





# A L G E B R A AN ELEMENTARY TEXT-BOOK

## INTRODUCTION TO ALGEBRA

TOE THE USE OF SECONDARY SCHOOLS AND TECHNICAL COLLEGES

Tourth Edition, Crown Sec. Cloth PRICE 78, 6d

#### EXCERPT from the PRULYCE

I have kept the fundamental penerpies of the subject well to the front from the very be image. At the same time I have not forgetten, what every in themset of and other) teacher should have perpetually as until, that a general proposition is a property of no value to one that has not mastered the particulars. The atmost regoin of accurate logical diclination has therefore been less my sum than a gradual development of algebraic does. In arranging the even less I have acted on a small r-principle of keeping out as far as possible questions that have no theoretical or practical interest.

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# ALGEBRA

## AN ELEMENTARY TEXT-BOOK

FOR THE

# HIGHER CLASSES OF SECONDARY SCHOOLS AND FOR COLLEGES

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### PART H

SECOND EDITION

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# PREFACE TO THE SECOND EDITION OF PART II.

The present edition of this volume has been carefully revised and corrected throughout. The principal alterations will be found in the Theory of Series; which has been developed a little in some places, with a view to rendering it more useful to students proceeding to study the Theory of Functions. In the interest of the same class of readers, I have added to the chapter on limits a sketch of the modern theory of irrational quantity, one of the most important parts of the purely Arithmetical Theory of Algebraic Quantity, which forms, as the fashion of mathematical thinking now runs, the most widely accepted basis for the great structure of Pure Analysis reared by the masters of our science.

I am indebted for proof-reading and for useful criticism to my friends Prof. G. A. GHISON and Mr. C. TWEEDIE, E.Sc. It is but right, however, to add that the careful and intelligent readers of the Pitt Press have rendered the work of correcting the proofs of this volume more of a sinceure than it often is when mathematical works are in question.

G. CHRYSTAL

EDINBURGH, 3rd March, 1900.

#### PREFACE TO FIRST EDITION.

The delay in the appearance of this volume finds an apology partly in circumstances of a private character, partly in public engagements that could not be declined, but most of all in the growth of the work itself as it progressed in my hands. I have not, as some one prophesied, reached ten volumes; but the present concluding volume is somewhat larger and has cost me infinitely more trouble than I expected.

The main object of Part II, is to deal as thoroughly as possible with those parts of Algebra which form, to use Euler's title, an Introductio in Analysin Infinitorum. A practice has sprung up of late (encouraged by demands for premature knowledge in certain examinations) of hurrying young students into the manipulation of the machinery of the Differential and Integral Calculus before they have grasped the preliminary notions of a Limit and of an Infinite Series, on which all the meaning and all the uses of the Infinitesimal Calculus are based. Besides being to a large extent an educational sham, this course is a sin against the spirit of mathematical progress. The methods of the Differential and Integral Calculus which were once an outwork in the progress of pure mathematics threatened for a time to become its grave. Mathematicians had fallen

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into a habit of covering their inability to solve many particular problems by a vague wave of the hand towards some generality, like Taylor's Theorem, which was supposed to give "an account of all such things," subject only to the awkwardness of practical inapplicability. Much has happened to remove this danger and to reduce d/dxand fdx to their proper place as servants of the pure mathematician. In particular, the brilliant progress on the continent of Function-Theory in the hands of Cauchy, Riemann, Weierstrass, and their followers has opened for us a prospect in which the symbolism of the Differential and Integral Calculus is but a minor object. For the proper understanding of this important branch of modern mathematics a firm grasp of the Doctrine of Limits and of the Convergence and Continuity of an Infinite Series is of much greater moment than familiarity with the symbols in which these ideas may be clothed. It is hoped that the chapters on Inequalities, Limits, and Convergence of Series will help to give the student all that is required both for entering on the study of the Theory of Functions and for rapidly acquiring intelligent command of the Infinitesimal Calculus. In the chapters in question, I have avoided trenching on the ground already occupied by standard treatises: the subjects taken up, although they are all important, are cither not treated at all or else treated very perfunctorily in other English text-books,

Chapters XXIX, and XXX, may be regarded as an elementary illustration of the application of the modern Theory of Functions. They are intended to pave the way

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for the study of the recent works of continental mathematicians on the same subject. Incidentally they contain all that is usually given in English works under the title of Analytical Trigonometry. If any one should be scandalised at this traversing of the boundaries of English examination subjects, I must ask him to recollect that the boundaries in question were never traced in accordance with the principles of modern science, and sometimes break the canon of common sense. One of the results of the old arrangement has been that treatises on Trigonometry, which is a geometrical application of Algebra, have been gradually growing into fragments more or less extensive of Algebra itself; so that Algebra has been disorganised to the detriment of Trigonometry; and a consecutive theory of the elementary functions has been impossible. The timid way, oscillating between illfounded trust and unreasonable fear, in which functions of a complex variable have been treated in some of these manuals is a little discreditable to our intellectual culture. Some expounders of the theory of the exponential function of an imaginary argument seem even to have forgotten the obvious truism that one can prove no property of a function which has not been defined. I have concluded chapter xxx, with a careful discussion of the Reversion of Series and of the Expansion in Power-Series of an Algebraic Functionsubjects which have never been fully treated before in an English text-book, although we have in Frost's Curre Tracing an admirable collection of examples of their use.

The other innovations call for little explanation, as they am merely at greater completeness on the old lines. In PREFACE

the chapter on Probability, for instance, I have omitted certain matter of doubtful soundness and of questionable utility; and filled its place by what I hope will prove a useful exposition of the principles of actuarial calculation.

I may here give a word of advice to young students reading my second volume. The matter is arranged to facilitate reference and to secure brevity and logical sequence; but it by no means follows that the volume should be read straight through at a first reading. Such an attempt would probably sicken the reader both of the author and of the subject. Every mathematical book that is worth anything must be read "backwards and forwards," if I may use the expression. I would modify the advice of a great French mathematician\* and say, "Go on, but often return to strengthen your faith." When you come on a hard or dreary passage, pass it over; and come back to it after you have seen its importance or found the need for it further on. To facilitate this skimming process, I have given, after the table of contents, a suggestion for the course of a first reading.

The index of proper names at the end of the work will show at a glance the main sources from which I have drawn my materials for Part II. Wherever I have consciously borrowed the actual words or the ideas of another writer I have given a reference. There are, however, several works to which I am more indebted than appears in the bond. Among these I may mention, besides Cauchy's

<sup>\* &</sup>quot;Allez en avant, et la loi vous viendra."

Analyse Algebrique, Serret's Algebre Supérieure, and Schlömilch's Algebraische Analysis, which have become classical, the more recent work of Stolz, to which I owe many indications of the sources of original information—a kind of help that cannot be acknowledged in footnotes.

I am under personal obligations for useful criticism, for proof-reading, and for help in working exercises, to my assistant, Mr. R. E. ALLARDICE, to Mr. G. A. GIBSON, to Mr. A. Y. FRASER, and to my present or former pupils—Messes. B. B. P. Brandford, J. W. Butters, J. Crockett, J. Goodwillie, C. Tweedie.

In taking leave of this work, which has occupied most of the spare time of five somewhat busy years, I may be allowed to express the hope that it will do a little in a cause that I have much at heart, namely, the advancement of mathematical learning among English-speaking students of the rising generation. It is for them that I have worked, remembering the searcity of aids when I was myself a student; and it is in their profit that I shall look for my reward.

G. CHRYSTAL

Edinburgh, 1st November 1889.

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# SUGGESTION FOR THE COURSE OF A FIRST READING OF PART II.

#### CHAPTER XXIII

#### Permutations and Combinations.

§ 1.] We have already seen the importance of the enumeration of combinations in the elementary theory of integral functions. It was found, for example, that the problem of finding the coefficients in the expansion of a binomial is identical with the problem of enumerating the combinations of a certain number of things taken 1, 2, 3, &c., at a time. Besides its theoretical use, the theory of permutations and combinations has important practical applications; for example, to economic statistics, to the calculus of probabilities, to fire and life assurance, and to the theory of voting.

Beginners usually find the subject somewhat difficult. This arises in part from the fineness of the distinctions between the different problems, distinctions which are not always easy to express clearly in ordinary language. Close attention should therefore be paid to the terminology we are now to introduce.

§ 2.] For our present purpose we may represent individual

things by letters.

By an r-permutation of n letters we mean r of those letters arranged in a certain order, say in a straight line. An n-permutation, which means all the letters in a certain order, is sometimes called a p-randation simply.

Example. The 2-permutations of the three letters a, b, c are bc, cb; ac, ca; ab, ba. The permutations of the three letters are abc, acb; bac, bca; cab, cba;

By an r-combination of n letters we mean r of those letters considered without reference to order.

Example. The 2-combinations of a, b, c are bc, ac, ab.

Unless the contemp is stated, the same letter is not supposed to occur more than one in each combination or permutation. In other words, if the *n* letters were printed on *n* separate counters each permutation or combination could be actually selected and set down before our eyes.

Another point to be attended to is that in some problems certain sets of the given letters may be all alike or indifferent; that is to say, it may be supposed that no alteration in any permutation or combination is produced by interchanging these letters.

§ 3.] The fundamental part of every demonstration of a theorem in the theory of permutations and combinations is an enumeration. It is necessary that this enumeration be systematic and exhaustive. If possible it should also be simplex, that is, each permutation or combination should occur only once; but it may be multiplex, provided the degree of multiplicity be ascertained (see § 8, below).

Along with the enumeration there often occurs the process of reasoning step by step, called mathematical induction.

The results of the law of distribution, as applied both to closed functions and to infinite series, are often used (after the manner of chap 18...; 5, 11, and exercise vi. 30) to lighten the labour of enumeration.

All these methods of proof will be found illustrated below. We have called attention to them here, in order that the student may know what tools are at his disposal.

#### PERMUTATIONS.

§ 4.] The number of r-permutations of a letters  $({}_{n}P_{r})$  is

$$n(n-1)(n-2)$$
 . .  $(n-r+1)$ .

 $1 \cdot t \cdot Pro \mathscr{J}$ . Suppose that we have r blank spaces, the problem is to find in how many different ways we can fill these with uletter—all different

We can fill the first blank in n different ways, namely, by putting into it any one of the n letters. Having put any one letter into the first blank, we have n=1 to choose from in filling

the second blank. Hence we can fill the second blank in n-1 different ways for each way we can fill the first. Hence we can fill the first two in n(n-1) ways.

When any two particular letters have been put into the first two blanks, there are n-2 left to choose from in filling the third. Hence we can fill the first three blanks in n(n-1) times (n-2)ways.

Reasoning in this way, we see that we can fill the r blanks in n(n-1) (n-2) . . . (n-r+1) ways.

Hence 
$$P_r = n(n-1) \dots (n-r+1)$$
.

2nd Proof.—We may enumerate, exhaustively and without repetition, the  ${}_{n}P_{r}$  r-permutations as follows:—

1st. All those in which the first letter a<sub>1</sub> stands first;

2nd. All those in which a2 stands first; and so on.

There are as many permutations in which  $a_1$  stands first as there are (r-1)-permutations of the remaining n-1 letters, that is, there are  $a_1 P_{r-1}$  permutations in the first class. The same is true of each of the other n classes.

Hence 
$${}_{n}P_{r} = n_{n-1}P_{r-1}$$
.

Now this relation is true for any positive integral values of n and r, so long, of course, as  $r \geqslant n$ . Hence we may write successively

$${}_{n}P_{r} = n_{n-1}P_{r-1},$$
 ${}_{n-1}P_{r-1} = (n-1)_{n-2}P_{r-2},$ 
 ${}_{n-r+2}P_{2} = (n-r+2)_{n-r+1}P_{1}.$ 

If now we multiply all these equations together, and observe that all the P's cancel each other except  $_{n}P_{r}$  and  $_{n-r+1}P_{1}$ , and observe further that the value of  $_{n-r+1}P_{1}$  is obviously n-r+1, we see that

$$_{n}P_{r} = n(n-1) . . . (n-r+2)(n-r+1)$$
 (1).

The second proof is not so simple as the first, but it illustrates a kind of reasoning which is very useful in questions regarding permutations and combinations.

Cor. 1. The number of different ways in which a set of n letters can be arranged in linear order is

$$n(n-1) \dots 3 \cdot 2 \cdot 1$$

that is, the product of the first n integral numbers.

This follows at once from (1), for the number required is the number of n-permutations of the n letters. Putting r = n in (1), we have

 $P = n(n-1) \dots 2 \cdot 1$ (2).

The product of the first n consecutive integers may be reaided as a function of the integral variable n. It is called factorial-n, and is denoted by uli\*.

Cor. 2. 
$${}_{n}P_{r}$$
  ${}_{n}!$   $(n=r)!$ .  
For  ${}_{n}P_{r}$   ${}_{n}(n-1)$  . . .  $(n=r+1),$ 

$$= n (n-1) . . . . (n-r+1)(n-r) . . . . 2 . 1$$

$$= \frac{n!}{(n-r)!}.$$

Cor. 3. The number of ways of arranging a letters in circular earler is  $(n-1)^n$ , or  $(n-1)^n/2$ , according as clock-order and

Since the circular order merely, and not actual position, is in question, we may select any one letter and keep it fixed. We have thus as many different arrangements as there are (n-1)permutations of the remaining n-1 letters, that is (n-1)!.

If, however, the letters written in any circular order clockwise he not distin uished from the letters written in the same order counter-clock-wile, it is clear that each arrangement will be counted twice over. Hence the number in this case is  $(n - 1)^n 2$ .

§ 5 ] When each of the whitters may be repeated, the number of r-permutations is n'.

<sup>.</sup> This is Kramp's a date on. Formerly a war u od in English works, but the is now being about loved on account of the difficulty in printing the | . The value of 1! is of come 1. Strictly speaking, 0! has no meaning. It is conversely, however to use it, with the understanding that its value is 1; by so doing we avoid the exceptional treatment of initial terms in many series.

Suppose that we have r blanks before us. We may fill the first in n ways; the second also in n ways, since there is now no restriction on the choice of the letter. Hence the first two may be filled in  $n \times n$ , that is,  $n^*$  ways. With each of these  $n^*$  ways of filling the first two blanks we may combine any one of the n ways of filling the third; hence we may fill the first three blanks in  $n^2 \times n$ , that is,  $n^3$  ways, and so on. Hence we can fill the r blanks in  $n^r$  ways.

§ 6.] The number of permutations of n letters of which a group of  $\alpha$  are all alike, a group of  $\beta$  all alike, a group of  $\gamma$  all alike,  $\delta c_n$  is

 $n!/a!\beta!\gamma!$ ...

Let us suppose that x denotes the number in question. If we take any one of the x permutations and keep all the rest of the letters fixed in their places, but make the a letters unlike and permutate them in every possible way among themselves, we shall derive a: permutations in which the a letters are all unlike. Hence the effect of making the a letters unlike is to derive xa: permutations from the x permutations.

If we now make all the  $\beta$  letters unlike, we derive  $xa!\beta!$  permutations from the xa!.

Hence, if we make all the letters unlike, we derive  $x\alpha!\beta!\gamma!\dots$  permutations. But these must be exactly all possible permutations of n letters all unlike, that is, we must have

$$x a! \beta! \gamma! \dots = n!$$

Hence

$$x = n!/\alpha!\beta!\gamma!$$
 . . .

Cor. The number of ways in which n things can be put into r pigeon-holes, so that a shall go into the first,  $\beta$  into the second,  $\gamma$  into the third, and so on, is

$$n!/a!\beta!\gamma!$$
 . . .

N.B.—The order of the pigeon-holes is fixed, and must be attended to, but the order of the things inside the holes is indifferent.

Putting the things into the holes is evidently the same as allowing them to stand in a line and affixing to them labels marked with the names of the holes. There will thus be  $\sigma$ 

labels each marked 1,  $\beta$  each marked 2,  $\gamma$  to h marked 3, and so on,

The problem is now to find in how many ways n labels,  $\alpha$  of which are alike,  $\beta$  alike,  $\gamma$  alike,  $\lambda c$ , can be distributed among n things standing in a given order. The number in question is  $n(\alpha/B)^{\gamma} + \cdots$ , by the above proposition.

Example 1. In arran sing the entw of an eight ear 1 beat the captain has four menthat can row only on the strokess le and four that can row only on the bowessle. In how many different ways can be arrange his boat—1st, when the stroke is factor?

In the first case, the captain may arrange his stroke-side in as many ways as there are 1/r mutations of 1 things, that is, in 4 ways, and he may arrange the bowested in just a many ways. Since the arrange ments of the two sides are independent, be has, therefore,  $12 \times 12 (-576)$  different ways of arranging the whole erew,

In the second case, sinke stroke is fixed, there are only 3! ways of arranging the stroke side. Hence, in this case, there are 3! x 4! (= 144) different ways of arranging the crew.

Example 2. Find the number of p rimitations that can be made with the letters of the word transalpine.

The letters are translapte, there being two sets, each containing two like betters. The number required as therefore (by § 6) 11!/2!2! 11, 10, 9, 8, 7, 6, 5, 3, 2, 2, 907, 200.

Example 3. In how many different ways can n different beads be formed into a brace int?

Since merely turning the bracelet over element a down-arrangement of the some into the erresponding bounce eleck-arrangement, it follows, by § 4, that the number requires  $(n-1)f_1/2$ .

#### COMPLYATIONS

[7] The number of ways in which s thirds can be selected by tellingene out of a set of n<sub>1</sub>, one out of a set of n<sub>2</sub>, &c., is n<sub>1</sub>n<sub>2</sub>...n<sub>n</sub>.

The first thing can be selected in  $n_t$  ways; the second in  $n_t$  ways; and so on. Hence, since the selection of each of the things does not depend in any way on the selection of the others, the number of ways in which the s things can be selected is  $n_t : n_t : n_t : n_t$ .

is s. [ The number of result nations of 
$$n \nmid tt$$
,  $r = ({}_{r}C_{r})$  is  $n \mid n-1 \rangle \dots (n-r+1) \mid 1 \mid 2 \mid \dots \mid r$ 

1st Proof.—We may enumerate the combinations as follows:

All those that contain the letter a<sub>1</sub>;

2nd. , , , , 
$$\alpha_2$$
 ,  $n$ th. , , , ,  $\alpha_n$ 

In each of these classes there is the same number of combinations; namely, as many combinations as there are (r-1)-combinations of n-1 letters; for we obviously form all the r-combinations in which  $a_1$  occurs by forming all possible (r-1)-combinations of  $a_2, a_3, \ldots, a_n$  and adding  $a_1$  to each of them.

This enumeration, though exhaustive, is not simplex; for each r-combination will be counted once for every letter it contains, that is, r times. Hence

$$r_n C_r = n_{n-1} C_{r-1}$$
 (1).

This relation holds for all values of n and r, so long as  $r \downarrow n$ . Hence we have successively—

$$nC_r = \frac{n}{r} _{n-1}C_{r-1},$$

$$n_{-1}C_{r-1} = \frac{n-1}{r-1} _{n-2}C_{r-2},$$

$$n_{-2}C_{r-2} \stackrel{\cdot}{=} \frac{n-2}{r-2} _{n-2}C_{r-3},$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$n_{-r+2}C_2 = \frac{n-r}{2} \stackrel{\cdot}{=} \frac{r+2}{2} _{n-r+1}C_1.$$

If we multiply these r-1 equations together, and observe that the C's cancel, except  ${}_{n}C_{r}$  and  ${}_{n-r+1}C_{1}$ , and that the value of  ${}_{n-r+1}C_{1}$  is obviously n-r+1, we have

$$_{n}C_{r} = \frac{n(n-1) \cdot \cdot \cdot \cdot (n-r+1)}{1 \cdot 2 \cdot \cdot \cdot \cdot r}$$
 (2)

 $2nd\ Proof.$ —Since every r-combination of n letters, if permutated in every possible way, would give r! r-permutations, and all the r-permutations of the n letters can be got once and only

once by dealing in this way with all the r-combinations, it follows that  ${}_{n}C_{r}r^{r} - {}_{n}P_{r}$ . Hence

$$nC_r = nP_r/r!$$
  $n(n-1)$  . . .  $(n-r+1)/1$  . 2 . . .  $r$ .

Cor. 1. If we multiply both numerator and denominator of the expression for  ${}_{n}C_{r}$  by  $(n-r)(n-r-1)\dots 2$ .1, we deduce

$$_{\mathbf{n}}C_{\mathbf{r}} = n!/r!(n-\mathbf{r})!$$
 (3).

Cor. 2. 
$${}_{n}C_{r} = {}_{n}C_{n-r}.$$

This follows at once from (3). It may also be proved by enumeration; for it is obvious that for every r-combination of the n things we select we leave behind an (n-r)-combination; there are, therefore, just as many of the latter as of the former.

Cor. 3. 
$${}_{n}C_{r} = {}_{n-1}C_{r} + {}_{n-1}C_{r-1}$$
 (4).

This can be proved by using the expressions for  ${}_{n}C_{r,n-1}C_{r,n-1}C_{r}$ , and the remark is important, because it shows that the property holds for functions of n having the form (2) irrespective of any restriction on the value of n.

The theorem (when n is a positive integer) also follows at once by classifying the r-combinations of n letters  $a_1, a_2, \ldots, a_n$  into, 1st, those that contain  $a_1, \ldots, a_{r-1}, c_{r-1}$  in number, and, 2nd, those that do not contain  $a_1, \ldots, a_{r-1}, c_r$  in number.

Cor. 4. 
$$n_{-1}C_s + n_{-2}C_s + n_{-2}C_s + n_{-2}C_s + \dots + sC_s - nC_{s+1}$$
 (5),

Since the order of letters in any combination is indifferent, we may arrange them in alphabetical order, and enumerate the (s+1)-combinations of n letters by counting, 1st, those in which  $a_1$  stands first; 2ml, those in which  $a_2$  stands first, 8c. This enumeration is clearly both exhaustive and simplex; and we observe that  $a_1$  cannot occur in any of the combinations of the 2nd class, neither  $a_1$  nor  $a_2$  in any of the 3rd class, and so on. Hence the number of combinations in the 1st class is  $n_{n+1}C_n$ ; in the  $2nd_{n+2}C_n$ ; in the  $3rd_{n+2}C_n$ ; and so on. Thus the theorem follows.

$$_{p}C_{s}^{\prime} + _{p}C_{s-1}{}_{q}C_{1} + _{p}C_{s-2}{}_{q}C_{2} + \dots + _{p}C_{1}{}_{1}C_{s-1} + _{q}C_{s} - _{p+q}C_{s}$$
 (6).

If we divide p+q letters into two groups of p and q respectively, the p+q letters may be classified exhaustively and simplexly as follows:—

1st. All the s-combinations of the p letters. The number of these is

2nd. All the combinations found by taking every one of the (s-1)-combinations of the p things with every one of the 1-combinations of the q things. The number of these is

$$_{n}C_{t-1}\times _{n}C_{1}$$

3rd. All the combinations found by taking every one of the (s-2)-combinations of the p things with every one of the 2-combinations of the q things. The number of these is  $\frac{C_{n-2} \times C_{n-2}}{C_{n-2} \times C_{n-2}}$ 

And so on. Thus the theorem follows.

It should be noticed that Cor. 4 and Cor. 5 furnish propositions in the summation of series. For example, we may write Cor. 5 thus—

$$\begin{array}{c} \frac{p(p-1) \dots (p-s+1)}{1 \cdot 2 \dots s} + \frac{p(p-1) \dots (p-s+2)}{1 \cdot 2 \dots (s-1)} \cdot \frac{q}{1} \\ + \frac{p(p-1) \dots (p-s+3)}{1 \cdot 2 \dots (s-2)} \cdot \frac{q(q-1)}{1 \cdot 2} \\ + \frac{p \cdot q(q-1) \dots (q-s+2)}{1 \cdot 2 \dots (s-2)} \\ + \frac{p \cdot q \cdot q-1}{1 \cdot 2 \dots (s-1)} \\ + \frac{q \cdot q-1}{1 \cdot 2 \dots (s-1)} \\ = \frac{(p+q) \cdot (p+q-1) \dots (p+q-s+1)}{1 \cdot 2 \dots s} \\ = \frac{(p+q) \cdot (p+q-1) \dots (p+q-s+1)}{1 \cdot 2 \dots s} \end{array}$$
 (7).

It is obvious that (7) is an algebraical identity which could be proved by actually transforming the left-hand side into the right (see chap. v.,  $\S$  16). If we take this view, it is clear that the only restriction upon p,q,s is that s shall be a positive integer. Thus generalised, (7) becomes of importance in the establishment of the Binomial Theorem for fractional and negative indices.

Cor. 6. If we multiply both sides of (7) by 1.2...s, and denote p(p-1)...(p-s+1) by  $p_s$ , we deduce

 $(p+q)_* = p_* + {}_sC_1p_{*-1}q_1 + {}_sC_2p_{*-2}q_2 + \ldots + q_*$  (8), which is often called Vandermonde's theorem, although the result was known before Vandermonde's day.

§ 9.] To find the number of r-combinations of p+q letters p of which are alike.

1st. With the q unlike letters we can form  ${}_qC_r$  r-combinations.

2nd. Taking one of the p letters, and r-1 of the q we can form  ${}_{q}C_{r-1}$  r-combinations.

3rd. Taking two of the p, and r-2 of the q, we can form  ${}_qC_{r-2}$  r-combinations; and so on, till at last we take r of the p (supposing p>r), and form one r combination.

We thus find for the number required

$$\begin{split} & _q C_{r+q} C_{r+1} + _q C_{r+2} + \ldots + _q C_1 + 1 \\ & = q! \left\{ \frac{1}{r! (q-r)!} + \frac{1}{(r-1)! (q-r+1)!} + \ldots \right. \\ & + \frac{1}{1! (q-1)!} + \frac{1}{q!} \right\}, \end{split}$$

Cor. The number of r-permutations of p+q things p of which are alike is

$$q!r! \left\{ \frac{1}{r!(q-r)!} + \frac{1}{1!(r-1)!(q-r+1)!} + \frac{1}{2!(r-2)!(q-r+2)!} + \cdots + \frac{1}{(r-1)!(r-1)!} + \frac{1}{r!ol} \right\}.$$

For, with the  ${}_qC_r$  combinations of the 1st class above we can form  ${}_qC_r$  r! permutations;

With the  $_qC_{r-1}$  combinations of the 2nd class,  $_qC_{r-1}r!$  permutations:

With the  $_qC_{r-2}$  combinations of the 3rd class (in each of which two letters are alike),  $_qC_{r-2}\tau!/2!$  permutations: and so on.

Hence the whole number of permutations is

$${}_qC_rr! + {}_qC_{r-1}r!/1! + {}_qC_{r-2}r!/2! + \dots + {}_qC_1r!/(r-1)! + 1,$$
 whence the result follows.

A similar process will give the number of r-combinations, or of r-permutations, when we have more than one group of like letters; but the general formula is very complicated.

§ 10.] The number of r-combinations of n letters ("H<sub>r</sub>), when each letter may be repeated any number of times up to r<sub>i</sub> is

$$n(n+1)(n+2) \dots (n+r-1), 1, 2, 3, \dots r$$
 (1).

In the first place, we remark that the number of (r+1)-combinations, in each of which the letter  $a_i$  occurs at least once, is the same as the number of r combinations not subject to this restriction. This is obvious if we reflect that every (r+1)-combination of the kind described leaves an r-combination when  $a_i$  is removed, and, conversely, every r-combination of the n-letters gives, when  $a_1$  is added to it, an (r+1)-combination of the kind described.

It follows, then, that if we add to each of the r-combinations of the theorem all the n letters, we get all the (n+r)-combinations of the n letters, in each of which each letter appears at least once, and not more than r+1 times. We may therefore enumerate the latter instead of the former.

This new problem may be reduced to a question of permutations as follows. Instead of writing down all the repeated letters. we may write down each letter once, and write after it the letter s (initial of same) as often as the letter is repeated. Thus, we write asssbsses . . . instead of annabbbee . . . With this notation there will occur in each of the (n + r)-combinations the n letters  $a_1, a_2, \ldots, a_n$  along with r s's. The problem now is to find in how many ways we can arrange these n + r letters. It must be remembered that there is no meaning in the occurrence of s at the beginning of the series; hence, since the order of the letters a. a. . . . a. is indifferent, we may fix a. in the first place. We have now to consider the different arrangements of the n-1letters  $a_2, a_3, \ldots, a_n$  along with r s's. In so doing we must observe that nothing depends on the order of  $a_2, a_3, \ldots, a_n$ inter se; so that in counting the permutations they must be regarded as all alike. We have, therefore, to find the number of permutations of n-1+r things, n-1 of which are alike, and rof which are alike. Hence we have

$$_{n}H_{r} - \frac{(n+r-1)!}{(n-1)!r!}$$
 (2),  
=  $\frac{n(n+1)...(n+r-1)}{1.2...r}$ .

Cor. 1.

$$_{n}H_{r}=_{n+r-1}C_{r}$$

This follows at once from (2).

Cor. 2. 
$${}_{n}H_{r-n-1}H_{r+n}H_{r-1}$$
.

For the r-combinations consist, 1st, of those in which  $a_i$  occurs at least once, the number of which we have seen to be  ${}_nH_{r-1}$ ; 2nd, of those in which  $a_i$  does not occur at all, the number of which is  ${}_{n-1}H_r$ .

Cor. 3.  ${}_{n}H_{r} = {}_{n-1}H_{r} + {}_{n-1}H_{r-1} + {}_{n-1}H_{r-2} + \dots + {}_{n-1}H_{1} + 1$ . This follows from the consideration that we may classify the r-combinations into

1st. Those in which  $a_i$  does not occur at all,  $a_{-1}H_r$  in number:

2nd. Those in which  $a_1$  occurs once,  $a_{-1}H_{r-1}$  in number;

3rd. Those in which  $a_1$  occurs twice,  $a_{-1}H_{r-1}$  in number: and so on.

Cor. 4. The number of different r-ary products that can be made with a different letters is  $n(n+1) \dots (n+r-1)/1 \cdot 2 \dots r$ ; and the number of terms in a complete integral function of the rth degree in a variables is  $(n+1)(n+2) \dots (n+r)/1 \cdot 2 \dots r$ .

The first part of the corollary is of course obvious. The second follows from the consideration that the complete integral function is the sum of all possible terms of the degrees  $0, 1, 2, \dots, r$  respectively. Hence the number of its terms is

$$1 + {}_{n}II_{1} + {}_{n}II_{2} + \dots + {}_{n}II_{n}$$

But, by Cor. 3, this sum is  $_{n+1}H_r$ .

We have thus obtained a general solution of the problems suggested in cleap, i.e., §§ 17, 19. As a verification, if we put n=2, we have for the number of terms in the general integral function of the thi degree in two variables  $3,1,\ldots,(r+2)[1,2,\ldots,r]$ , which reduces to (r+1)(r+2)[2], in agreement with our former result.

#### Exercises 1.

### Combinations and Permutations

- (L) How many different numbers can be made with the digits 11122333450?
- (2.) How many different permutations can be made of the letters of the sentence Ut tenno sic vis?

- (3.) How many different numbers of 4 digits can be formed with 0123456?(4.) How many odd numbers can be formed with the digits 3691?
- (5.) If  $_{\sim}C_{-}$ ,  $/_{\sim}_{\sim}C_{-}=132/35$ , find n.
- (6.) If  $m = {}_{n}C_{2}$ , show that  ${}_{m}C_{2} = 3_{n+1}C_{4}$ .
- (7.) In any set of n letters, if the number of r-permutations which contain a be equal to the number of those that do not contain a, prove that the same holds of r-combinations.
- (8.) In how many ways can the major pieces of a set of chess-men be arranged in a line on the board?

If the pawns be included, in how many ways can the pieces be arranged in two lines?

- (9.) Out of 13 men, in how many ways may a guard of 6 be formed in line, the order of the men to be attended to?
- (10.) In how many ways can 12 men be selected out of 17—1st, if there be no estriction on the choice; 2nd, if 2 particular men he always included; 3rd, if 2 particular men never be chosen together?

(11.) In how many ways can a bracelet be made by stringing together 5 like pearls, 6 like rubies, and 7 like diamonds?

How many different settings of 3 stones for a ring could be selected from the above?

What modification of the solution of the first part of the above problem is necessary when two, or all three, of the given numbers are even?

- (12.) In how many ways can an eight-cared boat be manned out of 31 men, 10 of whom can row on the stroke-side only, 12 on the bow-side only, and the rest on either side?
- (13.) In a regiment there are 10 captains, 20 lieutenants, 30 sergeants, and 60 corporals. In how many ways can a party be selected, consisting of 2 captains, 5 lieutenants, 10 sergeants, and 20 corporals?
- (14.) Three persons have 4 coats, 5 vests, and 6 hats between them; in how many different ways can they dress?
- (15.) A man has 12 relations, 7 ladies and 5 gentlemen; his wife has 12 relations, 5 ladies and 7 gentlemen. In how many ways can they invite a dinner party of 6 ladies and 6 gentlemen so that there may be 6 of the man's relations and 6 of the wife's?
- (16.) In how many ways can 7 ladies and 7 gentlemen be seated at a round table so that no 2 ladies sit together?
- (17.) At a dinner-table the host and hostess sit opposite each other. In how many ways can 2n guests be arranged so that 2 particular guests do not sit together?
- (18.) In how many ways can a team of 6 horses be selected out of a studof 16, so that there shall always be 3 out of the 6 ABCA'B'C', but never AA', BB', or CC' together?
- (19.) With 9 consonants and 7 vowels, how many words can be made, each containing 4 consonants and 3 vowels—1st, when there is no restriction on the arrangement of the letters; 2nd, when two consonants are never allowed to come together?
  - (20.) In how many ways can 52 cards, all different, be dealt into 4 equal

hands, the order of the hands, but not of the cards in the hands, to be attended to ?

In how many cases will 13 particular cards fall in one hand?

(21.) In how many ways can a set of 12 black and 12 white draught-men be placed on the black squares of a draught-board?

(22.) In how many ways can a set of chess-men be placed on a chess-board?
(23.) How many 3-combinations and how many 3-permutations can be made with the letters of parabola.

(21.) With an unlimited number of red, white, blue, and black balls at disposal, in how many ways can a bagful of 10 be selected?

In how many of these selections will all the colours be represented?

(25.) In an election under the cumulative system there were p candidates for q seats; (1) in how many ways can an elector give his votes; (2) if there be r voters, how many different states of the poll are there?

If there be 15 candidates and 10 seats, and a voter give one minute to the consideration of each way of giving his vote, how long would it take him to make up his mind how to vote?

#### BINOMIAL AND MULTINOMIAL THEOREMS.

§ 11.] It has already been shown, in chap. 1v., § 11, that

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + \dots + {}_nC_ra^{n-r}b^r + \dots + b^n$$

where  $u''_{1}, n''_{2}, \ldots, n''_{r}, \ldots$  denote the numbers of 1-, 2-, . . . , r-combinations of n things. Using the expressions just found for  $n''_{1}, n''_{2}, \&c_{r}$ , we now have

$$\frac{(a+b)^n - a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \dots}{\frac{n(n-1)}{1 \cdot 2} \cdot \dots \cdot (n-r+1)} a^{n-r}b^r + \dots + b^n$$

$$= \frac{1}{1 \cdot 2} \cdot \dots \cdot r$$
(1).

This is the Binomial Theorem as Newton discovered it, proved, of course, as yet for positive integral indices only.

§ 12.] We may establish the Binomial Theorem by a somewhat different process of reasoning, which has the advantage of being applicable to the expansion of an integral power of any multinomial.

Consider

$$(a_1 + a_2 + \dots + a_n)^n$$
 (2).

We have to distribute the product of n factors, namely,

$$(a_1 + a_2 + \dots + a_m)(a_1 + a_2 + \dots + a_m) \dots (a_1 + a_2 + \dots + a_m)(3);$$

and the problem is to find the coefficient of any given term, say

$$a_1^{a_1}a_2^{a_2} \dots a_m^{a_m}$$
 (4),

where of course  $a_1 + a_2 + \ldots + a_m = n$ . In other words, we have to find how often the partial product (4) occurs in the distributions of (3).

We may write out (4) in a variety of ways, such as

$$a_1a_1a_2a_2a_3a_4a_4...$$
 (5),

there being always  $a_1$   $a_1$ 's,  $a_2$   $a_3$ 's, &c.

Written as in (5) we may regard the partial product as formed by taking  $a_i$  from the 1st and 2nd brackets in (3);  $a_2$  from the 3rd, 4th, and 5th;  $a_3$  from the 6th; and so on. It appears, therefore, that the partial product (4) will occur just as often as we can make different permutations of the n letters, such as (5). Now, since  $a_i$  of the letters are all alike,  $a_2$  all alike,  $x \in \mathbb{R}$ , the number of different permutations is, by  $\S G$ ,  $n!/a_1!a_2! \ldots a_m!$ . Hence we have

$$(a_1 + a_2 + \dots + a_m)^n = \sum \frac{n!}{a_1! a_2! \dots a_m!} a_1^{a_1} a_2^{a_2} \dots a_m^{a_m} (6)$$
:

wherein  $a_1, a_2, \ldots, a_m$  assume all positive integral values consistent with the relation

$$a_1 + a_2 + \ldots + a_m = n$$
 (7).

This is the Multinomial Theorem for a positive integral index.

The Binomial Theorem is merely the particular case where m=2. We then have, since  $a_1+a_2=n$ , and therefore  $a_2=n-a_1$ ,

$$\begin{aligned} & (a_1 + a_2)^n = \Sigma \, \frac{n!}{a_1! \, (n - a_1)!} \, a_1^{a_1} \, a_2^{n - a_1}, \\ & = \Sigma \, \frac{n \, (n - 1)}{a_1!} \, \frac{(n - a_1 + 1)}{a_1!} \, a_1^{a_1} a_2^{n - a_2}, \end{aligned}$$

which agrees with (1).

$$(b_1 + b_2 x + \dots + b_m x^{m-1})^n$$
 (8)

we have simply to pick out all the terms which contain  $x^r$ . The general term is

$$\frac{n!}{a_1!a_2!\ldots a_m!}\,b_1^{a_1}b_2^{a_2}\ldots \,b_m^{a_m}x^{a_2+2a_2+\ldots+(m-1)a_m}$$

Hence we have to take all the terms which are such that

$$a_2 + 2a_3 + \dots + (m-1)a_m = r$$
 (9).

The coefficient of x" in the expansion of (8) is therefore

$$\Sigma_{a_{1}^{1}a_{1}^{1}, \dots, a_{m}^{1}} b_{1}^{a_{1}}b_{2}^{a_{2}}\dots b_{m}^{a_{m}}$$
 (10),

where  $a_1, a_2, \ldots, a_m$  have all positive integral values subject to the restrictions (7) and (9).

Example 1. The coefficient of  $a^3b^2$  in the expansion of  $(a+b+c+d)^6$  is

$$\frac{5!}{3! \cdot 2! \cdot 0! \cdot 0!} = 10.$$

Example 2. To find the coefficient of  $x^3$  in  $(1 + 2x + x^2)^4$ .

Here we must have  $a_1 + a_2 + a_3 = 4$ 

$$a_2 + 2a_3 = 5$$
.

Hence

 $a_1 = a_2 - 1$ ,  $a_2 = 5 - 2a_1$ ,

Since  $a_1$  and  $a_2$  must both be positive, the only two admissible values of  $a_2$ are 1 and 2. We have therefore the following table of values :-

$a_1$	a ;	$a_{s}$
0	3	1
1	1	2

The required coefficient is therefore

$$\frac{4!}{0!3!\overline{1!}}\,\mathbf{1}^{1}2^{3}\mathbf{1}^{4}+\frac{4!}{1!1!2!}\,\mathbf{1}^{1}2^{1}\mathbf{1}^{2}=\mathbf{56},$$

The correctness of the result may be easily verified in the present case; for  $(1 + 2x + x^2)^4 = (1 + x)^8$ , the coefficient of  $x^5$  in which is  ${}_{\theta}C_{\theta} = 56$ ,

Example 3. To find the greatest coefficient, or coefficients, in the expansion of  $(a_1 + a_n + \dots + a_m)^n$ .

This amounts to determining  $x, y, z, \ldots$  so that  $n!/x! y! z! \ldots$  shall be a maximum, where  $x + y + z + \dots = n$ . This, a ain, amounts to determining  $x, y, z, \dots$  so that

$$u = x!y!z!\dots$$
 (1)

shall be a minimum, subject to the condition

$$x + y + z + \ldots = n$$
 (2).

Let us first consider the case where there are only two variables, x and u. We obtain all possible values of  $x^iy!$  by giving y successively the values  $0, 1, 2, \ldots, n, x$  taking in consequence the values  $n, n-1, n-2, \ldots, 0$ . The consecutive value to x(y) is  $(x-1)^{\alpha}(y+1)!$ , and the ratio of the latter to the former is  $(y+1)_i x$ ; that is (since x+y-n),  $(n+1-x)_i x$ , that is,

(n+1)/x-1. This ratio is less than unity so long as (n+1)/x-2, that is, so long as x > (n+1)/2. Until x falls below this value the terms in the series above mentioned will decrease; and after x falls below this limit they will begin to increase.

If n be odd, =2k+1 say, then (n+1)/2=k+1, Hence, if we make x=k+1, the ratio (n+1)/x-1=1, and two consecutive values of x!u!, viz. (k+1)!k! and k!(k+1)!, are equal and less than any of the others.

If n be even, =2k say, then (n+1)/2=k+1. Hence, if we make x=k, we obtain a single term of the series, viz. k! k!, which is less than any of the others.

Returning now to the general case, we see that, if u be a minimum for all values of x, u, z, . . . subject to the restriction (2), it will also be a minimum for values such that x and y alone are variable, z, . . . being all constant. In other words, the values of x and y for which x!y!z!... is a minimum must be such as render x!y! a minimum. Hence, by what has just been proved, x and y must either be equal or differ only by unity. The like follows for every pair of the variables  $x, y, z, \ldots$  Let us therefore suppose that p of these are each equal to  $\xi$ ; then the remaining m-p must each be equal to  $\xi+1$ . Further, let q be the quotient and r the remainder when n is divided by m; so that n = mq + r. We thus have

$$p\xi + (m - p)(\xi + 1) = mq + r$$
.

Heuce so that

$$m\xi + (m - p) = mq + r$$
;  
 $\xi + (m - p)/m = q + r/m$ .

Now (m-p)/m and r/m are proper fractions; hence we must have

 $\xi = q$ , m - p = r. It follows, therefore, that r of the variables are each equal to q+1, and the rest are each equal to q. The maximum coefficient is therefore

$$n!/(q!)^{m-r}\{(q+1)!\}_{j=r}^{r};$$

that is.  $n!.(a!)^m(a+1)^r$ This coefficient is, of course, common to all terms of the type

$$a_1^q a_2^q \dots a_{m-r}^q a_{m-r+1}^{q+1} \dots a_m^{q+1}$$
.

As a special case, consider  $(a_1 + a_n + a_n)^4$ . Here  $4 = 3 \times 1 + 1$ ; q = 1, r = 1. Hence the terms that have the greatest coefficient are those of the type  $a_1a_2a_3^2$ , and the coefficient in question is  $4!/(1!)^32! = 12$ . This is right; for we find by distributing that

$$(a_1 + a_2 + a_3)^4 = \sum a_1^4 + 4\sum a_1^3 a_2 + 6\sum a_1^2 a_2^2 + 12\sum a_1^2 a_2 a_3^2$$

Example 4. Show that

$$1 - \frac{n}{1} \frac{1+x}{1+nx} + \frac{n(n-1)}{1\cdot 2} \frac{1+2x}{(1+nx)^2} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \frac{1+3x}{(1+nx)^3} + \dots = 0.$$
(Webstenholm.)

The left-hand side may be written

$$\begin{array}{lll} 1 - \frac{1}{1} \frac{1}{1 + nc} + \frac{n \, (n-1)}{1 \cdot 2} \frac{1}{(1 + nc)^2} & \frac{n \, (n-1) \, (n-2)}{1 \cdot 2 \cdot 3} & \frac{1}{(1 + nc)^4} + \dots \\ - \frac{n}{1} \frac{x}{1 + nc} + \frac{n \, (n-1)}{1 \cdot 2} \frac{2x}{(1 + nc)^2} & \frac{n \, (n-1) \, (n-2)}{(1 + nc)^2} & \frac{3x}{(1 + nc)^2} & \dots \end{array}$$

C. 11.

13.1 The Binomial Theorem can be used in its turn to establish identities in the theory of combinations; as the two following examples will show:—

Example 1 - We have

On the right band and of this if atty to each behalf of every power of x multivanals. These cash on any p -stove into (r,r) - Wain r, we have

If we make the problem of the total contracted, we see that the entractions of the  $n^n$  as  $n^n$ , which is concern of  $z^n$  on the letter  $x^n$  as 1.

$$1 = {n \choose 1} = {n \choose 2} =$$

A transfer that the control and thy the allocation of equating and the control can specimens

## 1'vi = 11

(1) 10 1 cm children in the special of (2 - 3)?

x = 1 (the term x = 1) x = 1 (the expansion of

(4.) Find the coefficient of x<sup>2r</sup> in the expansion of (x - 1/x)<sup>n</sup>.

(5.) Find the ratio of the coefficients of x<sup>2n</sup> in (1 + r)<sup>4n</sup> and (1 + r)<sup>2n</sup>.

(6.) Find the middle term in the expansion of (2 + \(\psi\))<sup>n</sup>.

(7.) The product of the coefficients in  $(1+x)^{n+1}$ : the product of the coefficients in  $(1+x)^n = (n+1)^n$ : n!.

(8.) The coefficient of  $x^r$  in  $\{(r-2) x^2 + nx - r\}$   $(x+1)^n$  is  $n_n C_{r-2}$ .

(9.) If I denote the integral part and F the proper fractional part of 3+√5)<sup>n</sup>, and if ρ denote the rational part and σ the irrational part of the ame, show that

$$\begin{split} &I=2\left\{3^{n}+_{n}e_{2}3^{n-2}\cdot5+_{n}C_{4}3^{n-4}\cdot5^{2}+\right.\right.\\ &F=1-(3-\sqrt{5})^{n},\\ &\rho=\frac{1}{2}\left(I+1\right),\\ &\sigma=\frac{1}{2}\left(I+2F-1\right). \end{split}$$

(10.) If  $(\sqrt{2}+1)^{2m+1} = I + F$ , where F is a positive proper fraction and I is integral, show that F(I+F) = 1.

(11.) Find the integral parts of  $(2\sqrt{3}+3)^{2m}$ , and of  $(2\sqrt{3}+3)^{2m+1}$ .

(12.) Show that the greatest term in the expansion of (a+x)<sup>n</sup> is the (r+1)th, where r is the integral part of (a+1)/(a/x+1).

Exemplify with  $(2+3)^{10}$  and with  $(2+1)^{9}$ .

(13.) Find the condition that the greatest term in (a+x)<sup>n</sup> shall have the greatest coefficient. Find the limits for x in order that this may be so in (1+x)<sup>n</sup>.

(14.) If the pth term be the greatest in  $(a+x)^m$ , and the qth the greatest in  $(a+x)^n$ , then either the (p+q)th or the (p+q-1)th or the (p+q-2)th is the greatest in  $(a+x)^{m+n}$ .

(15.) Sum the series

$$\frac{{}_{n}C_{1}}{1} + 2 \, {}_{n}C_{2} + 3 \, {}_{n}C_{3} + \ldots + n \, {}_{n}C_{n-1} \cdot \cdots + n \, {}_{n}C_{n-1$$

(16.) Sum the series

1 + 2 . C , + 3 . C , + 4 . C , + . . .

(17.) If p<sub>r</sub> denote the coefficient of x<sub>r</sub> in (1+x)<sup>n</sup>, prove the following relations:—

$$\mathbf{1}^{\circ},\quad p_{1}-2p_{2}+3p_{3}-\dots +n\;(-1)^{n-1}p_{n}=0.$$

**2°.** 
$$\frac{1}{2}p_1 - \frac{1}{5}p_2 + \dots + \frac{(-1)^{n-1}}{n+1} p_n = \frac{n}{n+1}$$
.

3°. 
$$1 + \frac{p_1}{2} + \frac{p_2}{3} + \dots + \frac{p_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$
.

(18.) If pr have the same meaning as in last question, show that

$$p_1 - \frac{1}{2}p_2 + \frac{1}{2}p_3 + \dots + \frac{(-1)^{n-1}}{n}p_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

(19.) Show that

$${}_{r}C_{s}\times 1+{}_{r-1}C_{s-1}\times{}_{r}C_{1}+{}_{r-2}C_{s-2}\times{}_{r}C_{2}+\ldots +{}_{r-s+1}C_{1}\times{}_{r}C_{s-1}+1\times{}_{r}C_{s}=,C_{s}2^{s},$$

(20.) Show that

$$(1 - {}_{n}C_{2} + {}_{n}C_{4} - \dots)^{2} + ({}_{n}C_{1} - {}_{n}C_{3} + \dots)^{2} = 1 + {}_{n}C_{1} + {}_{n}C_{2} + \dots$$

(21.) Show that

$$1 + {}_{n}C_{2} + {}_{n}C_{1} \times {}_{n}C_{3} + \dots + {}_{n}C_{n-2} \times {}_{n}C_{n} = (2n)!/(n+2)!(n-2)!$$

(22) Show that 
$$1 - n^2 + \left(\frac{n(n-1)}{2!}\right)^2 - \left(\frac{n(n-1)(n-2)}{3!}\right)^2 + \dots = 0$$
 if  $n$ 

be odd, and  $= (-1)^{n!} \cdot (n+2) \cdot (n+4) \cdot ... \cdot 2n/2 \cdot 4 \cdot ... \cdot n$  if n be even.

(23.) Show that

$$-1\cdot n\left( n+1\right) +\frac{n}{1!}\left( n-1\right) n+\frac{n\left( n+1\right) }{2!}\left( n-2\right) \left( n-1\right) +\frac{n\left( n+1\right) \left( n+2\right) }{3!}\left( n-3\right) \left( n-2\right) +\frac{n}{2!}\left( n-1\right) +\frac{n$$

$$+ \dots 2(2n+1)!/(n+2)!(n-1)!$$

(21) If u<sub>s</sub> stand for x' + 1'x', show that

$$u_{r+1} + {}_{r+1}C_1u_{r-1} + {}_{r+1}C_2u_{r-1} + \dots = u_1(u_r + {}_rC_1u_{r-2} + {}_rC_2u_{r-4} + \dots).$$

- (25.) If a, denote the coefficient of x! in  $(1+x)^{2/n-r}(1-x)^{2r}$  show that  $a_0 \cdot {}_{n}C_1a_1 + {}_{n}C_2a_2 - \dots = 0$  for all values of p except p = n, in which case the right hand side of the equation is 12,
  - (26.1 Show that

$$\frac{1}{x} - \frac{nC_1}{x+1} + \frac{nC_2}{x+2} - \dots + \frac{(-1)^n nC_n}{x+n} = \frac{n!}{x(x+1)\dots(x+n)}$$

- (27) Find the coefficient of  $x^r$  in  $(1 + x + x^2 + \dots)^2$ .
- (28) Find the coefficient of xis in (1 + x5 + x6 + x6)\*.
- (29) Find the coefficient of x<sup>n</sup> in (1 + x + 2x<sup>2</sup> + 3x<sup>3</sup> + . . . 12
- (30.) If a<sub>0</sub>, a<sub>1</sub>, . . . , a<sub>m</sub> be the coefficients of the powers of x in  $(1 + 2x + 2x^2)^n$ , show that  $a_0x_{2n} + a_1a_{2n-1} + \dots + a_{2n}a_0 = 0$  if n be odd, 2"n! / (4n)! | 2 if n be even.
- (31.) If a, be the coefficient of  $x^r$  in  $(1 + x + x^2 + \dots + x^p)^n$ , show that  $a_r = {}_nC_1a_{r-1} + {}_nC_2a_{r-2} + \dots = 0$ , unb. n + c a multiple of p + 1. What does the equation become in the latter case?
  - (32.) Find the coefficient of x<sup>11</sup> in (1 + 2x + 3x<sup>2</sup> + 4x<sup>2</sup>)<sup>12</sup>.
  - (33.) Write out the expansion of  $(a+b+c+d)^2$ .
  - (31) Show that

$$\Sigma \frac{1}{r(s)}^{\frac{1}{2}} \frac{2^{s}}{s} \cdots \frac{n^{k}}{k!} \equiv \frac{1}{\hat{p}!} \frac{\left\{n\left(n+1\right)\right\}}{2}^{p},$$

where r, r, . . . . I have all values between 0 and p, both inclusive, subject to the restriction  $r+s+\ldots+k-p$ .

(35.) If JI, have the meaning of \$ 10 above, prove that

$$1 = m_1 n H_1 - m_1 H_2 + m_2 H_3 + m_3 H_4 + m_4 H_5 + m_4 H_5 + m_4 H_5 + m_5 H_6 + m_5 H_6$$

$$-2^{r} \cdot 1 - {}_{n}e_{1} + {}_{n}H_{1} + {}_{n}e_{2} + {}_{n}H_{2} + {}_{n}e_{3} + {}_{n}H_{r} + \dots + (-1)^{n}{}_{n}e_{n,n}H_{n} = 0.$$

(36.) If  $x_r = x(x+1) + \dots + (x+r-1)$ , show that

$$(x+y)_r - x_r + {}_rC_1x_{r-1}y_1 + {}_rC_1x_r - y_1 + \dots + y_r.$$

(17) I will be harpest coefficient in the expansion of  $(a+b+c+d+c)^{2b}$ .

\$\$ 13-15

# EXAMPLES OF THE APPLICATION OF THE LAW OF DISTRIBUTION.

§ 14.] If we have r sets, consisting of n<sub>1</sub>, n<sub>2</sub>, . . . , n<sub>r</sub> different letters respectively, the whole number of different ways of making combinations by taking 1, 2, 3, . . . up to r of the letters at a time, but never more than one from each set, is

$$(n_1+1)(n_2+1)$$
 . . .  $(n_r+1)-1$ .

Consider the product

$$(1 + a_1 + b_1 + \dots + a_1 \text{ letters})$$
  
 $\times (1 + a_2 + b_2 + \dots + a_2 \text{ letters})$   
 $\times \dots \times (1 + a_r + b_r + \dots + a_r \text{ letters})$ 

In the distributed product there will occur every possible combination of the letters taken 1, 2, 3, . . . , r at a time, with the term 1 in addition. If we replace each letter by unity, each term in the distributed product will become unity, and the sum of these terms will exceed the whole number of combinations by unity. Hence the number required is

$$(1 + n_1) (1 + n_2) \dots (1 + n_r) - 1$$
  
=  $\sum n_1 + \sum n_2 n_2 + \dots + n_1 n_2 \dots n_r$ ,

This result might have been obtained by repeated use of § 7.

§ 15.] If we have r sets of counters, marked with the following numbers—

$$a_1, \beta_1, \ldots, \kappa_1,$$
  
 $a_2, \beta_2, \ldots, \kappa_2,$ 

the number of counters not being necessarily the same for each set, and the inscribed numbers not necessarily all different, then the number of different ways in which r counters can be drawn, one from each set, so that the sum of the inscribed numbers shall be n, is the coefficient of x\* in the distribution of the product

This theorem is an obvious result of the principles laid down in chap, iv.

Cor. 4. If in the first set there be  $a_i$  counters marked with the number  $a_i$  b, marked with  $\beta_1$ ,  $\delta c_i$ , in the second  $a_i$  marked with  $a_i$ ,  $b_i$  marked with  $\beta_1$ ,  $\delta c_i$ , the number of ways in which rcounters can be drawn so that the sum of the numbers on them is  $a_i$  is the coefficient of  $x^a$  in the distribution of

$$(a_1x^{a_1} + b_1x^{a_1} + \dots + k_2x^{a_4})$$
  
>  $(a_1x^{a_2} + b_2x^{a_3} + \dots + k_2x^{a_5})$   
×  $(a_2x^{a_2} + b_2x^{a_2} + \dots + k_2x^{a_7})$ .

Cox. 2. In a low there are a counters marked a, h marked β, &c. A counter is deaven v times, and each time replaced. The number of ways in which the sum of the drawings can amount to n is the ox-flecient of x in the distribution of

$$(ax^{\alpha} + bx^{\beta} + \dots)^{r}$$
.

## DISTRIBUTIONS AND DERANGEMENTS.

- § 16.] The variety of problems that arise in connection with the subject of the present chapter is endless, and it would be difficult within the limits of a text book to indicate all the methods that have been used in solving such of these problems as mathematicians have already discussed. The following have been selected as types of problems which are not, very readily at least reducible to the elementary cases above discussed.\*
- [17] To find the number of ways in which is different letters can be distributed among repipeon-boles, attention being paid to the order of the pipeon-boles, but not to the order of the letters in any one power bole, and no bole to contain less than one letter.
  - Let  $D_r$  denote the number in question.

<sup>.</sup> Los fueta conformation see Whitworth's Choice and Chance.

If we leave s specified holes vacant and distribute the letters among the remaining r-s holes under the conditions of the question, we should thus get  $D_r$  , distributions. Hence, if  $\mathcal{L}$ , have its usual meaning, the number of distributions when s of the holes are blank is -C. D....

Again, the whole number of distributions when none, one, two, &c., of the holes may be blank is evidently re, for we can distribute the n letters separately among the r holes in  $r^n$  ways.

Hence

$$D_r + {}_r C_1 D_{r-1} + {}_r C_2 D_{r-2} = \dots + {}_r C_{r-1} D_1 = r^n$$
 (A).

The equation (A) contains the solution of our problem, for by putting r = 2, r = 3, &c., successively, we could calculate  $D_2$ ,  $D_3$ . &c., and  $D_1$  is known, being simply 1.

We can, however, deduce an expression for  $D_r$  in terms of nand r, as follows. Writing r-1 in place of r we have

$$D_{r-1} + {}_{r-1}C_1 D_{r-2} + \dots + {}_{r-1}C_{r-2} D_1 = (r-1)^n$$
 (B).

From (A) and (B), by subtraction, remembering (18, Cor. 3) that

we derive

$$_{r}C_{s}-_{r-1}C_{s-1}=_{r-1}C_{s},$$

 $= r^n - (r - 1)^n$ 

$$D_r +_{r-1}C_1 D_{r-1} +_{r-1}C_2 D_{r-2} + \dots +_{r-1}C_{r-1} D_1$$

$$= r^a - (r-1)^a$$
(1)

From (1), putting r-1 in place of r, we derive

$$D_{r-1} + {}_{r-2}C_1 D_{r-2} + \dots + {}_{r-2}C_{r-2} D_1$$
  
=  $(r-1)^n - (r-2)^n$  (1').

From (1) and (1'), by subtraction, we derive

$$D_r + {}_{r-2}C_1D_{r-1} + {}_{r-2}C_2D_{r-2} + \dots + {}_{r-2}C_{r-2}D,$$
  
=  $r^n - 2(r-1)^n + (r-2)^n$  (2).

Treating now (2) exactly as we treated (1) we derive  $D_r + {}_{r-3}C_1 D_{r-1} + {}_{r-3}C_2 D_{r-2} + \dots + {}_{r-3}C_{r-3} D_3$ 

$$D_r +_{r-3}C_1 D_{r-1} +_{r-3}C_2 D_{3-2} + \dots +_{r-3}C_{r-2} D_3$$
  
=  $r^n - 3(r-1)^n + 3(r-2)^n - (r-3)^n$  (3),

The law of formation of the right-hand side is obvious, the coefficients being formed by the addition rule peculiar to the binomial coefficients (see chap. IV., § 11). We shall therefore finally obtain

$$\begin{split} D &= r^n \cdots_r \ell_r^r (r-1)^n +_r \ell_r^r (r-2)^n - \dots \cdot (-)^{r-1} r \ell_{r-1}^r \mathbf{I}^n, \\ &= r^n - \frac{r}{1} (r+1)^n + \frac{r(r-1)}{1-2} (r-2)^n - \dots \cdot (-)^{r-1} \frac{r}{1} \mathbf{I}^n. \end{split} \tag{4}.$$

Cor. If the order of the papen-holes be indifferent, the number of distributions is Derie. In other words, the number of partitions of a different letters into relats, no vacant lots being allowed, is Derie.

We shall discuss the closely-allied problem to find the number of r-partitions of n -that is, to find the number of ways in which n letters, all allies, may be distributed among r pigeon-holes, the order of the holes being indifferent, and no hole to be empty—when we take up the Theory of the Partition of Numbers.

\$48.) Given a series of n letters, to find in how many ways the order may be decraaged so that no one out of r assigned letters stall occupy its original position.

Let  ${}_{n}\Delta$ , denote the number in question.

The number of different derangements in which the r assigned betters do all (occupy their original places is <math>(n-r)!. Hence the number of derangements in which the r assigned letters do not all occupy their original places is n! + (n-r)!. Now, this last number is made up of r.

1st. The number of derangements in which no one of the r

letters occupas its original place, that is,  ${}_{n}\Delta_{r+}$ 

2nd. The number of derangements in which any one of the r letters occupies its original place, and no one of the remaining r = 1 does so, that  $r_{r+1} Z_{r+1} Z_{r+1}$ .

and The number of decaugements in which any two of the r letters of apy their original places, and no one of the remaining r = 2 does so, that is,  $rU_n = \Delta_{n-2}$ . And so on

Henry

$$a^{\prime} = a - i \beta^{\prime} = \Delta_{i} + {}_{i} U_{i,q} - \Delta_{i,q} + {}_{i} U_{i,q} - \Delta_{i,q} + \dots$$

$$+ {}_{i} U_{i,q,q} - \Delta_{i,q}$$
(A).

If we write in this equation n-1 for n, and r-1 for r, and subtract the new equation thus derived from (A), we deduce

$$= u - 1 \cdot l - u \Delta_{-1} \cdot l \cdot l \cdot l \cdot \Delta_{-1} \cdot l \cdot v \cdot l \cdot C_{-n} \cdot \Delta_{-1} \cdot k \cdot v \cdot v$$

$$= t \cdot v \cdot l \cdot l \cdot l \cdot v \cdot v \cdot \Delta_{1}$$

$$(1)$$

We can now treat this equation exactly as we treated equation (1) of § 16. We thus deduce

$$_{n}\Delta_{r} = n! - \frac{r}{1}(n-1)! + \frac{r(r-1)}{1 \cdot 2}(n-2)! - \dots (-)^{r}(n-r)!$$
 (2).

If we remember that (n-r)!, above, stands for the number of derangements in which the r letters all occupy their original positions, we see that, when r=n, (n-r)! must be replaced by 1. Hence

Cor. The number of derangements of a series of n letters in which no one of the original n occupies its original position is

$$n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right\}.$$
 (3).

The expression (3) may be written

 $n(\ldots, (4(3(2(1-1)+1)-1)+1)\ldots - (-1)^n) + (-1)^n$ 

Hence it may be formed as follows:—Set down 1, subtract 1; multiply by 2 and add 1; multiply by 3 and subtract 1; and so on. The function thus formed is of considerable importance in the present branch of mathematics, and has been called by Whitworth subfactoried n. He denotes it by \( \nu\). A more convenient notation would be \( n\_i\).

#### SUBSTITUTIONS.

§ 19.] Hitherto we have merely counted the permutations of a group of letters. If we direct our attention to the actual permutations, and in particular to the process by which these permutations are derived from each other, we are led to an order of ideas which forms the foundation of that important branch of modern algebra which is called the Theory of Substitutions.

Consider any two permutations, becala, beade, of the five letters a,b,c,d,e. The latter is derived from the former by replacing a by e,b by b,c by a,d by d,e by c. This process may be

represented by the operator 
$$\begin{pmatrix} ebude \\ abcde \end{pmatrix}$$
; and we may write

$$\binom{ebade}{abcde}$$
 becda – beade:

or, omitting the letters that are unaftered, and thus reducing the operator to it simplest for a,

$$\binom{var}{a\alpha} herda = harb.$$

The operator  $\binom{(m^2)}{m^2}$ , and the operation which it effects, are called a *Substitution*; and the operator is often denoted by a single capital letter S, T, Nc,

Since the number of different permutations of a group of n letter is n', it is obvious that the number of different substitutions is also n', if we include among them the identical substitution  $\binom{abcd}{abcd}$ . . . . . . . . , letter d by S or by 1), in which no letter is  $n \in \mathbb{N}$ .

We may effect two substitutions in succession upon the same permutation and represent the result by writing the two symbols representing the substitutions before the permutation in order

from right to left. Thus, if 
$$S = \left(\frac{cah}{abc}\right)$$
,  $T = \left(\frac{a}{ac}\right)$ ,

STached realid.

We may also effect the same substitution twice or three times over, and denote 88 by 82, 888 by 81, &c. Thus, 8 being as

S. a. hed Second Amend

It hould be observed that the multiplication of substitution symbols is not in a nead commutative. For example, 8 and T being a above, 8 Tached could, but Tsurbed could. If, when red cod to their simple totom, the spatial 8 and T have no better in common, they are obviously commutative. This condition, although utherient, is not necessary; for we have

$${h\cdot ah\choose al-l}{h\cdot ah\cdot a\choose ah\cdot a}\cdot ah\cdot b = e\cdot b\cdot b = {h\cdot a\cdot a\cdot a\choose a\cdot b\cdot a}\cdot ah\cdot b.$$

1.20 Since the new or of principation of n letters is fined 1 it is given but it we never the one of tration, S, the critical and led from new reports the principal day we start that. The made transfer of depetitions parabolic tall happens is called the new star substitution S. Hence we have  $S^a = 1$ , and  $S^{pa} = 1$ , where p is any positive integer.

We may define a negative index in the theory of substitutions by means of the equation  $S^{eq} \cdot S^{pa-q}$ ,  $\mu$  being the order of S, and p such that  $p\mu > q$ . From this definition we see that  $S^{pQ-q} = S^{ppa-q} = S^{pa} = 1$ . In other words,  $S^q$  and  $S^{eq}$  are inverse to each other; in particular, if

$$S = \begin{pmatrix} dabc \\ abcd \end{pmatrix}, \text{ then } S^{-1} = \begin{pmatrix} abcd \\ acdw \end{pmatrix} = \begin{pmatrix} badd \\ abcd \end{pmatrix}.$$

A set of substitutions which are such that the product of any number of them is always one of the set is called a group; and the number of distinct substitutions in the group is called the order of the group. The number of letters operated on is called the degree of the group.

It is obvious from what has been shown that all the powers of a single substitution, S, form a group whose order is the order of S.

§ 21.] A substitution such as  $\binom{hedic\hat{p}i}{ahodic\hat{p}i}$ , where each letter is replaced by the one that follows it, and the last by the first, is called a  $Cyclic\ Substitution$ , and is usually denoted by the symbol  $(abcde\hat{p}')^*$ 

The cyclic substitution (a), consisting of one letter, is an identical substitution; it may be held to mean that a passes into itself.

The cyclic substitution of two letters (ab), or what is the same thing (ba), is spoken of as a Transposition.

The effect of a cyclic substitution may be represented by writing the n letters at equal intervals round the circumference of a circle, and shifting each through 1  $\nu$ th of the circumference. Thus, or otherwise, it is obvious that the order of a cyclic substitution is equal to the number of the letters which it involves.

§ 22.] Every substitution either is cyclic or is the product of a number of independent cyclic substitutions (cycles).

Consider, for example, the substitution

<sup>.</sup> Or, of course, by (bedeja), (edejab), &c.

$$S = \begin{pmatrix} hfilegach \\ abcdefigh \end{pmatrix}$$
.

This replaces a by b, b by f, f by a; these together constitute the cyclic substitution (abf). Next, c is replaced by d, and d by c; this is equivalent to the cycle (cd). Again, c is replaced by g, and g by c; this gives the cycle (cd). Finally, b is unaltered. Hence we have the following decomposition of the substitution S into cycles—

$$S \supseteq (abf)(cd)(cg)(h).$$

The decomposition is obviously unique; and the reasoning by which we have arrived at it is perfectly general. It should be noticed that, since the cycles are independent, that is, have no betters in common, they are commutative, and it is indifferent in what order we write them.

\$23.] Every cyclic substitution of n letters can be decomposed into the product of n A transpositions.

For example, we have (abcd) - (ab)(bc)(cd); and the process is general.

Vor. Every substitution can be decomposed into n-r transpositions, where n is the number of letters which it displaces, and r the number of its proper cycles.

Thus.

$$\begin{pmatrix} hfdegach\\ abcdefigh \end{pmatrix} = (ahf)/cd)(eg)(h),$$

$$=(ah_{\mathbb{R}}(ht)(cd)\otimes g),$$

This decomposition into transpositions is not unique, as will be seen presently, but the above gives the minimum number.

§ 24.] The following properties of a product of two transpositions are of fundamental importance.

1. The product of two transpositions which have two letters in common is an identical substitution.

This is obvious from the meaning of (ab)

11. In the product of two transpositions, TT, which have a letter in common, T may be placed first, provided we replace the common letter in T by the other letter in T.

For we have 
$$(ab)(bc) \equiv \begin{pmatrix} bcd \\ abc \end{pmatrix}$$
,  $(bc)(ac) \equiv \begin{pmatrix} bcd \\ abc \end{pmatrix}$ , erefore  $(ab)(bc) = (bc)(ac)$ .

therefore (ab)(bc) = (br)(ac). Cor. 1. (cf)(af) = (ae)(cf).

Cor. 2. (ae)(af) = (af)(cf).

III. If two transpositions, T and T', have no letter in common. they are commutative.

This is a mere particular case of a remark already made regarding two independent substitutions.

§ 25.] The decomposition of a given substitution into transpositions is not unique.

For we can always introduce a pair of factors (ab)(ab), and then commutate one or both of them with the others, in accordance with the rules of § 24.

In this way we always increase the number of transpositions by an even number. In fact, we can prove the following important theorem—

The number of the transpositions which represent a given substitution is always odd or always even.

We may prove this by reducing the product of transpositions to a standard form as follows—

Select any one of the letters involved, say a; take the last transposition, T, on the right that involves a, and proceed to commutate this transposition successively with those to the left of it. So long as we come across transpositions that have no letter in common with T, neither T nor the others are affected. If we come to one that has a letter in common with T which is not a, we see (§ 24, IL, Cor. 1) that the a in T remains, the other letter being altered, and the transposition passed over remains unaltered. If we come to a transposition that has a, and a only. in common with T, by § 24, 1L, Cor. 2, T passes to the left unaltered, and the transposition passed over loses its a. Lastly, if we come to a transposition that has both a and its other letter in common with T, then both it and T may be removed. If this last happen, we must now take that remaining transposition containing a which is farthest to the right, and proceed as before.

The result of this process, so far as a is concerned, will be, either that all the transpositions containing a will have disappeared, or that some even number (including 0) will have done so, and one only, say (ab), will remain on the extreme left.

Consider now b. If among the remaining factors b does not occur, then we have obtained a cycle (ab) of the substitution;

and we now proceed to consider some other letter.

If, however, b does occur again, we take the factor farthest to the right in which it occurs, and commutate as before; the result being, either that all the transpositions (even in number) containing b disappear, or that an even number of them do, and we are left with, say (bc) in the second place. We now deal with c in like manner; and obtain in the third place, say (cd). This goes on until all the letters are exhausted, or until we come to a letter, say f, that disappears from the factors not yet finally arranged. We thus arrive at a product (ab)(bc)(cd)(dc)(cf) on the left,

Now 
$$(ab)(bc)(cd)(dc)(cf) \equiv \begin{pmatrix} bcdefa\\ abcdef \end{pmatrix}$$

 $\stackrel{.}{=} (abcdef).$ 

We have, in fact, arrived at one of the independent cycles of the substitution. If we now take any other letter that occurs in one of the remaining substitutions on the right, we shall in like manner arrive at the cycle to which it belongs, after losing an even number, if any, of the transpositions; and so on, until all the letters are exhausted, and all the cycles arrived at. Since the whole number of transpositions lost is even, the truth of the theorem is now obvious; and our proof furnishes a method for reducing to the minimum number of transpositions.

It appears, therefore, that we may divide all the substitutions of a set of n betters into two classes—namely, even substitutions, which are equivalent to an even number of transpositions, and odd substitutions, which are equivalent to an odd number of transpositions.

Vac. 1—If n be the number of letters aftered by a substitution, r the number of its cycles, and 2s an arbitrary even integer, the number of factors in an equivalent product of transpositions is u = r + 2s.

Cor. 2. The number of the even is equal to the number of the add substitutions of a set of n letters.

For any one transposition, applied in succession to all the different odd substitutions, will give as many even substitutions, all different. Hence there are at least as many even as there are at least as many even as the least as many odd as there are even. Hence the number of the even is equal to the number of the odd substitutions.

Cor. 3. A cyclic substitution is even or odd according as the number of the letters which it involves is odd or even.

For example,  $(abc) \equiv (ab)(bc)$  is even.

Cor. 4. The product of any number of substitutions is even or advantaged according as the number of odd factors is even or odd. In particular, any power what even of an even substitution, and any even nower of ann substitutions what even form even substitutions.

Cor. 5. All the even substitutions of a set of n letters form a

group whose order is u!/2.

§ 26.] If we select arbitrarily any one, say P, of the n! permutations of a set of n letters, and call it an even permutation, then we can divide all the n! permutations into two classes—1st, n!/2 even permutations, derived by applying to P the n! 2 even substitutions; [2nd, n!/2 odd permutations, derived by applying to P all the n!/2 odd substitutions.

The student who is familiar with the theory of determinants will observe that the above is precisely the classification of the permutations of the indices (or umbrae) which is adopted in

defining the signs of the terms in a determinant,

It is farther obvious, from the definitions given in chap, i.e., 5, co, that symmetric functions of a set of a certables are unaltered in value by any substitution what ever of the certables; or, as the phrase is, they are said to "admit any substitution whatever." Alternating functions, on the other hand, admit only even substitutions of their variables, the result of any old substitution being to alter their sign without otherwise affecting their value.

§ 27.] The limits of the present work will not permit us to enter farther into the Theory of Substitutions, or to discuss its applications to the Theory of Equations. The reader who desires

to pursue this subject farther will find information in the following works: Serret, Cours d'Algèbre Supérieure (Paris, 1879); Jordan, Traité des Substitutions (Paris, 1870); Netto, Substitutionen-theorie (Leipzig, 1882); Burnside, Theory of Groups (Cambridge, 1897).

### Exercises 111.

(1.) There are 10 count is in a box mathed 1, 2, . . . . , 10 respectively. Three drawns are made, the counter drawn loing topliced each time. In how many ways can the sum of the numbers drawn amount—1st, to 9 exactly; 2nd, to 9 at lost?

(2.) Out of the integers 1, 2, 3, . . . , 10 how many pairs can be selected

so that their sum shall be even?

(3.) How many different throws can be made with a dice?

(1.) In how many ways can 5 black, 5 white, 5 blue balls be equally distributed among three bags, the order of the bags to be attended to?

(5.) A selection of c thines is to be made partly from a group of a, the rest from a group of b. Prove that the number of ways in which such a set

can be made will never be greater than when the number of things taken from the group of a is next less than (a+1)(c+1)(c+b+2).

(6.) In how many ways can p + 1s and n > 1s be placed in a row so that no

(6.) In now many ways can p + 's and n - s be placed in a row so that he two - 's come together?'

(7.) In the Merse signalling system how many signals can be made without execuling 5 movements?

(8) In how many ways can 3 pairs of subscribers be set to talk in a telephone exchange having a subscribers?

(9.) There are 3 colears, and m fails of each. In how many ways can they be arrained in 3 bags each contamine m, the order of the bags to be attended to?

(10.) If of p+q+r things p be alike, q alike, and r different, the total number of combinations will be (p+1)(q+1) 2r - 1.

(11.) In how many ways can 2n than is be divided into n pairs?

(12.) The number of combinations of on things (n of which are alike), taken n at a time, is the coefficient of e<sup>n</sup> in (1 ⋅ x)<sup>2n</sup> (1 ⋅ x).

(13.) N boat clubs have a, b, c, 1, 1, . . . , 1 boats each. In how many ways can the beat be arranged s bp + to the restriction that the 1st boat of any club is to be always above its 2nd, its 2nd always above its 3rd, &c.?

(14.) If there is p through of the seriety q of another, r of another, &c., the number of commutations of the p ∈ q ∈ r ∈ ∈ through taken k at a time, is the coefficient of x in (1 − x<sup>2</sup>) (1 − x) (1 − x) (1 − x).

(15a) In how many way con an arrangement of n things in a row be deranged so that 1-st, each thing is moved one place; 2nd, no thing more than one place?

(16.) Given a things arranged in succession, the number of sets of 3

which can be formed under the condition that no set shall contain two things which were formerly contiguous is (n-2)(n-3)(n-1), the order in the the sets to be attended to.

(17.) In how many ways can m white and n black balls be arranged in a row so that there shall be 2r-1 contacts between white and black balls?

row so that there shall be 27 - 1 contacts between white and black balls?

(18.) In how many ways can an examiner give 30 marks to 8 que-tions without giving less than 2 to any one question?

(19.) The number of ways in which n letters can be arranged in r page onholes, the order of the holes and of the letters in each hole to be attended to and ennty holes admitted, is r(r+1)(r+2), r(r+n-1).

(20.) The same as last, no empty holes being admitted, n!(n-1)!(n-r)!

(20.) The same as last, no empty holes being admitted, n! (n − 1)!, (n − r)!
(r − 1)!.

(21.) The same as last, the order of the holes not being attended to, n!(n-1)!/(n-r)! r!(r-1)!.

(22.) The number of ways in which n letters, all alike, can be distributed into r pigeon-holes, the order of the holes to be attended to, empty holes to be excluded, is  $_{n-1}C_{r-1}$ .

(23.) Same as last, empty holes being admitted, n+r 1 Cr-1.

(24.) Same as last, no hole to contain less than q letters,  $_{n-1-r-l-1}C_{r-1}$ .

(25.) The number of ways of deranging a row of n letters so that no letter may be followed by the letter which originally followed it is n; + (n − 1);

(26.) The number of ways of deranging m+n terms so that m are displaced and n not displaced is  $(m+n)! m_0^n m! n!$ .

(27.) The number of ways in which r different things can be distributed among n+p persons so that certain n of these persons may each have one at least is

$$S_r = (n+p)^r - n(n+p-1)^r + \frac{n(n-1)}{2!}(n+p-2)^r - \dots$$

Hence prove that

C. 11.

$$S_1 = S_2 = \dots = S_{n-1} = 0, \quad S_n = n!, \quad S_{n+1} = \left(\frac{n}{2} + p\right)(n+1)!.$$

(1) 01 - 0 - an - an -

(28.) Fifteen school-girls walk out arranged in threes. How many times can they go out so that no two are twice together? (See Cayley's II other, vol. 1., p. 481.)

# Exercises IV.

# Topological.

- (1.) The number of sides of a complete n-point is \(\frac{1}{2}n\)(n-1), and the number of vertices of a complete n-side is the same.
- (2.) The number of triangles that can be formed with  $2n \lim_{n \to \infty} of \ \text{lengths}$  1, 2, . . . , 2n is n (n-1)(4n-5)/6.
  - (3.) There are n points in a plane, no three of which are collinear. How

<sup>\*</sup> Exercises 19-25 are solved in Whitworth's Choice and Chance and Chance

many closed r sided figures can be found by joining the points by straight him  $\tilde{r}$ 

(1) If m p rats in one staight less be poined to n points in another in every possible way, show that, exclusive of the m+n given points, there are mn m 10 to 1.2 points of intersection.

(5.) On three straight lines, A, B, C, are taken I, m, n points respectively, no one of which is a point or intersection. Show that the number of triangles which can be formed by taking three of the I+m+n points is \( \frac{1}{m} + n \) \( \frac{1}{n} + n \).

(l+m) = mn - nl - lm.

(c) There are ny different a plane, no three of which are collinear and no four concedes. Laronch every two of the points is drawn a straight line and threach every three a circle. Assuming each straight line to cut each circle in two distinct points, first the number of the intersections of straight lines with circles.

(7.) In a convex polygon of n sides the number of exterior intersections of diagonal as  $\beta$  in (n-3) (n-1) (n-5), and the number of interior intersections

18 Junior 11 (n. 2) (n. 3)

Fig. Three are separate in space, no three of which are collinear, and no four e-placear. A plane is due in throw he every three. Find, 1st, the number of 1 that this soft is treatment of the seplanes; 2nd, the number of three mass of interactions which pass through one of the given n points; 3nd, the number of direct points in the case of the constant points.

(a) Out of n straight lines 1, 2, . . . , n inches long respectively, four can be observed to form a principle quadrilateral in  $(2n(n-2)(2n-5)-3+3(-1)^n)/18$ 

Walte.

(10.) Shew that notice that more two of which are parallel and no three concurrent, divide a piane into york (n=2) regions. Hence, or otherwise, show that no panes through the centre of a sphere, no three of which are coard, could at surface into not no 2 regions.

(14) Show that two pends of ithat the lines lying in the same plane, one containing in the chart, divide the plane into min 2m+2m+1 regions, it has a suppresent that no two or the lines are parallel or connection.

(12.) If any number of closed curves be drawn in a plane each cutting all the others, near to, be the number of points through which r curves pass, the number 4 drawn to see clared sound by the places is

## CHAPTER XXIV.

# General Theory of Inequalities. Maxima and Minima.

§ 1.] The subject of the present chapter is of importance in many branches of algebra. We have already met with special cases of inequalities in the theory of Ratio and in the discussion of the Variation of Quadratic Functions of a single variable; and much of what follows is essential as a foundation for the theory of Limits, and for the closely allied theory of Infinite Series. In fact, the theory of inequalities forms the best introduction to the theory of infinite series, and, for that reason, ought to be set as

much as possible on an independent basis.

- § 2.) We are here concerned with real algebraical quantity merely. As we have already explained, no comparison of complex numbers as to relative magnitude in the ordinary sense can be made, because any such number is expressed in terms of two absolutely heterogeneous units. Strictly speaking, there is a similar difficulty in comparing real algebraical quantities which have not the same sign; but this difficulty is met (see chap, XIII., § 1) by an extension of the notion of inequality. It will be remembered that a is defined to be algebraically greater or less than b according as the reduced value of a b is positive or negative. An immediate consequence of this definition is that a positive quantity increases algebraically as it increases numerically, but a negative quantity decreases algebraically as it increases numerically. The neglect of this consideration is a fruitful source of mistakes in the theory of inequalities.
  - § 3.] From one point of view the theory of inequalities runs

parallel to the theory of conditional equations. In fact, the approximate numerical solution of equations depends, as we have seen, on the establishment of a series of inequalities.

The following theorems will bring out the analogies between the two theories, and at the same time indicate the nature of the restrictions that arise owing to the fact that the two sides of an inequality cannot, like the two sides of an equation, be interchanged without altering its nature. For the sake of brevity, we shall, for the most part, write the inequalities so that the greater quantity is on the left, and the sign > alone appears. The modifications necessary when the other sign appears are in all cases obvious.

I. If 
$$P > 0$$
,  $Q > R$ ,  $R > S$ , then  $P > S$ .

 $Proof. \rightarrow (P-Q) + (Q-R) + (R-S) \otimes P-S$ , hence, since P-Q, Q-R, R-S are all positive, P-S is positive, that is, P>S.

11. If P > Q, then  $P \pm R > Q \pm R$ . For  $(P + R) = (Q + R) \equiv P - Q$ ; he

For  $(P \pm R) - (Q \pm R) \equiv P - Q$ ; hence the sign of the former quantity is the same as the sign of the latter.

Cor. 1. 
$$IPP + Q > R + S$$
, then

$$P + Q - R > S$$
,  $-R - S > -P - Q$ ,  $-P - Q < -R - S$ .

It thus appears that we may transfer a term from one side of an inequality to another, provided we change its sign; and we may change the signs of all the terms on both sides of an inequality, provided we receive the symbol of inequality.

Cor. 2. Every inequality may be reduced to one or other of the forms P > 0 or P = 0.

In other words, every problem of inequality may be reduced to the determination of the sign of a certain quantity.

III. If 
$$P_1 \sim Q_1$$
,  $P_1 = Q_2$ , . . . ,  $P_n \sim Q_{n_1}$ 

$$\begin{array}{lll} the u & = P_1 + P_1 + \dots + P_n > Q_1 + Q_1 + \dots + Q_n; \\ \text{for} & = (P_1 + P_2 + \dots + P_n) - (Q_1 + Q_1 + \dots + Q_n). \end{array}$$

$$-(P_1-Q_1)+(P_2-Q_2)+\dots+(P_n-Q_n)$$

" whence the theorem follows

It should be noticed that it does not follow that, if  $P_1 > Q_1$ ,  $P_2 > Q_2$ , then  $P_1 + P_2 > Q_1 - Q_2$ .

<sup>.</sup> See, for example, the proof that every equation has a root.

IV. If P > Q, then PR > QR, and P/R > Q/R, provided R be positive; but PR < QR, P/R < Q/R, if R be negative.

For (P-Q)R and (P-Q)/R have both the same sign as P-Q if R be positive, and both the opposite sign if R be negative.

Cor. 1. If P > QR, and R > S, then P > QS, provided Q be positive.

Cor. 2. Every fractional inequality can be integralised.

For example, if P/Q > R/S, then, provided QS be positive, we have, after multiplying by QS, PS > QR; but, if QS be negative, PS < QR.

If there be any doubt about the sign of QS, then we may multiply by  $Q^2S^2$ , which is certainly positive, and we have  $QPS^2 > Q^2RS$ .

V. If  $P_1 > Q_1$ ,  $P_2 > Q_2$ , . . . ,  $P_n > Q_n$ , and all the quantities be positive, then

$$P_1P_2$$
 . . .  $P_n > -Q_1Q_2$  . . .  $Q_n$ .

For  $P_1P_2P_3 \dots P_n > Q_1P_2P_3 \dots P_n$ , since  $P_1 > Q_1$  and  $P_2P_3 \dots P_n$  is positive;

$$>Q_1Q_2P_3\dots P_n$$

since  $P_2 > Q_2$  and  $Q_1 P_3$  . . .  $P_n$  is positive; and so on. Hence, finally, we have

$$P_1P_2 \dots P_n > Q_1Q_2 \dots Q_n$$

Cor. 1. If P > Q, and both be positive, then  $P^n > Q^n$ , n being any positive integer.

Cor. 2. If P > Q, and both be positive, then  $P^{vw} > Q^{vw}$ , where  $Q^{vw}$  being any positive integer, and the real positive value of the  $P^{vw}$  the  $P^{vw}$  the  $P^{vw}$  the  $P^{vw}$  the  $P^{vw}$  to  $P^{vw}$  the  $P^{vw}$  to  $P^{vw}$  the  $P^{vw}$  to  $P^{vw}$  the  $P^{vw}$  the  $P^{vw}$  to  $P^{vw}$  the  $P^{vw}$  the  $P^{vw}$  to  $P^{vw}$  the  $P^{vw}$  the

For, if  $P^{n_0} \equiv Q^{n_0}$ , then, since both are real and positive,  $(P^{n_0})^n \equiv (Q^{n_0})^n$ , by Cor. 1; that is,  $P \equiv Q$ , which contradicts our hypothesis.

Cor. 3. If P>Q, both being positive, and n be any positive quantity, then  $P^{-n}< Q^{-n}$ , where, if the indices are fractional, there is the usual understanding as to the root to be taken.

Remark.- The necessity for the restrictions regarding the

sign of the members of the inequalities in the present theorem will appear if we consider that, although -2 > -3, and -3 > -4, yet it is not true that (-2)(-3)(-3)(-4).

These restrictions might be removed in certain cases; for example, it follows from -3>-4 that  $(-3)^{5}(-4)^{5}$ , in other words, that -27>-61; but the importance of such particular cases does not justify their statement at length.

Cor. 4. An inequality may be rationalised if due attention be paid to the above-mentioned restrictions regarding sign

§ 4.] By means of the theorems just stated and the help of the fundamental principle that the product of two real quantities is positive or negative according as these quantities have the same or opposite sign, and, in particular, that the square of any real quantity is positive, we can solve a great many questions regarding inequalities.

The following are some examples of the direct investigation of inequalities; the first four are chosen to illustrate the parallelism and mutual connection between inequalities and equations:—

Example 1. Under what circumstances is

(3x-1)/(x-2)+(2x-3)/(x-5) > or < 5?

1-t. Let us suppose that x does not lie between 2 and 5, and is not equal to either of these values. Then  $(x-2)\,(x-5)$  is positive, and we may multiply by this factor without reversing the signs of inequality.

Hence F=(3x-1)/(x-2)+(2x-3)/(x-5)><5, according as

(3x-1)(x-5) + (2x-3)(x-2) > < 5(x-2)(x-5), according as  $5x^2 - 23x + 11 > < 5x^3 - 35x + 50,$  according as 12x > < 33.

according as  $x > < 3\frac{1}{4}$ . Under our present supposition, x cannot have the value  $3\frac{1}{4}$ ; but we con-

clude from the above that if x > 5, T > 5, and if x > 2, F > 5, and. Suppose 2 < x > 5. In this case (x - 2)(x - 5) is negative, and we

must reverse all the signs of inequality after multiplying by it. We therefore infer that if  $2 < x < 3\frac{1}{4}$ , F < 5, and if  $3\frac{1}{4} < x < 5$ , then

We therefore infer that if  $2 \leqslant x \leqslant 3\frac{1}{4}$ ,  $F \leqslant 5$ , and if  $3\frac{1}{4} \leqslant x \leqslant 5$ , the  $F \leqslant 5$ .

The student should observe that, as x varies from  $-\tau$  to +x, the sign of the inequality is three reversed, namely, when x=2, when x=3, and when x=3, as the fit that but reversals occur because F changes sign by passage through an induite value; the second reversal occurs because F passes

through the value 5. The student should draw the graph of the func-

Example 2. Under what eigenmetances is

$$F \equiv (3x - 4)/(x - 2) > < 1?$$

Multiplying by the positive quantity  $(x-2)^2$ , we have

$$(3x-4)/(x-2) > < 1$$
,

according as 
$$(3x-4)(x-2) > < (x-2)^2$$
,

according as 
$$\{(3x-4)-(x-2)\}(x-2)><0$$
,

according as 
$$2(x-1)(x-2) > < 0$$
,  
Hence  $F > 1$ , if  $x < 1$  or  $> 2$ .

$$F < 1$$
, if  $1 < x < 2$ .

Example 3. Under what circumstances is  $x^3 + 25x > < 8x^2 + 26$ ?

$$x^3 + 25x > < 8x^2 + 26$$
,

according as 
$$x^3 - 8x^2 + 25x - 26 > < 0$$
,

according as 
$$(x-2)(x^2-6x+13) > <0,$$
  
according as  $(x-2)((x-3)^2+4) > <0.$ 

Now  $(x-3)^2+4$  is positive for all real values of x: hence

$$x^3 + 25x > < 8x^2 + 26$$
.

according as

$$x > < 2$$
.

Example 4. If the positive values of the square roots be taken in all cases, is

$$\sqrt{(2x+1)} + \sqrt{(x-1)} > \sqrt{(3x)}$$
?

Owing to the restriction as to sign, we may square without danger of reversing the inequality. Hence

$$(2x+1) + (x-1) > (3x),$$
  
according as  $2x+1+x-1+2$ ,  $(2x+1)(x-1) > (3x),$ 

according as

that is,

$$2\sqrt{(2x+1)(x-1)} > < 0$$
,  $2\sqrt{(2x+1)(x-1)} > < 0$ .

Now, provided x is such that the value of  $\sqrt{(2r+1)(x-1)}$  is real, that is, provided x>1,

$$2\sqrt{\{(2x+1)(x-1)\}} > 0$$
,

therefore  $\sqrt{(2x+1)} + \sqrt{(x-1)} > \sqrt{(3x)}$ , if x > 1.

Negative values of x less than  $-\frac{1}{2}$  would also make  $\sqrt{(2x+1)(x-1)}$  real; but such values would make  $\sqrt{(2x+1),\sqrt{(x-1)}}$ , and  $\sqrt{(3x)}$  imaginary, and, in that case, the original inequality would be meaningless.

Example 5. If x, y, z... be n real quantities  $(n-1) \sum_{i} e^{iz} \cdot 2\sum_{i} xy$ . Since all the quantities are real,  $\sum_{i} (x-y)^2 < 0$ .

Since all the quantities are real,  $\Sigma(x-y) \geq 0$ . Hence, since x will appear once along with each of the remaining n-1 letters, and the same is true of  $y_1 \geq \ldots$ , we have

$$(n-1)\sum_{x} x^2 - 2\sum_{x} y < 0,$$
  
 $(n-1)\sum_{x} x^2 < 2\sum_{x} y,$ 

The graphical study of inequalities involving only one variable will be found to be a good exercise.

In the case where x=y=z=..., we have  $\Sigma x^2=nx^2$ ,  $2\Sigma xy=2\cdot C\cdot x^2$  $= n(n-1)x^2$ , so that the inequality just becomes an equality.

When n=2, we have the theorem

$$x^2 + y^2 < 2xy$$
:

or, if we put  $x = \sqrt{a}$ ,  $y = \sqrt{b}$ , a and b being real and positive.

$$a+b < 2\sqrt{(ab)}$$
,

a theorem already established, of which the preceding may be regarded as a generalisation. A more important generalisation of another kind will be given presently.

Example 6. If x, y, z, ... be n real positive quantities, and p and q any two real quantities having the same sign, then

$$x^{p+q} + y^{p+q} < x^p y^q + x^q y^p$$
,  
 $n \Sigma x^{p+q} < \Sigma x^p \Sigma x^q$ .

We have seen that  $x^p - y^p$  and  $x^q - y^q$  will both have the same sign as x-y, or both opposite signs, according as p and q are both positive or both negative. Hence, in either case,  $(x^p - y^p)(x^q - y^q)$  has the positive sign. Therefore

$$(x^p - y^p)(x^q - y^q) < 0$$

whence

$$x^{p+q} + y^{p+q} \lessdot x^p y^q + x^q y^p$$
.

If we write down the "C" inequalities like the last, obtained by taking every possible pair of the n quantities x, y, z, . . ., and add, we obtain the following result -

$$(n-1) \sum_{x} p+q + \epsilon \sum_{x} p_{x} q_{x}$$

If we now add  $\sum rp+q$  to both sides, we deduce

 $n \sum_{x} p + q + \sum_{x} p \sum_{x} q$ N.P. —If p and q have opposite signs, then

These theorems contain a good many others as particular cases. For example, if we put a = -n, we deduce

 $\sum x^p \sum x^{-p} \prec n^2$ , which, when n=3, p=1 eives

$$(x + y + z) (1/x + 1/y + 1/z) < 9;$$

whence

$$(x + y + z) (yz + zx + xy) + 9xyz$$
;

and so on.

1. xample 7. If x, y, z be real and not all equal, then  $\Sigma x^z > < 3xyz$ , according as 1x - - 0

Lor  $\Sigma x^1 - 3xyz = \Sigma x (\Sigma x^2 - \Sigma xy),$ 

$$x^{r} = 3xyz = 2x(2x^{r} - 2xy)_{r}$$
  
=  $42x2(x - y)^{2}$ .

Hence the theorem, since  $\Sigma (x - y)^2$  is essentially positive.

Example 8 To show that 
$$\frac{1}{\sqrt{(2n+1)}} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{(2n-1)}{2n+1} > \frac{2(n+1)}{2n+1} \cdot \frac{1}{2}$$

where a is any residive interest.

whence

similarly

(n)'.

From the inequality a + b > 2 (ab) we deduce

$$(2n-1)+(2n+1)>2$$
,  ${(2n-1)(2n+1)}$ ;

$$(2n-1)/2n < \sqrt{\{(2n-1)/(2n+1)\}}$$
 (1);  
 $(2n-3)/2(n-1) < \sqrt{\{(2n-3)/(2n-1)\}}$  (2);

$$5/2, 3 < \sqrt{5.7}$$
 (n = 2)

$$5/2$$
,  $3 < \sqrt{5,7}$   $(n-2)$ ;  
 $3/2$ ,  $2 < \sqrt{3/5}$   $(n-1)$ ;

1/2,  $1 < \sqrt{\{1/3\}}$ Multiplying these inequalities together, we get

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)} < \frac{1}{\sqrt{(2n+1)}}$$
 (A).

Again, that is.

Hence

$$n + (n+1) > 2\sqrt{n(n+1)},$$
  
 $2n+1 > 2\sqrt{n(n+1)}.$ 

Hence we have the following inequalities-

$$(2n+1)/2n > \sqrt{(n+1)/n},$$
 (1)',  
 $(2n-1)/2(n-1) > \sqrt{(n/(n-1))},$  (2)',

$$7/2 \cdot 3 > \sqrt{4/3}$$
  $(n-2)',$   $5/2 \cdot 2 > \sqrt{3/2}$   $(n-1)',$ 

Multiplying these 
$$n$$
 inequalities together, we get 
$$\frac{1 \cdot 3 \cdot 5 \cdot \cdot \cdot \cdot (2n+1)}{2n} > \sqrt{(n+1)}.$$

$$\frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot 2n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} > \frac{\sqrt{(n+1)}}{2n+1}$$
(B).

(A) and (B) together establish the theorem in question.

(a) and (b) together establish the theorem in question. Since  $\sqrt{(n+1)/(2n+1)} > \sqrt{(n+1)/(2n+2)} > 1/2\sqrt{(n+1)}$ , we may state the above theorem more succinctly thus.

$$\frac{1}{\sqrt{(2n+1)}} > \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot \dots \cdot 2n} > \frac{1}{2\sqrt{(n+1)}}.$$

#### DERIVED THEOREMS.

§ 5.] We now proceed to prove several theorems regarding inequality which are important for their own sake, and will be of use to us in following chapters.

If  $b_1, b_2, \dots, b_n$  be all positive, the fraction  $(a_1 + a_2 + \dots + a_n)!$  $(b_1 + b_2 + \dots + b_n)$  is not less than the least, and not greater than the greatest, of the n-fractions  $a_1 b_1, a_2 b_2, \dots, a_n b_n$ .

Let f be the least, and f' the greatest of the n fractions, then

$$a_1/b_1 \not \triangleleft f$$
,  $a_2 b_2 \not \triangleleft f$ , . . . ,  $a_n b_n \not \triangleleft f$ .

Hence, since  $b_1, b_2, \dots, b_n$  are all positive.

$$a_1 \triangleleft fb_1, \ a_1 \triangleleft fb_2, \dots, a_n \triangleleft fb_n$$

Adding, we have

(
$$a_1 + a_2 + \ldots + a_n$$
)  $\Leftrightarrow f(b_1 + b_2 + \ldots + b_n)$ ;

whence

$$(a_1 + a_2 + \dots + a_n) (b_1 + b_2 + \dots + b_n) \lessdot f$$
.

In like manner, it may be shown that

$$(a_1 + a_2 + \ldots + a_n)/(b_1 + b_2 + \ldots + b_n) > f'.$$

Remark.—This theorem is only one among many of the same kind\*. The reader will find no difficulty in demonstrating the following:—

If  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$  be as before, and  $b_1, b_2, \dots, b_n$  be a positive quantities, then  $\Sigma b_1 a_n | \Sigma b_1 b_2$  is not less than the least, and not greater than the greatest, among the n fractions  $a_1 | b_1, a_2 | b_2$ , . . .  $a_n | b_n$ .

If  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n, l_1, l_2, \dots, l_n$  be all positive, then  $\|\Sigma_l(a_l^m) \Sigma_l(b_l^n)^{lon}\|$  and  $\|a_1a_1 \dots a_n\|_{l_1}b_2 \dots b_n\|_{l_1}^{l_2}$  are, each of them, not less than the least, and not greater than the greatest, among the n fractions  $a_1b_1, a_2b_1, \dots, a_nb_n$ .

Example, to prove that

$$\frac{1}{2} < \sqrt[n]{\frac{(1 \cdot 3) \cdot (2n-1)}{(2 \cdot 4) \cdot (2n-1)}} < 1.$$

Since the fractions 1/2, 3/4, . . . (2n-1)/2n are obviously in ascending order of magnitude, we have, in the second part of the last of the theorems just stated.

$$\frac{1}{2} < \sqrt[n]{\left\{\frac{1 \cdot 3}{2 \cdot 1} \cdot \frac{(2n-1)}{1 \cdot 1 \cdot 2n}\right\}} < \frac{2n-1}{2n}.$$

Now, (2n-1)(2n-1-1, 2n<1), hence the theorem follows; and it holds, be it observed, however great n may be.

§ 6.] If x, p, q be all positive, and p and q be integers, then  $(x^p-1)^p > <(x^q-1), q$  according as p><q.

Since p and q are positive,

$$(x^p - 1) p > < (x^q - 1)/q_*$$
  
 $q(x^p - 1) \cdots p(x^q - 1)_*$ 

according as

according as

$$(x-1)$$
 { $q(x^{p-1}+x^{p-2}+\ldots+1)-p(x^{d-1}+x^{d-2}+\ldots+1)$ } > < 0. If  $p>q$ , we have

$$\Xi(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+1)-p(x^{q-1}+x^{q-2}+\ldots+1)\},$$

$$\Xi(x-1)\{q(x^{p-1}+x^{p-2}+\ldots+x^q)-(p-q)(x^{q-1}+x^{q-2}+\ldots+1)\}.$$

Now, if x > 1.

$$x^{p-1} + x^{p-2} + \dots + x^q > (p-q) x^q;$$

$$x^{q-1} + x^{q-2} + \dots + 1 < qx^{q-1};$$
 therefore.

$$X > (x-1) \{q (p-q) x^{q} - (p-q) q x^{q-1} \},$$

$$> q (p-q) x^{q-1} (x-1)^{2},$$

Again, if x < 1,

$$x^{p-1} + x^{p-2} + \dots + x^{q} < (p-q)x^{q};$$
  
 $x^{q-1} + x^{q-2} + \dots + 1 > qx^{q-1};$ 

but, since x-1 is now negative, the rest of the above reasoning remains as before.

Hence, in both cases,

that is, if p < a.

$$(x^p-1)/p > (x^q-1)/q$$

By the same reasoning, if q > p,

$$(x^q - 1)/q > (x^p - 1)/p$$

 $(x^p - 1)/p < (x^n - 1)/q$ . § 7.] If x be positive, and  $\pm 1$ , then

$$mx^{m-1}(x-1) > x^m - 1 > m(x-1),$$

unless m lie between 0 and +1, in which case

$$mx^{m-1}(x-1) < x^{m} - 1 < m(x-1).$$

From § 6, we have

$$(\xi^p - 1) > \langle (p/q) (\xi^q - 1)$$
 (1),

according as p > < q, where  $\xi$  is any positive quantity +1, and p and q positive integers. In (1) we may put  $x^{eq}$  for  $\xi$ , where x is any positive quantity +1 (the real positive value of the qth root to be taken), and we may put m for p/q, where m is any positive commensurable quantity. (1) then becomes

$$x^m - 1 > < m(x - 1)$$
 (2),

according as m > -1, which is part of the theorem to be established.

In (2) we may replace x by 1/x, where x is any positive quantity #1, and the inequality will still hold.

Hence 
$$(1/x)^m - 1 > < m(1/x - 1)$$
 (3),

according as

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m > < 1If we multiply (3) by  $-x^m$ , we deduce

$$x^m - 1 < > mx^{m-1}(x-1),$$

 $m r^{m-1}(x-1) > < x^m - 1$ . that is. m > < 1.

according as We have thus established the theorem for positive values

of 20 Next, let m := n where n is any positive commensurable

quantity. Then  $x^{-n} = 1 > < (-n)(x-1).$ 

$$x = 1 > <(-n)(x - 1),$$
  
 $1 - x^n > < -nx^n(x - 1).$ 

according as according as

$$x^{n}-1 < > nx^{n}(x-1),$$
  
 $nx^{n+1} = nx^{n} > < x^{n}-1.$ 

Add  $x^{n+1} = x^n$  to both sides, and we see that

$$x^{-n}-1><(-n)(x-1),$$

according as  $(n+1)x^n(x-1) > < x^{n+1}-1.$ 

Now, since n is positive, n+1>1, therefore, by what we have already proved,

$$(n+1) x^n (x-1) > x^{n+1} - 1$$
.

Hence

$$x^{-n} = 1 > (-n)(x-1)$$
 (4).

In (4) we may write 1, x for x; and then we have  

$$(1, x)^{-n} - 1 \ge (-n)(1, x + 1).$$

If we multiply by  $-x^{-n}$ , this last inequality becomes

$$x^{-n} = 1 < (-n) x^{-n-1} (x-1),$$

 $(-n) x^{-n-1} (x-1) > x^{-n} - 1.$ that is.

Hence, if m be negative,  $m x^{m-1}(x-1) > x^m-1 > m(x-1)$ :

which completes the demonstration.

$$mx^{m-1}(x-y) \ge x^m - y^m \ge my^{m-1}(x-y)$$

Cor. If x and y be any two unequal positive quantities, we may replace x in the above theorem by x/y. On multiplying throughout by  $y^m$ , we thus deduce the following—

If x and y be positive and unequal, then

$$mx^{m-1}(x-y) > x^m - y^m > my^{m-1}(x-y),$$

unless m lie between 0 and +1, in which case

$$mx^{m-1}(x-y) < x^m - y^m < my^{m-1}(x-y),$$

We have been careful to state and prove the inequality of the present section in its most general form because of its great importance: much of what follows, and many theorems in the following chapter, are in fact consequences of it\*.

Example 1. Show that, if x be positive,  $(1+x)^m$  always lies between 1+mx and (1+x)/(1+(1-m)x), provided mx<1+x.

Suppose, for example, that m is positive and <1. Then, by the theorem of the present section,

 $m(1+x)^{m-1}x < (1+x)^m - 1 < mx$ .

Hence Also.

$$(1+x)^m < 1+mx$$
,  
 $(1+x)^m - 1 > mx (1+x)^m/(1+x)$ ,

 $\{1 - mx/(1+x)\} (1+x)^m > 1.$ 

If 
$$mx < 1 + x$$
,  $1 - mx/(1 + x)$  is positive, and we deduce  $(1 + x)^m > 1/(1 - mx/(1 + x))$ .

 $(1 + x)^{m} > 1/\{1 - mx/(1 + x)\},$ >  $(1 + x)/\{1 + (1 - m)x\},$ 

> (1+x)/(1+(1-m))xThe other cases may be established in like manner.

Remark.—It should be observed that  $(1 \pm x)^m > < 1 \pm mx$ .

according as m does not or does lie between 0 and +1.

Example 2. Show that, if  $u_1, u_2, ..., u_n$  be all positive, then  $(1 + u_1)(1 + u_2)...(1 + u_n) > 1 + u_1 + u_2 + ... + u_n$ ;

also that, if 
$$u_1, u_2, \ldots, u_n$$
 be all positive and each less than 1, then

 $(1-u_1)(1-u_2)$  . . .  $(1-u_n)>1$   $u_1-u_2-...-u_n$ .

The first part of the theorem is obvious from the identity

 $(1+u_1)(1+u_2)\dots(1+u_n)\equiv 1+\sum u_1+\sum u_1u_2+\sum u_1u_2u_4+\dots+u_1u_2\dots u_n.$ The latter part may be proved, step by step, thus—

 $1 - u_1 = 1 - u_1$ ,  $(1 - u_1) (1 - u_2) = 1 - u_1 - u_2 + u_1 u_2$ ,  $> 1 - u_1 - u_2$ .

<sup>•</sup> Several mathematical writers have noticed the unity introduced into the elements of algebraical analysis by the use of this mequality. See sepecially Schömlich's Hamblach der Algebraischen Analysis. The secret of its power lies in the fact that it contains as a particular case the fundamental limit theorem upon which depends the differentiation of an algebraic function. The use of the theorem has been considerably extended in the present volume.

Hence, since 1 - u, is positive,

$$(1-u_1)(1-u_2)(1-u_3)>(1-u_3)(1-u_1-u_2),$$
  
 $> 1-u_1-u_2-u_3+u_3(u_1+u_2),$   
 $> 1-u_1-u_2-u_3.$ 

And so on.

These inequalities are a generalisation of  $(1 \pm x)^n - 1 \pm nx$  (x < I and n a positive integer). They are us ful in the theory of infinite products.

\$ 8.\ The arithmetic mean of a positive quantities is not less than their geometric mean.

Let us suppose this theorem to hold for n quantities  $a, b, c, \ldots, k$ , and let l be one more positive quantity. By hypothesis.

$$(a + b + c + \dots + k)/n \leq (abc \dots k)^{1/n}$$

that is,

$$a+b+c+\ldots+k \leqslant n (abc \ldots k)^{\ln n}$$

Therefore

$$(t+b+e+\dots+k+l \leq n (ethe \dots k)^{+n}+l$$

Now.

$$n (abc ... k)^{(n+1)} + l \leq (n+1) (abc ... kl)^{(n+1)},$$

provided

$$n \{abc \dots k_l L_1^{(1)n} + 1 \leqslant (n+1) \{abc \dots k_l L_1^{(n+1)} \}_{k=1}^{(n+1)} \}$$
  
 $\iff (n+1) \{abc \dots k_l L_1^{(n+1)} \}_{k=1}^{(n+1)} \}$   
that is, provided

 $n\xi^{n+1} = 1 \blacktriangleleft (n+1)\xi^n$ .

En(n+1) also . . . 1 19. where

that is, provided
$$(n+1) \, \tilde{\varepsilon}^n (\tilde{\varepsilon} - 1) \, \vec{\varepsilon}^n \tilde{\varepsilon}^{n+1} = 1.$$

which is true by \$ 7

Hence, if our theorem hold for n quantities, it will hold for n+1. Now we have seen that  $(a+b) \ 2 \le (ab)^{\frac{1}{2}}$ , that is, the theorem holds for 2 quantities; therefore it holds for 3; therefore for 4; and so on. Hence we have in general

$$(a + b + c + \dots + k)/n \triangleleft (abc \dots + k)^{1/n}$$

It is, of course, obvious that the inequality becomes an equality when a b c . . . - h.

There is another proof of this theorem so interesting and fundamental in its character that it deserves mention here\*.

Consider the geometric mean  $(abc \dots k)^{+n}$ . If  $a, b, c, \dots$  be not all equal, replace the greatest and least of them, say a and k, by (a+k)/2; then, since  $\{(a+k)/2\}^2 > ak$ , the result has been to increase the geometric mean, while the arithmetic mean of the n quantities (a+k)/2,  $b, c, \dots, (a+k)/2$  is evidently the same as the arithmetic mean of  $a, b, c, \dots, k$ . If the new set of n quantities be not all equal, replace the greatest and least as before; and so on.

By repeating this process sufficiently often, we can make all the quantities as nearly equal as we please; and then the geometric mean becomes equal to the arithmetic mean.

But, since the latter has remained unaltered throughout, and the former has been increased at each step, it follows that the first geometric mean, namely,  $(abc + b + c + ... + k)^n$ , is less than the arithmetic mean, namely, (a + b + c + ... + k) n.

As an illustration of this reasoning, we have  $(1:3:5:9)^4 < (5:3:5:5)^4 < (5:4:5:4:5:4:5:4:5)^4 < (1:3:5:9)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^4 < (1:3:5:4:5)^$ 

Cor. If  $a, b, \ldots, k$  be n positive quantities, and  $p, q, \ldots, t$  be n positive commensurable quantities, then

$$\frac{pa+qb+\ldots+tk}{p+q+\ldots+t} \not < (a^pb^q \ldots k^t)^{(p+q)(\cdots+t)}.$$

It is obvious that we are only concerned with the ratios  $p:q:\ldots:t$ . Hence we may replace  $p,q,\ldots,t$  by positive integral numbers proportional to them. It is, therefore, sufficient to prove the theorem on the hypothesis that  $p,q,\ldots,t$  are positive integers. It then becomes a more particular case of the theorem of the present paragraph, namely, that the arithmetic mean of  $p+q+\ldots+t$  positive quantities, p of which are equal to a,q to  $b,\ldots,t$  to k, is not less than their geometric mean.

See also the ingenious proof of the theorem given by Cauchy (Analyse Algebrique, p. 457), who seems to have been the first to state the theorem in its most general form.

Example 1. Show that, if  $a, b, \ldots, k$  be n positive quantities,

The first part of the proposition follows from the above corollary by taking  $p=a, \ q=b, \dots, k=c$ .

The second inequality is obviously equivalent to

$$\left(\frac{\Sigma a}{na}\right)^a \left(\frac{\Sigma a}{nb}\right)^b \dots \left(\frac{\Sigma a}{n\hat{k}}\right)^k > 1,$$

which again is equivalent t

$$\binom{\Sigma_{pet}}{npa}^{pa}\binom{\Sigma_{pet}}{npb}^{pb} \dots \binom{\Sigma_{pet}}{npk}^{pk} \succ \mathbf{1},$$

where  $\rho$  is a positive integer which may be so chosen that  $\rho a, \rho b, \dots, \rho k$  are all positive integers. We shall therefore lose no generality by supposing  $a, b, c, \dots, k$  to be positive integers.

Consider now a positive quantities each equal to  $\Sigma a/na, b$  positive quantities cach equal to  $\Sigma b/nb, \Delta c$ . The geometric mean of these is not greater than their arithmetic mean. Hence

$$\left\{ \left( \frac{\Sigma a}{na} \right)^{a} \left( \frac{\Sigma a}{nb} \right)^{b} \cdot \cdot \cdot \cdot \left( \frac{\Sigma a}{nk} \right)^{b} \right\}^{1} \stackrel{\Sigma a}{\Longrightarrow} \frac{a \left( \Sigma a/na \right) + b \left( \Sigma a/nb \right) + \dots + k \left( \Sigma a/nk \right)}{a + b + \dots + k}$$

Therefore

$$\left(\frac{\sum a}{na}\right)^a \left(\frac{\sum a}{nb}\right)^b \dots \left(\frac{\sum a}{nk}\right)^b > 1$$
,

Example 2. Prove that I:3, , ,  $(2n-1) < n^n$ .

We have 
$$\{1+3+\ldots+(2n-1)\}/n > \{1\cdot 3 \ldots (2n-1)\}^{1/n}$$
,

that is,  $n^2/n > \{1, 3, ..., (2n-1)\}^{1/n}$ . Hence  $n^n > 1, 3, ..., (2n-1)$ .

tence 
$$n^n > 1 \cdot 3 \cdot \dots \cdot (2n-1)$$
.  
§ 9.] If  $a, b, \dots, k$  be a positive quantities, and  $p, q, \dots, k$ 

be a positive quantities, then

$$\frac{pu^m+qh^m+\dots+tk^m}{p+q+\dots+t} \Leftrightarrow \left(\frac{pu+qh+\dots+tk}{p+q+\dots+t}\right)^m \quad (1),$$

according as m does not or does lie between 0 and + 1.

If we denote

$$p_{i}(p+q+...+t), \quad q'(p+q+...+t), \&c.,$$

by  $\lambda$ ,  $\mu$ , . . .,  $\tau$ , and

$$a/(\lambda a + \mu b + \dots + \tau k), \quad b_i(\lambda a + \mu b + \dots + \tau k), \&c.,$$

by  $x, y, \ldots, w$ , so that

$$\lambda + \mu + \dots + \tau = 1$$
 (2),

$$\lambda x : \mu y + \dots + \tau w - 1$$
 (3),

4

then, dividing both sides of (1) by

 $\{(pa+qb+...+t/), (p+q+...+t)\}^m$ 

we have to prove that  $\lambda x^m + \mu y^m + \dots + \tau v^m < > 1$  (4).

according as m does not or does lie between 0 and  $\pm 1$ .

Now, by § 7, if m does not lie between 0 and  $\pm 1$ ,  $x^m - 1 \Leftrightarrow m(x-1)$ ,  $y^m - 1 \Leftrightarrow m(y-1)$ , &c. Therefore, since  $\lambda$ ,  $\mu$ , &c., are positive.

 $\Sigma \lambda (x^{m}-1) \leqslant \Sigma \lambda m (x-1),$   $\leqslant m \{\Sigma \lambda x - \Sigma \lambda\},$  $\leqslant m (1-1)$ 

by (2) and (3), that is,

$$\Sigma \lambda x^m - \Sigma \lambda \not = 0$$
.

Hence

Now.

$$\Sigma \lambda_{\mathcal{L}^m} \neq 1$$
.

In like manner, we show that, if m lies between 0 and +1,  $\Sigma \lambda_{\ell}^{m} \gg 1$ .

Cor. If we make p = q = ... = t, we have

$$\frac{a^m + b^m + \dots + k^m}{n} \Longleftrightarrow \binom{a + b + \dots + k}{n}^m \tag{5}^*,$$

that is to say, the arithmetical mean of the mth powers of n positive quantities is not less or not greater than the mth power of their arithmetical mean, according as m does not or does lie between 0 and + 1.

Remark.—It is obvious that each of the inequalities (1), (1), (5) becomes an equality if  $a = b = \ldots = k$ , if m = 0, or if m = 1.

Example. Show that  $\sum \chi^m$ , considered as a function of m, increases as m increases when m>+1, and decreases as m increases when m<-1,  $\lambda$ ,  $\mu$ ,  $\nu$ , . . . ,  $\chi$ ,  $\chi$ ,  $\chi$ , . . . . being as above.

1st. Let m>1. We have to show that  $\sum \lambda x^{m+r} > \sum \lambda x^m$ , where r is very

small and positive, that is,

$$\Sigma \lambda x^m (x^r - 1) > 0$$
,

$$\sum \lambda x^m (x^r - 1) > \sum \lambda x^m r x^{r-1} (x - 1),$$
  
 $> r \sum \lambda x^{m+r-1} (x - 1).$ 

<sup>•</sup> The earliest notice of this theorem with which we are acquainted is in Reynaud and Duhamel's Problemes et Directopments and Director Parties in Reynaud and Eulamest Problemes 1 Directopments are Director Artistics Mathématiques (1823), p. 155. Its surroundings seem to indicate that it was suggested by Cauchy's theorem of § 8. The original proof rest on a maximum or minimum theorem, established by means of the Differential Calculus; and the elementary proofs Intherto given have usually involved the use of indinity series.

Since m>1, m+r>1, therefore  $(m+r)x^{m+r-1}(x-1)>(m+r)(x-1)$ , that is,  $x^{m+r-1}(x-1)>(x-1)$ .

Hence

$$\Sigma \backslash x^m (x^r - 1) > r \Sigma \lambda (x - 1),$$
  
>  $r (\Sigma \lambda x - \Sigma \lambda),$ 

$$> 0$$
.  
 $\sum \{x^{m+r} > \sum \{x^m\}$ .

Therefore 2nd. Let m < -1.

$$\sum x^m (x^p - 1) < r \sum x^m (r - 1)$$
.

Now  $(m+1) x^m (x-1) > (m+1) (x-1)$ , since m+1 is negative. Hence, dividing by the negative quantity m+1, we have

$$x^m(x-1) < (x-1)$$
.

Hence

$$\sum \lambda x^{m} (x^{p} - 1) < r \ge \lambda (x - 1),$$

$$< r (\sum \lambda x - \sum \lambda),$$

$$< 0.$$

Therefore,

# $\sum \lambda x^{m+r} < \sum \lambda x^m$ . Exercises V \*

(1.) For what values of x/y is (a+b)xy/(ax+by) > (ax+by)/(a+b)?

(2.) If x, y, z be any real quantities, and x>y>z, then  $x^4y+y^4z+z^4x>xy^4+yz^4+zx^4$ .

(3) If  $x_i|y_i|z$  be any real quantities, then  $\Sigma(y-z)(z-x)>0$  and  $\Sigma yzf$   $\Sigma x^z>1.$ 

(4.) If  $x^2 + y^2 + z^2 + 2xyz = 1$ , then will all or none of the quantities x, y, z lie is tween -1 and +1.

(5.) It x and m be positive integers, show that

 $x^{2m+3} \sim x(x+1)(2x+1)(3x^2+3x+1)^m/2 \cdot 3^m < (x+1)^{2m+3}$ 

(6.)  $(a^2/b)^{\frac{1}{2}} + (b^2/a)^{\frac{1}{2}} \approx a^{\frac{1}{2}} + b^{\frac{1}{4}}$ .

(7.) If  $x_1,x_2,\ldots,x_n$  all have the same sign, and  $1+x_1,1+x_2,\ldots,1+x_n$  be all positive, then

If  $(1+x_1)>1+\Sigma x_1$ . (8.) Prove that  $8xyz > 11(y+z) > ^*\Sigma x^3$ .

(9.) If x, y, z, ..., a, b, c ... be two sets, each containing n real quantities positive or negative, show that

 $\Sigma a^{2}\Sigma x^{2} \triangleleft (\Sigma ax)^{2}$ ; also that, if all the quantities be positive,

$$\Sigma (x/a)/2x + 2x/2ax$$
;  
 $\Sigma 1/x + n^2$ .

and, if  $\Sigma x = 1$ ,

(10) If  $x_1, x_2, \ldots, x_n$  and also  $y_1, y_2, \ldots, y_n$  be positive and in ascending or in descending order of magnitude, then  $\sum x_1^2 y_1 (\sum x_1 y_1 - \sum x_1^2 / \sum x_1).$  (Laplace.)

Unless the contrary is stated, all letters in this set of exercises stand for real positive quantities.

(11.) If  $a, b, \ldots, l$  be in A.P., show that

$$a^{2}b^{2} \dots l^{2} > a^{n}l^{n}$$
.

- (12.) For what values of x is  $(x-3)/(x^2+x+1) > (x-4)/(x^2-x+1)$ ?
- (13.) Find the limits of x and y in order that

c > ax + by > d, a > cx + dy > b; ad - bc = 0

where

(14.)  $x^6 - x^5y + 4x^4y^2 - 2x^3y^3 + 4x^2y^4 - xy^5 + y^6 > 0$ , for all real values of x and y.

- and y. (15.) Is  $10x^2 + 5y^2 + 13z^2 > = < 8yz + 2xy + 18zx$ ?
  - (16.) If  $p < 2 \sqrt{2}$ , then  $\sqrt{(x^2 + y^2) + p\sqrt{(xy)}} > x + y$ . (17.) Is  $\sqrt{(a^2 + ab + b^2)} - \sqrt{(a^2 - ab + b^2)} > = < 2\sqrt{(ab)}$ ?
- (18.) If x and a be positive, between what limits must x lie in order that  $x+a>\sqrt{\{k(x^2+xa+a^2)\}+\sqrt{\{k(x^2-xa+a^2)\}}\}^2}$ 
  - (19.) If x < 1, then  $\{x + \sqrt{(x^2 1)}\}^{\frac{1}{2}} + \{x \sqrt{(x^2 1)}\}^{\frac{1}{2}} < 2$ .
  - (20.) If all the three quantities  $\langle \{a(b+c-a)\}, \langle \{b(c+a-b)\}, \langle \{c(a+b)\}, \langle$
- b c)} be real, then the sum of any two is greater than the third.
  (21.) If the sum of any two of the three x, y, z be greater than the third.
- then  $\frac{\alpha}{3} \sum x \sum x^2 > \sum x^3 + xyz$ .
  - (22.) ∑1/x > ∑x<sup>8</sup>/x<sup>3</sup>y<sup>3</sup>z<sup>3</sup>.
    (23.) If p<sub>r</sub> denote the sum of the products r at a time of a, b, c, d (each
- positive and <1), then  $p_2 + 2p_4 2p_3$ . (24.)  $\Sigma x^4 \lessdot xyz\Sigma x$ .
  - (24.)  $\Sigma x^* \triangleleft xyz \supseteq x$ . (25.) If s = a + b + c + ... n terms, then  $\Sigma s/(s - a) \triangleleft n^2/(n - 1)$ .
  - (25.) If s ≡ d + b + c + · · · · n terms, then 2s<sub>i</sub>(s − a) ∈ n · ((a − 1).
    (26.) If m > 1, x < 1, and mx < 1 + x, then 1/(1 + mx) > (1 ± x)<sup>m</sup> > 1 ± mx.
- If m < 1, x < 1, and mx < 1 + x, then (1 + x)/(1 + mx) > (1 + x). If m < 1, x < 1, mx < 1 + x, then  $(1 + x)/(1 + (1 - m)x) < (1 + x)^m < 1 + mx$ .
  - (27.) If  $z^n = x^n + y^n$ , then  $z^m > < x^m + y^m$  according as m > < n.
- (28.) If x and y be inequal, and x + y = 2a, then  $x^m + y^m > 2a^m$ , m being a positive integer.
  - (29.)  $n\{(n+1)^{1/n}-1\}<1+1/2+\ldots+1/n< n\{1-1/(n+1)^{1/n}+1/(n+1)\}$ . (Schlömilch, Zeitschr, f. Math., vol. III, p. 25.)
  - (30.) If  $x_1x_2$ ...  $x_n = y^n$ ,  $11(1 + x_1) < (1 + y)^n$ .
- (31.) If a, b, . . . , k be n positive quantities arranged in ascending order of magnitude, and if M<sub>r</sub> = {Σa<sup>r</sup>/n}<sup>1,τ</sup>, N<sub>r</sub> = {Σa<sup>1,τ</sup>}<sup>1,τ</sup>, then

 $(ab , ... k)^{1/n} < M_1 < M_2 < ... < k$ 

 $(ab ... k)^{1/n} < ... < N_3 < N_2 < N_1.$ 

(Schlömileh, Zeitschr. f. Math., vol. III. p. 301.)

- (32.) If p, q, r be all unequal, and  $x \neq 1$ , then  $\sum p x^{q-r} > \sum p$ .
- (33.) If n be integral, and x and n each >1, then
  - $x^{n} 1 > n \left(x^{(n+1)/2} x^{(n-1)/2}\right)$
- (34.) Prove for x, y, z that  $(2\Sigma yz \Sigma x^2)^{\Sigma z} \Leftrightarrow (\Sigma x)^{\Sigma z} \Pi (\Sigma x 2x)^z$ .
- (35.) If  $s = a_1 + a_2 + \dots + a_n$ , then II  $(s/a_r 1)^{a_r} > (n-1)^s$ .

(36.)  $3m (3m + 1)^2 > 4 (3m!)^{1/m}$ ,

(37.) If \*<sub>m</sub> be the sum of the nth powers of a<sub>1</sub>, a<sub>2</sub>, ..., a<sub>n</sub>, and p<sub>m</sub> the sum of their products m at a time, then (n-1)! \*<sub>m</sub> \* (n-m)! m! p<sub>m</sub>.

(38.) If  $a_1 > a_2 > \dots > a_n$ , then

$$(a_1 - a_n)^{n-1} > (n-1)^{n-1} (a_1 - a_2) (a_2 - a_3)$$
, . .  $(a_{n-1} - a_n)$ .

Hence, or otherwise, show that  $\{(n-1)!\}^2 > n^{n-2}$ .

(39.) Which is the greatest of the numbers 32, 23, 24, . . . ?

(40.) When is the greatest of the numbers  $(2, 2, 3, 3, 1, \dots, x_n)$ , each > 1, and if  $\xi_1, \xi_2, \dots, \xi_n$  be the arithmetic means, or the geometric means, of all but  $x_1, \dots, x_n$ , all but  $x_2, \dots$ , all but  $x_n$ , then  $\Pi x_1 \xi_1 > \Pi \xi_1 \xi_2$ .

(41.) If a, b, c be such that the sum of any two is greater than the third, and x, y, z such that  $\Sigma x$  is positive, then, if  $\Sigma a^2/z = 0$ , show that xyz is negative.

(12.) If  $A = a_1 + a_2 + \dots + a_n$ ,  $B = b_1 + b_2 + \dots + b_n$ , then  $\sum \{a_n | A - b_n | B \} (a_n | b_n)^n$  has the same sign as n for all finite values of n.

(Math. Trip., 1870.)

#### APPLICATIONS TO THE THEORY OF MAXIMA AND MINIMA.

§ 10.] The general nature of the connection between the theory of maxima and minima and the theory of inequalities may be illustrated as follows:—Let  $\phi(x, y, z)$ , f(x, y, z) be any two functions of x, y, z, and suppose that for all values consistent with the condition

$$f(x, y, z) = A \tag{1}$$

we have the inconslity

$$\phi(x, y, z) \triangleright f(x, y, z) \tag{2}.$$

If we can find values of x, y, z, say a, b, c, which satisfy the equation (1) and at the same time make the inequality (2) an equality, then  $\phi$  (a, b, c) is a maximum value of  $\phi$  (x, y, z). For, by hypothesis,  $\phi$  (a, b, c) = 1 and  $\phi$   $(x, y, z) \ge 1$ ; therefore  $\phi$  (x, y, z) cannot, for the values of x, y, z considered, be greater than A, that is, than  $\phi$  (a, b, c).

Again, if we consider all values of x, y, z for which

$$\phi(x, y, z) = A$$
 (1'),

if we have

$$f(x, y, z) \triangleleft \phi(x, y, z)$$
  
 $\triangleleft A$  (2'),

it follows in like manner that, if a, b, c be such that  $\phi(a, b, c) \cdot A_1$   $f(a, b, c) - A_1$  then f(a, b, c) is a minimum value of f(x, y, z).

The reasoning is, of course, not restricted to the case of three variables, although for the sake of brevity we have spoken of only three. The nature of this method for finding turning values may be described by saving that such values arise from exceptional or limiting cases of an inequality.

\$11.1 The reader cannot fail to be struck by the reciprocal character of the two theorems deduced in last section from the same inequality. The general character of this reciprocity will be made clear by the following useful general theorem:-

If for all values of x, y, z, consistent with the condition

$$f(x, y, z) = A,$$

 $\phi(x, y, z)$  have a maximum value  $\phi(a, b, c) = B$  say (where B depends, of course, upon A), and if when A increases B also increases, and vice versa, then for all values of x, y, z, consistent with the condition

$$\phi(x, y, z) = B$$
,

f(x, y, z) will have a minimum value f(a, b, c) = A.

**Proof.**—Let A' < A, then, by hypothesis, when f'(x, y, z) = A',

 $\phi(x, y, z) \triangleright B'$  where B' < B.

Hence, if  $\phi(x, y, z) = B$ , f(x, y, z) < A; for suppose if possible that f(x, y, z) = A' < A, then we should have  $\phi(x, y, z) \geqslant B'$ , that is, since B' < B,  $\phi(x, y, z)$  could not be equal to B as required. Hence, if a, b, c be such that  $\phi(a, b, c) = B$  and f(a, b, c) = A, f(a, b, c) is a minimum value of f(x, y, z).

By means of the two general theorems just proved, we can deduce the solution of a large number of maximum and minimum problems from the inequalities established in the present chapter.

§ 12.] From the theorem of § 8 we deduce immediately the two following:-

I. If  $x, y, z, \ldots$  be n positive quantities subject to the condition

$$\Sigma x = k$$

then their product ILx has a maximum value, (k/n), when x $y = \ldots = k/n$ .

II. If x, y, z, . . . be n positive quantities subject to the condition

$$\Pi x = k$$
,

then their sum  $\Sigma x$  has a minimum value,  $nk^{1/n}$ , when x = y = ...- A-1/n

The second of these might be deduced from the first by the reciprocity-theorem.

From the corollary in § 8 we deduce the following:-

III. If x, y, z, . . , be a positive quantities subject to the condition

$$\sum px = k$$
,

where p, q, r, . . . are all positive constants, then ILxp has a maximum value,  $\{k, \geq p\}^{\geq p}$ , when  $x = y = \dots = k/\sum p$ .

IV. If x, y, z, . . . be n positive quantities subject to the condition

where p, q, r, . . . are all positive constants, then Epr has a minimum value,  $(\Sigma_P) k^{1/\Sigma_P}$ , when  $x = y - \dots = k^{1/\Sigma_P}$ .

From the last pair we can deduce the following, which are still more general:-

V. If  $\lambda, \mu, \nu, \ldots, l, m, n, \ldots, p, q, r, \ldots$  be all positive constants, and x, y, z, . . . be all positive, then if

$$\sum \lambda_i x^i = k$$
.

 $\Pi_x^p$  is a maximum when

$$l\lambda x^{l}/p = m\mu y^{m}/q = mz^{n}/r = ...$$

VI todif  $\Pi x^p - k$ .  $\Sigma \lambda x^{l}$  is a minimum when

 $l\lambda x^{j}/p = may^{m}/q = nvz^{n}/r = ...$ 

Proof.—Denote p'l, q, m, r/n, . . . by  $a, \beta, \gamma, \dots$ 

 $\lambda x^{j} = a\xi, \quad \mu \eta^{m} = \beta \eta, \quad rz^{n} = \gamma \zeta, \ \&c.$ and let

 $x = (\alpha \xi/\lambda)^{1/2}$ , &c.;  $x^p = (\alpha \xi/\lambda)^n$ , &c.

So that

We then have in the first case

$$\Sigma a \xi = k \tag{1},$$

$$\Pi_i r^p = \Pi_i (n^i \lambda)^a \Pi \xi^a$$
 (2),

Hence, since  $(a/\lambda)^a$ ,  $(\beta'\mu)^\beta$ , . . . are all constant and all positive, H.r. is a maximum when HEa is a maximum. Now, under the condition (1),  $\Pi \xi^{\bullet}$  is a maximum when  $\xi = \eta = ... = k/\Sigma a$ .

Hence  $\Pi_{x}^{p}$  is a maximum when  $\lambda x^{r}/a = \mu y^{rr}/\beta = ...$  that is,  $L\lambda x^{I}/n = mun^{m}/a = ...$ when

The maximum value of  $\Pi_{\mathcal{X}}^p$  is  $\Pi(a/\lambda)^a (k \Sigma a)^{\Sigma a}$ , and the corresponding values of x, y, z, . . . are given by

$$x = (\alpha k/\lambda \Sigma \alpha)^{1.1}, \ldots$$

Applying the reciprocity-theorem, we see that, if

$$\Pi_i x^p = \Pi (\alpha/\lambda)^\alpha (k, \Sigma \alpha)^{\Sigma \alpha},$$

the minimum value of  $\Sigma \lambda x^{I}$  is k, corresponding to

$$\mathbf{r} = (\alpha k^{l} \lambda \Sigma \alpha)^{1l}$$
...

Whence, putting  $j = H(\alpha/\lambda)^{\alpha} (k \Sigma \alpha)^{\Sigma \alpha}$ , we see that, if  $Hx^n = i$ . the minimum value of  $\Sigma \lambda x^{j}$  is  $\Sigma a \{ j \prod (a, \lambda)^{a+2a}, \text{ corresponding} \}$ to

$$x = [\alpha \{j/\Pi (\alpha/\lambda)^{\alpha+1} \Sigma \alpha/\lambda]^{1/2} \dots$$

Cor. If we put l = m = n = ... = 1, p - q = r = ... = 1, we obtain the following particular cases, which are of frequent occurrence :-

If  $\Sigma \lambda x = k$ .  $\Pi x$  is a maximum when  $\lambda x = \mu y = \dots$ If  $\prod x = k$ ,  $\sum \lambda x$  is a minimum when  $\lambda x = \mu y = \dots$ 

Example 1. The cube is the rectangular parallelopined of maximum volume for given surface, and of minimum surface for given volume.

If we denote the lengths of three adjacent edges of a rectan whar parallelopiped by x, y, z, its surface is 2(yz+zx+xy) and its volume is xyz. If we put  $\xi = yz$ ,  $\eta = zx$ ,  $\zeta = xy$ , the surface is  $2(\xi + \eta + \zeta)$  and the volume  $\zeta(\xi \eta)$ . Hence, analytically considered, the problem is to make \$95 a maximum when  $\xi + \eta + \zeta$  is given, and to make  $\xi + \eta + \zeta$  a minimum when  $\xi \eta \zeta$  is given. Thus, by Th. I., is done in either case by making  $\xi = \eta = \zeta$ , that is, yz = zx = xy; whence x = y = z.

Example 2. The equilateral triangle has maximum area for given posimeter, and minimum perimeter for given area.

The area is  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ . Let x=s-a, y=s-b, z=s-c; then x+y+z=s; and the area is  $\sqrt{sxyz}$ . Since, in the first place, s is given, we have merely to make xyz a maximum subject to the condition x + y + z = s.

This leads to x = u = z (by Th. I.).

Next, let \Delta be given.  $(x + y + z) xyz = \Delta^2$ Then

 $s = \Delta^2/xyz$ If we put  $\xi = x^2yz$ ,  $\eta = xy^2z$ ,  $\zeta = xyz^2$ , we have

$$\xi + \eta + \zeta = \Delta^2 \qquad (1);$$

$$+\zeta = \Delta^2/(\xi \eta \zeta)^{-4}$$
 (2).

Hence, to make s a minimum when  $\Delta$  is given, we have to make  $\xi \eta_1^*$  a maximum, subject to the condition (1). This leads to  $\xi = \eta = \xi$ , that is,  $x^2\eta = xy^2\xi = xy^2\xi$ . Where  $x = \eta + z$ .

Example 3. To construct a right circular cylinder of given volume and minimum total surface.

Let x be the radius of the ends, and y the height of the cylinder. The total surface is  $2\pi (x^2 + xy)$ , and the volume is  $\pi x^2y$ .

We have, therefore, to make  $u = x^2 + xy$  a minimum, subject to the condition  $x^2u = c$ . We have

$$u = x^2 + xy = c/y + c/x$$
 (1);

$$x^2y = c$$
 (2).  
Let  $1/x = 2\xi$ ,  $1'y = \eta$ ;

then 
$$u = c (2\xi + \eta)$$
 (1'):

$$\xi^2 \eta = 1/4v$$
 (2)

We have now to make  $2\xi + \eta$  (that is,  $\xi + \xi + \eta$ ) a minimum, subject to the condition  $\xi \eta = 0$  onstant. This, by Th. II., leads to  $\xi = \xi = \eta$ , which gives 2x = y. Hence the height of the cylinder is equal to its diameter.

By the reciprocity-theorem (applied to the problem as originally stated in terms of x and y), it is obvious that a cylinder of this shape also has maximum volume for given total surface.

§ 13.] From the inequality of § 9 we infer the following :-

VII. If m do not lie between 0 and +1, and if p, q, r, ..., bcall constant and positive, then, for all positive values of x, y, z, ...such that

$$\Sigma px = k$$
,

 $\Sigma px^m$  (m unchanged) has a minimum value when  $x=y=z=\ldots$ .

If m lie between 0 and +1, instead of a minimum we have a maximum.

In stating the reciprocal theorem it is necessary to notice that, in the inequality,  $\Sigma px$  occurs raised to the mth power; so that, if m be nepative, a maximum of  $\Sigma px$  corresponds to a minimum of  $(\Sigma px)^n$ . Attending to this point, we see that—

VIII. If m > +1, and if p, q, r, ... be all constant and positive, then, for all positive values of x, y, z, ... such that

$$\Sigma px^m = k \ (m \ unchanged),$$

Spr has a maximum value when  $x - y - z - \dots$ 

If m < + 1, we have a minimum instead of a maximum,

Theorem VIII, might also be deduced from Theorem VIII, by the substitution  $\xi = x^n, \eta = y^n, \zeta = z^n, \chi_C, \dots$  Let

§ 14.] Theorem VII. may be generalised by a slight transformation into the following:—

1X. If m|n do not lie between 0 and +1, and if  $p, q, r, \ldots$ ,  $\lambda, \mu, \nu, \ldots$  be all constant and positive, then, for all positive values of  $x, y, z, \ldots$  such that

$$\sum \lambda x^n = k \ (n \ unchanged),$$

 $\Sigma px^m$  (m unchanged) has a minimum value when  $px^m/\lambda x^n = qy^m/\mu y^n = ...$ 

uy/vy = v. If m/n lie between 0 and +1, instead of a minimum we have a maximum.

The transformation in question is as follows:-

$$\lambda x^n = \rho \xi, \quad \mu y^n = \sigma \eta, \dots$$
 (1),

$$p_i x^m = \rho \xi^f$$
,  $q_i y^m = \sigma \eta^f$ , . . . (2).

From the first two equations in (1) and (2) we deduce  $\xi^{\mu-1} = p_{\mu} m^{\mu} - n/\lambda$ ,  $\rho^{\ell-1} = \lambda' x^{\mu_0 - m}/p$ , &c. Hence, if we take fn - m, that is, f = m/n,  $\rho, \sigma$ , . . . will be all constant and obviously all positive; we have, in fact,

$$\xi = (\rho x^{m-n}/\lambda)^{1/(f-1)}, \quad \eta = (qy^{m-n}/\mu)^{1/(f-1)}, \dots$$
 (3),

$$\rho = (\mathcal{N}/p)^{1/(f-1)}, \quad \sigma = (\mu^f/q)^{1/(f-1)}, ...$$
 (1);

and we have now to make  $\Sigma \rho \mathcal{E}'$  a maximum or minimum, subject to the condition

$$\Sigma \rho \xi = k$$
.

Now, by Th. VII.,  $\Sigma \rho \xi'$  is a minimum or maximum, according as f does not or does lie between 0 and +1, when  $\xi = \eta = \dots$ 

Thus the conditions for a turning value are

$$(px^{m-n}/\lambda)^{1/(f-1)} = (qy^{m-n}/\mu)^{1/(f-1)} = ...,$$

which lead at once to

$$px^m/\lambda x^n = qy^m/\mu y^n = \dots$$

Cor. A very common case is that where  $n=1,\ \lambda=\mu$  . . . = 1.

We then have, subject to the condition  $\Sigma x = k$ ,  $\Sigma px^m$ , a minimum or maximum when  $px^{m-1} = qy^{m-1} = \ldots$ , according as m does not or does lie between 0 and +1.

§ 15.] We have hitherto restricted  $p, q, r, \ldots$  in the in-

equality of § 9 to be constant. This is unnecessary; they may be functions of the variables, provided they be such that they remain positive for all positive values of x, y, z.

We therefore have the following theorem and its reciprocal

(the last omitted for brevity) :-

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X. If p, q, r, ... be functions of x, y, z, ... which are real and positive for all real and positive values of x, y, z, ..., then, for all positive values of x, y, z, ... which satisfy

$$\sum nx = k$$
.

 $(\Sigma px^m)(\Sigma p)^{m-1}$  (m unchanged) has a minimum or maximum value when  $x=y=\dots$ , according as m does not or does lie between 0 and +1.

For example, we may obviously put  $p = \lambda x^a$ ,  $q = \mu y^b$ , . . .

We thus deduce that if m>+1 or <0, then, for all positive values of  $x, y, z, \ldots$  consistent with  $\sum \langle x^{a+1} \equiv k, (\sum \langle x^{m+a} \rangle) (\sum \langle x^a \rangle)^{m-1}$  is a minimum when  $x=y=\ldots$ 

Theorem X, may again be transformed into others in appearance more general, by methods which the student will readily divine after the illustrations already given.

Also the inequalities of § 8 may be used to deduce maxima and minima theorems in the same way as those of § 9 were used in the proof of Theorem X.

Example 1. To find the minimum value of u=x+y+z, subject to the conditions a/x+b/y+c/z=1, x>0, y>0, z>0, a,b,c being positive constants. Let  $x=\rho \xi'$ ,  $y=c\eta'$ ,  $z=\tau \xi''$ ;

 $a/x = \rho \xi$ ,  $b/y = \sigma \eta$ ,  $c/z = \tau \tilde{\xi}$ .

Hence  $\rho^{\ell-1}$  and  $x^{\ell+1}$ . If we take  $\ell=-1$ , we therefore get

$$x = \sqrt{a_s^{r+1}}$$
,  $y = \sqrt{b_s^{r+1}}$ ,  $z = \sqrt{c_s^{r+1}}$ ;  
 $a(x) = \sqrt{a_s^r}$ ,  $by = \sqrt{b_r}$ ,  $cz = \sqrt{c_s^r}$ .

The problem now is to make  $u = \sum_{s} (s_s^{-1} \cdot \mathbf{a})$  minimum subject to the constant  $\sum_{s} a_s^2 - 1$ . By Th. VII. this is accomplished by making  $\xi = n = \xi$ . Hence  $\xi = n = \xi = 1/2$ ,  $\delta$ . The minimum value required is therefore  $(\sum_{s} a_s^2)$ ; the corresponding values of x, y, z are  $\sqrt{s} \sum_{s} a_s \sqrt{b} \sum_{s} a_s \sqrt{b} \sum_{s} a_s \sqrt{s} \sum_{s}$ 

Example 2. To find a point within a triangle such that the sum of the mth powers of its distances from the sides shall be a minimum (m>1).

Let a, b, c be the sides, x, y, z the three distances; then we have to make  $u = \sum_{x} a$  minimum, subject to the condition  $\sum dx = 2\Delta$ , where  $\Delta$  is the area of the transfel.

If  $\rho \xi^m = x^m$ ,  $\rho \xi = ax$ , then  $\rho^{m-1} = a^m$ ,  $\rho = a^{m \cdot (m-1)}$ .

Hence, if we put  $ax = a^{m(m-1)}\xi$ ,  $hy = b^{m(m-1)}\eta$ ,  $cz = c^{m(n-1)}\zeta$ , we have  $u = \sum_{n = -\infty} m(m-1)\xi^n$ 

$$2\Delta = \sum_i m_i(m-1) \varepsilon_i$$

The solution is therefore given by  $\xi = \eta = \tilde{\xi} = 2\Delta/2a^{m/(m-1)}$ .

Whence  $x = 2\Delta a^{1,(m-1)}/\sum a^{m} {m-1}, y = \&c., z = \&c.$ 

Example 3. Show that, if  $x^2 + y^4 + z^5 = 3$ , then  $(x^4 + y^5 + z^6)(x^2 + y^3 + z^4)$  has a minimum value for all positive values of x, y, z when x = y = z = 1.

This follows from Th. X., if we put m=2,  $p=x^2$ ,  $q=y^3$ ,  $r=z^4$ , which is legitimate since x, y, z are all positive.

Example 4. If  $x, y, z, \ldots$  be n positive quantities, and m do not lie between 0 and 1, show that the least possible value of  $(2\pi^{m-1})(21/x)^{m-1}$  is  $n^m$ . This follows at once from the inequality of § 9, if we put p=1/x, a=1/x.

§ 16.] The field of application of some of the foregoing theorems can be greatly extended by the use of undetermined multipliers in a manner indicated by Grillet\*.

Suppose, for example, it were required to discuss the turning values of the function

$$u \equiv (ax + p)^{l} (bx + q)^{m} (cx + r)^{n}$$
(1),

where l, m, n are all positive.

 $u \equiv (\lambda ax + \lambda p)^t (phx + \mu_I)^m (rex + rr)^n / \lambda^l \mu^m r^n$  (2), where  $\lambda$ ,  $\mu$ ,  $\nu$  are three arbitrary quantities, which we may subject to any three conditions we please.

Let the first condition be

$$l\lambda a + m\mu b + n\nu c = 0 \qquad (3);$$

then we have

$$l(\lambda ax + \lambda p) + m(\mu bx + \mu q) + n(\iota ex + \iota r)$$

$$= l\lambda p + m\mu q + n\nu r = k$$
(4),

where k is an arbitrary positive constant.

This being so, we see by Th. III. that  $\Pi : \lambda(\alpha x + \lambda p)^t$  is a maximum when

$$\lambda ax + \lambda p = \mu bx + \mu q = vcx + vr$$
  
=  $k/\Sigma l$  (5).

<sup>·</sup> Nouvelles Annales de Math., ser. 1., tt. 9, 16.

The four equations (3) and (5) are not more than sufficient to exhaust the three conditions on  $\lambda$ ,  $\mu$ ,  $\nu$ , and to determine  $\sigma$ .

We can easily determine x by itself. In fact, from (3) and (5) we deduce at once

$$l_{tt}/(qx+p) + mb/(bx+q) + nc/(cx+r) = 0$$
 (6).

This quadratic gives two values for  $x_1$  say  $x_1$  and  $x_2$ ; and the equations (5) give two corresponding sets of values for  $\lambda_1 \mu_2 \nu_3$  in terms of  $k_1$  say  $\lambda_1, \mu_2, \nu_3$  and  $\lambda_2, \mu_2, \nu_3$ .

If, then,  $\lambda_1^i \mu_1^m \nu_1^n$  be positive,  $x_1$  will correspond to a maximum value of u; if  $\lambda_1^i \mu_1^m \nu_1^n$  be negative,  $x_1$  will correspond to a minimum value of u; and the like for  $x_2$ .

Example 1. To discuss  $u \equiv (x+3)^2 (x-3)$ .

Example 1. To discuss  $u \equiv (x + 3)^{\alpha} (x - 3)$ . We have  $u \equiv (\lambda x + 3\lambda)^{\beta} (\mu x - 3\mu)/\lambda^{2}\mu$ .

Now  $2(\lambda x + 3\lambda) + (\mu x - 3\mu) = k,$ 

provided  $2\lambda + \mu = 0$ 

 $6\lambda - 3\mu = k$  (2).

Therefore  $(\lambda x + 3\lambda)^2 (\mu x - 3\mu)$  will be a maximum, provided

$$\lambda x + 3\lambda = \mu x - 3\mu \tag{3}.$$

(1),

Hence, by (1). 2/(x+3) + 1/(x-3) = 0;

which gives x=1. From (2) and (3) we deduce  $\lambda = k/12$ ,  $\mu = -k/6$ ; so that  $\lambda^2 \mu$  is negative.

We therefore conclude that u is a minimum when x = 1.

The student should trace the graph of the function u; he will thus find that it has also a maximum value, corresponding to x=-3, of which this method gives no account.

Example 2. For what values of x and y is

$$u = (a_1x + b_1y + c_1)^2 + (a_2x + b_2y + c_2)^2 + \dots + (a_nx + b_ny + c_n)^2$$

a minimum? Let  $\lambda_1, \lambda_2, \dots, \lambda_n$  be undetermined multipliers. Then we may write

$$u = 2\lambda_1^2 \{(a_1x + b_1y + c_1)/\lambda_1\}^2$$
 (1);

 $u = \sum_{1}^{\infty} \{ (a_1x + b_1y + c_1)/\lambda_1 \}^2$   $k = \sum_{1}^{\infty} \{ (a_1x + b_1y + c_1)/\lambda_1 \}$ (2),

and  $k = \sum \lambda_1^2 \{(a_1x + b_1y + c_1)/\Lambda_1\}$  (2), where k is an arbitrary positive constant, that is, independent of x and y, provided

 $\Sigma a_1 \lambda_1 = 0$ ,  $\Sigma b_1 \lambda_1 = 0$ ,  $\Sigma c_1 \lambda_1 = k$  (3).

This being so, by Th. VII., u is a minimum when

 $(a_1x + b_1y + c_1)/\lambda_1 - (a_2x + b_2y + c_2)/\lambda_2 = \dots - k_1 \sum_1 2$  (4). The n+2 equations, (3) and (1), just suffice for the determination of

 $\lambda_1, \lambda_2, \dots, \lambda_n, x, y$ . From the first two of (3), and from (t), we deduce

$$\Sigma a_1 (a_1x + b_1y + c_1) = 0,$$
  
 $\Sigma b_1 (a_1x + b_1y + c_1) = 0.$ 

Hence the values of x and y corresponding to the minimum value of n are given by the system

 $\Sigma a_1^2 x + \Sigma a_1 b_1 y + \Sigma a_1 c_1 = 0,$ 

 $\Sigma a_1 b_1 x + \Sigma b_1^2 y + \Sigma b_1 c_1 = 0.$ 

This is the solution of a well-known problem in the Theory of Errors of Observation.

§ 17.] Method of Increments.—Following the method already exemplified in the case of a function of one variable, we may define

 $I = \phi(x + h, y + k, z + l) - \phi(x, y, z)$ 

as the increment of  $\phi(x,y,z)$ . If, when x=a,y=b,z=c, the value of I be negative for all small values of h,k,l, then  $\phi(a,b,c)$  is a maximum value of  $\phi(x,y,z)$ ; and if, under like circumstances, I be positive,  $\phi(a,b,c)$  is a minimum value of  $\phi(x,y,z)$ .

Owing to the greater manifoldness of the variation, the examination of the sign of the increment when there are more variables than one is often a matter of considerable difficulty; and any general theory of the subject can scarcely be established without the use of the infinitesimal calculus.

We may, however, illustrate the method by establishing a case of the following general theorem, which includes some of those stated above as particular cases.

Purkiss's Theorem\*  $-Hf \phi(x, y, z, \ldots) f(x, y, z, \ldots)$  be symmetric functions of  $x, y, z, \ldots$ , and if  $x, y, z, \ldots$  be subject to an equation of the form

$$f(x, y, z, ...) = 0$$
 (1),

then  $\phi(x, y, z, \dots)$  has in general a turning value when x = y = z= . . . , provided these conditions be not inconsistent with the equation (1).

In our proof we shall suppose that there are only three variables; and so far as that is concerned it will be obvious that there is no loss of generality. But we shall also suppose both

Given with inadequate demonstration in the Oxford, Cambridge, and Dublin Messenger of Mathematics, vol. 1. (1862).

 $\phi\left(x,y,z\right)$  and  $f\left(x,y,z\right)$  to be integral functions, and this supposition, although it restricts the generality of the proof, renders it amenable to elementary treatment.

We remark, in the first place, that the conditions

$$x-y=z$$
 and  $f(x, y, z)=0$ 

are in general just sufficient to determine a set of values for x, y, z. In fact, if the common value of x, y, z be a, then a will be a root of the equation f(a, a, a) = 0.

Consider the functions

 $I = \phi\left(a+h, a+k, a+l\right) - \phi\left(a, a-a\right)$ , and  $f\left(a+h, a+k, a+l\right)$ . Each of them is evidently a symmetric function of h, k, l, and can therefore be expanded as an integral function of the elementary symmetric functions  $\Sigma h, \Sigma h l, h h l$ . We observe also that, since each of the functions vanishes when h = 0, k = 0, l = 0, there will be no term independent of h, k, l.

Let us now suppose  $h_c k_c I$  to be finite multiples of the same very small quantity  $r_c$  say  $h_c a r_c k_c B r_c + \gamma r_c$ . Then  $\Sigma h = r\Sigma a r_c + r \Sigma a \gamma_c \Sigma h t$  is distincted, and remembering that by the conditions of our problem  $f(a + h_c a + k_c a + t) = 0$ , we have, if we arrange according to powers of  $r_c$ .

$$I = Aur + (Bu^2 + Cr) r^2 + \&c.$$
 (1),

$$0 = Pur + (Qu^2 + Rv) r^2 + \&v.$$
 (2),

where the &c. stands for terms involving  $r^2$  and higher powers.

From (2) we have

$$ur = (Qu^2 + Rv) r_{\perp}^2 P + \&c_0$$
  
 $u^2r^2 = 0 + \&c_0$ 

$$2\sum_i \beta_i r^2 = \sum_i a^i r^i + \lambda c_{i+1}$$

 $\Delta c$ , as before including powers of r not under the 3rd.

Hence, substituting in (t) and writing out only such terms as contain no higher power of r than  $r^2$ , we have

$$I = (C - AR/P) vr^2 + \&c.,$$
  
=  $\frac{4}{3}r^2 (C - AR/P) \Sigma a^2 + \&c.$ 

Now (see chap, xv.,  $\S$  10), by taking r sufficiently small, we may cause the first term on the right to dominate the sign of I.

Hence I will be negative or positive according as (CP - AR)/P is positive or negative; that is,  $\phi(a, a, a)$  will be a maximum or minimum according as (CP - AR)/P is positive or negative.

Example. Discuss the turning values of  $\phi(x, y, z) \equiv xyz + b(yz + zx + xy)$ , subject to the condition  $x^2 + y^2 + z^2 = 3a^2$ .

The system

$$x = y = z$$
,  $x^2 + y^2 + z^2 - 3a^2 = 0$ 

has the two solutions  $x = y = z = \pm a$ ,

If we take x=y=z=+a, we find, after expanding as above indicated,

$$I\equiv (a^2+2ab)\;ur+(a+b)\;vr^2+\Delta v.,$$

$$0 = 2aur + (u^2 - 2r) r^2.$$

In this case, therefore,  $A=a^2+2ab,\ C=a+b,\ P=2a,\ R=-2$  ; and (CP-AR)/P=2a+3b.

Hence, when  $x=y=z=+a,\,\phi$  is a maximum or a minimum according as 2a+3b is positive or negative.

In like manner, we see that, when x=y=z=-a,  $\phi$  is a maximum or a minimum according as -2a+3b is positive or negative.

#### EXERCISES VI \*

- (1.) Find the minimum value of bex + eay + abz when xyz abc.
- (2.) Find the maximum value of xyz when x²/a² + y', b² + z², c = 1.
- (3.) If  $\Sigma x^2 = c$ ,  $\Sigma lx$  is a maximum when  $x : y : z : \ldots = l : m : n : \ldots$
- (4.) Find the turning values of  $\lambda x^{ma} + \mu y^{mb} + \nu z^{mc}$ , subject to the condition
- $px^a + qy^b + rz^c = d$ . (5.) Find the turning values of  $ax^p + by^q + cz^r$  when  $xyz = d^3$ .
- (6.) If  $xyz = a^2(x+y+z)$ , then yz + zc + xy is a manimum when  $x = y z = \sqrt{3}a$ .
  - (7.) Find the turning values of (x+l)(y+m)(z+n) where  $v^xb^yc^z=d$ .
  - (8.) Find the minimum value of axm + b/xn.
  - (9.) Find the turning values of (3x 2) (x 2)<sup>2</sup> (x 3)<sup>2</sup>.
  - (10.) If cx(b-y) = ay(c-z) = bz(a-x), find the maximum value of each.
- (11.) Find the turning values of x<sup>n</sup>/y<sup>n</sup> (m > n), subject to the condition x y = c. (Bonnet, Nowe, Ann., ser. I., t. 2.)
- (12.) If  $x^py^q + x^qy^p = a$ , then  $x^{p+q} + y^{p+q}$  has a minimum value when  $x = y = (a/2)^{1p+q}$ ; and, in general, if  $\Sigma x^py^q = a$ ,  $\Sigma x^{p+q}$  has a minimum value, x/(u-1), when  $x = y = z = \dots = \{a/(n-1) n\}^{1/(v+q)}$ . Discuss specially the case where p and a have opposite signs.
- (13.) If x<sup>p</sup>y<sup>q</sup>+x<sup>r</sup>y<sup>t</sup>=c, then x<sup>t</sup>y<sup>n</sup> is a maximum when x<sup>p-r</sup>/(ru st) = y<sup>r-q</sup>/(qt pu), the denominators, ru st and qt pu, being assumed to have the same sign. (Desboves, Questions d'Alybbre, p. 455. Paris, 1878.)

<sup>\*</sup> Here, unless the contrary is indicated, all letters denote positive quantities.

- (14.) If p > q, and  $x^p + y^p = a^p$ , then  $x^q + y^q$  is a minimum when  $x = y = a/2^{1p}$ . State the reciprocal theorem.
- (15.) Find the turning values of  $(ax^2 + by^2)/(\sqrt{a^2x^2 + b^2y^2})$  when  $x^2 + y^2 = 1$ . (16.) If  $x_1, x_2, \ldots, x_n$  be each > a, and such that  $(x_1 - a)(x_2 - a) \ldots (x_n - a) = b^n$ , the least value of  $x_1x_2 \ldots x_n$  is  $(a + b)^n$ , a and b being both positive.
- (17.) If f(m) denote the greatest product that can be formed with n integers whose sum is m, show that f(m+1)/f(m) = 1 + 1/q where q is the integral part of m/n.
- (18.) ARCP is a rectangle, APQ meets BC in P, and DC produced in Q. Find the position of APQ when the sum of the areas ABP, PCQ is a minimum.
- (19.) O is a given point within a circle, and POQ and ROS are two perpendicular chords. Find the position of the chords when the area of the quadrilateral PRQS is a maximum or a minimum.
- (20.) Two given circles meet orthogonally at A. PAQ meets the circles in P and Q respectively. Find the position of PAQ when PA.AQ is a maximum or minimum.
- (21.) To inscribe in a given sphere the right circular cone of maximum volume.
  - (22.) To circumscribe about a given sphere the right circular cone of minimum volume.
- (23.) Given one of the parallel sides and also the non-parallel sides of an isosceles trapezium, to find the fourth side in order that its area may be a maximum.
- (24.) To draw a line through the vertex of a given triangle, such that the sum of the projections upon it of the two sides which meet in that vertex shall be a maximum.

### CHAPTER XXV.

#### Limits.

§ 1.1 In laying down the fundamental principles of algebra, it was necessary, at the very beginning, to admit certain limiting cases of the operations. Other cases of a similar kind appeared in the development of the science; and several of them were discussed in chap, xv. In most of these cases, however, there was little difficulty in arriving at an appropriate interpretation; others, in which a difficulty did arise, were postponed for future consideration. In the present chapter we propose to deal specially with these critical cases of algebraical operation, to which the generic name of "Indeterminate Forms" has been given. The subject is one of the highest importance, inasmuch as it forms the basis of two of the most extensive branches of modern mathematics-namely, the Differential Calculus and the Theory of Infinite Series (including from one point of view the Integral Calculus). It is too much the habit in Earthish courses to postpone the thorough discussion of indeterminate forms until the student has mastered the notation of the differential calculus. This, for several reasons, is a mistake. In the first place, the definition of a differential coefficient involves the evaluation of an indeterminate form; and no one can make intelligent applications of the differential calculus who is not familiar beforehand with the notion of a limit. Again, the methods of the differential calculus for evaluating indeterminate forms are often less effective than the more elementary methods which we shall discuss below, and are always more powerful in combination with them. Moreover the notion of a limiting value can be applied to functions of an integral variable such as n! and to other functions besides, which cannot be differentiated, and are therefore not amenable to the methods of the Differential Calculus at all.

