3}{4}^\circ$ to a variable moment progressively diminishing, with $dM/di = -.04$ on the average in the range given here, when $i' = -5^\circ$, and further to a variable moment with $dM/di = -.10$ when $i' = -7^\circ$. Now dM is the restoring moment when i changes by di; for $i' = -2\frac{3}{4}^\circ$ it is practically zero, for $i' = -5^\circ$ it is .04 pound-inches per degree, and for $i' = -7^\circ$ it is −.10. The greater stability gained by increasing i' negatively is clear. If i' had been set at -1°, it is probable that the stability would altogether have vanished with $dM/di = +.06$ or thereabouts (by extrapolation).

As the moments have been reduced from the spindle to the C. G. a word about the method reduction is advisable. If the distance from the spindle to the C. G. be a, in the case of the JN-2 model

3.04″, and the angle between this line and the arm of R be θ, the moment M of R about the C. G. will be

$$M = R\left(\frac{M_s}{R} - a\cos\theta\right) = M_s - Ra\cos\theta = M_s - 3.04R\cos\theta.$$

Now the angle between the wing-chord and the line joining the C. G. to the spindle was in this case 1.6°. The angle between the wing-chord and R is $\tan^{-1}(L/D) + i$. Hence

$$\theta = 90^\circ - \tan^{-1}(L/D) - i + 1.6,$$
$$M = M_s - 3.04R\sin[\tan^{-1}(L/D) + i - 1.6].$$

By substituting in this formula, the value of M on the model may be found. To find M on the machine it is necessary to multiply by $\frac{1}{12}$, by the square of the speed-ratio, and by the cube of the linear dimension. The work involves, as so often in calculations with experimental data, differences of quantities nearly equal, so that the results are far from being as accurate as the data. Two persons working the slide-rule on the same figures may differ by $\frac{1}{2}$ to 1% and if they be required to subtract 4.00 from a computed 4.60 one may get 0.57 because of underestimating 4.60 by 0.03, and others may get 0.63 by overestimation — particularly when the "4.60" is the result of a series of operations. The following values are, therefore, but approximate.

	Model		M in pound-feet for machine				
i	M_s	M	79	51.8	47	45.2	43.7
$-\frac{1}{2}$	1.05	.15	1200	....	....	....	....
$1\frac{1}{2}$	1.65	.10	800	....	....	....	....
$5\frac{1}{2}$	2.71	−.08	....	−275	....	....	....
$7\frac{1}{2}$	3.17	−.18	....	−621	−508	....	....
$11\frac{1}{2}$	3.81	−.40	....	....	−1130	−935	....
$15\frac{1}{2}$	4.00	−.60	....	....	....	−1570	−1900

NOTE. The forces treated in this chapter are static forces. When the airplane is not in uniform motion other forces are brought in; these are treated with dynamic stability in Chaps. VI and VIII.

EXERCISES

1. Plot the values of L and D for the JN-2 from the table in Art. 75 indicating on the scales both the figures for the model and those for the machine. Note that the D curve becomes concave down at the extreme right.

2. Find a linear expression $L = a + bi$ to represent the lift over the range $i = -4°$ to $i = 8°$ and compare the differences between the formula and the tabulated values.

3. Write $L = .35 + 10.5i + ai^n$ or $L - .35 - 10.5i = ai^n$. Tabulate ai^n and determine a and n to represent the data. (NOTE. If $z = ai^n$, $\log z = n \log i + \log a$, and a plot of $\log i$ against $\log z$ should give a straight line from which the values of a and n may be read.)

4. Find by interpolation from the table in Art. 75 the values of L and D for $i = -\frac{1}{2}, 1\frac{1}{2}, 7\frac{1}{2}, 9\frac{1}{2}$ and compare with the values entered in the table of Art. 77 for the same model.

5. Find the values of dL/di as $\Delta L/\Delta i$ from the table at $i = 1$ and $i = 7$ by differencing the values at 0 and 2 and at 6 and 8. Find dL/di for $i = 10$, by using the values of L at $i = 8$ and $i = 12$. In like manner find dD/di for $i = 1$, $i = 7$, $i = 10$.

6. Find the values of dL/di and dD/di for the full machine (JN-2) at $i = 4$ and $i = 8$.

7. Show that the second differences in the parabola $y = a + bx + cx^2$ are constant and equal to $2c(\Delta x)^2$.

8. Model test, $\frac{1}{24}$ size, 30 mi/hr. Results as follows:

i	L	D	R	L/D	M_s	M/R
−4	−.21	.070	.22	−3.0	.15½	.7
−2	−.07	.053	.09	−1.3	.26½	3.0
−1	−.00	.045	.04½	.0	.31½	7.0
0	+.08	.042	.09	+1.9½	.31½	3.4½
1	.15½	.040	.16	3.8½	.35½	2.2
2	.22	.040	.23	5.5½	.37½	1.6½
4	.36	.045	.36	7.9½	.42	1.1½
6	.48	.051	.48	9.4	.46	.96
8	.61	.064	.61	9.6	.47½	.78
12	.76	.120	.77	6.3½	.37½	.48
16	.79	.205	.82	3.8½	.20½	.25
18	.80	.255	.84	3.1½	.19½	.23
20	.79	.296	.84	2.6½	.17	.20

Plot the curves for L, D, L/D, and M on one diagram.

9. Data as in Ex. 8. Difference L and D and discuss the results.

10. Find dL/di and dD/di in Ex. 8 at $i = -2, 0, 6, 8$.

11. Find L, D and M for full-size machine in Ex. 8 at 60 mi/hr, at $i = 0$, and at $i = 6$.

12. Model test, $\frac{1}{24}$ size, 20 mi/hr. Results as follows:

i	L	D	R	L/D	M_s	M/R
−7	−.286	.114	.31	....	−.82	−2.6½
−6	....	....	....	....	−.63	....
−4	−.099	.084	.13	....	−.23	−1.7½
−2	+.027	.072	.08	.37	+.19	+2.4½
0	.142	.067	.157	2.13	.57	3.6½
+1	.210	.065	.220	3.22	.79	3.6
2	.27½	.066	.28½	4.2	1.00	3.5
4	.40½	.071	.41	5.7	1.40	3.4
8	.64	.105	.65	6.1	2.11	3.2½
12	.78	.180	.80	4.3	2.56	3.2
16	.83½	.26½	.88	3.2	2.58	2.9½
18	.84½	.30½	.89	2.7	2.57	2.9
20	.85½	.34	....	....	....	....
22	.85½	.38	....	2.2	2.47	....

Fill in the data for $i = -6$ and derive those for $i = -1$ by interpolation. Plot curves of L, D, L/D, and M.

13. Find L, D, and M for full-size machine of Ex. 12 at 60 mi/hr, at $i = 0$, and at $i = 8$.

14. With data of table in Art. 75 calculate V, D, and H.P. for machine at $i = 7°$, $i = 10°$, $i = -1°$.

15. With data of table in Art. 75 calculate i and H.P. for $V = 75$ mi/hr and $V = 50$ mi/hr.

16. With same data figure D and H.P., on the assumption that D varies as $(lV)^{1.95}$, for $i = 2°$ and for $V = 60$ mi/hr.

17. If in scaling up from 30 mi/hr and $\frac{1}{24}$ size model to 120 mi/hr and full-size machine it is found that the drag comes out 15% too high (as checked by full-flight tests) find x in $(lV)^{2-x}$ if it be assumed that $D = k(lV)^{2-x}$.

18. Explain why the maximum of the L/D curve comes to the right of the minimum of the D curve on the model charts when $V = 30$ mi/hr, or other constant value. Is or is this not true on a performance chart, where the appropriate velocities are used? Why? How is the minimum of the drag curve related to the minimum of the H.P. curve and to the maximum of the L/D curve on the performance chart, and why?

19. With data of Ex. 8 find V, D, and H.P. for $i = 0$ and $i = 16$.

20. With the same data find i and H.P. for $V = 60$ mi/hr and $V = 90$ mi/hr.

21. Suppose the required H.P. is known, can the attitude and velocity be determined, and how? Take the data of Art. 75 and let H.P. = 80.

22. If the maximum "absolute" value K_y' for a wing is .65 find the maximum K_y. If $K_x = .0010$ find K_x'.

23. Plot on one diagram, and discuss, these two wing tests and that in Art. 76.

i	10^3K_y	10^3K_x	L/D	f	10^3K_y	10^3K_x	L/D	f
−5	−.70	.183	....	...	−.56	.137	...	...
−3	−.08	.073	....	...	−.13	.091	...	...
−1	+.40	.062	6.4	.54	+.30	.062	4.9	.65
0	.68	.065	10.5	.45	.58	.058	10	.48
1	.94	.067	14.0	.41	.86	.060	14.2	.44
2	1.17	.073	16.0	.38	1.08	.069	15.5	.41
3	1.38	.083	16.7	.35	1.28	.077	16.5	.37
4	1.55	.095	16.2	.34	1.47	.089	16.5	.35
6	1.89	.125	15.1	.31	1.83	.120	15.2	.33
8	2.29	.166	13.7	.30	2.20	.158	14	.32
10	2.56	.210	12.2	.29	2.57	.208	12.3	.31
12	2.79	.265	10.5	.28	2.90	.257	11.3	.30
14	2.85	.344	8.3	.28	3.01	.319	9.4	.30
16	2.92	.594	4.9	.32	2.86	.574	5.0	.34
21	2.44	.896	2.7	.38	2.50	.882	2.8	.40

24. Comparative body test, model $\frac{2}{25}$ size, 30 mi/hr. Effect of yaw. D is drag (down wind), P is side thrust (across wind).

Yaw	D_1	P_1	D_2	P_2
0	.036	0.	.028	0.
1	.036	.006	.028	.004
2	.036	.012	.028	.009
4	.038	.025	.029	.018
6	.042	.042	.029	.028
8	.047	.060	.031	.042

Plot these results. Explain why the drag remains constant at the start whereas thrust varies linearly. Which is the better body? Calculate the side thrust due to such body at 4° of yaw for a machine at 90 mi/hr.

25. Check the data in resultants in the table of Art. 77 for $i' = -\frac{1}{2}, 3\frac{1}{2}, 15\frac{1}{2}$, and draw the corresponding vectors from a common origin relative to a fixed direction assumed for wing-chord.

26. Take the data for wing test in Art. 76 including position of C. P. and draw the vectors relative to the wing-chord for $i = -1, 1, 3, 8, 12$.

27. Do as in Ex. 26 for one of the tests of Ex. 23.

28. Plot on one of the diagrams of Ex. 26 or 27 the vectors that would be obtained for a flat plate of aspect ratio 6, data of Chap. II.

29. Assume a point for the spindle and a direction through it as that of the wing-chord. Plot the vectors for the JN-2 for $i = -\frac{1}{2}, 1\frac{1}{2}, 3\frac{1}{2}, 7\frac{1}{2}$ and compare with the figure in the text.

30. Find from the table by interpolation the data for $i = -1, 0, 2, 6$, and plot the vectors.

31. In the table of moments of the "Clark," Art. 77, take differences to detect whether their irregularity throws any values of M strongly in suspicion. Plot the three curves for M on one diagram.

32. If the "Clark" test were run at 30 mi/hr on a $\frac{1}{26}$ size model, what would be the moments at $i = 0°$ and 77 mi/hr in the three cases?

33. In the second case of the "Clark" model with $i' = -5°$, find the speed when $i = 12$ and the moment in ft.lb at that speed and attitude.

34. Check the values for the moment M as calculated for the JN-2 from M_s for $i = -\frac{1}{2}, 7\frac{1}{2}, 15\frac{1}{2}$.

35. If the coördinates of the spindle center are (a, c) relative to assumed X and Z axes through the C. G. show that $M = M_s + cX - aZ$.

36. If L and D are relative to the wind which makes an angle θ with the X axis show that the X and Z forces are

$$X = D \cos \theta - L \sin \theta, \qquad Z = L \cos \theta + D \sin \theta.$$

Suppose the wing-chord makes an angle of $3\frac{1}{2}°$ with the X axis, calculate X and Z for the JN-2 with $i = 7\frac{1}{2}$ and use the result of Ex. 35 with $c = -.10$ and $a = 3.04$ to calculate M from M_s. Check with the value otherwise found.