§ 2.] The characteristic difficulty and the way of meeting it will be best explained by discussing a simple example. If in the function  $(x^2-1)$  (x-1) we put x-2, there is no difficulty in carrying out successively all the operations indicated by the synthesis of the function; the case is otherwise if we put x=1, for we have  $1^2-1-0$ , 1-1-0, so that the last operation indicated is 0.0-a case specially excluded from the fundamental laws; not included even under the case  $a \cdot 0$   $(a \neq 0)$  already discussed in chap.  $xx \in S$ . The first impulse of the learner is to assume that 0.0-1, in analogy with  $a \cdot a = 1$ ; but for this he has no warrant in the laws of algebra.

Strictly speaking, the function  $(x^2-1)(x-1)$  has no definite value when x - 1: that is to say, it has no value that can be deduced from the principles hitherto laid down. This being so and it being obviously desirable to make as general as possible the law that a function has a definite value corresponding to every value of its argument, we proceed to define the value of  $(x^2-1)/(x-1)$  when x=1. In so doing we are naturally guided by the principle of continuity, which leads us to define the value of  $(x^2-1)(x-1)$  when x=1, so that it shall differ infinitely little from values of  $(x^2-1)(x-1)$ , corresponding to values of x that differ infinitely little from 1. Now, so long as x + 1, no matter how little it differs from 1, we can perform the indicated division; and we have the identity  $(x^2-1)'(x-1) \equiv$ x+1. The evaluation of x+1 presents no difficulty; and we now see that for values of x differing infinitely little from 1, the value of  $(x^2 - 1)(x - 1)$  differs infinitely little from 2. We therefore define the value of  $(x^2-1)(x-1)$  when x=1 to be 2; and we see that its value is 2 in the useful and perfectly intelligible sense that, by bringing x sufficiently near to 1, we can cause  $(x^{i}-1)(x-1)$  to differ from 2 by as little as we please. The value of  $(x^2-1)$  (x-1) thus specially defined is spoken of as the limiting value, or the limit of  $(x^2-1)(x-1)$  for x = 1; and it is symbolised by writing

The resider should observe that the definition of the critical value just
prove has mother advantage, namely, it could not to assert the truth of the
prove that prove 1/2 = x + 1 = x +

$$L \frac{x^2-1}{x-1} = 2$$

where L is the initial of the word "limit." The subscript x - 1 may be omitted when the value of the argument for which the limiting value is to be taken is otherwise sufficiently indicated.

We are thus led to construct the following definition of the value of a function, so as to cover the cases where the value

indicated by its synthesis is indeterminate:—
When, by causing x to differ sufficiently little from a we can
make the value of f(x) approach as near as we please to a finite
definite quantity t, then t is said to be the limiting value, or limit,
of f(x) when x = a; and we write

 $\underset{\mathbf{z}=a}{L} f(x) = l.$ 

Cor. 1. A function is in general continuous in the neighbourhood of a limiting value; and, therefore, in obtaining that rathe we may subject the function to any transformation which is admissible on the hypothesis that the dramment x has any value in the neighbourhood of the critical value a.

We say "in general," because the statement will not be strictly true unless the phrase "differ infinitely little from " mean "differ cither in excess or in defect infinitely little from." It may happen that we can only approach the limit from one side; or that we obtain two different limiting values according as we increase x up to the critical value, or diminish it down to the critical value. In this last case, the graph of the function in the neighbourhood of x=a would have the peculiarity figured in chap. Xv., Fig. 5; and the function would be discontinuous. The latter part of the corollary still applies, however, provided the proper restriction on the variation of x be attended to.

When it is necessary to distinguish the process of taking a limit by increasing x up to a from the process of taking a limit by decreasing x down to a, we may use the symbol  $\underset{x=a+b}{L}$  for the former, and the symbol  $\underset{x=a+b}{L}$  for the latter.

Cor. 2. If L of (x) = l, then f(a + h) = l + d, where d is a function of a and h, whose value may be made as small as we please by sufficiently diminishing h.

5 - 2

This is simply a re-statement of the definition of a limit from another point of view.

Cor. 3. Ann ordinary value of a function satisfies the definition of a limiting value.

For example,  $L(x^2-1)/(x-1)=(2^2-1)/(2-1)-3$ . mark would be superfluous, were it not that attention to the point enables us to abbreviate demonstrations of limit theorems, by using the symbol L where there is no peculiarity in the evaluation of the function to which it is prefixed.

§ 3.1 It may happen that the critical value a, instead of being a definite finite quantity, is merely a quantity greater than any finite quantity, however great. We symbolise the process of taking the limit in this case by writing L f(x), or L f(x), according as the quantity in question is positive or negative.

For example,

$$L_{x=x}(x+1)/x = L_{x=x}(1+1/x) = 1.$$

In this case, we can, strictly speaking, as proach the limit from one side only; and the question of continuity on both sides of the limit does not arise. If, however, we, as it were, join the series of algebraical quantity  $x \dots -1 \dots 0 \dots +1 \dots +x$  through infinity, by considering + x and - x as consecutive values; then we say that f(x) is, or is not, continuous for the critical value x = x, according as |L| f(x) and |L| f(x) have, or have not, the same value. For example, (x+1)/x is continuous for  $x=\infty$ , for we have I: (x+1)/x=1=-I. (x+1)/x; but  $(x^2+1)/x$  is not continuous for x = x.

\$4.1 The value 0 may of course occur as a limiting value: for example,  $L(x, 1)^{\epsilon}$ ,  $(x^2 - 1) = 0$ . It may also happen, even for a finite value of a, that f(x) can be made greater than any finite quantity, however great, by bringing x sufficiently near to a. In this case we write L[f(x) = x]. In this admitting 0 and  $\infty$ as limiting values, the student must not forget that the general rules for evaluating limits are, as will be shown presently, subject in certain cases to exception when these particular limits occur.

#### ENUMERATION OF THE ELEMENTARY INDETERMINATE FORMS.

§ 5.] Let u and v be any two functions of x. We have already seen, in chap. XV., that u+e becomes indeterminate when u and v are infinite but of opposite sign; that  $u\times v$  becomes indeterminate if one of the factors become zero and the other infinite; and that u+v becomes indeterminate if u and v become both zero, or both infinite. We thus have the indeterminate forms—(L)  $\infty - \infty$ , (IL)  $0 \times \infty$ , (III.) 0 + 0, (IV.)  $\infty + \infty$ .

It is interesting to observe that all these really reduce to (III.). Take  $\infty - \infty$  for example. Since  $u + v \equiv (1 + v/u)/(1/u)$ , and  $L1/u = 1/\iota = 0$ , this function will not be really indeterminate unless Lv/u = 1. The evaluation of the form  $\mathbf{x} - \mathbf{x}$  therefore reduces to a consideration of cases (IV) and (III.) at most. Now, since  $u + v \equiv (1/v) + (1/u)$ , case (IV) can be reduced to (III.); and finally, since  $u \times v \equiv u + (1/v)$ , case (II.) can be reduced to (III.);

To exhaust the category of elementary algebraical operations we have to discuss the critical values of  $u^v$ . This is most simply done by writing  $u^v = u^{v \otimes x_v}$  where u is positive and >1. We thus see that  $u^v$  is determinate so long as  $r \log_a u$  is determinate. The only cases where  $v \log_a u$  ceases to be determinate are those where— $(V, v = 0, \log_a u = +\infty, that$  is v = 0, u = 0;  $(VIL) v = 0, \log_a u = -\infty, that$  is v = 0, u = 0;  $(VIL) v = \pm \infty, \log_a u = 0$ , that is  $v = \pm \infty, u = 1$ . There thus arise the indeterminate forms—(V, u) = 0, (VIL) =

All these depend on  $a^{a \times a}$ ; or, if we choose, upon  $a^{a \cdot a}$ ; so that it may be said that there is really only one fundamental case of indetermination, namely,  $0 \div 0$ .

### EXTENSION OF THE FUNDAMENTAL OPERATIONS TO LIMITING VALUES.

§ 6.] We now proceed to show that limiting values as above defined may, under some restrictions, be dealt with in algebraical

<sup>•</sup> The reader is already aware that 1° eives 1; and he may easily convince himself that  $0^{+\infty}$ ,  $0^{-x}$ ,  $x^{+y}$ ,  $x^{-x}$  give 0,  $\pm x$ ,  $\pm x$ , 0 is specifiely, no matter what their origin.

operations exactly like ordinary operands. This is established by means of the following theorems:—

I. The limit of a sum of functions of x is the sum of their limits, provided the latter does not take the indeterminate form  $\infty - \infty$ .

Consider the sum  $f(x) - \phi(x) + \chi(x)$  for the critical value x - a; and let Lf(x) = f',  $L\phi(x) - \phi'$ ,  $L\chi(x) = \chi'$ . Then, by § 2, Cor. 2,

$$f(x) = f' + \alpha, \ \phi(x) = \phi' + \beta, \ \chi(x) = \chi' + \gamma,$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$  can each be made as small as we please by bringing x sufficiently near to a.

Now, 
$$f'(x) - \phi(x) + \chi(x) = f' - \phi' + \chi' + (\alpha - \beta + \gamma).$$

But, obviously,  $a-\beta+\gamma$  can be made as small as we please by bringing x sufficiently near to a. Hence

$$L\{f(x) - \phi(x) + \chi(x)\} = f' - \phi' + \chi',$$
  
=  $Lf(x) - L\phi(x) + L\chi(x)$  (1).

that is,

This reasoning supposes f',  $\phi'$ ,  $\chi'$  to be each finite; but it is obvious that if one or more of them, all having the same sign, become infinite, then  $f' - \phi' + \chi'$  and  $L\{f(x) - \phi(x) + \chi(x)\}$  are both infinite, and the theorem will still be true in the peculiar sense, at least, that both sides of the equality are infinite. If, however, some of the infinities have one sign and some the opposite,  $f' - \phi' + \chi'$  ceases to be interpretable in any definite sense; and the proposition becomes meaningless.

II. The limit of a product of functions of x is the product of their limits, provided the latter does not take the indeterminate form 0 × x.

Using the same notation as before, we have

$$f(x) \phi(x) \chi(x) = (f' + a)(\phi' + \beta)(\chi' + \gamma)$$
$$= f''\phi'\chi' + \Sigma a\phi'\chi' + \Sigma a\beta\chi' + a\beta\gamma.$$

Now, provided none of the limits f',  $\phi'$ ,  $\chi'$  be infinite, since a,  $\beta$ ,  $\gamma$  can all be made as small as we please by bringing x sufficiently near to a, the same is true of  $\Sigma a\phi'\chi'$ ,  $\Sigma a\beta\chi'$ , and  $a\beta\gamma$ . Hence

$$Lf(x)\phi(x)\chi(x) = f'\phi'\chi' - Lf(x)L\phi(x)L\chi(x)$$
 (2).

If one or more of the limits f',  $\phi'$ ,  $\chi'$  be infinite, provided none of the rest be zero, the two sides of (2) will still be equal in the sense that both are infinite; but, if there occur at the same time a zero and an infinite value, then the right-hand side assumes the indeterminate form  $0 \cdot x$ ; and the equation (2) ceases to have any meaning.

III. The limit of the quotient of two functions of x is the quotient of their limits, provided the latter does not take one of the indeterminate forms 0.0 or x, x. We have

$$\frac{f(x)}{\phi(x)} = \frac{f' + \alpha}{\phi' + \beta} = \frac{f'}{\phi'} + \frac{f'' + \alpha}{\phi' + \beta} - \frac{f''}{\phi'} = \frac{f'}{\phi'} + \frac{\alpha\phi' - \beta f''}{\phi'(\phi' + \beta)}.$$

From this equation, reasoning as above, we see at once that, if neither f' nor  $\phi'$  be infinite, and  $\phi'$  be not zero,

$$L\frac{f(x)}{\phi(x)} = \frac{f'}{\phi'} = \frac{Lf(x)}{L\phi(x)}, \quad (3).$$

It is further obvious that if  $f' = \infty$ ,  $\phi' + \infty$ , both sides of (3) will be infinite; if  $\phi' = \infty$ ,  $f' + \infty$ , both sides will be zero; and if  $\phi' = 0$ , f' + 0, both sides will be infinite. In all these cases, therefore, the theorem may be asserted in a definite sense. If, however, we have simultaneously f' = 0,  $\phi' = 0$ , the right hand of (3) takes the form 0, 0; if  $f' = \infty$ ,  $\phi' = \infty$ , the form  $\infty / \infty$ ; and then the theorem becomes meaningless.

§ 7.] If the reader will compare the demonstrations of last paragraph with those of § 8, chap. xv., he will see that (except in the cases where infinities are involved) the conclusions rest merely on the continuity of the sum, product, and quotient. This remark immediately suggests the following general theorem, which includes those of last paragraph as particular cases:—

If F(u, v, w, ...) be any function of u, v, w, ..., which is determinate, and finite in value, and also continuous when

$$u = Lf(x), v = L\phi(x), w = L\chi(x), \dots$$

then

$$LF\{f(x), \phi(x), \chi(x), \ldots\} = F\{Lf(x), L\phi(x), L\chi(x), \ldots\}.$$

The reader will easily prove this theorem by combining \( \) 2. Cor. 2, with the definition of a continuous function given in chap. xv., \( \) 5, 14.

The most important case of this proposition which we shall have occasion to use is that where we have a function of a single function. For example,

$$\begin{split} & \underset{x=1}{L} \{(x^2-1)/(x-1)\}^2 = \{\underset{x=1}{L} \{(x^2-1)/(x-1)\}^2 = 1, \\ & \underset{x=1}{L} \log \{(x^2-1)/(x-1)\} = \log \{\underset{x=1}{L} \{(x^2-1)/(x-1)\} = \log 2. \end{split}$$

## ON THE FORMS 0.00 and z./z in connection with rylional functions.

§ 8.] The form 0.0 will occur with a rational function for the value x = 0 if the absolute terms in the numerator and denominator vanish. The rule for evaluating in this case is to arrange the terms in the numerator and denominator in order of ascending degree, divide by the lowest power of x that occurs in numerator or denominator, and then put x = 0. The limit will be finite, and  $\pm 0$ , if the lowest terms in numerator and denominator be of the same degree; 0 if the term of lowest degree come from the denominator;  $\infty$  if the term of lowest degree come from the numerator. All this will be best seen from the following examples:—

$$L_{x=y}^{-2|x^2|+3|x^3+|x^4|} = L_{x=0}^{-2|+3|x+|x^2|} = \frac{2}{3}.$$

Example 2.

$$L_{x=0} \frac{2x^3 + 3x^4 + x^5}{3x^7 + x^4 + x^6} = L_{x=0} \frac{2x + 3x^7 + x^3}{3 + x^7 + x^4} = \frac{0}{3} = 0.$$

Example 3.

$$\underset{x \sim e^{-2}}{L} \frac{2^{+4} + x^6}{x^8 + x^8} = \underset{x \sim e^{-2}}{L} \frac{2^{+} x^2}{x^2 + x^4} = \frac{2}{0} = \infty ,$$

§ 9.] The form  $\mathbf{z}_{\perp}\mathbf{z}$  can arise from a rational function when, and only when,  $\mathbf{z}_{\parallel}$  z. The limit can be found by dividing numerator and denominator by the highest power of  $\mathbf{z}_{\parallel}$  that occurs in either. If this highest power occur in both, the limit is finite; if it come from the denominator above, the limit is  $\mathbf{0}$ ; if from the numerator alone, the limit is  $\mathbf{z}_{\parallel}$ .

Example 1. 
$$\frac{I}{I} \frac{3\tau^3 + \tau^4}{2\tau - \tau^3 + z^4} = \frac{I}{I} \frac{3/x + 1}{2/x^2 + 1/x + 3} \frac{0 + 1}{0 + 0 + 3} \frac{1}{3}$$

Example 2.

$$L_{x=x} \frac{x^2 + 3x^3 + 4x^4}{2x + x^3 + 6x^6} = L_{x=\pm} \frac{1/x^4 + 3/x^3 + 1/x^2}{2/x^5 + 1/x^3 + 6} = \frac{0}{6} \cdot 0.$$

Example 3.

$$L_{x=x} \frac{x^2 + 3x^3 + 4x^6}{2x + 3x^2 + x^3} = L_{x=x} \frac{1}{2/x^3 + 3/x^3 + 4} = \frac{4}{0} - \infty.$$

§ 10.] If the rational function  $f(x)/\phi(x)$  take the form 0 0 for a finite value of  $x_i \neq 0$ , say for x = a, then, since f(a) = 0,  $\phi(a) = 0$ , it follows from the remainder-theorem that x = a is a common factor in f(x) and  $\phi(x)$ . If we transform the function by removing this factor, the result of putting x = a in the transformed function will in general be determinate; if not, it must be of the form 0/0, and x = a will again be a common factor, and must be removed. By proceeding in this way, we shall obviously in the end arrive at a determinate value, which will be the limit of  $f(x)/\phi(x)$  when x = a.

Example. Evaluate  $(3x^4 - 10x^3 + 3x^2 + 12x - 4)/(x^4 + 2x^3 - 22x^2 + 32x - 8)$  when x = 2. The value is, in the first instance, indeterminate, and of the form 0/0; hence x - 2 is a common factor. If we divide out this factor, we find that the value is still of the form 0/0; hence we must divide again. We then have a determinate result. The work may be arranged thus (see chap.  $v_c$ , § 13):—

The process of division is to be continued until we have two remainders which are not both zero. The quotient of these, 15/11 in the present case, is the limit required.

The evaluation of the limit in the present case may also be effected by changing the variable, an artifice which is frequently of use in the theory of limits. If we put x - a + z, then we have to evaluate  $Lf(a + z)/\phi(a + z)$  when z = 0. Since f(a + z) and  $\phi(a + z)$  are obviously integral functions of z, we can now apply the rule of § 8. It will save trouble in applying this method if it be remembered – 1st, that in arranging f(a + z) and  $\phi(a + z)$  according to powers of z we need not calculate the absolute

terms, since they must, if the form to be evaluated be 0/0, be zero in each case; 2nd, that we are only concerned with the lowest powers of z that occur in the numerator and denominator respectively.

$$\begin{split} L\frac{3x^4 + 10x^4 + 3x^2 + 12x + 4}{x^4 + 2x^3 + 32x^4 + 32x + 8} &= L\frac{3(2+z)^4 + 10(2+z)^3 + 3(2+z)^2 + 12(2+z) + 4}{x^4 + 2x^3 + 22x^2 + 32x + 8} &= L\frac{15z^4 + 12x^2 + 2y^2 + 2x(2+z)^2 + 32(2+z) + 8}{x^4 + 12x^2 + 2y^2 + 3xy} \\ &= L\frac{15z^4 + 12x^2 + 3xy}{x^4 + 12x^2 + 2x^2 + 3xy} \\ &= L\frac{15z^4 + 12x^2 + 3xy}{x^4 + 12x^2 + 2x^2 + 3xy} \end{split}$$

This method is of course at bottom identical with the former; for, since z = x - a, the division by  $z^2$  corresponds to the rejection of the factor  $(x - a)^2$ .

§ 11.] The methods which are applicable to the quotient of two integral functions apply to the quotient of two algebraic sums of constant multiples of fractional powers of x. Each of the two sums might, in fact, be transformed into an integral function of y by putting x = y', where d is the LC.M. of the denominators of all the fractional indices. It is, however, in general simpler to operate directly.

Example. Evaluate

$$l = L \frac{x^{\frac{1}{2}} + x^{\frac{3}{4}} + 3x^{\frac{1}{4}}}{x^{\frac{1}{4}} + 2x^{\frac{1}{4}} + x}.$$

If we divide by  $x^{\frac{1}{2}}$ , the lowest power of x that occurs, we have

$$\begin{split} & l = L \frac{x^{\frac{1}{5}} + x^{\frac{1}{3}} + 3x^{\frac{5}{19}}}{1 + 2x^{\frac{1}{5}} + x^{\frac{2}{3}}} \,, \\ & = \frac{0}{1} = 0. \end{split}$$

§ 12.] The following theorem, although partly a special case under the present head, is of great importance, because it gives the tundamental limit on which depends the "differentiation" of algebraic functions:—

If m be any real commensurable quantity, positive or negative,

$$-\frac{L}{x-1}(x^{n}-1),(x-1)-m$$
(1).

First, let m be a positive integer. Then we have

$$(x^{m}-1)/(x-1) \equiv x^{m-1} + x^{m-2} + \dots + x + 1$$

Hence

$$L_{x=1}(x^m-1)/(x-1) = 1+1+\dots+1+1 \text{ ($m$ terms),}$$

$$= m$$

Next, let m be a positive fraction, say p/q, where p and q are positive integers. Then the limit to be evaluated is  $L_{x=1}^{(p/q-1)}/(x-1)^n$ . If we put  $x=z^q$ , and observe that to x=1 corresponds z=1, the limit to be evaluated becomes  $L_x(z^p-1)^n/(z^q-1)$ . This may be evaluated by removing the common factor z=1; or thus

$$\begin{split} & \underset{r=1}{L} (z^{p}-1)/(z^{q}-1) = \underset{r=1}{L} \left( \frac{z^{p}-1}{z-1} \right) / \left( \frac{z^{q}-1}{z-1} \right), \\ & = \underset{r=1}{L} \left( \frac{z^{p}-1}{z-1} \right) / \underset{r=1}{L} \left( \frac{z^{q}-1}{z-1} \right), \quad \text{by § 6. III.} \\ & = p_{i}^{q} q = m. \end{split}$$

Finally, suppose m to have any negative value, say -n, where n is positive. Then

$$\begin{split} \underset{x=1}{L} & (x^{-n}-1)/(x-1) = \underset{x=1}{L} & (1-x^n)/x^n (x-1), \\ & = -\underset{x=1}{L} & (x^n-1)/(x-1)x^n, \\ & = -\{\underset{x=1}{L} & (x^n-1)/(x-1)\} \times \underset{x=1}{L} & 1/x^n. \end{split}$$

Now, by the last two cases, since n is positive,  $L(x^n-1)/(x-1) = n$ . Also  $L(1/x^n = 1)$ . Hence

$$L_{x=1}(x^{-n}-1)/(x-1)=-n$$
;

that is, in this case also,

$$L(x^m-1)/(x-1)=m.$$

Second Demonstration.—The above theorem might also be deduced at once from the inequality of chap. xxvv,  $\S$  7, as follows:—For all positive values of x, and all positive or negative values of m,  $x^m-1$  lies between  $nx^{m-1}(x-1)$  and m(x-1). Hence  $(x^m-1)/(x-1)$  lies between  $mx^{m-1}$  and m. Now, by

<sup>•</sup> There is here of course the usual understanding (see chap. x . § 2) as to the meaning of  $x^{p,q}$ .

bringing x sufficiently near to 1,  $mx^{m-1}$  can be made to differ as little from m as we please. The same is therefore true of  $(x^m-1)/(x-1)$ ; that is to say,

$$L\left(x^{m}-1\right)(x-1)=m$$
 for all real values of  $m$ .

Example 1. Find the limit of  $(x^p - a^p)/(x^q - a^q)$  when x = a. We have  $L_-(x^p - a^p)/(x^q - a^q) = L_-a^{p-q}\{(x/a)^p - 1\}/\{(x/a)^q - 1\}$ .

$$\begin{array}{ccc} L & (x^p - a^p)/(x^q - a^q) & \stackrel{L}{=} & a^{p-q} L & (y^p - 1)/(x^q - 1) \\ & = a^{p-q} L & (y^p - 1)/(y^q - 1) \\ & & = x^{p-q} L & (y^p - 1)/(y^q - 1) \end{array}$$

where y=x/a. Hence we have, by the theorem of the present paragraph  $L_{-}(x^{p}-a^{p})/(x^{q}-a^{q})=a^{p-q}p/q$ .

Example 2. Evaluate  $\log (x^{\frac{3}{2}}-1) - \log (x^{\frac{1}{2}}-1)$  when x=1.

$$\begin{split} L & \{ \log (x^{\frac{1}{2}} - 1) - \log (x^{\frac{1}{2}} - 1) \} = L \log \{ (x^{\frac{3}{2}} - 1) / (x^{\frac{1}{2}} - 1) \}, \\ & = \log \{ L (x^{\frac{3}{2}} - 1) / (x^{\frac{1}{2}} - 1) \}, \\ & = \log \left\{ L \left(x^{\frac{3}{2}} - 1\right), L \left(\frac{x^{\frac{1}{2}} - 1}{x - 1}\right) \right\}, \\ & = \log \left\{ l \}_{1}^{1} \right\}. \end{split}$$

Example 3. If lx,  $l^2x$ , . . . denote  $\log x$ ,  $\log (\log x)$ , . . . respectively, then, when x = x,  $Ll^p(x+1)/l^px = 1$ .

In the first place, we have

$$l(x + 1)/lx = \{l(x + 1) - lx + lx\}/lx,$$
  
=  $l(1 + 1/x)/lx + 1.$ 

 $-\log 3$ 

Now, when  $x = \infty$ , l(1+1/x) = l1 = 0 and lx = x. Hence Ll(x+1)/lx = 1.

If we assume that  $Ll^{T}(x+1)/l^{T}x = 1$ , we have

 $l^{r+1}(x+1)/l^{r+1}x = \{l^{r+1}(x+1) - l^{r+1}x + l^{r+1}x\}/l^{r+1}x,$ 

 $= l \{ l^r(x+1)/l^r x \}/l^{r+1}x + 1.$ 

Hence

$$Ll^{r+1}(x+1)/l^{r+1}x = l1/x + 1,$$

that is, the theorem holds for r + 1 if it holds for r. But it holds for r = 1, as we have seen therefore for r = 2,  $\Delta c$ . It is obvious that this theorem holds for any logaratimic base for which lx = x.

Example 4. If I have the same meaning as before, and  $\lambda$  have a similar meaning for the base  $\alpha$ , then

$$I. \ \ \mathbf{X}^rx/l^rx = \mathbf{1}/\log a.$$

Let  $\mu = 1/\log a$ . Since  $\Delta x = a/x$ , the theorem clearly holds when r = 1. It is therefore sufficient to show that, if it is true for r, it is true for r + 1. Now

$$\begin{split} \mathbf{X}^{t+1}x/l^{t+1}x &= \nabla (\mathbf{X}^tx)^*l^{t+1}x,\\ &= nl(\mathbf{X}^tx)^*l^{t+1}x,\\ &= n\left(I(\mathbf{X}^tx)^*l^{t+1}x + l^{t+1}x^1\right)^{t+1}x^1,\\ &= n\left(I(\mathbf{X}^tx)^*l^{t+1}x + l^{t+1}x^1\right)^{t+1}x^2, \end{split}$$

Hence, if we assume that  $I N^r x^{r} I^r x = \mu$ , we have

$$LX^{(+)}x_1l^{(+)}x = \mu_1^*/\mu/x + \Gamma_{11}^*$$

#### EXPONENTIAL LIMITS

\$ 13.1 The most important theorem in this part of the subject is the following, on which is founded the differentiation of exponential functions generally :-

The limit of  $(1 + 1/x)^x$  when x is increased without limit either positively or negatively is a finite number (denoted by e) bing between 2 and 3.

The following proof is due to Fort \*.

We have seen (chap. XXIV., § 7) that, if a and b be positive quantities, and m any positive quantity numerically greater than 1, then

 $ma^{m-1}(a-b) > a^m - b^m > mb^{m-1}(a-b)$ 

In this inequality we may put a = (y + 1), y, b = 1, m = y, x, where y>x>1. We thus have

where y>x.

Again, if in (1) we put a = 1, b = (y = 1)[y], (m, y, x) being as before), we have  $\frac{1}{3} > 1 - \left(\frac{y-1}{y}\right)^{y-1}$ .

Hence 
$$\left(1 - \frac{1}{y}\right)^{-} > 1 - \frac{1}{r},$$

$$\left(1 - \frac{1}{y}\right)^{y} > \left(1 - \frac{1}{r}\right)^{r},$$
and therefore 
$$\left(1 - \frac{1}{v}\right)^{-y} < \left(1 - \frac{1}{v}\right)^{-x}$$

and therefore

(3). where u > x. We see from (2) and (3) that, if we give a series of in-

Zeitschrift für Mathematik, vii., p. 46 (1502).

creasing positive values to x, the function  $(1+1/x)^x$  continually increases, and the function  $(1-1/x)^{-x}$  continually decreases. Moreover, since  $x^2 > x^2 - 1$ , we have

The values of  $(1-1/x)^{-x}$  and  $(1+1/x)^x$  cannot, therefore, pass each other. Hence, when x is increased without limit,  $(1-1/x)^{-x}$  must diminish down to a finite limit A, and  $(1+1/x)^x$  must increase up to a finite limit B. The two limits A and B must be equal, for the difference  $(1-1/x)^{-x} - (1+1/x)^x$  may be written  $\{x/(x-1)\}^x - \{(x+1)/x\}^x\}$ ; and by (1) we have

$$\frac{1}{x} \left( \frac{x}{x-1} \right)^x > \left( \frac{x}{x-1} \right)^x - \left( \frac{x+1}{x} \right)^x > \frac{1}{x \left( 1 - 1/x^2 \right)} \left( \frac{x+1}{x} \right)^x \quad (5).$$

But, since, as has already been shown,  $\{x/(x-1)\}^x$  and  $\{(x+1)/x\}^x$  remain finite when  $x=\infty$ , the upper and lower limits in (5) approach zero when x is increased without limit; the same is therefore true of the middle term of the inequality.

It has therefore been shown that  $L = (1 + 1/x)^x$  and  $L = (1 - 1/x)^{-x}$  have a common finite limit, which we may denote by the letter  $\epsilon$ .

Since  $(1+1/6)^6 - 2.521 \dots$  and  $(1-1/6)^{-6} = 2.985 \dots$ , e lies between 2.5 and 2.9. A closer approximation might be obtained by using a larger value of x; but a better method of calculating this important constant will be given hereafter, by which it is found that

The constant c is usually called Napier's Base\*; and it is the logarithmic or exponential base used in most analytical calculations. In future, when no base is indicated, and mere arith-

In honour of Napier, and not because he explicitly used this or indeed any other base.

metical computations are not in question, the base of a logarithmic or exponential function is understood to be  $e_i$  thus  $\log x$  and  $\exp x$  are in general understood to mean  $\log_e x$  and  $\exp_e x$  (that is,  $e^x$ ) respectively.

Cor. 1. 
$$L_{x=0}(1+x)^{1,x} = e$$
.

For  $L(1+1/z)^2=e$ ; and, if we put z=1/x, so that x=0 corresponds to  $z=\infty$ , we have  $L(1+x)^{1/2}=e$ .

Cor. 2. 
$$L_{x=x} \log_a \{(1+1/x)^x\} = L_{x=0} \log_a \{(1+x)^{1/\epsilon}\} = \log_a e$$
.

For, since  $\log_a y$  is a continuous function of y for finite values of y, we have, by  $\S$  7,

$$L \log_a \{(1+1/x)^r\} = \log_a \{L \atop x=x} (1+1/x)^r\},$$

$$= \log_a e.$$

The other part of the corollary follows in like manner.

Cor. 3. 
$$L_{x=\infty} (1+y/x)^x = L_{x=0} (1+xy)^{1/x} = e^y$$
.

If we put 1/z = y/x, then to  $x = \infty$  corresponds  $z = \infty$ ; hence

$$\begin{split} & \underbrace{L}_{z=w} \left( 1 + y_i[x] \right)^c = \underbrace{L}_{z=x} \left( 1 + 1_i z_i \right)^{2p}, \\ & = \underbrace{L}_{z=x} \left( (1 + 1_i z_i)^2 \right)^p, \\ & = \underbrace{L}_{z=x} \left( 1 + 1_i z_i \right)^{2p}, \end{split}$$
 by § 7,

Cor. 4.  $L(a^x-1)/x - \log a$ .

If we put  $y=a^x-1$ , so that  $x=\log_a{(1+y)}$ , and to x=0 corresponds y=0, we have

$$\begin{split} \underset{z = a}{L} & (\alpha' - 1)/\alpha = \underset{y = a}{L} y \cdot \log_{\alpha} (1 + y), \\ & = \underset{y = a}{L} 1 / \log_{\alpha} (1 + y)^{\circ y}, \\ & = 1 / \log_{\alpha} \left\{ \underset{y = a}{L} (1 + y)^{\circ y} \right\}, \\ & = 1 / \log_{\alpha} e - \log \alpha. \end{split}$$

It will be an excellent exercise for the student to deduce directly from the fundamental inequality (1) above, the important result that  $L_{\frac{x\to 0}{x}}(v^x-1)/x$  is

hinte; and thence, by transformation, to prove the lealing theorem of this paragraph.\*.

Cor. 5. If x be any positive quantity

$$e^x > 1 + x$$
,  $\log(1 + x) - x$ ;

and, if x be positive and less than 1,

$$e^{-x} > 1 - x$$
,  $\log(1 - x) - x$ .

Since  $\ell \ge (1+1/n)^n$ , when n may be as great as we please,

$$e^x = 1 \cdot (1 + 1 \cdot n)^{nx} - 1,$$

$$> nx \{(1 + 1/n) - 1\} > x$$
, by chap. xxiv., § 7,

for, however small x, we can by sufficiently increasing u make nx > 1.

Hence 
$$e^x > 1 + x$$
.

It follows at once that  $\log e^x > \log (1+x)$ , that is,  $x > \log (1+x)$ ,

Again, since  $e < (1 - 1/n)^{-n}$  and  $e^{-1} \ge (1 - 1/n)^n$ ,

$$e^{-x} - 1 > \frac{1}{2}(n - 1) n_1^{(n)} - 1,$$
  
 $> nx \frac{1}{2}(n - 1) n - 1...$ 

Hence  $e^{-x} > 1 - x$ , and therefore  $1/(1-x) > \epsilon^x$ .

It follows at once that  $\log (1/(1-x))$ , that is,  $-\log (1-x) > x$ .

Cor. 67. If  $lx_1 Px_2 \dots denote log x_i log (log x)_i$ , respectively, x>y>1, and r be any positive integer, then

$$(x-y) ylyl^2y$$
,  $l^ry \cdot l^{r+1}x - l^{r+1}y$   
 $\geq (x-y) xlyl^2x$ ,  $l^rx$ .

This may be proved by induction as follows, By Cor. 5.

$$lx - ly - l(x, y) = l(1 + (x - y)/y) < (x - y)/y$$

which proves the first inequality when r=0. Assume that it is true for r, i.e. that

 $l^{r+1}x \cdot l^{r+1}y \leq (x-y) y/y/y \cdot y$ , . .  $l^ry$ , then

$$l^{r+1}x \cdot l^{r+1}y = l(l^{r+1}x \cdot l^{r+1}y),$$
  
 $l(1 + (l^{r+1}x - l^{r+1}y) \cdot l^{r+1}y),$ 

$$\begin{array}{ll} U_{r} + (U \cap x - U \cap y) U \cap y_{1}, \\ \sim (U \cap x - U^{r+1}y) U^{r+1}y, & \text{by Cor. 5.} \end{array}$$

Hence the induction is complete.

<sup>\*</sup> See Schlonalch, Zeitschrift für Mathematik, vol. 111., p. 387 (1858).

t Malmsten, Grunert's Archie, viii (1846).

Again, we have by Cor. 5,

$$lx - ly = -l(y/x) = -l(1 - (x - y)/x) > (x - y)/x$$

Using this result, and proceeding by induction exactly as before, we establish the second inequality.

If we put x + 1 and x for x and y respectively we get the important particular result

$$1/xlxl^2x$$
 . . .  $l^rx > l^{r+1}(x+1) - l^{r+1}x$   
>  $1/(x+1) l(x+1) l^2(x+1)$  . . .  $l^r(x+1)$ ,

Cor. 7. From the inequality of Cor. 6, combined with the result of Example 3, § 12, we deduce at once the following important limits:-

$$\begin{array}{l} L _{x \sim x} \left\{ l^r (x+1) - l^r x \right\} = 0, \\ L _{x \sim x} \left\{ l^{r+1} (x+1) - l^{r+1} x \right\} x l x l^2 x \ldots l^r x = 1. \end{array}$$

Example 1. Show that the limit when n is infinite of  $1+1/2+\dots$  $+1/n - \log n$  is a finite quantity, usually denoted by  $\gamma$ , lying between 0 and 1. (Euler, Comm. Ac. Pet. (1734-5),)

Since, by Cor. 5, We have

$$\begin{split} &-\log{(1-1/n)} > 1/n & > \log{(1+1/n)}, \\ &\log{\{n/(n-1)\}} > 1/n & > \log{(n+1)/n}, \\ &\log{\{n-1\}/(n-2)\}} > 1/(n-1) > \log{\{n/(n-1)\}/,} \\ &\log{\{3/2\}} > 1/3 & > \log{\{4/3\}}, \\ &\log{\{2/1\}} > 1/2 & > \log{\{3/2\}}, \\ &1=1 & > \log{\{2/1\}}. \end{split}$$

Hence

 $1 + \log n > \sum 1/n > \log (n+1)$ , Therefore  $1 > \Sigma 1/n - \log n > \log (1 + 1/n)$ .

Now, when n = x,  $\log (1 + 1/n) = 0$ . Thus, for all values of n, however great,  $\Sigma 1/n - \log n$  lies between 0 and 1.

The important constant \( \gamma \) was first introduced into analysis by Euler, and is therefore usually called Euler's Constant. Its value was given by Euler himself to 16 places, namely, γ = 577215661901532(5). (See Inst. Calc. Deff., chap. vi.)\*

<sup>\*</sup> Euler's Constant was calculated to 32 places by Mascheroni in his Adnotationes ad Euleri Calculum Integralem. It is therefore sometimes called Ma-cheroni's Constant. His calculation, which was erroneous in the 20th place, was verified and corrected by Gauss and Nicolai. So. Gauss, Werke, Bd. III., p. 154. For an interesting historical account of the whole matter, see Glaisher, Mess. Math., vol. 1, (1872).

Example 2 Show that I.  $\{1/1 + 1/2 + \dots + 1/n\}/\log n = 1$ .

This follows at once from the inequality of last example.

From this result, or from Example 1, we see that  $L = \{1/1 + 1/2 + ... + 1/n\}$ 

z; and also that  $L = \{1/k + 1/(k+1) + \ldots + 1/n\} = x$ , where k is any finite resitive integer.

#### GENERAL THEOREMS.

- § 14.] Before proceeding further with the theory of the limits of exponential forms, it will be convenient to introduce a few general theorems, chiefly due to Cauchy. Although these theorems are not indispensable in an elementary treatment of limits, the student will find that occasional reference to them will tend to introduce brevity and coherence into the subject.
- 1. For any critical value of x,  $L\{f(x)\}^{\phi(x)} = \{Lf(x)\}^{L\phi(x)}$ , provided the latter form be not indeterminate

This is in reality a particular case of the general theorem of § 7. The only question that arises is as to the continuity of the tunctions of the limits. We may write

$$\{f(x)\}^{\phi(x)} = e^{\phi(x)\log f(x)}$$

Now  $w = \log u$  is a continuous function of u, so long, at least, as u lies between +1 and  $+\infty$ ; and  $e^{wx}$  is a continuous function of r and w. Hence, so long as  $L\phi(x)$  and  $L\log f(x)$  are neither of them infinite, we have

$$\begin{split} L \,_{b}^{t} f(x)_{i}^{(\delta,x)} &= L \,_{b}^{\delta(x) \log f(x)}, \\ &= e^{L\delta(x,L \log f(x)}, \\ &= e^{L\delta(x)\log Lf(x)}, \\ L \,_{b}^{t} f(x)_{i}^{\delta(x)} &= L L f(x)_{i}^{t} L^{\delta(x)}. \end{split} \tag{1}$$

Hence

determinate.

An examination of the special cases where either  $L\phi(x)$  or  $L\log f(x)$ , or both, become infinite, shows that, so long as  $(I_{\ell}f(x))^{LB(x)}$  does not assume one of the indeterminate forms  $\mathbf{0}^{0}$ ,  $\mathbf{1}^{-\delta}$ ,  $\mathbf{1}^{-\delta}$ , both sides of (1) become 0, or both  $\mathbf{z}$ ; so that the theorem may be stated as true for all cases where its sense is

that is.

Hence

H.  $L\{f(x+1)-f(x)\} = L[f(x)/x, provided \ L\{f(x+1)-f(x)\}]$ be not indeterminate\*. (Cauchy's Theorem.)

Since x is ultimately to be made as large as we please, we may put x = h + n, where h is a number not necessarily an integer, but as large as we please, and n is an integer as large as we please.

First, suppose that  $L\{f(x+1)-f(x)\}$  is not infinite, =k say. Since  $L\{f(x+1)-f(x)\}=k$ , we can always choose for h a definite value, so large that for x = h and all greater valuef(x+1) - f(x) - k is numerically less than a given quantity  $\alpha$ , no matter how small a may be. Hence we have numerically

f(b+1) = f(b) - k < a

$$f(h+2) - f(h+1) - k < a,$$

$$f(h+n) - f(h+n-1) - k < a;$$
and, by addition,  $f(h+n) - f(h) - nk < na;$ 
that is,  $f(x) - f(h) - (x-h)k < (x-h)a.$ 
Hence 
$$\frac{f(x)}{x} - \frac{f(h)}{x} - \left(1 - \frac{h}{x}\right)k < \left(1 - \frac{h}{x}\right)a,$$

$$\frac{f(x)}{x} - k < a + \frac{f(h)}{x} - \frac{h}{x}(h+a).$$

Since f(h), h, k, and a are, for the present, fixed, it results that, by making x sufficiently large, we can make f(x)/x - knumerically less than a. Now a can be made as small as we please by properly choosing h; hence the theorem follows.

Next, suppose that  $L\{f(x+1) - f(x)\} = +\infty$ ; then, by taking h sufficiently large, we can assume that

$$f(h+1) - f(h) > l,$$

$$f(h+2) - f(h+1) > l,$$

$$f(h+n) - f(h+n-1) > l,$$

where l is a definite quantity as large as we blease.

<sup>\*</sup> Theorems II, and III, are given by Cauchy in his Analyse Algebragu-(which is Part I, of his Cours d'Analyse de l'École Royale Polytechnique). Paris, 1821.

Hence

$$f(h+n)-f(h)>nl,$$
  
 $f(x)-f(h)>(x-h)l.$ 

that is

$$\frac{f(x)>(x-h)l}{x}>l+\frac{f(h)}{x}-\frac{hl}{x}$$

Since f(h), h, l are all definite, we can, by sufficiently increasing x, render f(h)/x = h/x as small as we please, therefore  $f(x)/x \geq l$ . Now, by properly choosing h, l can be made as large as we please; hence Lf(x)/x = x.

The case where  $L\{f(x+1) - f(x)\} = -\infty$  can be included in the last by observing that (-f(x+1)) - (-f(x)) has in this case  $+\infty$  for its limiting value.

III. 
$$L f(x+1)/f(x) = L \{f(x)\}^{t,x}$$
, provided  $L f(x+1)/f(x)$  be not indeterminate.

This theorem can be deduced from the last by transformation, as follows\*:—

We have 
$$L_{x=x} \{ \psi(x+1) - \psi(x) \} = L_{x=x=x} \frac{\psi(x)}{x}$$
,

where  $\psi(x)$  is any function such that  $L_{\parallel}\{\psi(x+1) - \psi(x)\}$  is not indeterminate. Let now  $\psi(x) - \log f(x)$ ; so that  $\psi(x+1) - \psi(x) = \log f(x+1) - \log f(x)$ ; and  $\psi(x)/x = \{\log f(x)\}/x = \log f(x)^{1/2}$ . Then we have

$$\frac{L}{x = x} \log \left\{ \frac{f(x+1)}{f(x)} \right\} = \frac{L}{x = x} \log \left\{ f(x) \right\}^{1/x}, \\
\log \left\{ \frac{L}{f(x+1)} \right\} = \log \left[ \frac{L}{x} \left\{ f(x) \right\}^{1/x} \right],$$

Hence

provided  $Lf(x+1)_x f(x)$  be not indeterminate. Hence, finally,

$$L \frac{f(x+1)}{f(x)} = L \{f(x)\}^{1/2}$$
.

Cauchy makes the important remark that the demonstrations of his two theorems evidently apply to functions of an integral variable such as x', where only positive integral values of x are admissible.

The reader will find it a good exercise to establish this theorem directly from first principles, as Cauchy does.

For example, we have  $L_{-}(x+1)!/e!=L_{-}(x+1)=o$ . Hence  $L_{-}(x!)^{1/x}-x$ , and consequently  $L_{-}(1/x!)^{1/x}=0$ .

# EXPONENTIAL LIMITS RESUMED.

§ 15.] If a > 1, then  $L = a^{\pi}/x = \infty$ ;  $L = \log_a x/x = 0$ ;  $L = x \log_a x = 0$ .

The first of these follows at once from Cauchy's Theorem (§ 14, II.) for we have

$$L\left(a^{x+1}-a^{r}\right)=La^{r}\left(a-1\right)=x\;.$$
 Hence 
$$La^{r}/x=x\;.$$

Hence

As the theorem is fundamental, it may be well to give an independent proof from first principles.

First, we observe that it is sufficient to prove it for integral values of x alone, for, however large x may be, we can always puts x = f + z where f is a positive proper fraction and z a positive integer. Then we have

$$\begin{split} & L \frac{a^{2}}{a} = L \frac{a^{f+z}}{a^{f+z}}, \\ & = L \frac{a^{f}}{f+z}, \frac{z}{z}, \\ & = e^{af} \frac{1}{L e^{af}}, \frac{1}{f+z}, \frac{a^{2}}{z}, \\ & = e^{f} \frac{1}{L e^{af}}, \frac{a^{2}}{z}, \\ & = e^{f} \frac{a^{2}}{L e^{af}}, \end{split}$$

$$(1).$$

where we have to deal merely with  $Le\bar{e}/z$ , z being a positive integer.

Let  $u_z = a^2/z$ , then  $u_{z+1}/u_z = az/(z+1) = a/(1+1/z)$ . Now, since  $\sum_{s=a} a/(1+1/z) = a > 1$ , we can always assign an integral value of z, say z = r, such that, for that and all greater values of z,  $u_{z+1}/u_z > b$ , where b > 1. We therefore have

$$u_r - u_r, \ u_{r+1}/u_r > b, \ u_{r+2}/u_{r+1} > b, \ \vdots \ u_{z}/u_{z-1} > b.$$

Hence, by multiplying all these inequalities together, we deduce  $u_r > b_r^{r-r} u_r > b_r^{r} u_s/b^{r}$ .

Now  $u_r/b'$  is finite, and, since b>1,  $b^z$  can be made as great as we please by sufficiently increasing z. Hence  $L_{z=z} u_z = \infty$ , on the supposition that z is always integral. But, since a' is finite, it follows at once from (1) that  $L_z a'/x = \infty$ , when x is unrestricted.

The latter parts of the theorem follow by transformation.

If we put a' = y, so that  $x = \log_a y$ , and to  $x = \infty$  corresponds y = x, we have

 $\mathbf{z} = \underset{\mathbf{z} = \mathbf{z}}{L} \frac{\alpha^{\mathbf{z}}/\mathbf{z} = L}{\mathbf{y} = \mathbf{z}} \frac{y}{\log_{\alpha} y}.$  Hence  $L \frac{\log_{\alpha} y}{y} = 1/\mathbf{z} = 0.$ 

If we put  $a^x = 1/y$ , so that  $x = -\log_a y$ , and to  $x = \infty$  corresponds y = 0, we have

$$x = L_{x=x} \alpha'/x = -L_{y\to +0} 1/y \log_a y.$$

$$L_y \log_a y = -1/x = 0.$$

Hence

Example 1. Show that, if a>1 and n be positive, then L,  $a^x/x^n=\infty$ ;  $L \log_a x/x^n=0$ ; L,  $x^n \log_a x=0$ .

for, since a > 1 and n is positive, we have  $a^{1:n} > 1$ , so that  $L(n^{1:n})^{r}/r = \infty$  and  $x^{n} = x$ .

The two remaining results can be established in like manner, if we put  $y = \log_a x$  in the one case, and  $y = -\log_a x$  in the other.

It should be noticed that if n be negative we see at once that L,  $a^x/x^n = \infty$ ; L,  $\log_a x/x^n = \infty$ ; L,  $x^n \log_a x = -\infty$ ,

Example 2. If x be any fixed finite quantity,  $L x^n/n! = 0$ .

Since n is to be made infinite, and x is finite, we may select some finite positive integer k such that x < k < n. Then we have

$$\begin{aligned} & \frac{x^n}{\tilde{n}!} = \frac{x^{k-1}}{(k-1)!} \cdot \frac{x}{\tilde{k}} \cdot \frac{x}{k+1} \cdot \dots \cdot \frac{x}{\tilde{n}}, \\ & \leq \frac{x^{k-1}}{(k-1)!} \left(\frac{x}{\tilde{k}}\right)^{n-k+1}, \end{aligned}$$

Now, since x < k,  $L(x/k)^{n-k+1} = 0$ , hence the theorem.

Example 3. Lm(m-1) . . . (m-n+1)/n! = 0 or x , according as m > 1or < -1,

First, let m > -1, then m+1 is positive. We can always find a finite positive integer k such that m+1 < k < n. Therefore we may write

$$_{m}C_{n} = \frac{m(m-1)\ldots (m-n+1)}{n!} = (-)^{n-k+1}{}_{m}C_{k-1}\left(1 - \frac{m+1}{k}\right)\left(1 - \frac{m+1}{k+1}\right)\ldots$$
 
$$= (-)^{n-k+1}{}_{m}C_{k-1}P, \text{say}.$$
 Now

Now

$$\log \mathbf{1}/P = -\log\left(1 - \frac{m+1}{k}\right) - \log\left(1 - \frac{m+1}{k+1}\right) - \dots - \log\left(1 - \frac{m+1}{n}\right),$$

$$> (m+1)/k + (m+1)/(k+1) + \dots + (m+1)/n,$$

by § 13, Cor. 5. Also, by § 13, Example 2, the limit of (m+1)/k + (m+1)/(k+1) + . . . + (m+1)/n is infinite when n = ∞ . It follows, therefore, that I.P = 0. and therefore that  $L_m C_n = 0$ .

Next, let m < -1, say  $m = -(1 + \alpha)$ , where  $\alpha$  is a positive finite quantity. We may now write

 ${}_{m}C_{n} = (-)^{n} \frac{(1+\alpha)(2+\alpha)}{1-2} \cdots \frac{(n+\alpha)}{n} = (-)^{n}P$ , say.

Now

$$\log P = -\log\left(1 - \frac{\alpha}{1+\alpha}\right) - \log\left(1 - \frac{\alpha}{2+\alpha}\right) - \dots - \log\left(1 - \frac{\alpha}{n+\alpha}\right),$$

 $> \alpha/(1 + \alpha) + \alpha/(2 + \alpha) + ... + \alpha/(n + \alpha)$ 

$$> \alpha/(1+p) + \alpha/(2+p) + . . . + \alpha/(n+p),$$

where p is the least integer which exceeds a. But the limit of a/(1+p) $+\alpha/(2+p)+\ldots+\alpha/(n+p)$  is infinite. Hence  $LP=\infty$ .

When m = -1,  ${}_{m}C_{n} = (-1)^{n}$ , and the question regarding the limiting value does not arise.

§ 16.] The fundamental theorem for the form 0° is that  $L x^{x} = 1.$ 

This follows at once from last paragraph; for we have

$$I_{\epsilon}r^{x} = I_{\epsilon}e^{x\log x} = e^{Lx\log x} = e^{0} = 1.$$

Example 1.  $L_{x^n}(x^n)^x = 1$ .

 $I_{-}(r^{n})^{x} = I_{-}r^{nx} = I_{-}(r^{x})^{n} = (I_{-}r^{x})^{n} = 1^{n} = 1$ For

Example 2.  $L_n x^{z^n} = 1$  (n positive).

 $Lx^{x^n} = Le^{x^n \log x} = e^{Lx^n \log x} = e^0 = 1$ , by § 15, Example 1. For

N.B.—If n be negative,  $L x^{x^n} = 0^x = 0$ .

§ 17.] If n and v be functions of x, both of which vanish when x = a, and are such that L, v, n° -1, where n is positive and neither 0 nor κ, and l is not infinite, then L, n° +1, provided the limit be so approached that u is positive\*.

For 
$$Lu^v = L(u^{u^n})^{v - v^n} - (Lu^{u^n})^{Lv - u^n}$$
.

Now, by § 16, Example 2, since n is positive,  $\underset{n\to\infty}{L} u^{n^n} = 1$ . Hence  $Lu^n + 1^{k} = 1$ .

If  $\sum_{x \in a} c_x u^x = x$ , this transformation leads to the form  $\mathbf{1}^*$ ; and therefore becomes illusory.

The above theorem includes a very large number of particular cases. We see, for example, that, if  $Lv_u$  be determinate and not infinite, then  $Lu^v=1$ . Again, since, as we shall prove in chapter XXX, every algebraic function vanishes in a finite ratio to a positive finite power of  $x-a_i$  it follows that every such function vanishes in a finite ratio to a positive finite power of every other such function. Hence  $Lu^v=1$  whenever u and vare algebraic functions of  $x^{\pm}$ .

 $\begin{array}{ll} \text{Example. Evaluate } L\{x-1+\sqrt{(x^2-1)}\}^{\sqrt[3]{x^2-1}} \text{ when } x=1.\\ \text{Here } u^{-x}\sqrt{(x-1)}\{\sqrt{(x-1)}+\sqrt{(x^2+x+1)}\}, \quad v\in \frac{\pi}{2}(x-1), \quad u^{2/2}/v \Rightarrow \{\sqrt{(x-1)}+\sqrt{(x^2+x+1)}\}^{2/2}.\\ \text{Hence } Lu^{2/2}/v = \frac{\pi}{2}, \quad \text{Therefore } Lu^{x}-L(u^{2/2})^{x}v^{2/2}=1^{1/\sqrt{2}}=1. \end{array}$ 

§ 18 ] In cases where the last theorem does not apply, the evaluation of the limit can very often be effected by writing n't in the form r\*isen, and then seeking by transformation to deduce the limit of r log n from some combination of standard cases.\*.

Example. Evaluate  $x^{(1)}\pi(e^x - 1)$  when x = 0.

It is obviously suggested to attempt to make this depend on  $L(\{(e^x-1)/x\}-1)$ . This may be effected as follows. We have

$$x^{(1)} \cdot (e^{x-1}) \equiv e^{\log x \cdot \log(e^{x}-1)}$$
.

See Franklin, American Journal of Mathematics, 1878.

<sup>†</sup> See Sprague, Proc. Fdinh. Math. Sec., vol. 111, p. 71 (1885).

At one time an erroneous impression prevailed that the indeterminate form 0° has always the value 1.—See Crelle's Jour., Ed. xtt.

Now

$$\frac{\log x}{\log \left(e^x - 1\right)} = \frac{\log x}{\log \left\{\left(e^x - 1\right) / x\right\} + \log x},$$

$$= \frac{1}{\log \ell(c^2-1)/x!/\log x + 1}.$$

Since  $L\log\{(c^x-1)/x\}=0$ , by § 13, Cor. 4, and  $L\log x=-x$ , we see that  $L\log x/\log(c^x-1)=1$ .

Hence  $I_{x^{1}\log(e^{x}-1)} = e$ .

§ 19.] Since  $u^{\bullet} = 1/(1 u)^{\bullet}$ , indeterminates of the form  $\infty$ : can always be made to depend on others of the form 0°, and treated by the methods already explained.

Example. Evaluate  $(1+x)^{1/2}$  when x=x.

Let 1+x=1/y, so that y=0 when x=x; then we have  $L_{x=x}(1+x)^{1:x}=L_{y=0}\{1/y^{y+1-y}\}=1/L\left(y^{y}\right)^{1/1-y}.$ 

Now 
$$Ly^y = 1$$
 and  $L1/(1-y) = 1$ ; hence  $L(1+x)^{1/x} = 1$ .

§ 20.] The fundamental case for the form 1' is  $\underset{x \to \infty}{L} (1 + 1/x)' = L(1 + x)^{1x} = e$ , already discussed in § 13. A great variety of other cases can be reduced to this by means of the following

theorem.

If u and v be functions of x such that u-1 and v=x when x=a, then  $Lu^v=e^{Lv(u-1)}$ , provided Lv (u-1) be determinate.

We have in fact

$$u^{v} = \{(1 + \overline{u-1})^{1} \cdot (u-1)^{1} \cdot (u-1) \}$$
Hence, by § 7,

$$Lu^v = L \{ (1 + \widehat{u - 1})^{1 \cdot (u - 1) \cdot (Lv(u - 1))}, \\ = e^{Lv(u - 1)}$$

provided Lv(u-1) be determinate.

Example 1. 
$$L_{x=1} x^{1/(x-1)} = L_{x-1} (1 + x - 1)^{1/(x-1)} = e$$
.

Example 2. Evaluate  $(1 + \log x)^{1/(x-1)}$  when x = 1. We have

$$t = L (1 + \log x)^{1/(x-1)} = L\{(1 + \log x)^{1/\log x}\}^{\log x \cdot (x-1)}$$

 $=e^{L\log x\cdot(x-1)},$  Now  $L\log x/(x-1)=L\log x^{1\cdot(x-1)}=\log Lx^{1\cdot(x-1)}=\log e=1$ . Hence l=e.

#### TRIGONOMETRICAL LIMITS.

§ 21.] We deal with this part of the subject only in so far as it is necessary for the analytical treatment of the Circular Functions in the following chapters. We assume for the present that these functions have been defined geometrically in the usual manner. We shall require the following inequality theorems:—

If x be the number of radians (circular units) in any positive analyless than a right angle, then

L 
$$\tan x > x > \sin x$$
:

11. 
$$x > \sin x > x - \{x^i\}$$

III. 
$$1 > \cos x > 1 - \frac{1}{2}x^2$$
.

If PQ be the arc of a circle of radius r, which subtends the central angle 2x, and if PT QT be the tangents at P and Q, then we assume as an axiom that

$$PT + TQ > \text{arc } PQ > \text{chord } PQ$$

Hence, as the reader will easily see from the geometric definition of the trigonometrical functions, we have

$$2r \tan x$$
,  $2rx > 2r \sin x$ ;

that is, which is I.

$$\tan x > x > \sin x$$
,

To prove II., we remark that  $\sin x = 2 \sin \frac{1}{2}x \cos \frac{1}{2}x$  =  $2 \tan \frac{1}{2}x \cos^2 \frac{1}{2}x = 2 \tan \frac{1}{2}x (1 - \sin^2 \frac{1}{2}x)$ . Hence, since, by L,  $\tan \frac{1}{2}x > \frac{1}{2}x$  and  $\sin \frac{1}{2}x < \frac{1}{2}x$ , we have

$$\sin x > 2 \cdot \frac{1}{2}x \left\{1 - (\frac{1}{2}x)^2\right\},$$
  
>  $x - \frac{1}{2}x^3$ .

The first part of III, is obvious from the geometric definition of  $\cos x$ . To prove the latter part, we notice that  $\cos x = 4 - 2 \sin^2 kx$ ; hence, by L.

$$\cos x > 1 - 2(\frac{1}{2}x)^2$$
  
>  $1 - \frac{1}{2}x^2$ .

§ 22.] The fundamental theorem regarding trigonometrical
limits is as follows:—

If x be the radian measure  $^*$  of an angle, then  $L_{x=0}(\sin x/x) = 1$ .

This follows at once from the first inequality of last paragraph. For, if  $x < k\pi$ , we have

$$\tan x > x > \sin x$$
;

therefore

$$\sec |x\!>\!x\!|\sin x +\!1.$$

In all that follows, and, in fact, in all analytical treatment of the trigonometrical functions, the argument is assumed to denote radian measure.

If we diminish x sufficiently, see x can be made to differ from 1 by as little as we please. Hence, by making x sufficiently small, we can make  $x/\sin x$  lie between 1 and a quantity differing from 1 as little as we please. Therefore

$$L r / \sin x = 1$$

Hence also

$$L \sin x/x = 1.$$

Cor. 1.

$$L_{r=0} \tan x/x = 1.$$

For  $L \tan x/x = L(\sin x/x)/\cos x$ .  $L \sin x/x \times L 1/\cos x = 1 \times 1 - 1$ .

Cor. 2. 
$$L \sin \frac{\alpha}{x} / \frac{\alpha}{x} = L \tan \frac{\alpha}{x} / \frac{\alpha}{x} - 1$$
 provided a is either a con-

stant, or a function of x which does not become infinite when  $x = \infty$ . This is merely a transformation of the preceding theorems. It should also be remarked that

$$L\left(\sin\frac{\alpha}{x}/\frac{\alpha}{x}\right)^{\beta} = L\left(\tan\frac{\alpha}{x}/\frac{\alpha}{x}\right)^{\beta} = 1,$$

provided  $\alpha$  and  $\beta$  are constants, or else functions of x which do not become infinite when  $x = \infty$ .

If, however,  $\alpha$  were constant, and  $\beta$  a function of x which becomes infinite when  $x = \infty$ , then each of the two limits would take the form  $1^*$ , and would require further examination.

§ 23.] Many of the cases excepted at the end of last paragraph can be dealt with by means of the following results, which we shall have occasion to use later on:—

If a be constant, or a function of x which is not infinite when  $x = \infty$ , then

$$L_{x=x} \left( \sin \frac{\alpha}{x} / \frac{\alpha}{x} \right)^{x} = 1;$$

$$L_{x=x} \left( \cos \frac{\alpha}{x} \right)^{x} - 1;$$

$$L \left( \tan \frac{\alpha}{x} / \frac{\alpha}{x} \right)^{x} = 1.$$

To prove the first of these, we observe that for all values of a/x less than  $\frac{1}{2}\pi$  we have, by § 21, II.,

$$1 > \left(\sin\frac{\alpha}{x} / \frac{\alpha}{x}\right)^x > \left\{1 - \frac{1}{x} \left(\frac{\alpha}{x}\right)^2\right\}^x.$$

Yow

$$\begin{split} \frac{L}{d-a} & (1-a^2) (L^a)^a - \frac{L}{L^a} (1-a^2) (L^2)^{-4a^2b} \Big\}^{-a^2b} \frac{L}{L^a} \\ & - \left\{ \frac{L}{L-a} (1-a^2) (L^2)^{-4a^2b} \right\}^{-L^2b^2b} \\ & = e^b - 1, \text{ by } \otimes 7 \text{ and } 13, \\ & L \left( \sin \frac{a}{L} / a \right)^b - 1. \end{split}$$

Hence

In exactly the same way we can prove that  $L_{x=x} \left(\cos \frac{a}{x}\right)^x = 1$ .

Finally, since

$$L\left(\tan\frac{a}{x}/\frac{a}{x}\right)^x - L\left(\sin\frac{a}{x}/\frac{a}{x}\right)^x + L1/\!\!\left(\cos\frac{a}{x}\right)^x,$$

the third result follows as a combination of the first two.

Example. Evaluate  $(\cos x)^{1/2}$  when x = 0. By § 20, we have  $L(\cos x)^{1/2} = L(\cos x - 1)/x^2 = -2\sin^2\frac{1}{2}x/x^2 = -\frac{1}{2}(\sin\frac{1}{2}x/\frac{1}{2}x)^2$ . Hence  $L(\cos x - 1)/x^2 = -\frac{1}{2}$ .

We therefore have

 $L(\cos x)^{1,x^2} = e^{-\frac{1}{2}}$ ,

#### SUM OF AN INFINITE NUMBER OF INFINITELY SMALL TERMS,

 $\S$  24.] If we consider the sum of n terms, say,  $u_1+u_2+\ldots+u_n$ , each of which depends on n in such a way that it becomes infinitely small when n becomes infinitely great, it is obvious that we cannot predict beforehand whether the sum will be finite or infinite. Such a sum partakes of the nature of the form 0 > x, for we cannot tell n priori whether the smallness of the individual terms, or the infiniteness of their number, will ultimately predominate. We shall have more to do with such cases in our next chapter; but the following instance is so famous in the history of the Infinitesimal Calculus before Newton and Leibnitz that it deserves a place here.

If 
$$r + 1$$
 be positive, then
$$L_{n-r} = \frac{L}{r} (1^r + 2^r + \dots + n^r) [n^{r+1} - 1] (r + 1).$$

In the case where r is an integer this theorem may be deduced from the formula of chap, xx, § 9.

The proofs usually given for the other cases are not very rigorous; but a satisfactory proof may be obtained by means of the inequality

$$(r+1) x^r (x-y) \gtrsim x^{r+1} - y^{r+1} \gtrsim (r+1) y^r (x-y)$$
 (1),

which we have already used so often.

If we put first x = p, y = p - 1, and then x = p + 1, y - p, we deduce

$$(p+1)^{r+1} - p^{r+1} \gtrsim (r+1) p^r \gtrsim p^{r+1} - (p-1)^{r+1}$$
 (2)

where the upper or the lower signs of inequality are to be taken according as the positive number r+1 is > or <1.

If in (2) we put for p in succession 1, 2, 3, . . . , n and add all the resulting inequalities we deduce

$$(n+1)^{r+1}-1 \ge (r+1)(1^r+2^r+...+n^r) \ge n^{r+1}$$

Hence

$$\{(1+1/n)^{r+1}-1/n^{r+1}\}/(r+1) \ge (1^r+2^r+...+n^r)/n^{r+1}$$

 $= \frac{1}{(r+1)}$ .

That is to say,  $(1^r + 2^r + \ldots + n^r)/n^{r+1}$  always lies between 1, (r+1) and  $4(1+1/n)^{r+1} - 1/n^{r+1}/(r+1)$ . But  $L = (1+1/n)^{r+1} = 1$ ; and  $L = 1/n^{r+1} = 0$ , since r+1 is positive. Hence the second of the two enclosing values ultimately coincides with the first, and our theorem follows.

It may be observed that, if r+1 were negative, the proof would fail, simply because in this case  $L|1/n^{r+1}| = \infty$ .

Cor. 1. If s be any finite integer, and r + 1 be positive,

$$L_{n=\infty} \{1^r + 2^r + \dots + (n-s)^r\}/n^{r+1} = 1/(r+1).$$

This is obvious, since  $L\{1^r+2^r+\ldots+(n-s)^r\}/u^{r+1}$  differs from  $L(1^r+2^r+\ldots+n^r)/u^{r+1}$  by a finite number of infinitely small terms.

Cor. 2. If a be any constant, and r + 1 be positive,

$$L_{n=\infty} \{ (a+1)^r + (a+2)^r + \dots + (a+n)^r \} / n^{r+1} = 1/(r+1).$$

This may be proved by a slight generalisation of the method used in the proof of the original theorem.

Cor. 3. If a and c be constants, and r + 1 + 0.

$$\begin{split} L & \{ (na+c)^r + (na+2r)^r + \dots + (na+nc)^r \} / n^{r+1} \\ & = \{ (a+c)^{r+1} - a^{r+1} \} / c \ (r+1). \end{split}$$

This also may be proved in the same way, the only fresh point being the inclusion of cases where r + 1 is negative.

§ 25.] Closely connected with the results of the foregoing paragraph is the following Limit Theorem, to which attention has been drawn by the researches of Dirichlet:—

If  $a,b,\rho$  be all positive, the limit, when  $n=\infty$ , of the sum of n terms of the series

$$\frac{1}{a^{1+\rho}} + \frac{1}{(a+b)^{1+\rho}} + \frac{1}{(a-2b)^{1+\rho}} + \dots + \frac{1}{(a+nb)^{1+\rho}} + \dots$$
 (1),

is finite for all finite values of  $\rho$ , however small; and, if  $\sum_{n=0}^{\infty} 1/(a+nb)^{1+\rho}$  denote this limit, then

$$L \rho \sum_{n=0}^{\infty} 1 \cdot (a + nb)^{1+\rho} = 1/b$$
 (2).

By means of the inequality (1) of last paragraph, we readily establish that

$$\{a + (p-1)|b\}^{-\rho} - \{a + pb\}^{-\rho} > pb^{-1}u + pb\}^{-\rho-1} > \{a + pb\}^{-\rho} - \{a + (p+1)|b\}^{-\rho} - \{a\}^{-\rho-1} > \{a + pb\}^{-\rho}$$
 (3).

Putting, in (3), 0, 1, 2, . . ., n successively in place of p, adding the resulting inequalities, and dividing by bp, we deduce

$$\frac{1}{h_{p}} \left\{ \frac{1}{(a-b)^{p}} - \frac{1}{(a+nb)^{p}} \right\} \ge \sum_{p=0}^{\infty} \frac{1}{(a+pb)^{p+p}} \ge \frac{1}{h_{p}} \left\{ \frac{1}{(a^{p})} - \frac{1}{(a+(n+1))b^{1p}} \right\}$$
(4)

Since  $-L|1/(a+nb)|^p = 0$ , and  $-L|1/(a+(n+1)|b|^p) = 0$ , when  $n = \infty$ , we deduce from (1),

$$\frac{1}{\rho b (a-b)^{\rho}} \ge \sum_{p=\nu}^{\infty} \frac{1}{(a+pb)^{\frac{1}{1-\rho}}} \ge \frac{1}{\rho b a p^{\rho}}$$
 (5).

From (5) the first part of the above theorem follows at once; and we see that  $1/\rho b (a-b)^\rho$  and  $1/\rho b^{\rho+1}$  are finite upper and lower limits for the sum in question.

We also have

whence it follows, since  $L|1/b|(a-b)^{\rho}=L|1/ba^{\rho}-1/b|$ , when  $\rho=0_{F}$  that

$$\underset{\rho=0}{L} \rho \overset{\infty}{\underset{p=0}{\Sigma}} \frac{1}{(a+pb)^{1+\rho}} - \frac{1}{b}.$$

From the theorem thus proved it is not difficult to deduce the following more general one, also given by Dirichlet:—

If  $k_1, k_2, \dots, k_n, \dots$  be a series of positive quantities, no one of which is less then any following one, and if they be such that L/T/t = a, where T is the number of the k's that do not exceed t, then  $\frac{2}{3}1/k_n^{-1/p}$  is finite for all positive finite values of p, however

small; and  $L_{\rho=0} \rho \sum_{n=0}^{\infty} 1/k_n^{-1+\rho} = \alpha^*$ . Cor. It follows from (5) that

$$\frac{1}{\rho (u-1)^{\rho}} > \underbrace{L}_{n=x} \left\{ \frac{1}{(u^{1+\rho}} + \frac{1}{(u+1)^{1+\rho}} + \dots + \frac{1}{(u+n)^{1+\rho}} \right\} > \frac{1}{\mu t^{\rho}} \quad (6),$$
an inequality which we shall have occasion to use hereafter.

# GEOMETRICAL APPLICATIONS OF THE THEORY OF LIMITS.

§ 26.] The reader will find that there is no better way of strengthening his grasp of the Analytical Theory of Limits than by applying it to the solution of geometrical problems. We may point out that the problem of drawing a tangent at any point of the graph of the function g = f(x) can be solved by evaluating the limit when h = 0 of  $\{f(x + h) - f(x)\}/h$ ; for, as will readily be seen by drawing a figure, the expression just written is the tangent of the inclination to the axis of x of the secant drawn through the two points on the graph whose abscissae are x and x + h; and the tangent at the former point is the limit of the

See Dirichlet, Crelle's Jour., Ed. 19 (1839) and 53 (1857); also Heine, ibid., Bd. 31.

secant when the latter point is made to approach infinitely close to the former\*.

Example. To find the inclination of the tangent to the graph of  $y = e^x$  at the point where this graph crosses the axis of y.

If  $\theta$  be the inclination of the tancent to the x-axis, we have

$$\tan \theta = L \left(e^{0+h} - e^{0}\right)/h,$$

$$= L \left(e^{h} - 1\right)/h,$$

$$= \log e = 1.$$

$$\theta = \frac{1}{2}\pi.$$

Hence

 $\S$  27.] The limit investigated in  $\S$  24 enables us to solve a problem in quadratures; and thus to illustrate in an elementary way the fundamental idea of the Calculus of Definite Integrals. We may in fact deduce from it an expression for the area included between the graph of the function  $y = x^r/V^{-1}$ , the axis of  $x_i$  and any two ordinates.

Let A and B be the feet of the two ordinates, a, b the corresponding abscisses, and b - a = c + 1. Dwide AB into a equal parts, draw the ordinates through A, B, and the n - 1 points of division; and construct—1st, the series of rectangles whose bases are the n parts, and whose altitudes are the lat,  $2nd_1 = -1$ , and n but ordinates respectively;  $2nd_1$  the series of rectangles whose bases are as b fore, but whose altitudes are the  $2nd_1$   $3rd_1 = -1$ , (n+1)th ordinates.  $4R_1$  and d, be the sums of the areas of the first and second series of rectangles, and A the area neleosed between the curve, the axis of x and the ordinates through d and B, then obliviously  $I_{x} < d < 1$  solutions  $I_{x} < d < 1$ .

Now

$$I_n = c \{ar + (a + c/n)^r + (a + 2c/n)^r + \dots + (a + n - 1c/n)^r\}/nl^{r-1};$$
  
 $J_n = c \{(a + c/n)^r + (a + 2c/n)^r + \dots + (a + nc/n)^r\}/nl^{r-1}.$ 

Since  $J_n - I_n - c (b^r - a^r)/n(r^{-1})$ , which vanishes when n = x,  $LI_n = LJ_n$ , and therefore  $A - LJ_n$ , when n = x. Hence

$$\begin{split} A &= \frac{c}{l^{r-1}} L \frac{(na+4c)^r + (na+2c)^r + \dots + \frac{4}{n^{r+1}}}{n^{r+1}}, \\ &= \frac{c}{l^{r-1}} \frac{4(a+c)^{r+1} - a^{r+1}}{c(r+1)} \frac{1}{4}, \text{ by § 21, Cor. 3.} \end{split}$$

Hence  $A = \left( L^{r+1} - a^{r+-} \right) / (r+1) \, l^{r-1}.$ 

This gives, when  $r = \frac{1}{2}$ , and a = 0, the Archimedian rule for the quadrature of a parabolic segment.

We would carnestly recommend the barner at this stage to begin (if
the has not air ody done so; the study of Prox's Curre Tracing, a work which
should be in the hands of every one who aims at becoming a mathematician,
either practical or sea utific.

<sup>+</sup> The reaster should draw the figure for himself.

#### NOTION OF A LIMIT IN GENERAL. ABSTRACT THEORY OF IRRATIONAL NUMBERS.

§ 28.] In the earlier part of this chapter limiting values have, been associated with the supply of values for a function in special cases where its definition fails owing to the operations indicated becoming algebraically illegitimate. This view naturally suggested itself in the first instance, because we have been more concerned with the laws of operation with algebraic quantity than with the properties of quantity regarded as continuously variable.

It is possible to take a wider view of the notion of a limit; and in so doing we shall be led to several considerations which are interesting in themselves, and which will throw light on the following chapter.

Although in what precedes we defined a limit, it will be observed that no general criterion was given for the existence of a finite definite limit. All that was done was to give a demonstration of the existence of a limit in certain particular cases. When the limit is a rational number, the demonstrations present no logical difficulty; but when this is not the case we are brought face to face with a fundamental arithmetical difficulty, viz. the question as to the definition of irrational number. For example in proving the existence of a finite definite limit for (1 + 1/x) when x is increased indefinitely, what we really proved was not that there exists a quantity e such that  $e - (1 - 1)xy^e$  can be made smaller than any assignable quantity, but that two rational numbers A and B can be found differing by as little as we please such that  $(1+1/x)^r$  will lie between them if only x be made sufficiently large. From this we infer without farther proof that a definite limit exists, whose value may be taken to be either A or B. For practical purposes this is sufficient, because we can make A and B agree to as many places of decimals as we choose; but the theoretical difficulty remains that the fimit e, of whose definite existence we speak, is any one of an infinite number of different rational numbers, the particular one to be differently selected according to circumstances, there being in fact \* no single

<sup>\*</sup> See chap. xxvIII., § 3,

rational number which can claim to be the value of the limit. The introduction of a definite quantity e as the value of the limit under these circumstances is justified by the fact that we thus cause no algebraic contradiction. Such quantities as  $\sqrt{2}$ ,  $\frac{1}{2}$ 4, &c, have already been admitted as algebraic operands on similar grounds.

§ 29.1 The greater refinement and rigour of modern mathematics, especially in its latest development—the Theory of Functions—have led mathematicians to meet directly the logical difficulties above referred to by giving a priori an abstract definition of irrational real quantity and building thereon a purely arithmetic theory. There are three distinct methods, commonly snoken of as the theories of Weierstrass, Dedekind and Cantor. A mixture of the two last, although perhaps not the most elegant method of exposition, appears to us best suited to bring the issues clearly before the mind of a beginner. We shall omit demonstrations, except where they are necessary to show the sequence of ideas, the fact being that the initial difficulties in the Theory lie not in framing demonstrations, but in seeing where new definitions and where demonstrations are really necessary. a similar reason we shall at once assume the properties of the onefold of Rational Numbers as known; and also the theory of

The theory of Weierstrass, earliest in point of time, was given in his learners, but not published by himself. An account of it will be found in Biermann, Phorm of a tanalytic shear Functional (Leipzig, 1887), pp. 19-33.

A brief but excellent account of Dedekand's theory is given by Weber, Leibbuch der Alphei (Branes-liveig, 1895, 1898), pp. 4-16; see also Pedekind's two tracts, Stripplent und resitionale Zahlen (Braunechweig, 1872, 1892); and War send and was sellen die Zahlen? (Braunechweig, 1888, 1899). For expositions of Cantro's theory see Math. Ann., Bd. 5 (1872), p. 128, and He Ed. 21 (1883), p. 565; also Heme, Crelle's Jour., Bd. 74 (1872); and Stolz, Allgemeine Arithmetik, I. Th. (Leipzig, 1883), pp. 97-124.

Meray, in his Nonvoia Precess d'Analyne Infinitésimale (Paris, 1872), published independently a the sy very similar to Cantor's, which will be found set forth in the first volume of his Legons Neuvelles sur l'Analyse Infinitésimale (Paris, 1894).

A good general sketch of the whole subject is given by Pringsheim in his article on Irrationalizablen, Ac., Lings I podic der Mathematischen Wissenschaften (Leping, 1839), 181, 41, p. 47.

terminating and repeating decimals, which depends merely on the existence of rational limits.

§ 30.] Starting with 1 and confining our operations to the four species +, -,  $\times$ ,  $\div$ , we are led to the onefold of Kational Quantity

..., 
$$-m/n$$
, ... -1, ... 0, ... +1, ... +  $m/n$ , ... ( $R$ )

in which every number is of the form  $\pm m/n$ , where m and n are finite integral numbers.

The one-fold R possesses the following properties.

(i) It is an ordered onefold, in the sense that each number is either greater or less than every other. The onefold may therefore be arranged in a line so that each number occupies a definite place, all those that are less being to the left, all greater to the right.

(ii) R is an arithmetic onefold, in the sense that any concatenation of the operations +, -, \*, + in which the operated are rational numbers (excepting always division by 0) leads to a number in R.

(iii) a and b being any two positive quantities in R, such that 0 < a < b, we can always find a positive integer n so that

 $na > b^*$ ; and consequently b/n < a.

(iv) Between any two unequal quantities in R, however nearly equal, we can insert as many other quantities belonging to R as we please. We express this property by saying that R is a compact onefold. This follows at once from (iii), since the rational numbers

 $a, a+(b-a)/n, a+2(b-a)/n, \ldots, a+(n-1)(b-a)/n, b$  are obviously in order of magnitude, and the integer n may be chosen as large as we please.

§ 31.] Dedekind's Theory of Sections. Any arrangement of all the rational numbers into two classes A and B, such that every number in A is less than every number in B, we may call a section of R. We denote such a section by the symbol (A, B).

It is obvious that to every rational number a corresponds a

<sup>·</sup> This is sometimes spoken of as the Axiom of Archimedes.

<sup>+</sup> Dedekind uses the word Schnett.

section of R; for we may take A to include all the rational numbers which are not greater than a, and B to include the rest, viz. all that are greater than a. Conversely, if in the class A there be a number a which is not exceeded by any of the others in A, then the section may be regarded as generated by a. The same is true if in the class B there be a number a which is not greater than any of the others in B; for we might without essential alteration transfer a to the class A, in which it would then be the greatest number. The case where there is a greatest number a in A and a least number b in b is obviously impossible. For a and b must be different, since the two classes A and B are exhaustive and number b is compact, insert numbers between them which must belong either to A or to B; so that a and b could not be greatest and least in their respective classes as apposed.

But it may happen that there is no greatest rational number in A, and no least rational number in B. There is then no rational number which can be said to generate the section. Such a section is called an empty or irrational section. It is not difficult to prove that, if m/n be any positive rational number which is not the quotient of two integral square numbers, and A denote all the rational numbers whose squares are less than m/n, and B all those whose squares are greater than m/n, then the section (A, B) is empty.

§ 32.] An ordered one-fold which has no empty sections is said to be continuous. It will be observed that the one-fold of rational numbers is discontinuous although it is compact.

Starting with the discontinuous onefold of rational numbers  $R_i$  we construct another onefold S by assigning to every empty or irrational section a symbol which we shall call by anticipation a number, adding the adjective irrational to show that it is not a number in R. As the section and the number are coordinated, we may use the symbol  $(A_1, R)$  to denote the number as well as the section. We can also without contradiction re-name all the rational numbers by attaching to each the corresponding sectional symbol.

Naturally we define the number (A, B) as being greater than

the number (A',B') when A contains all the (rational) numbers in A' and more besides; and consequently B' contains all the numbers in B and more besides. The numbers (A,B) (A',B') are equal when A' contains all the numbers in A, neither more nor less, and the like is consequently true of B' and B.

0 is the section in which A consists of all the negative and B of all the positive rational numbers.

(A, B) is positive when some of the numbers in A are positive; negative when some of the numbers in B are negative. Also, if we understand -A to mean all the numbers in A each with its sign changed, then (-B, -A) = (A, B).

The new manifold S is therefore obviously an ordered manifold; and it is clearly compact, since R is compact. It is also continuous, i.e. every section in S is generated by a number in S; for, if a,  $\beta$  be a classification of all the numbers (or sections) of S such that every number in a is less than every number in B, then  $(a, \beta)$  determines a section in S of the most general kind. But, if A contain all the rational sections in a and B all the rational sections in B, then (A, B) is a section in B, i.e. a number in B; and it is obvious that every number in S < (A, B) is a number in B < (B, B). Hence  $(a, \beta)$  corresponds to the number (A, B), which is a number in B.

§ 33.] Systematic representation of a number, rational or irrational. Consider any number defined by means of a section (A, B) of the rational onefold R. We are supposed to have the means, direct or indirect, of settling whether any rational number belongs to the class A or to the class B. Suppose (A, B) positive. Consider the succession of positive integers  $0, 1, 2, \ldots$ ; and select the greatest of these which belongs to A, say  $a_0$ . Then  $b_0 = a_0 + 1$  belongs to B. The two rational numbers  $a_0$ ,  $b_0$  determine two sections in R between which there is a gap of width 1. Within this gap the section (A, B) lies, i.e.  $a_0 < (A, B) < b_0$ .

Next divide the unit gap into ten parts by means of the rational numbers  $a_s + 1/10$ ,  $a_s + 2/10$ , . . . ,  $a_s + 9/10$ , and select the greatest of these numbers, say  $a_1 = a_s + p/10$ , which belongs to  $A_1$ ; then  $b_1 = a_1 + 1/10$  belongs to B. We have now a gap in

R of width 1/10, determined by the numbers  $a_i$ ,  $b_i$  within which (A, B) lies.

We next divide the gap of 1/10 into ten parts by means of the numbers  $a_i + 1/10^a$ ,  $a_i + 2/10^a$ , . . . ,  $a_i + 9/10^a$ ; and so on, Proceeding in this way, we can determine two rational numbers (terminating decimals in fact),

$$a_n = a_0 + p_1/10 + \dots + p_n/10^n$$
,  $b_n = a_n + 1/10^n$  (1)

between which (A,B) lies, the width of the gap between  $a_n$  and  $b_n$  being  $1/10^n$ . It is obvious that  $a_n, a_1, \ldots, a_n$  are a non-decreasing succession of positive rational numbers; and it can easily growed that  $b_0, b_1, \ldots, b_n$  are a non-increasing succession.

1°. At any stage of the process it may happen that a<sub>n</sub> is the greatest possible number in A, in other words that p<sub>n+1</sub>, and all successive p's are zero. The section (A, B) is then determined by the number a<sub>n</sub>; and (A, B) is the rational number a<sub>n</sub>.

If the process does not stop in this way, two things may

happen.

- 2°. The digits  $p_1, p_2, \ldots, p_n$ , may form an endless secsion but repeat, say in the cycle  $p_r, p_{r+1}, \ldots, p_n$ . In this case there exists a rational number a to which  $a_s = a_t + p_t/10 + \ldots + p_{n_t}/10^+$  approximates more and more closely as we increase n; and, since  $b_n = a_n + 1/10^n$ ,  $b_n$  also approaches the same limit. It follows that the rational numbers of class A might be defined as the numbers none of which exceeds every number of the succession  $a_v, a_1, \ldots, a_n$ , however large n be taken. Hence, if we agree to attach the number a to the class A, it will be the greatest number of that class, and the section (A, B) is generated by a.
- 3". The digits  $p_1, p_2, \ldots, p_n$  may form an endless non-repeating succession. Since the gap  $b_n a_n = 1/10^n$  can be made as small as we please, it follows as before that the rational numbers of class A may be defined as all the rational numbers none of which exceeds every number in the endless succession  $a_n, a_1, \ldots, a_n, \ldots$ . This statement does not as in last case enable us to identify (A,B) with any rational number; but, since n may be as large as we please, we can by calculating a sufficient number of the digits  $p_1, p_2, \ldots$  separate (A,B) from every other

number, rational or irrational, no matter how near that number may be to (A, B).

Conversely, it is obvious from the above reasoning that every terminating or repeating decimal determines a rational section in R, and therefore a rational number; and every non-terminating non-repeating decimal an irrational section in R. i.e. an irrational number.

It is an obvious consequence of the foregoing discussion that between any two distinct numbers, rational or irrational, we can find as many other numbers, rational or irrational, as we please.

§ 34.] Cantor's Theory. The rational numbers  $a_0, a_1, \ldots, a_n, \ldots$  in § 33 evidently possess the following property. Given any positive rational number  $\epsilon$ , however small, we can always find an integer  $\nu$  such that  $|a_n - a_{n+r}| < \epsilon$  when  $n \not< r$ , r being any positive integer whatever.

We are thus naturally led to consider an infinite sequence of rational numbers

$$u_1, u_2, \ldots, u_n, \ldots$$
 (2)

which has the property that for every positive rational value of  $\epsilon$ , however small, there is an integer v such that  $|u_n - u_{n+r}| < \epsilon$  when  $n \not\leftarrow v$ , r being any positive integer whatever.

Such a sequence is called a convergent sequence; and  $u_1, u_2$ , &c. may be called its convergents. It should be observed that we no longer, as in § 33, confine the convergents to be all (or even ultimately all) of the same sign; nor do we suppose that they form a non-decreasing or a non-increasing (monochine) sequence.

To every convergent sequence corresponds a definite section of the onefold of rational numbers (R): so that every such sequence defines a real number, rational or irrational.

We may prove this important theorem as follows.

Let  $\epsilon_1$  be any positive rational number whatever; then we can find  $\nu_1$  such that, when  $n \lessdot \nu_1$ ,  $\lfloor u_n - u_{n+r} \rfloor < \epsilon_1$ . In particular, we shall have, if  $m > \nu_1$ ,  $\lfloor u_{\nu_1} - u_m \rfloor < \epsilon_1$ , whence

$$u_{\nu_1} - \epsilon_1 < u_m < u_{\nu_1} + \epsilon_1$$
 (2).

In other words, the two rational numbers  $a_1$ ,  $u_{r_1} + \epsilon_1$ ,  $b_1$ ,  $u_{r_1} + \epsilon_1$  determine two sections in R such that all the numbers of the

sequence  $\Sigma$  on and atter  $u_{r_i}$  lie in the gap of width  $2\epsilon_i$  between those two sections

Next choose any rational number  $\epsilon_i < \epsilon_i$ . We can then establish a gap of width  $2\epsilon_i$ , whose bounding sections are given by  $a_j \neq u_{i_1} + \epsilon_j$ ,  $b_j \in u_{i_1} + \epsilon_j$ . The number  $\epsilon_j$  will in general be greater than  $\epsilon_i$ ; but it might be less. Also the gap  $a_ib_j$  might partly overlap the gap  $a_ib_j$ . But, since all the convergents on and after u, the within the gap  $a_ib_j$ , we can throw saide the part of  $a_ib_j$ , if any, that lies outside  $a_ib_j$ , and determine a number  $\epsilon_i \ll \epsilon_j$  such that

$$a_1 < u_m - b_2$$

Hence every convergent sequence determines a section of R; and therefore defines a number, rational or irrational.

Conversely, as we have seen in § 33, every number, rational or introducing may be defined by means of a convergent sequence. If the sequence is  $u_1, u_2, \dots, u_n$ , we shall often denote both the sequence and the corresponding number by  $(u_n)$ . Since it is only the ultimate convergents that determine the section, it is clay that we may omit any finite number of terms from a convergent sequence without affecting the number which it defines. In particular, the sequences  $u_1, u_2, \dots, u_n, \dots, u_n, \dots$  and  $u_n, \dots, u_n, \dots, u_n$  define the same number. It should be noticed that in the  $c_1 < 0$  rational numbers the convergents on and after

a particular rank may be all equal: in fact we may define any rational number a by the sequence  $a, a, \ldots, a, \ldots$ , and call it (a).

Since each gap in the above process lies within all preceding gaps, and the section in R which is finally determined within them all, we have, if v be such that  $|u_n-u_{n+r}|<\varepsilon$  when  $u\leqslant v$ ,

$$u_{\nu} - \epsilon > (u_n) > u_{\nu} + \epsilon$$
 (3),

an important inequality which enables us to obtain rational approximations as close as we please to the number which is defined by the sequence  $u_1, u_2, \ldots, u_n, \ldots$ 

§ 35.] Null-sequence. If by taking n sufficiently great we can make  $\{n_n\}$  less than any given positive quantity, however small, it follows from (3) that  $(n_n)$  must be between 0 and a rational number which is as small as we please. We therefore conclude that in this case the sequence  $n_1, n_2, \dots, n_{n_1}, \dots$  corresponds to 0; and we call it a null-sequence.

§ 36.] Definition of the four species for the generalised onefold

of real numbers S.

If  $(u_n)(r_n)$  be any two numbers, rational or irrational, defined by convergent sequences, it is easy to prove that the sequences  $(u_n + r_n)$ ,  $(u_n - r_n)$ ,  $(u_n r_n)$ ,  $(u_n r_n)$ , are convergent sequences\*, provided in the case of  $(u_n / c_n)$  that  $(r_n)$  is not a null-sequence. We may therefore define these to mean  $(u_n) + (r_n)$ ,  $(u_n) - (r_n)$ ,  $(u_n) \times (r_n)$ ,  $(u_n) + (r_n)$  respectively. For it is easy to verify that, if we give these meanings to the symbols  $+, -, \times, +$  in connection with the numbers  $(u_n)$  and  $(r_n)$ , then the Fundamental Laws of Algebra set forth in clap. 1. § 28 will all be satisfied.

For examplet,

$$(u_n) - (v_n) + (v_n) = (u_n - v_n) + (v_n)$$
, by definitions  
 $= (\langle u_n - v_n \rangle + v_n)$ , by def.  
 $\equiv (u_n)$ , by laws of operation for  $R$ .

The reasoning is much the same as in § 6 above.

<sup>+</sup> The plain bracket () is appropriated to the definition of the number by a sequence; the crooked bracket has reference to operations in R.

Again.

$$(u_n) \times \{(v_n) + (w_n)\} \subseteq (u_n) \times (v_n + w_n), \text{ by def.}$$
  
 $\cong (u_n \{v_n + w_n\}), \text{ by def.}$   
 $\cong (u_n v_n + u_n w_n), \text{ by laws of operation for } (R),$   
 $\cong (u_n v_n) + (u_n w_n), \text{ by def.}$   
 $\cong (u_n) (v_n) + (u_n) (w_n), \text{ by def.}$ 

and so on.

In order that two numbers  $(u_n)$  and  $(r_n)$  may be equal it is formally necessary and sufficient that  $(u_n) - (v_n) \equiv 0$ , in other words, that  $(u_n - v_n) = 0$ , that is, that  $u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n$ . ... shall be a null-sequence. This from the point of view of our exposition might also be deduced from the fact that  $(u_n)$  and  $(r_n)$  must correspond to the same section in R. We can also readily show that all null-sequences are equal, as they ought to be, since they all correspond to 0.

We have now shown that the one-old of real quantity (8) built upon R by the introduction of irrational numbers is an arithmetic namifold. The proof that S has the property iii, of  $\S$  30 is so simple that it may be left to the reader. Henceforth, then, we may operate with the numbers of S exactly as we do with rational numbers.

§ 37.] It is worthy of remark that the properties of the rational onefold R can, by means of appropriate abstract definitions, be established on a purely arithmetical basis. It is not even necessary to introduce the idea of measurement in terms of a unit. The numbers may be regarded as ordinal; and addition and subtraction, greateness and lessness, &c. interpreted merely as progress backwards and forwards among objects in a row, which are not necessarily placed at equal or at any determinate distances apart\*.

Following the older mathematicians since Descartes, we have in the earlier part of this work assumed that, if we choose any point on a straight line as origin, every other point on it has for

See, for example, Harkness and Modey, Introduction to the Thing of Analytic Functions. (Macmillan, 1808)

its coordinate a definite real quantity: and conversely that every real quantity, rational or irrational, can be represented in this way by a definite point. The latter part of this statement, viz. that to every irrational number in general\* there corresponds a definite point on a straight line, is regarded by the majority of recent mathematicians who have studied the theory of irrationals as an axiom regarding the straight line, or as an axiomatic definition of what we mean by "points on a straight line."

§ 38.] Generalisation of the notion of a Convergent Nequence. It is now open to us to generalise our definition of a convergent sequence by removing the restriction that  $\epsilon$  and  $u_1, u_2, \ldots, u_n, \ldots$  shall be rational numbers. Bearing in mind that we can now operate with all the quantities in S just as if they were rational, we can, exactly as in § 34, establish the theorem that every convergent sequence of real numbers  $u_1, u_2, \ldots, u_n, \ldots$  defines a real number  $(u_n)$ .

Also we can show that, if  $\epsilon$  be any real positive quantity, however small, we can always determine  $\nu$  so that

$$u_m - \epsilon < (u_n) < u_m + \epsilon$$
 (4),

when  $m \not < \nu$ .

For we have merely, as in § 34, to determine  $\nu$  so that  $|u_m - u_{m+r}| < \epsilon' < \epsilon$ , when  $m < \nu$ .

Then we have

$$u_m - \epsilon' \geqslant (u_n) \geqslant u_m + \epsilon'$$
;

and therefore

$$u_m - \epsilon < (u_n) < u_m + \epsilon$$

when  $m \not \ll \nu$ .

§ 39.] General Definition of a Limit and Criterion for its Existence.

Returning now to the point from which this discussion started, we define the limit of the infinite sequence of real quantities

$$u_1, u_2, ..., u_n, ...$$
 ( $\Sigma$ ),

as a quantity u such that, if \(\epsilon\) be any real quantity however small,

We do not speak of special irrationalities, such as \$\sqrt{2}\$, which arise in elementary geometrical constructions.

then there exists always a positive integer v such that  $|u_n-u|<\epsilon$  when n < v. And we prove the following fundamental theorem.

The necessary and sufficient condition that the sequence, Z, have a finite definite limit is that it be a convergent sequence; and the limit is the real number which is than defined by the sequence.

The condition is necessary; for, if a limit u exist, then

$$[u_n - u_{n+r} \ge u_n - u + u - u_{n+r}],$$
  
 $\Rightarrow [u_n - u] + [u_{n+r} - u].$ 

Now, since n is the limit of the sequence, we can find r such that  $\lfloor n_n - n \rfloor < \frac{1}{2} \epsilon$  when n < r; and,  $\hat{n}$  fortheri,  $\lfloor n_n - r \rfloor < \frac{1}{2} \epsilon$  when n < r thence we can always find r so that  $\lfloor n_n - n_n r \rfloor < \epsilon$ , where  $\epsilon$  is any positive quantity as small as we choose. Hence  $\mathbf{X}$  is convergent.

Also the condition is sufficient. In fact, we can show that  $(u_n)$ , the number defined by the sequence when it is convergent, satisfies the definition of a limit. For, given  $\epsilon$ , we have seen that we can find r so that

$$u_m - \epsilon < (u_n) < u_m + \epsilon$$

when  $m \lessdot \nu$ : whence it follows that  $|u_m - (u_n)| \leq \epsilon$  when  $m \lessdot \nu$ .

Moreover there cannot be more than one finite limit; for, if there were two such, say u and r, we should have

But, since both u and v are limits we could, by sufficiently increasing  $u_i$  make  $u_n \cdot u^i$  and  $|u_n - v|$  cach less than  $\frac{1}{2}\epsilon_i$  and therefore  $|u - r| \cdot v_i$ ,  $i, \epsilon_i$  as small as we please. Hence u and v cannot be unequal.

The reader will readily prove that, if  $u_1, u_2, \ldots, u_n, \ldots$  be a nanobercosing (non-one cossing) infinite sequence, no number of which is greater than these than the finite number  $l_i$  then this sequence has a finite limit not greater than (not less than)  $l_i$ .

§ 40.] Let us now consider any function of x, say f(x), which is well defined in the sense that, for all values of x that have to be considered, with the possible exception of a finite number of isolated critical values, the value of f(x) is determined when the value of x is given. We define the limiting value,  $f_{ij}(f(y))$  when

x is increased up to the value a, by the property that, when any positive quantity  $\epsilon$  is given, there exists a finite quantity  $\xi < a$  such that

 $|f(x)-l|<\epsilon$ 

when  $\xi > x < \alpha$ .

This obviously includes our former definition of a limiting value; and we may denote l by L f(x).

Let  $a_1, a_2, \ldots, a_n, \ldots$  be any ascending convergent sequence which defines the number  $a_i$ ; and let us suppose, as we obviously may, that there is no critical value of x in the interval  $a_i \ge x < a$ . Then, if we consider the sequence  $u_1 = f(u_1)$ ,  $u_2 = f(u_2), \ldots, u_n = f(u_n), \ldots$ , the results of last paragraph lead us at once to the following theorem.

The necessary and sufficient condition that L f(x) be finite and definite is that it be possible to find a finite quantity  $\xi < a$  such that, when  $\xi > x < x' < a$ ,

$$|f(x) - f(x')| < \epsilon$$

where ∈ is any finite positive quantity however small.

The reader will easily formulate the corresponding proposition regarding  $L \circ f(x)$ .

§ 41.] There is one more point to which it may be well to direct attention before we leave the theory of limits.

 $\int_{x=4\pi i}^{x} f(x)$  is not necessarily equal to the value of f(x) when x=a. For example,  $\int_{x=1\pi i}^{x} (x^2-1)/(x-1) = 2$ ; but  $(x^2-1)/(x-1)$  has no value when x=1.

A more striking case arises when f(x) is well defined when x=a, but is discontinuous in the neighbourhood of x=a. Thus, if

$$f(x) \equiv L \{ \sin x/1 - \sin 2x/2 + \dots + (-1)^{n-1} \sin nx/n \},$$

then it is shown in chap. XXIX., § 40, that  $L_{x=7-9}f(x) = + \pi/2$ .  $L_{x=7-9}f(x) = -\pi/2$ ; whereas  $f(\pi) = 0$ .

#### EXERCISES VII.

#### Limits

Find the limiting values of the following functions for the given values of the variables:---

- (1.)  $(3x^{\frac{1}{4}} + 2x^{\frac{2}{3}} + 3x^{\frac{1}{2}})/(x^{\frac{1}{4}} + x^{\frac{1}{3}} + x^{\frac{1}{2}}), x = 0, \text{ and } x = x.$
- (1.)  $(3x^4 + 2x^5 + 3x^2)/(x^4 + x^5 + x^2)$ , x = 0, and x = x(2.)  $(x^4 - x^3 - 9x^2 + 16x - 4)/(x^3 - 2x^2 - 1x + 8)$ , x = 2.
- (2.)  $(x^3 x^2 9x^2 + 16x 4)/(x^2 2x^2 1x + 5), \quad x = 2.$ (3.)  $\log (x^3 - 2x^2 - 2x - 3) - \log (x^3 - 4x^2 + 4x - 3), \quad x = 3.$
- (4.)  $\{x (n+1)x^{n+1} + nx^{n+2}\}/(1-x)^2$ , x = 1 (n a positive integer). (Euler, D(t), Calc.)
- (5.)  $\{x'(x-1)-(x-1)\}/\{2'(x-1)-x'(x-1)\}, x=1.$
- (6.)  $(x^{m+n} a^m x^n)/(x^{p+q} a^p x^q), \quad x = a.$
- (7.)  $\{(a+x)^m (a-x)^m\}/\{(a+x)^n (a-x)^n\}, x = 0.$ (8.)  $\{(x^m-1)^p - (x^n-1)^q\}/\{(x-1)^p - (x-1)^q\}, x = 1.$
- $(9.) \frac{(x^m-1)^2 (x^m-1)^n/((x-1)^n (x-1)^n}{(x^m-1)^2 (x^m-1)(x^n-1) + (x^n-1)^2}$
- (9.)  $\frac{1}{(x^m-1)^2+(x^m-1)}\frac{2f(x)}{(x^m-1)+(x^m-1)}\frac{2f(x)}{(x^m-1)^2}, \quad x=1,$ (10.)  $\{a-\sum_{i=1}^n(a^2-x^2)\}^2/x^2, \quad x=0, \quad \text{(Euler, Diff. Calc.)}$
- (10.)  $\{a \zeta(a + x^2)\}/x^2$ , x = 0. (Euler, Fig. Catc.) (11.)  $\{\xi'(a + x) - \xi'(a - x)\}/\{\xi'(a + x) + \xi'(a - x)\}$ , x = 0.
- (12.)  $\{(a^2 + ax + x^2)^{\frac{1}{2}} (a^2 ax + x^2)^{\frac{1}{2}}\}\}\{(a + x)^{\frac{1}{2}} (a x)^{\frac{1}{2}}\}, x = 0.$  (Euler,
- [Page Calc.]

  (13.)  $\{(2a^2x x^4)^{\frac{1}{2}} a(a^2x)^{\frac{1}{4}}\}/\{a (ax^2)^{\frac{1}{4}}\}, x = a.$  (Grewory, Examples in
- D(ff, Calc.) $(14.) \{a + \sqrt{(2a^2 - 2ax)} - \sqrt{(2ax - x^2)}\}/\{a - x + \sqrt{(a^2 - x^3)}\}, x = a.$  (Euler,
- Digr. Cate.)
- (15.)  $x \sqrt{(x^2 + y^2)}$ , when x = x, y x, but  $y^2/x$  finite = 2p.
- (16.)  $\sum x^n (y-z)/\Pi (y-z), \quad x=y=z.$
- (17.)  $\sum x^m (y^n z^n) / \sum x^p (y^q z^q), \quad x = y = z = a,$
- (18.)  $nx^{n-1}/(x^n a^n) 1$   $(x a), \quad x = a$ . (19.)  $2^x(a^{\frac{n-2}{2}} - 1), \quad x = x$ . (20.)  $x^{1/2}, \quad x = x$ .
- (21.)  $(1+1/x^2)^x$ , x=x. (22.)  $x^{2x}/(1+x^2)^x$ ,  $x=\infty$ .
- (23.)  $(1+1/x)^x$ , x=0.  $(21) (1+1/x)^{x^2}$ ,  $x=\infty$ .
- $(25.) \quad x^{1/(x-1)^2}, \quad x=1. \qquad \qquad (26.) \quad x^{1/(x^2-1)}, \quad x=1.$
- (27.)  $a^{x^n}/x$ , x = x. (28.)  $(\log x)^{1/x}$ ,  $x = \infty$ . (29.)  $(\log x/x)^{1/x}$ ,  $x = \pi$ . (30.)  $(\log^m x/\log^m x)$
- (31.)  $a^{x}f(x)$ , x = x, where f(x) is a rational function of x, and a a constant.
- (32.)  $(ax^n + bx^{n-1} + ...)^{1x}$ , x = x. (Cauchy.)
- (33.)  $x^{(j+1)(i+2)(i-2)}$ , x = 0.
- (31.)  $\{(x^2+x+1)/(x^2-x+1)\}^x$ ,  $x=\infty$ .
- (35)  $\left(\frac{1}{2}(a^{2}+b^{2})\right)^{1/2}$ , x=0,
- (36.)  $\{1+2/\sqrt{(x^2+1)}\}^{\frac{3}{N-1}(x^2+1)}$ ,  $x = \epsilon$ . (Longehamps.)
- (37.)  $\begin{pmatrix} d_{+} + d_{+}r + \dots + d_{p}r^{p} & \lambda_{0} + \lambda_{1}x \\ h_{0} + h_{1}x + \dots + h_{p}x^{p} \end{pmatrix}$ ,  $x = \mathbf{x}$  (Math. Trip., 1886.)

- (38.)  $\{1/(e^x-1)\}^{1/x}$ ,  $x = \infty$ .
- (39.)  $\{\log (1+x)\} \log (1+x^2), \quad x = 0.$
- (40.)  $\log (1 + ax)/\log (1 + bx)$ , x = 0.
- (41.)  $(e^x e^{-x})/\log (1+x)$ , x = 0. (Euler,  $D(\mathcal{F}|Calc.)$ )
- (13.)  $\tan^{-1}x/x$ , x = 0. (12.)  $(\frac{1}{2}\pi - x) \tan x$ ,  $x = \frac{\pi}{2}$ .
- (14.)  $(1 \sin x + \cos x)/(\sin x + \cos x 1)$ ,  $x = \frac{1}{2}\pi$ . (Euler, Diff. Calc.)
- (16.1  $x \{\cos ta(x) 1\}$ , t = x. (45.)  $\sin x/(1-x^2/\pi^2)$ ,  $x=\pi$ .
- (18.)  $\sec x \tan x$ ,  $x = k\pi$ . (47.)  $(\sin x - \sin a)/(x - a)$ , x = a. (49.)  $(\sin^4 x - \tan^4 x)/(1 + \cos x)(1 - \cos x)^3$ , x = 0.
  - (51.)  $(\cosh x 1)Ix^2$ , x = 0.  $(50.)^*$   $\sinh x/x$ , x = 0.
  - (53.)  $\sin x/\log (1+x)$ , x=0. (52.)  $\tanh^{-1} x/x$ , x = 0.
  - (55.)  $\cos x \log \tan x$ ,  $x = \frac{1}{2}\pi$ . (54.)  $\sin x \log x$ , x = 0.
  - (56.)  $\log \tan mx/\log \tan nx$ , x=0.
- (57.)  $(\log \sin mx \log x)/(\log \sin nx \log x), x = 0.$ (59.) sm  $x^{\tan x}$ , x = 0,
- (58.)  $\sin x^{\sin x}$ , x = 0.
- (60.)  $(\sinh x)^{\tan x}$ , x = 0.
- (61.)  $\{(x/a) \sin (a/x)\}^{x^m} (m < 2), x = \infty$ .
- (63.)  $(\cos mx)^{-\cos^2nx} = 0$ . (62.)  $(\cos mx)^{n}x^{2}$ , x = 0.
- (64.)  $(2 x/a) \tan \pi x/a$ , x = a,
- (65.)  $\log_e (\log_e x) / \cos \frac{\pi e}{\sigma_e}$ , x = e.
- (66.) Show that sin x cot (a/x) log (1 + tan (a/x)) has no determinate limit when  $x = \infty$ .
- (67.) If l<sub>a</sub><sup>2</sup>x stand for log<sub>a</sub> (log<sub>a</sub>x), l<sub>a</sub><sup>3</sup>x for log<sub>a</sub> (log<sub>a</sub> (log<sub>a</sub>x)), &c., show that  $L = [1 - \{l_a p_x / l_a p_x (x+1)\}^m] x l_a x l_a x \dots l_a p_x = m (l_a \epsilon)^p$ . (Schlomilch, Algebraische Analysis, chap, 11.)
  - (68.) Show that  $L \sum_{n=\infty}^{s-n} \sum_{s=s}^{(n+s)^{1/n}/n} (n+s)^{1/n}/n = 1$ .
  - (69.) Show that  $L \underset{n=\infty}{\overset{s-n}{\sum}} \{(a+s)/n\}^n \text{ lies between } e^a \text{ and } e^{a+1}.$
  - (70.) Show that  $L \sum_{n=-\infty}^{\infty} \{(a+sc/n)/(a+c)\}^n$  is finite if a+c be numerically greater than a, and that  $L \sum_{n=0}^{s-n} \{(a+sc/n)/a\}^n$  is finite if a+c be numerically less than a.
    - (71.) Trace the graph of y = (a<sup>x</sup> 1)/x, when a > 1, and when a < 1.</p>
- (72.) Trace the graph of y x1/x for positive values of x; and find the direction in which the graph approaches the origin.

For the definition and elementary properties of the hyperbolic functions cosh x, sinh x, tanh x, &c., see chap. xxix. All that is really want d here is  $\cosh x = \sqrt{(e^x + e^{-x})}$ ,  $\sinh x = \sqrt{(e^x - e^{-x})}$ .

(73.) Trace the graph of  $y = (1 + 1/r)^x$ ; and find the angle at which it crosses the  $u, \bowtie of y$ .

(7t.) Find the orders of the zero and intuity values of y when determined as a function of x by the following equation:  $\cdot$ :—

- (a)  $x(x^2-ay)^2-y^5=0$ . (Frost's Curve Tracing, § 155, Ex. 3.)
- (β)  $x^2y^6 + a^4y^5 x^5y^3 + ax^6y a^6x^2 = 0$ . (1b., Ex. 7.)
- (a)  $(x-1)y^3 + (x^2-1)y^2 (x-2)^2y + x(x-2) = 0$ ,

(75.) If u and v be functions of the integral variable n determined by the equations  $u_n = u_{n-1} + v_{n-1}$ ,  $v_n = u_{n-1}$ , show that L,  $u_n/v_n = (1 \pm \sqrt{5})/2$ . How ought the subligators spin to be settled when  $u_n$  and  $u_n$  are both positive?

(76.) Show that

$$2^{n(n+1)} > (n+1)^{n+1} {n \choose 1}^n {n-1 \choose 2}^{n-1} \cdot \cdot \cdot {2 \choose n-1}^2 {1 \choose n}^1.$$

- (77.) Show that  $L = \left\{ \frac{(m+1)(m+2)}{1 \cdot 2 \cdot \dots \cdot n} \right\}^{1/n} = 1$ ,
- (78.)  $L \log (1-x) \log x = 0$ , when x = 0,

<sup>.</sup> For a general method for dealing with such problems, see chap. xxx.

# CHAPTER XXVI

# Convergence of Infinite Series and of Infinite Products.

- § 1.] The notion of the repetition of an algebraical operation upon a series of operands formed according to a given law presents two fundamental difficulties when the frequency of the repetition may exceed any number, however great, or, as it is shortly expressed, become infinite. Since the mind cannot overlook the totality of an infinite series of operations, some defimition must be given of what is to be understood as the result of such a series of operations; and there also arises the further question whether the series of operations, even when its meaning is defined, can, consistently with its definition, be subjected to the laws of algebra, which are in the first instance laid down for chains of operations wherein the number of links is finite. That the two difficulties thus raised are not imaginary the student will presently see, by studying actual instances in the theory of sums and products involving an infinite number of summands and multiplicands.
- § 2.] One very simple case of an infinite series, namely, a geometric series, has already been discussed in chap, xx., § 15. The fact that the geometric series can be summed considerably simplifies the first of the two difficulties just mentioned \*; nevertheless the leading features of the problem of infinite series are all present in the geometric series; and it will be found that most questions regarding the convergence of infinite series are ultimately referred to this standard case.

<sup>.</sup> The second was not considered.

The consideration of the infinite geometric series suggests the following definitions.

Consider a succession of finite real summands  $u_1, u_2, u_3, \ldots, u_n, \ldots$  unlimited in number, formed according to a given law, so that the *n*th term  $u_n$  is a finite one-valued function of u; and consider the successive sums

When n is increased more and more, one of three things must happen :—

1st. S<sub>n</sub> may approach a fixed finite quantity S in such a way that by increasing a sufficiently we can make S<sub>n</sub> differ from S by as little as we please; that is, in the notation of last chapter, L. S<sub>n</sub> = S. In this case the series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

is said to be convergent, and to converge to the value S, which is spoken of as the sum to infinity.

Example. 
$$1 + \frac{1}{2} + \frac{1}{1} + \dots + \frac{1}{2^n} + \dots$$
 Here  $S = \underset{n=\infty}{L} S_n = 2$ .

2nd.  $S_n$  may increase with n in such a way that by increasing n sufficiently we can make the numerical value of  $S_n$  exceed any quantity, however large; that is,  $L S_n \pm \omega$ . In this case the series is said to be INVERGENT.

Example, 
$$1+2+3+\ldots$$
 Here  $L_nS_n=\infty$ 

3rd. S<sub>n</sub> may neither become infinite nor approach a definite limit, but oscillate between a number of finite values the selection among which is determined by the integral character of n, that is, sysuch considerations as whether n is odd or even; of the form 3m, 3m+1, 3m+2, &c. In this case the series is said to oscillation.

N.B. If all the terms of the series have the same sign, then  $S_n$  continually increases (or at least never decreases) in numerical value as n increases; and the series cannot oscillate.

Example,  $3 - 1 - 2 + 3 - 1 - 2 + 3 - 1 - 2 + 3 - 1 - 2 + \dots$ . Here  $L = S_n = 0$ , 3, or 2, according as n is of the form 3m, 3m + 1, or 3m + 2.

In cases 2 and 3 the series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

is also said to be non-convergent\*. In many important senses non-convergent series cannot be said to have a sum; and it is obvious that infinite series of this description cannot, except in special cases, and under special precautions, be employed in mathematical reasoning.

Series are said to be more or less rapidly convergent according as the number of terms which it is necessary to take in order to get a given degree of approximation to the sum is smaller or larger. Thus a geometric series is more rapidly convergent the smaller its common ratio. Rapid convergency is obviously a valuable quality in a series from the arithmetical point of view.

It should be carefully noticed that the definition of the convergeucy of the series

$$u_1 + u_2 + u_1 + \dots + u_n + \dots$$

involves the supposition that the terms are taken successively in a given order. In other words, the sum to infinity of a convergent series may be, so far as the definition is concerned, dependent upon the order in which the terms are written. As a matter of fact there is a class of series which may converge to one value, or to any other, or even become divergent, according to the order in which the terms are written.

§ 3.] Two essential conditions are involved in the definition of a convergent series—1st, that  $N_n$  shall not become infinite for any value of n, however great; 2nd, that, as n increases, there shall be continual approach to a definite limit N. If we introduce the symbol  ${}_{m}R_n$  to denote  $u_{n+1} + u_{n+2} + \dots + u_{n+m}$ , that is, the sum of m terms following the nth, following Cauchy we may state the following criterion:—

The necessary and sufficient condition for the convergence of a series of real terms is that, by taking n sufficiently great, it be possible to make the absolute value of  $_{m}R_{n}$  as small as we please, no matter what the value of m may be.

Some writers use divergent as equivalent to non-convergent. On the whole, especially in elementary exposition, this practice is inconvenient.

This condition may be amplified into the following form. Given in advance any positive quantity  $\epsilon_i$  however small, it must be possible to assign an integer r such that for n=r and all greater values  $\lceil_m R_m \rceil < \epsilon$  or it may be contracted into the form  $L_m R_m = 0$  when  $n = \infty$ , for all values of m.

The condition is necessary; for, by the definition of convergency, we have  $L(S_n - S_n)$  where S is a finite definite quantity; therefore also, whatever  $m_i(L(S_{n+m} - S_n))$ . Hence

$$L_{n+m} - S_n) = S - S \cdot 0:$$

that is,  $L_m R_a = 0$ .

Also the condition is sufficient: for, if we assign any positive quantity  $\epsilon$ , it is possible to find a finite integer r such that, when  $n \lessdot r$ ,  $\lfloor nR_n \rfloor \le \epsilon$ , that is  $\langle S_{n+m} - S_n \rangle \le \epsilon$ . In particular, therefore,  $\langle S_{n+m} - S_n \rangle \le \epsilon$ . Since  $S_n$ , being the sum of a finite number of finite terms, is finite, and m may have any value we please, it follows that for no value of n exceeding r can  $S_n$  become infinite. Hence  $L(S_n)$  cannot be infinite.

Also the limit of  $S_n$  cannot have one finite value when n has any particular integral character, and another value when n has a different integral character; for any such result would involve that for certain values of m L  $S_n$  and L  $S_{n+m}$  should have different values; but this cannot be the case, since for all values of m L  $(S_{n+m} - S_n) = L$  m n n.

It should be noticed that, when all the terms of a series have the same sign, there is no possibility of oscillation; and the condition that  $S_n$  be finite for all values of n however great is sufficient. In case the subtlety of Cauchy's single criterion should puzzle the beginner, he should notice that the proof which shows that  $L_N R_n = 0$  can usually be readily modified so as to show that  $L_S$ , is not infinite. In fact some of our earlier

<sup>•</sup> A more rigorous demonstration of the above criterion is obtained by applying the result of § 39, chap, xxv. to the sequence S<sub>1</sub>, S<sub>2</sub>, . . . . , S<sub>n+++</sub>. We have given the above demonstration for the sake of readers who have not mastered the Theory given in chap, xxv., §§ 28-40.

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demonstrations are purposely made redundant, by proving both  $L_m R_n = 0$ , and  $LS_n$  not infinite.

Cor. 1. In any convergent series  $L_{-}u_{n}=0$ ,

For  $u_n = S_n - S_{n-1} = R_{n-1}$ , and, by the criterion for convergency, we must have  $L_{-1}R_{n-1}=0$ . This condition, although necessary, is not of itself sufficient, as will presently appear in many examples.

If  $R_n = L_{-m}R_n$ , and S and  $S_n$  have the meanings above assigned to them, then  $S_n = S - R_n$ ,

For  $S_{n+m} = S_n + {}_m R_n$ , therefore  $\underset{m=\infty}{L} S_{n+m} = S_n + \underset{m=\infty}{L} {}_m R_n$ ; and  $L S_{n+m} = S$ , hence the theorem.

 $R_n$  is usually called the residue of the series, and  ${}_mR_n$  a partial residue.

Obviously, the smaller  $R_n/S_n$  is for a given value of n, the more convergent is the series; for  $R_a$  is the difference between S, and the limit of S, when n is infinitely great.

 $R_n$  is, of course, the sum of the infinite series

$$u_{n+1} + u_{n+2} + u_{n+3} + \dots$$
;

and it is an obvious remark that the residue of a convergent series is itself a convergent series.

Cor. 3. The convergency or divergency of a series is not affected by neglecting a finite number of its terms.

For the sum of a finite number of terms is finite and definite; and the neglect of that sum alters  $L S_n$  merely by a finite determinate quantity; so that, if the series was originally convergent, it will remain so; if originally oscillating or divergent, it will remain so.

Example 1. Consider the series  $1/1 + 1/2 + 1/3 + \dots + 1/n + \dots$ Here  $R_n = 1/(n+1) + 1/(n+2) + ... + 1/(n+m)$ ,

 $> 1/(n+m) + 1/(n+m) + \dots + 1/(n+m)$ 

> m/(n+m),

> 1/(n/m + 1).

Now, however great n may be, we can always choose m so much greater that n/m shall be less than any quantity, however small. Hence we cannot cause mRn to vanish for all values of m by sufficiently increasing n. We therefore conclude that the series is not convergent; hence since all the terms are positive it must diverge, notwithstanding the fact that the terms ultimately become infinitely small. We shall give below a direct proof that  $LS_n = \infty$ .

Example 2.

$$\frac{1}{1} \log \frac{2^2}{1 \cdot 3} + \frac{1}{2} 1 \cdot \frac{3^2}{2 \cdot 4} + \dots + \frac{1}{n} \log \frac{(n+1)^2}{n(n+2)}$$

Since  $(n+1)^2/n (n+2) = (1+1/n)/\{1+1/(n+1)\}$ , we have

nice 
$$(n+1)!/n (n+2) = (1+1/n)(1+1/(n+1))$$
, we have
$$mR_n = \frac{1}{n+1} \log \frac{1}{1+1/(n+2)} + \frac{1}{n-2} \log \frac{1}{1+1/(n+2)}$$

$$+ \dots + \frac{1}{n-m} \log \frac{1}{1+1/(n+m)}$$

$$\leq \frac{1}{n+1} \left[ \log \frac{1}{1+1/(n+1)} + \log \frac{1}{1+1/(n+m)} + \dots + \log \frac{1}{1+1/(n+m)} \right]$$

$$\leq \frac{1}{n+1} \left[ \log \frac{1}{1+1/(n+1)} + \log \frac{1}{1+1/(n+m)} + \dots + \log \frac{1}{1+1/(n+m+1)} \right]$$

$$\leq \frac{1}{n+1} \log \frac{1}{1+1/(n+1)} + \dots + \log \frac{1}{1+1/(n+m+1)}$$

$$\leq \frac{1}{n+1} \log \frac{1}{1+1/(n+m)} + \dots + \log \frac{1}{1+1/(n+m+1)}$$
(1).

Now, whatever m may be, by making n large enough we can make 1/(n+1), and, a fortion, 1/(n+m+1), as small as we please, therefore  $L_{-m}R_n=0$  for all values of m.

If in (1) we put 0 in place of n, and n in place of m, and observe that  $S_n = {}_{n}B_m$ , we see that

$$S_n < \log_2 \frac{1 + 1/1}{1 + 1/(n + 1)};$$

so that S can never exceed log 2 whatever n may be.

Both conditions of conversency are therefore satisfied.

Putting  $m = \mathbf{x}$  in (1), we find for the residue of the series  $R_* < \lceil \log (1 + 1/(n + 1)) \rceil / (n + 1);$ 

a result which would enable us to estimate the rapidity of the convergency, and to settle how many terms of the series we ought to take to get an approximation to its limit accurate to a given place of decimals.

- § 4.] The following theorems follow at once from the criterion for convergency given in last paragraph. Some of them will be found very useful in discussing questions regarding convergence. We shall use  $\Sigma u_a$  as an abbreviation for  $u_1 + u_2 + \dots + u_a + \dots$  that is, "the series whose ath term is  $u_a$ ."
- If u<sub>n</sub> and v<sub>n</sub> to positive, u<sub>n</sub> < v<sub>n</sub> for all values of n, and \(\Sigma \), convergent the \(\text{\text{\$\sigma}}\) \(\Su\_n\) is convergent.

If  $u_n$  and  $v_n$  be positive,  $u_n > v_n$  for all values of n, and  $\Sigma v_n$  divergent, then  $\Sigma u_n$  is divergent.

For, under the first set of conditions, the values of  $S_n$  and  ${}_nR_n$  belonging to  $\Sigma u_n$  are less than the values of the corresponding functions  $S_n$  and  ${}_nR_n$  belonging to  $\Sigma r_n$ . Hence we have  $0 < S_n < S_{n+1}^c 0 < mR_n < mR_n$ . But, by hypothesis,  $S_n^c$  is finite for

all values of  $u_n$  and  $L_{m}R'_n=0$ ; hence  $S_n$  is finite for all values of  $u_n$  and  $L_{m}R_n=0$ ; that is,  $\Sigma u_n$  is convergent.

Under the second set of conditions,  $S_n > S'_n$ . Hence, since  $L = S'_n = \infty$ , we must also have  $L = S_n = \infty$ ; that is,  $\Sigma u_n$  is divergent.

II. If, for all values of n,  $r_n > 0$ , and  $u_n/r_n$  is finite, then  $\Sigma u_n$  is convergent if  $\Sigma r_n$  is convergent, and divergent if  $\Sigma r_n$  is divergent.

By chap. XXIV., § 5, if A be the least, and B the greatest of the fractions,  $u_{n+1}/r_{n+1}$ ,  $u_{n+2}/r_{n+2}$ , . . . ,  $u_{n+m}/r_{n+m}$ , then

$$A < \frac{u_{n+1} + u_{n+2} + \dots + u_{n+m}}{v_{n+1} + v_{n+n} + \dots + v_{n+m}} \setminus B.$$

Now, since  $u_n/v_n$  is finite for all values of n, A and B are finite. Hence we must have in all cases  ${}_mR_n-C_m\tilde{L}_n$ , where C is a finite quantity whatever values we assign to m and n. Hence  $S_n$  (that is,  ${}_nR_0$ ) will be finite or infinite according as  $S_n'$  is finite or infinite; and if  $L_{mR}R_n=0$ , we must also have  $L_{mR}n=0$ .

III. If  $u_n$  and  $v_n$  be positive, and if, for all values of n,  $u_{n+1}|u_n < v_{n+1}|v_n$ , and  $\Sigma v_n$  is convergent, then  $\Sigma u_n$  is convergent; and if  $u_{n+1}|u_n > v_{n+1}|v_n$ , and  $\Sigma v_n$  is divergent, then  $\Sigma u_n$  is divergent.

We have, if  $u_{n+1}/u_n < v_{n+1}/v_n$ ,

$$\begin{split} S_n &= u_1 \left\{ 1 + \frac{u_2}{u_1} + \frac{u_2}{u_2} \cdot \frac{u_2}{u_1} + \dots \right\}, \\ &< u_1 \left\{ 1 + \frac{v_2}{v_1} + \frac{v_3}{v_2} \cdot \frac{v_2}{v_1} + \dots \right\}, \\ &< u_1 S_n', \end{split}$$

Now, by hypothesis,  $LS'_n$  is finite: hence  $LS_n$  must be finite. Also, since all the terms of  $\Sigma u_n$  are positive, the series cannot oscillate, therefore  $\Sigma u_n$  must be convergent.

In like manner, we can show that, if  $u_{n+1}/u_n > v_{n+1}/v_n$ , and  $\Sigma v_n$  be divergent, then  $\Sigma u_n$  is divergent.

N.B.—In Theorems I., II., III. we have, for simplicity, stated that the conditions must hold for all values of n; but

we see from § 3. Cor. 3, that it is sufficient if they hold for all values of a exceeding a certain finite value r: for all the terms up to the rth in both series may be neglected.

Also, when all the terms of a series have the same sign, we suppose, for simplicity of statement, that they are all positive. This, clearly, in no way affects the demonstration.

It is convenient to speak of  $u_{n+1}/u_n$  as the Ratio of Convergence of  $\Sigma u_n$ . Thus we might express Theorem III, as follows: -- Any series is convergent (divergent) if its ratio of convergence is always less (greater) than the ratio of convergence of a convergent (divergent) series.

IV. If a series which contains negative terms be convergent when all the negative terms have their signs changed, it will be convergent as it stood originally.

For the effect of restoring the negative signs will be to diminish the numerical value both of  $S_n$  and of  $\square R_n$ .

Definition. - A series which is convergent when all its terms are taken positively is said to be Absolutely Convergent.

It will be seen immediately that there are series whose convergency depends on the presence of negative signs, and which become divergent when all the terms are taken positively. Such series are said to be semi-convergent. In \$\infty\$ 5 and 6, unless the contrary is indicated, we suppose any series of real terms to consist of positive terms only, and convergence to mean absolute convergence.

## SPECIAL TESTS OF CONVERGENCY FOR SERIES WHOSE TERMS ARE ULTIMATELY ALL POSITIVE.

§ 5.] If we take for standard series a geometric progression. say  $\Sigma r^n$ , which will be convergent or divergent according as r < or > 1, and apply § t, Th. I., we see that  $\Sigma u_n$  will be convergent if, on and after a certain finite value of  $u_n u_n < r^n$ . where r < 1; divergent if, on and after a certain finite value of  $u, u_n > r^n$ , where  $r \cdot t$ . Hence

1. Yu., is convergent or divergent according as union is ultimately less or greater than unity.

This test settles nothing in the case where  $u_n^{\text{tr}}$  is ultimately unity, or where  $L_{n=0}^{\text{tr}}$  fluctuates between limits which include units.

Example.  $\Sigma 1/(1+1/n)^{n^2}$  is a convergent series; for  $L_{-n-1}^{-1} = 1/L (1+1/n)^n = 1/c,$ 

by chap. xxv., § 13, where e > 2, and therefore 1/e < 1.

If, with the series  $\Sigma^{p^n}$  for standard of comparison, we apply § 4, Th. III., we see that  $\Sigma u_n$  is convergent or divergent according as  $u_{n+1}/u_n$  is, on and after a certain finite value of  $u_n$  always >1. Hence

II.  $\Sigma u_n$  is convergent or divergent according as its ratio of convergency is ultimately < or > 1.

Nothing is settled in the case where the ratio of convergency is ultimately equal to 1, or where L until un fluctuates between limits which include units.

The examination of the ratio  $u_{n+1}/u_n$  is the most useful of all the tests of convergence\*. It is sufficient for all the series that occur in elementary mathematics, except in certain extreme cases where these series are rarely used. In fact, this test, along with the Condensation Test of § 6, will suffice for the reader who is not concerned with more than the simpler applications of infinite series.

Notwithstanding their outward difference, Tests I, and II, are fundamentally the same when  $L = u_{n+1}/u_n$  is not indeterminate. This will be readily seen by recalling the theorem of Cauchy, given in chap,  $\mathbf{x}\mathbf{x}\mathbf{v}$ , § 14, which shows that  $L = u_{n+1}/u_n = L = u_n^{-1}n$ . It is useful to have the two forms of test, because in certain cases I, is more easily applied than II.

Example 1. To test the convergence of  $\sum n^r x^n$ , where r and x are constants. We have in this case

$$u_{n+1}/u_n = (n+1)^r x^{n+1}/n^r x^n,$$
  
=  $(1+1/n)^r x.$ 

Hence  $Lu_{n+1}/u_n = x$ . The series is therefore convergent if x < 1, and divergent if x > 1.

 $<sup>\</sup>bullet$  We here use (as is often convenient) "convergence" to mean "the quality of the series as regards convergency or divergency."

If x = 1, we cannot settle the question by means of the present test.

Example 2. If  $\phi$  (n) be any algebraical function of n,  $\Sigma \phi$  (n)  $x^n$  is convergent if x < 1, divergent if x > 1.

This hardly needs proof if  $L \phi(n)$  be finite. If  $L \phi(n)$  be infinite, we know (see chap. xxx.) that we can always find a positive value of r, such that  $L \phi(n)/r'$  is finite, = 4 say. We therefore have

 $Lu_{-1}/u_{-} = xL\phi(n+1)/\phi(n)$ .

$$=x\left[L\frac{\phi\left(n+1\right)}{\left(n+1\right)^{r}}\right]L\frac{\phi\left(n\right)}{n^{r}}\right]L\frac{\left(n+1\right)^{r}}{n^{r}},$$

$$=x\left\{A/A\right\}\times1,$$

$$=x.$$

This very general theorem includes, among other important cases, the integro-geometric series

$$\phi(1) x + \phi(2) x^2 + \dots + \phi(n) x^n + \dots$$

where  $\phi(n)$  is an integral function of n: and the series

$$-\frac{x}{1} + \frac{x^2}{2} + \dots + \frac{x^n}{n} + \dots$$
 (1),

which, as we shall see in chap.  $xxv_{HH}$ , represents (when it is convergent)  $-\log(1-x)$ . It follows, by § 1, Th. IV., that, since the series (1) is convergent when x<1, the series

$$\frac{x}{1} - \frac{x^2}{2} + \dots + (-)^{n-1} \frac{x^n}{n} + \dots$$
 (2)

is also convergent when x < 1.

When (2) is convergent, it represents  $\log (1+x)$ ,

Example 3.  $\Sigma x^n/n!$  (the Exponential Series) is convergent for all values of x.

$$u_{n+1}/n_n = \{x^{n+1}/(n+1)!\}/\{x^n/n!\},$$

$$= x_n(n+1)$$

Hence, however great x may be, since it is independent of n, we may always choose x so great that, for all values of  $n \circ x$ , x/(n+1) < 1. Since the limit of the ratio of convergence is zero in this case, we should expect the convergency for mederate values of x to be very tapel; and this is so, as we shall show by examining the residue in a later chapter. We have supposed x to be positive; if x be negative the sense is convergence is in fact absolute, § 4. Th, IV.

Example 4.  $\Sigma (-1^n m (m-1) \dots (m-n+1) x^n / n! (x positive),$  where m has any real value  $^n$ , is convergent if x < 1, divergent if x < 1.

If m were a positive integer, the series would terminate, and the question of convergency would not arise.

$$\begin{split} Lu_{n+1}/u_n &= -xL \; \frac{m-n}{n+1} \; , \\ &= -xL \; \frac{m/n-1}{1+1/n} \; , \end{split}$$

Hence the theorem.

The series just examined is the expansion of  $(1-x)^m$  when x<1. It follows, by § 4. Th. IV., that the series  $\sum m(m-1) \dots (m-n+1) x^n/n!$ . whose terms are ultimately alternately positive and negative, is convergent if x < 1; this series is, as we shall see hereafter, the expansion of  $(1+x)^m$ when x < 1.

§ 6.] Cauchu's Condensation Test.—The general principle of this method, upon which many of the more delicate tests of convergence are founded, will be easily understood from the following considerations :-

Let  $\Sigma u_n$  be a series of positive terms which constantly decrease in value from the first onwards. Without altering the order of these, we may associate them in groups according to some law. If  $v_1, v_2, \dots, v_m$  be the 1st, 2nd, ... mth, ... of these groups, the series  $\Sigma v_m$  will contain all the terms of  $\Sigma u_n$ ; and it is obvious from the definition of convergency that \(\Sigmu\_u\) is convergent or divergent according as \(\Sigma\_m\) is convergent or divergent; we have in fact  $L \sum_{n=\infty}^{\infty} \Sigma u_n = L \sum_{m=\infty}^{\infty} \Sigma r_m$ . It is clear that the convergency or divergency of  $\Sigma r_m$  will be more apparent than that of  $\Sigma u_n$ , because in  $\Sigma v_n$  we proceed by longer steps towards the limit, the sum of n terms of  $\Sigma v_m$  being nearer the common limit than the sum of n terms of  $\Sigma u_n$ . Finally, if  $\Sigma v'_n$  be a new series such that  $v'_n \gtrsim v_n$ , then obviously  $\Sigma u_n$  is convergent if  $\Sigma v'_n$ 

convergent

divergent

We shall first apply this process of reasoning to the following case :--

Example. The series 1/1 + 1/2 + ... + 1/n + ... is divergent.

Arrange the given series in groups, the initial terms in which are of the following orders, 1, 2,  $2^2$ , . . .  $2^m$ ,  $2^{m+1}$ , . . . The numbers of terms in the successive groups will be 2-1,  $2^2-2$ ,  $2^3-2^2$ , . . .  $2^{m+1}-2^m$ ,  $2^{m+2}-2^{m+1}$ , . . . respectively. Since the terms constantly decrease in value, if 2m+1 be the greatest power of 2 which does not exceed n, then

 $>1+\frac{m}{2}$ .

$$\begin{split} S_{\mathbf{a}} &: \frac{1}{1} + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2^2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \ldots + \left(\frac{1}{2^m} + \frac{1}{2^m + 1} + \ldots + \frac{1}{2^{m+1} - 1}\right), \\ &> 1 + \left(2^2 - 2\right) \frac{1}{2^2} + \left(2^3 - 2\right) \frac{1}{2^3} + \ldots + \left(2^{m+1} - 2^m\right) \frac{1}{2^{m+1}}, \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}, \end{split}$$

Hence, by making a sufficiently great, we can make  $S_n$  as large as we please. The series 1/1 + 1/2 + 1/3 + ... is therefore divergent. This might also be deduced from the inequality 60 of chap, xxy = 2.5.

Cauchy's Condensation Test, of which the example just discussed is a particular case, is as follows:—

If f(u) be positive for all values of n, and constantly decrease as n increases, then  $\Sigma f(n)$  is convergent or divergent according a  $\Sigma a^n f(a^n)$  is convergent or divergent, where a is any positive integer 4/2.

The series  $\Sigma f(n)$  may be arranged as follows:—

$$\left\{ f(1) + \dots + f(a-1) \right\} + \left\{ f(a) + f(a+1) + \dots + f(a^2-1) \right\}$$

$$+ \left\{ f(a^2) + f(a^2+1) + \dots + f(a^2-1) \right\}$$

$$+ \left\{ f(a^m) + f(a^m+1) + \dots + f(a^{m+1}-1) \right\}$$

Hence, neglecting the finite number of terms in the square brackets, we see that  $\Sigma f(n)$  is convergent or divergent according as

$$\Sigma \{ f(a^m) + f(a^m + 1) + \dots + f(a^{m+1} - 1) \}$$
 (1)

is convergent or divergent. Now, since  $f(a^m) > f(a^m+1) > \dots > f(a^{m+1}-1) > f(a^{m+1})$ , we have

$$(a^{m+1}-a^m)f(a^n) > f(a^m) + f(a^m+1) + \dots + f(a^{m+1}-1)$$

$$> (a^{m+1}-a^m)f(a^{m+1}).$$

that is,

$$(a + 1) a^m f(a^n) \ge f(a^m) + f(a^m + 1) + \dots + f(a^{m+1} - 1)$$
  
  $\ge \{(a - 1)/a\} a^{m+1} f(a^{m+1}).$ 

Hence, by § 4. Th. L. the series (1) is convergent if  $\Sigma(a-1)$   $a^{m}f(a^{-1})$  is convergent, divergent if  $\Sigma_{i}^{(a-1)}(a-1)_{i}a_{i}^{m+1}f(a^{m+1})$  is

divergent. Now, by § 4, Th. II.,  $\Sigma (a-1) u^m f'(u^n)$  is convergent if  $\Sigma a^m f(a^m)$  is convergent, and  $\Sigma \{(a-1)/a', a^{m+1}f'(a^{m+1})\}$  is divergent if  $\Sigma a^{m+1}f(a^{m+1})$  is divergent; and for our present purpose  $\Sigma a^m f(a^m)$  and  $\Sigma a^{m+1}f(a^{m+1})$  are practically the same series, say  $\Sigma a^n f'(a^n)$ . Hence Cauchy's Theorem is established.

N.B.—It is obviously sufficient that the function f(n) be positive and constantly decrease for all values of n greater than

a certain finite value r.

Cor. 1. The theorem will still hold if a have any positive value not less than 2\*.

Let a lie between the positive integers b and b+1, (b < 2). If  $\Sigma a^n f(a^n)$  be convergent, then  $\underset{n \to \infty}{L} a^n f(a^n) = 0$ , that is,  $\underset{x \to \infty}{L} x^n f(x) = 0$ .

Hence, on and after some finite value of x, the function xf'(x) will begin to decrease constantly  $\dagger$  as x increases. We must therefore have  $(b+1)^nf'(b+1)^nf < a^nf'(a^n)$ , on and after some finite value of n. If, therefore,  $\sum a^nf(a^n)$  is convergent, a factor, will  $\sum (b+1)^nf'(b+1)^nf$ 

If  $\Sigma a^n f(a^n)$  be divergent,  $xf(x) = 1^{\circ}$  may, or  $2^{\circ}$  may not decrease

as x increases.

In case 1°,  $b^n f(b^n) > a^n f(a^n)$ . Hence the divergence of  $\Sigma a^r f(a^n)$  involves the divergence of  $\Sigma b^r f(b^n)$ ; and the divergence of  $\Sigma f(u)$  follows by the main theorem.

In case 2°, the divergence of  $\Sigma f(n)$  is at once obvious; for, if  $L x f(x) \neq 0$ , then ultimately  $x f'(x) \geq A$ , where  $A \geq 0$ . Hence  $f'(x) \geq A/x$ . Now  $\Sigma A/n$  is divergent, since  $\Sigma 1/n$  is divergent; therefore  $\Sigma f(n)$  is divergent.

In what follows we shall use  $\epsilon x$ ,  $\epsilon^2 x$ , . . . to denote  $a^r$ ,  $a^{s^*}$ , . . , a being any positive quantity  $\leq 2$ ; and  $\lambda x$ ,  $\lambda^* \epsilon_r$ , . . .  $\lambda r_r F x_r$ , . . . to denote  $\log_2 x_r \log_a(\log_a x_r)$ , . .  $\log_e x_r \log_e(\log_a x_r)$ , . . , where e is Napier's Base.

Also if 1<a<2, see Kohn, Grunert's Archiv, Bd. 67 (1882) and Hill, Mess. Math., N. S., 307 (1896).

<sup>+</sup> This assumes that xf(x) has not an infinite number of turning values; so that we can take x so great that we are past the last turning value, which must be a maximum.

Cor. 2.  $\Sigma f(n)$  is convergent or divergent according as  $\Sigma cac^2n$  . . . Caf(Cn) is convergent or divergent.

This follows, for integral values of the base a, by repeated application of Cauchy's Condensation Test; and, for non-integral values of a, by repeated applications of Cor. 1. Thus  $\Sigma f(a)$  is convergent or divergent according as  $\Sigma a f(a)$  is convergent or divergent. Again,  $\Sigma a f(a)$  is convergent or divergent according as  $\Sigma a n \epsilon(a) f(\epsilon a)$ , that is  $\Sigma a n \epsilon f(\epsilon a)$ , is convergent or divergent; and so on

Cor. 3.  $\Sigma f(u)$  is convergent or divergent according as the first of the functions

$$T_o = \lambda f(x)/x$$

$$T_1 = \lambda \{xf(x)\}/\lambda x$$
,

$$T = \lambda \{x\lambda xf(x)\}/\lambda^2 x$$

$$T_r = \lambda \{x\lambda x\lambda^2 x ... \lambda^{r-1} xf(x)\} \lambda^r x$$

which does not vanish when x = x, has a negative or a positive limit.

By Cor. 2,  $\Sigma f'(n)$  is convergent or divergent according as  $\Sigma cn \tilde{c}n$  . . . Cnf(cn) is convergent or divergent.

Now the latter series is (by § 5, Th. I.) convergent or divergent according as

$$L_{i}(\epsilon n\vec{c}n : ..., \epsilon^{r}nf(\epsilon^{r}n))^{+n} < \text{or} > 1;$$

that is, according as

$$L \, \log_a \left\{ \epsilon n \epsilon^* n \, \ldots \, , \, \epsilon^r n f(\epsilon^r n) \right\}^{1/n} < > 0 \, ;$$

that is, 
$$\sum_{n=\infty}^{\infty} \log_4 \{ \alpha \epsilon^n n \ldots \epsilon^n t^r (\epsilon^n) \}^r n < > 0.$$

If we put  $x = \epsilon'n$ , so that  $\lambda x = \epsilon'^{-1}n$ ,  $\lambda^2x = \epsilon'^{-2}n$ , . . .  $\lambda^{r-1}x + \epsilon n$ ,  $\lambda^rx = n$ , and x = z when n = z, the condition for convergency or divergency becomes

$$L_{-}\lambda_{-}^{+}x\lambda_{-}x\lambda_{-}^{+}x, \quad , \quad \lambda_{-}^{+}xf(x)_{+}^{+}\lambda_{-}^{+}x<>0$$
 (1).

If, on the strength of Cor. 1, we take c for the exponential base, the condition may be written

$$L |l| x l x l \cdot x , \qquad |l^{r-1} x t(x)| / |l^{r} x < > 0$$
 (2),

where all the logarithms involved are Napicrian logarithms.

We could establish the criterion (2) without the intervention of Cor. 1 by first establishing (1) for integral values of a, and then using the theorem of chap. XXV., § 12, Example 4, that L |Xx|Fx = 1/a.

Cor. 4. Each of the series

$$\sum I/n^{1+\alpha}$$
 (1),

$$\Sigma 1/n [/n]^{1+a}$$
 (2),

$$\sum 1/n \ln \left[l^2 u\right]^{1+\alpha} \tag{3},$$

$$\Sigma 1/nln l^{2}n \dots l^{r-1}n \{l^{r}n\}^{1+\alpha} \qquad (r+1),$$

is convergent if a>0, and divergent if a=or<0.

As the function  $nlnl^n n \dots l^n$  frequently occurs in what follows, we shall denote it by  $P_\tau(n)$ ; so that  $P_{\theta}(n) = n$ ,  $P_1(n) - nln$ , &c.

1st Proof.—Apply the criterion that  $\Sigma f(n)$  is convergent or divergent according as  $LU_1^3P_r(x)f(x)_1^s/l^{r+1}x <> 0$ . In the present case,  $f(x) = 1/P_r(x)(l^rx)^a$ . Hence

$$l\{P_r(x)f(x)_i^t/l^{r+1}x = l\{1/(l^rx)_{i}^{\alpha_i}/l^{r+1}x,$$

It follows that (r+1) is convergent if a>0, and divergent if a<0. If a=0, the question is not decided. In this case, we must use the test function one order higher, namely,  $\{\{P_{r+1}(x)f(x)\}\}^{p+2}x$ . Since  $f(x)=1/P_r(x)$ , we have

$$l\{P_{r+1}(x)f(x)\}/l^{r+2}x = l\{l^{r+1}x\}/l^{r+2}x,$$
= 1 >0.

Hence, when a = 0, (r + 1) is divergent.

2nd Proof.— By the direct application of Cauchy's Condensation Test, the convergence of (1) is the same as the convergence of  $\Sigma a^{n}(a^{n})^{1+n}$ , that is,  $\Sigma (1/a^{n})^{n}$ . Now the last series is a geometrical progression whose common ratio is  $1/a^{n}$ ; it is therefore convergent if a > 0, and divergent if  $a \cdot$  or < 0. Hence (1) is convergent if a > 0, and divergent if a = or < 0.

Again, the convergence of (2) is by Cauchy's rule the same as the convergence of  $\Sigma a^n/a^n\{la^n\}_{l=1}^{n}$ , that is,  $\Sigma 1/(la)^{1+\alpha}n^{1+\alpha}$ ;

and the convergence of this last the same as that of  $\Sigma 1/n^{1+\alpha}$ . Hence our theorem is proved for (2).

 $1_{\sigma}$ t us now assume that the theorem holds up to the series (r). We can then show that it holds for (r+1). In fact, the convergence of (r+1) is the same as that of  $2\sigma^{\alpha}a^{\alpha}b^{\alpha}r^{2}a^{\alpha}$ ,  $t^{\alpha}r^{\alpha}a^{\alpha}t^{\alpha}r^{2}a^{\alpha}$ ,  $t^{\alpha}r^{\alpha}a^{\alpha}t^{\alpha}r^{2}a^{\alpha}$ ,  $t^{\alpha}r^{\alpha}a^{\alpha}t^{\alpha}r^{2}a^{\alpha}$ , that is,  $21 (nda)t^{\alpha}(nda)$ ,  $t^{\alpha}r^{\alpha}(nda)t^{\alpha}r^{\alpha}$ .

First suppose a>0, and a>e. Then la>1, nla>n. Hence

$$1/(nla) l(nla) \dots l^{r-2} (nla) {l^{r-1} (nla)}^{i+n}$$
  
 $< 1/(nla) \dots l^{r-2} n {l^{r-1} n}^{i+n}$ 

But, since a>0,  $\Sigma 1/P_{r-1}(n) (l^{r-1}n)^n$  is convergent, a fortiori,  $\Sigma 1/P_r(n) (l^rn)^n$  is convergent.

Next suppose  $a \geqslant 0$ , and 2 < a < e. Then nla < n; and, proceeding as before, we prove  $\Sigma 1/P_r(n) \langle l^r n \rangle^a$  more divergent than the divergent series  $\Sigma 1/P_{r-1}(n) \langle l^{r-1} n \rangle^a$ .

Legarithmic Scale of Convergency.—The series just discussed are of great importance, instanch as they form a scale with which we can compare series whose ratio of convergence is ultimately unity. The scale is a descending one; for the least convergent of the convergent series of the rth order is more convergent than the most convergent of the convergent series of the (r+1)th order. This will be seen by comparing the nth terms,  $u_n$  and  $u^*_{n-1}$  of the rth and (r+1)th series. We have  $u^*_n|u_n+(I^{r+1})^{n-1}|u^*_n|^{r}$ , where u is very small but >0, and  $u^*$  is very large.

If we put  $x \cdot U^{-1}n$ , we may write  $L_{n-n}u_n \in L_{n-n}(x^{n+n})$ ,  $Lx_1^{n+n}$ . Hence, however small a, so long as it is greater than 0, and however large a',  $Lu'_n/u_n = \mathbb{Z}$ .

If we suppose the character of the logarithmic scale established by means of the second demonstration given above, we may, by comparing  $\Sigma \alpha_s$  with the various series in the scale, and using § 1, Th. L, obtain a fresh demonstration of the criterion of Cor. 3. We leave the details as an exercise for the student. This is perhaps the best demonstration, because, apart from the criterion itself, nothing is presupposed regarding f(x), except that it is positive when x is greater than a certain finite value.

By following the same course, and using § 4, Th. 111., we can establish a new criterion for series whose ratio of convergence is ultimately unity, as follows, where  $\rho_x = f(x+1)/f(x)$ .

Cor. 5. If f(x) be always positive when x exceeds a certain finite value,  $\Sigma f(n)$  is convergent or divergent according as the first of the following functions—

$$\begin{split} & \tau_0 = \rho_x - 1 \ ; \\ & \tau_1 = P_v \left( (x+1) \rho_x - P_v \left( x \right) \right) ; \\ & \tau_2 = P_1 \left( (x+1) \rho_x - P_1 \left( x \right) \right) ; \\ & \vdots \\ & \tau_r = P_{r-1} \left( (x+1) \rho_x - P_{r-1} \left( x \right) \right) ; \end{split}$$

which does not vanish when x = x has a negative or a positive limit.

Comparing  $\Sigma f(n)$  with  $\Sigma 1/P_r(n)\{l^r n\}^a$ , we see that  $\Sigma f(n)$  will be convergent if, for all values of x greater than a certain finite value,

where 
$$a > 0$$
.  $(1), P_{\tau}(x) \{ P_{\tau}(x) \} P_{\tau}(x+1) \{ P_{\tau}(x+1) \} P_{\tau}(x+1) \}$ 

Now (1) is equivalent to

$$P_{r}(x+1)\rho_{x} - P_{r}(x) < P_{r}(x) \left[ \{l^{r}x/l^{r}(x+1)\}^{\alpha} - 1 \right].$$

$$LP_{r}(x) \left[ \{l^{r}x/l^{r}(x+1)\}^{\alpha} - 1 \right].$$

$$\begin{aligned} & Also & LP_{r}(x)\left[\{l^{r}x/l^{r}(x+1)\}^{\alpha} - 1\right] \\ = & -LP_{r-1}(x)\left\{l^{r}(x+1) - l^{r}x\right\} \cdot \frac{l^{r}x}{l^{r}(x+1)} \cdot \frac{|l^{r}x|^{r}(x+1)|^{\alpha} - 1}{|l^{r}x|^{r}(x+1)|^{r} - 1} \end{aligned}$$

 $=-1 \times 1 \times \alpha = -\alpha$ , by chap. XXV..  $\lesssim$  12 and 13.

Hence a sufficient condition for the convergency of  $\Sigma f(n)$  is  $L \{P_r(x+1)\rho_x - P_r(x)\} < -\alpha$  (a positive),

In like manner, the condition for divergency is shown to be  $L_{-}\{P_{r}(x+1)\rho_{x}-P_{r}(x)\}>-\alpha \text{ (a negative)},$ 

$$>0$$
.  
Example 1. Discuss the convergence of  $\sum e^{-1+4\pi}\cdots^{-1}n_{j/s}$ .

Here  $T_0 = l \left\{ f(n) \right\} / n,$ 

$$= -\frac{1+1/2+...+1/n+rln}{n}$$

Now, by chap. xxv., § 13, Example 1,

$$1 + (r+1) \ln > 1 + 1/2 + \dots + 1/n + r \ln > r \ln + l (n+1).$$

C. 11.

Hence  $LT_{-}=0$ . We must therefore examine  $T_{+}$ . Now

$$T_1 = l \{ nf(n) \} / ln,$$
  
=  $-\{1 + 1/2 + ... + 1/n + (r - 1) ln \} / ln,$   
=  $-\{1 + 1/2 + ... + 1/n \} / ln - (r - 1).$ 

By chap. xxv., § 13, Example 2, L(1+1/2+...+1/n)/ln = 1. Hence  $LT_{r} = -1 - r + 1 = -r$ . The given series is therefore convengent or divergent according as r > or < 0.

If r = 0,  $LT_0 = 0$ , and  $LT_1 = 0$ . But we have

$$T_2 = l \{ nlnf(n) \} / l^2n,$$

=1-21+1/2+...+1/n In 3/En. Now, when n is very large, the value of 1 + 1/2 + ... + 1/n - ln approaches Euler's Constant. Hence  $I.T_n=1>0$ . In this case, therefore, the series

under discussion is divergent. Example 2. To discuss the convergence of the hypergeometric series.

$$1 + \frac{\alpha \cdot \beta}{\gamma \cdot \delta} x + \frac{\alpha (\alpha + 1) \cdot \beta (\beta + 1)}{\gamma (\gamma + 1) \cdot \delta (\delta + 1)} x^2 + \dots$$

The general term of this series is

$$f(n) = \frac{\alpha (\alpha + 1) \dots (\alpha + n - 1) \cdot \beta (\beta + 1) \dots (\beta + n - 1)}{\gamma (\gamma + 1) \dots (\gamma + n - 1) \cdot \delta (\delta + 1) \dots (\delta + n - 1)} x^{n}.$$

The form of f(n) renders the application of the first form of criterion somewhat troublesome. We shall therefore use the second. We have

$$\rho_{n} = \frac{(\alpha + n)(\beta + n)}{(\gamma + n)(\delta + n)}x,$$

$$\tau_{0} = \frac{(\alpha + n)(\beta + n)}{(\gamma + n)(\delta + n)}x - 1,$$

Hence the series is convergent if x < 1, divergent if x > 1.

If x = 1,  $L\tau_0 = 0$ , and we have

$$\tau_1 = \frac{(n+1)(\alpha+n)(\beta+n)}{(\gamma+n)(\beta+n)} - n,$$

$$= \frac{(\alpha+\beta-\gamma-\delta+1)n^2 + An + B}{n^2 + Cn + D};$$

 $I.\tau_1 = a + \beta$   $\gamma - \delta + 1$ .

If, therefore, x-1, the hypergeometric series is convergent or divergent according as  $\alpha \cdot \beta \cdot \gamma = \delta + 1 \sim \text{ or } > 0$ .

If  $\alpha + \beta = \gamma = \delta + 1 = 0$ ,  $L\tau_1 = 0$ . But we have

$$\tau_{I} = (n+1) l (n+1) \frac{(\alpha+n) + \beta+n}{(\gamma+n) \cdot (\delta+n)} \quad nln,$$

$$= \{n \{l(n+1) - ln\} + (\alpha + \beta + 1)\} \{l(n+1) - ln\} + \{ltl(n+1) + Bln\} / n$$

Hence, since 
$$Ln\{l(n+1)-ln\}=1$$
,  $L\{l(n+1)-ln\}=0$ ,  $L(n+1)/n^2=0$ .

Hence, since 
$$Ln\{l(n+1) - ln\} = 1$$
,  $L\{l(n+1) + ln\} = 0$ ,  $L(n+1)/n^2 = 0$ ,  $L(n+1)$ 

Example 3. Consider the series

$$1 - \frac{m}{1} + \frac{m(m-1)}{1 \cdot 2} + \dots + (-1)^n \frac{m(m-1)}{1 \cdot 2 \cdot \dots \cdot n} + \dots + (-1)^n \frac{m(m-1)}{1 \cdot 2 \cdot \dots \cdot n} + \dots$$

This may be written

This may be written 
$$1 + \frac{m}{1} + \frac{(-m)(-m+1)}{1 \cdot 2} + \dots + \frac{(-m)(-m+1) \cdot \dots \cdot (-m+n-1)}{1 \cdot 2 \cdot \dots \cdot n} + \dots$$

It is therefore a hypergeometric series, in which  $\alpha = -m$ ,  $\beta = \gamma$ ,  $\delta = 1$ , x=1. It follows from last article that the series in question is convergent or divergent according as -m < >0, that is, according as m is positive or negative.

This series is the expansion of  $(1-x)^m$ , when x=1.

Example 4. Consider the series

$$1 + \frac{m}{1} + \frac{m(m-1)}{1 \cdot 2} + \dots + \frac{m(m-1)}{1 \cdot 2} \cdot \frac{(m-n+1)}{1 \cdot 2} + \dots$$
 (I).

In this series the terms are ultimately alternatively positive and negative in sign. Hence the rules we have been using are not directly applicable.

1st. Let m be positive; and let m − r be the first negative quantity among m, m-1, m-2, . . . Ac., then, neglecting all the terms of the series before the (r+1)th, we have to consider

$$\frac{m(m-1)\dots(m-r+1)}{1\cdot 2\cdot \dots r} \left\{ 1 + \frac{m-r}{r+1} + \frac{(m-r)(m-r-1)}{(r+1)(r+2)} + \dots \right\}$$

If we change the signs of the alternate terms of the series within brackets, it becomes

$$1 + \frac{r - m}{r + 1} + \frac{(r - m)(r - m + 1)}{(r + 1)(r + 2)} + \dots$$
(3).

Now (3) is a hypergeometric series, in which  $\alpha = r - m$ ,  $\beta = \gamma$ ,  $\delta = r + 1$ , x=1. Hence  $a + \beta - \gamma - \delta + 1 = r - m - (r+1) + 1 = -m < 0$ . Therefore (3) is convergent. Hence (2), and therefore (1), is absolutely convergent.

2nd. Let m be negative,  $= -\mu$  say. The series (1) then becomes

$$1 - \frac{\mu}{1} + \frac{\mu(\mu+1)}{1 \cdot 2} - \dots + (-1)^n \frac{\mu(\mu+1) \cdot \dots \cdot (\mu+n-1)}{1 \cdot 2 \cdot \dots \cdot n} + \dots (1).$$

Since \( \mu \) is positive, the hypergeometric series

$$1 + \frac{\mu}{1} + \frac{\mu(\mu+1)}{1 \cdot 2} + \dots + \frac{\mu(n+1) \cdot \dots \cdot (\mu+n-1)}{1 \cdot 2 \cdot \dots \cdot n} + \dots$$
 (5).

is divergent.

Hence (4) cannot be absolutely convergent in the present case,

Since  $\rho_n = -(\mu + n)/(n + 1)$ , the terms will constantly increase in numerical value if  $\mu > 1$ . Hence the series cannot be even semi-convergent unless  $\mu < 1$ . If  $\mu$  be less than 1,  $\rho_{\mu} < 1$ , and the series will be semi-convergent provided  $Lu_n = 0$ .

Now 
$$\log u_n = \sum \log \frac{\mu + n}{1 + n} = \sum \log \left\{ 1 + \frac{\mu - 1}{\mu - 1} \right\}$$
.

Since  $L \log \{1 + (\mu - 1)/(n + 1)\}/\{(\mu - 1)/(n + 1)\} = 1$  (see chap. xxv., § 13), the series  $\Sigma \log \{1 + (\mu - 1)/(n+1)\}$  and  $\Sigma (\mu - 1)/(n+1)$  both diverge to an infinity of the same sign. But the latter series diverges to  $-\infty$  or  $+\infty$ . according as  $\mu < \text{ or } > 1$ . Hence  $Lu_n = 0$  or  $\infty$ , according as  $\mu < \text{ or } > 1$ .

Hence the series 1 is livercont if a>1, a=1 one reset if a<1. It is a large two decision of a=1. Hence, it is also up the series (I)

is also lately convergent if  $0 = -\pi$ :

semi-convergent,  $d = -1 + m \le 0$ ; cosculating, d = -1 = m;

divergent, if - z < m < -1.

SERIES WHERE THEMS HAVE PAUL DIPALLY LIPTUREING NEGATIVE FIGNS OF CONTAIN A PERIODIC FACTOR SUCH AS SIN MO.

8.7.] Series which contain an infinite number of negative terms may or may not be also butby convergent. The former class falls under the cases abrudy the useed. We propose now to give a few the runs regarding the latter class of series, whose convergency depends on the distribution of negative signs through ut the series.

The only cases of much provided importance are those—1st, where the infinity of negative signs has a periodic arrangement;

How will Nove-Howevery to notice of someod theorems, given comply Warne in his Mediatories do not a la Giuss in his great member in the Hypercoem that S mee, it may be said that Canchy was the for the of the military to relefic mercet sense; an impact of the general principles of the should were not not his Release's Analytiques and in in other Mily or over In his Force was a Markemanipaes tom (1817), he gave to following integral orders in from which most of the higher emteria have ring —If fir larg, values of the end of positive and increases, this  $\Sigma \to \infty$  is a property of  $I = \int_{-\infty}^{\infty} dx = 0$  in whitten, otherwise divergent. The second state of the relations was first given by hands. Credit's Fourt, For xiii 1807 | Pa M. Sanger and Jersey at lall Lang. 310 (1869), first rave the Lorentzmin Salar of Fig. 1 mai line some established the Distribute. Some of Could to the first first stated entiting equivalent A character than the about firms of the second of a particular 5. Continental writers rise of these and standard account the the will a theory to Bertrand. Bertran, T. J. Nov. 1842, p. 1941, Morror stating matche had obtained the real of a care of Is Morra, who also Has Morrar is very important, by suscited filling a costless in livery to forms of the criteria and a demonstraton of the requires care, we have to refer attained his name, along with De Morean's to the to constitute of their Edment India Cartalistic gave er in litary for contrations of a range doctors on and Malmstein Granert's all the even gar can say interest into your measures in a gending essentially

2nd, where the occurrence of negative signs is caused by the presence in the nth term of a factor, such as  $\sin n\theta$ , which is a periodic function of n.

In the former case (which might be regarded as a particular instance of the latter) we can always associate into a single term every succession of positive terms and every succession of negative terms. Since the recurrence of the positive and negative terms is periodic, we thus reduce all such series to the simpler case, where the terms are alternately positive and negative.

We may carry the process of grouping a step farther, and associate each negative with a preceding or following positive term, and the result will in general be a series whose terms are ultimately either all positive or all negative.

The process last indicated often enables us to settle the convergence of the series, but it must be remembered that the series derived by grouping is really a different series from the original one, because the sum of n terms of the original series does not always correspond to the sum of m terms of the derived series. The difference between the two sums will, however, never exceed

on the inequality of chap, xxv., § 13, Cor. 6, that  $\Sigma 1^{r}P_{r}(m+n)\{l^{r}(m+n)\}^{q}$ (where l'm is positive) is convergent or divergent, according as a < or < 0; and thence deduces Cor. 3. Pancker, Crelle's Jour., Bd. xLII. (1851), deduces both Cor. 3 and Cor. 5 from Cauchy's Condensation Test, much as we have done, except that the actual form in which we have stated the rule of Cor. 5 is taken from Catalan, Traité Él. d. Séries (1860). Dn Bois-Reymond, Crelle's Jour., Bd. LXXVI. (1873), gives an elegant general theory embracing all the above criteria, and also those of Kummer, Crelle's Jour., XIII. (1835). Abel had shown that, however slightly divergent  $\sum u_n$  may be, it is always possible to find  $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$  such that  $L\gamma_n=0$  and yet  $\Sigma \gamma_n u_n$  shall be divergent. Dn Bois-Reymond shows that, however slowly  $\Sigma u_n$  converge, we can always find  $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$  such that  $L\gamma_n = \infty$  and  $\Sigma \gamma_n u_n$  nevertheless shall be convergent. He shows that functions can be conceived whose ultimate increase to infinity is slower than that of any step in the locarithmic scale; and concludes definitely that there is a domain of convergency on whose borders the logarithmic criteria entirely fail-a point left doubtful by his predecessors. Finally, Kohn, Grunert's Archiv (1882), continuing Du Bois-Reymond's researches, gave a new criterion of a mixed character; and Pringsheim (Math. Ann. 1890, 1891) has discussed the whole theory from a general point of view. The whole matter, although not of great importance as regards the ordinary applications of mathematics, illustrates an exceedingly interesting phase in the development of mathematical thought,

the sum of a finite number of terms of the original series; and this difference must vanish for  $n = \infty$ , if the terms of the original series ultimately become infinitely small.

Example. Consider the series

$$\frac{1}{1} \cdot \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{3n - 2} - \frac{1}{3n - 1} - \frac{1}{3n} + \dots$$
 (1)

Compare this with the series

$$\frac{1}{1} - \left(\frac{1}{2} + \frac{1}{3}\right) + \frac{1}{4} - \left(\frac{1}{5} + \frac{1}{6}\right) + \dots + \frac{1}{3n - 2} - \left(\frac{1}{3n - 1} + \frac{1}{3n}\right) + \dots$$
(2),

that is, the series whose (2n-1)th term is 1/(3n-2), and whose (2n)th term is -(1/(3n-1)+1/3n).

If  $S_n$ ,  $S_n'$  denote the sums of n terms of (1) and (2) respectively, then  $S_{2n-2} = S_{2n'-1}$ ,  $S_{2n-1} = S_{2n'-1} - 1/(3n-1)$ ,  $S_{2n} = S_{2n'}$ . Since L1/(3n-1) = 0, we have in all cases  $LN_n = LS_n'$ . Hence (1) is convergent or divergent according as (2) is convergent or divergent. That (1) is really divergent may be shown by comparing it with the series.

$$\Sigma \{1/(3n-2)-1/(3n-1)-1/3n\}$$
 (3),

If  $S_n^{(n)}$  denote the sum of n terms of this last series, we can show as before that  $LS_n^{(n)} = LS_n^{(n)}$ . But the nth term of (3) can be written in the form  $(-9+12/n-2/n^2)/(3-2/n)/(3-1/n)3n$ ; and therefore bears to the nth term of  $\Sigma 1/n$  a ratio which is never infinite. But  $\Sigma 1/n$  is divergent.

By § 4, II., (3) is therefore also divergent. Hence (1) is divergent.

It should be noticed that in the case of an oscillating series, where Lu, +0, the grouping of terms may convert a non-convergent into a convergent series; so that we cannot in this case infer the convergency of the original from the convergency of the derived series.

Example.

$$\left(1+\frac{1}{2}\right)^2 - \left(1+\frac{1}{3}\right)^2 + \dots + \left(1+\frac{1}{2n}\right)^2 - \left(1+\frac{1}{2n+1}\right)^2 + \dots$$

is obviously a non-convergent oscillating gries. By

$$\left\{ \left(1 + \frac{1}{2}\right)^2 - \left(1 + \frac{1}{3}\right)^2 \right\} + \left\{ \left(1 + \frac{1}{1}\right)^2 - \left(1 + \frac{1}{3}\right)^2 \right\} + \dots + \left\{ \left(1 + \frac{1}{2n}\right)^2 - \left(1 + \frac{1}{3}\right)^2 \right\} + \dots \right\}$$

whose nth term is  $(8n^2 + 8n + 1)/(4n^2 + 2n)^2$ , i.e.  $(8 + 8/n + 1/n^2)/16(1 + 1/2n)^2 n^2$ , is convergent, being comparable in the scale of convergency with  $\Sigma 1/n^2$ .

 This remark is all the more important because the converse process of splitting up the afte term of a series into a group of terms with alternating spins, and using the rules of § 8, often gives a simple me ans of deciding as to its convergency. These ries 1/1, 2 + 1/3, 1 + 1/5, 6 + 1/7, 8 + . . . . may be tested in time way. § 8.] The following rule is frequently of use in the discussion of semi-converging series:—

If 
$$u_1 > u_2 > u_3 > \dots > u_n > \dots$$
 and all be positive, then
$$u_2 - u_2 + u_2 - \dots - (-)^{n-1}u_n + (-)^n u_{n+1} + \dots$$
 (1)

converges or oscillates according as  $L u_n = or + 0$ .

Using the notation of § 3, we have

$$u_n R_n = \pm (u_{n+1} - u_{n+2} + \dots \pm u_{n+m}),$$
  
 $= \pm \{u_{n+1} - (u_{n+2} - u_{n+2}) - \dots \},$   
 $= \pm \{(u_{n+1} - u_{n+2}) + (u_{n+2} - u_{n+4}) + \dots \}.$ 

Hence we have

$$u_{n+1} > {}_{m}R_{n} > u_{n+1} - u_{n+2}$$
 (2),

numerical values being alone in question. If, therefore,  $Lu_n = 0$ , we have  $Lu_{n+1} = Lu_{n+2} = 0$ ; and it follows that  $L_m R_n' = 0$  for all values of m. Also

 $u_1 > u_2 = S_n > u_1 - u_2$ 

so that  $S_n$  is finite for all values of n. The series (1) is therefore convergent if  $Lu_n=0$ .

If  $Lu_n = a + 0$ , then  $L_{n,n}R_n = a$  or = 0 according as m is odd or even. Hence the series is not convergent. We have, in fact,  $L(S_{m+1} + S_m) = Lu_{m+1} = a$ , which shows that the sum of the series oscillates between S and S + a, where  $S = LS_{\infty}$ .

Cor. The series

$$(u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}) + \dots$$

where u1, u2, . . . are as before, is convergent.

Example 1. The series  $\Sigma \{-1\}^{n-1}/n$  is convergent, notwith-standing the fact, already proved, that  $\Sigma 1/n$  is divergent.

Example 2.  $\Sigma(-1)^{n-1}(n+1)/n$  is an oscillating series; but  $\Sigma(-1)^{n-1}(n+1)/n - (n+2)/(n+1)$ } is convergent.

§ 9.] The most important case of periodic series is  $\Sigma a_n \cos(n\theta + \phi)$ , where  $a_n$  is a function of n, and  $\phi$  is independent of n, commonly spoken of as a *Trisponmetrical* or *Fourier's Series*. The question of the convergence of this kind of series is one of great importance owing to their constant application in mathematical physics.

We observe in the first place that

If Σi<sub>n</sub> be an absolutely converging series then Σa<sub>n</sub>cos(nθ + φ) is convergent.

This follows from § 4, I.

II. If  $\theta = 0$  or  $2k\pi/(k$  being an integer),  $\Sigma a_n \cos(n\theta + \phi)$  is convergent or divergent according as  $\Sigma a_n$  is convergent or divergent.

nvergent or direction taccording as  $\Sigma a_n$  is convergent or diver. This is obvious, since the series reduces to  $\Sigma a_n \cos \phi$ .

III. If  $\theta \neq 0$  or  $2k\pi$ , then  $2a_{+}\cos\left(n\theta \neq \phi\right)$  is convergent if, for all radius of n greater than a certain finite value,  $a_{n}$  has the same sign and never increases as n increases, and if  $L(a_{n}=0)$ .

This is a particular case of the following general theorem, which is founded on an inequality given by Abel:-

IV. If  $\Sigma u_n$  be convergent or oscillatory, and  $a_1, a_2, \ldots, a_n, \ldots$ by a series of positive quantities, which never increase as n increases, and if  $L(a_n = 0)$ , then  $\Sigma a_n u_n$  is convergent.

Abel's Inequality is as follows:--If, for all values of n,

$$A > u_1 + u_2 + \dots + u_n > B$$

where  $u_1, u_2, \ldots, u_n$  are any real quantities whatever, and if  $u_1, u_2, \ldots, u_n$  be a series of positive quantities which never increase as n increases, then

$$a_1.1 > a_1u_1 + a_2u_2 + \dots + a_nu_n > a_1B.$$

This may be proved as follows:—Let  $S_n=u_1+u_2+\ldots+u_n$ ,  $S_n'=a_1u_1+a_2u_2+\ldots+a_nu_n$ . Then  $u_1=S_1,\ u_2=S_2-S_1,\ \&c.$ ; and

$$S_n^+ = a_1 S_1 + a_2 (S_1 - S_1) + \dots + a_n (S_n - S_{n-1}),$$

$$= S_1 \cdot a_1 - a_2 + S_2 (a_1 - a_2) + \dots + S_{n-1} (a_{n-1} - a_n) + S_n a_n$$

Hence, since  $S_i$ ,  $S_1$ , . . . ,  $S_i$  are each  $\leq A$  and  $\geq B_i$  and  $(a_1-a_2)_i$ ,  $(a_1-a_2)_i$ , . . . ,  $(a_{n-1}-\alpha_i)_i$ ,  $a_i$  are all positive or zero,

$$\{(a_1 + a_2) + (a_1 - a_2) + \dots + (a_{r-1} - a_n) + a_{n1}^{-1}A$$

 $> S_n > \{(a_1 - a_2) + (a_2 - a_1) + \dots + (a_{n-1} - a_n) + a_n\} B;$ 

$$a_{i}A = S_{n} \rightarrow a_{i}B \tag{1}.$$

Theorem IV follows at once, for, since  $\Sigma u_n$  is not divergent,

 $S_n$  is not infinite for any value of n. Hence, by (1),  $S_n'$  is not infinite. Also, by Abel's Inequality,

$$\begin{aligned} &a_{n+1}C >_m R_n' = a_{n+1}u_{n+1} + a_{n+2}u_{n+2} + \dots + a_{n+m}u_{n+m} \\ &= S_{n'+m}' - S_n' > a_{n+1}D \end{aligned} \tag{2},$$

where C and D are the greatest and least of the values of  ${}_{m}R_{n} (=u_{n+1}+u_{n+2}+\ldots +u_{n+m}=S_{n+m}-N_{n})$  for all different positive values of m. Now, since  $\Sigma u_{n}$  is convergent or oscillatory,  $S_{n+m}-S_{n}$  is either zero or finite, and  $L_{n}u_{n+1}=0$ , by hypothesis. Therefore, it follows from (2), that  $L_{n}R_{n}'=0$  for all values of m. Hence  $\Sigma u_{n}u_{n}$  is convergent.

We shall prove in a later chapter that, when

$$u_n = \cos(n\theta + \phi),$$
  
 $S_n = \sin\frac{1}{2}n\theta\cos\frac{1}{2}(n+1)\theta + \phi\}/\sin\frac{1}{2}\theta.$ 

If, therefore, we exclude the cases where  $\theta = 0$  or  $2k\pi$ , we see that  $S_a$  cannot be infinite. Theorem III, is thus seen to be a particular case of Theorem IV.

Cor. If  $a_n$  be as  $a^{\dagger}_n$   $c_n \leq (-1)^{n-1}a_n \cos(n\theta + \phi)$ ,  $\sum a_n \sin(n\theta + \phi)$ , and  $\sum (-1)^{n-1}a_n \sin(n\theta + \phi)$  are all convergent.

## CONVERGENCE OF A SERIES OF COMPLEX TERMS.

§ 10.] If the *n*th term of a series be of the form  $x_n + y_n i$ , where *i* is the imaginary unit, and  $x_n$  and  $y_n$  are functions of n, we may write the sum of n terms in the torm  $S_n + T_n i$ , where

$$S_n = x_1 + x_2 + \dots + x_n,$$
  
 $T_n = y_1 + y_2 + \dots + y_n.$ 

By the sum of the infinite series  $\Sigma(x_n + y_n i)$  is meant the limit when  $n = \infty$  of  $S_n + T_n i$ ; that is,  $(LS_n) + (LT_n) i$ .

The necessary and sufficient condition for the convergency of  $\Sigma(x_n + y_n i)$  i - crefore that  $\Sigma x_n$  and  $\Sigma y_n$  be both convergent.

For, if the series  $\Sigma x_n$  and  $\Sigma y_n$  converge to the values S and T respectively,  $\Sigma (x_n + y_n t)$  will converge to the value S + T t; and, if either of the series  $\Sigma x_n$ ,  $\Sigma y_n$  diverge or oscillate, then  $(\mathcal{B}S_n) + (LT_n)^T$  will not have a finite definite value.

§ 11.] Let  $z_n$  denote  $x_n + y_n i$ ; and let  $|z_n|$  be the modulus of  $z_n^{-2}$ ; so that  $|z_n| = |x_n|^2 + |y_n|^2$ . We have the following theorems i, which are sufficient for most elementary purposes:—

1. The complex  $s_n i_n S_n^{-1}$  is composite if the real series  $S_n^{-1} = 1$ .

The complex series Σz<sub>n</sub> is convergent if the real series Σ |z<sub>n</sub>| is convergent.

For, since  $\Sigma[z_n]$  is convergent, and  $[x_n]$  and  $[y_n]$  are each less than  $[z_n]$ , it follows from § 4, 1., that  $\Sigma[x_n]$  and  $\Sigma[y_n]$  are both convergent; that is,  $\Sigma x_n$  and  $\Sigma[y_n]$  are both convergent. Hence, by § 10,  $\Sigma z_n$  is convergent.

It should be noticed that the condition thus established, although sufficient, is not necessary. For example, the series  $(1-i)/1 + (1-i)/2 + (1-i)/3 + \ldots$  is convergent since  $1/1 - 1/2 + 1/3 + \ldots$  and  $-1/1 + 1/2 - 1/3 + \ldots$  are both convergent; but the series of moduli, namely,  $\sqrt{2}/1 + \sqrt{2}/2 + \sqrt{2}/3 + \ldots$ , is divergent.

When  $\Sigma z_n$  is such that  $\Sigma_n | is convergent$ ,  $\Sigma z_n$  is said to be absolutely convergent. Since the modulus of a real quantity  $u_n$  is simply  $u_n$  with its sign made positive, if need be, we see that the present definition of absolute convergency includes that formerly given, and that the theorem just proved includes  $\S_n$ ,  $\S_n$ , as a particular case.

Cor. 1. If  ${}_mR_n$  denote  $z_{n+1}+z_{n+2} \dots + z_{n+m}$ , then the necessary and sufficient condition that the complex series  $\Sigma z_c$  converge is that it be possible, by taking a sufficiently great, to make  $|{}_mR_n|$  as small as we please, whatever the value of m.

Cor. 2. If  $\lambda_n$  be real or complex, and  $z_n$  a complex number whose modulus is not infinite for any value of n, however great, then  $\Sigma(\lambda_n z_n)$  will be absolutely convergent if  $\Sigma \lambda_n$  is absolutely convergent.

For  $|\lambda_n z_n| = |\lambda_n| |z_n|$ ; and, since  $\Sigma \lambda_n$  is absolutely convergent,  $\Sigma |\lambda_n|$  is convergent. Hence, since  $|z_n|$  is always finite,  $\Sigma |\lambda_{n+1} z_n|$  is convergent by § 4, II.; that is,  $\Sigma |\lambda_n z_n|$  is convergent. Hence  $\Sigma (\lambda_n z_n)$  is absolutely convergent.

Example 1. The series  $\Sigma_{i,j}(n)$  is absolutely convergent for all finite values of  $z_i$ 

Example 2. The series  $\Sigma_{+}^{n}/n$  is absolutely convergent provided |z|<1.

See chap, xit., § 13.

<sup>†</sup> Cauchy, Resumes Analytiques, § xiv.

Example 3. The series  $\Sigma(\cos\theta + i\sin\theta)^n/n$  is convergent if  $\theta + 0$  or  $2k\pi$ . For the series  $\Sigma\cos n\theta/n$  and  $\Sigma\sin n\theta/n$  are convergent by § 9. III.

Example 4. The series  $(\cos \theta + i \sin \theta)^n/n^2$  is absolutely convergent. For the series of moduli is  $\Sigma 1/n^2$ , which is convergent.

II. Let  $\Omega$  be the fixed limit or the greatest of the limits\* to which  $|z_n|^m$  tends when n is increased indefinitely, or a fixed limit to which  $|z_{n+j}/z_n|$  tends when n is increased indefinitely: then the series  $\Sigma_{z_n}$  will be convergent if  $\Omega < 1$  and divergent if  $\Omega > 1$ .

For, if  $\Omega < 1$ , then, by § 5, I. and II., the series  $\Sigma \mid z_n \mid$  is convergent; and therefore, by § 11, I.,  $\Sigma z_n$  is convergent.

If  $\Omega>1$ , then either some or all of the terms of the series  $\Sigma \|z_n\|$  ultimately increase without limit. In any case, it will be possible to find values of n for which  $\|z_n\|$  exceeds any value however great; and, since  $\|z_n\| = (\|x_n\|^2 + \|y_n\|^2)^{1/2}$ , the same must be true of one at least of  $\|x_n\|$  and  $\|y_n\|$ . Hence one at least of the series  $\Sigma x_n$ ,  $\Sigma y_n$  must diverge; and consequently  $\Sigma (x_n + y_n)$ , i.e.  $\Sigma z_n$  must diverge.

## APPLICATION OF THE FUNDAMENTAL LAWS OF ALGEBRA TO INFINITE SERIES.

§ 12.] Law of Association.—We have already had occasion to observe that the law of association cannot be applied without limitation to an infinite series; see the remarks at the end of § 7. It can, however, be applied without limitation provided the series is convergent. For let  $S_n$  denote the sum of m terms of the new series obtained by associating the terms of the original series into groups in any way whatever. Then, if  $S_n$  denote the sum of n terms of the original series, we can always assume m so great that  $S_n'$  includes at least all the terms in  $S_n$ . Hence  $S_n' = S_{n-n} p R_n$ , where p is a certain positive integer. Now, since the original

<sup>•</sup> It will be noticed that this includes the case where  $L_{\parallel}|z_n|^{-\alpha}$  has different values according to the integral character of n: but the corresponding case where  $L_{\parallel}|z_n|z_n|$  oscillates is not metaboled. We have retained Cauchy's original councilation; but it is easy to see that some additions might be made to the theorem in the latter case.

series is convergent, by taking n sufficiently large we can make  $_pR_n$  as small as we please. It follows therefore that  $LS_n = LS_n$ . Hence the association of terms produces no effect on the sum of the infinite convergence excise.

§13.] Law of Commutation.—The law of commutation is even more restricted in its application than the law of association. We may however prove that the law of commutation can be

applied to absolutely convergent series.

We shall consider here merely the case where each term of the series is displaced a finite number of steps. Let  $\Sigma u_n$  be the original series,  $\Sigma u_n'$ , the new series obtained by commutation of the terms of  $\Sigma u_n$ . Since each term is only displaced by a finite number of steps, we can, whatever n may be, by taking m sufficiently great always secure that  $S_n'$  contains all the terms of  $S_n$  at least. Under these circumstances  $S_n' = S_n$  contains fewer terms than  ${}_2B_n$ , where p is finite, since m is finite. Now, since  $\Sigma u_n$  is absolutely convergent, even if we take the most unfavourable case and suppose all the terms of the same sign, we shall have  $L_{p}R_n = 0$ ; and, a faction,  $L_{p}S_n' = L_{p}S_n = 0$ . Hence  $L_{p}S_n' = L_{p}S_n$ ; which establishes our theorem.

The above reasoning would not apply to a semi-convergent series because the vanishing of  $L_pR_n$  does not depend solely on the individual magnitude of the terms, but partially on the alternation of nestitive and negative signs.

Cauchy, in his Resources Analytiques, § vit. (1833), seems to have been the first to call explicit attention to the fact that the convergence of a semi-convergent series is essentially dependent on the order of its terms. Dirichlet and Ohm gave examples of the effect of the order of the terms upon the sum.

Finally Riemann, in his famous memoir on Fourier's Seriest, showed that the series  $\Sigma (-1)^{n-1}u_n$ , where  $Lu_n = 0$ , and  $\Sigma u_{2n+1}$  and  $\Sigma u_{2n}$  are both divergent, can, by proper commutation of its terms,

See below, \$ 33, Cor. 2.

<sup>+</sup> Written in 1854 and published in 1897. See his Gesammelte Math Barke, p. 211.

be made to converge to any sum we please; and Dirichlet has shown that commutation may render a semi-convergent series divergent.

When the sum of an infinite series is independent of the order of its terms it is said to converge unconditionally. It is obvious from what has been said that unconditional convergence and absolute convergence are practically synonymous,

Example 1. The series

$$-\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{(2n-1)}} - \frac{1}{\sqrt{(2n)}} + \dots$$
(1)

is convergent by § 8; but the series

$$\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - \frac{1}{\sqrt{1}}\right) + \dots + \left(\frac{1}{\sqrt{(4m+1)}} + \frac{1}{\sqrt{(4m+2)}} - \frac{1}{\sqrt{(2m+2)}}\right)^{+} \dots$$
 (2)

which is evidently derivable from (1) by commutation (and an association which is permissible since the terms ultimately vanish), is divergent. For if  $u_m = 1/\sqrt{(4m + 1) + 1/\sqrt{(4m + 3) - 1/\sqrt{(2m + 2)}}}$ , and  $r_m = 1/\sqrt{m}$ , from  $Lu_m/r_m = L \{1/\sqrt{(4 + 1/m) + 1/\sqrt{(1 + 3/m) - 1/\sqrt{(2 + 2/m)})}} = 1/2 + 1/2 - 1/\sqrt{2} = 1 - \frac{1}{2}\sqrt{2}$ . Hence  $u_m/r_m$  is always finite; and  $\Sigma r_m$  is divergent, by § 6, Cor. 4. Hence  $\Sigma u_m$  is divergent. (Dirichlet.)

Example 2. The series

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{1}{(2n-1)} - \frac{1}{(2n)} + \dots \qquad (1),$$

$$\left(\frac{1}{1} + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \dots + \left(\frac{1}{4m+1} + \frac{1}{4m+3}\right) - \frac{1}{2m+2} + \dots (2),$$

are both convergent; but they converge to different sums. For, by taking successively three and four terms of each series, we see that the sum of (1) lies between '583 and '833; whereas the sum of (2) lies between '926 and 1-176.

Addition of two infinite series. If  $\Sigma u_n$  and  $\Sigma v_n$  be both convergent, and converge to the values S and T respectively, then  $\Sigma (u_n + v_n)$  is convergent and converges to the value S + T.

We may, to secure complete generality, suppose  $u_n$  and  $v_n$  to be complex quantities. Let  $S_n$ ,  $T_n$ ,  $U_n$  represent the sums of n terms of  $\Sigma u_n$ ,  $\Sigma e_n$ ,  $\Sigma (u_n + v_n)$  respectively; then we have, however great n may be,  $U_n = S_n + T_n$ . Honce, when n - z.  $LU_n = LS_n + LT_{n_n}$  which proves the proposition.

§ 11.] Law of Distribution.—The application of the law of distribution will be indicated by the following theorems:—

If a be any finite quantity, and  $\Sigma u_n$  converge to the value  $S_n$ , then  $\Sigma au_n$  converges to as.

The proof of this is so simple that it may be left to the reader.

If  $\sum u_n$  and  $\sum v_n$  converge to the values S and T respectively, and at least one of the two series be absolutely convergent, then the series  $u_1v_1 + (u_1v_2 + u_2v_1) + \dots + (u_1v_n + u_1v_{n-1} + \dots + u_nv_n) + \dots$  (1) converges to the value  $ST^*$ .

Let  $S_n$ ,  $T_n$ ,  $U_n$  denote the sums of n terms of  $\Sigma u_n$ ,  $\Sigma v_n$ ,  $\Sigma (u_n v_n + u_2 v_{n-1} + \dots + u_n v_1)$  respectively; and let us suppose that  $\Sigma u_n$  is absolutely convergent. We have

$$S_n T_n = U_n + L_n$$

where

$$L_n = u_v r_n + u_v r_{n-1} + \dots + u_n r_2 + u_v r_n + \dots + u_n r_3$$

$$= u_z r_n + u_z (r_n + r_{n-1}) + \dots + u_n (r_n + \dots + r_z)$$
 (2).

If therefore u be even, -2m say,

$$L_n = \left[ u_2 v_{2m} + u_3 \left( v_{2m} + v_{,m+1} \right) + \dots + u_m \left( r_{,m} + \dots + r_{m+2} \right) \right]$$

+  $[u_{m+1}(r_{2m} + \ldots + r_{m+1}) + \ldots + u_{2m}(r_{2m} + \ldots + r_2)]$  (3). If n be odd, -2m + 1 say,

$$L_n \left[ u_i r_{im+1} + u_{\lambda} \left( r_{im+1} + r_{im} \right) + \dots + u_m \left( r_{im+1} + \dots + r_{m+1} \right) \right]$$

+ 
$$[u_{m+1}(v_{2m+1}+...+v_{m+2})+...+u_{2m+1}(v_{2m+1}+...+v_2)]$$
 (4).

Now, since  $\Sigma c_n$  is convergent it is possible, by making m sufficiently great, to make each of the quantities  $c_{2n}$ ,  $|c_{2m-1} + c_{2m}|$ , . . . ,  $|c_{m+2} + \cdots + c_{2m-1} + c_{2m+1} + \cdots + c_{$ 

<sup>•</sup> The original demonstration of this theorem given by Cauchy in his Analyse Algebrague equived that both the series \( \text{Tag.} \), \( \text{Tag.} \) is absolutely convergent. All 8% demonstrates is subject to the same restriction. The more peneral form was given by \( Methods, \text{Colored Jourge, LAXIV.} \) (1875). Abel had, however, proved a more general from (see § 20, Cot.), which partly includes the roal transparence of the more reported through (see § 20, Cot.).

+  $r_{2m+1}$  | as small as we please. Also, since  $|T_1|$ ,  $|T_2|$ ,  $|T_3|$ , . . . ,  $|T_m|$ , . . . are all finite, and  $|T_r - T_4| < |T_r| + |T_8|$ , therefore

$$|v_{m+1}+\ldots+v_{2m}|, \ldots, |v_2+\ldots+v_{2m}|,$$
  
 $|v_{m+2}+\ldots+v_{2m+1}|, \ldots, |v_2+\ldots+v_{2m+1}|,$ 

are all finite. Hence, if  $\epsilon_m$  be a quantity which can be made as small as we please by sufficiently increasing m, and  $\beta$  a certain finite quantity, we have, from (3) and (4), by chap. XII., § 11,

$$|L_n| < \epsilon_m (|u_2| + |u_3| + \ldots + |u_m|) + B(|u_m|) + |u_m| + |u_m| + |u_m|)$$

If, therefore, we make n infinite, and observe that, since  $\Sigma a_m$  is absolutely convergent,  $[u_z + |u_y| + \ldots + |u_n|]$  is finite, and  $L(|u_{m+1}| + |u_{m+2}| + \ldots + |u_n|) = 0$ , we have (seeing that  $L\epsilon_m = 0$ )  $L(L_\alpha) = 0$ . Hence  $LS_n T_\alpha = LU_\alpha$ , that is,  $LU_m = ST$ .

Cauchy has shown that, if both the sries involved be semiconvergent, the multiplication rate does not necessarily apply.

Suppose, for example,  $u_n = v_n = (-1)^{n-1}/\sqrt{n}$ . Then both  $\Sigma u_n$  and  $\Sigma v_n$  are semi-convergent series. The general term of (1) is

$$w_n = \pm \left(\frac{1}{\sqrt{\{n\}}} + \frac{1}{\sqrt{\{(n-1)/2\}}} + \dots + \frac{1}{\sqrt{\{2(n-1)\}}} + \frac{1}{\sqrt{\{n\}}}\right)$$
 (5).

Now, since  $r(n-r+1)=\frac{1}{2}(n+1)^2-\frac{1}{4}2(n+1)-r\frac{1}{2}$ , therefore, for all values of r,  $r(n-r+1)-\frac{1}{2}(n+1)^2$ , except in the case where r=1, (n+1), and then there is equality. It follows that  $m_n>m_1^2(n+1)-2/(1+1/n)$ . The terms of  $\Sigma m_n$  are therefore ultimately numerically greater than a quantity which is infinitely nearly equal to 2. Hence  $\Sigma m_n$  cannot be a convergent script.

UNIFORMITY AND NON-UNIFORMITY IN THE CONVERGENCE OF SERIES WHOSE TERMS ARE FUNCTIONS OF A VARIABLE.

§ 15.] Let x for the present denote a real variable. If the nth term of an infinite series be f(n, x), where f(n, x) is a single valued function of n and of x, and also for all integral values of n a continuous function of x within a certain interval, then the infinite series  $\geq f(n, x)$  will, if convergent, be a single valued finite function of x, say  $\phi(x)$ . At first sight, it might be supposed that  $\phi(x)$  must necessarily be continuous, seeing that each term of f(n, x) is so. Cauchy took this view; but, as

Abel\* first pointed out,  $\phi(x)$  is not necessarily continuous. No doubt  $\Sigma/(n, x + h)$  and  $\Sigma/(n, x)$ , being each convergent, have each definite finite values, and therefore  $\Sigma \setminus f(n, x + h) - f(n, x)$  is convergent, and has a definite finite value; but this value is not necessarily zero when h = 0 for all colors of x. Suppose, for example, following Du Bois-Reymond, that f(n, x) = x/(nx + 1) (nx - x + 1). Since f(n, x) = nx/(nx + 1) - (n - 1)x/(n - 1x + 1), we have, in this case,  $S_n = nx/(nx + 1)$ . Hence, provided  $x \neq 0$ ,  $LS_n = 1$ . If, however, x = 0 then  $S_n = 0$ , however great n may be. The function  $\phi(x)$  is, therefore, in this case, discontinuous when x = 0.

The discontinuity of the above series is accompanied by another peculiarity which is often, although not always, associated with discontinuity. The Residue of the series, when x+0, is given by

$$R_n = 1 - S_n = 1/(nx + 1).$$

Now, when x has any given positive value, we can by making n large enough make 1.(nx+1) smaller than any given positive quantity a. But, on the other hand, the smaller x is, the larger must we take n in order that 1.(nx+1) may fall under c; and, in general, when x is variable, there is no finite lower limit for n, and pendent of x, say v, such that if n>v then  $R_u < c$ . Owing to this peculiarity of the residue, the series is said to be non-uniformly convergent in any interval which includes 0; and, since, when x approaches 0, the number of terms required to secure a given degree of approximation to the limit becomes infinite, the series is said to Converge Infinitely Slowly near <math>x = 0.

These considerations lead us to establish the following important definition, where we no longer restrict ourselves to functions of a real variable. If, for all values of z within a given region R in Argand's Diagram, we can for every positive value of s, however small, assign a positive integer v induces then the z-is such that, when u > v,  $R_u > s$ , then the series

<sup>•</sup> Recherches sur la Serie  $1+\frac{r_0}{4}x+\frac{m(r_0-1)}{1+2}x^2+\ldots$  Crelle's Jour. Bd. r. (1826).

 $\Sigma f(n, x)$  is said to be Uniformly Convergent within the region in question.

Stokes\*, who in his classical paper on the Critical Values of the Sums of Periodic Series was the first to make clear the fundamental principle underlying the matter now under discussion, has pointed out that the question of uniformity or non-uniformity of convergence always arises when we consider the limiting value of a function of more than one variable. Consider, for example, the function f(x, y); and let us suppose that, for all values of y in a given region R, f(x, y) approaches a finite definite limit when x approaches the value  $\alpha$ ; and let us call this limit  $f(\alpha, y)$ . Then if we assign in advance any positive quantity  $\epsilon$ , however small, we can always find a positive quantity  $\lambda$ , such that, when  $|x-\alpha| < \lambda$ ,  $|f(x, y) - f(\alpha, y)| < \epsilon$ . If it be possible to determine  $\lambda$  so that the inomality

 $|f(x, y) - f(a, y)| < \epsilon$ 

shall hold for all values of y contained in R, then the approach or convergence to the limit is said to be uniform within R. If, on the other hand,  $\lambda$  depends on y, the convergence to the limit is said to be non-uniform.

Example 1. Consider the series  $1+z+2^2+\ldots+z^n+\ldots$ ; and let  $|z|<\rho-1$ . We have  $|R_n|=|z^{n+1}/(1-z)|<\rho^{n+1}/(1-\rho)$ . Hence, in order to secure that  $R_n<\epsilon$ , we have merely to choose  $n>-1+\log(\epsilon-\epsilon\rho)\log\rho$ . Since  $-1+\log(\epsilon-\epsilon\rho)\log\rho$ , is independent of z, we see that within any circle whose centre is the origin in Argand's Diagram, and whose radius is less than unity by however little, the series  $\mathbb{E}^{2n}$  is uniformly convergent.

On the other hand, as  $\rho$  approaches unity low  $(\epsilon - \epsilon \rho)/\log \rho$  becomes larger and larger. Hence the convergence of  $\Sigma^{\alpha}$  becomes infinitely slow when |z| approaches unity. We infer that the convergence of  $\Sigma^{\alpha}$  is not uniform within and upon the circle of radius unity. And, in fact, when the upper limit of |z| is 1, it is obviously impossible when  $\epsilon$  is given to assign a finite value of n such that  $|z^{\alpha+1}/(1-z)| < \sinh$  the true for all values of z.

<sup>•</sup> Trans. Camb. Phil. Soc., Vol. VIII. (1847). Continental writers have generally overlooked Stokes' work; and quote Stodel, Abbl. d. Rugerischen Abad. d. Wiss. Bd. v. (1850). For exceptions, see Reifi, Geschiebt, was under the Reifier, p. 207 (1889); and Pringsheim. Enc. d. Math. Wiss. Bd. u. p. 95 (1899). In his first edition the writer, although well acquainted with Stokes great paper, by an unfortunate lapse of memory, fell into the same mistake. The question of uniformity of convergence is now a fundamental point in the Theory of Functions.

Example 2. Osgood \* has shown that, if

$$\phi_n(x) = \sqrt{(2e)} n \sin^2 \pi x \cdot e^{-n^2 \sin^4 \pi x},$$

the infinite series which has  $\phi_n(x) + \phi_n(2!|x|/2! + \dots + \phi_n(n!|x)/n!$  for the sam of n terms converges non-uniformly in every interval.

From the definition of Uniform Convergence we can at once draw the following conclusions.

Cor. 1. If the terms of  $\Sigma | f(n,z)|$  are ultimately less than the terms of a converging series of positive terms whose values are independent of z, then  $\Sigma f(n,z)$  converges uniformly.

For, if  $\Sigma u_n$  be the series of positive terms in question, and  $R_n$  the residue of  $\Sigma f(n, z)$ , then

$$|R_n| \geqslant |f(n+1,z)| + |f(n+2,z)| + \dots,$$
  
 $\leq u_{n+1} + u_{n+2} + \dots,$ 

Since  $\sum u_n$  is convergent, we can find an integer  $\nu$  so that, when  $n > \nu$ ,  $u_{n+1} + u_{n+2} + \ldots < \epsilon$ ; and  $\nu$  will be independent of z, since  $u_{n+1}$ ,  $u_{n+2}$ , ... are independent of z. Hence we can find  $\nu$  independent of z so that  $|R_n| < \epsilon$ , when  $n > \nu$ ,  $\epsilon$  having the usual meaning.

Cor. 2. If  $\Sigma |f(n, z)|$  is uniformly convergent, then  $\Sigma f(n, z)$  is uniformly convergent.

§ 16.] We now proceed to establish a fundamental theorem regarding the Continuity of a Uniformly Converging Series.

Let f(n, z) be a finite single valued function of the complex variable z and the integral variable n, which is continuous as regards z for all values of n, however further, and for all values of z within a region R in Argand's Diagram. Farther, let  $\Sigma f(n, z)$  converge endformly within R, say to  $\phi(z)$ . Then  $\phi(z)$  is a continuous function of z at all points within the region R.

Let the sum to n terms and the residue after n terms of  $\Sigma_{\ell}(n,z)$  be  $S_n$  and  $R_n$ ; and let  $S_n'$  and  $R_n'$  be the like for  $\Sigma_{\ell}(n,z')$ , where z and z' are any two points within the region R. Then we have

$$\phi(z) = S_n + R_n, \quad \phi(z') = S_n' + R_n'$$
 (1).

Bull, Am. Math. Soc., Ser. 2, m. (1/26). This paper is well worthy of study on account of the interesting geometrical methods which the author uses.

Since  $\Sigma f(n,z)$  is uniformly convergent within R, given any positive quantity  $\epsilon$ , however small, we can find a finite integer r, independent of z, such that for all values of z within R,  $R_n < \epsilon$  and  $R_n' < \epsilon$ , when n > r. Let us suppose n in the equations (1) chosen' to satisfy this condition. Since the choice of z is unrestricted we can by making |z-z'| sufficiently small cause the absolute value of each of the differences  $f'(1,z)-f'(1,z'),\ldots,f'(n,z)-f'(n,z')$  to become as small as we please, and, therefore, since n is inite we can choose |z-z'| so small that  $|S_n - S_n'|$ , which is not greater

than  $\sum_{i=1}^{n} |f(n, z) - f(n, z')|$ , shall be less than  $\epsilon$ .

Now

$$\begin{aligned} \mid \phi\left(z\right) - \phi\left(z'\right) \mid &= \mid S_{n} - S_{n}' + R_{n} - R_{n}' \mid \\ &\Rightarrow \mid S_{n} - S_{n}' \mid + \mid R_{n} \mid + \mid R_{n}' \mid \\ &< 3\epsilon_{n} \end{aligned}$$

which proves our theorem; for  $\epsilon$ , and therefore  $3\epsilon$ , can be made as small as we please.

It follows from what has been proved that discontinuity of  $\Sigma f(n,z)$  is necessarily accompanied by non-uniformity of convergence; but it does not follow that non-uniformity of convergence is necessarily accompanied by discontinuity. In fact, Du Bois-Reymond has shown by means of the example

$$\Sigma \{x/n (nx+1) (nx-x+1) - x^2/(nx^2+1) (nx^2-x+1)\}$$

that infinitely slow convergence may not involve discontinuity. The sum of this series is always zero even when x=0; and yet, near x=0, the convergence is infinitely slow.

It should also be noticed that the fact that a series converges at a point of infinitely slow convergence, does not involve that the sum is continuous at that point. Thus the series

$$\Sigma x/(nx+1)(nx-x+1)$$

converges at x=0; but, owing to the infinite slowness of convergency at x=0, the sum is discontinuous there, being in fact 0 at x=0, and 1 for points infinitely near to x=0. In such cases it is necessary to state the region of uniform convergence with some care. The fact is that the series in question is convergent in the real interval  $\rho\!\!>\!\!x\!\!>\!\!b$ , where b is any finite

positive quantity and  $\rho$  is a positive quantity as small as we please but not evanescent. This is usually expressed by saying that the series is uniformly convergent in the interval 0 < x > b. Such an interval may be said to be 'open' at the lower and 'closed' at the upper end.

Example†. If  $\mu_n$  be independent of z, and  $w_n(z)$  be a single valued function of n and z, finite for all values of n, lowever great, and finite and continuous as regards z within a region R, then, if  $\Sigma u_n$  be absolutely convergent,  $\Sigma u_n v_n(z)$  is a continuous function of z within R.

It will be sufficient to prove that the series  $\sum \mu_n w_n(z)$  is uniformly convergent within R.

Since  $w_n(z)$  is finite for all points within R, we can assign a finite positive quantity, g, independent of z, such that, for all points within R,  $\{x_i\}_{i=1}^n$ 

Consider  $R_n$ , the residue of  $\Sigma \mu_n w_n(z)$  after n terms. We have

Hence

$$R_n = \mu_{n+1} w_{n+1}(z) + \mu_{n+2} w_{n+2}(z) + \dots$$
  
 $|R_n| > |\mu_{n+1}| |w_{n+1}(z)| + |\mu_{n+2}| |w_{n+2}(z)| + \dots$ ,  
 $\leq q(|\mu_{n+1}| + |\mu_{n+2}| + \dots),$ 

Since  $\Sigma u_n$  is absolutely convergent,  $\Sigma \mid \mu_n \mid$  is convergent, and we can assign an integer r such that, when n > r,  $|\mu_{n+1}| + |\mu_{n+2}| + \ldots < \epsilon |g|$ , where  $\epsilon$  is a positive quantity as small as we please.

Both  $\mu_n$  and g being independent of z, it is clear that r is independent of z. It noe we have, when n > r,  $|R_n| < r$ , r being independent of z. The series is therefore uniformly convergent: and it follows from the main theorem of this paragraph that its sum is a continuous function of z.

## Special discussion of the power series $\sum a_n z^n$ .

§ 17.] As the series  $\Sigma a_n z^n$  is of great importance in Algebraic Analysis and in the Theory of Functions, we shall give a special discussion of its properties as regards both convergence and continuity. We may speak of it for shortness as the Power Series; and we shall consider both  $a_n$  and z to be complex numbers, say  $a_n = r_n (\cos a_n + i \sin a_n)$ ,  $z = \rho (\cos \theta + i \sin \theta)$ , where  $r_n$  and  $a_n$  are functions of the integral variable u, but  $\rho$  and  $\theta$  are independent of u.

Harkness and Morley use these convenient words in their Introduction to the Theory of Analytic Functions, Macmillan (1898).

<sup>†</sup> Du Bos-Reymond, Math. Ann. iv. (1871).

The leading property of the Power Series is that it has what is called a  $Circle^*$  of Converg.nee, whose centre is the origin in Argand's Diagram, and whose radius (Iadius of Convergence) may be zero, finite, or infinite. For all values of z within (but not upon) this circle the series is absolutely and uniformly convergent; and (if the radius be finite) for all values of z without divergent. On the circumference of the circle of convergence the series may converge either absolutely or conditionally, oscillate, or diverge; but on any other circle it must either converge absolutely or else diverge.

The proof of these statements rests on the following theorem.

If the series  $\Sigma a_n z^n$  be at least semi-convergent when  $z=z_0$ , then it is absolutely and uniformly convergent at all points within a circle whose radius  $<|z_n|$ .

Since  $\Sigma a_n z_q^n$  is convergent, none of its terms can be infinite in absolute value, hence it is possible to find a finite positive quantity g such that  $|a_n z_q^n| < g$ , for all values of n however large.

Hence 
$$|a_n z^n| = |a_n z_0^n (z/z_0)^n|,$$
  
 $= |a_n z_0^n| |(z/z_0)^n|,$   
 $< g |(z/z_0)^n|.$ 

Now, since z is within the circle  $|z_0|$ ,  $|z/z_0| < 1$ . Hence the series  $g\Sigma(z/z_0)^n$  is absolutely convergent. Therefore (§ 4, I.)  $\Sigma |a_n z^n|$  is absolutely convergent.

The convergence is uniform. For, since  $|z| < |z_0|$ , we can find z' such that  $|z| < |z'| < |z_0|$ . Now, by the theorem just established,  $\mathbf{X} \|a_nz^n\|$  will be convergent, and its terms are independent of z. But, since |z| < |z'|,  $|a_nz^n| < |a_nz^n|$ . Hence, by § 15, Cor. 1,  $\mathbf{X} a_nz^n$  is uniformly convergent.

Circle of Convergence. Three cases are in general possible.

1st. It may not be possible to find any value  $z_0$  of z for which the series  $\Sigma a_n z^n$  converges. We shall describe this case by saying that the circle of convergence and the radius of convergence are infinitely small. An example is the series  $\Sigma n! z^n$ .

2nd. The series may converge for any finite value of z

When in what follows we speak of a circle (R), we mean a circle of radius R whose centre is the origin in Argand's Diagram.

however large. We shall then say that the circle and the radius of convergence are infinite. An example of this very important class of series is  $\Sigma x^{\alpha}/n!$ .

3rd. There may be finite values of z for which  $\Sigma a_z z^a$  converges, and other finite values for which it does not converge. In this case there must be a definite upper limit to the value of  $|z_o|$  such that the series converges for all points within the circle  $|z_o|$  and diverges for all points without. For the series converges when  $|z|<|z_o|$ ; and it must diverge when  $|z|>|z_o|$ ; for, if it converged even conditionally for  $|z'|>|z_o|$ , then it would converge when |z|<|z'|. We could, therefore, replace the circle  $|z_o|$  by the greater circle |z'|, and proceed in this way until we either arrive at a maximum circle of convergence, beyond which there is only divergence, or else fall back upon case 2, where the series converges within any circle however great.

We shall commonly denote the radius of the circle of convergence, or as it is often called the Radius of Convergence, by R. It must be carefully noticed that both as regards uniformity and absoluteness of convergency the Circle of Convergence is (so far as the above demonstration goes) an open region, that is to say, the points on its circumference are not to be held as being within it. Thus, for example, nothing is proved as regards the convergence of the power series at points on the circumference of the Circle of Convergence; and what we have proved as regards uniformity of convergence is that  $\Sigma a_n z^n$  is uniformly convergent within any circle whose radius is less than R by however little.

§ 18.] Cauchy's Rules for determining the Radius of Convergence of a Power Series.

 Let ω be the fixed limit or the greatest of the limits to which |a<sub>n</sub>|<sup>1m</sup> tends when n is increased indefinitely, then 1/ω is the radius of convergence of Σa<sub>n</sub>z<sup>n</sup>.

For, as we have seen in § 11, II.,  $\Sigma a_n z^n$  is convergent or divergent according as  $L |a_n z^n|^{+n} < or > 1$ ; that is, according as  $\omega |z| < or > 1$ ; that is, according as  $|z| < or > 1/\omega$ .

 Let whe a fixed limit to which |a<sub>n+1</sub>|a<sub>n</sub>| tends when n is increased indefinitely; then 1/w is the radius of convergence of \(\text{\sigma}\) a<sub>n</sub>z<sup>n</sup>.

The proof is as before. The second of these rules is often easier of application than the first; but it is subject to failure in the case where  $L |a_{n+1}/a_n|$  is not definite.

Example 1.  $1+z/1+z^2/2+...$ 

Here, by the first rule,  $\omega = L (1/n)^{1/n} = L m^m \cdot 1$  (chap. xxv., § 16). Hence R=1.

By the second rule,  $\omega = L_{n-\infty} \{1/(n+1)\}/\{1/n\} = L_{n-\infty} n/(n+1) = 1$ . Hence R=1, as before.

Example 2.  $z + 2z^2 + z^3 + 2z^4 + \dots$ 

Here if n = 2m,  $L |a_n^{-1/n}| = L |a_n^{-1/n}| = 1$ ,

if 
$$n = 2m + 1$$
,  $L \mid a_n^{1/n} \mid = L \mid 2^{1/n} = 1$ .

Hence  $\omega = 1$ , and R = 1. The second rule would fail.

§ 19.] Convergence of a Power Series on its Circle of Convergence.

The general question as to whether a power series converges, oscillates or diverges at points on its circle of convergence is complicated. For series whose coefficients are real the following rule covers many of the commoner cases.

 Let a<sub>n</sub> be real, such that ultimately a<sub>n</sub> has the same sign and never increases; also that  $La_n = 0$ , and  $La_{n+1}/a_n = 1$ , when  $n = \infty$ . Then the radius of convergence of  $\sum a_n z^n$  is unity; and

1st. If  $\Sigma a_n$  is convergent,  $\Sigma a_n z^n$  converges absolutely at every point on its circle of convergence.

2nd. If  $\Sigma a_n$  is divergent,  $\Sigma a_n z^n$  is semi-convergent at every point on its circle of convergence, except z=1, where it is divergent.

If we notice that on the circle of convergence  $\Sigma a, z^n$  reduces to  $\Sigma a_n(\cos n\theta + i\sin n\theta) = \Sigma a_n\cos n\theta + i\Sigma a_n\sin n\theta$ , we deduce the above conclusions at once from \$ 9.

Cor. Obviously the above conclusions hold equally for  $\Sigma (-1)^n a_n z^n$ , except that the point z=-1 takes the place of the point z = 1.

The following Rule, given by Weierstrass in his well-known memoir Ueber die Theorie der Analytischen Facultaten\*, applies

Crelle's Jour., Bd. 51 (1850).

to the more general case where the coefficients of the power series may be complex. By  $\S$  6, Cor. 5, it is easy to show that it includes as a particular case the greater part of the rule already given.

II. If on and after a certain value of n we can expand antilan in the form

$$\frac{a_{n+1}}{a_n} = 1 + \frac{g + hi}{n} + \frac{a_2}{n^2} + \dots$$

where g and h are real, then the behaviour of  $\Sigma a_n z^n$  on its circle of convergence, the radius of which is obviously unity, is as follows:—

1st. If a \d 0 the series diverges.

2nd. If a < -1 the series converges absolutely.

3rd. If  $-1 \geqslant g < 0$  the series is semi-convergent, except at the point z = 1, where it oscillates if g = -1 and h = 0, and diverges if g > -1.

For the somewhat lengthy demonstration we refer to the original memoir.

§ 20.] Abel's Theorems\* regarding the continuity of a power series.

Since (§ 18)  $\Sigma \sigma_n z^n$  converges uniformly at every point within its circle of convergence, we infer at once that

 The sum of the power series Σa<sub>n</sub>z<sup>n</sup> is a continuous function of z, say φ(z), at all points within its circle of convergence.

This theorem tells us nothing as to what happens when we pass from within to points on the circumference of the circle of convergence, or when we pass from point to point on the circumference. Much, although not all, of the remaining information required is given by the following theorem.

11. If the power series  $\sum a_n z^n$  be convergent at a point  $z_0$  on its circle of convergence, and z be any point within the circle, then

$$L \sum_{r=r_{-}}^{\infty} d_{n}z^{n} = \sum_{1}^{\infty} d_{n}z_{0}^{n};$$

provided the order of the terms in  $\Sigma a_n z_0^n$  be not deranged in cases where it is only semi-convergent.

<sup>.</sup> Crelle's Jour., Bd 1. (1826).

In the first place, we can show that in proving this theorem we need only consider the case where z and  $z_0$  lie on the same radius of the circle of convergence. For, if z and  $z_0$  be not on the same radius, describe a circle through  $z_0$ , and let it meet the radius  $\partial z_0$  in  $z_1$ . Then it is obvious that, by making  $|z-z_0|$  sufficiently small, we can make  $|z-z_1|$  and  $|z_1-z_0|$  each smaller than any assigned positive quantity however small.

Since z and  $z_1$  are both within the circle of convergence, we can, by making  $|z-z_1|$  sufficiently small, make  $|\phi(z)-\phi(z_1)|$  less than any assigned positive quantity  $\epsilon$ , however small. But

$$| \phi(z) - \phi(z_0) | = | \phi(z) - \phi(z_1) + \phi(z_1) - \phi(z_0) |,$$

$$\Rightarrow | \phi(z) - \phi(z_1) | + | \phi(z_1) - \phi(z_0) |,$$

$$< \epsilon + | \phi(z_1) - \phi(z_0) |.$$

If, therefore, we could prove that by making  $|z_1 - z_a|$  sufficiently small we could make  $|\phi(z_1) - \phi(z_0)|$  as small as we please, it would follow that by making  $|z - z_a|$  sufficiently small we could make  $|\phi(z) - \phi(z_0)|$  as small as we please.

Let us suppose then that z and  $z_0$  have the same amplitude  $\theta$ , then we may put  $z = \rho(\cos\theta + i\sin\theta)$ ,  $z_0 = \rho_0(\cos\theta + i\sin\theta)$ , and we take  $a_n = r_n(\cos a_n + i\sin a_n)$ . Hence

$$\begin{split} a_n z^n &= r_n \left(\cos a_n + i \sin a_n\right) \rho^n \left(\cos n\theta + i \sin n\theta\right), \\ &= \left(\frac{\rho}{\rho_\theta}\right)^n r_n \rho_0^n \left\{\cos \left(n\theta + a_n\right) + i \sin \left(n\theta + a_n\right)\right\}, \\ &= z^n \left(U_n + i V_n\right), \end{split}$$

where  $x = \rho/\rho_0$ , and becomes unity when  $z = z_0$ ; and  $U_n$  and  $V_n$  are real and do not alter when z is varied along the radius of the circle of convergence.

It is now obvious that all that is required is to prove that if the series of real terms  $\Sigma x^n U_n$  remains convergent when x=1, then  $L = \sum_{x=1}^n Z_n = \sum_{n=1}^\infty U_n$ , or, what is practically the same thing, to prove that, if  $\Sigma U_n$  be a convergent series, then

$$\begin{split} & L \sum_{x_1=-0}^{\infty} \frac{1}{3} (1-x^n) \ U_n = 0. \\ \text{Let} & S_n = (1-x) \ U_1 + (1-x^n) \ U_2 + \ldots + (1-x^n) \ U_n, \\ & = (1-x^n) \ U_n + (1-x^{n-1}) \ U_{n-1} + \ldots + (1-x) \ U_1. \end{split}$$

Since  $x \triangleright 1$ ,  $1-x^a$ ,  $1-x^{a-1}$ , ..., 1-x satisfy the conditions imposed on  $a_{i_1}, a_{i_2}, \ldots, a_{n}$  in Abel's Inequality (§ 9). Also, since  $\Sigma U_n$  is convergent,  $U_n, U_{n-1}, \ldots, U_i$  satisfy the conditions imposed on  $u_i, u_{i_2}, \ldots, u_{n}$ . Hence, A and B being two finite quantities, we have

$$(1-x^n) A > S_n > (1-x^n) B$$
.

This inequality will hold however large we may choose n; and we may cause x to approach the value 1 according to any law we please. Let us put  $x = 1 - 1/n^2$ . Then we have, for all values of n, however great.

Therefore, since A and B are finite, L  $S_n = 0$ ; that is,

$$L \stackrel{\pi}{\leq} (1-x^n) U_n = 0,$$

It will be observed that, in the above proof, each term of  $\Sigma r^* U_n$  is coordinated with the term of the same order in  $\Sigma U_n$ . Hence the order of the terms in  $\Sigma U_n$  must not be deranged, if it converges conditionally,

It follows from the above, by considering paths of variation within the circle of convergence and along its circumference, that, if a power series converge at all points of the circumference of its circle of convergence, then as regards continuity of the sum the circle of convergence may be regarded as a closed region. This does not exclude the possibility of points of infinitely slow convergence on the circumference of the circle of convergence, because such points are not necessarily points of discontinuity.

On the other hand, if at any point P on the circumference of the circle of convergence the series either ceases to converge or is discontinuous, then the series cannot at such points be continuous for paths of variation which come from within. If however the series converge on both sides of P at points on the circumference infinitely near to  $P_s$ , it must converge to the same values.

It would thus appear to be impossible that a power series

should converge infinitely near any point P of the circumference of its circle of convergence to one finite value and to a different finite value at P itself. It follows that, if a power series is convergent, generally speaking, along the circumference of its circle of convergence, it cannot become discontinuous at any point on the circumference unless it cease to converge at that point.

By considering the series  $\Sigma u_n z^n$ ,  $\Sigma v_n z^n$ , and the series

 $\sum (y_{-}v_{1} + y_{-}, v_{2} + \dots + y_{n}v_{n}) x^{-11}$ 

which is their product when both of them are absolutely convergent, and applying the second of the two theorems in the present paragraph, we easily arrive at the following result, also due to Abel

Cor. If each of the series  $\Sigma u_n$  and  $\Sigma v_n$  converge, san to limits u and v respectively, then, if the  $s_i$  ries  $\geq (u_n r_1 + u_{n-1} v_2 + \ldots + u_n r_n)$ be convergent, it will converge to uv; and this will hold even if all the three series be only semi-conversant.

Example 1. The series  $1+z+\ldots+z^n+\ldots$  has for circle of convergence the circle of radius unity. Within this circle the series converges to 1/(1-z). On the circumference the series becomes  $\Sigma(\cos n\theta + i \sin n\theta)$ . which oscillates for all values of  $\theta$ , except  $\theta = 0$  for which it diverges. At peints within and infinitely near to the circle of convergence the series converges to  $\frac{1}{2} + i \cot \frac{\theta}{2}$ .

Example 2. The radius of convergence for  $z/1 + ... + z^n/n + ...$  is unity. Within the unit circle, as we shall prove later on, the series converges to - Log (1 - z). On the circumference of the unit circle the series reduces to  $\Sigma(\cos n\theta + i \sin n\theta)/n$ . This series (see § 9, III.) is convergent when  $\theta \neq 0$ ; but only semi-convergent, since  $\Sigma 1/n$  is divergent. When  $\theta = 0$ . the series diverges. The sum is therefore continuous everywhere at and on the circle of convergence, except when  $\theta=0$ . At points within the circle infinitely near to z=1 the series converges to a definite limit, which is very great; but at z=1 the series diverges to  $+\infty$ .

Example 3. \(\Sigma z^n/n^2\) converges absolutely at every point on the circumference of its circle of convergence (R=1); and consequently represents a function of z which is continuous everywhere within that circle and upon its circumference.

Example 4. Enzn is divergent at every point on its circle of convergence (R=1); and its sum is a continuous function at all points within its circle of convergence, but not at points upon the circumference.

Example 5. Pringsheim\* has established the existence of a large class of series which are semi-convergent at every point on the circumference of their circle of convergence; a particular case is the series  $\frac{\Sigma}{2}(-1)^{k_1}z^{k_1}n \log n$ , where  $\lambda_n = 1$  when  $2^{m_1}p_1 + 2^{m_1}1$ ,  $\lambda_n = 0$  when  $2^{m_1}p_1 + 2^{m_1}1$ .

§ 21.] Principle of Indeterminate Coefficients.

If  $a_0 \pm 0$ , there is a circle of non-evanescent radius within which the convergent power series  $\Sigma a_n z^n$  cannot vanish.

Since the evanescence of the series implies  $a_{\sigma} = -a_1 z - a_2 z^2 - \dots$ , it will be sufficient to show that there exists a finite positive quantity  $\lambda$  such that, if  $\rho = |z| < \lambda$ , then

$$|-a_1z-a_2z^2-\ldots|<|a_0|$$

Now, since the series  $\Sigma a_n z^n$  is absolutely convergent at any point  $z_0$  within its circle of convergence, there exists a finite positive quantity g such that for all values of  $u_1 | a_n z_0^n | = a_n p_0^n < g$ . Hence  $|a_n| < g/p_0^n$ .

Now

$$|-a_1z - a_2z^2 - ...| \ge |a_1z| + |a_2z^2| + ...$$
  
 $\ge |a_1|\rho + |a_2|\rho^2 + ...$   
 $\le g\{(\rho/\rho_0) + (\rho/\rho_0)^2 + ...\}$   
 $\le \rho\rho((\rho_0 - \rho).$ 

Hence, if we choose  $\lambda$  so that  $g\lambda/(\rho_0 - \lambda) = |a_0|$ , that is  $\lambda = |a_0| \rho_0/(g + |a_0|)$ , we shall have

$$|-a_1z-a_2z^2-...|<|a_0|$$

for all values of z within the circle  $\lambda$ .

Cor. 1. If  $a_m + 0$ , there is a circle of non-evanescent radius within which the convergent power series  $a_m z^m + a_{m+1} z^{m+1} + \dots$  vanishes only when z = 0.

For 
$$a_m z^m + a_{m+1} z^{m+1} + \dots$$
  
=  $z^m (a_m + a_{m+1} z + \dots).$ 

Now, since  $a_m + 0$ , by the theorem just proved there is a circle of non-evanescent radius within which  $a_n + a_{m+1}z + ...$  cannot vanish: and  $z^m$  cannot vanish unless z = 0.

Math. Ann., Bd. xxv. (1885).

Cor. 2. If  $a_0 + a_1z + a_2z^2 + \dots$  vanish at least once at some point distinct from z = 0 within every circle, however small, then must  $a_0 = 0$ ,  $a_1 = 0$ ,  $a_2 = 0$ , . . , that is, the series vanishes identically.

Cor. 3. If for one value of z at least, differing from 0, the series  $\Sigma a_n z^n$  and  $\Sigma b_n z^n$  converge to the same sum within every circle, however small, then must  $a_0 = b_0$ ,  $a_1 - b_1$ , . . ., that is, the series must be identical.

### INFINITE PRODUCTS.

§ 22.] The product of an infinite number of factors formed in given order according to a definite law is called an Infinite Product. Since, as we shall presently see, it is only when the factors ultimately become unity that the most important case arises, we shall write the nth factor in the form  $1 + u_n$ .

By the value of the infinite product is meant the limit of

$$(1 + u_1)(1 + u_2) . . . (1 + u_n),$$

(which may be denoted by  $\Pi (1 + u_n)$ , or simply by  $P_n$ ), when n is increased without limit.

It is obvious that if  $Lu_n$  were numerically greater than unity, then  $LP_n$  would be either zero or infinite. As neither of these cases is of any importance, we shall, in what follows, suppose  $|u_n|$  to be always less than unity. Any finite number of factors at the commencement of the product for which this is not true, may be left out of account in discussing the convergency. We also suppose any factor that becomes zero to be set aside; the question as to convergency then relates merely to the product of all the remaining factors.

Four essentially distinct cases arise-

1st.  $LP_n$  may be 0.

2nd.  $L\ddot{P}_n$  may be a finite definite quantity, which we may denote by  $\Pi\left(1+u_n\right)$ , or simply by P.

3rd.  $LP_n$  may be infinite.

4th.  $LP_n$  may have no definite value; but assume one or other of a series of values according to the integral character of n.

In cases 4 and 2 the infinite product might be said to be convergent, it is, however, usual to confine the term convergent to the 2nd case, and to this convenient usage we shall adhere; in case 3 divergent; in case 4 oxillatory.

 $\S$  23.] If, instead of considering  $P_n$ , we consider its logarithm, we reduce the whole theory of infinite products (so far as real positive factors are concerned) to the theory of infinite series; for we have

$$\log P_n = \log (1 + u_1) + \log (1 + u_2) + \dots + \log (1 + u_n)$$
  
=  $\sum_{n=1}^{\infty} \log (1 + u_n)$ ;

and we see at once that

1st. If  $\Sigma \log (1 + u_n)$  is divergent, and  $L^{\frac{n}{2}} \log (1 + u_n) = -\infty$ , then  $\mathbb{H}(1 + u_n) = 0$ ; and conversely.

2nd. If  $\Sigma \log (1 + u_n)$  be convergent, then If  $(1 + u_n)$  converges to a limit which is finite both ways: and conversely.

3rd. If  $\Sigma \log (1+u_n)$  is divergent, and  $L\Sigma \log (1+u_n) = +\infty$ , then  $\Pi(1+u_n)$  is divergent; and conversely.

4th. If  $\Sigma \log (1+u_n)$  oscillates, then II  $(1+u_n)$  oscillates; and conversely.

§ 24.] If we confine ourselves to the case where  $u_n$  has ultimately always the same sign, it is easy to deduce a simple criterion for the convergency of  $\Pi (1 + u_n)$ .

If  $Lu_n < 0$ , then  $\Sigma \log (1 + u_n) = -\infty$ , and  $\Pi (1 + u_n) = 0$ .

If  $Lu_n>0$ ,  $\Sigma \log (1+u_n)=+\infty$ , and  $\Pi (1+u_n)$  is divergent, R is therefore a necessary condition for the convergency of  $\Pi (1+u_n)$  that  $Lu_n=0$ .

Since  $Lu_n = 0$ ,  $L(1 + u_n)^{1/n} = c$ ; hence  $L \log (1 + u_n)/u_n = 1$ . It therefore follows from  $\beta$  a that  $\Sigma \log (1 + u_n)$  is convergent or divergent according as  $\Sigma u_n$  is convergent or divergent. Moreover, if  $u_n$  be ultimately negative, the last and infinite parts of  $\Sigma u_n$  and  $\Sigma \log (1 + u_n)$  will be negative; and if  $u_n$  be ultimately

<sup>•</sup> The logarithm of a complex number has not yet been defined, much by discussed. Given, however, the theory of the locarithm of a complex variable there is nothing illogical in making it the basis of the theory of intuite products, as the former does not presuppose the latter.

positive, the last and infinite parts of  $\Sigma u_n$  and  $\Sigma \log (1 + u_n)$  will be positive. Hence the following conclusions—

If the terms of  $\Sigma u_n$  become ultimately infinitely small, and have ultimately the same sign, then

1st.  $\Pi(1+u_n)$  is convergent, if  $\Sigma u_n$  be convergent; and conversely.

2nd. If  $(1 + u_n) = 0$ , if  $\sum u_n$  diverge to  $-\infty$ ; and conversely.

3rd.  $\Pi\left(1+u_n\right)$  diverges to  $+\infty$ , if  $\Sigma u_n$  diverge to  $+\infty$ ; and conversely.

Since in the case contemplated, where  $u_a$  is ultimately of invariable sign, the convergency of  $\Pi\left(1+u_a\right)$  does not depend on any arrangement of signs but merely on the ultimate magnitude of the factors, the infinite product, if convergent, is said to be absolutely concerpent. It is obvious that any infinite product in which the sign of  $u_a$  is not ultimately invariable, but which in which the signs of  $u_a$  are made all alike, will be, a fortiori, convergent in its original form, and is therefore said to be absolutely convergent; and we have in general, for infinite products of real factors, the theorem that  $\Pi\left(1+u_a\right)$  is absolutely convergent when  $\Sigma u_a$  is absolutely convergent; and conversely.

Cor. If either of the two infinite products  $\Pi(1 + u_n)$ ,  $\Pi(1 - u_n)$  be absolutely convergent, the other is absolutely convergent.

For, if  $\Sigma u_n$  is absolutely convergent, so is  $\Sigma (-u_n)$ ; and conversely.

Example 1.  $(1+1/1^2)(1+1/2^2)\dots(1+1/n^2)\dots$  is absolutely convergent since  $\Sigma 1/n^2$  is absolutely convergent.

Example 2.  $(1-1/2)\,(1-1/3)$  , . . . (1-1/n) , . . has zero for its value since  $\Sigma\,(-1/n)$  diverges to  $-\infty$  .

Example 3.  $(1+1/\surd^2)\,(1+1/\surd^3)$  . . .  $(1+1/\surd^n)$  . . . diverges to  $+\infty$  since  $\Sigma\,(1/\surd^n)$  diverges to  $+\infty$  .

Example 4.  $(1+1)_n^{(1)}(1-1)_n^{(2)}(1+1)_n^{(2)}(1-1)_n^{(4)})$ . . Since the  $a_n$  is not ultimately invariable, and since the series  $\Sigma(-1)^{n+1}_n^{(1)}$  is not also intely convergent, the rules of the pre-set paragraph do not apply. We must therefore examine the series  $\Sigma \log (1+(-1)^{n+1}_n/n)$ . The terms of this series become ultimately infinitely small; therefore we may (see § 12) associate every odd term with the following even term. We thus replace the series by the equivalent series

$$\Sigma\log\big\{1+1/\sqrt{(2n-1)}-1/\sqrt{(2n)}-1/\sqrt{(4n^2-2n)}\big\}.$$

It is easy to show that

 $1/\sqrt{(2n-1)} = 1/\sqrt{(2n)} - 1/\sqrt{(4n^2-2n)} < 0$ 

for all values of n>1. Hence the terms of the series in question altimately been negative. Moreover,  $1/\sqrt{(2n-1)-1/\sqrt{(2n-1)}-1/\sqrt{(4n^2-2n)}}$  is ultimately comparable with -1/2n. Hence  $2\log(1+(-1)^{n-1}/\sqrt{n})$  diverges to  $-\infty$ . The value of  $(1+1/\sqrt{1})(1-1/\sqrt{2})(1+1/\sqrt{3})(1-1/\sqrt{4})$ ... is therefore 0. This is an example of a semi-convergent product

Example 5.  $e^{1+1}e^{-1-\frac{1}{2}e^{1+\frac{1}{4}}e^{-1-\frac{1}{4}}}$ . The series  $\Sigma \log (1+u_n)$  in this case becomes  $(1+1)-(1+\lambda)+(1+\lambda)-(1+1)+\dots$ .

which oscillates. The infinite product therefore oscillates also.

Example 6. If  $(1-x^{n-1}/n)$  is absolutely convergent if x < 1, and has 0 for its value when x = 1.

§ 25.] We have deduced the theory of the convergence of infinite products of real factors from the theory of infinite series by means of logarithms; and this is probably the best course for the learner to follow, because the points in the new theory are suggested by the points in the old. All that is necessary is to be on the outbook for discrepancies that arise here and there, mainly owing to the imperfectness of the analogy between the properties of 0 (that is, +a-a) and 1 (that is, +a+a).

It is quite easy, however, by means of a few simple inequality theorems, to deduce all the above results directly from the definition of the value of  $\Pi$  (1 +  $u_n$ ).

If  $P_n$  have the meaning of § 22, then we see, by exactly the same reasoning as we used in dealing with infinite series, that the necessary and sufficient conditions for the convergency of  $\Pi(1+u_n)$  are that  $P_n$  be not infinite for any value of n, however large, and that  $L_n(P_{n+m}-P_n)=0$ ; and that the latter condition includes the former.

If we exclude the exceptional case where  $L = P_n = 0$ , then, since  $P_n$  is always finite, the condition  $L = (P_{n+n} - P_n) = 0$  is equivalent to  $L = (P_{n+n}/P_n - 1) = 0$ , that is,  $L = (P_{n+n}/P_n - 1) = 0$ .

See Weierstrass, Abhandlungen aus d. Functioneulehre, p. 203; or Crelle's Jour., Bd. 51.

If, therefore, we denote  $(1 + u_{n+1}) (1 + u_{n+2}) \dots (1 + u_{n+m})$  by  ${}_{m}Q_{n}$ , we may state the criterion in the following form, where  $u_{n}$  may be complex:—

The necessary and sufficient condition that  $\Pi\left(1+n_{n}\right)$  converges to a finite limit differing from zero is that  $\underset{n=n}{L}_{\lfloor m \choose n}-1 \rfloor=0$ , for all values of m.

For, since  $\sum_{n=\mu} |_{mQ_n} - 1| = 0$ , given any quantity  $\epsilon$  however small, we can determine a finite integer  $\nu$  such that, if  $u \lessdot \nu |_{mQ_n} - 1| < \epsilon$ . Therefore, since  $_{mQ_n} - P_{n+m}/P_n$ , we have in particular

$$1 - \epsilon < P_{w+m}/P_w < 1 + \epsilon$$

Since  $\nu$  is finite,  $P_{\nu}$  is finite both ways by hypothesis. Therefore  $(1 - \epsilon) P_{\nu} < P_{\nu + \nu} < (1 + \epsilon) P_{\nu}$ .

Since m may be as large as we please, the last inequality shows that  $P_n$  is finite for all values of n however large.

Again, since  $P_n$  is not infinite, however large n, the condition  $L \mid_n Q_n - 1 \mid = 0$ , which is equivalent to  $L \mid_n q_n - 1 \mid_n q_n - 1$  leads to  $L \mid_n q_n - 1 \mid_n q_n - 1$ . The possibility of oscillation is thus excluded. The sufficiency of the criterion is therefore established. Its necessity is obvious.

We shall not stop to re-prove the results of § 24 by direct deduction from this criterion, but proceed at once to complete the theory by deducing conditions for the absolute convergence of an infinite product of complex factors.

§ 26.] If  $(1 + u_n)$  is convergent if If  $(1 + u_n)$  is convergent.

Let  $\rho_n = \lfloor u_n \rfloor$ , so that  $\rho_n$  is positive for all values of n, then since  $\Pi(1 + \rho_n)$  is convergent,

$$L\{(1 + \rho_{n+1})(1 + \rho_{n+2})...(1 + \rho_{n+m}) - 1\} = 0$$
 (1).

Now  ${}_{\mathfrak{m}}Q_{n}-1=(1+u_{n+1})(1+u_{n+1})\dots(1+u_{n+m})-1,$  $=\Sigma u_{n+1}+\Sigma u_{n+1}u_{n+2}+\dots+u_{n+1}u_{n+2}\dots u_{n+m}.$ 

Hence, by chap. XII., §§ 9, 11, we have

 $0 \geqslant |_{m}Q_{n} - 1| \geqslant \sum \rho_{n+1} + \sum \rho_{n+1}\rho_{n+2} + \dots + \rho_{n+1}\rho_{n+2} \dots + \rho_{n+m},$   $\geqslant (1 + \rho_{n+1}) (1 + \rho_{n+2}) \dots (1 + \rho_{n+m}) - 1.$ 

c. n, 1t

Hence, by (1),  $L|_{m}Q_{n}-1|=0$ .

Remark.—The converse of this theorem is not true; as may be seen at once by considering the product  $(1+1)(1-\frac{1}{2})(1+\frac{1}{2})$   $(1-\frac{1}{4})$ , ..., which converges to the finite limit 1; although  $(1+1)(1+\frac{1}{2})(1+\frac{1}{2})(1+\frac{1}{2})$ , ... is not convergent.

When  $\Pi\left(1+u_n\right)$  is such that  $\Pi\left(1+|u_n|\right)$  is convergent,  $\Pi\left(1+u_n\right)$  is said to be absolutely convergent. If  $\Pi\left(1+u_n\right)$  be convergent, let  $\Pi\left(1+|u_n|\right)$  non-convergent,  $\Pi\left(1+u_n\right)$  is said to be smi-convergent. The present use of these terms includes as a particular case the use formerly made in § 24.

§ 27.] If  $\Sigma_{\parallel}u_n \parallel be \ convergent, \ then \ \Pi(1+u_n)$  is absolutely

convergent; and conversely.

For, if  $\Sigma[|u_n|]$  be convergent, it is absolutely convergent, seeing that  $|u_n|$  is by its nature positive. Hence, by § 24, If  $(1+|u_n|)$  is convergent. Therefore, by § 26, If  $(1+u_n)$  is absolutely convergent.

Again, if  $\Pi(1+u_n)$  be absolutely convergent,  $\Pi(1+|u_n|)$  is convergent; that is, since  $|u_n|$  is positive,  $\Pi(1+|u_n|)$  is absolutely convergent. Therefore, by § 21,  $\Sigma |u_n|$  is absolutely convergent.

Cor. If  $\Sigma u_n$  be absolutely convergent,  $\Pi\left(1+u_nx\right)$  is absolutely convergent, where x is either independent of n or is such a function of n that  $L\left[x\right] \neq \infty$  when  $n=\infty$ .

Example 1. If  $(1-x^n/n)$  is absolutely convergent for all complex values such that  $|x| \in \mathbb{N}$ , but is not absolutely convergent when |x| = 1.

Example 2. If  $(1-x/n^2)$ , where x is independent of n, is absolutely convergent.

§ 28.] After what has been done for infinite series, it is not necessary to discuss in full detail the application of the laws of algebra to infinite products. We have the following results—

1. The law of association may be safely applied to the factors

of H (1 + un) provided Lun = 0; but not otherwise.

11. The accessary and sufficient condition that  $\mathbf{u}(1 + \mathbf{u}_n)$  shall converge to the same limit (finite both ways), whatever the order of its factors, is that the series  $\sum \mathbf{u}_n$  be absolutely convergent.

When  $u_n$  is real, this result follows at once by considering the series  $\Sigma \log (1 + u_n)$ ; and the same method of proof applies when  $u_n$  is complex, the theory of the logarithm of a complex variable being presumosed \*.

An infinite product which converges to the same limit whatever the order of its factors is said to be unconditionally convergent. The theorem just stated shows that unconditional convergence and absolute convergence may be taken as equivalent terms. A conditionally convergent product has a property analogous to that of a conditionally convergent series; viz. that by properly arranging the order of its terms it may be made to converge to any value we please, or to diverge.

III. If both  $\Pi$   $(1 + u_n)$  and  $\Pi$   $(1 + v_n)$  be absolutely convergent, then  $\Pi$   $\{(1 + u_n) (1 + v_n)\}$  is absolutely convergent, and has for its limit  $\{\Pi(1 + u_n)\} \times \{\Pi(1 + v_n)\}$ ; also  $\Pi$   $\{(1 + u_n), (1 + v_n)\}$  is absolately convergent, and has for its limit  $\{\Pi(1 + u_n)\}/\{\Pi(1 + v_n)\}$ ; varied none of the inclors of  $\Pi(1 + v_n)$  emish.

If  $Q_n$  denote  $(1 + u_{n+1})(1 + u_{n+2}) \dots$ , the total residue of the infinite product  $\Pi(1 + u_n)$  after n factors, then, if the product converges to a finite limit which is not zero, given any positive quantity  $\epsilon$ , however small, we can always assign an integer r such that  $\|Q_n - 1\| < \epsilon$ , when n < r.

If  $u_n$  be a function of any variable z, then, when  $\epsilon$  is given,  $\nu$  will in general depend on z.

If, however, for all values of z within a given region R in Argand's diagram an integer v independent of z can be assigned such that

$$|Q_n-1|<\epsilon$$
,

when  $n \not = v$ , then the infinite product is said to be uniformly convergent within R.

IV. If f(n, z) be a finite single valued function of the integral variable n and of z, continuous as requests z within a region R, and if  $\Pi\{1+f(n,z)\}$  converges uniformly for all values of z within R to a finite limit  $\phi(z)$ , then  $\phi(z)$  is a continuous function of z within R.

Let z and z' be any two points within R, then, since

<sup>\*</sup> See Harkness and Morley, Treatise on the Theory of Functions (1869), § 79; or Stolz, Allgemeine Arithmetik, Thl. II. (1886), p. 238.

 $\phi(z)$  and  $\phi(z')$  are each finite both ways, it is sufficient to prove that  $L \mid \phi(z')/\phi(z) \mid = 1$ .

Let

$$\phi(z) \cdot P_{z}O_{z} \cdot \phi(z') = P' \cdot O'.$$

where  $P_n$ ,  $Q_n$ , &c. have the usual meanings.

Since the product is uniformly convergent, it is possible to determine a finite integer  $\nu$  (independent of z or z) such that, when  $\mu \preceq \nu$ , we have

$$|Q_n-1|<\epsilon$$
, and  $|Q_n-1|<\epsilon$ 

where  $\epsilon$  is any assigned positive quantity however small. Hence, in particular, we must have

$$|Q_n| = 1 + \theta \epsilon, \quad |Q'_n| = 1 + \chi \epsilon;$$

where  $\theta$  and  $\chi$  are real quantities each lying between -1 and +1.

Now 
$$\left|\frac{\phi(z')}{\overline{\phi}(z)}\right| = \frac{P_{\nu}}{P_{\nu}} \left|\frac{Q_{\nu}}{Q_{\nu}}\right|.$$

Also, since  $L_{r}/P_{r}/P_{r} = 1$ ,  $\nu$  being a finite integer, and, z being at our disposal, we can without disturbing  $\nu$  choose |z-z'| so small that  $|P_{r}/P_{r}| = 1 + \psi \epsilon$ , where  $-1 < \psi < +1$ .

Hence

$$\begin{aligned} \left| \left| \frac{\phi\left( \frac{\epsilon}{\epsilon} \right)}{\phi\left( \frac{\epsilon}{\epsilon} \right)} \right| - 1 \right| & \cdot \left| \frac{1 + \psi \, \epsilon \left( 1 + \chi \, \epsilon \right)}{1 + \theta \, \epsilon} - 1 \right|, \\ & = \left| \frac{(\psi + \chi - \theta) \, \epsilon + \psi \chi \, \epsilon^2}{1 + \theta \, \epsilon} \right|, \\ & < \epsilon \frac{3 + \epsilon}{1 - \epsilon}. \end{aligned}$$

Since  $\epsilon(3+\epsilon)/(1-\epsilon)$  can, by sufficiently diminishing  $\epsilon$ , be made as small as we please, it follows that  $L_{-1}(\phi(z))/\phi(z) = 1$ .

Cor. 1. If  $\mu_n$  and  $w_n(z)$  satisfy the conditions of the example in §16, then  $W\{1 + \mu_n w_n(z)\}$  is a continuous function of z within the region R.

For, if we use dashes to denote absolute values, we have

$$[Q_i \sim 1] \sim (1 + \mu_{n+1} w_{n+1}) (1 + \mu'_{n+2} w_{n+2})$$
... - 1.

Since  $w_n(z)$  is finite for all values of n and z, we can find a finite upper limit, q, for  $w'_{n+1}, w'_{n+2}, \ldots$  Therefore

$$|Q_n-1|<(1+g\mu'_{n+1})(1+g\mu'_{n+2})...-1.$$

Since  $\Sigma \mu'_n$  is absolutely convergent,  $\Sigma_i \mu'_n$  is absolutely convergent. Hence  $\Pi(1 + g\mu'_n)$  is absolutely convergent; and we can determine a finite integer  $\nu$  (evidently independent of z, since g and  $\mu'_n$  do not depend on z), such that, when  $n \not < \nu$ ,  $(1 + g\mu'_{n+1})(1 + g\mu'_{n+2}) \dots -1 < \epsilon$ . Hence we can determine  $\nu$ , independent of z, so that  $|Q_n - 1| \ne \epsilon$ , where  $\epsilon$  is a positive quantity as small as we please. It follows that  $\Pi\{1 + \mu_n \nu_n(z)\}$  is uniformly convergent, and therefore a continuous function of z within R.

Cor. 2. If  $\Sigma a_n z^n$  be convergent when |z| = R, then  $\Pi(1 + a_n z^n)$  converges to  $\phi(z)$ , where  $\phi(z)$  is a finite continuous function of z for all values of z such that |z| < R.

Cor. 3. If f(n, y) be finite and single-valued as regards n, and finite, single-valued, and continuous as regards y within the region T, and if  $\Sigma f(n, y) z^n$  be absolutely convergent when |z| = R, then, so long as |z| < R,  $\Pi(1 + f(n, y) z^n)$  converges to  $\psi(y)$ , where  $\psi(y)$  is a finite continuous function of y within T.

Cor. 4. If  $\Sigma a_n$  be absolutely convergent, then  $\Pi(1+a_nz)$  converges to  $\psi(z)$ , where  $\psi(z)$  is a finite and continuous function of z for all finite values of z.

We can also establish for infinite products the following theorem, which is analogous to the principle of indeterminate coefficients.

V. If, for a continuum of values of z including 0,  $\Pi(1+a_nz^n)$  and  $\Pi(1+b_nz^n)$  be both absolutely convergent, and  $\Pi(1+a_nz^n) = \Pi(1+b_nz^n)$ , then  $a_1 = b_1$ ,  $a_2 = b_2$ , . . . ,  $a_n = b_n$ , . . .

For we have

$$\Sigma \log (1 + a_n z^n) = \Sigma \log (1 + b_n z^n),$$

both the series being convergent,

Hence for any value of z, however small, we have, after dividing by z.

$$\sum a_n z^{n-1} \log (1 + a_n z^n)^{1/t_n z^n} = \sum b_n z^{n-1} \log (1 + b_n z^n)^{1/t_n z^n}$$

Since  $\underset{z=0}{L} \log \left(1 + a_n z^n\right)^{\lfloor a_n z^n \rfloor} = 1$ , we have, for very small values of z

$$a_1A_1 + a_2A_2z + a_3A_3z^2 + \dots = b_1B_1 + b_2B_2z + b_3B_3z^2 + \dots$$
 (1), where  $A_1, A_2, \dots, B_3$ ,  $B_2$  differ very little from unity, and all have unity for their limit when  $z = 0$ .

Hence, since  $\sum a_n z^{n-1}$  and  $\sum b_n z^{n-1}$  are, by virtue of our hypotheses, absolutely convergent, we have

$$L_{r=0} (a_2 A_1 z + a_3 A_3 z^2 + \dots) = 0$$

$$L_{z=0}^{L} (b_{z}B_{z}z + b_{3}B_{z}z^{2} + \dots) = 0,$$

Hence, if in (1) we put z = 0, we must have

$$a_1 L A_1 = b_1 L B_1$$

But  $LA_1 \circ LB_1 = 1$ ; therefore  $a_1 \circ b_1$ . Removing now the common factor  $1 + a_1 z$  from both products, and proceeding as before, we can show that  $a_2 \circ b_2$ ; and so on.

§ 29.] The following theorem gives an extension of the Factorisation Theorem of chap, v., § 15, to Infinite Products.

If  $\psi(z) = \Pi\left(1 + a_n z\right)$  be convergent for all values of z, in the sense that  $L_{\geq n}(l_n - 1) = 0$ , when n = x, no matter what value m may have, then  $\psi(z)$  will vanish if z have one of the values  $-1/a_1, -1/a_2, \ldots, -1/a_r, \ldots$  and, if  $\psi(z) = 0$ , then z must have one of the values  $-1/a_1, -1/a_2, \ldots, -1/a_r, \ldots$ 

In the first place, we remark that, by our conditions, the vanishing of  $L_aQ_a$  when n-z is procluded. The exceptional case, mentioned in § 23, where  $\Sigma \log (1+a_nz)$  diverges to  $-\infty$ , and  $\Pi(1+a_nz)$  converges to 0 for all values of z, is thus excluded.

Now, whatever n may be, we have

$$\psi(z) = P_n Q_n$$
 (1).

Suppose that we cause z to approach the value  $-1/a_r$ . We can always in the equation (1) take n greater than r; so that  $1+a_rz$  will occur among the factors of the integral function  $P_{\bf a}$ . Hence, when  $z=-1/a_r$ , we have  $P_{\bf a}=0$ , and therefore, since  $Q_{\bf a}+\infty$ ,  $\psi$  ( $-1/a_r$ ) 0.

Again, suppose that  $\psi(z) = 0$ . Then, by (1),  $P_nQ_n = 0$ . But, since n may be as large as we please, and  $LQ_n = 1$  when  $n = \infty$ , we can take n so large that  $Q_c \neq 0$ . Hence, if only n be large enough, the integral function  $P_n$  will vanish. Hence z must have a value which will make some one of the factors of  $P_n$  vanish; that is to say, z must have some one of the values  $-1/a_1, -1/a_2, \ldots, -1/a_{r_1}, \ldots$ 

It should be noticed that nothing in the above reasoning prevents any finite number of the quantities  $a_1, a_2, \ldots, a_r, \ldots$  from being equal to one another; and the equal members of the series may, or may not, be contiguous. If there be  $\mu_n$  contiguous factors identical with  $1 + a_n z$ , the product  $\psi(z)$  will take the form  $\Pi(1 + a_n z)^m$ ; and it can always be brought into this form if it be absolutely convergent, for in that case the commutation of its factors does not affect its value.

Cor. 1. If z lie within a continuum (z) which includes all the values

$$-1/a_1$$
,  $-1/a_2$ , . . . ,  $-1/a_n$ , . . . ( $\Delta$ ),

and 
$$-1/b_1$$
,  $-1/b_2$ , ...,  $-1/b_n$ , ... (B),

if  $\Pi(1 + a_n z)^{\mu_n}$  and  $\Pi(1 + b_n z)^{\mu_n}$  be absolutely convergent for all values of z in (z), if f(z) and g(z) be definite functions of z which become neither zero nor infinite for any of the values  $(\Lambda)$  or (B), and if, for all values of z in (z),

$$f(z) \Pi (1 + a_n z)^{\mu_n} = g(z) \Pi (1 + b_n z)^{\nu_n}$$
 (1),

then must each factor in the one product occur in the other raised to the same power; and, for all the values of z in (z),

$$f(z) = g(z)$$
 (2).

For, since, by (1), each of the products must vanish for each of the values (A) or (B), it follows that each of the quantities (A) must be equal to one of the quantities (B); and vice versa. The two series (A) and (B) are therefore identical.

Since the two infinite products are absolutely convergent, we may now arrange them in such an order that  $a_1 = b_1, a_2 = b_4, \ldots$ , &c., so that we now have

$$f(z)(1 + a_1z)^{a_1}(1 + a_2z)^{a_2}... = q(z)(1 + a_1z)^{a_1}(1 + a_2z)^{a_2}...(3).$$

Suppose that  $\mu_1 \neq \nu_1$ , but that  $\mu_1$ , say, is the greater; then we have, from (3),

$$f(z) (1 + a_1 z)^{\mu_1 - \nu_1} (1 + a_2 z)^{\mu_2} \dots = g(z) (1 + a_2 z)^{\nu_2} \dots$$
 (4).

Now this is impossible, because the left-hand side tends to  $\mathbf{0}$  as limit when  $z=1/a_1$ , whereas the right-hand side does not vanish when  $z=1/a_1$ . We must therefore have  $\mu_1=\nu_1$ ; and, in like manner,  $\mu_2=\nu_2$ ; and so on.

We may therefore clear the first n factors out of each of the products in (1), and thus deduce the equation

$$f(z) Q_n = g(z) Q_n$$
 (5),

where  $Q_n$  and  $Q_n$  have the usual meaning. The equation (5) will hold, however large n may be. Hence, since  $LQ_n + LQ_n = 1$ , we must have

$$f(z) = g(z)$$
.

Cor. 2. From this it follows that a given function of z which vanishes for any of the values  $(\Delta)$  and for no others within the continuum (z), can be expressed within (z) as a convergent infinite product of the form f(z) if  $(1 + a_z z)^{-\epsilon}$  (where f(z) is finite and not zero for all finite values of z within (z)), IF AT ALL, in one way only.

If the infinite product be only semi-convergent, the above demonstration fails.

It may be remarked that it is not in general possible to cross-a function having given zero points, in the form described in the corollary. On this subject the student should consult Weierstrass, Athandlangen and der Functionehlere, p. 13 et sep.

# ESTIMATION OF THE RESIDUE OF A CONVERGING SERIES OR INFINITE PRODUCT,

§ 30.] For many theoretical, and for some practical purposes, it is often required to assign an upper limit to the residue of an infinite series. This is easily done in what are by far the two most important cases, namely:—(1) Where the ratio of convergence  $(\rho_n \sim u_{n+1}/u_n)$  ultimately becomes less than unity, and the terms are all ultimately of the same sign; (2) Where the terms ultimately continually diminish in numerical value, and alternate in sign.

Circ (1). It is essential to distinguish two varieties of series

under this head, namely:—(a) That in which  $\rho_n$  descends to its limit  $\rho$ : (b) That in which  $\rho_n$  ascends to its limit  $\rho$ .

In case (a), let u be taken so large that, on and after n,  $\rho_n$  is always numerically less than 1, and never increases in numerically value. Then

$$R_n = u_{n+1} + u_{n+2} + u_{n+3} + \dots,$$
  

$$= u_{n+1} \left\{ 1 + \frac{u_{n+2}}{u_{n+1}} + \frac{u_{n+2}}{u_{n+2}} \cdot \frac{u_{n+2}}{u_{n+1}} + \dots \right\},$$

$$= u_{n+1} \left\{ 1 + u_{n+1} + u_{n+1} + u_{n+2} + u_{n$$

Therefore, if dashes be used to denote the numerical values, or moduli, of the respective quantities, we have

$$R'_{n} \geqslant u'_{n+1} \{1 + \rho'_{n+1} + \rho'_{n+1}^{2} + \dots \},$$
  
 $\geqslant u'_{n+1}/(1 - \rho'_{n+1}),$   
 $\geqslant u'_{n+2}/(1 - u'_{n+2}/\mu_{n+1})$ 
(1)

And also, for a lower limit.

$$R'_n \ll u'_{n+1}/(1-\rho)$$
 (2).

In case (b), let n be so large that, after n,  $\rho_n$  is numerically less than 1, and never decreases in numerical value. Then

$$R_n = u_{n+1} \{1 + \rho_{n+1} + \rho_{n+2} \rho_{n+1} + \dots \},$$
  
 $R'_n \geqslant u'_{n+1} \{1 + \rho + \rho^2 + \dots \},$   
 $\geqslant u'_{n+1}/(1 - \rho)$  (3);

and we have also

$$R'_n \not\leftarrow u'_{n+1}/(1 - \rho'_{n+1}),$$
  
 $\not\leftarrow u'_{n+1}/(1 + u'_{n+2}/u'_{n+1})$  (4).

Case (2). When the terms of the series ultimately decrease and alternate in sign, the estimation of the residue is still simpler. Let n be so large that, on and after n, the terms never increase in numerical value, and always alternate in sign. Then we have

$$R'_n = u'_{n+1} - u'_{n+2} + u'_{n+3} - \dots$$
  
 $\geqslant u'_{n+1}$  (5);

$$\langle u'_{n+1} - u'_{n+2} \rangle$$
 (6).

(2).

§ 31.] Residue of an Infinite Product. Let us consider the infinite products, If  $(1+u_n)$  and If  $(1-u_n)$ , in which  $u_n$  becomes ultimately positive and best thou anity. If the series  $\Sigma u_n$  converge in such a way that the limit of the convergency-ratio  $\rho_n$  is a positive quantity  $\rho$  less than 1, then it is easy to obtain an estimate of the residue. Let  $Q_n$ ,  $Q'_n$  denote the products of all the factors after the nth in If  $(1+u_n)$  and If  $(1-u_n)$  respectively, so that  $Q_n > 1$ , and  $Q'_n < 1$ . We suppose n so great that, on and after n,  $u_n$  is positive,  $\rho_n$  less than 1, and either  $(a) \rho_n$  never increases, or cless  $(b) \rho_n$  never decreases. In case (a),  $\Sigma u_n$  falls under case (1) (a) of last paragraph; in case (b),  $\Sigma u_n$  falls under case (1) (b) of last paragraph. We shall, as usual, denote the residue of  $\Sigma u_n$  by  $R_n$ ; and we shall suppose that n is so large that  $|R_n| < 1$ .

Now (by chap. xxiv., § 7, Example 2),

 $>1-R_*$ 

$$Q_{n} = (1 + u_{n+1})(1 + u_{n+2}) \dots,$$

$$> 1 + u_{n+1} + u_{n+2} + \dots,$$

$$> 1 + R_{n} = (1).$$

$$Q'_{n} - (1 - u_{n+1})(1 - u_{n+2}) \dots,$$
(1).

Also,

$$\begin{split} &1/Q_n = \{1 - u_{n+1}/(1 + u_{n+1})\} \{1 - u_{n+2}/(1 + u_{n+2})\} \dots, \\ &> 1 - u_{n+1}/(1 + u_{n+1}) - u_{n+2}/(1 + u_{n+2}) - \dots, \\ &> 1 - u_{n+1} - u_{n+2} - \dots, \\ &> 1 - R_n. \end{split}$$

Whence

$$Q_n - 1 < R_n/(1 - R_n)$$
 (3).

In like manner,

$$\begin{split} 1/Q_n &= \{1 + u_{n+1}/(1 - u_{n+1})\} \cdot \{1 + u_{n+2}/(1 - u_{n+1})\} \cdot \dots \cdot, \\ &> 1 + u_{n+1}/(1 - u_{n+1}) + u_{n+2}/(1 - u_{n+2}) + \dots \cdot, \\ &> 1 + u_{n+1} + u_{n+2} + \dots \cdot, \\ &> 1 + R_n. \end{split}$$

Whence

$$1 - Q_n > E_n (1 + E_n)$$
 (4).

From (1), (2), (3), and (4) we have

$$R_n < Q_n - 1 < R_n/(1 - R_n)$$
 (5);

$$R_n/(1+R_n) < 1-Q'_n < R_n$$
 (6),

Since upper and lower limits for  $R_n$  can be calculated by means of the inequalities of last paragraph, (5) and (6) enable us to estimate the residues of the infinite products  $\Pi (1 + u_n)$  and  $\Pi (1 - u_n)$ .

Example. Find an upper limit to the residue of II  $(1-x^n/n)$ , x<1.

Here  $u_n=x^n|n$ ,  $p_n=x/(1+1/n)$ , p=x. The series has an ascending convergency ratio: and we have  $R_n=u_n+y(1-\rho)<x^{n+1}/(n+1)$  (1-x). Therefore,  $1-Q_n=x^{n+1}/(n+1)$  (1-x). Hence, if  $P'_n$  be the nth approximation to  $\Pi(1-x^n/n)$ ,  $P'_n$  differs from the value of the whole product by less than  $100x^{n+1}/(n+1)$  (1-x)  $q_0$  of  $P'_n$  itself.

### CONVERGENCY OF DOUBLE SERIES,

§ 32.] It will be necessary in some of the following chapters to refer to certain properties of series which have a doubly initie number of terms. We proceed therefore to give a brief sketch of the elementary properties of this class of series. The theory originated with Cauchy, and the greater part of what follows is taken with slight modifications from note VIII. of the Analyse Alufbrique, and § 8 of the Késumés Analytiques.

Let us consider the doubly infinite series of terms represented in (1). We may take as the general, or specimen term,  $u_{m,n}$ , where the first index indicates the row, and the second the column, to which the term belongs. The assemblage of such terms we may denote by  $\Sigma u_{m,n}$ ; and we shall speak of this assemblage as a Double Series\*.

A great variety of definitions might obviously be given of the sum to a finite number of terms of such a series; and, corresponding to every such definition, there would arise a definite question regarding the sum to infinity, that is, regarding the convergency of the series.

There are, however, only four ways of taking the sum of the double series which are of any importance for our purposes.

<sup>.</sup> Sometimes the term "Series of Double Entry" is used.

First Way,—We may define the finite sum to be the sum of all the mn terms within the rectangular array OKMN. This we denote by  $S_{m_1n}$ . Then we may take the limit of this by first making m and finally n infinite, or by first making n infinite and finally m infinite. If the result of both these limit operations is the same definite quantity  $S_i$  then we say that  $\Sigma u_{m_1n}$  converges to S in the first way.

	111.9	u <sub>1,2</sub>		 21,10	u <sub>1, n+1</sub>	
u <sub>2-1</sub>	u <sub>52</sub>	$u_{z_*,i}$	$u_{2,+}$	 u <sub>2, n</sub>	$u_{2,n+1}$	
$u_{3,1}$	u.,2	$u_{343}$	$u_{3i+1}$	 u <sub>3, n</sub>	$u_{3,n+1}$	
$u_{4r1}$	u4.2	и 143	u <sub>is i</sub>	 и <sub>1, п</sub>	<i>u</i> <sub>4+n+1</sub>	
:	:	:	:	:	:	
:						
u <sub>101</sub>	и.,	1/ -	<i>u</i> <sub>n,4</sub>	 11 <sub>n</sub> , n	u <sub>n•n+1</sub>	
		14.3	71.	8.9	L	
:	:	:	:	:	:	
$u_{m_*1}$	u <sub>m, 2</sub>	u <sub>m, 7</sub>	$u_{m,4}$	 $u_{m_{\pi}n}$	<i>u</i> <sub>m, n+1</sub>	
"m+1.1	$u_{m+1,2}$	u <sub>m+1,3</sub>	u <sub>m+1,4</sub>	u <sub>m+1</sub> ,	Mm+1:n+1	

It may, however, happen—1st, that both these operations lead to an infinite value; 2nd, that neither leads to a definite value; 3rd, that one leads to a definite finite value, and the

other not; 4th, that one leads to one definite finite value, and the other to another definite finite value\*. In all these cases we say that the series is non-convergent for the first way of summing.

Second Way.—Sum to n terms each of the series formed by taking the terms in the first m horizontal rows of (1); and call the sums  $T_{1_1n_1} T_{2_1n_2} \dots T_{n_{n_1}n_n}$ . Define

$$S'_{m_1,n} = T_{1,n} + T_{2,n} + \dots + T_{m_1,n}$$
 (2)

as the finite sum.

Then, supposing each of the horizontal series to converge to  $T_1,\ T_2,\ \ldots,\ T_m$  respectively, and  $\Sigma T_m$  to be a convergent series, define

$$S' = T_1 + T_2 + \dots + T_m + \dots$$
 ad  $x$  (3)

as the sum to infinity in the second way.

Third Way.—Sum to m terms each of the series in the first n columns; and let these sums be  $U_{1, m}, U_{2, m}, \ldots, U_{n, m}$ . Define

$$S''_{m,n} = U_{1,m} + U_{2,m} + \dots + U_{n,m}$$
 (4)

as the finite sum.

Then, supposing these vertical series to converge to  $U_1$ ,  $U_2$ , ...,  $U_n$  respectively, and  $\Sigma U_n$  to be a convergent series, define

$$S'' = U_1 + U_2 + \dots + U_n + \dots$$
 at  $x$  (5)

as the sum to infinity in the third way.

So long as m and n are finite, it is obvious that we have

$$S'_{m,n} = S''_{m,n} = S_{m,n}$$
;

so that, for finite summation, the second and third ways of summing are each equivalent to the first.

The case is not quite so simple when we sum to infinity. I is clear, however, that

$$S' = L_{m=n} \{ L S_{m,n} \}$$
 (6);

and 
$$S'' = \underbrace{L}_{n=x} \underbrace{L}_{m=x} S_{m,n}$$
 (7);

<sup>.</sup> Examples of some of these cases are given in § 35 below.

so that S' and S'' will be equal to each other and to S when the two ways of taking the limit of  $S_{m,n}$  both lead to the same definite finite result\*.

Fourth Way.—Sum the terms which lie in the successive diagonal lines of the array, namely, AA', BB', CC', . . . , KK'; and let these sums be  $D_1, D_2, \ldots, D_{n+1}$  respectively; that is,  $D_2 = u_{1,1}, D_2 = u_{1,2} + u_{n,1}, \ldots, D_{n+1} - u_{1,n} + u_{2,n-1} + \ldots + u_{n,1}$ . Define

Define  $S' = D_2 + D_4 + \dots + D_n \tag{8}$ 

as the finite sum; and, supposing  $\Sigma D_n$  to be convergent, define

$$S''' = D_2 + D_3 + \dots + D_n + \dots \text{ ad } \infty$$
 (9)

as the sum to infinity in the fourth way.

The finite sum according to this last definition includes all the terms in the triangle OKN'; it can therefore never (except for m=n=1) coincide with the finite sum according to the former definitions. Whether the sum to infinity (S''') according to the fourth definition will coincide with S, S', or S'', depends on the nature of the series. It may, in fact, happen that the limits S, S', S'' exist and are all equal, and that the limit S''' is infinite f.

§ 33.] Double series in which the terms are all ultimately of the same sign. By far the most important kind of double series is that in which, for all values of m and n greater than certain fixed limits,  $u_{m,n}$  has always the same sign, say always the positive sign. Since, by adding or subtracting a finite quantity to the sum (however defined), we can always make any finite number of terms have the same sign as the ultimate terms of the series, we may, so far as questions regarding convergency are concerned, suppose all the terms of  $\Sigma u_{m,n}$  to have the same (say positive) sign from the beginning. Suppose now (1) to represent the array of terms under this last supposition; and let us farther suppose that  $\Sigma u_{m,n}$  is convergent in the first way.

Then, since  $L(S_{m+p,n+q} - S_{m,n})$  N-N=0, when m=x, n=x whatever p and q may be, it follows that the sum of all

<sup>.</sup> For an illustration of the case when this is not so, see below, § 35,

<sup>†</sup> See below, § 35.

the terms in the gnomon between NMK and two parallels to NM and MK below and to the right of these lines respectively, must become as small as we please when we remove NM sufficiently far down and MK sufficiently far to the right.

From this it follows, a fortieri, seeing that all the terms of the array are positive, that, if only m and n be sufficiently great, the sum of any group of terms taken in any way from the residual terms lying outside OKMN will be as small as we please.

Hence, in particular,

1st. The total or partial residue of each of the horizontal series vanishes when  $n=\infty$ .

2nd. The same is true for each of the vertical series.

3rd. The same is true for the series  $\Sigma D_n$ .

The last inference holds, since  $S'''_n$  obviously lies between  $S_{a,n-a}$  and  $S_{a-1,n-1}$ .

Hence

Theorem I. If all the terms of  $\sum u_{m,n}$  be positive, and if the series be convergent in the first sense, then each of the horizontal series, each of the vertical series, and the diagonal series will be convergent, and the double series will be convergent in the remaining three ways, always to the same limit.

If we commutate the terms of a double series so that the term  $u_{m,n}$  becomes the term  $u_{m,n}$ , where m = f(m,n), n' = g(m,n), f(m,n) and g(m,n) being functions of m and n, each of which has a distinct value for every distinct pair of values of m and n (say non-repeating functions), and each of which is finite for all finite values of m and n (Restriction  $\Lambda^*$ ), then we shall obviously leave the convergency of the series unaffected. Hence

Cor. 1. If  $\Sigma u_{m,n}$  be a series of positive terms convergent in the first way, then any commutation of its terms (under Restriction A) will leace its convergency unaffected; that is to say, it will converge in all the four ways to the same limit S as before.

No such restriction is usually mentioned by writers on this subject; but some such restriction is obviously implied when it is said that the terms of an absolutely convergent series are commutative; otherwise the characteristic property of a convergent series, namely, that it has a vanishing residue, would not be conserved.

Cor. 2. If the terms (all positive) of a convergent single series  $\Sigma u_n$  be arranged into a double series  $\Sigma u_{n,m',n'}$  where m' and n' are functions of n subject to Restriction A, then  $\Sigma u_{m,m'}$  will converge in all four ways to the same limit as  $\Sigma u_n$ .

It should be noticed that this last corollary gives a further extension of the laws of commutation and association to a series of positive terms; and therefore, as we shall see presently, to any absolutely convergent series.

Let us next assume that the series  $\Sigma n_{m,n}$  is convergent in the second way. Then, since  $\Sigma T_m$  is convergent, we can, by sufficiently increasing  $m_i$  make the residue of this series, that is, the sum of as many as we choose of the terms below the infinite horizontal line  $NM_i$  less than  $\frac{1}{2}\epsilon_i$  where  $\epsilon$  is as small as we phease. Also, since each of the horizontal series is, by our hypothesis, convergent, we can, by sufficiently increasing  $n_i$  make the residue of each of them, less than  $\epsilon 2m_i$  and therefore the sum of their residues, that is, as many as we phease of the terms above  $NM_i$  produced and right of  $MK_i$  less than  $\frac{1}{4}\epsilon_i$ . Hence, by sufficiently increasing both m and  $n_i$  we can make the sum of the terms outside  $OKM_iN_i$  less than  $\epsilon_i$  that is, as small as we phease. From this it follows that  $\Sigma n_{n,n}$  is convergent in the first way, and, therefore, by Theorem 1, in all the four ways.

In exactly the same way, we can show that, if  $\Sigma u_{m,n}$  is convergent in the third way, it is convergent in all four ways.

Finally, let us assume that  $\Sigma u_{|||,||}$  is convergent in the fourth way. It follows that the residue of the diagonal series  $\Sigma D_p$  can, by making p large enough, be made as small as we please. Now, if only m and n be each large enough, the residue of  $S_{m,n}$  that is, the sum of as many as we please of the terms outside OKMN, will contain only terms outside OKMN, all of which are terms in the residue of  $S_{|||}$ . Hence, since all the terms in the array (1) are positive, we can make the sum of as many as we please of the terms outside OKMN as small as we please, by

sufficiently increasing both m and n. Therefore  $\Sigma n_{m,n}$  is convergent in the first way, and consequently in all four ways.

Combining these results with Theorem I., we now arrive at the following:-

Theorem II. If a double series of positive terms converge in any one of the four ways to the limit 8, it also converges in all the other three ways to the same limit 8; and the subsidiary single series, horizontal, vertical, and diagonal, are all convergent.

Cor. Any single series  $\Sigma u_n$  consisting of terms solveted from  $\Sigma u_{m,n}$  (under Restriction A) will be a convergent series, if  $\Sigma u_{m,n}$  be convergent.

Restriction A will here take the form that n' must be a function of m and n whose values do not repeat, and which is finite for finite values of m and n.

Example. The double series  $\sum x^m y^n$  is convergent for all values of x and y, such that 0 < x < +1, 0 < y < +1.

For the (m+1)th horizontal series is  $x^m \Sigma y^n$ , which converges to  $x^m/(1-y)$  since 0 < y < +1. Also  $\Sigma x^m/(1-y)$  converges to 1/(1-x)(1-y) since 0 < x < +1.

§ 34.] Absolutely Convergent Double Series.—When a double series is such that it remains convergent when all its terms are taken positively, it is said to be Absolutely Convergent.

Any convergent series whose terms are all ultimately of the same sign is of course an absolutely convergent series according to this definition.

It is also obvious that all the propositions which we have proved regarding the convergency of double series consisting solely of positive terms are, a fortiori, true of absolutely convergent double series, for restoring the negative signs will, if it affect the residues at all, merely render them less than before.

In particular, from Theorem II. we deduce the following, which we may call Cauchy's test for the absolute concernency of a double series.

Theorem III. If  $u'_{m,n}$  be the numerical or positive value of  $u_{m,n}$ , and if all the horizontal series of  $\Sigma u'_{m,n}$  be convergent, and the sum of their sums to infinity also convergent, then

1st. The Horizontal Series of  $\Sigma a_{m,n}$  are all absolutely con-

vergent, and the sum of their sums to infinity converges to a definite finite limit S.

2nd. \(\sum\_{m,n}\) converges to S in the first way.

3rd. All the Vertical Series are absolutely convergent, and the sum of their sums to infinity converges to S.

4th. The Diagonal Series is absolutely convergent, and converges to S.

5th. Any series formed by taking terms from Σu<sub>m,n</sub> (under Restriction A) is absolutely convergent.

The like conclusions also follow, if all the vertical series, or if the diagonal series of  $\Sigma u'_{m,n}$  be convergent.

Cor. If  $\geq u_n$  and  $\geq v_n$  be each absolutely convergent, and converge to a and v respectively, then  $\geq (u_nv_1+u_{n-1}v_1+\dots+u_1v_n)$  is absolutely convergent, and converges to u. For the series in question is the diagonal series of the double

series  $\Sigma u_m r_n$ , which, as may be easily shown, satisfies Cauchy's conditions.

This is, in a more special form, the theorem already proved

in § 14. Example 1. Find the condition that the double series  $\Sigma(-)^m_{-C_mx^{n-m}y^m}$ 

(n · t m, oCo = 1) be absolutely convergent; and find its sum. The series may be arranged thus:—

If x' and y' be the moduli, or positive values, of x and y, then the series  $\Sigma u'_{m,n}$  corresponding to the above will be

In order that the horizontal series in this last may be convergent, it is necessary and sufficient that x' < 1.

Also  $T_{m+1} = y''m/(1-x')^{m+1}$ ; hence the necessary and sufficient condition that  $\Sigma I'_m$  be convergent is that y' < 1-x', which implies, of course, that y' < 1.

The given series will therefore satisfy Canchy's conditions of absolute convergency if |x|+1, |x|+|y|+1, and consequently also |y|<1.

These being fulfilled, we have  $T_{m+1} = (-)^m y^m/(1-x)^{m+1}$ ;

and

$$S' = \frac{1}{1-x} \left\{ 1 - \frac{y}{1-x} + \dots + (-)^m \left( \frac{y}{1-x} \right)^m \dots \right\},$$

$$= \frac{1}{1-x} \frac{1}{1-x} \cdot \dots;$$

and the sum of the series, in whatever order we take its terms, is 1/(1-x+y).

Example 2. If  $u_r = x^{2r} + x^{2r+1} + x^{2r+2} + \dots$ , where x < 1, show that

$$\frac{u_0}{20} + \frac{u_1}{21} + \frac{u_2}{22} + \dots = 2u_0 - \frac{x^{2^0}}{20} - \frac{x^{2^1}}{21} - \frac{x^{2^2}}{24} - \dots$$

Let S denote the series on the left. Then S may be written as a double series thus,

$$\begin{split} &\frac{1}{2^{0}}(x^{20}+x^{21}+x^{22}+\ldots+x^{2n}+\ldots)\\ &+\frac{1}{2^{1}}(0+x^{21}+x^{22}+\ldots+x^{2}+\ldots)\\ &+\frac{1}{2^{2}}(0+0+x^{2^{2}}+\ldots+x^{2^{n}}+\ldots) \end{split}$$

Now each of the vertical series is absolutely convergent, and we have  $U_n = x^{n} (1-1)^{2n+1})/(1-\frac{1}{2}) = x^{n} (2-1)^{2n})$ .  $\Sigma U_n$  is of the same order of convergence as  $\Sigma x^{n}$ , hence it is absolutely convergent. Also all the terms of the double series are positive. The double series therefore satisfies Cauchy's conditions; and its sum is the same as that of  $\Sigma U_n$  or of  $\Sigma T_n$ . Now

$$\Sigma T_n = u_0/2^0 + u_1/2^1 + u_2/2^2 + \dots;$$
  
 $\Sigma U_n = \Sigma x^{2n} (2 - 1/2^n),$   
 $= 2\Sigma x^{2n} - \Sigma x^{2n}/2^n.$ 

 $= 22x^2 - 2x^3/2^2,$   $= 2u_0 - x^{2^0}/2^0 - x^{2^1}/2^1 - \dots$ Hence the theorem.

§ 35.] Examples of the exceptional cases that arise when a double series is not absolutely convergent. It may help to accentuate the points of the foregoing theory if we give an example or two of the anomalies that arise when the conditions

of absolute convergency are not fulfilled.

Example 1. It is easy to construct double series whose horizontal and vertical series are absolutely convergent, and which nevertheless have not a definite sum of the first kind; but, on the other hand, have one definite sum of the second kind and another of the third kind.

If the finite sum of the first kind,  $S_{m_1n}$ , of a double series be A + f(m, n), where A is independent of m and n, then it is easy to see that

$$u_{m,n} = f(m, n) - f(m-1, n) - f(m, n-1) + f(m-1, n-1).$$
  
Hence we have only to give  $f(m, n)$  such a form that

$$L_{m=x} \{ L_{n=x} f(m, n) \} + L_{n=x} \{ L_{m=x} f(m, n) \},$$

and we shall have a series whose sums of the second and third kind are not alike, and which consequently has no definite sum of the first kind.

Suppose, for example, that f(m, n) = (m+1)/(m+n+2), then

$$u_{\mathbf{m}_n} = (m+1)/(m+n+2) - m/(m+n+1) - (m+1)/(m+n+1) + m/(m+n),$$
  
=  $(m-n)/(m+n) (m+n+1) (m+n+2).$ 

It is at once obvious that the sums of the second, third, and fourth kind for this series are all different. For in the first place we observe that  $u_{mn} = -u_{mn}$ . Hence there is a "skew" arrangement of the terms in the array (1), such that the terms equidistant from the dexter diagonal of the array and on the same perpendicular to this diagonal are equal and of opposite size, those on the diagonal are so the diagonal are for  $2D_{n}$  is therefore zero; and the same of the fourth kind is  $2D_{n}$  is therefore zero; and the sam of the fourth kind is

Also, owing to the arrangement of signs, we have  $T_{m_n} = -U_{m_n}$ ; and, since each of the horizontal and each of the vertical series in this case is convergent,  $T_n = -U_n$ , and therefore S' = -S'.

$$T_{m,n} = \sum_{n=0}^{\infty} \left\{ (m+1) \left\{ 1/(m+n+2) - 1/(m+n+1) \right\} - m \left\{ 1/(m+n+1) - 1/(m+n) \right\} \right\}.$$

$$= (m+1) \{1/(m+n+2) - 1/(m+2)\} - m \{1/(m+n+1) - 1/(m+1)\}.$$
 Hence

$$T_m = -(m+1)/(m+2) + m/(m+1) = -1 (m+1) (m+2).$$

The series  $\Sigma T_m$  is therefore absolutely convergent; and its sum to infinity is obviously -1+1/2=-1/2. Hence the double series has for its sum -1/2, +1/2, or 0, according as we sum it in the second, third, or fourth way,

At first sight, the reader might suppose (seeing that the horizontal series as all absolutely convergent, and that the sum of their actual sums is also absolutely convergent), that this case is a violation of Canchy's criterion. But it is not so. For, if we take all the terms in the mth horizontal series positively, and notice that the terms begin to be negative first m = n, then we see that  $T_m$  the sum of the positive values of the terms in the mth series is given by

$$\begin{split} T_m &\sim \sum_{n=1}^m u_{m,n} - \sum_{n=1+1}^n u_{m,n}, \\ &= (m+1) \left\{ 1/(2m+2) - 1/(m+2) \right\} - m \left\{ 1/(2m+1) - 1/(m+1) \right\} \\ &= (m+1) \left\{ 1/(2m+2) - 1/(m+2) \right\} + m \left\{ 6 - 1/(2m+2) \right\} \\ &= 1 - 2m/(2m+1) - (m+1)/(m+2) + m/(m+1), \\ &= (m^2 + m + 1)/(m+1) + (m+2) (2m+1), \end{split}$$

Now the convergence of  $\Sigma T_m$  is of the same order as that of  $\Sigma I/m$ , that is to say,  $\Sigma T_m$  is divergent. Hence Cauchy's conditions are not fully satisfied; and the anomaly pointed out above coases to be surprising. The present case is an excellent example of the care required in dealing with double series which are wont to be used some what recklessly by beginners in mathematics.

Before Cauchy the reckless use of double series and consequent perplexity was not confined to beginners. See a curious paper by Babbags, Plat. Trans. R.S.L. (1849).

Example 2. The double series  $\Sigma (-)^{m+n}1/mn$ , whose horizontal and vertical series are each semi-convergent, converges to the snm (log 2)2 in the second, third, or fourth way (see chap, xxviii., § 9, and Exercises xiii, 11). But alteration in the order of the terms in the array would alter the snm (see chap, xxviii., § 4, Example 3).

Example 3. If the two series  $\Sigma a_a$  and  $\Sigma b_a$  converge to a and b respectively. and at least one of them be absolutely convergent, then it follows from § 11 that the double series  $\Sigma a_n b_n$  converges to the same sum, namely  $ab_n$  in all the four ways, although it is not absolutely convergent, and its sum is not independent of the order of its terms.

The same also follows by § 20, Cor., provided  $\Sigma a_n$ ,  $\Sigma b_n$ ,  $\Sigma (a_n b_1 + a_{n-1} b_n)$  $+ \dots + a_1 b_n$ ) be all convergent, even if no one of the three be absolutely convergent\*.

If, however, both  $\Sigma a_n$  and  $\Sigma b_n$  be semi-convergent, then the diagonal series may be divergent, although the series converges to the same limit in the second and third way. This happens with the series  $\Sigma (-)^{m+a}1/(mn)^a$  where σ is a quantity lying between 0 and 1. This series obviously converges to the finite limit  $(1-1/2\alpha+1/5\alpha-...)$ : in the second and third ways. For the diagonal series we have

$$D_n = \sum_{r=1}^{n} 1/r^{\alpha} (n-r)^{\alpha}.$$

Now, since  $0 < \alpha < 1$ , we have, by chap. xxiv., § 9,  $r^{\alpha} + (n-r)^{\alpha} < 2^{1-\alpha} \{r$  $+(n-r)!^{\alpha} < 2^{1-\alpha}n^{\alpha}$ Therefore

$$\begin{split} D_{\mathbf{n}} &= \frac{1}{2^{1-\alpha}n^{\alpha}} \sum_{r^{\alpha}} \frac{2^{1-\alpha}n^{\alpha}}{(n-r)^{\alpha}} & = 4 \frac{1}{2^{1-\alpha}n^{\alpha}} \sum_{r} \frac{r^{\alpha} + (n-r)^{\alpha}}{r^{\alpha}(n-r)^{\alpha}} \,, \\ & \leq \frac{2}{2^{1-\alpha}n^{\alpha}} \sum_{r=1}^{n} \frac{1}{r^{\alpha}} & = \frac{2}{2^{1-\alpha}n^{\alpha}} \cdot \frac{n}{n^{\alpha}} \,, \end{split}$$

Hence, if  $\alpha = \frac{1}{2}$ ,  $LD_n < 2^{\alpha}$ ; and, if  $\alpha < \frac{1}{2}$ ,  $LD_n = \infty$ , when  $n = \infty$ . Therefore  $\Sigma D_{\alpha}$  diverges if  $0 < \alpha > \frac{1}{4}$ .

#### IMAGINARY DOUBLE SERIES.

36.1 After what has been laid down in § 10, it will be obvious that, in the first instance, the convergency of a double series of imaginary terms involves simply the convergency of two double series, each consisting of real terms oulv.

It is at once obvious that each of the two double series,  $\Sigma_{\alpha_{m,n}}$ ,  $\Sigma_{\beta_{m,n}}$ , will be absolutely convergent if the double series

See Stolz, Allgemeine Arithmetik, Th. 1., p. 248.

 $\Sigma \sqrt{(a^n_{m,n} + \beta^n_{m,n})}$  is convergent. Hence, if  $u'_{m,n}$  denote the modulus of  $u_{m,n} - a_{m,n} + i\beta_{m,n}$ , we see that  $\Sigma u_{m,n}$  will converge to the same limit in all four ways if  $\Sigma u'_{m,n}$  be convergent. In this case we say that the imaginary series is absolutely convergent.

Since all the terms  $u'_{m,n}$  are positive, we deduce from Theorem II, the following:—

Theorem IV. If all the horizontal series in the double series formed by the moduli of the terms of  $\Sigma u_{m,n}$  be convergent, and the sum of their sums to infinity be also convergent, then the series  $\Sigma u_{m,n}$  is absolutely convergent, and all its subsidiary series are also absolutely convergent.

Here subsidiary series may mean any series formed by selecting terms from  $\Sigma u_{m,n}$  under Restriction A. Theorem IV., of course, includes Theorem III. as a particular case.

§ 37.] The following simple general theorem regarding the convergency of the double series  $\sum u_{m,n}x^my^n$  will be of use in a later chapter.

If the moduli of the coefficients of the series  $\Sigma a_{m,n} x^m y^n$  have a finite upper limit  $\lambda$ , then  $\Sigma a_{m,n} x^m y^n$  is absolutely convergent for all values of x and y such that |x| < 1, |y| < 1.

For, if dashes be used to indicate moduli, we have, by hypothesis,  $a'_{m,n} \succ \lambda$ . Hence the series  $\Sigma a'_{m,n} r'm'$  is, a fortiori, convergent if the series  $\Sigma \lambda r'my''$  is convergent; that is, if  $\Sigma r''my''$  is convergent. Now, as we have already seen (§ 33), this last series is convergent provided x' < 1 and y' < 1. Hence the theorem in question.

## EXERCISES VIII.

Examine the convergency of the series whose nth terms are the following:—

- (1.)  $(1+n)/(1+n^2)$ . (2.)  $n^p/(n^p+a)$ .
- (3.)  $e^{-n^2x}$ , (4.)  $1/(n^2+1)$ , (5.)  $1/\sqrt{(n^2-n)}\{\sqrt{n}-\sqrt{(n-1)}\}$ , (6.)  $a^n/(a^n+x^n)$ ,
- (a)  $x_1 \in (n-n) \setminus (n-2)(n-2)$  (b)  $x_1 \in (n-2)$  (c) (n-2)(n-2) (d) (n-2)(n-2) (e) (n-2)(n-2) (f) (n
- (9.)  $\{(y+x^n)/(z-x^n)\}^{1/n}$ . (10.)  $n \log \{(2n+1)/(2n-1)\} 1$ .
- (11.) 1.3.5... (2n-1)/2.4.6...2n.
- (12.)  $\{1/1^a + 1/2^a + \dots + 1/n^a\}/n^a$ .

(13.) 
$$1/(an+b)$$
.

(14.) 
$$n/(an^2 + b)$$
.

(15.) m(m-1), . .  $(m-n+1)/n^n$ .

(16.)  $\{(n+1)/(n+2)\}^n/n$ ,

(17.) Show that  $\frac{m}{n} + \frac{m(m+1)}{n(n+1)} + \frac{m(m+1)(m+2)}{n(n+1)(n+2)} + \dots$  is convergent or

divergent according as n - m > or > 1.

(18.) Show that  $a^{1/m} + a^{1/m+1}(m+1) + a^{1/m+1}(m+1) + 1, (m+2) + ...$  is convergent or divergent according as a < or < 1/c. (Bourguet, Nouv, Ann., ser. 11., t. 18.)

(19.) Examine the convergency of Σ1/n(n+1)/n.

(20.) Show that Σn<sup>w</sup>/(n+1)<sup>w+a</sup> is convergent or divergent according as a > or ▶1. (Bertrand,)

(21.) Show that Σ1/n log n {log log n}<sup>a</sup> is convergent or divergent according as  $\alpha > \text{ or } < 1$ .

(22.) Show that Σ1/(n+I+cos nπ)<sup>2</sup> is convergent. (Catalan, Traité Ét. d. Séries, p. 28.)

Examine the convergency of the following infinite products:-

(23.) If {1+f(n)r<sup>n</sup>}, where f(n) is an integral function of n.

(24.) 
$$\Pi\{(x^{2n}+x)/(x^{2n}+1)\}.$$
 (25.)  $\Pi\{n^{z+1}/(n-1)^z(n+z)\}.$ 

(26.) If Σf(n) be convergent, show that, when n = ∞,

$$L\left\{ \prod_{i=1}^{n}\left( x+f\left( n\right) \right)\right\} ^{1/n}=x.$$

(27.) If n denote one of the series of primes 2, 3, 5, 7, 11, . . ., then  $\Sigma f(p)$  is convergent if  $\Sigma f(p)/\log p$  is convergent. (Bonnet, Liouville's Jour., vin. (1843), and Tehebiehef, ib., xvii. (1852).)

(28.) If x < I, show that the remainder after n terms of the series</p>

$$1^r x + 2^r x^2 + 3^r x^3 + \dots$$

is  $<(n+1)^r x^{n+1}/\{1-(1+1/n)^r x\}$ .

(29.) If u<sub>n</sub>, u<sub>1</sub>, . . . , u<sub>n</sub> be all positive, and Σu<sub>n</sub>x<sup>n</sup> be convergent for all values of  $x^2 < a^2$ , then

$$\Sigma x^{n} \left\{ u_{n} - (n+1) \, a u_{n+1} + \frac{(n+1) \, (n+2)}{1 \cdot 2} \, a^{2} u_{n+2} - \&c. \right\}$$

will be convergent between the same limits of x.

(30.) Point out the fallacy of the following reasoning :-

Let  $\Sigma = 1 + 1 + 1 + \dots$  ad  $\infty$ . then

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
  
=  $(1 + \frac{1}{2} + \frac{1}{3} + \dots) - 2(\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \dots)$ 

 $= \Sigma - 2\Sigma/2 = 0$ 

(31.) If ρ and ρ' be the ratios of convergence of Σ1/P<sub>r=1</sub> (n) (P<sup>-1</sup>n) 1+α and  $\Sigma 1/P_r(n) \{l^r n\}^{1+\alpha}$  (see § 6), then  $L(\rho'_n - \rho_n) P_{r-1}(n) = \alpha$ , when  $n = \infty$ . What conclusion follows regarding the convergence of the two series?

(32.) If Σu<sub>n</sub> is divergent, then Σu<sub>n</sub>/S<sub>n</sub>, a is divergent if a · 1 (where  $S_n = u_1 + u_2 + \dots + u_n$ ), and  $\sum u_n / S_n^{\alpha+1}$  is convergent if  $\alpha > 0$ . Hence show that there can be no function  $\phi(n)$  such that every series  $\Sigma u_n$  is convergent or divergent, according as  $L_-\phi(n)u_n=$  or +0. (Abel, \*Eurres\*, II., p. 197.)

(33.) If Σu<sub>n</sub> be any convergent series whose terms are ultimately positive, we can always find another convergent series, Σv<sub>n</sub>, whose terms are ultimately positive, and such that Lv<sub>n</sub>|u<sub>n</sub> = x.

If  $\Sigma u_n$  be any divergent series whose terms are ultimately positive, we can always find another divergent series whose terms are ultimately positive, and such that  $Lu, tr_n = \infty$ .

(These theorems are due to Du Bois-Reymond and Abel respectively; for concise demonstrations, see Thomae, Elementere Theorie der Analytischen Functionen, Halle, 1880)

(34.) If u<sub>n+1</sub>/u<sub>n</sub> = (n<sup>n</sup> + An<sup>n-1</sup> + . . .)/(n<sup>n</sup> + A<sup>n</sup>n<sup>n-1</sup> + . . .), then ∑u<sub>n</sub> will be convergent or divergent according as .1 - A<sup>r</sup> > or > 1. (Gauss, Werke, Ed. III., p. 139.)

(35.) If u<sub>n+1</sub>|u<sub>n</sub> = a − β|n + γ|n<sup>2</sup> + δ|n<sup>3</sup> + . . . , then Σu<sub>n</sub> is convergent or divergent according as a < or > 1. If a = 1, Σu<sub>n</sub> is convergent only if β>1, (Schlomich, Zeitsche, f. Math., x<sub>n</sub>, p. 74.)

(36.) Σ1/u<sub>n</sub> is convergent if u<sub>n+2</sub> - 2u<sub>n+1</sub> + u<sub>n</sub> is constant or ultimately increases with n. (Laurent, Nouv. Ann., ser. 11, t. 8.)

(37.) If the terms of Σu, are ultimately positive, then-

(I.) If  $\psi(n)$  can be found such that  $\psi(n)$  is positive,  $L\psi(n)u_n=0$ , and  $L\{\psi(n)u_n/u_{n+1}-\psi(n+1)\}>0$ ,  $\Sigma u_n$  is convergent.

(11.) If  $\psi$  (n) be such that  $L\psi$  (n)  $u_n = 0$ ,  $L\{\psi$  (n)  $u_n | u_{n+1} - \psi$  (n+1)\} = 0, and  $L\psi$  (n)  $u_n | u_n | \psi$  (n)  $u_n | u_{n+1} - \psi$  (n+1)\} = 0.

(III.) If  $u_n/u_{n+1}$  can be expanded in descending powers of n,  $\Sigma u_n$  is convergent or divergent according as  $L\{nu_n/u_{n+1}-(n+1)\} > 0 r > 0$ .

(IV.) If u<sub>n</sub>/u<sub>n+1</sub> can be expanded in descending powers of n, Σu<sub>n</sub> is convergent or divergent according as Luu<sub>n</sub> = or ±0. (Kummer's Criteria, Crelle's Jour., xiii. (1835) and xxi.)

(3x) If the terms of Σu<sub>n</sub> be ultimately positive, and if, on and after a certain value of n, a<sub>n</sub>u<sub>n</sub>|u<sub>n+1</sub> - u<sub>n+1</sub> > μ, where a<sub>n</sub> is a function of n which is always positive for values of n in question, and μ is a positive constant, then Σu<sub>n</sub> is convergent.

From this rule can be deduced the rules of Cauchy, De Morgan, and Bertrand. (Jensen, Comptex Rendus, c. vi., p. 729. 1888.)

Discuss the convergence of the following double series:-

- (39.)  $\Sigma (-1)^{n-1} r^m / n$ . (10.)  $\Sigma (-1)^{n-1} r^m / n l_*$
- (11)  $\Sigma \{(n-1)^m/n^{m+1} n^m/(n+1)^{m+1}\}$ , (12)  $\Sigma \epsilon^m \eta^n/(m+n)$ , (13.)  $\Sigma 1/(m+n)^2$ ,
- (11)  $\Sigma 1/(m+n)$ . (15.)  $\Sigma 1/(m^2-n^2)$ .
- (46.) Under what restrictions can 1/(1+x+y) be expanded in a double serie of the form  $1+\sum A_{m,n}x^my^n$ ?
- (47) If \(\Sigma\_{m,n}\) converge to S in the first way, and if its diagonal series be convergent, show that the diagonal series converges to S also.

Deduce Abel's Theorem regarding the product of two semi-convergent series. (See Stolz, Math. Ann., xxiv.)

(48.) If  $u_n/u_{n-1}$  can be expanded in a series of the form  $1 + a_1/n + a_2/n^2 + \ldots$ , show that

show that

1°. If  $a_1=0$ ,  $a_2=0$ , . . . ,  $a_{\mu-1}=0$ ,  $a_{\mu}+0$ , then  $u_n=u+v_n/n$ , where u is a 'definite constant  $\pm 0$  and  $\pm \infty$ , and  $Lv_n$  is finite when  $n=\infty$ .

2°. If  $a_1 \neq 0$ , and the real part of  $a_1$  be positive, then  $Lu_n = \infty$  when  $n = \infty$ .

3°. If  $a_1 \neq 0$ , and the real part of  $a_1 = 0$ , then  $Lu_n$  is not infinite, but is not definite.

4°. If  $a_1 \neq 0$ , and the real part of  $a_1$  be negative, then  $Lu_n = 0$ .

Apply these results to the discussion of the convergency of  $\Sigma u_n x^n$ , and, in particular, to the Hypergeometric Series, and to the following series:—

$$\Sigma_{\mu+\nu i} C_n (x+yi)^n, \quad \Sigma^n [n^{\mu+\nu i}, \quad \Sigma_m C_n] (m+n)^p, \quad \Sigma (-)^n {}_m C_n] (m+n)^p.$$

(See Weierstrass, Ueber die Theorie der Analytischen Facultät.—Crelle's Jour., 11.)

(49.) Discuss the convergence of ∑<sub>m</sub>C<sub>n</sub> (a − nβ)<sup>n-1</sup> (x + nβ)<sup>n</sup>.

(50.) If u<sub>n</sub> and v<sub>n</sub> be positive for all values of n, never increase when n increases, and be such that Lu<sub>n</sub> = 0, Lv<sub>n</sub> = 0, when n = x, find the necessary and sufficient condition that Σ (u<sub>n</sub>v<sub>1</sub> + u<sub>n-1</sub>v<sub>2</sub> + · · · + u<sub>1</sub>v<sub>n</sub>) = Σu<sub>n</sub> × Σv<sub>n</sub>. (See Pringsheim, Math. Ann., Bd. xxt.)

(51.) If 0 < M<sub>n</sub> < M<sub>n+1</sub> and LM<sub>n</sub>=0 when n = ∞, show that every divergent series of real positive terms can be expressed in the form ∑ (M<sub>n+1</sub> − M<sub>n</sub>); and every convergent series of real positive terms in the form ∑ (M<sub>n+1</sub> − M<sub>n</sub>)| M<sub>n</sub> M<sub>n+1</sub>.

Also that the successions of series

$$\Sigma (M_{n+1} - M_n)/P_r(M_n)$$
,  $r = 0, 1, 2, ...$ 

$$\Sigma (M_{n+1} - M_n)/P_r(M_{n+1}) (l_r M_{n+1})^{\rho}, r = 0, 1, 2, ...$$

where  $0 , and <math>P_p(x)$  has the meaning of § 6 above, form two scales, the first of slower and slower divergency; the second of slower and slower convergency. (Pringsheim, Math. Ann., Edd. 222v., 222v., 222v.)

## CHAPTER XXVII.

# Binomial and Multinomial Series for any Index.

#### RINOMIAL SERIES

 $\S$  1.] We have already shown that, when m is a positive integer,

$$(1 + x)^m = 1 + {}_{\sim}C_{\sim}x + {}_{\sim}C_{\sim}x^2 + ... + {}_{\sim}C_{\sim}x^n + ... + {}_{\sim}C_{\sim}x^m$$
 (1).

where 
$${}_{m}C_{n} = m(m-1) \dots (m-n+1)/n!$$
 (2)

When m is not a positive integer,  ${}_{m}C_{n}$ , although it has still a definite analytical meaning, can no longer be taken to denote the number of n-combinations of m things; hence our former demonstration is no longer applicable. Moreover, the right-hand side of (1) then becomes an infinite series, and has, according to the principles of last chapter, no definite meaning unless the series be convergent. In cases where the series is divergent there cannot be any question, in the ordinary sense at least, regarding the equivalence of the two sides of (1).

As has already been shown (pp. 122, 131), the series

$$1 + {}_{m}C_{1}x + {}_{m}C_{2}x^{2} + \dots + {}_{m}C_{n}x^{n} + \dots$$
 (3)

is convergent when x has any real value between -1 and +1; also when x = +1, provided m > -1; and when x = +1, provided m > 0. We propose now to inquire, whether in these cases the series (3) still represents  $(1 + x)^n$  in any legitimate sense.

In what follows, we suppose the numerical value of m to be a commensurable number\*; also, for the present, we consider

If m be incommensurable we must suppose it replaced by a commensurable approximation of sufficient accuracy.

only real values of x, and understand  $(1+x)^m$  to be real and positive.

§ 2.] If we assume that  $(1+x)^m$  can be expanded in a convergent series of ascending powers of x, then it is easily shown that the coefficient of  $x^n$  must be  $m(m-1) \dots (m-n+1)/n!$ .

For, let

 $(1 + x)^m = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$  (1)

where  $a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$  (2)

is convergent so long as |x| < R (it will ultimately appear that R = 1). Then, if h be so small that |x + h| < R, we have

$$(1+x+h)^n = a_0 + a_1(x+h) + a_2(x+h)^2 + \dots + a_n(x+h)^n + \dots$$
 (3), the series in (3) being convergent by hypothesis.

Hence by the principles of last chapter, we have

$$\frac{(1+x+h)^n - (1+x)^n}{(1+x+h) - (1+x)} = a_1 \frac{(x+h) - x}{(x+h) - x} + a_2 \frac{(x+h)^2 - x^2}{(x+h) - x} + \dots + a_n \frac{(x+h)^n - x^n}{(x+h) - x} + \dots$$

$$(4),$$

the series in (4) being still convergent. Hence, if we take the limit when  $\hbar=0,$  and observe that

$$L\frac{(1+x+h)^m-(1+x)^m}{(1+x+h)-(1+x)}=m(1+x)^{m-1}, \qquad L\frac{(x+h)^n-x^n}{(x+h)-x}=nx^{n-1},$$

by chap. xxv., § 12, we have

$$m(1+x)^{m-1} = a_1 + 2a_2x + \dots + na_nx^{n-1} + \dots$$
 (5),

where the series on the right must still be convergent, since  $L(n+1) \, a_{n+1}/n a_n = L a_{n+1}/a_n$  when  $n=\infty^*$ . Hence, multiplying by 1+x, we deduce

 $m(1+x)^m = a_1 + (a_1 + 2a_2)x + \dots + \{na_n + (n+1)a_{n+1}\}x^n + \dots,$  that is,

$$ma_0 + ma_1x + \dots + ma_nx^n + \dots = a_1 + (a_1 + 2a_2)x + \dots + \{na_n + (n+1)a_{n+3}\}x^n + \dots$$
 (6).

We here make the farther assumption that the limit of the sum of the inite number of terms on the right of (4) is the sum of the limits of these terms.

By chap, xxvi.,  $\S$  21, the coefficients of the powers of x on both sides of (6) must be equal. Hence

$$a_1 = ma_n - 2a_1 - (m-1)a_1, \ldots, (n+1)a_{n+1} = (m-n)a_n, \ldots$$
 (7).

From (7) we deduce at once

$$a_1 - ma_0$$
,  $a_2 = m(m-1)a_0/2!$ , . . .  $a_n = m(m-1)$  . . .  $(m-n+1)a_0/n!$ , . . .

To determine  $a_0$  we may put x=0. We then get from (1),  $a_0=1^m=1$  (if we suppose, as usual, the real positive value of any root involved to be alone in question). We therefore have

$$(1 + x)^m = 1 + \sum_m C_n x^n$$
 (8).

The theorem is therefore established; and we see that the hypothesis under which we started is not contradicted provided |x| < 1, this being a sufficient condition for the convergency of  $\Sigma_m C_n x^n$ .

§3.] Although the assumption that  $(1+x)^m$  can be expanded in a series of ascending powers of x leads to no contradiction in the process of determining the coefficients, so long as |x| < 1; this fact can scarcely be regarded as sufficient evidence for the validity of a theorem so fundamentally important. We proceed, therefore, to establish the following theorem, in which we start from the series in the first instance,

Whenever the series  $1 + \sum_{n} C_{n}x^{n}$  is convergent, its sum is the real positive value of  $(1 + x)^{m}$ .

The fundamental idea of the following demonstration is due to Euler\*; but it involves important additions, due mainly to Cauchy, which were necessary to make it accurate according to the modern view of the nature of infinite series.

Let us denote the series

$$1 + {}_{m}C_{1}x + {}_{m}C_{2}x^{3} + \dots + {}_{m}C_{n}x^{n} + \dots$$
 (1)

by the symbol f(m).

So long as -1 < x < +1, f(m) is an absolutely convergent series, and (by chap. XXVI., § 20) is a continuous function both of m and of x.

Nov. Comm. Petrop., t. xix. (1775).

Hence,  $m_1$  and  $m_2$  being any real values of m, we have

$$f'(m_1)f'(m_2) = \{1 + \sum_{m_1} C_n x^n\} \{1 + \sum_{m_2} C_n x^n\},$$

$$= 1 + \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n} C_{n} + \sum_{m=1}^{n} C_{n-1} + \sum_{m=1}^{n} C_{n-2} + \sum_{m=1}^{n} C_{m} \right) x^{n}$$
 (2),

where the last written series is convergent (by chap. xxvt., § 11), since the two series,  $1 + \sum_{m_1} C_n x^n$  and  $1 + \sum_{m_2} C_n x^n$ , are absolutely convergent.

Now, by chap. XXIII., § 8, Cor. 5,

 $_{m_1}C_n + {}_{m_2}C_{1\,m_1}C_{n-1} + {}_{m_2}C_{2\,m_1}C_{n-2} + \dots + {}_{m_2}C_n = {}_{m_1+m_2}C_n ;$  hence  $f(m_1)f(m_2) = 1 + \sum_{m_1+m_2}C_{n_1+m_2}C_{n_2+m_3}$ 

$$=f(m_1 + m_2)$$
 (3).

In like manner, we can show that

$$f(m_1 + m_2) f(m_1) = f(m_1 + m_2 + m_2).$$

Hence  $f(m_1) f(m_2) f(m_3) = f(m_1 + m_2 + m_3)$ ; and in general, v being any positive integer,

$$f(m_1) f(m_2) \dots f(m_r) = f(m_1 + m_2 + \dots + m_r)$$
 (4).

This result may be called the Addition Theorem for the Binomial Series.

If in (4) we put  $m_1 = m_2 = \dots = m_v = 1$ , then we deduce

$$\{f(1)\}^{c} = f(c) \tag{5},$$

where v is any positive integer.

If in (4) we put  $m_1 = m_2 = \ldots = m_v - p/q$ , where p and q are any positive integers, and also put v = q, we deduce

$$\{f(p/q)\}^{q} = f(p)$$
 (6).

Hence, by (5),  $\{f(p/q)\}^q = \{f(1)\}^p$  (7).

Again, if in (3) we put  $m_1 = m$ ,  $m_2 = -m$ , we deduce

$$f(m)f(-m) = f(m-m) = f(0)$$
 (8).

Hence 
$$f(-m) = f(0)/f(m) \tag{9}.$$

These properties of the series (1) hold so long as -1 < x < +1, and they are sufficient to determine its sum for all real commensurable values of m.

For, since  ${}_{1}C_{1} = 1$ ,  ${}_{1}C_{2} = 0$ , ...,  ${}_{1}C_{n} = 0$ , ...,  ${}_{0}C_{1} = 0$ ,  ${}_{0}C_{2} = 0$ , ..., we have

$$f(1) = 1 + x$$
,  $f(0) = 1$ ,

Suppose, now, m to be a positive integer. Then, by (5),

$$(1 + x)^m = f'(m) = 1 + {}_mC_1x + {}_mC_2x^2 + \dots + {}_mC_mx^m$$
 (10),

where the series terminates, since  ${}_{m}C_{m+1}=0$ ,  ${}_{m}C_{m+2}=0$ , . . . , when m is a positive integer. This is another demonstration of that part of the theorem with which we are already familiar.

Next, let m be any positive commensurable quantity, say p/q, where p and q are positive integers. Then, by (7),

$$\{f(p/q)\}^{q} = (1+x)^{p}$$
 (11).

Hence f(p/q) is one of the qth roots of the positive  $^{\bullet}$  quantity  $(1+x)^p$ . But f(p/q) is necessarily real; hence, if  $(1+x)^{pq}$  denote, as usual, the real positive qth root of  $(1+x)^p$ , we must have

$$f(p/q) = \pm (1 + x)^{p \cdot q}$$
 (12).

The only remaining question is the sign of the right-hand side of (12).

Since f(p/q) is a continuous function both of p/q and of x, its equivalent  $\pm (1+x)^{p,q}$  must be a continuous function both of p/q and of x. Now  $(1+x)^{p,q}$  does not vanish (or become infinite) for any values of p/q or of x admissible under our present hypothesis; and being the equivalent of a continuous function it cannot change sign without passing through 0. Hence only one of the two possible signs is admissible; and we can settle which by considering any particular case. Now, when x=0, f(p/q)=1. Hence the positive sign must be taken; and we establish finally that

$$f(p,q) = +(1+x)^{q}$$

that is,

$$(1 + x)^m = 1 + {}_mC_1x + {}_mC_2x^2 + \dots + {}_mC_nx^n + \dots$$
 (13),

when m is any positive commensurable quantity.

Positive, since -1 < x < 1, by hypothesis.</li>

(16).

Finally, let m be any negative commensurable quantity, say m = -m', where m' is a real positive commensurable quantity.

By (9) we have

$$f(-m') = f(0)/f(m') = 1/f(m')$$
.

Hence, by (13),

$$f(-m') = 1/(1+x)^{m'},$$
  
=  $(1+x)^{-m'}.$ 

that is.

$$(1+x)^m = 1 + {}_mC_1x + {}_mC_2x^2 + \dots + {}_mC_nx^n + \dots$$
 (14),

where m is any commensurable negative quantity.

The results of (10), (13), and (14) establish the Binomial Theorem for all values of x such that -1 < x < +1. It remains to consider the extreme cases.

When x = +1, the series (1) reduces to

$$1 + {}_{m}C_{1} + {}_{m}C_{2} + \ldots + {}_{m}C_{n} + \ldots$$

This series is semi-convergent if -1 < m < 0, absolutely convergent if m > 0. Hence, by Abel's Second Theorem, chap. xxvi.,  $\S 20$ ,

$$(1+\overline{1-0})^m = L_{m-1-n} \{1 + {}_{m}C_{1}x + {}_{m}C_{2}x^2 + \ldots + {}_{m}C_{n}x^n + \ldots \},$$

that is.

$$2^{m} = 1 + {}_{m}C_{1} + {}_{m}C_{2} + \dots + {}_{m}C_{n} + \dots$$
 (15),

provided m>-1, with the condition that, when -1 < m < 0, the order of the terms in the series of (15) must not be altered.

If 0 < x < 1, we have, by the general case already established,

$$(1-x)^m = 1 - {}_m C_1 x + {}_m C_2 x^2 - \dots (-)^n {}_m C_n x^n + \dots$$

Hence, since the series

$$1 - {}_{m}C_{1} + {}_{m}C_{2} - \dots (-)^{n}{}_{m}C_{n} + \dots$$

is convergent if m > 0, we have, by Abel's Theorem,

$$(1-\overline{1-0})^m = L_{x-1-0}(1-{}_mC_1x+{}_mC_2x^2-\ldots (-)^n{}_mC_nx^n+\ldots),$$
 that is,

 $0 = 1 - {}_mC_1 + {}_mC_2 - \dots (-)^n {}_mC_n + \dots$  provided m be positive.

The results of (15) and (16) complete the demonstration of

the Binomial Theorem in all cases where its validity is in question.

Cor. If x + y, it follows from the above result that we can always expand  $(x + y)^m$  in an absolutely convergent series. We have in fact, if |x| > |y|, that is, |y/x| < 1,

$$(x + y)^m = x^m (1 + y/x)^m,$$

$$= x^m \{1 + {}_mC_1(y/x) + {}_mC_2(y/x)^2 + \dots + {}_nC_n(y/x)^n + \dots \},$$

$$= x^m + {}_mC_1x^{m-1}y + {}_mC_2x^{m-2}y^2 + \dots + {}_nC_nx^{m-n}y^n + \dots$$
(17);
and if  $|x| < |y|$ , that is,  $|x/y| < 1$ ,
$$(x + y)^m = y^m (1 + x/y)^m + (x + y)^m + \dots \},$$

$$= y^m \{1 + {}_mC_1(x/y) + {}_mC_2(x/y)^2 + \dots + {}_mC_n(x/y)^n + \dots \},$$

$$= y^m + {}_mC_1x^m + x + {}_mC^m x^{m-1}x^2 + \dots + {}_mC_nx^{m-1}x^n + \dots$$
(18),

If m be a positive integer, both the formulae (17) and (18) will be admissible because both series terminate. But, if m be not a positive integer, only one of the two series will be convergent.

§ 4.] The general formulae of last paragraph contain a vast number of particular cases. To help the student to detect these particular cases under the various disguises which they assume, we proceed to draw his attention to several of the more commonly occurring. The difficulties of identification are in reality in most cases much smaller than they at first sight appear. We assume in all cases that the values of the variables are such that the series are convergent.

```
Example 1.
                  (1+x)^{-1} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots
                  (1-x)^{-1} = 1 + x + x^2 + \dots + x^n + \dots
For
                  (1+x)^{-1}=1+\sum_{i=1}^{n}C_{n}x^{n};
                      C_n = -1(-1-1)(-1-2), , (-1-n+1)/n!,
and
                            = (-1)^n 1 \cdot 2 \cdot 3 \cdot \dots \cdot n! n!
                           = (-12.1)
                 (1-x)^{-1} = 1 + \sum_{i=1}^{n} C_{ij} (-x)^{n};
              =_1C_n(-x)^n = (-)^n(-)^nx^n = (-)^nx^n
and
                          - rn
    Example 2.
              (1+x)^{-2}=1-2x+3x^2-\ldots+(-)^n(n+1)x^n+\ldots
               (1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (n+1)x^n + \dots
                   =_2C_n = -2(-2-1) . . . (-2-n+1)/n!,
For
```

 $= (-)^n (n+1).$ 

$$(1+x)^{-3} = 1 - 3x + 6x^2 - \dots + (-)^n \frac{1}{2} (n+1) (n+2) x^n + \dots;$$
  
 $(1-x)^{-2} = 1 + 3x + 6x^2 + \dots + \frac{1}{2} (n+1) (n+2) x^n + \dots;$ 

Example 4.

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^{2} + \frac{1}{16}x^{3} - \dots + (-)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^{n} + \dots;$$

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots - \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^n - \dots$$
  
Example 5.

Example 5. 
$$(1+x)^{-1} = 1 - \frac{1}{2}x + \frac{6}{8}x^2 - \frac{6}{16}x^3 + \dots + (-)^n \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n}x^n + \dots;$$

 $(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{5}{8}x^2 + \frac{5}{16}x^3 + \dots + \frac{1}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n-1)}x^n + \dots$ Example 6.

 $(1+x)^{m/2} = 1 + \frac{m}{1} \frac{x}{2} + \frac{m(m-2)}{2!} \left(\frac{x}{2}\right)^2 + \dots$ 

$$+ \frac{m(m-2)(m-4) \dots (m-2n+2)}{n!} \left(\frac{x}{2}\right)^{n} + \dots;$$

$$= 1 + \frac{m}{2}x + \frac{m(m-2)}{2}x^{2} + \dots + \frac{m(m-2)(m-4) \dots (m-2n+2)}{2 + 1 \cdot 6 \cdot \dots \cdot 2n}x^{n} + \dots;$$

$$(1+x)^{-m/2} = 1 + \Sigma (-)^n \frac{m(m+2)(m+1) \dots (m+2n-2)}{2 + 1 + 6} x^n$$
.

Example 7.

$$(1+x)^{pq} = 1 + 2 \frac{p (p-q) (p-2q) \dots (p-nq+q)}{q \cdot 2q \cdot 3q \cdot \dots nq} x^n;$$

$$(1-x)^{-pq} = 1 + 2 \frac{p (p+q) (p+2q) \dots (n+nq-q)}{q \cdot 2q \cdot 3q \cdot \dots nq} x^n.$$

Example 8.

$$(1-x)^{-m} = 1 + \sum \frac{m(m+1) \cdot (m+n-1)}{n!} x^n.$$

It will be observed that the coefficient of  $x^n$  in this last expansion, when m is integral, is (see chap. xxrv., § 10) the number  $(_m H_n)$  of n-combinations of m things when repetition is allowed. It is therefore usual to denote this coefficient by the symbol  $_m H_n$ , m being now unrestricted m value. We shall return to this function later on.

Example 9.

$$\frac{1}{2}\left\{(1+x)^m+(1-x)^m\right\}=1+{}_{m}C_{2}x^2+{}_{m}C_{4}x^4+\ldots+{}_{m}C_{2n}x^{2n}+\ldots\;;$$

Ultimate Sign of the Terms.—Infinite Binomial Series belong to one or other of two classes as regards the ultimate sign of the terms—1st, those in which the signs of the terms are ultimately alternately positive and negative; 2nd, those in which all the terms are ultimately of the same sign. If x and m denote positive quantities (m of course not a positive integer),

1st. The expansions of  $(1+x)^m$  and  $(1+x)^{-m}$  both belong to the first class. In  $(1+x)^m$  the first negative term will be that containing  $x^{m+1}$ , where n is the least integer which exceeds m. In  $(1+x)^{-m}$  the first negative term is of course the second.

and. The expansions of  $(1-x)^m$ ,  $(1-x)^{-m}$ , both belong to the second class. In  $(1-x)^m$  the terms will have the same sign on and after the term in  $x^n$ , n being the least integer which exceeds m, and this sign will be + or - according as n is even or odd. In  $(1-x)^{-m}$  all the terms are positive after the first.

§ 5.] A great variety of series suitable for various purposes can be readily deduced from the Binomial Series; and, conversely, many series can be summed by identifying them with particular cases of the Binomial Series itself, or with some series deducible from it.

The following cases deserve special attention, because they include so many of the series usually treated in elementary textbooks as particular cases, and because the methods by which the summation is effected are typical.

Consider the series  $\Sigma \phi_r(n)_n C_n x^n$ , where  $\phi_r(n)$  is any integral function of n of the rth degree. Such a series stands in the same relation to the simple Binomial Series as does the Integro-Geometric to the simple Geometric Series. We may therefore speak of it as an Integro-Binomial Series.

We may always, by the process of chap. v., § 22, establish an identity of the following kind,

$$\phi_r(n) \equiv A_v + A_1 n + A_1 n(n-1) + \dots + A_r n(n-1) + \dots + (n-r+1)$$
 (1), where  $A_n, A_1, A_2, \dots, A_r$  are constants, that is, are independent of  $n$ .

We can therefore write the general term of the Integro-Binomial Series in the following form:—

$$\begin{split} \phi_{t}(n)_{m}C_{n}x^{n} &= A_{+m}C_{n}x^{n} + A_{+}n_{m}C_{n}x^{n} + \dots \\ &+ A_{+}n_{-}(n-1) \dots (n-r+1)_{m}C_{n}x^{n} \\ & \equiv A_{-}C_{n}x^{n} + mA_{-}(r_{n-1}C_{n-1}x^{n-1}) \\ &+ m_{-}(m-1)A_{+}x^{2} \dots C_{n-r}C_{n-r}x^{n-r} & \dots - m_{-}(m-1) \dots \\ & (m-r+1)A_{+}x^{r}_{m-r}C_{n-r}x^{n-r} & (2). \end{split}$$

$$\S 4, 5 \qquad \Sigma \phi_r(n)_m C_n x^n / (n + a)(n + b) \dots (n + k)$$
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Hence, if the summation proceed from 0 to  $\infty$ , we evidently have

$$\overset{\infty}{\underset{0}{\circ}} q_r(n)_m C_n x^n = A_0 \overset{\infty}{\underset{0}{\circ}} {_mC_n} x^n + m A_1 x \overset{\infty}{\underset{1}{\circ}} {_{m-1}C_{n-1}} x^{n-1} + \dots 
+ m(m-1) \dots (m-r+1) A_r x^r \overset{\infty}{\underset{1}{\circ}} {_{m-r}C_{n-r}} x^{n-r} (3),$$

$$= A_0 (1 + x)^m + m A_1 x (1 + x)^{m-1} + \dots$$

$$+ m (m-1) \dots (m-r+1) A_r x^r (1 + x)^{m-r},$$

since all the Binomial Series are evidently complete\*. Hence  $\overset{\infty}{\Sigma} \phi_r(n)_m C_n x^n = \{A_0 + m A_1 x/(1+x) + m (m-1) A_2 x^2/(1+x)^2 + \dots$ 

$$+ m (m-1) \dots (m-r+1) A_r x^r / (1+x)^r (1+x)^m (4)$$
;

and the summation to infinity of the Integro-Binomial Series is effected.

The formula will still apply when *m* is a positive integer, although in that case the series on the left of (4) has not an infinite number of terms. The only peculiarity is that a number of the terms within the crooked bracket on the right-hand side of (4) may become zero.

Cor. We can in general sum the series  $\stackrel{\otimes}{\Sigma} \phi_r(n)_m C_n x^n/(n+a)(n+b)$ ... (n+k), where  $a, b, \ldots, k$  are unequal positive integers, in ascending order of magnitude.

For, by introducing the factors n+1, n+2, . . . , n+a-1. n+a+1, n+a+2, . . . , n+b-1, &c., we can reduce the general term to the form

$$\psi(n)_{m+k}C_{n+k}x^{n+k}/(m+1)(m+2)...(m+k)x^{k}$$
 (5)

where  $\psi(n)$  is an integral function of n, namely,  $\phi_r(n)$  multiplied by all the factors introduced which are not absorbed by  $_{m+k}C_{n+k}$ .

If the lower limit of summation be not 0, then the Binomial Series on the right-hand side of (3) will not all be complete, and the sum will not be quite so simple as in (4).

<sup>+</sup> It may be remarked that the series is evidently convergent when x < 1. The examination of the convergence when x = 1 will form a good exercise on chap. Axv.

Hence

$$\stackrel{\circ}{\Sigma} \phi_r(n)_m C_n x^n / (n + \alpha) (n + b) \dots (n + k)$$

$$= \{ \sum_{k=0}^{\infty} (n)_{m+k} C_{n+k} x^{n+k} \} / (m+1) (m+2) \dots (m+k) x^{k}$$
 (6).

The summation of the series inside the crooked bracket may be effected; for it is an Integro-Binomial Series. Hence the summation originally proposed is always possible.

We have not indicated the lower limit of the summation, and it is immaterial what it is. Even if the lower limit of summation be 0, the Binomial Series into which the righthand side of (6) is decomposed will not all be complete (see Example 6, below).

It should also be noticed that this method will not apply if m be such that any of the factors  $m+1, m+2, \ldots, m+k$  vanish. In such cases the right-hand side of (6) would become indeterminate, and the evaluation of its limit would be trouble-some

The above method can be varied in several ways, which need not be specified in detail. It is sufficient to add that by virtue of Abel's Second Theorem (chap. xxvt., § 20) all the above summations hold when  $x=\pm 1$ , provided the series involved remain convergent.