CHAPTER XIII

STREAM FUNCTION AND VELOCITY POTENTIAL

78. Stream Function. The equations for planar motion of a fluid without viscosity were found to be

$$\frac{du}{dt} = X - \frac{g}{\rho}\frac{\partial p}{\partial x}, \quad \frac{dv}{dt} = Y - \frac{g}{\rho}\frac{\partial p}{\partial y}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the case of a liquid where ρ is constant, the third, that is, the kinematic equation becomes simply

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The condition that $M\,dx + N\,dy$ shall be an exact differential

$$M\,dx + N\,dy = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x\,\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The equation of continuity for the liquid shows that $u\,dy - v\,dx$, or $v\,dx - u\,dy$ is an exact differential, because the equation of continuity is precisely the condition for an exact differential. Let

$$d\psi = v\,dx - u\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Then ψ is called the stream function of the motion. The reason for this is as follows:

Consider an element, ds of any curve in the plane, and its projections dx, dy. As the fluid is incompressible, the amount of flow across the element ds is equal to the amount of flow across the two elements dx and dy — otherwise fluid would be collecting in the triangular area, contrary to the condition of constant density. Now, the flow across dy is $u\,dy$, and the flow across dx is $-v\,dx$. Hence,

$u\,dy - v\,dx$ is the total flow across ds from left to right, as the point advances along ds, and

$$v\,dx - u\,dy = d\psi$$

is the amount of flow across ds from right to left. Hence, $d\psi$ is the flow across an element of arc ds, and the difference of ψ at any two points of the plane, namely,

$$\int d\psi = \int (v\,dx - u\,dy) = \psi(x, y) - \psi(x_0, y_0) \quad . \quad . \quad . \quad (7)$$

is the total flow across any curve joining these two points. (The total flow across any two curves is necessarily the same because of the incompressibility). One reason for calling ψ the stream function is that it determines the amount of fluid streaming across a curve.

Consider next the curves ψ = const. If

$$\psi = C, \qquad d\psi = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and, consequently, there is no streaming across any element ds of arc on the curve $\psi = C$, and the equation $\psi = C$ must, therefore, be the equation of the stream line; for the stream line is by definition one along which the fluid is flowing, and across which no fluid is flowing. (This is a second reason for calling ψ the stream function.) The velocities in the fluid may be expressed in terms of the stream function; namely, since

$$v\,dx - u\,dy = d\,\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

it follows that

$$v = \frac{\partial\psi}{\partial x}, \qquad u = -\frac{\partial\psi}{\partial y}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

A knowledge of the stream function, therefore, determines the stream lines and the velocities throughout the fluid at any time.

The stream lines themselves may change, and, in general, do change from instant to instant if the motion is not steady. The stream line is a line of flow at an instant. In steady motion, where the flow is the same at all times at each point, ψ does not depend upon time, and the stream lines are the actual paths of the particles in the fluid. Thus, when a hose is playing and is held steady the stream lines are constant in position, and the motion of the individual particles is along one; but if the hose be swept from side to side the stream lines at any instant are changing, and over an interval of time change a great deal in position; but no particle flows exactly along a stream line.

79. Circulation. The circulation of a fluid along a curve may be calculated as follows: The velocity u multiplied by dx may be taken as the measure of the flow along (not across) the element dx. In the same way, $v\,dy$ may be taken as the flow in dy. The flow in or along ds may be defined as the product of ds by the component of the velocity along ds, or as the product of the velocity by the component of ds along the velocity. This definition of the flow along a curve is exactly similar to the definition in mechanics of the work done, namely, as the displacement times the component force in the direction of the displacement, or as the force times the component displacement in the direction of the force. In elementary mechanics it is known that the differential work is

$$dW = X\,dx + Y\,dy;$$

that is, that the work may be calculated by resolving both the displacement and the force. In the same way, the circulation may be calculated by resolving both the displacement and the velocity, for the mathematical statement of the definitions is entirely similar in the two cases, and the analytical expression of the definition must, therefore, also be similar. Hence, the circulation along ds is

$$dC = u\,dx + v\,dy, \quad \dots\dots\dots \quad (11)$$

and the circulation along any curve joining the point (x_0, y_0) to the point (x, y) in the plane is

$$\int dC = \int (u\,dx + v\,dy), \quad \dots\dots \quad (12)$$

where the integration goes from (x_0, y_0) to (x, y), and is along the path; that is, the values of dx and dy at each point of the path must be those appropriate to the path.

(Note. The fact that the product of the displacement and the component of the velocity in the direction of the displacement takes the form $u\,dx + v\,dy$ has been inferred from the corresponding fact for force and displacement. It is very important for any student of science to see the analogies between different branches of science, and thereby to be able to write down a result in one domain from a known result in another. Whenever such an analogy is seen to exist, and is used in a demonstration, time is saved; but the fact of the existence, and the possibility for the use, of the analogy are attributable to an underlying identity of mathematical ideas. The above proposition stated as a purely mathematical theorem is this: Two directed magnitudes OP and OF are given, and are resolved along two perpendicular directions into $OP = OM + MP$, $OF = ON + NF$. The product of either OP or OF by the projection of the other upon it is the sum of the products of the appropriate projections, namely, $OM \times ON + MP \times NF$. This geometric proposition is very easy to prove by the principle of similar triangles combined with the fact the projection of a broken line on a fixed direction is the sum of the projections on that direction of the segments of the broken line.)

The value of an integral such as the circulation taken around a closed path may be expressed as a double integral over the area enclosed by the path. This may be proved first for an infinitesimal rectangle with sides dx, dy. The flow in the bottom side of the rectangle is $u\,dx$. That in the top side is $-(u + d_y u)dx$ where $d_y u$ is the change in u corresponding to a change in y alone, and is $d_y u = (\partial u/\partial y)dy$. The flow up on the right-hand side is $v\,dy$, whereas the flow on the left-hand side is $-(v + d_x v)dy$, where $d_x v$ is $(\partial v/\partial x)(-dx)$. The total flow is, therefore,

$$u\,dx - (u + \partial u/\partial y \cdot dy)dx + v\,dy - (v - \partial v/\partial x \cdot dx)dy$$
$$= (\partial v/\partial x - \partial u/dy)dx\,dy. \quad . \quad (13)$$

Now, if there be given any closed curve, and the plane be ruled into rectangles, the flow around each little rectangle may be expressed as $\partial v/\partial x - \partial u/\partial y$ times the area of that rectangle. When two adjacent rectangles are considered the flow around the two rectangles may be computed without reference to the common side, because the common side is regarded as described in opposite directions for the two different rectangles, and the flow on this side for two rectangles is equal and opposite, and cancels out of itself. By piecing together any number of adjacent rectangles, the total circulation around all those rectangles is seen to be equal to the circulation around the perimeter bounding the totality of the rectangles. This circulation, however, is equal to the sum of the expressions $(\partial v/\partial x - \partial u/\partial y)dx\,dy$ for all the rectangles, and this sum would be written as a double integral. Hence

$$\int (u\,dx + v\,dy) = \int\int (\partial v/\partial x - \partial u/\partial y)\,dx\,dy. \quad . \quad . \quad (14)$$

(NOTE. This result is, of course, a geometric result independent of the actual fact that u and v are velocities in a fluid, and that $u\,dx + v\,dy$ is the flow in a curve. The general formula would be

$$\int (X\,dx + Y\,dy) = \int\int (\partial Y/\partial x - \partial X/\partial y)dx\,dy, \quad . \ (15)$$

where X, Y are any functions whatsoever of the coördinates x, y; and where the integration on the left is performed around a closed curve, and that on the right over the region bounded by the curve. The theorem is really merely a theorem in partial integration, because, clearly,

$$\int (\partial Y/\partial x)dx = (Y_1 - Y_0),$$

and

$$\int\int (\partial Y/\partial x)\,dx\,dy = \int (Y_1 - Y_0)dy.$$

With respect to the u term care must be exercised to get the right sign.)

The circulation around a closed curve may be expressed in terms of the stream function; namely,

$$\int u\,dx + v\,dy = \int\int (\partial v/\partial x - \partial u/\partial y)dx\,dy$$

$$= \int\int (\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2)dx\,dy \quad . \ (16)$$

The circulation around any closed curve will vanish if

$$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (17)$$

at every point in the plane. Conversely, if the circulation around every closed curve vanishes, the equation (17) must hold; for if the circulation be applied to an infinitesimal rectangle the double integral (16) reduces to a single term, and its vanishing gives (17).

80. Velocity Potential. If the circulation about every closed curve vanishes, the circulation along a curve joining two points A and P is the same for every curve joining those points; for consider two curves joining A and P, and making together a closed curve. The circulation around the two curves APA vanishes, but the circulation along either curve from A and P is the negative of that along the same curve from P to A. Therefore, the circulation around the closed curve is the difference of the circulations along the two curves, which must, therefore, be equal. If the circulation around every curve vanishes, then

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0; \qquad (18)$$

for the circulation may be computed around an infinitesimal rectangle where the double integral reduces to a single term. This condition is, however, the condition that $u\,dx + v\,dy$ be an exact differential; and hence when the circulation around every closed curve vanishes $u\,dx + v\,dy$ is an exact differential.

In this case let

$$u\,dx + v\,dy = -d\phi. \qquad (19)$$

Then

$$u = -\partial\phi/\partial x, \qquad v = -\partial\phi/\partial y. \qquad (20)$$

Hence, if the circulation around every closed curve vanishes the velocities may be written as the negatives of the derivatives of a function $\phi(x, y)$. This function is called the velocity potential. The circulation along a curve is the negative of the change in the velocity potential, for

$$\int u\,dx + v\,dy = -\int d\phi = -(\phi_1 - \phi_0) \qquad (21)$$

This result holds only when there is a velocity potential; that is, only when the circulation depends solely upon the initial and final points, and not upon the curve joining them.

In every case for the motion of an incompressible fluid, that is, a liquid, in a plane there is a stream function ψ; but there is a velocity potential only when $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$. In some cases there is a velocity potential for the motion of a fluid which is not incompressible; namely, when the component velocities satisfy the equation (18). In treating the motion of liquids in a plane the stream function may always be used whether or not there is a velocity potential. In treating the motion of incompressible fluids the velocity potential is used provided there is a velocity potential; that is, provided the circulation along all closed curves is zero.

When there is a velocity potential for the motion of a fluid, whether compressible or not, the motion is said to be irrotational. The reason for this name depends upon the analysis of infinitesimal displacements in a moving fluid. A small particle of fluid has a resultant velocity. This is the amount observable in looking at the motion. If, however, the motion of the different points in the small particle of fluid be observed, it is seen that superposed upon the general velocity of the whole particle there is a deformation of the particle, and a rotation of the particle. The detailed analysis of the deformation and rotation will not be given here. It will merely be stated that the angular velocity of rotation when examined is found to be

$$\omega = \tfrac{1}{2}(\partial v/\partial x - \partial u/\partial y); \quad . \; . \; . \; . \; . \; . \; . \; . \quad (22)$$

and consequently, the condition for the velocity potential is $\omega = 0$, which means that there is no rotation — the motion is irrotational.

The importance of irrotational motion is partly due to the analytic simplicity which arises when there is a velocity potential; and partly due to the physical fact that owing to the viscosity in the fluid any rotation that exists in the general body of the fluid tends to disappear. The effect of viscosity on rotation is very interesting. Near a boundary of the fluid the fluid sticks, and the general rush of the fluid further away from the boundary brings viscous forces into action, which make the motion near the boundary rotational or even full of eddies (turbulent). These eddies peel off from the boundary, so to speak, and move out into the fluid. They may be observed moving off into the fluid on the down-stream side of a bridge which is supported on piers or piles that obstruct the flow of the river. Once the eddies are well into the stream the effect of the boundaries subsides, and then the viscous forces tend to reduce the relative motion, and the eddies die out. This dying out is also readily observed. The importance of the study of irrotational motion lies mainly in the physical fact that, in the main body of the stream, motion does tend to be irrotational.

Care must be taken to distinguish between rotational and circulatory motion. By circulatory motion is meant motion of a fluid in circles around a center. The velocity is then perpendicular to the radii, and a function of the distance from the center. It may be proved that circulatory motion is irrotational, provided the velocity falls off inversely as the distance from the center; for consider the circulation in any circle. If q be the constant velocity and r the radius, the circulation is $2\pi rq$, and the circulation in an arc of the circle which subtends the small angle $d\theta$ at the center is $rq\,d\theta$. Now consider the polar element of area bounded by two radii subtending an angle $d\theta$ and two circular arcs. The circulation along the radii is zero because the velocity is perpendicular to the radii. The circulation along the circular arc is $rq\,d\theta$, and if the total circulation around the infinitesimal figure is zero, the value of $rq\,d\theta$ must be the same on both circular arcs; one being described in the direction opposite to the other. Hence, rq must be the same for r and for $r + dr$; that is,

$$d(rq) = 0, \qquad rq = C, \qquad q = C/r. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

81. Irrotational Motion. When fluid motion is irrotational, the dynamical equations of the motion may be integrated. The equations are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= X - \frac{g}{\rho}\frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= Y - \frac{g}{\rho}\frac{\partial p}{\partial y}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

If the velocity potential be introduced by (20), the equations become

$$\begin{aligned} -\frac{d^2\phi}{\partial x\,\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y} &= X - \frac{g}{\rho}\frac{\partial p}{\partial x}, \\ -\frac{\partial^2\phi}{\partial y\,\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} &= Y - \frac{g}{\partial}\frac{\partial p}{\rho y}. \end{aligned}$$

If the first equation be multiplied by dx, and the second equation by dy, and the results be added and simplified by the use of the formula for the total differential, then

$$-d\frac{\partial\phi}{\partial t} + \tfrac{1}{2}d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = X\,dx + Y\,dy - \frac{g}{\rho}\,dp. \quad . \quad . \quad (25)$$

The external applied accelerations X, Y generally satisfy the condition that $X\,dx + Y\,dy$ is an exact differential because the forces are ordinarily conservative. Let it be assumed that

$$X\,dx + Y\,dy = -dV. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Then V is an acceleration potential, just as ϕ is a velocity potential.