Example 1. To expand  $(x+y)^m$  in a highly convergent series when x and y are nearly equal. From the obvious identities

$$\begin{aligned} & \{(x+y)/2x\}^m = \{2x/(x+y)\}^{-m} = \{1+(x-y)/(x+y)\}^{-m}, \\ & \{(x+y)/2y\}^m = \{2y/(x+y)\}^{-m} = \{1-(x-y)/(x+y)\}^{-m}, \end{aligned}$$

 $(x+y)^m (1/2x)^m \pm 1/(2y)^m$  =  $\{1 + (x-y)/(x+y)\}^{-m} \pm \{1 + (x-y)/(x+y)\}^{-m}$ , we deduce at once

$$\begin{split} (x+y)^m &= 2^m x^m \left\{ 1 + \sum \left( -\right)^n {}_m H_n \left( \begin{matrix} x-y \\ x+y \end{matrix} \right)^n \right\} \,, \\ &= 2^m y^m \left\{ 1 + \sum_m H_n \left( \begin{matrix} x-y \\ x+y \end{matrix} \right)^n \right\} \,, \end{split}$$

 $= 2 \cdot y \cdot \left(1 + -m \ln \left(x + y\right)\right)$   $= 2 \cdot y \cdot \left(1 + -m \ln \left(x + y\right)\right)$   $= 2 \cdot y \cdot \left(1 + -m \ln \left(x + y\right)\right)$ 

$$\begin{array}{ll} \text{where} & _{m}H_{n}=m\left(m+1\right), \quad , \quad \left(m+n-1\right)\!\!/\!n!, \\ & = \frac{2^{m+1}x^{m}y^{m}}{x^{m}+y^{m}} \left\{1+\frac{m\left(m+1\right)\left(x-y\right)^{2}}{2!}\!\!\left(\frac{x-y}{x+y}\right)^{2}\!\!+\!\frac{m\left(m+1\right)\left(m+2\right)\left(m+3\right)}{4!}\!\!\left(\frac{x-y}{x+y}\right)^{4} \right. \\ & + \dots \left\{x-x\right\}, \end{array}$$

$$=\frac{x^{m+1}x^m\eta^m}{x^m-y^m}\left\{\frac{m}{1!}\left(\frac{x-y}{x+y}\right)+\frac{m\left(m+1\right)\left(m+2\right)}{3!}\left(\frac{x-y}{x+y}\right)^3+\ldots\right\}\;,$$

All these series are highly convergent, since (x-y)/(x+y) is small.

Example 2. To sum the series

$$\frac{2}{9} + \frac{2}{2!} \left(\frac{2}{9}\right)^2 + \frac{2.5}{3!} \left(\frac{2}{9}\right)^3 + \frac{2.5.8}{4!} \left(\frac{2}{9}\right)^4 \dots$$

If we denote this series by  $u_1 + u_2 + u_3 + \dots$ , we see that

$$\begin{split} u_n &= \frac{2 \cdot 5 \cdot \dots \cdot \left\{2 + (n-2)\beta\right\} \cdot 2^n}{n!} \\ &= \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \dots \cdot \left(-\frac{1}{2} + n - 1\right)}{n!} \cdot \binom{2}{3}^n, \\ &= - \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} + 1\right)\left(-\frac{1}{2} + 2\right) \cdot \dots \cdot \left(-\frac{1}{2} + n - 1\right)}{n!} \cdot \binom{2}{3}^n, \\ &= - \left(-\right)^n \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{n!} \cdot \dots \cdot \left(\frac{1}{2} - n + 1\right)} \cdot \binom{2}{3}^n, \end{split}$$

Hence

Therefore.

Example 3. To sum the series 
$$m + \frac{m(m-1)}{1} + \frac{m(m-1)(m-2)}{1} + \dots$$

whenever it is convergent

Here we have

$$\begin{split} &u_{n+1} = \frac{m \, (n-1) \, (m-2) \, \ldots \, (m-n)}{n!}, \\ &= \frac{m \, (m-1) \, (m-1-1) \, \ldots \, (m-1-n+1)}{n!}, \\ &= m \, _{m-1} \, C_n. \end{split}$$

Hence

Hence 
$$u_1+u_2+u_3+\ldots=m\left\{1+{}_{m-1}C_1+{}_{m-1}C_2+\ldots\right\}$$
  
=  $m\left\{1+1\right\}^{m-1}=m2^{m-1}$ ,  
provided  $m-1>-1$ , that is  $m>0$ .

It should be observed that we have at once from \$ 2 (5) the equation  $m(1+x)^{m-1}=1$ ,  $C_1+2$ ,  $C_2x+\dots+n_mC_nx^{n-1}+\dots$ 

from which the above result follows by putting x=1.

By repeating the process of § 2, we should deduce the equation m(m-1)... $(m-k+1)(1+x)^{m-k}=1.2...k_mC_k+2.3...(k+1)$ 

 ${}_{m}C_{k+1}x + \ldots + (n-k+1)(n-k+2) \ldots n_{m}C_{n}x^{n-k} + \ldots$ 

whence it follows that m(m-1) . . .  $(m-k+1) 2^{m-k} = 1 . 2 . . . . k_m C_k$ 

 $+2.3...(k+1)_{ka}C_{k+1}+...$  (3), provided m > k - 1. These results might also be easily established by the method first used.

Example 4. To sum the serie

$$\frac{1}{1.2...k} + \frac{mC_1x}{2.3...(k+1)} + \frac{mC_2x^2}{3.4...(k+2)} + \dots$$

Here we have

$$u_{n+1} = \frac{m \cdot n^{2}}{(n+1)(n+2) \cdot \cdot \cdot \cdot (n+k)}$$

$$= \frac{m + k C_{n+k} x^{n+k}}{(m+1)(m+2) \cdot \cdot \cdot \cdot (m+k) x^{k}}$$

Hence

Therefore  $u_1 + u_2 + u_3 + \dots$ 

$$= \frac{(1+x)^{m+k} - 1 - {}_{m+k}C_1x - {}_{m+k}C_2x^2 - \dots - {}_{m+k}C_{k-1}x^{k-1}}{(m+1)(m+2)\dots(m+k)x^k}$$
(i)

If m > -k-1, this gives as a particular case

 $\sum_{m} C_{n}/(n+1)(n+2)$  . . . (n+k) =

$${2^{m+k}-1-\sum_{m=k}^{s-k-1} {m+k \choose s}/(m+1)(m+2) . . . (m+k)}$$
 (5)

The formulæ (1), (2), (3), (1), and (5) contain of course a considerable variety of particular cases.

Example 5. Evaluate  $\sum_{n=0}^{\infty} n^3 {}_m C_n x^n$ .

Let  $n^3 \equiv A_0 + A_1 n + A_2 n (n-1) + A_3 n (n-1)(n-2)$ , then we have the following calculation to determine  $A_0$ ,  $A_1$ ,  $A_2$ ,  $A_3$  (see chap. v., § 22).

Hence

$$\begin{split} &\sum\limits_{0}^{\infty} n_{m}^{1} C_{n} x^{n} = 0 \text{ . } \sum\limits_{0}^{\infty} m^{n} c_{n} x^{n} + 1 m x \sum\limits_{1}^{\infty} c_{m-1}^{1} C_{n-1} x^{n-1} + 3 m \left( m - 1 \right) x^{2} \sum\limits_{0}^{\infty} c_{m-2}^{1} C_{n-2} x^{n-2} \\ &+ m \left( m - 1 \right) \left( m - 2 \right) x^{3} \sum\limits_{0}^{\infty} c_{n-2}^{1} C_{n-1} x^{n-3}, \end{split}$$

 $= mx (1+x)^{m-1} + 3m (m-1) x^2 (1+x)^{m-2} + m (m-1) (m-2) x^3 (1+x)^{m-2},$   $= \{m^7x^3 + m (3m-1) x^2 + mx\} (1+x)^{m-3},$ 

Example 6. Evaluate  $\sum_{m} C_n x^n/(n+2) (n+4)$ .

$$\frac{{}_{m}C_{n}x^{n}}{\left(n+2\right)\left(n+3\right)} = \frac{\left(n+1\right)\left(n+3\right){}_{m+1}C_{n+4}x^{n+4}}{x^{4}\left(m+1\right)\left(m+2\right)\left(m+3\right)\left(m+3\right)},$$

$$\left(n+1\right)\left(n+3\right) \equiv n^{2} + 4n + 3,$$

$$=A_{0}+A_{1}\left(n+1\right)+A_{2}\left(n+4\right)\left(n+3\right).$$
 
$$=4\begin{bmatrix}1&+4&+3\\0&-4&+0\\1&+01+3\end{bmatrix}$$
 
$$A_{0}=3,$$

$$A_1 = -3, A_2 = 1,$$

We therefore have

### Exercises IX.

Expand each of the following in ascending powers of x to 5 terms; and in each case write down and simplify the coefficient of  $x^r$ .

(1.)  $(1+x)^{3/2}$ .

(2.)  $(1-x)^{-1}$ :

(3.)  $(1-x)^{-34}$ .

(4.)  $(2 - \frac{1}{2}x)^{2/3}$ . (7.) 2/(1 - nx). (5.) (a + 3x)<sup>1/3</sup>.
 (8.) 1/(1 - 3x<sup>2</sup>)<sup>1/3</sup>.

(6.) <sup>3</sup>(a<sup>3</sup> − x<sup>3</sup>),
(9.) (x = 1/x) · n

(10.) Write down the first four terms in the expansion of  $\{(a+x)/(a-x)\}^{1/2}$  in ascending powers of x.

Determine the numerically greatest term in

(11.)  $(3+x)^{2/3}$ , x < 3.

3. (12.) (2 - 3/2)<sup>11/2</sup>.

(13.)  $(1 - 5/7)^{-13/5}$ .

(14.) Find the greatest term in  $(1+x)^{-n}$ , when  $x=\frac{s}{7}$ , n=4.

(15.) If n be a positive integer, find the greatest term in (n - 1/n)<sup>2n+1</sup>.

(16.) The sum of the middle terms of (1+x)<sup>m</sup> for all even values of m (including 0) is (1 − 4x)<sup>-1,2</sup>.

(17.) 
$$x^n = 1 + n\left(1 - \frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1 - \frac{1}{x}\right)^2 + \dots$$

(18.) Show that, if m exceed a certain value, then  $2^{m} = 1 + \frac{(m+1)m}{2} + \frac{(m+1)m(m-1)(m-2)}{1} + \dots$ 

(19.) Sum the series

$$a - (a + b) m + (a + 2b) {m(m - 1) \over 2!} - (a + 3b) {m(m - 1)(m - 2) \over 2!} + \dots$$

for such values of m as render the series convergent.

(20.)  $\sqrt{27} = 2 + \frac{5}{2!3} + \frac{5 \cdot 7}{3!3^2} + \dots$ 

(21.) 
$$\frac{23}{24} - \frac{2}{3}\sqrt{2} = \frac{1}{2 \cdot 3!} - \frac{1 \cdot 3}{2^4 \cdot 4!} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 5!} - \cdots$$

(22.) Sum to infinity

$$\frac{1}{6} + \frac{1 \cdot 4}{6 \cdot 12} + \frac{1 \cdot 4 \cdot 7}{6 \cdot 12 \cdot 15} + \dots$$

(23.) Sum the series

$$m(m-1) + \frac{m(m-1)(m-2)}{1!} + \dots + \frac{m(m-1)\dots(m-r+1)}{(r-2)!} + \dots$$

for such values of m as render the series convergent.

(21.) If n be even, show that

n(n+2) . . . (2n-2)/1 , 3 . . .  $(n-1)=2^{n-1}$ .

(25.) In the expansion of  $(1-x)^{-m}$  no coefficient can be equal to the next following unless all the coefficients are equal.

(26.) Prove by induction that

$$1 + m + \frac{m(m+1)}{2!} + \dots + \frac{m(m+1) \cdot \dots \cdot (m+r-1)}{r!} = \frac{(m+r)!}{m! \cdot r!}$$

where r is a positive integer. Hence show that, if x < 1,

$$(1-x)^{-m} = \sum \frac{(m+r-1)! x^r}{(m-1)! r!}$$
.

(27.) The sum of the first r coefficients in  $1/\sqrt[r]{(1-x)}$ ; the coefficient of the rth term = 1 + n(r-1); 1.

(28.) If 
$$F(a) = 1 + \frac{ax}{1!} + \frac{a(a+k)}{2!}x^2 + \frac{a(a+k)(a+2k)}{3!}x^3 + \dots$$
, the series

being absolutely convergent, then

$$F(a) F(b) = F(a + b),$$

What is the condition for the convergency of the series?
(29.) Show that

$$\frac{x^2}{2} - {}_{n}C_{1}\frac{x^3}{3} + {}_{n}C_{2}\frac{x^4}{6} - \dots = [1 - \{(n+1)x + 1\}(1-x)^{n+1}]/(n+1)(n+2),$$

Sum the following series, so far as they are convergent:-

(30.)  $\Sigma (n-1)^2 m(m-1)$  . . .  $(m-n+1)x^n/n!$ , from n=1 to  $n=\infty$ .

(4.1.)  $\Sigma(-1)^{n-1}(n+1)(n+2)(1,3,5,...,(2n-5)x^n/n!, \text{ from } n=0 \text{ to } n=\infty$ .

(32.)  $\sum m(m+1) \dots (m+n-1)x^n/(n+3)n!$ , from n=0 to  $n=\infty$ .

(33.)  $\Sigma (n-1)^2 1 \cdot 1 \cdot 7 \cdot \dots \cdot (3n-2)/(n+2) (n+3) n l$ , from n=1 to  $n=\infty$ .

(34.) Why does the method of summation given in § 5 not apply to  $\frac{7}{2}\pi^n, (n+1)$ ?

## SERIES DEDUCED BY EXPANSION OF RATIONAL FUNCTIONS OF x.

[56.] Since every rational function of x can be expressed in the form I · F, where I is an integral function of x, and F a proper rational fraction, and since F can, by chap VIII., § 7, be expressed in the form  $\Sigma A(x-a)^{-n}$ , where A is constant, it follows that for certain values of x a rational function of x can be expanded in a series of ascending powers of x, and for certain other values of x in a series of descending powers of  $x^*$ . We shall have occasion to dwell more on the general consequences of this result in a later chapter, where we deal with the theory of Reenring Series. There are, however, certain particular cases which may with advantage be studied here.

§ 7.] Series for expressing  $a^n + \beta^n$  and  $(a^{n+1} - \beta^{n+1})/(a - \beta)$  in

terms of  $\alpha\beta$  and  $\alpha + \beta$ , n being a positive integer.

If we denote the elementary symmetric functions  $\alpha + \beta$  and  $\alpha\beta$  by p and q respectively, it follows from chap. XVIII.,  $\S 2$ , that we can express the symmetric functions  $\alpha^n + \beta^n$ ,  $(\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta)$  as follows:—

$$a^n + \beta^n = a_0 p^n + a_1 p^{n-2} q + \dots + a_r p^{n-2r} q^r + \dots$$
 (1),  
 $(a^{n+1} - \beta^{n+1})/(a - \beta) = b_0 p^n + b_1 p^{n-2} q + \dots + b_r p^{n-2r} q^r + \dots$  (2),

where both series terminate.

By the methods of chap. VIII., § 8, or by direct verification we can establish the identity

$$\frac{2 - px}{1 - px + qx^2} \equiv \frac{2 - (\alpha + \beta) \cdot x}{(1 - \alpha x) \cdot (1 - \beta x)} \equiv \frac{1}{1 - \alpha x} + \frac{1}{1 - \beta x}$$
(3).

Now if x be (as it obviously always may be) taken so small that  $px-qx^2<1$ , we have by the Binomial Theorem

$$\frac{2 - px}{1 - px + qx^2} = (2 - px) \left\{ 1 - (px - qx^2) \right\}^{-1} = (2 - px) \left\{ 1 + (px - qx^2) + (px - qx^2)^2 + \dots + (px - qx^2)^n + \dots \right\}$$
(4).

Now (by chap. xxvl., § 34) if x be taken between -a and +a, a being such that the numerical value of  $\pm pa\pm qc^3<1$ , that arrangement of signs being taken which makes  $\pm pa\pm qa^3$  greatest, the each of the terms on the right-hand side may be expanded in powers of x and the whole rearranged as a convergent series proceeding by ascending powers of x.

Strictly speaking, this is as yet established only for cases where a is real. The cases where a is imaginary will, however, be covered by the extension of the Binomial Theorem given in chap. XXIX.

We thus find that

$$\frac{2 - px}{1 - px + qx^2} = (2 - px) \left\{ 1 + \sum \left( p^n - {n - 1 \choose 1} p^{n-2} q + {n - 2 \choose 2} p^{n-4} q^2 - \dots + \left( - \right)^n {n - r \choose r} p^{n-2r} q^r + \dots \right\} x^n \right\}$$
(5),

$$= 2 \{1 + \Sigma \&c.\} - px \{1 + \Sigma \&c.\}$$
(6).

The coefficient of  $x^n$  on the right-hand side of (6) is

$$\begin{split} 2 &\{p^n - {}_{n-1}C_1p^{n-2}q + {}_{n-2}C_2p^{n-4}q^2 + \dots + (-)^n {}_{n-r}C_rp^{n-2r}q^r \\ &+ \dots \} = p &\{p^{n-1} - {}_{n-2}C_1p^{n-2}q + {}_{n-2}C_2p^{n-2}q^2 + \dots \\ &+ (-)^r {}_{n-r-1}C_rp^{n-2r-1}q^r + \dots \}. \end{split}$$

Now

$$2_{n-r}C_r - {n-r-1}C_r = n\left(n-r-1\right)\left(n-r-2\right)$$
 . . .  $(n-2r+1)/r!$  . Hence

$$\frac{2-px}{1-px+qx^2} = 2 + \sum \left\{ p^n - \frac{n}{1!} p^{n-2} q + \frac{n(n-3)}{2!} p^{n-4} q^2 - \dots + (-)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!} p^{n-2r} q^r + \dots \right\} x^n (7).$$

Again

$$\frac{1}{1-ax} + \frac{1}{1-\beta x} = \{1 + ax + a^2x^3 + \dots + a^nx^n + \dots \} + \{1 + \beta x + \beta^2x^2 + \dots + \beta^nx^n + \dots \},$$

$$= 2 + \sum (a^n + \beta^n)x^n$$
(8).

All the series involved in (s) will be absolutely convergent, provided x be taken so small that |ax| and  $|\beta x|$  are each <1. Now, by (3), the series in (7) and (8) must be identical. Hence, comparing the coefficients of  $x^n$ , we must have (by chap. xxvi., § 21)

$$a^{n} + \beta^{n} - p^{n} - \frac{n}{1!} p^{n-2} q + \frac{n(n-3)}{2!} p^{n-4} q^{2} - \dots + (-1)^{r} \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!} p^{n-2r} q^{r} + \dots$$
(9).

As we have indicated (by using  $\equiv$ ), the equation (9) is an algebraical identity, on the understanding that p stands for  $\alpha + \beta$ 

and a for aB. The last term will or will not contain p according as a is odd or even.

In like manner, from the identity

$$\frac{x}{1 - px + qx^2} \equiv \frac{x}{1 - (\alpha + \beta)x + \alpha\beta x^2} \equiv \left\{ \frac{1}{1 - ax} - \frac{1}{1 - \beta x} \right\} \frac{1}{a - \beta}$$

we deduce

$$(a^{n+1}-\beta^{n+1})/(a-\beta) \equiv p^n - \frac{n-1}{1!} p^{n-2}q + \frac{(n-2)(n-3)}{2!} p^{n-4}q^2 - \dots + (-1)^r \frac{(n-r)(n-r-1)}{r!} \dots (n-2r+1) p^{n-2r}q^r + \dots (10),$$

subject to the same remarks as (9).

If we write the series (9) in the reverse order, and observe that, when n is even, = 2m say, only even powers of p occur, and that the term which contains p<sup>2s</sup> is

$$(-)^{m-s} \frac{2m(m+s-1)(m+s-2)...(2s+1)}{(m-s)!} p^{2s} q^{m-s},$$
 that is,

$$(-)^{m-s} \frac{2m(m+s-1)(m+s-2)\dots(m+1)m(m-1)\dots(m-s+1)}{(2s)!} p^{2s}q^{m-s},$$

that is.

$$(-)^{m-s}2\frac{m^2(m^2-1^2)\dots(m^2-s-1^2)}{(2s)!}p^{ns}q^{ms-s},$$

then we have

$$\alpha^{2m} + \beta^{2m} \equiv (-)^m 2 \left\{ q^m - \frac{m^2}{2!} p^2 q^{m-1} + \frac{m^2 (m^2 - 1^2)}{4!} p^4 q^{m-3} - \dots \right\}$$

$$+(-)^{s}\frac{m^{2}(m^{2}-1^{2})...(m^{2}-s-1^{2})}{(2s)!}p^{2s}q^{m-s}-...$$
 (9').

Similarly, we have

$$a^{2m+1} + \beta^{2m+1} \equiv (-)^m (2m+1) \left\{ p \cdot q^m - \frac{(m+1)m}{3!} p^2 q^{m-1} + \frac{(m+2)m(m^2-1^2)}{5!} p^2 q^{m-2} - \dots \right\}$$
  
 $+ (-)^{s-1} \frac{(m+s-1)m(m^2-1^2) \dots (m^2-\overline{s-2}^2)}{(2s-1)!} p^{2s-1} q^{m-s+1} \dots \right\}$ 
  
 $(9^n)$ .

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$$\frac{a^{2n} - \beta^{2m}}{a - \beta} \equiv (-)^{n-1} \left\{ \prod_{1}^{m} pq^{m-1} - \frac{m(m^2 - 1^2)}{3!} p^3 q^{m-1} + \dots \right. \\
+ (-)^{s-1} \frac{m(m^2 - 1^2) \cdot \dots \cdot (m^2 - s - 1^2)}{(2s - 1)!} p^{5g-1} q^{m-s} + \dots \right\} (10^5).$$

$$\frac{a^{2m+1} - \beta^{2m+1}}{a - \beta} \equiv (-)^m \left\{ q^m - \frac{(m+1)m}{2!} p^2 q^{m-1} + \frac{(m+2)m(m^2 - 1^2)}{4!} \right. \\
p^s q^{m-1} - \dots + (-)^s \frac{(m+s)m(m^2 - 1^2) \cdot \dots \cdot (m^2 - s - 1^2)}{(2s)!} \\
p^{2s} q^{m-s} + \dots \right\} (10^n).$$

Since a and  $\beta$  are the roots of the quadratic function  $z^z - pz + q$ , we may replace a and  $\beta$  in the above identities by  $\frac{1}{2} (p + \sqrt{(p^2 - 4q)})$ , and  $\frac{1}{2} \{p - \sqrt{(p^2 - 4q)}\}$  respectively. If this be done, and we at the same time put p = x and  $-4q = y^2$ , we deduce the following:—

$$\begin{cases} x + \sqrt{(x^2 + y^2)!^n} + \{x - \sqrt{(x^2 + y^2)!^n} \\ \equiv 2^n \left\{ x^n + \frac{n}{1! \ 2!} \ x^{n-2} y^2 + \frac{n \ (n-3)}{2! \ 2!} \ x^{n-4} y^4 + \dots \right. \\ + \frac{n \ (n-r-1) \ (n-r-2) \dots \ (n-2r+1)}{r! \ 2^{r}} \ x^{n-2r} y^{rr} + \dots \right\},$$

$$\equiv 2 \left\{ y^n + \frac{n^2}{2!} \ x^2 y^{n-2} + \frac{n^2 \ (n^2 - 2^2)}{4!} \ x^4 y^{n-4} + \dots \right. \\ + \frac{n^2 \ (n^2 - 2^2) \ (n^2 - 4^2) \dots \ (n^2 - 2s - 2^2)}{(2s)!} \ x^2 y^{n-r} + \dots \right\},$$
if  $n$  be even;
$$\equiv 2 \left\{ nxy^{n-1} + \frac{n \ (n^2 - 1^2)}{3!} \ x^3 y^{n-r} + \frac{n \ (n^2 - 1^2) \ (n^2 - 3^2)}{(2s + 1)!} \right\},$$

$$x^{2s+1} y^{n-2s-1} + \dots + \frac{n \ (n^2 - 1^2) \ (n^2 - 3^2)}{(2s + 1)!} \dots \left(n^{r-2s-1^2} \right)$$

$$x^{2s+1} y^{n-2s-1} + \dots \right\},$$
 if  $n$  be odd.

§ 7, 8 SERIES FOR 
$$\{x + \sqrt{(x^2 + y^2)}\}^n - \{x - \sqrt{(x^2 + y^2)}\}^n$$
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$$\begin{cases} x + \sqrt{(x^2 + y^2)}\}^n - \{x - \sqrt{(x^2 + y^2)}\}^n \\ \equiv 2^n \sqrt{(x^2 + y^2)} \left\{ x^{n-1} + \frac{(n-2)}{1! \, 2^2} \, x^{n-2} y^2 + \frac{(n-3)}{2! \, 2^4} \right\} \\ x^{n-5} y^4 + \dots + \frac{(n-r-1)(n-r-2)\dots(n-2r)}{r! \, 2^2} \\ x^{n-2r-1} y^{2r} + \dots \right\},$$

$$\equiv 2\sqrt{(x^2 + y^2)} \left\{ \frac{n}{1!} \, xy^{n-2} + \frac{n(n^2 - 2^2)}{3!} \, x^2 y^{n-4} + \dots \right\},$$

$$\text{if } n \text{ be even;}$$

$$\equiv 2\sqrt{(x^2 + y^2)} \left\{ y^{n-1} + \frac{(n^2 - 1^2)}{2!} \, x^2 y^{n-2} + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!} \right\},$$

$$\text{if } n \text{ be even;}$$

$$\equiv 2\sqrt{(x^2 + y^2)} \left\{ y^{n-1} + \frac{(n^2 - 1^2)(n^2 - 3^2) \dots (n^2 - 2s - 1^2)}{2!} \right\},$$

$$x^2 y^{n-5} + \dots + \frac{(n^2 - 1^2)(n^2 - 3^2) \dots (n^2 - 2s - 1^2)}{(2^2s)!}$$

$$x^2 y^{n-2r-1} + \dots \right\}, \text{ if } n \text{ be odd.}$$

These series are important in connection with the theory of the circular and hyperbolic functions.

§ 8.] A slight extension of the method of last paragraph enables us to find expressions for the sum and for the number of r-ary products of n letters (repetition of each letter being allowed).

The inverse method of partial fractions gives us the identity  $1/(1-a_0x)(1-a_0x)\dots(1-a_0x)\dots(1-a_0x)=\sum_{s=1}^n (1-a_sx)^{-1}$  (1).

where 
$$A_s = a_s^{n-1}/(a_s - a_1)$$
 ( $a_s - a_2$ ). . .  $(a_s - a_n)$ .

Also, since  $(1-a_vx)^{-1}=1+\Sigma a_v^xx^x$ , we have (by chap. XXVI., § 14), provided x be taken small enough to secure the absolute convergency of all the series involved,

$$1/(1 - \alpha_1 x) (1 - \alpha_2 x) ... (1 - \alpha_n x)$$
  
=  $(1 + \sum \alpha_1^r x^r) (1 + \sum \alpha_2^r x^r) ... (1 + \sum \alpha_n^r x^r)$  (2),  
=  $1 + \sum_n K_r x^r$  (3),

where  ${}_{n}K_{r}$  is obviously the sum of all the r-ary products of  $a_{1}, a_{2}, \ldots a_{n}$ . Since the coefficients of  $x^{r}$  on the right-hand sides of (1) and (3) must be equal, we have

$${}_{n}K_{r} = \sum a_{s}^{n+r-1}/(a_{s} - a_{1})(a_{s} - a_{2}) ...(a_{s} - a_{n})$$
 (4).

If, for example, there be three letters, a1, a2, a3, we have

$${}_{3}h'_{r} \equiv \frac{\alpha_{1}^{r+2}}{(\alpha_{1} - \alpha_{1})(\alpha_{1} - \alpha_{2})} + \frac{\alpha_{2}^{r+2}}{(\alpha_{2} - \alpha_{1})(\alpha_{2} - \alpha_{2})} + \frac{\alpha_{3}^{r+2}}{(\alpha_{3} - \alpha_{1})(\alpha_{3} - \alpha_{2})}$$

$$\equiv -\frac{\alpha_{1}^{r+2}(\alpha_{2} - \alpha_{2}) + \alpha_{2}^{r+2}(\alpha_{3} - \alpha_{1}) + \alpha_{3}^{r+2}(\alpha_{1} - \alpha_{2})}{(\alpha_{2} - \alpha_{3})(\alpha_{3} - \alpha_{1})(\alpha_{1} - \alpha_{2})}$$
(5).

If we put  $a_1 = a_2 = \dots = a_n = 1$ , then each of the terms in  ${}_n K'_r$  reduces to 1, and  ${}_n K'_r$  becomes  ${}_n H_r$ . Hence, from (3),

$$(1-x)^{-n} = 1 + \sum_{n} II_{r} x^{r}$$
 (6).

Equating coefficients of  $x^r$  on both sides of (6), we have  $H_r = n(n+1) \dots (n+r-1)/r^r$ 

a result already found by another method in chap, XXIII., § 10.

§ 9.] Some interesting results can be obtained by expanding 1/(y+x)(y+x+1), . . . (y+x+n) in descending, and in ascending powers of y.

If we write

$$1/(y+x)(y+x+1)$$
... $(y+x+n)=\sum_{n=0}^{r-1}A_r(y+x+r)^{-1}$ 

then we find, by the method of chap. VIII., § 6, that

$$1 = A_r(-r)(-r+1) \dots (-1) 1 \cdot 2 \dots (n-r).$$

Hence  $A_r = (-)^r {}_n C_r/n!$ .

Therefore

$$n!/(y+x)(y+x+1)...(y+x+n) = \sum_{n=0}^{\infty} (-)^{r} {n \choose r} (y+x+r)^{-1}$$
 (1).

Hence, if  $P_1, P_2, P_3, \ldots$  denote respectively the sum of  $x, x+1, \ldots, x+n$ , and of their products taken  $2, 3, \ldots$  at a time (without repetition), we have

$$\frac{n!}{\bar{y}^n} \left\{ 1 + \frac{P_1}{y} + \frac{P_2}{y^2} + \dots \right\}^{-1} = \Sigma \left( -\right)^n C_r \left( 1 + \frac{x + r}{y} \right)^{-1},$$

$$\frac{n!}{y^n} \left\{ 1 - \left( \frac{P_1}{y} + \frac{P_2}{\bar{y}^2} + \dots \right) + \left( \frac{P_1}{y} + \frac{P_2}{\bar{y}^2} + \dots \right) - \dots \right\}$$

$$= \Sigma \left( -\right)^n C_r \left\{ 1 + \Sigma \left( -\right)^r {\binom{x + r'}{y}} \right\}$$
(2),

where we suppose y to have a value so large that all the series involved are convergent.

Since there is no power of 1/y less than the nth on the left of (2), the coefficient of any such power on the right must vanish. Therefore

$$(x+n)^s - {}_nC_1(x+n-1)^s + {}_nC_2(x+n-2)^s - \dots (-)^nx^s = 0$$
 (3), where s is any positive integer  $< n$ .

Equating coefficients of  $1/y^n$ ,  $1/y^{n+1}$ , and  $1/y^{n+2}$ , we find

$$(x + n)^n - aC_1(x + n - 1)^n + aC_2(x + n - 2)^n - ...$$
  
 $(-)^n x^n = n!$  (1);  
 $(x + n)^{n+1} - aC_1(x + n - 1)^{n+1} + aC_2(x + n - 2)^{n+1} - ...$   
 $(-)^n x^{n+1} = n! P_1,$   
 $= (n + 1)!(x + l, n)$  (5)

$$= (n+1)! (x + \frac{1}{2}n)$$

$$(x+n)^{n+2} - {}_{n}C_{1}(x+n-1)^{n+2} + {}_{n}C_{2}(x+n-2)^{n+2} + \dots$$

$$(-)^{n} x^{n+2} = n! (P_{1}^{2} - P_{2}),$$

$$= \frac{1}{2} (n+2)! \{x^2 + nx + \frac{1}{12}n (3n+1)\}$$
 (6);

and so on.

Again from (1) we have

$$\frac{n!}{x(x+1)\dots(x+n)}\{1+(\ell_{1}y+\ell_{2}y^{2}+\dots\}^{-1}\}$$

$$=\sum_{r=0}^{r-n}(-)^{r}\frac{n}{x+r}\{1+\frac{y}{x+r}\}^{-1}$$
(7),

where  $Q_1$ ,  $Q_2$ ,  $Q_3$ , ... are respectively the sum of 1/x, 1/(x+1), ..., 1/(x+n), and the sums of their products taken 2, 3, ... at a time. From (7), by expanding and equating coefficients of y, we get

$$\frac{n!}{x(x+1)\dots(x+n)} \left\{ \frac{1}{x} + \frac{1}{x+1} + \dots + \frac{1}{(x+n)} \right\}$$

$$= \frac{1}{x^3} - \frac{nC_1}{(x+1)^2} + \frac{nC_2}{(x+2)^2} - \dots + (-)^n \frac{1}{(x+n)^2}$$
 (8).

If we put x=1, we get the following curious relation between the sum of the reciprocals of  $1, 2, \ldots, n+1$ , and the reciprocals of their squares:—

$$\frac{1}{n+1} \left\{ \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n+1} \right\} = \frac{1}{1^2} = \frac{n'}{2^2} + \frac{n'^2}{3^2} = \dots$$

$$(-)^n \frac{1}{(n+1)^2} \tag{9}.$$

§ 10.] We have now exemplified most of the elementary processes used in the transformation of Binomial Series. The following additional examples may be useful in helping the student to thread the intricacies of this favourite field of exercise for the tyro in Mathematics.

Example 1. Find the coefficient of  $x^n$  in the expansion of  $(1-x)^2/(1+x)^{3/6}$  in a-cending powers of x.

If  $(1+x)^{-2}=1+2a_nx^n$ , then  $(1-x)^2/(1+x)^{2}=(1-2x+x^2)(1+2a_nx^n)$ . Hence the coefficient required is  $a_n-2a_{n-1}+a_{n-2}$ . If we substitute the actual values of  $a_{n-1}a_{n-1}, a_{n-2}$ , we find that

$$a_n - 2a_{n-1} + a_{n-2} = (-)^n (16n^2 - 8n - 1) \frac{3 \cdot 5}{2 \cdot 4 \cdot 6} - \frac{(2n - 3)}{2n}$$

Example 2. If  $f(x) = a_0 + a_1 x + a_0 x + \dots$ , then the coefficient of  $x^r$  in the expansion of  $f(x)/(1-x)^m$  in ascending powers of x is  $a_{0-m}H_r + a_{1-m}H_{r-2} + \dots + a_r$ . This follows at once from the equation

 $f(x)/(1-x)^m = (a_0 + \sum i_r x^r) (1 + \sum_m H_r x^r),$ 

In particular, if we put 
$$f(r) \equiv (1-x)^{-n}$$
 and  $m=1$ , we deduce that
$$\lim_{n \to 1} H_r = \lim_n H_{r+1} + \lim_{n \to 1} H_{r+1} + \dots + 1;$$

and, if we put  $f(r) \equiv (1-x)^{-n}$ , we deduce that

Now

$$= m + n H_r = m H_r + m H_{r-1 \ n} H_1 + m H_{r-2 \ n} H_2 + \dots + n H_r$$
,

results which have already appeared, in the particular case where m and n are integral (see chap, xxiii., § 10).

Example 3. Show that

$$_{\mathbf{m}}C_{\mathbf{n}}/2 + _{\mathbf{m}+1}c_{\mathbf{n}}/2^{2} + _{\mathbf{m}+2}C_{\mathbf{n}}/2^{3} + \ldots$$
 and  $\mathbf{z} = \mathbf{1} + _{\mathbf{m}}C_{1} + _{\mathbf{m}}C_{2} + \ldots + _{\mathbf{m}}C_{\mathbf{n}}$  (1).  
The left-hand side of (1) is obviously the coefficient of  $\mathbf{z}^{n}$  in

$$X = (1+x)^m/2 + (1+x)^{m+1}/2^2 + (1+x)^{m+1}/2^3 + \dots$$
 and  $x = 1$ 

$$X = \frac{1}{2} (1 + x)^m \{1 + \{(1 + x), 2\} + \{(1 + x)/2\}^2 + \dots \text{ ad } \alpha \},$$
  
=  $(1 + x)^m/2 \{1 - (1 + x)/2\}$ , if we suppose  $x < 1$ .

$$=(1+x)^{m}/(1-x),$$

$$=1+\Sigma\left(1+{}_{\mathbf{m}}C_1+{}_{m}C_2+\ldots\; .\; +{}_{m}C_n\right)x^n,$$

by last example. Hence the theorem follows.

Example 4. Sum the series

$$S = 1 - \frac{n-3}{2!}$$
,  $\frac{(n-4)(n-5)}{3!} - \frac{(n-5)(n-6)(n-7)}{4!} + \dots$ 

a being a positive integer.

The equations (9''') of § 7 being also-braical identities, we may substitutherein any values of x and y we choose, so long as no ambiguity arises in the determination of the functions involved. We may, for example, put x = -1 and y = 2i. We thus find

$$\left\{\frac{-1+\sqrt{3}i}{2}\right\}^n + \left\{\frac{-1-\sqrt{3}i}{2}\right\}^n \equiv (-)^n\{1-nS\}.$$

Hence, if  $\omega$  and  $\omega^2$  denote, as usual, the two imaginary cube roots of +1, we have

$$S = \{1 + \ell + (n^{-1} \ell \omega^n + \omega^{2n})\}/n.$$

If we evaluate  $\omega^n + \omega^{cn}$  for the four cases where n has the forms 6m,  $6m \pm 1$ ,  $6m \pm 2$ ,  $6m \pm 3$  (remembering that  $\omega^{6m} = 1$ ,  $\omega^{-1} = \omega^2$ ,  $\omega^{-2} = \omega$ ), we find that S has the values -1/n, 0, 2/n, and 3/n respectively.

Example 5. Snm the series

$$S = 1 + \frac{n\left(n - 1\right)}{2\left(2r + 1\right)} + \frac{n\left(n - 1\right)\left(n - 2\right)\left(n - 3\right)}{2 \cdot 4\left(2r + 1\right)\left(2r + 3\right)} + \frac{n\left(n - 1\right)\left(n - 2\right)\left(n - 3\right)\left(n - 3\right)\left(n - 4\right)\left(n - 3\right)}{2 \cdot 4 \cdot 6\left(2r + 1\right)\left(2r + 3\right)\left(2r + 5\right)}$$

n being a positive integer.

If we denote the series by  $1+u_1+u_2+u_3+\ldots$ , then

$$\begin{aligned} u_s &= \frac{n \left( n - 1 \right) \dots \left( n - 2s + 1 \right)}{2 \cdot 4 \dots 2s \cdot \left( 2r + 1 \right) \left( 2r + 3 \right) \dots \left( 2r + 2s - 1 \right)}, \\ &= \frac{n! \left( 2r)! \left( r + 1 \right) \left( r + 2 \right) \dots \left( r + s \right)}{\left( n - 2s \right)! \left( 2r + 2s \right)! s!}, \end{aligned}$$

restricting r for the present to be a positive integer. We may therefore write

$$u_s = \frac{n! (2r)!}{(n+2r)!} {}_{n+2r} C_{2r+2s} \cdot {}_{r+s} C_s$$

Now  $_{r+s}C_s$  is the coefficient of  $x^{2r}$  in the expansion of  $x^{2r+2s}(1+1)x^s)^{r+s}$ ; that is, in the expansion of  $x^{2r+2s}(\sqrt{(1+1/x^s))^{-2r+2s}}$ . Hence  $2u_s$  is one part of the coefficient of  $x^{2r}$  in the expansion of

$$\frac{n! (2r)!}{(n+2r)!} [\{1+x\sqrt{(1+1/x^2)}\}^{n+2r} + \{1-x\sqrt{(1+1/x^2)}\}^{n+2r}].$$

Hence 2S is the whole coefficient of  $x^{2r}$  in the expansion of

$$\frac{n! \; (2r)!}{(n+2r)!} \big[ \big\{ 1 + \sqrt{(1+x^2)} \big\}^{n+2r} + \big\{ 1 - \sqrt{(1+x^2)} \big\}^{n+2r} \big].$$

Now, by § 7,

 $\{1+\sqrt{(1+x^2)}\}^{n+2r} + \{1-\sqrt{(1+x^2)}\}^{n+2r}$ 

$$= 2^{u+2r} \left\{ 1 + \sum \frac{(n+2r)(n+2r-s-1)(n+2r-s-2) \cdot \cdot \cdot \cdot (n+2r-2s+1)}{(s)!} \cdot \frac{x^{u_0}}{(s)!} \right\}$$

the coefficient of  $x^{2r}$  in which is

$$\frac{\left(n+2r\right)\left(n+r-1\right)\left(n-r-2\right)}{r!}\,2^{\frac{r}{2r}}\,. \quad . \quad . \quad (n+1)$$

Hence

$$\begin{split} S &= 2^{n+2r-1} \frac{n! \left(2r\right)! \left(n+2r\right) \left(n+r-1\right)!}{\left(n+2r\right)! r \left(n! 2^{2r}\right)}, \\ &= 2^{n-1} \frac{\left(n+r-1\right) \left(n+r-2\right)}{\left(n+2r-1\right) \left(n+2r-2\right)}, \quad \left(r+1\right)}{\left(n+2r-1\right) \left(n+2r-2\right)}. \end{split}$$

The summation is thus effected for all integral values of r. So far, however, as r is concerned, the formula arrived at might be reduced to an identity between two integral functions of r of finite degree. Since we have shown that this identity holds for an infinite number of particular values of r, it must (clap. v., § 16) hold for all values of r. The summation is therefore generals of ara ar is concerned.

### EXERCISES X.

Find the coefficient of  $x^r$  in the expansion of the following in ascending powers of x.

- (1.) x/(x-a)(x-b)(x-c). (2.)  $x^{r+2}/(x-a)(x-b)(x-c)$ .
- (3.)  $x^{m+3}/(x-a)(x-b)(x-c)$ , where m is a positive integer < r-3.
- (1.)  $(3-x)/(2-x)(1-x)^2$ . (5.)  $2x^2/(x-1)^2(x^2+1)$ .
- (6.)  $(1-px)^m (1-qx)^{-n}$ .
- (7.) If (1-3x)<sup>n</sup>/(1-2x)<sup>2</sup> be expanded in ascending powers of x, the coefficient of x<sup>n+r-1</sup> is (-1)<sup>n</sup> (r-2n) 2<sup>r-1</sup>, n and r being positive integers.
- (8.) Find the numerically greatest term in the expansion of  $(a-x)/(b+x)^3$  in ascending powers of x.
  - (9.) Show that
- $(x + \beta)(x + 2\beta) \dots (x + n\beta)$

$$\frac{(x-\beta)(x-2\beta)\dots(x-n\beta)}{z-1+\sum_{r=1}^{n}(-)^{n-r}}\frac{n(n+r)(n^2-1^2)(n^2-2^2)\dots(n^3-r-1^2)}{(r!)^2}\frac{r\beta}{z-r\beta}$$

and hence show that

$$\sum_{r=1}^{r-n} (-)^{n-r} \frac{rn(n+r)(n^2-1^2)(n^2-2^2)}{(r!)^4} \cdot \dots \cdot (n^2-r-1^2) = n(n+1).$$

(10.) If n be a positive integer, show that

$$1 - {}_{m}C_{1} + {}_{m}C_{2} - \ldots (-)^{n}{}_{m}C_{n} = (-)^{n}{}_{m-1}C_{n}$$

(11.) If n be an even positive integer,

 ${}_{m}C_{n} - {}_{m}C_{n-1} \cdot {}_{m}C_{1} + {}_{m}C_{n-2} \cdot {}_{m}C_{2} - \dots + {}_{m}C_{n} = (-)^{n,2} {}_{m}C_{n/2}.$ (12.) If m and n be positive integers, show that

 $\begin{array}{ll} {}_{m}C_{\bullet} & {}_{m'1}C_{n} + {}_{m}C_{2} \cdot ({}_{m-2/2}C_{n-1} + {}_{m}C_{4} \cdot ({}_{m-2/2}C_{n-2} + \dots + {}_{m}C_{2n} \cdot ({}_{m-2n/2}C_{\bullet} \\ & = \frac{m^{2} \cdot (m^{2} - 2^{2})}{(2n)!} ; \end{array} ;$ 

$$\sum_{m=1}^{m} \frac{(2n)!}{(m-1)!} \frac{(2n)!}{(2n-1)!} \frac{(2n)!}{(m-1)!} \frac{(2n)!}{(2n-1)!} \frac{(2n)!}{(m-1)!} \frac{(2n)!}{(2n-1)!} \frac$$

(2n + 1)! (See Schlomitch, Handb, d. Alg. Anal., § 38.) (13.) Show, by equating coefficients in the expansion of  $(1-x^{-1})^m(1-x)^{-m}$ , where m is a positive integer, that

$$1 - m^2 + \frac{m^2 (m^2 - 1)}{(2!)^2} + \dots + (-1)^m \frac{m^2 (m^2 - 1^2)}{(m!)^2} \cdot \dots \cdot \frac{(m^2 - m - 1^2)}{(m!)^2} = 0.$$

(14.) If n be a positive multiple of 6, then

$${}_{n}C_{1} - {}_{n}C_{3}3 + {}_{n}C_{5}3^{2} + \dots = 0;$$
  
 ${}_{n}C_{1} - {}_{n}C_{3}\frac{1}{2} + {}_{n}C_{5}\frac{1}{\cdots} - \dots = 0.$ 

$$a = 1 - a_1 \cdot a_2 \cdot \frac{1}{3} + a_1 \cdot a_2 \cdot \frac{1}{32} + \dots$$
 = 0.  
(15.) If  $(1+x)^{-3} = 1 + a_1 \cdot x + a_2 \cdot x^2 + \dots$ , sum the series  $1 - a_1 + a_2 - a_2 + \dots$ 

(15.) If  $(1+x)^{-3} = 1 + a_1x + a_2x^2 + \dots$ , sum the series  $1 - a_1 + a_2 - a_3 + \dots$  to n terms. (16.) If  $(1+x)^{2n} = 1 + a_1x + a_2x^2 + \dots$ , then  $1 - a_1^2 + a_2^2 - \dots =$ 

(16.) If  $(1 + x)^{2n} = 1 + a_1x$  $(-1)^n 2n (2n - 1) \dots (n + 1)/n!$ .

(17.) 
$$\frac{r!}{r! \cdot 1!} - \frac{2^2(r+1)!}{(r-1)! \cdot 3!} + \dots + \frac{(-1)^n 2^{2r} (2r)!}{0! \cdot (2r+1)!} = \frac{(-1)^r}{2r+1}.$$

(18.) 
$$\sum_{r=0}^{r=n} 1/4^r (r!)^2 (2n-2r)! = (4n)!/1^n \{(2n)!\}^3$$
.

(19.) Sum to n terms  $\Sigma (2n-2)!/2^{2n-1}n \{(n-1)!\}^{9}$ .

(20.) Sum the series

$$n+(n-1)\frac{1}{3}+(n-2)\frac{1\cdot 4}{3\cdot 6}+(n-3)\frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}+\cdots+\frac{1\cdot 4\cdot \cdot \cdot \cdot (3n-5)}{3\cdot 6\cdot \cdot \cdot \cdot \cdot (3n-3)}$$

(21.) Find for what values of n the following series are convergent; and show that when they are convergent their sums are as given below.

$$\frac{1}{m} - \frac{n}{1!} \frac{1}{m+1} + \frac{n(n-1)}{2!} \frac{1}{m+2} - \dots = \frac{(m-1)!}{(n+1)(n+2)\dots(n+m)};$$

 $\frac{1}{m} + \frac{n}{1!} \frac{1}{m+1} + \frac{n(n-1)}{2!} \frac{1}{m+2} + \dots = \frac{(m-1)!}{(n+1)(n+2) \dots (n+m)} \{ \frac{m+n}{m+n} C_{m-1} 2^{n+1} + \dots + (-)^{m-1} 2^{n+m} + (-)^{m} 1 \}$ 

m in both cases being a positive integer.

(22.) 
$$\sum_{s=0}^{s-n} \frac{(r+s)! (m+n-r-s-1)!}{r! \, s! (m-r-1)! (n-s)!} = \frac{(m+n)}{m! \, n!}.$$

(23.) 
$$\sum_{r=0}^{r=m} \sum_{s=0}^{s=n} \frac{(r+s)!}{r!} \frac{(m+n-r-s)!}{(m-r)!} = \frac{(m+n+1)!}{m!}.$$

(24.) The number of the r-ary products of three letters, none of which is to be raised to a power greater than the nth, where n < r < 2n, is

$$r(3n-r)+1-\frac{n}{2}n(n-1).$$

(25.) Prove, for a, b, c, that Σa<sup>r</sup>/(a − b) (a − c) ≡ 0, if r = 0, or r = 1; ≡ 1, if r = 2; and generalise the theorem.

(26.) Show that

$$\frac{a(b-c)(bc-aa')(a^m-a'^m)}{a-a'} + \frac{b(c-a)(ca-bb')(b^m-b'^m)}{b-b'}$$

$$+\frac{c}{a}(a-b)(ab-cc')(c^m-c'^m)$$

 $=(b-c)(c-a)(a-b)(bc-aa')(ca-bb')(ab-cc')S_{m-3}bb',$ where aa'=bb'=cc', and  $S_{m-3}$  is the snm of the  $\{m-3\}$ -ary products of  $\{a,b,c,a',b',c'\}$ . (Math. Trip., 1886.) (27.) If  $S_r$  be the sum of the r-ary products of the roots of the equation  $x^n + a_0 x^{n-1} + a_0 x^{n-1} + \dots + a_n = 0$ , then

$$0 = S_2 + S_1 a_1 + a_2,$$

$$0 = S_n + S_{n-1} a_1 + S_{n-2} a_2 + \dots + a_n,$$

$$0 = S_n + S_{n-1} a_1 + S_{n-2} a_2 + \dots + S_{n-n} a_n.$$

(Wronski.)

(23.) If S<sub>r</sub> be the sum of the r-ary products of n letters, P<sub>r</sub> the sum of the products r at a time, S<sub>r</sub> the sum of their rth powers, then

$$\begin{split} & \Sigma_r = nS_r - (n-1)P_1S_{r-1} + \ldots + (-1)^r(n-r)P_r, & \text{if } r < n-1, \\ & = nS_r - (n-1)P_1S_{r-1} + \ldots + (-1)^{n-1}P_{n-1}S_{r-n+1}, & \text{if } r > n-1. \end{split}$$

(29.) If  $\mathbf{r} = (1 - \alpha \mathbf{r})^{-1} (1 - \beta \mathbf{r})^{-1} \dots$ , the number of ways of distributing n things,  $\lambda$  of which are of one sort,  $\mu$  of another sort, . . . , into p boxes placed in a row is the coefficient of  $x^{-\alpha}\lambda\beta^{\mu}$ . . . in the expansion of  $\{v-1\}^p$  in ascending powers of x, namely,

$$u_1 = pC_1u_2 + pC_2u_2 + \dots,$$
  
where  $u_s = (p + \lambda - s)! (p + \mu - s)! \dots f(p - s)! \lambda! (p - s)! \mu! \dots$ 
(Math. Trip., 1888.)

(30.) With the same data as in last question, show that the whole number of ways of distributing the things when the order in which they are arranged inside each box is attended to is

$$n! (n-1)!/(n-p)! (p-1)! \lambda! \mu! \nu!$$
 . . . (Math. Trip., 1888.)

(Math. Trip., 1882.)

Show that

(31.) 
$$1+1/2+...+1/x=x^{C_1}-1x^{C_2}+1x^{C_3}-...$$

$$(32.) \ \ 1 - \frac{(m+1)}{3!} \frac{m}{2^2} + \frac{(m+2)(m+1)m(m-1)}{5!} 2^4 - \dots = \frac{(-1)^m}{2m+1}.$$

$$(33.) \quad 1 - \frac{m^2}{2!} \frac{2^2 + \frac{m^2 \left(m^2 - 1^2\right)}{4!} 2^4 - \frac{m^2 \left(m^2 - \frac{1^2\right) \left(m^2 - 2^2\right)}{6!} 2^6 + \dots = (-1)^m,$$

(34.) If m and n are both positive integers, and m > n, then

$$\frac{2^{-n}}{n!} + \frac{(m-n)\left(m-n-1\right)}{1!\left(n+1\right)!} 2^{-n-2} + \frac{(m-n)\left(m-n-1\right)\left(m-n-2\right)\left(m-n-3\right)}{2!\left(n+2\right)!} 2^{-n-4} + \frac{(m-n)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)}{2!\left(n+2\right)!} 2^{-n-4} + \frac{(m-n)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)}{2!\left(n+2\right)!} 2^{-n-4} + \frac{(m-n)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)\left(m-n-1\right)}{2!\left(n+2\right)!} 2^{-n-4} + \frac{(m-n)\left(m-n-1\right)\left(m-1\right)$$

$$\dots = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m-1)}{(m+n)!}$$

(35.) If r be a positive integer,

$$r \left. \left\{ 1 + \frac{r^2 - 1^2}{3!} x + \frac{\left(r^2 - 1^2\right)\left(r^2 - 2^2\right)}{5!} x^2 + \frac{\left(r^2 - 1^2\right)\left(r^2 + 2^2\right)\left(r^2 + 3^2\right)}{7!} x^2 + \dots \right\} \right.$$

$$=(x+2)^{r-1}-\frac{1}{r-2}C_1(x+2)^{r-2}+\frac{1}{r-2}C_2(x+2)^{r-3}-\frac{1}{r-4}C_4(x+2)^{r-7}+\cdots$$

### MULTINOMIAL THEOREM FOR ANY INDEX.

§ 11.] Consider the integral function  $a_1x + a_2x^2 + \ldots + a_rx^r$ , whose absolute term vanishes, the rest of the coefficients being real quantities positive or negative. Confining ourselves in the meantime to real values of x, we see, since the function vanishes when x = 0, that it will in all cases be possible to assign a positive quantity  $\rho$  such that for all values of x between  $-\rho$  and  $+\rho$  we shall have

$$|a_1x + a_2x^2 + \dots + a_rx^r| < 1$$
 (1).

In fact, it will be sufficient if  $\rho$  be such that

$$a\rho + a\rho^2 + \dots + a\rho^r < 1$$

where a is the numerical value of the numerically greatest among  $a_1, a_2, \ldots, a_r$ . That is, it will be sufficient if

$$a\rho (1-\rho^r)/(1-\rho) < 1$$
;

a fortiori (supposing  $\rho < 1$ ) it will be sufficient if

$$\alpha\rho/(1-\rho)<1$$
;

that is, if

$$\rho < 1/(a + 1)^*$$
(2).

 $\rho$  is, in fact, the numerically least among the roots of the two equations

$$a_r x^r + \dots + a_1 x \pm 1 = 0$$
,

as may be seen by considering the graph of  $a_rx^r + \ldots + a_1x$ . Therefore, whether m be integral or not, provided -p and -p we can always expand  $(1 + a_1x + a_2x^2 + \ldots + a_rx^r)^m$  by the form

$$1 + \sum_{m} C_{\bullet} (a_{1}x + a_{m}x^{2} + ... + a_{r}x^{r})^{s}$$
 (3)

and the series (3) will be absolutely convergent whether m be positive or negative. Hence, since  $a_1x + a_2x^2 + \ldots + a_rx^r$  is a terminating series and therefore has a finite value for all values of x positive or negative, it follows from the principle established in chap.  $\mathbf{x}\mathbf{x}\mathbf{v}\mathbf{t}_1$ , § 34, that we may arrange (3) according to powers

<sup>•</sup> This is merely a lower limit for  $\rho$ ; in any individual case it would in general be much greater.

of x, and the result will be a power series which will converge to the sum  $(1 + a_1x + a_2x^2 + \dots + a_rx^r)^m$  so long as  $-\rho < x < +\rho$ .

Since s is a positive integer, we can expand  ${}_{m}C_{s}(a_{s}x + a_{s}x^{s} + \ldots + a_{s}x^{s})^{s}$  by the formula of chap. XXIII., § 12. The coefficient of  $x^{s}$  in this expansion will be

$$\Sigma = C \cdot s! a^{-a_1} a^{-a_2} \dots a^{-a_r} / a^{-1} a^{-1} \dots a^{-1}$$

that is.

$$\Sigma a_1^{a_1} a_2^{a_2} \dots a_r^{a_r} m(m-1) \dots (m-s+1)/a_1! a_2! \dots a_r!$$
 (4),

where the summation extends over all positive integral values of  $a_1, a_2, \dots, a_r$ , including 0, which are such that

$$\frac{a_1 + a_2 + \dots + a_r = s}{a_1 + 2a_2 + \dots + ra_r = n}$$
 (5).

In order, therefore, to find the coefficient of  $x^n$  in (3) we have merely to extend the summation in (4) so as to include all values of s; in other words, to drop the first of the two restrictions in (5).

Hence, whether m be integral or not, provided x be small enough, we have

$$(1 + a_1x + a_2x^2 + \dots + a_rx^r)^n$$
  
=  $1 + \sum_{\alpha_1, \alpha_2, \dots, \alpha_r} \frac{m(m-1) \dots (m - \sum_{\alpha_1} + 1)}{a_1!} a_1^{\alpha_1} a_2^{\alpha_2} \dots a_r^{\alpha_r} x^n$  (6),

the summation to be extended over all positive integral values of  $a_1, a_2, \ldots, a_r$ , including 0, such that

$$a_1 + 2a_2 + \dots + ra_r - n$$
.

The details of the evaluation of the coefficient in any particular case are much the same as in chap XXIII., § 12. Example 2, and need not be farther illustrated. It need scarcely be added that when n is very large the calculation is tedious. In some cases it can be avoided by transforming  $1 + a_1x + a_2x^2 + \dots + a_rx^r$ before applying the Binomial Expansion, but in most cases the application of the above formula is in the end both quickest and most conducive to accuracy. Example. To find the coefficient of  $x^n$  in  $(1+x+x^2+\ldots+x^r)^m$ . We have

$$(1+x+x^2+\ldots+x^r)^m \equiv \{(1-x^{r+1})/(1-x)\}^m,$$
  
 $\equiv (1-x^{r+1})^m (1-x)^{-m},$   
 $\equiv (1-x^{r+1})^m (1+\sum_{i=1}^{m} x^i),$ 

Hence, if n < r + 1, the coefficient of  $x^n$  is simply

 $_{m}H_{n}=m\;(m+1)$  . . (m+n-1)/n!;

but, if  $n \triangleleft r + 1$ , the coefficient of  $x^n$  is

$$_{m}H_{n} - _{m}C_{1} \cdot _{m}II_{n-r-1} + _{m}C_{2} \cdot _{m}II_{n-2r-2} - . . .$$

# NUMERICAL APPROXIMATION BY MEANS OF THE BINOMIAL THEOREM.

§ 12.] The Binomial Expansion may be used for the purpose of approximating to the numerical value of  $(1 + x)^m$ . According as we retain the first two, the first three, . . . , the first n+1 terms of the series  $1 + {}_{R}C_{1}x + {}_{R}C_{2}x^{2} + \ldots$ , we may be said to take a first, a second, . . . an nth approximation to  $(1 + x)^m$ .

The principal points to be attended to are-

1st, To include in our approximation the terms of greatest numerical value; in other words, to take u so great that the numerically greatest term, at least, is included.

2nd, To take n so great that the residue of the series is certainly less than half a unit in the decimal place next after that to which absolute accuracy is required.

3rd, To calculate each of the terms retained to such a degree of accuracy that the accumulated error from the neglected digits in all the terms retained is less than a unit in the place next after that to which absolute accuracy is required.

The last condition is easily secured by a little attention in each particular case. We proceed to discuss the other two.

§ 13.) The order of the numerically greatest term.

In the case of the Binomial Series  $(1+x)^m$ , if  $\xi$  denote the numerical value of x, so that  $0 < \xi < 1$ , we have for the numerical value of the convergency-ratio  $u_{n+1}/u_n$ 

$$\sigma_{n} = \frac{m-n}{n+1} \xi_{n} \text{ or } = \frac{n-m}{n+1} \xi_{n}$$
 (1),

according as m - n is positive or negative.

Hence it is obvious, in the first place, that, if  $-1 \le m < + 1$ , that is, if m be a positive or negative proper fraction, the condition  $a_n < 1$  is satisfied from the very beginning, and the first term will be the greate-5.

If m>+1, the condition  $\sigma_n<1$  is obviously satisfied for any value of n which exceeds m; in fact, the condition will be satisfied as soon as

$$(m-n)\xi < n+1$$
,

that is,  $n > (m\xi - 1)/(1 + \xi)$  (2),

the right-hand side of which is obviously less than m. This condition is satisfied from the beginning if  $\xi < 2/(m-1)$ .

If m be  $<-1=-\mu$ , say, where  $\mu>1$ , the condition  $\sigma_n<1$  will be satisfied as soon as

$$(\mu + n)\xi < n+1$$
,

that is, 
$$n > (\mu \xi - 1)/(1 - \xi)$$
 (3).

This condition is satisfied from the beginning if  $\xi < 2/(\mu + 1)$ .

§ 14.] Upper limit of the residue. We have seen that, ultimately, the terms of a Binomial Series either (1) alternate in sign or (2) are of constant sign.

To the first of these classes belong the expansions of  $(1+x)^m$ 

and  $(1+x)^{-m}$ , where x and m are positive.

If n be greater than the order of the numerically greatest term, and in the case of  $(1+x)^n$  (see § 4) also > m, then the residue may be written in the form

$$R_{n-1} + (u_{n+1} - u_{n+2} + u_{n+3} - \dots)$$
 (1),

where  $u_{n+1}, u_{n+2}, u_{n+3}, \ldots$  are the numerical values of the various terms, and we have  $u_{n+1} > u_{n+2} > u_{n+3} > \ldots$ 

Hence, in the present case, the error committed by taking an uth approximation is numerically less than  $u_{n+1}$ . In other words,

if we stop at the term of the nth order, the following term is an unner limit for the error of the approximation.

Cor. A lower limit for the error is obviously  $u_{n+1} - u_{n+2}$ .

The expansions of  $(1-x)^m$  and  $(1-x)^{-m}$  belong to the second class of series, in which the terms are all ultimately of the same sign. It will be convenient to consider these two expansions separately.

In the case of  $(1-x)^m$ , if we take n>m, then we shall certainly include the numerically greatest term; and  $\sigma_n$ , the numerical value of the convergency-ratio, will be  $(n-m) \ x/(n+1)$ , that is,  $\{1-(m+1)/(n+1)\} \ x$ . This continually increases as n increases, and has for its limit x, when  $n=\infty$ . Hence

$$\sigma_{n+1} < \sigma_{n+2} < \ldots < x < 1$$
.

Therefore,  $u_{n+1}$ ,  $u_{n+2}$ , . . . having the same meaning as before,

$$\begin{split} R_n &= \pm \; (u_{n+1} + u_{n+2} + u_{n+3} + \dots), \\ &= \pm \; u_{n+1} \, (1 + \sigma_{n+1} + \sigma_{n+1} \sigma_{n+2} + \sigma_{n+1} \sigma_{n+2} \sigma_{n+3} + \dots). \end{split}$$

Therefore

$$|R_n| < u_{n+1}(1+x+x^2+x^3+\dots),$$
  
 $< u_{n+1}/(1-x)$  (2).

Hence the error in this case is numerically less than  $u_{n+1}/(1-x)$ , and it is in excess or in defect according as the least integer which exceeds m is even or odd (see § 4).

Cor. A lower limit for the error is obviously  $u_{n+1}/(1-\sigma_{n+1})$ , that is,  ${}_{m}C_{n+1}x^{n+1}/(1-(n+1-m).x/(n+2))$ .

In the expansion of  $(1-x)^{-m}$ , all the terms are positive; and, in order to include the greatest term, we have merely to take n > (mx-1)/(1-x).

We have, in this case,

$$\sigma_n = (n + m) x/(n + 1) = \{1 - (1 - m)/(n + 1)\} x,$$
  
=  $\{1 + (m - 1)/(n + 1)\} x.$ 

Hence, if m < 1

$$\sigma_{n+1} < \sigma_{n+2} < \ldots < x < 1$$

and an upper limit of  $R_n$  will be  $u_{n+1}/(1-x)$  as in last case, a lower limit  $\ln ing |u_{n+1}/(1-\sigma_{n+1})|$ , that is,  $_mH_{n+1}x^{n+1}/[1-(n+1+m)x/(n+2)]$ .

If m > 1.

$$1 > \sigma_{n+1} > \sigma_{n+2} > \dots > r$$

and an upper limit of  $R_n$  will be  $u_{n+1}/(1-\sigma_{n+1})$ , that is,  ${}_{m}H_{n+1}x^{n+1}/\{1-(n+1+m)x/(n+2)\}$ , a lower limit being  $u_{n+1}/(1-x)$ .

The error for  $(1-x)^{-m}$  is, of course, always in defect.

Example 1. To calculate the cube root of 29 to 6 places of decimals. The nearest cube to 29 is 27. We therefore write

$$\mathcal{J}/29 = (3^{\circ} + 2)^{1/3} = 3 (1 + 2/3^{\circ})^{1/3},$$
  
=  $u_u + u_1 + u_2 + u_3 + u_4 + \dots$ 

The first term is here the greatest; and the terms alternate in sign after  $\mathbf{m}_1$ .

Also  $\mathbf{u}_r$ , written in the most convenient form for calculating successive terms, is

$$u_r = 3 \begin{pmatrix} z_1^2 \end{pmatrix} \begin{pmatrix} z_1^4 & z_2 \end{pmatrix} \begin{pmatrix} z_1^4 & z_1 \end{pmatrix} \begin{pmatrix} z_1^2 & z_1 \end{pmatrix} \begin{pmatrix} z_1^2 & z_2 \end{pmatrix}$$
.  $\begin{pmatrix} 6r - 8 \\ z_1 & z_2 \end{pmatrix}$ .

Therefore

	+	-
$u_v =$	3 000,000,00	
$u_1 = u_0 2/81 =$ $u_2 = u_1 4/162 =$ $u_3 = u_n 10/243 =$	71,071,07	-001,828,99
$u_3 = u_2 10/24.5 \equiv u_4 = u_3 16/324 \equiv$	15,21	3,72
	3 071,119,31	001,832,71
	001,832,71	001,102,11

 $u_s = u_s 22/105$  3 072,316,03 20

Hence the error in defect, due to nealect of the residue, amounts to less than 2 in the seventh place. The error for neglect of divits does not exceed 1 in the seventh place. Therefore, the best 6-place approximation to \$29 is 3-072,317. In Barlow's Tables we find 3-072,316,8 given as the value to 7 places.

Example 2. To calculate  $(1-x)^m/(1+x+x^2)^m$  to a second approximation, x being small,

$$\begin{aligned} & (1-x)^m (1+x+r^i)^{-m} \\ &= \left\{ 1-mx + \frac{m+r-i-1}{2} x^i \right\} \times \left\{ 1-m(x+x^i) + \frac{m(m+1)}{2} x^2 \right\}, \end{aligned}$$

where we have already ne-deeted all powers of x above the second in each of the two series:

$$\begin{split} &= \left\{ 1 - mx + \frac{m\left(m-1\right)}{2} x^2 \right\} \left\{ 1 - mx + \frac{m\left(m-1\right)}{2} x^2 \right\} \text{,} \\ &= 1 + \left( - m - m \right) x + \left\{ \frac{m\left(m-1\right)}{2} + m^2 + \frac{m\left(m-1\right)}{2} \right\} x^2, \end{split}$$

where higher powers of x than  $x^2$  have again been neglected in distributing the product;

$$= 1 - 2mx + m(2m - 1)x^2$$

### Exercises XI.

(1.) The general term in the expansion of (1+x+y+xy)/(1+x+y) is  $(-1)^{m+n}(m+n-2)! x^m y^n/(m-1)! (n-1)!.$ 

Determine limits for x within which the following multinomials can be expanded in convergent series of ascending powers of x; and find the coefficients of

- (2.)  $x^4$  in  $(1-2x+x^3-3x^3)^{-1/4}$ . (3.)  $x^5$  in  $(1-3x-7x^2+x^3)^{-3/2}$ . (4.)  $x^6$  and  $x^7$  in  $(x+3x^3+5x^5+7x^7+\dots)^{-2}$ .
- (6.)  $x^r$  in  $(2 + 3x + x^2)^{-2}$ , (5.)  $x^7$  in  $(1-3x+x^3-x^5)^{-3/2}$ .
- (7.) Show that in (9a<sup>2</sup> + 6ax + 4x<sup>2</sup>)<sup>-1</sup> the coefficient of x<sup>3r</sup> is 2<sup>3r</sup> (3a)<sup>-3r-2</sup>: and that the coefficient of every third term vanishes.
  - (8.) The coefficient of x<sup>m</sup> in (1 + x x<sup>2</sup>)<sup>m</sup> (m a positive integer) is

$$1 + \frac{m(m-1)}{(1!)^2} + \frac{m(m-1)(m-2)(m-3)}{(2!)^2} + \dots$$

(9.) The coefficient of x<sup>3r+1</sup> in (1+x)/(1+x+x<sup>2</sup>)<sup>2</sup> is - (r+1).

(10.) Evaluate 2/(100/99), and 5/(1002/998), each to 10 places of decimals; and demonstrate in each case the accuracy of your approximation,

Find a first approximation to each of the following, x being small:—

(11.) 
$$\frac{\{x + \sqrt{(x^2 + 1)}\}^{2m} - \{x - \sqrt{(x^2 + 1)}\}^{2m}}{\{x + \sqrt{(x^2 + 1)}\}^{2m+1} - \{x - \sqrt{(x^2 + 1)}\}^{2m+1}},$$

(12.)  $(1+x)(1+rx)(1+rx).../(1-x)(1-x)^r(1-x)^{r^2}...$ 

(13.)  $\sqrt{(2-\sqrt{(2-\sqrt{(2-\ldots-\sqrt{(1+x)\ldots)})})}}$ ; where  $\sqrt{\ }$  is repealed n times.

(14.) If x be small compared with N2, then \( \sqrt{(N^2+x)} = N + x/1N \) + Nx/2 (2N<sup>2</sup>+x), the error being of the order  $x^4/N^7$ . For example, show that  $\sqrt{(101)} = 10_{5045}^{4045}$ , to 8 places of decimals.

(15.) If p differ from N<sup>3</sup> by less than 1 per cent. of either, then p differs

from  $\frac{2}{3}N + \frac{1}{3}p/N^2$  by less than N/90000. (Math. Trip., 1882.)

when m is a positive integer.

(16.) If  $p = N^4 + x$  where x is small, then approximately

$$\sqrt[4]{p} = \frac{51}{56} N + \frac{5}{56} p/N^3 + \frac{27}{11} Nx/(7p + 5N^4);$$

show that when N=10, x=1, this approximation is accurate to 16 places of decimals. (Math. Trip., 1886.)

- (17.) Show that  $L_{n-\infty} \{1/\sqrt{n^2+1}/(n^2+1) + \dots + 1/\sqrt{(n^2+2n)}\} = 2$ .
- (Catalan, Nouv. Ann., sec. t., t. 17.)
  (18.) Find an upper limit for the residue in the expansion of (1+x)<sup>m</sup>

## CHAPTER XXVIII.

# Exponential and Logarithmic Series.

#### EXPONENTIAL SERIES.

§ 1.] We have already attached a definite meaning to the symbol a\* when a is a positive real quantity, and x any positive or negative commensurable quantity. We propose now to discuss the possibility of expanding a\* in a series of ascending powers of x.

If we assume that a convergent expansion of  $a^x$  in ascending powers of x exists, then we can easily determine its coefficients.

For, let

$$\alpha^{2} = A_{0} + A_{1}x + A_{2}x^{2} + \dots + A_{n}x^{n} + \dots$$
 (1),

then, proceeding exactly as in chap, XXVII., § 2, we have

$$L(a^{x+h}-a^x)/h = A_1 + 2A_2x + \dots + nA_nx^{n-1} + \dots;$$

and the series on the right will be convergent so long as x lies within limits for which (1) is convergent. Now (by chap. xxv., §13)

$$L (\alpha^{x+h} - \alpha^x)/h = \alpha^x \lambda L (e^{\lambda h} - 1)/\lambda h,$$
  
=  $\lambda \alpha^x$ 

where  $\lambda = \log_e a$ , and e is Napier's Base, namely, the finite quantity  $L (1 + 1/n)^n$ . Hence

$$\lambda u^x = 1A_1 + 2A_2x + \dots + nA_nx^{n-1} + \dots$$
 (2).

Therefore, by (1),

$$\lambda (A_0 + A_1x + ... + A_{n-1}x^{n-1} + ...)$$
  
=  $(A_0 + A_1x + ... + nA_nx^{n-1} + ...)$ 

Since both the series in (3) are convergent, we must have

$$1.1_1 = \lambda A_0$$
,  $2.1_2 = \lambda A_1$ , ...,  $nA_n - \lambda A_{n-1}$ .

Using these equations, we find, successively,

 $A_1 = A_0 \lambda/1!$ ,  $A_2 = A_0 \lambda^2/2!$ , ...  $A_n = A_0 \lambda^n/n!$  (4).

Also, since, by the meaning attached to  $a^x$ ,  $a^0 = \pm 1$ , putting x = 0 on both sides of (1), we have

$$+1=2l_{o} (5).$$

Hence, finally,

$$a^{x} = 1 + \lambda x/1! + (\lambda x)^{2}/2! + \dots + (\lambda x)^{n}/n! + \dots$$
 (6).

We see, a posteriori, that the expansion found is really convergent for all values of x (chap. XXVI., § 5), and also that the series in (2) is convergent for all values of x. Our hypotheses are therefore justified.

This demonstration is subject to the same objection as the corresponding one for the Binomial Series: it is, however, interesting, because it shows what the expansion of  $a^*$  must be, provided it exist at all. We shall next give two other demonstrations, each of which supplies the deficiency of that just given, and each of which has an interest of its own.

§2.] Deduction of the Exponential from the Binomial Expansion, By the binomial theorem\*, we have, provided z be numeric-

By the binomial theorem', we have, provided z be fining ally greater than 1.

$$\begin{pmatrix} 1+\frac{1}{z} \end{pmatrix}^{ex} = 1 + zx \frac{1}{z} + \frac{zx^{e}(zx-1)}{2!} \frac{1}{z^{e}} + \dots$$

$$+ \frac{zx(zx-1)}{n!} \frac{1}{z^{e}} + \dots$$

$$+ \frac{z^{e}(1-1/zx)}{2!} + \dots + \frac{x^{e}(1-1/zx)}{n!} \frac{1}{n!} + R_{n-1}(1),$$

where

$$R_n = \frac{x^{n+1}}{(n+1)!} \frac{(1-1/zx)\dots(1-n/zx)}{(n+1)!} + \frac{x^{n+2}(1-1/zx)\dots(1-n+1/zx)}{(n+2)!} + \dots (2)$$

In what follows we have re-tricted the value of the index zx. Since z is to be ultimately made infinite, there is no objection to our supposing it always so chosen that zx is a positive integer. We then depend merely on the binomial expansion for positive integral indices. This will not affect the value of  $L(4+1)z^{1/2}$ , for it has been shown (chap, xxx, § 13) that this has the same value when z becomes + or  $-\infty$ , and whether z increases by integral or by fractional intermentate.

Suppose now x to be a given quantity; and give to n any fixed integral value whatever. Then, no matter what positive or negative commensurable value x may have, we can always choose z as large as we please, and at the same time such that zx is a positive integer, p say, where p > n. The series (2) will then terminate; and we shall have 1/zx < 2/zx < ... < n/zx < ... < (p-1)/zx < 1. With this understanding, it follows that

$$R_n < \frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \dots + \frac{x^n}{p!},$$
  
 $< \frac{x^{n+1}}{(n+1)!} \left\{ 1 + \frac{x}{n+2} + \frac{x^2}{(n+2)!} + \dots \text{ ad } x \right\},$   
 $< x^{n+1}/(n+1)! (1 - x/(n+2))!$  (3):

and we have

$$\left(1 + \frac{1}{z}\right)^{\infty} = 1 + x + \frac{x^{2}(1 - 1/p)}{2!} + \dots + \frac{x^{n}(1 - 1/p) \dots (1 - (n - 1/p))}{n!} + R_{n-1}(1).$$

where  $R_n$  satisfies the condition (3).

Now let z, and therefore also p, increase without limit (n remaining fixed as before). Then, since

$$L_{p-p}(1-1/p)$$
 . .  $(1-n-1/p)=1$ ,

we have

$$L_{t=\infty} \left( 1 + \frac{1}{z} \right)^{zz} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_n$$
 (5),

 $R_n$  being still subject to (3).

We may now, if we choose, consider the effect of increasing n. When this is done,  $x^{n+1}/(n+1)!\{1-x/(n+2)\}$  (see chap. xxv., § 15) continually diminishes, having zero for its limit when  $n=\infty$ ; we may therefore write

$$L_{z=\infty} \left(1 + \frac{1}{z}\right)^{2x} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \text{ ad } \infty \quad (6).$$

Thus the value of  $L(1+1/z)^{2\varepsilon}$  is obtained in the form of an infinite series, which converges for all values of x. For most purposes the form (5) is, however, more convenient, since it gives an upper limit for the residue of the series.

§ 3.] The conditions of the demonstration of last paragraph will not be violated if we put x=1. Hence, using e, as in chap. xxv., to denote  $\sum_{n=0}^{\infty} (1+1/z)^n$ , we have

$$e = \frac{L}{z} \left( 1 + \frac{1}{z} \right)^2 = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + R_n$$
 (7)

where

$$R_a < (n+2)/(n+1)(n+1)!$$
 (8).

This formula enables us to calculate e with comparative rapidity to a large number of decimal places. We have merely to divide 1 by 2, then the quotient by 3; and so on. Proceeding as far as n-12, we have

Here the error in the last figure owing to figures neglected in the arithmetical calculation could not exceed the carriage from  $10 \times 5$ , that is, 5. Also the residue  $R_{12} \leqslant \frac{1}{13} (1/13!) \leqslant \frac{1}{3} \cdot 0000000002 \leqslant 000000003$ , so that the neglect of  $R_{12}$  would certainly not affect the eighth place. Hence we have as the nearest 7-place approximation for e

$$e = 2.7182818$$
.

It is usual to give a demonstration that the numerical constant e is incommensurable. The ordinary demonstration is as follows:—

Let us suppose that  $\epsilon$  is commensurable, say =p/q, where p and q are finite positive integers. Then we have by (7)

$$p/q = 2 + 1/2! + \dots + 1/q! + R_{qq}$$
  
 $R_q < (q + 2)/(q + 1)^2 q!$ 

where

Hence, multiplying by q!, we get

 $p \cdot (q-1)! = I + q! R_{d}$ 

where p (q-1)! and I are obviously integral numbers. Hence  $q!R_q$  must be integral.

Now

$$q!R_q < (q+2)/(q+1)^2$$
,  
 $< (q+2)/\{q(q+2)+1\}$ ,

that is,  $q!R_a$  is a positive proper fraction.

The assumption that e is commensurable therefore leads to an arithmetical absurdity, and is inadmissible.

Another demonstration which gives more insight into the nature of this and some other similar cases of incommensurability in the value of an infinite series is as follows:—

If  $r_1, r_2, \ldots, r_n$ , . . . be an infinite series of integers given in magnitude and in order, then it can be shown (see chap.  $m, g \ge 1$ ) that any commensurable number p/q (where p and q are prime to each other, and p < q) can be expanded, and that in one way only, in the form

$$\frac{p}{q} = \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{p_3}{r_1 r_2 r_3} + \dots + \frac{p_n}{r_1 r_2 \dots r_n} + \dots$$
(9),

where  $p_1 < r_1, p_2 < r_2, \dots, p_n < r_n, \dots$ ; and that the series will always terminate when either q or all its factors occur among the factors of the integers  $r_1, r_2, \dots, r_n$ . Hence no infinite series of the form (9) can represent any vulgar fraction whose denominator consists of factors which occur among  $r_1, r_2, \dots, r_n$ .

In particular, if  $r_1, r_2, \dots, r_n$ , contain all the natural primes, and, a fortiori, if they be the succession of natural numbers (excepting 1), namely, 2, 3, 4, 5, ...,  $\overline{n+1}$ , ..., then the series in (9) cannot represent any commensurable number at all.\*

The incommensurability of e is a mere particular case of the last conclusion; for we have in the series representing e-2

$$r_1=2$$
,  $r_2=3$ , . . . ,  $r_n=n+1$ , . . . ;  $p_1=1$ ,  $p_2=1$ , . . . ,  $p_n=1$ , . . . .

Hence e-2 is incommensurable, and therefore e also.

§ 4.] Returning to equation (5) of § 2, since  $L(1+1/z)^z$  has a finite value e, we have  $L(1+1/z)^z = \{L(1+1/z)^z\}^2 = \mathcal{E}$ , therefore

\* It should be noticed that an infinite series of the form (9) may represent a fraction whose denominator contains a factor not occurring among  $r_1, r_2, \ldots, r_n, \ldots$ , for example,

$$\frac{1}{2} = \frac{1}{3} + \frac{2}{3 \cdot 5} + \frac{3}{3 \cdot 5 \cdot 7} + \frac{4}{3 \cdot 5 \cdot 7 \cdot 9} + \dots \text{ ad } x.$$

This point seems to have been overlooked by some mathematical writers.

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \dots + \frac{x^{n}}{n!} + R_{n}$$
 (10),

where  $R_n$  is subject to the inequality (3).

Finally, since  $a^x = e^{\lambda x}$ , where  $\lambda = \log_e a$ , we have

$$a^{x} = 1 + \frac{(\lambda x)}{1!} + \frac{(\lambda x)^{2}}{2!} + \dots + \frac{(\lambda x)^{n}}{n!} + R_{n}$$
 (11),

where

$$R_n < (\lambda x)^{n+1}/(n+1)!\{1 - \lambda x/(n+2)\}$$
 (12).

Since  $LR_n = 0$  when  $n = \infty$ , the series (10) and (11) may of course each be continued to infinity.

This completes our second demonstration of the exponential theorem.

§ 5.] Summation of the Exponential Series for real values of x.

A third demonstration was given by Cauchy in his Analyse Algebrique. It follows closely the lines of the second demonstration of the binomial theorem; and no doubt it was suggested by the elegant process, due to Euler, on which that demonstration is founded. This third demonstration is of great importance, because we shall (in chap, xxix.) use the process involved in it to settle the more general question regarding the summation of the Exponential Series when x is a complex number.

Denote the intinite series

$$1 : x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots$$

by the symbol f(x). Then, since the series is convergent for all values of x, f(x) is a single valued, finite, continuous function of x (chap. XXVI., § 19).

Also, since f(x) and f(y) are both absolutely convergent series, we have, by the rule for the multiplication of series (chap. XXVI., § 14).

$$f(x)f'(y) = 1 + (x + y) + {x^2 + xyy + y^2 \choose 2! + 1!1! + 2!} + \dots + {x^n + y + x^{n+1}y + x^{n+2}y^2 \choose n! + (n-1)!1! + (n-2)!2! + \dots + y^n \choose n!} + \dots$$

Now

$$\frac{x^n}{n!} + \frac{x^{n-1}y}{(n-1)! \, 1!} + \frac{x^{n-2}y^2}{(n-2)! \, 2!} + \dots + \frac{y^n}{n!}$$

$$= (x^n + nC_1 x^{n-1}y + nC_2 x^{n-2}y^2 + \dots + y^n)/n!,$$

$$= (x + y)^n/n!,$$

by the binomial theorem for positive integral exponents.

Неисе

$$f(x)f(y) = 1 + \sum (x+y)^n/n!,$$
  
=  $f(x+y)$ 

Hence

$$= f(x+y)$$

$$f(x)f(y)f(z) = f(x+y)f(z),$$
(1).

= f(x + y + z);and in general  $x, y \in \mathbb{R}$  being any real x

and, in general,  $x, y, z, \ldots$  being any real quantities positive or negative,  $f(x) f(y) f(z) \ldots = f(x+y+z+\ldots) \qquad (2).$ 

J(x)J(y)J(z) ... = J(x+y+z+...) (2).

This last result is called the Addition Theorem for the Exponential Series.

From (2), putting x = y = z, . . . , = 1, and supposing the number of letters to be n, we deduce

$$\{f(1)\}^n = f(n)$$
 (3).

Also, taking the number of the letters to be q, and each to be p/q, we deduce

$$\{f(p/q)\}^{q} = f(p)$$
 (4),

where p and q are any positive integers. From (4), by means of (3), we deduce

$$\{f(p/q)\}^q = \{f(1)\}^p$$
 (5).

Finally, from (1), putting y = -x, we deduce

$$f(x)f(-x) = f(0)$$
 (6).

The equations (5) and (6) enable us to sum the series f(x) for all commensurable values of x.

From (5) we see that f'(p/q) is a qth root of  $\{f(1)\}^p$ . Now, since p/q is positive, the value of f'(p/q) is obviously real and positive. Also f'(1), that is,  $1+1/1+1/21+\dots$ , is a finite positive quantity, which we may call e. Therefore  $\{f'(1)\}^p$ , or  $e^p$ , is real and positive. Hence f'(p/q) must be the real positive f'(1) where f'(1) is f'(1) in f'(1) is f'(1). Therefore f'(1) is f'(1) in f'

$$1 + \frac{p_{eq}}{1!} + \frac{(p/q)^{2}}{2!} + \dots = e^{p_{eq}}$$
 (7),

p and q being any positive integers,

Finally, since f(0) = 1, we see from (6) that

$$f(-p/q) = 1/f(p/q),$$
  
=  $1/e^{p/q},$   
=  $e^{-p/q}$ 

Hence

$$1 + \frac{(-p/q)}{1!} + \frac{(-p/q)^2}{2!} + \dots = e^{-p/q}$$
 (8),

where p/q is any positive commensurable number.

By combining (7) and (8) we complete the demonstration of the theorem, that

$$e^x = 1 + \frac{x^2}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots,$$

for all commensurable values of x, e being given by

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots$$

The student will not fail to observe that e is introduced and defined in the course of the demonstration.

The extension of the theorem to the case where the base is any positive quantity a is at once effected by the transformation  $a^{x} = e^{\lambda x}$ , as in last demonstration.

§ 6.] From the Exponential Series we may derive a large number of others; and, conversely, by means of it a variety of series can be summed.

Bernoulli's Numbers.—One of the most important among the series which can be deduced from the exponential theorem is the expansion of x (1  $-e^{-x}$ ), the coefficients in the even terms of which are closely connected with the famous numbers of Bernoulli.

We shall first give Cauchy's demonstration, which shows, **a** priori, that  $x(1-e^{-x})$  can be expanded in an ascending series of powers of x, provided x lie within certain limits.

§§ 5, 6

We have 
$$\frac{x}{1-e^{-x}} = \frac{1}{(1-e^{-x})/x} = \frac{1}{1-u}$$
 (1),

where

$$y = 1 - (1 - e^{-x})/x$$
 (2).

Now, from (1), we have

$$x/(1-e^{-x}) = 1 + y + y^2 + \dots \text{ ad } \infty$$
 (3);

and this series will be absolutely convergent provided -1 < y < +1. Also, from (2), using the exponential theorem, we have

$$y = x/2! - x^2/3! + x^3/4! - \dots$$
 and  $\infty$  (4);

and this series is absolutely convergent for all values of x, and therefore remains convergent when all the signs are taken alike.

If, therefore, we can find a value of  $\rho$  such that

$$\rho/2! + \rho^2/3! + \rho^3/4! + \dots$$
 ad  $x < 1$  (A),

then, for all values of x between  $-\rho$  and  $+\rho$ , Cauchy's conditions of absolute convergency (chap. XXVI., § 34) will be fulfilled for the double series which results, when we substitute in (3) the value of y given by (4). This double series may therefore be arranged according to powers of x, and the result will be a convergent expansion for  $x/(1 - e^{-x})$ .

It is easy to show that a value of  $\rho$  can be found to satisfy the condition (A); for we have

$$\rho/2! + \rho^2/3! + \dots = (e^{\rho} - 1)/\rho - 1.$$

We have, therefore, merely to choose  $\rho$  so that

$$e^{\rho} - 1 < 2\rho$$
 (5).

 between 1 and 2.\* It will, therefore, certainly be possible to expand  $x_i(1-e^{-x})$  in a convergent series of powers of x if -1 < x < +1.

If we make the substitution for y, and calculate the coefficients of the first few terms, we find that

$$\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{6}\frac{x^2}{2!} - \frac{1}{30}\frac{x^4}{4!} + \frac{1}{42}\frac{x^6}{6!} - \dots$$
 (6).

Knowing, a priori, that the expansion exists, we can easily find a recurrence formula for calculating the successive coefficients. Let

$$x/(1-e^{-x}) = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots$$
 (7).

Then, putting -x in place of x, we must have, since  $-x/(1-e^x) = e^{-x}x/(1-e^{-x})$ ,

$$e^{-x}x/(1-e^{-x}) = A_0 - A_1x + A_2x^2 - A_3x^3 + \dots$$
 (8).

Since both the series are convergent, we have, by subtracting,

$$x = 2 A_1 x + 2 A_2 x^3 + \dots$$
 (9).

Hence  $A_1 = \frac{1}{2}$ ; and all the other coefficients of odd order must vanish.

Therefore, from (7), we have

$$\begin{split} x &= (A_0 + \frac{1}{2}x + A_1x^2 + A_4x^4 + \dots)(1 - e^{-x}), \\ &= (A_0 + \frac{1}{2}x + A_2x^2 + A_4x^4 + \dots + A_{2n}x^{2n} + \dots) \\ &\times \left(\frac{x}{1!} - \frac{x^n}{2!} + \frac{x^2}{3!} - \dots - \frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} - \dots\right). \end{split}$$

The product of these two convergent series will be another convergent series, all of whose coefficients, except the coefficient of x, must vanish. Hence, equating coefficients of odd powers of x, we deduce  $A_0$  1, and

$$\frac{A_{2n}}{1!} + \frac{A_{2n-2}}{3!} + \dots + \frac{A_{2}}{(2n-1)!} - \frac{1}{2(2n)!} + \frac{1}{(2n+1)!} = 0,$$

More nearly, the root is 1:250....; but the actual value, as will be seen presently, is not of much importance.

that is, 
$$\frac{A_{2n}}{1!} + \frac{A_{2n-2}}{3!} + \dots + \frac{A_2}{(2n-1)!} = \frac{2n-1}{2(2n+1)!}$$
 (10).

In like manner, if we equate the coefficients of even powers of x, we deduce

$$\frac{A_{2n}}{2!} + \frac{A_{2n-2}}{4!} + \dots + \frac{A_2}{(2n)!} = \frac{2n}{2(2n+2)!}$$
(11).

If, as is usual, we put  $A_{2n} = (-)^{n-1} B_n/(2n)!$ , our expansion becomes

$$\frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \frac{B_3}{6!}x^6 - \dots$$
 (12)

and the equations (10) and (11) may be written

$$\sum_{2n+1} C_{2n} B_n - \sum_{2n+1} C_{2n-2} B_{n-1} + \dots + \sum_{n-1} (-1)^{n-1} \sum_{2n+1} C_2 B_1 = (-1)^{n-1} (n-\frac{1}{2})^{n-1}$$
 and

$$\sum_{2n+2} C_{2n} B_n - \sum_{2n+2} C_{2n-2} B_{n-1} + \dots (-)^{n-1} \sum_{2n+2} C_2 B_1 = (-)^{n-1} n$$
 (11') respectively.

If we put n = 1, n = 2, n = 3, ..., successively, either in (10) or in (11), we can calculate, one after the other, the numbers  $B_1$ ,  $B_2$ , ...,  $B_3$ , ..., which are called Bernoulli's numbers. Since we know, a priori, that the expansion exists, the two equations (10) and (11) must of necessity be consistent. Neither of them furnishes the most convenient method for calculating the numbers rapidly to a large number of decimal places; but it is easy to deduce from them exact values for a few of the earlier in the series, namely.

$$\begin{split} B_1 &= \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \\ B_5 &= \frac{5}{66}, \quad B_6 = \frac{691}{2730}, \quad B_7 = \frac{7}{6}, \quad B_8 = \frac{3617}{510}, \\ B_9 &= \frac{4887}{798}, \quad B_{19} = \frac{(222277)}{2810}, \quad \&c. \end{split}$$

<sup>•</sup> There is considerable divergence among mathematical writers as to the notation for Bronodil's numbers. What we have denoted by B<sub>n</sub> is often denoted by B<sub>n</sub>, or by B<sub>2m-1</sub>. For further properties of these numbers, and for tables of their values, see Euler, Inst. Diff. Calc. Cap. 5, § 122; Ohn. Celle's Jour., Bd. xx. p. 11; J. C. Adams, Brit. Assoc. Rep., 1877, p. 8, also Cambridge Observations, 1890, App. 1; Standt, Crelle's Jour., Bd. xxi.; Boole's Finite Differences (cd. by Moulton); and, for a useful bibliography of the relative literature, Els., Am. Jour. Math. (1882).

We shall return to the properties of these numbers in chap, XXX.

Remark regarding the limits within which the expansion of  $x/(1-e^{-x})$  is valid.—If we denote the series

$$1 + \frac{1}{2}x + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \dots$$

by  $\phi(x)$ , we may state the problem we have just solved as follows:—To finda convergent vertix  $\phi(x)$  such that  $(1 - e^{-x}) \phi(x) = x$ , that is, such that  $(x - x^2]2! + x^2/3! - ... + \phi(x) = x$ .

Now, since  $x - x/(2! + x^2/3! - is$  absolutely convergent for all values of x, and the coefficients of  $\phi(x)$  satisfy (10) and (11),  $\phi(x)$  will satisfy the condition  $(x - x^2/2! + x^2/3! - ...)$ ,  $\phi(x) = x$  so long as  $\phi(x)$  is convergent. Hence, so long as  $\phi(x)$  is convergent, it will be the expansion of  $x/(1 - c^2)$ . As a matter of fact, it follows from an expression for Bernoulli's numbers given in chap, xxx. that  $\phi(x)$  is convergent so long as  $-2\pi < x < +2\pi$ . The actual limits of the validity of the expansion are therefore much wider than those originally assigned in the a priority proof of its existence.

Cor. 1. Since  $x(e^x + e^{-x})/(e^x - e^{-x}) = x/(1 - e^{-2x}) - x/(1 - e^{2x})$ , we deduce from (12)

$$x\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=1+\frac{B_{1}}{2!}2^{2}x^{2}-\frac{B_{2}}{4!}2^{4}x^{4}+\frac{B_{1}}{6!}2^{3}x^{6}+\dots$$
 (13).

Cor. 2. Since 
$$x/(1+e^{-x}) = 2x/(1-e^{-2x}) - x (1-e^{-x})$$
,

$$\frac{x}{1+e^{-x}} = \frac{1}{2}(2^{1}-1)x + \frac{B_{1}}{2!}(2^{2}-1)x^{2} - \frac{B_{2}}{4!}(2^{4}-1)x^{4} + \dots (14).$$

§ 7.] Bernoulli's Theorem.—We have already seen that the sum of the rth powers of the first n integers ( ${}_{n}S_{r}$ ) is an integral function of n of the r + 1th degree (see chan.  $\mathbf{x}\mathbf{x}$ . § 9).

We shall now show that the coefficients of this function can be expressed by means of Bernoulli's numbers.

From the identity

$$(e^{-x}-1)/(e^x-1) \equiv 1 + e^x + e^{2x} + \dots + e^{(n-1)x}$$

that is,

$$(e^{nx}-1)/(1-e^{-x}) = e^{x} + e^{2x} + e^{3x} + \dots + e^{nx},$$

we deduce at once

$$\begin{cases} \frac{(nx + n^2x^2 + \dots + \frac{n^2x^2 + \dots + n^2x^2 + \dots + \frac{n^2x^2 + \dots + \frac{n^2x^2 + \dots + n^2x^2 + \dots +$$

wherein all the series are absolutely convergent, so long as n is finite, provided x do not exceed the limits within which  $1 + \frac{1}{2}x + B_1x^2/2! - B_2x^2/4! + \dots$  is convergent. The coefficient of  $x^{r+1}$  on the right of (1) must therefore be equal to the coefficient of  $x^{r+1}$  in the convergent series which is the product of the factors on the left. Hence

$$\frac{{}_{n}S_{r}}{r!} = \frac{n^{r+1}}{(r+1)!} + \frac{n^{r}}{2 \cdot r!} + \frac{B_{1}n^{r-1}}{2! \cdot (r-1)!} - \frac{B_{2}n^{r-3}}{4! \cdot (r-3)!} + \frac{B_{3}n^{r-6}}{6! \cdot (r-5)!} \cdot \dots$$

Therefore

$$_{n}S_{r} = \frac{n^{r+1}}{r+1} + \frac{1}{2}n^{r} + \frac{r}{2} \underbrace{B_{1}n^{r-1} - \frac{r(r-1)(r-2)}{4!} B_{2}n^{r-3}}_{5!} + \frac{r(r-1)(r-2)(r-3)(r-4)}{6!} B_{2}n^{r-4} - \dots$$
 (2),

the last term being  $(-)^{\frac{1}{2}(r-2)}B_{\frac{1}{2}r}n$ , or  $\frac{1}{2}(-)^{\frac{1}{2}(r-3)}rB_{\frac{1}{2}(r-1)}n^2$ , according as r is even or odd.

This formula was first given by James Bernoulli (Ars Conjectandi, p. 97, publi-hed posthumously at Basel in 1713). He gave no general demonstration; but was quite aware of the importance of his theorem, for he boasts that by means of it he calculated intra semi-quadrantem horse! the sum of the 10th powers of the first thousand integers, and found it to be

## 91,409,924,241,424,243,424,241,924,242,500.

It will be a good exercise for the reader to check Bernoulli's result.

# SUMMATION OF SERIES BY MEANS OF THE EXPONENTIAL THEOREM.

§ 8.] Among the series which can be summed by means of the Exponential Series, two, related to it in the same way as the series of chap. XXVII., § 5, are related to the Binomial Series, deserve special mention.

We can always sum the series  $\overset{\sim}{\Sigma}\phi_r(n)\,x^n/n!$ , where  $\phi_r(n)$  is an integral function of n of the rth degree. (Integro-Exponential Series.)

234  $\hat{\Sigma}\phi_r(n)/n!, \hat{\Sigma}\phi_r(n)/n!(n+a)(n+b)...(n+k)$  CII. XXVIII

For, as in chap. XXVII., § 5, we can always establish an identity of the form

$$\phi_r(n) = A_0 + A_1 n + A_2 n (n-1) + \dots + A_r n (n-1) \dots (n-r+1).$$

Then we have, taking, for simplicity of illustration, the lower limit of summation to be 0.

$$\begin{split} \frac{z}{b} \frac{\varphi_r(n) \, x^n}{n!} &= A_0 \frac{z}{b} \frac{x^n}{n!} + A_1 x \frac{z}{b} \frac{x^{n-1}}{(n-1)!} + A_2 x^2 \frac{z}{b} \frac{x^{n-2}}{(n-2)!} + \cdots \\ &\quad + A_1 x^2 \frac{z}{b} \frac{x^{n-r}}{(n-r)!} \\ &= (A_0 + A_1 x + A_2 x^2 + \cdots + A_r x^r) \, e^x. \end{split}$$

Cor. We can in general sum the series  $\Sigma \Phi_r(n) x^n/n! (n+a)$  $(n+b) \dots (n+k)$ , where  $a, b, \dots, k$  are unequal positive integers.

The process is the same as that used in the corollary of chap. xxvii., § 5, only the details are a little simpler. (See Example 5, below.)

Example 1. To deduce the formulæ (3), (4), (5) of chap. xxvn., § 9, by means of the exponential theorem,

 $(x+n)^s-{}_nC_1(x+n-1)^s+\dots \ (-)^r{}_nC_r\ (x+n-r)^s+\dots \ (-)^nx^s$  is evidently the coefficient of  $z^s$  in

$$s! \{e^{(x+n)z} - {}_{n}C_{1}e^{(x+n-1)z} + \dots (-)^{r}{}_{n}C_{r}e^{(x+n-r)z} + \dots (-)^{n}e^{xz}\}$$

$$\begin{split} &= \cdot ! \, e^{xy} \, (e^x - 1)^n, \\ &= s! \, \left\{ 1 + xz + \frac{x^2z^2}{2z^2} + \dots \right\} \, \left\{ 1 + \frac{z}{2z} + \frac{z^2}{3!} + \dots \right\}^n z^n, \\ &= s! z^n \, \left\{ 1 + xz + \frac{x^2z^2}{2z^2} + \dots \right\} \left\{ 1 + \frac{z}{n} z + \frac{n(2n+1)}{n(2n+1)} z^2 + \dots \right\}. \end{split}$$

The lowest power of z in the product last written is  $z^n$ , and the coefficients of  $z^n$ ,  $z^{n+1}$ ,  $z^{n+2}$  are sl, sl  $(x + \frac{1}{2}n)$ ,  $\frac{1}{2}sl$   $\{x^2 + nx + \frac{1}{2}n (3n + 1)\}$  respectively.

Hence 
$$(x+n)^s = {}_{-}C_1(x+n-1)^s + \dots + (-)^r {}_{-}C_r(x+n-r)^s + \dots + (-)^n x^s$$

=0, if s < n;

= n!, if s = n;

= (n+1)! (x+1n), if s = n+1;

 $=\frac{1}{2}(n+2)!\{x^2+nx+1!n(3n+1)\}, \text{ if } s=n+2.$ 

Example 2. If n and r be positive integers, show that

$$e^{x}\left\{\frac{1}{\{r!} + \frac{n}{1!} \frac{n}{(r+1)!} x + \dots + \frac{n(n-1) \dots (n-s+1)}{s! (r+s)!} x^{s} + \dots + \frac{n(n-1) \dots 1}{n! (r+n)!} x^{s}\right\}$$

$$r! + \frac{n+r+1}{1!(r+1)!}r + \cdots + \frac{(n+r+1)(n+r+2)}{s!(r+s)!}\cdots + \frac{(n+r+s)}{s!}\frac{x^{s}}{s!} + \cdots$$
 and  $x$ .

The right-hand side is the coefficient of  $z^{n+r}$  in

$$(z+x)^n + \frac{(z+x)^{n+1}}{1!} + \dots + \frac{(z+x)^{n+r}}{r!} + \dots + \frac{(z+x)^{n+r+s}}{(r+s)!} + \dots$$

$$=(z+x)^n e^{z+x},$$

$$= e^{x} \left\{ z^{n} + {}_{n}C_{1}z^{n-1} \ x + \ldots + {}_{n}C_{n}x^{n} \right\} \times \left\{ 1 + \frac{z}{1!} + \frac{z^{2}}{2!} + \cdots + \frac{z^{n}}{n!} + \ldots \right\}.$$

Now the coefficient of zn+r in this product is

$$e^{x}\left\{\frac{1}{r!} + \frac{n}{1!(r+1)!}x + \dots + \frac{n(n-1)\dots 1}{n!(r+n)!}x^{n}\right\}.$$

Hence the theorem.

If we put r=0, and x=1, we have

$$\begin{aligned} \mathbf{1} + \frac{n+1}{(1!)^2} + \frac{(n+1)(n+2)}{(2!)^2} + \dots & \text{ad } \infty \\ &= e^{\left\{1 + \frac{n}{(1!)^2} + \frac{n(n-1)}{(n!)^2} + \dots + \frac{n(n-1)}{(n!)^2} + \dots + \frac{n(n-1)}{(n!)^2} + \dots + \frac{n(n-1)}{(n-1)^2} + \dots + \frac{n(n$$

Example 3. Sum the series

$$\frac{1^3}{1!}x + \frac{1^3 + 2^3}{2!}x^2 + \dots + \frac{1^3 + 2^3 + \dots + n^3}{n!}x^n + \dots$$
 ad  $\infty$ .

We have (by chap. xx., § 7)

 $1^3 + 2^3 + \dots + n^3 = (n^4 + 2n^3 + n^2)/4$ ,

=  $\frac{1}{4}$  { $A_0 + A_1 n + A_2 n (n-1) + A_3 n (n-1) (n-2) + A_4 n (n-1) (n-2) (n-3)$ }, where  $A_0, A_1, \ldots, A_4$  may be calculated as follows:—

Hence

$$\begin{split} \Sigma \, \frac{1^3 + 2^3 + \ldots + n^3}{n!} x^n &= x \, \Sigma \, \frac{x^{n-1}}{(n-1)!} + \frac{7}{2} x^2 \, \Sigma \, \frac{x^{n-2}}{(n-2)!} + 2 x^2 \, \Sigma \, \frac{x^{n-3}}{(n-3)!} + \frac{1}{4} x^4 \, \Sigma \, \frac{x^{n-4}}{(n-4)!}, \\ &= (x + \frac{\pi}{4} x^2 + 2 x^3 + \frac{1}{4} x^4) \, e^x. \end{split}$$

If we put x=1, we have

$$\Sigma (1^3 + 2^3 + ... + n^3)/n! = 27 e/4.$$

Example 4. Show that  $\sum_{n=1}^{n=\infty} n^3/n! = 5e$ .

Since 
$$n^3 \equiv n + 3n(n-1) + n(n-1)(n-2)$$
,  

$$\sum n^2/n! = \sum 1/(n-1)! + 3\sum 1/(n-2)! + \sum 1/(n-3)!$$

$$=5e.$$

Example 5. Evaluate  $\sum_{n=0}^{\infty} (n-1) x^n / (n+2) n!$ .

$$\Sigma \frac{(n-1) x^n}{(n+2) n!} = \frac{1}{x^2} \Sigma \frac{(n^2-1) x^{n+2}}{(n+2)!}.$$

Now

$$n^2 - 1 \equiv 3 - 3(n + 2) + (n + 2)(n + 1).$$

Therefore

$$\frac{\sum_{1}^{n} (n-1) x^{n}}{(n+2) n!} = \frac{1}{x^{2}} \left\{ 3 \sum_{1}^{\infty} \frac{x^{n+2}}{(n+2)!} - 3x \sum_{1}^{\infty} \frac{x^{n+1}}{(n+1)!} + x^{2} \sum_{1}^{\infty} \frac{x^{n}}{n!} \right\},$$

$$\begin{split} \frac{\sum_{i=1}^{N} (n+2) n!}{(n+2)!} &= \frac{3x}{2} \sum_{i=1}^{N} \frac{(n+2)!}{(n+1)!} + \frac{x^2}{2} \sum_{i=1}^{N} n! \, v, \\ &= \{3 (e^x - 1 - x - \frac{1}{2}x^2) - 3x (e^x - 1 - x) + x^2 (e^x - 1)\} / x^2, \\ &= \{ \{x^2 - 3x + 3\} e^x + (\frac{1}{2}x^2 - 3)\} / x^2. \end{split}$$

#### EXERCISES XII

- (1.) Evaluate 1/e to six places of decimals,
- (2.) Calculate x to a second approximation from the equation  $50 \log_x (1+x) = 49x$ .
  - (3.) If  $e^x = 1 + xe^{hx}$ , and  $x^a$  be negligible, show that

$$h = 1/2! + x/4! - x^2/4!5!$$

- (4.) Show that, if a be any positive integer.
- $(1-1/n)^{-n} > 1+1/1!+1/2!+\dots+1/n! > (1+1/n)^n$ (5.) Sum from 0 to x Σ (1 − 3n + n<sup>2</sup>) x<sup>n</sup>/nt.

## Sum to infinity

- (6.)  $1^{2}/2! + 2^{2}/3! + 3^{2}/4! + . . . .$
- $17.1 \quad 13/2! + 23/3! + 33/4! + \dots$
- (8.)  $1 2^3/1! + 3^3/2! 4^3/3! + \dots$
- (9.)  $1^4 + 2^4/2! + 3^4/3! + . . .$

# Show that

- (10.)  $1/(2n)! 1/1!(2n-1)! + 1/2!(2n-2)! \dots 1/1!(2n-1)! + 1/(2n)! = 0$
- (11.) If n > 3,  $n^3 + nC_2(n-2)^3 + nC_4(n-4)^3 + ... = n^2(n+3)2^{n-4}$ .
- (12.)  $n^n = C_1(n-2)^n + {}_nC_2(n-1)^n ... = 2^n n!,$
- (13.) By expanding cl (1-x), or otherwise, show that, if

$$A_r = \sum_{n=1}^{n-r} (n+r-1)!/n! (n-1)!, \text{ then } A_{r+1} - (2r+1) A_r + r (r-1) A_{r-1} = 0,$$
(Math. Trip., 1882.)

 $(x - x^3/3! + x^5/5! - \dots)(1 - x^5/2! + x^4/4! - \dots) = \sum_{i=1}^{n} (-1)^n 2^{2n} x^{2n+1}/(2n+1)!$ 

- (15.) Solve the equation x<sup>2</sup> x 1/n = 0; and show that the nth power of its greater root has e for its limit when n = x.
  - (16.) For all positive integral values of n

$$\begin{split} & n^{n-1} \binom{n-2}{2}^{n-1} \cdot \binom{n-2}{3}^{n-2} \cdot \cdots \binom{n-2}{n-1} < r^{n(n-1)/2}, \\ & (17.) \quad \text{If} \\ & x^n - s_n + \frac{s_1}{11} (x-1) + \frac{s_2}{2!} (x-1) \cdot (x-2) + \ldots + \frac{s_n}{n} (x-1) \cdot (x-2) \cdot \ldots \cdot (x-n), \end{split}$$

 $A_s = (s+1)^n - {}_sC_1 \cdot {}^n + {}_sC_2 (s-1)^n - {}_s \cdot {}_s \cdot {}_s (-)^s {}_sC_s 1^n,$ show that

- (18.) Show that  $\sum_{i=0}^{\infty} (n^3 + 2n^2 + n 1)/n! = 9c + 1$ .
- (19.) Sum  $\Sigma (n+a)(n+b)(n+c) x^n/n!$  from n=0 to  $n=\infty$ .
- (20.) Show that e cannot be a root of a quadratic equation having finite rational coefficients.
  - (21.) Sum the series  $\sum x^n/(n+3) n!$  from n=0 to  $n=\infty$ .
  - (22.) Sum to infinity the series 13/3. 1! + 33/4. 2! + 53/5. 3! + . . .
  - If B., B., . . . . B. denote Bernoulli's numbers, show that
  - (23.)  $_{2n+1}C_{2n-1}B_n _{2n+1}C_{2n-3}B_{n-1} + \dots (-)^{n-1}{_{2n+1}C_1}B_1 = (-1)^{n-1}$ .
  - (24.)  $_{2n+1}C_{2n}B_n \frac{2n+1}{2}C_{2n-2}\frac{B_{n-1}}{2^2} + \dots + (-)^{n-1}\frac{2n+1}{2}C_2\frac{D_1}{2} = (-)^{n-1}\frac{2n}{2^{2n}}$
- (25.)  $\frac{1}{2} {}_n C_1 B_1 \frac{1}{4} {}_n C_2 B_2 + \frac{3}{4} {}_n C_3 B_2 \ldots = (n-1)/2 (n+1),$  the last term on the left being  $(-)^{k(n-2)} B_{n/2}$ , or  $\frac{1}{2} (-)^{k(n-2)} n B_{(n-1)/2}$ , according as n is even or odd.
- (26.) By comparing Bernoulli's expression for 1<sup>r</sup>+2<sup>r</sup>+...+n<sup>r</sup> with the expressions deducible from Lagrange's Interpolation Formula, show that

$$\begin{array}{l} \overset{t=2p+1}{\overset{\sim}{\Sigma}}(-)^{t-1}_{-2p+1}C_t \, \overset{t\overset{\sim}{S}_{2p}}{\overset{\leftarrow}{t}} = (-)^{p-1} \, B_p; \\ \overset{t=2p+2}{\overset{\sim}{\Sigma}}(-)^{t-1}_{-2p+1}C_t \, \overset{t\overset{\sim}{S}_{-2p+1}}{\overset{\leftarrow}{t}} = 0. \end{array}$$

Also that

$$\sum_{1}^{t-2p} (-)^{t-1} {}_{2p}C_{t} \frac{t^{S_{2p}}}{t(t+1)} = (-)^{p-1} B_{p};$$

$$\frac{t^{=2p+1}}{1} (-)^{t-1}_{2p+1} C_t \frac{t^{S_{2p+1}}}{t(t+1)} = 0.$$
(Kronecker, Crelle's Jour., Ed. LXXXIV.; 1887.)

 $(27.) x(e^{x}-e^{-x})/(e^{x}+e^{-x}) = \frac{B_{1}}{ct}(2^{2}-1)2^{2}x^{2} + \frac{B_{2}}{ct}(2^{4}-1)2^{4}x^{4} + \frac{B_{3}}{ct}(2^{6}-1)2^{6}x^{8} + ...$ 

# LOGARITHMIC SERIES.

§ 9.] Expansion of  $\log (1 + x)$ .—It is obvious that no function of x which becomes infinite in value when x = 0 can be expanded in a convergent series of ascending powers of x. For, if we suppose

$$f(x) = A_0 + A_1x + A_2x^2 + \dots$$

then on putting x=0 we have  $\mathbf{z}=A_{\theta}$ ; and the attempt to determine even the first coefficient fails.

There can therefore be no expansion of  $\log x$  of the kind mentioned. We can, however, expand by (1+x) in a series of ascending powers of x, provided x be numerically less than unity.

The base in the first instance is understood to be e as usual.

By § 4, we have

$$(1+x)^{x} = 1+z \{\log (1+x)\} + z^{2} \{\log (1+x)\}^{2}/2! + \dots (1);$$

and this series is convergent for all values of z.

Again, by the binomial theorem, we have, provided the numerical value of x be less than 1,

$$(1+x)^{s} = 1 + zx + z(z-1)x^{2}/2! + z(z-1)(z-2)x^{3}/3! + \dots,$$

 $= 1 + zx - z (1 - z/1) x^{2} + z (1 - z/1) (1 - z/2) x^{2}/3 + \dots (2).$ 

If we arrange this as a double series, we have

$$(1+x)^z = 1 + z.x - \{z.x^2/2 - z^2x^2/2\} + \{z.x^2/3 - (1+\tfrac{1}{2})z^2x^3/3 + \tfrac{1}{2}z^3x^3/3\} + \frac{1}{2}z^3x^3/3\} + \frac{1}{2}z^3x^3/3\} + \frac{1}{2}z^3x^3/3$$

$$(-)^{n-1} \{zx^n/n - {}_{n-1}P_1 z^2x^n/n + {}_{n-1}P_2 z^3x^n/n - \dots$$

where  $_{n-1}P_r$  stands for the sum of all the r-products of 1/1, 1/2, . . . , 1/(n-1), without repetition.

In order that Cauchy's criterion for the absolute convergency of the double series (3) may be satisfied, it will be sufficient if the series

$$zx^{n}/n + {}_{n-1}P_{1}z^{2}x^{n}/n + \ldots + {}_{n-1}P_{n-1}z^{n}x^{n}/n$$
 (4)

and

$$1 + zx + z (1 + z/1) x^2/2 + z (1 + z/1) (1 + z/2) x^2/3 + ...$$
 (5)

be both convergent when z and x are positive.

Now the sum of (1) is always  $z(z+1) \dots (z+n-1)x^n/n!$ ; and this has 0 for its limit when n-x, provided x<1. Also, the series (5) is absolutely convergent when x<1.

Hence, by chap. XXVI., § 34, we may rearrange the series (3)

according to powers of z, and it will still converge to  $(1+x)^a$ .

Confining our attention to the first power of z, for the present, we thus find

$$(1+x)^{z} = 1 + \{x, 1-x^{2}, 2+x^{2}, 3-\dots\} z + \dots$$
 (5).

Now, since there can only be one convergent expansion of

 $(1+x)^x$  in powers of z, the series in (1) and (5) must be identical. Therefore

$$\log (1+x) = x/1 - x^2/2 + x^3/3 - \dots (-)^{n-1} x^n/n + \dots$$
 (6).

The series on the right of (6) is usually called the logarithmic series. It is absolutely convergent so long as -1 < x < 1, and it is precisely under this restriction that the above demonstration is valid.

If we put x=1 on the right of (6), we get the series  $1/1-1/2+1/3-\ldots$  ( $-1)^{n-1}/n+\ldots$ , which is semi-convergent. Hence, by Abel's Theorem (chap. XXVI., § 20), equation (6) will still hold in this case; and we have

$$\log 2 = 1/1 - 1/2 + 1/3 - \dots + (-1)^{n-1}/n + \dots$$
 (7),

provided the order of the terms as written be adhered to.

If we put x = -1 in (6), the series becomes divergent. It diverges, however, to  $-\infty$ ; so that, since  $\log 0 = -\infty$ , the theorem still holds in a certain sense.

Cor. If we arrange the coefficients of the remaining powers of z in (5), and compare with (1), we find

$$\{\log (1+x)\}^2 = 2! \{ {}_1P_1 x^2/2 - {}_2P_1 x^3/3 + {}_3P_1 x^4/4 - \dots \},$$

$$\{\log (1+x)\}^n = n! \{n-1P_{n-1}x^n/n - {}_nP_{n-1}x^{n+1}/(n+1) + {}_{n-1}P_{n-1}x^{n+2}/(n+2) - \dots \}$$
 (8).

These formulæ and the above demonstration are given by Cauchy in his Analyse Algebrique.

§ 10.] A variety of expansions can be deduced from the logarithmic theorem. The following are some of those that are most commonly met with:—

We have

$$\log(1+x) = x/1 - x^2/2 + x^2/3 - \dots + (-)^{n-1}x^n/n + \dots;$$

also  $\log (1-x) = -x/1 - x^2/2 - x^3/3 - \dots - x^n/n - \dots$ 

Hence, by subtraction, since  $\log{(1+x)} - \log{(1-x)} \equiv \log{\{(1+x)/(1-x)\}}$ , we deduce

$$\log \left\{ (1+x)/(1-x) \right\} = 2 \left\{ x/1 + x^{n}/3 + \dots + x^{2n-1}/(2n-1) + \dots \right\}$$
 (9).

Putting in (9) y = (1 + x)(1 - x), and therefore x - (y - 1)/(y + 1), we get

$$\log y = 2 \left\{ \frac{1}{1} \left( \frac{y-1}{y+1} \right) + \frac{1}{3} \left( \frac{y-1}{y+1} \right)^3 + \dots + \frac{1}{2n-1} \left( \frac{y-1}{y+1} \right)^{2n-1} + \dots \right\}$$
(10)

an expansion for  $\log y$  (but not, be it observed, in powers of y) which will be convergent if y be positive—the only case at present in question.

Again, since  $1+x\equiv x(1+1/x)$ , and  $\log(1+x)\equiv \log x+\log(1+1/x)$ , putting in (10) y=1+1/x, so that  $(y-1),(y+1)\equiv 1/(2x+1)$ , we have

$$\log (1+x) = \log x + 2 \{1/1 (2x+1) + 1 \ 3/2x + 1)^3 + \dots \}$$
 (11).

Finally, since  $x + 1 = x^2 (1 - 1/x^2)'(x - 1)$ ,

$$\log (x+1) = 2 \log x - \log (x+1)$$

$$= 2 (1 + 1)(2x^2 - 1) + 1 + 3(2x^2 - 1)^2 + \dots$$
 (12).

If, in any of the above formulæ, we wish to use a base a different from c, we have simply to multiply by the "modulus"  $Hog_{c,a}$  (see chap. XXI., § 9). Thus, for example, from (10) we derive

$$\log_{a} y = \frac{2}{\log_{a} a} \left\{ \left( \frac{y-1}{y+1} \right) + \frac{1}{3} \left( \frac{y-1}{y-1} \right)^{2} + \dots \right\}$$
 (13).

## ON THE CALCULATION OF LOGARITHMS

§ 11.] The early calculators of logarithms largely used methods depending on the repeated extraction of the square root. This process was combined with the Method of Differences, which seems to have aris no out of the practical necessities of the Logarithmic Calculator\*.

See G1. Next Art of Point of the proof of Britain on, 9th ed., from which there of what fore we to be see

Thus, Briggs used the approximate formula

$$\log_{10} 2 = (2^{10/2^{47}} - 1) 2^{47}/10 \log_e 10$$

depending on the accurate formula

$$L_{r=0}(x^z-1)/z = \log_e x,$$

which we have already established in the chapter on Limits, and which might readily be deduced from the exponential theorem. The calculation of  $\log_{10} 2$  in this way, therefore, involved the raising of 2 to the tenth power and the subsequent extraction of the square root 47 times!

Calculations of this kind were infinitely laborious, and nothing but the enthusiasm of pioneers could have sustained the calculators. If it were necessary nowadays to calculate a logarithmic table afresh, or to calculate the logarithm of a single number to a large number of places, some method involving the use of logarithmic series would probably be adopted.

The series in  $\S$  10 enable us to calculate fairly rapidly the Napierian Logarithms of the small primes, 2, 3, 5, 7.

Thus, putting y = 2 in (10) we have

$$\log 2 = 2 \{1/1 \cdot 3 + 1/3 \cdot 3^3 + 1/5 \cdot 3^5 + \dots \}.$$

The calculation to nine places may be arranged thus:-

1/3	.333,383,333	1/1 . 3	333,333,333	
1/33	37,037,037	1/3 . 33	12,345,679	
1/35	4,115,226	1/5 , 35	823,045	
$1/3^7$	457.247	1/7 . 37	65,321	
1/39	50,805	1 9 . 39	5,645	
1/311	5,645	1/11 . 3 <sup>n</sup>	513	
1/313	627	1/13 . 313	48	
1/315	70	1/15 . 315	5	
1/317	8	1 17 . 317	0	
			216 572 589	

 $\begin{array}{c|c} 316,573,589 & \pm 4 \\ \hline & 2 \\ \hline 693,147,178 & \pm 8 \end{array}$ 

By the principle of chap. XXVI., § 30, the residue of the series is less than

$$\{1/19 : 3^{19}\}/(1-\frac{1}{\nu}),$$

that is, less than '000,000,000,06; and the utmost error from the carriage to the last line is ±4. The utmost error in our calculation is ±8. Hence, subject to an error of 1 at the utmost in the last place, we have

$$\log 2 = 693,147,18.$$

Having thus calculated  $\log 2$ , we can obtain  $\log 3$  more rapidly by putting x=2 in (11). Thus

$$\log 3 - \log 2 + 2 \{1/1, 5 + 1/3, 5^3 + 1/5, 5^5 + \dots \}$$

Knowing  $\log 2$  and  $\log 3$ , we can deduce  $\log 4 - 2 \log 2$ , and  $\log 6 - \log 3 + \log 2$ . Then, putting x = 4 in (12), we have

$$\log 5 = 2 \log 4 - \log 3 - 2 \{1/31 + 1/3 \cdot 31^3 + \dots \}$$

Also, putting x = 6 in (12), we have

$$\log 7 = 2 \log 6 = \log 5 = 2 \{1,71 + 1,3,71^3 + \dots \}.$$

It will be a good exercise in computation for the student to calculate by means of these formula the Napierian Logarithms of the first 10 integers. The following table of the results to ten places will serve for verification:—

Lourrithm

140.	Logaritim,			
1	0,000,000,000,0			
2	0.693,147,180.65			
::	13098,612,288,7			
4	1/386,294,361,1			
5	1:609,137,912.4			
6	1:791.759.169,2			
7	1:945,910,149.1			
8	2:079,111,541,7			
9	2:197,221,577,3			
10	2°302,585,093,0			

From the value of log,10 we deduce the value of its reciprocal, namely, M = 431,294,481,903,251; and, by multiplying by this number, we can convert the Napierian Logarithm of

 <sup>6</sup> means that the 10th digit has been increased by a unit, because the 11th excee is 4.

any number into the ordinary or Briggian Logarithm, whose base is 10.

Much more powerful methods than the above can be found for calculating log 2, log 3, log 5, log 7, and M.

By one of these (see Exercises XIII., 2, below) Professor J. C. Adams has calculated these numbers to 260 places of decimals

§ 12.] The Factor Method of calculating Logarithms\* is one of the most powerful, and at the same time one of the most instructive, from an arithmetical point of view, of all the methods that have been proposed for readily finding the logarithm of a given number to a large number of decimals.

This method depends on the fact that every number may, to any desired degree of accuracy, be expressed in the form

$$10^{m}p_{0}/(1-p_{1}/10)(1-p_{2}/10^{2})(1-p_{3}/10^{3})$$
 . . . (1),

where  $p_0$ ,  $p_1$ ,  $p_2$ , . . . each denote one of the 10 digits, 0, 1, 2, . . . , 9,  $p_0$  being of course not 0.

Take, for example, 314159 as the given number. First divide by 105.3, and we have

Next multiply 1'047,196,666,666 by  $1-4/10^{\circ}$ , that is, cut off two digits from the end of the number, then multiply by 4 and subtract the result from the number itself. The effect of this will be to destroy the first significant figure after the decimal point. We have in fact

$$1.047,196,666,666 \times (1 - 4/10^{\circ}) = 1.005,308,800,000$$

Next multiply 1 005,308,800,000 by  $1-5/10^\circ$ , and so on till the twelve figures after the point are all reduced to zero. The actual calculation can be performed very quickly, as follows:—

For a full history of this method see Glaisher's article above quoted;
 or the Introduction to Gray's Tables for the Formation of Logarithms and Anti-Logarithms to Twenty-jour Places (1876).

9 2, 9 6 9 9/108, 2/109, 9/1010, 6/1011, 9/1019.

The remaining factors being obvious without farther calculation. Hence we have

$$314159 \times (1 - 4/10^{\circ})(1 - 5/10^{\circ}) \dots (1 - 9/10^{\circ})$$
  
=  $10^{\circ}$ ,  $3(1 + x/10^{\circ})$ ,  $x \geqslant 9$ .

Therefore

$$314159 = 10^{\circ}$$
,  $3(1 + x/10^{\circ})/(1 - 4/10^{\circ})(1 - 5/10^{\circ})$ , ...  $(1 - 9/10^{\circ})$ 

Since  $\log (1 + x/10^{19}) < x \cdot 10^{19}$ , it follows from (2) that, as far as the twelfth place of decimals,

$$\begin{split} \log 311159*5 & \log 10 + \log 3 - \log (1 - 0.10^{\circ}) - \log (1 - 5/10^{\circ}) \\ & - \log (1 - 2.10^{\circ}) - \log (1 - 8/10^{\circ}) - \log (1 - 2/10^{\circ}) \\ & - \log (1 - 1/10^{\circ}) - \log (1 - 9.10^{\circ}) - \log (1 - 2/10^{\circ}) \\ & - \log (1 - 9/10^{\circ\circ}) - \log (1 - 6/10^{\circ}) - \log (1 - 9/10^{\circ}). \end{split}$$

All, therefore, that is required to enable us to calculate log 344459 to twelve places is an auxiliary table containing the logarithms of the first 10 integers, and the logarithms of  $1-p/10^\circ$  for all integral values of p from 1 to 9, and for all integral values of r from 1 to 12. To make quite sure of the last figure this auxiliary table should go to at least thirteen places.

§ 13.] It should be noticed that a method like the above is suitable when only solitary logarithms are required. If a complete table were required, the Method of Differences would be employed to find the great majority of the numbers to be entered. A full discussion of this method would be out of place here\*; but we may, before leaving this part of the subject, give an analytical view of the method of interpolation by First Differences, already discussed graphically in chan, XXI.

We have

$$\log_{10}(x+h) - \log_{10}x = \log_{10}(1+h/x)$$

$$= M\{h/x - \frac{1}{2}(h/x)^2 + \frac{1}{2}(h/x)^3 + \dots \}$$
(1).

Hence, if h < x, we have approximately

$$\log_{10}(x + h) - \log_{10}x = Mh/x$$
 (2),

the error being less than  $\frac{1}{2}M(h/x)^2$ .

The equation (2) shows that, if  $\frac{1}{2}M(k/x)^2$  do not affect the nth place of decimals, then, so long as  $h \geqslant k$ , the differences of the values of the function are proportional to the differences of the values of the argument, provided we do not tabulate beyond the nth place of decimals.

Take, for example, the table sampled in chap. xxi., where the numbers are entered to five and the logarithms to seven places. Suppose x=30000; and let us inquire within what limits it would certainly be safe to apply the rule of proportional parts. We must have

$$\frac{1}{2} \times \cdot 4343 (h/30000)^3 < 5/10^8$$
,

if the interpolated logarithm is to be correct to the last figure, that is,

$$h < 3\sqrt{23.04}$$
, < 14.

It would therefore certainly be safe to apply the rule and interpolate to seven places the logarithms of all numbers lying between 30040 and 30014. This agrees with the fact that in the table the tabular difference has the constant value 144 within, and indeed beyond, the limits mentioned.

### SUMMATION OF SERIES BY MEANS OF THE LOGARITHMIC SERIES.

§ 14.] A great variety of series may, of course, be summed by means of the Logarithmic Series. Of the simple power series that can be so summed many are included directly or indirectly under the following theorem, which stands in the same relation

<sup>\*</sup> For sources of information, see Glaisher, l.c.

to the logarithmic theorem as do the theorems of chap. xxvii., § 5, and chap. xxviii., § 8, to the binomial and exponential theorems:—

The series whose general term is  $\phi(n) x^a/(n+a)(n+b)$ . (n+k), where  $\phi(n)$  is an integral function of n, and a, b, a are positive or negative unequal integers, can always be summed to infinity provided the series is convergent.

It can easily be shown that the series is convergent provided x be numerically less than unity, and divergent if x be numerically greater than unity.

If the degree of  $\phi(n)$  be greater than the degree of (n+a) (n+b) . . . (n+k), the general term can be split into

$$\psi(n) x^n + \chi(n) x^n/(n+a)(n+b) \dots (n+k)$$
 (1),

where  $\psi(n)$  and  $\chi(n)$  are integral functions of n, the degree of the latter being less than the degree of  $(n+a)(n+b) \dots (n+k)$ .

Now  $\Sigma \psi\left(n\right)x^{n}$  is an integro-geometric series, and can be summed by the method of chap. xx., § 13.

By the method of Partial Fractions (chap. viii.) we can express  $\chi(n)/(n+a)\,(n+b)\,\ldots\,(n+k)$  in the form

$$A/(n+a) + B/(n+b) + \dots + K/(n+k),$$

where  $A,\,B,\,\ldots,\,K$  are independent of n. Hence the second part of (1) can be split up into

$$Ax^{n}/(n+a) + Bx^{n}/(n+b) + \dots + Kx^{n}/(n+k)$$
 (2);

and we have merely to sum the series

$$A \sum x^{n}/(n+a), \quad B \sum x^{n}/(n+b), \quad \dots, \quad K \sum x^{n}/(n+k)$$
 (3).

Now, supposing, for simplicity of illustration, that the summation extends from n=1 to  $n=\infty$ , we have

$$A \sum_{1}^{\infty} x^{a}/(n+a) = Ax^{-a} \sum_{1}^{\infty} x^{a+a}/(n+a),$$
  
=  $-Ax^{-a} \{x/1 + x^{2}/2 + \dots + x^{a}/n + \log(1-x)\}$  (4).

Each of the other series (3) may be summed in like manner. Hence the summation can be completely effected.

When any of the integers a, b, . . . , k are negative, the method requires the evaluation of limits in certain cases.

If x = 1, the series under consideration will not be convergent unless the degree of  $\phi(n)$  be less than the degree of (n + a)  $(n + b) \dots (n + k)$ . It will be absolutely convergent if the degree of  $\phi(n)$  be less than that of  $(n + a)(n + b) \dots (n + k)$  by two units. If the degree of  $\phi(n)$  be less than that of (n + a)  $(n + b) \dots (n + k)$  by only one unit, then the series is semi-convergent if the terms ultimately alternate in sign, and divergent if they have ultimately all the same sign.

In all cases, however, where the series is convergent we can, by Abel's Theorem, find the sum for x=1 by first summing for x<1, and then taking the limit of this sum when x=1.

In the special case where  $\phi(n)$  is lower in degree by two units than  $(n+a)(n+b) \dots (n+k)$ , and  $a, b, \dots, k$  are all positive, an elegant general form can be given for  $\sum_{i=1}^{k} \phi(n)/(n+a)$   $(n+b) \dots (n+k)$ .

From the identity

$$\phi(n)/(n+a)(n+b)$$
 . . .  $(n+k)$   
 $\equiv A/(n+a) + B/(n+b) + \dots + K_n(n+k)$ ,

we have

$$\phi(n) \equiv A(n+b)(n+c) \dots (n+k) + B(n+a)(n+c) \dots (n+k) + \dots + K(n+a)(n+b) \dots (n+j)$$
(5),

and, bearing in mind the degree of  $\phi(n)$ , we have

$$A + B + \dots + K = 0$$
 (6).

Also, putting in succession  $n=-a, n=-b, \ldots, n=-k$ , we have

$$A = \phi(-a)/(b-a)(c-a) \dots (k-a)$$

$$B = \phi(-b)/(a-b)(c-b) \dots (k-b)$$

$$\vdots$$

$$K = \phi(-k)/(a-k)(b-k) \dots (j-k)$$
(7).

Reverting to the general result, we see from (4) that

$$\sum_{1}^{\infty} \phi(n) x^{n}/(n+a) (n+b) \dots (n+k) \\ = -\sum_{1} A x^{-a} (x/1 + x^{2}/2 + \dots + x^{a}/a) - \log(1-x) \sum_{1} A x^{-a}$$
(8),

where the  $\Sigma$  on the right hand indicates summation with respect to  $a, b, \ldots, k$ .

Now, since  $A + B + \dots + K = 0$ ,  $\Sigma A x^{-n}$  is an algebraical function of x which vanishes when x - 1. Also 1 - x is an algebraical function of x having the same property. Therefore, by chap, xxv., § 17, we have

Hence, taking the limit on both sides of (8), we have, by Abel's Theorem.

$$\sum_{1}^{8} \frac{b}{(n)} (n)/(n+a)(n+b) \dots (n+k) = -\sum_{1} 4(1/1+1/2+\dots+1/a),$$

$$= -\sum_{1} \frac{b}{(b-a)} \frac{(-a)(1/1+1/2+\dots+1/a)}{(b-a)(c-a)\dots(c-k)}$$
(9),

the 2 on the right denoting summation with respect to a, b, c, ..., k

Example 1. Evaluate  $\sum_{n=0}^{\infty} n^3 x^n / (n-1) (n+2)$ .

We have 
$$n^2x^n/(n-1)(n+2) \equiv (n-1)x^n + \frac{1}{2}x^n/(n-1) + \frac{\pi}{2}x^n/(n+2)$$
.  
Now  $\sum_{n=0}^{\infty} (n-1)x^n = 1x^2 + 2x^3 + 3x^4 + \dots$ ,  $(1-x)^{-\frac{\pi}{2}}(n-1)x^n = 1x^2 + 2x^3 + 3x^4 + \dots$ ,  $-2 \cdot 1x^2 - 2 \cdot 2x^4 - \dots$ ,  $+1x^4 + \dots$ ,  $=x^7$ .  
Hence  $\sum_{n=0}^{\infty} (n-1)x^n = x^7/(1-x)^3$ .

 $\frac{1}{3}\sum_{n=1}^{\infty} x^{n}/(n-1) = \frac{1}{3}x\sum_{n=1}^{\infty} x^{n-1}/(n-1)$ Also

$$\frac{1}{3}\sum_{n=1}^{\infty} x^{n}/(n+2) = \frac{1}{3}x^{-2}\sum_{n=1}^{\infty} x^{n+2}/(n+2),$$

 $= - \xi x^{-2} \{x/1 + x^2/2 + x^3/3 + \log (1-x)\}.$ 

Hence the whole sum is

$$x^2/(1-x)^2 = {}^n x^{-1} - \frac{1}{2} - \frac{1}{2} x - \frac{1}{3} (x + 8x^{-1}) \log (1-x)$$

Evaluate  $\sum 1/(n-1)(n+2)$ . Example 2

By the same process as before, we find

$$\sum_{x=1}^{\infty} \frac{1}{(n-1)(n+2)} = \frac{1}{2}x^{-1} + \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}(x^{-2} - x) \log (1 - x).$$

Now, since  $L_{z=1} (1-x)^{z^{-2}-z} = 1$  (chap. xxv., § 17),  $L_{z=1} (x^{-2}-x) \log (1-x) = 0$ .

Therefore  $\sum_{n=0}^{\infty} 1/(n-1)(n+2) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{1}{1} \frac{1}{5}$ .

This result might be obtained in quite another way.

It happens that  $\Sigma 1/(n-1)(n+2)$  can be summed to n terms. In fact, we have

$$1/(n-1)(n+2) \equiv \frac{1}{3}\{1/(n-1)-1/(n+2)\}.$$

Hence, since the series is now finite and commutation of terms therefore permissible,

$$3\frac{5}{2}I((n-1)(n+2)) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-4} + \frac{1}{n-3} + \frac{1}{n-2} + \frac{1}{n-1}$$

$$-\frac{1}{4} - \dots - \frac{1}{n-4} - \frac{1}{n-3} - \frac{1}{n-2} - \frac{1}{n-1}$$

$$-\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2},$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2},$$

Hence, taking the limit for  $n=\infty$  , we have

$$\sum_{2}^{\infty} = \frac{1}{3} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right) = \frac{11}{18}.$$

Example 3. To sum the series

$$\left(\frac{1}{1} + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \dots \text{ ad } \infty.$$
(Lionnet, Now. Ann., ser. 11., t. 18.)

Let the (n+1)th term be  $u_n$ , then, since  $u_\infty=0$ , association is permitted (see chapter xxv., § 7), and we may write

$$\begin{split} &u_n \equiv \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, \\ &\equiv \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} + \frac{1}{4n+2} - \frac{1}{4n+4}, \\ &\equiv \left(\frac{1}{4n} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4}\right) + \frac{1}{2} \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right), \\ &\equiv v_n + w_n, \text{ say.} \end{split}$$

Now, as may be easily verified,  $v_n$  and  $w_n$  are rational functions of n, in which the denominator is higher in degree than the numerator by two units at least. Hence (chap. xxv.,  $\S$  6)  $\Sigma v_n$  and  $\Sigma w_n$  are absolutely convergent series. Therefore (chap. xxv.,  $\S$  13)

$$\begin{split} &\overset{\sim}{\sum} u_n = \overset{\sim}{\sum} (v_n + w_n), \\ &= \overset{\sim}{\sum} v_n + \overset{\sim}{\sum} w_n. \end{split}$$

Hence, again dissociating v., and w., (as is evidently permissible) we have

$$\frac{\xi}{6}u_n - 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots$$

$$+ \frac{1}{2} \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \dots \right)$$

$$= \log_2 2 + \frac{1}{2} \log_2 2, \qquad \text{by § 9 above,}$$

$$= \frac{\pi}{6} \log_2 2 + \frac{1}{3} \log_2 2 + \frac{1$$

This example is an interesting specimen of the somewhat delicate operation of evaluating a semi-convergent series. The process may be described as consisting in the conversion of the semi-convergent into one or more absolutely convergent series, whose terms can be commutated with safety. It should be observed that the terms in the given series are merely those of the series  $1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$ , written in a different order. We have thus a striking instance of the truth of Abel's remark that the sum of a semi-convergent series may be altered by commutating its terms.

#### APPLICATIONS TO INEQUALITY AND LIMIT THEOREMS,

 $\S$  15.] The Exponential and Logarithmic Series may be applied with effect in establishing theorems regarding inequality. Thus, for example, the reader will find it a good exercise to deduce from the logarithmic expunsion the theorem, already proved in chapter xxv., that, if x be positive, then

$$x-1 > \log x > 1 - 1/x$$
 (1).

It will also be found that the use of the three fundamental series—Binomial, Exponential, and Logarithmic—greatly facilitates the evaluation of limits. Both these remarks will be best brought home to the reader by means of examples.

Example 1. Show that

$$\log \frac{n}{m-1} > \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} > \log \frac{n+1}{m}$$

If we put 1-1/x=1/m, that is, x=m/(m-1), in the second part of (1) above, and then replace m by  $m+1, m+2, \ldots, n$  successively, we get

$$\log m - \log (m-1) > 1/m,$$
  
 $\log (m+1) - \log m > 1/(m+1),$ 

 $\log n - \log (n-1) - 1/n.$ 

Hence, by addition,  

$$\log n - \log (m+1) > 1/m + 1/(m+1) + \dots + 1/n$$
 (2),

Next, if we put x - 1 = 1/m in the first part of (I), and proceed as before, we get

$$\log (m+1) - \log m < 1/m$$
,  
 $\log (m+2) - \log (m+1) < 1/(m+1)$ ,  
 $\log (n+1) - \log n < 1/n$ .

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$$\log (n+1) - \log m < 1/m + 1/(m+1) + \dots + 1/n$$
 (3).

From (2) and (3),

 $\log \{n/(m-1)\} > 1/m + 1/(m+1) + \dots + 1/n > \log \{(n+1)/m\}$ 

Example 2. If p and q be constant integers, show that

$$L\{1/m+1/(m+1)+...+1/(pm+q)\} = \log p.$$

(Catalan, Traité Élémentaire des Séries, p. 58.)

Put n = pm + q in last example, and we find that

$$\log\{(pm+q)/(m-1)\} > 1/m+1/(m+1)+\ldots+1/(pm+q) > \log\{(pm+q+1)/m\}.$$
  
Now  $L \log\{(pm+q)/(m-1)\} = \log p,$ 

and

$$L_{m=\infty} \log \{(pm+q+1)/m\} = \log p.$$

Hence the theorem.

Example 3. Evaluate  $L(c^x-1)^2/\{x-\log(1+x)\}$  when x=0.

Since 
$$(e^x - 1)^2 = (x + \frac{1}{2}x^2 + \dots)^2 = x^2(1 + \frac{1}{2}x + \dots)^2$$
;

$$x - \log (1 + x) = \frac{1}{2}x^2 - \frac{1}{5}x^3 + \dots = \frac{1}{2}x^2(1 - \frac{2}{3}x + \dots)$$

Therefore

$$(e^x - 1)^2/\{x - \log(1+x)\} = 2(1 + \frac{1}{2}x + \dots)^2/(1 - \frac{2}{3}x + \dots)$$

Since the series with the brackets are both convergent, it follows at once  $L(e^x - 1)^2/\{x - \log(1 + x)\} = 2,$ that

#### Exercises XIII.

(1.) If 
$$P = 1/31 + 1/3 \cdot 31^3 + 1/5 \cdot 31^5 + \dots$$
,  $Q = 1/49 + 1/3 \cdot 49^3 + 1/5 \cdot 49^5 + \dots$ ,

 $R = 1/161 + 1/3 \cdot 161^3 + 1/5 \cdot 161^5 + \dots$ 

then 
$$\log 2 = 2 (7P + 5Q + 3R),$$
  
 $\log 3 = 2 (11P + 8Q + 5R),$ 

 $\log 5 = 2(16P + 12O + 7K)$ (See Glaisher, Art. "Logarithms," Encu. Brit., 9th ed.)

(2.) If  $a = -\log(1 - 1/10)$ ,  $b = -\log(1 - 4/100)$ ,  $c = \log(1 + 1/80)$ , d = $-\log (1-2/100)$ ,  $e = \log (1+8/1000)$ , then  $\log 2 = 7a - 2b + 3c$ ,  $\log 3 = 11a - 3b$ +5c,  $\log 5 = 16a - 4b + 7c$ ,  $\log 7 = \frac{1}{2}(39a - 10b + 17c - d) = 19a - 4b + 8c + e$ .

(Prof. J. C. Adams, Proc. R.S.L.; 1878.) (3.) Calculate the logarithms of 2, 3, 5, 7 to ten places, by means of the

formulæ of Example 1, or of Example 2, (4.) Find the smallest integral value of x for which (1.01)x > 10x.

Sum the series :-

 $(5.1 - 2)H(x^3 - 3x)^3 + 2 \cdot /3(x^3 - 3x)^3 + ...$ 

(6.) 
$$1 + \left(\frac{1}{2} + \frac{1}{3}\right) \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right) \frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right) \frac{1}{4^3} + \dots$$

- (7)  $x^{1}/1$ ,  $2 x^{2}/2$ ,  $3 + x^{3}/3$ ,  $4 \dots$ ,  $(-)^{n-1}x^{n}/n$  (n+1).
- (8.)  $x^2/3 + x^4/15 + \dots + x^{2n}/(4n^2 1) + \dots$ (9.)  $x/1^2 + x^2/(1^2 + 2^2) + x^3/(1^2 + 2^2 + 3^2) + ... + x^n/(1^2 + 2^2 + ... + n^2) + ...$
- also  $1/1^2 + 1/(1^2 + 2^2) + 1/(1^2 + 2^2 + 3^2) + \dots + 1/(1^2 + 2^2 + \dots + n^2) + \dots$ (10.) 4/1 2 3 + 6/2 3 4 + 8/3 4 5 + . . .
- (11.) If x > 100, then, to seven places of decimals at least  $\log(r + 8) =$  $2 \log (x + 7) - \log (x + 5) - \log (x + 3) + 2 \log x - \log (x - 3) - \log (x - 5) + 2 \log x$  $(x-7) - \log (x-8)$ .
  - (12.) Expand log (1+x+x2) in ascending powers of x.
  - (13.) From  $\log (x^3 + 1) \equiv \log (x + 1) + \log (x^2 x + 1)$ , show that, if m be a positive integer, then

$$1 - \frac{6m - 2}{2!} + \frac{(6m - 3)(6m - 4)}{3!} - \frac{(6m - 4)(6m - 5)(6m - 6)}{4!} + \dots = 0.$$

(Math. Trip., 1882.)

(14.)  $\{\log_{+}(1+x)\}^{2} = 2x^{2}/2 - 2(1/1+1/2)x^{3}/3 + \dots + (-)^{n}2\{1/1+1/2+\dots$ 1/(n-1)  $x^n/n$ ... Does this formula hold when x=1?

(15.)  $\log (1+x)^{\log(1-x)} = -Q_1x^2/1 - Q_2x^4/2 - \dots - Q_{2n-1}x^{2n}/n - \dots$  $Q_{2n-1} = 1/1 - 1/2 + 1/3 - ... + 1/(2n-1).$ where

(16.) If x < 1, show that

 $x + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{16}x^{16}... = \log\{1/(1-x)\} - \frac{1}{2}P_3 - \frac{1}{6}P_6 + \frac{1}{6}P_6 - \frac{1}{7}P_7 - \frac{1}{6}P_6 + \frac{1}{16}P_{16}...;$ where  $P_{-} = x^{n} + x^{2n} + x^{4n} + x^{4n} + x^{5n} + x^{16n} + \dots$ , and the general term is  $(-)^{n}P_{-}/n$ , unless n is a power of 2, in which case there is no term.

(Trin. Coll., Camb., 1878.)

(17.) If  $e^x \times e^{x^3} \times e^{x^3} \times e^{x^3} = A_0 + A_1 x + \dots$ , then  $A_{2r} = A_{2r+1} = 1.3.5 \dots$ (2r-1)/2.4.6...2r

(18.) If  $x + a_x x^3 + a_5 x^5 + ... + y + a_3 y^3 + a_5 y^5 + ... = \{(x + y)/(1 - xy)\}^1 +$  $a_*\{(x+y)/(1-xy)\}^3 + a_*\{(x+y)/(1-xy)\}^5 + \dots$ , for all values of x and y which render the various series convergent, find an, and . . .

#### Show that

- (19.)  $\log (4/\epsilon) = 1/1 \cdot 2 1/2 \cdot 3 + 1/3 \cdot 4 1/4 \cdot 5 + \dots$
- (20.)  $\log 2 = 4(1/1, 2, 3 + 1/5, 6, 7 + 1/9, 10, 11 + 1/13, 14, 15 + ...)$  (Euler.)
- (21.)  $(1-1/2-1/4)+(1/3-1/6-1/8)+(1/5-1/10-1/12)+...= k \log 2$ (See Lionnet, Nouv. Ann., ser. II., t. 18.)
- (22.)  $\sigma_1/11 n\sigma_2/2! + n(n-1)\sigma_2/3! ... \text{ to } n+1 \text{ terms } = 1/(n+1)^2$ , where  $\sigma_r = 1/1 + 1/2 + 1/3 + \dots + 1/r$ . (Math. Trip., 1888.)
- (23.)  $e \sim (1+1/m)^m$  lies between e/(2m+1) and e/(2m+2), whatever m niay be. (Nouv. Ann., ser. ii., t. 11.)
  - (21.) L {x/(x − 1) − 1/log x} = ½, when x = 1. (Euler, Inst. Calc. Diff.)
  - (25.)  $L_{\epsilon} \{e^{x} 1 \log(1 + x)\}/x^{2} = 1$ , when x = 0. (Euler, l.c.)
  - (26.)  $L(x^x x)/(1 x + \log x) = -2$ , when x = 1, (Euler, i.e.)

and

- (27.)  $L(1+1/n)^{1/n}(1+2/n)^{1/n}$ ,  $(1+n/n)^{1/n}=4/e$ , when  $n=\infty$ ,
- (28.)  $L\{(2n-1)!/n^{2n-1}\}^{1/n}=4/c^2$ , when  $n=\infty$ .
- (29.) e<sup>x</sup>>1+x, for all real values of x.
   (30.) x-1>log x>1-1/x, for all positive values of x; to be deduced from the locarithmic expansion.
  - (31.)  $c^n > (1+n)^n/n!$ , n being any integer.
  - (32.) If n be an integer > c, then  $n^{n+1} > (n+1)^n$ .
- (33.) If A, B, a, b be all positive, then (a b)/(A B) + (A Bb) log (B/A)/(A B)<sup>2</sup> is negative. (Tait.)
  - (34.) If x > y > a, then  $\{(x+a)/(x-a)\}^x < \{(y+a)/(y-a)\}^y$ .
  - (35.)  $L\{1/(n+1)+1/(n+2)+...+1/2n\} = \log 2$ , when  $n=\infty$ . (Catalan.)
  - (36.)  $\log \{(n+\frac{1}{2})/(m-\frac{1}{2})\} > 1/m+1/(m+1)+...+1/n > \log \{(n+1)/m\}.$  (Bourguet, Nouv. Ann., ser. 11., t. 18.)
  - (37.)  $\log 3 = 5/1 \cdot 2 \cdot 3 + 14/4 \cdot 5 \cdot 6 + \dots + (3n-4)/(3n-2)(3n-1)3n + \dots$
- (38.) If  $\sum_{1}^{\infty} (-)^{n-1} \phi(n)/(n+a)$  (n+b), ... (n+k), where  $a,b,\ldots,k$  are all positive integers and  $\phi(n)$  is an integral function of n, be absolutely convergent, its sum is

 $S = \sum_{a,b,...,k} \phi(-a) \{1/a - 1/(a - 1) ... (-)^{a-1} 1/1 \} / (b-a) (c-a) ... (k-a);$ and, if it be semi-convergent, its sum is

$$S + \log 2 \sum_{a,b,\dots,k} (-)^a \phi(-a)/(b-a) (c-a) \dots (k-a).$$

(39.) Show that the residue in the expansion of  $\log \{1/(1-x)\}$  lies between

$$x^{n+1}\{1+(n+1)x/(n+2)\}/(n+1)$$

$$x^{n+1}\{1+(n+1)x/(1-x)(n+2)\}/(n+1).$$

- (40.) In a table of Eriggian Logarithms the numbers are entered to 5 significant figures, and the mantisse of the logarithms to 7 figures. Calculate the tabular difference of the logarithms when the number is near 30000; and find through what extent of the table it will remain constant.
  - (41.) Show that  $(1+1/x)^{x+1}$  continually decreases as x increases.
    - (42.) Show that  $\sum_{n=1}^{\infty} 1/n (4n^2 1)^2 = \frac{\pi}{3} 2 \log 2$ ,

# CHAPTER XXIX.

# Summation of the Fundamental Power Series for Complex Values of the Variable.

# GENERALISATION OF THE ELEMENTARY TRANSCENDENTAL FUNCTIONS.

§ 1.] One of the objects of the present chapter is to generalise certain expansion theorems established in the two chapters which precede. In doing this, we are led to extend the definitions of certain functions such as  $a^x$ ,  $\log_a x$ ,  $\cos x$ , &c., already introduced, but hitherto defined only for real values of the variable x; and to introduce certain new functions analogous to the circular functions.

Seeing that the circular functions play an important part in what follows, it will be convenient here to recapitulate their leading properties. This is the more necessary, because it is not uncommon in English elementary courses so to define and discuss these functions that their general functional character is lost or greatly obscured.

§ 2.] Definition and Properties of the Direct Circular Functions. Taking, as in chap. XII., Fig. 1, a system of rectangular axes, we can represent any real algebraical quantity θ, by causing a radius vector OP of length r to rotate from OX through an angle containing θ radians, counter-clockwise if θ be a positive, clockwise if it be a negative quantity. If (x, y) be the algebraical values of the coordinates of P, any point on the radius vector of θ, then (x, y, y, x, y, x, x, x, y are obviously all functions of θ, and of θ alone. The functions thus geometrically defined are called

 $\cos \theta$ ,  $\sin \theta$ ,  $\tan \theta$ ,  $\cot \theta$ ,  $\sec \theta$ ,  $\csc \theta$  respectively, and are spoken of collectively as the circular functions.

All the circular functions of one and the same argument,  $\theta$ , are algebraically expressible in terms of one another, for their definition leads immediately to the equations

$$\begin{array}{l} \tan\theta = \sin\theta/\cos\theta, \quad \cot\theta = \cos\theta/\sin\theta; \\ \sec\theta = 1/\cos\theta, \quad \csc\theta = 1/\sin\theta; \\ \cos^2\theta + \sin^2\theta = 1, \quad \sec^2\theta - \tan^2\theta = 1; \end{array}$$
 (1);

from which it is easy to deduce an expression for any one of the  $\sin x$ ,  $\cos \theta$ ,  $\sin \theta$ ,  $\tan \theta$ ,  $\cot \theta$ ,  $\sec \theta$ ,  $\csc \theta$ , in terms of any other.

When  $F(\theta)$  is such a function of  $\theta$  that  $F(-\theta) = F(\theta)$ , it is said to be an even function of  $\theta$ ; and, when it is such that  $F(-\theta) = -F(\theta)$ , it is said to be an odd function of  $\theta$ . For example,  $1 + \theta$  is an even, and  $\theta - \frac{1}{3}\theta$  is an odd function of  $\theta$ .

It is easily seen from the definition of the circular functions that  $\cos \theta$  and  $\sec \theta$  are even, and  $\sin \theta$ ,  $\tan \theta$ ,  $\cot \theta$ , and  $\csc \theta$  odd functions of  $\theta$ .

When  $F(\theta)$  is such that for all values of  $\theta$ ,  $F(\theta + n\lambda) = F(\theta)$ , where  $\lambda$  is constant, and n any integer positive or negative, then  $F(\theta)$  is said to be a periodic function of  $\theta$  having the period  $\lambda$ .

It is obvious that the graph of such a function would consist of a number of parallel strips identical with one another, like the sections of a wall paper; so that, if we knew a portion of the graph corresponding to all values of  $\theta$  between a and  $a + \lambda$ , we could get all the rest by simply placing side by side with this an infinite number of repetitions of the same.

Since the addition of  $\pm 2\pi$  to  $\theta$  corresponds to the addition or subtraction of a whole revolution to or from the rotation of the radius vector, it is obvious that all the circular functions are periodic and have the period  $2\pi$ . This is the smallest period, that is, the period par excellence, in the case of  $\cos \theta$ ,  $\sin \theta$ ,  $\sec \theta$ ,  $\csc \theta$ . It is easily seen, by studying the defining diagram, that  $\tan \theta$  and  $\cot \theta$  have the smaller period  $\pi$ . Thus we have

$$\begin{array}{ll} \cos \left(\theta + 2n\pi\right) = \cos \theta, & \sin \left(\theta + 2n\pi\right) = \sin \theta, \\ \sec \left(\theta + 2n\pi\right) = \sec \theta, & \csc \left(\theta + 2n\pi\right) = \csc \theta, \\ \tan \left(\theta + n\pi\right) = \tan \theta, & \cot \left(\theta + n\pi\right) = \cot \theta. \end{array}$$
 (2).

Besides these relations for whole periods, we have also the following for half and quarter periods:—

$$\cos (\pi \pm \theta) = -\cos \theta, \quad \sin (\pi \pm \theta) = \mp \sin \theta;$$

$$\cos (\frac{1}{2}\pi \pm \theta) = \mp \sin \theta, \quad \sin (\frac{1}{2}\pi \pm \theta) = +\cos \theta;$$

$$\tan (\frac{1}{2}\pi \pm \theta) = \mp \cot \theta, \quad \cot (\frac{1}{2}\pi \pm \theta) = \mp \tan \theta;$$

$$\mathring{X}e_{+}$$

$$(3)$$

all easily deducible from the definition.

We have the following table of zero, infinite, and turning

	θ	0	<u>1</u> π	π	2π	2π	Ac.	
-	$\cos \theta$	+ 1	0	- 1	0	+ 1		
	$\sin \theta$	0	+ 1	0	- 1	-0		
	$\tan \theta$	-0	x	0	x	0	&c.	(4)
	$\cot \theta$	x	0	x	0	x		
	$\sec \theta$	+ 1	x	- 1	x	+ 1		
	$cosec \theta$	x	+ 1	x	- 1	x.	1	

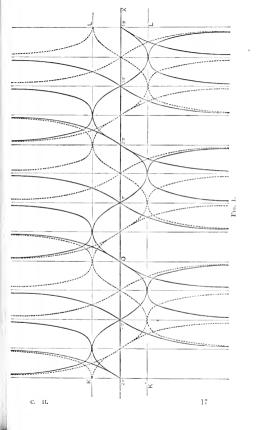
which might of course be continued forwards and backwards by adding and subtracting whole periods

Hence  $\cos \theta$  has an infinite number of zero values corresponding to  $\theta = \frac{1}{2}(2n+1)\pi$ , where n is any positive or negative integer, no infinite values; an infinite number of maxima and of minima values corresponding to  $\theta = 2n\pi$  and  $\theta = (2n+1)\pi$  respectively; and is susceptible of all real algebraical values lying between -1 and +1.

Sin  $\theta$  is of like character.

But  $\tan\theta$  is of quite a different character. It has an infinite number of zero values corresponding to  $\theta = n\pi$ ; an infinite number of infinite values corresponding to  $\theta = \frac{1}{2}(2n+1)\pi$ ; no turning values; and is susceptible of all real algebraical values between  $-\infty$  and  $+\infty$ .

Cot  $\theta$  is of like character.



See  $\theta$  and cosee  $\theta$  have again a distinct character. Each of them has infinite and turning values, and is susceptible of all real algebraical values not lying between -1 and +1. The graphs of the functions  $y=\sin x, y=\cos x$ , &c., are given in Fig. 1. The curves lying wholly between the parallels  $KL_t$ , K'L', belong to  $\cos x$  and  $\sin x$ , the cosine graph being dotted; all that lies wholly outside the parallels  $KL_t$ , K'L', belongs either to  $\sec x$  or to  $\csc x$ , the graph of the former being dotted. The curves that lie partly between and partly outside the parallels  $KL_t$ , K'L', belong either to  $\tan x$  or to  $\cot x$ , the graph of the latter being dotted.

Again, from the geometrical definition combined with elementary considerations regarding orthogonal projection are deduced the following Addition Formulæ:—

$$\cos (\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi; 
\sin (\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi; 
\tan (\theta \pm \phi) = (\tan \theta \pm \tan \phi)/(1 \mp \tan \theta \tan \phi).$$
(5).

As consequences of these, we have the following :-

$$\cos\theta + \cos\phi = 2\cos\frac{1}{2}(\theta + \phi)\cos\frac{1}{2}(\theta - \phi);$$

$$\cos\phi - \cos\theta = 2\sin\frac{1}{2}(\theta + \phi)\sin\frac{1}{2}(\theta - \phi);$$

$$\sin\theta \pm \sin\phi = 2\sin\frac{1}{2}(\theta \pm \phi)\cos\frac{1}{2}(\theta \mp \phi).$$
(6)

$$\cos \theta \cos \phi = \frac{1}{2} \cos (\theta + \phi) + \frac{1}{2} \cos (\theta - \phi);$$
  

$$\sin \theta \sin \phi - \frac{1}{2} \cos (\theta - \phi) - \frac{1}{2} \cos (\theta + \phi);$$
  

$$\sin \theta \cos \phi = \frac{1}{2} \sin (\theta + \phi) + \frac{1}{2} \sin (\theta - \phi).$$
(7)

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta - 2\cos^2 \theta - 1 + 1 - 2\sin^2 \theta \\ = (1 - \tan^{-1}\theta) \cdot (1 + \tan^2 \theta), \\ \sin 2\theta - 2\sin \theta \cos \theta - 2\tan \theta \cdot (1 + \tan^2 \theta), \\ \tan 2\theta - 2\tan \theta \cdot (1 - \tan^2 \theta).$$
(8).

§ 3.] Inverse Circular Functions. When, for a continuum (continuous stretch) of values of y, denoted by (y), we have a relation

$$x = \Gamma(\beta)$$
 (1),

which enables us to calculate a single value of x for each value of y, and the resulting values of x form a continuum (x), then the graph of F(y) is continuous; and we can use it either to find x when y is given, or y when x is given. We thus see that (1) not only determines x as a continuous function of y, but also y as a continuous function of x. The two functions are said to be inverse to each other; and it is usual to denote the latter function by  $F^{-1}(x)$ . So that the equation

$$y = F^{-1}(x)$$
 (2)

is identically equivalent to (1).

It must be noticed, however, that, although  $F^{-1}(x)$  is continuous, it will not in general be single-valued, unless the values in the continuum (x) do not recur. This condition, as the student is already aware, is not fulfilled even in some of the simplest cases. Thus, for example, if  $x=y^p$ , for  $-\infty < y < +\infty$ , the continuum (x) is given by  $0 \geqslant x < +\infty$ ; and each value of x occurs twice over. We have, in fact,  $y=\pm x^3$ ; that is, the inverse function is two-valued.

It is also important to notice that, even when the direct function, F(y), is completely defined for all real values of y, the inverse function,  $F^{-1}(x)$ , may not be completely defined for all values of x.  $F^{-1}(x)$  is, in fact, defined by (1) solely for the values in the continuum (x). Take, for example, the relation  $x=y^2$ , for  $-\infty < y < +\infty$ . The continuum (x) is given by  $0 \geqslant x < +\infty$ ; hence y is defined, by the above relation, as a function of x for values of x between 0 and  $+\infty$  and for no others.

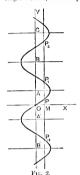
The application of the above ideas to the circular functions leads to some important remarks. It is obvious from the geometrical definition of  $\sin y$  that the equation

$$x = \sin y \tag{3}$$

completely defines x as a single-valued continuous function of y, for  $-\infty < y < +\infty$ . Hence, we may write

$$y = \sin^{-1} x \tag{4},$$

where the inverse function,  $\sin^{-1}x^{\bullet}$ , is continuous, but neither single-valued, nor completely defined for all real values of x.



Since, by the properties of  $\sin y$ , x lies between -1 and +1 for all real values of y,  $\sin^{-1}x$  is, in fact, defined by (3) only for values of x lying between -1 and +1. For other values of x the meaning of  $\sin^{-1}x$  is at present arbitrary.

By looking graphically at the problem "to determine y for any value of x lying between -1 and +1," we see at once that  $\sin^{-1}x$  is multiple-valued to an infinite extent.

If, however, we confine ourselves to values of  $\sin^{-1}x$  lying between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ , we see at once from the graph (Fig. 2) that for any value of x lying between -1 and +1 there is one, and only one, value of  $\sin^{-1}x$ . If we draw parallels to the axis of x through the points A, B, C, ..., A', B', ...

whose ordinates are  $+\frac{1}{2}\pi$ ,  $+\frac{3}{2}\pi$ ,  $+\frac{5}{2}\pi$ , ...,  $-\frac{1}{2}\pi$ ,  $-\frac{3}{2}\pi$ , ..., then between every pair of consecutive parallels we find, for a given value of x (-1 > x > +1), one, and only one, value of  $y = \sin^{-1}x$ .

The values of y corresponding to points between the parallels A and A constitute what we may call the *Principal Branch* of the function. Similarly, the part of the graph between A and B represents the 1st positive branch; the part between B and C the 2nd positive branch; the part between A' and B' the 1st negative branch; and so on.

If, as is usual, we understand the symbol  $\sin^{-1}x$  to give the value of y corresponding to x, for the principal branch only, and use  $y_n$  or  $n \sin^{-1}x$  for the nth branch, then it is easy to see that

$$y_n = \sin^{-1} x = n\pi + (-1)^n \sin^{-1} x$$
 (5),

<sup>&#</sup>x27;This may be read "angle whose sine is x" or "arc-sine x." In Continental work the latter name is contracted into arc-sin x; and this is used instead of sin—x.

where n is a positive or negative integer according as the branch in question is positive or negative.

It is obviously to some extent arbitrary what portion of the graph shall be marked off as corresponding to the principal branch of the function; in other words, what part of the function shall be called the principal branch. But it is clearly necessary, if we are to avoid ambiguity—and this is the sole object of the present procedure—that no value of y should recur within the part selected; and, to secure completeness, all the different values of y should, if possible, be represented. Attending to these considerations, and drawing the corresponding figures, the reader will easily understand the reasons for the following conventions regarding  $\cos^{-1}x$ ,  $\tan^{-1}x$ ,  $\cot^{-1}x$ ,  $\sec^{-1}x$ ,  $\csc^{-2}x$ , wherein y and the inverse functional symbols  $\cos^{-1}x$ , &c., relate to the principal branch only, and  $y_n$  to the nth branch, positive or negative.

$$y = \cos^{-1} x, \quad y \text{ between 0 and } + \pi;$$

$$y_n = (n + \frac{1}{2} + (-)^{n-1} \frac{1}{2})\pi + (-)^{n} \cos^{-1} x.$$

$$y_n = n\pi + \tan^{-1} x.$$

$$y \text{ between } -\frac{1}{2}\pi \text{ and } +\frac{1}{2}\pi;$$

$$y = n\pi + \tan^{-1} x.$$

$$y = \cot^{-1} x. \quad y \text{ between 0 and } \pi;$$

$$y_n = n\pi + \cot^{-1} x.$$

$$y = \sec^{-1} x. \quad y \text{ between 0 and } \pi;$$

$$y_n = (n + \frac{1}{2} + (-)^{n-1} \frac{1}{2})\pi + (-)^{n} \sec^{-1} x.$$

$$y = \csc^{-1} x. \quad y \text{ between 0 and } \pi;$$

$$y_n = (n + \frac{1}{2} + (-)^{n-1} \frac{1}{2})\pi + (-)^{n} \sec^{-1} x.$$

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$$y = \cos^{-1} x$$

Since every function must, in practice, be unambiguously defined, it is necessary, in any particular case, to specify what branch of an inverse function is in question. If nothing is specified, it is understood that the principal branch alone is in question.

It is obvious that all the formulæ relating to direct circular functions could be translated into the notation of inverse circular functions. In this translation, however, close attention must be paid to the points just discussed. Thus If x be positive, the formula  $\cos \theta = \pm \sqrt{(1 - \sin^2 \theta)}$  becomes  $\sin^{-1} x = \cos^{-1} \sqrt{(1 - x^2)}$ :

but if x be negative it becomes

$$\sin^{-1} x = -\cos^{-1} J(1-x^2).$$

If  $0 < x < 1/\sqrt{2}$ ,  $0 < y < 1/\sqrt{2}$ , we deduce from the addition formulæ for the direct functions

$$\sin^{-1} x + \sin^{-1} y = \cos^{-1} \left[ \sqrt{(1 - x^2)(1 - y^2)} - xy \right];$$
if  $0 < x < 1, 0 < y < 1$ .

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} [(x + y)/(1 - xy)].$$

If x and y be both positive, but such that xy > 1, then  $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} [(x+y)/(1-xy)] \circ$ 

and, in general, it is easy to show that

where  $\rho=1$ , 0, or -1, according as  $\tan^{-1}x+\tan^{-1}y$  is greater than  $\frac{1}{2}\pi$ , lies between  $\frac{1}{2}\pi$  and  $-\frac{1}{2}\pi$ , or is less than  $-\frac{1}{2}\pi$ .

# ON THE INVERSION OF $w = z^n$ .

§ 4.] When the argument, and, consequently, in general, the value of the function are not restricted to be real, the discussion of the inverse function becomes more complicated, but the fundamental notions are the same.

For the present it will be sufficient to confine ourselves to the case of a binomial algebraical equation. Let us first consider the case

$$w = z^n \tag{1},$$

where n is a positive integer, z is a complex number, say z = x + yi, and, consequently, w also in general a complex number, say w = u + vi.

To attain absolute clearness in our discussion it will be

<sup>•</sup> In English Text-Books equations of this kind are often loosely stated; and the result has been some confusion in the higher branches of mathematics, such as the integral calculus, where these inverse functions pay an important part.

necessary to pursue a little farther the graphical method of chap. xv., § 17.

It follows from what has there been laid down, and from the fact that any integral function of x and y is continuous for all finite values of x and y, that, if we form two Argand Diagrams. one for x + yi (the z-plane), and one for u + vi (the w-plane), then, whenever the graphic point of  $z^*$  describes a continuous curve, the graphic point of walso describes a continuous curve. In this sense, therefore, the equation (1) defines w as a continuous function of z for all values, real or complex, of the latter. For simplicity in what follows we shall suppose the curve described by z to be the whole or part of a circle described about the origin of the z-plane. We shall also represent z by the standard form  $r(\cos\theta + i\sin\theta)$ , and w by the standard form  $s(\cos \phi + i \sin \phi)$ ; but we shall, contrary to the practice followed in chap. XII., allow the amplitudes  $\theta$  and  $\phi$  to assume negative values. Thus, for example, if we wish to give z all values corresponding to a given modulus r, without repetition of the same value, we shall, in general, cause  $\theta$  to vary continuously from  $-\pi$  to  $+\pi$ , and not from 0 to  $2\pi$ , as heretofore. In either way we get a complete single revolution of the graphic radius; and it happens that the plan now adopted is more convenient for our present purpose.

It is obvious that by varying the amplitude in this way, and then giving all different values to r from 0 to  $+\infty$ , we shall get every possible complex value of z, once over; and thus effect a complete exploration of any one-valued function of z.

Substituting in (1) the standard forms for w and z, and taking, for simplicity, n=3, we have

$$s(\cos \phi + i \sin \phi) = r^3(\cos \theta + i \sin \theta)^3$$
  
=  $r^3(\cos 3\theta + i \sin 3\theta)$  (2)

by Demoivre's Theorem. Hence we deduce

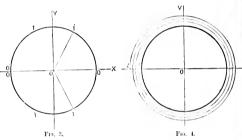
$$s = r^3$$
,  $\phi = 3\theta + 2n\pi$ ;

<sup>\*</sup> For shortness, in future, instead of "graphic point of z" we shall say "z" simply.

or, if (as will be sufficient for our purpose) we confine ourselves to a single complete revolution of the graphic radius of  $z_i$ 

$$s=r^3, \quad \phi = 3\theta$$
 (3).

If, therefore, we give to r any particular value, s has the fixed value  $r^2$ ; that is to say, w describes a circle about the origin of the w-plane (Fig. 4). Also, if we suppose z to describe its circle (Fig. 3) with uniform velocity, since  $\phi = 3\theta$ , w will describe the corresponding circle with a uniform velocity three times as great. To one complete revolution of z will therefore



correspond three complete revolutions of w. In other words, the values in the (w)-continuum which correspond to those in the (z)-continuum are each repeated three times over.\*

The actual course of w is the circle of radius  $r^a$  taken three times over. We may represent this multiple course of w by drawing round its actual circular course the spiral (0, 1, 1, 0, 1', 1, 0'), which re-enters into itself at  $\vec{0}'$  and  $\vec{0}'$ . The actual course may then be imagined to be what this spiral becomes when it is shrunk tight upon the circle.

To indicate this positivity of ic we shall occasionally use the term "Repeating Function." A repeating function need not, however, be periodic as we π<sup>2</sup> is.

If we now letter the corresponding points on the z-circle with the same symbols we have the circle  $\vec{0} \cdot \vec{1} \vec{1}'$  in the w-plane, corresponding to the circular are  $\vec{0} \cdot \vec{1} \vec{1}'$  in the z-plane, and so on, in this sense that, when z describes the are  $\vec{0} \cdot \vec{1} \vec{1}'$ , then w describes the complete circle  $\vec{0} \cdot \vec{1} \vec{1}'$ , and so on.

It follows from this graphical discussion that the equation  $w = z^2$ , which defines w as a one-culted continuous function of z for all values of z, defines z as a three-valued continuous function of w for all values of w.

In other words, since, in accordance with a notation already defined, (1) may be written

$$z = \sqrt[3]{w} \tag{1'},$$

we have shown that the cube root of w is a three-valued continuous function of w for all values of w.

It is obvious that there is nothing in the above reasoning peculiar to the case n=3, except the fact that we have a triple spiral in the w-plane, and a trisected circumference in the z-plane. Hence, if we consider the equation

$$w = z^n$$
 (4),

and its equivalent inverse form

$$z = \sqrt[n]{w} \tag{4'},$$

all the alteration necessary is to replace the triple by an *n*-ple spiral, returning into itself on the negative or positive part of the *u*-axis, according as *n* is odd or even; and the trisected circumference by a circumference divided into *n* equal parts.

Thus we see that the equation (4), which defines w as a continuous one-valued function of z for all values of z, defines z (that is, the nth root of w) as a continuous n-valued function of w for all values of w.

§ 5.] Riemann's Surjace. It may be useful for those who are to pursue their mathematical studies beyond the elements, to illustrate, by means of the simple case at some set, a beautiful method for representing the continuous variation of a repeating function which was devised by the German mathematician Riemann, who ranks, along with Cauchy, as a founder of that branch of modern algebra whose fundamental conceptions we are now explaining.

Instead of supposing all the spires of the w-path in Fig. 4 to lie in one plane, we may conceive each complete spire to lie in a separate plane superposed on the ic-plane. Instead of the single ic-plane, we have thus three separate planes, P., P., P., superposed upon each other. To secure continuity between the planes, each of them is supposed to be slit along the u-axis from 0 to - x; and the three joined together, so that the upper edge of the slit in Pa is joined to the lower edge of the slit in Pa; the lower edge of the slit in Po to the upper edge of the slit in P, the lower edge of the slit in P, to the upper edge of the slit in  $\overline{P}$ , this last junction taking place across the two intervening, now continuous, leaves. We have thus clothed the whole of the w plane with a three-leaved continuous flat belicoidal \* surface, any continuous path on which must, if it circulates about the origin at all, do so three times before it can return into itself. This surface is called a Riemann's Surface. The origin, about which the surface winds three times before returning into itself, is called a Winding Point, or Branch Point, of the Third Order. Upon this three-leaved surface w will describe a continuous single path corresponding to any continuous single path of z, provided we suppose that there is no continuity between the leaves except at the junctions above described.

§ 6.] If we confine  $\theta$  to that part  $\Gamma(01')$  of its circle which is bisected by OX, and  $\phi$  to the corresponding spire  $\Gamma(01')$  of its path, so that  $\phi$  lies between  $-\pi$  and  $+\pi$ , and  $\theta$  between  $-\pi/n$  and  $+\pi/n$ , then z becomes a one-valued function of w for all values of w. We call this the principal branch of the n-valued function  $\gamma'w'$ ; and, as we have the distinct notation  $w^{3n}$  at our disposal, we may restrict it to denote this particular branch of the function z. In other words, if

$$w = s (\cos \phi + i \sin \phi), \quad -\pi < \phi < +\pi$$

we define  $w^{1/n}$  by the equation

$$w^{1,n} = s^{1,n} (\cos, \phi/n + i \sin, \phi/n);$$

and we also restrict  $(\cos\phi + i\sin\phi)^{\nu n}$  to mean  $\cos$ ,  $\phi/n + i\sin$ ,  $\phi/n$ . Just as in § 4, we take the next spire after Tol' in the positive direction (counter-clock) to represent the first positive branch of  $\zeta'w$ ; the next in the negative direction to represent the first negative branch of  $\zeta'w$ ; and so on, the last positive and the last negative brainful spires, or only half spires, according as n is add or even

If, as is usual, we represent the actual analytical value of  $\boldsymbol{w}$ 

Like a spiral staircase.

that is.

by the form  $s(\cos\phi+i\sin\phi)$ , where  $\phi$  is always taken between  $-\pi$  and  $+\pi$ , then it is easy to find expressions for the values of z, belonging to the n-1 positive and negative branches of  $\frac{\pi}{2}/m$  and corresponding to any given value of m, in terms of the value belonging to the principal branch. We have, obviously, merely to add or subtract multiples of  $2\pi$  to represent the successive positive and negative whole revolutions of the graphic radius of m. Thus, if z,  $z_t$ ,  $z_{-t}$  relate to the principal, tth positive, and tth negative branches of  $z = \sqrt[n]{m}$  respectively, we have

$$\begin{aligned} &z = s^{1/n} \{\cos \cdot \phi/n + i \sin \cdot \phi/n\}; \\ &z_t = s^{1/n} \{\cos \cdot (\phi + 2t\pi)/n + i \sin \cdot (\phi + 2t\pi)/n\}; \\ &z_{-t} = s^{1/n} \{\cos \cdot (\phi - 2t\pi)/n + i \sin \cdot (\phi - 2t\pi)/n\}. \end{aligned}$$
 (5).

We have thus been led back by a purely graphical process to results equivalent to those already found in chap. XII., § 18.

Cor. 1. Hence, if z denote the principal value of the nth root of w, and  $\omega_n = \cos \cdot 2\pi/n + i \sin \cdot 2\pi/n$ , then

$$\begin{aligned} z_t &= z \omega_n^t, & z_{-t} &= z \omega_n^{-t}; \\ z_t &= w^{1/n} \omega_n^t, & z_{-t} &= w^{1/n} \omega_n^{-t} \end{aligned}$$
 (6).

Cor. 2. The principal value of the nth root of a positive real number r is the real positive nth root, that is, what has already been denoted by  $r^{th}$  (see chap. x, § 2).

For, in this case, we have w=r (cos  $0+i\sin 0$ ), that is,  $\phi=0$ . Hence  $\sqrt[n]{w}=r^{1/n}$ .

Cor. 3. There is continuity between the last values of any branch of \*\(^1\)\nu and the first values of the next in succession, and between the last values of the last positive branch and the first values of the last negative branch; but elsewhere two values of \*\(^1\)\nu belonging to different branches, and corresponding to the same value of \(^n\), differ by a finite amount.

It should be noticed as a consequence of the above that the principal value of the nth root of a real negative number, such as -1, is not definite until its amplitude is assigned. For we may write  $-1 = \cos x + i \sin x$  or  $= \cos x - n + i \sin (-\pi)$ ; and the principal value in the former case is  $\cos x / n + i \sin x / n$ . in the latter  $\cos (-x / n) + i \sin (-x / n)$ . This ambiguity does not exist for complex numbers differing from -1, even when they differ infinitely little, as will be at once soen by referring to Figs. 3 and 4.

§ 7.] It should be observed that if, instead of restricting  $\phi$  in the expression  $z:s^{i,\eta}\{\cos,\phi/n+i\sin,\phi/n\}$  to lie between  $-\pi$  and  $+\pi$ , we cause it to vary continuously from  $-n\pi$  to  $+n\pi$ , then  $s^{i,\eta}\{\cos,\phi/n+i\sin,\phi/n\}$  varies continuously and passes once through every possible value of  $\sqrt[n]{w}$ , where |w| is given -s.

It follows also that, if w describe any continuous path starting from P and returning thereto, the value of  $\mathbb{C}_{p^{n}}$  will vary continuously; and will return to its original value, if w have circulated round the origin of the w-plane pn times, where p is 0 or any integer; and, in general, will return to its original value multiplied by  $\omega_{n}$ , where t is the algebraical value of  $+\mu - \nu$ ,  $\mu$  and  $\nu$  being the number of times that w has circulated round the origin in the positive and negative directions respectively. On account of this property, the origin is called a Branch Point of  $\mathbb{C}^{lw}$ .

§ 8.] Let us now consider briefly the equation

$$u^{p} = z^{q} \tag{1},$$

where p and q are positive integers. We shall suppose p and q to be prime to each other, because that is the only case with which we shall hereafter be concerned \*.

Our symbols having the same meanings as before, we derive from (1)

$$s^p(\cos p\phi + i\sin p\phi) = r^q(\cos q\theta + i\sin q\theta)$$
 (2),

Hence, taking the simplest correspondence that will give a complete view of the variation of both sides of the equation last written, we have

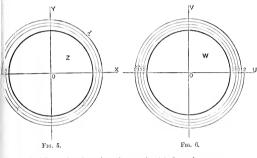
$$s^p = r^q$$
,  $p\phi = q\theta$  (3).

If, then, we fix r, and therefore s, the paths of z and w will be circles about the origins of the z- and w-planes respectively; and, since p is prime to q, if z and w start from the positive part

<sup>•</sup> If p and q had the G.C.M. k, so that p > lp', q > kq', where p' and q' are mute ally prime, then the equation (1) could be written (κ<sup>m</sup>)<sup>1</sup> = (ε<sup>n</sup>), which is equivalent to the k equations, w<sup>1</sup> = ε<sup>n</sup>, w<sup>1</sup> = ω<sub>1</sub> : ε<sup>n</sup>, w<sup>1</sup> = ω<sub>1</sub> : ε<sup>n</sup>, ..., w<sup>2</sup> = ω<sub>2</sub> : ε<sup>n</sup>, ..., w<sup>2</sup> = ω<sub>1</sub> : ε<sup>n</sup>, ..., w<sup>2</sup> = ω<sub>1</sub> : ε<sup>n</sup>, who is a primitive kin root of + 1. Each of these k equations fulls unject the case above describes d.

of the x- and u-axes simultaneously, they will not again be simultaneously at the starting place before z has made p, and w has made q revolutions.

To get a complete representation of the variation we must therefore cause  $\theta$  to vary from  $-p\pi$  to  $+p\pi$ , and  $\phi$  from  $-g\pi$  to  $+q\pi$ . The graphs of z and w will therefore be spirals having p and q spires respectively. To each whole spire of the q-spiral will correspond the p/qth part of the p-spiral. The case where p = 3 and q = 4 is illustrated by Figs. 5 and 6.



It follows, therefore, that the equation (1) determines w as a continuous p-valued function of z, and z as a continuous q-valued function of w. Taking the latter view, and writing (1) in the form

$$z = \sqrt[q]{w^p} \tag{1'},$$

and (3) in the form

$$r = s^{p/q}, \quad \theta = p\phi/q$$
 (3'),

we see that, if we cause  $\phi$  to vary continuously from  $-q\pi$  to  $+q\pi$ , then  $s^{p,q}\left(\cos\frac{p}{q}\phi+i\sin\frac{p}{q}\phi\right)$  will vary continuously through all the values which  $\sqrt[q]{w^p}$  can assume so long as |w|=s, and will return to the same value from which it started. In fact, we

see in general that, if w start from any point and return to the same point again after circulating  $\mu$  times round the origin in the positive direction, and v times in the negative direction, then  $\sqrt[p]{v^w}$  returns to its original value multiplied by  $\cos \sqrt{2pt\pi/q} + i\sin \sqrt{2pt\pi/q}$  where  $t = +\mu - v$ ; that is, by  $\omega_q^{pq}$ , where  $\omega_q$  denotes a primitive qth root of +1.

If, as usual, we divide up the circular graph of w into whole spires, counting forwards and backwards as before, and consider the separate branches of the function  $\frac{f(w)}{2}$  corresponding to these, then each of these branches is a single-valued function of  $\theta$ .

The spire corresponding to  $-\pi < \phi < +\pi$  is taken as the principal spire, and corresponding thereto we have the principal branch of the function  $z = \xi' w^p$ , namely,

$$z = s^{pq} \left\{ \cos \frac{p}{q} \phi + i \sin \frac{p}{q} \phi \right\}, \quad -\pi < \phi < +\pi.$$

For the (+t)th and (+t)th branches respectively, we have  $z_t = s^{p \cdot q} (\cos_{+} p \cdot (\phi + 2t\pi)/q + i \sin_{+} p \cdot (\phi + 2t\pi)/q_D^2)$ 

$$\begin{aligned} &z_t = s \cdot \frac{1}{2} \cos s \cdot p \left( \phi + 2t\pi \right) / q + t \sin s \cdot p \left( \phi + 2t\pi \right) / q_0 \\ &= \omega_q^{pt} z; \\ &z_{-t} = s^{p \cdot q} \left[ \cos s \cdot p \left( \phi - 2t\pi \right) / q + t \sin s \cdot p \left( \phi - 2t\pi \right) / q_0^2 \right] \end{aligned}$$

 $= \omega_q^{-1/2} z.$   $= \omega_q^{-1/2} z.$ 

As before, we may use  $w^{p,q}$  to stand for the principal branch of  $\frac{q}{\delta}w^p$ , and we observe, as before, that the principal value of  $\frac{q}{\delta}w^p$  when w is a real positive quantity is the real positive value of the qth root, that is, what we have, in chap.  $\mathbf{x}$ , denoted by  $w^{p,q}$ .

§ 9. It must be observed that, when p is not prime to q, the expressions of ⟨cos, p⟨φ, 2 trs |q + i sin, p(φ, 2 trs |q) | 1 to longer furnish all the q values of ⟨cos, but us may be easily verified) only q/k of them, where k is the G.C.M. of p and q. The appropriate expression in this case would be set (cos, p(φ, 2 trs |q + i sin, p(q, 2

This has expression gives in all cases the q-different values of  $\xi''''$  but it has this great meanwhilence, that, if we arrange the branches by taking accessively  $t = 0, t = 1, t = 2, \dots$  the end value of each branch is equal, not to the initial value of the successing branch, but to the initial value of a brunch several orders further on. There will therefore be more than one crossing in the graphic epital. The investigation from this point of view will

be a good exercise for the student. When p is prime to q, the two expressions for  $\chi^0 e^p$  are equivalent; and we have preferred to use the one which leads to the simpler graphic spiral.

If we adopt Riemann's method for the graphical representation of the equation  $w^{\mu}=x^{\mu}$ , then we shall have to cover the z-plane with a p-leaved Riemann's surface, having at the origin a winding point of the pth order; and the w-plane with a g-leaved surface, having at the origin a winding point of the q-th order.

#### EXERCISES XIV.

# (1.) Solve the equation

$$\tan^{-1}\{(x+1)/(x-1)\} + \tan^{-1}\{(x+2)/(x-2)\} = \frac{1}{4}\pi$$
,

and examine whether the solutions obtained really satisfy the equation when  $\tan^{-1}$  denotes the principal branch of the inverse function.

(2.) If  $27r^2 \le 4q^3$ , show that the roots of the equation  $x^3 - qx - r = 0$  are  $2(q/3))^{1/2} \cos a$ ,  $2(q/3)^{1/2} \cos (\frac{2}{3}\pi + a)$ ,  $2(q/3)^{1/2} \cos (\frac{2}{3}\pi - a)$ , where a is determined by the equation  $\cos 3a = \frac{1}{2}r(3/q)^{1/2}$ .

Show that the solution of any cubic equation, whose roots are all real, can be effected in this way; and work out the roots of  $x^3-5x+3=0$  to six places of decimals. (See Lock's Higher Trigonometry, § 135, or Todhunter's Trigonometry, 7th ed., § 200.)

Trace the graphs of the following, x being a real argument:-

- (3.)  $y = \sin x + \sin 2x$ .
- $(4.) \quad y = \sin x + \cos 2x.$

(5.)  $y = \sin x \sin 2x$ . (7.)  $y = x \sin x$ . (6.)  $y = \tan x + \tan 2x$ , (8.)  $y = \sin x/x$ ,

- (9.)  $y = \sin 3x/\cos x$ .
- (10.)  $y = \sin^{-1} x^2$ ,
- (11.)  $y^2 = \sin^{-1} x$ . (12.)  $\sin y = \tan x$ .

Discuss graphically the following functional equations connecting the complex variables w and z. In particular, trace in each case the w-paths when the z-paths are circles about the origin of the z-plaue, or parallels to the real and to the imaginary axis.

(13.)  $w^2 = z^3$ .

(14.) w = 1/z.

(15.)  $w = 1/z^3$ .

- (16.)  $w^2 = 1/z^3$ . (18.)  $w^2 = (z - a)^2 (z - b)$ .
- (17.)  $w^2 = (z a) (z b)$ . (19.)  $w^3 = (z - a)^2$ .
- (18.)  $w^2 = (z a)^2 (z b)$ (20.)  $w^2 = (z - a)^3$ .
- (21.) w = (az + b)/(cz + d).
- (22.)  $w^2 = 1/(z a)(z b)$ ,

§ 10.] We can now extend to their utmost generality some of the theorems regarding the summation of series already established in previous chapters.

It is important to remark that the peculiar difficulties of this

part of the subject do not arise where we have to deal merely with a finite summation; that is to say, the summation of a scries to n terms. For any such summation involves merely a statement of the identity of two chains of operations, each containing a finite number of links, and any such identity rests directly on the fundamental laws of algebra, which apply alike to real and to complex quantities.

Even when the series is infinite, provided it be convergent, and its sum be a one-valued function, the difficulty is merely one that has already been fully settled in chap. xxvi.

The fresh difficulty arises when the sum depends upon a multiple-valued function. We have then to determine which branch of the function represents the series; for the series, by its nature, is always one-valued.

We commence with some cases where the last-mentioned point does not arise.

### GEOMETRIC AND INTEGRO-GEOMETRIC SERIES.

# § 11.] The summation

$$1+z+z^{2}+\ldots+z^{n}=(1-z^{n+1})/(1-z)$$
 (1),

since it depends merely on a finite identity, holds for all values of z. We may therefore suppose that  $z = x \circ yi - r(\cos\theta + i\sin\theta)$ , and the equation (1) will still hold.

Also, since  $L_{n=1} z^{n+1} - Lr^{n+1} (\cos n + 1\theta + i \sin n + 1\theta) = 0$ , when r < 1, we have, provided |z| < 1, the infinite summation

$$1 + z + z^2 + \dots$$
 ad  $x + 1/(1-z)$  (2)

for complex as well as for real values of z.

In like manner, the finite summation of the integro-geometric series  $\tilde{\Sigma}\phi(n)z^n$ , which we have seen can always be effected for real values of z (see chap.  $XX_+$ , § 11), holds good for all values of z; and, since  $\tilde{\Sigma}\phi(n)z^n$  is convergent provided |z|<1, the infinite summation deducible from the finite one will hold good for all complex values of z such that |z|<1. By substituting in (1) or (2), and in the corresponding equations for  $\Sigma \phi(n) z^n$ , the value  $r(\cos \theta + i \sin \theta)$  for z, and then equating the real and imaginary parts on both sides, we can deduce a large number of summations of series involving circular functions of nultiples of  $\theta$ .

Example 1. To sum the series

 $S_n = 1 + r \cos \theta + r^2 \cos 2\theta + \dots + r^n \cos n\theta,$ 

 $T_n = r \sin \theta + r^2 \sin 2\theta + \dots + r^n \sin n\theta,$  $U_n = \cos \alpha + r \cos (\alpha + \theta) + r^2 \cos (\alpha + 2\theta) + \dots + r^n \cos (\alpha + n\theta).$ 

 $V_n = \sin \alpha + r \sin (\alpha + \theta) + r^2 \sin (\alpha + 2\theta) + \dots + r^n \sin (\alpha + n\theta),$ 

to n terms; and to  $\infty$  when r < 1. Starting with equation (1), let us put  $z = r(\cos \theta + i \sin \theta)$ , and equate real

and imaginary parts on both sides. We find  $1+r(\cos\theta+i\sin\theta)+r^2(\cos2\theta+i\sin2\theta)+\ldots+r^n(\cos n\theta+i\sin n\theta)$ 

 $1+\tau(\cos\theta+t\sin\theta)+\tau(\cos(\theta+t\sin\theta)+\dots+\tau(\cos(\theta+t\sin\theta))$   $=\{1-\tau^{n+1}(\cos(n+1)\theta+t\sin(n+1)\theta)\}/\{1-\tau(\cos\theta+t\sin\theta)\}$ (3);
where

 $S_n = \{1 - r \cos \theta - r^{n+1} \cos (n+1) \theta + r^{n+2} \cos n\theta\} / \{1 - 2r \cos \theta + r^2\}$  (i);

 $T_n = \{r \sin \theta - r^{n+1} \sin (n+1) \theta + r^{n+2} \sin n\theta \} / \{1 - 2r \cos \theta + r^2\}$  (5). Again, since  $U_n = \cos aS_n - \sin aT_n$ ,

 $V_n = \sin \alpha S_n + \cos \alpha T_n$ ,

we deduce from (4) and (5) the following:—

 $U_n = \{\cos \alpha - r\cos(\alpha - \theta) - r^{n+1}\cos(\overline{n+1}\theta + \alpha) + r^{n+2}\cos(n\theta + \alpha)\}\}$   $\{1 - 2r\cos\theta + r^2\}$  (6),

 $V_n = \left\{ \sin \alpha - r \sin (\alpha - \theta) - r^{n+1} \sin \overline{(n+1\theta + \alpha)} + r^{n+2} \sin (n\theta + \alpha) \right\} / \left\{ 1 - 2r \cos \theta + r^2 \right\}$ (7).

From these results, by putting r = +1, or r = -1, we deduce several important particular cases. For example, (6) and (7) give  $\cos \alpha + \cos (\alpha + \theta) + \cos (\alpha + 2\theta) + \dots + \cos (\alpha + n\theta)$ 

 $=\cos\frac{1}{2}\left(\alpha+(\alpha+n\theta)\right)\sin\frac{1}{2}(n+1)\theta/\sin\frac{1}{2}\theta \quad (6');$  $\sin\alpha+\sin(\alpha+\theta)+\sin(\alpha+2\theta)+...+\sin(\alpha+n\theta)$ 

 $= \sin \frac{1}{2} \left\{ a + (a + n\theta) \right\} \sin \frac{1}{2} \left( n + 1 \right) \theta / \sin \frac{1}{2} \theta$  (7)

Finally, if r < 1, we may make n infinite in (4), (5), (6), (7); and we thus find

 $S_{\infty} = (1 - r \cos \theta)/(1 - 2r \cos \theta + r^2)$  (1");

 $T_{\infty} = r \sin \theta / (1 - 2r \cos \theta + r^2) \qquad (5'');$ 

 $U_x = \{\cos a - r \cos (a - \theta)\}/\{1 - 2r \cos \theta + r^2\}$  (6");

 $V_{\infty} = \{ \sin \alpha - r \sin (\alpha - \theta) \} / \{ 1 - 2r \cos \theta + r^2 \}$ 

(7'').

For brevity, and in order to keep the attention of the reader as closely
as possible to the essentials of the matter, we leave it to him, or to his teacher,
to supply the details of the analysis.

Example 2. Sum to infinity the series

$$S = 1 - 2r \cos \theta + 3r^2 \cos 2\theta - 4r^3 \cos 3\theta + \dots$$
 (r < 1).

If  $z = r (\cos \theta + i \sin \theta)$ , then S is the real part of the sum of the series  $T = 1 - 2x + 3x^2 - 4z^2 + \dots$ 

Now, by chap, xx., § 14. Example 2.

 $T = 1/(1+z)^2$ .

Hence  $S = R \left\{ \frac{1}{(1 + r \cos \theta + r i \sin \theta)^2} \right\},$ 

= 
$$R \{ (1 + r \cos \theta - ri \sin \theta)^2 / (1 + r \cos \theta^2 + r^2 \sin^2 \theta)^3 \},$$
  
=  $(1 + 2r \cos \theta + r^2 \cos 2\theta) / (1 + 2r \cos \theta + r^2)^2$ .

Example 3. Exemplify the fact that every algebraical identity leads to two trigonometrical identities in the particular case of the identity

$$-(b-c)(c-a)(a-b) = bc(b-c) + ca(c-a) + ab(a-b).$$

In the given identity put  $a = \cos a + i \sin a$ ,  $b = \cos \beta + i \sin \beta$ ,  $c = \cos \gamma + i \sin \gamma$ , and observe that

 $\cos\beta+i\sin\beta-\cos\gamma-i\sin\gamma=2i\sin\frac{1}{2}\left(\beta-\gamma\right)\left\{\cos\frac{1}{2}\left(\beta+\gamma\right)+i\sin\frac{1}{2}\left(\beta+\gamma\right)\right\}.$  We thus get

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$$\sin \frac{1}{2} (\beta - \gamma) \{\cos \frac{1}{2} (\beta + \gamma) + i \sin \frac{1}{2} (\beta + \gamma)\}$$

$$= \sum \sin \frac{1}{2} (\beta - \gamma) \{\cos \beta + i \sin \beta\} \{\cos \gamma + i \sin \gamma\} \{\cos \frac{1}{2} (\beta + \gamma) + i \sin \frac{1}{2} (\beta + \gamma)\},$$

whonce

$$\begin{array}{l} 4\cos\left(\alpha+\beta+\gamma\right) \prod \sin\frac{1}{2}\left(\beta-\gamma\right) = \sum \sin\frac{1}{2}\left(\beta-\gamma\right)\cos\frac{3}{4}\left(\beta+\gamma\right); \\ 4\sin\left(\alpha+\beta+\gamma\right) \prod \sin\frac{1}{2}\left(\beta-\gamma\right) = \sum \sin\frac{1}{2}\left(\beta-\gamma\right)\sin\frac{3}{4}\left(\beta+\gamma\right). \end{array}$$

FORMULE CONNECTED WITH DEMONVE'S THEOREM AND THE BINOMIAL THEOREM FOR AN INTEGRAL INDEX.

§ 12.] By chap. xtt., § 15 (3), we have

$$\cos (\theta_1 + \theta_2 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n)$$

$$= (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_3) \dots (\cos \theta_n + i \sin \theta_n).$$

If we expand the right-hand side, and use  $P_r$  to denote  $\Sigma \cos \theta_1 \cos \theta_2 \dots \cos \theta_r \sin \theta_{r+1} \dots \sin \theta_n$ , that is, the sum of all the partial products that can be formed by taking the cosines of r of the angles  $\theta_1, \theta_2, \dots, \theta_n$  and the sines of the rest, then we find that

$$\cos (\theta_1 + \theta_3 + \dots + \theta_n) + i \sin (\theta_1 + \theta_2 + \dots + \theta_n)$$

$$= P_n + i P_{n-1} - P_{n-2} - i P_{n-3} + P_{n-4} + i P_{n-5} - \dots$$

<sup>•</sup> We use I(f(x+yi)) and If(x+yi) to denote the real and imaginary parts of f(x+yi) respectively.

Hence

$$\cos (\theta_1 + \theta_2 + \dots + \theta_n) = P_n - P_{n-2} + P_{n-4} - P_{n-6} + \dots$$
(1);  

$$\sin (\theta_1 + \theta_2 + \dots + \theta_n) = P_{n-1} - P_{n-3} + P_{n-5} - P_{n-7} + \dots$$
(2).

From these, or, more directly, from

$$\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1\cos\theta_2$$

$$\ldots \cos \theta_n (1 + i \tan \theta_1) (1 + i \tan \theta_2) \ldots (1 + i \tan \theta_n),$$
 we derive

 $\tan (\theta_1 + \theta_2 + \dots + \theta_n) = (T_1 - T_2 + T_5 - \dots)/(1 - T_2 + T_4 - \dots)$  (3),  $T_{-} = \Sigma \tan \theta_{1} \tan \theta_{2} \dots \tan \theta_{-}$ where

The formulæ (1), (2), (3) are generalisations of the familiar addition formulæ for the cosine, sine, and tangent,

From the usual form of Demoivre's Theorem, namely,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$$
,

we derive, by expansion of the right-hand side.

 $\cos n\theta + i \sin n\theta = \cos^n \theta + i \cdot nC_1 \cos^{n-1} \theta \sin \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta$  $-i \cdot C_n \cos^{n-3} \theta \sin^3 \theta + C_n \cos^{n-4} \theta \sin^4 \theta + \dots$ 

Hence

$$\cos n\theta = \cos^n \theta - {}_{n}C_2 \cos^{n-2} \theta \sin^2 \theta + {}_{n}C_4 \cos^{n-4} \theta \sin^4 \theta - \dots (4)^*;$$
  

$$\sin n\theta = {}_{n}C_1 \cos^{n-1} \theta \sin \theta - {}_{n}C_2 \cos^{n-2} \theta \sin^2 \theta$$

$$+ {}_{n}C_{5}\cos^{n-5}\theta\sin^{5}\theta - \dots$$
 (5);

$$\tan n\theta = \frac{nC_1 \tan \theta - nC_2 \tan^3 \theta + nC_2 \tan^3 \theta - \dots}{1 - nC_2 \tan^2 \theta + nC_4 \tan^4 \theta - \dots}$$
(6).

These are generalisations of the formula (8) of § 2.

The formulæ (4) and (5) above at once suggest that  $\cos n\theta$ can always be expanded in a series of descending powers of  $\cos \theta$ ; that, when n is even,  $\cos n\theta$  can be expanded in a series of even powers of  $\sin \theta$  or of  $\cos \theta$ ;  $\sin n\theta / \sin \theta$  in a series of odd powers of  $\cos \theta$ ; and  $\sin n\theta/\cos \theta$  in a series of odd powers of  $\sin \theta$ : and, when n is odd,  $\cos n\theta$  in a series of odd powers of  $\cos \theta$ ;  $\cos n\theta/\cos \theta$  in a series of even powers of  $\sin \theta$ ;  $\sin n\theta$  in a series of odd powers of  $\sin \theta$ ;  $\sin n\theta / \sin \theta$  in a series of even powers of  $\cos \theta$ .

The formulæ (4), (5), (6), (8) were first given by John Bernoulli in 1701 (see Op., t. 1., p. 387).

Knowing, a priori, that these series exist, we could in various ways determine their coefficients; or we could obtain certain of them from (1) and (2) by direct transformation, and then deduce the rest by writing  $\frac{1}{2}\pi - \theta$  in place of  $\theta$ . (See Todhunter's Trigonom try, § 286–288.)

We may, however, deduce the expansions in question from the results of chap. xxvii., § 7. If in the equations (9), (10), (97), (10"), (10") there given we put  $a = \cos \theta + i \sin \theta$ ,  $\beta = \cos \theta - i \sin \theta$ , and therefore  $p = 2 \cos \theta$ , q = 1, we deduce

$$\begin{split} 2\cos n\theta &\cdot (2\cos \theta)^{\mathbf{a}} - \frac{n}{1!} (2\cos \theta)^{\mathbf{a}-2} + \frac{n(n-3)}{2!} (2\cos \theta)^{\mathbf{a}-4} - \dots \\ (-)^{\mathbf{r}} \frac{n(n-r-1)(n-r-2) \dots (n-2r-1)}{r!} (2\cos \theta)^{\mathbf{a}-2} + \dots (7)^{\mathbf{a}};\\ \sin n\theta/\sin \theta &= (2\cos \theta)^{\mathbf{a}-1} - \frac{n-2}{1!} (2\cos \theta)^{\mathbf{a}-2} + \frac{(n-3)(n-4)}{2!} \\ &\quad (2\cos \theta)^{\mathbf{a}-2} - \dots (-)^{\mathbf{r}} \frac{(n-r-1)(n-r-2) \dots (n-2r)}{r!} \\ &\quad (2\cos \theta)^{\mathbf{a}-2} - \dots (-)^{\mathbf{r}} \frac{(n-r-1)(n-r-2) \dots (n-2r)}{r!} \\ &\quad (-)^{\mathbf{r}} \frac{n^2(n^2-2^2) \dots (n^2-2s-2^2)}{(2s)} \cos^2 \theta + \dots } (n \text{ even}) \end{aligned}$$

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which he established by an induction based on the previous results of Vieta
regarding the multi-section of an angle.

<sup>•</sup> The series (7), (2), (10) were first given by James Bernoulli in 1702 (see  $Op_+$ ), (4), p. 926). He deduced them from the formula  $2 \sin^2 n\theta = \frac{n^2}{n^2} (2 \sin \theta)^2 - \frac{n^2(n^2 - 1^2)}{n^2} (2 \sin \theta)^2 + \frac{n^2(n^2 - 1^2)}{n^2} (2 \sin \theta)^2 - \dots$ 

$$\begin{split} \sin n\theta / \sin \theta &= (-)^{(n-1)2} \left\{ 1 - \frac{n^2 - 1}{2!} \cos^2 \theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!} \cos^4 \theta - \dots \right. \\ &\left. (-)^s \frac{(n^2 - 1^2)(n^2 - 3^2) \dots (n^2 - 2s - 1^2)}{(2s)!} \cos^{ss} \theta + \dots \right\} (n \text{ odd) (12)}. \end{split}$$

If in the above six formulæ we put  $\frac{1}{2}\pi - \theta$  in place of  $\theta$ , we derive six more in which all the series contain sines instead of cosines. In this way we get, inter alia, the following:—

$$\cos n\theta = 1 - \frac{n^2}{2!} \sin^2 \theta + \frac{n^2 (n^2 - 2^2)}{4!} \sin^4 \theta - \dots (n \text{ even})$$
 (9');

$$\sin n\theta = \frac{n}{1!}\sin \theta - \frac{n(n^2 - 1^2)}{3!}\sin^3\theta + \frac{n(n^2 - 1^2)(n^2 - 3^2)}{5!}\sin^5\theta - \dots$$
(n odd) (10');

$$\sin n\theta/\cos\theta = \frac{n}{1!}\sin\theta - \frac{n(n^2 - 2^2)}{3!}\sin^3\theta + \frac{n(n^2 - 2^2)(n^2 - 4^2)}{5!}\sin^5\theta - \dots$$
(n even) (11');

$$\cos n\theta/\cos \theta = 1 - \frac{n^2 - 1^2}{2!} \sin^2 \theta + \frac{(n^2 - 1^2)(n^2 - 3^2)}{4!} \sin^4 \theta - \dots$$
(n odd) (12').

The formulæ of this paragraph are generalisations of the familiar expressions for  $\cos 2\theta$ ,  $\sin 2\theta$ ,  $\cos 3\theta$ , and  $\sin 3\theta$ , in terms of  $\cos \theta$  and  $\sin \theta$ .

§ 13.] The converse problem to express  $\cos^n \theta$ ,  $\sin^n \theta$ , and, generally,  $\sin^m \theta \cos^n \theta$  in a series of sines or cosines of multiples of  $\theta$ , can also be readily solved by means of Demoivre's Theorem.

If, for shortness, we denote  $\cos \theta + i \sin \theta$  by x, then we have, by Demoivre's Theorem, the following results:—

$$\begin{aligned} x &= \cos \theta + i \sin \theta, \quad 1/x = \cos \theta - i \sin \theta; \\ x^n &= \cos n\theta + i \sin n\theta, \quad 1/x^n = \cos n\theta - i \sin n\theta; \\ \cos \theta &= \frac{1}{2} \left( x + 1/x \right), \quad \sin \theta = \frac{1}{2i} \left( x - 1/x \right); \\ \cos n\theta &= \frac{1}{2} \left( x^n + 1/x^n \right), \quad \sin n\theta = \frac{1}{2i} \left( x^n - 1/x^n \right). \end{aligned}$$

Hence

$$\begin{aligned} \cos^{2m}\theta &= \frac{1}{2^{2m}}(x+1/x)^{2m}, \\ &= \frac{1}{2^{2m}}\{(x^{2m}+1/x)^{2m}\} + \frac{1}{2m}C_1(x^{2m-2}+1/x^{2m-1}) \\ &+ \frac{1}{2m}C_2(x^{2m-4}+1/x^{2m-4}) + \dots + \frac{1}{2m}C_n\}, \\ &= \frac{1}{2^{2m-1}}\{\cos 2m^{2n} + \frac{1}{2m}C_1\cos (2m-2)\theta + \frac{1}{2m}C_2\cos (2m-4)\theta + \dots + \frac{1}{2}\frac{1}{2m}C_n\}. \end{aligned}$$
 Similarly.

 $\cos^{2m+1}\theta = \frac{1}{2^{2m}} \left\{ \cos(2m+1)\theta + \frac{1}{2m+1}C_1\cos(2m-1)\theta + \frac{1}{2m+1}C_2\cos(2m-3)\theta + \dots + \frac{1}{2m+1}C_n\cos\theta \right\}$ (3)

$$\sin^{2m}\theta = \frac{(-1)^m}{2^{2m-1}} \left\{ \cos 2m\theta - \frac{1}{2m}C_1 \cos (2m-2)\theta + \frac{1}{2m}C_2 \cos (2m-4)\theta + \dots (-)^m \frac{1}{2} \frac{1}{2m}C_n \right\} \quad (4);$$

$$\sin^{2m+1}\theta = \frac{(-1)^m}{2^{-m}} \{ \sin(2m+1)\theta - \frac{1}{2^{m+1}}C_1\sin(2m-1)\theta + \frac{1}{2^{m+1}}C_2\sin(2m-3)\theta + \dots , (-)^m \frac{1}{2^{m+1}}C_2\sin\theta \}$$

formulæ  $\sin^2\theta = -\frac{1}{2}(\cos 2\theta - 1)$ ,  $\cos^2\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta)$ , &c. In any particular case, especially when products, such as

 $\sin^{\alpha}\theta\cos^{\alpha}\theta$ , have to be expanded, the use of detached coefficients after the manner of the following example will be found to conduce both to rapidity and to accuracy.

Example 1. To expand  $\sin^2\theta\cos^3\theta$  in a series of sines of multiples of  $\theta$ .

$$\sin^5 \theta \cos^3 \theta = \frac{1}{2^6 i^5} (x - 1/x)^3 (x + 1/x)^3.$$

Starting with the coefficients of the highest power which Lappens to be remembered, say the 4th, we proceed thus—

Coefficients of Multiplier	Confficients of Product.
	1 - 4 + 6 - 4 + 1 1 - 5 + 10 - 10 + 5 - 1
1 - 1	1 - 5 + 10 - 10 + 5 - 1
1+1	1-4+5+0-5+4-1
1 + 1	1 - 3 + 1 + 5 - 5 - 1 + 3 - 1
1 . 1	1 9 9 6 6 10 - 6 19 19 .

The coefficients in the last line are those in the expansion of  $(x-1/x)^2(x+1/x)^3$ . Hence, arranging together the terms at the beginning and end, and replacing

$$\begin{split} \frac{1}{2i}(x^8-1/x^8) \text{ by sin } 8\theta, \frac{1}{2i}(x^8-1/x^9) \text{ by sin } 6\theta, \text{ and so on, we find} \\ \sin^6\theta \cos^3\theta &= \frac{1}{2i}\{\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta + \frac{1}{2}, 0\}, \\ &= \frac{1}{162}\{\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta\}. \end{split}$$

The student will see that  $\sin^m \theta \cos^n \theta$  can be expanded in a series of sines or of cosines of multiples of  $\theta$ , according as m is odd or even. The highest multiple occurring will be  $(m + n) \theta$ .

Example 2. If  $\theta = 2\pi/n$ , and  $\alpha$  any angle whatever, and

$$_{m}U_{n}=\cos^{m}\alpha+\cos^{m}(\alpha+\theta)+...+\cos^{m}(\alpha+\overline{n-1}\theta),$$

 $_{m}V_{n} = \sin^{m}\alpha + \sin^{m}(\alpha + \theta) + ... + \sin^{m}(\alpha + \overline{n-1}\theta).$ 

where m is any positive integer which is not of the form r + sn/2, then

$$_{2m}U_n = {}_{2m}V_n = n.1.3...(2m-1)/2.4...2m;$$
  
 $_{2m+1}U_n = {}_{2m+1}V_n = 0.$ 

This will be found to follow from a combination of the formula of the present paragraph with the summation formulæ of § 11.

### EXERCISES XV

Sum the following series to n terms, and also, where admissible, to infinity:-

- (1.)  $\cos \alpha \cos (\alpha + \theta) + \cos (\alpha + 2\theta) \dots$
- (2.)  $\sin \alpha \sin (\alpha + \theta) + \sin (\alpha + 2\theta) \dots$
- (3.) Σ sin³ nθ. (4.)  $n\cos\theta + (n-1)\cos 2\theta + (n-2)\cos 3\theta + \dots$ (5.)  $\sum \sin n\theta \cos (n+1) \theta$ . (6.)  $\sum \sin n\theta \sin 2n\theta \sin 3n\theta$ .
- (7.)  $\sin \alpha \cos \alpha \sin (\alpha + \theta) + \cos^2 \alpha \sin (\alpha + 2\theta) \dots$ (8.)  $1 + \cos \theta / \cos \theta + \cos 2\theta / \cos^2 \theta + \cos 3\theta / \cos^3 \theta + ...$  to n terms, where
- $\theta = n\pi$ . (9.)  $1 - 2r \cos \theta + 3r^2 \cos 2\theta - 4r^3 \cos 3\theta + ...$ 
  - (10.)  $\sin \theta + 3 \sin 2\theta + 5 \sin 3\theta + 7 \sin 4\theta + . . .$
  - (11.)  $\Sigma n^2 \cos(n\theta + a)$ . (12.)  $\sum n (n+1) \sin (2n+1) \theta$ ,
- (13.)  $\sin 2n\theta 2nC_1 \sin (2n-2)\theta + 2nC_2 \sin (2n-4)\theta . . . (n a positive)$ integer).
- (14.)  $\sin (2n+1) \theta + 2n+1 C_1 \sin (2n-1) \theta + 2n+1 C_2 \sin (2n-3) \theta + ... (n a)$
- positive integer). (15.)  $\sum m(m+1) \dots (m+n-1) r^n \cos(a+n\theta)/n!$  to infinity, m being a positive integer.
  - (16.) Does the function

 $(\sin^2\theta + \sin^2 2\theta + ... + \sin^2 n\theta)/(\cos^2\theta + \cos^2 2\theta + ... + \cos^2 n\theta)$ 

approach a definite limit when  $n = \infty$ ?

(17.) Expand  $1/(1-2\cos\theta.x+x^2)$  in a series of ascending powers of x.

- (18.) Expand  $1/(1-2\cos\theta, x+x^2)^2$  in a series of ascending powers of x.
- (19.) Expand  $(1+2x)/(1-x^3)$  in a series of a-cending powers of x; and show that

$$1 - 3n + \frac{(3n-1)(3n-2)}{2!} - \frac{(3n-2)(3n-3)(3n-4)}{3!} + \dots = (-1)^n.$$

(20.) Show that  $1/(1+x+x^2)=1-x+x^3-x^4+x^6-x^7+x^9-x^{10}+\dots$ ; and that, if the sum of the even terms of this expansion be  $\phi(x)$ , and the sum of the odd terms  $\psi(x)$ , then  $\{\phi(x)\}^2-\{\psi(x)\}^2=\phi(x^2)+\psi(x^2)$ .

Prove the following identities by means of Demoivre's Theorem, or otherwise.  $\Sigma$  and  $\Pi$  refer to the letters  $\alpha, \beta, \gamma$ :—

- (21.)  $\Sigma \sin \alpha/(1 + \Sigma \cos \alpha) = -11 \tan \frac{1}{2}\alpha$ , where  $\alpha + \beta + \gamma = 0$ .
- (22.)  $\sum \sin(\theta \beta) \sin(\theta \gamma) / \sin(\alpha \beta) \sin(\alpha \gamma) = 1$ .
- (23.)  $\sum \sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha + \gamma) \cos \alpha / \sin \frac{1}{2}(\alpha \beta) \sin \frac{1}{2}(\alpha \gamma) = \cos (\alpha + \beta + \gamma).$ (24.)  $\cos \alpha \cos (\alpha - 2\alpha) \cos (\alpha - 2\beta) \cos (\alpha - 2\gamma) + \sin \alpha \sin (\alpha - 2\alpha) \sin (\alpha - 2\beta).$

 $\sin (\sigma - 2\gamma) = \cos 2\alpha \cos 2\beta \cos 2\gamma$ , where  $\sigma = \alpha + \beta + \gamma$ . Expand in series of cosines or sines of multiples of  $\theta$ :—

- (25.)  $\cos^{10} \theta$ . (26.)  $\sin^{7} \theta$ . (27.)  $\sin^{8} \theta$ .
  - (28.)  $\cos^6 \theta \sin^3 \theta$ . (29.)  $\cos^6 \theta \sin^4 \theta$ .
  - Expand in series of powers of sines or cosines:-
- (30.)  $\cos 10\theta$ , (31.)  $\sin 7\theta$ . (32.)  $\sin 3\theta \cos 6\theta$ , (33.)  $\cos m\theta \cos n\theta$ .
  - EXPANSION OF  $\cos \theta$  AND  $\sin \theta$  IN POWERS OF  $\theta$

 $\S$  14.] We propose next to show that, for all finite real values of  $\theta_s$ 

$$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \theta^2/6! + \dots \text{ ad } \infty$$
 (1);

$$\sin \theta = \theta - \theta^3/3! + \theta^5/5! - \theta^7/7! + \dots \text{ ad } \infty$$
 (2).

These expansions\* are of fundamental importance in the part of algebraical analysis with which we are now concerned. They may be derived by the method of limits either from the formulae of § 12, or from two or more of the equivalent formulae of § 13. We shall here choose the former course. It will appear, however, afterwards that this is by no means the only way in which these important expansions might be introduced into algebra.

First given by Newton in his tract Analysis per equationes numero terminorum infinitas, which was shown to Barrow in 1669. The leading idea of the above demonstration was given by Euler (Introd. in Anal. Inf., 1. 1, § 152), but his demonstration was not ricorous in its details.

From (4) and (5) of § 12, writing, as is obviously permissible,  $\theta/m$  in place of  $\theta$ , and taking n=m, we deduce, after a little rearrangement,

$$\cos \theta = \cos^{m} \frac{\theta}{m} \left\{ 1 - \frac{1 - 1/m}{2!} \theta^{2} \left( \tan \frac{\theta}{m} / \frac{\theta}{m} \right)^{2} + \frac{(1 - 1/m) (1 - 2/m) (1 - 3/m)}{4!} \theta^{4} \left( \tan \frac{\theta}{m} / \frac{\theta}{m} \right)^{4} - \dots \right\}$$
(3),  
$$= \cos^{m} \frac{\theta}{m} \left\{ 1 - u_{2} + u_{4} - \dots \right\}, \text{ suy,}$$
(3);

and

$$\sin \theta = \cos^m \frac{\theta}{m} \left\{ \theta \left( \tan \frac{\theta}{m} \middle/ \frac{\theta}{m} \right) - \frac{(1 - 1/m) (1 - 2/m)}{3!} \theta^p \left( \tan \frac{\theta}{m} \middle/ \frac{\theta}{m} \right)^3 + \dots \right\}$$
(4),

= 
$$\cos^m \frac{\theta}{m} \{u_1 - u_3 + \dots\}$$
, say, (4').

Here, from the nature of the original formula, m must be a positive integer; but nothing hinders our giving it as large a value as we please, and we propose in fact ultimately to increase it without limit. On the other hand, we take  $\theta$  to be a fixed finite real quantity, positive or negative.

The series (3), as it stands, terminates; and its terms alternate in sign.

We have

$$\left| \frac{u_{2n+2}}{u_{2n}} \right| = \frac{(1-2n/m)(1-\overline{2n+1}/m)}{(2n+1)(2n+2)} \theta^2 \left( \tan \frac{\theta}{m} / \frac{\theta}{m} \right)^3.$$

Hence, so long as n is finite,

$$L_{m=\infty} \left| \frac{u_{2n+2}}{u_{2n}} \right| = \frac{\theta^2}{(2n+1)(2n+2)}.$$

If, therefore, we take  $2n+1>\theta^*$ , we can always, by taking m large enough, secure that, on and after the term  $u_{2n}$ , the numerical value of the convergency-ratio of the series (3) shall be less than unity.

Strictly speaking, it is sufficient if θ < √{(2n+1)(2n+2)}.</li>

From this it follows that, if  $2n+1>\theta$ , and m be only taken large enough,  $\cos\theta$  will be intermediate in value between

$$\cos^{m}\frac{\theta}{m}\left\{1-u_{2}+u_{4}-\ldots(-)^{n}u_{2n}\right\} \tag{5},$$

and

$$\cos^{m}\frac{\theta}{m}\left\{1-u_{2}+u_{4}-\ldots\left(-\right)^{n}u_{2n}+\left(-\right)^{n+1}u_{2n+2}\right\} \quad (6).$$

Therefore  $\cos \theta$  will always lie between the limits of (5) and (6) for  $m = \infty$ .

Now (see chap. xxv., § 23)

$$L\cos^{m}(\theta/m) = 1$$
,  $Lu_{x} = \theta^{2}/2!$ ,  $Lu_{4} = \theta^{4}/4!$ , . . .  $Lu_{2n} = \theta^{2n}/(2n)!$ ,  $Lu_{2n+2} = \theta^{2n+2}/(2n+2)!$ ,

Hence  $\cos \theta$  lies between

$$1 - \theta^2/2! + \theta^4/4! - \dots (-)^n \theta^{2n}/(2n)!$$

and

$$1 - \theta^2/2! + \theta^4/4! - \dots (-)^n \theta^m/(2n)! + (-)^{n+1} \theta^{2n+2}/(2n+2)!$$
. In other words, provided  $2n+1 > \theta$ ,

$$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \dots - (-)^n \theta^{2n}/(2n)! + (-)^{n+1} R_{2n}$$
 (7),  
where  $R_{2n} < \theta^{2n+2}/(2n+2)!$ .

Here 2n may be made as large as we please, therefore since  $L \xrightarrow{\theta^{m+2}/(2n+2)!} 0$  (chap. xxv., § 15, Example 2), we may write

$$\cos \theta = 1 - \theta^2/2! + \theta^4/4! - \dots$$
, ad  $\infty$  (7').

By an identical process of reasoning, we may show that, provided  $2n+2>\theta^*, \ then$ 

$$\sin \theta = \theta - \theta^{r}/3! + \dots + (-)^{n} \frac{\theta^{2n+1}}{(2n+1)!} + (-)^{n+1} R_{2n+1}$$
(8), where 
$$R_{2n+1} < \frac{\theta^{2n+2}}{(2n+3)!} + \dots + (-)^{n+1} R_{2n+1}$$
(9).

and therefore

$$\sin \theta = \theta - \theta^2/3! + \theta^{-1}5! - \dots \text{ad } \infty$$
 (8').

It has already been shown, in chap. XXVI., that the series (7) and (8) are convergent for all real finite values of  $\theta$ ; they are

More closely, if θ < √{(2n+2)(2n+3)}.</li>

therefore legitimately equivalent to the one-valued functions  $\cos\theta$  and  $\sin\theta$  for all real values of  $\theta$ , that is, for all values of the argument for which those functions are as yet defined. From this it follows that the two series must be periodic functions of  $\theta$  having the period  $2\pi$ . This conclusion may at first sight startle the reader; but he can readily verify it by arithmetical calculation through a couple of periods at least.

When  $\theta$  is not very large, say  $\Rightarrow \frac{1}{4}\pi$ , which is the utmost value of the argument we need use for the purposes of calculation\*, the series converge with great rapidity, five or six terms being amply sufficient to secure accuracy to the 7th decimal place.

We shall not interrupt our exposition to dwell on the many uses of these fundamental expansions. A few examples will be sufficient, for the present, on that head.

Example 1. To calculate to seven places the cosine and sine of the radian.

We have

 $\cos 1 = 1 - 1/2! + 1/4! - 1/6! + 1/8! - 1/10! + R_{10},$ 

$$R_{10} < 1/12!$$
,

 $=\mathbf{1}-\cdot 500,000,0+\cdot 041,666,7-\cdot 001,388,9+\cdot 000,024,8-\cdot 000,000,3+R_{10},\\R_{10}<\cdot 000,000,003,$ 

= .540,302,3.

Similarly,

$$\sin 1 = 1 - 1/3! + 1/5! - 1/7! + 1/9! - R_0,$$

$$R_0 < 1/11! < .000,000,03,$$

= 841,471.0.

The error in each case does not exceed a unit in the 7th place.

Example 2. If  $\theta < 3$ , then  $\theta > \sin \theta > \theta - \frac{1}{2}\theta^3$ ,  $1 - \frac{1}{2}\theta^2 < \cos \theta < 1 - \frac{1}{2}\theta^2 + \frac{1}{2}4\theta^4$ . These inequalities follow at once from (7) and (8) above. They are extensions of those previously deduced, in chap. xxv., § 21, from geometrical considerations.

Example 3. Expand  $\cos(\alpha + \theta)$  in powers of  $\theta$ .

Result.  $\cos (a + \theta) = \cos a \cos \theta - \sin a \sin \theta$ ,

 $=\cos \alpha - \sin \alpha \theta - \cos \alpha \theta^2/2! + \sin \alpha \theta^3/3! + \cos \alpha \theta^4/4! - \dots$ 

<sup>\*</sup> Seeing that the cosine or sine of every angle between  $\frac{1}{4}\pi$  and  $\frac{1}{2}\pi$  is the sine or cosine of an angle between 0 and  $\frac{1}{4}\pi$ .

Example 1. Find the limit of

$$\theta (1 - \cos \theta)/(\tan \theta - \theta)$$
 when  $\theta = 0$ .

 $\theta (1 - \cos \theta)/(\tan \theta - \theta) \text{ when } \theta = 0,$   $L\theta (1 - \cos \theta)/(\tan \theta - \theta) = L \sec \theta L\theta (1 - \cos \theta)/(-\sin \theta - \theta \cos \theta),$ 

$$= 1 \times L\theta (\theta^{2}, 2 - \theta^{4}, 1! + \dots) (\theta - \theta^{2}) \Omega^{4} + \dots - \theta + \theta^{3} [2 - \dots),$$

$$= L (\theta^{3}/2 - \theta^{3}/4! + \dots)/(\theta^{7}/3 + \dots),$$

$$= L (1/2 + P\theta^{2} + \dots)/(1/3 + Q\theta^{2} + \dots),$$

#### EXERCISES XVI.

(1.) Expand  $\sin (\alpha + \theta) \sin (\beta + \theta)$  in powers of  $\theta$ .

(2.) Calculate sin 45 32' 30" to five places of decimals.

(3.) Given  $\tan \theta/\theta = 1001/1000$ , calculate  $\theta$ .

(4.) Expand cos<sup>2</sup> θ, sin<sup>2</sup> θ, and sin<sup>3</sup> θ cos θ in powers of θ; and find the general term in each case.

(5.) Show that cos<sup>m</sup> θ (m a positive integ r) can be expanded in a convergent series of even powers of θ; and that the coefficient of θ<sup>∞</sup> in this expansion is

$$(-)^n \{m^{2n} + {}_mC_1(m-2)^{2n} + {}_mC_2(m-4)^{2n} + ...\}/2^{m-1}(2n)!,$$

(6.) Show that, if m and n be positive integers, and 1 < n < m, then  $m^n - C_1(m-2)^n + C_2(m-4)^n - \ldots = 0$ .

Examine how this result is modified when n = 1, or n = m.

Evaluate the following limits:-

- (7.)  $(\sin^2 m\theta \sin^2 n\theta)/(\cos p\theta \cos q\theta)$ ,  $\theta = 0$ .
- (8.)  $\{\sin p (\alpha + \theta) \sin p\alpha\}/\theta$ ,  $\theta = 0$ .
- (9.)  $\{\sin^n p (a+\theta) \sin^n na\}/\theta$ ,  $\theta = 0$ .
- (10.)  $\{\sin^n p(\alpha + \theta)\cos(\alpha + \theta) \sin^n pa\cos\alpha\}/\theta$ ,  $\theta = 0$ .
- (11.)  $(a^{\theta} \sin a\theta b^{\theta} \sin b\theta)/(b^{\theta} \tan a\theta a^{\theta} \tan b\theta), \quad \theta = 0.$
- (12.)  $1/2x^2 \pi/2x \tan \pi x 1/(1-x^2)$ , x = 1 (Euler).
- (13.)  $\{\sin x/x\}^{1/x}$ , x=0.
- (14.)  $\{(x/a)\sin(a/x)\}^{x^m}$ , x = x, (m = >2).
- (15.) Show, by employing the process used in chap, xxvii., § 2, that the series for  $\sin n\theta/\cos \theta$  in powers of  $\sin \theta$  can be derived from the series for  $\cos n\theta$  in powers of  $\sin \theta$ ; and so on.
  - (16.) Show, by using the process of chap.  $xxvn... \S 2$ , twice over, that, if  $\cos n\theta = 1 + A_1 \sin^2 \theta + A_2 \sin^4 \theta + \dots + A_r \sin^{rr} \theta + \dots$ ,

then

$$\begin{array}{l} -n^2\cos n\theta = 2A_1 + \left(3\cdot 4A_2 - 2^zA_1\right)\sin^2\theta + . \quad . \\ & + \left\{(2r+1)\left(2r+2\right)A_{r^4} - (2r)^2A_r\right\}\sin^{2r}\theta + . \quad . \quad . \end{array}$$

Hence determine the coefficients  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_4$ ,  $A_5$ ,  $A_6$ ; and, by combining Exercise 15 with Exercise 16, deduce all the series (7), . . . . (12) of § 12.

(17) Show (from § 13) that cos<sup>n</sup>θ and sin<sup>n</sup>θ can each be expanded in a convergent series of powers of θ; and find an expression for the coefficient of the general term in each case.

In particular, show that

$$\sin^2 x/3! = x^2/3! - (1+3^2)x^5/5! + (1+3^2+3^4)x^7/7! - (1+3^2+3^4+3^6)x^7/9! + \dots$$

BINOMIAL THEOREM FOR ANY COMMENSURABLE INDEX.

§ 15.] If, as in chap. xxvII., § 3, we write

$$f(m) = 1 + \sum_{m} C_{n} z^{n}$$
 (10),

where m is any commensurable number as before, but z is now a complex variable, then, so long as |z| < 1,  $\Sigma_m C_u z^n$  will (chap. XXVI., § 3) be an absolutely convergent series; and f(m) will be a one-valued continuous function both of m and of z. Hence the reasoning of chap. XXVII., § 3, which established the addition theorem  $f(m_i)f(m_2) = f(m_1 + m_2)$  will still hold good; and all the immediate consequences of this theorem—for example, the equations (4), (5), (6), (7), (8), (9) in the paragraph referred to—will hold for the more general case now under consideration.

In particular, if p and q be any positive integers (which for simplicity, we suppose prime to each other), then

$$\{f(p/q)\}^q = \{f(1)\}^p,$$
  
=  $(1+z)^p$  (11).

It follows that f(p/q) represents part of the q-valued function  $\mathcal{J}(1+z)^p$ ; and it remains to determine what part.

Let  $z = r (\cos \theta + i \sin \theta)$ , then, since we have merely to explore the variation of the one-valued function f(p/q), it will be sufficient to cause  $\theta$  to vary between  $-\pi$  and  $+\pi$ .

Also, let

$$\begin{aligned} w &= 1 + z = 1 + x + yi, \\ &= 1 + r\cos\theta + ir\sin\theta, \\ &= \rho(\cos\phi + i\sin\phi), \end{aligned}$$
 (a)

so that

$$\rho = \{(1+x)^2 + y_\perp^{211/2} = (1+2r\cos\theta + r^2)^{1/2}; \\ \tan\phi = y/(1+x) = r\sin\theta/(1+r\cos\theta),$$
 (b).

If we draw the Argand diagram for w=1+x+yi, we see that when r is given w describes a circle of radius r, whose centre is the point (1, 0). Since r<1, this circle falls short of the origin. Hence  $\phi$ , the inclination to the x-axis of the vector drawn from the origin to the point w, is never greater than

 $\tan^{-1}\{r/(1-r^2)^{-2}\}$ , and never less than  $-\tan^{-1}\{r/(1-r^2)^{-2}\}$ . Hence  $\phi$  lies in all cases between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ . Therefore, since f(p,q) is continuous, only one branch of the function  $\sqrt[q]{(1+z)^p}$  is in question. Now, if we denote the principal branch by  $(1+z)^{p,q}$ , so that

$$(1+z)^{p/q} = \rho^{1/q} (\cos p \phi/q + i \sin p \phi/q),$$

we have, by § 8,

where

$$\sqrt[q]{(1+z)^p} = (1+z)^{p \cdot q} \omega_q^{pt}$$
 (12),

where  $t=0, \pm 1, \pm 2, \ldots$ , according to the branch of the function which is in question. Hence we have

$$f(p/q) = (1+z)^{p-q} \omega_q^{pq}$$
,

where t has to be determined.

Now, when z = 0, we have f(p/q) = 1, hence we must have  $1 = \omega^{-p}$ .

Hence t = 0, and we have

$$f(p/q) = (1+z)^{p \cdot q} = \rho^{p \cdot q} (\cos \cdot p \phi/q + i \sin \cdot p \phi/q),$$
$$-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi.$$

Next consider any negative commensurable quantity, say p/q. Then (by chap. XXVII., § 3 (9)),

$$f(-p'q) = f(0)/f(p/q),$$
  
=  $1/f(p/q),$ 

If, therefore, we define  $(1+z)^{-p\cdot q}$  to mean the reciprocal of the principal value of  $(1+z)^{p\cdot q}$ , we have

$$f(-p/q) - (1+z)^{-p/q} = 1/(1+z)^{p/q}$$

$$= \rho^{-p/q} \{\cos(-p\phi/q) + i\sin(-p\phi/q)\} \quad (13)$$

To sum up: We have now established the following expansion for the principal value of  $(1+z)^n$ , in all cases where m is any commensurable number, and |z| < 1:—

$$(1+z)^m - 1 + \sum_m C_n z^n \tag{14},$$

The theorem may also be written in the following forms:—  $1+\sum_{m}t'_{n}(x+yi)^{n}=\{(1+x)^{2}+y\gamma^{n+2}\left[\cos x\right] \max^{-1}\{y/(1+x)\}$ 

+ 
$$i \sin . m \tan^{-1} \{y_i(1+x)\}$$
 (15);

 $1 + \sum_{m} C_n r^n (\cos n\theta + i \sin n\theta)$ 

$$= (1 + 2r \cos \theta + r^2)^{m/2} (\cos m\phi + i \sin m\phi),$$

where  $-\frac{1}{2}\pi < \phi = \tan^{-1} \left\{ r \sin \theta / (1 + r \cos \theta) \right\} < +\frac{1}{2}\pi$  (16).

§16.] The results of last paragraph were first definitely established by Cauchy\*. In a classical memoir on the present subject<sup>†</sup>, Abel demonstrated the still more general theorem

$$1+\Sigma_{m+ki}C_n(x+yi)^n$$

$$= [(1+x)^2 + y^2]^{-2} [\cos \{m \tan^{-1} \{y/(1+x)\} + \frac{1}{2}k \log \{(1+x)^2 + y^2\}\} + i \sin \{m \tan^{-1} \{y/(1+x)\} + \frac{1}{2}k \log \{(1+x)^2 + y^2\}\}]$$

$$= \exp [-k \tan^{-1} \{y/(1+x)\}].$$

Into the proof of this theorem we shall not enter, as the theorem itself is not necessary for our present purpose.

§ 17.] The demonstration of § 15 fails when |z|=1. Here, however, the second theorem of Abel, given in chap. XXVI, § 20, comes to our aid. From it we see that the summation of, say, (16) will hold, provided the series on the left hand remain convergent when r=1.

Now the series  $1 + \sum_{m} C_{m} (\cos n\theta + i \sin n\theta)$  will be convergent if, and will not be convergent unless, each of the series

$$S = 1 + \sum_{m} C_n \cos n\theta,$$
  

$$T = \sum_{m} C_n \sin n\theta$$

be convergent.

In the first place, we remark that, if m < -1,  $L_m C_n = \pm \infty$  when  $n = \infty$ , so that neither of the series S, T can be convergent.

If m=-1, then  ${}_{m}C_{n}=(-1)^{n}$ ,  $S=1+\Sigma(-1)^{n}\cos n\theta$ ,  $T=\Sigma(-1)^{n}\sin n\theta$ , neither of which is convergent (see chap. xxvi., § 9).

If -1 < m < 0, then  $L_m C_n = 0$ ; and the coefficients ultimately alternate in sign. Hence, by chap. XXVI., § 9, both the series S and T are convergent, provided  $\theta + \pm \pi$ . When  $\theta$  has one or other of these excepted values, then  $S = 1 + \Sigma (-1)^n {}_m C_n$ , which is divergent when m lies between -1 and 0 (see chap. XXVI., § 6, Example 3).

<sup>•</sup> See his Analyse Algébrique

<sup>†</sup> Cluvres Completes (ed. by Sylow & Lie), t. 1., p. 235.

If m>0, then, as we have already proved (see chap. xxvi., § 6, Example 4),  $\Sigma_m C_n$  is absolutely convergent, and, a fortiori,  $1+\Sigma_m C_n\cos u\theta$  and  $\Sigma_m C_n\sin u\theta$  are both absolutely convergent.

It follows, therefore, that the equation

$$(1+z)^m = 1 + \sum_m C_n z^m$$

will hold when |z|=1, in all cases where m>0; and also when m lies between -1 and 0, provided that in this last case the imaginary part of z do not vanish, that is, provided the amplitude of z is not  $\pm \pi$ .

In other cases where |z|=1, the theorem is not in question, owing to the non-convergency of  $\Sigma_m C_n z^n$ .

In all cases where |z| > 1, the series  $\Sigma_m C_n z^n$  is divergent, and the validity of the theorem is of course out of the question.

EXPONENTIAL AND LOGARITHMIC SERIES—GENERALISATION
OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

§ 18.1 The series

$$1+z+z^2/2!+z^3/3!+\ldots$$

is absolutely convergent for all complex values of z having a finite modulus (see chap. xxvi., § 10). Hence it defines a single-valued continuous function of z for all values of z. We may call this function the Exponential of z, or shortly  $\operatorname{Exp} z^{\bullet}$ ; so that  $\operatorname{Exp} z$  is defined by the equation

Exp 
$$z = 1 + z + z^2$$
,  $2! + z^3/3! + ...$  (1).

The reasoning of chap. XXVIII., § 5, presupposes nothing but the absolute convergence of the Exponential Series, and is therefore applicable when the variable is complex. We have therefore the following addition theorem for the function Expz:—

<sup>•</sup> When it is necessary to distinguish between the general function of a complex variable r and the ordinary exponential function of a real variable x, we shall use Exp with a capital letter for the former, and either d'origin for the latter. After the student fully understands the theory, he may of course drop this distinction. It seems to be forgotten by some writers that the cin c<sup>41</sup> is a mera nomina numbra—a contraction for the name of a function, and not 2.71825... Oblivious of this fact has fed to some strange pieces of mathematical logic.

$$\operatorname{Exp} z_1 \operatorname{Exp} z_2 \dots \operatorname{Exp} z_m = \operatorname{Exp} (z_1 + z_2 + \dots + z_m) \quad (2),$$

where  $z_1, z_2, \ldots, z_m$  are any values of z whatever.

In particular, we have, if m be any positive integer,

$$(\operatorname{Exp} z)^m = \operatorname{Exp} (mz) \tag{3}.$$

Also

$$\operatorname{Exp} z \operatorname{Exp} (-z) = \operatorname{Exp} 0,$$

$$=1:$$

and therefore

$$\operatorname{Exp}(-z) = 1/\operatorname{Exp} z \tag{4}.$$

We have, further,

Exp 1 = 1 + 1 + 1/2! + 1/3! + . . .,  
= 
$$e$$
 (5):

and, if x be any real commensurable number,

Exp 
$$x = 1 + x + x^2/2! + x^3/3! + \dots,$$
  
=  $e^x$  (6),

by chap. xxviii, where  $e^x$  denotes, of course, the principal value of any root involved if x be not integral.

It appears, therefore, that Exp. coincides in meaning with ex, so far as ex is yet defined.

We may, therefore, for real values of x and for the corresponding values of y, take the graph of y = Exp.x to be identical with the graph of  $y = e^x$ , already discussed in chap, XXI. Hence the equation

$$y = \operatorname{Exp} x$$
 (7)

defines x as a continuous one-valued function of y, for all positive real values of y greater than 0. We might, in fact, write (7) in the form

$$x = \operatorname{Exp}^{-1} y \tag{8}$$

and it is obvious that Exp-1y may, for real calues of y greater than 0, be taken to be identical with logy as previously defined.

If we consider the purely imaginary arguments +iy and -iy, we have, by the definition of Exp z,

$$\begin{aligned} & \operatorname{Exp} \left( + iy \right) = 1 + iy - y^2 / 2! - iy^2 / 3! + y^4 / 4! + iy^2 / 5! - \dots, \\ & = \left( 1 - y^2 / 2! + y^4 / 4! - \dots \right) \\ & + i \left( y - y^2 / 3! + y^2 / 5! - \dots \right), \\ & = \cos y + i \sin y \end{aligned} \tag{9} ; \\ & \operatorname{Exp} \left( - iy \right) = \left( 1 - y^2 / 2! + y^4 / 4! - \dots \right) \\ & - i \left( y - y^2 / 3! + y^2 / 5! - \dots \right), \\ & = \cos y - i \sin y \end{aligned} \tag{9},$$

Finally, by the addition theorem,

$$\begin{aligned} & \operatorname{Exp} (x + yi) = \operatorname{Exp} (x) \operatorname{Exp} (yi), \\ & = e^{x} (\cos y + i \sin y) \end{aligned} \tag{10}.$$

The General Exponential Function is therefore always expressible by means of the Elementary Transcendental Functions  $e^x$ ,  $\cos y$ ,  $\sin y$ , already defined.

Inasmuch as the function  $\operatorname{Exp} z$  possesses all the characteristics which  $e^x$  has when z is real, and is identical with  $e^x$  in all cases where  $e^x$  is already defined, it is usual to employ the notation  $e^x$  for  $\operatorname{Exp} z$  in all cases. This simply amounts to defining  $e^x$  in all cases by means of the equation

$$e^z = 1 + z + z^2/2! + z^3/3! + \dots$$

which, as we now see, will lead to no contradiction.

§ 19.] Graphic Discussion of the General Exponential Function—Definition of the General Logarithmic Function. Let w be defined as a function of z by the equation

$$w \in \operatorname{Exp} z$$
 (1);

and let z = x + yi, and  $w - u + ri = s(\cos\phi + i\sin\phi)$ . Then, since Exp  $(x + yi) = e^x(\cos y + i\sin y)$ , we have

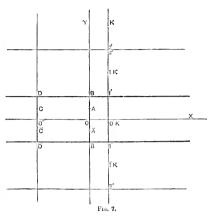
$$s(\cos\phi + i\sin\phi) = e^{\epsilon}(\cos y + i\sin y) \tag{2}.$$

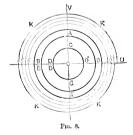
Hence

$$s = e^x$$
,  $\phi = y$  (3),

where we take the simplest relation between the amplitudes that will suit our purpose.

Suppose now that in the z-plane (Fig. 7) we draw a straight line 2 TT2 parallel to the y-axis, and at a distance x from it.





Then, if we cause z to describe this line, x will remain constant, and therefore ex will remain constant; that is to say, the point w will describe a circle (K) (Fig. 8) whose radius is  $e^x$  about the origin in the w-plane. If we draw parallels to the x-axis in the z-plane, at distances  $0.1' = \pi$ ,  $0.2' = 3\pi$ , ..., above, and  $0.1' = \pi$ ,  $0.2' = 3\pi$ . . . . below, then, as y varies from  $-\pi$  to  $+\pi$ , z travels from  $\bar{1}'$ to 1': as y varies from  $+\pi$  to  $+3\pi$ , z travels from 1' to 2', and so on : and each of these pieces of the straight line corresponds to the circumference of the circle & taken once over. To make the correspondence clearer, we may, as heretofore, replace the repeated circle K by a spiral supposed ultimately to coincide with it. Then to the infinite number of pieces, each equal to  $2\pi$ , on the line K corresponds an infinite number of spires of the spiral K.

In like manner, to every parallel to the y-axis in the z-plane corresponds a spiral circle in the w-plane concentric with the circle K. To the axis of y itself corresponds the spiral circle  $B\overline{A}OAB$  of radius unity; to the parallel  $\overline{D}O''D$  to the left of

the y-axis the spiral circle  $\bar{D}O'D$ ; and so on.

To the whole strip between the infinite parallels DB and DB corresponds the whole of the w-plane taken once over: namely, to the right half of the infinite strip corresponds the part of the w-plane outside the circle BAOAB; to the left half of the strip the part of the w-plane inside the circle BAOAB

To each such parallel strip of the z-plane corresponds the

whole of the w-plane taken once over.

Hence the values of w are repeated infinitely often, and we see that the equation (1) defines in as a continuous periodic function of z having the period 2 mi.

Conversely, the above graphic discussion shows that the equation (1) delines z a a continuous x -ple valued function of w.

Taking the latter view, we might write the equation in the torm

$$z = \text{Lxp}^{-1} w$$
 (1').

Instead of  $\exp^{-i} w$  we shall, for the most part, employ the more usual notation  $\log w$ , using, however, for the present at least, a capital letter to distinguish from the one-valued function  $\log y$ , which arises from the inversion of  $y = e^x$ , when x and y are both restricted to be real.

In accordance with the view we are now taking, we may write (3) in the form

$$x = \log s$$
,  $y = \phi$ .

Hence z = Log w

gives  $x + yi = \text{Log} \{s(\cos \phi + i \sin \phi)\},$ where  $x = \log s$ , and  $y = \phi$ .

In other words, we have

Log  $w = \log |w| + i \operatorname{amp}(w)$  (2'):

and, if we cause  $\phi$  (that is, amp (w)) to vary continuously through all values between  $-\infty$  and  $+\infty$ , then the left-hand side of the equation (2') will vary continuously through all values which  $\log w$  can assume for a given value of |w|.

If we confine  $\phi$  to lie between  $-\pi$  and  $+\pi$ , then Log w becomes one-valued: and we have

$$\text{Log } w = \log s + i\phi$$
 (4),

where  $s = |w| = \sqrt{(u^2 + v^2)}$ , and  $\cos \phi = u/\sqrt{(u^2 + v^2)}$ ,  $\sin \phi = v/\sqrt{(u^2 + v^2)}$ ,  $-\pi > \phi > +\pi$ .

This is called the principal branch of Log w; and we may denote it by z.

It is obvious from the graphic discussion that, if  $z_t$  or  $_tLog\ w$  denote the value of  $Log\ w$  in its t-th branch, z being the value in the principal branch corresponding to the same value of w (that is, a value of w whose amplitude differs by an integral multiple of  $2\pi$ ), then

$$_{t}\operatorname{Log} w = z_{t} = z + 2t\pi i,$$

$$= \log s + i (\phi + 2t\pi)$$
(5),

where  $\phi$  is the amplitude (confined between the limits  $-\pi$  and  $+\pi$ ) of w, and t is any integer positive or negative.

If w be a real positive quantity, =u say, then s=|w|=u,  $\phi=$  amp w=0; and we have, for the principal value of  $\log u$ ,

$$\text{Log } u = \log u.$$

Hence, for real positive values of the argument, log u is the principal value of Log u. The other values are of course given by Log u = log u + 2t $\pi$ i, t being the order of the branch.

We have also the following particular principal values:-

Log 
$$(+i) = \frac{1}{2}\pi i$$
,  
Log  $(-i) = -\frac{1}{2}\pi i$ ,

 $Log(-1) = -\frac{1}{2}\pi i,$  $Log(-1) = +\pi i:$ 

the principal value in the last case is not determinate until we know the amplitude; and the same applies to all purely real negative arguments.

§ 20.] Definition of  $Exp_{a}z$ . The meaning of  $a^{z}$ , or, as it is sometimes written,  $Exp_{a}z$ , has not as yet been defined for values of z which are not real and commensurable.

We now define it to mean  $\text{Exp}(z, A\log a)$ , where  $A\log a$  is the *t*-th branch of the inverse function  $\log a$ , and *t* may have any positive or negative integral value including 0.

Thus defined,  $a^{2}$  is in general multiple-valued to an infinite extent. In fact, since  $A \log a = \log s + i (\phi + 2t\pi)$ , where s = |a|, and  $\phi = \sup a (-\pi < \phi < +\pi)$ , we have, if z = x + yi,

$$a^{c+pi} = \operatorname{Exp}\left[ (x+yi) \left\{ \log s + i \left( \phi + 2t\pi \right) \right\} \right],$$

$$=\operatorname{Exp}\big[\big\{x\log s-\left(\phi+2t\pi\right)y\big\}+i\left\{y\log s+\left(\phi+2t\pi\right)x\big\}\big],$$

$$= \exp \{x \log s - (\phi + 2t\pi) y\} \cdot [\cos \{y \log s + (\phi + 2t\pi) x\}]$$

+ 
$$i \sin \{y \log s + (\phi + 2t\pi) x\}$$
] (1).

If we put t = 0, that is, take the principal branch of Log a, in the defining equation, then we get what may be called the principal branch of  $a^{r+yi}$ , namely,

$$a^{x+yi} = \operatorname{Exp} (z \operatorname{Log} a),$$

$$=\exp\{x\log s-\phi y\},\left[\cos\{y\log s+\phi x\}+i\sin\{y\log s+\phi x\}\right]\ \ (2),$$

The value given in (1) would then be called the t-th branch, and might for distinction be denoted by  $e^{x^{x+yt}}$  or by  $t \text{Exp.}_{\alpha}(x+yt)$ .

It is important to notice that the above definition of  $a^*$  agrees with that already given for real commensurable values of z provided we take the corresponding branches. In fact, when y=0, (1) gives

$$u^x = \exp(x \log s) \cdot [\cos(\phi + 2t\pi) x + i \sin(\phi + 2t\pi) x];$$

that is, if x = p/q,

 $[s(\cos\phi + i\sin\phi)]^{p/q}$ 

$$= s^{p \cdot q} [\cos \cdot (\phi + 2t\pi) p/q + i \sin \cdot (\phi + 2t\pi) p/q]$$
 (3);

the right-hand side of which is the t-th branch of the left as ordinarily defined.

Cor. It follows from the above that when x is an incommensurable number the function at has an infinite number of values even when both a and x are real.

The principal value of  $a^x$ , however, when both a and x are real and a is positive, is  $\exp(x \log a)$ , which differs infinitely little from the principal value of  $a^x$ , if  $x^t$  be a commensurable anantity differing infinitely little from x.

§ 21.] The Addition Theorem for Log z.

By the result of § 19 we have

 $_{m}$ Log  $w_{1} + _{n}$ Log  $w_{2}$ 

$$= \log |w_1| + \log |w_2| + i \operatorname{amp} w_1 + i \operatorname{amp} w_2 + 2(m+n) \pi i.$$

Now (chap. XIL, § 15)  $|w_1|$   $|w_2| = |w_1|w_2|$ , and, if amp  $(w_1|w_2)$  were not restricted in any way, we should have amp  $w_1$ + amp  $w_2$  = amp  $(w_1|w_2)$ . Since, however, amp  $(w_1|w_2)$  is restricted in the definition of  $\log(w_1|w_2)$  to lie between  $-\pi$  and  $\pi$ , we have

$$amp \ w_1 + amp \ w_2 = amp \ (w_1 \ w_2) + 2\rho \pi$$

where  $\rho = +1$ , 0, or -1 according as amp  $w_1 + \text{amp } w_2 > +\pi$ , lies between  $+\pi$  and  $-\pi$ , or  $<-\pi$ . Hence we have

$$_{m}\text{Log }w_{1} + _{n}\text{Log }w_{2} = _{m+n+p}\text{Log }(w_{1}w_{2})$$
 (1),

where  $\rho$  is as defined.

In like manner, it may be shown that

$${}_{m}\operatorname{Log} w_{1} - {}_{n}\operatorname{Log} w_{2} = {}_{m-n+\rho}\operatorname{Log} (w_{1}/w_{2}) \qquad (2)$$

where  $\rho=+1$ , 0, or -1 according as amp  $w_1$  – amp  $w_2>+\pi$ , between  $+\pi$  and  $-\pi$ , or  $<-\pi$ .

Taking the definition of  $a^{x+yt}$  given in § 20, and making use of equation (1) of that paragraph, we have

 $_{i}\operatorname{Log}_{\mathcal{A}}a^{r+yi} = \operatorname{log}_{\mathcal{A}}a^{r+yi} + (\operatorname{amp}_{\mathcal{A}}a^{r+yi} + 2k\pi) i.$ 

 $=x\log s-(\phi+2t\pi)\,y+\{y\log s+(\phi+2t\pi)\,x|\ i+2\ (k+l)\,\pi i,$  where l is an integer, positive or negative, chosen so that

$$-\pi < y \log s + (\phi + 2t\pi) x + 2t\pi < +\pi$$

Hence

$$_{k}\text{Log }_{\ell}e^{x+yi} = (x+yi)\{\log s + (\phi + 2t\pi) \ i\} + 2(k+l)\pi i,$$

$$= (x+yi)\{\log a + 2(k+l)\pi i\}$$
(3).

The equations (1), (2), (3) are generalisations of formulæ for  $\log x$  with which the reader is already familiar.

If we confine each of the multiple-valued functions 'Log and 'Expa to its principal branch, we have

$$\operatorname{Log} a^{x+yi} = (x+yi)\operatorname{Log} a + 2l\pi i \tag{3'},$$

where I is so chosen that

$$-\pi < y \log s + \phi x + 2l\pi < +\pi$$
.

§ 22.] Expansion of Log (1+z) in powers of z.

Consider first the principal branch of the function Log (1+z). By the definition and discussion of  $\lesssim 20$ , we see that, when x is any real quantity, the principal branch of  $(1+z)^z$  has for its value Exp  $(x \log (1+z))$ . Hence we have

$$(1+z)^x = 1 + \{x \operatorname{Log}(1+z)\} + \{x \operatorname{Log}(1+z)\}^2/2! + \dots;$$

and, since the series  $1 + \Sigma_x C_n z^n$  represents the principal branch of  $(1+z)^x$ , we have

$$1 + \sum_{z} C_{n} z^{n} = 1 + \{x \operatorname{Log} (1 + z)\} + \dots$$

Now all the conditions involved in the reasoning of chap, xxviii., § 9, will be fulfilled here, provided the complex variable z be so r -stricted that |z| < 1.

Hence, if |z| < 1, we must have, as before,

$$\text{Log}(1+z) = z - z^2 \cdot 2 + z^3 \cdot 3 - z^4 / 4 + \dots$$
 (1).

In other words, so long as |z| < 1, the series  $z - z^2/2 + z^3/3 + \dots$  represents the principal branch of  $Exp^{-1}(1+z)$ .

Cor. Since  $_{t}\text{Log}\left(1+z\right)=\text{Log}\left(1+z\right)+2t\pi i$ , we have

$$_{i}\log(1+z) = 2t\pi i + z - z^{n}/2 + z^{n}/3 - z^{4}/4 + \dots$$
 (2),

which gives us an expansion for the t-th branch of  $\exp^{-1}(1+z)$  within the region of the z-plane for which |z| < 1.

It follows readily, from the principles of chap, XXVL, § 9, that when |z|=1 the series  $z-z^2/2+z^2/3-\ldots$  is convergent, provided amp  $z\pm\pm\pi$  (other odd multiples of  $\pi$  are not in question here). Hence, by the theorem of Abel so often quoted already, the expansion-formulae (1) and (2) will still hold when |z|=1, provided amp  $z\pm\pm\pi$ .

GENERALISATION OF THE CIRCULAR FUNCTIONS—INTRO-DUCTION OF THE HYPERBOLIC FUNCTIONS.

§ 23.] General definition of Cosz, Sinz, Tanz, Cotz, Secz, Cosecz. Since the series  $1-z^2/2!+z^2/4!-\ldots$ ,  $z-z^2/3!+z^2/5!-\ldots$  are convergent for all values of z having a finite modulus, however large, they are each single-valued continuous functions of z throughout the z-plane. Let us call the functions thus defined Cosz and Sinz, using capital initial letters, for the present, to distinguish from the geometrically defined real functions cosx and sinz. We thus have

Cos 
$$z = 1 - z^2/2! + z^4/4! - \dots$$
 (1),

Sin 
$$z = z - z^3/3! + z^5/5! - \dots$$
 (2).

We also define  $\operatorname{Tan} z$ ,  $\operatorname{Cot} z$ ,  $\operatorname{See} z$ ,  $\operatorname{Cosec} z$  by the following equations:—

$$\operatorname{Tan} z = \operatorname{Sin} z/\operatorname{Cos} z; \quad \operatorname{Cot} z = \operatorname{Cos} z/\operatorname{Sin} z; 
\operatorname{Sec} z = 1/\operatorname{Cos} z; \quad \operatorname{Cosec} z = 1/\operatorname{Sin} z.$$
(3).

In the first place, we observe that when z is real, =x say, we have, by § 14.

Cos 
$$x = 1 - x^3/2! + x^4/4! - \dots = \cos x$$
,  
Sin  $x = x - x^3/3! + x^5/5! - \dots = \sin x$ ;

so that, when the argument is real, the more general functions Cos., Sin., Tan., Cot., Sec., Cosec. coincide with the functions cos., sin., tan., cot., sec., cosec. already geometrically defined for real values of the argument. Since

$$\begin{split} &1 - z^2/2! + z^4/4! - \dots = \frac{1}{2} \{ \text{Exp } (iz) + \text{Exp } (-iz) \}, \\ &z - z^3/3! + z^6/5! - \dots = \frac{1}{2} \{ \text{Exp } (iz) - \text{Exp } (-iz) \}, \end{split}$$

it follows from (1) and (2) that we have for all values of z

$$\begin{aligned} &\operatorname{Cos} z = \frac{1}{2} \left\{ \operatorname{Exp} \left( iz \right) + \operatorname{Exp} \left( - iz \right) \right\} \\ &\operatorname{Sin} z = \frac{1}{2i} \left\{ \operatorname{Exp} \left( iz \right) - \operatorname{Exp} \left( - iz \right) \right\} ; \end{aligned} \tag{4)*}$$

with corresponding expressions for  $\operatorname{Tan} z$ ,  $\operatorname{Cot} z$ ,  $\operatorname{Sec} z$ , and  $\operatorname{Cosec} z$ .

By (4) we have

 $\cos^2 z + \sin^2 z$ 

$$= \frac{1}{4} \left[ \{ \exp{(iz)} \}^2 + \{ \exp{(-iz)} \}^2 + 2 \exp{(iz)} \exp{(-iz)} \right]$$

$$- \{ \exp{(iz)} \}^2 - \{ \exp{(-iz)} \}^2 + 2 \exp{(iz)} \exp{(-iz)} \}.$$

Hence, bearing in mind that we have, by the exponential addition theorem.

$$\operatorname{Exp}(iz)\operatorname{Exp}(-iz) = \operatorname{Exp}(iz - iz) = \operatorname{Exp}0 = 1.$$

we see that

$$\cos^2 z + \sin^2 z = 1 \tag{5},$$

from which we deduce at once, for the generalised functions, all the algebraical relations which were formerly established for the circular functions properly so called.

We also see, from (1), that  $\cos(-z) = \cos z$  and  $\sin(-z) = -\sin z$ ; that is to say,  $\cos z$  is an even, and  $\sin z$  an odd function of z.

Since, by (4), we have

$$\cos z + i \operatorname{Sin} z - \operatorname{Exp}(iz),$$
  
$$\operatorname{Cos} z - i \operatorname{Sin} z = \operatorname{Exp}(-iz).$$

These formulæ were first given by Euler. See Int. in Anal. Inf., t. 1, § 138. He gave, however, no sufficient justification for their usage, resting merely on a hold analogy, as Bernoulli and Demovre had done before him.

it follows from the exponential addition theorem, namely,

$$\operatorname{Exp}(iz_1 + iz_2) = \operatorname{Exp}(iz_1) \operatorname{Exp}(iz_2).$$

that

$$Cos(z_1 + z_2) + i Sin(z_1 + z_2) = (Cos z_1 + i Sin z_1) (Cos z_2 + i Sin z_2)$$

$$= (\operatorname{Cos} z_1 \operatorname{Cos} z_2 - \operatorname{Sin} z_1 \operatorname{Sin} z_2) + i (\operatorname{Sin} z_1 \operatorname{Cos} z_2 + \operatorname{Cos} z_1 \operatorname{Sin} z_2)^*.$$

Hence, changing the signs of  $z_1$  and  $z_2$ , and remembering that Cos. is even and Sin. odd, we have

$$\begin{aligned} &\cos\left(z_1+z_2\right)-i\operatorname{Sin}\left(z_1+z_2\right)=\left(\operatorname{Cos} z_1\operatorname{Cos} z_2-\operatorname{Sin} z_1\operatorname{Sin} z_2\right)\\ &-i\left(\operatorname{Sin} z_1\operatorname{Cos} z_2+\operatorname{Cos} z_1\operatorname{Sin} z_2\right). \end{aligned}$$

Therefore, by addition and subtraction, we deduce

$$\frac{\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2}{\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2};$$
(6).

In other words, the addition theorem for Cos. and Sin. in general is identical with that for cos. and sin.

By (6) we have

 $Cos(z + 2n\pi) = Cos z Cos 2n\pi - Sin z Sin 2n\pi$ 

that is, if n be any positive or negative integer, so that  $\cos 2n\pi = \cos 2n\pi = 1$ , and  $\sin 2n\pi = \sin 2n\pi = 0$ , then

$$Cos(z + 2n\pi) = Cos z$$
.

In like manner,  $Sin(z + 2n\pi) = Sin z$ ;  $Tan(z + n\pi) = Tan z$ ; &c. That is to say, the Generalised Circular Functions have the same real periods as the Circular Functions proper.

Just in the same way, we can establish all the relations for half and quarter periods given in equations (3) of  $\S$  2. Thus, for example,

$$\begin{aligned} \cos \left(\pi + z\right) &= \cos \pi \cos z - \sin \pi \sin z, \\ &= \cos \pi \cos z - \sin \pi \sin z, \\ &= -\cos z. \end{aligned}$$

Also all the equations (5), (6), (7) of § 2 will hold for the generalised functions; for they are merely deductions from the addition theorem.

<sup>•</sup> We cannot here equate the coefficient of i, &c., on both sides, because  $Sin(z_1+z_2)$ , &c., are no longer necessarily real.

§ 24.1 We proceed next to discuss briefly the variation of the generalised circular functions.

Consider first the case where the argument is wholly imaginary, say z ig. In this case we have

Cos 
$$(iy) = \frac{1}{2} \{ \text{Exp} (iiy) + \text{Exp} (-iiy) \},$$
  

$$= \frac{1}{2} (e^{-y} + e^{y})$$
(1);  
Sin  $(iy) = \frac{1}{2i} (e^{-y} - e^{y}),$   

$$= \frac{1}{2i} (e^{y} - e^{-y})$$
(2).

(2).

We are thus naturally led to introduce and discuss two new functions, namely,  $\frac{1}{2}(e^y + e^{-y})$  and  $\frac{1}{2}(e^y - e^{-y})$ , which are called the Hyperbolic Cosine and the Hyperbolic Sine. These functions are usually denoted by  $\cosh y$  and  $\sinh y$ ; so that, for real values of u, cosh u and sinh u are defined by the equations

$$\cosh y = \frac{1}{2} (e^y + e^{-y}), \quad \sinh y = \frac{1}{2} (e^y - e^{-y})$$
 (3).

In general, when y is complex, we define the more general functions Cosh z and Sinh z by the equations

We also introduce  $\tanh y$ ,  $\coth y$ ,  $\operatorname{sech} y$ , and  $\operatorname{cosech} y$  by the definitions

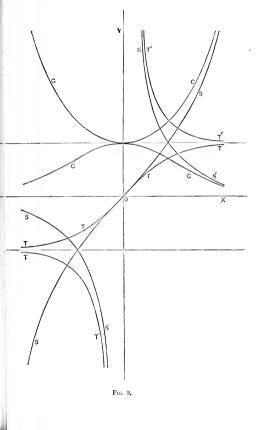
$$\tanh y = \sinh y \cosh y$$
,  $\coth y = \cosh y/\sinh y$ ;  
 $\operatorname{sech} y = 1/\cosh y$ ,  $\operatorname{cosech} y = 1/\sinh y$ ;

and the more general functions Tanh z, Coth z, &c., in precisely the same way.

From the equations (1) and (2) we have

$$\begin{array}{ll} \operatorname{Cos}\left(iy\right) & \operatorname{cosh} y, & \operatorname{Sin}\left(iy\right) & i \operatorname{sinh} y; \\ \operatorname{Tan}\left(iy\right) = i \operatorname{tanh} y, & \operatorname{Cot}\left(iy\right) = -i \operatorname{coth} y; \\ \operatorname{Sec}\left(iy\right) = \operatorname{soch} y, & \operatorname{Cose}\left(iy\right) = -i \operatorname{cosech} y; \end{array}$$

and, of course, in general, Cos iz Cosh z, &c.



of the hyperbolic functions for purely real arguments.  $\S 25$ .] Variation of the Hyperbolic Functions for real arguments. The graphs of  $y = \cosh x$ ,  $y = \sinh x$ , &c., are given in Fig. 9 as follows:—

cosh x, CC; sinh x, SOS; coth x, T'T'T'T; tanh x, TTOTT; sech x, C'C': cosech x, S'S'S'S.

By studying these curves the reader will at once see the truth of the following remarks regarding the direct and inverse hyperbolic functions of a real argument.

(1)  $\cosh x$  is an even function of x, having two positive infinite values corresponding to  $x = \pm x$ , no zero value, and a minimum value 1 corresponding to x = 0.

 $\cosh^{-1}y$  is a two-valued function of y, defined for the continuum  $1 \gg y \gg \infty$ , having a zero value corresponding to  $y \sim 1$ , and infinite values corresponding to  $y \sim \infty$ , but no turning value.

(2)  $\sinh x$  is an odd function of x, having a zero value when x = 0, and positive and negative infinite values when  $x = +\infty$  and  $x = -\infty$  respectively.

 $\sinh^{-1}y$  is one-valued, and defined for all values of y; it has a zero value for y + 0, and positive and negative infinite values when  $y + \infty$  and  $y = -\infty$  respectively.

(3)  $\tanh x$  is an odd function, has a zero value for x = 0, positive maximum + 1, and negative minimum - 1, corresponding to  $x = +\infty$  and  $x = -\infty$  respectively.

 $\tanh^{-1}y$  is a one-valued odd function, defined for  $-1 \gg y \gg + 1$ ; has zero value for y = 0, positive and negative infinite values corresponding to  $y \approx +1$  and  $y \approx -1$ .

(4) coth x is an odd function, having no zero value, but an infinite value for x = 0, and minimum + 1, and maximum - 1, for  $x = +\infty$  and  $x = -\infty$  respectively.

 $\coth^{-1}y$  is a one-valued old function, defined, except for the continuum  $-1 \gg y > +1$ , having positive and negative infinite values corresponding to y = +1 and y = -1 respectively, and a zero value for  $y = \infty$ .

(5) sech x is an even function, having a maximum + 1 for x = 0, and a zero value for  $x = +\infty$ .

sech<sup>-1</sup>y is a two-valued function, defined for 0 > y > 1, having a zero value for y = 1, and infinite values for y = 0.

(6) cosech x is an odd function, having zero values for

 $x=\pm\infty$ , and an infinite value for x=0.  $\operatorname{cosech}^{-1}y$  is one-valued and defined for all values of y, having zero values for  $y=\pm\infty$ , and infinite values for y=0.

§ 26.] Logarithmic expressions for cosh-1y, sinh-1y, &c.

If  $x = \cosh^{-1}y$ , we have

$$y = \cosh x = \frac{1}{2} (e^x + e^{-x})$$
 (1),

From (1) and (2),

$$\pm \sqrt{(y^2 - 1)} = \frac{1}{2} (e^x - e^{-x})$$
 (2).

Hence

Therefore

$$e^x = y \pm \sqrt{(y^2 - 1)}.$$
  
 $x = \log (y + \sqrt{(y^2 - 1)}):$ 

that is,  $\cos h^{-1}y = \log \{y + x\}$ 

$$\cosh^{-1} y = \log \{ y \pm \sqrt{(y^2 - 1)} \}$$
 (3),

the upper sign corresponding to the positive or principal branch of  $\cosh^{-1}y$ , the lower sign to the negative branch.

In like manner we can show that

$$\sinh^{-1}y = \log \{y + \sqrt{(y^2 + 1)}\}\$$
 (4);

$$\tanh^{-1} y = \frac{1}{2} \log \{ (1+y) (1-y) \}$$
 (5);

$$\coth^{-1} y = \frac{1}{2} \log \{(y+1)/(y-1)\}\$$
 (6);

$$\operatorname{sech}^{-1} y = \log \left[ \left\{ 1 \pm \sqrt{(1 - y^2)} \right\} / y \right]$$
 (7);

$$\operatorname{cosech}^{-1} y = \log \left[ \{ 1 + \sqrt{(1 + y^2)} \} | y \right]$$
 (8).

§ 27.] Properties of the General Hyperbolic Functions analogous to those of the Circular Functions.

We have already seen that the properties of the circular functions, both for real and for complex values of the argument, might be deduced from the equations of Euler, namely,

$$\begin{aligned} &\cos z = \frac{1}{2} \left\{ \operatorname{Exp} \left( + iz \right) + \operatorname{Exp} \left( - iz \right) \right\}; \\ &\sin z = \frac{1}{2i} \left\{ \operatorname{Exp} \left( + iz \right) - \operatorname{Exp} \left( - iz \right) \right\} \end{aligned} \tag{A}.$$

In like manner, the properties of the general hyperbolic functions spring from the defining equations

$$\frac{\cosh z - \frac{1}{2} \left\{ \exp(+z) + \exp(-z) \right\}}{\sinh z - \frac{1}{2} \left\{ \exp(+z) - \exp(-z) \right\}}$$
(B).

We should therefore expect a close analogy between the functional relations in the two cases. In what follows we state those properties of the hyperbolic functions which are analogous to the properties of the circular functions tabulated in §2. The demonstrations are for the most part omitted; they all depend on the use of the equations (B), combined with the properties of the general exponential function, already fully discussed.

The demonstrations might also be made to depend on the relations connecting the general circular functions with the general hyperbolic functions given in § 24\*, namely,

$$\begin{array}{cccc} \operatorname{Cosh} z & -\operatorname{Cos} iz, & i \operatorname{Sinh} z - \operatorname{Sin} iz; \\ +i \operatorname{Tanh} z & -\operatorname{Tan} iz, & -i \operatorname{Coth} z & \operatorname{Cot} iz; \\ \operatorname{Sech} z & -\operatorname{Sec} iz, & -i \operatorname{Cosech} z - \operatorname{Cosec} iz; \\ \end{array} \right)$$
 (C).

Algebraic Relations.

Periodicity.—All the hyperbolic functions have the period  $2\pi i$ ; and  $\mathrm{Tanh}\,z$  and  $\mathrm{Coth}\,z$  have the smaller period  $\pi i$ .

Thus

$$\begin{array}{ll} \operatorname{Cosh}\left(z+2n\pi i\right) & \operatorname{Cosh}z\;;\;\&e.\\ \operatorname{Tanh}\left(z+n\pi i\right) & \operatorname{Tanh}z\;;\;\&e. \end{array} \right\} \tag{2}.$$

Also,

$$\begin{aligned} & \operatorname{Cosh}\left(\pi i \pm z\right) = -\operatorname{Tosh}z, & \operatorname{Sinh}\left(\pi i \pm z\right) = \mp \operatorname{Sinh}z; \\ & \operatorname{Cosh}\left(\frac{1}{2}\pi i \pm z\right) = \pm i\operatorname{Sinh}z, & \operatorname{Sinh}\left(\frac{1}{2}\pi i \pm z\right) = i\operatorname{Cosh}z; \\ & \operatorname{Tanh}\left(4\pi i \pm z\right) = \pm \operatorname{Coth}z, & \operatorname{Coth}\left(4\pi i \pm z\right) = \pm \operatorname{Tanh}z; \end{aligned} \tag{3}.$$

Addition Formula.

$$\begin{array}{l} \operatorname{Cosh}\left(z_1\pm z_2\right) - \operatorname{Cosh}\left(z_1\operatorname{Cosh}\left(z_2\pm\operatorname{Sinh}\left(z_1\operatorname{Sinh}\left(z_1\right)\right)\right) \\ \operatorname{Sinh}\left(z_1+z_2\right) - \operatorname{Sinh}\left(z_1\operatorname{Cosh}\left(z_2\pm\operatorname{Cosh}\left(z_1\operatorname{Sinh}\left(z_1\right)\right)\right) \\ \operatorname{Tanh}\left(z_1\pm z_2\right) - \left(\operatorname{Tanh}\left(z_1\pm\operatorname{Tanh}\left(z_1\right)\right)\left(1\pm\operatorname{Tanh}\left(z_1\operatorname{Tanh}\left(z_1\right)\right)\right) \\ \end{array} \right) \end{array} \right)$$

This connection furnishes the simplest memoria technica for the hyperbolic formulae.

Cosh 
$$z_1 + \text{Cosh } z_2 = 2 \text{ Cosh } \frac{1}{2} (z_1 + z_2) \text{ Cosh } \frac{1}{2} (z_1 - z_2);$$
  
Cosh  $z_1 - \text{Cosh } z_2 = 2 \text{ Sinh } \frac{1}{2} (z_1 + z_2) \text{ Sinh } \frac{1}{2} (z_1 - z_2);$   
Sinh  $z_1 + \text{Sinh } z_2 = 2 \text{ Sinh } \frac{1}{2} (z_1 + z_2) \text{ Cosh } \frac{1}{2} (z_1 - z_2);$ 

Cosh 
$$z_1$$
 Cosh  $z_2 = \frac{1}{2}$  Cosh  $(z_1 + z_2) + \frac{1}{2}$  Cosh  $(z_1 - z_2)$ ;  
Sinh  $z_1$  Sinh  $z_2 = \frac{1}{2}$  Cosh  $(z_1 + z_2) - \frac{1}{2}$  Cosh  $(z_1 - z_2)$ ;  
Sinh  $z_1$  Cosh  $z_2 = \frac{1}{2}$  Sinh  $(z_1 + z_2) + \frac{1}{2}$  Sinh  $(z_1 - z_2)$ . (7).

Cosh 
$$2z = \text{Cosh}^2 z + \text{Sinh}^2 z = 2 \text{ Cosh}^2 z - 1,$$
  
 $= 1 + 2 \text{ Sinh}^2 z = (1 + \text{Tanh}^2 z)/(1 - \text{Tanh}^2 z).$   
Sinh  $2z = 2 \text{ Sinh} z \text{ Cosh } z = 2 \text{ Tanh } z/(1 - \text{Tanh}^2 z).$   
Tanh  $2z = 2 \text{ Tanh } z/(1 + \text{Tanh}^2 z).$ 
(8).

Inverse Functions.—Regarding the inverse functions Cosh<sup>-1</sup>, Sinh<sup>-1</sup>, &c., it is sufficient to remark that we can always express them by means of the functions Cos<sup>-1</sup>, Sin<sup>-1</sup>, &c. Thus, for example, if we have Cosh<sup>-1</sup>2 = w, say, then

$$z = \operatorname{Cosh} w = \operatorname{Cos} iw.$$
 Hence 
$$iw = \operatorname{Cos}^{-1}z;$$
 that is, 
$$w = -i\operatorname{Cos}^{-1}z.$$
 So that 
$$\operatorname{Cosh}^{-1}z = -i\operatorname{Cos}^{-1}z;$$
 and so on.

In the practical use of such formula, however, we must attend to the multiple-valuedness of Cosh<sup>-1</sup> and Cos<sup>-1</sup>. If, for example, in the above equation, the two branches are taken at random in the two inverse functions, then the equation will take the form

$$\label{eq:cosh-1} {\rm Cosh^{-1}}z = 2m\pi i \pm i \; {\rm Cos^{-1}}z,$$

where m is some positive or negative integer, whose value and the choice of sign in the ambiguity  $\pm$  both depend on circumstances.

§ 28.] Formulæ for the Hyperbolic Functions analogous to Demoivre's Theorem and its consequences.

We have at once, from the definition of Cosh z and Sinh z,

$$\begin{aligned} & \cosh \left( \left( z_{1} + z_{2} + \ldots + z_{n} \right) \pm \operatorname{Sinh} \left( z_{1} + z_{2} + \ldots + z_{n} \right) \\ & + \operatorname{Exp} \pm \left( z_{1} + z_{2} + \ldots + z_{n} \right), \\ & + \operatorname{Exp} \pm z_{1} \operatorname{Exp} \pm z_{2} - \ldots \operatorname{Exp} \pm z_{n}, \\ & + \operatorname{Cosh} z_{1} \pm \operatorname{Sinh} z_{2} \right), & \left( \operatorname{Cosh} z_{2} + \operatorname{Sinh} z_{2} \right), \\ & + \operatorname{Cosh} \left( z_{1} + \operatorname{Sinh} z_{2} \right), & \left( \operatorname{A} \right) \end{aligned}$$

and, in particular, if n be any positive integer,

$$\cosh nz \pm \sinh nz = (\cosh z \pm \sinh z)^n$$
 (B).

These correspond to the Demoivre-formulæ, with which the reader is already familiar\*.

We can deduce from (A) and (B) a series of formulæ for the hyperbolic functions analogous to those established in § 12 for the circular functions.

Thus, in particular, we have

$$\cosh(z_1+z_2+\ldots+z_n)=P_n+P_{n-2}+P_{n-4}+\ldots$$
 (1'),

where  $P_r = \Sigma \cosh z_1 \cosh z_2$  . . . Cosh  $z_r \sinh z_{r+1}$  . . . Sinh  $z_n$ . Tanh  $(z_1 + z_2 + \dots + z_n)$ 

$$= (T_1 + T_3 + T_5 + \dots)/(1 + T_2 + T_4 + \dots) \quad (3'),$$

where

$$T_r = \Sigma \operatorname{Tanh} z_1 \operatorname{Tanh} z_2$$
 . . . Tanh  $z_r$ .

 $\cosh nz = \cosh^n z + {}_nC_z \cosh^{n-z}z \sinh^2 z$ 

$$+ {}_{n}C_{4} \operatorname{Cosh}^{n-4} z \operatorname{Sinh}^{4} z + \dots$$
 (4').

 $\operatorname{Sinh} nz = {}_{n}C_{1}\operatorname{Cosh}^{n-1}z\operatorname{Sinh}z + {}_{n}C_{3}\operatorname{Cosh}^{n-2}z\operatorname{Sinh}^{3}z$ 

$$+ {}_{n}C_{5} \operatorname{Cosh}^{n-5}z \operatorname{Sinh}^{5}z + \dots$$
 (5')

Cosh 
$$nz = (-)^{n/2} \left\{ 1 - \frac{n^2}{2!} \cosh^2 z + \frac{n^2(n^2 - 2^2)}{4!} \cosh^4 z + \dots \right\}$$
  

$$(-)^x \frac{n^x(n^2 - 2^2)}{2!} + \dots + \frac{(n^2 - 2^2 - 2^2)}{(2^2)!} \cosh^2 z + \dots \right\} (9),$$
(n even);

<sup>•</sup> As a matter of history, Demoivre first found (B) in the form y = {1(x | x | (1+v) - v)} - x'[x'(1+v') - v]}, where y is the ordinate of P in Fig. 10 below, and r the ordinate of Q, overteeponding to a vector OQ such that the area AOQ is n times AOP, and OA is taken to be 1. He then deduced the corresponding formula for the circle by an imaginary transformation, Csc. Macelliana Analytica, Lib. H., cap. 1.)

Sinh 
$$nz/\sinh z = (-)^{(n-2)/2} \left\{ \frac{n}{1!} \cosh z - \frac{n(n^2 - 2^2)}{3!} \cosh^2 z + \dots \right\}$$

$$(-)^{*} \frac{n(n^2 - 2^2) \dots (n^2 - 2^2)^2}{(2s + 1)!} \cosh^{2s+1} z + \dots \right\} (11'),$$

$$(n \text{ even}):$$

and so on.

We may also deduce formulæ analogous to those of  $\S~13,$  such as

$$\sinh^{2m+1} z = \frac{1}{2^{n}} \left( \sinh (2m+1) z - \frac{1}{2m+1} C_1 \sinh (2m-1) z + \dots \right)$$

$$(-)^{n/2} \frac{1}{2m+2} C_m \sinh z \right).$$

§ 29.] Fundamental Inequality and Limit Theorems for the Hyperbolic Functions of a real argument.

If u be any positive real quantity, then

$$\tanh u < u < \sinh u < \cosh u$$
 (1).

By the definitions of § 24 we have

$$\sinh u = \frac{1}{2} \left\{ \exp(u) - \exp(-u) \right\}$$
:

$$= u + u^3/3! + u^5/5! + \dots$$
 (2);

$$\cosh u = 1 + u^2/2! + u^4/4! + \dots$$
 (3):

whence it appears at once that  $\sinh u > u$ .

Again,  $\cosh u = +\sqrt{(1 + \sinh^2 u)}$ , so that  $\cosh u > \sinh u$ .

Finally, since

 $\tanh u = \sinh u/\cosh u$ 

= 
$$u(1 + u^2/3! + u^4/5! + \dots)/(1 + u^2/2! + u^4/4! \dots),$$

and  $u^2/3! < u^2/2!$ ,  $u^4/5! < u^4/4!$ , &c.,

we see that  $\tanh u < u$ .

Cor. When u = 0,  $L \sinh u/u = 1$ , and  $L \tanh u/u = 1$ . This may either be deduced from (1) or established directly by means of the series (2) and (3).

If a be a quantity which is either finite and independent of n or else has a finite limit when  $n = \infty$ , then, when  $n = \infty$ ,

$$L\left(\cosh\frac{a}{n}\right)^n = 1$$
,  $L\left(\sinh\frac{a}{n}\left/\frac{a}{n}\right)^n = 1$ ,  $L\left(\tanh\frac{a}{n}\left/\frac{a}{n}\right)^n = 1$ .

We have

$$\left(\cosh\frac{a}{n}\right)^n = \left(\frac{e^{a/n} + e^{-a/n}}{2}\right)^n = e^a \left(\frac{1 + e^{-2a/n}}{2}\right)^n.$$

Hence, if we put  $1+e^{-2\pi n}-2-2z,$  so that z=0 corresponds to  $n-\infty,$  then we have

$$L\left(\cosh\frac{\alpha}{n}\right)^n = e^{\alpha} L_{\frac{n-\alpha}{2}}((1-z)^{-\frac{1}{n}})^{\frac{\alpha}{2}z\log(1-2z)}.$$

Now,  $L(1-z)^{-1/z} = e$ , and  $L2z/\log(1-2z) = -1$ . Hence, by chap. xxv., § 13,

$$L\left(\cosh\frac{a}{\nu}\right)^n = e^{\alpha}e^{-\alpha} = 1.$$

We leave the demonstration of the second limit as an exercise for the reader. The third is obviously deducible from the other two,

A very simple proof of these theorems may also be obtained by using the convergent series for  $\cosh \cdot a/n$  and  $\sinh \cdot a/n$ .

§ 30.] Geometrical Analogies between the Circular and Hyperbolic Functions.

If  $\theta$  be continuously varied from  $-\pi$  to  $+\pi$ , and we connect x and y with  $\theta$  by the equations

$$x = a \cos \theta, \quad y = a \sin \theta$$
 (1),

then we have

$$a^2 + y^2 = a^2 (\cos^2 \theta + \sin^2 \theta) = a^2$$
 (2).

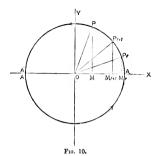
Hence, if (x, y) be the co-ordinates of a point P, as  $\theta$  varies continuously from  $-\pi$  to  $+\pi$ , P will describe continuously the circle A'AA'' (of radius a) in the direction indicated by the arrow-heads (Fig. 10).

Let P be the point corresponding to  $\theta$ ; and let  $\Theta$  denote the area AOP, to be taken with the sign + or - according as  $\theta$  is positive or negative. Then  $\Theta$  is obviously a function of  $\theta$ . We can determine the form of this function as follows:—

Divide  $\theta$  into n equal parts, and let  $P_1, P_2, \ldots, P_r, \ldots P$  be the points corresponding to  $\theta/n, 2\theta/n, \ldots, r\theta/n, \ldots, n\theta/n$  respectively. Then we have, by the lemmas of Newton,

Area 
$$AOP = L \sum_{n=\infty}^{r=n-1} \sum_{r=0}^{r-n-1} P_r OP_{r+1}$$
.

(1').



Now

$$= OM_{r+1}P_{r+1} + M_{r+1}P_{r+1}P_rM_r - OM_rP_r.$$

$$= \frac{1}{2} \left\{ x_{r+1}y_{r+1} + (y_{r+1} + y_r) \left( x_r - x_{r+1} \right) - x_r y_r \right\},$$

$$= \frac{1}{2} (x_r y_{r+1} - x_{r+1} y_r),$$

$$= \frac{1}{2}a^2 \{\cos \cdot r\theta/n \sin \cdot (r+1)\theta/n - \sin \cdot r\theta/n \cos \cdot (r+1)\theta/n\},$$
  
=  $\frac{1}{2}a^2 \sin \cdot \theta/n$ .

Hence

$$\Theta = \frac{1}{2}a^{2} Ln \sin \theta n,$$
  
=  $\frac{1}{2}a^{2} \theta L (\sin \theta n)/(\theta n),$ 

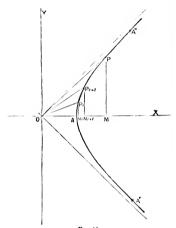
$$= \frac{1}{2}a^2 \theta. \tag{3}.$$

Hence, if  $\theta = 2\Theta/a^2$ , we have  $\cos \theta = x/a$ ,  $\sin \theta = y/a$ ,  $\tan \theta = y/a$ ,  $\cot \theta = x/y$ , &c.

Next, let u be continuously varied from  $-\infty$  to  $+\infty$ ; and let

Then 
$$x = a \cosh u, \quad y = a \sinh u \tag{1'}.$$
 
$$x^2 - y^2 = a^2 (\cosh^2 u - \sinh^2 u) = a^2 \tag{2'}.$$

Hence, if (x, y) be the co-ordinates of P, as  $u^{\bullet}$  varies continuously from  $-\infty$  to  $+\infty$ , P will describe continuously the right-hand branch A'AA'' of the rectangular hyperbola, whose



F10. 11.

semi-axis-major is OA = a, in the direction indicated by the arrow-heads in Fig. 11.

If P be the point corresponding to u,  $P_r$ ,  $P_{r+1}$  the points corresponding to ru/n and (r+1)u/n, and U the area AOP agreeing in sign with u, then, exactly as before,

Adopting an astronomical term, we may call u the hyperbolic excentric anomaly of P. The quantity u plays in the theory of the hyperbola, in general, the same part as the executric angle in the theory of the ellipse.

$$U = \frac{1}{2} L \sum_{r=1}^{r=n-1} (x_r y_{r+1} - x_{r+1} y_r);$$

and

 $x_r y_{r+1} - x_{r+1} y_r$ 

 $= a^{2} \{\cosh. ru/n \sinh. (r+1) u/n - \sinh. ru/n \cosh. (r+1) u/n\},$ =  $a^{2} \sinh. u/n$ .

Therefore

$$U = \frac{1}{2} a^{2} L n \sinh u / n,$$

$$= \frac{1}{2} a^{2} u L (\sinh u / n) / (u / n),$$

$$= \frac{1}{2} a^{2} u, \quad \text{by § 29,}$$
(3').

Hence, if the area AOP = U, and  $u = 2U/a^2$ , then, x and y being the co-ordinates of P, we might give the following geometric definitions of  $\cosh u$ ,  $\sinh u$ , &c.:—

$$\cosh u = x/a$$
,  $\sinh u = y/a$ ,  
 $\tanh u = y/x$ ,  $\coth u = x/y$ , &c.

It will now be apparent that the hyperbolic functions are connected in the same way with one half of a rectangular hyperbola, as the circular functions are with the circle. It is from this relation that they get their name.

We know, from elementary geometrical considerations, that the area  $\Theta$  is the product of  $\frac{1}{2}a^2$  into the number of radians in the angle AOP. It therefore follows from (3) that the variable  $\theta$  introduced above is simply the number of radians in the angle AOP. Our demonstration did not, however, rest upon this fact, but merely on the functional equation  $\cos\theta \theta + \sin^2\theta = 1$ . This is an interesting point, because it shows us that we might have introduced the functions  $\cos\theta = \sin\theta = \frac{1}{2}\left\{ \operatorname{Exp}\left(i\theta\right) - \operatorname{Exp}\left(-i\theta\right)\right\}$ ; and then, by means of the

above reasoning, have deduced the property which is made the basis for their geometrical definition. When this point of view is taken, the theory of the circular and hyperbolic functions attains great analytical symmetry, to becomes merely a branch of the general theory of the exponential function as defined in 8 1s.

When we attempt to get for u a connection with the arc AP, like that which subsists in the case of the circle, the parallel ccases to run on the same elementary line. To understand its nature in this respect we must resort to the theory of Elliptic Integrals.

§ 31.] Expression of Real Hyperbolic Functions in terms of Real Circular Functions.

(3).

Since the range of the variation of  $\cosh u$  when u varies from  $-\infty$  to  $+\infty$  is the same as the range of  $\sec \theta$  when  $\theta$  varies from  $-\frac{1}{2}\pi$  to  $+\frac{1}{2}\pi$ , it follows that, if we restrict  $\theta$  and u to have the same sign, there is always one and only one value of u between  $-\infty$  and  $+\infty$  and of  $\theta$  between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$  such that

$$\cosh u = \sec \theta$$
 (1).

If we determine  $\theta$  in this way, we have

$$\sinh u = \pm \sqrt{(\cosh^2 u - 1)},$$
  
=  $\pm \sqrt{(\sec^2 \theta - 1)}$ :

hence, bearing in mind the understanding as to sign, we have  $\sinh u = \tan \theta$  (2).

From these we deduce

$$e^{\mathbf{u}} = \cosh \mathbf{u} + \sinh \mathbf{u},$$
  
=  $\sec \theta + \tan \theta;$   
$$\mathbf{u} = \log (\sec \theta + \tan \theta),$$
  
=  $\log \tan (4\pi + \frac{1}{2}\theta)$ 

Also, as may be easily verified,

$$\tanh \frac{1}{2}u = \tan \frac{1}{2}\theta \tag{4}.$$

When  $\theta$  is connected with u by any of the four equivalent equations just given, it is called the Gudermannian\* of u, and we write  $\theta = \gcd u$ .

<sup>.</sup> This name was invented by Cayley in honour of the German mathematician Gudermann (1798-1852), to whom the introduction of the hyperbolic functions into modern analytical practice is largely due. The origin of the functions goes back to Mercator's discovery of the logarithmic quadrature of the hyperbola, and Demoivre's deduction therefrom (see p. 306). According to Houel, F. C. Mayer, a contemporary of Demoivre's, was the first to give shape to the analogy between the hyperbolic and the circular functions. The notation co-h, sinh, seems to be a contraction of coshyp, and sinhyp,, proposed by Lambert, who worked out the hyperbolic trigonometry in considerable detail, and gave a short numerical table. Many of the hyperbolic formulæ were independently deduced by William Wallace (Professor of Mathematics in Eduburgh from 1819 to 1838) from the geometrical properties of the rectangular hyperbola, in a little-known memoir entitled New Series for the Quadrature of Conic Sections and the Computation of Logarithms (Trans. R.S.E., vol. vi., 1812). For further historical information, see Gunther, Die Lehre von den gewohnlichen und verallgemeinerten Huperbelfunktionen (Hallo, 1881); also, Beitrage zur Geschichte der Neueren Mathematik (Programmschrift, Ansbach, 1881).

It is easy to give a geometrical form to the relation between  $\theta$  and u. If, in Fig. 11, a circle be described about O with a as radius, and from M a tangent be drawn to touch this circle in Q (above or below OX according as u is positive or negative), then, since  $MQ^2 = OM^2 - OQ^2 = x^2 - a^2 = y^2$ , we have  $a \cosh u = x = a \sec QOM$ . Therefore  $QOM = \theta$ , and we have  $y = MQ = a \tan \theta$ . From this relation many interesting geometrical results arise which it would be out of place to pursue here. We may refer the reader who desires further information regarding this and other parts of the theory of the hyperbolic functions to the following authorities:-Greenhill, Differential and Integral Calculus (Macmillan, 1886), and also an important tract entitled A Chapter in the Integral Calculus (Hodgson, London, 1888); Laisant, "Essai sur les Fonctions hyperboliques," Mém, de la Soc. Phys. et Nat. de Bordeaux, 1875; Heis, Die Hyperbolischen Functionen (Halle, 1875). Tables of the functions have been calculated by Gudermann, Theorie der Potential- oder Cyclischhuperbolischen Functionen (Berlin, 1833); and by Gronau (Dantzig, 1863). See also Cayley, Quarterly Journal of Mathematics, vol. xx.; and Glaisher, Art, Tables, Encyclopædia Britannica, 9th Ed.

## EXERCISES XVII.

(1.) Write down the values of the six hyperbolic functions corresponding to the arguments ξπi, πi, 2πi.

Draw the graphs of the following, x and y being real:-

(2.)  $y = \sinh x/x$ .

(3.)  $y = x \coth x$ .

- (4.) y = gd x.
- (5.)  $y = \sinh^{-1} \{1/(x-1)\}$ , (6.) Express Sinh<sup>-1</sup>z, Tanh<sup>-1</sup>z, Sech<sup>-1</sup>z, Cosech<sup>-1</sup>z, by means of Sin<sup>-1</sup>z.
- Cos-1z, &c. (7.) Show that  $\cosh^6 u - \sinh^6 u = 1 + 3 \sinh^2 u \cosh^2 u$ ,
  - (8.) Show that

 $4 \cosh^3 u - 3 \cosh u - \cosh 3u = 0$ :

 $4 \sinh^3 u + 3 \sinh u - \sinh 3u = 0.$ 

(9.) Show that any cubic equation which has only one real root can be numerically solved by means of the conations of last exercise. In particular, show that the roots of  $x^3 - qx - r = 0$  are  $\sqrt{(q/3)} \cosh u$ ,  $2\sqrt{(q/3)} (\cos \frac{\pi}{2})$  $\cosh u \pm i \sin 2\pi \sinh u$ ), u being determined by  $\cosh 3u = 3r \sqrt{3/2} \sqrt{a^3}$ .

(10.) Solve by the method of last exercise the equation  $x^3 + 6x + 7 = 0$ .

## Express

- (11.) tanh-1x + tanh-1y in the form tanh-1z.
- (12.)  $\cosh^{-1}x + \cosh^{-1}u$  in the form  $\cosh^{-1}z$ .
- (13.)  $\sinh^{-1}x \sinh^{-1}y$  in the form  $\cosh^{-1}z$ .

Expand in a series of hyperbolic sines or cosines of multiples of u:-

- (14.) Cosh<sup>10</sup> u.
- (15,) sinh7 u,
- (16.) cosh "u sinh "u.

Expand in a series of powers of hyperbolic sines or cosines of u:-

- (17.) Cosh 10u.
- (18.) sinh 7u.

(12.) cosh 6u sinh 3u.

(20.) sinh mu cosh nu.

Establish the following identities:-

- (21.)  $\tanh \frac{1}{2}(u+r) \tanh \frac{1}{2}(u-r) = 2 \sinh r/(\cosh u + \cosh r)$
- (22.)  $\frac{\sinh(u-r) + \sinh u + \sinh(u+r)}{\cosh(u-r) + \cosh u + \cosh(u+r)} = \tanh u.$
- (23.)  $\tanh u + \tanh (\frac{1}{2}\pi i + u) + \tanh (\frac{2}{2}\pi i + u) = 3 \tanh 3u$ ,
- $\cosh 2u + \cosh 2v + \cosh 2w + \cosh 2(u + v + w) = 411 \cosh (v + w).$
- (24.)  $\operatorname{Tan} \frac{1}{2} (u + iv) = (\sin u + i \sinh v)/(\cos u + \cosh v).$
- (25.) Express  $\cosh^2(u+ir) + \sinh^2(u+ir)$  in terms of functions of u and v.

Eliminate u and v from the following equations:-

- (26.)  $x = a \cosh(u + \lambda), \quad y = b \sinh(u + \mu).$
- (27.)  $y \cosh u x \sinh u = a \cosh 2u$ ,
- $y \sinh u + x \cosh u = a \sinh 2u$ .
- (25.)  $x = \tanh u + \tanh r$ ,  $y = \coth u + \coth r$ , u + r = c.
- (29.) Expand sinh (u + h) in powers of h.
- (30.) Expand tanh<sup>-1</sup>x in powers of x; and deduce the expansions of cosh<sup>-1</sup>x and sinh<sup>-1</sup>x. Discuss the limits within which your expansions are valid.
  - (31.) Given  $\sinh u/u = 1001/1000$ , calculate u.
- (32) Show that the series  $\sum_{i=1}^{n} \frac{1}{2^{n}} \left( \frac{x^{i \times n-1} 1}{x^{i \times n-1} + 1} \right)$  is convergent, and that its sum is  $(x^2 + 1)/(x^2 1) 1/\log x$  (Wallace, l.c.).
- (33.) Prove that the infinite product  $\cosh \frac{u}{2i} \cosh \frac{u}{2i} \cosh \frac{u}{2^2}$ ... is convergent, and that its value is  $\sinh u/u$ .
  - (34.) Show that

$$\log x = \frac{x - x^{-1}}{2}, \frac{2}{x^{1/2} + x^{-1/2}}, \frac{2}{x^{1/4} + x^{-1/4}}, \frac{2}{x^{1/4} + x^{-1/4}}, \dots \text{ ad } \infty.$$

(35) If 
$$\frac{1}{P_n} = \frac{x - x^{-1}}{2} \cdot \frac{2}{x^{1/2} + x^{-1/2}} \cdot \cdot \cdot \cdot \cdot \frac{2}{x^{1/2} + x^{-1/2}}$$
, show that  $P_n$  differs

from 1/log x (in defect) by less than

$$\{1 + \frac{1}{2}(x^{1/2^{m+1}} + x^{-1/2^{m+1}})\}/3, x^{m+1}P$$

Evaluate the following limits:-

- (36.)  $(\sinh x \sin x), x^3, x = 0.$
- (37.)  $(\sinh^2 mx \sinh^2 nx)'(\cosh px \cosh qx), x = 0.$
- (34.)  $(\tan^2 x \tanh^3 x)/(\cos x \cosh x), x = 0.$

Show that, when h=0,

- (39.)  $L (\cosh a (x+h) \cosh ax)/h = a \sinh ax$
- (40.)  $L \{ \sinh a (x+h) \sinh ax \} / h = a \cosh ax.$
- (41.)  $L \{ \tanh a (x + h) \tanh a x \} / h = a \operatorname{sech} a x$ .
- (42.)  $L \left\{ \operatorname{coth} a (x+h) \operatorname{coth} ax \right\} / h = -a \operatorname{cosech}^2 ax$
- (43.) Show that

$$\begin{split} \frac{1}{2^n} \coth \frac{u}{2^n} &= \coth u - \sum\limits_{1}^n \frac{1}{2^n} \tanh \frac{u}{2^n}, \\ \frac{1}{u} &= \coth u - \sum\limits_{1}^n \frac{1}{2^n} \tanh \frac{u}{2^n}, \end{split}$$

and state the corresponding formulæ for the circular functions (Wallace, Trans. R.S.E., vol. vi.).

(44.) From the formulæ of last exercise, derive, by the process of chap. xxvn., § 2, the following:—

$$\begin{split} \frac{1}{2^{2n}} \coth^2 \frac{u}{2^n} &= \coth^2 u - \sum_{1}^n \frac{1}{2^{2n}} \tanh^2 \frac{u}{2^n}, \\ \frac{1}{u^2} &= \coth^2 u - \sum_{2}^n \frac{1}{2^{2n}} \tanh^2 \frac{u}{2^n}. \end{split}$$

(Wallace, 1.c.)

In the following, 0 is the centre of the hyperbola  $x^i|a^2 - y^2|b^2 = 1$ ; A one of its vertices; F the corresponding focus; P and P' any two points on the curve, whose excentric anomalies are u and u', and whose co-ordinates are (x, y)(x, y), so that  $x = a \cos h u$ ,  $y = b \sinh u$ ,  $\delta c$ .; and N is the projection of P on the axis a. Show that

- (45.) Area  $ANP = \frac{1}{4}ab (\sinh 2u 2u)$ .
- (46.) Area of the right segment cut off by the double ordinate of P

$$= \frac{b}{a} x \sqrt{(x^2 - a^2)} - ab \cosh^{-1} \frac{x}{a},$$

$$= \frac{b}{a} x \sqrt{(x^2 - a^2)} - ab \log \frac{x + \sqrt{(x^2 - a^2)}}{a}.$$

- (47.) Area of the segment cut off by  $PP' = \frac{1}{2}ab \{\sinh (u'-u) (u'-u)\}$ .
- Express this in terms of x, y, x', y'.
- (48.) If R be the middle point of PP', and OR meet the hyperbola in S, the co-ordinates of S are {a cosh ½ (u+u'), b sinh ½ (u+u')}.
  - (49.) OS bisects the hyperbolic area POP'.
- (50.) If PP move parallel to itself, the locus of R is a straight line passing through O.
- (51.) If PP' cut off a segment of constant area, the locus of R is a hyperbola.

GRAPHICAL DISCUSSION OF THE GENERALISED CIRCULAR FUNCTIONS.

§ 32.] Let us now consider the general functional equation  $w = \cos z$ , or, as we may write it,

$$u + iv = \operatorname{Cos}(x + yi)$$
 (1),

where u, v, x, y are all real.

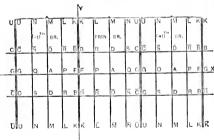
Since  $\cos(x+yi) = \cos x \cos yi - \sin x \sin yi = \cos x \cosh y - i \sin x \sinh y$ , we have

$$u = \cos x \cosh y$$
,  $v = -\sin x \sinh y$  (2):

and therefore

$$u^2/\cos^2 x - v^2/\sin^2 x$$
 (3),

$$u^2/\cosh^2 y + v^2/\sinh^2 y = 1$$
 (4).

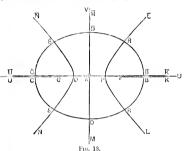


F10. 12.

In order to avoid repetition of the values u and v, arising from the periodicity of  $\cos x$  and  $\sin x$ , we confine z, in the first instance, to lie between the axis of y and a parallel  $I = \overline{C}GCU$  to this axis at a distance from it equal to  $\pi$  (Fig. 12).

If we draw a series of parallels to the y-axis within this strip, we see, from equation (3), that to each of these will belong half

of a hyperbola in the w-plane (Fig. 13), having its foci at the fixed points F and G, which are such that OF = OG - 1. Thus, for example, if in the z-plane  $FP = \frac{1}{4}\pi$  and  $FQ = \frac{3}{4}\pi$ , then to the parallels  $LP\overline{L}_1$ ,  $NQ\overline{N}$  correspond the two halves  $LP\overline{L}_2$ ,  $NQ\overline{N}$  of a hyperbola whose transverse axis is  $PQ = \sqrt{2}$ .



								Į¥								
F	qĸ	JL I	M	IN	U	O	Ŋ	М	F	ĸ	K	L	М	N	U	Ū
		(-13°	BR.				PRIN.	BR.				(+1) <sup>TH</sup>	BR.			
į	В	R	D_	s	С	₫	ŝ	D	R	B	В		D_	s	С	<u>c</u>
	F	P	A	Q	G	G	Q	A_	P	F	F	P	A	Q	G	FΧ
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Fig. 11.

To the parallel  $MA\overline{M}$ , which bisects the strip, corresponds the axis of v (which may be regarded as that hyperbola of the confocal system which has its transverse axis equal to 0); and to the parallels KFK and  $UG\bar{U}$  which bound the strip, correspond the parts  $KF\overline{K}$  and UGU of the u-axis, each regarded as a double line (flat hyperbola).

Again, if we draw parallels to the x-axis across the strip, to each of these will correspond one of the halves of an ellipse belonging to a confocal system having F and G for common foci. Thus to BRDSC and  $\overline{BRDSC}$  equidistant from the x-axis correspond the two halves BRDSC and  $\overline{BRDSC}$  of the same ellipse whose semi-axes are cosh y and sinh y. In particular, to FPAOG on the x-axis itself corresponds the double line (flat ellipse) FPAQG.

Thus, to the whole of the first parallel strip between  $KO\overline{K}$ and  $U\overline{U}$  corresponds uniquely the whole of the u-plane. Hence, if we confine ourselves to this strip. (1) defines w and z each as a continuous one-valued function of the other. To each succeeding or preceding strip corresponds the w-plane again taken once over, alternately one way or the opposite, as indicated by the lettering in Fig. 12. w is therefore a periodic function of z, having the real period  $2\pi$ ; and z is a multiple-valued function of w of infinite multiplicity, having two branches for each period of ic.

The value of z corresponding to the first strip on the right of the y-axis is called the principal branch of Cos-1 w, and the others are numbered as usual. We therefore have for the t-th branch

$$_{t}$$
Cos<sup>-1</sup>  $w = z_{t} = (t + \frac{1}{2} + (-)^{t-1} \frac{1}{2}) \pi + (-)^{t}$ Cos<sup>-2</sup>  $w$  (5),

where Cos-1 w is the principal value as heretofore; and Cos-1 w = x + ui, x and y being determined by (3) and (4), when y and v are given.

It should be noticed that for the same branch of a there is continuity from B to  $\overline{B}$  not directly across the u-axis, but only by the route BFB; whereas there is continuity from B to  $\hat{B}$ 

(2);

directly, if we pass from one branch to the next. This may be represented to the eye by slitting the w-axis from F to  $+\infty$  and from G to  $-\infty$ , as indicated in Fig. 13. If we were to construct a Riemann's surface for the w-plane, so as to secure unique correspondence between every w-point and its z-point, then the junctions of the leaves of this surface would be along these slits. The reader will find no difficulty in constructing the model.

Since to the line KFPAQGU (the whole of the u-axis) corresponds in the z-plane the three lines KF, FPAQG, GU taken in succession, we see that as w varies first from  $+\infty$  to 1, then from 1 to -1, and finally from -1 to  $-\infty$ ,  $\cos^{-1}w$  varies first from  $\infty i$  to 0, then from 0 to  $\pi$ , and finally from  $\pi$  to  $\pi + \infty i$ ; so that an angle whose cosine is greater than 1 is either wholly or partly inaginary.

§ 33.1 If  $w = \sin z$ , say

$$u + iv = \operatorname{Sin}(x + yi) \tag{1},$$

then, as in last paragraph,

$$u = \sin x \cosh y$$
,  $v = \cos x \sinh y$ 

$$u^2/\sin^2 x - v^2/\cos^2 x = 1$$
 (3);

$$u^2/\cosh^2 y + v^2/\sinh^2 y = 1$$
 (4).

The graphical representation is, as the student may easily verify, obtained by taking Fig. 13 for the w-plane and Fig. 14 for the z-plane.

We have also, for the t-th branch of the inverse function,

$$_{t}Sin^{-1}w = z_{t} = t\pi + (-)^{t}Sin^{-1}w,$$

where  $Sin^{-1}w = x + yi$ , x and y being determined by equations (3) and (4), under the restrictions proper to the principal branch of the function.

§ 34.] If  $w = \operatorname{Tan} z$ , say

$$u + iv = \operatorname{Tan}(x + yi) \tag{1},$$

then  $(u + iv) \operatorname{Cos} (x + yi) = \operatorname{Sin} (x + yi),$ 

that is,  $(u\cos x \cosh y + v\sin x \sinh y) + i (-u\sin x \sinh y + v\cos x \cosh y)$ 

$$= \sin x \cosh y + i \cos x \sinh y.$$

Therefore

 $u\cos x\cosh y + v\sin x\sinh y = \sin x\cosh y$ ,

 $-u \sin x \sinh y + v \cos x \cosh y = \cos x \sinh y.$ 

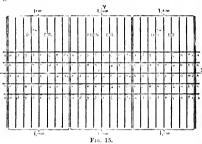
From the last pair of equations it is easy, if we bear in mind the formula of § 27, to deduce the following:—

$$u = \sin 2x/(\cos 2x + \cosh 2y), \quad v = \sinh 2y/(\cos 2x + \cosh 2y)$$
 (2);

$$u^2 + v^2 + 2u \cot 2x - 1 = 0$$
 (3):

$$u^2 + v^2 - 2v \coth 2v + 1 = 0 (4).$$

The graphical representation of these results is given by Figs. 15 and 16.



When x is kept constant, the equation to the path of w is given by (3), which evidently represents a series of circles passing through the points (0, +1) and (0, -1).

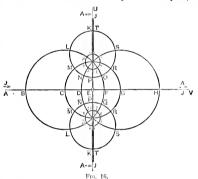
When y is constant, the equation to the path of w is (4), which represents a circle having its centre on the reaxis; and it is easy to verify that the square of the distance between the centres of the circles (3) and (1) is equal to the sum of the squares of their radii, from which it appears that they are orthotomic.

If we consider a parallel strip of the z-plane bounded by  $x=-\frac{1}{2}\pi$ ,  $x=+\frac{1}{2}\pi$ , we find that to this corresponds the whole

w-plane taken once over. The corresponding values of z are said to belong to the principal branch of the function Tan-1 w.

To the vertical parallels in the z-plane correspond the circles passing through I and I in the w-plane, and to the horizontal parallels correspond the circles in the w-plane which cut the former orthogonally.

It should be noticed that I and  $\tilde{I}$  in the w-plane correspond to  $+\infty$  and  $-\infty$  in the direction of the y-axis in the z-plane, and



that to A and J in the z-plane correspond the points at  $\infty$  on the u- and v-axes in the w-plane; also that there is no continuity directly across  $IK \times v$  or  $IK \times v$  in the w-plane, except in passing from one branch of Tan<sup>-1</sup>w to the next.

For the t-th branch of the inverse function we have

$$_{t}\operatorname{Tan}^{-1}w = z_{t} = t\pi + \operatorname{Tan}^{-1}w$$
 (5),

where the principal value  $\operatorname{Tan}^{-1} w$  is given by  $\operatorname{Tan}^{-1} w = x + yi$ , x and y being determined, under the restrictions proper to the principal branch, by means of (3) and (4).

§ 35.] It will be a useful exercise for the student to discuss directly the graphical representation of  $w = \sec z$ ,  $w = \csc z$ , and  $w = \cot z$ . The figures in the w-plane for these functions may, however, be derived from those already given, by means of the following interesting general principle.

If Z be any z-path,  $\hat{W}$  and  $\hat{W}$  the corresponding w-paths for w = f(x + yi) and w' = 1/f(x + yi), then  $\hat{W}$  is the image with respect to the u-axis of the inverse of  $\hat{W}$ , the centre of inversion being the w-plane and the radius of inversion being unity.

This is easily proved; for, if  $(\rho, \phi)$ ,  $(\rho', \phi')$  be the polar co-ordinates of points on W and W' corresponding to the point (x, y) on Z, then we have

$$\rho(\cos\phi + i\sin\phi) = f(x + yi),$$
  
$$\rho'(\cos\phi' + i\sin\phi') = 1/f(x + yi).$$

Hence

$$\rho(\cos\phi + i\sin\phi) = 1/\rho'(\cos\phi' + i\sin\phi'),$$
  
=  $(1/\rho')(\cos(-\phi') + i\sin(-\phi')).$ 

Therefore  $\rho = 1/\rho'$ ,  $\phi = -\phi' + 2k\pi$ , which is the analytical expression of the principle just stated.

From this it appears at once that, if we choose for our standard r-paths a double system of orthoronic parallels to the zz and y-axes, then the x-paths for w = Cotx will be a double system of orthotomic circles, and the w-paths for w = Sec x and w = Cose x a double system of orthotomic Bicircular Quarties.

Example 1. If 
$$u + vi = Sec(x + yi)$$
, show that

$$\begin{split} u &= 2\cos x \cosh y/(\cos 2x + \cosh 2y);\\ v &= 2\sin x \sinh y/(\cos 2x + \cosh 2y);\\ (u^2 + v^*)^2 &= u^2/\cos^2 x + v^2 \sin^2 x;\\ (u^2 + v^2)^2 &= u^2/\cosh^2 y + v^2/\sinh^2 y. \end{split}$$

Discuss the graphical representation of the functional equation, and show how to deduce the 6-th branch from the principal branch of the function.

The curves represented by the last two equations are most easily traced from their polar equations, which are

$$\rho^2 = 2 (\cos 2\phi - \cos 2x)/\sin^2 2x,$$
  
 $\rho^2 = 2 (\cosh 2y - \cos 2\phi)/\sinh^2 2y,$ 

reaportively.

Example 2. The same problem for u + vi = Cosec(x + yi).

§ 36.] Before leaving the present part of our subject, it will be well to point out the general theorem which underlies the fact that to the orthogonal parallels in the z-plane in the six cases just discussed correspond a system of orthogonal paths in the w-plane.

Let us suppose that f(z) is a continuous function of the complex variable z, such that for a finite area round every point z=a within a certain region in the z-plane f(z) can always be expanded in a convergent series of powers of z-a, so that we have

$$f(z) = f(a) + A_1(z - a) + A_2(z - a)^2 + \dots$$
 (1),

where  $A_1, A_2, \ldots$  are functions of a and not of z.

Then we have the following general theorem, which is fundamental in the present subject.

If A<sub>1</sub> ± 0, the angle between any two z-paths emanating from a is the same as the angle between the corresponding w-paths emanating from the point in the w-plane which corresponds to a.

**Proof.**—Let z be any point on any path emanating from a,  $(r, \phi)$  the polar co-ordinates of z with respect to a as origin, the prime radius being parallel to z and z. Let w and b be the w-points corresponding to z and a,  $(\rho, \phi)$  the polar co-ordinates of w with respect to b. Then we have

 $\rho (\cos \phi + i \sin \phi)$ 

$$= w - b - f(z) - f'(a),$$

$$= A_1(z - a) + A_2(z - a)^2 + \dots, \quad \text{by (1)},$$

$$= A_1 r(\cos \theta + i \sin \theta) + A_2 r^2(\cos \theta + i \sin \theta)^2 + \dots (2).$$

Let now  $A_1 = r_1 (\cos a_1 + i \sin a_1)$ ,  $A_2 = r_2 (\cos a_2 + i \sin a_2)$ , . . ., then (2) may be written

$$\rho (\cos \phi + i \sin \phi) = r_1 r \{\cos (a_1 + \theta) + i \sin (a_1 + \theta)\}$$
  
  $+ r_2 r^2 \{\cos (a_2 + 2\theta) + i \sin (a_2 + 2\theta)\} + \dots$  (3).

Whence

$$\rho \cos \phi = r_1 r \cos (a_1 + \theta) + r_2 r^2 \cos (a_2 + 2\theta) + \dots$$
 (4).

$$\rho \sin \phi = r_1 r \sin (a_1 + \theta) + r_2 r^2 \sin (a_2 + 2\theta) + \dots$$
 (5).

In the limit, when r and consequently  $\rho$  are made infinitely small, (1) and (5) reduce to

 $(\rho/r)\cos\phi = r_1\cos(a_1 + \theta), \quad (\rho/r)\sin\phi = r_1\sin(a_1 + \theta)$  (6). Since  $\rho$  and r are both positive, these conations lead to

$$\rho(r-r_1, \text{ and } \phi - 2k\pi + a_1 + \theta) \tag{7}.$$

Hence, if we take any two paths emanating from a in directions determined by  $\theta$  and  $\theta'$ , we should have  $\phi - \phi' = \theta - \theta'$ , which proves our theorem.

We see also, from the first of the equations in (7), that if we construct any infinitely small triangle in the z-plane, having its vertex at a, to it will correspond an infinitely small simihar triangle in the w-plane having its vertex at b.

Hence, if we establish a unique correspondence between points (u, v) and (x, y) in any two planes by means of the relation

$$u + vi = f(x + \eta i) = \chi(x, y) + i\psi(x, y),$$

then to any diagram D in the one plane corresponds a diagram D in the other which is similar to D in its infinitesimal detail.

The propositions just stated show that, if we have in the z-plane any two families of curves A and B such that each curve of A cuts each curve of B at a constant angle a, then to these correspond respectively in the w-plane families A' and B' such that each curve of A' cuts each curve of B' at an angle a. Since the six circular functions satisfy the preliminary condition regarding the function  $f(x \cdot yi)$ , the theorem regarding the w-v-curves for these functions which correspond to x = const,  $y \cdot \text{const}$ , follows at once.

If  $A_1 = 0, A_2 = 0, \dots, A_{n-1} = 0, A_n \neq 0$ , then the above conclusions fail. In fact, the equations (7) then become

$$\rho/r^n = r_n, \quad \phi = 2k\pi + a_n + n\theta \tag{7'}$$

and we have  $\phi = \phi' - n(\theta - \theta')$ .

In this case, as the point z circulates once round a, the point w circulates u times round b. That is to say, b is a winding point of the uth order tor z; and the Riemann's surface for the w-plane has an u-fold winding point at b. We have a simple example of this in the case of u = z; already discussed, for which

w = 0 is a winding point of the third order. The points  $w = \pm 1$ and  $z = \pm 0$  are corresponding points of a similar character for  $w = \cos z$ 

The theorem of the present paragraph is of great importance in many parts of mathematics. From one point of view it may be regarded as the geometrical condition that  $\phi(x, y) + i\gamma(x, y)$  may be, according to a certain definition. a function of x + yi. In this way it first made its appearance in the famous memoir entitled Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse, in which Riemann laid the foundations of the modern theory of functions, which has borne fruit in so many of the higher branches of mathematics.

From another point of view the theorem is of great importance in geometry. When the points in one plane are connected with those in another in the manner above described, so that corresponding figures have infinitesimal similarity, the one plane is said by German mathematicians to be conform abachildet, that is, conformably represented (Cayley has used the phrase "orthomorphically transformed") upon the other; and there is a corresponding theory for surfaces in general. Many of the ordinary geometrical transformations are particular cases of this; for example, the student will readily verify that the equation  $w = a^2/z$  corresponds to inversion.

Lastly, the theory of conjugate functions, as expounded by Chrk-Maxwell in his work on electricity (vol. 1. chap. x11.), depends entirely on the theorem which we have just established. In fact, the curves in Figs. 12. 13. 15, and 16 may be taken to represent lines of force and lines of equal potential: so that every particular case of the equation u + vi = f(x + yi) gives the solution of one or more physical problems.

## Exercises XVIII.

(1.) Discuss the variation of sin-1u and sin-1iv, where u and v are real. and vary from  $-\infty$  to  $+\infty$ .

Draw the Argand diagrams for the following, giving in each case, where they have not been given above, the w-paths when the z-paths are eircles about the origin and parallels to the real and imaginary axes:-

- (3.)  $w = \exp z$ . (2.)  $w = \log z$ .
- (1.)  $w = \cosh z$ .

where

(5.) w = tanh z.

(6.) Show that eos<sup>-1</sup> (u + iv) = cos<sup>-1</sup> U - i eosh<sup>-1</sup> V;  $\sin^{-1}(u + iv) = \sin^{-1}U + i \cos^{-1}V$ .

 $2U = \sqrt{\{(n+1)^2 + v^2\}} - \sqrt{\{(n-1)^2 + v^2\}},$ 

 $2V = \sqrt{\{(u+1)^2 + v^2\}} + \sqrt{\{(u-1)^2 + v^2\}},$ 

the principal branch of each function being alone in question.

(7.) Show that the principal branch of tan-1 (u + ir) is given by x + yi.  $u = 1 \tanh^{-1} \left( 2u/(u^2 + v^2 + 1) \right)$ ; whore

and  $x = 1 \tan^{-1} \{2u/(1 - u^2 - v^2)\}$ , if  $u^2 : v^2 < 1$ :  $=\pm 4\pi + 1 \tan^{-1} \{2n/(1 - u^2 - v^2)\}, \text{ if } u^2 + v^2 - 1,$ 

the paper or lower sign being taken according as a is positive or negative. (8.) If  $u + ri = \cot(x + yi)$ , show that

 $u = \sin 2x/(\cosh 2u - \cos 2x)$ ,  $v = -\sinh 2u/(\cosh 2u - \cos 2x)$ ;  $u^2 + v^2 - 2u \cot 2x - 1 = 0$ ,  $u^2 + v^2 + 2v \coth 2y + 1 = 0$ .

(9.) If  $u + ri = \operatorname{cosec}(x + yi)$ , show that  $u = 2 \cdot \operatorname{in} x \cosh u / (\cosh 2u - \cos 2x)$ ,  $v = -2 \cos x \sinh u / (\cosh 2u - \cos 2x)$ ;  $(u^2 + v^2)^2 = u^2/\cos^2 x + v^2/\sin^2 y$ ,  $(u^2 + v^2)^2 = u^2/\cosh^2 y + v^2/\sinh^2 y$ 

Express the following in the form u + vi, giving both the principal branch and the general branch when the function is multiple-valued:-

(10.)  $\cosh^{-1}(x + ui)$ . (11.) Tanh-1 (x + yi).

(12.)  $\frac{1}{2i}$ Log  $\{(x+yi)/(x-yi)\}$ , (13.) Log Sin (x+yi).

(11.)  $(\cos \theta + i \sin \theta)^{i}$ . (15.)  $\text{Log}_{a+i\beta}(x+yi)$ .