An integration of the differential equation (25) with respect to space, since the differentials are space differentials, gives

$$-\frac{1}{g}\frac{\partial\phi}{\partial t} + \frac{1}{2g}(u^2 + v^2) + \frac{V}{g} + \int\frac{dp}{\rho} = C(t), \quad . \quad . \quad . \quad (27)$$

where C is a constant of integration, independent of space, but possibly dependent on the time. In steady motion $\partial\phi/\partial t$ vanishes, and C is a constant independent of the time. This integral is generally called Kelvin's theorem. It corresponds to Bernoulli's equation with this difference — that Bernoulli's equation holds only along a stream line in steady motion; the value of the constant is different for different stream lines, that is, is a function of space. Kelvin's theorem has the additional term $-\partial\phi/\partial t$, and holds only for irrotational motion; the value of the constant varies in time but does not vary in space, that is, maintains the same value at any instant over the whole fluid. In the special case that the motion is both steady and irrotational, Kelvin's theorem and Bernoulli's are identical, and there is the added information that the constant is the same for all stream lines.

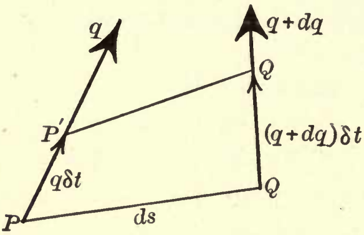

Fig. 30.

An important quantity is the rate of change of the circulation in the curve, where the rate is a fluid rate, that is, the curve is supposed to move with the fluid. Let $PQ = ds$ be an element of arc at any time, and $P'Q'$ the element at the time δt later, namely, $ds + \delta\, ds$. Let q be the velocity in the fluids, regarded as a vector. Then (Fig. 30)

$$PP' = q\,dt, \qquad QQ' = (q + dq)dt,$$

where dq represents the change in q in space alone, and since the perimeter of the infinitesimal quadrilateral $PQQ'P'$ is zero, the relation

$$\begin{aligned} P'Q' \quad + \quad Q'Q \quad + \quad QP + PP' &= \\ ds + \delta\, ds - (q + dq)\delta t - \ ds \ + \ q\,\delta t &= 0 \end{aligned}$$

must hold when the additions are considered as geometric; that is, as vectorial. Hence, $\delta ds = dq\,\delta t$, and

$$\delta\, dx = du\,\delta t, \qquad \delta\, dy = dv\,\delta t, \quad . \ . \ . \ . \ . \ . \ . \quad (28)$$

may be obtained by taking components, regarding ds and dq as vectors. Here δ represents the kind of change that has been designated as *fluid*. Now, the element of circulation is $u\,dx + v\,dy$, and the fluid change in this element may be obtained by the rules for differentiation, namely,

$$\delta(u\,dx + v\,dy) = \delta u\,dx + \delta\,vdy + u\,\delta dx + v\,\delta dy,$$

and

$$\frac{\delta}{\delta t}(u\,dx + v\,dy) = \frac{\delta u}{\delta t}dx + \frac{\delta v}{\delta t}dy + u\,du + v\,dv.$$

The expressions $\delta u/\delta t$, $\delta v/\delta t$ are the accelerations, and may be substituted from the dynamical equations. Then

$$\frac{\delta}{\delta t}(u\,dx + v\,dy) = -\frac{g}{\rho}dp - dV + \tfrac{1}{2}d(u^2 + v^2),$$

inasmuch as X and Y are generally derivable from a potential V. Integrate along a curve from A to B. Then (using d/dt for fluid differentiations)

$$\frac{d}{dt}\int_A^B (u\,dx + v\,dy) = \left[\tfrac{1}{2}(u^2 + v^2) - V - g\int\frac{dp}{\rho}\right]_A^B. \quad . \quad (29)$$

Hence, the rate of change of the circulation in a curve which moves with the fluid may be obtained by subtracting the value of a certain expression at one end of the curve from the value of that same expression at the other end of the curve. If, then, the curve be a closed curve, the rate of change of the circulation must be zero because the points A and B coincide. Hence, the important theorem: The circulation in a closed curve for any non-viscous fluid subject to external accelerations derivable from a potential remains constant for all time as the curve moves with the fluid.

For example, suppose water to be running out of a circular sink. The particles which lie upon a circle concentric with the outlet remain upon a circle concentric with that outlet; but as the water runs out the circle shrinks in diameter. Hence, the velocity in the circle must speed up as the circle shrinks if the circulation is to remain constant. In fact, if the velocity is q when the radius is r, and if the velocity of flow at that radius makes an angle i with the circle, the circulation in the circle is $2\pi rq \cos i$, and must be constant. If i were to remain constant, q would vary inversely as r; but as a matter of fact i becomes larger and larger as r diminishes, being nearly equal to zero for large values of r, and not far from 90° for small values. Thus, $\cos i$ diminishes as r diminishes, and q increases more rapidly than $1/r$. Of course, the efflux of water from a circular sink is not strictly a two-dimensional problem, because the water is drawn by its tendency to fall under the action of gravity; but the problem may be imagined to be two-dimensioned by considering the fluid as sucked out from the orifice, gravity not to act. In this case the fluid would flow out if confined between two parallel horizontal planes.

The stream function for liquid motion had a simple geometrical interpretation because the difference of the stream function at any two points gave the flow of the incompressible fluid across any curve joining those points. The velocity potential, or rather its negative, had likewise a simple geometric meaning, namely, that of the flow along any curve joining two points. This flow would be the same along any two curves joining the same points in case there was a velocity potential. The velocity potential, therefore, was seen to be the negative of the circulation in case the circulation was independent of the path of integration; that is, in case the circulation vanished around a closed path.

The theorem on the constancy of the circulation along the closed curve shows that if the circulation in any particular closed curve is zero at any time, the circulation in that curve as it moves with the fluid must always remain zero. If, now, irrotational motion be defined as that for which the circulation in every closed curve is zero, it follows that in a perfect fluid, compressible or not, the motion if once irrotational must remain always irrotational, and if once rotational, must remain always rotational. This constancy of the irrotational or rotational character of the motion applies not to a particular region of space, but to a particular part of the fluid, for the theorem on the rate of change of the circulation applies to fluid curves. It follows that whenever eddies are observed to be generated in the body of a moving fluid or whenever they are observed to die out there is evidence by this very fact that the fluid is not free from viscosity.

It is important to note the difference between the expression which occurs in the formula for the rate of change of circulation and that which occurs in Bernoulli's or Kelvin's theorem; namely,

$$\frac{1}{g}\frac{d}{dt}\int_A^B (u\,dx + v\,dy) = \left[\frac{1}{2g}(u^2+v^2) - \frac{V}{g} - \int\frac{dp}{\rho}\right]_A^B, \quad (29)$$

and

$$\frac{1}{g}\frac{\partial\phi}{\partial t} + C(t) = \frac{1}{2g}(u^2+v^2) + \frac{V}{g} + \int\frac{dp}{\rho}. \quad (30)$$

The signs of the last two terms in the second are opposite to those in the first. It is possible to transform the first expression into the second by using the velocity potential; for $u\,dx + v\,dy = -d\phi$, and

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial t} - (u^2+v^2). \quad (31)$$

One application of the circulation theorem may be made to the case of a rotating fluid. Suppose a fluid rotating with constant angular velocity ω about an axis, and suppose further that the circulation theorem applies for three-dimensional motion, as is the fact; namely, that

$$\frac{d}{dt}\int_A^B (u\,dx + v\,dy + w\,dz) = \left[\tfrac{1}{2}(u^2 + v^2 + w^2) - V - g\int\frac{dp}{\rho}\right]_A^B. \quad (32)$$

The external acceleration is $-g$ if the Z-axis be taken upward, and the acceleration potential $V = gz$. Now, consider any curve running from the axis of rotation out in a plane passing through that axis. In the case that the motion is circulatory, that is, the particles of fluid trace circles concentric with the axis, $w = 0$. Moreover, the motion at any point of the curve is perpendicular to the curve, and hence the circulation in the curve lying in the plane through the axis is necessarily zero, and its rate of change is zero. Hence,

$$0 = \left[\tfrac{1}{2}(u^2 + v^2) - gz - \frac{gp}{\rho}\right]_A^B, \quad \ldots\ldots \quad (33)$$

provided the density be considered constant. The curves of constant pressure are

$$\tfrac{1}{2}\omega^2 r^2 - gz = \text{const.} \quad \ldots\ldots\ldots \quad (34)$$

if r be the distance from an axis. In particular, the free surface must be a surface of constant pressure, namely, the atmospheric. The free surface is, therefore, a parabola. This could also be proved by treating the problem as one in hydrostatics, where the external accelerations were $-g$ downward, and $\omega^2 r$ radially.

In the above illustration Bernoulli's theorem is applicable for any particular stream line; that is, for any particular circle concentric with the axis because the motion is steady; but this theorem gives no information which is not obvious at once from the symmetry of the motion; it merely states that the velocity must be constant around such a circle because both the pressure and the potential are constant. Kelvin's theorem is not applicable to the problem because the motion is not irrotational. There is no velocity potential ϕ. The problem, therefore, is one which must either be handled by the methods of hydrostatics or by some theorem such as the circulation theorem, which goes further than either Bernoulli's or Kelvin's.

82. Irrotational Liquid Motion. When motion is irrotational there is velocity potential. When the fluid is incompressible there is a stream function. Therefore, for the irrotational motion of a liquid there is a stream function and a velocity potential. The equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \text{gives} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad . \quad . \quad . \quad (35)$$

as a condition which must be satisfied by the velocity potential in irrotational liquid motion; and the condition for irrotational motion, namely,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \quad \text{gives} \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \quad . \quad . \quad . \quad (36)$$

as a condition imposed upon the stream function. This equation for ψ or ϕ is called by the name Laplace's equation. Hence, for the irrotational motion of an incompressible fluid the velocity potential and stream function satisfy Laplace's equation. Let

$$z = x + iy, \qquad w = -u + iv, \qquad i = \sqrt{-1}. \quad . \quad . \quad . \quad (37)$$

The complex number z determines, therefore, a point (x, y) in the plane and the complex number w determines a pair of components, u, v; and if these are regarded as component velocities in a fluid, the complex number w determines the velocity just as z determines the position. Let $w = f(z)$ be any function of the complex number z. For example,

$$w = z^2 = (x + iy)^2,$$

or

$$-u + iv = x^2 - y^2 + 2ixy,$$

or

$$u = y^2 - x^2, \qquad v = 2xy.$$

On trial it is found that these values of u and v satisfy the relations (35, 36), which are the conditions satisfied by a liquid moving irrotationally.

It may be proved that if $w = f(z)$, the real and imaginary parts of w, namely, $-u, v$ are always such functions of (x, y) that equations (35, 36) are satisfied. For let

$$-u + iv = f(z) = f(x + iy).$$

Differentiate with respect to x, and denote by $f'(z)$ the derivative of $f(z)$ with respect to the variable z. Then

$$-\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'(z)\frac{\partial z}{\partial x} = f'(z).$$

And

$$-\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = f'(z)\frac{\partial z}{\partial y} = if'(z).$$

Hence,

$$-\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=i\left(-\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)=-i\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}.$$

Equate the real and imaginary parts in these two equations, and the proof is complete. The result is that by writing down any function of a complex number and by separating this function into real and imaginary parts and prefixing the negative sign to the real part, possible velocities for irrotational liquid motion are obtained.

Another use of the complex number is related not primarily to the velocity but to the velocity potential and stream function. The relations between these two are

$$-u=\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}, \quad \text{and} \quad -v=\frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}. \quad . \quad . \quad (38)$$

Now, if the function

$$w=f(z)=\phi+i\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

be regarded as having for real part the velocity potential, and for imaginary part the stream function, precisely the conditions (38) on the derivatives are fulfilled. Therefore, a possible case of fluid motion may be obtained by using for the velocity potential the real part of any function f of the complex number $z = x + iy$, and for the stream function the imaginary part (omitting i). In treating liquid irrotational motion it is generally simpler to deal with the stream function or the velocity potential, and to determine from them the velocities rather than to deal with the velocities themselves. Consequently, this second usage of complex numbers is more common than the first.

When the stream function is set equal to a constant, the stream lines are obtained, and this is one of the most important uses of the stream function. When the velocity potential is set equal to a constant, an equipotential line is obtained. These lines are important, but not so important as the stream lines. The equipotential lines are perpendicular to the stream lines because along an equipotential line

$$d\phi=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial y}dy=0$$

determines the slope of the equipotential line as

$$\frac{dy}{dx}=-\frac{\partial \phi}{\partial x}\bigg/\frac{\partial \phi}{\partial y},$$

whereas the slope on a stream line is determined from the equation

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0,$$

from which

$$\frac{dy}{dx} = - \frac{\partial \psi}{\partial x} \bigg/ \frac{\partial \psi}{\partial y},$$

which is the negative reciprocal of the value for the equipotential line.

If the velocity potential ϕ is given, the stream function ψ may be determined if it is possible to recognize what function of the complex number $z = x + iy$ has its real part equal to ϕ. If, however, this function cannot be determined by inspection, the stream function may be found by integration because

$$d\psi = v\, dx - u\, dy = - \frac{\partial \phi}{\partial y} dx + \frac{\partial \phi}{\partial x} dy.$$

When the values for the partial derivatives are substituted the resulting expression is an exact differential, and may be integrated by the methods applicable to exact differentials.

83. Sources and Sinks. Thus far it has been assumed that the fluid was neither generated nor destroyed; but a considerable number of cases of fluid motion can best be treated by imagining that at certain points liquid is being produced, or is disappearing, the former being called sources, the latter sinks. A source or sink may be realized in two-dimensional motion approximately by supplying fluid to the region between two near-by parallel planes through a pipe perpendicular to one of the planes, and opening into the region between the two. If the fluid is pumped in there is a source; if it is sucked out there is a sink. If from the point O in a liquid so much fluid is radiating as will cause the liquid to cross a unit circle concentric with that point with unit velocity, the source at O is said to be a unit source. The equation of continuity does not apply at the point O itself, but does apply throughout the mass of the liquid, and as the same amount of fluid must cross every circle concentric with O, the velocity of the fluid motion must vary inversely with the distance from the source. For a unit source the velocity is radial, and equal to $1/r$. For a source of strength m, that is, a source which radiates m times as much fluid as a unit source, the velocity is everywhere m times as great, and is m/r.

Now, if polar coördinates with origin at O be used

$$\phi = -\int (u\,dx + v\,dy) = -\int (u_1\,dr + v_1\,r\,d\theta),$$

if u_1, v_1 are radial and normal velocities respectively. Hence,

$$\phi = -\int m\frac{dr}{r} = -m\log r, \quad \ldots\ldots \quad (40)$$

may be taken as the velocity potential of the flow from the source of strength m. This form of velocity potential suggests the logarithm of a complex number; in fact, if

$$w = \phi + i\psi = -m\log z = -m\log(re^{i\theta}), \quad \ldots\ldots \quad (41)$$

then

$$\phi = -m\log r, \qquad \psi = -m\theta, \quad \ldots\ldots \quad (42)$$

where θ is the angle made by the radius with the X-axis. The stream function for a source is, therefore, $-m\theta$. If the source is at the point (a, b), the velocity potential is

$$\phi = -m\log r = -m\log\sqrt{(x-a)^2 + (y-b)^2}, \quad \ldots \quad (43)$$

and the stream function is

$$\psi = -m\theta = -m\tan^{-1}\left(\frac{y-b}{x-a}\right). \quad \ldots\ldots \quad (44)$$

For a sink it is only necessary to take m as a negative quantity. The liquid motion which arises from any combination of sources and sinks is obtained by adding the velocity potentials for the different sources and sinks to obtain the velocity potential for the combination of sources and sinks, and by adding the stream functions to obtain the total stream function. The actual velocities in the fluid may then be obtained by taking the proper derivatives of either the velocity potential or the stream function.

The uniform motion of the fluid with velocity U in the x direction obviously has the velocity potential $\phi = -Ux$, because the derivative of ϕ with respect to x changed in sign is equal to U, and the derivative of ϕ with respect to y is zero. This suggests that for the complex variable

$$w = \phi + i\psi = -U(x + iy), \quad \ldots\ldots \quad (45)$$

and, indeed, this function does give

$$\phi = -Ux, \quad \text{and} \quad \psi = -Uy. \quad \ldots\ldots \quad (46)$$

The stream function, therefore, for the uniform motion is $-Uy$.

EXERCISES

1. Show that for the steady motion of a fluid (not necessarily liquid) there is a flux function $f(x,y)$ such that

$$u = -\frac{1}{\rho}\frac{\partial f}{\partial y}, \qquad v = \frac{1}{\rho}\frac{\partial f}{\partial x};$$

and that $f = C$ gives the stream lines and $f_1 - f_0$ the amount of flow per second across any curve (measured by amount of matter).

2. What are the dimensions of circulation, velocity potential, stream function, and flux function as defined in Ex. 1?

3. In the following cases sketch the stream lines, determine whether or not the motion is rotational, calculate the velocities, and the pressure, and also the velocity potential when there is one, and the angular velocity when the motion is rotational. Assume $X = Y = 0$.

(*a*) $\psi = x^2 - y^2$, (*b*) $\psi = (x^2 + y^2)$, (*c*) $\psi = \tan^{-1} y/x$,
(*d*) $\psi = \log(x^2 + y^2)$, (*e*) $\psi = (x^2 + y^2)^{-\frac{1}{2}}$, (*f*) $\psi = \sin x \sin y$,
(*g*) $\psi = e^x \cos y$, (*h*) $\psi = \cos x \cosh y$, (*i*) $\psi = (xy)^{\frac{1}{2}}$.

4. Prove that (30) may be derived from (29) instead of by direct integration of the equations of motion.

5. Show that in the steady motion of formula (33) the pressure is greater where the velocity is greater and less where the velocity is less. Compare with the statement of Art. 66.

6. Obtain the velocities which correspond to these functions ($w = -u + iv$):

(*a*) $w = 1/z$, (*b*) $w = e^z$, (*c*) $w = z^{\frac{1}{2}}$,
(*d*) $w = \log z$, (*e*) $w = z + 1/z$, (*f*) $w = \cos^{-1} z$.

[Remember that $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$,

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \qquad \log z = \log r + i\theta, \qquad \frac{1}{z} = \frac{1}{r}e^{-i\theta}, \text{ etc.}]$$

7. If the polar element of arc be used, what is the expression for the flow $d\psi$ across a curve from right to left? Let u, v be radial and normal velocities. Show that $v = \partial\psi/\partial r$, $u = -\partial\psi/r\,\partial\theta$.

8. What is the formula for circulation along a curve in polar coördinates? What is the condition that the differential be exact? If it is exact show that $u = -\partial\phi/\partial r$ and $v = -\partial\phi/r\,\partial\theta$.

9. Prove that if ds is any arc and $d\phi$ is the increase of ϕ along that arc, then $d\phi/ds$ is the negative of the component velocity along ds.

10. If $w = f(z) = \phi + i\psi$ is defined by any one of the expressions in Ex. 6, find the velocities in the liquid.

11. If $w = f(z) = \phi + i\psi$ show that

$$f'(z) = \frac{dw}{dz} = -u + iv = -\frac{u + iv}{u^2 + v^2}.$$

Hence show that $f'(z)$ is the velocity vector reversed in direction and with reciprocal magnitude.

12. A source discharges one gallon per second between two planes 3 in apart. Find the velocity at any distance, the strength, and the velocity potential and stream function.

13. If a source at $A(a, b)$ and equal sink at $A'(a', b')$ be combined, show that the stream lines are circular arcs passing through the source and sink. (Hint: Show that by (44) the angle APA' subtended at any point P by the segment AA' joining source and sink is ψ/m.)

14. A source of strength m is in a general stream of velocity U. Find and sketch the stream lines. Calculate the velocity at any point.

15. A source and equal sink 6 in apart discharge and take in one gallon per minute. Find the velocity (*a*) halfway between them, (*b*) six inches from the center on the line joining them, (*c*) six inches from the center on the line perpendicular to the junction. (Assume the thickness of the stream is 1 inch and that the liquid is water, $\rho = 62\frac{1}{2}$ lb/ft^3).

16. Find the pressure defect at each of the three points in Ex. 15 as compared with the "pressure at infinity."

17. Given $\psi = x^2 + xy - y^2 + x - y$. Show that the motion is irrotational and find the velocity potential. Are there any points at which the liquid is at rest? Find the pressure difference of between (0,0) and (10, 10) if the liquid is water.

CHAPTER XIV

MOTION OF A BODY IN A LIQUID

84. Sources and Sinks. If a body moves in a liquid the fluid motion cannot be steady, for the position of the body changes with the time, and, consequently, the motion of the fluid cannot be the same at each point for all time, even if the body moves uniformly in a straight line. When the motion of the body is uniform and in a straight line it is possible to regard the body as at rest, and the fluid as flowing past the body with the opposite velocity. The motion in the fluid is then steady. In treating the motion of a body in a fluid it is convenient always to bear in mind the possibility of considering the motion of the fluid about the body rather than the motion of the body through the fluid. One of the most important problems to solve for the motion of a body in a fluid is the distribution of pressures due to the motion about the different portions of the body, so that by integration the resultant pressure of the fluid on the body may be obtained. The method of solution consists in determining the velocity potential or the stream function for the motion, from which the velocities of the fluid may be calculated, from which in turn the pressure in the fluid may be found by applying Bernoulli's theorem in the case of steady motion, or Kelvin's theorem in the general case.

The direct method of treating the motion of the body in the fluid would call for the determination, from the given motion of the body and the figure of the body, of the velocity potential or stream function, using the condition that the contour of the body must necessarily be a stream line, because the fluid can only flow tangentially. An indirect method would consist in setting up different types of fluid motion, and determining their stream lines. Any stream line might then be taken as the contour of the body moving in the fluid. This indirect method does not, except accidentally, determine the motion of the fluid about a given body. Rather it determines a type of fluid motion and the shape of body which could give rise to it. The indirect method is, however, much simpler than the direct for it depends on differentiation, whereas the direct method depends on integration.

Suppose that there is a source of strength m at the point $(a, 0)$ and a sink of strength $-m$ at the point $(-a, 0)$. The stream lines of this motion will be

$$\psi = -m \tan^{-1} \frac{y}{x - a} + m \tan^{-1} \frac{y}{x + a} = C \quad . \quad . \quad . \quad . \quad (1)$$

The equation may be freed from antitangents by taking the tangent of both sides. The resulting stream lines are similar to the magnetic lines about a magnet with poles at (a, O) and $(-a, O)$. The motion of the fluid along the X-axis may be obtained either from the stream function or from the velocity potential, which is

$$\phi = -m \log \sqrt{(x - a)^2 + y^2} + m \log \sqrt{(x + a)^2 + y^2}, \quad . \quad (2)$$

and in particular the velocity at the point x is

$$u = \frac{2ma}{x^2 - a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This shows that the velocity is negative for points between the source and sink, and positive for other points on the axis. The physical considerations connected with source and sink would show this same result.