(16.) Show that the general value of Sin<sup>-1</sup> (cosec θ) is (t+t) π + i log  $\cot \frac{1}{2}(t\pi + \theta)$ , where t is any integer.

(17.) Show that the real part of Exp. (Log (1+i)) is e<sup>-π<sup>2</sup>/8</sup> cos (1π log 2). (18.) Prove, by means of the series for Cos θ and Sin θ, that Sin 2θ = 2 Sin θ

Cos # (19.) Deduce Abel's generalised form of the binomial theorem from

£\$ 20, 22, (20.) Show that

 $1 + m + m^{2}C_{1}x + m + m^{2}C_{2}x^{2} + \dots$  ad x $= (1 + x)^m [\cos (n \log (1 + x))] + i \sin (n \log (1 + x))].$ 

(21.) Show that the families of curves repre-cuted by

 $\sin x \cosh u = \lambda$ .  $\cos x \sinh u - \mu$ are orthotomic.

(22.) Find the equation to the family of curves orthogonal to re- $\cos n\theta = \lambda$ .

(23.) Find the condition that the two families

 $Ax^{2} + 2Bxy + Cy^{2} = \lambda$ ,  $A'x^{2} + 2B'xy + C'y^{2} = \mu$ 

be orthotomic.

(24.) If  $\tan (x + iu) = \sin (u + iv)$ , prove that  $\coth v \sinh 2u = \cot u \sin 2x$ .

## SPECIAL APPLICATIONS OF THE FOREGOING THEORY TO THE CIRCULAR PUNCTIONS.

§ 37.1 In order to avoid breaking our exposition of the general theory of the elementary transcendents, we did not stop to deduce consequences from the various fundamental theorems.

To this part of the subject we now proceed; and we shall find that many of the ordinary theorems regarding series involving the circular functions are simple corollaries from what has gone before.

Let us take, in the first place, the generalised form of the binomial theorem given in § 15. So long as  $1 + \Sigma_m C_n z^n$  is convergent, we have seen that it represents the principal value of  $(1 + z)^m$ . Hence, if  $z = r (\cos \theta + i \sin \theta)$ , where r is positive, and  $-\pi \gg \theta \gg +\pi$ , we have

$$1 + \sum_{m} C_n r^n (\cos n\theta + i \sin n\theta)$$

$$= (1 + 2r\cos\theta + r^2)^{m/2}(\cos m\phi + i\sin m\phi),$$

where  $-\frac{1}{2}\pi \geqslant \phi = \tan^{-1}\{r\sin\theta/(1+r\cos\theta)\} \geqslant +\frac{1}{2}\pi$ . Hence, equating real and imaginary parts, we must have

$$1 + \sum_{n} C_n r^n \cos n\theta = (1 + 2r \cos \theta + r^2)^{m/2} \cos m\phi$$
 (1);

$$\sum_{m} C_n r^n \sin n\theta = (1 + 2r \cos \theta + r^2)^{m/2} \sin m\phi \quad (2).$$

These formula will hold for all real commensurable values of m, provided r < 1.

When r = 1, we have

$$\phi = \tan^{-1} \{ \sin \theta / (1 + \cos \theta) \} = \frac{1}{2} \theta,$$

and (1) and (2) become

$$1 + \sum_{m} C_{n} \cos n\theta = 2^{m} \cos^{m} \frac{1}{2}\theta \cos \frac{1}{2}m\theta \tag{1'},$$

$$\sum_{m} C_{m} \sin n\theta = 2^{m} \cos^{m} 1\theta \sin \frac{1}{2}m\theta \tag{2'}.$$

$$\Sigma_m C_n \sin n\theta = 2^m \cos^m \frac{1}{2} \theta \sin \frac{1}{2} m\theta \tag{2}$$

These formulæ hold for all values of  $\theta$  between  $-\pi$  and  $+\pi^*$ , when m>-1; and also for the limiting values  $-\pi$  and  $+\pi$  themselves, when m>0.

§ 38.] Series for  $\cos m\phi$  and  $\sin m\phi$ , when m is not integral. If in (1) and (2) of last paragraph we put  $\theta = \frac{1}{2}\pi$ , and  $\tau = \tan \phi$ , so that  $\phi$  must lie between  $-\frac{1}{4}\pi$  and  $+\frac{1}{4}\pi$ , then  $(1 + 2\tau \cos \theta + \tau^{2})^{m/2} = \sec^{m}\phi$ ; and we find

$$\cos m\phi = \cos^{m}\phi \left(1 - {}_{m}C_{2}\tan^{2}\phi + {}_{m}C_{4}\tan^{4}\phi - \dots\right)$$
(3),  
 
$$\sin m\phi = \cos^{m}\phi \left({}_{m}C_{1}\tan\phi - {}_{m}C_{3}\tan^{2}\phi + \dots\right)$$
(4).

Since the left-hand sides of (1') and (2') are periodic, it is easy to see that, for 2ρπ −π > ∂ + 2, π +π, the right-hand sides will be 2<sup>m</sup> cos<sup>m</sup>; ∂ cos ½m (∂ − 2ρπ) and 2<sup>m</sup> cos<sup>m</sup>; ∂ sin ½m (∂ − 2ρπ) respectively, where 2<sup>m</sup> cos<sup>m</sup>½σ, being the value of a modulus, must be made real and positive.

Whence

$$\tan m\phi = \frac{{}_{m}C_{1}}{1-{}_{n}C_{2}}\frac{\tan \phi - {}_{m}C_{3}}{\tan^{2}\phi + {}_{m}C_{4}}\frac{1}{\tan^{4}\phi - \dots}$$
(5).

These formula: are the generalisations of formula (4), (5), (6) of § 12. They will hold even when  $\phi$  has either of the limiting values  $\pm \frac{1}{4}\pi_1$  provided m>-1; so that we have

$$2^{m/2}\cos\frac{1}{4}m\pi = 1 - {}_{m}C_{2} + {}_{m}C_{4} - \dots;$$
  
$$2^{m/2}\sin\frac{1}{2}m\pi = {}_{m}C_{1} - {}_{m}C_{2} + \dots.$$

Since

$$\cos^{m-2r}\phi = (1 - \sin^2\phi)^{(m-2r)/2} = 1 + \sum_{m=2r} (-1)^n \sin^{2r}\phi$$

and the terms of this series are ultimately all positive, it follows that the double series deducible from (3), that is to say, from  $\Sigma(-)^*\pi_0 C_{\infty}^{m-n} C_{\infty}^{m-n} C_{\infty}^{m})^* \Phi$  by substituting expansions for the cosines, satisfies Cauchy's conditions (chap. XXVI., § 34), for there is obviously absolute convergency everywhere under our present restriction that  $-\frac{1}{4}\pi \gg \Phi + \frac{1}{4}\pi$ .

Hence we may arrange this double series according to powers of  $\sin \phi$ .

The coefficient of  $(-)^r \sin^{2r} \phi$  is

$$\stackrel{\leftarrow}{\underset{s=0}{\text{erg}}} \sum_{m=0}^{m-r} (m-2) \cdot C_{r-s,m} C_{2s}$$

$$= \frac{m(m-2) \cdot \dots \cdot (m-2r+2)}{1 \cdot 3 \cdot \dots \cdot (2r-1)} \cdot \Sigma_{(m-1)^m} C_{r(2r-1)^n} C_{r-s}.$$

Now, by chap. XXIII., § 8, Cor. 5,

$$\sum_{(m+1)^2} C_{s,(2r-1)^2} C_{r-s} = \frac{1}{(m+2r-2)^2} C_r$$

Hence the coefficient of ( ) sin \$\pi\$ \$\ph\$ is

$$\frac{m(m-2) \dots (m-2r+2)(m+2r-2) \dots (m+2) m}{1 \cdot 3 \dots (2r-1) 2 \dots (2r-2) 2r}$$

$$\frac{m^r(m^2-2^r) \dots (m^2-2r-2^r)}{(2r)!}$$

Hence

$$\cos m\phi = 1 - \frac{m^2}{2}, \sin^2\phi + \frac{m^2(m^2 - 2^2)}{4!} \sin^4\phi + \dots$$
 (6).

In like manner, we can show that

$$\sin m\phi = \frac{m}{1!}\sin \phi - \frac{m(m^2 - 1^2)}{3!}\sin^3\phi + \frac{m(m^2 - 1^2)(m^2 - 3^2)}{5!}\sin^5\phi - \dots$$
 (7)

Also

$$\cos m\phi = \cos\phi \left\{ 1 - \frac{m^2 - 1}{2!} \sin^2\phi \right\}$$

$$+\frac{(m^2-1^2)(m^2-3^2)}{4!}\sin^4\phi - \dots$$
 (8);

 $\sin m\phi = \cos \phi \, \left\{ \frac{m}{1!} \sin \phi - \frac{m \, (m^2 - 2^2)}{3!} \sin^3 \phi \right.$ 

+ 
$$\frac{m(m^2-2^2)(m^2-4^2)}{5!}\sin^5\phi - \dots$$
 (9).

The demonstration above given establishes these formula under the restriction  $-\frac{4}{4}\pi \Rightarrow \phi \Rightarrow \frac{1}{4}\pi$ . It can, however, be shown that they hold so long as  $-\frac{1}{2}\pi \Rightarrow \phi \Rightarrow \frac{1}{2}\pi$ ; that is to say, so long as the series involved are convergent.

Cauchy, from whom the above is taken, shows that by expanding both sides in powers of m and equating coefficients we obtain expansions for  $\phi$ ,  $\phi^2$ ,  $\phi^3$ ,  $\chi_0$ , in powers of  $\sin \phi$ .

Thus, for example, we deduce

$$\phi = \sin \phi + \frac{1}{2} \frac{\sin^3 \phi}{3} + \frac{1}{2} \frac{.3}{.4} \frac{\sin^5 \phi}{.5} + \frac{1}{2} \frac{.3}{.4} \frac{.5}{.6} \frac{\sin^7 \phi}{.7} + \dots$$

If we put  $x = \sin \phi$ , this gives

$$\sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \frac{x^5}{5} + \frac{1}{2} \cdot \frac{3 \cdot 5}{4 \cdot 6} \frac{x^7}{7} + \dots$$
 (10).

In particular, if we put  $x = \frac{1}{2}$ , we obtain

$$\pi = 6 \left\{ \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \dots \right\}$$
 (11)

from which the value of  $\pi$  might be calculated with tolerable rapidity to a moderate number of places. The result to 10 places is  $\pi=3.1415926536\ldots$ .

FY AMPLES The important series (10) for expanding  $\sin^{-1}x$  is here demonstrated for values of x lying between  $-1/\sqrt{2}$  and  $+1/\sqrt{2}$ . It can be shown that it is

valid between the limits x = -1 and x = +1. The series was discovered by Newton, who gives it along with the series for sin x and cos x in powers of x in a small tract entitled Analysis per Equationes Numero Terminorum Infinitas. Since this tract was shown by Newton to Barrow in 1669, the series (10) is one of the oldest examples of an infinite series applicable to the quadrature of the circle.

Example 1. If m > 0, and

$$\begin{split} &C = 2^{-m} \sum_{n=0}^{\infty} {}_{m}C_{n}\cos\left(m-2n\right)x,\\ &S = 2^{-m} \sum_{n=0}^{\infty} {}_{m}C_{n}\sin\left(m-2n\right)x,\\ &C = 2^{-m} \sum_{n=0}^{\infty} {}_{(-)}^{n-1} {}_{m}C_{n}\cos\left(m-2n\right)x,\\ &S' = 2^{-m} \sum_{n=0}^{\infty} {}_{(-)}^{n-1} {}_{m}C_{n}\sin\left(m-2n\right)x, \end{split}$$

then, a being any integer,

1°.  $C = (\cos x)^m \cos 2m\rho \pi$ ,  $S = (\cos x)^m \sin 2m\rho \pi$ .

from  $x = (2\rho - \frac{1}{2}) \pi$  to  $x = (2\rho + \frac{1}{2}) \pi$ .

2°.  $C = (-\cos x)^m \cos m (2\rho + 1) \pi$ ,  $S = (-\cos x)^m \sin m (2\rho + 1) \pi$ , from  $x = (2\rho + \frac{1}{2}) \pi$  to  $x = (2\rho + \frac{3}{2}) \pi$ .

3°.  $C' = (\sin x)^m \cos m (2\rho + \frac{1}{2}) \pi$ ,  $S' = (\sin x)^m \sin m (2\rho + \frac{1}{2}) \pi$ , from  $x = 2\rho \pi$  to  $x = (2\rho + 1) \pi$ .

4°.  $C' = (-\sin x)^m \cos m (2\rho + \frac{\pi}{2}) \pi$ ,  $S' = (-\sin x)^m \sin m (2\rho + \frac{\pi}{6}) \pi$ , from  $x = (2\rho + 1)\pi$  to  $x = (2\rho + 2)\pi$ . These formula will also hold when m lies between - 1 and 0, only that

the extreme values of x in the various stretches must be excluded. (Abel, (Eurres, t. 1., p. 249.)

If we multiply (1') and (2') above by cos a and sin a respectively, and add, we obtain the formule

 $\cos \alpha + \sum_{m} C_{m} \cos (\alpha - n\theta) = 2^{m} \cos^{m} \frac{1}{2} \theta \cos (\alpha - \frac{1}{2} m\theta + m\rho\pi),$ 

wherein it must be observed that  $\cos^m k\theta$  is the modulus of  $(1 + 2r\cos\theta + r^2)^{m/2}$ when r = 1, and must therefore be always so adjusted as to have a real positive value.

From the equation just written, Abel's formula can at once be deduced by a series of substitutions,

Example 2. Show, by taking the limit when m=0 on both sides of (1) and (2) above, that the series (1) and (2) of § 40 can be deduced from the generalised form of the binomial theorem.

Example 3. Sum to infinity the series  $\Sigma n^3 = C_n \sin^n \theta \cos n\theta$ . This series is the real part of  $\Sigma n^1 = C_n \sin^n \theta (\cos n\theta + \sin n\theta)$ . Hence

 $S = R \left( \sum_{i=1}^{n} C_{i} \sin^{n} \theta \left( \cos \theta + i \sin \theta \right)^{n} \right).$ 

 $E\left[\left\{m^3\sin^2\theta\left(\cos\theta+i\sin\theta\right)^2+m\left(3m-1\right)\sin^2\theta\left(\cos\theta+i\sin\theta\right)^2\right]\right]$ + m = in # (cos # + 1 = in #) { ] . sin # (cos # + i sin #) } m-1]. FORMULÆ FROM EXPONENTIAL & LOGARITHMIC SERIES 331

by Example 5 of chap. xxvii., § 5,

 $= [m^3 \sin^2 \theta \cos \{3\theta + (m-3)\phi\} + m(3m-1)\sin^2 \theta \cos \{2\theta + (m-3)\phi\} + m\sin \theta \cos \{\theta + (m-3)\phi\}](1 + 2\sin \theta \cos \theta + \sin^2 \theta)^{(m-3)^2},$ where  $\phi = \tan^{-1} \{\sin^2 \theta / (1 + \sin \theta \cos \theta)\}.$ 

§ 39.] Formula deduced from the Exponential Series. From the equation

$$e^{x}(\cos y + i \sin y) = 1 + \sum (x + yi)^{n}/n!$$

putting  $x = r \cos \theta$ ,  $y = r \sin \theta$ , we deduce

 $e^{r\cos\theta}\left\{\cos\left(r\sin\theta\right)+i\sin\left(r\sin\theta\right)\right\}=1+\Sigma r^n\left(\cos n\theta+i\sin n\theta\right)/n!.$ 

Hence

$$e^{r\cos\theta}\cos(r\sin\theta) = 1 + \sum r^n\cos n\theta/n!$$
 (1);

$$e^{r \cos \theta} \sin (r \sin \theta) = \sum r^n \sin n\theta/n!$$
 (2):

which hold for all values of r and  $\theta$ .

In like manner, many summations of series involving cosines and sines of multiples of  $\theta$  may be deduced from series related to the exponential series in the way explained in chap. xxvIII. § 8.

Thus, for instance, from the result of Example 3, in the paragraph just quoted, we deduce

$$\sum_{1}^{\infty} \left(1^{3}+2^{3}+\ldots+n^{3}\right) x^{n}/n! = e^{r \cos \theta} \left\{r \cos \left(\theta+r \sin \theta\right)+\frac{\pi}{2} r^{2} \cos \left(2\theta+r \sin \theta\right)\right. \\ \left.+2 r^{3} \cos \left(3\theta+r \sin \theta\right)+\frac{1}{2} \cos \left(4\theta+r \sin \theta\right)\right\}.$$

§ 40.] Formulæ deduced from the Logarithmic Series. Since the principal value of Log (1+z) is given by Log  $(1+z) = \log |1+z| + i \operatorname{amp} (1+z)$ , and since the series  $z-z^2/2+z^2/3-\ldots$  represents the principal value of Log (1+z), if we put  $z=r(\cos\theta+i\sin\theta)$ , we have

$$\log (1 + 2r \cos \theta + r^2)^{1/2} + i \tan^{-1} \{r \sin \theta / (1 + r \cos \theta)\}$$

$$= \Sigma (-)^{n-1} r^n (\cos n\theta + i \sin n\theta) / n,$$

where  $-\frac{1}{2}\pi \geqslant \tan^{-1} \{r \sin \theta/(1 + r \cos \theta)\} \geqslant \frac{1}{2}\pi$ , that is, the principal value of the function  $\tan^{-1}$  is to be taken.

Hence we have the following :-

$$\frac{1}{2} \log (1 + 2r \cos \theta + r^2) = \sum (-1)^{n-1} r^n \cos n\theta / n \qquad (1):$$

$$\tan^{-1} \{r \sin \theta_i (1 + r \cos \theta)\} = \sum_{i=1}^{n-1} r^n \sin n\theta/n$$
 (2).

Although, strictly speaking, we have established these results for values of  $\theta$  between  $-\pi$  and  $+\pi$  both inclusive, yet, since both sides are periodic functions of  $\theta$ , they will obviously hold for all values of  $\theta$ , provided r < 1.

If r=1, (1) and (2) will still hold, provided  $\theta + \pm \pi$ ; for the series in (1) and (2) are both convergent, and we have, by Abel's Theorem.

$$\begin{array}{ll} \cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots &= L_{-\frac{1}{2}}\log{(1+2r\cos{\theta} + r^2)},\\ &\quad + \log{(2\cos{\frac{3}{2}\theta})}\\ &\sin{\theta} - \frac{1}{2}\sin{2\theta} + \frac{1}{3}\sin{3\theta} - \ldots &= \tan^{-1}{\sin{\theta}/(1+\cos{\theta})}, \end{array} \label{eq:energy}$$

$$\sin \theta - \frac{1}{2} \sin 2\theta + \frac{1}{3} \sin 3\theta - \dots = \tan^{-1} \left\{ \sin \frac{\theta}{2} (1 + \cos \theta) \right\},$$

$$= \tan^{-1} \left\{ \tan \frac{1}{2} (\theta + 2k\pi) \right\},$$

$$= \frac{1}{2} \theta + k\pi$$
(4),

where k must be so chosen that  $\frac{1}{2}\theta + k\pi$  lies between  $-\frac{1}{2}\pi$  and  $+\frac{1}{2}\pi$ . Thus, if  $\theta$  lie between  $-\pi$  and  $+\pi$ , k=0, and we have simply

$$\sin \theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \dots = \frac{1}{3}\theta \tag{4'}.$$

In particular, if we put  $\theta = \frac{1}{2}\pi$ , we get

$$\frac{1}{4}\pi = 1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} + \dots$$
 (5),

which is Gregory's quadrature; see § 41.

When  $\theta = \pm (2p+1) \pi$ , the series in (3) diverges to  $-\infty$ , and the right-hand side becomes  $\log \theta$ , that is  $-\infty$ , so that (3) still holds in a certain sense.

The behaviour of the series in (4) when  $\theta=\pm (2p+1)r$  is very curious. Let us take, for simplicity, the case  $\theta=\pm r$ . With this value of  $\theta$  we have for values of r as near unity as we please  $\tan^{-1}(r\sin\theta)(1+r\cos\theta)=0$ . Hence, by Abel's Theorem, when  $\theta=J r$ ,  $\sin\theta-\frac{1}{2}\sin2\theta+\dots=0$ , as is otherwise sufficiently obtains.

On the other hand, for any value of  $\theta$  differing from  $\pm \tau$  by however little, we have  $L - \tan^{-1}\{r\sin\theta[(1+r\cos\theta)\} - \frac{1}{2}\theta\}$ . Hence, again, by Abel's Theorem, for  $\theta = \frac{1}{2}\tau + \phi$ , where  $\phi$  is infinitely small, we have

$$\sin \theta - \frac{1}{2} \sin 2\theta + \dots = \pm \frac{1}{2}\pi \mp \frac{1}{2}\phi$$

The erries  $y=\sin\theta-\frac{1}{2}\sin 2\theta+$ , i. is therefore discontinuous in the neighbourhood of  $\theta=\pm\pi$ ; for, when  $\theta=\pm\pi$ , y=0, and when  $\theta$  differs infinitely little from  $\pm\pi/2$ . This discontinuity is accompanied by the phenomenon of minitely slow convergence in the neighbourhood of  $\tau=1$ ,  $\theta=\pm\pi$ ; and the sudden alteration of the value of the sum v=0 are exacted with the fact that the values of the double limits

L L tan-1  $\{r\sin\theta/(1+r\cos\theta)\}$  and L L tan-1  $\{r\sin\theta/(1+r\cos\theta)\}$  are not alike.

When  $\theta$  lies between  $\pi$  and  $3\pi$ , we may put  $\theta = 2\pi + \theta'$ , where  $\theta'$  lies between  $-\pi$  and  $+\pi$ , then, for such values of  $\theta$ , we have

$$y = \sin \theta' - \frac{1}{2} \sin 2\theta' + \dots$$
  
=  $\frac{1}{2}\theta'$ , as we have already shown,  
=  $\frac{1}{2}\theta - \pi$ .

Hence, however small  $\phi$  may be, we have, for  $\theta = \pi + \phi$ ,  $y = \frac{1}{2}\phi - \frac{1}{2}\pi$ . But, as have just seen, for  $\theta = \pi - \phi$  we have  $y = -\frac{1}{2}\phi + \frac{1}{2}\pi$ . Hence, as  $\theta$  varies from  $\pi - \phi$  to  $\pi + \phi$ , y varies abruptly from  $-\frac{1}{2}\phi + \frac{1}{2}\pi$  to  $\frac{1}{2}\phi - \frac{1}{2}\pi$ . In other words, as  $\theta$  passes through the value  $\pi$ , y suffers an abrupt decrease amounting to  $\pi$ .

We have discussed this case so fully because it is probably the first instance that the student has met with of a function having the kind of discontinuity figured in chap. xv., Fig. 5. It ought to be a good lesson regarding the necessity for care in handling limiting cases in the theory of infinite series.

§ 41.] Gregory's Series. If in equation (2) of last paragraph we put  $\theta = 2\pi$ , we deduce the expansion

$$\tan^{-1} r = r - \frac{1}{2}r^3 + \frac{1}{2}r^5 - \dots$$
 (6),

where  $\tan^{-1}r$  represents, as usual, the principal value of the inverse function, and  $-1 \gg r \gg 1$ .

In particular, if r = 1, we have

$$\pi = 4 (1 - 1 + 1 - \dots)$$

The series (6), which is famous in the history of the quadrature of the circle, was first published by James Gregory in 1670; and independently, a few years later, by Leibnitz. About the be-imining of the 18th century, two English calculators, Abraham Sharp and John Machin (Professor of Astronomy at Gresham Coblege), used the series to calculate  $\pi$  to a large number of places. Sharp, using the formule  $\frac{1}{2}\pi = \tan^{-1}1/\sqrt{3} = (1/\sqrt{3})\{1-1/3,3+1/5,3^2,\ldots\}$ , suggested by Halley, carried the calculation to 71 places; that is, about twice as far as Ludolph van Coulon had gone. Machin, using a formula of his own, for long the best that was known, namely,  $\frac{1}{2}\pi = 4\tan^{-1}1/\sqrt{3} - \tan^{-1}1/\sqrt{3}$ , and to 160 places. Euler, apparently unaware of what the English calculators had done, used the far less effective formula  $\frac{1}{2}\pi = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}$ . Gauss (Werke, Ed. In, p. 501) found, by means of the theory of numbers, two remarkable formulae of this kind, namely .—

$$\frac{1}{4}\pi = 12 \tan^{-1} \frac{1}{1}8 + 8 \tan^{-1} \frac{1}{57} - 5 \tan^{-1} \frac{1}{299},$$
  
 $= 12 \tan^{-1} \frac{1}{38} + 20 \tan^{-1} \frac{1}{57} + 7 \tan^{-1} \frac{1}{239} + 24 \tan^{-1} \frac{1}{268},$ 

<sup>\*</sup> The reader should now draw the graph of the function y, for all real values of  $\theta$ .

by means of which  $\pi$  could be calculated with great rapidity should its value ever be required beyond the 707th place, which was reached by Mr Shanks in 18731.

#### EXERCISES XIX.

Sum the following series to infinity, pointing out in each case the limits within which the summation is valid:—

(1.) 
$$1 - \frac{1}{2}\cos\theta + \frac{1}{2} \cdot \frac{3}{4}\cos 2\theta - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6}\cos 3\theta + \dots$$

(2.) 
$$x \frac{\cos \theta}{1} + \frac{1}{2} \cdot x^3 \frac{\cos 3\theta}{3} + \frac{1 \cdot 3}{2 \cdot 4} x^2 \frac{\cos 5\theta}{5} + \dots$$

(3.) 
$$\frac{\cos \theta}{1} + \frac{1}{2} \frac{\cos 3\theta}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{\cos 5\theta}{5} + \dots;$$

result 
$$\frac{1}{2} \cos^{-1} (1 - 2 \sin \theta)$$
.

- (4.)  $\Sigma (2n-1)(2n-3)\cos n\theta/n!$  (5.)  $\Sigma \sin n\theta/(n+2)$  nl
- (6.)  $e^{-a} \sin \theta \frac{1}{2}e^{-3a} \sin 3\theta + \frac{1}{8}e^{-5a} \sin 5\theta \dots$

(7.) 
$$\sin \theta - \frac{1}{2-3} \sin 2\theta + \frac{1}{3-4} \sin 3\theta - \dots$$

- 2.3 3.4 (8.)  $\sin^2 \theta 4 \sin^2 2\theta + 4 \sin^2 3\theta . . . :$
- result  $\frac{1}{2} \log \sec \theta$ . (9.)  $\sum \cos 2n\theta \ln \ln n - 1$ . (10.)  $\sum \sin n\theta \ln^2 - 1$ .
- (11.)  $\frac{1}{2} \sin \theta \sin \theta 1 \sin 2\theta \sin^2 \theta + 4 \sin 3\theta \sin^2 \theta$  . . .
- (11.) \(\frac{1}{2}\sin \theta \sin \theta \frac{1}{2}\sin 2\theta \sin^2 \theta + \frac{1}{2}\sin 3\theta \sin^2 \theta \quad \qu
- (12.)  $\cos(\alpha + \beta) \cos(\alpha + 3\beta)/3! + \cos(\alpha + 5\beta)/5! \dots$ (13.)  $\cos \theta - 1\cos 2\theta + 1\cos 3\theta - \dots$
- result  $\frac{1}{2} \log (2 + 2 \cos \theta)$ , except when  $\theta = (2p + 1) \pi$ .
- (14.)  $\cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{3} \cos 3\theta + \dots$ ; result  $-\frac{1}{2} \log (2 - 2 \cos \theta)$ , except when  $\theta = 2p\pi$ .
- (15.)  $\sin \theta + \frac{1}{2} \sin 2\theta + \frac{1}{4} \sin 3\theta + \dots$ ; result =0, if  $\theta = 0$ ; =\frac{1}{2} (\pi - \theta), if  $\theta \sim \theta$ ; \pi. \Delta \text{\$\text{\$\text{\$\text{\$\text{\$\theta\$}}}\$} = \frac{1}{2} \text{\$\texit{\$\text{\$\text{\$\text{\$\text{\$\tex{\$\}\$}}}\text{\$\text{\$\text{\$\text{\$\texi{\$\text{\$\texi{\$\text{
- result =0, if  $\theta$  =0; =  $\frac{1}{2}(\pi \theta)$ , if  $\theta \sim \theta$ ;  $\pi$ , (16.)  $\sin \theta = \frac{1}{4} \sin 3\theta + \frac{1}{4} \sin 5\theta = \dots$ ,
- (17.)  $x \cos \theta = \frac{1}{3}x^3 \cos 3\theta + \frac{1}{3}x^3 \cos 5\theta + \dots$ ; result  $\frac{1}{2} \tan^{-1} \{2x \cos \theta/(1-x^2)\}$ .
- (18.)  $\cos\theta\cos\phi = \frac{1}{2}\cos2\theta + \frac{1}{2}\cos3\theta\cos\phi = \frac{1}{2}\cos\frac{1}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2}\cos\frac{1}{2$
- (19.)  $x \cos \theta \cos \phi \frac{1}{4}x^3 \cos 3\theta \cos 3\phi + \frac{1}{4}x^2 \cos 5\theta \cos 5\phi \dots;$ result  $\frac{1}{4} \tan^{-1} \left\{ 4x \left(1 - x^2\right) \cos \theta \cos \phi / \left\{ (1 + x^2)^2 - 4x^3 \left(\cos^2 \theta - \cos^2 \phi\right) \right\} \right\}.$

(20.) Show that log (1 + x + x<sup>2</sup>j = 2∑ ( - )<sup>n-1</sup> cos ½nx x<sup>n</sup>jn, provided |x| < 1, and examine whether the result holds when |x = 1.</p>

<sup>\*</sup> For the instory of this subject see Ency. Erit., art. "Squaring the Circle," by Muir.

(21.) Show that, under certain restrictions upon θ,

$$\log (1 + 2 \cos \theta) = -2\Sigma \cos \frac{1}{2} n\pi \cos n\theta/n;$$
  
 $\theta = -\Sigma \cos \frac{1}{2} n\pi \sin n\theta/n.$ 

(22.) Show that

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} + \dots$$
(Newton, Second Letter to Oldenburg, 1676.)

#### EXPREISES XX

(1.) Calculate π to 10 places by means of Machin's formula.

(2.) Show that, if x < 1,  $(\tan^{-1} x)^2$ 

 $= x^2 - (1+1/3) x^4/2 + \dots + (-1)^{n-1} \{1+1/3 + \dots + 1/(2n-1)\} x^{2n}/n + \dots$ Does the formula hold when x = 1?

(3.) Expand tan<sup>-1</sup> (x + cot α) in powers of x.

- (4.) Deduce the series for sin<sup>-1</sup>x from Gregory's series by means of the addition theorem for the binomial coefficients.
  - (5.) If x lie between 1/\(\sigma^2\) and 1, show tha

$$\sin^{-1} x = \pi - \frac{\sqrt{(1-x^2)}}{x} \left\{ 1 - \frac{1}{3} \frac{1-x^2}{x^2} + \frac{1}{5} \frac{(1-x^2)^2}{x^4} - \dots \right\}.$$

- Show that § 38 (10) is merely a particular case of (7).
- (7.) Show that

$$\frac{\theta}{\cos \theta} = \sin \theta + \frac{2}{3}\sin^3 \theta + \frac{2 \cdot 4}{3 \cdot 5}\sin^5 \theta + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\sin^7 \theta + \cdots$$
(Pfaff.)

(8.)  $\frac{1}{2}\theta^2 = \frac{\sin^2\theta}{2} + \frac{2}{3}\frac{\sin^4\theta}{4} + \frac{2}{3}\frac{4}{5}\frac{\sin^6\theta}{6} + \dots$  (Stainville.)

(9.)  $\theta^2 = \sin^2 \theta + \frac{3}{4} \cdot \frac{5}{5} \left(1 + \frac{1}{3^2}\right) \sin^2 \theta + \dots + \frac{3 \cdot 5 \dots (2n-1)}{4 \cdot 6 \dots 2n} \frac{2n+1}{2n+1} \left(1 + \frac{1}{3^2} + \dots + \frac{1}{(2n-1)^2}\right) \sin^{2n+1} \theta + \dots$ 

(10.)  $\theta^4 = \sin^4 \theta + \frac{4}{5} \cdot \frac{7}{5} \left(1 + \frac{1}{2^2}\right) \sin^6 \theta + \cdots + \frac{4 \cdot 6 \cdot \cdots (2n-2)}{5 \cdot 7 \cdot \cdots (2n-1)} \frac{2}{n} \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{(n-1)^2}\right) \sin^{2n} \theta + \cdots$ 

- (11.) Deduce from § 38 (6) and (7) an expression for θ<sup>m</sup>/sin<sup>m</sup>θ in powers of sin θ.
  - (12.) If  $\sin \theta = x \sin (\theta + a)$ , show that  $\theta + r\pi = \sum x^n \sin na/n$ ,
  - (13.) If  $c^2 = a^2 2ab \cos C + b^2$ , then

 $\log c = \log a - (b/a) \cos C - \frac{1}{2} (b/a)^2 \cos 2C - \frac{1}{3} (b/a)^3 \cos 3C - \dots$ 

(14.) Show that

 $1 - \frac{n-3}{2} + \frac{(n-4)(n-5)}{2 \cdot 3} - \frac{(n-5)(n-6)(n-7)}{2 \cdot 3 \cdot 4} + \cdots = \frac{1 + (-)^{n+1} 2 \cos \frac{2}{3} n \pi}{n}.$ 

Show that

(15.)\* 
$$\theta^2 = \sin^2\theta + 2^2 \sin^4\frac{\theta}{2} + 2^4 \sin^4\frac{\theta}{2^2} + 2^6 \sin^4\frac{\theta}{2^2} + \dots$$

(16.)\* 
$$u^2 = \sinh u - 2^2 \sinh^4 \frac{u}{2} - 2^4 \sinh^4 \frac{u}{2^2} - 2^6 \sinh^4 \frac{u}{2^3} - \dots$$

(17.)\* 
$$\frac{3}{1}\theta = \sin \theta + 3\sin^3 \frac{\theta}{2} + 3^2\sin^3 \frac{\theta}{2} + \dots$$
.

$$(18.)^{\bullet} = \frac{3}{4} \sin \theta = \frac{1}{4 \cdot 3^{m-1}} \sin 3^m \theta + \sum_{i=3m-1}^{m} \sin^2 3^{m-1} \theta_{\bullet}$$

$$(19.)^{\bullet} = \frac{3}{4} \cos \theta = \sum_{1}^{\infty} \frac{(-1)^{m-1}}{3^{m-1}} \cos^{3} 3^{m-1} \theta.$$

See Laisant, "Essai sur les Fonctions hyperboliques," Mém. de la Soc. de Bordeaux, 1875.

## CHAPTER XXX

## General Theorems regarding the Expansion of Functions in Infinite Forms.

#### EXPANSION IN INFINITE SERIES.

§ 1.1 Caucha's Theorem regarding the Expansion of a Function of a Function. If

$$y = \alpha_0 + \Sigma \alpha_n x^n \qquad (1),$$

the series being convergent so long as |x| < R, and if

 $z = b_0 + \Sigma b_n y^n$ this series being convergent so long as |y| < S, then from (1) and (2) we can derive the expansion

$$z = C_n + \Sigma C_m x^n$$
.

provided x be such that |x| < R, and also

$$|\alpha_n| + \sum |\alpha_n| |x|^n < S.$$

This theorem follows readily from chap. xxvi., §§ 14 and 34. We have already used particular cases of it in previous chapters.

§ 2.] Expansion of an Infinite Product in the form of an Infinite Series.

If  $\Sigma u_n$  be an absolutely convergent series, and  $u\Sigma u_1, u\Sigma u_1 u_2$ . . . , " \Su\_1 u\_2 . . . u\_r , . . . denote the sums of the products of its first n terms taken one, two, . . . , r, . . . , at a time, then

$$L_{n} \Sigma u_{1} = T_{1}, \quad L_{n} \Sigma u_{1} u_{2} = T_{2}, \quad \ldots, \quad L_{n} \Sigma u_{1} u_{2} \ldots u_{r} = T_{r}, \ldots$$

where  $T_1, T_2, \ldots, T_r, \ldots$  are all finite.

Also the infinite series  $1 + \sum T_n$  is convergent: and converges to the same limit as the infinite product  $\Pi(1 \pm u_n)$ .

After what has been laid down in chap, xxvi., it will obviously be sufficient if we prove the above theorem on the assumption that all the symbols  $u_1, u_2, \ldots, u_n, \ldots$  represent positive quantities. In the more general case where these are complex numbers the moduli alone would be involved in the statements of inequality, and the statements of equality would he true as under.

Since  $u_1, u_2, \ldots, u_n, \ldots$  are all positive, we see, by the Multinomial Theorem (chap. XXIII., § 12), that

$$\begin{aligned} 0 <_n \ge u_1 \, u_2 \, , \, \dots \, u_r < (u_1 + u_2 + \dots + u_n)^r / r! \\ < (u_1 + u_2 + \dots + u_n + \dots + u_n + \dots )^r / r! \\ < N^r / r! \, , \end{aligned} \tag{1)}$$

where S is the finite limit of the convergent series  $\Sigma u_a$ : and the inequality (1) obviously holds for all values of r up to r = n, however great n may be.

Therefore  $_{n}\Sigma u_{1}u_{2}$ ...  $u_{r}$  has always a finite limit,  $T_{r}$  say, such that

$$0 \geqslant T_r \geqslant \overline{S}^r/r!$$
 (2).

By (2), we have

 $0 < 1 + T_1 + T_2 + \dots$  and  $x < 1 + S/1! + S^2/2! + \dots$  and  $x < 1 + S/1! + S^2/2! + \dots$ that is.

$$0 < 1 + \sum_{1}^{\infty} T_{n} < e^{S}$$
 (3).

Hence  $1 + \sum T_n$  is a convergent series, whose limit cannot exceed e8.

Again, since  $L_n \Sigma u_1 u_2 \dots u_r = T_r$  when  $n = \infty$ , we may write

$$_{n}\Sigma u_{1}u_{2}$$
 . . .  $u_{r}=(1+_{r}A_{n})T_{r}$  (4),

where  $L_r A_n = 0$  when  $n = \infty$ .

Hence, An being a mean among 1An, 2An, . . ., An, and therefore such that  $LA_n = 0$  when  $n = \infty$ , we have

$$\prod_{i=1}^{n} (1 + u_n) = 1 + {}_{n} \sum u_1 + {}_{n} \sum u_1 u_2 + \dots + {}_{n} \sum u_1 u_2 + \dots + u_n$$

$$= 1 + (1 + A_n) \sum_{i=1}^{n} T_n \quad (5).$$

If in (5) we put  $n = \infty$ , we get

$$\prod_{1}^{\infty} (1 + u_n) = 1 + L \{ (1 + A_n) \sum_{1}^{n} T_n \}, 
= 1 + \sum_{1}^{\infty} T_n$$
(6),

since  $LA_n = 0$ , and  $\Sigma T_n$  is finite.

This completes the proof of our proposition.

Cor. 1. If  $\Sigma u_n$  be absolutely convergent, then,  $T_n$  having the above meaning,  $1 + \Sigma x^n T_n$  will be convergent for all finite values of x; and we shall have

$$\prod_{n=1}^{\infty} (1 + x u_n) = 1 + \sum_{n=1}^{\infty} x^n T_n$$
 (7).

This follows at once by the above, and by chap. XXVI., § 27. Cor. 2. Let

$$u_n = {}_n v_0 + {}_n v_1 \cdot x + {}_n v_2 x^2 + \dots$$
 (S),

where  $_{n}v_{0}$ ,  $_{n}v_{1}$ , &c., are independent of x, and the series on the right of (8) may either terminate or not; and let

$$u_n' = |_n v_0| + |_n v_1| ||x| + |_n v_2| ||x|^2 + \dots$$
 (9).

Then, if  $\Sigma u_n'$  be convergent for all values of x such that  $|x| < \rho$ , it follows that for all such values  $\Pi(1 + u_n)$  is convergent, and can be expanded in a convergent series of ascending powers of x.

For, if  $T_n^{\dagger}$  have the meaning above assigned to it, then it will obviously be possible to arrange  $T_n$  as an ascending series of powers of x. Moreover, if we consider the double series that thus arises from  $1+\Sigma T_n$ , we see that all Cauchy's conditions (see chap. XXVI., § 35) for the absolute convergence of this double series are satisfied. Hence we may arrange  $1+\Sigma T_n$  as a convergent series of ascending powers of x.

Example 1. To expand  $(1+x)(1+x^2)(1+x^4)(1+x^8)$ ... in an ascending series of powers of x. (Euler, Introd. in Anal. Inf., § 328.)

The series  $\Sigma |x|^{2^n}$  is obviously convergent so long as |x| < 1. Hence, so long as |x| < 1, we may write

$$(1+x)(1+x^2)(1+x^4)(1+x^5)$$
.  $\cdot = 1 + C_1x + C_2x^2 + ... + C_nx^n + ...$  (10).

To determine the coefficients  $C_1$ ,  $C_2$ ,  $C_n$ , we observe that, if we multiply both sides of (10) by 1-x, the left-hand side becomes L ( $1-x^{2n}$ ), that is, 1, since |x|<1. We must therefore have

$$1/(1-x) = 1 + C_1x + C_2x^2 + \dots + C_nx^n + \dots,$$

that is.

$$1 + x + x^2 + \dots + x^n + \dots = 1 + C_1 x + C_2 x^2 + \dots + C_n x^n + \dots = 0$$
  
ore  $C_1 = C_2 = \dots = C_n = \dots = 1$ ,

Another way is to put x2 for x on both sides of (10), and then multiply by (1+x). We thus get

$$1 + \Sigma C_n x^n = 1 + x + C_1 x^2 + \dots + C_n x^{2n} + C_n x^{2n+1} + \dots;$$
thence
$$C_{n} = C_{n+1} = C_n, \quad C_1 = 1,$$

whence from which it is easy to prove that all the coefficients are unity.

Example 2. To show that

 $(1 + xz)(1 + x^2z) \dots (1 + x^mz)$ 

$$=1+\sum_{n=1}^{\infty}\frac{(1-r^n)(1-x^{m-1})\dots(1-x^{m-n+1})}{(1-x)(1-x)}x^{n(n+1)/2}z^n \qquad (1).$$
(Cauchy, Comptes Rendus, 1810.)

Let

$$(1+xz)(1+x^2z)\dots(1+x^mz)$$
  
=  $1+A_1z+A_2z^2+\dots+A_nz^n+\dots+A_nz^n$  (2),

where  $A_1, A_2, \ldots$  are functions of x which have to be determined.

Put xz in place of z on b th sides of (2), then multiply on both sides by  $(1 + xz)/(1 + x^{m+1}z)$ , and we get

$$(1+zz)(1+x^2z)\dots(1+x^mz)$$

$$= \{1 + (1 + A_1)xz + (A_1 + A_2)x^2z^2 + \dots + (A_{n-1} + A_n)x^nz^n + \dots + A_mx^{m+1}z^{m+1}\},$$

$$\times \{1 - x^{m+1}z + x^{n(n+1)}z^2 + \dots + (-)^nz^{n(m+1)}z^n + \dots \}$$
(3)

Hence, arranging the right-hand side of (3) according to powers of z, replacing the left-hand side by its equivalent according to (2), and then equating the coefficients of zn on the two sides, we get

whence

$$\frac{1 - x^n}{x^{n+1} - x^m} A_n = A_{n-1} - A_{n-1} x^m + A_{n-2} x^{2m} - \dots (-)^{n-1} x^{(n-1)m}$$
(4).

Putting n-1 in place of n in (1), we have

$$\frac{1-x^{n-1}}{x^{n-1}(1-x^m)}A_{n-1} = A_{n-2} - A_{n-2}x^m + A_{n-4}x^{2m} - \dots \cdot (-)^{n-2}x^{(n-2)m}$$
 (5)

If we multiply (5) by xm and add (1), we derive, after an obvious reduction,

$$(1 - x^n) A_n = (x^n - x^{n_1+1}) A_{n-1}$$
 (6).

In like manner.

$$\begin{array}{ll} (1-x^{n-1}) \cdot I_{n-1} = (x^{n-1}-x^{m+1}) \cdot I_{n-2} & \textbf{(G_2),} \\ (1-x^{n-r}) \cdot I_{n-2} = (x^{n-r}-x^{(n+1)}) \cdot I_{n-3} & \textbf{(G_3),} \end{array}$$

$$x^{n-1}$$
). $I_{n-2} = (x^{n-1} - x^{(n+1)}).I_{n-3}$  (6<sub>3</sub>)

$$(1-x)A_1=(x-x^{m+1})$$
 (6<sub>n</sub>).

(7).

Multiplying  $(6_1)$ ,  $(6_2)$ , . . . ,  $(6_n)$  together, we derive

$$(x-x^{m+1})(x^2-x^{m+1})\dots(x^n-x^{m+1})$$

$$= \frac{(1-x^n)(1-x^{m-1})\dots(1-x^{m-n+1})}{(1-x^m)(1-x^{m-1})\dots(1-x^{m-n+1})} x^{n(n+1)/2}$$
(8)

$$A_n = \frac{(x - x^{m+1})(x^* - x^{m+1})}{(1 - x)(1 - x^*)}, \dots (x^n - x^{m+1})$$

$$= \frac{(1 - x^m)(1 - x^m)}{(1 - x)(1 - x^m)}, \dots (1 - x^m)}$$

$$= \frac{(1 - x^m)(1 - x^m)}{(1 - x)(1 - x^m)}, \dots (1 - x^m)}$$

$$(8),$$

which establishes our result

If |x| < 1, the product  $(1 + xz)(1 + x^2z)$ , will be convergent when continued to infinity, and will, by the theorem of the present paragraph, be expansible in a series of powers of z. The series in question will be obtained by putting  $m = \infty$  in (1). We thus get

$$(\mathbf{I} + xz) (\mathbf{I} + x^2z) \dots \text{ad} = \mathbf{I} + \sum_{n=1}^{\infty} \frac{x^{n(n+1)/2}}{(1-x)(1-x^2)\dots(1-x^n)} z^n \quad (9),$$

an important theorem of Euler's (Introd. in Anat. Inf., § 306).

§ 3.] Expansion of Sech x and Sec x.

We have, by the definition of Exp.z.

$$2/(\text{Exp }x + \text{Exp} - x) = 1/(1 + \sum_{n=1}^{\infty}/(2n)!)$$
 (1)

Hence, if

$$y = \sum_{n} x^{2n}/(2n)!$$
 (2)

$$2/(\operatorname{Exp} x + \operatorname{Exp} - x) = 1/(1+y),$$
  
=  $1 + \sum_{i=1}^{n} (-1)^n u^n$  (3)

The expansion (3) will be valid provided |y| < 1; and the series (2) is absolutely convergent for all finite values of x. Hence, if  $\xi = |x|$ , it follows from § 1 that the series (3) can be converted into a series of ascending powers of a provided

$$\sum_{n=1}^{\infty} |\xi^{2n}/(2n)! < 1$$
 (4).

This last condition involves that

$$\frac{1}{2} (e^{\xi} + e^{-\xi}) - 1 < 1;$$

that is, that  $\xi < \log (2 + \sqrt{3})$ .

This condition can obviously be satisfied; and we conclude that  $2/(\operatorname{Exp} x + \operatorname{Exp} - x)$  can be expanded in a series of ascending powers of x provided |x| do not exceed a certain finite limit.

Since the function in question is obviously an even function of x, only even powers of x will occur in the expansion. We may therefore assume

$$2 (\text{Exp } x + \text{Exp } - x) = 1 + \Sigma (-)^n E_n x^{2n} / (2n)!$$
 (5)

To determine  $E_1, E_2, \ldots$ , we multiply one side of (5) by

 $\frac{1}{2}$  (Exp. x + Exp - x) and the other by its equivalent  $1 + \sum x^{2n}/(2n)!$ ; we thus have

$$1 = \{1 + \Sigma (-)^n E_n x^{2n}/(2n)!\} \{1 + \Sigma x^{2n}/(2n)!\}$$
 (6).

 $E_1, E_2, \ldots$  must be so determined that (6) becomes an identity. We must therefore have

$$\frac{1}{(2n)!0!} = \frac{E_1}{(2n-2)!2!} + \frac{E_2}{(2n-4)!4!} - \dots + (-)^n \frac{E_n}{0!(2n)!} = 0$$
 (7)

$$E_n = {}_{2n}C_1E_{n-1} = {}_{2n}C_4E_{n-2} + \dots + (-)^{n-2}{}_{2n}C_{n-2}E_1 + (-1)^{n-1}$$
 (8). The last equation enables us to calculate  $E_1, E_2, E_3, \dots$  successively. We have, in fact,

$$\vec{E_1} = 1$$
;  $E_1 = 6E_1 - 1$ ;  $E_3 = 15E_2 + 15E_1 + 1$ ;  
 $E_4 = 28E_4 - 70E_2 + 28E_1 - 1$ ; &c.

whence

$$\begin{array}{llll} E_i = & 1, & E_s = & 2702765, \\ E_z = & 5, & E_z = & 199360981, \\ E_z = & 61, & E_s = & 19391512445, \\ E_s = & 1385, & E_s = & 2101879675441, \\ E_s = & 50524. & \end{array}$$

These numbers were first introduced into analysis by Euler\*, and the above table contains their values so far as he calculated them.

Since the constants  $E_1$ ,  $E_2$ , ..., are determined so as to make (6) an identity, (6), and therefore also (5), will be valid for all values of x, real or complex, which render all the series involved convergent. Hence, since  $1 + 2x^{2\alpha}(2n)!$  is convergent for all values of x, (5) will be valid for all values of x which render the series  $1 + 2 (-\gamma)^{\alpha} E_{\alpha}x^{2\alpha}(2n)!$  convergent. We shall determine the radius of convergency of this series presently. Meantime we observe that (5) as it stands may be written

Such 
$$x = 1 + \sum_{i=1}^{n} (-i)^n E_n x^{2n}/(2n)!$$
 (9);

and, if we put ix in place of x, it gives

Sec 
$$x = 1 + \sum E_n x^{2n} / (2n)!$$
 (10).

See Inst. Calc. Diff., § 221: the last five digits of F<sub>g</sub> are incorrectly given by fader as 61671.

For a number of curious properties of the Eulerian numbers see Sylvester, Comptex Rendus, 1, 52; and Stern, Crelle's Jour., B I. Exxix.

Cor. Sech and Secax can each be expanded in a series of even powers of x.

The possibility of such an expansion follows at once from the above. The coefficients may be expressed in terms of Euler's numbers. We may also use the identity  $1 = (1 + \sum_{i=1}^{n} A_n x^{2n}/(2n)!)$  $\cos^n x$ ; expand  $\cos^n x$  first as a series of cosines of multiples of x; finally in powers of x: and thus obtain a recurrence formula for calculating A1, A2, . . . The convergency of any expansion thus obtained will obviously be co-extensive with the convergency of (10).

§ 4.1 Erpansion of Tanh x, x Coth x, Cosech x: Tan x, x Cot x, Cosec x\*.

We have already shown, in chap. XXVIII., § 6, for real values of x, that

$$x/(1-e^{-x}) = 1 + \frac{1}{2}x + \Sigma (-)^{n-1}B_nx^{2n}/(2n)!,$$

the expansion being valid so long as the series on the right is convergent. In exactly the same way we can show, for any value of x real or complex, that

$$x/(1 - \text{Exp} - x) = 1 + \frac{1}{2}x + \Sigma(-)^{n-1}B_nx^{2n}/(2n)!$$
 (1)

where Exp - x is defined as in chap, xxix, and x is such that |x| is less than the radius of convergency of the series in (1). From (1) we derive the following, all of which will be valid so long as the series involved are convergent:

$$x (\operatorname{Exp} x - \operatorname{Exp} - x) / (\operatorname{Exp} x + \operatorname{Exp} - x) = 4x / (1 - \operatorname{Exp} - 4x) - 2x / (1 - \operatorname{Exp} - 2x) \cdot x,$$
  
=  $\mathbf{\Sigma} (-)^{n-1} 2^{2n} (2^{2n} - 1) B_n x^{2n} / (2n)!$  (2);

$$x (\operatorname{Exp} x + \operatorname{Exp} - x)/(\operatorname{Exp} x - \operatorname{Exp} - x)$$
  
=  $x/(1 - \operatorname{Exp} - 2x) - x/(1 - \operatorname{Exp} 2x)$ ,  
=  $1 + \Sigma (-)^{n-1} 2^{2n} B_n x^{2n}/(2n)$ ! (3)

$$= 1 + \Sigma (-)^{n-1} 2^{2n} B_n x^{2n}/(2n)!$$
(3);  

$$2x/(\operatorname{Exp} x - \operatorname{Exp} - x) = 2x/(1 - \operatorname{Exp} - x) - 2x/(1 - \operatorname{Exp} - 2x),$$

= 
$$1 + 2\Sigma (-)^n (2^{2n-1} - 1) B_n e^{2n}/(2n)!$$
 (4). From these equations, we have at once

Tauh 
$$x = \sum (-1)^{n-1} 2^{2n} (2^{2n} - 1) B_n x^{2n-1}/(2n)!$$
 (5)

$$x \operatorname{Coth} x = 1 + \Sigma (-)^{n-1} 2^{2n} B_n x^{2n}/(2n)!$$
 (6);

$$x \operatorname{Cosech} x = 1 + 2\Sigma (-)^n (2^{2n-1} - 1) B_n x^{2n} / (2n)!$$
 (7)

If in (2), (3), and (4), we replace x by ix, we deduce

$$\operatorname{Tan} x = \Sigma 2^{2n} (2^{2n} - 1) B_n x^{2n-1}/(2n)!$$
 (8);

$$x \text{ Cot } x = 1 - \Sigma 2^{2n} B_n x^{2n} / (2n)!$$
 (9);

$$x \operatorname{Cosec} x = 1 + 2\Sigma (2^{2n-1} - 1) B_n x^{2n}/(2n)!$$
 (10).

Cor. Each of the functions (Tanh x)". (x Coth x)". (x Cosech x)". (Tan x)", (x Cot x)", (x Cosec x)" can be expanded in an ascending

## Exercises XX1.

If θ = c·l u (see chap, xxix., § 31), show that

$$\theta = a_1 u - a_3 n^3 + a_6 u^5 - \dots$$
  
 $u = a_1 \theta + a_2 \theta^1 + a_4 \theta^5 + \dots$ 

where  $a_{2n+1} = E_n/(2n+1)!$ .

series of powers of x.

(2.) Find expressions for the coefficients in the expansions of Sin" x and Cost r

Find recurrence-formulæ for calculating the coefficients in the expansions of (x cosec x)" and (sec x)". In particular, show that

$$S_{CC^{2p+1}}x = \sum_{n=0}^{\infty} \frac{S_{P}E_{n} + S_{P-1}E_{n+1} + \dots + S_{1}E_{n+p-1} + E_{n+p}}{(2p)!} \frac{x^{2n}}{(2p)!}$$
  
where  $S_{c}$  denotes the sum of the products  $r$  at a time of  $\{2,3^{2},5^{2},\dots,(2p-1)^{2},2^{2},2^{2},\dots,(2p-1)^{2},2^{2},2^{2},\dots,(2p-1)^{2},2^{2},2^{2},\dots,(2p-1)^{2},2^{2},2^{2},\dots,(2p-1)^{2}\}$ 

(Elv. American Jour. Math., 1882.)

(1.) If |x| < 1, show that

 $(1+x^2)(1+x^4)(1+x^6)\dots$  ad  $x=1+\sum x^{n^2+n}/(1-x^2)(1-x^4)\dots(1-x^{2n})$ .

(5.) If |x|>1, and p be a positive integer, show that

$$1 + \sum_{n=1}^{\infty} \frac{x^{n(n+1-2r)/2}}{(x-1)(x^2-1)\dots(x^n-1)} = 0.$$

(6.) Show that the Pinomial Theorem for positive integral exponents is a particular case of § 2, Example 2.

(7.) Show that

$$(1 + xz)(1 + x^2z) \dots (1 + x^{2m-1}z)$$
  

$$= 1 + \sum_{n=1}^{m} \frac{(1 - x^{2m})(1 - x^{2m-2}) \dots (1 - x^{2m-2n+2})}{(1 - x^2)(1 - x^2) \dots (1 - x^{2m})} z^{n}z^{n}z^{n}$$
(Cauchy, Comptex Rendus, 1840.)

(8.) Show that

$$\frac{1}{(1-xz)(1-x^2z)\dots(1-x^mz)} = 1 + \sum_{x} x^n z^n \frac{(1-x^n)(1-x^{m+1})\dots(1-x^{m+n-1})}{(1-x)(1-x^2)\dots(1-x^n)};$$
also that, if  $|x| < 1$ ,  $|x| < 1$ ,

$$1/(1-xz)(1-x^2z)$$
...ad  $x = 1 + \sum x^n z^n/(1-x)(1-x^2)$ ... $(1-x^n)$ .  
(Euler, Int. in Anal. Inf., § 313.)

(9.) If m be a positive integer  $(1-x^m)(1-x^{m-1})$ ,  $(1-x^{m-n+1})$  is exactly divisible by  $(1-x)(1-x^2)$  . . .  $(1-x^n)$ .

(Gauss, Summatio quarumdam serierum singularium.

(10.) If  $f(x, m) = 1 + \sum_{n=1}^{\infty} \frac{(1 - x^m)(1 - x^{m-1}) \dots (1 - x^{m-n+1})}{(1 - x)(1 - x^n) \dots (1 - x^n)}$ , where |x|>1, show that

 $f(x, m) = f(x, m - 2\lambda) (1 - x^{m-1}) (1 - x^{m-3}) \dots (1 - x^{m-3\lambda+1})$   $= \frac{1 - x^{m-1}}{1 - x^{-1}} \frac{1 - x^{m-3}}{1 - x^{-3}} \frac{1 - x^{m-5}}{1 - x^{-5}} \dots \text{ ad } x.$ 

Hence show that, if |x| < 1, then

$$1 + \sum x^{a_1(n+1)/2} = \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^3} \cdot \frac{1-x^6}{1-x^5} \cdot \dots \text{ ad } \infty.$$
(Gauss. 1b.)

(11.) Show that, if m be a positive integer,

$$(1+x)(1+x^2)\dots(1+x^m) = 1 + \sum x^n \frac{(1-x^{2m})(1-x^{2m-2})\dots(1-x^{2m-2n+2})}{(1-x^2)(1-x^4)\dots(1-x^{2n})}$$
(Gauss. Ib.)

(12.) Show that

$$\frac{1}{(1-xz)(1-x^2z)\dots(1-x^{2m-1}z)}$$

$$=1+\sum x^n z^n \frac{(1-x^{2m})(1-x^{2m+2})\dots(1-x^{2m+2n-2})}{(1-x^2)(1-x^2)\dots(1-x^{2m})}.$$

Also that, if |x| < 1, and |zx| < 1,

 $1/(1-xz)(1-x^3z)$ , ad  $x = 1 + \sum x^n z^n/(1-x^2)(1-x^3)$ ,  $(1-x^3)$ (13.) Show that, if |x|<1.</p>

 $1/(1-x)(1-x^3)(1-x^5)$ ...ad  $x = (1+x)(1+x^2)(1+x^3)$ ...ad  $x = (1+x)(1+x^2)(1+x^3)$ ... (Euler 1 c., 8 325 )

(14.) If |x| < 1,

$$(1-x)(1-x^2)(1-x^3)$$
 . . . ad  $\omega = \sum_{-\infty}^{+\infty} (-)^n x^{(2n^2+n)/2}$ .  
(Euler, Nov. Comm. Pet., 1760.)

(15.) If |x| < 1, log  $\{(1-x)(1-x^2)(1-x^3)...ad \infty\} = -\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (n)x^n/n$ ,

where 
$$\{n\}$$
 denotes the sum of all the divisors of the positive integer  $n$ ; for example,  $\{4\} = 1 + 2 + 4$ .

Hence show that

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} \int (n) x^n.$$

(Euler, 16.)

(16.) If d(n) denote the number of the different divisors of the positive integer n, and |x| < 1, show that

$$\sum_{1}^{x} d(n) x^{n} = \sum_{1}^{\infty} \frac{x^{n}}{1 - x^{n}}.$$
(Lambert, Essai d'Architectonique, p. 507.)

Also that

$$\frac{x}{\Sigma} d(n) x^n = \sum_{1}^{\infty} x^{n} \begin{pmatrix} 1 + x^n \\ 1 - x^n \end{pmatrix}.$$
(Clausen, Crelle's Jour., 1827.)

(17.) If |x| < 1, show that

$$\frac{x}{1-x} - \frac{x^3}{1-x} + \frac{x^3}{1-x^5} - \dots = \frac{x}{1+x^2} + \frac{x^3}{1+x^6} + \dots$$

(18.)  $\sum x^{2n+1}/(1-x^{2n+1})^2 = \sum nx^n/(1-x^{2n}).$ 

 $\Sigma (-)^{n-1}nx^n/(1+x^n) = \Sigma (-)^{n-1}x^n/(1+x^n)^2$ .

(19.) The sum of the products r at a time of x,  $x^2$ , ...,  $x^n$  is  $x^{r(r+1)/2}(x^{r+1}-1)(x^{r+2}-1)$ ...  $(x^n-1)/(x-1)(x^2-1)$ ...  $(x^{n-r}-1)$ .

(20.) If  $S_r$  be the sum of the products r at a time of  $1, x, \ldots, x^{n-1}$ , then  $S_r = S_{n-r} x^{-(n-1)(n-2r)/2}$ .

(21.) Show that, if x lie between certain limits, and the roots of  $ax^2 + bx + c$  be real, then  $(px + q)/(ax^2 + bx + c)$  can be expended in the form  $u_0 + \sum_{i=1}^{n} \frac{1}{2} + c_i x^{-i}$ ; and that, if the roots be imaginary, no expansion of this kind is possible for any value of x.

# ON THE EXPRESSION OF CERTAIN FUNCTIONS IN THE FORM OF FINITE AND INFINITE PRODUCTS.

§ 5.] The following General Theorem covers a variety of cases in which it is possible to express a given function in the form of an infinite product; and will be of use to the student because it accentuates certain points in this delicate operation which are often left obscure if not misunderstood.

Let f(n, p) be a function (with real or imaginary coefficients) of the integral variables n and p, such that L[f(n, p)] is finite for all finite values of n, say L[f(n, p)] = f(n); and let us suppose that for all values of n and p (n < p), however great, which exceed a certain finite value,  $f(n, p) \mid f(n) \mid$  is not infinite.

Then 
$$L = \prod_{p=0}^{p} \{1 + f(n, p)\} = \prod_{1 \leq 1} \{1 + f(n)\}$$
 (1), provided  $\Sigma^{q}(n)$ , be convergent (that is, provided  $\Pi\{1 + f(u)\}$  be absolutely convergent).

Let us denote  $\prod_{n=1}^{p} \{1 + f(n, p)\}$  by  $P_p : L \prod_{p=n-1}^{p} \{1 + f(n, p)\}$  by P : |f(n, p)| by  $f_1(n, p)$ ; and |f(n)| by  $f_1(n)$ .

We may write

$$P_p = \prod_{n=1}^{m} \{1 + f(n, p)\} \prod_{n=m+1}^{p} \{1 + f(n, p)\},$$
  
=  $P_m Q_m$ , say, (2).

Just as in chap. XXVI., § 26, we have

$$|Q_m-1| \gg \frac{p}{11} \{1+f_1(n,p)\} - 1.$$

Now, by one of our conditions, if m, and therefore p, exceed a certain finite value, we may put  $f_1(n, p)/f_1(n) = A_n$ , where  $A_n$  is not infinite. If, therefore, A be an upper limit to  $A_n$ , and therefore finite and positive, we have  $f_1(n, p) > Af_1(n)$ . Hence

$$|Q_m - 1| \geqslant \prod_{n=m+1}^{p} \{1 + Af_1(n)\} - 1,$$
  
 $\Rightarrow \prod_{n=1}^{\infty} \{1 + Af_1(n)\} - 1,$  (3).

Let us now put  $p = \infty$  in (2). Since m is finite, and  $\underset{\mathbf{p}=\infty}{L} f(n, p) = f(n)$ , we have

$$L P_{m} = \prod_{1}^{m} \{1 + f'(n)\},$$

$$P = \prod_{1}^{m} \{1 + f'(n)\} Q_{m} \tag{4},$$

Therefore

where  $Q_m$  is subject to the restriction (3).

Let us, finally, consider the effect of increasing m.

Since  $\Pi \left\{1+\hat{f_1}(n)\right\}$  is absolutely convergent,  $\Pi \left\{1+A[f_1(n)]\right\}$  is absolutely convergent. It therefore follows that, by sufficiently increasing m, we can make  $\prod_{m+1}^m \left\{1+A[f_1(n)]-1\right\}$ , and, a fortiori,  $|Q_m-1|$  as small as we please. Hence, by taking m sufficiently great, we can cause  $Q_m$  to approach 1 as nearly as we please. In other words, it follows from (4) that

$$P = \prod_{i=1}^{\infty} \{1 + f(n)\}$$
 (5).

In applying this theorem it is necessary to be very careful to see that both the conditions in the first part of the enunciation regarding the value of f(n, p) are satisfied. Thus, for example, it is not sufficient that  $L \int_{T} f(n, p) = \int_{T} f(n, p$ 

absolutely convergent. This seems to be taken for granted by many mathematical writers; but, as will be seen from a striking example given below, such an assumption may easily lead to fallacious results.

# § 6.] Factorisation of sinh pu, sinh u, sin pθ, and sin θ .

From the result of chap. XII.,  $\S$  20, we have, p being any positive integer,

$$x^{2p} - 1 \equiv (x^2 - 1) \prod_{n=1}^{p-1} \left(x^2 - 2x \cos \frac{n\pi}{p} + 1\right)$$
 (1).

From this we have

$$\frac{x^{2p}-1}{x^2-1} \cong \prod_{n=1}^{p-1} \left( x^2 - 2x \cos \frac{n\pi}{p} + 1 \right);$$

whence, putting x = 1, and remembering that  $L(x^{2p} - 1)_{i}(x^{2} - 1) - p$ , we have

$$p = 2^{p-1} \prod_{i=1}^{p-1} (1 - \cos n \pi^{i} p)$$
 (2);

$$=4^{p-1}\prod_{i=1}^{p-1}\sin^2(n\pi_i)2p\tag{3}$$

and, since  $\sin \pi/2p$ ,  $\sin 2\pi/2p$ , . . . ,  $\sin (p-1)\pi/2p$  are obviously all positive.

$$\sqrt{p} = 2^{p-1} \prod_{i=1}^{p-1} \sin_i n\pi' 2p$$
 (4).

If we divide both sides of (1) by  $x^p$ , we deduce

$$x^p - x^{-p} = (x - x^{-1}) \text{ II } (x + x^{-1} - 2 \cos \pi \pi/p)$$
 (5).

where for brevity we omit the limits for the product, which are as before.

If in (5) we put  $x = e^u$ , we get at once

$$\sinh pu = 2^{p-1} \sinh u \coprod (\cosh u - \cos \cdot n\pi/p) \tag{6},$$

$$-4^{p-1}\sinh n \Pi \left(\sin^2 n \pi/2p + \sinh^2 n/2\right)$$
 (7).

Using (3), we can throw (7) into the following form :-

 $\sinh pn - p \sinh n \, \Pi \left\{1 + \sinh^2 n \, 2 / \sin^2 n \pi / 2p\right\}$  (8). Finally, since (8) holds for all values of n, we may replace u

by n/p, and thus derive

\* The results in §§ 6-9 we reall riven in one form or another by Euler in his Introductio in Analysia Infiniterium. His demonstrations of the funda-

The results in §§ 6.9 were all given in one form or another by Euler in his Introductio Manijean Infinitorium. His demonstrations of the fundamental theorems were not satisfactory, although they are still to be found unaitered in many of our elementary text-books.

$$\sinh u = p \sinh \frac{u}{p} \prod_{n=1}^{p-1} \left\{ 1 + \frac{\sinh^2 u/2p}{\sin^2 u\pi/2p} \right\}$$
(9).

We shall next apply to (9) the general theorem of § 5. Before doing so, we must however, satisfy ourselves that the requisite conditions are fulfilled.

In the first place, so long as n is a finite integer, we have

$$L \frac{\sinh^2 u/2p}{\sin^2 u\pi^2} = \frac{u^2}{u^2\pi^2}$$
 (10).

This can be deduced at once, for complex values of u, from the series for  $\sinh u/2p$  and  $\sin u\pi/2p$ . When u is real it follows readily from chap, xxv., § 22.

The product II  $(1+u^2/u^2\pi^2)$  is obviously absolutely convergent. We have, therefore, merely to show that, for all values of n and p exceeding a certain finite limit,

$$\begin{vmatrix} \sinh^2 \cdot u/2p / \frac{u^2}{n^2 - u^2} \end{vmatrix} < A \tag{11},$$

where A is a finite positive constant. That is to say, we have to show that

$$\left| \frac{\sinh \cdot u/2p}{u/2p} \right| / \left( \frac{\sin \cdot n\pi/2p}{n\pi/2p} \right)$$

remains finite.

Now

$$\begin{vmatrix} \sinh \cdot u/2p \\ u/2p \end{vmatrix} = \left| 1 + \frac{1}{3!} \left( \frac{u}{2p} \right)^2 + \dots \right|$$

$$\Rightarrow 1 + \left| \frac{1}{3!} \left( \frac{u}{2p} \right)^2 + \dots \right| \tag{12}.$$

Since the series within the bracket is absolutely convergent, its modulus can be made as small as we please by taking p sufficiently great.

Again we know, from chap. XXIX., § 14, that, if  $\theta \gg \sqrt{(6 \times 7)} \Rightarrow 6.48$ , and, a fortion, if  $\theta \gg 2\pi$ , then

$$\sin \theta \triangleleft \theta - \frac{1}{6}\theta^3$$
,

that is, if  $\theta$  be positive,

$$\sin \theta/\theta \ll 1 - 16^{\circ}$$
.

Now, since  $n \geqslant p-1$ ,  $n\pi/2p \geqslant \frac{1}{2}\pi$ . Therefore

$$\frac{\sin \frac{n\pi/2p}{n\pi/2p}}{n\pi/2p} \ll 1 - \frac{1}{b} \left(\frac{n\pi}{2p}\right)^{4}$$

$$\ll 1 - \frac{\pi^{2}}{24} \ll 58 \tag{13}.$$

From (12) and (13) it is abundantly evident that the condition (11) will be satisfied if only p be taken large enough; and it would be easy, if for any purpose it were necessary, to assign a numerical estimate for A. All the conditions for the applicability of the General Limit Theorem being fulfilled, we may make p infinite in (9). Remembering that  $Lp \sinh u/p = u$ , we thus get

$$\sinh u = u \prod_{n=1}^{r} (1 + u^{n}/u^{2}\pi^{2})$$
 (14).

To get the corresponding formulæ for  $\sin p\theta$  and  $\sin \theta$ , we have simply to put in (5)  $x = \exp i\theta$ . The steps of the reasoning are, with a few trilling modifications, the same as before. It will therefore be sufficient to write down the main results with a corresponding numbering for the equations.

$$\sin p\theta - 2^{p-1}\sin\theta \prod_{n=1}^{p-1}(\cos\theta - \cos n\pi/p) \tag{6'};$$

$$4^{p-1}\sin\theta\Pi(\sin^2,n\pi/2p-\sin^2,\theta/2)$$
 (7').

$$\sin p\theta = p\sin\theta \Pi \left(1 - \sin^2\theta / 2 / \sin^2\theta / n\pi / 2p\right) \qquad (8').$$

$$\sin \theta = p \sin \frac{\theta}{p} \prod_{n=1}^{p-1} \left\{ 1 - \frac{\sin^2 \theta / 2p}{\sin^2 \theta / 2p} \right\}$$
 (9').

$$\sin \theta = \theta \lim_{n \to 1} \{1 - \theta^2/n^2 \pi^2\} \tag{14'}$$

It should be noticed that, inasmuch as (6), (7), (8), (9), and (14) were proved for all values of u, real and complex, we might have derived (6), (7), (8), (9), and (14) at once, by putting  $u \cdot i\theta$ .

Cor. 1. The following finite products for sin pt and sinh pu should be noticed:—

 $\sin p\theta = 2^{p-1} \sin \theta \sin (\theta + \pi/p) \sin (\theta + 2\pi/p)$ ...

$$\sin\left(\theta + \overline{p-1}\pi/p\right) \quad (15):$$

 $\sinh pu = (-2i)^{p-1} \sinh u \sinh (u + i\pi/p) \sinh (u + 2i\pi/p) \dots$ 

$$\sinh \left(u + \overline{p - 1}i\pi/p\right)$$
 (16).

The first of these may be deduced from (6'), as follows :- $\sin p\theta = 2^{p-1} \sin \theta \Pi (\cos \theta - \cos n\pi/p)$ 

= 
$$2^{p-1} \sin \theta \Pi \{ 2 \sin (n\pi/2p + \theta/2) \sin (n\pi/2p - \theta/2) \}$$
,

$$= 2^{p-1} \sin \theta \Pi \{ 2 \sin (n\pi/2p + \theta/2) \cos (\overline{p - n\pi/2p + \theta/2}) \}.$$

Hence, rearranging the factors, we get

$$\sin p\theta = 2^{p-1} \sin \theta \Pi \{ 2 \sin (n\pi/2p + \theta/2) \cos (n\pi/2p + \theta/2) \},$$

$$=2^{p-1}\sin\theta \prod_{n=1}^{p-1}\sin(\theta + n\pi/p).$$

We may deduce (16) from (15) by patting  $\theta = -in$ .

Wallis's Theorem.

If in (14') we put  $\theta = \frac{1}{2}\pi$ , we deduce

$$1 = \frac{1}{2} \pi \Pi (1 - 1/2^2 n^2) \qquad (17);$$

 $1 = \frac{1}{2} \pi \prod_{1}^{n} (1 - 1/2^{n})^{2}$   $\frac{\pi}{2} = \frac{2^{2}}{1 \cdot 3} \cdot \frac{4^{2}}{3 \cdot 5} \cdot \dots \cdot \frac{(2n)^{2}}{(2n-1)(2n+1)} \cdot \dots \text{ ad } \infty,$ 

$$1.33.5 (2n-1)(2n+1)$$

$$= \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \cdot \dots \text{ ad } \infty (18).$$

This formula was given by Wallis in his Arithmetica Infinitorum, 1656. It is remarkable as the earliest expression of π by means of an infinite series of rational operations. Its publication probably led to the investigations of Bronneker, Newton, Gregory, and others, on the same subject.

§ 7.] Factorisation of  $\cos p\theta$ ,  $\cos \theta$ ,  $\cosh pu$ ,  $\cosh u$ . Following the method of chap. XII., § 20, and using the roots of -1, we can readily establish the following identity:-

$$x^{2p} + 1 \equiv \prod_{n=1}^{p} \left(x^2 - 2x \cos \frac{(2n-1)\pi}{2p} + 1\right)$$
 (1).

Putting herein x = 1, we get

$$2 = 2^{p} \Pi \left(1 - \cos \left(2n - 1\right) \pi / 2p\right)$$
 (2);

$$= 4^{p} \prod \sin^{2} \cdot (2n - 1) \pi / 4p$$
 (3).

Hence, since all the sines are positive,

$$\sqrt{2} = 2^p \prod \sin_+(2n-1) \pi/4p$$
 (4).

From (1).

$$x^{p} + x^{-p} \equiv 11 (x + x^{-1} - 2 \cos \cdot (2n - 1) \pi/2p)$$
 (5);

whence, putting x - Exp  $i\theta$ , we deduce

$$\cos p\theta = \frac{1}{2} \cdot 2^{p} \Pi \left(\cos \theta - \cos \cdot (2n - 1) \pi/2p\right)$$
 (6);

$$=\frac{1}{2}$$
,  $4^{p}\Pi \left(\sin^{2}\left(2n-1\right)\pi/4p-\sin^{2}\left(\theta/2\right)\right)$  (7).

From (7), by means of (3), we derive

$$\cos p\theta = \Pi (1 - \sin^2 \theta/2 / \sin^2 \theta/2 - 1) \pi/4p)$$
 (8).

From (8), putting  $\theta/p$  in place of  $\theta$ , we get

$$\cos \theta = \prod_{n=1}^{P} \left\{ 1 - \frac{\sin^2 \theta/2p}{\sin^2 (2n-1)\pi/4p} \right\}$$
 (9).

For any finite value of n we have

$$L_{p,\sigma x} \frac{\sin^2 \theta/2p}{\sin^2 (2n-1)\pi/4p} = \frac{4\theta^2}{(2n-1)^2\pi^2}$$
 (10).

Also the product  $\Pi \left(1 + 4\theta^2/(2n-1)^2\pi^2\right)$  is absolutely convergent.

Moreover,

$$\begin{vmatrix} \sin \theta/2p \\ \theta/2p \end{vmatrix} = \left[1 - \frac{1}{3!} \left(\frac{\theta}{2p}\right)^2 + \dots \right]$$

$$\Rightarrow 1 + \frac{1}{3!} \left(\frac{\theta}{2p}\right)^2 - \dots$$
(12);

so that  $|\sin i\theta/2p/\theta|/2p$  can be brought as near to 1 as we please by sufficiently increasing p.

Also, since  $(2n-1) \pi/(4p \gg \frac{1}{2}\pi)$ , we have, exactly as in last paragraph.

$$\frac{\sin (2n-1)\pi/4p}{(2n-1)\pi/4p} \ll 58 \tag{13}.$$

We may, therefore, put p = x in (9); and we thus get

$$\cos \theta = \prod_{n=1}^{r} \{1 - 4\theta_{J}^{r} (2n - 1)^{2} \pi^{J}\}$$
 (14).

In like manner, putting  $x = e^*$  in (5), we get

$$\cosh p u = \frac{1}{2} \cdot 2^{p} \prod_{n=1}^{p} (\cosh u - \cos \cdot (2n-1) \pi/2p)$$
 (6')

$$=\frac{1}{2}$$
,  $4^{p}\Pi \left(\sin^{2}\left(2n-1\right)\pi/\Gamma p+\sinh^{2}\left(u/2\right)\right)$  (7').

$$\cosh pu = \Pi (1 + \sinh^2 u/2 / \sin (2n - 1) \pi/4p)$$
 (8').

$$\cosh u = \prod_{i=1}^{p} \left\{ 1 + \frac{\sinh^2 (u_i 2p)}{\sin^2 (2u - 1) \pi / \ln i} \right\}$$
 (9').

$$\cosh u = \prod_{n=1}^{\infty} \left\{ 1 + 4u^2/(2n - 1)^2 \pi^2 \right\}$$
 (14').

We might, of course, derive the hyperbolic from the circular formulæ by putting  $\theta=iu$ .

It is also important to observe that we might deduce (14) from the corresponding result of last paragraph, as follows:—

From (14') and (17) of last paragraph, we have

$$\begin{split} \sin\theta &= \frac{2\theta}{\pi} \Pi \left\{ \frac{1 - \theta^2/n^2\pi^2}{1 - 1/(2n)^2} \right\}, \\ &= \frac{2\theta}{\pi} \Pi \left\{ \frac{2n\pi - 2\theta}{(2n - 1)\pi}, \frac{2n\pi + 2\theta}{(2n + 1)\pi} \right\}. \end{split}$$

Hence, putting  $\frac{1}{2}\pi - \theta$  in place of  $\theta$ , we deduce

$$\begin{split} \cos\theta &= \frac{\pi - 2\theta}{\pi} \Pi \left\{ \frac{(2n-1)\pi + 2\theta}{(2n-1)\pi}, \frac{(2n+1)\pi - 2\theta}{(2n+1)\pi} \right\}, \\ &= (1 - 2\theta/\pi) \Pi \left\{ (1 + 2\theta/2(2n-1)\pi) (1 - 2\theta/2(2n+1)\pi) \right\}, \\ &= (1 - 2\theta/\pi) (1 + 2\theta/\pi) (1 - 2\theta/3\pi) (1 + 2\theta/3\pi), \dots ... \end{split}$$

Written in this last form the infinite product is only semiconvergent, and the order of its terms may not be altered without risk of changing its value; we may, however, associate them as they stand in groups of any finite number. Taking them in pairs, we have

$$\begin{split} \cos\theta &= (1 - 4\theta^2/\pi^2) \, (1 - 4\theta^2/3^2 \pi^2) \ . \ . \ . \ . , \\ &= \prod_{n=1}^{\infty} \{1 - 4\theta^2/(2n-1)^2 \pi^2\}. \end{split}$$

§ 8.] From the above results we can deduce several others which will be useful presently.

C. 11.

We have, since all the products involved are absolutely convergent,

$$\frac{\sin\left(\theta+\phi\right)}{\sin\theta} = \frac{\theta+\phi}{\theta} \frac{11\left\{1-(\theta+\phi)^{2}n^{2}\pi^{2}\right\}}{11\left\{1-\theta^{2}/n^{2}\pi^{2}\right\}},$$

provided  $\theta + n\pi$ .

Hence, provided  $\theta \neq n\pi$ ,

$$\cos \phi + \sin \phi \cot \theta = \left(1 + \frac{\phi}{\theta}\right) \Pi \left\{1 - \frac{2\theta \phi + \phi^2}{n^2 \pi^2 + \theta^2}\right\}$$
(1).

In like manner, starting with  $\cos{(\theta + \phi)}/\cos{\theta}$ , we deduce

$$\cos \phi - \sin \phi \tan \theta - \Pi \left\{ 1 - 4 \frac{2\theta \phi + \phi^2}{(2n-1)^2 \pi^2 - 4\theta^2} \right\}$$
 (2),

provided  $\theta + \frac{1}{2}(2n-1)\pi$ . Also, from the identity

$$\frac{\sin \phi + \sin \theta}{\sin \theta} = \frac{\sin \frac{1}{2} (\phi + \theta) \cos \frac{1}{2} (\phi - \theta)}{\sin \frac{1}{2} \theta \cos \frac{1}{2} \theta},$$

we derive

 $1 + \csc \theta \sin \phi$ 

$$\begin{split} &=\left(1+\frac{\phi}{\theta}\right)\Pi\left[\frac{(1-(\phi+\theta)^2/4n^2\pi^2)\{1-(\phi-\theta)^2/(2n-1)^2\pi^2\}}{(1-\theta^2/4n^2\pi^2)\{1-\theta^2/(2n-1)^2\pi^2\}}\right]\\ &=\left(1+\frac{\phi}{\theta}\right)\Pi\left\{1-\frac{(-)^22\theta\phi+\theta^2}{(2n-1)^2\pi^2}\right\} \end{split} \tag{3},$$

provided  $\theta + n\pi$ .

A great variety of other results of a similar character could be deduced; but these will suffice for our purpose.

§ 9.] Before leaving the present subject, it will be instructive to discuss an example which brings into prominence the necessity for one of the least obvious of the conditions for the applicability of the General Theorem of § 5.

We have,  $\theta$  being neither 0 nor a multiple of  $\pi$ ,

$$x^{2p} - 2x^p \cos \theta + 1 = (x^p - (\cos \theta + i \sin \theta))(x^p - (\cos \theta - i \sin \theta)),$$

The  $\rho$ th roots of  $\cos \theta + i \sin \theta$  are given by

$$\cos_{-i}(2n\pi + \theta)/p + i \sin_{-i}(2n\pi + \theta)/p, \quad n = 0, 1, \dots, \overline{p-1}$$
 (1).

The pth roots of  $\cos \theta - i \sin \theta$ , that is, of  $\cos (-\theta) + i \sin (-\theta)$ , by

$$\cos (2n\pi - \theta) p + i \sin (2n\pi - \theta) p$$
,  $n = 0, 1, ..., p-1$  (2).

Since

$$\cos \cdot (2n\pi - \theta)/p = \cos \cdot \{2(p-n)\pi + \theta/p, \\ \sin \cdot (2n\pi - \theta)/p = -\sin \cdot \{2(p-n)\pi + \theta/p, \\$$

(2) may be replaced by

$$\cos (2n\pi + \theta)/p - i \sin (2n\pi + \theta)/p$$
,  $n = 0, 1, ..., p - 1$  (2').

We have, therefore,

$$x^{2p} - 2x^p \cos \theta + 1$$

$$\equiv (x^2 - 2x\cos\theta/p + 1)\prod_{n=1}^{p-1} \{x^2 - 2x\cos(2n\pi + \theta)/p + 1\}$$
 (3)

Since  $\cos \cdot (2n\pi + \theta)/p = \cos \cdot (2(p-n)\pi - \theta)/p$ , we may, if p be odd, arrange all the factors of the product on the right of (3) in pairs. Thus, if p = 2q + 1, we have

 $x^{4q+2} - 2x^{2q+1}\cos\theta + 1 \equiv$ 

$$\left(x^2 - 2x\cos\frac{\theta}{2q+1} + 1\right) \prod_{n=1}^{q} \left\{ \times \frac{(x^2 - 2x\cos, (2n\pi + \theta)/(2q+1) + 1)}{(x^2 - 2x\cos, (2n\pi - \theta)/(2q+1) + 1)} \right\}$$

If we now put x = 1, we get

$$4\sin^{2}\frac{\theta}{2} = 4^{2q+1}\sin^{2}\frac{\theta}{4q+2}\prod_{n=1}^{n=q}\left\{\sin^{2}\cdot\frac{2n\pi+\theta}{4q+2}\sin^{2}\cdot\frac{2n\pi-\theta}{4q+2}\right\} \quad (5).$$

If we divide both sides of (4) by  $x^{cq+1}$ , and put  $x = \text{Exp } i\phi$ , we deduce

$$2(\cos(2q+1)\phi - \cos\theta)$$

$$= 2^{2q+1} \{\cos \phi - \cos \theta/(2q+1)\} \Pi \{\cos \phi - \cos (2n\pi \pm \theta)/(2q+1)\}$$

where the double sign indicates that there are two factors to be taken.

Transforming (6), and using (5), &c., just as in the previous paragraphs, we get, finally,

 $\cos \phi - \cos \theta$ 

$$= 2 \sin^2 \frac{1}{2} \theta \left\{ 1 - \frac{\sin^2 \cdot \phi/(4q+2)}{\sin^2 \cdot \theta/(4q+2)} \right\} \prod_{n=1}^{q} \left\{ 1 - \frac{\sin^2 \cdot \phi/(4q+2)}{\sin^2 \cdot (2n\pi \pm \theta)/(4q+2)} \right\}$$
(7)

Since  $n \geqslant q$ ,  $(2n\pi \pm \theta)/(4q+2) \geqslant (2q\pi \pm \theta)/(4q+2)$ ; and the limit of this last when  $q=\infty$  is  $\frac{1}{2}\pi$ . Hence, by taking q large enough we can secure that  $(2n\pi \pm \theta)/(4q+2)$  shall have for its

upper limit a quantity which differs from  $\frac{1}{2}\pi$  by as little as we please; and therefore (see § 6) that  $\sin ((2n\pi \pm \theta)/(4\eta + 2))$  ( $(2n\pm \theta)/(4\eta + 2)$  shall have for its lower limit a quantity not less than '58.

We may, therefore, put q = x, &c., in (7). We then get

$$\cos \phi - \cos \theta = 2 \sin^2 \frac{1}{2} \theta \left( 1 - \phi^2 / \theta^2 \right) \prod_{n=1}^{\infty} \left\{ 1 - \phi^2 / (2n\pi \pm \theta)^3 \right\}$$
 (8),

that is,  $\cos \phi = \cos \theta$ 

$$= 2 \sin^2 \frac{1}{2} \theta \left\{ 1 - \frac{\phi^2}{\theta^2} \right\} \left\{ 1 - \frac{\phi^2}{(2\pi - \theta)^2} \right\} \left\{ 1 - \frac{\phi^2}{(2\pi + \theta)^2} \right\} \dots$$

Putting  $\phi = iu$  in (8), we deduce

$$\cosh u - \cos \theta = 2 \sin^2 \frac{1}{2} \theta \left( 1 + u^2 / \theta^2 \right) \prod_{n=1}^{\infty} \left\{ 1 + u^2 / (2n\pi \pm \theta)^2 \right\}$$
 (9).

The formula (8) might have been readily derived from those of previous paragraphs by using the identity  $\cos \phi - \cos \theta = 2 \sin \frac{1}{2} (\theta + \phi) \sin \frac{1}{2} (\theta - \phi)$  and proceeding as in the latter part of § 7.

Remark.—At first sight, it seems as if we might have dispensed with the transformation (4) and reasoned directly from (3), thus.—

From (3) we deduce

$$2\left(\cos p\phi - \cos\theta\right) = 2^{p}\left(\cos\phi - \cos\theta\right) \prod_{n=1}^{p-1} \{\cos\phi - \cos\theta(2n\pi + \theta)/p\},$$

Hence

 $\cos \phi - \cos \theta$ 

$$= 2\sin^2\frac{1}{2}\theta\left\{1 - \frac{\sin^2\phi}{\sin^2\theta} \frac{2p_1}{2p_1^2}\prod_{n=1}^{p-1}\left\{1 - \frac{\sin^2\frac{\phi}{2p_1^2}}{(2n\pi+\theta)/2p_1^2}\right\}.$$

Put now p = x, &c., and we get

$$\cos\phi - \cos\theta - 2\sin^2\frac{1}{2}\theta\left(1 - \phi^2/\theta^2\right) \prod_{i=1}^{n} \{1 - \phi^2/(2n\pi + \theta)^2\}.$$

This result is manifestly in contradiction with (8), although the reasoning by which it is established is the same as that often considered sufficient in such cases.

In point of fact, however, the condition of § 5, that  $M = f_1(n, p)/f_1(n)$  must remain finite when n and p exceed certain limits, is not satisfied.

In the present case the upper limit of  $(2n\pi + \theta)/2p$ , namely,  $\{2(p-1)\pi + \theta\}/2p$ , can be made to approach as near to  $\pi$  as we please. Hence in this case M may become infinite. We have, in fact.

$$M = \left| \frac{\sin \cdot (\phi/2p) \int (\phi/2p)}{\sin \cdot (2n\pi + \theta)/2p \int (2n\pi + \theta)/2p} \right|^2;$$

hence, if we give n its extreme value p-1, and put  $p=\infty$ , M becomes infinite. No finite upper limit to the modulus M can therefore be assigned; and the General Theorem of § 5 cannot be applied.

This is an instructive example of the danger of reasoning rashly concerning the limits of infinite products.

## Exercises XXII.

(1.) If (1+ix/a)(1+ix/b)(1+ix/c), A+iB, then  $\Sigma \tan^{-1}(x/a) = \tan^{-1}(B/A)$ .

 $\overset{\circ}{\Sigma} \tan^{-1}(2/n^2) = 3\pi/4$ . Hence show that

(Glaisher, Quart. Jour. Math., 1878.)

(2.) Find the n roots of  $x^{n}-nx^{n-2}+\frac{n(n-3)}{n!}x^{n-4}-...$ 

$$(-)^r \frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!} x^{n-2r} + \dots = 0.$$

(3.) If n be an odd integer, find the n roots of the equation

 $x + \frac{n^2 - 1^2}{3!} x^3 + \frac{(n^2 - 1^2)(n^2 - 3^2)}{5!} x^5 + \frac{(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}{7!} x^7 + \dots = a.$ 

(1.) Solve completely

$$x^n + {}_nC_1\cos a \ x^{n-1} + {}_nC_2\cos 2a \ x^{n-2} + \dots + \cos na = 0.$$
 (Math. Trip., 1882.)

(5.) The roots of

$$x^n \sin n\theta - {}_nC_1x^{n-1}\sin \left(n\theta + \phi\right) + {}_nC_2x^{n-2}\sin \left(n\theta + 2\phi\right) - \ldots = 0$$
  
are given by  $x = \sin \left(\theta + \phi - k\pi/n\right) \csc \left(\theta - k\pi/n\right)$ , where  $k = 0, 1, \ldots$ , or  $(n-1)$ .

If  $a = \pi/2p$ , prove the following relations:—

(6.) 
$$p = 2^{p-1} \sin 2a \sin 4a \dots \sin (2p-2)a;$$
  
 $1 = 2^{p-1} \sin a \sin 3a \dots \sin (2p-1)a,$ 

- (7.)  $_{\bullet}/p = 2^{p-1}\cos a \cos 2a \dots \cos (p-1)a$ .
- (8.)  $1 = 2^{p-1} \sin \alpha/2 \sin \beta a/2 \dots \sin (2p-1)\alpha/2;$  $= 2^{p-1} \cos \alpha/2 \cos \beta a/2 \dots \cos \beta(2p-1)\alpha/2.$
- (9.)  $\sin p\theta = 2^{p-1} \sin \theta \sin (2\alpha + \theta) \sin (4\alpha + \theta) \dots \sin (2p-2\alpha + \theta);$  $\cos p\theta = 2^{p-1} \sin (\alpha + \theta) \sin (3\alpha + \theta) \sin (5\alpha + \theta) \dots \sin (2p-1\alpha + \theta).$
- (10.)  $\tan p\theta = \tan \theta \tan (\theta + 2a) \dots \tan (\theta + (2p 2)a)$ , where p is odd,
- (11.)  $\tan \theta \tan (\theta + 2\alpha) \dots \tan (\theta + (2p 2)\alpha) = (-1)^{p/2}$ , where p is even,
- (12.) Show that the modulus of

$$\cos{(\theta + i\phi)}\cos{(\theta + i\phi + \pi/p)}\dots\cos{(\theta + i\phi + (p-1)\pi/p)}$$
  
is  $\left\{\cosh{p\phi} - \cos{(p\pi + 2p\theta)}\right\}/2^{p-\frac{1}{2}}$ .

(13.) If n be even, show that

$$\sin^2\frac{\theta}{2} = (-1)^{n/2} 2^{n-2} \cos\frac{\theta}{n} \cos\frac{\theta+2\pi}{n} \cos\frac{\theta+1\pi}{n} \cdot \cdot \cdot \cos\frac{\theta+(2n-2)\pi}{n}.$$

(14.) Show that 
$$\prod_{\theta} (1 + \sec 2^n \theta) = \tan 2^n \theta / \tan \theta;$$

and cvaluate

$$\left\{\frac{11}{0}\left\{\frac{1+\sec\left(\theta/2^n\right)}{2}\right\}.$$

(15.) Show that

$$\begin{split} & \prod_{1}^{m} \left( 1 - \frac{4}{3} \sin^2 \frac{\theta}{3^n} \right) = \frac{\sin \theta}{\theta}; \\ & \prod_{1}^{m} \left( 1 - 4 \sin^2 \frac{\theta}{3^n} \right) = \cos \theta; \end{split}$$

and write down the corresponding formulæ for the hyperbolic functions.
(Laisant.)

Prove the following results (Euler, Int. in Anal. Inf., chap. Ix.):-

$$\begin{aligned} & (16.) \quad \frac{e^{b+x} + e^{c+x}}{e^b + e^c} = \Pi \ \left\{ 1 + \frac{4 \left(b - c\right) x + 4 x^2}{\left(2n - 1\right)^2 \, \pi^2 + \left(b - c\right)^2} \right\}; \\ & \frac{e^{b+x} - e^{c-x}}{e^b - e^c} = \left( 1 + \frac{2x}{b - c} \right) \Pi \left\{ 1 + \frac{4 \left(b - c\right) x + 4 x^2}{\left(2n\right)^2 \, \pi^2 + \left(b - c\right)^2} \right\}. \end{aligned}$$

(17.) 
$$\frac{\cosh y + \cosh r}{1 + \cosh r} = \Pi \left\{ 1 + \frac{d^2 e^y + y^2}{(2n - 1)^2 r^2 + c^2} \right\},$$

$$\frac{\cosh y - \cosh c}{1 - \cosh c} = \left( 1 - \frac{y^2}{c^2} \right) \Pi \left\{ 1 - \frac{2y + y^2}{(2n)^2 r^2 + c^2} \right\};$$

$$\frac{\sinh y + \sinh c}{\sinh c} = \left( 1 + \frac{y}{c} \right) \Pi \left\{ 1 + \left( - \right)^n 2cy + y^2 \right\};$$

$$\frac{\sinh y - \sinh c}{c} = -\left( 1 - \frac{y}{c} \right) \Pi \left\{ 1 + \left( - \right)^{n-2} 2cy + y^2 \right\};$$

$$\frac{\sinh y - \sinh c}{c} = -\left( 1 - \frac{y}{c} \right) \Pi \left\{ 1 + \left( - \right)^{n-2} 2cy + y^2 \right\}.$$

Write down the corresponding formula for the circular functions, and deduce them by transformation from  $\S [9]$ .

(18.) 
$$\frac{\cos\phi + \cos\theta}{1 + \cos\theta} = \Pi \left\{ I - \frac{\phi^2}{((2n-1)\pi \pm n)^2} \right\}.$$

(19.) 
$$\cos \phi + \tan \frac{1}{2}\theta \sin \phi = \Pi \left\{ \left( 1 + \frac{2\phi}{(2n-1)\pi - \theta} \right) \left( 1 - \frac{2\phi}{(2n-1)\pi + \theta} \right) \right\}$$

(20.) 
$$\frac{\cos(\theta - \phi)}{\cos \theta} = \Pi \left\{ \left( 1 + \frac{2\phi}{(2n-1)\pi - 2\theta} \right) \left( 1 - \frac{2\phi}{(2n-1)\pi + 2\theta} \right) \right\};$$

$$\frac{\sin(\theta - \phi)}{\sin(\theta - \phi)} = \left( 1 - \frac{\phi}{\phi} \right) \Pi \left\{ \left( 1 + \frac{2\phi}{\phi} \right) - \frac{\phi}{\phi} \right\} \left( 1 - \frac{\phi}{\phi} \right) \right\}.$$

(21.) Show that

we that 
$$\begin{aligned} &\cosh 2v - \cos 2u = 2\left(u^2 + v^2\right) \prod_{\substack{n = r \\ n \neq r}}^{\left(\ln \pi \pm u\right)^2 + \frac{r^2}{r^2}} \right\}; \\ &\cosh 2v + \cos 2u = 2\Pi \\ &\left\{\frac{\left((2n - 1)\pi \pm 2u\right)^2 + 4v^2}{\left(2n - 1\right)^2 \pi^2}\right\}; \\ &\cosh 2u - \cos 2u = 4u^2 \prod_{\substack{n = 1\\ n \neq r}}^{\left(4n + \frac{4u^4}{r^2}\right)}; \end{aligned}$$

 $\cosh 2u + \cos 2u = 2\Pi \left\{ 1 + \frac{2^4 u^4}{(2n-1)^4 \pi^4} \right\}.$ (Schlömilch, Handb, d, Alg, Anal, chap. xi.)

(22.) Evaluate 
$$\prod_{n=0}^{\infty} \left( \frac{4n^2 - 4n + 5}{4n^2 - 4n + 1} \right)$$
.

(23.) If 
$$\varpi = \log (1 + \sqrt{2})$$
, show that 
$$\sqrt{2} = \left(1 + 4 \frac{\varpi^2}{1^2 \pi^2}\right) \left(1 + 4 \frac{\varpi^2}{3^2 \pi^2}\right) \dots \text{ ad } \varpi.$$

EXPANSION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS IN AN INFINITE SERIES OF PARTIAL FRACTIONS.

§ 10.] By § 8 we have, provided  $\theta \pm \frac{1}{2}(2n-1)\pi$ ,

$$\cos \phi - \sin \phi \tan \theta = \Pi \left\{ 1 - 4 \frac{2\theta \phi + \phi^2}{(2n-1)^2 \pi^2 + 4\theta^2} \right\}$$
 (1).

Now, referring to § 2, Cor. 2, we have here

$$\begin{aligned} &u_n' = 8 \left| \frac{\theta}{(2n-1)^2 \pi^2 - 4\theta^2} \right| |\phi| + 4 \left| \frac{1}{(2n-1)^2 \pi^2 - 4\theta^2} \right| |\phi|^2, \\ &\Rightarrow \frac{8\theta'}{(2n-1)^2 \pi^2 - 4\theta^2} |\phi'|^2 + \frac{4}{(2n-1)^2 \pi^2 - 4\theta^2} |\phi'|^2, \end{aligned}$$

$$\Rightarrow \frac{1}{|(2n-1)^2\pi^2 - 4\theta'^2|} \frac{\phi' + \frac{1}{|(2n-1)^2\pi^2 - 4\theta'^2|}}{|(2n-1)^2\pi^2 - 4\theta'^2|} \frac{\phi^2}{|(2n-1)^2\pi^2 - 4\theta'^2|} + \frac{1}{|(2n-1)^2\pi^2 - 4\theta'^2|} \frac{\phi^2}{|(2n-1)^2\pi^2 - 4\theta'^2|} \frac{\phi^2$$

where  $\theta' = |\theta|$ ,  $\phi' = |\phi|$ . It follows, therefore, that the product in (1) may be expanded as an ascending series of powers of  $\phi$ .

Expanding also on the left of (1), we have

$$1 = \frac{\phi^{-1}}{2!} + \dots + \tan \theta \left( \phi - \frac{\phi^{2}}{3!} + \dots \right)$$

$$= 1 - 4 \left( 2\theta \phi + \phi^{2} \right) \frac{1}{2(n-1)^{2}\pi^{2} - 4\theta^{2}} + 16 \left( 2\theta \phi + \phi^{2} \right)^{2} \frac{1}{2(2m-1)^{2}\pi^{2} - 4\theta^{2}} \left\{ (2m-1)^{2}\pi^{2} - 4\theta^{2} \right\} \left\{ (2n-1)^{2}\pi^{2} - 4\theta^{2} \right\}$$
(2),

Since the two series in (2) must be identical, we have, by comparing the coefficients of  $\phi$ ,

$$\tan \theta = 8\theta \sum_{1}^{\infty} \frac{1}{(2n-1)} \frac{1}{\pi^2 - 4\theta^2}$$
 (3).

This series, which is analogous to the expansion of a rational function in partial fractions obtained in chap. vii., is absolutely convergent for all values of  $\theta$  except  $\frac{1}{2}\pi$ ,  $\frac{\pi}{2}\pi$ ,  $\frac{\pi}{2}\pi$ , . . . It should be observed, however, that when  $\theta$  lies between  $\frac{1}{2}(2n-1)\pi$  and  $\frac{1}{2}(2n+1)\pi$ , the most important terms of the series are those in the neighbourhood of the nth term, so that the convergence diminishes as  $\delta$  increases,

We may, if we please, decompose  $8\theta/((2n-1)^2\pi^2-4\theta^2)$  into  $2\lceil ((2n-1)\pi-2\theta_1^2-2)/(2n-1)\pi+2\theta_0^2$  and write the series (3) in the semi-convergent form

$$\tan \theta = \frac{2}{\pi - 2\theta} - \frac{2}{\pi + 2\theta} + \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} + \frac{2}{5\pi - 2\theta} - \frac{2}{5\pi + 2\theta} + \dots$$
 (3').

In exactly the same way, we deduce from (1) and (3) of  $\S 8$  the following:—

$$\theta \cot \theta = 1 - 2\theta^2 \Sigma \frac{1}{n^2 \pi^2 - \theta^2}$$
 (4),

. . 1

$$\begin{array}{lll} \theta \cot \theta & 1 & \frac{\theta}{\pi - \theta} + \frac{\theta}{\pi + \theta} + \frac{\theta}{2\pi} - \frac{\theta}{2\pi - \theta} + \frac{\theta}{2\pi + \theta} \\ & - \frac{\theta}{2\pi - \theta} + \frac{\theta}{2\pi + \theta} - \dots \end{array} \tag{4.}$$

provided  $\theta \neq \pi$ ,  $2\pi$ ,  $3\pi$ , . . . ; and

$$\theta \csc \theta = 1 + 2\theta^2 \sum_{\mu', \pi^2 - \theta^2}^{(-1)^{n-1}}$$
 (5),

or

$$\theta \csc \theta = 1 + \frac{\theta}{\pi - \theta} - \frac{\theta}{\pi + \theta} - \frac{\theta}{2\pi - \theta} + \frac{\theta}{2\pi + \theta}$$

$$+\frac{\theta}{2-4}-\frac{\theta}{2-4}-\dots$$
 (5),

provided  $\theta \pm \pi$ ,  $2\pi$ ,  $3\pi$ , . . . .

We might derive (4) from (3) by writing  $(\frac{1}{2}\pi - \theta)$  for  $\theta$  on both sides, multiplying by  $\theta$ , decomposing into a semi-convergent form like (3'), and then reassociating the terms in pairs; also (5) might be deduced from (3) and (4) by using the identity  $2 \csc \theta = \tan \frac{1}{2}\theta + \cot \frac{1}{2}\theta$ .

When we attempt to get a corresponding result for  $\sec \theta$ , the method employed above ceases to work so easily; and the result obtained is essentially different. We can reach it most readily by transformation from (5). If we put (5) into the form

cosec 
$$\theta = \frac{1}{\theta} + \frac{1}{\pi - \theta} - \frac{1}{\pi + \theta} - \frac{1}{2\pi - \theta} + \frac{1}{2\pi + \theta} + \frac{1}{2\pi - \theta} - \frac{1}{2\pi + \theta} - \dots$$

which we may do, provided  $\theta \neq 0$ , and then put  $\frac{1}{2}\pi - \theta$  in place of  $\theta$ , we get

$$\sec \theta = \frac{2}{\pi - 2\theta} + \frac{2}{\pi + 2\theta} - \frac{2}{3\pi - 2\theta} - \frac{2}{3\pi + 2\theta} + \frac{2}{5\pi - 2\theta} + \frac{2}{5\pi + 2\theta} - \dots$$
 (6');

or, if we combine the terms in pairs,

$$\sec \theta = 4\Sigma \left(-\right)^{n-1} \frac{(2n-1)\pi}{(2n-1)^2\pi^2 - 4\theta^2} \tag{6},$$

where  $\theta = \frac{1}{2}\pi$ ,  $\frac{3}{2}\pi$ ,  $\frac{5}{2}\pi$ , . . .

The series (6), unlike its congeners (3), (4), and (5), is only

semi-convergent; for, when n is very large, its nth term is comparable with the nth term of the series  $\Sigma\Gamma(2n-1)$ .

We might, by pairing the terms differently, obtain an absolutely convergent series for  $\sec \theta$ , namely,

$$\sec \theta = \frac{2}{\pi - 2\theta} + 4 \left( \pi - 2\theta \right) \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{4\pi^2 \pi^2 - (\pi - 2\theta)^2}$$
 (7)

but this is essentially different in form from (3), (4), and (5).

Cor. 1. The sum of all the products two and two of the terms of the series  $\Sigma 1/((2n-1)^2\pi^2 + 4\theta^2)$  is  $(\tan\theta - \theta)/128\theta^2$ ; and the like sum for the series  $\Sigma 1/(n^2\pi^2 + \theta^2)$  is  $(3 - \theta^2 - 3\theta \cot\theta)/8\theta^4$ .

This may be readily established by comparing the coefficients of  $\phi^2$  in (2) above, and in the corresponding formula derived from § 8 (1).

Cor. 2. The series  $\Sigma 1/\{(2n-1)^{\epsilon}\pi^{2}-4\theta^{\epsilon}\}^{2}$  converges to the value  $(\theta \tan^{2}\theta - \tan\theta + \theta)/(64\theta^{\epsilon})$ ; and  $\Sigma 1/(n^{2}\pi^{2} - \theta^{\epsilon})^{2}$  to the value  $(\theta^{\epsilon}\cos c^{2}\theta + \theta \cot\theta - 2)/4\theta^{\epsilon}$ .

Since the above series have been established for all values of  $\theta$ , real and imaginary, subject merely to the restriction that  $\theta$  shall not have a value which make  $\theta$  the function to be expanded infinite, we may, if we choose, put  $\theta = ui$ . We thus get, inter-alia,

$$\tanh u = 8u \Sigma 1/\{(2n-1)^2 \pi^2 + 4u^2\}$$
 (8);

$$u \coth u = 1 + 2u^2 \Sigma 1/(n^2 \pi^2 + u^2)$$
 (9);

$$u \operatorname{cosech} u = 1 - 2u^2 \Sigma (-1)^{n-1} \{ (u^2 \pi^2 + u^2) \}$$
 (10);

$$\operatorname{sech} u = 4\Sigma \left(-\right)^{n-1} \left(2n-1\right) \pi_{I}^{\left(3\right)} \left(2n-1\right)^{2} \pi^{2} + 4u^{2}\right\} \tag{11}.$$

Expressions for the numbers of bernoulli and eulerradius of convergency for the expansions of  $\tan \theta$ ,  $\cot \theta$ ,  $\csc \theta$ , and  $\sec \theta$ .

§ 11.] If  $\|\theta\| < \pi$ , then every term of the infinite series  $\Sigma \theta^{2}/(n^{2}\pi^{2} - \theta^{2})$  can be expanded in an absolutely convergent series of ascending powers of  $\theta$ . Also, when all the powers of  $\theta$  are replaced by their moduli, the series arising from  $1/(n^{2}\pi^{2} - \theta^{2})$  will simply become  $1/(n^{2}\pi^{2} - \theta^{2})$ , which is positive, since  $|\theta| < \pi$ . The double series

$$\sum_{n=1}^{r} \left\{ \frac{\theta^{2}}{n^{2}\pi^{2}} + \frac{\theta^{3}}{n^{4}\pi^{4}} + \dots + \frac{\theta^{2n}}{n^{2n}\pi^{2m}} + \dots \right\}$$

therefore satisfies Cauchy's criterion, and may be arranged according to powers of  $\theta$ . Hence, if

$$\sigma_{\text{om}} = 1/1^{2m} + 1/2^{2m} + 1/3^{2m} + \dots$$
 (1).

we have, by § 10 (4),

$$\theta \cot \theta = 1 - 2 \Sigma \theta^2 / (n^2 \pi^2 - \theta^2),$$
  
=  $1 - 2 \Sigma \sigma_{2m} \theta^{2m} / \pi^{2m}$  (2).

Since  $\sigma_{2m}$  ( $<\sigma_2$ ) is certainly finite\*, the series (2) will be convergent so long as, and no longer than,  $\theta < \pi$ .

Now, by § 4 (9), we have

$$\theta \cot \theta = 1 - \sum_{n=0}^{\infty} \frac{1}{2^{2m}} B_m \theta^{2m} / (2m)!$$
 (3),

provided  $\theta$  be small enough.

The two series (2) and (3) must be identical. Hence we have

$$B_m = \frac{2(2m)! \, \sigma_{2m}}{(2\pi)^{2m}} = \frac{2(2m)!}{(2\pi)^{2m}} \left\{ \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \dots \right\}$$
(4).

§ 12.] If, instead of using the expansion for  $\theta \cot \theta$ , we had used in a similar way the expansion for  $\tan \theta$ , we should have arrived at the formula

$$B_m = \frac{2(2m)!}{(1-1/2^{2m})(2\pi)^{2m}} \left\{ \frac{1}{1^{2m}} + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \dots \right\}$$
 (5).

This last result may be deduced very readily from (4); it is, indeed, merely the first step in a remarkable transformation of the formula (4), which depends on a transformation of the series  $\sigma_m$  due to Euler†. We observe that the result of multiplying the convergent series  $\sigma_{2m}$  by  $1-1/2^{2m}$  is to deprive the series of all terms whose denominators are multiples of 2. Thus

$$(1 - 1/2^{2m}) \sigma_{2m} = 1 + 1/3^{2m} + 1/5^{2m} + \dots$$

<sup>•</sup> It may, in fact, be easily shown that  $L\sigma_{2m}=1$  when  $m=\infty$ ; for, by chap.  $\mathbf{xxv.}$ , § 25, we have the inequality  $1/(2m-1)>1/2^{2m}+1/i^{2m}+1/1^{2m}+1$ ...> $1/(2m-1)\cdot 2^{2m-1}$ , which shows that  $L(1/2^{2m}+1/i^{2m}+1/i^{2m}+1)\cdot 0$ . when  $m=\infty$ .

<sup>+</sup> See Introd. in Anal. Inj., § 253.

If we take the next prime, namely 3, and multiply  $(1-1/2^{2m})\sigma_{2m}$  by  $1-1/3^{2m}$ , we shall deprive the series of all terms involving multiples of 3; and so on. Thus we shall at last arrive at the conation

$$(1-1/2^{2m})(1-1/3^{2m})(1-1/5^{2m})$$
, . .  $(1-1/p^{2m})\sigma_{2m}$   
=  $1+1/q^{2m}$  + . . . (6),

where  $2, 3, 5, \ldots, p$  are the succession of natural primes up to p, and q is the next prime to p. We may, of course, make q as large as we please, and therefore 1/g + . . . (which is less than the residue after the q-1th term of the convergent series σ<sub>m</sub>) as small as we please. Hence

$$\sigma_{2m} = 1/(1-1/2^{-m})(1-1/3^{2m})(1-1/5^{2m})$$
 . . . (7),

where the succession of primes continues to infinity. Hence  $B_m = 2 (2m)!/(2\pi)^{2m} (1 - 1/2^{2m}) (1 - 1/3^{2m}) (1 - 1/5^{2m}) ...$ 

$$B_m = 2(2m)!/(2\pi)! \cdot (1 - 1/2!!)(1 - 1/3!!)(1 - 1/3!!!) \cdot ... (8)$$

§ 13.] Bernoulli's Numbers are all positive: they increase after B<sub>3</sub>; and have ∞ for an upper limit.

That the numbers are all positive is at once apparent from § 11 (1). The latter part of the corollary may also be deduced from (4) by means of the inequality of chap. xxv., § 25. For we have

$$1/(2m-1) > 1/2^{2m} + 1/3^{2m} + 1/4^{2m} + \dots > 1/(2m-1) 2^{2m-1}$$
 (9).

Hence

$$\begin{split} B_{m+1} &= \frac{(2m+2)\left(2m+1\right)\sigma_{m+1}}{(2\pi)^3\sigma_{m}}, \\ &> \frac{(2m+2)\left(2m+1\right)\left\{1+1\left/\left(2m+1\right)\right\}2^{2n+1}\right\}}{\left(2\pi\right)^3\left\{1+1\left(2m-1\right)\right\}}, \\ &> \frac{(2m)^3-1}{4-2}. \end{split}$$

Hence  $B_{m+1}/B_m > 1$ , provided  $m > \sqrt{(\pi^2 + \frac{1}{4})}$ , that is, if m > 3.16. Now  $B_s > B_s$ , hence  $B_s < B_s < B_s < \ldots$ 

Again, it follows from (9) that  $L\sigma_{2m} = 1$  when  $m = \infty$ , and  $L(2m)!/(2\pi)^{2m}$  is obviously infinite; hence  $LB_m$  is infinite.

Cor.  $B_m/(2m)!$  ultimately decreases in a geometrical progression having for its common ratio 1/4\pi^2. From which it follows that the series for  $\tan \theta$ ,  $\theta$  and  $\theta$  cosec  $\theta$ , given in § 4, have for their radii of convergence  $\theta = k\pi$ ,  $\pi$  and  $\pi$  respectively.

§ 14.] Turning now to the secant series, we observe that  $4c^{-1} - (2n-1)\pi_{\parallel}^{2}/(2n-1)^{2}\pi^{2} - 4\theta^{2}$  does not, if treated in the above way as it stands, give a double series satisfying Cauchy's criterion, for, although when  $|\theta| < \frac{1}{2}\pi$  the horizontal series are absolutely convergent after we replace  $\theta$  by  $|\theta|$ , yet the sum of the sums of the horizontal series, namely,  $4\Sigma(-p^{n-1}(2n-1)\pi)^{2} + 4(\theta^{n})$ , is only semi-convergent. We can, however, pair the positive and negative terms together, and deal with the series in the form

$$4\Sigma \left\{ \frac{(4n-3)\pi}{(4n-3)^2\pi^2 - 4\theta^2} - \frac{(4n-1)\pi}{(4n-1)^2\pi^2 - 4\theta^2} \right\}$$
(10),

that is,  $8\pi \Sigma \frac{(4n-3)(4n-1)\pi^2 + 4\theta^2}{\{(4n-3)^2\pi^2 - 4\theta^2\}^4(4n-1)^2\pi^2 - 4\theta^2\}}$  (11)

Since (11) remains convergent when for  $\theta$  we substitute  $|\theta|$ , it is clear that we may expand each term of (10) in ascending powers of  $\theta$ , and rearrange the resulting double series according to powers of  $\theta$ . In this way we get

$$\begin{split} \sec\theta &= 4\sum_{n=n}^{\infty} \left[\sum_{n=1}^{\infty} \left\{ \left(\frac{1}{(4n-3)^{2m+1}} - \frac{1}{(4n-1)^{2m+1}} \right) \right] \frac{2^{2m}\theta^{2m}}{\pi^{2m+1}}, \\ &= \sum_{n=0}^{\infty} 2^{2m+2} \tau_{2m+1}\theta^{2m} / \pi^{2m+1} \end{split}$$
 (1

where  $\tau_{2m+1} = 1/1^{2m+1} - 1/3^{2m+1} + 1/5^{2m+1} - \dots$  (13).

Comparing (12) with the series  $\sec \theta = 1 + \sum E_m \theta^{2m}/(2m)!$ .

obtained in § 3, we see that

 $E_m = \frac{2^{2m+2}(2m)! \tau_{2m+1}}{\tau_{2m+1}}$ 

$$= 2 \left(2m\right)! \left(\frac{2}{\pi}\right)^{2m+1} \left\{ \frac{1}{1^{2m+1}} - \frac{1}{3^{2m+1}} + \frac{1}{5^{2m+1}} - \dots \right\}$$
(14),

which may be transformed into

$$E_m = 2 \left(2m\right)! \left(\frac{2}{\pi}\right)^{2m+1} / \left(1 + \frac{1}{3^{2m+1}}\right) \left(1 - \frac{1}{5^{2m+1}}\right) \left(1 - \frac{1}{7^{2m+1}}\right) \dots$$
 in the same way as before. (15)\*

See again Euler, Introd. in Anal. Inf., § 284.

Cor. 1. Euler's numbers are all positive; they continually increase in magnitude, and have infinity for their upper limit.

For we have

$$1 > \tau_{2m+1} > 1 - 1/3^{2m+1}$$
 (16).

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Hence

$$\begin{split} \frac{E_{m+1}}{E_m} &= \frac{(2\,m+2)\,(2\,m+1)\,4\,\tau_{2\,m+3}}{\pi^2\,\tau_{2\,m+1}}, \\ &> \frac{(2\,m+2)\,(2\,m+1)\,4\,\left(1-1/3\,^{2\,n+3}\right)}{\pi^2}. \end{split}$$

But this last constantly increases with  $m_i$  and is already greater than 1, when m=1. Hence  $E_1 < E_2 < E_2 < \ldots$  Also, from (16), we see that  $L\tau_{m+1} = 1$  when  $m=\infty$ , and  $L(2m)!(2m)!^{(2m)} = \infty$ , hence  $LE_m = \infty$ .

Cor. 2.  $E_{al}(2m)!$  ultimately decreases in a geometrical progression whose common ratio is  $4/\pi^2$ . Hence the radius of convergence of the secant series is  $\theta = \frac{1}{2}\pi$ .

§ 15.1 We have, by § 11 (1),

$$\sigma_{2m} = \frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \dots = \frac{2^{2m-1}B_m}{(2m)!} \pi^{2m}$$
 (1)\*;

and hence

$$\sigma'_{2m} = \frac{1}{1^{2m}} + \frac{1}{3^{2m}} + \frac{1}{5^{2m}} + \dots = \left(1 - \frac{1}{2^{2m}}\right) \frac{2^{2m-1} B_m}{(2m)!} \frac{1}{n^{2m}},$$

$$= \frac{(2^{2m} - 1) B_m}{2^{2m} m!} \frac{1}{n^{2m}},$$

$$= \frac{(2^{2m} - 1) B_m}{2^{2m}} \frac{1}{n^{2m}},$$
(2);

aml

$$\sigma :_{2m-\frac{1}{4},n} = \frac{1}{2^{2n}} + \frac{1}{5^{2n}} + \dots \qquad \left(1 - \frac{2}{2^{2m}}\right) \frac{2^{2m-1}B_n}{(2m)!} \pi^{2n},$$

$$= \frac{(2^{2m-1} - 1)B_n}{(2m)!} \pi^{2m}.$$
(3).

Again from (11) of last paragraph

$$\tau_{2m+1} - \frac{1}{1^{2+\alpha_1}} - \frac{1}{3} \frac{1}{\frac{1}{2^{\alpha_1}}} + \frac{1}{5^{\alpha_1+1}} + \dots = \frac{E_m}{2^{\alpha_m} \cdot (2m)!} \frac{\pi^{2m+1}}{\pi^{2m+1}}$$
 (1).

The remarkable summations involved in the formulæ (1), (2), (3) were
discovered independently by John Fermulli (see e.g., t. iv., p. 10), and by
Euler (Comm. As. P. Crop., 1710).

Inasmuch as we have independent means of calculating the numbers  $B_m$  and  $E_m$ , the above formulæ enable us to sum the various series involved. It does not appear that the series  $\sigma_{2m+1}$  can be expressed by means of  $B_m$  or  $E_m$ ; but Euler has calculated (to 16 decimal places) the numerical values of  $\sigma_{2m+1}$  in a number of cases, by means of Maclaurin's formula for approximate summation\*. As the values of  $\sigma_m$  are often useful for purposes of verification, we give here a few of Euler's results. It must not be forgotten that the formulæ involving  $\pi$  for  $\sigma_m$  are accurate when m is even; but only approximations when m is odd.

$$\begin{array}{lll} \sigma_2 = 1^\circ 6449340668 & = \pi^2/6, \\ \sigma_3 = 1^\circ 2020569031 & = \pi^2/25^\circ 79436 & \dots \\ \sigma_4 = 1^\circ 9823232337 & = \pi^4/90, \\ \sigma_5 = 1^\circ 9369277551 & = \pi^2/295^\circ 1215 & \dots \\ \sigma_6 = 1^\circ 9173430620 & = \pi^2/945, \\ \sigma_7 = 1^\circ 9183492774 & = \pi^2/2995^\circ 286 & \dots \\ \sigma_8 = 1^\circ 91450, \\ \sigma_8 = 1^$$

## EXPANSIONS OF THE LOGARITHMS OF THE CIRCULAR FUNCTIONS.

§ 16.] From the formulæ of §§ 6 and 7, we get, by taking logarithms,

$$\log \sin \theta = \log \theta + \sum_{n=1}^{\infty} \log \left(1 - \theta^{2}/n^{2}\pi^{2}\right),$$

$$= \log \theta - \sum_{n=1}^{\infty} \sigma_{2n} \theta^{2n}/n\pi^{2n}$$
(1),

since the double series arising from the expansions of the logarithms is obviously convergent, provided  $|\theta| < \pi$ .

If we express  $\sigma_{2m}$  by means of Bernoulli's numbers, (1) may be written

log sin 
$$\theta = \log \theta - \sum_{m=1}^{\infty} 2^{2m-1} B_m \theta^{2m} / m (2m)!$$
 (1').

<sup>\*</sup> Inst. Calc. Diff., chap. vi.

The corresponding formula for  $\cos \theta$  are

$$\log \cos \theta = -\Sigma (2^{m} - 1) \sigma_{2m} \theta^{2m} / m \pi^{2m}$$

$$= -\Sigma 2^{2m-1} (2^{2m} - 1) B_{-} \theta^{2m} / m (2m)!$$
(2),

The like formulae for log tau  $\theta$ , log cot  $\theta$ , log sinh u, log cosh u,  $\infty$ ... can be derived at once from the above.

If a table of the values of  $\sigma_{2m}$  or of  $B_m$  be not at hand, the first few may be obtained by expanding  $\log (\sin \theta/\theta)$ , that is,  $\log (1 - \theta^2/3! + \theta^4/5! + \dots)$ , and comparing with the series  $-2\sigma_{2m}e^{2m}/m\pi^{2m}$ . For example, we thus find at once that  $\sigma_2 = \pi^2/6$ .

#### STIRLING'S THEOREM.

§ 17.] Before leaving this part of the subject, we shall give an elementary proof of a theorem of great practical importance which was originally given by Stirling in his Methodus Differentialis (1730).

When n is very great, n', approaches equality with  $\sqrt{(2n\pi)(n/e)^n}$ ; or, more accurately, when n is a large number, we have

$$n! = \sqrt{(2\pi n)} (n/e)^n \exp\{1/12n + \theta\}$$
 (1),

where  $-1/2 \ln^2 < \theta < 1/2 \ln (u-1)$ .

Since  $\log \{n_i(n-1)\} = -\log (1-1/n)$ , we have

$$\log \frac{n}{n} = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{2n^3} + \frac{1}{1n^4} + \dots + \frac{1}{nn^n} + \dots$$

We can deprive this expansion of its second term by multiplying by n-k. We thus get

$$(n-\frac{1}{2})\log\frac{n}{n-1}=1+\frac{1}{12n^2}+\frac{1}{12n^3}+\ldots+\frac{m-1}{2m(m+1)n^m}+\ldots$$

Hence, taking the exponential of both sides, and writing successively  $n, n-1, n-2, \ldots, 2$  in the resulting equation, we deduce

$${n \choose n-1}^{-\frac{1}{2}} - \exp\left(1 + \frac{1}{12n^2} + \frac{1}{12n^2} + \dots + \frac{m-1}{2m(m+1)n^n} + \dots\right),$$

$$\left(\frac{n-1}{n-2}\right)^{n-1-\frac{1}{2}} = \exp\left(1 + \frac{1}{12\frac{1}{(n-1)^2}} + \frac{1}{12\frac{(n-1)^3}} + \cdots + \frac{m-1}{2m\frac{(m-1)}{(m-1)^m}} + \cdots\right),$$

$$\left(\frac{n-2}{n-3}\right)^{n-2-\frac{1}{2}} = \exp\left(1 + \frac{1}{12\frac{1}{(n-2)^2}} + \frac{1}{12\frac{(n-2)^3}} + \cdots + \frac{m-1}{2m\frac{(m-1)}{(m-1)^m}} + \cdots\right),$$

$$\left(\frac{3}{2}\right)^{3-\frac{1}{2}} = \exp\left(1 + \frac{1}{12.3^2} + \frac{1}{12.3^3} + \cdots + \frac{m-1}{2m\frac{(m-1)}{(m-1)^2}} + \cdots\right),$$

$$\left(\frac{2}{1}\right)^{2-\frac{1}{2}} = \exp\left(1 + \frac{1}{12.2^2} + \frac{1}{12.2^2} + \cdots + \frac{m-1}{2m\frac{(m-1)}{(m-1)^2}} + \cdots\right).$$

By multiplying all these together, we get

$$\frac{n^{n-\frac{1}{2}}}{(n-1)!} = \exp \left\{ (n-1) + \frac{1}{12} S'_2 + \frac{1}{12} S'_3 + \dots + \frac{m-1}{2m(m+1)} S'_m + \dots \right\}$$
(2),

where  $S'_m = 1/2^m + 1/3^m + 1/4^m + ... + 1/n^2$ Now

$$S'_m = S_m - 1/(n+1)^m - 1/(n+2)^m - \dots$$
 (3),  
 $S_m = 1/2^m + 1/3^m + \dots + 1/n^m + \dots$  ad  $\infty$ 

 $1/(m-1)n^{m-1} > 1/(n+1)^m + 1/(n+2)^m + ... > 1/(m-1)(n+1)^{m-1}$ Hence

$$S_m - 1/(m-1)(n+1)^{m-1} > S'_m > S_m - 1/(m-1)n^{m-1}$$

By the inequality (6) of chap, xxv., § 25, we have

Therefore

$$\frac{1}{12}S'_2 + \frac{1}{12}S'_2 + \dots + \frac{m-1}{2m(m+1)}S'_m + \dots \\
\geq \frac{1}{2}\frac{\pi}{2}\frac{(m-1)N_m}{(m+1)} - \frac{1}{2}\frac{\pi}{2}\frac{1}{m(m+1)n^{m-1}} \qquad (1) \\
\leq \frac{1}{2}\frac{\pi}{2}\frac{(m-1)S_m}{m(m+1)} - \frac{1}{2}\frac{\pi}{2}\frac{1}{m(m+1)(m+1)^{m-1}} \qquad (5)$$

C. 11.

Since  $S_m < 1/(m-1)$ , the series  $\Sigma(m-1) S_m/m (m+1)$  converges to a finite limit which is independent both of m and of n. Again,

$$\frac{2}{2} \frac{1}{m(m+1) n^{m-1}} = \frac{1}{2 \cdot 3n} + \frac{1}{3 \cdot 4n^2} + \frac{1}{4 \cdot 5n^3} + \dots$$

$$< \frac{1}{6n} + \frac{1}{12n^2} \left\{ 1 + \frac{1}{n} + \frac{1}{n^2} + \dots \right\},$$

$$< \frac{1}{6n} + \frac{1}{12n(n-1)}$$
(7).

Also, by (6),

$$\frac{\tilde{S}}{2} \frac{1}{m (m+1)(n+1)^{m-1}}$$

$$= \frac{\tilde{S}}{2} \left( \frac{1}{m} - \frac{1}{m+1} \right) \frac{1}{(n+1)^{m-1}},$$

$$= (n+1) \frac{\tilde{S}}{2} \frac{1}{m (n+1)^{m}} - (n+1)^{2} \frac{\tilde{S}}{2} \frac{1}{(m+1)(n+1)^{m+1}},$$

$$= (n+1) \left\{ -\frac{1}{n+1} - \log \left( 1 - \frac{1}{n+1} \right) \right\}$$

$$= (n+1)^{2} \left\{ -\frac{1}{(n+1)} - \frac{1}{2} \frac{1}{(n+1)^{2}} - \log \left( 1 - \frac{1}{n+1} \right) \right\},$$

$$= \frac{1}{2} + n - (n^{2} + n) \log \left( 1 + \frac{1}{n} \right);$$

$$= \frac{1}{2} + n - n + \frac{1}{2} - \frac{1}{3} + \frac{1}{4n^{2}} - \dots - 1 + \frac{1}{2n} - \frac{1}{3n^{2}} + \dots,$$

$$= \frac{1}{2 \cdot 3n} - \frac{1}{3 \cdot 3n^{2}} + \frac{1}{4 \cdot 5n^{2}} - \dots,$$

$$= \frac{1}{2 \cdot 3n} - \frac{1}{3 \cdot 3n^{2}} + \frac{1}{4 \cdot 5n^{2}} - \dots,$$
(8)

Combining (2), (1), (5), (7), and (8), we have

$$\frac{n^{\frac{n+\frac{1}{2}}}}{n!} > \exp\left\{n - 1 + \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m-1)^{N_m}}{(m+1)} - \frac{1}{12n} - \frac{1}{2\ln(n-1)}\right\}$$
(9);

$$< \exp\left\{n + 1 + \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m-1)}{m} \cdot \frac{S_m}{12n} + \frac{1}{12n} + \frac{1}{24n^2}\right\}$$
 (10).

Hence, putting

$$C = \exp \left\{ 1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(m-1) S_m}{m(m+1)} \right\}$$
 (11),

so that C is a finite numerical constant, we have

$$n! > Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} - \frac{1}{94n^2}\right)$$
 (12),

$$< Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} + \frac{1}{24n(n-1)}\right)$$
 (13);

or, since the exponential function is continuous,

$$n! = Ce^{-n}n^{n+\frac{1}{2}}\exp\left(\frac{1}{12n} + \theta\right)$$
 (14),

where  $-1/24n^2 < \theta < 1/24n(n-1)$ . Hence, putting  $n = \infty$  on both sides of (14), we have

Ln! =  $CLe^{-n}n^{n+\frac{1}{2}}$  (15)

The constant C may be calculated numerically by means of the equation (11). Its value is, in fact,  $\sqrt{(2\pi)}$ , as may be easily shown by using Wallis's Theorem, § 6 (18).

Thus we have, when  $n = \infty$ ,

$$\frac{\pi}{2} = L \frac{2^{2n} (n!)^2 (2n+1)}{1^2 3^2 \dots (2n+1)^2} = L \frac{2^{4n} (n!)^4 (2n+1)}{\{(2n+1)!\}^2}.$$

Hence, using (15), we get

$$\begin{split} &\frac{\pi}{2} = C^2 L \frac{2^{4s} e^{-4n} n^{4n+2} \left(2n+1\right)}{e^{-4n-2} \left(2n+1\right)^{4n+3}} \,, \\ &= \frac{C^2}{4} L \frac{e^{2s}}{\left(1+1/2n\right)^{-n+2} \left(1+1/2n\right)^{2n+2}}{\left(1+1/2n\right)^{2n+2} \left(1+1/2n\right)^{2n+2}} \\ &= \frac{C^2}{4} \frac{e^2}{\sqrt{s}} \,. \end{split}$$

Therefore, since C is obviously positive,

$$C = \sqrt{(2\pi)}$$
 (16).

Using this value of C in (14), we get finally

$$n! = \sqrt{(2\pi n)(n/e)^n} \exp\{1/12n + \theta\}^*$$
 (17),

where  $-1/24n^2 < \theta < 1/24n (n-1)$ .

An elementary proof that Ln!=L<sub>N</sub>(2πn) (n/e)<sup>n</sup> was given by Glaisher (Quart. Jour. Math., 1878). In an addition by Cayley a demonstration of the approximation (17) is also given; but inasmuch as it assumes that series

Cor. By combining (11) and (16), we deduce that

$$1 - \frac{1}{2} \sum_{m=0}^{\infty} \frac{(m-1)}{m} \frac{S_m}{(m+1)} = \frac{1}{2} \log 2 + \frac{1}{2} \log \pi$$
 (18),

where  $S_m = 1/2^m + 1/3^m + 1/4^m + ...$ , ad  $\infty$ ,

## Exercises XXIII.

- (1.) Show that, when |x|>π, x cot x can be expanded in the form A<sub>0</sub> + Σ (R<sub>n</sub>x<sup>n</sup> + C<sub>n</sub>x<sup>n</sup>); and determine the coefficients in the particular case where π < x < 2π.</p>
- (c). Show that the sum of the products r at a time of the squares of the reciprocals of all the integral numbers is  $\pi^{-r}/(2r+1)!$ ; and find the like sum when the odd integers alone are considered.
  - (3.) Sum to n terms

$$\tan \theta + \tan (\theta + \pi/n) + \tan (\theta + 2\pi/n) + \dots;$$
  
 $\tan^2 \theta + \tan^2 (\theta + \pi/n) + \tan^2 (\theta + 2\pi/n) + \dots;$ 

Sum the following:-

- (1.)  $1/(1^2 + x^2) + 1/(2^2 + x^2) + 1/(3^2 + x^2)$  . . .
- (5)  $1/x^2 1/(x^2 \pi^2) + 1/(x^2 2^2\pi^2) \dots$
- (6.)  $1/x + 1/(x-1) + 1/(x+1) + 1/(x-2) + 1/(x+2) + \dots$
- (7.)  $1/(1-\epsilon) + 1/(4-\epsilon) + 1/(9-\epsilon) + \dots + 1/(n^2-\epsilon) + \dots$
- (8.) 1/1 · 2 + 1/2 · 4 + 1/3 · 6 + 1/4 · 8 + · · · · Show that
- (9.)  $(\pi^2 6)/6 = 1/1^2$ ,  $2 + 1/2^2$ ,  $3 + 1/3^2$ ,  $4 + \dots$
- (10.)  $\pi/8 1/3 = 1/1$ , 3.5 1/3, 5.7 + 1/5, 7.9 ...
- (11.) If f<sub>r</sub> (n) be an integral function of n whose degree is r, show that <sup>∞</sup>/<sub>Σ</sub> f<sub>r</sub>(n)/(2n 1)<sup>2n</sup> can be expressed in terms of Bernoulli's numbers, provided r > 2m 2; and <sup>∞</sup>/<sub>Σ</sub> (-)<sup>n-1</sup> f<sub>r</sub> (n)/(2n 1)<sup>2n+1</sup> in terms of Euler's numbers, provided r > 2m 1.

In particular, show that

$$\frac{1}{3^4} + \frac{1+2}{5^4} + \frac{1+2+3}{7^4} + \dots := \frac{\pi^2}{64} \left( 1 - \frac{\pi^2}{12} \right).$$

(12.) Show that

$$\begin{array}{l} \overset{\pi}{\sum} 1/(n\pi+\theta)^3 = \csc^2\theta \; ; \\ \overset{\pi}{\sum} 1/(n\pi+\theta)^4 = \csc^4\theta - \frac{3}{2} \csc^2\theta, \end{array}$$

n = 0 being included among the values to be given to n. (Wolstenholme.)

of the form of  $1, 2^m + 1/3^m + \dots$  can be expanded in powers of 1/m, it cannot be said to be elementary. The proofs usually given by means of the Mao-laurin-sum-formula are unsatisfactory, for they depend on the use of a series which does not in general converge when continued to infinity, and which can only be used in conjunction with its residue. See Baake, Creftle's Jour., xxv.

(13.) 
$$\sum_{1}^{\infty} \frac{1}{n^4 + x^4} = \frac{\pi \sqrt{2} \sinh \pi x \sqrt{2 + \sin \pi x} \sqrt{2}}{\cosh \pi x \sqrt{2 - \cos \pi x} \sqrt{2}} - \frac{1}{2x^4}.$$
(Math. Trip. 1888)

(14.) Show that

$$\sum_{m=1}^{\infty} \frac{1}{\{(2n)^2 - (2m-1)^2\}^2} = \frac{\pi^2}{16(2m-1)^2} - \frac{1}{2(2m-1)^4}$$

$$\sum_{m=1}^{\infty} \frac{1}{\{(2n-1)^2 - (2m)^2\}^2} = \frac{\pi^2}{64m^2}$$

Also that the sum of the reciprocals of the squares of all possible differences between the square of any even and the square of any odd number is  $\frac{24}{384}$ .

(15.) If n < n, show that

$$\frac{\cos^{p}\theta}{\cos n\theta} = \frac{1}{n}\sum_{r=0}^{n-1} (-)^{r} \frac{\sin_{+}(2r+1)\pi/2n \cdot \cos_{+} p(2r+1)\pi/2n}{\cos\theta - \cos_{+}(2r+1)\pi/2n}.$$

(16.) Show that

$$\begin{split} \tan^{-1}\frac{v}{u} &= \sum_{n=1}^{\infty} \left\{ \tan^{-1}\frac{v}{n\pi - u} - \tan^{-1}\frac{v}{n\pi + u} \right\} = \tan^{-1}\left(\tanh v \cot u\right); \\ \sum_{n=1}^{\infty} \left\{ \tan^{-1}\left(\frac{2v}{2u - 1}\right)\frac{2v}{\pi - 2u} - \tan^{-1}\left(\frac{2v}{2n - 1}\right)\frac{1}{n + 2u} \right\} = \tan^{-1}\left(\tanh v \tan u\right). \end{split}$$

(Schlömilch, Handb, d, Alg, Anal., cap. xr.)

(17.) If  $\lambda(x) \equiv x \prod_{1}^{m} \{1 - (x/na)^{2}\}, \quad \mu(x) \equiv \prod_{1}^{m} \{1 - (2x/2n - 1a)^{2}\}, \quad \text{express}$ 

 $\lambda$  (x+a/2) in terms of  $\mu$  (x), and also  $\mu$  (x+a/2) in terms of  $\lambda$  (x). Hence evaluate L 1 . 3 . 5 . . .  $(2m-1)\sqrt{(2m+1)/2^m m!}$ .

(Math. Trip., 1882.)

(18.) Show that, if r be a positive integer,

$$\underset{r = \infty}{L} \left( 1 - \frac{1}{r} \right)^{1/1} \left( 1 - \frac{2}{r} \right)^{1/2} \dots \left( 1 - \frac{r-1}{r} \right)^{1/(r-1)} = e^{-\pi^2/6}.$$

(19.) Show that

$$L_{x=\infty} \left( \frac{x}{x^2 + 1^2} + \frac{x}{x^2 + 2^2} + \frac{x}{x^2 + 3^2} + \dots \right) = \frac{\pi}{2} .$$

(20.) If n, p, x be all integers, prove

$$L_{x=\infty} \frac{(n+x)(n+x+1) \cdot (n+p+x-1)}{(1+x)(2+x) \cdot (p+x)} = 1.$$

# REVERSION OF SERIES—EXPANSION OF AN ALGEBRAIC FUNCTION.

§ 18.] The subject which we propose to discuss in this and the following paragraphs originated, like so many other branches of modern analysis, in the works of Newton, more especially in his tract De Analysi per E-quationes Numero Terminorum Infinitus. Let us consider the function

 $\Sigma(m,n)x^my^n \equiv (1,0)x + (0,1)y + (2,0)x^2 + (1,1)xy + (0,2)y^2 + \dots$ , where the indices m and n are positive integers, and we use the symbol (m,n) to denote the coefficient of  $x^my^n$ , so that (m,n) is a constant. We suppose the absolute term (0,0) to be zero; but the coefficients (1,0) (0,1) are to be different from zero. The rest of the coefficients may or may not be zero; but, if the number of terms be infinite, we suppose the double series to be absolutely convergent when |x| = |y|. 1°. From this it follows that the coefficient (m,n) must become infinitely small when m and n become infinitely great; so that a positive quantity  $\lambda$  can in all cases be assigned such that  $((m,n)) \geqslant \lambda$  whatever values we assign to m and n. It also follows (see clap. XXVI., § 37) that  $\Sigma(m,n)x^my^n$  is absolutely convergent for all values of x and y such that  $(x|z) \ge 1$ ,  $y \ge 1$ .

We propose to show that one value, and only one value, of y as a function of x can be found which has the following properties:—

 y is expansible in a convergent series of integral powers of x for all values of x bying within limits which are not infinitely narrow.

- 2', y has the initial value 0 when x = 0.
- 3°. y makes the equation

Let us assume for a moment that a convergent series for y of the kind demanded can be found. Its absolute term must vanish by condition 2. Hence the series will be of the form

$$y = h_1 x + h_2 x^2 + h_3 x^3 + \dots$$
 (2).

In order that this value of y may make (1) an intelligible identity, it must be possible to find a value of x<1 such that (2) gives a value of y<1. The series (1), when transformed by means of (2), will then satisfy Cauchy's criterion, and may be arranged according to powers of x. All that is further necessary

The more general c · · · , when the series is convergent so long as (x | > a and y · · β, can carly be brought under the above by a simple transformation.

to satisfy condition 3° is simply that the coefficients of all the powers of x shall vanish.

It will be convenient for what follows to assume that (0, 1) = -1 (which we may obviously do without loss of generality), and then put (1) into the form-

$$y = \{(1, 0) x + (2, 0) x^2 + (3, 0) x^3 + \dots \}$$

$$+ \{(1, 1) x + (2, 1) x^2 + (3, 1) x^3 + \dots \} y$$

$$+ \{(0, 2) + (1, 2) x + (2, 2) x^2 + (3, 2) x^4 + \dots \} y^2$$

$$+ \{(0, n) + (1, n) x + (2, n) x^2 + (3, n) x^3 + \dots \} y^5$$
(3).

Using (2), we get
$$b_1x + b_2x^2 + b_3x^3 + \dots$$

$$= \{(1, 0)x + (2, 0)x^2 + (3, 0)x^3 + \dots\}$$

$$+ \{(1, 1)x + (2, 1)x^2 + (3, 1)x^3 + \dots\} \{b_1 + b_2x + b_2x^3 + \dots\} x^4 + \{(0, 2) + (1, 2)x + (2, 2)x^2 + (3, 2)x^4 + \dots\} \{b_1 + b_2x + b_2x^3 + \dots\}^2 x^3 + \{(0, n) + (1, n)x + (2, n)x^2 + (3, n)x^4 + \dots\} \{b_1 + b_2x + b_2x^3 + \dots\}^n x^4 + \dots\}^n x^4 \{(0, n) + (1, n)x + (2, n)x^2 + (3, n)x^4 + \dots\} \{b_1 + b_2x + b_2x^3 + \dots\}^n x^4 \{(1, n)x^4 + \dots\}^n x^4 \{(1,$$

Hence, equating coefficients, we have

$$b_1 = (1, 0),$$

$$b_2 = (2, 0) + (1, 1) b_1 + (0, 2) b_1^2,$$

$$b_3 = (3, 0) + (1, 1)b_3 + (2, 1)b_1 + (1, 2)h_1^2 + 2(0, 2)h_1h_2 + (0, 3)h_1^3,$$

$$b_n = (n, 0) + (1, 1)b_{n-1} + (2, 1)h_{n-2} + \dots + (0, n)h_1^n$$

$$b_n = (n, 0) + (1, 1)b_{n-1} + (2, 1)b_{n-2} + \dots + (0, n)b_1$$
(5).

Here it is important to notice that each equation assigns one of the coefficients as an integral function of all the preceding coefficients. Hence, since the first equation gives one and only one value for b, all the coefficients are uniquely determined. There is therefore only one value of y, if any.

In order to show that (5) really affords a solution, we have to show that for a value of x whose modulus is small enough, but not infinitely small, the conditions for the absolute convergency of (2) and (4) are satisfied when  $b_1, b_2, \ldots$  have the values assigned by (5).

(9):

This, following a method invented by Cauchy, we may show by considering a particular case. The moduli of the coefficients of the series (3) have, as we have seen, a finite upper limit  $\lambda$ . Suppose that in (3) all the coefficients are replaced by  $\lambda$ , and that x has a positive real value < 1. Then we have

This series is convergent so long as x<1 and |y|<1. It can, in fact, be summed; for, adding  $\lambda + \lambda y$  to both sides, we get

that is, 
$$(1 + \lambda) y + \lambda = \lambda/(1 - x) (1 - y),$$
  
 $(1 + \lambda) y^2 - y + \lambda x/(1 - x) = 0.$ 

Hence, remembering that the value of y with which we are concerned vanishes when x=0, we have

$$y = [1 - \sqrt{1 - 4\lambda (1 + \lambda) x/(1 - x)}]/2 (\lambda + 1)$$

Now, provided  $4\lambda (1+\lambda) x/(1-x) < 1$ , that is,  $x < 1/(2\lambda + 1)^2$ , the right-hand side of (7) can be expanded in an absolutely convergent series of integral powers of x, the absolute term in which vanishes. Also, when  $x < 1/(2\lambda + 1)^2$ , the value of y given by (7) is positive and < 1, therefore the absolute convergency of (6) is assured.

It follows that the problem we are considering can be solved in the present particular case. If we denote the series for y in this case by

$$y = C_1x + C_2x^2 + C_3x^3 + \dots$$
 (8),

then the equations for determining  $C_1$ ,  $C_2$ ,  $C_3$ , . . . will be found by putting  $(1, 0) = (2, 0) = (1, 1) = . . . = \lambda$  in (5), namely,

$$\begin{split} &C_1 - \lambda, \\ &C_2 = \lambda \left(1 + C_1 + C_1^2\right), \\ &C_3 = \lambda \left(1 + C_2 + C_1 + C_1^2 + 2C_1C_2 + C_1^2\right), \\ &\cdots \\ &C_n = \lambda \left(1 + C_{n-1} + C_{n-2} + \dots + C_1^n\right), \end{split}$$

from which it is seen that  $C_1$ ,  $C_2$ ,  $C_3$ , . . . are all real and positive.

Returning now to the system (5), and denoting moduli by attaching dashes, we have, since (1, 0)', (2, 0)', &c., are all less than  $\lambda$ ,

$$\begin{split} b_1' &= (1,\,0)' < \lambda < C_1, \\ b_2' &\geq (2,\,0)' + (1,\,1)' h_1' + (0,\,2)' h_1'^2 < \lambda \left(1 + C_1 + C_1'^2\right) < C_2, \\ b_2' &\geq (3,\,0)' + (1,\,1)' h_2' + (2,\,1)' h_1' + (1,\,2)' h_1'^2 + 2 \, (0,\,2)' h_1' h_2' + (0,\,3)' h_1'^2, \\ &< \lambda \left(1 + C_2 + C_1 + C_1'^2 + 2 C_1' C_2 + C_1'^2\right) < C_3', \end{split}$$

Hence the moduli of the coefficients in (2) are less than the moduli in the series (8), which is known to be absolutely convergent. It therefore follows that the series (2) will certainly be absolutely convergent, provided  $|x| \le 1/(2\lambda + 1)^2$ .

It only remains to show that x may be so chosen (and yet not infinitely small) that y as given by (2) shall be such that y' < 1. We have

$$y' < h_1'x' + h_2'x'^2 + h_2'x'^3 + \dots, < C_1x' + C_2x'^2 + C_2x'^3 + \dots, < [1 - \sqrt{1 - 4\lambda (1 + \lambda) x'/(1 - x')}]/2 (\lambda + 1)$$
 (11).

Now the right-hand side of (11) is less than 1, provided  $x' < 1/(2\lambda + 1)^2$ . If, therefore,  $|x| < 1/(2\lambda + 1)^2$ , the absolute convergency of the double series (3) or (4) will be assured; and (2) will convert (1) into an intelligible identity.

We have thus completely established that one and only one value of y expansible within certain limits as a convergent series of integral powers of x can be found to satisfy the equation  $(1)_1$  and the like follows for x as regards y. The functions of x and y thus determined, being representable by power-series, are of course continuous. The limits assigned in the course of the demonstration for the admissibility of the solution are merely lower limits; and it is easy to see that the solution is valid so long as (2) itself and the double series into which it converts the left-hand side of (1) remain absolutely convergent.

It should be remarked that we have not shown that no other power-series whose absolute term does not vanish can be found to satisfy (1); nor have we shown that no other function having zero initial value, but not expansible in integral powers of x, can be found to satisfy (1). We shall settle these questions presently in the case where the series  $\Sigma_{-}(m,n)\,x^my^n$  terminates.

§ 19.] The problem of the Reversion of Series properly so
called is as follows:—

Given the equation

$$x = a_0 + a_m y^m + a_{m+1} y^{m+1} + \dots$$
 (1),

where  $a_n \neq 0$ , but  $a_n$  may or may not be zero, and the series  $a_n y^1 + a_{n+1} y^{n+1} + \dots$  is absolutely convergent so long as  $\{y_i \geq a\}$  fixed positive quantity  $p_i$  to find a convergent expansion, or convergent expansions, for y in ascending powers of  $x - a_n$ .

Let  $\xi$  denote  $((x-a_n)^n a_n)^{km}$ , that is, the principal value of the mth root of  $(x-a_n) a_m$ , and  $a_m$  a primitive mth root of unity, then (1) is equivalent to m equations of which the following is a type:—

$$\omega_m^T \xi = y \left( 1 + \frac{a_{m+1}}{a_m} y + \frac{a_{m+2}}{a_m} y^2 + \dots \right)^{1/m}$$
 (2).

Now, the series inside the bracket in (2) being absolutely convergent for all values of y such that  $y \gg p$ , it follows from the binomial theorem combined with §1 that we can, by taking y within certain limits, expand the right-hand side of (2) in an ascending series of powers of y. We thus get, say,

$$-\omega_m^{-r}\xi + y + C_2y^2 + C_2y^3 + \dots = 0$$
 (3),

It follows, therefore, from the general theorem of last paragraph that we have, provided  $|\xi|$  does not exceed a certain limit.

$$y = b_1 \omega_m^r \xi + b_2 \omega_m^{2r} \xi^2 + b_3 \omega_m^{3r} \xi^3 + \dots$$
 (4).

We have, at course, m such results, in which the coefficients  $b_1, b_2, b_3, \ldots$  will be the same, but r will have the different values  $0, 1, 2, \ldots, (m-1)$ .

Each of these solutions is, by chap. XXVI. § 19, a continuous function of x. If we cause x to circulate about  $a_0$  in Argand's Diagram, the m branches of y will pass continuously into each other; and after m revolutions the branches will recur. The point  $a_0$  is therefore a Branch Point of the mth order for the function y, just as the point 0 is for the function  $x^{0.8}$  in chap. XXIX. § 5, 6.

Cor. In the particular case where  $a_0 = 0$ , m = 1, we get the single solution

$$y = b_1 x + b_2 x^2 + b_3 x^3 + \dots$$
 (5).

Example. To reverse the series

$$z = 1 + y/1! + y^2/2! + y^3/3! + . . .$$
 (6).

Let z=1+x, then we have

$$x = \frac{y}{1!} + \frac{y^3}{2!} + \frac{y^3}{3!} + \dots$$
 (7).

Hence, provided |x| lie within certain limits, we must have by the general theorem  $y = b_x x + b_x x^2 + b_x x^3 + \dots$ (8).

Knowing the existence of the convergent expansion (8), we may determine the coefficients as follows.

Give y a small increment k, and let the corresponding increment of x be k; then, from (7), we have

$$h = \frac{(y+k)-y}{1!} + \frac{(y+k)^2 - y^2}{2!} + \frac{(y+k)^3 - y^3}{3!} + \dots$$

Hence, since  $L\{(y+k)^n-y^n\}/k=ny^{n-1}$  when k=0, and since, owing to the continuity of the series as a function of y, k=0 when k=0, we have

$$L \frac{h}{k} = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \dots,$$
  
= 1 + x (9).

Again, from (8), we have, in like manner,

$$L_{\bar{h}}^{k} = b_1 + 2b_2x + 3b_3x^2 + \dots$$
 (10).

Combining (9) and (10), we have  $b_1 + 2b_2x + 3b_3$ .

$$b_1 + 2b_2x + 3b_3x^2 + \dots = 1/(1+x),$$
  
=  $1 - x + x^2 - \dots$ 

We must therefore have

$$b_1 = 1$$
,  $b_2 = -1/2$ ,  $b_3 = 1/3$ , . . .

Therefore

$$y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$= \frac{z - 1}{1} - \left(\frac{z - 1}{2}\right)^2 + \frac{(z - 1)^3}{3} - \dots$$
(11).

It must be remembered that (11) gives only that branch of the function y which is expansible in powers of z-1 and which vanishes when z=1. We have, in fact, merely given another investigation of the expansion of the principal value of  $\log z$ .

§ 20.] Expansions of the various branches of an Algebraic Function.

The equation

$$\Sigma(m, n) x' y^n + (0, 0) - 0$$
 (1),

where the series on the left terminates, gives for any assigned value of  $\boldsymbol{x}$  a finite number of values of  $\boldsymbol{y}$ . If the highest power of  $\boldsymbol{y}$  involved be the nth, we might, in fact, write the equation in the form

 $A_n y^n + A_{n-1} y^{n-1} + \dots + A_1 y + A_n = 0$  (2),

where  $A_0, A_1, \ldots, A_n$  are all integral functions of x. If, then, we give to x any particular value, a, real or complex, it follows from chap. XII., § 23, that we get from (2) n corresponding values of y, say  $b_1, b_2, \ldots, b_n$ . If we change x into a value a + b differing slightly from a, then  $b_1, b_2, \ldots, b_n$  will change into  $b_1 + k_1, b_2 + k_2, \ldots, b_n + k_n$ ; that is to say, we shall get n values of y which will in general be different from the former set. We may therefore say that (2) defines y as an n-valued function of x; and we call y when so determined an aly braic function of x;

Since every equation of the form y = F(x), where F(x) is an ordinary synthetic irrational algebraic function (as defined in chap. XIV., § 1), can be rationalised, it follows that every ordinary irrational algebraic function is a branch of an algebraic function as now defined. Since, however, integral equations whose degree is above the 4th cannot in general be formally solved by means of radicals, it does not follow, conversely, that every algebraic function is expressible as an ordinary synthetic irrational algebraic function.

In what follows we assume that the equation (2) contains (so long as x and y are not specialised) no factor involving x alone y alone. We also suppose that, so long as x is not assigned, the equation is Irreducible, that is to say, that it has not a root in common with an integral equation of lower degree in y whose coefficients are integral functions of x. If this were so, a factor could (by the process for obtaining the G.C.M. of two integral functions of x, and the roots of the equation formed by equating this factor to 0 would be the common root or roots in question. Therefore the equation (2) could be broken up into two integral equations in y whose coefficients would be integral functions of x; and each of these would define a separate algebraic function of x. The condition of irreducibility involves that (2) cannot have

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two or more of its roots equal for all values of x. For, if (2) had, say, r equal roots, then, denoting all the roots by  $y_1, y_2, \ldots, y_n$ , the equation

 $\Sigma(y-y_i)(y-y_2)$  . . .  $(y-y_{s+1})(y-y_{s+1})$  . . .  $(y-y_n)=0$  (3) would have r-1 roots in common with (2), for r-1 equal factors would occur in each of the terms comprehended by  $\Sigma$ . Now the coefficients of (3) are symmetric functions of the roots of (2); therefore (3) could be exhibited as an equation whose coefficients are integral functions of  $A_0, A_1, \ldots, A_n$ , and therefore integral functions of  $x^*$ . Hence (2) would be reducible, which is supposed not to be the case.

It must, however, be carefully noticed that irreducibility in general (that is, so long as x is not specialised) does not exclude reducibility or multiplicity of roots for particular values of x. In fact, we can in general determine a number of particular values of x for which (2) and (3) may have a root in common! In other words, it may happen that the n branches of y have points in common; but it cannot happen that any two of the n branches wholly coincide.

When, for x = a, the n values  $b_1, b_2, \ldots, b_n$  are all different, a (or its representative point in an Argand-diagram) is called an ordinary point of the function y, and  $b_1, b_2, \ldots, b_n$  single values. If  $b_1 = b_2 = \ldots = b_r$ , each  $= b_r$  say, then a is called an r-ple point of the function, and b an r-ple value.

For every value of x (zero point) which makes  $A_0 = 0$ , one branch of y has a zero value; for every value of x (double zero point) which makes  $A_0 = 0$  and  $A_1 = 0$ , two branches have a zero value; and so on. These are called single, double, . . . , zero rathes.

For every value of x (pole) which makes  $A_n = 0$ , one branch of y has an infinite value; for every value of x (double pole) which makes  $A_n = 0$  and  $A_{n-1} = 0$ , two branches have an infinite

have a root in common.

and

<sup>\*</sup> See chap. xvIII., § 4.

<sup>†</sup> These are the values of x for which

 $<sup>\</sup>begin{split} A_ny^n + A_{n-1}y^{n-1} + \ldots + A_1y + A_n &= 0 \\ nA_ny^{n-1} + (n-1)A_{n-1}y^{n-2} + \ldots + A_1 &= 0 \end{split}$ 

value; and so on. These may be called single, double, . . ., infinities of the function.

The main object of what follows is to show that every branch of an algebraic function is (within certain limits), in the neighbourhood of every point, expansible in an assending or descending power series of a particular kind; and thus to show that every branch is, except at a pole, continuous for all finite values of x.

§ 21.] If, at the point x = a, the algebraic function y has a single value y - b, then y - b is, within certain limits, expansible in an absolutely convergent series of the form

$$y-b=C_1(x-a)+C_2(x-a)^2+C_3(x-a)^3+\dots$$
 (4).

Let  $x = a + \xi$ ,  $y = b + \eta$ , then the equation (1) becomes, after rearrangement.

$$(0, 0) + (1, 0) \xi + (0, 1) \eta + (2, 0) \xi^2 + \&c. = 0$$
 (5).

Since y = h is a single root of (1) corresponding to x = a, it follows that when  $\xi = 0$  (5) must give one and only one zero value for  $\eta$ . Therefore we must have (0, 0) = 0 and  $(0, 1) \neq 0$ .

It follows, from the general theorem of § 18, that within certain limits the following convergent expansion,

$$n = C_1 \dot{\xi} + C_2 \dot{\xi}^2 + C_2 \dot{\xi}^3 + \dots$$

and no other of the kind will satisfy the equation (5); that is,

$$y = b + C_1(x - a) + C_2(x - a)^2 + C_3(x - a)^3 + \dots$$
 (6)

will satisfy (1). The function y determined by (6) is continuous so long as |x-a| is less than the radius of convergency of the series involved; and it has the value y=b when x=a.

If we suppose all the values of  $y_i$  say  $b_i$ ,  $b_2$ , . . . ,  $b_n$ , corresponding to x=a to be single, then we shall get in this way for each one of them a value of the function y of the form (6). Hence we infer that

Cor. So hing as no two of the branches of an algebraic function have a point in common, each branch is a continuous function of x; and the increment of y at any point of a particular branch is expansible in an ascending series of positive integral powers of the increment of x so long as the modulus of the increment of x does not exceed a certain limite value.

§ 22.] We proceed to discuss the modification to which the conclusions of last paragraph are subject when x = a is a multiple

point of the function v.

We shall prove that for every multiple point of the 9th order, to which corresponds a q-ple value y = b, we can find q different convergent expansions for y of the form  $y = b + \Sigma C_r(x - a)^r$ , where the exponents r form a series of increasing positive rational numbers.