Now, if there be superposed upon the motion due to the source and sink a general uniform motion along the X-axis equal to $-U$, the velocity upon the X-axis will vanish at the two points where

$$\frac{2ma}{x^2 - a^2} = U \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The stream function for this superposed motion is

$$\psi = -m \tan^{-1} \frac{y}{x - a} + m \tan^{-1} \frac{y}{x + a} + Uy, \quad . \quad . \quad . \quad (5)$$

and the stream lines are given by the equation

$$-m \tan^{-1} \frac{y}{x - a} + m \tan^{-1} \frac{y}{x + a} + Uy = C, \quad . \quad . \quad . \quad (6)$$

or

$$\frac{\dfrac{y}{x + a} - \dfrac{y}{x - a}}{1 + \dfrac{y^2}{x^2 - a^2}} = \tan \frac{(C - Uy)}{m}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

or

$$\frac{2ay}{x^2 + y^2 - a^2} = \tan \left(\frac{Uy}{m} - \frac{C}{m} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For different values of C these stream lines may be plotted. The noteworthy characteristic of the system of stream lines is, however, this: The plane is divided into two regions; there is an oval stream line which passes through the two points where the velocity is zero ($C = 0$); the stream lines outside of this oval ($C > 0$) are open and run from infinity to infinity parallel at great distances with the X-axis; the stream lines inside the oval ($C < 0$) run from the source to the sink. The flow in the fluid is, therefore, of two distinct types; a general flow past the oval stream line, and a circulation within the oval stream line. The equation of the oval stream line itself is

$$\frac{2ay}{x^2 + y^2 - a^2} = \tan\frac{Uy}{m}. \qquad (9)$$

A body with the shape of this oval would move through the fluid with a velocity potential and stream function equal to the values (2) and (1), provided x and y are coördinates referred to the center of the body, and hence moving coördinates. For different values of the constants a, U, and m, this oval takes a variety of shapes. It may be elongated, as when the source and sink are very far apart, or it may be practically circular, as when the source and sink are very near together. There is a general resemblance of the oval when fairly elongated to the shape of a ship.

The facts with regard to the stream lines have been stated; the proof is straightforward. Take a figure with source at $A(a, 0)$ and sink at $B(-a, 0)$. Let P (x, y) be any point. Let

$$\measuredangle PBA = \theta_2, \qquad \measuredangle PAx = \theta_1, \qquad \measuredangle APB = \theta.$$

Then (6) is

$$-m\theta_1 + m\theta_2 + Uy = C, \qquad (10)$$

or

$$\theta = \theta_1 - \theta_2 = Uy/m - C/m. \qquad (11)$$

If $U = 0$, the stream lines are circles passing through A and B because the vertical angle θ is constant. If U/m is sufficiently small the stream lines will still be nearly circular arcs joining A and B unless y is large so that Uy/m is not appreciable compared with C/m. The stream lines cannot cross the x-axis ($y = 0$) except at A and B (when θ is indeterminate) unless $C = 0$. But if $C = 0$ the stream line (8) for small enough values of y may be written

$$\frac{2ay}{x^2 + y^2 - a^2} = \frac{Uy}{m} \quad \text{or} \quad \frac{2a}{x^2 - a^2} = \frac{U}{m},$$

as in (4). Hence the oval stream line must be $C = 0$. For any given y the angle θ is greater inside the oval and less outside than upon it. Hence, if $C > 0$ there are points outside the oval, but if $C < 0$ there are points inside.

85. The Doublet. If the source and sink are very near together almost all the fluid emitted from the source is sucked into the sink, and there is very little motion except in the immediate neighborhood of the source and sink unless the source and sink are exceedingly powerful. This is similar to the case of a small magnet. Unless the magnetic poles are very strong there is practically no magnetic effect except very near the magnet. Let the equation of the oval be written

$$\frac{2may}{x^2 + y^2 - a^2} = m \tan \frac{Uy}{m}, \quad \ldots \ldots \quad (12)$$

and let the polar strength m be very large, and a very small. Then $\tan (Uy/m)$ is nearly equal to Uy/m; and the equation may be written

$$\frac{Sy}{x^2 + y^2} = Uy, \qquad S = 2ma, \quad \ldots \ldots \quad (13)$$

or

$$x^2 + y^2 = S/U \quad \ldots \ldots \quad (14)$$

It is, therefore, seen that the combination of an infinitely strong source and sink infinitely near together (called a doublet) with a general stream of velocity U in the negative x direction gives rise to an oval stream line which is a circle; and, consequently, this case corresponds to the motion of a cylinder in a fluid, or, rather, to the motion of a fluid around the cylinder. The stream function and velocity potential which correspond to this limiting case can be calculated as follows:

$$\psi = - m \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2} + Uy,$$

$$\psi = - \frac{2may}{x^2 + y^2} + Uy = Uy - \frac{Sy}{x^2 + y^2}, \quad \ldots \quad (15)$$

$$\phi = - m \log \sqrt{\frac{x^2 + y^2 - 2ax + a^2}{x^2 + y^2 + 2ax + a^2}} + Ux,$$

$$= - \frac{m}{2} \log \frac{1 - 2ax/(x^2 + y^2)}{1 + 2ax/(x^2 + y^2)} + Ux,$$

$$\phi = \frac{2max}{x^2 + y^2} + Ux, \qquad = \frac{Sx}{x^2 + y^2} + Ux \quad \ldots \quad (16)$$

If the radius of the cylinder is $R = (S/U)^{\frac{1}{2}}$, the stream function and velocity potential are

$$\psi = Uy\left(1 - \frac{R^2}{x^2 + y^2}\right) = U\left(r - \frac{R^2}{r}\right)\sin\theta, \quad . \; . \; . \quad (17)$$

$$\phi = Ux\left(1 + \frac{R^2}{x^2 + y^2}\right) = U\left(r + \frac{R^2}{r}\right)\cos\theta, \quad . \; . \; . \quad (18)$$

where (r, θ) are polar coördinates referred to the center of the cylinder.

86. Moving Cylinder. To find the stream function and velocity potential for the cylinder moving in the fluid with the velocity U it is merely necessary to subtract from the values above found the uniform velocity $-U$; that is, to add to (15) and (16) or to (17) and (18) the values of ψ and ϕ which correspond to the velocity U; and these values are respectively $-Uy$ and $-Ux$. Hence, for the moving cylinder,

$$\psi = -\frac{UR^2y}{x^2 + y^2} = -\frac{UR^2}{r}\sin\theta, \quad . \; . \; . \; . \; . \; . \quad (19)$$

$$\phi = +\frac{UR^2x}{x^2 + y^2} = \frac{UR^2}{r}\cos\theta \quad . \; . \; . \; . \; . \; . \; . \quad (20)$$

But in these formulas it must be remembered that (x, y) and (r, θ) are coördinates referred to the moving center of the cylinder. If the cylinder be moving uniformly along the x-axis, and if the center of the cylinder were at the origin when $t = 0$, then at any time the stream function and velocity potential would be

$$\psi = -\frac{UR^2y}{(x - Ut)^2 + y^2}, \qquad \phi = +\frac{UR^2(x - Ut)}{(x - Ut)^2 + y^2} \quad . \; . \quad (21)$$

These formulas contain the time explicitly, and show that in case the cylinder moves the motion is not steady.

(1) Given the velocity potential or stream function, the next step is to calculate the velocities and substitute in Bernoulli's or Kelvin's formulas. The first case to be considered is that of steady motion, with equations (17) and (18). Here

$$\frac{\partial\phi}{\partial x} = -u = U\left(1 - \frac{R^2}{r^2}\cos 2\theta\right),$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{UR^2}{r^2}\sin 2\theta.$$

Hence,

$$q^2 = u^2 + v^2 = U^2\left(1 - \frac{2R^2}{r^2}\cos 2\theta + \frac{R^4}{r^4}\right), \quad . \; . \; . \; . \quad (22)$$

$$\frac{p}{\rho} = C - \frac{q^2}{2g} = C - \frac{U^2}{2g} + \frac{U^2R^2}{gr^2}\cos 2\theta - \frac{U^2R^4}{2gr^4}. \quad . \quad . \quad (23)$$

To find the pressure of the stream on the cylinder past which it flows it is merely necessary to calculate the pressure in the direction of the stream because the symmetry of the figure shows that there can be no resultant pressure perpendicular to the stream. Then

$$P = -2\int_0^{\pi} p \cos\theta\, R\, d\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

whence $P = 0$, by substitution from (23) with $r = R$; that is, the stream does not exert a pressure on the cylinder.

(NOTE. This result is, of course, contrary to ordinary experience. It is valuable, however, in showing that in a perfect fluid, where there is no viscous drag, and where the stream lines close in symmetrically behind the body, there is no resultant pressure. Any pressure that there is on a cylinder must, therefore, arise either from the viscosity of the fluid directly, or from the viscosity indirectly, through the breaking up of the fluid behind the object into eddies, or from cavitation, that is, discontinuous motion, which theoretically could exist in a perfect fluid.)

(2) If the cylinder be moving in a fluid, formulas (19) or (20) must be used, and Kelvin's theorem, which contains the term $\partial\phi/\partial t$ is needed. To calculate $\partial\phi/\partial t$ from (20), the condition must be expressed that a point (r, θ) shall be fixed in space. In fact, r and θ being coördinates relative to the center of the cylinder, that is, moving coördinates, will designate different points at different times. Now,

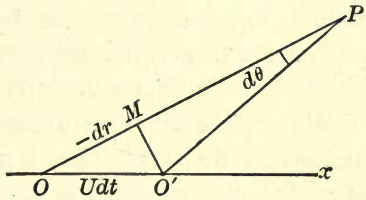

Fig. 31. Moving Cylinder.

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial t}.$$

Consider the infinitesimal figure formed by any fixed point P in space, and two positions O, O' of the moving center. Strike an arc with center PO, and radius OP', meeting P in M. Then $OO' = U\,dt$, and

$$dr = -OM = -OO'\cos\theta = -U\cos\theta\, dt, \quad . \quad . \quad . \quad (25)$$

$$d\theta = \measuredangle P = O'M/MP = (U\,dt\sin\theta)/r. \quad . \quad . \quad . \quad . \quad (26)$$

Hence,

$$\frac{\partial\phi}{\partial t} = -\frac{UR^2}{r^2}\cos\theta\,(-u\cos\theta) - \frac{UR^2}{r}\sin\theta\left(U\frac{\sin\theta}{r}\right),$$

or

$$\frac{\partial\phi}{\partial t} = \frac{U^2R^2}{r^2}\cos 2\theta \quad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

The radial velocity may be taken as $-\partial\phi/\partial r$, and the velocity perpendicular to the radius as $-\partial\phi/r\,\partial\theta$; and this form is more convenient than to use the velocities along x and y. Then

$$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{r\,\partial\theta}\right)^2 = \frac{U^2R^4}{r^4}.$$

From Kelvin's theorem

$$\frac{p}{\rho} = C - \frac{q^2}{2g} + \frac{\partial\phi}{g\,\partial t},$$

or

$$\frac{p}{\rho} = C - \frac{U^2R^4}{2gr^4} + \frac{U^2R^2}{gr^2}\cos 2\theta \quad \ldots\ldots \quad (28)$$

The pressure is again found by (24), and is seen to vanish as before. In fact, the two expressions for the pressure differ only by the term $U^2/2g$, which may be absorbed into the undetermined constant C.

(NOTE. It is particularly instructive to note that the term $\partial\phi/\partial t$ does contribute a necessary term to the expression p/ρ; and, that when this correction is applied the expressions for the pressures (23 and 28) on the cylinder become identical except for the undetermined constant. These pressures should, of course, be identical, because the uniform motion of a cylinder in a fluid should set up the same pressures as the uniform motion of the fluid past the cylinder, it being only the relative motion which counts.)

(3) If the cylinder moving in the fluid with a velocity U has an acceleration, that is, if U is not constant but variable, there is another term in $\partial\phi/\partial t$, owing to the necessity of differentiating U; so that

$$\frac{\partial\phi}{\partial t} = \frac{U^2R^2}{r^2}\cos 2\theta + \frac{dU}{dt}\frac{R^2}{r}\cos\theta \quad \ldots\ldots \quad (29)$$

This additional term in $\partial\phi/\partial t$ contributes to the pressure the additional expression

$$P = -2\int_0^{\pi}\frac{\rho}{g}\frac{R^2}{R}\frac{dU}{dt}\cos\theta\,(\cos\theta\,R\,d\theta) = \frac{-\pi\rho}{g}R^2\frac{dU}{dt}. \quad (30)$$

Hence, the acceleration of the cylinder is opposed by a pressure. The expression $\pi\rho R^2$ is the amount of fluid displaced. Let $W' = \pi\rho R^2$ be this amount. Then the back force is

$$P = -\frac{W'}{g}\frac{dU}{dt} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

If the cylinder were acted on by an external force X, the equation of motion of a cylinder would be

$$W\frac{dU}{dt} = gX - W'\frac{dU}{dt}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32)$$

which may be written

$$(W + W')\frac{dU}{dt} = gX \quad . \; . \; . \; . \; . \; . \; . \; . \quad (33)$$

Thus, the effect of the fluid is to produce an apparent increase in the mass of the moving cylinder, the increase being equal to the mass of the liquid displaced. The effect is, in other words, not a resistance to motion in the ordinary sense of an expression dependent on the velocity, but a resistance to motion in the sense of a quantity depending on the acceleration, and thus in the nature of an inertia reaction.

A word should be said, too, about the force X. The external force may be of either of two sorts: it may be due to a force applied as by a string, that is, a direct mechanical force applied to the cylinder; or it may be due to a force like that of gravity, which is applied at a distance, and which acts not only on the cylinder, but on the surrounding fluid. This latter kind of force gives rise to the ordinary hydrostatic buoyancy. It gives no net force on the object in the fluid when the density of that object is the same as the density of the fluid. Let the first type of force be designated by X_1, and the second type of force by X_2. The equation becomes

$$\frac{W + W'}{g}\frac{dU}{dt} = X_1 + X_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (34)$$

In the particular case where a uniform force like gravity acts,

$$X_2 = (\sigma - \rho)Vf/g = (W - W')f/g, \quad . \; . \; . \; . \; . \quad (35)$$

where σ is the density of the cylinder, and ρ the density of the fluid, and V is the volume of the cylinder, and f is the acceleration due to the external action. The equation of motion then becomes

$$\frac{W + W'}{g}\frac{dU}{dt} = X_1 + (W - W')\frac{f}{g} \quad . \; . \; . \; . \; . \quad (36)$$

In case there is no direct force X_1 acting, the acceleration of the body is

$$\frac{dU}{dt} = \frac{W - W'}{W + W'} f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

A dirigible of the Zeppelin type (or a submarine, well submerged) when moving transversely approximates to the case of a cylinder moving in a fluid. If the dirigible were in equilibrium, W and W' would be equal, and there would be no net buoyant effect. A force applied by a rope in a similar manner tending to move the dirigible through the air perpendicular to its axis would produce an acceleration equal only to the force multiplied by g and divided by twice the weight, because the weight of the displaced air and the weight of the Zeppelin are equal. This added apparent inertia would be different in amount for motion in the direction of the axis. Moreover, the moments and products of inertia which have to do with rotary displacements of the dirigible would be altered by the presence of the surrounding fluid. As the air certainly would not behave quite like a perfect fluid, it might be that the added apparent inertia in the case of transverse motion would not in reality be as much as the weight of the air displaced. These changes in apparent inertia make the problem of the motion of a dirigible distinctly more difficult than the corresponding problem for motion of an airplane; for the airplane has such a great average density that whatever may be the effect of the displaced air, the mass of the air involved is so small compared with that of the airplane that the effect would appear to be negligible; and thus throughout the theory of motion of an airplane, and in particular in the discussion of stability, no allowance is made of the inertia effects of the fluid in modifying the coefficients which must be applied to the accelerations.

87. Spinning Cylinder. If the cylinder were spinning on its axis as it moved, there would be no effect of the spin transmitted to the fluid in case the fluid were non-viscous; but in case the fluid be viscous it must necessarily adhere to the cylinder; and thus there must be set up by the spinning of the cylinder a circulatory motion of the fluid about the axis of the cylinder. It may be imagined that in some way this circulatory motion were set up in a perfect fluid, and the question would then arise: what are the effects of the fluid upon a cylinder moving with the velocity U when there is superposed upon the ordinary motion of the fluid a circulation about the cylinder? It may be assumed for simplicity that the circulatory motion is irrotational. It has been seen that there is one type, and only one, of circulatory motion which is rotational; namely, that in which the fluid velocity dies off inversely as the distance from the center, and for which the velocity potential is of the form $-A\theta$. The motion to be discussed, therefore, has the velocity potential

$$\phi = -A\theta + \frac{UR^2}{r}\cos\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In these expressions r and θ are polar coördinates referred to a moving region. As the motion is irrotational, Kelvin's theorem may be applied.

The velocity in a fluid may be resolved along the radius and perpendicular to it. Then

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = \frac{UR^2\cos\theta}{r^2},$$

$$\frac{r\,d\theta}{dt} = -\frac{\partial\phi}{r\,\partial\theta} = \frac{UR^2\sin\theta}{r^2} + \frac{A}{r}.$$

Hence

$$q^2 = U^2\frac{R^4}{r^4} + \frac{2AUR^2}{r^2}\sin\theta + \frac{A^2}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

$$\frac{\partial\phi}{\partial t} = U^2\frac{R^2}{r^2}\cos 2\theta + \frac{dU}{dt}\frac{R^2}{r}\cos\theta - \frac{AU}{r}\sin\theta, \quad . \quad . \quad (40)$$

if the intensity of the circulatory irrotation motion as measured by A be treated as constant though the translatory motion be allowed an acceleration. The pressure in the fluid is, then, for values on the cylinder where $r = R$, as follows:

$$\frac{p}{\rho} = C - \frac{q^2}{2g} + \frac{1}{g}\frac{\partial\phi}{\partial t} = C - \frac{1}{2g}\left(U^2 + \frac{A^2}{R^2} + 2AU\sin\theta\right)$$
$$+ \frac{1}{g}\left(\frac{dU}{dt}R\cos\theta + U^2\cos 2\theta - \frac{AU}{R}\sin\theta\right).$$

To find the resultant pressure on the cylinder it is necessary to calculate in this case both the pressure along x and the pressure along y because the circulatory motion has destroyed the symmetry so that the pressure along y cannot be guaranteed in advance to vanish. These pressures are

$$P_x = -\int_0^{2\pi} pR\cos\theta\,d\theta, \qquad P_y = -\int_0^{2\pi} pR\sin\theta\,d\theta.$$

The value for P_x turns out to be exactly what it was before. The added terms in p contribute nothing to the integral. Hence,

$$P_x = -\frac{W'}{g}\frac{dU}{dt}.$$

In calculating P_y most of the terms in p may be neglected because they obviously give a zero value to the integral. The result is

$$P_y = \frac{2\rho\pi AU}{g} = \frac{2W'AU}{R^2 g}, \quad \ldots \ldots \quad (41)$$

where as before W' is the weight of the fluid displaced. When, therefore, there is forward motion of the cylinder combined with the circulation (irrotational) in the fluid around the cylinder, equations of motion in the x and y directions become

$$\frac{W}{}\frac{dU}{dt} = gX - \frac{W'}{}\frac{dU}{dt}, \qquad \frac{W}{}\frac{dV}{dt} = gY + \frac{2W'AU}{R^2} \quad . \ . \quad (42)$$

This shows that in addition to the inertia reaction to motion through the fluid and in addition to the external applied forces there is a lift exerted upon the cylinder urging it in the direction of the Y-axis, the amount of the lift being $2W'AU/R^2$.

It follows, therefore, that there is no longer the possibility of neglecting the motion in the Y direction in calculating the equations of motion for the cylinder in the fluid. The lift due to the circulation produces a component acceleration in the Y direction, and this will set up a velocity in the Y direction, so that it will be necessary to go over the whole problem from the start, and take account of the possibilities when there is motion U in the X direction, and V in the Y direction. For motion with a velocity V in the Y direction the velocity potential is

$$\phi = \frac{VR^2}{r}\sin\theta \quad \ldots \ldots \quad (43)$$

The total velocity potential is, therefore,

$$\phi = \frac{UR^2}{r}\cos\theta + \frac{VR^2}{r}\sin\theta - A\theta \quad \ldots \ldots \quad (44)$$

This allows for velocities U and V along the two axes, and a circulation about the cylinder. It is again necessary to calculate q^2, and $\partial\phi/\partial t$, and by integration to determine the resultant pressures in the X and Y directions. The final equations of motion are then found to be

$$\left.\begin{aligned} \frac{W\,dU}{dt} &= gX - \frac{W'\,dU}{dt} - \frac{2W'AV}{R^2} \\ \frac{W\,dV}{dt} &= gY - \frac{W'\,dV}{dt} + \frac{2W'AU}{R^2} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (45)$$

The circulation has produced not only a lift, but a drag. If the equations be transformed they become

$$\left.\begin{aligned} (W + W')\frac{dU}{dt} &= gX - \frac{2W'AV}{R^2} \\ (W + W')\frac{dV}{dt} &= gY + \frac{2W'AU}{R^2} \end{aligned}\right\} \quad \ldots\ldots \quad (46)$$

Here, again, the external applied forces X and Y should be divided into forces due to actual mechanical actions directly upon the cylinder and to the indirect actions due to forces acting at a distance, which also affect the fluid, and result in the phenomenon of buoyancy.

If there be no direct mechanical forces acting, but only the buoyancy, the equations of motion become

$$(W + W')\frac{dU}{dt} + \frac{2W'AV}{R^2} = 0,$$

$$(W + W')\frac{dV}{dt} - \frac{2W'AU}{R^2} = -g(W - W'),$$

provided it be understood that the x-axis is horizontal, the y-axis vertical, and the external force that due to gravity. These equations are of the form

$$\frac{dU}{dt} + nV = 0, \qquad \frac{dV}{dt} - nU = -g', \quad \ldots\ldots \quad (47)$$

where

$$n = \frac{2W'A}{W + W'}, \qquad g' = g\left(\frac{W - W'}{W + W'}\right). \quad \ldots\ldots \quad (48)$$

If the first equation be differentiated and the value dV/dt be substituted from the second, the resultant equation in U is

$$\frac{d^2U}{dt^2} + n^2U = ng', \quad \ldots\ldots\ldots\ldots \quad (49)$$

of which the solution is

$$U = C'\cos nt + C''\sin nt + g'/n.$$

The value for V may be obtained from the first equation as

$$V = C'\sin nt - C''\cos nt.$$

The position of the cylinder at any time is then obtained by integration as

$$\left.\begin{aligned} x &= +\frac{C'}{n}\sin nt - \frac{C''}{n}\cos nt + \frac{g't}{n} + C''', \\ y &= -\frac{C'}{n}\cos nt - \frac{C''}{n}\sin nt + C^{\text{iv}}. \end{aligned}\right\} \quad . \; . \; . \quad (50)$$

It will be observed that these equations are not at all of the form appropriate to the parabolic path; that is, the motion of a cylinder in the fluid when circulation takes place around the cylinder is very different from the motion of the cylinder in a vacuum. As a particular case it may be supposed that the cylinder is projected horizontally with a velocity U from the origin of coördinates. The initial conditions are, therefore, when $t = 0$ as follows:

$$x = 0, \qquad y = 0, \qquad dx/dt = U, \qquad dy/dt = 0.$$

Hence,

$$C' = U - \frac{g'}{n}, \qquad C'' = C''' = 0, \qquad C^{\text{iv}} = \frac{U}{n} - \frac{g'}{n^2},$$

and the equation for the path of the center of the cylinder is

$$x = \left(\frac{U}{n} - \frac{g'}{n^2}\right)\sin nt + \frac{g't}{n},$$

$$y = \left(\frac{U}{n} - \frac{g'}{n^2}\right)(1 - \cos nt).$$

These are the equations not of a parabola, but of a trochoid. When the circulation in the fluid is zero the value of n is also zero. If the values of $\sin nt$ and $\cos nt$ be expanded into series, the results may be written

$$x = Ut + \text{higher powers in } n, \quad y = -\frac{g't^2}{2} + \text{higher powers in } n.$$

Therefore, as n approaches the limit zero, the equations of the path go over into the equations for the parabola, with this difference — that the effective value of gravity has been reduced from g to g', owing to the inertia of the fluid.

If a cylinder were spun in a fluid such as the air, and launched rotating with its axis horizontal, the viscosity of the air would set the fluid into circulation about the cylinder. To solve the problem of the motion, taking account of the viscosity of the fluid would be too complicated, but if it be granted that the effect of the viscosity is manifested chiefly in setting up the circulatory motion, and if it be granted further that the motion is irrotational, it would follow that except for the diminution of the angular velocity of the cylinder, owing to the frictional reaction, the path of the center would be a trochoid. This departure from the parabolic path is well known in the case of pitched balls. As the rotation is about the horizontal axis, in this case only the "rise" and "drop" are considered, not the "in" or "out." In order, however, to obtain a mathematical treatment of the pitched ball along the general lines here given for the cylinder, it would be necessary to have available the expressions for the velocity potential of a sphere moving in a three-dimensional liquid with circulation about an axis of the sphere. The solution of this problem will not be undertaken, but the result is similar to that just found for a cylinder. The apparent increase of inertia in the case of the sphere is not the weight of the displaced fluid, nor the mass of the displaced fluid, but one-half that mass.

EXERCISES

1. A source spills 50 ft^3/min of water; there is an equal sink 2 ft away, and a general stream of 3 ft/sec in the line joining the two. (The third dimension is 1 ft.) Where is the velocity zero? What is the velocity halfway between source and sink? What is the pressure difference between the two points?

Ans. $\frac{1}{2}$ in beyond source or sink; 3.27 ft/sec; $10\frac{1}{3}$ lb/ft^2

2. A source and sink (5 gal/sec) are 6 in apart in a stream of 2 ft/sec in the line joining them. (Let the third dimension be figured as 1 ft for the given rate of discharge.) Where is the water at rest and what is the velocity halfway between source and sink?