It will probably help the reader to keep the thread of the somewhat delicate analysis that follows if we premise the following remarks regarding expansibility in ascending power-series in general:—

If  $\eta$  be expansible in an absolutely convergent ascending series of positive powers of  $\xi$ , of the form

$$\eta = C_1 \xi^{\alpha} + C_2 \xi^{\alpha} \cdot \alpha + C_2 \xi^{\alpha} \cdot \alpha + C_3 \xi^{\alpha} \cdot \alpha + \dots$$
 (A),

where  $a_1, a_2, \ldots$  are all positive, then we can establish a series of transformations of the following kind:—

$$\eta = \xi^{\alpha_j} \left( C_1 + \eta_i \right), \quad \eta_1 = \xi^{\alpha_j} \left( C_2 + \eta_j \right), \quad \eta_2 = \xi^{\alpha_j} \left( C_3 + \eta_j \right), \dots,$$

$$\eta_{n-1} = \xi^{\alpha_n} \left( \ell'_n + \eta_n \right)$$
 (B),

where  $\eta_1, \eta_2, \ldots, \eta_n$  all vanish when  $\xi = 0$ ;  $C_1, C_2, \ldots, C_n$  are all independent of  $\xi$ , and all different from zero; and

$$C_1 = L\eta/\xi^{\alpha_1}$$
,  $C_2 = L\eta_1/\xi^{\alpha_2}$ , . . . ,  $C_n = L\eta_{n-1}/\xi^{\alpha_n}$  when  $\xi = 0$ .

Conversely, if we can establish a series of transformations of the form (B), and if we can show that  $\eta_{\eta}$  is expansible in a series of ascending positive powers of  $\xi$ , it will obviously follow that  $\eta$ is expansible in the form (A).

Let now y=b be a q-ple value of y corresponding to x=a, and put as before  $x=a+\dot{\xi},\ y=b+\eta,$  then the equation (1) becomes

$$\Sigma(m, n) \xi^m \eta^n = 0 \qquad (7).$$

(11).

Since q values of g become h when x = a, q values of  $\eta$  must become 0 when  $\xi = 0$ . Hence the lowest power of  $\eta$  in (7) which is not multiplied by a power of  $\xi$  must  $he \eta^{\eta}$ . There must also be a power of  $\xi$  which is not multiplied by a power of  $\eta$ , otherwise (7) would be divisible in general by some power of  $\eta$ , which is impossible since (1) is irreducible. Let the lowest such power of  $\xi$  be  $\xi^{\mu}$ .

Put now

$$\eta = \xi^{\lambda} \left( C_1 + \eta_1 \right) = \xi^{\lambda} v \tag{8},$$

and let us seek to determine a positive value of  $\lambda$  such that  $C_1 = L\tau = L\eta/\xi^{\lambda}$  is finite both ways\* when  $\xi = 0$ .

The equation (7) gives

$$\Sigma (m, n) \xi^{m+\lambda n} r^n = 0$$
 (9).

Now (9) will furnish values of r which are finite both ways when  $\xi = 0$ , provided we can so determine  $\lambda$  that at least two terms of (9) are of the same positive degree in  $\xi$ , and lower in degree than all the other terms.

Assume for the present that we can find a value of  $\lambda$  for which a group of r terms has the character in question, so that

$$\delta = m_1 + \lambda n_1 = m_2 + \lambda n_2 = \ldots = m_r + \lambda n_r$$
 (10),

where and

$$n_1 \geqslant n_2 \geqslant \dots \geqslant n_r$$
;  
 $\lambda = (m_1 - m_r)/(n_r - n_1) = g/h$ , say,

where g is prime to h,

$$\delta = (m, h + n, a)/h.$$

Then, putting  $\xi_1 = \dot{\xi}^{1,h}$ , so that  $\xi_1 = 0$  when  $\xi = 0$ , and dividing out  $\xi_1^{m_1h + n_2}$ , we deduce an equation of the form

$$\phi\left(\xi_{1}, v\right) \xi_{1} + \left(m_{r}, n_{r}\right) v^{n_{r}} + \left(m_{r-1}, n_{r-1}\right) v^{n_{r-1}} + \dots + \left(m_{1}, n_{1}\right) v^{n_{1}} = 0$$
(12)

where  $\phi(\xi_1, r)$  is an integral function of  $\xi_1$  and r.

For our present purpose we are concerned only with those

<sup>.</sup> That is, neither zero nor infinite - a useful phrase of De Morgan's.

<sup>1</sup> It is sufficient for our purpose to take the principal value merely of the hth root of \(\delta\).

roots of (12) whose initial values are finite both ways. There are evidently  $n_r - n_1$  such roots, and their initial values are given by

$$(m_r, n_r) \bar{v}^{n_r-n_1} + (m_{r-1}, n_{r-1}) \bar{v}^{n_{r-1}-n_1} + \dots + (m_1, n_i) = 0$$
(13).

If the roots of (13) are all different, then we get  $u_r - u_1$  transformations of the form (8); and the corresponding values of e, that is, of  $C_1 + \eta_1$ , are given by the algebraical equation (12). Moreover, since all the values of v are single, we shall get for each value of  $\eta_1$  an expansion of the form

$$\eta_1 = d_1 \xi_1 + d_2 \xi_1^2 + \dots,$$

$$= d_1 \xi^{1/\hbar} + d_2 \xi^{2/\hbar} + \dots$$
(11);

and each of these will give for  $\eta$  a corresponding expansion of the form

$$\eta = C_1 \xi^{gh} + d_1 \xi^{(g+1)h} + d_2 \xi^{(g+2)h} + \dots$$
(14').

If a group of the roots of (13) be equal, then we must proceed by means of a second transformation,

$$\eta_1 = \dot{\xi}_1^{\mu} (C_2 + \eta_2)$$
(15),

to separate those roots of (12) which have equal values. If the next step succeeds in finally separating all the initial values, then we have for each of the group of equal roots of (13) two transformations (8) and (15), and finally an expansion like (14'), the result being the final separation of all the  $n_r - n_1$  roots of (12), with convergent expansions for each of them.

Moreover, we must in every case be able, by means of a finite number of transformations like (8) and (15), to separate the initial values, otherwise we should have two branches of y coincident up to any order of approximation, which is impossible, since (1) is irreducible.

The indices in the series (14') may be all integral or else partly or wholly fractional (see Examples 2 and 1 below).

In the former case the corresponding branch of the function  $\eta$  is single-valued in the neighbourhood of the point  $\xi = 0$ ; that is to say, if we cause  $\xi$  to circulate about the point  $\xi = 0$  and

return to its original position,  $\eta$  returns to the value with which we started.

If some or all of the indices be fractional, the series will take the form

 $\eta = C_1^{\prime} \xi^{\beta/q} + C_2 \xi^{\beta/q} + C_3 \xi^{\gamma/q} + \dots$  (14"),

where one at least of the fractions a/q,  $\beta/q$ , . . . , is at its lowest terms. The function  $\eta$  is then q-valued and the series (14") will as in § 19 lead to a cycle, as it is called, of q branches which pass continuously into each other when  $\xi$  is made to circulate q times round  $\xi=0$ . At any multiple point there may be one or more such cycles; and for each of them the point is said to be a branch point of the qth order, q being the number of branches belonging to the cycle.

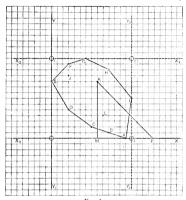
All that now remains is to show that we can in all cases select a number of groups of terms satisfying the conditions (10) sufficient to give us q expansions corresponding to the q branches which meet at the q-ple point x = a.

The best way, both in theory and in practice, of settling this point is to use Newton's Parallelogram, which is constructed as follows:  $-4 \cot OX$  and OY (Fig. 1) be a pair of rectangular axes, the first quadrant of which is ruled into squares (or rectangles) for convenience in plotting points whose co-ordinates are positive integers. For each term  $(m, n) \xi^m \eta^n$  in equation (7) we plot a point K (degree-point) whose co-ordinates are OM = m, MK = n. We observe that, if KP be drawn so that  $cot KPO = \lambda$ , then  $OP = OM + MP - m + n\lambda$ . Hence OP is the degree in  $\xi$  of the term in (9) which corresponds to  $(m, n)\xi^m \eta^n$ . If, therefore, we select any group of terms whose degree-points lie on a straight line  $\lambda$ , these will all have the same degree in  $\xi$ , namely, the intercept of  $\lambda$  on OX.

The necessary and sufficient conditions, therefore, that a group of two or more terms furnish the initial values of a group of expansions, let us say be an effective group, are:—

1°. That the line  $\Lambda$  containing the degree-points shall cut O.V to the right of  $O_t$  and O.V above  $O_t$ . This secures that  $\lambda$  be positive.

2°. That all the other degree-points shall lie on the opposite side of A to the origin. This secures that all the other terms in (9) be of higher degree in \(\xi\) than those of the selected group.



Fro. L.

We have thus the following rule for selecting the effective groups :-

Let A and E be the degree-points of the terms that contain  $\xi$  and  $\eta$  alone, so that OA = p, OE = q. Let a radius vector. coinciding originally with A.V., turn clock-wise about A as centre until it passes through another of the degree-points B. If it passes through others at the same time as B, let the last of them taken in order from A be C. Next, let the radius turn about C as centre in the same direction as before, until it passes through another point or points, and let the last of this group taken in order from C be D. Then let the radius turn about D:

and so on, until at last it passes through E, or through a group of which E is the last.

We thus form a broken line convex towards O, beginning at A and ending at E, every part of which contains a group of degree-points the terms corresponding to which satisfy the conditions (10).

Now the degree of the equation (13) corresponding to any group CD is the difference between the degrees of  $\eta$  in the first and last terms C and D; but this difference is the projection of CD on OY. The sum of all the projections of AC, CD, &c., on OY is OE, that is to say, g. Hence we shall get, by taking all the groups AC, CD, &c., q different expansions for y corresponding to the q different branches that meet at the multiple point x=a. Each one of these has the same initial value b, and each is represented by a separate expansion in positive ascending rational powers of x=a.

Example 1. To separate the branches of the function  $\eta$  at the point  $\xi = 0$ ,  $\eta$  being determined by

$$\begin{split} D\xi^{1}\eta^{5} + C\xi^{7}\eta^{2} + E\eta^{10} + B\xi^{10}\eta + A\xi^{15} + L\xi^{9}\eta^{4} \\ + J\xi^{3}\eta^{11} + F\xi^{5}\eta^{13} + K\xi^{8}\eta^{10} + G\xi^{6}\eta^{14} + I\xi^{14}\eta^{7} \\ + H\xi^{10}\eta^{12} = 0. \end{split} \tag{16}$$

The lowest term in  $\eta$  alone is  $\eta^{in}$ , so that  $\xi$ =0 is a multiple point of the 10th order. Plotting the decrees of the terms in Newton's diagram, and naming the points by affixing the coefficients, we find (see Fig. 1) that the effective groups are  $AB^{ii}$ , CD, DE. Taking, for simplicity of illustration,

$$A=\pm 2$$
,  $B=\pm 3$ ,  $C=\pm 1$ ,  $D=\pm 1$ ,  $E=\pm 1$ , we get from the group  $AEC$ 

 $\lambda = 6/2$ . 3/4, so that h = 1, and  $v^2 - 3v + 2 = 0$  gives the initial values of  $v_s$  that is, v = 1, or 2, the corresponding expansions being

$$\eta = \xi^{\pm} (1 + d_1 \xi + d_2 \xi^{\pm} + \dots),$$
  
 $\eta = \xi^{\pm} (2 + d_1' \xi + d_1' \xi^2 + \dots).$ 

From the group CD, we get

 $\lambda = 1.3$ ,  $c^3 - 1 = 0$  gives the initial values of  $r_*$ 

that is,  $r = 1, \omega, \omega^2$ , where  $\omega$  is a primitive imaginary cube root of I, the corresponding expansions being

$$\xi^{+\beta} (1 + d_1 \xi^{1\beta} + d_2 \xi^{2\beta} + \dots),$$
  
 $\eta = e^{\pm i \beta} (\omega + d_1^{-1} \xi^{1\beta} + d_2^{-1} \xi^{2\beta} + \dots),$   
 $\eta = \xi^{4\beta} (\omega^{-1} d_1^{-1} \xi^{1\beta} + d_2^{-1} \xi^{2\beta} + \dots),$ 

(18).

In like manner, DE gives five expansions of the type

$$n = \xi^{3.5} (\alpha + d_1 \xi^{1.5} + d_2 \xi^{2.5} + \dots),$$

where a is any one of the five 5th routs of 1

All the ten branches are thus accounted for; and they fall into eyeles of the orders 1, 1, 3, 5.

Example 2. To separate the branches of  $\eta$  at the point  $\xi=0$ ,  $\eta$  being determined by

$$4\xi^5 - 3\xi^4 - 4\xi^2(\eta - \xi) + 4(\eta - \xi)^2 = 0$$
 (17),

The effective group for (17) at the point  $\xi=0$  corresponding to branches which have the initial value  $\eta=0$  is  $4(\eta-\xi)^2$ ; as will be readily seen from Newton's diagram.

$$\lambda = 1$$
,  $h = 1$  and, if  $\eta = \xi (C_1 + \eta_1) = \xi c$ , we have

$$4\xi^3 - 3\xi^2 - 4\xi(v-1) + 4(v-1)^2 = 0$$

Hence two branches have the same initial value for v, viz. v=1. For each of these  $\eta=\xi\left(1+\eta_1\right)$ ; and we have for  $\eta_1$  the equation

$$4\xi^3 - 3\xi^2 - 4\xi\eta_1 + 4\eta_1^2 = 0$$
 (18),

If we draw Newton's diagram for (18), we find that the effective group is  $4\eta_1^2 - 4\xi\eta_1 - 3\xi^2$ ; and that  $\lambda = 1$ . Put now  $\eta_1 = \xi(C_2 + \eta_2) = \xi p_1$ ; and we get

$$4\xi + (2v_1 - 3)(2v_1 + 1) = 0$$
 (19).

The initial values of  $v_1$  are given by  $(2v_1-3)(2v_1+1)=0$ , which give the single values  $v_1=3/2$ ,  $v_1=-1/2$ . Hence for the two branches we have

$$\eta_1 = \xi (3/2 + \eta_0); \quad \eta_1' = \xi (-\frac{1}{2} + \eta_0');$$

and the farther procedure will lead to internal power series for  $q_2$  and  $\eta_3$ .

We have therefore for the two branches

$$\eta = \xi + 3\xi^2/2 + C_1\xi^4 + \dots;$$

$$\eta' = \xi - \xi^2/2 + C_3 \xi^3 + \dots$$
;  
and the double point is not a branch point on either,

It should be observed that, if we form an integral equation be selecting from any given one a series of terms which form an effective group, the new equation gives an algebraic function. Those branches of this function that have zero initial values coincide to a first approximation (that is, as far as the first term of the expansion) with certain of the branches of the algebraic function determined by the original equation which have initial zero values. Thus, reverting to Example 1 just discussed, from the group ABC we have

$$A \xi^{13} + B \xi^{10} \eta + C \xi^7 \eta^2 = 0.$$

This gives, when we drop out the irrelevant factor \$7,

$$C\eta^2 + B\xi^3\eta + A\xi^6 = 0,$$

which breaks up into two equations,

$$\eta + p\xi^3 = 0$$
,  $\eta + q\xi^3 = 0$ ;

and thus determines two functions, each of which has a branch coincident to a first approximation with a branch of n (as determined by (16)) which has zero initial value.

In like manner, CD gives  $C\xi^4 + D\eta^3 = 0$ ; and DE gives  $D\xi^3 + E\eta^5 = 0.$ 

We thus get a number of binomial equations, each of which gives an approximation for a group of branches of the function n determined by (16). We shall return to this view of the matter in § 24.

§ 23.1 Before leaving the general theory just established, we

ought to point out that Newton's Parallelogram enables us to obtain, at every point (singular or non-singular), convergent expansions for every branch of an algebraic function in ascending or descending power-series, as the case may be.

To establish this completely, we have merely to consider the remaining cases where x or y or both become infinite.

1st. Let us suppose that the value of the function y tends towards a finite limit b when x tends towards  $\infty$ . Then, if we put  $\eta = y - b$ ,  $x = \xi$ , we shall get an equation of the form

$$\Sigma(m, n) \xi^m \eta^n = 0 \tag{17},$$

which gives  $\eta = 0$  when  $\xi = \infty$ .

Let us suppose that Fig. 1, as originally constructed, is the Newton-diagram for (17), and let  $\xi^*$  be the highest power of  $\xi$ that occurs in (17) so that OO = k. Now in (17) put  $\xi = 1/\xi'$ . and multiply the equation by  $\xi^*$ ; we then get the equation

$$\Sigma(m, n) \xi^{(k-m)} \eta^n = 0 \tag{18},$$

which is obviously equivalent to (17).

But the Newton-diagram for (18) is obviously still Fig. 1. provided O.Y. and OY, be taken, instead of OX and OY, as the positive parts of the axes.

Hence, if we make a boundary convex towards O: in the same way as we did for O, we shall obtain a series of branches of q all of which are expansible in ascending powers of £, that is, in descending powers of  $\xi$ , and all of which give  $\eta = 0$  when  $\xi = \infty$ . For each such branch we have

$$\eta = \xi'^{\lambda} (c + d\xi'^{\alpha} + \epsilon \xi'^{\beta} + \dots),$$

that is.

$$(y-b) x^{\lambda} = c + d/x^{\alpha} + e/x^{\beta} + \dots$$
 (19)

where  $\lambda$ ,  $\alpha$ ,  $\beta$ , . . . are all positive, and c is finite both ways.

2nd. Suppose that  $x = \alpha$  is a pole of y, so that  $y = \infty$  when x=a; and put y=y,  $\xi=x-a$ , so that we derive an equation

$$\Sigma (m, n) \xi^m \eta^n = 0 \qquad (20),$$

for which Fig. 1 is the Newton-diagram with OX and OY as axes. Then, putting  $\eta = 1/\eta'$ , we get an equation of the form

$$\Sigma(m, n) \xi^m \eta^{d-n} = 0$$
 (21),

l being the highest exponent of η in (20). The Newton-diagram for (21) is then Fig. 1 with  $O_1X_1$ and O, Y, as axes; and we construct, as before, a boundary, EFG say, convex towards O1, every part of which gives a series of branches of  $\eta'$ , that is, of  $1/\eta$ , expansible in ascending powers of £. For every such branch we shall have

$$\eta_{\alpha}^{\dot{c}\lambda} = 1/(c + il_{\dot{c}}^{\dot{c}\alpha} + \epsilon \dot{c}^{\dot{c}\beta} + ...),$$

where  $\lambda$ ,  $\alpha$ ,  $\beta$ , . . , are all positive, and c is finite both ways. Hence also, by the binomial theorem combined with § 1,

$$n\xi\lambda = 1/c + d'\xi^{\alpha} + e'\xi^{\beta'} + \dots$$

that is,

$$y(x-a)^{\lambda} = 1/c + d'(x-a)^{\alpha} + e'(x-a)^{\beta} + \dots$$
 (22),

where  $\lambda$ ,  $\alpha$ ,  $\beta'$ , . . . are all positive, and c is finite both ways.

3rd. Suppose that y has an infinite value corresponding to  $x = \infty$  (pole at infinity). Then, if we put  $x = \xi = 1/\xi'$ ,  $y = \eta \cdot 1/\eta'$ , we shall get, by exactly the same kind of reasoning as before, a boundary GIII convex to O2, each part of which will give a group of expansions of the torm

$$\eta' = \xi'^{\lambda} \{ c + d\xi'^{\alpha} + c\xi'^{\beta} + \dots \}$$

Whence, as before, for every such branch

$$y_{-}x^{\lambda} = 1/(c + d/x^{\alpha} + e_{/}x^{\beta} + \dots),$$
  
 $1/c + d'/x^{\alpha} + e'/x^{\beta} + \dots$  (23)

where  $\lambda$ ,  $\alpha$ ,  $\beta$ , . . . are all positive, and c is finite both ways.

If we combine the results of the present with those of the foregoing paragraphs, we arrive at the following important general theorem regarding any algebraic function y:—

If y=0 when x=a  $(a \neq \infty)$ , then  $\underset{x \neq a}{L} y'(x-a)^{\lambda}$  is finite both ways.

If y=0 when  $x=\infty$ , then  $L(y/x^{-\lambda})$  is finite both ways.

If  $y = \infty$  when x - a  $(a \neq \infty)$ , then  $\sum_{x=a} y_i(x-a)^{-\lambda}$  is finite both ways.

If y = x when x = x, then  $L[y]x^{\lambda}$  is finite both ways.

λ is in all cases a finite positive commensurable number which may be called the ORDER of the particular zero or infinite value of g.

This theorem heals as naturally to speak of algebraical zero- or infinity values of functions in peneral, meaning such as how the property just stated. Thus  $\sin x = 0$  when x = 0; but  $L\sin x/x = 1$  when x = 0; therefore we say that  $\sin x$  has an algebraic zero of the first order when x = 0. Again,  $\tan x = x$  when x = |x| but  $L\tan x/(x - |x|)^{-1}$  is finite when x = |y|; the infinity of  $\tan x$  is therefore algebraical of the first order. On the other hand,  $x^2 = x$  when x = x but this is not an algebraical infinity, since no finite value of  $\lambda$  can be found such that  $Lx^2/x^{\lambda}$  is finite when x = x. (See chap. xxx., § 15.)

§ 24.] Application of the method of successive approximation to the expansion of functions. This method, when applied in conjunction with Newton's diagram, greatly increases the practical usefulness of the general theorems which have just been established. The method is, moreover, of great historical interest, because it appears from the scanty records left to us that it was in this form that the general theorems which we have been discussing originated in the mind of Newton.

Let us suppose that the terms of an equation (which may be an infinite series) have been plotted in Newton's diagram, and that an effective group of terms has been found lying on a line  $\Lambda$ ; and let  $\eta^{\alpha} = \xi^{\alpha}$  (the coefficients are taken to be unity for simplicity of illustration) be a factor in the group thus selected, repeated, say, a times, so that the whole group is  $\phi_1(\xi, \eta)(\eta^{\alpha} = \xi^{\alpha})$ . Let  $\Lambda$  be moved parallel to itself, until it meets a term or group of terms  $\phi_2(\xi, \eta)$ ; then again until it meets a group  $\phi_3(\xi, \eta)$ ; and so on.

The complete equation may now be arranged thus-

$$\phi_1\left(\xi,\,\eta\right)\left(\eta^m-\xi^n\right)^a+\phi_2\left(\xi,\,\eta\right)+\phi_3\left(\xi,\,\eta\right)+\dots=0,$$

or thus-

$$(\eta^m - \xi^n)^a + \frac{\phi_2(\xi, \eta)}{\phi_1(\xi, \eta)} + \frac{\phi_3(\xi, \eta)}{\phi_1(\xi, \eta)} + \dots = 0$$
 (24),

say,  $(\eta^m - \xi^n)^a + \tau_1 + \tau_3 + \dots = 0$ .

Now, by the properties of the diagram, when  $\eta = \xi^{n,m}$ ,  $\phi_2(\xi, \eta)$ ,  $\phi_2(\xi, \eta)$ ,  $\phi_3(\xi, \eta)$ , . . . are in ascending or descending order as regards degree in  $\xi$ , and the same is true of  $\tau_2$ ,  $\tau_3$ , . . . Let us suppose that  $\xi$  and  $\eta$  are small, so that  $\tau_2$ ,  $\tau_3$ , . . . are in ascending order.

As we have seen,  $\eta^m = \xi^m$ , that is,  $\eta = \xi^{m/m}$ , gives a first approximation. To obtain a second, we may neglect  $\tau_3$ ,  $\tau_4$ , . . . , and substitute in  $\tau_2$  the value of  $\eta$  as determined by the first approximation. To get a third approximation, neglect  $\tau_4$ , . . . , substitute in  $\tau_2$  the value of  $\eta$  as given by the second approximation, and in  $\tau_3$  the value of  $\eta$  as given by the first approximation.

We may proceed thus by successive steps to any degree of approximation; the only points to be attended to are never to neglect any terms of lower degree than the highest retained, and not to waste labour in calculating at any stage the coefficients of terms of higher degree than those already neglected.

There is a special case in which this process of successive substitution requires modification. We have supposed above that the substitution of the first approximation,  $\eta = \xi^{nm}$ , in  $\tau_2$ does not cause  $\tau_2$  to vanish, which will happen, for example, when  $\phi_2(\xi, \eta)$  contains  $\eta^m - \xi^n$  as a factor. In such a case the beginner might be tempted to put  $\tau_2 = 0$  and go on to substitute the first approximation in  $\tau_3$ . This would probably lead to error. For, if we were to substitute the complete value of  $\eta$  in  $\tau_2$ , it would not in general vanish, but simply become of higher order than is indicated in Newton's diagram, of the same order possibly as  $\tau_3$ . The best course to follow in such cases may be learned from Example 5 below, which deals with a case in point.

Example 1. Taking the equation (16), to find a third approximation to one of the branches of the group CD.

Next in order to C and D a parallel to CD meets successively B and A. Hence, putting, for simplicity, D = +1, C = B = A = -1, the equation (16) may be written

$$\xi^{3} \eta^{2} (\eta^{3} - \xi^{4}) - \xi^{10} \eta - \xi^{13} + \dots = 0.$$
Whence
$$\eta^{3} - \xi^{4} - \xi^{7} / \eta - \xi^{10} / \eta^{2} + \dots = 0$$
(25).

The first approximation is  $n = \xi^{4/3}$ ; hence, neglecting  $\xi^{1/2}/n^3$  in (25), we get for the second

$$\eta^{3} - \xi^{4} - \xi^{7}/\xi^{4/3} = 0. \tag{20}$$

 $n = \xi^{4/3} (1 + \xi^{5/3})^{1/3} = \xi^{4/3} (1 + 1\xi^{5/3})$ Whence (26)

 $\eta^3 - \xi^4 - \xi^{17/3} - \xi^{22/3} = 0$ 

If we use this second approximation in 
$$\xi^{2}/\eta$$
, and the first approximation in  $\xi^{10}/\eta^{2}$  now to be retained, (25) gives for the third approximation  $\eta^{2} = \xi^{2} + \xi^{2}/(3\cdot 1 + 4\xi^{23}) = \xi^{10}/(3\cdot 2 - 1)$ 

Whence, if all terms higher than the last retained be neglected.

which gives

$$\begin{array}{l} \eta = \xi^{4/3} \left(1 + \xi^{5/3} + \frac{\pi}{3} \xi^{10/3}\right)^{\frac{1}{3}}, \\ = \xi^{4/3} \left(1 + \frac{1}{3} \xi^{5/3} + \frac{1}{9} \xi^{10/3}\right) \end{array} \tag{27},$$

which is the required third approximation,

This might of course be obtained by successive applications of the method of transformation employed in the demonstration of § 22, or by the method of indeterminate coefficients, but the calculations would be laborious. It will be observed on comparing (27) with the theoretical result in \$ 22 that  $d_1 = d_2 = d_3 = d_4 = d_4 = d_5 = d_6 = d_6 = 0$ ; a fact which, in itself, shows the advantages of the mesent method.

The other branches of the cycle to which (27) belongs are given by  $\eta = (\omega \xi^{1/3})^4 \{1 + 1 (\omega \xi^{1/3})^5 + 1 (\omega \xi^{1/3})^{10} \},$ 

where w is any imaginary cube root of unity.

Example 2. To find a second approximation for the branches corresponding to ABC in equation (16), in the special case where A = +1, B = -2, C = +1, D = -1.

The terms concerned in this approximation are (ABC) and (D). We therefore write

$$\xi^{7}(\eta - \xi^{3})^{2} - \xi^{7}\eta^{5} = 0,$$
  
 $(\eta - \xi^{2})^{3} - \eta^{5}/\xi^{4} = 0.$ 

or

The first approximation is  $\eta = \xi^2$ ; hence the second is given by

$$(\eta - \xi^{1})^{2} - \xi^{13}/\xi^{4} = 0,$$
  
 $(\eta - \xi^{1})^{2} - \xi^{11} = 0.$ 

that is, 
$$(\eta - \xi^*)^2 - \xi^{11} = 0$$
.  
Whence  $\eta - \xi^3 + \xi^{11} = 0$ .

which gives the two second approximations corresponding to the group, There are two, because to a first approximation the branches are coincident. This, therefore, is a case where a second approximation is necessary to distinguish the branches.

Example 3. To find a second approximation, for large values both of £ and η, to the branch corresponding to HI in equation (16).

Referring to Fig. 1, we see that, if III move parallel to itself towards O. the next point which it will meet is G. Hence, so far as the approximation in question is concerned, we may replace (16) by

$$(H \xi^{10} n^{12} + I \xi^{14} n^7) + G \xi^6 n^{14} = 0,$$

For simplicity, let us put H = 1, I = G = -1, and write the above equation in the form

$$n^5 = 54 = n^7/34 = 0$$
.

Confining ourselves to one of the five first approximations, say  $\eta = \xi^{4,5}$ , we get for the second approximation

$$\eta^5 - \xi^4 - \xi^{9,5} = 0$$

which gives

ch gives 
$$\eta = \xi^{4.5} (1 + \frac{1}{3} \xi^{-12/5}).$$

The other branches of the cycle are given by
$$\mathbf{r} = (\omega^{\xi^{1/5}})^4 \{1 + 1 (\omega^{\xi^{1/5}})^{-12} \},$$

where ω is any imaginary fifth root of unity.

Example 4. Given

$$x = y + y^2/2! + y^3/3! + y^4/4! + \cdots$$

to find u to a fourth approximation. We have

$$y = x - y^2/2! - y^3/3! - y^4/4! - \dots$$

Hence

1st approx. 
$$y = x$$
.  
2nd ,  $y = x - \frac{1}{2}x^2$ .  
3rd ,  $y = x - \frac{1}{2}(x - \frac{1}{2}x^2)^2 - \frac{1}{6}x^3$ ,  
 $= x - \frac{1}{2}x^2 + \frac{1}{6}x^2$ .  
4th ,  $y = x - \frac{1}{6}(x - \frac{1}{6}x^2 + \frac{1}{6}x^2)^2 - \frac{1}{6}(x - \frac{1}{2}x^2)^2 - \frac{1}{6}x^4$ .

Example 5. To separate the branches of  $\eta$  at  $\xi=0$ , where

$$4\xi^5 - 3\xi^4 - 4\xi^2(\eta - \xi) + 4(\eta - \xi)^2 = 0.$$

If we plot the terms in Newton's diagram, and arrange them in groups corresponding to their order of magnitude, we find

$$\{4(\eta - \xi)^2\}_1 - \{4\xi^2(\eta - \xi)\}_2 - \{3\xi^4\}_3 + \{4\xi^5\}_4 = 0,$$

where the suffixes attached to the brackets indicate the orders of the groups.

The first approximation  $n = \xi$  is common to the two branches.

Since  $\eta - \xi$  is a factor in  $\{\ \}_2$ , we cannot obtain a second approximation by neglecting { }2 and putting  $\eta = \xi$  in { }2. In obtaining the second approximation we therefore retain { }, treating  $\eta - \xi$  as a variable to be found. We thus get

 $4(\eta - \xi)^2 - 4\xi^2(\eta - \xi) = 3\xi^4;$ 

 ${2(\eta - \xi) - \xi^2}^2 = 4\xi^4$ , whence  $n = \xi + 3\xi^2/2$ : which gives  $\eta' = \xi - \xi'/2$ .

The branches are thus separated.

If a third approximation were required, we should now retain { }4, and write

$$\{2(\eta - \xi) - \xi^*\}^2 - 4\xi^4 - 1\xi^4.$$
 Hence 
$$2(\eta - \xi) - \xi^2 = \pm 2\xi^2(1 - \xi)^4,$$
 We thus cet

We thus get

$$\begin{split} & \eta = \xi + 3\xi^2/2 - \xi^3/2 \ ; \\ & \eta' = \xi - \xi^2/2 + \xi^3/2. \end{split}$$

Historical Nate,—As has already been remarked, the fundamental idea of the reversion of series, and of the expansion of the roots of algebraical or other equations in power-series, originated with Navton. His fermous "Brafalledgram" is first mentioned in the second letter to Oldenburg; but is more fully explained in the Grounderier Analytics see Heavily's (altion of Newton's Works, U. p. 200). The method was well understood by Newton's followers, String and Taylor; but seems to have been lost sight of in Ingland after their time. It was much used (in a modified form of De Gua's by Cramer in his well-known Analyse des Liques Courbes Analysis, (1759). Lagrange gave a complete analytical form to Newton's method in his "M-noire sur U. Sago des Fractions Continues," Nover, Min. d. C. Le, royd, d. Seienees d. Berlin (1756). (See Elimes de Lagrange, t. rv.)

Notwith-standing its great utility, the method was everywhere all but forgotten in the early part of this century, as has been pointed out by De Morgan in interesting account of it given in the Cambridge Philosophical Transactions.

vol. ix, (1855),

The idea of demonstrating, a pri i, the possibility of expansions such as the reversion-formal of § 18 originated with Cauch iy, and to hum, in effect, are due the methods employed in § 18 and 19. See his memoirs on the Integration of Partial Differential Equations, on the Calculus of Lamits, and on the Nature and Properties of the Rods of an Equation which contains a Variable Parameter, Exercises of Jouleve et al. Physique Mathématique, 1. a, (1800, p. 327; 1. 1. (1801, pp. 41, 102). The form of the demonstrations given in § 18, 19 has been borrowed partly from Thomes, II. Theorie der. Jauditechte Functionen viace Complexen Premodelichon (1416), 1880, p. 107; partly from Stolz, Allgeminic Arthurchy, I. Th. Lopping, 1885, p. 200;

The Parallebyran of Newton was need for the theoretical purpose of establishing the expansibility of the branches of an algebraic function by Phisenx in his Classical Memoir on the Algebraic Functions (Lioux, Math. Jour., 1850). Phisenx and Berfort and Boungh (1 theor. in Fort functions) (Lioux, Math. Jour., 1850), p. use Canchy's Theorem regarding the number of the rost of an algebraic equation in a given contour; and thus infer the continuity of the roots. The demonstration given in § 21 depends upon the proof, a prior, of the possibility of an expansion in a power series; and in this respect follows the original idea of

Newton.

The reader who desires to pursue the subject further may consult Durège, the reader of the tracest der Fine tower once Complexen Verinderlichen Grosse, for a good introduction to this great brach of modern function-theory.

The English student has now at his disposal the two treatises of Harkness and Morley, and the work of Forsyth, which deal with function-theory from various points of view.

The applications are very numerous, for example, to the finding of curvatures and curves of closest contact, and to curve-tracing generally. A number of beautiful examples will be found in that mach-to-be-recommended text-book, Frost's Curve Tracing.

#### Exercises XXIV.

Revert the following series and find, so far as you can, expressions for the coefficient of the general term in the Reverse Series:—

(1.) 
$$y = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{5!}x^3 + \dots$$
  
(2.)  $y = x - 4x^3 + 3x^5 - 3x^7 + \dots$ 

(3.) 
$$y = x - \frac{x^3}{21} + \frac{x^5}{51} - \frac{x^7}{71} + \dots$$

(4.)  $y = x + x^2/2^2 + x^3/3^2 + x^4/4^2 + \dots$ 

(5.) If  $y = \sin x/\sin (x+a)$ , expand x in powers of y.

x and y being determined as functions of each other by the following equations, find first and second approximations to those branches, real or imaginary, for which |x| or |y|, or both, become either infinitely small or infinitely great:—

(6.)  $y^2 - 2y = x^4 - x^2$ .

(7.) 
$$a^3(y+x) - 2a^2x(y+x) + x^4 = 0$$
, (F. 69\*).

(8.) 
$$(x-y)^2 - (x-y) x^2 - \frac{1}{2}x^4 - \frac{1}{2}y^4 = 0$$
, (F. 82).  
(9.)  $a(y^2 - x^2)(y - 2x) - y^4 = 0$ , (F. 88).

(10.) 
$$ax(y-x)^2-y^4=0$$
, (F. 96).

(11.) 
$$x(y-x)^2 - a^3 = 0$$
, (F. 115).

(12.) 
$$x^3y^2 - 2a^2x^2y + a^4x - b^5 = 0,$$
 (F. 121).  
(13.)  $y(y-x)^2(y+2x) = 9cx^3,$  (F. 131).

(14.) 
$$\{x(y-x)-a^2\}^2y^3=a^7$$
, (F. 140).

(15.) 
$$x^7 - x^4y^3 + a^3y^4 - axy^5 = 0$$
, (F. 143).

(16.) 
$$a(x^5+y^5) - a^2x^3y + x^2y^4 = 0$$
, (F. 143).  
(17.)  $x^3y^4 + ax^2y^3 + bx^4y + cx + dy^2 = 0$ , where  $a, b, c, d$  are all positive,

(18.) If  $c_n$  be any constant whatever when n is a prime number, and such that  $c_n = c_p c_q c_r \dots$  when n is composite and has for its prime factors  $p, q, r, \dots$ , then show that

(F. 155).

If  $a, b, c, \ldots$  be a given succession of primes finite or infinite in number, s any integer of the form  $a^ab^ac^a \cdots t$  any integer of the forms  $a, ab, ab, \ldots$  (where none of the prime factors are nowes), and b.

then 
$$F(x) = \sum e_x f(x^i),$$

$$f(x) = \sum (-)^n e_x F(x^i).$$

where u is the number of factors in t.

(This remarkable theorem was given by Mobius, Crelle's Jour., Ix. p. 105. For an elegant proof and many interesting consequences, see an article by J. W. L. Ghaisber, Phil. Mag., ser. 5, xviii., p. 518 (1854).)

<sup>\*</sup> F. 69 means that a discussion of the real branches of this function, with the corresponding graph, will be found in Frost's Carve Tracing, § 69.

# CHAPTER XXXI

## Summation and Transformation of Series in General.

#### THE METHOD OF FINITE DIFFERENCES.

§ 1.] We have already touched in various connections upon the summation of series. We propose in the present chapter to bring together a few general propositions of an elementary character which will still further help to guide the student in this somewhat intricate branch of algebra.

It will be convenient, although for our immediate purposes it is not absolutely necessary, to introduce a few of the elementary conceptions of the Calculus of Finite Differences. We shall thus gain clearness and conciseness without any sacrifice of simplicity; and the student will have the additional advantage of an introduction to such works as Boole's Finite Differences, where he must book for any further information that he may require regarding the present subject.

Let, as heretofore,  $u_n$  be the nth term of any series; in other words, let  $u_n$  be any one-valued function of the integral variable n;  $u_{n+1}$ ,  $u_{n+2}$ , . . . ,  $u_i$  the same functions of n-1, n-2, . . . , 1 respectively.

Farther, let  $\Delta u_{n1} = \Delta u_{n+1}$ , ...,  $\Delta u_1$ denote  $u_{n+1} - u_n$ ,  $u_n \cdot u_{n+1}$ , ...,  $u_2 - u_1$ ; also  $\Delta (\Delta u_n)$ ,  $\Delta (\Delta u_{n-1})$ , ...,  $\Delta (\Delta u_l)$ , which we may write, for shortness,

$$\Delta^2 u_n$$
,  $\Delta^2 u_{n-1}$ , . . . ,  $\Delta^2 u_1$ ,

denote

$$\Delta u_{n+1} - \Delta u_n$$
,  $\Delta u_n - \Delta u_{n-1}$ , ...,  $\Delta u_2 - \Delta u_1$ ;

and so on. Thus we have the successive series,

$$u_1, \quad u_2, \quad u_3, \quad \dots, \quad u_8, \quad \dots \quad (1);$$
  
 $\Delta u_1, \quad \Delta u_2, \quad \Delta u_3, \quad \dots, \quad \Delta u_4, \quad \dots \quad (2);$   
 $\Delta^2 u_1, \quad \Delta^2 u_3, \quad \Delta^2 u_3, \quad \dots, \quad \Delta^2 u_8, \quad \dots \quad (3);$   
 $\Delta^2 u_1, \quad \Delta^2 u_2, \quad \Delta^2 u_3, \quad \dots, \quad \Delta^2 u_n, \quad \dots \quad (4);$ 

where each term in any series is obtained by subtracting the one immediately above it from the one immediately above and to the right of it.

The series (2), (3), (4), . . . are spoken of as the series of 1st, 2nd, 3rd, . . . differences corresponding to the primary series (1).

Example 1. If  $u_n = n^2$ , the series in question are

1, 4, 9, 16, ..., 
$$n^2$$
, ...;  
3, 5, 7, 9, ...,  $2n+1$ , ...;  
2, 2, 2, 2, ..., 2, ...;  
0, 0, 0, 0, ..., 0, ...;

where, as it happens, the second differences are all equal, and the third and all higher differences all vanish.

Cor. If we take for the primary series

$$\Delta^r u_1$$
,  $\Delta^r u_2$ ,  $\Delta^r u_3$ , . . . ,  $\Delta^r u_n$ , . . . ,

then the series of 1st, 2nd, 3rd, . . . differences will be

In other words, we have, in general,  $\Delta^r \Delta^s u_n = \Delta^{r+s} u_n$ . This is sometimes expressed by saying that the difference operator  $\Delta$  obeys the associative law for multiplication.

Although we shall only use it for stating formulæ in concise and easily-remembered forms, we may also introduce at this stage the operator E, which has for its office to increase by unity the variable in any function to which it is prefixed. Thus

$$E\phi(n) = \phi(n+1)$$
;  $Eu_n = u_{n+1}$ ;  $Eu_1 = u_2$ ;

and so on

In accordance with this definition we have  $E(Eu_n)$ , which we contract into  $E^*u_{n+1} = Eu_{n+1} = u_{n+2}$ ; and, in general,  $E^mu_n = u_{n+m}$ . We have also, as with  $\Delta$ ,  $E^*L^*u_n = E^{r+s}u_n$ ; for each of these is obviously cound to  $u_{n+m+1}$ .

Example 2. 
$$E^{r}n^{3} = (n + r)^{3}$$
.

Example 3. The *m*th difference of an integral function of *n* of the *r*th degree is an integral function of the (r-m)th degree if m < r, a constant if r = m, zero if m > r.

Let 
$$\phi_{r}(n) = an^{r} + bn^{r-1} + cn^{r-2} + \dots$$

then

$$\begin{split} \Delta \varphi_r \left( n \right) &= a \left( n+1 \right)^{r+b} \left( n+1 \right)^{r-1} + c \left( n+1 \right)^{r-2} + \dots \\ &= a n^r - b n^{r-1} - c n^{r-2} + \dots \\ &= r n^{r-1} + \left\{ \frac{1}{2} r \left( r-1 \right) a + \left( r-1 \right) b \right\} n^{r-2} + \dots \\ &= \varphi_{r-1} \left( n \right), \end{split}$$

say, where  $\phi_{r-2}(n)$  is an internal function of n of the (r-1)th degree. Then, in the manner, we have  $\Delta \psi_{r-1}(n) = \psi_{r-2}(n)$ . Dut  $\Delta \psi_{r-1}(n) = \Delta \psi_r n$ ; hence  $\Delta^2 \psi_r (n) = \psi_{r-2}(n)$ . Similarly,  $\Delta^2 \psi_r (n) = \psi_{r-3}(n)$ ; and, in general,  $\Delta^2 \psi_r (n) = \psi_{r-3}(n)$ . We see also that  $\Delta^2 \psi_r (n)$  will reduce to a constant, namely,  $r! a_1$  and that all differences whose order exceeds r will be zero.

The product of a series of factors in arithmetical progression, such as  $(a + b) \dots (a + (m - 1)b)$ , plays a considerable part in the summation of series. Such a product was called by Kramp a Faculty, and be introduced for it the notation  $a^{m,b}$ , calling a the base, m the exponent, and b the difference of the faculty. This notation we shall occasionally use in the slightly modified form  $a^{m,b}$ , which is clearer, especially when the exponent is compound. Since

$$a(a+b)$$
 . . .  $(a+(m-1)b) = b^m (a/b)(a/b+1)$  . . .  $(a/b+m-1)$ , from the can always be reduced to a multiple of another whose different

any faculty can always be reduced to a multiple of another whose difference is unity, that is, to another of the form  $c^{(m)}$ , which, omitting the l, we may write  $c^{(m)}$ . In this notation the ordinary factorial m! would be written  $1^{(m)}$ .

The reader should carefully verify and note the following properties of the differences of Faculties and Factorials. In all cases  $\Delta$  operates as usual with respect to n.

$$\Delta(a+bn) \stackrel{m-b}{=} mb \{a+b(n+1)\} \stackrel{m-1}{=} b$$
.

Example 5.

$$\Delta \{1/(a+bn)^{-m/b}\} = -mb/(a+bn)^{-m+1/b}$$
.

Example 6.

$$\Delta \left\{ \frac{a^{-n-b}}{(a^{-n-b})} = \frac{a^{-c}}{a-b} \cdot \frac{(a-b)^{-n+1-b}}{c^{-n+1+b}} = .$$

Example 7.

$$\Delta \cos (\alpha + \beta n) = -2 \sin \frac{1}{2}\beta \sin (\alpha + \frac{1}{2}\beta + \beta n);$$
  

$$\Delta \sin (\alpha + \beta n) = +2 \sin \frac{1}{2}\beta \cos (\alpha + \frac{1}{2}\beta + \beta n).$$

§ 2.] Fundamental Theorems. The following pair of theorems\* form the foundation of the methods of differences, both direct and inverse:—

$$I. \quad \Delta^m u_n = u_{n+m} - {}_{m}C_1u_{n+m-1} + {}_{m}C_2u_{n+m-2} + ... + (-)^m u_n.$$

II. 
$$u_{m+n} = u_m + {}_{n}C_1\Delta u_m + {}_{n}C_2\Delta^2 u_m + ... + \Delta^n u_m$$

To prove I, we observe that

$$\Delta u_n = u_{n+1} - u_n;$$

$$\Delta^2 u_n = u_{n+2} - u_{n+1} - u_{n+1} + u_n;$$

$$= u_{n+2} - 2u_{n+1} + u_n;$$

hence

$$\Delta^{3} u_{n} = u_{n+3} - 2u_{n+2} + u_{n+1} - u_{n+2} + 2u_{n+1} - u_{n},$$

$$= u_{n+3} - 3u_{n+2} + 3u_{n+1} - u_{n};$$

and so on.

Here the numerical values of the coefficients are obviously being formed according to the addition rule for the binomial coefficients (see chap. IV., § 11); and the signs obviously alternate. Hence the first theorem follows at once.

To prove II. we observe that we have, by the definition of  $\Delta u_{a_n}$ ,  $u_{m+1} = u_m + \Delta u_m$ . Hence, since the difference of a sum of functions is obviously the sum of their differences, we have, in like manner,  $u_{m+2} = u_{m+1} + \Delta u_{m+1} = u_m + \Delta u_n + \Delta (u_n + \Delta u_m) = u_m + \Delta u_m + \Delta u_m + \Delta^2 u_m$ . We therefore have in succession

<sup>•</sup> The second of these was given by Newton, Principia, lib. III., lemma v. (1687); and is sometimes spoken of as Newton's Interpolation Formula. See his tract, Methodus Differentialis (1711); also Demoivre, Miscellanea Analytica, p. 152 (1730), and Stirling, Methodus Differentialis, &c., p. 97 (1730).

$$\begin{array}{lll} u_{n+1} - u_m + & \Delta u_m, \\ u_{n+2} - u_m + & \Delta u_n, \\ & + & \Delta u_m + & \Delta^2 u_m, \\ u_m + & 2\Delta u_m + & \Delta^2 u_m, \\ u_{n+3} - & u_m + 2\Delta u_m + & \Delta^2 u_m, \\ & + & \Delta u_m + 2\Delta^2 u_m + & \Delta^2 u_m, \\ u_{n+3} \Delta u_{n+3} \Delta^2 u_$$

and so on.

The second theorem is therefore established by exactly the same reasoning as the first, the only difference being that the signs of the coefficients are now all positive.

If we use the symbol E, and separate the symbols of operation from the subjects on which they operate, the above theorems may be written in the following easily-remembered symbolical forms:—

$$\Delta^m u_n = (E-1)^m u_n = (I.);$$
  $u_{m+n} = (I+\Delta)^n u_m = (II.).$ 

§ 3.] The following theorem enables us to reduce the summation of any series to an inverse problem in the calculus of finite differences.

If  $r_n$  be any function of n such that  $\Delta r_n = u_n$ , then

$$\sum_{i=1}^{n} u_n = r_{n+1} - r_d \tag{1}.$$

This is at once obvious, if we add the equations

$$\begin{array}{lll} u_n & \Delta v_n &= v_{n+1} - v_n, \\ \\ u_{n-1} &= \Delta v_{n-1} = v_n & - v_{n-1}, \\ \\ & & \\ & & \\ u_s &= \Delta v_s &= v_{s+1} - v_s. \end{array}$$

The difficulty of the summation of any series thus consists entirely in finding a solution (any solution will do) of the finite difference equation  $\Delta v_n - u_n$ , or  $v_{n+1} - v_n = u_n$ . This solution can be effected in finite terms in only a limited number of cases, some of the more important of which are exemplified below.

On the other hand, the above theorem enables us to con-

struct an infinite number of finitely summable series. All we have to do is to take any function of a whatever and find its first difference; then this first difference is the ath term of a summable series. It was in this way that many of the ordinary summable series were first obtained by Leibnitz, James and John Bernoulli, Demoivre, and others.

Example 1. 
$$\sum_{n=0}^{n} \{a+nb\} \{a+(n+1)b\} \dots \{a+(n+m-1)b\}$$
.

Using Kramp's notation, we have here to solve the equation

$$\Delta v_n = \{a + nb\}^{(m-b)}$$
 (2).

Now we easily find, by direct verification, or by putting m+1 for m and r-1 for n in § 1, Example 4, that

$$\Delta [\{a + (n-1)|b\}^{\lfloor m+1 \rfloor b}/(m+1)|b] = \{a + nb\}^{\lfloor m \rfloor b}.$$

Hence  $v_n = \{a + (n-1)b\}^{\lfloor m+1 \rfloor b}/(m+1)b$  is a value of  $v_n$  such as we require.

Therefore

$$\sum_{s}^{5} \{a+nb\}^{-m+b} = \frac{\{a+nb\}^{-m+1-b} - \{a+(s-1)/b\}^{-(m+1)/b}}{(m+1)/b}$$
(3)

Hence the well-known rule

$$\overset{\mathbf{n}}{\succeq} \{a+nb\} \{a+(n+1)b\} \dots \{a+(n+m-1)b\}$$

$$= C + \{a+nb\} \{a+(n+1)b\} \dots \{a+(n+m-1)b\} \{a+(n+m)b\}/(m+1)b$$

$$(4).$$

where C is independent of n, and may be found in practice by making the two sides of (4) agree for a particular value of n.

Example 2. To sum any series whose nth term is an integral function of n, say f(n).

By the method of chap. v., \$ 22 (2nd ed.), we can express f(n) in the form  $a+bn+cn(n+1)+dn(n+1)(n+2)+\ldots$  Hence

$$\sum_{i=1}^{n} f(n) = C + an + \frac{1}{2}bn(n+1) + \frac{1}{2}cn(n+1)(n+2) + \frac{1}{4}dn(n+1)(n+2)(n+3) + \dots$$
(5),

where the constant C can be determined by giving n any particular value in (5).

Example 3.  $\Sigma 1/\{a+bn\}^{m+b}$ .

Proceeding exactly as in Example 1, and using § 1, Example 5, we deduce

$$\sum_{s}^{n} \frac{1}{\{a+bn\}^{\lfloor m+b}} = \frac{1/\{a+bs\}^{\lfloor m-1+b} - 1/\{a+b(n+1)\}^{\lfloor m-1+b}}{(m-1)b}$$
(6).

Hence a rule for this class of series like that given in Example 1.

Example 4. To sum the series  $\Sigma f(n)/\{a+bn\}^{-m+b}$ , f(n) being an integral function of n.

Decompose f(n), as in Example 2, into

$$a + \beta (a + bn)^{-1/b} + \gamma (a + bn)^{-2+b} + \delta (a + bn)^{-3/b} + \dots$$
 (7).

Then we have to evaluate

$$a\sum_{a=1}^{n} 1/(a+bn)^{-m-b} + s\sum_{a=1}^{n} 1/(a+b(n+1))^{-m-1-b} + \dots$$
 (8).

which can at once be done by the rule of Example 3\*.

Example 5.

$$\sum_{c=n-b}^{n} \frac{a^{-n-b}}{c^{-n-b}} = \frac{a}{a-c+b} \left\{ \frac{(a+b)^{-n-b}}{c^{-n-b}} - \frac{(a+b)^{-s-1-b}}{c^{-s-1-b}} \right\}$$
 (9).

This can be deduced at once from § 1, Example 6, by writing a+b for b and n-1 for n.

Example 6. To sum the series whose terms are the Figurate Numbers of the mth rank.

The figurate numbers of the let, 2nd, 3rd, . . . ranks are the numbers in the 1st, 2nd, 3rd, . . , vertical columns of the table (11.) in chap. rv, § 25. Hence the (n+1)th figurate number of the mth rank is  $\underset{n+m-1}{n}C_{n-1} = \underset{n+m-1}{n}C_n = m (m+1) \dots (m+n-1)/n!$ . Hence we have to sum the series

$$1 + \sum_{1}^{n} \frac{m(m+1) \dots (m+n-1)}{1 \cdot 2 \cdot \dots n}$$
.

Now if in (9). Example 5, we put a = m, b = 1, c = 1, we get

$$\frac{\sum_{n=1}^{n} \frac{m^{-n}}{1^{-n}}}{\sum_{n=1}^{n} \frac{(m+1)^{-n}}{1^{-n+1}}} - \frac{m+1}{1}$$
.

Hence

$$1+m+\frac{m\ (m+1)}{1,\ 2}+\ldots+\frac{m\ (m+1)\ \ldots\ (m+n-1)}{1,\ 2\ \ldots\ n} = \frac{(m+1)\ (m-2)\ \ldots\ (m+1+n-1)}{1,\ 2\ \ldots\ n} (10);$$

that is to say, the sum of the first n figurate numbers of the mth rank is the nth figurate number of the (m+1)th rank.

This theorem is, however, morely the property of the function  ${}_{m}H_{n}$ , which we have already established in chap. xxiii. § 10. Cor. 4. The present demonstration of (10) is of course not restricted to the case where m is a positive integer.

Many other well-known results are included in the formula of Example 5, some of which will be found among the exercises below.

The method of Examples 1 to t are all to be found in Stirling's Methodus Inglerentialis. He applies them in a very remarkable way to the approximate evaluation of series which cannot be summed. (See Exercises MART, 17.)

Example 7. To sum the series

$$S_n = \cos \alpha + \cos (\alpha + \beta) + \dots + \cos (\alpha + (n-1)\beta);$$

 $T_n = \sin \alpha + \sin (\alpha + \beta) + \dots + \sin (\alpha + (n-1)\beta).$ 

From § 1, Example 7, we have  $\cos(\alpha + \beta n) = \Delta \{ \sin(\alpha - \frac{1}{2}\beta + \beta n)/2 \sin \frac{1}{2}\beta \}$ . Hence

S<sub>n</sub> = 
$$\{\sin\left(\alpha - \frac{1}{2}\beta + \beta n\right) - \sin\left(\alpha - \frac{1}{2}\beta\right)\}/2\sin\frac{1}{2}\beta$$
,  
=  $\frac{\sin\frac{1}{2}\beta n}{\sin\frac{1}{2}\beta}\cos\left\{\alpha + \frac{1}{2}\beta\left(n - 1\right)\right\}$ .

Similarly,

$$T_n = \frac{\sin \frac{1}{2}\beta n}{\sin \frac{1}{2}\beta} \sin \left\{\alpha + \frac{1}{2}\beta \left(n - 1\right)\right\}.$$

§ 4.] Expression for the sum of n terms of a series in terms of the first term and its successive differences.

Let the series be  $u_1 + u_2 + \ldots + u_n$ ; and let us add to the beginning an arbitrary term  $u_0$ . Then if we form the quantities

$$S_0 = u_0$$
,  $S_1 = u_0 + u_1$ ,  $S_2 = u_0 + u_1 + u_2$ ,  
...,  $S_n = u_0 + u_1 + u_2 + \dots + u_n$ , ...

we have

$$\Delta S_n = u_{n+1}, \quad \Delta^2 S_n = \Delta u_{n+1}, \quad \dots, \quad \Delta^m S_n = \Delta^{m-1} u_{n+1}, \quad \dots$$

Hence, putting n = 0,

$$\Delta S_0 = u_1, \quad \Delta^2 S_0 = \Delta u_1, \quad \dots, \quad \Delta^m S_0' = \Delta^{m-1} u_1, \quad \dots$$
 (1).

Now, by Newton's formula (§ 2, II.),

$$S_n = S_0 + {}_nC_1\Delta S_0 + {}_nC_2\Delta^2 S_0 + ... + \Delta^n S_0$$
 (

If, therefore, we replace  $S_0$ ,  $\Delta S_0$ ,  $\Delta^2 S_0$ , . . . by their values according to (1), we have

$$\sum_{n=0}^{n} u_{n} = u_{0} + {}_{n}C_{1}u_{1} + {}_{n}C_{2}\Delta u_{1} + {}_{n}C_{3}\Delta^{2}u_{1} + \dots + \Delta^{n-1}u_{1} \quad (3);$$

or, if we subtract uo from both sides,

$$\sum_{1}^{n} u_n = {}_{n}C_1u_1 + {}_{n}C_2\Delta u_1 + {}_{n}C_3\Delta^2 u_1 + \dots + \Delta^{n-1}u_1 \quad (4)^*.$$

The formula (4) is simply an algebraical identity which may be employed to transform any series whatsoever; for example, in the case of the geometric series  $\Sigma x^n$  it gives

<sup>•</sup> This formula, which, as Demoivre (Miscell, An., p. 153) pointed out, is an immediate consequence of Newton's rule, seems to have been first explicitly stated by Montmort, Journ. d. Savans (1711). It was probably independently found by James Bernoulli, for it is given in the Ars Conjectandi, p. 98 (1713).

$$x^{1} + x^{2} + \dots + x^{n}$$

$$= ux + \frac{n(n-1)}{2!}x(x-1) + \frac{n(n-1)(n-2)}{3!}x(x-1)^{2} + \dots$$

$$+ x(x-1)^{n-1}$$

which can be easily verified independently by transforming the right-hand side. The transformation (4) will, however, lead to the sum of the series, in the proper sense of the word sum, only when the mth differences of the terms become zero, m being a finite integer. The sum of the series will in that case be given by (4) as an integral function of n of the mth degree. Since the nth term of the series is the first difference of its finite sum, we see conversely that any series whose sum to n terms is an integral function of n of the mth degree must have for its nth term an integral function of n of the (m-1)th degree. We have thus reproduced from a more general point of view the results of chap, xX, s 10.

Example. Sum the scries

$$\sum_{n=0}^{\infty} (n+1)(n+2)(n+3)$$
.

If we tabulate the first few terms and the successive differences, we get

Hence, by (1),

$$\sum_{n=0}^{\infty} (n+1)(n+2)(n+3)$$

$$= n \cdot 24 + \frac{n(n-1)}{2} \cdot 36 + \frac{n(n-1)(n-2)}{6} \cdot 24 + \frac{n(n-1)(n-2)(n-3)}{24} \cdot 6,$$

$$= \frac{1}{2} (n^4 + 10n^3 + 35n^2 + 50n).$$

§ 5.] Monomore's Theorem reparding the summation of  $\Sigma u_n x^n$ . An elegant formula for the transformation of the power-series  $\Sigma u_n x^n$  may be obtained as follows. Let us in the first place consider  $S = \sum_{i=1}^n u_i x^n$ , which we suppose to be convergent when |x| < 1; and let us further suppose that |x| < 1 - x|. Put |x| < y > 1 is that

and

$$|y/(1+y)| = |x| < 1,$$
  
 $|y| = |x/(1-x)| < 1.$ 

Then, since

$$(1+y)^{-m} = 1 - {}_{m}C_{1}y + {}_{m+1}C_{2}y^{2} - {}_{m+2}C_{3}y^{3} + \dots,$$

we have

$$\begin{split} S &= \sum_{\mathbf{x}} u_{n} y^{n} / (1 + y)^{\mathbf{x}}, \\ &= u_{1} y - u_{1} y^{2} + \quad u_{1} y^{3} - \quad u_{1} y^{4} + \quad u_{1} y^{6} - \dots \\ &+ \quad u_{2} y^{2} - _{2} C_{1} u_{2} y^{7} + _{2} C_{2} u_{2} y^{4} + _{1} C_{1} u_{2} y^{7} + \dots \\ &+ \quad u_{2} y^{3} - _{2} C_{1} u_{2} y^{4} + _{1} C_{2} u_{2} y^{6} - \dots \\ &+ \quad u_{4} y^{4} - _{4} C_{1} u_{4} y^{6} + \dots \end{split}$$

This double series evidently satisfies Cauchy's criterion, for both |y| < 1 and |y/(1+y)| < 1. Hence we may rearrange it according to powers of y. If we bear in mind § 2, 1., we find at once

$$S = u, u + \Delta u, u^2 + \Delta^2 u, u^3 + \Delta^3 u, u^4 + \Delta^4 u, u^5 + \dots$$

Hence, replacing y by its value, namely, x/(1-x), we get

$$\sum_{n=0}^{\infty} u_n x^n = \frac{u_1 x}{1-x} + \frac{\Delta u_1 x^2}{(1-x)^2} + \frac{\Delta^2 u_1 x^3}{(1-x)^4} + \dots$$
(1)\*.

When the differences of a finite order m vanish, Montmort's formula gives a closed expression for the sum to infinity; and, if the differences diminish rapidly, it gives in certain cases a convenient formula for numerical approximation.

Cor. 1. We have for the finite sum

$$\sum_{l=1}^{n} u_n x^n = (u_1 - u_{n+1} x^n) \frac{x}{1-x} + (\Delta u_1 - x^n \Delta u_{n+1}) \frac{x^2}{(1-x)^2} + (\Delta^2 u_1 - x^n \Delta^2 u_{n+1}) \frac{x^2}{(1-x)^2} + \dots (2).$$

For, if we start with the series  $u_{n+1}x^{n+1} + u_{n+2}x^{n+2} \dots$ , and proceed as before, we get

$$\sum_{n+1}^{\infty} u_n x^n = \frac{u_{n+1} x^{n+1}}{(1-x)^1} + \frac{\Delta u_{n+1} x^{n+2}}{(1-x)^2} + \frac{\Delta^2 u_{n+1} x^{n+3}}{(1-x)^3} + \dots (3).$$

From (1) and (3) we get (2) at once by subtraction.

<sup>\*</sup> First given by Montmort, Phil. Trans. R.S.L. (1717). Demoivre gave in his Miscellanea a demonstration very much like the above.

The formula (2) will furnish a sum in the proper sense only when the differences vanish after a certain order. The summation of the integro-geometric series, already discussed in chap  $xx_*$ ,  $\lesssim 13$  and 14, may be effected in this way. It should be observed that, inasmuch as (2) is an algebraic identity between a finite number of terms, its truth does not depend on the convergency of  $\Sigma u_n x^n$ , although that supposition was made in the above demonstration.

Cor. 2. If  $u_n$  be a real positive quantity which constantly diminishes as n increases, and if  $Lu_n=0$ , then

$$u_1 - u_2 + u_3 - \dots = \frac{1}{2} u_1 - \frac{1}{2^2} \Delta u_1 + \frac{1}{2^2} \Delta^2 u_1 - \dots$$
 (1)\*.

This is merely a particular case of (1); for, if in (1) we put -x for x, we get

$$\sum_{1}^{\infty} (-)^{n} u_{n} x^{n} = \sum_{1}^{\infty} (-)^{n} \Delta^{n-1} u_{1} \cdot \left(\frac{x}{1+x}\right)^{n}$$
 (5).

Since the differences must ultimately remain finite, the right-hand side of (5) will be convergent when x = 1. Also, by Abel's Theorem (chap, XXVI, § 20), since  $\Sigma (-)^n u_n$  is convergent, the limit of the left-hand side of (5) when x = 1 is  $\sum_{i=1}^{n} (-)^n u_n$ . Hence the theorem follows.

The transformation in formula (4) in general increases the convergency of the series, and it may of course, in particular cases, lead to a finite expression for the sum.

Cor. 3. We get, by subtraction, the following formula:-

$$u_1 - u_2 + \dots$$
 (-)<sup>n-1</sup> $u_n = \frac{1}{2} (u_1 - (-)^n u_{n+1}) - \frac{1}{2^2} (\Delta u_1 - (-)^n \Delta u_{n+1}) + \frac{1}{2^2} (\Delta^2 u_1 - (-)^n \Delta^2 u_{n+1}) - \dots$  (6),

in which the restrictions on  $u_n$  will be unnecessary if the right-hand side be a closed expression, which it will be if the differences of  $u_n$  vanish after a certain order.

Euler, Inst. Diff. Calc., P. at H., cap. 1. (1757).

Example 1. We have (Gregory's Series)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \tag{7}.$$

If we apply (4), we have  $u_n = 1/(2n-1)$ . Hence

$$\Delta^r u_n = (-)^r 2 \cdot 4 \cdot ... \cdot 2r/(2n-1)(2n+1)(2n+3) \cdot ... \cdot (2n+2r-1);$$
  
 $\Delta^r u_1 = (-)^r 2 \cdot 4 \cdot ... \cdot 2r/1 \cdot 3 \cdot 5 \cdot ... \cdot (2r+1),$ 

Therefore

$$= (-)^{r} 2^{r} \cdot 1 \cdot 2 \cdot \dots \cdot r/1 \cdot 3 \cdot 5 \cdot \dots \cdot (2r+1).$$

$$= \frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{2}{5} \cdot \frac{3}{7} + \dots$$
(8).

Example 2. To sum the series

$$S_{-} = 1^{2} - 2^{2} + 3^{2} - \dots (-)^{n-1} n^{2}$$

Since 
$$\Delta u_{n+1} = 2n + 3$$
,  $\Delta u_1 = 3$ ,  
 $\Delta^2 u_{n+1} = 2$ ,  $\Delta^2 u_1 = 2$ ,  
 $\Delta^3 u_{n+1} = 0$ ,  $\Delta^3 u_1 = 0$ ,

 $\Delta^{\circ} u_{n+1} = 0$ , we have, by (6),

$$S_n = \frac{1}{2} \left\{ 1 - (-)^n (n+1)^2 \right\} - \frac{1}{4} \left\{ 3 - (-)^n (2n+3) \right\} + \frac{1}{8} \left\{ 2 - (-)^n 2 \right\},$$
  
=  $(-)^{n-1} \ln (n+1).$ 

# Exercises XXV.

(1.) Sum to n terms the series whose nth term is the nth r-gonal number\*.

Sum the following series to n terms, and, where possible, also to infinity:—

(2.)  $\Sigma n (n+2) (n+4)$ .

- (3.)  $\sum_{n=0}^{n} 1/(n^2-1)$ .
- (4.) 1/3.8+1/8.13+1/13.18+...
- (5.) 1/1.3.5+1/3.5.7+1/5.7.9+... (6.) 1/1.2.3.4+1/2.3.4.5+1/3.4.5.6+...
- (7.)  $\Sigma (an+b)/n (n+1) (n+2)$ .
- (8.) 1/1.3.5+2/3.5.7+3/5.7.9+...
- (9.) 1/1 · 2 · 4 + 1/2 · 3 · 5 + 1/3 · 4 · 6 + · · · ·
- (10.)  $1/1 \cdot 3 \cdot 7 + 1/3 \cdot 5 \cdot 9 + 1/5 \cdot 7 \cdot 11 + \dots$
- (11.)  $\Sigma (n+1)^2/n (n+2)$ .
- (12.) 4/1.3.5.7+9/2.4.6.8+16/3.5.7.9+...
- (13.)  $\Sigma \sec n\theta \sec (n+1) \theta$ . (14.)  $\Sigma \tan (\theta/2^n)/2^n$ .
- (15.)  $\sum \tan^{-1} \{(na-n+1) a^{n-1}/(1+n(n-1) a^{2n-1})\}.$
- (16.) ∑ tan<sup>-1</sup> {2/n<sup>2</sup>}.
- (17.)  $m! + (m+1)!/1! + (m+2)!/2! + \dots$ (18.)  $1!/m! + 2!/(m+1)! + 3!/(m+2)! + \dots$
- (18.) 1!/m! + 2!/(m+1)! + 3!/(m+2)! + ...

<sup>•</sup> The sums to n terms of arithmetical progressions whose first terms are all unity, and whose common differences are 0, 1, 2, ..., (r-1), ... respectively, are called the nth polygonal numbers of the 1st, 2nd, 3rd, ..., rth, ... order. The numbers of the first, second, third, fourth, ... orders are spoken of as linear, triangular, square, pentagonal, ... numbers.

(19.)  $1 - {}_{m}C_{1} + {}_{m}C_{2} - ... (-)^{n}{}_{m}C_{n}$ .

(20.) Show that the figurate numbers of a given rank can be summed by the formula of \$3. Example 1.

(21.) 
$$1 + \frac{1}{m} + \frac{1 \cdot 2}{m(m+1)} + \frac{1 \cdot 2 \cdot 3}{m(m+1)(m+2)} + \dots$$

(22.) 
$$\frac{a(a+1) - \dots (a+r)}{c} + \frac{a(a+1) \dots (a+r+1)}{c(c+1)} + \dots$$

(23.) 
$$\frac{a}{c(c+1)\dots(c+r)} + \frac{a(a+1)}{c(c+1)\dots(c+r+1)} + \dots$$

(24.) 
$$\sum (a+n)^{-m-2}/(c+n)^{-m}$$
.

(25.) 
$$\frac{1 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 5 \cdot 6} + \dots$$

$$(1+r)(1+2r) = (1+r)(1+2r)(1+3r)$$

$$(27.) \quad \frac{2}{1}m - \frac{2^2}{1-3}m(m-1) + \frac{2^3}{1-3}m(m-1)(m-2) - \dots$$

(29.) Show that

1 + 2(1-a) + 3(1-a)(1-2a) + ... + n(1-a)(1-2a) ... (1-(n-1)a) $= a^{-1} \{1 - (1 - a) (1 - 2a) \dots (1 - na)\},$ 

$$(30.) \quad \frac{1}{x+1} = \frac{1}{x-1} - \frac{2!}{(x-1)(x-2)} + \frac{3!}{(x-1)(x-2)(x-3)} - \dots$$

$$(x-1)(x-2) \dots \dots (x-n) \left(1 - \frac{n+1}{x+1}\right).$$

(31.) If a + b + 2 = c + d, then  $\frac{a}{a} \frac{a}{n} \frac{b}{b} \frac{n}{n} = \frac{ab}{(a+1)(b+1)-cd} \frac{y(a+1)}{c} \frac{n}{n} \frac{(b+1)^{(n)}}{d} = \frac{(a+1)^{(n+1)}(b+1)}{c^{(n+1)}(d-n)}$ 

$$1 - \frac{q \cdot r}{(p-q+1) \cdot (p+r-1)} + \frac{q \cdot (q-1) \cdot r \cdot (r-1)}{(p-q+2) \cdot (p+r-1) \cdot (p+r-2)} - \cdots$$

(Educational Times Reprint, vol. x11., p. 98.)

(33.) Transform the constion

by \$ 5, Cor. 2,

(34.) Show, by means of § 2, 1., that, if m be a positive integer, then

$$1 = {}_{m}C_{1}\frac{a}{b} + {}_{m}C_{2}\frac{a(a-1)}{b(b-1)} + {}_{m}C_{3}\frac{a(a-1)(a-2)}{b(b-1)(b-2)} + \dots$$

$$= \left(1 - \frac{a}{b}\right) \left(1 - \frac{a}{b-1}\right) \cdot \cdot \cdot \left(1 - \frac{a}{b-m+1}\right).$$

#### RECURRING SERIES.

§ 6.] We have already seen that any proper rational fraction such as  $(a + bx + cx^2)/(1 + px + qx^2 + rx^2)$  ° can always be expanded in an ascending series of powers of x. In fact, if |x| be less than the modulus of that root of  $rx^2 + qx^2 + px + 1 = 0$  which has the least modulus, we have (see chap. XXVII., § 6 and 7)

$$\frac{a + bx + cx^2}{1 + px + qx^2 + rx^3} = u_0 + u_1x + u_2x^2 + \dots + u_nx^n + \dots$$
 (1).

We propose now to study for a little the properties of the series (1).

If we multiply both sides of the equation (1) by  $1+px + qx^2 + rx^2$ , we have

$$a + bx + cx^2 = (1 + px + qx^2 + rx^2) (u_0 + u_1x + u_2x^2 + \dots + u_nx^n + \dots)$$
(2).

Hence, equating coefficients of powers of x, we must have

$$u_0 = a$$
 (3<sub>1</sub>);

$$u_1 + pu_0 = b$$
 (3<sub>2</sub>);  
 $u_1 + pu_1 + qu_0 = c$  (3<sub>3</sub>);

$$u_z + pu_2 + qu_1 + ru_0 = 0$$
 (34);

$$u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$$
 (3<sub>n+1</sub>).

Any power-series which has the property indicated by the equation  $(3_{n+1})$  is scalled a Recurring Power-Series  $\dagger$ ; and the equation  $(3_{n+1})$  is spoken of as its Scale of Relation, or, briefly, its Scale. The quantities p, q, r, which are independent of n, may be called the Constants of the Scale. According as the scale has  $1, 2, 3, \ldots, r, \ldots$  constants, the recurring series is said to be of the 1st, 2nd, 3rd, ..., rth, ... order. When x = 1, so that we have simply the series  $u_0 * u_1 * u_2 * u_3 * u_4 * u_$ 

For simplicity, we confine our exposition to the case where the demoninator is of the 3rd degree; but all our statements can at once be generalised.

<sup>+</sup> The theory of Recurring Series was originated and largely developed by Demoivre.

the series as a recurring series simply •; so that every recurring series may be regarded as a particular case of a recurring power-series.

It is obvious from our definition that all the coefficients of a recurring power-series of the rth order can be calculated when the values of the first r are given and also the constants of its scale. Hence a recurring series of the rth order depends upon 2r constants; namely, the r constants of its scale, and r others.

From this it follows that if the first 2r terms of a series (and these only) be given, it can in general be continued as a recurring series of the rth order, and that in one way only; as a recurring series of the (r+1)th order in a two-fold infinity of ways; and so on

On the other hand, if the first 2r terms of the series be given, two conditions must be satisfied in order that it may be a recurring series of the (r-1)th order; four in order that it may be a recurring series of the (r-2)th order; and so on.

Example. Show that

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \dots$$

is a recurring series of the 2nd order. Let the scale be  $u_n + pu_{n-1} + qu_{n-2} = 0$ . Then we must have

3+2p+q=0, 4+3p+2q=0, 5+4p+3q=0, 6+5p+4q=0.

The first two of these equations give p=-2, q=+1; and these values are consistent with the remaining two equations. Hence the theorem,

§ 7.] The rational fraction  $(a+bx+cx^2)(1+px+qx^2+rx^2)$ , of which the recurring power-series  $u_a \cdot u_1x + u_2x^2 + \dots$  is the development when |x| is less than a certain value, is called the Generating Function of the series. We may think of the series and its generating function without regarding the fact that the one is the equivalent of the other under certain restrictions. If we take this view, we must look at the denominator of the function as furnishing the scale, and consider the coefficients

We might of course to end a recurring power-series as a particular case of a recurring series in general. Thus, if we put \$U\_n = u\_n x^n\$, we might regard the series in (1) as a recurring series whose scale is.

 $U_n + PxU_{n-1} + qx \cdot U_{n-2} + rx \cdot U_{n-3} = 0$ 

as determined by the equations  $(3_1)$ ,  $(3_2)$ , . . . ,  $(3_{n+1})^*$ . No question then arises regarding the convergence of the series.

Given the scale and the first r terms of a recurring powerseries of the rth order, we can always find its generating function.

Taking the case r = 3, we see, in fact, from the equations  $(3_i)$ ,  $(3_2)$ , . . .  $(3_{n+1})$ , . . . of  $\S$  6, that

 $\{u_0 + (u_1 + pu_0)x + (u_2 + pu_1 + qu_0)x^2\}/\{1 + px + qx^2 + rx^2\}$ 

is the generating function of the series  $u_0 + u_1x + u_2x^2 + \dots$ whose scale is  $u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0$ .

Cor. 1. Every recurring power-series may, if |x| be small enough, be regarded as the expansion of a rational fraction.

Cor. 2. The general term of any recurring series can always be found when its scale is given and a sufficient number of its initial terms.

For we can find the generating function of the series itself or of a corresponding power-series; decompose the generating function into partial fractions of the form  $A(x-a)^{-\delta}$ ; expand each of these in ascending powers of x; and finally collect the coefficient of  $x^n$  from the several expansions.

Example. Find the general term of the recurring series whose scale is  $u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0$ , and whose first three terms are 1 + 0 - 5. Consider the corresponding power-series. Here p = -4, q = 5, r = -2; so that  $a = u_0 = 1$ ,  $b = u_1 + pu_0 = -4$ ,  $c = u_2 + pu_1 + qu_0 = 0$ .

The generating function is therefore

$$\frac{1-4x}{1-4x+5x^2-2x^3} \stackrel{\boxtimes}{=} \frac{1-4x}{(1-x)^2(1-2x)},$$

$$\stackrel{\boxtimes}{=} \frac{2}{1-x} + \frac{4}{(1-x)^2} - \frac{4}{(1-2x)},$$

Expanding, we have

$$\frac{1 - 4x}{1 - 4x + 5x^2 - 2x^2} = 2\{1 + \Sigma x^n\} + 3\{1 + \Sigma(n+1)x^n\} - 4\{1 + \Sigma 2^n x^n\},$$

$$= 1 + \Sigma 13n + 5 - 2^{n+2}\}x^n.$$

The general term in question is therefore  $3n + 5 - 2^{n+2}$ .

§ 8.] If  $u_n$  be any function of an integral variable u which satisfies an equation of the form

$$u_n + pu_{n-1} + qu_{n-2} + ru_{n-3} = 0,$$

or, what comes to the same thing,

$$u_{n+3} + pu_{n+2} + qu_{n+1} + ru_n = 0$$
 (1),

<sup>.</sup> We might also regard the series as deduced from the generating function by the process of ascending continued division (see chap. v., § 20).

we see from the reasoning of last paragraph that  $u_n$  is uniquely determined by the equation (1), provided its three initial values  $u_e$ ,  $u_1$ ,  $u_2$  are given; and we have found a process for actually determining  $u_n$ .

It is not difficult to see that we might assign any three values of  $u_n$  whatever, say  $u_n$ ,  $u_s$ ,  $u_s$ ,  $u_s$ , and the solution would still be determinate. We should, in fact, by the process § 7, determine  $u_n$  as a function of n linearly involving three arbitrary constants  $u_s$ ,  $u_1$ ,  $u_2$ , say  $f(u_s, u_1, u_2, n)$ ; and  $u_s$ ,  $u_1$ ,  $u_2$  would be uniquely determined by the three linear equations

 $f(u_0, u_1, u_2, \alpha) = u_\alpha, f(u_0, u_1, u_2, \beta) = u_\beta, f(u_0, u_1, u_2, \gamma) = u_\gamma$  (2).

An equation such as (1) is called a Linear Difference-Equation of the 3rd order with constant coefficients; and we see generally that a linear difference-quation of the rth order with constant coefficients has a unique solution when the values of the function involved are given for r different values of its integral argument.

Example. Find a function  $u_n$  such that  $u_{n+2} - 4u_{n+2} + 5u_{n+1} - 2u_n = 0$ ; and  $u_0 = 1$ ,  $u_1 = 0$ ,  $u_2 = -5$ .

We have simply to repeat the work of the example in § 7.

 $\S 9.]$  To sum a recurring series to n+1 terms, and (when convergent) to infinity.

Taking the case of a power-series of the 3rd order, let  $S_n = u_n + u_1 x + u_2 x^2 + \dots + u_n x^n$ .

then

 $pxS_n = pu_ox + pu_1x^2 + \dots + pu_{n-1}x^n + pu_nx^{n+1},$   $qx^2S_n = qu_ox^2 + \dots + qu_{n-2}x^n + qu_{n-1}x^{n+1} + qu_nx^{n+2},$ 

 $rx^{p}S_{n}=$  ...  $+ru_{n-2}x^{n}+ru_{n-2}x^{n+1}+ru_{n-1}x^{n+2}+ru_{n}x^{n+3}$ Hence adding, and remembering that  $u_{n}+pu_{n-1}+qu_{n-2}$ 

 $+ ru_{n-3} = 0$  for all values of *n* which exceed 2, we have  $(1 + px + qx^2 + rx^3) S_n = u_0 + (u_1 + pu_0) x + (u_2 + pu_1 + qu_0) x^3$ 

+  $(pu_n+qu_{n-1}+ru_{n-2})x^{n+1}$ +  $(qu_n+ru_{n-1})x^{n+2}+ru_nx^{n+2}$  (1); where  $S_n$  can in general be at once determined by dividing by  $1+px+qx^2+rx^2$ .

The only exceptional case is that where for the particular value of x in question, say x , a, it happens that

$$1 + p\alpha + q\alpha^2 + r\alpha^3 = 0,$$

In this case the right hand of (1) must, of course, also vanish, and  $S_n$  takes the indeterminate form 0/0.  $S_n$  may in cases of this kind be found by evaluating the indeterminate form by means of the principles of chap. xxv. This, however, is often much more troublesome than some more special process applicable to the particular case.

If the series  $\Sigma u_n x^n$  be convergent, then  $Lu_n x^n = 0$  when  $n = \infty$ ; therefore the last three terms on the right of (1) will become infinitely small when  $n = \infty$ . We therefore have for the sum to infinity in any case where the series is convergent

$$S_{\infty} = \frac{u_{0} + \left(u_{1} + pu_{0}\right) x + \left(u_{2} + pu_{1} + qu_{0}\right) x^{2}}{1 + px + qx^{2} + rx^{2}} \tag{2}.$$

The particular cases

$$u_0 + u_1 + u_2 + \dots + u_n + \dots$$
 (3),

$$u_0 - u_1 + u_2 - \dots + (-)^n u_n + \dots$$
 (4),

are of course deducible from (1) and (2) by putting x=+1 and x=-1. Exceptional cases will arise if 1+p+q+r=0, or if 1-p+q-r=0.

It is needless to give an example of the above process, for Examples 1 and 2, chap.  $XX_n \$  14, are particular instances,  $\Sigma n^2 x^n$  and  $1 + \Sigma (-)^{n-1} 2n x^n$  being, in fact, recurring series whose are  $u_n - 3u_{n-1} + 3u_{n-2} - u_{n-3} = 0$  and  $u_n + 2u_{n-1} + u_{n-2} = 0$  respectively.

#### Exercises XXVI.

Sum the following recurring series to n+1 terms, and, where admissible, to infinity:—

- (1.) 2+5+13+35+97+...
  - (2.) 2+10+12-24+2+10+12+...
- (3.)  $2 + 17x + 95x^2 + 461x^3 + \dots$
- (4.)  $5+12x+30x^2+78x^3+210x^4+\cdots$ (5.)  $1+4x+17x^2+76x^3+353x^4+\cdots$
- (6.)  $1 + 4x + 10x^2 + 22x^3 + 46x^4 + \cdots$
- (7.) If a series has for its rth term the sum of r terms of a recurring series, it will itself be a reenring series with one more term in the scale of relation.

Find the sum of the series whose rth term is the sum of r terms of the recurring series 1+6+40+258+...

(8.) If  $T_n$ ,  $T_{n+1}$ ,  $T_{n+2}$  be consecutive terms of the recurring series whose scale is  $T_{n+2} = aT_{n+1} - bT_n$ , then

 $(T_{n+1})^2 - aT_nT_{n+1} + bT_n^2)/(T_{n-r+1}^2 - aT_{n-r}T_{n-r+1} + bT_{n-r}^2) = b^r$ .

(9.) Form and sum to n terms the series each term in which is half the difference of the two preceding terms.

(10.) Show that every integral series (chap. xx., § 4) is a recurring series; and show how to find its scale,

(11.) If  $u_n = u_{n-1} + u_{n-2}$ , and  $u_2 = \alpha u_1$ , show that

$$u_{-}^{2} - u_{-+1}u_{--1} = (-)^{n}(\alpha^{2} - \alpha - 1)u_{1}^{2}$$

(12.) If the series  $u_1, u_2, u_3, \ldots, u_n, \ldots$  be such that in every four consecutive terms the sum of the extremes exceeds the sum of the means by a constant quantity c, find the law of the series; and show that the sum of 2m terms is

$$\frac{1}{6}m(m-1)(4m-5)c-m(m-2)u_1+mu_2+m(m-1)u_3.$$

(13.) If  $u_{n+2} = u_{n+1} + u_n$ ,  $u_1 = 1$ ,  $u_2 = 1$ , sum the series

$$\frac{1}{1 \cdot 2} + \frac{2}{1 \cdot 3} + \dots + \frac{u_{n+2}}{u_{n+1}u_{n+3}}.$$

(14.) By French law an illegitimate child receives one-third of the portion of the inheritance that he would have received had be been legitimate. If there be I legitimate and n illegitimate children, show that the portion of inheritance 1 due to a legitimate child is

$$\frac{1}{l} - \frac{n}{3l(l+1)} + \frac{n(n-1)}{3^{2}l(l+1)(l+2)} - \dots (-) \frac{n(n-1)\dots 2.1}{3^{n}l(l+1)\dots (l+n)}.$$
(Catalan, Now. Ann., ser. 11., t. 2.)

SIMPSON'S METHOD FOR SUMMING THE SERIES FORMED BY TAKING EVERY &TH TERM FROM ANY POWER-SERIES WHOSE SUM IS KNOWN

§ 10.] This method depends on the theorem that the sum of the pth powers of the kth roots of unity is k if p be a multiple of k, but otherwise zero.

This is easily seen to be true; for, if  $\omega$  be a primitive kth root of 1, then the k roots are  $\omega^0$ ,  $\omega^1$ ,  $\omega^2$ , . . . ,  $\omega^{k-1}$ . If  $p - \mu k$ , then  $(\omega^*)^p = \omega^{nk} = (\omega^k)^{n^*}$ . If p be not a multiple of k, then we have

$$(\omega^{c})^{p} + (\omega^{l})^{p} + \dots + (\omega^{k-1})^{p} = 1 + (\omega^{p})^{1} + (\omega^{p})^{2} + \dots + (\omega^{p})^{k-1},$$
  
 $- \frac{1}{2} 1 - (\omega^{c})^{k}, (1 - \omega^{p}),$   
 $= 0.$ 

for  $(\omega^{\dagger})^k = (\omega^k)^k = 1$ , and  $\omega^j + 1$ .

Let us suppose now that f(x) is the sum of n terms of the power-series  $u_0 + \Sigma u_n x^n$ , n being finite, or, it may be, if the series is convergent, infinite.

Consider the expression

 $U_m \equiv \underbrace{(\omega^0)^{k-m} f(\omega^0 x) + (\omega^1)^{k-m} f(\omega^1 x) + (\omega^2)^{k-m} f(\omega^0 x) + \ldots + (\omega^{k-1})^{k-m} f(\omega^{k-1} x)}_{k}$ (1).

where m is 0 or any positive integer < k.

The coefficient of x" in the equivalent series is

$$u_r \{ (\omega^{\circ})^{k-m+r} + (\omega^{\circ})^{k-m+r} + (\omega^{\circ})^{k-m+r} + \ldots + (\omega^{k-1})^{k-m+r} \}, \ k = (2).$$

Now, by the above theorem regarding the kth roots of unity, the quantity within the crooked brackets vanishes if k-m+r be not a multiple of k, and has the value k if k-m+r be a multiple of k. Therefore we have

$$U_m = u_m x^m + u_{m+k} x^{i_{1+k}} + u_{m+2k} x^{i_{1+2k}} + ...$$
 (3),

where the series extends until the last power of x is just not higher than the nth, and, in particular, to infinity if f(x) be a sum to infinity.

If we put m = 0, we get

$$\{f(x) + f(\omega^{1}x) + f'(\omega^{2}x) + \dots + f'(\omega^{k-1}x)\} \ k$$
  
=  $u_{\alpha} + u_{k}x^{k} + u_{\alpha k}x^{2k} + u_{\alpha k}x^{2k} + \dots$ , (4).

Example 1.

$$1 + x + x^2 + ... + x^n = (1 - x^{n+1}) (1 - x).$$

Hence, if  $\omega$  be a primitive cube root of 1, we have

$$1 + x^3 + x^6 + \dots + x^{3s} = \left\{ \begin{cases} \frac{1 - x^{n+1}}{1 - x} + \frac{1 - \omega^{n+1} x^{n+1}}{1 - \omega x} + \frac{1 - \omega^{2n+2} x^{n+1}}{1 - \omega x} \end{cases} \right\},$$

where 3s is the greatest multiple of 3 which does not exceed n.

Example 2. To sum the series

$$\frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \dots$$
 and  $\infty$ .

<sup>\*</sup> This method was given by Thomas Simpson, Phil. Trans. R. S. L. Kov. 16, 1758 (see De Morçan's Trigonometry and Double Algebra (1819), p. 159). It was used apparently independently by Waring (see Phil. Trans. R. S. L. 1784).

We have

$$e^x = 1 + x + \frac{x^2}{21} + \frac{x^3}{21} + \dots$$
 and  $x$ .

Hence, if  $\omega$  be a primitive 4th root of unity, say  $\omega = i$ , then, since here k = 4, m = 3, k - m = 1,  $\omega^2 = -1$ ,  $\omega^3 = -i$ , we get

$$\frac{1}{4} \left( e^x + ie^{4x} - e^{-x} - ie^{-4x} \right) = \frac{x^3}{3!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \dots$$

$$\frac{1}{4} \left( \sinh x - \sin x \right) = \frac{x^3}{7!} + \frac{x^7}{7!} + \frac{x^{11}}{11!} + \dots$$

that is,

### MISCELLANEOUS METHODS.

§ 11.] When the nth term of a series is a rational fraction, the unite summation may often be effected by merely breaking up this term into its constituent partial fractions; and even when summation cannot be effected, many useful transformations can be thus obtained. In dealing with infinite series by this method, close attention must be paid to the principles laid down in chap. XXVI., especially § 13; otherwise the tyro may easily fall into mistakes. As an instance of this method of working, see chap. XXVIII., § 14, Examples 1 and 2.

Example 1. Show that

$$\begin{cases} \frac{1}{(\bar{x}+1)^2(r+2)} + \frac{1}{(\bar{x}+2)^2(r+3)} + \frac{1}{(\bar{x}+3)^2(r^2+1)} + \cdots \\ + \frac{1}{(\bar{x}+1)(r+2)^2} + \frac{1}{(\bar{x}+2)^2(r+3)^2} + \frac{1}{(\bar{x}+3)(\bar{x}+4)^2} + \cdots \end{cases} \} = \frac{1}{(\bar{x}+1)^2}$$

Denote the sums of n terms of the two given series by  $S_n$  and  $T_n$  respectively, and their nth terms by  $u_n$  and  $v_n$  respectively. Then

$$u_n = -1 (x+n) \cdot 1/(x+n)^2 + 1/(x+n+1);$$
  
 $v_n = 1/(x+n) - 1/(x+n+1)^2 - 1/(x+n+1).$ 

Whence we get at once

$$S_n + T_n = 1/(x+1)^2 - 1/(x+n+1)^2$$
,  
 $S_n + T_n = 1/(x+1)^2$ ,

Therefore

Example 2. Resolution into partial fractions will always effect the

Example 2. Resolution into partial fractions will always effect th summation of the series

$$\sum_{i=1}^{n} f(n)/(n+a)(n+b)$$
 . . .  $(n+k)$ 

where  $a, b_1, \ldots, k$  are positive or negative integers, and f(n) is an integral function of n whose degree is less by two at least than the degree of  $(n+a)/(n+b) + \cdots + (n+k)$ .

For we have

f(n)/(n+a) (n+b) . . . (n+k)  $\equiv \Sigma A/(n+a)$ ,

hna

$$f(n) \equiv \sum A(n+b)(n+c) \dots (n+k)$$

Since the degree of f(n) is less by one at least than the degree of the right-hand side of this last identity, we must have

$$A + B + ... + K = 0$$
.

But, since  $a, b, \ldots, k$  are all integral, any partial fraction whose denominator p is neither too small nor too great will occur with all the numerators  $A, B, \ldots, K$ , so that we shall have  $A/p + B/p + \ldots + K/p = 0$ .

On collecting all the fractions belonging to all the terms of the series we shall be left with a certain number at the beginning and a certain number at the end; so that the sum will be reduced to a closed function of n.

 $\ 12.]$   $Euler's\ Identity.$  The following obvious identity\*

$$1 - a_1 + a_1 (1 - a_2) + a_1 a_2 (1 - a_3) + \dots + a_1 a_2 \dots a_n (1 - a_{n+1})$$

$$= 1 - a_1 a_2 \dots a_{n+1} \quad (1)$$

is often useful in the summation of series. It contains, in fact, as particular cases a good many of the results already obtained above.

If in (1) we put

$$a_1 = \frac{x}{y}$$
,  $a_2 = \frac{x + p_1}{y + p_1}$ ,  $a_3 = \frac{x + p_2}{y + p_2}$ , ...,  $a_{n+1} = \frac{x + p_n}{y + p_n}$ ,

and multiply on both sides by y/(y-x), we get

$$1 + \frac{x}{y + p_1} + \frac{x(x + p_1)}{(y + p_1)(y + p_2)} + \dots + \frac{x(x + p_1) \dots (x + p_{n-1})}{(y + p_1)(y + p_2) \dots (y + p_n)}$$

$$= \frac{y}{y - x} - \frac{x}{y - x} - \frac{(x + p_1)(x + p_2) \dots (x + p_n)}{(y + p_1)(y + p_2) \dots (y + p_n)}$$

$$(2).$$

If the quantities involved be such that

$$L_{n=\infty} \frac{(x+p_1)(x+p_2)...(x+p_n)}{(y+p_1)(y+p_2)...(y+p_n)} = 0$$
(3)

then

1 + 
$$\frac{x}{y+p_1}$$
 +  $\frac{x(x+p_1)}{(y+p_1)(y+p_2)}$  + . . . ad  $\infty = \frac{y}{y-x}$  (4)

Used in the slightly different form,

 $<sup>(1 +</sup> a_1) (1 + a_2) (1 + a_3) (1 + a_4) . . .$ 

by Euler, Nov. Comm. Petrop. (1760).

If in (2) we put y = 0, we get

$$1 + \frac{x}{p_1} + \frac{x(x + p_1)}{p_1 p_1} + \dots + \frac{x(x + p_1) \dots (x + p_{n-1})}{p_1 p_2 \dots p_n} = \left(1 + \frac{x}{p_1}\right) \left(1 + \frac{x}{p_2}\right) \dots \left(1 + \frac{x}{p_n}\right) (5).$$

From (5) a variety of particular cases may be derived by putting n=x, and giving special values to  $p_1, p_2, \ldots$  Thus, for instance, if the infinite series  $\Sigma 1/p_n$  diverge to  $+\infty$ , then (see chap. xxvi.,  $\S 24$ ) we have

$$1 = \frac{x}{p_1} + \frac{x(x - p_1)}{p_1 p_2} - \dots \text{ ad } x = 0$$
 (6).

In general, if the continued product  $\Pi(1+x/p_n)$  converge to any

definite limit, then the series  $1 + \sum_{1}^{\infty} x(x+p_1) \dots (x+p_{n-1})/p_1p_2 \dots p_n$ converges to the same limit.

Example. Find when the infinite series

$$S \equiv 1 + \frac{x}{y+p} + \frac{x(x+p)}{(y+p)(y+2p)} + \frac{x(x+p)(x+2p)}{(y+p)(y+2p)(y+3p)} + \dots$$
 (7)

converges, and the limit to which it converges.

If in (2) above we put 
$$p_1 = p$$
,  $p_2 = 2p$ , &c., . . . , we have
$$S = \frac{y}{y - x} - \frac{x}{y - x} \frac{L}{n - x} \frac{(x + p)(x + 2p) \cdot . \cdot (x + np)}{(y + p)(y + 2p) \cdot . \cdot \cdot (y + np)}$$
(8).

Now the limit in question may be written

$$\prod_{i=1}^{\infty} \left\{ 1 + \frac{(x-y)/np}{1+y/np} \right\};$$

but this diverges to x if (x-y)/p be positive, and converges to 0 if (x-y)/p be negative (chap. xxvi., § 21).

Hence, if p denote in all cases a positive quantity, we see that

$$1 + \frac{x}{y+p} + \frac{x(x+p)}{(y+p)(y+2p)} + \dots \text{ ad } x = \frac{y}{y-x}.$$

if y > x; and x = x(x-y)

$$1 + \frac{x}{y - p} + \frac{x(x - p)}{(y - p)(y - 2p)} + \dots \text{ ad } x = \frac{y}{y - x},$$

if y < x.

#### EXERCISES XXVII.

(1.) Given  $1/(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + \dots$ sum  $1 + 4x^3 + 7x^6 + 10x^9 + \dots$ 

(2.) Sum the series 
$$1 + x^3/1 + x^6/7 + \dots ;$$

$$1 + x^4/3! + x^6/6! + \dots ;$$

(3.) If  $f(x) = u_0 + u_1x + u_2x^2 + \dots$ , and  $\alpha, \beta, \gamma, \dots$  be the nth roots of -1, show that

$$\frac{1}{n} \{ a^{2n-m} f(\alpha x) + \beta^{cn-m} f(\beta x) + \dots \} = u_{mc} r^m - u_{m+n} x^{m+n} + u_{m+2n} x^{m+2n} - \dots$$
where  $m < n$ . (De Morgan,  $D(\beta, Calc., p. 319 (1839).)$ )

Sum the following series, and point out the condition for convergency when the summation extends to infinity:-

- (4.) 1 − x³/4 + x6/7 − . . . ad ∞ ;
- $x = x^4/4! + x^7/7! \dots$  ad  $\infty$ . (5.) 1+mC3+mC6+mC9+... ad x:
- $1 {}_{m}C_{o} + {}_{m}C_{o} {}_{m}C_{o} + \dots$  ad  $\infty$ .
  - (6.) 1/1.3+1/1.2.4+1/1.2.3.5+... to n terms.
- (7.) 1/1,  $2 \cdot 3 + mC_1/2 \cdot 3 \cdot 4 + mC_2/3 \cdot 4 \cdot 5 + ... ad <math>\infty$ .
- (8.)  $1 2x/1 + 3x^2/2 4x^3/3 + \dots$  ad  $\infty$ .
- (9.) cos θ/1, 2, 3 + cos 2θ/2, 3, 4 + cos 3θ/3, 4, 5 + . . . ad ∞.
- (10.)  $1/1^2$ ,  $2^2 + 7/2^2$ ,  $3^2 + \dots + (2n^2 + 4n + 1)/(n + 1)^2 (n + 2)^2$ .
- (11.)  $1/1^2 \cdot 2^2 1/2^2 \cdot 3^2 + \dots + (-)^{n-1} 1/n^2 (n+1)^2 + \dots$  ad  $\infty$ .
- (12.) If n be a positive integer, show that

 $\frac{n}{m+n} + \frac{1}{2} \frac{n(n-1)}{(m+n)(m+n-1)} + \frac{1}{3} \frac{n(n-1)(n-2)}{(m+n)(m+n-1)(m+n-2)} + \dots$ 

$$=\frac{n}{m+1}-\frac{1}{2}\frac{n(n-1)}{(m+1)(m+2)}+\frac{1}{3}\frac{n(n-1)(n-2)}{(m+1)(m+2)(m+3)}-\ldots$$

(13.) Show that

$$\frac{{}_{n}C_{1}}{1-x/1} - \frac{{}_{n}C_{2}}{(1-x/1)(1-x/2)} + \frac{{}_{n}C_{3}}{(1-x/1)(1-x/2)(1-x/3)} - \dots = \frac{n}{n-x};$$
 and hence show that

$$_{n}C_{1}\sigma_{1} - _{n}C_{2}\sigma_{2} + ... (-)^{n}{}_{n}C_{n}\sigma_{n} = 1/n,$$
  
where  $\sigma_{r} = 1/1 + 1/2 + ... + 1/r.$ 

(14.) Sum the series

$$\begin{aligned} &1 - \frac{m^2}{1^2} + \frac{m^2 \left(m^2 - 1^2\right)}{1^2 \cdot 2^2} - \frac{m^2 \left(m^2 - 1^2\right) \left(m^2 - 2^2\right)}{1^2 \cdot 2^2 \cdot 3^2} + \dots \text{ ad } \varpi; \\ &1 + \frac{m^2}{1^2} + \frac{m^2 \left(m^2 + 1^2\right)}{1^2 \cdot 3^2} + \frac{m^2 \left(m^2 + 1^2\right) \left(m^2 + 3^2\right)}{1^2 \cdot 3^2 \cdot 5^2} + \dots \text{ ad } \varpi. \end{aligned}$$

(15.) Show that

$$\begin{split} \frac{a_1}{a_1+p_1} + \frac{p_1 a_2}{(a_1+p_1)} & + \frac{p_1 p_2 a_3}{(a_1+p_2)} + \frac{p_1 p_2 a_3}{(a_1+p_2)} (a_2+p_2) (a_2+p_3) + \cdots \\ & + \frac{p_1 p_2 \dots p_{n-1} a_n}{(a_1+p_1)(a_2+p_2) \dots (a_n+p_n)} = 1 - \frac{p_1 p_2 \dots p_n}{(a_1+p_2)(a_2+p_2) \dots (a_n+p_n)} \end{split}$$

(16.) Show that

$$\tan^2 \frac{1}{2} \pi x = \frac{1^4 - (1^2 - x^2)^2}{(1^2 - x^2)^2} + \frac{3^4 - (3^2 - x^2)^2}{(1^2 - x^2)^2(3^2 - x^2)} + \dots$$
(Glaisher, Math. Mess., 1873, p. 138.)

(17.) Show that

$$\frac{1}{n^2} = \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \frac{1 \cdot 2}{n(n+1)(n+2)(n+3)} + \dots;$$

and apply this result to the approximate calculation of \(\pi^2\) by means of the formula  $\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + \dots$ 

- (18) Show that  $\Sigma 1/(m^n-1)=1$  and  $\Sigma 1/(a^n-1)=\log 2$ , where m and n have all possible positive integral values differing from unity, a is any even positive integer, and each distinct fraction is counted only once, (Goldbach's Theorem, see Liour, Math. Jour., 1842.)
- (19.) If a have any positive integral value except unity, and r be any positive integer which is not a perfect power, show that  $\sum (n-1)/(r^n-1)$  $=\pi^{1}/6$ ; and, if d(n) denote the number of divisors of n, that  $\Sigma (d(n)-1)/r^{n}$ =1: also that  $\Sigma (n-1)/r = \Sigma 1/(r-1)^2$ . (16.)

# CHAPTER XXXII.

## Simple Continued Fractions.

NATURE AND ORIGIN OF CONTINUED FRACTIONS.

§ 1.] By a continued fraction is meant a function of the form

$$a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_4 + \frac{b_4}{a_4}}}$$
 (1):

the primary interpretation of which is that  $b_2$  is the antecedent of a quotient whose consequent is all that lies under the line immediately beneath  $b_2$ , and so on.

There may be either a finite or an infinite number of links in the chain of operations; that is to say, we may have either a terminating or non-terminating continued fraction.

In the most general case the component fractions  $\frac{b_2}{a_2}$ ,  $\frac{b_3}{a_1}$ , ..., as they are sometimes called, may have either positive or negative numerators and denominators, and succeed each other

without recurrence according to any law whatever. If they do recur, we have what is called a recurring or periodic continued fraction.

For shortness, the following abbreviative notation is often used instead of (1),

 $a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \frac{b_4}{a_4 + 1}}} \cdot \cdot \cdot$  (2),

the signs + being written below the lines to prevent confusion with

$$a_1 + \frac{b_2}{a_2} + \frac{b_3}{a_2} + \frac{b_4}{a_4} + \dots$$

Examples have already been given (see chap. III., Exercises III., 15) of the reduction of terminating continued fractions; and from these examples it is obvious that every terminating continued fraction whose constituents  $a_1, a_2, \ldots, b_2, b_3, \ldots$  are communicated numbers reduces to a communicated number of the second continued fraction.

§ 2.] In the present chapter we shall confine ourselves mainly to the most interesting and the most important kind of continued fraction, that, namely, in which each of the numerators of the component fractions is +1, and each of the denominators a positive integer. When distinction is necessary, this kind of continued fraction, namely,

$$a_1 + \frac{1}{a_2 + a_3 + a_4 + \dots}$$
 (1),

may be called a *simple continued fraction*. Unless it is otherwise stated, we suppose the continued fraction to terminate.

In this case, for a reason that will be understood by and by, the numbers  $a_1, a_2, a_5, \ldots$  are called the first, second, third,  $\ldots$  partial quotients of the continued fraction.

§ 3.] Every number, commensurable or incommensurable, may be expressed uniquely as a simple continued fraction, which may or may not terminate.

For, let X be the number in question, and  $a_1$  the greatest integer which does not exceed X; then we may write

$$X = a_1 + \frac{1}{X},\tag{1},$$

where  $\mathcal{N}_i > 1$ , but is not necessarily integral, or even commensurable.

Again, let  $a_2$  be the greatest integer in  $X_1$ , so that  $a_2 < 1$ ; then we have

$$X_1 = a_9 + \frac{1}{X_9} \tag{2},$$

where  $\mathcal{N}_2 > 1$ , as before.

The notation  $a_1 + \frac{b_1}{a_2} + \frac{b_3}{a_3} + \frac{b_4}{a_4} + \dots$ , is frequently used by Continental writers.

Again, let  $a_3$  be the greatest integer in  $X_2$ ; then

$$X_2 = \alpha_3 + \frac{1}{X_2}$$
 (3)

and so on.

This process will terminate if one of the quantities V, say  $X_{n-1}$ , is an integer; for we should then have

$$X_{n-1} = a_n$$
 (n)

Now, using (2), we get from (1)

$$X = a_1 + \frac{1}{a_2 + \frac{1}{X_2}}$$
.

Thence, using (3), we get

$$X = a_1 + \frac{1}{a_2 + \frac{1}{a_4 + \frac{1}{V_*}}};$$

and so on.

Finally, then,

$$X = a_1 + \frac{1}{a_2 + a_3 +} \cdot \cdot \cdot \frac{1}{a_n}$$
 (a).

It may happen that none of the quantities  $\mathcal{X}$  comes out mtegral. In this case, the quotients  $a_1, a_2, \ldots$  either recur, or go on continually without recurrence; and we then obtain in place of (a) a non-terminating continued fraction, which may be periodic or not according to circumstances.

To prove that the development is unique, we have to show that, if

$$a_1 + \frac{1}{a_2 + 1} \frac{1}{a_3 + 1} \dots = a_1' + \frac{1}{a_2' + 1} \frac{1}{a_3' + 1} \dots$$
 ( $\beta$ )

then  $a_1 = a_1'$ ,  $a_2 = a_2'$ ,  $a_3 = a_3'$ , &c.

Now, since  $a_2$  and  $a_2'$  are positive integers, and  $\frac{1}{a_2+}$  . . . and  $\frac{1}{a_2'+}$  . . . are both positive, it follows that  $\frac{1}{a_2+}\frac{1}{a_3+}$  . . . and  $\frac{1}{a_2'+}$  . . . are both proper fractions. Hence, by chap. III., § 12,

we must have

$$a_1 = a_1'$$
  $(\gamma)$ ,

and

$$\frac{1}{a_1} \cdot \frac{1}{a_1} + \dots = \frac{1}{a_2} \cdot \frac{1}{a_3} + \dots$$
 (8).

Again, from  $(\delta)$ , we have

$$a_1 + \frac{1}{a_2 + a_3 + a_4} + \dots + a_2' + \frac{1}{a_2' + a_3 + a_4} + \dots$$
 (c).

From (c), by the same reasoning as before, we have

$$a_2 - a_2'$$
 (ζ),

and

and 
$$\frac{1}{a_1} + \frac{1}{a_4} + \frac{1}{a_5} + \dots = \frac{1}{a_1} + \frac{1}{a_4} + \frac{1}{a_5} + \dots$$
 (9).

Proceeding in this way, we can show that each partial quotient in the one continued fraction is equal to the partial quotient of the same order in the other\*.

This demonstration is clearly applicable even when the continued fraction does not terminate, provided we are sure that the fractions in  $(\beta)$ ,  $(\delta)$ , (n),  $\delta c$ , have always a definite meaning. This point will be settled when we come to discuss the question of the convergency of an infinite continued fraction.

Cor. If  $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$  be all positive integers,  $x_{n+1}$  and  $y_{n+1}$  any positive quantities rational or irrational each of which is greater than unity, and if

$$a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + x_{n+1}}} = b_1 + \frac{1}{b_2 + \dots + \frac{1}{b_n + y_{n+1}}},$$

then must

$$a_i = b_1$$
,  $a = b_2$ , . . . ,  $a_n = b_n$ , and also  $x_{n+1} = y_{n+1}$ .

§ 4.] As an example of the general proposition of § 3, we may show that every commensurable number may be converted into a terminating continued fraction.

Let the number in question be A B, where A and B are integers prime to each other. Let a, be the quotient and C the remainder when A is divided by B;  $a_2$  the quotient and D the

<sup>.</sup> We suppose, as is clearly allowable, that, if the fraction terminates, the last quotient is -1. It should also be noticed that the first partial quotient may be zero, but that none of the others can be zero, as the process is arranged above.

remainder when B is divided by C;  $a_s$  the quotient and E the remainder when C is divided by D; and so on, just as in the arithmetical process for finding the G.C.M. of A and B. Since A and B are prime to each other, the last divisor will be 1, the last quotient  $a_n$ , say, and the last remainder 0. We then have

$$\begin{split} \frac{A}{B} & = a_1 + \frac{C}{B} = a_1 + \frac{1}{B_1^2 C}; \\ \frac{B}{C} & = a_2 + \frac{D}{C} - a_2 + \frac{1}{C/D}; \\ \frac{C}{C} & = a_3 + \frac{E}{D} - a_3 + \frac{1}{D_1^2 E}; \\ \frac{A}{B} & = a_1 + \frac{1}{a_2 + a_3 + a_4} + \frac{1}{a_4} + \frac{1}{a_4}. \end{split}$$

Hence

It should be noticed that, if A = B, the first quotient  $a_1$  will be zero.

Example 1.

To convert 167/81 into a continued fraction.

Going through the process of finding the G.C.M. of 167 and 81, we have

Hence

$$\frac{167}{81} = 2 + \frac{1}{16 +} \frac{1}{5}$$

Example 2.

Consider ·23 = 23/100.

We have

$$\begin{array}{c} 100)23 (0) \\ 0 \\ \hline 23)100 (4) \\ -\frac{92}{8)23 (2)} \\ -\frac{16}{7)8 (1)} \\ -\frac{7}{1)7 (7)} \\ 7 \\ -\frac{7}{0} \end{array}$$

Hence

$$\cdot 23 = 0 + \frac{1}{4 + 2} \cdot \frac{1}{1 + 1} \cdot \frac{1}{7} \cdot \frac{1}{7}$$

Cor. If we remove the restriction that the last partial quotient shall be greater than unity, we may declop any commensurable number as a continued fraction which has, at our pleasure, an even or an odd number of partial quotients.

For example,  $2 + \frac{1}{16 + \frac{1}{5}}$  has an odd number of partial quotients; but we may write it  $2 + \frac{1}{1 + \frac{1}{14}} + \frac{1}{16}$ , which has an even number.

§ 5.] Any simple surd, and, in fivet, any simple surd number, such as A · Bp<sup>n</sup> · Cp<sup>n</sup> · . . . . + Kp<sup>n-1n</sup>, can be converted into a continued fraction, although not, of course, into a terminating continued fraction.

The process consists in finding the greatest integer in a series of surd numbers, and in rationalising the denominator of the reciprocal of the residue. Methods for effecting both these steps are known (see chap. x.), but both, in any but the simplest cases, are very laborious. It will be sufficient to give two simple examples, in each of which the result happens to be a periodic continued fraction.

Example 1.

To convert \sqrt{13 into a continued fraction.

We have, 3 being the greatest integer < \$\15\$,

$$\sqrt{13} = 3 + (\sqrt{13} - 3) = 3 + \frac{1}{1/(\sqrt{13} - 3)},$$
  
=  $3 + \frac{1}{(\sqrt{13} + 3)/1}$  (1).

Again, since the greatest integer in  $(\sqrt{13} + 3)/4$  is 1, we have

$$\frac{\sqrt{13} + 3}{4} = 1 + \frac{\sqrt{13} - 1}{4} = 1 + \frac{1}{4 \cdot (\sqrt{13} - 1)},$$
  
=  $1 + \frac{1}{(\sqrt{13} + 1)/3}$  (2)

Similarly, we bay

$$\sqrt{13} + 1 = 1 + \sqrt{\frac{13}{3}} = 1 + \frac{1}{3/(\sqrt{13} - 2)},$$

$$= 1 + \frac{1}{(\sqrt{13} + 2)/3}$$
(3):

$$\begin{split} & \frac{\sqrt{13} + 2}{3} = 1 + \frac{13 - 1}{3} = 1 + \frac{1}{3/(\sqrt{13} - 1)}, \\ & = 1 + \frac{1}{(\sqrt{13} + 1)/4}, \\ & \frac{\sqrt{13} + 1}{4} = 1 + \frac{\sqrt{13} - 3}{4} = 1 + \frac{1}{4/(\sqrt{13} - 3)}, \end{split} \tag{1};$$

$$=1+\frac{1}{\sqrt{13}+3}$$
 (5);

$$\sqrt{13} + 3 = 6 + \sqrt{13} - 3 = 6 + \frac{1}{1/(\sqrt{13} - 3)},$$

$$= 6 + \frac{1}{(\sqrt{13} + 3)/4}$$
(6)

after which the process repeats itself.

From the equations (1)...(6) we derive

$$\sqrt{13} = 3 + \frac{1}{1} + \frac{1}{1+} + \frac{1}{1+} + \frac{1}{6+} + \frac{1}{1+} + \cdots,$$

where the \*\* indicate the beginning and end of the cycle of partial quotients Example 2.

To convert  $\frac{\sqrt{3}-1}{\alpha}$  into a continued fraction.

We have

$$\begin{split} &\sqrt{3}-1 = 0 + \frac{1}{2/(\sqrt{3}-1)},\\ &= 0 + \frac{1}{\sqrt{3}+1};\\ &\sqrt{3}+1 = 2 + \sqrt{3}-1 = 2 + \frac{1}{1/(\sqrt{3}-1)},\\ &= 2 + \frac{1}{(\sqrt{3}+1)/2};\\ &\frac{\sqrt{3}+1}{2} = 1 + \frac{\sqrt{3}-1}{2} = 1 + \frac{1}{2/(\sqrt{3}-1)},\\ &= 1 + \frac{1}{\sqrt{3}+1}; \end{split}$$

after which the quotients recur. We have, therefore,

$$\frac{\sqrt{3}-1}{2} = 0 + \frac{1}{2+} \frac{1}{1+} \dots$$

It will be proved in chap, XXXIII, that every positive number of the form  $(\sqrt{P} + Q)|R$ , where P is a positive integer which is not a perfect square, and Q and R are positive or negative integers, can be converted into a periodic continued fraction; and that every periodic continued fraction represents an irrational number of this form.

E press t. I make as state out i fractions, terminating or 

$$(1 - 1) = (11) \times (12)$$
  $(12) \times (13) \times (13) \times (14)$ 

(15.) Shi 6 that 
$$1 + \frac{1}{s} = 1 - \frac{1}{s} - \frac{1}{s} - \frac{1}{s} + \dots$$

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$$2\sqrt{1-\frac{1}{1+4}}\frac{1}{4\frac{1}{1+4}}\dots$$

$$\frac{1}{2}\sqrt{-2^2+12} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$$

$$\frac{1}{4}$$
 11 =  $-\frac{1}{4}$   $\frac{1}{2}$   $\frac{1}{4}$  ....

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PROPERTIES OF THE CONVERGENTS TO A CONTINUED PRACTION.

 $\S$  6.] Let us denote the complete continued fraction by  $x_1$ , so

$$x_1 = a_1 + \frac{1}{a_2 + a_3 + a_4 + a_4 + \cdots + a_n} \cdot \frac{1}{a_n}$$
 (1);

and let

$$x_2 = a_2 + \frac{1}{a_2 + a_3 + a_4 + \dots + a_n}$$
 (2);

$$\alpha_s - \alpha_s + \frac{1}{\alpha_s + 1} \cdot \frac{1}{\alpha_s}$$
and so on.

Then  $x_2, x_3, \ldots$  are called the complete quotients corresponding to a<sub>2</sub>, a<sub>3</sub>, . . . , or, simply, the second, third, . . . complete quotients, The fraction itself, or x1, may be called the first complete quotient. It will be observed that  $a_1, a_2, a_3, \ldots$  are the integral parts of  $x_1, x_2, x_3, . . . .$ 

Let us consider, on the other hand, the fractions which we obtain by first retaining only the first partial quotient, second by retaining only the first and second, and so on; and let us denote the fractions thus obtained, when reduced (without simplification, as under) so that their numerators and denominators are integral numbers, by  $p_1/q_1$ ,  $p_2/q_2$ , . . . Then we have

$$\alpha_1 = \frac{\alpha_1}{1} = \frac{p_1}{\alpha}$$
(a)

$$a_1 + \frac{1}{1} = a_1 a_2 + 1 = \frac{p_2}{p_2}$$
 (β),

$$\begin{array}{ll} a_1 + \frac{1}{a_2} &= \frac{a_1 a_2 + 1}{a_2} &= \frac{p_2}{q_2} \\ a_1 + \frac{1}{a_2 + a_3} &= \frac{a_1 a_2 a_3 + a_1 + a_2}{a_2 a_3 + 1} &= \frac{p_3}{q_3} \\ \end{array} \tag{$\beta$)},$$

$$a_1 + \frac{1}{a_2 + a_3 + \cdots} \cdot \frac{1}{a_n} =$$
 &c.  $= \frac{p_n}{q_n}$  (5)

where

$$p_1 = a_1,$$
  $q_1 = 1$   $(a'),$   $p_2 = a_1a_2 + 1,$   $q_2 = a_2$   $(\beta'),$ 

$$p_3 = a_1 a_2 a_3 + a_1 + a_3, \quad q_3 = a_2 a_3 + 1$$
  $(\gamma'),$ 

and so on.

The fractions  $p_1(q_1, p_1, q_2, \dots)$  are called the first, second, . . . convergents to the continued fraction.

Cor. If the continued fraction terminates, the last convergent is, by its definition, the continued fraction itself.

§ 7.] It will be seen, from the expressions for  $p_1$ ,  $p_2$ ,  $p_3$  and  $q_1$ ,  $q_2$ ,  $q_3$  in § 6 (a'), ( $\beta$ '), ( $\gamma$ '), that we have

$$p_1 = a_2p_2 + p_1$$
 (1);  
 $q_3 = a_1q_2 + q_1$  (2).

This suggests the following general formulae for calculating the numerator and denominator of any convergent when the numerators and denominators of the two preceding convergents are known, namely,

$$p_n = \alpha_n p_{n-1} + p_{n-2}$$
 (3);  
 $q_n = \alpha_n q_{n-1} + q_{n-2}$  (4).

Let us suppose that this formula is true for the *n*th convergent. We observe from the definitions (a),  $(\beta)$ , . . ,  $(\delta)$  of S G, that the n-1th convergent,  $p_{n+1}$ ,  $p_{n+1}$ , is derived from the *n*th if we replace  $a_n$  by  $a_n + 1$   $a_{n+1}$ . Hence, since  $p_{n+1}$ ,  $q_{n+1}$ ,  $p_{n+1}$ ,  $q_{n+2}$ ,  $q_{n+2}$ ,  $q_{n+2}$  do not contain  $a_n$ , and since, by hypothesis,

$$\frac{p_n}{q_n} = \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}},$$

it follows that

$$\frac{p_{n-1}}{q_{n+1}} = \frac{(a_n + 1 \ a_{n+1}) \ p_{n-1} + p_{n-2}}{(a_n + 1 \ a_{n+1}) \ q_{n-1} + q_{n-2}}$$

or, after reduction.

$$\begin{split} \frac{p_{n+1}}{q_{n+1}} &= \underbrace{\frac{d_{n+1}}{d_{n+1} + d_n}}_{\substack{q_{n+1} + q_{n-1} \\ = \frac{d_{n+1}p_n + p_{n-1}}{d_{n+1}q_n + q_{n-1}}}_{= \underbrace{\frac{d_{n+1}p_n + p_{n-1}}{d_{n+1}q_n + q_{n-1}}}_{= \underbrace{1}$$

Hence it is sufficient if we take by (3) and 4).

$$p_{n+1} = a_{n+1}p_n + p_{n-1};$$
  
 $q_{n+1} = a_{n+1}q_n + q_{n-1}.$ 

In other words, if the rule hold for the nth convergent, it holds for the n+1th Now, by (1) and (2), it holds for the third; hence, by what has just been proved, it holds for the fourth; hence for the fifth; and so on. That is to say, the rule is general.

Cor. 1. Since  $a_n$  is a positive integral number, it follows from (3) and (4) that the numerators of the successive convergents form an increasing series of integral numbers, and that the same is true of the denominators.

Cor. 2. From (3) and (4) it follows that

$$\frac{P_n}{p_{n-1}} = a_n + \frac{1}{a_{n-1} + a_{n-2} +} \cdot \cdot \cdot \frac{1}{a_1}$$

$$\frac{q_n}{q_{n-1}} = a_n + \frac{1}{a_{n-1} + a_{n-2} +} \cdot \cdot \cdot \frac{1}{a_2}$$
(5):

and

For, dividing (3) by  $p_{n-1}$ , and writing successively n-1, n-2, ..., 3 in place of n, we have

$$p_n/p_{n-1} = \alpha_n + \frac{1}{p_{n-1}/p_{n-2}};$$

$$p_{n-1}/p_{n-2} = \alpha_{n-1} + \frac{1}{p_{n-2}/p_{n-2}};$$

$$p_n/p_2 = \alpha_3 + p_1/p_2;$$

$$= \alpha_3 + \frac{1}{\alpha_4 + \alpha_5};$$

From these equations, by successive substitution, we derive (5); and (6) may be proved in like manner.

Example 1.

The continued fraction which represents the ratio of the circumference of a circle to the diameter is  $3 + \frac{1}{7} + \frac{1}{15 + \frac{1}{1} + \frac{1}{292 + \frac{1}{1 + \frac{1}{1 + \cdots}}} + \frac{1}{1 + \frac{1}{1 + \cdots}}$ . It is required to calculate the successive convergents.

The first two convergents are 3 and  $3 + \frac{1}{7}$ , that is,  $\frac{3}{1}$ ,  $\frac{22}{7}$ .

Hence, using the formulæ (3) and (1), we have the following table:-

ı	n	a	p	q
Ì	1	3	3	1
1	2	7	22	7
ı	3	15	333	106
	4	1	355	113
	5	292	103993	33102
	6	1	104318	33215
	7	1	208341	66317

where p<sub>4</sub>=355, for example, is obtained by multiplying the number over it, namely 333, by 1, and adding to the product the number one place higher still, namely 22. The successive convergence are therefore

Example 2.

If  $p_1'|_{L_1}$ ,  $p_2'|_{L_2}$ , ... be the convergents to  $1 + \frac{1}{9} + \frac{1}{8} + \frac{1}{4} + \cdots + \frac{1}{9} + \cdots$ ol r . show that

$$p_{n-1} = (n-1)p_{n-1} + (n-1)p_{n-2} + (n-2)p_{n-2} + \dots + 3p_2 + 2p_1 + 2p_2 + 2p_3 + 2p_4 + 2p_4 + 2p_5 + 2p_5$$

By the re urrene. I imula we have

$$\begin{split} p_n &= np_{n-1} + p_{n-2};\\ p_{n-1} &= (n-1)p_{n-1} + p_{n-3}\\ p_{n-1} &= (n-2)p_{n-3} + p_{n-4} \end{split}$$

 $p_3 = 3p_1 + p_3$ and (since r = 1, r, - ')

Adding all these equations, and observing that Pn-2, Pn-2, . . . . Ps each ecur three times, once on the left multipood by I, once on the right must about by 1, and again on the right multipared by n-1, n-2, . . . , 3

 $p_n = (n-1) \int_{-n-1}^{n} + (n-1) p_{n-2} + (n-2) \int_{-n-2}^{n} + \dots + 3p_2 + 2p_1 + (p_1+1),$ which cives the required result since  $p_i = 1$ .

In the case of the certine I fraction  $(|\cdot| - \frac{1}{t_0 + t_0}, \dots, \frac{1}{t_0 + t_0})$ 

$$\frac{r_{2n+1}}{t_{n+1}} = i_1 + \frac{1}{i_2} + \dots + \frac{1}{a_1}$$
 (a),

smo every odd parted que tent is a..

$$\frac{I_{(n+1)}}{I_{(n)}} = a_1 + \frac{1}{1} + \dots + \frac{1}{I_1}$$
 (3).

w) have 
$$P_{ext} = 0$$
 (5)

$$a = -i F_{i,n-k} - I_{2n} + I_{2n-1}$$
 (8).

Now, if we very a little of the we have proceedings in honce (8) gives

§ 8.] From equations (3) and (1) of last section we can prove the following important property of any two consecutive convergents:—

 $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$  (1).

For, by § 7 (3) and (4),

$$p_{n+1}q_n - p_n q_{n+1} = (a_{n+1}p_n + p_{n-1}) q_n - p_n (a_{n+1}q_n + q_{n-1}),$$
  
=  $-(p_n q_{n-1} - p_{n-1}q_n).$ 

Hence, if (1) hold, we have

$$p_{n+1}q_n - p_nq_{n+1} = -(-1)^n,$$
  
=  $(-1)^{n+1}.$ 

In other words, if the property be true for any integer n, it holds for the next integer n+1. Now

$$p_2q_1 - p_1q_2 = (a_1a_2 + 1) 1 - a_1a_2,$$
  
= 1,  
=  $(-1)^2$ ;

that is to say, the property in question holds for n=2, hence it holds for n=3; hence for n=4; and so on.

Cor. 1. The convergents, as calculated by the rule of § 7, are fractions at their lowest terms.

For, if  $p_n$  and  $q_n$ , for example, had any common factor, that factor would, by § 8 (1), divide (-1)<sup>n</sup> exactly. Hence  $p_n$  is prime to  $q_n$ ; and  $p_n/q_n$  is at its lowest terms.

Cor. 2.

$$\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^n}{q_n q_{n-1}}$$
(2)

Cor. 3.

$$\begin{split} & \frac{P^{n}}{q_{n}} = \frac{P_{1}}{q_{1}} + \left(\frac{P_{2}}{q_{2}} - \frac{P_{1}}{q_{1}}\right) + \left(\frac{P_{3}}{q_{2}} - \frac{P_{2}}{q_{2}}\right) + \dots + \left(\frac{P_{n}}{q_{n}} - \frac{P_{n-1}}{q_{n-1}}\right), \\ &= a_{1} + \frac{1}{q_{1}q_{2}} - \frac{1}{q_{2}q_{3}} + \dots + \frac{1}{q_{n-1}q_{n}} \end{split}$$

$$(3)$$

Cor. 4.

For

$$p_n q_{n-2} - p_{n-2} q_n = (-)^{n-1} a_n$$
 (4)

$$\begin{array}{l} p_n q_{n-2} - p_{n-2} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}), \\ = (p_{n-1} q_{n-2} - p_{n-2} q_{n-1}) \alpha_n, \\ = (-)^{n-1} \alpha_n, \text{ by Cor. 1.} \end{array}$$

Cor. 5.

$$p_n|q_n - p_{n-2}/q_{n-2} = (-)^{n-1}a_n/q_nq_{n-2}$$
 (5).

The odd convergents continually increase in value, the eren convergents continually decrease: every even convergent is greater than every odd convergent; and every odd convergent is less than, and every even convergent greater than, any following convergent.

These conclusions follow at once from the conations (2) and (5).

Cor. 7. Given two positive integers p and a which are prime to each other, we can always find two positive integers p' and a' such that  $ya' - ya = \pm 1$  or = -1, as we whase.

For, by § 4, Cor., we can always convert p/q into a continued fraction having an even or an odd number of partial quotients. as we please. If p'/q' be the penultimate convergent to this continued fraction, we have in the former case pq' - p'q = +1, in the latter pq' - p'q = -1.

Example. If  $I_n I_n$  be the nth convergent to  $a_1 + \frac{1}{a_1 + a_2 + a_3 + \dots + \frac{1}{a_n}}$ , and  $_{s}P_{n}/_{s}Q_{n}$  the convergent to  $a_{s}+\frac{1}{a_{n+1}+1}$  . . .  $\frac{1}{a_{n}}$  which corresponds to the partial quotient a., show that

 $p_n q_{n-r} - p_{n-r} q_n = (-1)^{n-r+1}_{n-r+1} Q_n$ 

We have, by our data,

$$\frac{p_n}{q_n} = a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}$$
 (a),

$$\frac{p_{n-r}}{q_{n-r}} = a_1 + \frac{1}{a_2 - \dots - \frac{1}{a_{n-r}}}$$

$$= \frac{1}{1} \qquad (\beta);$$

hence

$$\frac{p_n}{q_n} = a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_{n-r+1}} P_n \int_{n-r+1} Q_n}$$
 (7).

Now

$$\frac{p_{n-r}}{q_{n-r}} = \frac{a_{n-r}p_{n-r-1} + p_{n-r-2}}{a_{n-r}q_{n-r-1} + q_{n-r-2}}.$$

Hence, by (a) and (-1),

$$\begin{split} P_n &= \frac{(q_{n+r} + q_{n+r})Q_n(q_{n+r})P_n(p_{n+r} + p_{n+r-1})}{q_n + (q_{n+r} + q_{n+r})Q_n(q_{n+r-1})P_n(q_{n+r-1})(q_{n+r-1})}, \\ &= \frac{P_{n+r} + q_{n+r}(Q_n(p_{n+r-1})q_n(q_{n+r-1})q_{n+r-1})}{q_{n+r+1}Q_n(q_{n+r-1})q_n(q_{n+r-1})P_n}, \\ &= \frac{q_{n+r}(P_n(p_{n+r-1})q_n(q_{n+r-1})q_n(q_{n+r-1})}{q_{n+r-1}Q_n(q_{n+r-1})q_n(q_{n+r-1})}, \end{split}$$
(5).

Now it is easy to see that the numerator and denominator of the fraction last written are mutually prime: therefore

From  $(\epsilon)$  we derive

$$\begin{split} p_{n}q_{n-r} - p_{n-r}q_{n} &= -(p_{n-r}q_{n-r-1} - p_{n-r-1}q_{n-r})_{n-r+1}Q_{n}, \\ &= (-1)(-1)^{n-r}_{n-r+1}Q_{n}, \\ & \text{by (1) above,} \\ &= (-1)^{n-r+1}_{n-r+1}, O_{n}; \end{split}$$

as was to be shown.

§ 9.] The convergents of odd order are each less than the whole continued fraction, and the convergents of even order are each greater; and each convergent is nearer in value to the whole continued fraction than the preceding.

We have, by § 7,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}};$$

and the whole continued fraction  $x_1$  is derived from  $p_{n+1}/q_{n+1}$  by replacing the partial quotient  $a_{n+1}$  by the complete quotient  $x_{n+1}$ . Hence

$$x_1 = \frac{x_{n+1}p_n + p_{n-1}}{x_{n+1}q_n + q_{n-1}}$$
.

From this value of  $x_1$  we obtain

$$x_{1} - \frac{p_{n}}{q_{n}} = \frac{x_{n+1}p_{n} + p_{n-1}}{x_{n+1}q_{n} + q_{n-1}} - \frac{p_{n}}{q_{n}},$$

$$= \frac{p_{n-1}q_{n} - p_{n}q_{n-1}}{q_{n}(x_{n+1}q_{n} + q_{n-1})}$$

$$= \frac{p_{n}q_{n}q_{n-1}q_{n-1}}{q_{n}(x_{n+1}q_{n} + q_{n-1})}$$
(1).

Similarly

$$x_1 - \frac{p_{n-1}}{q_{n-1}} = \frac{x_{n+1} \left( p_n q_{n-1} - p_{n-1} q_n \right)}{q_{n-1} \left( x_{n+1} q_n + q_{n-1} \right)}$$
(2)

From (1) and (2) we deduce

$$\frac{x_1 - \frac{p_n}{q_n}}{x_1 - \frac{p_{n-1}}{q_{n-1}}} = -\frac{q_{n-1}}{q_n x_{n+1}}$$
(3).

Now  $q_{n-1}$ ,  $q_n$  are positive integers;  $x_{n+1} \not< 1$ ; and, by § 7, Cor. 1,  $q_{n-1} < q_n$ . It follows, therefore, from (3) that  $x_1 - p_n/q_n$  is opposite in sign to, and numerically less than,  $x_1 - p_{n-1}/q_{n-1}$ . In other words,  $p_n/q_n$  differs from  $x_1$  by less than  $p_{n-1}/q_n$ ; does; and if the one be less than  $x_1$ , the other is greater, and cice versá.

Now the first convergent is obviously less than  $\sigma_0$ , hence the second is greater, the third less, and so on; and the difference between  $x_0$  and the successive convergents continually decreases.

Vor. 1. The difference between the continued fraction and the nth convergent is less than  $1/q_nq_{n+1}$ , and greater than  $a_{n+1}q_nq_{n+2}$ .

For, by what has just been proved,

$$\frac{l^{\gamma_n}}{q_n}, \quad \frac{l^{\gamma_{n+2}}}{q_{n+2}}, \quad x_1, \quad \frac{l^{\gamma_{n+1}}}{q_{n+1}}$$

are, in order of magnitude, either ascending or descending. Hence

$$\begin{split} \frac{P_n}{q_n} &\sim x_1 < \frac{P_n}{q_n} \sim \frac{P_{n+1}}{q_{n+1}}, \\ &< \frac{1}{q_n q_{n+1}}, \text{ by } \S \text{ S } (2). \end{split}$$

Again,

$$\begin{split} \frac{P_n}{q_n} \sim x_1 > & \frac{P_n}{q_n} \sim \frac{P_{n+2}}{q_{n+2}}, \\ > & \frac{a_{n+2}}{q_n q_{n+2}}, \text{ by § 8 (5)}. \end{split}$$

Since  $q_{n+1} > q_n$ , and since  $q_{n+2}/q_{n+2} - (a_{n+2}q_{n+1} + q_n)/a_{n+2} - q_{n+1} + q_n/a_{n+2} - q_{n+1} + q_n/a_{n+2} = \text{being} < 1$ ), it follows that the upper and lower limits of the error committed by taking the nth convergent instead of the whole continued fraction may be taken to be  $1/q_s$  and  $1/q_s/q_s + q_{n+1}$ ). These, of course, are not so close as those given above, but they are simpler, and in many cases they will be found sufficient.

Cor. 2. In order to obtain a good approximation to a continued fraction, it is advisable to take that convergent whose corresponding partial quotient immediately precedes a very much larger partial quotient.

For, if the next quotient be large, there is a sudden increase in  $g_{n(t)}$ , so that 1  $g_ng_{n(t)}$  is a very small fraction.

The same thing appears from the consideration that, in taking  $p_{\perp}q_{a}$  instead of the whole fraction, we take  $a_{a}$  instead of

§ 9 condition that  $p_n/q_n$  be a convergent to  $x_1$  439

 $a_k + \frac{1}{a_{n+1} + \dots}$ , that is, we neglect the part  $\frac{1}{a_{n+1} + \dots}$ ... of the complete quotient. Now, if  $a_{n+1}$  be very large, this neglected part will of course be very small.

Cor. 3. The odd convergents form an increasing series of rational fractions continually approaching to the value of the whole continued fraction; and the even convergents form a decreasing series luxuing the same property.

Cor. 4. If  $p_n|q_n-x_1<1/q_n(q_n+q_{n-1})$ , where  $q_{n-1}$  is the denominator of the penultimate convergent to  $p_n|q_n$  who reoncerted into a simple continued fraction having an even number of quotients, then  $p_n|q_n$  is one of the convergents to the simple continued fraction which represents  $x_1$ ; and the like holds if  $x_1-p_n|q_n<1/q_n(q_n+q_{n-1})$ , where  $q_{n-1}$  is the denominator of the penultimate convergent to  $p_n|q_n$  when converted into a simple continued fraction having an old number of quotients.

Let  $a_1, a_2, \ldots, a_n$  be the *n* partial quotients of  $p_n/q_n$ , when converted into a simple continued fraction having an even number of quotients, and let  $p_{n-1}/q_{n-1}$  be the penultimate convergent. Then  $p_n q_{n-1} - p_{n-1} q_n = 1$ .

Let  $x_{n+1}$  be determined by the equation

$$x_1 = a_1 + \frac{1}{a_n + \dots + \frac{1}{a_n + x_{n+1}}}$$

Then we have

$$x_1 = (x_{n+1}p_n + p_{n-1})/(x_{n+1}q_n + q_{n-1}),$$

whence

$$x_{n+1} = (x_1q_{n-1} - p_{n-1})/(p_n - x_1q_n),$$

<sup>•</sup> The value of every simple continued fraction lies, of course, between 0 and ∞; and we may, in fact, regard those as the first and second convergents respectively to every continued fraction. If we write 0 = q, and ∞ = \(\frac{1}{2}\), and denote these by \(\frac{P}{2}\) and \(\frac{P}{2}\) as that we understand \(\frac{P}{2}\), to be 0, \(\frac{P}{2}\) by to be 1, \(\frac{q}{2}\) to be 0, \(\text{then }\) \(\frac{P}{2}\) and \(\frac{P}{2}\) and \(\frac{P}{2}\) and \(\frac{P}{2}\) and \(\frac{P}{2}\) and the series \(\frac{P}{2}\), \(\frac{P}{2}\),

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or, if we put  $\xi - p_n/q_n - x_1$ ,

$$\begin{split} x_{n+1} &= \{(p_n q_{n-1} - p_{n-1} q_n)' q_n - q_{n-1} \xi\}/q_n \xi, \\ &= (1/q_n - q_{n-1} \xi) q_n \xi. \end{split}$$

Hence the necessary and sufficient condition that  $x_{n+1}>1$  is that  $1 \ y_n = q_{n-1}\xi > q_n\xi,$ 

that is,

$$\xi < 1/q_n (q_n + q_{n-1}),$$

which is fulfilled by the condition in the first of our two theorems.

Let now  $b_1, b_2, \ldots, b_n$  be the first n partial quotients in the simple continued fraction that represents  $x_1$ . Then we have

$$x_1 = b_1 + \frac{1}{b_2 + \dots + \frac{1}{b_n + y_{n+1}}}$$

where  $y_{n+1} > 1$ .

Hence

$$a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n + x_{n+1}}} - b_1 + \frac{1}{b_2 + \dots + \frac{1}{b_n + y_{n+1}}}$$

Therefore, by § 3, Cor., we must have

$$a_1 = b_1, \quad a_2 = b_2, \quad \dots, \quad a_n = b_n, \quad x_{n+1} = y_{n+1},$$

$$a_1 = \frac{1}{n} \quad \text{that is } p_n \text{ is the arth convergent}$$

Hence  $a_1 + \frac{1}{a_2 +} \dots + \frac{1}{a_n}$ , that is,  $\frac{p_n}{q_n}$  is the *n*th convergent to  $x_1$ .

The second theorem is proved in precisely the same way.

Since  $q_{n-1} < q_n$ , the conditions above are a fortinri fulfilled if  $x_1 \sim p_n/q_n < 1/2q_n^2$ .

§ 10.] The propositions and corollaries of last section show that the method of continued fractions possesses the two most important advantages that any system of numerical calculation can have, namely, 1st, it furnishes a regular series of rational approximations to the quantity to be evaluated, which increase step by step in complexity, but also in exactness; 2nd, the error committed by arresting the approximation at any step can at once be estimated. The student should compare it in these respects with the decimal system of notation.

§ 11.] It should be observed that the formation of the successive convergents virtually determines the meaning we attach to the chain of operations in a continued fraction.

If the continued fraction terminate, we might of course proceed to reduce it by beginning at the lower end and taking in the partial quotients one by one in the reverse order. The reader may, as an exercise, work out this treatment of finite continued fractions, and he will find that, from the arithmetical point of view, it presents few or none of the advantages of the ordinary plan developed above.

In the case of non-terminating continued fractions, no such alternative course is, strictly speaking, open to us. Indeed, the further difficulty arises that, a priori, we have no certainty that such a continued fraction has any definite meaning at all. The point of view to be taken is the following:-If we arrest the continued fraction at any partial quotient, say the sth, then, in the case of a simple continued fraction, however great s may be, we have seen that the two convergents,  $p_{2n-1}/q_{2n-2}$ ,  $p_{2n}/q_{2n}$ , include the fraction  $p_s/q_s$  between them. Hence, if we can show that  $p_{2n-1}/q_{2n-1}$  and  $p_{2n}/q_{2n}$  each approach the same finite value when n is increased without limit, it will follow that as s is increased without limit, that is, as more and more of the partial quotients of the continued fraction are taken into account,  $p_{e}/q_{s}$ approaches a certain definite value, which we may call the value of the whole continued fraction. Now, by § 8, Cor. 5, p<sub>m-1</sub> q<sub>m-1</sub> continually increases with n, and  $p_{en}/q_{en}$  continually decreases, and  $p_{2n}/q_{2n} > p_{2n-1}/q_{2n-1}$ . Hence, since both are positive, each of the two must approach a certain finite limit. Also the two limits must be the same; for by § 8, Cor. 2,  $p_{2n}/q_{2n} - p_{2n-1}/q_{2n-1}$ =  $1/q_{2n}q_{2n-1}$ , and by the recurrence formula for  $q_n$  it follows that  $q_{2n}$  and  $q_{2n-1}$  increase without limit with n; therefore  $\rho_{2n}/q_{2n}$  $-p_{2n-1}/q_{2n-1}$  may be made as small as we please by sufficiently increasing n.

It appears, therefore, that every simple continued fraction has a definite finite value.

Example.

To obtain a good commensurable approximation to the ratio of the

circumference of a circle to the diameter. Referring to Example 1, § 7, we have the following approximations in defect:—

and the following in excess:-

$$\frac{22}{7}$$
,  $\frac{355}{113}$ ,  $\frac{101318}{33215}$ , &e.

Two of these \*, namely 22/7 and 355/113, are distinguished beyond the others by preceding large partial quotients, namely, 15 and 292.

The latter of these is exceedingly accurate, for in this case  $1/q_s q_{s+1} = 1/113 \times 33102 = 0000002673$ , and  $\alpha_{s+1}/q_s q_{s+2} = 1/113 \times 33102 = 0000002605$ . The error therefore has between 000000266 and 000000267; that is to say, 355/113 is accurate to the 6th decimal place. In point of fact, we have

$$\pi = 3.14159265358 \dots$$
 $355/113 = 3.11159292935 \dots$ 
Difference =  $00000026677 \dots$ 

#### EXERCISES XXIX.

- (1.) Calculate the various convergents to \$\frac{769}{1111}\$, and estimate the errors committed by taking the first, second, third, \$\delta \circ\,\ \text{instead}\$ of the fraction.
- (2.) Find a convergent to the infinite continued fraction  $\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\dots$  which shall represent its value within a millionth.
- (3) Find a commensurable approximation to \(\sigma'(17)\) which shall be accurate within 1/100-00, and such that no nearer fraction can be found not having a greater denominator.
- (4) The sidereal period of Venus is 2217 days, that of the earth 365-25 days; calculate the various cycles in which transits of Venus may be expected to occur. Calculate the number of degrees in each case by which Venus is displaced from the node, when the earth is there, at the end of the first cycle after a former central transit.
- (5.) Work out the same problem for Mercury, whose sidercal period is 87:97 days.
- (6.) According to the Northampton table of mortality, out of 3635 persons who reach the age of 40, 3559 reach the age of 41. Show that this is expressed very accurately by saying that 47 out of 48 survive.

The first of them, 22,7, was given by Archimedes (212 n.e.). The second, 355,113, was given by Adrian Metins (published by his son, 1610 And): it is in great favour, not only on account of its accuracy, but because it can be easily remembered as consisting of the first three odd numbers each repeated twice in a certain successing.

(7.) Find a good rational approximation to \( \sqrt{19} \) which shall differ from it by less than 1/100000; and compare this with the rational approximation obtained by expressing \( \sqrt{19} \) as a decimal fraction correct to the 6th place.

(8.) If  $\alpha$  be any incommensurable quantity whatever, show that two integers, m and n, can always be found, so that  $0 < \alpha n - m < \kappa$ , however small

κ may be.

(5.) Show that the numerators and also the denominators of any two consecutive convergents to a simple continued fraction are prime to each other; also that if p<sub>n</sub> and p<sub>n-2</sub> have any common factor it must divide a<sub>n</sub> exactly.

(10.) Show that the difference between any two consecutive odd convergents to Ma<sup>2</sup> + 1) is a fraction whose numerator, who n at its lowest terms, is 2a.

(11.) Prove directly, from the recursive relation connecting the numerators and denominators, that every convergent to a simple continued fraction is intermediate in value to the two preceding.

(12.) Prove that

$$q_n x_1 - p_n = (-1)^{n+1} / x_1 x_3 \dots x_{n+1}$$

Show that  $p_n|q_n$  differs from  $x_1$  by less than  $1/a_2a_3\ldots a_{n+1}q_n$ . Is this a better estimate of the error than  $1/q_nq_{n+1}$ ?

(13.) If the integers x and y be prime to each other, show that an integer u can always be found such that

$$(x^2 + y^2) u = z^2 + 1$$

where z is an integer.

(14.) Prove that

$$\begin{split} \{p_n^2 - q_n^2\} & (p_{n-1}^2 - q_{n-1}^2) = (p_n p_{n-1} - q_n q_{n-1})^2 - 1 \ ; \\ & \frac{p_n^2 + q_n^2}{p_{n-2}^2 + q_{n-2}^2} = \frac{(p_n p_{n-1} + q_n q_{n-1})^2 + 1}{(p_{n-1} p_{n-2} + q_{n-1} q_{n-1})^2 + 1} \ , \end{split}$$

(15.) Prove that p<sub>n-1</sub>p<sub>n</sub> - q<sub>n-1</sub>q<sub>n</sub>x<sub>1</sub><sup>2</sup> is positive or negative according as n is even or odd.

(16.) If P/Q, P'/Q', P''/Q'' be the nth, n-1th, n-2th convergents of

$$\frac{1}{a_1+} \frac{1}{a_2+} \frac{1}{a_3+} \frac{1}{a_4+} \cdots,$$

$$\frac{1}{a_2+} \frac{1}{a_3+} \frac{1}{a_4+} \cdots,$$

$$\frac{1}{a_3+} \frac{1}{a_4+} \cdots$$

respectively, show that

$$P=\alpha_zP'+P^{\prime\prime},\quad Q=(\alpha_1\alpha_2+1)\; P'+\alpha_1P^{\prime\prime}.$$

(17.) If the partial quotients of x<sub>1</sub> = p<sub>d</sub>q<sub>n</sub> form a reciprecal scries (that is, a series in which the first and last terms are equal, the scond and second last equal, and so on), then p<sub>n+1</sub>=q<sub>n</sub>, and (q<sub>n</sub><sup>2</sup> = 1)p<sub>n</sub> is an integer; and, conversely, if these conditions he satisfied, the quotients will form a reciprocal series.

(18.) Show, from last exercise, that every integer which divides the sum of two integral squares that are prime to each other is itself the sum of two sonares. (See Serr.t. Alo. Sup., 4m; ed.), t. 1, p. 29.) (19.) Show that

(20.) If  $x_1 = \frac{1}{a+} \frac{1}{a+} \frac{1}{a+} \dots$ , show that  $p_n = q_{n-1}$ .

(21.) The successive convergents of  $2a + \frac{1}{a} + \frac{1}{a} + \frac{1}{a} + \frac{1}{a} + \frac{1}{a} + \cdots$  are always double those of  $a + \frac{1}{2a + \frac{1}{2a + 1}} \cdot \frac{1}{a}$ .

(22.) If the reduced form of the nth complete quotient, r., in  $a_1 + \frac{1}{a_1 + a_2 + a_3 + \cdots}$  be  $\zeta_n/\eta_n$ , show that

$$\zeta_n = a_n \zeta_{n+1} + \zeta_{n+2},$$

(23.) Find the numerically least value of ax - by for positive integral values of x and y, a and b being positive integers, which may or may not be prime to each other.

### CLOSEST COMMENSURABLE APPROXIMATIONS OF GIVEN COMPLEXITY.

§ 12.] One commensurable approximation to a number (commensurable or incommensurable) is said to be more complex than another when the denominator of the representative fraction is greater in the one case than in the other. The problem which we put before ourselves here is to find the fraction, whose denominator does not exceed a given integer D, which shall most closely approximate (by excess or by defect, as may be assigned) to a given number commensurable or incommensurable. solution of this problem is one of the most important uses of continued fractions. It depends on a principle of great interest in the theory of numbers, which we proceed to prove,

Lemma. — If p'q and p''q' be two fractions such that pq'-p'q=1, then no fraction can lie between them unless its denominator is greater than the denominator of either of them.

Proof. - Let a b be a fraction intermediate in magnitude to p/q and p/q. Then

$$\frac{p}{q} - \frac{a}{b} < \frac{p}{q} - \frac{p'}{\bar{q}'}$$

$$\frac{a}{b} - \frac{p'}{q'} < \frac{p}{q'} - \frac{p'}{q'}.$$
(1);

$$p'$$

$$\frac{P}{a'} \leq \frac{P}{a} = \frac{P}{a'}$$
 (2).

From (1), 
$$\frac{pb-qa}{qb} < \frac{tq^j-p^iq}{qq^i};$$
 
$$\frac{pb-qa}{qb} < \frac{1}{qq^i}$$
 Hence 
$$qb>qr^j(pb-qa);$$

and

b > (pb - qa)q'Now p/q - a/b is positive, hence pb - qa is a positive integer. It follows, therefore, that b > q'.

Similarly it follows from (2) that b>q.

Hence no fraction can lie between p/q and p'/q' unless its denominator is greater than both q and q'. In other words, if pq' - p'q = 1, no commensurable number can lie between p, q and p'/g' which is not more complex than either of them.

§ 13.] The nth convergent to a continued fraction is a nearer approximation to the value of the complete fraction than any fraction whose denominator is not greater than that of the conrergent. For any fraction a/b which is nearer in value to the continued fraction than  $\rho_n/q_n$  must, a fortiori, be nearer than  $p_{n-1}/q_{n-1}$ . Hence, since  $p_n | q_n$  and  $p_{n-1}/q_{n-1}$  include the value of the continued fraction between them, it follows that a/b must lie between these two fractions. Now we have, by § 8, either  $p_nq_{n-1} - p_{n-1}q_n = 1$ , or  $p_{n-1}q_n - p_nq_{n-1} = 1$ . Hence, by § 12, b must be greater than  $q_n$ , which proves our proposition.

Example.

Consider the continued fraction  $x_1 = 3 + \frac{1}{1 + 3} + \frac{1}{4 + 2 + 5} + \frac{1}{5}$ .

The successive convergents are  $\frac{3}{1}$ ,  $\frac{4}{1}$ ,  $\frac{15}{4}$ ,  $\frac{64}{17}$ ,  $\frac{113}{38}$ ,  $\frac{779}{207}$ . any one of these, say 64/17, the statement is, that no fraction whose denominator does not exceed 17 can be nearer in value to  $x_1$  than 64/17.

\$ 14.1 The result of last section is a step towards the solution of the general problem of § 12; but something more is required.

Consider, for example, the successive convergents  $p_{n-2}q_{n-2}$ ,  $p_{n-1}/q_{n-1}$ ,  $p_n/q_n$  to  $x_1$ , and let n be odd, say. Then

$$\frac{p_{n-2}}{q_{n-2}}$$
,  $\frac{p_n}{q_n}$ ,  $x_1$ ,  $\frac{p_{n-1}}{q_{n-1}}$ 

are in increasing order of magnitude. We know, by last

section, that no fraction whose denominator is less than  $q_{n-1}$  can lie in the interval  $p_{n-2}|_{f_{n-1}}, p_{n-1}|_{f_{n-1}}$  and also that no fraction whose denominator is less than  $q_n$  can lie in the interval  $p_n|_{f_{n-1}}, p_{n-1}|_{f_{n-1}}$  but we have no assurance that a fraction whose denominator is less than  $q_n$  may not lie in the interval  $p_{n-1}, q_{n-2}, p_n|_{f_n}$ , for  $p_n|_{f_{n-1}}-p_{n-2}|_{f_n}$  a<sub>n</sub>, where a<sub>n</sub> may be >1.

This lacuna is filled by the following proposition :-

1°. The series of fractions

 $\frac{p_{-2}}{q_{n-2}}$ ,  $\frac{p_{n-2}+p_{n-1}}{q_{n-2}+q_{n-1}}$ ,  $\frac{p_{n-2}+2p_{n-1}}{q_{n-2}+2q_{n-1}}$ , ...

$$\frac{p_{n-2} + a_n - 1}{q_{n-2} + a_n - 1} \frac{p_{n-2} + a_n p_{n-1}}{q_{n-2} + a_n q_{n+1}} \begin{pmatrix} p_n \\ q_n \end{pmatrix} (1),$$

form (according as n is odd or even) an increasing or a decreasing series.

2'. Each of them is at its lovest terms; and each consecutive pair, say P.Q. P.Q. satisfies the condition PQ-PQ ±1; so that no common usurable quantity less complex than the more complex of the two can be inserted between them.

The first and last of these fractions (formerly called Convergents merely) we now call, for the sake of distinction, Principal Convergents; the others are called Intermediate Convergents to the continued fraction. To prove the above properties, let us consider any two consecutive fractions of the series (1), say  $P_iQ$ ; then

$$\begin{split} \frac{P}{Q} &= \frac{P}{Q} = \frac{p_{s/2} + rp_{s/1}}{q_{s/2} + rq_{s/1}} - \frac{p_{s/2} + r + 1p_{s/1}}{q_{s/2} + r + 1q_{s/1}} \\ &\qquad \qquad \text{(where } r = 0, \text{ or } 1, \text{ or } 2, \dots, \text{ or } a_{\mathsf{a}} - 1), \\ &\qquad \qquad \vdots \qquad \qquad \frac{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}, \\ &\qquad \qquad = \frac{1}{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}, \\ &\qquad \qquad = \frac{1}{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}, \\ &\qquad \qquad = \frac{1}{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}, \\ &\qquad \qquad = \frac{1}{(q_{\mathsf{a}+2} + rq_{\mathsf{a}+1})(q_{\mathsf{a}+2} + r + 1q_{\mathsf{a}+1})}, \end{split} \tag{2}$$

Hence

$$PQ' - P'Q = -1 \text{ if } n \text{ be odd,}$$

$$= -1 \text{ if } n \text{ be even.}$$
(3)

(2) and (3) are sufficient to establish 1° and 2°.

3°. Since P<sub>j</sub>(Q − p<sub>n-j</sub>/q<sub>n-1</sub> = ± 1/q<sub>n-1</sub> (q<sub>n-2</sub> + rq<sub>n-1</sub>), and since x<sub>i</sub> obviously lies between P<sub>j</sub>(Q and p<sub>n-1</sub> 'q<sub>n-1</sub>, it follows that the intermediate convergent P<sub>j</sub>(Q differs from the continued fraction by less than Γ'q<sub>n-1</sub>, Q, a fortiori by less than Γ'q<sub>n-1</sub>, ?...

§ 15.] If we take all the principal convergents of old order with their intermediates wherever the partial quotients differ from

unity, and form the series

$$\frac{0}{1}$$
, ...,  $\frac{p_1}{q_1}$ , ...,  $\frac{p_s}{q_s}$ , ...,  $\frac{p_{n-2}}{q_{n-2}}$ , ...,  $\frac{p_n}{q_n}$ , ... (A),

and likewise all the principal convergents of even order with their intermediates, and form the series

$$\frac{1}{0}$$
, ...,  $\frac{p_2}{q_2}$ , ...,  $\frac{p_4}{q_4}$ , ...,  $\frac{p_{n-3}}{q_{n-3}}$ , ...,  $\frac{p_{n-1}}{q_{n-1}}$ , ... (B),

then (A) is a series of commensurable quantities, increasing in complexity and increasing in magnitude, which continuedly approach the continued fraction; and (B) is a series of commensurable quantities, increasing in complexity and decreasing in magnitude, which continuedly approach the same; and it is impossible between any consecutive pair of either series to insert a commensurable quantity which shall be less complex than the more complex of the two.

If the continued fraction be non-terminating, each of the two series  $(\Lambda)$  and (B) is non-terminating.

If the continued fraction terminates, one of the series will terminate, since the last member of one of them will be the last convergent to  $x_1$ ; that is to say,  $x_i$  itself. The other series may, however, be prolonged as far as we please; for, if  $p_{n-1}/q_{n-1}$  and  $p_n/q_n$  be the last two convergents, the series of fractions

$$\frac{p_{n-1}}{q_{n-1}}$$
,  $\frac{p_{n-1}+p_n}{q_{n-1}+q_n}$ ,  $\frac{p_{n-1}+2p_n}{q_{n-1}+2q_n}$ , ...

For a rule for estimating the errors of principal and intermediate convergents to a continued fraction, see Hargreaves, Mess. Math., Feb. 1898.

forms either a centinually increasing or a continually decreasing series, in which no principal convergent occurs, but whose terms approach more and more nearly the value  $p_n q_n$ , that is,  $x_1^*$ .

§ 16.] We are now in a position to solve the general problem of § 121. Suppose, for example, that we are required to find the fraction, whose denominator does not exceed D, which shall approximate most closely by defect to the quantity  $x_i$ . What we have to do is to convert  $x_i$  into a simple continued fraction, form the series  $(\Lambda)$  of birst section, and solve that fraction from it whose denominator is either D, or, fictling that, is sthan but nearest to D, say P, Q. For, if there were any fraction nearer to  $x_i$  than  $P_iQ_i$  it would lie to the right of P Q in the series; that is to say, ould fall between P Q and the next fraction P Q of the series, or between two fractions still more complex. Hence the denominator of the supposed fraction will be greater than  $Q_i$  and hence greater than D.

Similarly, the fraction which most nearly approximates to  $x_1$  by excess, and whose denominator does not exceed  $D_i$  is obtained

<sup>•</sup> This may also be seen from the fact that the continued fraction  $a_1+\frac{1}{a_2+}\dots\frac{1}{a_{n-1}}$  may also be written  $a_1+\frac{1}{a_2+}\dots\frac{1}{a_{n-1}}\frac{1}{x}$ ; that is to say, we may c such the last quetient to be x, and the last convergent  $(p_{n-1}+p_{n-1})(p_{n-1})(p_{n-1})$ .

t The first general's lution of this problem was given by Wallis (see his Alachra (1685), chap, v. r. Huychens also was led to discuss it when designing the toothed | heels of his Princtarium (see his Descriptio Automati Planetarn, 1682). One of the earlier appearances of continued fractions in math mates was the value of 1 v given by Lord Bronneker (about 1655). While discussing Productor - I raction in his Arithmetica Infinitorum (1656). Wallis gives a g od many of the elementary properties of the convergents to a general continue i fraction, including the rule for their formation. Saunder-on, Euler, and Lambert all helped in developing the theory of the subject. See two interesting bibliographical papers by Gunther and Franco, Lalletino di Rella circi hia e di Storia delle Scienze Mathematiche e I is che, t. vii. In this chapter we have mainly followed Lacrange, who gave the first full expectation of it in his a librous to the French edition of Euler's Algebra (1795). We may here direct the attention of the reader to a series of comprehensive artists on continue I fractions by Stern, Crelle's Jour., x., \$1. 33101.

by taking that fraction in series (B) of last section whose denominator most nearly equals without exceeding D.

N B — If the denominator in the (A) series which most nearly equals without exceeding D be the denominator of an intermediate convergent, the denominator in the (B) series which most nearly equals without exceeding D will be the denominator of a principal convergent.

Example 1.

To find the fraction, whose denominator does not exceed 60, which approximates most closely to  $\frac{779}{207}$ .

We have 
$$\frac{779}{207} = 3 + \frac{1}{1+} \frac{1}{3+} \frac{1}{4+} \frac{1}{2+} \frac{1}{5}$$
.

The odd convergents are  $\frac{0}{1}$ ,  $\frac{3}{1}$ ,  $\frac{15}{4}$ ,  $\frac{113}{38}$ ;

 $\frac{1}{0}$ ,  $\frac{4}{1}$ ,  $\frac{64}{17}$ ,  $\frac{779}{207}$ . the even convergents

The two series are

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{7}{2}, \frac{11}{3}, \frac{15}{4}, \frac{79}{21}, \frac{113}{38}, \frac{922}{245}, \frac{1701}{452}, \frac{2480}{659}, \dots (A),$$

$$\frac{1}{0}, \frac{4}{1}, \frac{19}{5}, \frac{34}{9}, \frac{49}{13}, \frac{64}{17}, \frac{207}{55}, \frac{350}{93}, \frac{493}{131}, \frac{636}{169}, \frac{779}{207}$$
 (B).