Ans. $\frac{3}{4}$ in from source or sink; 2.87 ft/sec.

3. Plot (6) or (8) if $U/m = 1$, $a = 1$, $C = 0$. The second may be solved for x as a function of y.

4. Plot (6) or (8) if $U/m = 1$, $a = 1$, $C/m = -1$.

5. Plot (6) or (8) if $U/m = 1$, $a = 1$, $C/m = +1$.

6. Plot (6) or (8) if $U/m = 1$, $a = \frac{1}{2}$, $C = 0$.

7. Plot (6) or (8) if $U/m = 1$, $a = 2$, $C = 0$.

8. In (17) show that $\psi = 0$ is the circle. Plot $\psi = U$.

9. A cylinder is at rest in a stream. Find where the velocity of the liquid is greatest and how much it then is.

10. Find how much the pressure at infinity must be, if the pressure in the stream about a fixed cylinder is to be everywhere positive, except where it is least (zero). What happens if the pressure at infinity is not so great?

11. A cylinder moves in a liquid otherwise at rest. Show that the velocity of the liquid at the periphery of the cylinder is always equal to the velocity of the

cylinder, — and find its direction. Why is the pressure not constant if the velocity is?

12. A Zeppelin (assumed cylindrical) is 500 ft long by 40 ft in diameter. If the Zeppelin remaining horizontal descends uniformly with a velocity of 10 ft/sec, how many ft.lb of work would be needed to stop it (on the assumption of the laws of perfect liquids). *Ans.* 156,000.

13. If the Zeppelin of Ex. 12 floats at rest and a force of 150 lb is applied vertically, what is the acceleration?

14. The Zeppelin of Ex. 13 throws over 200 lb of ballast. With what acceleration does it start to rise?

15. A cylinder of radius R moving sidewise with velocity U spins with angular velocity ω. On the assumption that the irrotational circulatory motion of the fluid outside the cylinder has at the periphery of the cylinder the same velocity as that of the cylinder due to its rotation, find the lift on the cylinder.

16. A cylinder of the same density as water moves 10 ft/sec in water and turns on its axis once per second. The radius is 1 ft. Calculate, on the assumptions of Ex. 15, the lift on the cylinder in pounds and the radius of curvature of its path (the tangent is supposed to be horizontal).

CHAPTER XV

MOTION IN THREE DIMENSIONS

88. General Equations. The equations of motion are

$$\left.\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{g}{\rho}\frac{\partial p}{\partial x},\\ \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{g}{\rho}\frac{\partial p}{\partial y},\\ \frac{dw}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{g}{\rho}\frac{\partial p}{\partial z}.\end{aligned}\right\} \quad . \; . \quad (1)$$

The proof of the equations is obtained by considering the forces acting upon an infinitesimal volume $dx\,dy\,dz$. There are three independent variables u, v, w, which are the component velocities along the axis. The formula for the total differential is

$$d = \frac{\partial}{\partial t}dt + \frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

the first term dealing with the variation in time at a specific point, the last three terms with the variation in space at a particular time. The fluid derivative is obtained by dividing the differential by dt, and using the fact that if the motion of the fluid be followed,

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w \quad . \; . \; . \; . \; . \; . \quad (3)$$

In addition to these dynamical equations there is the equation of continuity, which for three dimensions is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho z}{\partial z} = 0 \quad . \; . \; . \; . \; . \; . \; . \quad (4)$$

The circulation along any curve is for three dimensions

$$\int (u\,dx + v\,dy + w\,dz), \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

just as the work done by a particle is

$$\int (X\,dx + Y\,dy + Z\,dz) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

The geometric proof that the circulation is a product of the velocity by the element of arc by the cosine of the angle between the two is much as in the case of two dimensions. The calculation of the rate of change of the circulation is also carried on in a manner similar to that used before. The result is that for a curve moving with the fluid

$$\frac{1}{g}\frac{d}{dt}\int_A^P (u\,dx + v\,dy + w\,dz) = -\left[\frac{V}{g} + \int\frac{dp}{\rho} - \frac{q^2}{2g}\right]_A^P \quad (7)$$

The conclusion follows as before that the circulation around a closed fluid curve must be constant. Motion is again defined as irrotational in case the circulation around all closed fluid curves is zero. If motion is once rotational it remains always rotational, and if once irrotational, it remains always irrotational when the fluid is followed.

For irrotational motion the circulation around closed curves is zero, and therefore, the circulation along any two curves joining point A to any point P must be the same. Consequently, the circulation depends only upon the point P if A be regarded as a fixed lower limit of integration. The circulation is thus a function of the coördinates x, y, z, of P. Let

$$-d\phi = u\,dx + v\,dy + w\,dz, \; (\phi_P - \phi_A) = -\int_A^P (u\,dx + v\,dy + w\,dz). \quad (8)$$

The function ϕ is called the velocity potential, and

$$u = -\frac{\partial\phi}{\partial x}, \qquad v = -\frac{\partial\phi}{\partial y}, \qquad w = -\frac{\partial\phi}{\partial z} \quad (9)$$

As in the previous case there is a velocity potential when the motion is irrotational. If the fluid is a liquid, that is, if ρ is constant, the equation of continuity expressed in terms of ϕ is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (10)$$

This is the equation of Laplace.

For irrotational motion, that is, when there is a velocity potential, the dynamical equations may be integrated in the terms of the velocity potential. It is merely necessary to substitute for u, v, w, the derivatives of ϕ changed in sign, to multiply the three equations respectively by dx, dy, dz, and to add. The result is a perfect differential if X, Y, Z be derivable from a potential V. Then,

$$-\frac{1}{g}\frac{\partial\phi}{\partial t}+\frac{q^2}{2g}=-\frac{V}{g}-\int\frac{dp}{\rho}+C(t)\,.\,.\,.\,.\,.\quad(11)$$

The constant of integration depends on the time, because the integration is performed with respect to space only, assuming time to be constant. The equation may be written

$$\frac{q^2}{2g}+\frac{V}{g}+\int\frac{dp}{\rho}=\frac{1}{g}\frac{\partial\phi}{\partial t}+C(t)\,.\,.\,.\,.\,.\,.\quad(12)$$

This is again Kelvin's theorem, which states that the total head, namely, the sum of the kinetic head, the static head, and the pressure head is equal to the rate of change of the velocity potential divided by g, except for a constant, which may be a function of the time. The theorem holds only for irrotational motion. If it happens that the motion is steady, the total head is constant all over the fluid at all times, for then $\partial\phi/\partial t = 0$ and C cannot depend on t. If the fluid is a liquid, p/ρ may replace the integral of dp/ρ. All these theorems apply, of course, only to motion in a liquid without viscosity, but as has been explained for the two-dimensional case, a great many types of fluid motion go on very much as though there were no viscosity, even when fluids are slightly viscous.

89. Sources and Sinks. For the motion of a body in a liquid in three dimensions there is available both the direct and indirect attack. For the direct attack it would be necessary to integrate the equations of motion, fitting the integrals to the known motion of the fluid around the periphery of the moving body, and to the known fluid velocity (generally assumed to be zero or constant) at an infinite distance. For the indirect method of attack, a start is made by considering simple types of motion due to sources and sinks. In three dimensions a source is a point in the liquid which radiates liquid in every direction with equal velocity. The strength of the source is unity when the velocity of the fluid at a unit distance is always one. As the same amount of fluid must flow across every sphere concentric with the source, and as the surface of the sphere varies with the square of the radius, the velocity at any distance from the source must vary inversely as the square of the distance. (This is a very different law of variation from that found for two-dimensional flow, where the velocity varied inversely as the distance.) If r denotes the distance from the source, then

$$q = -\frac{\partial\phi}{\partial r}, \qquad -\frac{\partial\phi}{\partial r} = \frac{m}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, the velocity potential for a simple source is

$$\phi = \frac{m}{r} = \frac{m}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}} \quad . \quad . \quad . \quad (14)$$

if the source of strength m is at the point (a, b, c). A sink is the negative of the source, and the velocity potential is, therefore, the same as that for a source except that m is negative. The lines of flow about a source and sink are the same as the lines of electric force about a positive and negative charge, or the lines of magnetic course about a magnet idealized to consist of two magnetic poles.

If there is a general flow of the liquid along the x-axis from $+\infty$ to $-\infty$, with a velocity $-U$, the velocity potential for this uniform flow must clearly be $\phi = Ux$. Hence, the velocity potential for a source at $(a, 0, 0)$ and an equal sink at $(-a, 0, 0)$ in the presence of a generally uniform stream $-U$ is

$$\phi = \frac{m}{\sqrt{(x-a)^2 + y^2 + z^2}} - \frac{m}{\sqrt{(x+a)^2 + y^2 + z^2}} + Ux. \quad (15)$$

The velocity in this combination of source and sink in a stream must vanish at two points in the line of the source and sink, and outside both. This may be seen from the fact that to the right of $(a, 0, 0)$ the source is discharging liquid in the positive x direction, the sink is sucking the liquid back, but with a smaller velocity, and the stream is carrying the liquid back. There is a point at which these three velocities, taken together, vanish, namely, when

$$\frac{m}{(x-a)^2} - \frac{m}{(x+a)^2} - U = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Passing through the two points where the velocity is zero, there is a surface of revolution about the x-axis, which is a particular stream surface; that is, it is a surface such that the velocity of the fluid at any point of the surface is parallel to the surface. Thus, in the three-dimensional case, as in that of two dimensions, the liquid is divided into two regions, one in which the stream lines issue from the source and return to the sink, and the other in which the stream lines are open, passing from $+\infty$ around the oval stream surface on to $-\infty$. The oval surface is, therefore, a possible surface for a body moving in a fluid.

It is necessary to find the shape of this surface. In the two-dimensional case the stream function was known, and the surface was obtained by setting the stream function equal to a constant. In this case the stream function is not known, and in general in three dimensions there is no stream function (though, as a matter of fact, for motion symmetric about an axis of rotation, as here, such a function can be defined). The component velocities u, v, w may, however, be obtained from the velocity potential, and the stream line is one in which the differential displacements dx, dy, dz are proportional to the velocities; that is, the differential equations of a stream line are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \ldots \ldots \ldots \ldots \quad (17)$$

As the motion is one in which there is symmetry around the X-axis, it is sufficient to solve the problem of the determination of the stream lines in the x, y plane. Now,

$$u = -\frac{\partial \phi}{\partial x} = \frac{m(x-a)}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} - \frac{m(x+a)}{[(x+a)^2 + y^2 + z^2]^{\frac{3}{2}}} - U$$

$$v = -\frac{\partial \phi}{\partial y} = \frac{my}{[(x-a)^2 + y^2 + z^2]^{\frac{3}{2}}} - \frac{my}{[(x+a)^2 + y^2 + z^2]^{\frac{3}{2}}}$$

Let $z = 0$. Then the stream lines in the x, y plane are given by

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad v\,dx - u\,dy = 0,$$

or

$$\left[\frac{1}{[(x-a)^2 + y^2]^{\frac{3}{2}}} - \frac{1}{[(x+a)^2 + y^2]^{\frac{3}{2}}}\right] y\,dx$$
$$- \left[\frac{x-a}{[(x-a)^2 + y^2]^{\frac{3}{2}}} - \frac{x+a}{[(x+a)^2 + y^2]^{\frac{3}{2}}} - \frac{U}{m}\right] dy = 0.$$

As this stands, it is not an exact differential, but becomes so if multiplied through by y. Multiplying and integrating, the result is

$$\frac{x-a}{\sqrt{(x-a)^2 + y^2}} - \frac{x+a}{\sqrt{(x+a)^2 + y^2}} + \frac{Uy^2}{2m} = C. \quad . \quad . \quad (18)$$

The revolution of this curve about the x-axis gives the symmetrical stream surfaces. The only one of these curves which can cut the y-axis (elsewhere than at the source or sink) may be found by putting $y = 0$, which shows that $C = 0$. Hence the closed oval curve, the revolution of which gives the surface of revolution which may move in the fluid with a velocity U, is

$$\frac{x-a}{\sqrt{(x-a)^2+y^2}} - \frac{x+a}{\sqrt{(x+a)^2+y^2}} + \frac{Uy^2}{2m} = 0. \quad . \quad . \quad (19)$$

If a is large, this surface is elongated something like a long ellipse, and much the shape of a dirigible, except that in practice, owing to the viscosity of the fluid, it is advisable for the nose of the dirigible to be blunter, and the tail of the dirigible to be sharper than would be determined by this symmetrical curve, which is suitable for a perfect fluid only.

90. Moving Sphere. It was found that in case the source and sink approached each other, the strength of each becoming infinite, the oval curve approached as its limit a circle of definite radius. Consider, therefore, the problem of determining the velocity potential for a source and sink of infinite strength, and infinitely near together. It will be convenient to use polar coördinates in the plane, measuring the angle from the x-axis, and the radius from the center of the source and sink. Then

$$\begin{aligned}\phi &= \frac{m}{\sqrt{r^2-2ar\cos\theta+a^2}} - \frac{m}{\sqrt{r^2+2ar\cos\theta+a^2}} \\ &= \frac{m}{r}\left[\left(1-\frac{2a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-\frac{1}{2}} - \left(1+\frac{2a}{r}\cos\theta+\frac{a^2}{r^2}\right)^{-\frac{1}{2}}\right] \\ &= \frac{m}{r}\left[1+\frac{a\cos\theta}{r} - 1 - \frac{a\cos\theta}{r}\right] = \frac{2\,ma\cos\theta}{r^2}.\end{aligned}$$

In this calculation a is assumed small, and higher powers of a/r have been neglected. If the product $2ma$ be denoted by S, the velocity potential is simply

$$\phi = \frac{S}{r^2}\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

The combination of an infinitely strong source and sink infinitely near together is called a doublet.

If there be superposed upon the doublet a uniform stream, the resultant velocity potential is

$$\phi = \frac{S\cos\theta}{r^2} + Ur\cos\theta \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The velocity in the radial direction is

$$\frac{d\phi}{-dr} = \frac{2S\cos\theta}{r^3} - U\cos\theta = \left(\frac{2S}{Ur^3} - 1\right)U\cos\theta.$$

This radial velocity will vanish for all values of θ when

$$2S = Ur^3, \quad \text{or} \quad r^3 = 2S/U \quad . \; . \; . \; . \; . \; . \quad (22)$$

Hence, for a sphere of radius $R = (2S/U)^{\frac{1}{3}}$ the velocity in the fluid is wholly tangential to the spherical surface, and the velocity potential ϕ is that due to the motion of a liquid flowing around a sphere. If R be introduced into the formula for the potential, the result is

$$\phi = \left(\frac{R^3}{2r^2} + r\right)U\cos\theta \quad . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

The velocity potential for a sphere moving in the liquid is

$$\phi = \frac{R^3}{2r^2}U\cos\theta \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

With these values of the velocity potential it is possible to calculate from Kelvin's theorem the reaction of the fluid on the sphere when the fluid moves by the sphere, or when the sphere moves in a fluid which is otherwise at rest.

The stream lines for the motion of the liquid around the sphere close in symmetrically behind the sphere, and the result is that the resultant pressure upon the sphere is zero, as it was in the case of the cylinder. For the motion of the sphere in the liquid, it is necessary to calculate not only q^2, but $\partial\phi/\partial t$. From

$$-\frac{\partial\phi}{\partial r} = \frac{R^3}{r^3}U\cos\theta, \qquad -\frac{\partial\phi}{r\,\partial\theta} = \frac{R^3}{2r^3}U\sin\theta,$$

$$q^2 = \left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{r\,\partial\theta}\right)^2 = \frac{R^6}{r^6}U^2(\cos^2\theta + \tfrac{1}{4}\sin^2\theta) \; . \; . \; . \quad (25)$$

Next

$$\frac{\partial\phi}{\partial t} = \frac{\partial\phi}{\partial r}\frac{dr}{dt} + \frac{\partial\phi}{r\,\partial\theta}\frac{r\,d\theta}{dt} + \frac{R^3}{2r^2}\frac{dU}{dt}\cos\theta$$

$$= -\frac{R^3}{r^3}U^2\cos^2\theta + \frac{R^3}{2r^3}U^2\sin^2\theta + \frac{R^3}{2r^2}\frac{dU}{dt}\cos\theta \; . \; . \; . \quad (26)$$

The resultant force on the sphere is calculated as

$$P = -\int_0^\pi (p\cos\theta \times 2\pi R^2\sin\theta)\,d\theta,$$

the value for p being taken from

$$\frac{p}{\rho} = C - \frac{q^2}{2g} + \frac{1}{g}\frac{\partial \phi}{\partial t}.$$

$$= C + \frac{1}{2g}\frac{dU}{dt}R\cos\theta + \frac{1}{8g}U^2(9\cos^2\theta - 5) \quad . \quad . \quad . \quad (27)$$

The only term that contributes anything to P is the term in dU/dt. The integration and transformation give

$$P = -\tfrac{1}{2}\frac{W'}{g}\frac{dU}{dt}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where W' is the mass of the liquid displaced. As in the case of the cylinder, this reaction varies with the acceleration, and does not depend at all upon the velocity. The coefficient, however, contains one-half the mass of the liquid displaced, instead of the whole mass. The equations for motion of the sphere in the x-direction are, therefore,

$$\frac{W + \frac{1}{2}W'}{g}\frac{dU}{dt} = X = X_1 + X_2,$$

where the external force has been resolved into the directly applied force X_1, and the net buoyant force X_2 due to action at a distance. In case the sphere moving in the liquid is of the same density as the liquid, the buoyant force vanishes in the simple case when the external acceleration is that due to gravity or any uniform force. In the general case where the external acceleration is f, the result is

$$\frac{W + \frac{1}{2}W'}{g}\frac{dU}{dt} = X_1 + (W - W')\frac{f}{g} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Thus, when a sphere is projected in a liquid in a horizontal direction with the velocity U under the acceleration of gravity, the effective downward acceleration is $g(W - W')/(W + \frac{1}{2}W')$, and the path is a parabola.

If the sphere were spinning about an axis, there would be, owing to the viscosity, some circulation about the sphere, but it is not at all certain what the circulatory motion would be. The liquid in contact with the spherical surface would be dragged with the surface, and thus have a large velocity around the equator, and a small velocity near the poles of the spinning axis. It would undoubtedly be true, however, that in addition to the retarded action dependent upon the acceleration there would be a lift perpendicular to the direction of motion.

91. Sustentation. The results found for the cylinder and the similar results which undoubtedly hold for the sphere are closely connected with the theory of sustentation of the airplane, according to Lanchester and Kutta. The straight mathematical analysis of this motion is too difficult to undertake at this point, but a description of the phenomenon may be given. Suppose, first, that the airplane wing is of infinite lateral extent, so that the motion is strictly two-dimensional. Instead of considering the motion of the airplane through the air, let it be supposed that the air flows by the aerofoil. The general motion of the stream past the aerofoil may be considered in the first instance to be irrotational, because if the air were a perfect fluid, and were moving irrotationally at a great distance in front of the aerofoil, it must continue to move irrotationally when passing the wing. This irrotational motion, at any rate if it were discontinuous, so that there were a dead wake over the upper surface of the wing, would give rise to a resultant force, largely normal to the aerofoil; and thus there would be a lift and drag as observed. As a matter of fact, the hypothesis of Lanchester and Kutta is that in addition to the general irrotational stream motion there is a circulatory motion of the air about the aerofoil whereby the air passes forward under the wing, up past the leading edge, back across the top of the wing, and down the trailing edge. Such a circulatory motion in the fluid superposed upon the general stream would increase the velocity of the stream above the wing, and decrease the velocity below, as compared with the velocities which would be present if there were no circulation superposed on the stream. If it be assumed that the circulatory motion is irrotational, Kelvin's theorem may be applied with the special restriction that the motion is steady, so that $\partial\phi/\partial t$ vanishes, and the pressure may be calculated from the formula

$$\frac{p}{\rho} = C - \frac{q^2}{2g}.$$

The presence of the circulation increases the velocity q above the wing, and thus diminishes the pressure, or increases the suction, giving a greater lift on the top of the wing. It also diminishes the velocity below the wing, and thus augments the pressure there. The result is that the circulatory motion produces an increase in the lift on the wing. Certain assumptions may be made for the detailed mathematical calculation, and when these are made the results as found by Kutta bring the lift on the wing more closely in accord with the observed values than when corresponding calculations are made without the assumption of circulation.

Of course, with an airplane wing of finite lateral extent, there is considerable spilling of the air from the ends of the wing, and one may imagine that vortices are set up at the lateral ends of the wing, and that these vortices peel off and flow down the fluid. The matter may be pursued further by referring to Lanchester. The detailed theory of vortices is left for more advanced lectures. It was, however, stated in Art. 80 that the angular velocity in plane motion is $\omega = \frac{1}{2}(\partial v/\partial x - \partial u/\partial y)$. It may, therefore, be surmised that the three component angular velocities in three dimensions are

$$p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

Motion in space is rotational wherever p, q, r do not all vanish, irrotational when $p = q = r = 0$.

EXERCISES

1. Write out the details of the proof of one of the set (1).
2. Write out the proof of (4).
3. Write out the proof of (7).
4. Show that (9) are equivalent to $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.

5. Show that if r, ω, θ are polar coördinates in space, θ being the colatitude and ω the longitude, component velocities along the radius, along the meridian and along a circle of constant latitude are respectively

$$u = \frac{dr}{dt}, \quad v = r\frac{d\theta}{dt}, \quad w = r\sin\theta\frac{d\omega}{dt}.$$

6. Consider the polar element of volume; show that its faces are: perpendicular to the radius $r^2 \sin\theta\, d\theta\, d\omega$, $(r + dr)^2 \sin\theta\, d\theta\, d\omega$; perpendicular to the meridian, $r\sin\theta\, d\omega\, dr$, $r\sin(\theta + d\theta)\, d\omega\, dr$; perpendicular to the latitude circle, $r\, d\theta\, dr$, $r\, d\theta\, dr$; and that the element of volume is $r^2 \sin\theta\, dr\, d\omega\, d\theta$.

7. Show that in polar coördinates the equation of continuity is

$$\frac{\partial\rho}{\partial t} + \frac{\partial u r^2 \sin\theta}{r^2 \sin\theta\, \partial r} + \frac{\partial\rho v r\sin\theta}{r^2 \sin\theta\, \partial\theta} + \frac{\partial\rho w r}{r^2 \sin\theta\, \partial\omega} = 0,$$

where u, v, w are defined as in Ex. 5. Reduce the expression.

8. If there is a velocity potential $\phi(r, \omega, \theta)$ show that

$$u = -\frac{\partial \phi}{\partial r}, \quad v = -\frac{\partial \phi}{r\,\partial \theta}, \quad w = -\frac{\partial \phi}{r \sin \theta\, d\omega}.$$

9. Show that if liquid motion has a velocity potential

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \phi}{\partial \omega^2} = 0.$$

10. Give the steps in the proof of (12) from the equations of motion.

11. Derive (12) from (7) when there is a velocity potential.

12. The discharge from a source and absorption by a sink are equal to 2 gallons per second. The source and sink are 1 ft apart. Find the defect of pressure in the fluid (as compared with the pressure at great distances) at the following points (a) halfway between source and sink, (b) half a foot beyond the source or sink in the line joining them, (c) half a foot from the mid-point in any direction perpendicular to the line.

13. If the source and sink of Ex. 12 lie in a general stream of velocity $U = \frac{1}{2}$ ft/sec in the line joining them, find the points of zero velocity.

14. A spherical balloon, 50 ft. in diameter, floats at rest in the air. If a force of 100 lbs is applied downwards, what acceleration would theoretically be set up?

15. The balloon of Ex. 14 throws over 50 lb of ballast. With what acceleration does it start to rise?

16. If the balloon of Ex. 14 is not at rest but rising at a uniform rate of 3 ft/sec, how many ft.lb of work are needed to stop it? Assume perfect fluids.

17. A sphere of radius R is moving with uniform velocity U in a liquid, the radius is not constant but is increasing at the rate dR/dt. What is the velocity potential and what the resultant pressure?

18. A liquid of great depth is moving about an axis from $r = 0$ to $r = a$ with a constant angular velocity ω, and from $r = a$ to $r = \infty$ irrotationally. Find the shape of the free surface. (See Arts. 80 and 81. Two equations are necessary, one from $r = 0$ to $r = a$, the other from $r = a$ to $r = \infty$; but the surfaces should have the same slope at $r = a$.) This is known as Rankine's combined vertex; if the angular velocity is high the free surface is funnel shaped.

19. Show that in cylindrical coördinates the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho r u}{r\,\partial r} + \frac{\partial \rho v}{r\,\partial \theta} + \frac{\partial \rho w}{\partial z} = 0;$$

and that for the motion of a liquid when $v = 0$ and all motion is in the planes through the z-axis, $\partial(ru)/\partial x + \partial(rw)/\partial z = 0$. Hence infer that for this type of motion there is a stream function,

$$\psi = \int r(w\,dz - u\,dr).$$

Apply in Art. 89.

INDEX

[Numbers refer to pages]