Hence, of the fractions whose denominators do not exceed 60, 143/38 is the closest by defect and 207/55 the closest by excess to 779/207.

Of these two it happens that 143/38 is the closer, although its denominator is less than that of 207/55; for we have 143/38 = 3.76315.... 207/55 = 3.76363 . . . , and 779/207 = 3.76328 . . . For a rule enabling us in most cases to save calculation in deciding between the closeness of the (A) and (B) approximations, see Exercises xxx., 10.

### Example 2.

Adopting La Caille's determination of the length of the tropical year as 3654 5h 48' 49", so that it exceeds the civil year by 5h 48' 49", we are required to find the various ways of rectifying the calendar by intercalating an integral number of days at equal intervals of an integral number of years, (Lagrange.)

The intercalation must be at the rate of  $\frac{20929^4}{86400}$  per year; that is to say, at the rate of 20929 days in 86400 years. If, therefore, we were to intercalate 20929 days at the end of every 864 centuries we should exactly represent La Caille's determination. Such a method of rectifying the calendar is open to very obvious objections, and consequently we seek to obtain an approximate rectification by intercalating a smaller number of days at shorter intervals,

If we turn 86400/20020 into a continued fraction and form the (A) and (P) series of convergents, we have (counting the earlier terms)

Hence, if we take approximations which err by excess, we may with increasing accuracy intercalate 1 day every 4 years, 8 every 33, 39 every 161, and so on 1; and be assured that each of these gives us the greatest accuracy obtainable by taking an internal number of days less than that indicated in

the text of the series.

The (B) series may be used in a similar manner t.

Example 3.

An eclipse of the sun will happen if at the time of new moon the earth be within about 13° of the line of nobes of the orbits of earth and moon. The peri, alletwen two new moons is on the average 29.5306 days, and the mean syr-like period of the earth and moon is 3466496 days. It is required to calculate the simpler periods for the recurring of eclipses.

Suppose that after any the same time from a new moon the moon and earth have made respectively the multiples x and y of a revolution, then  $x \times 29 \cdot 5306 =$ 

y = 3466196. Thence  $y/x = 295306(3466196 = 0 + \frac{1}{11 + 1} + \frac{1}{1 + 2 + 1} + \frac{1}{4 + 3 + \cdots}$ . The successive convergents to this fraction are 1/11, 1/12, 3/35, 4/47, 19/223, 61.746.

Suppose we take the convergent 4/47, the error incurred thereby will be 1/47 × 223 in excess, and we may write on the most unfavourable supposition

$$\frac{y}{x} = \frac{4}{47} - \frac{1}{47 \times 223} \,.$$

<sup>•</sup> The fraction 141 corresponds to the Julian intercalation, introduced by June; e.Core (15) = 1.33 Spires the second of Persian intercalation, said to be due to the mathematician Omar Akhayami (1979 a.c.). The method in presentine among the effective and time is the Grayerian, which corrects the Januar international to the fraction 400 (3), who has not one in the above series; in fact, 70 days very 283 years would be more ascenar. The Grayerian method has, however, the advantage of preserving ty multiples of a century. The Greeks and Base-mas with use the Julian method more, and in consequence there is a different extlation of the distribution of the distribution of the consequence of the first production of the production of the distribution of the color of the color of the distribution of the color of the distribution of the color of the distribution of the color of the co

See Lagrange s a rithe neste O e French edition of Euler's Algebra (Paris, 1997) (t. 10, p. 1997).

Hence, if x=47, y=4-1/223. But  $360^\circ/223=1^\circ$  61. Hence, 47 lumations after total eclipse, new moon will happen when the earth is less than  $1^\circ$ -61 from the line of nodes, 47 lumations after that again when the earth is less than  $3^\circ$ -2 from the line of nodes, and so on. Hence, since 47 lumations = 1388 days, eclipses will recur after a total eclipse for a considerable number of periods of 1388 days.

If we take the next convergent we find for the period of recurrence 223 lumations, which amounts to 18 years and 10 or 11 days, according as five or four leap years occur in the interval. The displacement from the node in this case is certainly less than 367/716, that is, less than half a degree, so that this is a far more certain cycle than the hax'; in fact, it is the famous "saros" of antiquity which was known to the Chaldean astronomers.

Still more accurate results may of course be obtained by taking higher convergents.

#### Exercises XXX.

- (1.) Find the first eight convergents to  $1 + \frac{1}{2+3} + \frac{1}{4+1} + \frac{1}{1+1} + \dots$ , and find the fraction nearest to it whose denominator does not exceed 600.
- (2.) Work out the problem of Exercise xxix., 4, using intermediate as well as principal convergents.
- (3.) Work out all the convergents to  $2\pi$  whose denominators do not exceed 1000.
- (4.) Solve the same problem for the base of the Napierian system of logarithms  $e=2.71828183\ldots$
- (5.) Two seales, such that 1×73 parts of the one is equal to 1860 parts of the other, are superposed so that the zeros coincide: find where approximate coincidences occur and estimate the divergence in each case.
- (6.) Two pendulums are bung up, one in front of the other. The first beats seconds exactly; the second lease 5 min, 37 sec, in 24 hours. Thy pass the vertical together at 12 o'clock noon. Find the times during the day at which the first passes the vertical, and the second does so approximately at the same time.
- (7.) Along the side AB and diagonal AC of a square field round posts are creeted at equal intervals, the interval in the two cases being the same. A person looking from a distance in a direction perpendicular to AB sees in the perspective of the two rows of posts places where the posts seem very close together ("sphosts") and places where the intervals are clear owing to approximate coincidences. Calculate the distances of the centres of the ghosts from A, and show that they grow bronder and sparser as they recede from A.
- (8.) Show that between two given fractions p/q and p'/q', such that pq' -p'q = 1, an infinite number of fractions in order of magnitude can be inserted such that between any consecutive two of the series no fraction can be found less complex than either of them.

(2) In the serie, of fractions whose denominators are 1, 2, 3, ..., a there is at least one whose denominator is r, say, such that it differs from a given irrational quantity x by less than 1/ms. (For a proof of this theorem, due to Dutchiet, not depending on the theory of continued fractions, see Seriet, Alb. Sup., 425 (4, t. p. 2.75).

(10.) If the nearest rational approximation in excess or defect (see § 16) be an intermediate convergent P/Q, where Q = M<sub>n-1</sub> + q<sub>n-p</sub>, show that the approximation in detector excess will be mearer unless Q > A<sub>n+1</sub> + q<sub>n-q</sub>. Thus,

(11.) If zero partial quotients he contrary to the usual understanding) admitted, show that every continued fraction may be written in the form  $0+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_2}+\dots$ , where  $a_1,a_2,a_3,\dots$  are each either 0 or 1. Show

the bearing of this on the theory of the so-called intermediate convergents.

(12.)  $\varpi_0 = 0$ ,  $\varpi_1 = 1$ ,  $\varpi_r = a_n$ ,  $\pi_{r+1} = \varpi_{r+2}$ ; show that  $p_{n+r}/q_{n+r} \sim p_n/q_n = \varpi_r/q_{n+r} + r_1 - p_n/q_n = (\varpi_r/q_n, -\pi_n)/(|q_{n+r} + f_{n+r}|_{n+r})$ , where  $f_n = x_n - a_n$ . (11a) items  $q_n = q_n$ ,  $q_n = q_n$ .

## CHAPTER XXXIII.

## On Recurring Continued Fractions.

EVERY SIMPLE QUADRATIC SURD NUMBER IS EQUAL
TO A RECURRING CONTINUED FRACTION.

§ 1.] We have already seen in two particular instances (chap. XXXII., § 5) that a simple surd number can be expressed as a recurring continued fraction. We proceed in the present chapter to discuss this matter more closely\*.

Let us consider the simple surd number  $(P_1 + \sqrt{R})/Q_1$ . We suppose that its value is positive; and we arrange, as we always may, that  $P_1$ ,  $Q_1$ , R shall be integers, and that  $\sqrt{R}$  shall have the positive sign as indicated. It will of course always be positive; but  $P_1$  and  $Q_1$  may be either positive or negative. It is further supposed that  $R - P_1^2$  is exactly divisible by  $Q_1$ . This is allowable, for, if  $R - P_1^2$  were, say, prime to  $Q_1$ , then we might write  $(P_1 + \sqrt{R})/Q_1 = (P_1Q_1 + \sqrt{Q_1^2R})/Q_1^2 = (P_1' + \sqrt{R'})/Q_1'$ , where  $R' - P_1^{-2} = Q_1^2(R - P_1^2) = (R - P_1^2) Q_1'$  is exactly divisible by  $Q_1'$ .

For example, to put  $\frac{1}{4}\left(2-\sqrt{\frac{5}{2}}\right)$  into the standard form contemplated, we must write

 $\frac{1}{4}\left(2-\sqrt{\frac{3}{2}}\right) = \frac{1}{4}\left(2-\frac{\sqrt{6}}{2}\right) = \frac{-4+\sqrt{6}}{-8} = \frac{-16+\sqrt{96}}{-32};$ 

so that in this case  $P_1 = -16$ ,  $Q_1 = -32$ , R = 96;  $R - P_1{}^2 = 96 - 256 = -100$ , which is exactly divisible by  $Q_1 = -32$ .

 $<sup>^{\</sup>circ}$  The following theory is due in the main to Lagrange. For the details of its exposition we are considerably indebted to Serret, Alg. Sup., chap. II

 $\leq 2$  ]. If we adopt the process and notation of chap, XXXII,  $\lesssim 3$  and 5, the calculation of the partial and complete quotients of the continued fraction which represents  $(P_1 + \sqrt{R})/Q_1$  proceeds as follows:—

$$x_{1} = \frac{P_{1} + \sqrt{R}}{Q_{1}} - a_{1} + \frac{1}{x_{2}};$$

$$x_{2} = \frac{P_{2} + \sqrt{R}}{Q_{2}} = a_{2} + \frac{1}{x_{3}};$$

$$\vdots$$

$$x_{n} = \frac{P_{n} + \sqrt{R}}{Q_{n}} = a_{n} + \frac{1}{x_{n+1}};$$
(1),

where it will be remembered that  $a_1, a_2, \ldots$  are the greatest integers which do not exceed  $x_1, x_2, \ldots$  respectively; and  $x_3, x_4, \ldots$  are each positive, and not less than unity.

It should be noticed, however, that since we keep the radical  $\sqrt{R}$  unaltered in our arrangement of the complete quotients, it by no means follows that  $P_2$ ,  $Q_3$ ,  $P_4$ ,  $Q_5$ ,  $\aleph c$ , are integers, much less that they are positive integers.

The connection between any two consecutive pairs, say  $P_n$ ,  $Q_n$  and  $P_{n-1}$ ,  $Q_{n+1}$ , follows from the equation

$$\frac{P_n - \sqrt{R}}{Q_n} = a_n + \frac{1}{(P_{n+1} + \sqrt{R})} \frac{Q_{n+1}}{Q_{n+1}}$$
(2),

or

$$\{(P_n - a_n Q_n) | P_{n+1} - Q_n Q_{n+1} + R_1^* + \{P_n - a_n Q_n + P_{n+1}\} \sqrt{R} = 0$$
(3).

It follows from (3) by chap, xi., § 8, that

$$(P_n - a_n Q_n) P_{n+1} - Q_n Q_{n+1} + R = 0.$$
  
 $P_n - a_n Q_n + P_{n+1} = 0;$ 

whence

$$P_{n+1} = a_n Q_n = P_n \tag{4},$$

$$P_{n+1}^{2} + Q_{n}Q_{n+1} = R$$
 (5).

If we write n = 1 for n in (5), we have

$$P_{n'} + Q_{n-1}Q_{n} = R \tag{6}.$$

From (5), by means of (4) and (6), we have

$$Q_n Q_{n+1} = R - (a_n Q_n - P_n)^2,$$
  
 $= P_n^2 + (I_{n-1} Q_n - (a_n Q_n - P_n)^2,$   
 $Q_{n+1} = Q_{n-1} + 2a_n P_n - a_n^2 Q_n,$   
 $= Q_{n-1} + a_n (P_n - P_{n+1})$  (7).

so that

The formula (4) and (7) give a convenient means of calculating  $P_2$ ,  $P_3$ ,  $Q_i$ ,  $P_4$ ,  $Q_i$ , &c., and hence the successive complete quotients  $x_2$ ,  $x_3$ , . . .

Qo is given by the equation

namely,

$$\begin{split} &P_{2}^{2}+Q_{2}Q_{1}=R,\\ &Q_{2}=\frac{R-(a_{1}Q_{1}-P_{1})^{2}}{Q_{1}},\\ &=\frac{R-P_{1}^{2}}{Q_{1}}+2a_{1}P_{1}-a_{1}^{2}Q_{1}. \end{split}$$

From this last equation it follows, since by hypothesis  $(R-P_3)^2/Q_1$  is an integer, that  $Q_2$  is an integer. Hence, since  $P_1$ ,  $Q_4$  are integers, it follows, by (4) and (7), that  $P_2$ ,  $P_3$ , . . . ,  $P_n$ ,  $Q_2$ , . . . ,  $Q_n$  are also all integers.

§ 3.] We shall now investigate formulae connecting  $P_n$  and  $Q_n$  with the numerators and denominators of the convergents to the continued fraction which represents  $(P_1 + \sqrt{R})_i Q_i$ .

We have (chap. XXXII., § 9)

$$\begin{split} \frac{P_1 + \sqrt{R}}{Q_1} &= \frac{p_{n-1} x_n + p_{n-2}}{q_{n-1} r_n + q_{n-2}} \\ &= \frac{p_{n-1} P_n + p_{n-2} Q_n + p_{n-1} \sqrt{R}}{q_{n-1} P_n + q_{n-2} Q_n + q_{n-1} \sqrt{R}}, \end{split} \tag{A},$$

Hence

$$(P_1 + \sqrt{R}) (q_{n-1}P_n + q_{n-2}Q_n + q_{n-1}\sqrt{R})$$
  
=  $Q_1(p_{n-1}P_n + p_{n-2}Q_n + p_{n-1}\sqrt{R})$  (1).

From (1) we derive

$$q_{n-1}P_n + q_{n-2}Q_n = Q_1p_{n-1} - P_1q_{n-1}$$
 (2);

$$p_{n-1}P_n - p_{n-2}Q_n = P_1p_{n-1} + \frac{R - P_1^2}{Q_1}q_{n-1}$$
 (3)

From (2) and (3) we obtain, since  $p_{n-1}q_{n-2} + p_{n-2}q_{n-1} = (-1)^{n-1}$ ,

$$\begin{array}{l} (-1)^{n-1}P_n - P_1\left(p_{n-1}q_{n-2} + p_{n-2}q_{n-1}\right) \\ + \frac{R}{Q_1}\frac{P_1^2}{q_{n-1}q_{n-2}} - Q_1p_{n-1}p_{n-2} - (4); \\ (-1)^{n-1}Q_n = -2p_{n-1}q_{n-1}P_1 - \frac{R}{Q_1}\frac{P_1^2}{q_{n-1}^2} + Q_1p_{n-1}^2 - (5). \end{array}$$

The formula (t) and (5) give us the required expressions, and furnish another proof that  $P_1, P_2, \ldots, P_n, Q_2, Q_3, \ldots, Q_n$  are all integral.

§ 4.] If in equation (2) of last paragraph we replace  $P_1$  by its value  $Q_1(p_{n-1}x_n-p_{n-2}),(q_{n-1}x_n+q_{n-2})-\varsigma'\hat{R}$ , derived from equation (A), we have

$$q_{n-1}P_n + q_{n-2}Q_n = \frac{(-1)^{n-1}Q_1}{q_{n-1}x_n + q_{n-2}} + q_{n-1}\sqrt{R}$$
(1).

Also, since  $x_n = (P_n + \sqrt{R})/Q_n$ , we have

$$P_n - x_n Q_n := \sqrt{R}$$
 (2).

From equations (1) and (2) we derive, by direct calculation, the tollowing four:—  $P_n$  =

$$\frac{x_n}{(q_{n-2}, q_{n-2})} \left( \left(q_{n-1} x_n - q_{n-2}\right) \left(q_{n-1} + \frac{q_{n-2}}{x_n}\right) \sqrt{R} + (-1)^{n-1} Q_i \right)$$
 (3);

 $Q_n$ 

$$\frac{1}{(q_{n-2}, \dots q_{n-1})} \{q_{n-1}, q_{n-1}, q_n, x_n + q_n\} 2 \sqrt{R} + (-1)^{n-1} Q_1 \} \quad (4);$$

 $\frac{\sqrt{R} - P_n}{\frac{C_n}{(q_{n-1} \ell_{n-2} \ell_{n-1})^n} \left\{ 2q_{n-2} \left( q_{n-1} + \frac{q_{n-2}}{x_n} \right) \sqrt{R} - (-1)^{n-1} Q_i \right\}$  (5);

2 × 11 - Qn =

$$\frac{1}{(q_{n-1}x_n+q_{n-2})} \leq r_{-1}(q_{n-1}+q_{n-2})(q_{n-1}x_n+q_{n-2})2\sqrt{R} + (-1)^{n-1}Q_i\} (6).$$

The coefficients of  $\sqrt{k}$  and  $2\sqrt{k}$  in these four formula are positive and increase without limit when n is increased without limit. Thence, since  $Q_i$  is a fixed quantity, it follows that for some value of n, say n=r, and for all greater values,  $P_n$ ,  $Q_n$ ,  $\sqrt{R}-P_n$ ,  $2\sqrt{R}-Q_n$  will all be positive. In other words, on and after a certain value of n, n=v say,  $P_n$  and  $Q_n$  will be positive; and  $P_n < \sqrt{R}$ , and  $Q_n < 2\sqrt{R}$ .

Cor. 1. Since  $P_n$  and  $Q_n$  are integers, it follows that after n=v  $P_n$  cannot have more than  $\sqrt{R}$  different values, and  $Q_n$  cannot have more than  $2\sqrt{R}$  different values; so that  $x_n=(P_n+\sqrt{R})/Q_n$  cannot have more than  $\sqrt{R}\times 2\sqrt{R}=2R$  different values. In other words, after the (the complete quotient, the complete quotients must recur within 2R steps at most.

Hence the continued fraction which represents  $(P_1 + \sqrt{R})_T Q_1$ must recur in a cycle of 2R steps at most.

Since ever after  $n = v P_n$  and  $Q_n$  remain positive, it is clear that in the cycle of complete quotients there cannot occur any one in which  $P_n$  and  $O_n$  are not both positive.

It should be noticed that it is merely the fact that  $P_n$  and  $Q_n$  ultimately become positive that causes the recurrence.

If we knew that, on and after  $n = r_* P_n$  remains positive, then it would follow, from § 2 (4), that  $Q_e$  and all following remain positive; and it would follow, from § 2 (5), that  $P_{r+1}$  and all following are each  $<\sqrt{R}$ : and hence, from (4), that  $Q_{e+1}$  and all following are each  $<2\sqrt{R}$ : and we should thus establish the recurrence of the continued fraction by a somewhat different process of reasoning.

Cor. 2. Since  $a_n$  is the greatest integer in  $(P_n + \sqrt{R})/Q_n$ , and since, if n > r,  $P_n$  and  $Q_n$  are both positive, and  $P_n < \sqrt{R}$ , and  $Q_n > 1$ , it follows that, if n > r,  $a_n < 2\sqrt{R}$ .

It follows, therefore, that none of the partial quotients in the cycle can exceed the greatest integer in  $2\sqrt{R}$ .

Cor. 3. By means of (3) and (4), we can show that ultimately

$$P_n + Q_n > \sqrt{R}$$
 (7),

Cor. 4. From § 2 (5), we can also show that ultimately

$$P_n + Q_{n-1} > \sqrt{R}$$
 (8).

Cor. 5. Since  $\sqrt{R} > P_m$ , it follows from Cor. 3 and Cor. 4 that ultimately

$$P_{m} - P_{n} < Q_{n},$$

$$< Q_{n-1}$$
(9).

EVERY RECURRING CONTINUED FRACTION IS EQUAL TO A SIMPLE OUADRATIC SURD NUMBER.

§ 5.] We shall next prove the converse of the main proposition which has just been established, namely, we shall show that every recurring continued fraction, pure or mixed, is equal to a simple quadratic surd number.

First, let us consider the pure recurring continued fraction

$$x_1 = \frac{a_1}{\bullet} + \frac{1}{a_2 + \dots + \frac{1}{a_r + \dots + +$$

Let the two last convergents to

$$a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_r}}$$

be p'/q' and p,q.

From (1) we have

$$x_1 = a_1 + \frac{1}{a_2 + 1} \cdot \cdot \cdot \cdot \frac{1}{a_r + 1} \cdot \frac{1}{x_1},$$
  
=  $\frac{p x_1 + p'}{q x_1 + q'};$ 

whence

$$qx_1^2 + (q'-p)x_1 - p' = 0$$
 (2).

The quadratic equation (2) has two real roots; but one of them is negative and therefore not in question, hence the other must be the value of  $x_1$  required.

We have, therefore,

$$x_1 = \frac{p - q' + \sqrt{(p - q')^2 + 4p''}q}{2q}$$

$$= \frac{L + \sum_{M} N}{M}, \text{ say };$$
(3),

which proves the proposition in the present case.

It should be noticed that, since  $a_i + 0$ ,  $p_i(q > 1)$ ; so that p > q > q'. Hence p - q' cannot vanish, and a pure recurring fraction can never represent a surd number of the form  $\sqrt{N}/M$ .

Next, consider the general case of a mixed recurring continued fraction.

Let

$$x_1 = a_1 + \frac{1}{a_2 +} \cdot \cdot \cdot \frac{1}{a_r + \frac{1}{a_1 +}} \frac{1}{a_2 +} \cdot \cdot \cdot \frac{1}{a_s +} \cdot \cdot \cdot (4).$$

Also let

Then, by (3),

$$y_1 = \frac{L + \sqrt{N}}{M}$$
.

From (4) we have

$$x_1 = a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_r + y_1}}$$

whence, if P/Q and P/Q be the two last convergents to  $a_1 + \frac{1}{a_1 + \dots + \frac{1}{a_n}}$ ,

$$x_1 = \frac{Py_1 + P'}{Qy_1 + Q'},$$

$$\frac{PL + PM + P\sqrt{N}}{QL + QM + Q\sqrt{N}}$$

$$= \frac{QL + QM + Q\sqrt{N}}{QL + QM + Q\sqrt{N}}$$
(6)

Hence, rationalising the denominator, we deduce

$$x_1 = \frac{U + V \sqrt{N}}{W}$$
.

Example 1.

Evaluate  $x_1 = \frac{1}{2} + \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \dots$ 

The two last convergents to  $1 + \frac{1}{2+1} \frac{1}{1}$  are 3/2 and 4/3; hence

$$x_1 = \frac{4x_1 + 3}{3x_1 + 2}$$
.

We therefore have

$$3x_1^2 - 2x_1 - 3 = 0,$$

the positive root of which is

$$x_1 = \frac{1 - \sqrt{10}}{3}$$
.

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Example 2. Evaluate  $y_1 = 1 + \frac{1}{1+1} + \frac{1}{2+1+1} + \frac{1}{1+1} + \cdots$ .

The two last convergents to  $3 + \frac{1}{1}$  are 3/1 and 13/1; and, by Example 1 above.

$$\frac{1}{2} + \frac{1}{2+1} + \dots = \frac{1+\sqrt{10}}{3}$$

We have, therefore,

$$\begin{split} y_1 &= 3 + \frac{1}{4 + (1 + \sqrt{10)/3}}, \\ &= \frac{13 \left(1 + \sqrt{10)/3 + 1}, \right. \\ \left. + \left(\frac{1}{4}\right) \left(\frac{1}{3} + \frac{1}{4}\right), \\ &= \frac{22 + 13 \sqrt{10}}{7 + 1\sqrt{10}}, \\ &= \frac{366 - 3 \sqrt{10}}{11}, \\ &= \frac{122 - \sqrt{10}}{2}. \end{split}$$

## on the continued fraction which represents $\sqrt{(C'D)}$ .

§ 6.] The square root of every positive rational number, say  $\chi(CD)$ , where C and D are positive integers, and C/D is not the square of a common-unable number, can be put into the form  $\chi'N/M$  where N/CD and M-D. Since N/M=C is an integer, we know from what precedes that  $\sqrt{N/M}$  can be developed, and that in one way only, as a continued fraction of the form

$$x_1 = a_1 + \frac{1}{a_{2} + \cdots} + \frac{1}{a_{r} + a_{1} + a_{2} + \cdots} + \frac{1}{a_{2} + \cdots} + \cdots$$
 (1).

We have, in fact, merely to put  $P_i$  o, R = N,  $Q_i = M$  in our previous formulae.

We suppose that  $\sqrt{N}/M$  is greater than unity, so that  $a_1 \neq 0$ . If  $\sqrt{N}/M$  were less than unity, then we have only to consider  $M\sqrt{N}/\sqrt{M}/N/N$ , which is greater than unity.

The acyclic part  $a_1 + \frac{1}{a_r} + \dots + \frac{1}{a_r}$  must consist of one term at

least, for we saw, in § 5, that a pure recurring continued fraction cannot represent a surd number of the form  $\sqrt{N_f}M$ . Let us suppose that there are at least two terms in this part of the fraction; and let  $P_fQ$ ,  $P_fQ$  be the two last convergents to

 $a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_r}}$ ; and p', q',  $p'_i q$  the two last convergents to

$$a_1 + \frac{1}{a_2 +} \cdot \cdot \cdot \frac{1}{a_r + a_1 + a_2 +} \cdot \cdot \cdot \frac{1}{a_s}$$
. Then, if  $y_1 = \frac{1}{a_1 +} \cdot \frac{1}{a_s +} \cdot \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot \cdot \frac{1}{a_{s+1} +} \cdot \cdot$ 

we have

$$x_1 = a_1 + \frac{1}{a_2 + \dots} \cdot \frac{1}{a_r + y_1},$$
  
=  $a_1 + \frac{1}{a_r + y_1} \cdot \frac{1}{a_r + y_1 - a_r + \dots} \cdot \frac{1}{a_r + y_r}.$ 

Hence

$$x_1 - \frac{Py_1 + P'}{Qy_1 + Q'} = \frac{py_1 + p'}{qy_1 + q'}$$
 (2).

Eliminating y, from the equations (2), we have

$$(Qq' - Q'q) x_1^2 - (Qp' - Q'p + Pq' - P'q) x_1 + (Pp' - P'p) = 0$$
 (3).

Now, if  $x_1 = \sqrt{N}$ , M, we must have

$$M^2x_1^2 - N = 0$$
 (4),

In order that the equations (3) and (1) may agree, we must have

$$Qp' - Q'p + Pq' - P'q - 0$$
 (5);

and

$$\frac{Pp'-P'p}{Qq'-Q'q} = -\frac{N}{M^2}$$
 (6).

It is easy to show that equation (6) cannot be satisfied. We have, in fact,

$$\frac{Pp' - P'p}{Qp' - Q'q} = \frac{P'p'}{Q'q'} \frac{P}{Q'Q'} \frac{P' - p}{Q'Q' - q'q'} \frac{p'}{q'}$$
(7)

But, by chap. xxxn. § 7,

$$\frac{P}{P} = \frac{p}{p'} - a_r + \frac{1}{a_{r-1} + \dots + \frac{1}{a_1} - a_s - \frac{1}{a_{s-1} + \dots + \frac{1}{a_1}}, \dots \frac{1}{a_1}$$

$$= a_r - a_s \pm f_s^*$$

where f is a proper fraction.

Similarly

$$Q' = \frac{q}{q'}$$
  $\frac{d_r + \frac{1}{d_{r-1} + \cdots} + \frac{1}{d_2} - a_e - \frac{1}{a_{s-1} + \cdots} + \frac{1}{d_2}}{d_{s-1} + \cdots}$ 

where f' is a proper fraction.

Now  $a_t - a_t$  cannot be zero, for, if that were so, we should have  $a_r - a_t$ , that is to say, the cycle of partial quotients would begin one place somer, and would be  $a_t$ ,  $a_1$ ,  $a_2$ , . . . ,  $a_{r-1}$ , and not  $a_1$ ,  $a_2$ , . . . ,  $a_r$ , as was supposed. It follows then that  $a_r - a_t$  is a positive or negative integral number. Hence the signs of P/P - p/p' and Q/Q - q/q' are either both positive or both negative, and the sign of the quotient of the two is positive. Hence the left-hand side of (6) is positive, and the right-hand side negative.

There cannot, therefore, be more than one partial quotient in the acyclic part of (1).

Let us, then, write

$$x_1 - a + \frac{1}{a_1 + a_2 + \dots} \cdot \frac{1}{a_4 + a_1 + \dots} \cdot (s),$$

$$= a + \frac{1}{a_1 + a_2 + \dots} \cdot \frac{1}{a_n + 1} \cdot \frac{1}{a_n + 1} \cdot \frac{1}{a_n + a_2 + \dots} \cdot (s),$$

Hence

$$x_1 = \frac{p_1(x_1 - a) + p'}{q_1(x_1 - a) + a'},$$

which gives

$$q[x_i^2 - (p' + q'a - q)x_i - (p - ap') = 0$$
 (9).

From (9) we obtain

$$x_1 = \frac{p' + q'a - q}{2q'} + \frac{\sqrt{(p' + q'a - q)^2 + 1(p - ap')q'}}{2q'}$$
 (10).

In order that (10) may agree with  $x_i = \sqrt{N/M}$ , we must have

$$p' + q'a + q = 0 \tag{11}$$

and

$$q^{\gamma}N^{\gamma}M^{\gamma} = (p + ap^{\gamma})q^{\prime}$$
 (12).

Cor. I. By equation (11) we have

$$p, q' + a = q, q'$$

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Hence, by chap. XXXII., § 7, Cor. 2,

$$2a + \frac{1}{a_1 + a_2 + \dots + \frac{1}{a_{s-1}}} = a_s + \frac{1}{a_{s-1} + \dots + \frac{1}{a_1}}$$

It follows, therefore, by chap. XXXII., § 3, that

 $a_{s}=2at, \quad a_{s-1}=a_{1}, \quad a_{s-2}=a_{2}, \quad \dots \quad a_{1}=a_{s-1}.$ 

In other words, the last partial quotient of the coclical part of the continued fraction which represents  $\sqrt{N} M$  is double the unique partial quotient which forms the acyclical part; and the rest of the cycle is reciprocal, that is to say, the partial quotients equidistant from the two extremes are equal.

In short, we may write

$$\frac{\sqrt{N}}{M} = a + \frac{1}{a_1 + a_2 + \dots} \cdot \frac{1}{a_2 + a_1 + 2a + a_1 + \dots} \cdot (13)$$

Cor. 2. If we use the value of q'a given by (11), we may throw (12) into the form

$$q'^2N_1M^2 = pq' - p'(q - p')$$
;

whence.

$$q'^2 N M^2 - p'^2 = pq' - p'q,$$
  
=  $\pm 1$  (11),

the upper sign being taken if p,q be an even convergent, the lower if it be an odd convergent.

§ 7.] All the results already established for  $(P_1 + \sqrt{N}), Q_1$  apply to  $\sqrt{N}/M$ . For convenience, we modify the notation as follows:—

$$\begin{aligned} & \mathbf{a}_1 &= a, & x_1 = (P_1 + \sqrt{R}) \ Q_1 = (0 + \sqrt{N}) \ M; \\ & a_2 &= a_{1s}, & x_2 \ (P_2 + \sqrt{R}) \ Q_2 = (L_s + \sqrt{N}) \ M_1; \\ & a_3 &= a_{2s}, & x_3 = (P_s + \sqrt{R}) \ Q_3 = (L_s + \sqrt{N}) \ M_2; \\ & a_s &= a_{s-1}, & x_s = (P_s + \sqrt{R}) \ Q_3 = (L_{s-1} + \sqrt{N}) \ M_{s-1}; \\ & a_{s+1} &= 2a_s, & a_{s+2} &= a_1, \end{aligned}$$

From § 2 (4), we then have

$$L_n = a_{n-1} M_{n-1} - L_{n-1}$$
 (1);

and, in particular, when n = 1.

$$L_{\perp} = \alpha_s M$$
 (1').

From § 2 (5), we have

$$L_{n}^{-} + M_{n-1}M_{n} = N$$
 (2);

and, in particular,

$$L_c^2 + MM = N$$
 (2).

From § 3 (1) and (5), we have

$$(-1)^n L_n = (N, M) g_n g_{n-1} - M \rho_n p_{n-1}$$
 (3):

$$(-)^n M_n = M \rho_n^2 + (N_c M) q_n^2$$
 (4).

These formulæ are often useful in particular applications. It will be a good exercise for the student to establish them directly.

§ 8 ] Let us call L<sub>1</sub>, L<sub>2</sub>, &c., the Rational Dividends and M, M , M<sub>1</sub>, &c., the Divisors belonging to the development of √N/M. Then, from the results of § 1, we see that

None of the rational dividends can exceed  $\sqrt{N}$ ; none of the partial quotients and none of the divisors can exceed  $2\sqrt{N}$ .

All the rational dividends, and all the divisors, are positive.

It is, of course, obvious that the rational dividends and the divisors form eyels collatered with the cycle of the partial and total quotients; manely, just as we have

 $\alpha_{s+1} = a_1, \qquad \alpha_{s+2} = a_2,$  so we have

L.  $L_{ij}$   $L_{ij}$   $L_{ij}$   $L_{ij}$ 

and

$$M_{-1} = M_1, \quad M_{+2} = M_2,$$
 (2).

(1),

We can also show that the cycles of the rational dividends and of the divisors have a reciprocal property like the cycle of the partial quotients; manely, we have

For. by 5.7 (2).

$$L_{i+1} + M = M_i - L_i + M_i M_i$$

but  $L_{i+i} = L_i$  and  $M_{i+1} = M_{i+i}$  bence

(5).

Again, by § 7 (1).

$$L_{\cdots} = a_{\cdot}M_{\cdot} - L_{\cdot}$$
:

but  $L_{s+1} = L_1$ ,  $\alpha_s = 2\alpha$ ,  $M_s = M$ , hence we have

$$L_0 = 2aM - L_{\odot}$$

Now, by § 7 (1'),  $L_1 = aM$ , hence

$$L_1 = 2L_1 - L_{12}$$

therefore

refore 
$$L_{\bullet} = L_1$$
  
Again, by § 7 (2),

 $L_s^2 + M_s M_{s-1} = L_s^2 + M_s M_s$ 

whence, bearing in mind what we have already proved, we have
$$M_{i-1} = M_i$$
(6).

Once more, by § 7 (1),

$$L_s = a_{s-1} M_{s-1} - L_{s-1}$$
,  
 $L_o = a_1 M_1 - L_0$ .

Now  $M_{s-1} = M_1$  and  $a_{s-1} = a_1$ , hence

$$L_s - L_z = L_1 - L_{s-1}$$

But  $L_s = L_1$ , hence  $L_{\cdot,\cdot} = L_{\cdot,\cdot}$ 

Proceeding step by step, in this way, we establish all the equations (3).

It appears, then, that we may write the cycles of the rational dividends and of the divisors thus-

$$L_1, L_2, L_3, \ldots, L_3, L_2, L_1;$$
  
 $M_1, M_2, M_3, \ldots, M_3, M_2, M_1, M.$ 

Since M precedes  $M_1$ , we may make the cycle of the divisors commence one step earlier, and we thus have for partial quotients, rational dividends, and divisors the following cycles:-

That is to say, the cycle of the rational dividends is collateral with the cycle of the partial quotients, and is completely reciprocal; the cook of the divisors begins one step earlier\* (that is, from the gery beginning), and is reciprocal after the first term.

[8–9.] The following theorem forms, in a certain sense, a converse to the propositions just established regarding the cycles of the continued fraction which represents  $\sqrt{N_L M_*}$ .

$$\begin{array}{lll} I_{i}f & L_{n_{i}} = L_{n+1}, & M_{m_{i}} = M_{n_{i}}, & a_{m_{i}} = a_{n_{i}}, \\ t \ln n & L_{m-1} = L_{n+2}, & M_{m-1} = M_{n+1}, & a_{m-1} = a_{n+1} \end{array} \tag{1}.$$

We have, by § 7 (2),

$$L_{n}^{2} + M_{n}M_{m-1} = L_{n+1}^{2} + M_{n+1}M_{n+1}$$

whence, remembering our data, we deduce

$$M_{n-1} = M_{n+1}$$
 (2)  
Again, by § 7 (1),  
 $L_{n+1} + L_{n-1} = a_{n-1} M_{n-11}$   
 $L_{n+2} + L_{n+3} = a_{n-1} M_{n-13}$ 

whence, since  $L = L_{ext}$  by data.

$$L_{m-1} - L_{n+2} \cdot (a_{m-1} - a_{n+1}) M_{m-1},$$
  
=  $(a_{m-1} - a_{n+1}) M_{n+1}$  (3).

If  $L_{++} > L_{n+2}$ , we may write (3)

$$(L_{n-1} - L_{n+1}) M_{n+1} = a_{m-1} - a_{n+1}$$
 (4);

if  $L = L_{n+2}$ , we may write

so that

$$(L_{n+2}-L_{n-1})M_{m+1}=a_{n+1}-a_{m-1}$$
 (5).

But, by ( 4 (9), the left-hand sides of (4) and (5) (if they differ to in (9) are each < 1, while the right-hand sides are each positive integers (if they differ from 9).

positive integers (if they differ from 0).
It follows, then, that each side of equation (3) must vanish,

$$L_{m-1} = L_{m+2} \tag{6}.$$

$$a_{m-1} = a_{n+1}$$
 (7).

which completes the proof. (4),

<sup>•</sup> The fact that the cycle of the dissours begins one step earlier than the cycle of the partial quotients and rational dividends is true for the general recurring continued fraction. Several other propositions proved for the spould case two under one-detailed have a more general application. The creamstance, are left by it water houseful to discover.

Cor. 1. Starting with the equations in the second lim of (1) as data, we could in like manner prove that

$$L_{m-2} = L_{n+3}$$
,  $M_{m-1} = M_{n+3}$ ,  $a_{m-2} = a_{n+3}$ ;

and so on, forwards and backwards.

Cor. 2. If we put m = n, the conditions in (1) become

$$L_n = L_{n+1}$$
,  $M_n = M_n$ ,  $a_n = a_n$ ;

in other words, the conditions reduce to

$$L_n = L_{n+1}$$
;

and the conclusion becomes

$$L_{n-1} = L_{n+2}$$
,  $M_{n-1} = M_{n+1}$ ,  $a_{n-1} = a_{n+1}$ .

Hence, if two consecutive rational dividends be equal, they are the middle terms of the cycle of rational dividends, which must therefore be an even cycle; and the partial quotient and divisor corresponding to the first of the two rational dividends will be the middle terms of their respective endes, which must therefore be odd cycles.

Cor. 3. If we put m = n + 1, the conditions in (1) reduce to

$$M_{n+1} = M_n$$
,  $\alpha_{n+1} = \alpha_n$ ;

and the conclusion gives

$$L_n = L_{n+2}$$
,  $M_n = M_{n+1}$ ,  $\sigma_n = a_{n+1}$ .

Using this conclusion as data in (1), we have as conclusion

$$L_{n-1} = L_{n+3}$$
,  $M_{n-1} = M_{n+2}$ ,  $a_{n-1} = a_{n+2}$ ;

and so on.

Hence, if two consecutive divisors  $(M_n, M_{n+1})$  be equal, and also the two corresponding partial quadrats  $(a_n, a_{n+1})$  be equal, these two pairs are the middle terms of their respective cycles, which are both even; and the rational dividend  $(L_{n+1})$  corresponding to the second member of either pair is the middle term of its cycle, which is odd.

These theorems enable us to save about half the labour of calculating the constituents of the continued fraction which represents  $\sqrt{N}/M$ . In certain cases they are useful also in reducing surds of the more general form  $(L+\sqrt{N})/M$  to continued fractions.

Example 1.

Express  $\sqrt{8463}/39$  as a simple continued fraction; and exhibit the cycles of the rational dividends and of the divisors.

We have

$$\begin{array}{c} \sqrt{5163} = 2 + \frac{-78 + \sqrt{8163}}{39} = 2 + \frac{1}{(78 + \sqrt{8163})/61} \\ 78 + \sqrt{8163} = 2 + \frac{-44 + \sqrt{8163}}{61} = 2 + \frac{1}{(14 + \sqrt{8163})/107} \\ \frac{44 + \sqrt{8463}}{107} = 1 + \frac{-63 + \sqrt{8163}}{107} = 1 + \frac{1}{(63 + \sqrt{8163})/12} \\ \frac{63 + \sqrt{8463}}{42} = 3 + \frac{-63 + \sqrt{8463}}{42} = 3 + \frac{1}{(63 + \sqrt{8463})/107} \\ \frac{63 + \sqrt{8463}}{107} = 1 + \Delta c. \end{array}$$

Since we have now two successive rational dividends each equal to 63, we know that the cycle of partial quotients has culminated in 3. Hence the cycles of partial quotients, rational dividends, and divisors are—

Divisors . . . 39, 61, 107, 42, 107, 61; and we have

 $\frac{\sqrt{8463}}{39} = 2 + \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{1+2+4+} + \cdots$ 

Example 2.

If c denote the number of partial quotients in the cycle of the continued fraction which represents  $\sqrt{N/M_s}$  prove the following formulæ:—

If c = 2t,

$$\frac{p_{f}}{q_{s}} = \frac{p_{t+1} q_{t} + p_{t} q_{t-1}}{q_{t} (q_{t+1} + q_{t-1})}$$
(I.);

if c = 2t + 1.

$$\frac{p_{e}}{q_{e}} = \frac{p_{t+1}q_{t+1} + p_{t}l_{t}}{q_{t+1}^{2} + q_{t}^{2}}$$
(II.):

of m be any positive integer.

$$\frac{p_{zme}}{q_{mne}} = \frac{p_{me}^2 + (N/M^2) q_{me}^2}{2p_{me} q_{me}}$$
(III.).

For brevity we shall prove (III.) alone. The reader will find that (L) stration, so chips  $x_1x_2 = x_1x_2 = x_1x_2 = x_2 = x_1x_2 = x_2 = x_1x_2 = x_1x_2$ 

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$$\begin{split} P_{-mc} &= a_1 + \frac{1}{a_1} + \frac{1}{a_1} + \frac{1}{2a_1} + \frac{1}{a_1} \frac{2m \text{ cycles}}{a_1}, \\ &= \frac{1}{a_1} + \frac{1}{a_1} + \frac{1}{a_1} + \frac{1}{a_1} + \frac{1}{a_1} + \frac{1}{a_1 + a_1 + p_{out}/j_{out}} (m \text{ cycles}), \\ &= (a + p_{out}/j_{out} + p_{out} + p_{out}) \\ &= (a + p_{out}/j_{out} + p_{out} + p_{out}) \\ &= (a + p_{out}/j_{out} + p_{out} + p_{out}) \\ &= (a + p_{out}/j_{out} + p_{out} + p_{out}) \end{aligned}$$
(a)

Now the equations (2) and (3) of § 3 give us

$$q_{me}P_{me+1} + q_{me-1}Q_{me+1} = Mp_{me}$$
  
 $p_{me}P_{me+1} + p_{me-1}Q_{me+1} = (N/M)q_{me}$ 
( $\beta$ ).

In the present case,

$$P_{mc+1} = P_{c+1} = L_c = L_1 = aM,$$
  
 $Q_{mc+1} = Q_{c+1} = M_c = M.$ 

The equations (B) therefore give

Also

$$\begin{cases}
 aq_{me} + q_{me-1} = p_{me} \\
 ap_{me} + p_{me-1} = (N/M^2) q_{me}
\end{cases}$$
(7)

From (a) and (γ) (III.) follows at once.

The formulæ (1.), (III.), (III.) enable us, after a certain number of convergents to  $\sqrt{N}/M$  have been calculated, to calculate high convergents without finding all the intermediate ones.

Consider, for example,

$$\frac{\sqrt{8463}}{39} = 2 + \frac{1}{2+} \frac{1}{1+} \frac{1}{3+} \frac{1}{1+} \frac{1}{2+} \frac{1}{4+} .$$

Here c=6, t=3, and we have for the first four convergents 2/1, 5/2, 7/3. 26/11: hence

$$\begin{split} & \underbrace{P_6}_{q_6} = \underbrace{P_8 (q_1 + P_2 q_2)}_{q_1}, \\ & \underbrace{q_6}_{q_1} = \underbrace{q_2 (q_1 + q_2)}_{q_1}, \\ & \underbrace{26 (3.3 + 7 \times 2)}_{q_2} = \underbrace{92}_{q_2}, \\ & \underbrace{P_{12}}_{p_1} = \underbrace{P_6^2 + (N_1 M^2)}_{p_2} q_2^2, \\ & \underbrace{P_{12}}_{q_2} = \underbrace{p_{12}^2 + (N_1 M^2)}_{2 \times 10^2 \times 30} = \underbrace{16027}_{7176}; \\ & \underbrace{P_{21}}_{2 \times 10^2 \times 30^2} = \underbrace{P_{12}^2 + (N_1 M^2)}_{2 \times 30^2 \times 16927 \times 7176}, \\ & \underbrace{16027 \times 30^2 \times 16927 \times 7176}_{2 \times 30^2 \times 16927 \times 7176}. \end{split}$$

The rapidity and elegance of this method of forming rational approximations cannot fail to strike the reader.

#### EXERCISES XXXI.

Express the following surd numbers as simple continued fractions, and exhibit the cycles of the partial quotients, rational dividends, and divisors :-

(3.) 
$$\sqrt{(\frac{1}{4})}$$
.

(4.) 
$$\sqrt{(61)}$$

(5.) 
$$\frac{2+\sqrt{(29)}}{5}$$
. (6.)  $1+\sqrt{\frac{2}{3}}$ .

(7.) Express the positive root of x²-x-4=0 as a continued fraction, and find the 6th convergent to it.

(8.) Express both roots of 2x2-6x-1=0 as continued fractions, and point out the relations between the various cycles in the two fractions.

(9.) Show that

$$(a^2 + b) = a + \frac{b}{2a + 1}$$
...;  
 $(a^2 - b) = a - \frac{b}{2a + 1}$ ....

(10.) Express  $\varsigma'(a^2+1)$  as a simple continued fraction, and find an expression for the nth convergent,

Evaluate the following recurring continued fractions, and find, where you can, closed expressions for their nth convergents; also obtain recurring formula for simplifying the calculation of high convergents:—

(11.) 
$$a + \frac{1}{a+} \cdot \cdot \cdot \cdot$$
  
(12.)  $\frac{1}{a-} \cdot \cdot \cdot \cdot$   
(13.)  $\frac{1}{a+b+} \cdot \cdot \cdot \cdot$ 

Show, in this case, that

(14.) 
$$p_{2n+2} - 2p_{2n} + p_{2n-2} = ahp_{2n},$$

$$\frac{1}{1+} \frac{1}{1+} \cdots \frac{1}{2+} \cdots,$$

where the cycle consists of n units followed by 2.

(15.) Show that

$$\begin{pmatrix} 1 & 1 \\ \tilde{x}_{+} & 1x_{+} & \cdots \end{pmatrix} \begin{pmatrix} \underline{1} \\ 2x_{+} & \cdots \end{pmatrix}$$

is independent of x.

(16.) Show that

$$(x + \frac{1}{2x_{+}}, \dots)^{2} - (x - \frac{1}{2x_{-}}, \dots)^{2} = 2.$$
(17.) If  $x = a + \frac{1}{b_{+}}, \frac{1}{c_{+}}, \dots, y = b + \frac{1}{c_{+}}, \frac{1}{a_{+}}, \dots, y = b + \frac{1}{a_{+}}, \dots, y = a + b + c_{+}, \dots, y =$ 

show that

$$-\frac{2\left(x+\frac{a+c}{2n},\frac{a+c}{(a+c)},\frac{(a+b+c)}{a+c}\right)}{2n}=\frac{1}{bc+1}+\frac{1}{ca+1}+\frac{1}{ab+1},$$

(15.) Snow that

$$\left(\frac{a}{b+1}\dots\right)^2 = \frac{a^2}{2a+b^2}\dots$$

(19.) If p be the numerator of any convergent to √2, then 2p<sup>2</sup> +1 will also be the numerator of a convergent, the upper or lower sign being taken according as p(q) is an old or an even convergent; also, if q, q' be two consecutive denominators, a<sup>2</sup> + a<sup>2</sup> will be a denominator.

(20.) Evaluate

$$\frac{1}{1+}$$
  $\frac{1}{1+}$   $\cdots$   $\frac{1}{n-}$   $\cdots$  ,

where the cycle consists of  $\overline{n-1}$  units followed by n.

(21.) In the case of  $\frac{1}{1+}\frac{1}{4+}\dots$ , prove that

$$p_{2n} = q_{2n+1} = \{(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}\}/2\sqrt{2},$$
  
 $p_{2n+1} = \frac{1}{4}q_{2n} = \{(\sqrt{2}+1)^{2n} - (\sqrt{2}-1)^{2n}\}/4\sqrt{2}.$ 

(22.) Convert the positive root of ax²+akx-b=0 into a simple continued fraction; and show that p<sub>n</sub> and q<sub>n</sub> are the coefficients of x<sup>n</sup> in (x + bx²-x¹)(1 − ab+2·x²+x¹) and (ax + ab+1·x²+x¹)/(1 − ab+2·x²+x¹) respectively.

Hence, or otherwise, show that if  $\alpha$ ,  $\beta$  be the roots of  $1 - (ab + 2)z + z^2 = 0$ , then

$$\begin{split} ap_{2n} &= bq_{2n-1} = ab \, \frac{a^n - \beta^n}{a - \beta} \,, \\ p_{2n+1} &= q_{2n} \quad = \frac{(a^{n+1} - \beta^{n+1}) - (a^n - \beta^n)}{a - \beta} \,. \end{split}$$

(23.) If the number of quotients in the cycle of

$$\frac{\sqrt{N}}{M} = a + \frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_2 + \frac{1}{a_1} + \frac{1}{2a_1} + \dots + \frac{1}{a_n} + \frac{1}{a_n}} \dots \text{ be } c,$$

show that

$$\alpha + \frac{1}{\alpha_1 +} \cdot \cdot \cdot \cdot \frac{1}{\alpha_1 +} \frac{1}{2\alpha +} \frac{1}{\alpha_1 +} \cdot \cdot \cdot \cdot \frac{1}{\alpha_1 +} \frac{1}{\alpha} (m \text{ cycles}) = \frac{N \eta_{mc}}{M^2 p_{mc}}.$$

(24.)\* If c be the number of quotients in the cycle of  $\sqrt{N/M}$ , show that if c = 2t + 1,

$$\frac{p^2_{t-r-1} + p^2_{t+r}}{q^2_{t-r-1} + q^2_{t+r}} = \frac{N}{M^2},$$

 $r=0, 1, \ldots, t-1;$ and if c=2t,

$$\frac{p_{t-r-1}p_{t-r-1}+p_{t+r-1}p_{t+r}}{q_{t-r-2}t_{t-r-1}+q_{t+r-1}q_{t+r}} = \frac{N}{M}.$$
(25.)† If  $\sqrt{Z} = a + \frac{1}{a_1 + a_2 + \cdots} + \frac{1}{a_2 + \frac{1}{a_1 + 2} + \cdots} + \frac{1}{a_1 + 2} + \cdots$ , and if the convergent

For solutions of Exercises 24 and 26-29 see Muir's valuable little tract on The Expression of a Quadratic Surd as a Continued Fraction, Glaspow (Machense), 1874.

<sup>†</sup> In connection with Exercises 25 and 30-32 see Serret's Cours d'Algèbre Supérieure, 3me éd., t. I., chaps. I. and II.

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# APPLICATIONS TO THE SOLUTION OF DIOPHANTINE PROBLEMS.

§ 10.] When an equation or a system of equations is indeterminate, we may limit the solution by certain extraneous conditions, and then the indeterminateness may become less in degree or may cease, or it may even happen that there is no solution at all of the kind demanded.

Thus, for example, we may require (I.) that the solution be in rational numbers: (II.) that it be in integral numbers; or, still more particularly. (III.) that it be in positive integral numhers Problems of this kind are called Diophantine Problems. in honour of the Alexan-Irine mathematician Diophantos, who, so far as we know, was the first to systematically discuss such problems, and who showed extraordinary skill in solving them\*. We shall confine ourselves here mainly to the third class of Diophantine problems, where positive integral solutions are required, and shall consider the first and second classes merely as stepping-stones toward the solution of the third. We shall also treat the subject merely in so far as it illustrates the use of continued fractions: its complete development belongs to the higher arithmetic, on which it is beyond the purpose of the present work to enter+.

# Equations of the 1st Degree in Two Variables.

\$ 11.1 Since we are ultimately concerned only with positive integral solutions, we need only consider equations of the form ax + by = c, where a, b, c are positive integers. We shall suppose that any factor common to the three coefficients has been

See Heath's Diophantos of Abzandria (Camb. 1555).

<sup>+</sup> The reader who wishes to pursue the study of the higher arithmetic should first read Theory of Numbers, Part I. (1892) by G. B. Mathews, M.A.; then the late Hanry Smith's series of Deports on the Theory of Numbers, published in the Annual Reports of the British Association (1850-60-61-62); then Lagendre, Thiorie des Nombres; Princhet's Vorlesungen über Zahlenthrore. . i. by Delekind; and finally Gauss's Dieg sintrores Arithmetica. He will then be in a p sition to master the various special memoirs in which Jac. bi, Hermite, Kummer, Henry Sm.th, and others have developed this great branch of place mathematica-

removed. We may obviously contine ourselves to the cases where a is prime to  $b_1$  or, if x and y be integers, any factor common to a and b must be a factor in c. In other words, if a be not prime to b, the equation  $ax \pm by - c$  has no integral solution,

§ 12 | To find all the integral solutions of ax - by = c; and to separate the positive integral solutions.

We can always find a particular integral solution of

$$ax - by - c (1).$$

For, since a is prime to b, if we convert a b into a continued fraction, its last convergent will be a b. Let the penultimate convergent be p q, then, by chap. XXXII., § 8,

Therefore 
$$a_I - ph = \pm 1$$
 (2)

 $a\left(\pm cq\right) - b\left(\pm cp\right) = c \tag{2}$ 

$$x' = \pm cq, \quad y' = \pm cp$$
 (4)

is a particular integral solution of (1),

Next, let (x, y) be any integral solution of (1) whatever. Then from (1) and (3) by subtraction we derive

$$a^{-1}x = -\pm cp_1^2 + b^{-1}y - (\pm cp_1^2 - 0)$$

Therefore

Hence

$$|x - (\pm cq)| |y - (\pm cp)| = b a$$
 (5).

Since a is prime to b, it follows from (5), by chap. III., Exercises IV.. 1, that

$$x - (\pm cq) = ht, \quad y - (\pm cp) = at,$$

where t is zero or some integer positive or negative. Hence every integral solution of (1) is included in

$$x = \pm cq + ht$$
,  $y = \pm cp + at$  (6),

where the upp r or lower sign must be taken according as the upper or lower sign is to be taken in (2).

Finally, let us discuss the number of possible integral solutions, and separate those which are positive

1° It a h = f then the upper sign must be taken in (2), and we have

$$x = (q : bt, \quad g = p : at)$$
 (6).

There are obviously an infinity of integral solutions. To get positive values for x and y we must (since cp/n < cq/h) give to t values such that  $-cp \ a > t > + \infty$ . There are, therefore, an infinite number of positive integral solutions.

2°. If a/b < p/q, so that cp/a > cq/b, we must write

$$x = -cy + bt$$
,  $y = -cp + at$  (6").

All our conclusions remain as before, except that for positive solutions we must have  $ep/a \gg t \gg +\infty$ .

We see, therefore, that ax - by = c has in all cases an infinite number of positive integral solutions.

§ 13.] To find all the integral solutions of

$$ax + by = c (7),$$

and to separate the positive integral solutions.

We can always find an integral solution of (7); for, if p and q have the same meaning as in last paragraph, we have

$$(\pm cq)\alpha + (\mp cp)b = c$$
 (8),

that is, 
$$x' = \pm cq$$
,  $y' = \mp cp$  is a particular integral solution of (7).

By exactly the same reasoning as before, we show that all the integral solutions of (7) are given by

$$x = +cq - ht$$
,  $y = \mp c\rho + at$  (9);

so that there are in this case also an infinity of integral solutions.

To get the positive integral solutions :-

1°. Let us suppose that a/b > p/q, so that cp/a < cq/b. Then the general solution is

$$x = cq - bt$$
,  $y = -cp + at$  (9').

Hence for positive integral solutions we must have cp/a > t $\Rightarrow cq/b$ .

2°. Let us suppose that a/b < p/q, so that cp/a > cq/b, then

$$x = -cq - bt$$
,  $y = cp + at$  (9").

Hence for positive integral solutions we must have  $-cp/a \geqslant t \geqslant -cq/b$ .

In both these cases the number of positive integral solutions is limited. In fact, the number of such solutions cannot exceed  $1 + cq(b + cp)x_1$ ; that is, since  $\lfloor (q - pb) \rfloor = 1$ , the number of positive integral solutions of the equation ax + by = c cannot exceed 1 + cab.

Example 1. To find all the integral and all the positive integral solutions of 8x + 15y = 159.

We have

$$\frac{8}{13} = \frac{1}{1+1} \frac{1}{1+1} \frac{1}{1+1} \frac{1}{2} \frac{1}{2}$$

The penaltunate convergent is 3.5; and we have

$$5 \times 5 - 13 + 3 = 1$$
,

 $S(797) + 13 \left(-477\right) = 159.$  Hence a particular solution of the given equation is  $x' = 795, \ y' = -477;$  and

ie general solution is 
$$x = 795 - 13t$$
,  $y = -477 + 5t$ .

For positive interal solutions we must have 795/13 < t < 177/8, that is,  $61 \frac{1}{3} < t < 15 < 15 < 177/8$ . The only admissible values of t are therefore 60 and 61; t suggive x = 15, y = 3, and x = 2, y = 11, which are the only positive integral solutions.

Example 2. Find all the positive integral solutions of 3x + 2y + 3z = 8. We may write this equation in the form

$$3x + 2y = 4 - 3z,$$

from which it appears that these solutions alone are admissible for which z>0. It or 2.

The general integral solution of the given equation is obviously

x = 5 - 3z - 2t, y = -8 + 3z + 3t.

In order to obtain  $p(\cdot,\cdot)$  is values for x and y, we must give to t integral values bying between 4s,  $\{t$  and 4s + t. The admissible values of t are 3 and 4, when t = 0, 2, when t = 1; and 1, when t = 2. Hence the only restrict mineral so oftons are

In a singlet way we may treat any single equation involving more than

[14] Any system of equations in which the number of variables exceeds the number of equations may be treated by methods which depend ultimately on what has been already there.

Consider, for example, the system

$$ax + by + cz = d \tag{1},$$

$$a'x + b'y + c'z = d'$$
 (2),

where a, b, c, d, a', &c. denote any integers positive or negative.

This system is equivalent to the following:—

$$-(ca') x + (bc') y = (dc')$$
 (3),

$$ax + by + cz = d \tag{4},$$

where (ca') stands for ca' - c'a, &c.

Let  $\delta$  be the G.C.M. of the integers (ae'), (he'). Then, if  $\delta$  be not a factor in (de'), (3) has no integral solution, and consequently the system (1) and (2) has no integral solution.

If, however,  $\delta$  be a factor in (de'), then (3) will have integral solutions the general form of which is

$$x = x'' + (bc') t/\delta, \quad y = y'' + (ca') t/\delta$$
 (5),

where (x'', y'') is any particular integral solution of (3), and t is any integer whatever.

If we use (5) in (4), we reduce (4) to

$$cz - c (ab') t/\delta = d - ax'' - by'$$
(6),

where  $c(ab')/\delta$  is obviously integral.

In order that the system (1), (2) may be soluble in integers, (6) must have an integral solution. Let any particular solution of (6) be z = z', t = t'. Then

$$\frac{z-z'}{t-t'}=\frac{(ab')}{\delta}.$$

Hence, if  $\epsilon$  be the G.C.M. of (ab') and  $\delta$ , that is, the G.C.M. of (bc'), (ca'), (ab'), then

$$z = z' + (ab') u/\epsilon, \quad t = t' + \delta u/\epsilon \tag{7},$$

where u is any integer.

From (5) and (7) we now have

$$x = x' + (bc') u/\epsilon$$
,  $y = y' + (ca') u/\epsilon$ ,  $z = z' + (ab') u/\epsilon$  (S),  
where  $x' = x'' + (bc') t'/\delta$ ,  $y' = y'' + (ca') t'/\delta$ .

If in (8) we put u=0, we get x=x', y=y', z=z'; therefore (x',y',z') is a particular integral solution of the system (1), (2). A little consideration will show that we might replace (x',y',z'). A particular integral solution whatever. Hence (8) gires all

the integral solution  $e^{i\phi}(1)$ , (2), (x', y', z') being any particular integral solution, e the G.C.M. of (bx'), (ca'), (ab'), and u any integer whatever.

The positive integral solutions can be found by properly

limiting u.

Example: 3x + 4y + 27z = 34, 3x + 5y + 21z = 29.

Here  $(h^*)=-51$ , (va')=18, (ah')=3. Hence  $\epsilon=3$ ; a particular integral solution is (1,1,1); and we have for the general integral solution

x = 1 - 17u, y = 1 + 6u, z = 1 + u.

The only positive integral solution is  $x=1,\ y=1,\ z=1.$ 

# Equations of the 2nd Degree in Two Variables.

§ 15.] It follows from § 7 (1) that, if  $p_n|q_b$  be the nth convergent and M, the (n+1)th rational divisor belonging to the development of  $\mathbf{x}(t^*|D)$  as a simple periodic continued fraction, then

 $D_{P_n}^2 = C_{P_n}^2 - (-)^n M_n$  (1).

Hence the equation  $Dx^2 + Cy' = +H$ , where C, D, H are positive integers, and C|D is not a perfect square, admits of an infinite number of integral solutions provided its right-hand side occurs among the quantities  $(-)^*M$  belonging to the simple continued fraction which represents  $\gamma(C|D)$ ; and the same is true of the equation  $Dx^2 + Cy' = H$ .

The most important case of this proposition arises when we suppose D = 1. We thus get the following result:—

The equation x - Cy = H, where C and H are positive integers, and U is not a perfect square, admits of an infinite name x of intervals bottoms provided its right-hand side occurs among the quarticles (-)  $M_x$  belonging to the development of C as a simple continued fraction.

Cor. 1. The equal of x = Cg = 1, where C is positive and not a perfect square always a ! its of an infinite number of solutions.

Fy what the label of a line connective equation is commonly
that note that Production I have originally proposed by Fermat
as a chalance to the large notation maticians. Solutions were obtained by

\$\\$14-16 Lagrange's theorem regarding  $x^2 - Cy^2 = \pm II + 479$ 

For, if the number of quotients in the period of  $\sqrt{U}$  be even, =2s say, then  $(-)^{\omega}M_{\omega}$  will be +1 (since here M=+1). Therefore we have

 $p_{2ts}^2 - C_{fats}^2 = +1$ ,

where t is any positive integer; that is to say, we have the system of solutions

 $x = p_{2ts}, y = q_{2ts}$  (A),

for the equation  $x^2 - Cy^2 = 1$ .

If the number of quotients in the period be odd, =2s-1 say, then  $(-)^{2s-1}M_{2s-1}$  will be =1, but  $(-)^{4s-2}M_{4s-2}$ ,  $(-)^{4s-4}M_{8s-4}$ ,  $\dots$  will each be +1. Hence we shall have the system of solutions

$$x = p_{4ts-2t}, y = q_{4ts-2t}$$
 (B),

for the equation  $x^2 - Cy^2 = 1$ .

Cor. 2. The equation  $x^2 - Cy^2 = -1$  admits of an infinite number of integral solutions provided there be an odd number of quotients in the period of  $\mathcal{J}C$ .

§ 16.] In dealing with the equation

$$x^2 - Cy^2 = \pm II \qquad (1)$$

we may always confine ourselves to what are called *primitive* solutions, that is, those for which x is prime to y. For, if x and y have a common factor  $\theta$ , then  $\theta^2$  must be a factor in H, and we could reduce (1) to  $x'^2 - Cy'^2 = \pm H/\theta^2$ . In this way, we could make the complete solution of (1) depend on the primitive solutions of as many equations like  $x'^2 - Cy'^2 = \pm H/\theta^2$  as H has source divisors.

We shall therefore, in all that follows, suppose that x is prime to y, from which it results that x and y are prime to H.

With this understanding, we can prove the following im-

portant theorem:—
If  $H < \sqrt{C}$ , all the solutions of (1) are furnished by the

If  $H \leq \mathcal{NC}$ , all the solutions of (1) are furnished by the convergents to  $\mathcal{NC}$  according to the method of § 15.

This amounts to proving that, if x = p, y = q be any primitive

integral solution of (1), then p/q is a convergent to  $\sqrt{C}$ .

Brouncher and Wallis. The complete theory, of which the solution of this equation is merely a part, was given by Lagrange in a series of memoirs which form a landmark in the theory of numbers. See especially Eurres, t. II, p. 377.

Now we have, if the upper sign be taken,

Harris

$$\begin{array}{cccc} p^{2} - Cq^{2} & H, \\ p | q - \sqrt{C} & H | q (p + \sqrt{C}q), \\ < \sqrt{C} | q (p + \sqrt{C}q), \\ < 1 | q^{2} (p | q | \sqrt{C} + 1) \end{array}$$

Now  $p'q - \sqrt{\ell'}$  is positive, therefore  $p/q \sqrt{\ell'} > 1$ . Hence  $p'q - \sqrt{\ell'} < 1/2q^2$  (3).

It follows, therefore, by chap. xxxii., § 9, Cor. 4, that p/q is

one of the convergents to \( \sigma C. \)

If the lower sign be taken, we have

$$q^2 = (1/C) p^2 = H/C$$
,

where  $H(C \leq \sqrt{(1/C)})$ . We can therefore prove, as before, that q/p is one of the convergents to  $\sqrt{(1/C)}$ , from which it follows that p/q is one of the convergents to  $\sqrt{C}$ .

Cor. 1. All the solutions of 
$$x^2 + Cr^2 = 1$$
 (4)

are furnished by the posultimate convergents in the successive or alternate periods of CC.

Cor. 2. If the number of quotients in the period of AC be even, the equation

 $x^2 - Cg^2 = -1$ has no integral solution. If the number of quotients in the

pass no integral soution. If the number of quotients in the period be odd, all the integral solutions are furnished by the possibility convergents in the alternate periods of M.

§ 17.] We have so in that all the integral solutions of the equation (1) are derivable from the convergents to  $\sqrt{C_3}$  it is easy to give a general expression for all the solutions in terms of the first one, say  $(p_i, q_i)$ . If we put

we have

$$r = \ell[r] - (p^2 - \ell[q^2])^n - 1.$$

Hence (6) gives a solution of (4).

In like manner, if n be any integer, and (p, q) the first solution of  $C_0$ , a more general solution is given by

Finally, if (p, q) be the first solution of (1), we may express all the solutions derivable therefrom\* by means of the general solution (6) of the equation (4). For, if (r, s) be any solution whatever of (4), we have

$$p^2 - Cq^2 = \pm H,$$
  
 $p^2 - Cq^2 = \pm I,$   
 $q^2 - Cs^2 = 1;$   
 $(p^2 - Cq^2)(p^2 - Cs^2) = \pm II,$   
 $(pr + Cqs)^2 - C(ps \pm qr)^2 = \pm II.$ 

Therefore

$$\begin{cases}
x = pr \pm Cqs \\
y = ps \pm qr
\end{cases}$$
(8)

is a solution of (1).

The formulæ (6), (7), (8) may be established by means of the relations which connect the convergents of C (see Exercises XXII., 25, and Serret, Alg. Sup., § 27 et  $s_{21}$ ). This method of demonstration, although more tedious, is much more satisfactory, because, taken in conjunction with what we have established in § 16, it shows that (6), (7), and (8) contain all the solutions in question.

Example 1. Find the integral solutions of  $x^2 - 13y^2 = 1$ .

If we refer to chap, xxxii., § 5, we find the following table of values for 1/13:-

n	$a_n$	$p_n$	$I_n$	$M_n$	
1	3	3	1	4	
2	1	4	1	3	
3	1	7	2	3	
4	1	11	3	4	
5	1	18	5	1	
6	6	119	33	4	
7	1	137	38	3	
8	1	256	71	3	
9	1	393	109	4	
10	1	619	180	1	
11	6	4287	1189	4	

Hence the smallest solution of  $x^2 - 13y^2 = 1$  is x = 649, y = 180. We have, in fact,

 $<sup>649^2 - 13 \</sup>cdot 180^2 = 421201 - 421200 = 1.$ 

It must not be forgotten that there may be more than one solution in the first period. For every such prunary solution there will be a general group like (8).

From Manalow we that the general solution is given by

 $\begin{array}{l} x = \frac{1}{4} \left\{ (0.19 + 180) \sqrt{13} \right\}^n + (619 + 180 \sqrt{13})^n \right\}, \\ y = \frac{1}{4} \left\{ (0.19 + 180 \sqrt{13})^n + (619 + 180 \sqrt{13})^n \right\} / \sqrt{13}, \end{array}$ 

where ni care cave into or.

In particular, taken n=2, we get the solution

$$x = 6496 + 13 \cdot 180^{3} = 842401,$$
  
 $y = -2 \cdot 649 \cdot 180 = 233640.$ 

If your book in the line  $\tau$  if solutions of  $x^2 - 1^{\alpha}y^2 = -1$ .

The pro- ex-solution is given by the 5th convergent to 13, as may be seen by the table given in last example.

The general solution is, by (7),

$$\begin{split} x &= \frac{1}{2} \left\{ (13 + 5\sqrt{13})^{2n-1} + (13 - 5\sqrt{13})^{2n-1} \right\}, \\ y &= \frac{1}{2\sqrt{13}} \left\{ (13 + 5\sqrt{13})^{2n-1} + (13 - 5\sqrt{13})^{2n-1} \right\}, \end{split}$$

old an area to be into be

1 and 3. Fir I all the integral solutions of  $x^2 - 1/(y^2 = 3)$ .

The primary sector is x=1 y=1, as may be seen from the table above.

The creatis late in is the flat, by est,

$$x = 1r \pm 4.5$$
,  $y = 1r \pm r$ ,

here (\*, s) i any lute n whatever of (\*-13y\* = 1

1. For r = r + r = 649 and s = 180, we get the two solutions, z = 256, k = 74, at k = -4,  $m_s = 1.500$ .

\$ 18. Let us noxt on sider the equation

$$x = Cy = \pm H$$
 (9),

where C is precise and not a perfect square, and H is positive A , C

We propose so show that the solution of (9) can always be notice to a condition the solution of an equation of the same form  $\hat{p}_1$  with  $H \in \mathcal{L}^{\infty}(\mathbb{R}^2)$  that is, upon the case already completely  $\mathbb{L}^{\infty}(\mathbb{R}^2)$  is in (4.5) if

Let (x,y) be any permittive solution of (9), so that x is prime to  $y \in T$  on we can always determine  $(x_0,y_0)$  so that

$$xy_1 = y_1x_1 + 1$$
 (10)\*.

In first if possible the possible and first tion, we have, by § 12,

$$x = \langle x + p, y - ty \pm q \rangle$$
 (11).

<sup>\* 1.</sup> Ph. 1 - , between the cubble signs here and in (9).

(14).

§ 18

If we multiply both sides of (9) by  $x_1^2 - Cy_1^2$ , and rearrange the left-hand side, we get

$$(xx_1 - Cyy_1)^2 - C(xy_1 - yx_1)^2 = \pm H(x_1^2 - Cy_1^2).$$

This gives, by (10),  $(xx_1 - Cuu_1)^2 - C = \pm H(x_1^2 - Cu_1^2)$ (12).

Now

$$xx_1 - Cyy_1 = t(x^2 - Cy^2) + (xp - Cyq)$$

But we may put  $xp - Cyq = SH \pm K_1$ , where  $K_1 > \frac{1}{2}H$ . Hence

$$xx_1 - Cyy_1 = (t \pm S) H \pm (\pm K_1)$$

Now t and the double sign in (13) are both at our disposal; and we may obviously so choose them that

$$xx_1 - Cyy_1 = K_1$$
  
where

$$K_1 \gg \frac{1}{2}H$$
. (16).

We therefore have, from (12),

$$K_{\gamma}^{2} - C = + II/(x_{\gamma}^{2} - Cu_{\gamma}^{2})$$
 (17).

Now, by hypothesis,  $\sqrt{C} < H$ , therefore  $C < H^2$  and  $K_1^2 \sim C$ 

 $< H^2$ . Since  $(x_i, y_i)$  are integers, it follows from (17) that, if (9) have an integral solution, then it must be possible to find an integer  $K_i > \#H$  such that

$$(K_1^2 - C)/H = H_1$$
 (18),

where  $H_1$  is some integer which is less than  $H^2/H$ , that is, < H. If no value of  $K_1 < \frac{1}{2}H$  can be found to make  $(K_1^2 - C)/H$ 

integral (and, be it observed, we have only a limited number of possible values to try, since  $K_1 \geq \frac{1}{2}H$ ), then the equation (9) has no integral solution.

Let us suppose that one or more such values of  $K_i$ , say  $K_i$ ,  $K_i''$ ,  $K_i''$ , ..., can be found, and let the corresponding values of  $H_i$  be  $H_i$ ,  $H_i'$ ,  $H_i''$ , ... Then it follows from our analysis that for every integral solution of (9) we must be able to find an integral solution of one of the limited group of equations

$$x_1^2 - Cy_1^2 = \pm H_1$$
  
 $x_1^2 - Cy_1^2 = \pm H_1'$   
 $x_1^2 - Cy_1^2 = \pm H_1''$ 
(19)

where  $H_1$ ,  $H_1'$ ,  $H_1''$ , . . . are all less than  $H_2$ 

If it also happens that in all the equations (19) the numerical value of the right-hand side is  $<\sqrt{C}$ , then these equations can all be completely solved, as already explained.

If  $(x_1, y_1)$  be a solution of any one of them, we see, by (10) and (15), that

If in any of the equations (19), say, for instance, in the first, the condition  $H_1 < \sqrt{C}$  is not yet fulfilled, we can repeat the above transformation, and deduce from it a new system,

$$\begin{array}{c} x_2^2 + Cy_2^2 = \pm H_2 \\ x_2^2 + Cy_2^2 = \pm H_2 \end{array}$$
 (21),

where  $H_i$  and  $H_i'$  are each less than  $H_i$ ; and we have

$$x_{1} = (K, x_{1} \mp Cy_{1}) H_{2}, \quad y_{1} = (K, y_{2} \mp x_{2}) H_{2}$$

$$x_{1} = (K, x_{1} \mp Cy_{1}) H_{2}, \quad y_{1} = (K, y_{2} \mp x_{2}) H_{2}$$
(22)

Since the H's are all integers, the chain of successive operations thus indicated must finally come to an end in every branch.

Thus we see that any integral solution of (9) must be deducible from the solution of one or other of a finite group of equations of the tope

$$u^{2} = Cy^{2} - H_{\mathbf{a}}^{(r)}$$

$$where H_{\mathbf{c}}^{(r)} + \sqrt{C_{\mathbf{c}}}$$

$$(23),$$

The practical method of solution thus suggested is as follows:—

Find all the integral values of  $K_i < \frac{1}{2}H$  for which  $(K_1^2 - C)_iH$  is an integer. The any one of these, say  $K_1$ ; and let  $H_1$  be the corresponding value of  $(K_1^2 - C)^2H$ . Then, if  $H_1 < \mathcal{J}C_i$  solve the equation  $x_i^2 + C_{ij}^2 - \pm H_i$  generally; take the formula (20); and find which of the solutions  $(x_{ij}, y_i)$ , if any, make  $(x_{ij})$  integral. We thus get a group of solutions of (9). If  $H_1 > \mathcal{J}C_i$  then we find all the values of  $K_i > \mathcal{I}H_i$  for which  $(K_i^2 - C_i)H_i$  is integral.

<sup>\*</sup>Since the  $\pm 0$  of x and y or unsufer intuit the solutions of  $x^2 + Cy^2 = 4 (H/x)$ , conversely to the coordinate double lights of  $H_1$ ,  $H_1$ , we have the coordinate  $\pi_1$  are an energy for the coordinate  $\pi_2$  are independent.

=  $H_2$  say, and, if  $H_2 < \sqrt{C}$ , solve the equation  $x_+^p + Cy_+^p \pm H_1$ ; then pass back to x through the two transformations (20) and (22); and, finally, select the integral values of x and y thus resulting, if there be any

By proceeding in this way until each branch and twig, as it were, of the solution is traced to its end, we shall get all the possible integral solutions of (9), or else satisfy ourselves that there are none.

The straightforward application of these principles is illustrated in the following example. Into the various devices for shortening the labour of calculation we cannot enter here.

Example. Find the integral solutions of

where  $K_1 > 30$ .

Then 
$$K_1^2 = 15 + 61H_1$$
.

Since  $K_1^2 > 900$ , we have morely to select the perfect squares among the numbers 15, 76, 137, 198, 250, 320, 381, 412, 503, 564, 625, 686, 747, 808, 869. The only one is 625, corresponding to which we have  $K_1 = 25$  and  $H_1 = 10$ .

Since  $H_1 > \sqrt{15}$ , we must repeat the process, and put

$$(K_2^2 - 15)_i 10 = II_2$$
 (18")

where  $K_2 > 5$ , and therefore  $K_2 > 25$ .

Since  $K_2^2 = 15 + 10H_2$ , the only values of  $K_2^2$  to be examined here are 5, 15, 25. Of these the last only is suitable, corresponding to which we have  $K_2 = 5$ ,  $H_2 = 1$ .

We have now arrived at the equation

$$x_2^2 - 15y_2^2 = 1$$
 (21'),

the first solution of which is easily seen to be (i, 1). Hence the general solution of (21') is

$$x_{z} = \frac{1}{2} \{ (1 + \sqrt{15})^{n} + (4 - \sqrt{15})^{n} \}$$
  
 $y_{z} = \frac{1}{15} \{ (4 + \sqrt{15})^{n} - (4 - \sqrt{15})^{n} \}$ 
(24).

The general solution of (9') is connected with this by the relations

$$x_1 = (5x_2 \mp 15y_2)/1,$$
  $y_1 = (5y_2 \mp x_2)/1$  (22');  
 $x = (25x_1 \mp 15y_1)/10,$   $y = (25y_1 \mp x_1)/10$  (20').

Hence 
$$x = 11x_2 \mp 45y_2$$
,  $y = \pm 3x_2 + 14y_1$   
 $x = 11x_3 \mp 30y_2$ ,  $y = \pm 2x_2 + 11y_3$ 

where  $x_2$  and  $y_2$  are given by (21). The question regarding the integrality of x and y does not arise in this case.

As a verification put  $x_2=4$ ,  $y_2=1$ , and we get the solutions (11, 2), (101, 26), (14, 5) and (71, 11) for (9), which are correct.

, 19.] There remain two cases of the binomial equation  $x^*$   $Cy=\pm H$  which are not covered by the above analysis—

$$x^2 - Cy^+ + H$$
 (26),  
where  $C$  is a perfect square, say  $C = R^2$ ; and

 $x^{2} + Cy^{2} = 0$ 

$$x^2 + Cy^2 = + II$$
 (27).

The equation (26) may be written

$$(x - Ry)(x + Ry) + II.$$

Hence we must have

$$\begin{aligned}
x - Ry &= u \\
x + Ry &= v
\end{aligned} (28),$$

where u and r are any pair of complementary factors of  $\pm H$ . We have therefore simply to solve every such pair as (28), and select the integral solutions. The number of such solutions is clearly limited, and there may be none.

In the case of equation (27) also the number of solutions is obviously limited, since each of the two terms on the left is positive, and their sum cannot exceed H. The simplest method of solution is to give p all integral values  $\gg \sqrt{t}H_c C_p$ , and examine which of these, if any, render  $H + C_p \vec{r}$  a perfect square.

[5 20.] In conclusion, we shall briefly indicate how the solution of the general equation of the 2nd degree,

$$-ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
 (29),

where a, b, c, f, g, h are integers, can be made to depend on the solution of a binomial equation.

By a slight modification of the analysis of chap, vii., § 13, the reader will easily verify that, provided a and b be not both zero, and c be not zero, (29) may be thrown into one or other of the forms

$$(\ell^*y + F^*)^2 + \ell^*(ax + hy + g)^2 = -a\Delta \qquad (30);$$

$$(\ell'x + \ell i) = \ell'(hx + by + f')^2 = b\Delta \qquad (31)$$

where  $\Delta = abx - 2i\partial t - aj' - bj' - cb'$ , C - bi - ab, F - gh = af, G - bj - bj', say into the form (30). If, then, we put

$$\frac{Cy \cdot F \cdot \xi}{ax + hy + g - q}$$
(32),

(30) reduces to

$$\dot{\xi}^2 - Ca^2 = -a\Delta \qquad (33)$$

which is a binomial form, and may be treated by the methods already explained.

If  $k^2 > ab$ , then C is positive, and, provided C be not a perfect square, we fall upon cases (1) or (9).

If C be a positive and a perfect square, we have case (26).

It should be noticed that, if either a=0 or b=0, or both a=0 and b=0, we get the leading peculiarity of this case, which is that the left-hand side of the equation breaks up into rational factors (see Example 2 below).

If  $k^2 < ab$ , then C is negative, and we have case (27).

If  $h^2 = ab$ , then C = 0, and the equation (29) may be written  $(ax + ha)^2 + 2aax + 2ata + ac = 0$  (34),

which can in general by an obvious transformation be made to depend upon the equation  $n^2 = O\hat{\xi}$  (35),

 $\gamma' = Q\xi$ 

which can easily be solved.

Example 1. Find all the positive integral solutions of

 $3x^2 - 8xy + 7y^2 - 4x + 2y = 100.$ 

This equation may be written

$$(3x-4y-2)^2+5(y-1)^2=3.36$$
,  
 $\xi^2+5y^2=336$ .

say

Here we have merely to try all values of  $\eta$  from 0 to S, and find which of them makes  $336 - 5\eta^2$  a perfect square. We thus find

 $\xi = \pm 16$ ,  $\eta = \pm 4$ ;  $\xi = \pm 4$ ,  $\eta = \pm 8$ .

Hence

$$3x - 4y - 2 = \pm 16$$
,  $y - 1 = \pm 4$  (1);  
 $3x - 4y - 2 = \pm 4$ ,  $y - 1 = \pm 8$  (2).

It is at once obvious that in order to get  $p_0$ -tive values of y the upper sign must be taken in the second equation in each case. Hence y=5 or y=9. To get corresponding positive integral values of x, we must take the lower sign in the first of (1), and the upper sign in the first of (2). Hence the only positive integral solutions are

$$x=2$$
,  $y=5$ , and  $x=14$ ,  $y=9$ .

Large to 1 I there may internal solution of

 $3 \ln q + 2 q^2 - 4x - 3z = 12$ .

The same of the 2nd degree break uninto two rational for the West of put the obstation into the form

Since 2: ( i six ask to a than 9x + 6y - 1 when both x and y are 1 - 1 . . . 1 : ist be a sed to a m nor factor of 112, that is, to 1, 2, 4, 7, y=2 mily 3. To get the corresponding values of x, we have 9x+6y+1=56 p.  $1.9 \cdot ... \cdot 6 = 1 = 14$ , that is to say, 9x = 45 and 9x = -9. Hence the can perfore the disolation if x=5, y=2.

1.: ample 3. First all the internal solutions of

 $9x^2 - 12xy + 4y^2 + 3x + 2y = 12.$ 

Here the term of the 2nd de nee form a complete square, and we may

$$(3x - 2y)^2 + (3x - 2y) + 4y = 12,$$

4 (3x - 2u) + 4 (3x - 2u) + 1 + 16u = 49 3 $(6x - 4a + 1)^2 = 10 - 16a$ .

$$u = 6x - 4y + 1$$
 (1)

$$y = (40 - u^2)/16$$

Notes a computer 1 half s, where s is a positive integer \$8, 1 de la la la that a sail not be integral notess (19 - se)/16 be integral,

1! Therefore

 $u = 16a \pm 1$ Honce, by (1), (2), as in 0, we must have

$$x = 2 + 4 \cdot (1 - 8\pi)/3, \quad y = 3 + 2\mu - 16\mu^3$$
 (1

 $x = 1 + (5 - 32\pi)/3$ ,  $y = 3 + 2\pi - 16\mu^2$ 

Transition to declare to a so that x shall be integral. The (i), a see that μ(1-8μ), 3 will be integral when and only when

$$x = 2 + 4x + 90x^2$$
,  $y = 3 - 6x + 144x^2$  (6):

$$10 + 68 = 166$$
,  $y = -11 + 90\nu - 144\nu^2$  (7).

T. with the total of 52 and is integral when and only when p a 1 ra a 1.

$$x = -5 - 72\nu - 96\nu$$
 ,  $y = -11 - 90\nu - 144\nu^2$ 

$$x = 1.76\nu - 96\nu^2$$
,  $y = -15 + 102\nu - 114\nu^2$  (9),

The fernicibe (C), (7), (8), (9), who are any have any integral value, may or reduce, contain all the integral's dutions of the given equation,

#### Exercises XXXII.

Find all the integral and also all the positive integral solutions of the following conations:—

(1.) 5x + 7y = 29.

(2.) 16x - 17y = 27.

(3.) 11x + 7y = 1103.

7y = 1103. (i.) 1367x - 1013y = 16246.

(5.) If £x. ys. be double £y. xs., find x and y.

(6.) Find the greatest integer which can be formed in nine different ways and no more, by adding together a positive integral multiple of 5 and a positive integral multiple of 7.

(7.) In how many ways can £2:15:6 be paid in half-crowns and florins?

(8.) A has 200 shilling-coins, and B 200 franc-coins. In how many ways can A pay to B a debt of 4s,?

(9.) 4 apples cost the same as 5 plums, 3 pears the same as 7 apples, 8 apricots the same as 15 pears, and 5 apples cost twop-nee. How can I buy the same number of each fuit so as to spend an exact number of pence and spend the least possible sum?

(10.) A woman has more than 5 dozen and less than 6 dozen of eggs in her basket. If she counts them by fours there is one over, if by fives there are four over. How many eggs has she?

(11.) A woman counted her eggs by threes and found that there were two over; and again by sixes and found there were three over. Show that she made a mistake.

(12.) Find the least number which has 3 for remainder when divided by 8, and 5 for remainder when divided by 7.

(13.) Find the least number which, when divided by 28, 19, 15 respectively, gives the remainders 15, 12, 10 respectively.

(14.) In how many ways can  $\pounds 2$  be paid in half-crowns, shillings, and sixpences?

(15.) A bookease which will hold 250 volumes is to be filled with 3-volumed novels, 5-volumed poems, 12-volumed histories. In how many ways can this be done? If novels cost 10s. 6d. per volume, poems 7s. 6d., and histories 5s., show that the cheapest way of doing it will cost £129. 15s.

Solve the following systems, and find the positive integral solutions:-

(16.) x + 2y + 3z = 120.

(18.) 2x + 5y + 3z = 324.)

(17.) x+y+z+u=4. 5y+6z+9u=1<sup>3</sup>.

6x - 4y + 14z = 190.

(19.) 5x - 6y + 7z = 173, 17x - 4y + 3z = 510. (20.) 17x + 10y + 21z = 400.

(21.) x + y + z + u = 26. 3x + 2y + 4z + u = 63,

3x + 2y + 4z + u = 63,2x + 3y + 2z + 4u = 74.

(22.) Show how to express the general integral solution of the system  $a_{11}x_1 + a_{11}x_2 + \dots + a_{1n}x_n = d_1$ ,

 $a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = d_2,$ 

 $a_{n-1,1}x_1 + a_{n-1,2}x_2 + \dots + a_{n-1,n}x_n = d_{n-1}$ 

by means of determinants, when a particular solution is known.

Find the value of x which make the values of the following functions

(23.)  $2x^2 + 2x$ . (24.)  $(x^2 - x)/5$ . (25.) x + 11 and x + 20, simultaneously.

(26.) 7x + 6 and 4x + 3, simultaneously, 127 1 r2+r+8

Solve the following equations, giving in each case the least integral

solution, and indicates how all the other integral solutions may be found:-(29.)  $x^2 - 44y^2 = +5$ ,

(31.)  $x^2 - 44u^2 = +4$ .

(35.)  $x^2 - 47y^2 = -1$ .

(37.)  $x^2 - 7y^2 = 186$ .

(33.)  $x^2 - 69n^2 = -11$ .

(39.)  $x^2 - (a^2 - 1)y^2 = 1$ ,

(11.)  $x^2 - (a^2 - a)y^2 = 1$ .

(43.) xy - 2x - 3y = 15. (15.)  $3x^2 + 2xy + 5y^2 = 590$ .

(28.)  $x^2 - 41y^2 = -8.$ 

(30.)  $x^2 - 44\pi^2 = -7$ , (32.)  $x^2 + 3x^2 - 628$ 

(34.)  $x' - 47u^2 = +1$ . (36.)  $x^2 - 26u^2 = -1105$ .

(38)  $x^2 - (w + 1) u = 1$ . (40.)  $x^{q} = a + a + a + c = 1$ . (44.)  $x^2 + 5xy - 2x + ay = 853$ .

(11.)  $x^2 - u^2 + 4x - 5y = 27$ .

(46.)  $x^2 + 4xy - 11y^2 + 2x - 86y - 140 = 0$ , (17.)  $x^2 - xy - 72y + 2x - 440y - 659 = 0$ .

(18.)  $x^2 + 2xy - 17y^2 + 72y - 75 = 0$ .  $(49.) \quad 64x^2 + 28xy + 251y^2 + 264x + 526y + 260 = 0.$ 

(50) Show that all the primitive solutions of Dx2 - Cy2 = + 11 are furnished by the convergents to  $\mathcal{N}(C/D)$ , provided  $H < \mathcal{N}(CD)$ . Show also how to reduce the equation  $Dx^2 - Cy^2 = \pm H$ , when H > (CD),

(51) Find all the solutions of

$$4x^2 - 7y^2 = -3$$
,  
and of  $4x^2 - 7y^2 = 53$ .

(52) If D, E, F, H be integers, and H < √(E2-DF) (real), show that all the solutions of

$$Dx^2 - 2Exy + Fy^2 = \pm II$$

are furnished by the convergents to one of the roots of

$$Dz^1 + 2Ez + F = 0.$$
  
(See Serret, Alg. Sup., § 35.)

(53) If Un=pa-xqn, where x is a periodic fraction having a cycle of c quotients, and p, and q, have their usual meanings, then

$$U_{r,\epsilon_{T}} = (a - \beta x_{r+1})^{n} U_{r},$$
 $x_{r+1} = \frac{a_{r+1}}{a_{r+2}} + \frac{1}{a_{r+c}} + \cdots,$ 

where

$$\frac{a}{B} = i_{r+1} + \frac{1}{a_{r+2} + \cdots + \frac{1}{a_{r+d}}},$$

In particular, if  $x = \zeta(C/D)$ , then

 $DP_{ne+r} = \langle CD \rangle q_{ne+r} = \{\alpha M_r - \beta L_r - \beta \chi'(CD)\}^n (Dp_r - \chi'(CD))q_r\}/M_r^n$ Point out the bearing of this result on the solution of  $Px^2 + Cy^2 = \pm II_*$ 

# CHAPTER XXXIV

# General Continued Fractions.

### FUNDAMENTAL FORMULE.

§ 1.] The theory of the general continued fraction

$$x_1 = a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}$$
 (A),

where  $a_1, a_2, a_3, \ldots, b_2, b_3, \ldots$  are any quantities whatever, is inferior in importance to the theory of the simple continued fraction, and it is also much less complete. There are, however, a number of theorems regarding such fractions so closely analogous to those already established for simple continued fractions that we give them here, leaving the demonstrations, where they are like those of chap. XXXII., as exercises for the reader. There are also some analytical theories closely allied to the general theory of continued fractions which will find an appropriate place in the present chapter.

In dealing with the general continued fraction, where the numerators are not all positive units, and the denominators not necessarily positive, it must be borne in mind that the chain of operations indicated in the primary definition of the right-hand side of (A) may fail to have any definite meaning even when the number of the operations is finite. Thus in forming the third convergent of  $1+\frac{1}{1}-\frac{1}{1-1}-\frac{1}{1}$ . . . we are led to 1+1/(1-1); and in forming the fourth to 1+1/(1-1/(1-1)); and in forming the fourth to 1+1/(1-1/(1-1)); this obvious that we could not suppose the convergents of this fraction formed by the direct process of chap. XXXII., § 6 (a), (b),

( ) It neast also be remembered that no piece of reasoning that involves the use of the value of a non-terminating continued fraction is legitimate till we have shown that the value in question is finite and definite.

In cases where any difficulty regarding the meaning or convergeneral of the continued fraction taken in its primary sense arises, we regard the form on the right of (A) merely as representing the assemble we of convergents  $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$  whose denominature are constructed by means of the recurrence-formular (2) and

That is to say, when the primary definition fails, we make the formulæ (2) and (3) the definition of the continued fraction.

In what follows we shall be most concerned with two varieties of continued fraction, namely,

$$a_1 + \frac{b_2}{a_1 + a_2 + a_3} + \cdots$$
 (B),  
 $a_1 + \frac{b_2}{a_3 - a_3 - a_3} + \cdots$  (C),

$$a_1 + \frac{b_2}{a_2 - a_3} = \frac{b_3}{a_3}$$
 (C),

where  $a_1, a_2, a_3, \ldots, b_2, b_3, \ldots$  are all real and positive. We shall speak of (B) and (C) as continued fractions of the first and second class respectively.

§ 2.1 If p. To, P. To, &c. be the successive convergents to

$$x_1 = a_1 + \frac{b_2}{a_1 + a_2 + \cdots} + \frac{b_3}{a_1 + a_2 + \cdots}$$
 (1)

then

$$p_n = a_n p_{n-1} + b_n p_{n-2} (2)$$

$$q_n = a_n q_{n-1} + h_n q_{n-2}$$
 (3),

with the initial conditions  $p_i = 1$ ,  $p_i = a_1$ ;  $q_1 = 1$ ,  $q_2 = a_2$ ,

Cor. 1. In a continued fraction of the first class pn and qn are both positive; and, provided and 1, each of them continually increases with no.

In a continued fraction of the second class, subject to the restriction  $a_n \triangleleft 1 + b_n$ ,  $p_n$  and  $q_n$  are positive, and each of them continually increases with no.

It do not not not any follow that Lp = x and Lq, x, for the succonvenients here are not pentive integral numbers, as in the case of

These conclusions follow very readily by induction from such formulæ as

$$p_n - p_{n-1} = (a_n - 1) p_{n-1} + b_n p_{n-2}$$
 (4).

Cor. 2.

$$\frac{p_n}{p_{n-1}} = a_n + \frac{b_n}{a_{n-1}} + \frac{b_{n-1}}{a_{n-2}} \cdot \cdot \cdot \cdot \cdot \frac{b_2}{a_1}$$
 (5);

$$\frac{q_n}{q_{n-1}} = a_n + \frac{b_n}{a_{n-1}} + \frac{b_{n-1}}{a_{n-2}} + \dots \cdot \frac{b_3}{a_2}$$
 (6).

§ 3.1 From (2) and (3) we deduce

$$p_n q_{n-1} - p_{n-1} q_n = (-)^n b_z b_z$$
, , ,  $b_n$  (1).

Cor. 1. The convergents, as calculated by the recurrence-rule, are not necessarily at their lowest terms.

Cor. 2.

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-)^n \frac{b_1 b_2 \dots b_n}{q_n q_{n-1}}$$
 (2).

Cor. 3.

$$\frac{p_n}{q_n} = a_1 + \frac{b_2}{q_1 q_2} - \frac{b_2 b_3}{q_2 q_3} + \dots (-)^n \frac{b_2 b_3 \dots b_n}{q_{n-1} q_n}$$
(3).

Cor. 4.

$$p_n q_{n-2} - p_{n-2} q_n = (-)^{n-1} a_n b_2 b_3 ... b_{n-1}$$
 (4);

$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-)^{n-1} \frac{a_n b_2 b_1 \dots b_{n-1}}{q_n q_{n-2}}$$
 (5).

Cor. 5.

$$\binom{p_n}{q_n} - \binom{p_{n-1}}{q_{n-1}} / \binom{p_{n-1}}{q_{n-1}} - \binom{p_{n-2}}{q_{n-2}} = -\frac{b_n q_{n-2}}{q_n},$$
  

$$= -\frac{b_n q_{n-2}}{a_n q_{n-2} + b_n q_{n-2}}$$
(6).

Cor. 6. In a continued fraction of the first class, the old convergents form an increasing series, and the even convergents a decreasing series; and every old convergent is less than, and every even convergent greater than, following convergents.

In a continued fraction of the second class, subject to the restriction  $a \not\in 1 + b_n$ , all the convergents are positive, and form an increasing series.

These conclusions follow at once from (2) and (5), if we remember that, for a fraction of the second class, we have to replace  $b_1, \ldots, b_n$  by  $-b_1, \ldots, -b_n$ .

#### CONTINUANTS

§ 4.] The functions  $p_n$ ,  $q_n$  of  $a_1$ ,  $a_2$ , . . . ,  $a_n$ ;  $b_2$ ,  $b_1$ , . . . ,  $b_n$  which constitute the numerators and denominators of the continued fraction

$$a_1 + \frac{b_2}{a_2 + a_3 + \dots + \frac{b_n}{a_n}}$$

belong to a common class of rational integral functions.

In fact,  $p_n$  is determined by the set of equations

$$p_2 = a_1 p_1 + b_2 p_2$$
,  $p_3 - a_1 p_2 + b_2 p_3$ , ...,  $p_n - a_n p_{n-1} + b_n p_{n-2}$  (1),

together with the initial conditions  $p_0 = 1, \ p_1 = a_1;$  while  $q_n$  is determined by the system

$$q_3 - a_n q_1 + b_n q_1, \quad q_4 = a_4 q_5 + b_4 q_2, \quad \dots \quad q_n = a_n q_{n-1} + b_n q_{n-2}$$
 (2),

together with the initial conditions  $q_1 = 1$ ,  $q_2 = a_3$ .

It is obvious, therefore, that  $q_n$  is the same function of  $a_2, a_7, \ldots, a_n$ ;  $b_2, b_4, \ldots, b_n$  as  $p_n$  is of  $a_1, a_2, \ldots, a_n$ ;  $b_2, b_3, \ldots, b_n$ .

We denote the function  $p_n$  by

$$p_n = K\left(\frac{b_2, \dots, b_n}{a_1, a_1, \dots, a_n}\right) \tag{3},$$

and speak of it as a continuant of the nth order whose denominators are  $a_1, a_2, \ldots, a_n$  and whose numerators are  $b_1, \ldots, b_n$ . We have then

$$q := K\left(\frac{b_3, \dots, b_n}{a_1, \dots, a_n}\right)^{-1} \tag{1}$$

<sup>•</sup> This was for equal to a by Euler in his incinor cutified "Specimen Attack within Specimen A. ." Not.  $e = 1 + e_B$ , (1774). Liquard demonstrations of folder constances we have by an hydrogen correct source (1880). The theory has been 10. § 1 of Li and connective with determinants by Sylvester and Moor.

When the numerators of the continuant are all unity, it is usual to omit them altogether, and write simply  $K(a_1, a_2, \ldots, a_n)$ . A continuant of this kind is called a *simple continuant*.

When it is not necessary to express the numerators and denominators it is convenient to abbreviate both

$$K\begin{pmatrix} b_2, \dots, b_n \\ a_1, a_2, \dots, a_n \end{pmatrix}$$
 and  $K(a_1, a_2, \dots, a_n)$ 

into K(1, n). In this notation we should have, if r < s.

$$K(r, s) = K\begin{pmatrix} b_{r+1}, \dots, b_s \\ a_{r+1}, \dots, a_s \end{pmatrix}$$
 (5);

$$K(s, r) = K\begin{pmatrix} b_s, \dots, b_{r+1} \\ a_{r+1} & a_{r+1} \end{pmatrix}$$
 (6).

In particular, K(r,r) means simply  $a_r$ , so that  $p_i = K(1,1) = a_1$ . To make the notation complete, we shall denote  $p_0$  and  $q_1$  by K(-), which therefore stands for unity; and, in general, when the statement of any rule requires us to form a continuant for which the system of numerators and denominators under consideration furnishes no constituents, we shall denote that continuant by K(-) and understand its value to be unity. It will be found that this convention introduces great simplicity into the enunciation of theorems regarding continuants.

§ 5.] A continuant of the nth order is an integral function of the nth degree of its constituents.

This follows at once from the definition of the function, for we have, by § 4 (1),

$$K(l, n) = a_n K(l, n - 1) + b_n K(l, n - 2),$$

$$K(l, n - 1) = a_{n-1} K(l, n - 2) + b_{n-1} K(l, n - 3),$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$K(l, l + 1) = a_{l+1} K(l, l) + b_{l+1} K(l, l),$$

$$K(l, l) = a_{l}, K(l, l) = 1,$$

$$K(l, l) = a_{l}, K($$

The following rule of Hindenburg's gives a convenient process for writing down the terms of a series of continuants, say K(1, 1), K(1, 2), K(1, 3), . . . :—

11	11	113	$-it_4$	- O	
1.		u,	$ct_4$	et;	
$cI_{\pm}$		$-b_i$	$-\ell\ell_4$	(t <sub>2</sub>	
$\epsilon t_1$	11.		$-h_{4}$	(t <sub>b</sub>	
h.			$-h_4$	115	1
141	u <sub>1</sub>	$(\ell_3$		1,5	
$b_2$		$(\ell_3$		100	
$\alpha_1$		$b_s$		b,	1

1st. Write down  $a_i$ , and enclose it in the rectangle 1, 1. The term in 1, 1 is K(1, 1).

2nd. Write  $a_2$  to the right of all the rows in 1, 1; and write  $b_2$  undermeath. Enclose all the rows thus constructed in the rectangle 2, 2. Then the rows in 2, 2 give the products in K(4, 2), namely,  $a_1a_2+b_2$ .

and. Write  $a_i$  at the ends of all the rows of 2, 2; repeat under 2, 2 all the rows in 1, 1, and write  $b_i$  at the end of each of them. Enclose all the rows thus constructed in 3, 3. Then the rows in 3, 3 give the products in K(1, 3), namely,  $a_ia_i a_i + b_ia_i + a_ib_i$ .

The law for continuing the process will now be obvious. The scheme is, in fact, merely a graphic representation of the continual application of the recurrence-formula

$$K(1, n) = a_n K(1, n-1) + b_n K(1, n-2)$$
 (8).

By considering Hindenburg's scheme we are led to the following rule of Euler's \* for writing down all the terms of a continuant of the ath order.

Write d wave a  $a_{++}$ ,  $a_{-}$ ,  $a_{++}$ . This is the first term. To get the rest on  $\mathbb{Z}_{+}$  from this product in every possible way one or more pair of constants of a dways replacing the second  $a_{-}$  of the privilegal hand  $\mathbb{Z}_{+}$  streamly.

I for the logical terms of the enquire continuant morely. Cayley
 I for Maps, I cope are the resolution form.

For example, to get the terms of K(1, 4). The first is  $a_1a_2a_3a_4$ . By omitting from this, first  $a_1a_2$ , then  $a_2a_3$ , then  $a_3a_4$ , and replacing by  $b_2$ ,  $b_3$ ,  $b_4$  respectively, we get three more terms,  $b_2a_3a_4$ ,  $a_1b_2a_4$ ,  $a_1a_2b_4$ . Then, omitting two pairs, we get  $b_3b$ . We thus set all the terms of K(1, 4)

It is easy to verify this rule up to K(1, 5); and a glance at the recurrence-formula (8) shows that, if it holds for any two consecutive orders of continuants, it will hold for all orders.

From Euler's rule we deduce at once the following :-

Cor. 1. The value of a continuant is not altered by reversing the order of its constituents, that is to say,

$$K\begin{pmatrix} b_2, \dots, b_n \\ a_1, a_2, \dots, a_n \end{pmatrix} = K\begin{pmatrix} b_{n_1}, \dots, b_2 \\ a_{n_1}, a_{n-1}, \dots, a_1 \end{pmatrix} \quad (9).$$

We could obviously form the continuant K(1, n) by starting with  $a_n a_{n-1} \dots a_{n+1}$  instead of  $a_n a_n \dots a_{n-1} a_n$  and replacing each consecutive pair of a's in every possible way by a b of the same order as the first a of the pair. In this way we should get precisely the same terms as before. Hence the theorem. We may express it in the form

$$K(l, m) = K(m, l)$$
 (10).

Cor. 2. We have the following recurrence-formula:-

$$K(l, m) = a_l K(l+1, m) + b_{l+1} K(l+2, m)$$
 (11).

For, by Cor. 1,

$$K(l, m) = K(m, l),$$
  
=  $a_l K(m, l+1) + b_{l+1} K(m, l+2)$ , by (7),  
=  $a_l K(l+1, m) + b_{l+1} K(l+2, m)$ , by Cor. 1.

§ 6.] The theorems (1) and (4) of § 3 may be written in continuant notation as follows:—

$$K(1, n) K(2, n-1) - K(1, n-1) K(2, n)$$
  
=  $(-)^n b_2 b_3 ... b_n K() K()$  (12)

$$K(1, n) K(2, n-2) - K(1, n-2) K(2, n)$$
  
=  $(-)^{n-2} b_1 b_2 ... b_{n-1} K(-) K(n, n)$  (13).  
These are particular cases of the following general theorem,

originally due to Euler\*:—

Euler stated it, however, only for simple continuants. It has been stated in the above general form and proved by Stern, Muir, and others.

where  $1 \le m \le n$ .

11 - descen is easily remembered by means of the following elegant tests at technical given by its discoverer:—

$$T_1, 2, \ldots, l-2, l-1, l, \ldots, m, m-1, m+2, \ldots, n$$

Dues then v is, all lines ends one the indices belonging to K(l, m); then two homomodal lines as above; and put dots over the indices immediately outside  $\Omega$ ; two vertexillines. The indices for the first e attinuant on the left of (14) are the whole row; thus e of the second are in-ide the vertical lines; those of  $\Omega$ ; the dual the orbit units the upper and over the lower horizontal lines; the office of the two continuous so on the right outside the two vertical lines, the distribution is E in  $\Omega$ , and the more at  $\Omega$ , and the modes of the minus sign is the number of constituents in K(l, m).

The proof of the theorem is very simple. We can show, by means of the recurrence-formula (7) and (11), that, if the formula hold for l, m+2, and for l, m+1, or for l-2, m, and for l-1, m, it will hold for l, m. Now (12) asserts the truth of the theorem for l-2, m+n-1; and it is easy to deduce from (12), by means of (7) and (11), that the theorem holds for l-3, m+n-1, and also for l-2, m-n-2. The general case is therefore established by a double mathematical induction based on the particular case (12).

The theorem (11) might be made the basis of the whole theory of continued fractions; and it leads at once to a variety of important particular results, some of which have already been given in the two preseding chapters. Among these we shall nearly mention the following regarding what may be called retrieved: simple continuants:—

$$K(a_1, a_2, \ldots, a_i, a_i, \ldots, a_i, a_i)$$
  
 $K(a_1, a_2, \ldots, a_i)^2 + K(a_1, a_2, \ldots, a_{i-1})^{i-1}(\Lambda);$ 

$$K(a_1, a_2, \ldots, a_{i-1}, a_i, a_{i-1}, \ldots, a_2, a_1)$$
  
=  $K(a_1, a_2, \ldots, a_{i-1}) \{K(a_i, a_2, \ldots, a_i) + K(a_1, a_2, \ldots, a_{i-1})\}$ 

Example. Show that every prime p of the form 1 + 1 can be exhibited as the sum of two integral squares.

Let  $\mu_1, \mu_2, \dots, \mu_k$  be all the inte ers prime to p and  $-\frac{1}{2}p$ ; and let simple continued fractions be formed for  $p(a_1, p/a_2, \dots, p/a_k, cash$  terminating so that the last partial quotient > 1. Then each of these continued fractions has for its last convergent the value  $K(a_1, a_2, \dots, a_k)$   $K(a_2, a_1, \dots, a_k)$ , where the two continuants are of course prime to each other, and  $a_1 > 1$ ,  $a_2 > 1$ .

From this it appears that there are as many ways, and no more, of representing p by a simple continuant (whose constituents are positive integers the first and the last of which are each greater than unity) as there are integers prime to p and  $4 \pm h$ .

Now, since  $K(a_1, a_2, \dots, a_n) = K(a_n, \dots, a_2, a_1)$ , and  $a_n > 1$ , it is obvious that  $K(a_n, \dots, a_2, a_1)$  must arise from one of the other fractions  $p|\mu$ . Hence, given any fraction  $p|\mu$ , it is possible to find another also belonging to the series which shall have the same partial quotients in the reverse order.

Let p be a prime of the l-rm 4+1, then the greatest integer in  $\frac{1}{2}p$  is M, which is even. Since, therefore, the number of continuous which are equal to p must be even, and since K(p) is one of them, there must, among the remaining odd number, be one at least which gives rise to no near fraction when we reverse its constituents, that is to say, which is redipocal. Now the reciprocal continuant in question cannot be of the form  $K(a_1, a_2, \dots, a_{s-1}, a_s)$ , b-r or it follows from (B) that such a continuant cannot represent a prime, unless i=1, or clse i=2, and  $a_1=1$ , all of which are obviously excluded.

We must therefore have an equation of the form

The must interest that a an equation of the form 
$$p = K(a_1, a_2, \ldots, a_i, a_i, a_i, \ldots, a_2, a_3),$$

$$K(a_1, a_2, \ldots, a_i)^2 + K(a_1, a_2, \ldots, a_{i-1})^2,$$
by (A), which proves the theorem in question.

As an example, take  $13 = 3 \times 4 + 1$ .

We have 
$$\frac{13}{1} = 13$$
;  $\frac{13}{2} = 6 + \frac{1}{2}$ ;  $\frac{13}{3} = 4 + \frac{1}{3}$ ;  $\frac{13}{4} = 3 + \frac{1}{4}$ ;  $\frac{13}{5} = 2 + \frac{1}{11 + 1 + 2}$ ;  $\frac{13}{6} = 2 + \frac{1}{6}$ . So that  $13 = K(13) = K(6, 2) = K(1, 3) = K(3, 4) = K(2, 1, 1, 2) = K(2, 1)$ ; and, in particular,  $13 = K(2, 1, 1, 1, 2) = K(2, 1)^2 + K(2)^2 = 3^2 + 2^3$ .

§ 7.] By considering the system of equations (1) of § 4, it is easy to see that, if we multiply a<sub>r</sub>, b<sub>r</sub>, b<sub>r+1</sub> by c<sub>r</sub>, the result is the same as if we multiplied the continuant K(1, n) (n>r) by c<sub>r</sub>. Hence we have

$$K \begin{pmatrix} c_1b_2, & c_2c_2b_2, & c_2c_1b_4, & \dots, & c_{n-1}c_nb_n \\ a_1, & c_2a_2, & c_2a_2, & c_2a_4, & \dots, & c_{n}c_nb_n \end{pmatrix}$$

$$= c_2c_2\dots c_n K \begin{pmatrix} b_2, & \dots, b_n \\ a_1, & a_2, & \dots, a_n \end{pmatrix} (15).$$

<sup>\*</sup> The following elegant proof of this well-known theorem of Fermat's was given by the late Professor Henry Smith of Oxford (Crelle's Jour., 1855).

We may so determine  $c_2, c_3, \ldots, c_n$  that all the numerators of the continuant become equal. In fact, if we put

we get 
$$c_1 - \lambda_1 - c_2 c_1 b_2 - \lambda_1 - \ldots - c_{n-1} c_n b_n = \lambda_1$$
  
 $c_1 - \lambda_1 b_2 - c_2 - b_2 b_3 - c_1 = \lambda b_1 b_1 b_2 - c_3 - b_1 b_2 b_3 b_3$   
 $c_1 = \lambda b_2 b_2 b_2 b_2 b_3 - c_3 - b_1 b_2 b_3 b_3$ 

Henre

$$\frac{K\left(\frac{b_{2},\ldots,b_{n}}{a_{1},a_{2},\ldots,a_{n}}\right)}{(1^{2}\lambda)^{2}b_{n}b_{n}\cdot b_{n-4}\cdot\ldots \times K\left(\frac{\lambda_{1}}{a_{1},\lambda_{1}a_{2}}b_{2},a_{1}b_{2}^{2}b_{2},\lambda_{1}a_{2}b_{2}^{2}b_{1}b_{2},\ldots\right)}$$
(16b)

where  $\rho$  is the number of even integers (excluding 0) which do not exceed n.

Cor. Every continuant can be reduced to a simple continuant, or to a continuant each of whose numerators is -1.

Thus, if we put  $\lambda = +1$  and  $\lambda = -1$ , we have

$$K \begin{pmatrix} h_2 & \dots & h_n \\ a_1, a_2, \dots & a_n \end{pmatrix}$$

$$= h \cdot h_{n-2} \cdot \dots \times K \begin{pmatrix} a_1, a_1h_2, a_2h_2h_3, a_1h_2h_3h_4, \dots & a_nh_{n-1}h_{n-2} \dots \end{pmatrix} (17),$$

$$= \begin{pmatrix} -1, &$$

§ 8.] The connection between a continuant and a continued fraction follows readily from (11). For we have, provided  $K(2, n), K(3, n), K(4, n), \ldots$  are all different from zero,

$$\begin{split} & \underbrace{K\left(1,\,n\right)}_{K\left(2,\,n\right)} = a_1 + \frac{b_2}{K\left(2,\,n\right)} \underbrace{K\left(3,\,n\right)^*}_{K\left(3,\,n\right)} & \underbrace{k\left(2,\,n\right)}_{d_2 + K\left(3,\,n\right) \cdot K\left(1,\,n\right)^*} \end{split}$$

Hence

$$K(1, n) = a_1 - b - b_2 - b_r - b_$$

If in this last equation we put r = n, and remember that here K(n+1, n) = K(-1) = 1, we get

$$\frac{K(1, n)}{K(2, n)} = a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots + \frac{b_n}{a_n}}}$$
 (20)

a result which was obvious from the considerations of § 4.

§ 9.1 When the continuant equation

$$K(1, n) = a_n K(1, n-1) + b_n K(1, n-2),$$

or

$$p_n = a_n p_{n-1} + b_n p_{n-2}$$

which may be regarded as a finite difference equation of the second order, can be solved, we can at once derive from (20) an expression for

$$a_1 + \frac{b_2}{a_1} + \frac{b_3}{a_2} + \dots + \frac{b_n}{a_n}$$

When  $a_n$  and  $b_n$  are constants, the problem is simply that of finding the general term of a recurring series, already solved in chap, XXXI., § 7.

Example. To find an expression for the nth convergent to

$$F = 1 + \frac{1}{1+} \frac{1}{1+} \cdot \cdot \cdot \cdot \frac{1}{1+} \cdot \cdot \cdot \cdot$$

Here we have to solve the equation  $p_n = p_{n-1} + p_{n-2}$ , with the initial conditions  $p_0 = 1$ ,  $p_1 = 1$ . The result is

Hence

$$\begin{split} &K(1,\,n) = p_n = \{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}|^{2n+1}\sqrt{5},\\ &p_n = \frac{K(1,\,n)}{K(2,\,n)} = \frac{\{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\}|^{2n+1}\sqrt{5}}{\{(1+\sqrt{5})^n - (1-\sqrt{5})^n\}|^{2n}\sqrt{5}},\\ &= \frac{1}{2}\frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{(1+\sqrt{5})^n - (1-\sqrt{5})^n}. \end{split}$$

$$=\frac{1}{3}\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{(1-\sqrt{5})^{n+1}}$$

From the expression for K(1, n) (all the terms in which reduce in this case to + 1) we see incidentally that the number of different terms in a continuant of the nth order is

$$\frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{\frac{2n+1}{5}\frac{1}{5}}=\frac{1}{2n}\left\{_{n+1}C_1+5_{n+1}C_3+5_{n+1}^2C_5+\dots\right\}.$$

§ 10.] When two continued fractions F and F' are so related that every convergent of F is equal to the convergent of F' of the same order, they are said to be equivalent\*.

We may also have an (m, n)-equivalence, that is, p<sub>rm</sub>/q<sub>rm</sub> = p<sub>rn</sub>/q<sub>rm</sub>. See Exercises xxx: 11., 2, 17, &c.

It follows at once from  $\S$  7 and 8 (and is, indeed, otherwise obvious, provided the continued fraction has a definite meaning according to its primary definition) that we may multiply  $a_r, b_r$ , and  $b_{r+1}$  by any quantity  $m(\pm 0)$  without disturbing the equivalence of the fraction. Hence we may reduce every continued fraction to an equivalent one which has all its numerators equal to  $\pm 1$  or to  $\pm 1$ . Thus we have

$$a_{1} + \frac{b_{2}}{a_{2}} + \frac{b_{3}}{a_{3}} + \frac{b_{4}}{a_{4}} + \dots + \frac{b_{n}}{a_{n}} + \dots$$

$$= a_{1} + \frac{1}{a_{2}|b_{2}} + \frac{1}{a_{2}|b_{2}} + \frac{1}{a_{3}|b_{4}|b_{2}} + \dots + \frac{1}{a_{n}b_{n-1}b_{n-2}} \dots + \frac{1}{|b_{n}b_{n-2}\dots + b_{n-2}|}$$
(21)

§ 11.] If we treat the equations (1) as a linear system to determine K(1, 1), K(1, 2), . . . , K(1, n), and use the determinant notation, we get

which gives an expression for a continuant as a determinant. The theory of continuants has been considered from this point of view by Sylvester and Muir\*; and many of the theorems regarding them can thus be proved in a very simple and natural manner.

## EXERCISES XXXIII.

(1) Assuming that both the fractions

$$x = \frac{a}{a+b+c+\cdots}$$
,  $y = \frac{a}{b+c+d+\cdots}$ ...

are convergent, show that

$$x(i+1+y) = a + y$$
.

<sup>.</sup> See Muir's Theory of Determinants, chap. 111.

(Clausen.)

(2.) If p/q and p'/q' be the ultimate and penultimate convergents to  $a+\frac{1}{h+1}\dots\frac{1}{L}$  , show that

where the quotient q' + p is repeated n - 1 times, and the upper or the lower sign is to be taken according as p/q is an even or an odd convergent.

- (3.) Evaluate  $a + \frac{1}{a+} \frac{1}{a+} \dots$  to n quotients, a being any real quantity positive or negative. Slow from your result that the continued fraction in
- question always converges to the numerically greatest root of  $x^2 ax 1 = 0^*$ .

  (4.) Deduce from the results of (2) and (3) that a recurring continued fraction whose numerators and denominators are real quantities in general converges to a finite limit; and indicate the nature of the excentional cases.
  - (5.) Evaluate  $2 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \dots$  to *n* terms.
- (6.) Show that the *n*th convergent to  $\frac{1}{3-}\frac{4}{3-}\frac{2}{3-}\frac{2}{3-}\frac{2}{3-}\frac{2}{3-}\dots$ , every subsequent component being  $\frac{2}{3}$ , is  $(2^n-1)/(2^n+1)$ .
  - (7.) Show that  $\frac{x}{x+1-x} \frac{x}{x+1-x} \dots$  to  $n \text{ terms} = \frac{x^{n+1}-x}{x^{n+1}-1}$ .
  - (8.)  $\frac{1}{1-a} \frac{a}{a+1} \frac{a+1}{a+2} \cdots \frac{(n+1)}{(n+1)}$  components)
  - $= 1 + a + a (a + 1) + \dots + a (a + 1) + \dots + (a + n 1).$ (9.) If  $\phi(n) = \frac{1}{a + a + \dots}$ , n quotients, then  $\phi(m + n) = \{\phi(m) + \phi(n) a\phi(m)\phi(n)\}/\{1 + \phi(m)\phi(n)\}.$
  - (10.) Show that
    - $K(0, a_2, a_3, \ldots, a_n) = K(a_3, \ldots, a_n);$  $K(\ldots a, b, c, 0, e, f, g, \ldots) = K(\ldots a, b, c + e, f, g_1 \ldots);$
  - $K(\ldots a, b, c, 0, 0, 0, e, f, g, \ldots) = K(\ldots a, b, e+c, f, g, \ldots);$  $K(\ldots a, b, c, 0, 0, e, f, \ldots) = K(\ldots a, b, c, c, f, \ldots);$
  - (Muir, Determinants, p. 159.)

    (11.) Show that the number of terms in a continuant of the nth order is
  - $1 + (n-1) + \frac{(n-2)(n-3)}{2!} + \frac{(n-3)(n-4)(n-5)}{3!} + \cdots$ (Sylvester.)
  - (12.) If  $p_n = K\begin{pmatrix} b_2, b_3, \cdots, b_n \\ a_1, a_2, a_3, \cdots, a_n \end{pmatrix}$ , show that there exists a relation of the form

the form 
$$Ap_n^2 + Bp_{n-1}^2 + Cp_{n-2}^2 + Dp_{n-3}^2 = 0$$
, where  $A$ ,  $B$ ,  $C$ ,  $D$  are integral functions of  $a_n$ ,  $b_{n,1}$ ,  $a_{n-1}$ ,  $b_{n-1}$ ,

\* This is a particular case of the theorem (due to Euler?) that the numerically greatest root of  $x^2 - px + q = 0$  is  $p = \frac{q}{x} - \frac{q}{x} + \cdots$ .

(13) Show that

$$K\left(\frac{l_{11}}{l_{11}},\frac{(b_1+a_1)}{a_2},\frac{l_{11}}{a_2},\frac{(b_2+a_2)}{a_2},\dots\right) = (b_1+a_1)(b_2+a_2)(b_3+a_3)\dots;$$
and deduce the theorem of § 19. (Muir, l.c.)

Taking  $(a, b, c, \ldots, k)$  to denote the continued fraction  $\frac{1}{a-b-1}\frac{1}{c-1}$ 1. . . . and  $[a, b, c, \ldots, k]$ , or, when no confusion is likely, [a, k], to

is note  $K\begin{pmatrix} -1, -1, \dots, -1 \\ a, b, c, \dots, k \end{pmatrix}$ , prove the following theorems\*:—

(11.) If 
$$x = (a, b, c, \dots, k)$$
, then  $y = (c, \dots, c, b, a, x)$ ;  
 $xy = (c, \dots, c)$ ,  $x = (a, \dots, c)$ ,  $x = (c, \dots, a)$ ,  $x = (a, \dots, c)$ ,  $x = (a, \dots, a)$ ,  $x = (a,$ 

 $= (c_1, \ldots, a_n)^2 (d_1, \ldots, a_n)^2 (d_n, \ldots, d_n)^2 \ldots (d_n)^2$ . (17.)  $(a_1, ..., c) - (a_1, ..., d) = (c_1, ..., a) (d_1, ..., a)^2 (c_1, ..., a)^2 ... (d)^3$ .

(16.)  $[a, b, c, d, e] = 1_i(a, b, c, d, e) (b, c, d, e) (c, d, e) (d, e) (e),$ (17.) Prove the following equivalence theorem :-

$$= \frac{1}{m} \left\{ am + 1 - \frac{1}{2 + bm - 2 + cm - 1} \cdot \dots \right\}.$$

 $(2^{(1)})$   $\sqrt{2} \cdot 1 + \frac{1}{2+2+} \cdot \dots - \frac{1}{5} \left\{ 7 + \frac{1}{11} + \frac{1}{14+14+} \cdot \dots \right\}$ 

(21.) (a...e, f, a..., e, f', a,...e, f'', ...nd ∞) - (c, . . . , a, f, e, . . . , a, f', e, . . . , a, f'', . . . ad x)

- $= (a, \ldots, e) (e, \ldots, a),$ (122) Show that the successive constituents a, B, 5, . . . . . \ a, \rm may be ometed from the continued fraction  $(...a, b, a, \beta, \gamma, ..., \lambda, \mu, \nu, c, d, ...)$ with at altering its value, provided  $[\beta, \dots, \mu] = \pm 1, \alpha = \pm [\gamma, \dots, \mu]$ and  $r = \pm [\beta, \dots, \lambda]$ ; and c in tract examples.
- (23) If x=11,..., c, t,...), the other root of the quadratic equation to which this bods is  $x = (i, e, \dots, a, \dots)$ .
  - (24.) If  $b+\frac{1}{l_{j+1}}\cdots \frac{1}{l_{j_1+1}\cdots l_{j_n}}\cdots \frac{1}{l_{m}}$  , . . be one root of a quadratic

<sup>.</sup> The rotation and the orier of riens used in (11) to (23), as well as I me of the special results, are due to Mobius (Crelle & Jour., 1830).

equation, the other is

$$b + \frac{1}{b_1 +} \cdots \frac{1}{b_m - a_m -} \frac{1}{a_{m-1} +} \frac{1}{a_{m-2} +} \cdots \frac{1}{a +} \frac{1}{a_m +} \cdots \frac{1}{a +} \cdots \frac{1}{a$$

(25.) If q > p, show that

w that
$$1 = \frac{q - p}{q - \frac{pq}{q^2 - p^2}} \frac{pq(q - p)}{q^2 - p^2} \frac{pq(q - p)^2}{q^2 - p^2} \dots;$$

$$(q-p) q = q^2 - p^2 - \frac{pq}{q} \frac{(q-p)^2}{q^2 - p^2}$$
.

#### CONVERGENCE OF INFINITE CONTINUED FRACTIONS.

§ 12.] By the value or limit of an infinite continued fraction is meant the limit, if any such exist, towards which the convergent  $p_n/q_n$  approaches when n is made infinitely great. It may happen that this limit is finite and definite; the fraction is then said to be convergent. It may happen that  $L_p p_n/q_n$  fluctuates between a certain number of finite values according to the integral character of n; the fraction is then said to oscillute. Finally, it may happen that  $L_p p_n/q_n$  tends constantly towards  $\pm \infty$ ; in this case the fraction is said to be divergent.

We have already seen that all simple continued fractions are convergent. The fraction  $1 - \frac{1}{1-1} - \frac{1}{1-1} - \dots$  is an obvious example of oscillation, its value being 1, 0, or  $-\infty$  according as n = 3m + 1, 3m + 2, or 3m + 3.

value being 1, 0, or  $-\infty$  according as n=3m+1, 3m+2, or 3m+3. The fraction  $1-\frac{1}{2+\frac{1}{2}\sqrt{5}-1}\frac{1}{1+\frac{1}{4}-1}$ . diverges to  $-\infty$ , for  $\frac{1}{1+1}\frac{1}{1+1}$ . ... somereges to  $-\frac{1}{2+\frac{1}{2}\sqrt{3}}$ , as may be easily seen from the expression for its nth convergent given in s

The last example brings into view a fact which it is important to notice, namely, that the divergence of an infinite continued fraction is something quite different from the divergence of an infinite series. The divergence of the fraction is, in fact, an accidental phenomenon, and will in general disappear if we modify the fraction by omitting a constituent. It is therefore

<sup>\* (23)</sup> and (24) are generalisations of an older theorem of Galois'. See Gergonne Ann. d. Math., t. xix.

not safe in general to argue that a continued fraction does not diverge because the continued fraction formed by taking all its constituents after a certain order converges.

With the exception of simple continued fractions and recurring continued fractions (whether simple or not), the only cases where rules of any generality have been found for testing convergency are continued fractions of the "first" and "second class." To these we shall confine ourselves in what follows.

§ 13.] A continued fraction of the first class cannot be divergent; and it will be convergent or oscillating if any one of the residual fractions  $x_2, x_3, \ldots, x_n, \ldots$  converge or oscillate.

The latter part of this proposition is at once obvious from the equation

 $x_1 = a_1 + \frac{b_2}{a_2 + a_3 + \cdots + a_n} \cdot \frac{b_n}{x_n}$ 

Again, since (§ 3, Cor. 6) the odd convergents continually increase and the even convergents continually decrease, while any even convergent is greater than any following odd convergent, it follows that  $Lp_{2n}/l_{2n} - A$  and  $Lp_{2n-1}/l_{2n-1} = B$ , where A and B are two finite quantities, and  $A \not \subset B$ . If A = B, the fraction is convergent; if A > B, it oscillates; and no other case can arise.

§ 14.] A continued fraction of the first class is convergent if the series Σa<sub>n+1</sub>a<sub>n</sub> b<sub>n</sub> be divergent.

We have, since all the quantities involved are positive,

$$\begin{array}{lll} q_n \cdot a, q_{n-1} + h_n q_{n-1}; \\ q_{n-1} = a_{n-1} q_{n-2} + b_{n-1} q_{n-2}, & q_{n-1} \geq a_{n-1} q_{n-2}; \\ q_{n-2} \cdot a_{n-2} q_{n-3} + b_{n-2} q_{n-4}, & q_{n-2} \geq a_{n-2} q_{n-3}; \\ \vdots & \vdots & \vdots & \vdots \\ q_s \cdot a_s q_s + b_s q_1, & q_s \geq a_s q_s; \\ q_s \cdot a_s q_s + b_s q_1, & q_s \geq a_s q_s; \\ q_s \cdot a_s q_1, & \vdots & \vdots \\ \end{array}$$

Our knowledge of the convergence of continued fractions is chiefly due to Schömilch, Handle A. Algebrauchen Analysis (1845); Arall, Loquantions, Normalls de Fester undue Continuo, Sunda, (1843); Scholl, Unternubungen über die Convergini und Divergenz der Kitteibruche Habilitationsschrift München 1846); also Abhaellingen d. Math. Claise d. K. Bayerischen Akad. d. Wiers, Bl. M. (1866); also Schomilton Schomilton (1846); also Abhaellingen d. Math. Claise d. K. Bayerischen Akad. d. Wiers, Bl. M. (1866); also Schomilton (1886).

Hence

Therefore

 $q_n q_{n-1} > q_1 q_n (b_3 + a_2 a_3) (b_4 + a_3 a_4) \dots (b_n + a_{n-1} a_n),$ and, since  $a_1 = 1$ ,  $a_2 = a_2$ ,

$$\frac{g_n q_{n-1}}{b_n b_n \dots b_n} > \frac{a_2}{b_2} \left( 1 + \frac{a_1 a_1}{b_2} \right) \left( 1 + \frac{a_3 a_4}{b_4} \right) \dots \left( 1 + \frac{a_{n-1} a_n}{b_n} \right)$$
 (1).

Now, since  $\sum a_{n-1}a_n b_n$  is divergent,  $\Pi (1 + a_{n-1}a_n b_n)$  diverges to +  $\infty$  (chap. XXVI., § 23), therefore  $Lq_nq_{n=1}b_xb_x$ , . .  $b_n + \infty$ . Hence

$$L\left(\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}}\right) - L\frac{b_1b_2...b_{2n}}{q_{2n}q_{2n-1}} = 0,$$

that is, the continued fraction is convergent.

Cor. 1. The fraction in question is convergent if  $La_{n-1}a_n/b_n > 0$ .

Cor. 2. Also if  $La_nb_n > 0$ , and  $\Sigma a_n$  be divergent.

Cor. 3. Also if  $La_{n+1}b_n/a_{n-1}b_{n+1} > 1$ .

The above criterion is simple in practice; but it is not complete, inasmuch as it is not proved that oscillation follows if  $\sum a_{n-1}a_n/b_n$  be convergent. The theorem of next paragraph supplies this defect.

§ 15.] If a continued fraction of the first class be reduced to the form

$$d_1 + \frac{1}{d_2 + d_3 + d_4 + \dots} \cdot \frac{1}{d_n + \dots}$$
 (1),

so that 
$$d_1 = a_1, \quad d_2 = \frac{a_2}{b_2}, \quad d_3 = \frac{a_2b_2}{b_2}, \quad d_4 = \frac{a_4b_3}{b_4b_2}, \quad \dots$$

$$d_n = \frac{a_nb_{n-1}b_{n-1}}{b_nb_{n-2}}. \quad (5)$$

then it is convergent if at least one of the series

$$d_2 + d_3 + d_7 + \dots$$
 (6)  
 $d_2 + d_1 + d_2 + \dots$  (7)

$$d_2 + d_4 + d_6 + \dots (7)$$

be divergent, oscillating if both these series be convergent.

This proposition depends on the following inequalities between the q's and d's of the fraction (4):—

$$0 < q_n < (1 + d_i) (1 + d_i) . . . (1 + d_n)$$
 (8);

$$q_{2n} > d_2 + d_4 + \dots + d_{2n}$$
 (9);

 $q_{:n-1} > 1$ 

These follow at once from Euler's law for the formation of the terms in  $q_{2i}$  which, in the present case, runs as follows:— Write down  $d_2d_3 \dots d_n$  and all the terms that can be formed therefrom by omitting any number of pairs of consecutive d's. We thus see that  $q_2$  contains fewer terms than the product  $(1+d_2)(1+d_2),\dots (1+d_n)$ ; and, since the terms are all positive, (8) follows. Again, in forming the terms of the 1st degree in  $q_{2n}$ , we can only have letters that stand in odd places in the succession  $d_2d_4,\dots d_{2n}$ ; hence (9); and (10) is obvious from a similar consideration.

To apply this to our present purpose, we observe that, since the numerators are all equal to 1, we have

$$\frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} = \frac{1}{q_{2n}q_{2n-1}}$$
(11).

If we suppose  $d_2 + 0$ , neither  $q_{2n}$  nor  $q_{2n+1}$  can vanish. Hence, if both  $Lq_{2n}$  and  $Lq_{2n-1}$  be finite, the fraction will oscillate, and if one of them be infinite it will converge.

Now, if both the series (6) and (7) converge, the series  $d_x \circ d_3 \circ d_4 + \dots + d_n$  will converge; and the product on the right of (8) will be finite when  $n = \pi$ . In this case, therefore, both  $q_{2n}$  and  $q_{2n+1}$  will be finite; and the fraction (4) will oscillate.

If the series  $d_+ + d_+ + d_+ + \dots$ , diverge, then by (9)  $Lq_{29} = \infty$ , and the fraction (4) will converge.

By the same reasoning, if the series  $d_3+d_4+d_7+\ldots$  diverge, then the fraction

$$d_2 + \frac{1}{d_1 + d_1 + \dots} \cdot \frac{1}{d_n + \dots}$$

will converge; and consequently the fraction (4) will converge.

Remark.—We might deduce the criterion of last paragraph from the above. For we have

$$d_1d_2 = a_1a_2/b_2$$
,  $d_2d_3 = a_2a_3/b_3$ , ...,  $d_{n-1}d_n = a_{n-1}a_n/b_n$ .

Now, if the series  $\Sigma d_n$  converge, the series formed by adding together the products of every possible pair of its terms must, by chap,  $\mathbf{XXX}$ , § 2, converge: a fortiori, the series  $\Sigma d_{n-1}d_n$ , that is,  $\Sigma a_{n-1}a_n/b_n$ , must converge. Hence, if this last series diverge,  $\Sigma d_n$  cannot converge.  $\Sigma d_n$  must therefore diverge, since it cannot oscillate, all its terms being positive. Therefore either (6) or (7) must diverge, that is to say, the fraction (4) must converge.

Example 1. Consider the fraction

Here

It may be shown, by the third criterion of chap. xxvi. § 6, Cor. 5, that the series  $\Sigma d_{2n+1}$  is divergent. Or we may use Stoling's Theorem. Thus, when n is very great, we have very nearly

$$\begin{aligned} d_{2n+1} &= 2 \ (2n!)^2 / 2^{4n} \ (n!)^4, \\ &= 2 \left[ \left\{ \sqrt{(2\pi 2n)} \ (2n/e)^{2n} \right\} / \left\{ 2^{2n} \ (2\pi n) \ (n/e)^{2n} \right\} \right]^2, \\ &= 2/\pi n. \end{aligned}$$

The convergence of  $\Sigma d_{2n+1}$  is therefore comparable with that of  $\Sigma 1/n$ , which is divergent.

Hence the continued fraction in question converges, Example 2.

$$a + \frac{x}{a+} \cdot \frac{x^2}{a+} \cdot \frac{x^3}{a+} \cdot \cdot \cdot$$

oscillates or converges according as x>1 or :- I.

Example 3.

$$\frac{1}{2+} \frac{2}{3+} \frac{3}{4+} \cdots \cdots$$

Here  $La_{n-1}a_n/b_n = L(n-1)n/(n+1) = \infty$ , therefore the fraction is convergent.

§ 16.] There is no comprehensive criterion for the convergence of fractions of the second class; but the following theorem embraces a large number of important cases:—

If an infinite continued fraction of the second class of the form

$$F = \frac{b_2}{a_1 - a_2} = \frac{b_3}{a_2 - \cdots a_n} = \cdots$$
 (1)

be such that

$$a_n \geq b_n + 1$$
 (2)

for all values of n, it converges to a finite limit F not greater than unity.

If the sign > occur at least once among the conditions (2), then F < 1.

If the sign = glove occur, then F = 1 - 1/S, where

 $S = 1 + b_a + b_b b_1 + b_a b_b b_4 + \dots + b_b b_a \dots b_n + \dots \text{ ad } \infty$  (A), so that F = or < 1 according as the series in (A) is divergent or converment.

These results follow from the following characteristic properties of the restricted fraction (1):-

$$p_n - p_{n-1} \ge b_1 b_2 \dots b_n$$
 (3);

acc. as  $a_n \ge b_n + 1$ :

$$p_n \ge b_1 + b_2b_3 + b_3b_4 + \dots + b_2b_3 \dots b_n$$
 (4)

$$q_n - q_{n-1} \ge b_2 b_3 \dots b_n$$
 (5)

$$q_n \ge 1 + b_2 + b_2 b_3 + \dots + b_2 b_3 \dots b_n$$
 (6)

$$q_n - p_n \ge q_{n-1} - p_{n-1} \ge \dots \ge q_2 - p_2 \ge 1$$
 (7)

To prove (3) we observe that

 $p_n = p_{n-1} \ge h_n (p_{n-1} - p_{n-n}),$ 

$$p_n - p_{n-1} = (a_n - 1) p_{n-1} - b_n p_{n-2}$$

Hence, since  $p_n$ ,  $q_n$  are positive and increase with n (§ 2, Cor. 1),

$$\begin{array}{cccc} p_{n-1} & p_n & \vdash b_{n-1}(p_{n-2} & p_n), \\ & & & & \text{acc. as } a_{n+1} \geq b_{n+1} + 1; \\ & & & & & \text{acc. as } a_{\delta} \geq b_{\delta} + 1. \end{array}$$
 Therefore  $p_n - p_{n+1} \geq b_{\delta} b_{\delta}$ ,  $b_n$ , where the upper sign must

be taken if it occur anywhere among the conditions to the right of the vertical line.

To prove (1), we have merely to put in (3) n-1, n-2, . . . , 3 in place of n, adjoin the equation  $p_4 = b_2$ , and add all the resulting equations.

(5) and (6) are established in precisely the same way.

It follows, of course, that  $p_n$  and  $q_n$  both remain finite or both become infinite when u = x, according as the series in (6) is convergent or divergent

To prove (7), we have

$$q_n - p_n = \alpha_n (q_{n-1} - p_{n-1}) - h_n (q_{n-2} - p_{n-2}),$$
  
 $\geq (q_{n-1} - p_{n-1}) + h_n (q_{n-1} - p_{n-1}) - (q_{n-2} - p_{n-2}).$ 

according as  $a_n \ge b_n + 1$ , provided  $a_{n-1} - \rho_{n-1}$  is positive.

This shows that, if any one of the relations in (7) hold, the next in order follows. Now  $q_1-p_1+q_2-b_1 \ge 1$ , according as  $a_2 \ge b_2+1$ ; and  $q_2-p_3-a_2a_3-b_2-b_1a_2 \ge (a_2-b_2)(b_1+1)-b_2 \ge (a_2-b_2)+b_1(a_2-b_1-1)$ , according as  $a_3 \ge b_1+1$ ; hence the theorem. It is important to observe that the first > that occurs among the relations  $a_2 \ge b_2+1$ ,  $a \ge b_2+1$ , . . . determines the first > that occurs among the relations  $a_1 \ge b_2+1$ , and the signs to the right of this one will be =, all those to the left >.

The convergency theorems for the restricted fraction of the second class follow at once. In the first place, as we have already seen in § 3, the convergents to (1) form an increasing series of positive quantities, so that there can be no oscillation. Also, since  $q_n - p_n \ge 1$ , it follows that

$$p_n^{-1}q_n \le 1 - 1/q_n$$
 (8).

Therefore, since  $q_n > 1$ , it follows that F converges to a finite limit > 1.

If the sign > occur at least once among the relations (2), the sign < must be taken in (8); that is, F < 1.

If the sign = occur throughout, we have

$$Lp_n/q_n = 1 - L1 \cdot q_n - 1 - 1/S$$
.

where S is the sum to infinity of the series (6). Hence, if (6) converge, F < 1; if it diverge, F - 1.

If we dismiss from our minds the question of convergency, and therefore remove the restriction that  $b_2$ ,  $b_2$ , . . . ,  $b_n$  be positive, but still put  $a_n - b_n + 1$ ,  $a_{n-1} - b_{n-1} + 1$ , . . . ,  $a_n - b_2 + 1$ ,  $a_n - b_n + 1$ , we get by the above reasoning

$$p_n/q_n = 1 - 1/q_n$$
 (8)

$$q_n - 1 + b_2 + b_2 b_3 + \dots + b_2 b_3 \dots b_n$$
 (6').

Now (8') gives us  $q_n = 1/(1 - p_n/q_n)$ . Hence the following remarkable transformation theorem:—

Cor. If  $b_1, \ldots, b_n$  be any quantities whatsoever, then

$$1 + b_1 + b_2 b_3 + \dots + b_3 b_4 \dots b_n$$

$$= \frac{1}{1 - b_1 + 1 - b_3 + 1} - \frac{b_3}{b_3 + 1} - \frac{b_n}{b_n + 1} \quad (9),$$

from which, putting  $u_1=b_2$ ,  $u_2=b_2h_3$ , . . . ,  $u_n=b_2h_3$  . . .  $b_{n+1}$ , we readily derive

an important theorem of Euler's to which we shall return presently.

INCOMMENSURABILITY OF CERTAIN CONTINUED FRACTIONS.

§ 17.] If  $a_2, a_3, \ldots, a_n, b_1, b_2, \ldots, b_n$  be all positive integers, then

1. The infinite continued fraction

$$\frac{b_1}{a_1+\frac{b_2}{a_1+}}\dots\frac{b_n}{a_n+}\dots$$
 (1)

converges to an incommensurable limit provided that after some finite value of a the condition  $a_n \not \sim b_n$  be always satisfied.

11. The infinite continued fraction

$$\frac{b_1 - b_3}{a_2 - a_3} \cdot \cdot \cdot \cdot \frac{b_n}{a_n - a_n} \cdot \cdot \cdot$$
 (2)

converges to an incommensurable limit provided that after some finite value of n the condition  $a_n \ge b_n + 1$  be always satisfied, where the sign > n od not always occur but must occur infinitely often.\*.

To prove 11., let us first suppose that the condition  $a \ge b_x + 1$  holds from the first. Then (2) converges, by § 16,

<sup>.</sup> The e the creme are due to Legendre, Éléments de Géométrie, note iv.

to a positive value < 1. Let us assume that it converges to a commensurable limit, say  $\lambda_{z_i}\lambda_i$ , where  $\lambda_1$ ,  $\lambda_2$  are positive integers, and  $\lambda_1 > \lambda_2$ .

Let now

$$\rho_3 = \frac{b_3}{a_4} - \frac{b_4}{a_4} - \dots$$

Since the sign > must occur among the conditions  $a_i \ge b_3 + 1$ ,  $a_4 \ge b_4 + 1$ , . . . ,  $\rho_2$  must be a positive quantity < 1. Now, by our hypothesis,

therefore

$$\lambda_z/\lambda_1 = b_z/(\alpha_2 - \rho_\beta),$$
  
 $\rho_3 = (\alpha_z\lambda_2 - b_z\lambda_1)/\lambda_2,$   
 $= \lambda_z/\lambda_2, \text{ say,}$ 

where  $\lambda_3 = a_2\lambda_3 - b_2\lambda_1$  is an integer, which must be positive and  $<\lambda_2$ , since  $\rho_3$  is positive and <1.

Next, put

$$\rho_4 = \frac{b_4}{a_1 - a_2} \frac{b_5}{a_5} \dots$$

Then, exactly as before, we can show that  $\rho_4 = \lambda_4/\lambda_3$ , where  $\lambda_4$  is a positive integer  $< \lambda_3$ .

Since the sign > occurs infinitely often among the conditions  $a_n \ge b_n + 1$ , this process can be repeated as often as we please. The hypothesis that the fraction (2) is commensurable therefore requires the existence of an infinite number of positive integers  $\lambda_1, \lambda_2, \lambda_1, \lambda_4, \ldots$  such that  $\lambda_1 > \lambda_2 > \lambda_1 > \lambda_4 > \ldots$ ; but this is impossible, since  $\lambda_1$  is finite. Hence (2) is incommensurable.

Next suppose the condition  $a_n \ge b_n + 1$  to hold after n = m. Then, by what has been shown,

$$y = \frac{b_{m+1}}{a_{m+1}} - \frac{b_{m+2}}{a_{m+2}} \cdot \cdot \cdot$$

is incommensurable.

Now we have

$$F = \frac{b_2}{a_2 - a_3} \cdot \frac{b_m}{a_m - g},$$

$$F = \frac{(a_m - g)}{(a_m - g)} \frac{p_{m-1} - b_{m}p_{m-2}}{q_{m-1} - b_{m}q_{m-2}},$$

$$= \frac{p_m}{q_m - g} \frac{gp_{m-1}}{g_{m-1}}$$
(3)

consequently

where  $p_m/q_m$ ,  $p_{m-1}/q_{m-1}$  are the ultimate and penultimate convergents of

$$\frac{b_3}{a_2-\overline{a_3}-\cdots \cdot \frac{b_m}{a_m}}.$$

It results from (3) that

$$y(F_{f_{m-1}} - p_{m-1}) = F_{f_m} - p_m$$
 (4).

Now  $F_{I_m-1} - p_{m-1}$  and  $F_{I_m} - p_m$  cannot both be zero, for that would involve the equality  $p_m/p_m = p_{m-1}/p_{m-1}$ , which is inconsistent with the equation (2) of § 3. Hence, if F were commensurable, (4) would give a commensurable value for the incommensurable y. F must therefore be incommensurable.

The proof of I. is exactly similar, for the condition  $a_n \not\leftarrow b_n$  secures that each of the residual fractions of (1) shall be positive and less than unity.

These two theorems do not by any means include all cases of incommensurability in convergent infinite continued fractions.

Brouncker's fraction, for example,  $1 + \frac{1^2}{2 + 2}, \frac{3^2}{2 + 2}, \frac{5^2}{2 + 2}, \dots$ , converges to the incommensurable value  $4/\pi$ , and yet violates the condition of Proposition I.

CONVERSION OF SERIES AND CONTINUED PRODUCTS INTO
CONTINUED FRACTIONS.

§ 18.] To convert the series

into an "equivalent" continued fraction of the form

$$\frac{b_1}{a_2} = \frac{b_2}{a_2} = \cdots = \frac{b_n}{a_n} = \cdots$$
 (1).

A continued fraction is said to be "equivalent" to a series when the nth convergent of the former is equal to the sum of n terms of the latter for all values of n.

Since the convergents merely are given, we may leave the denominators  $q_1, q_2, \ldots, q_n$  arbitrary (we take  $q_0 = 1$ , as usual).

For the fraction (1) we have

$$p_n/q_n - p_{n-1}/q_{n-1} = b_1b_2...b_n/q_{n-1}q_n$$
 (2);

$$q_1 = a_1$$
,  $q_2 = a_2q_1 - b_2$ , . . . ,  $q_n = a_nq_{n-1} - b_nq_{n-2}$  (3);

Since

$$p_1/q_1 = b_1/q_1$$
 (4),  
 $p_2/q_1 = u_1 + u_2 + \dots + u_n$  (5)

we get from (2) and (5)

$$u_n = b_1 b_2 \dots b_n / q_{n-1} q_n,$$
  
 $u_{n-1} = b_1 b_2 \dots b_{n-1} / q_{n-2} q_{n-1},$   
 $u_2 = b_1 b_2 / q_1 q_2,$   
 $u_1 = b_1 / q_1,$ 
(6).

From (6), by using successive pairs of the equations, we get  $b_1 = q_1 u_1$ ,  $b_2 = q_2 u_2/u_1$ ,  $b_3 = q_2 u_3/q_1 u_2$ , . . . ,  $b_n = q_n u_n/q_{n-2} u_{n-1}$ 

Combining (3) with (7), we also find

$$a_1 = q_1$$
,  $a_2 = q_2(u_1 + u_2)/q_1u_1$ ,  $a_3 = q_3(u_2 + u_3)/q_2u_2$ , ...,  
 $a_n = q_n(u_{n-1} + u_n)/q_{n-1}u_{n-1}$  (8).

Hence

$$S_n = u_1 + u_2 + \dots + u_n$$

$$= \frac{q_1 u_1}{q_1 - q_2(u_1 + u_2)/q_1 u_1} - \frac{q_2 u_2/q_1 u_2}{q_2(u_2 + u_2)/q_2 u_2} - \frac{q_2 u_2/q_2 u_2}{q_2 u_2 - q_2 u_2/q_2 - q_2 u_2}.$$

$$= \frac{q_1 u_1}{q_2(u_2 - u_2)/q_2 - q_2 u_2}.$$
(9).

It will be observed that the q's may be cleared out of the fraction. Thus, for example, we get rid of  $q_1$  by multiplying the first and second numerators and the first denominator by  $1/q_1$ , and the second and third numerators and the second denominator by  $q_1$ ; and so on. We thus get for  $S_n$  the equivalent fraction

$$S_{n} = \frac{u_{1}}{1 - \frac{u_{2}/u_{1}}{(u_{1} + u_{2})/u_{1} - \frac{u_{3}/u_{2}}{(u_{2} + u_{3})/u_{2}}} \cdots \frac{u_{n}/u_{n-1}}{(u_{n-1} + u_{n})/u_{n-1}}$$
(10),

which may be thrown into the form

$$S_{n} = \frac{u_{1}}{1 - u_{1} + u_{2} - u_{2} + u_{4} - u_{4}} \cdot \cdot \cdot \frac{u_{n-2} u_{n}}{u_{n-1} + u_{n}}$$
(11).

This formula is practically the same as the one obtained incidentally in § 16; it was first given, along with many applications, by Euler in his memoir, "De Transformatione Serierum in Fractiones Continuas," Opuscula Analytica, t. H. (1785).

It is important to remark that, since the continued fraction (10) or (11) is *equivalent* to the series, it must converge if the series converges, and that to the same limit.

By giving to  $u_1, u_2, \ldots, u_n$  various values, and modifying the fraction by introducing multipliers as above, we can deduce a variety of results, among which the following are specially useful:—

$$\frac{v_1 x + v_2 x^2 + \dots + v_n x^n}{z_1 x - v_1 + v_2 x} = \frac{v_1 x - v_2 x}{1 - v_1 + v_2 x} - \frac{v_1 v_2 x}{v_2 + v_2 x} - \dots - \frac{v_{n-2} v_n x}{v_{n-1} + v_n x} \quad (12);$$

$$\frac{x}{v_1} + \frac{x^2}{v_2} + \dots + \frac{x^n}{v_n} \\
= \frac{x}{v_1} - \frac{v_1^2 x}{v_2 x + v_2} - \frac{v_2^2 x}{v_2 x + v_2} - \dots - \frac{v_{n-1}^2 x}{v_{n-1} x + v_n} \quad (13);$$

$$\frac{a_1}{b_1} x - \frac{a_1 a_2}{b_1 b_2} x^2 + \dots + \frac{a_1 a_2}{b_1 b_2} \dots - \frac{b_n a_n x}{b_n} \\
= \frac{a_1 x - b_1 a_2 x}{b_1 - b_2 + a_2 x} - \frac{b_2 a_2 x}{b_2 + a_n x} - \dots - \frac{b_{n-1} a_n x}{b_n + a_n x} \quad (14).$$

Example 1.  $H + \frac{1}{2}\pi < x < \frac{1}{2}\pi$ , then

$$\tan^{-3}x = x - x^{j}/3 + x^{k}/5 - x^{j}/7 + \dots ,$$

$$= \frac{x - 1^{2}x^{2} - 3^{2}x^{2} - 5^{2}x^{2}}{1 + 3 - x^{2} + 5 - 3x^{2} + 7 - 5x^{2} + \dots };$$

$$x,$$

and, in particular,

 $\frac{\pi}{1} = \frac{1}{1 + 2} + \frac{1^2 - 3^2 - 5^2}{2 + 2 + 2 \cdot 6} \dots$ 

which is Brouncker's formula for the quadrature of the circle.

Example 2, H x - 1,

$$(1+x)^m = 1 + \frac{mx}{1-2} + \frac{1(m-1)x}{2+n-1} + \frac{2(m-2)x}{3+(m-2)x-4+(m-3)x-1} + \dots$$

Also, if m > -1.

$$2^m = 1 + \frac{m}{1 - m} \frac{1(m-1)}{m+1} \frac{2(m-2)}{m+1} \frac{3(m-3)}{m+1} \dots;$$

and, if m > 0,

$$0 = 1 - \frac{m}{1 + 3 - m + 5 - m + 7 - m + 2} \cdot \dots \cdot$$

§ 19.1 The analysis of last paragraph enables us to construct a continued fraction, say of the form (1), whose first a convergents shall be any given quantities  $f_1, f_2, \dots, f_n$  respectively.

All we have to do is to replace  $u_1, u_2, \dots, u_n$  in (10) or (11)

by  $f_1$ ,  $f_2 - f_1$ , . . . ,  $f_n - f_{n-1}$  respectively.

ty 
$$f_{11}, f_{22} - f_{31}, \dots, f_{n-1} - f_{n-2}$$
 respectively.

The required fraction is, therefore,
$$f_{1} - f_{2} - f_{1} - f_{1}(f_{2} - f_{2}) \cdot (f_{2} - f_{1}) \cdot (f_{1} - f_{2}) \cdot \dots \cdot (f_{n-2} - f_{n-1}) \cdot (f_{n-1} - f_{n-1}) \cdot \dots \cdot (f_{n-2} - f_{n-2}) \cdot (f_{n-1} - f_{n-2}) \cdot \dots \cdot (f_{n-2} - f_{n-$$

Hence we can express any continued product, say

$$P_{\pi} = \frac{d_1 d_2 \dots d_n}{e_1 e_2 \dots e_n},$$

as a continued fraction.

We have merely to put  $f_1 = d_1/e_1$ ,  $f_2 = d_1d_2/e_1e_2$ ... effect some obvious reductions, and we find

$$P_{n} = \frac{d_{1}}{e_{1}} \frac{e_{1}(d_{2} - e_{2})}{d_{2}} \frac{d_{2}e_{2}(d_{3} - e_{3})}{d_{3}e_{3}(d_{2} - e_{3})} \frac{d_{2}e_{4}(d_{4} - e_{4})}{d_{2}e_{4}e_{4}e_{5}} \frac{d_{2}e_{4}(d_{3} - e_{3})}{d_{3}e_{4}e_{2}e_{5}} - \frac{d_{3}e_{4}e_{4}e_{5}}{d_{3}e_{4}e_{4}e_{5}} \frac{d_{3}e_{4}e_{4}e_{5}}{d_{3}e_{4}e_{4}e_{5}} \frac{d_{n-1}e_{n-1}(d_{n-2} - e_{n-2})}{e_{n-2}(d_{n-2}e_{n})} \frac{d_{n-1}e_{n-1}(d_{n-2} - e_{n-2})}{d_{n-2}e_{3}e_{4}e_{5}} \frac{d_{3}e_{5}e_{5}e_{5}}{d_{3}e_{5}e_{5}e_{5}}.$$
(16)\*.

§ 20.] Instead of requiring that the continued fraction be equivalent to the series, or to the function f(n, x), which it is to represent, we may require that the sum to infinity of the series (or  $f(\infty, x)$ ) be reduced to a fraction of a given form, say

$$\frac{\beta_o}{1-}\frac{\beta_1 x}{1-}\frac{\beta_2 x}{1-}\cdots\frac{\beta_n x}{1-}\cdots$$
(1),

where  $\beta_0$ ,  $\beta_1$ , . . . ,  $\beta_{\pi}$  are all independent of x.

There is a process, originally given in Lambert's Beytrage

A similar formula, given by Stern, Crelle's Jour, x., p. 267 (1833), may be obtained by a slight modification of the above process.

(th. 11., p. 75), for reducing to the form (1) the quotient of two convergent series, say F(1, x)/F(0, x).

We suppose that the absolute terms of F(1, x) and F(0, x) do not vanish, and, for simplicity, we take each of these terms to be 1. Then we can establish an equation of the form

$$F(1, x) - F(0, x) = \beta_1 x F(2, x)$$
 (2<sub>1</sub>)

where  $F_{\mathbb{Q}^2}$ , x) is a convergent series whose absolute term we suppose again not to vanish, and  $\beta_1$  is the coefficient of x in  $F(1, x) \cdot F(0, x)$ , which also is supposed not to vanish.

In like manner we establish the series of equations

$$F'(2, x) - F'(1, x) = \beta_2 x F'(3, x)$$
 (22)

$$F(3, x) - F(2, x) = \beta_3 x F(1, x)$$
 (23),

$$F(n+1, x) - F(n, x) = \beta_{n+1} x F(n+2, x) - (2_{n+1}).$$

Let us, in the meantime, suppose that none of the functions F becomes 0 for the value of x in question. We may then put

$$G(n, x) = F(n+1, x)/F(n, x)$$
 (3),

where G(n, x) is a definite function of n and x which becomes neither 0 nor  $\infty$  for the value of x in question.

The equation  $(2_{n+1})$  may now be written

the following: -

$$G(n, x) - 1 = \beta_{n+1}xG(n+1, x)G(n, x),$$
  
 $G(n, x) = 1 \{1 - \beta_{n+1}xG(n+1, x)\}$  (4).

If in (4) we put successively  $n = 0, n = 1, \ldots$ , we derive

$$G(0, x) = \frac{1}{1 - 1} \frac{\beta_1 x}{1 - 1} \cdots \frac{\beta_n x}{1 - (1 - 1/G(n, x))}$$
 (5);

$$1 - \frac{1}{G(n, x)} = \frac{\beta_{n+1} x}{1 - 1} \cdot \cdot \cdot \frac{\beta_{n+m} x}{1 - (1 - 1) G(n + m, x)}$$
(6).

General expressions have been found for  $\beta_0$ ,  $\beta_1$ , . . . by Heilermann, Crelle's Jew. (1846), and by Mair, Proc. I. M.S. (1876).

<sup>\*</sup> The vanishing of one or more of these coefficients would lead to a more general  $\ell$  -rin than (1), namely,

In order that we may be able to assert the equality

$$G(0, x) = \frac{1}{1-1} \frac{\beta_1 x}{1-1} \dots \frac{\beta_n x}{1-1} \dots \text{ ad } x$$
 (7)

it is necessary, and it is sufficient, that it be possible by making m sufficiently great to cause 1-1 G(n,x) to differ from the mth convergent of the residual fraction

$$\frac{\beta_{n+1}x}{1-\frac{\beta_{n+2}x}{1-\cdots}} \cdot \cdot \cdot \frac{\beta_{n-n}x}{1-\cdots} \cdot \cdot \cdot$$
 (8)

by as little as we please.

Let us denote the convergents of (8) by  $p_1/q_1$ ,  $p_2/q_2$ , . . .,  $p_m/q_m$ . Then, from (6), we see that

$$\{1 - 1/G(n, x)\} - p_m/q_m$$

$$= \frac{\{1 - 1/G(n + m, x)\} \beta_{n+1}\beta_{n+2} \dots \beta_{n+m}x^{m}}{q_{m}[q_{m} - q_{m-1}\{1 - 1/\ell \hat{r}(n + m, x)\}]}$$
(10)

The necessary and sufficient condition for the subsistence of (7) is, therefore, that the right-hand side of (9), or of (10), shall vanish when  $m = \infty$ .

Concerning these conditions it should be remarked that while either of them secures the convergence of the infinite continued fraction in (7), the convergence of the fraction is not necessarily by itself a sufficient condition for the subsistence of the equation (7).

In what precedes we have supposed that none of the functions F(n, x) vanish. This restriction may be partly removed. It is obvious that no two consecutive F's can vanish, for then (by the equations (2)) all the preceding F's would vanish, and G(0, x) would not be determinate. Suppose, however, that F(r+1, x') = 0, so that G(r, x') = 0; then (5) furnishes for G(0, x') the closed continued fraction

$$G\left(0,\,x'\right) = \frac{1}{1-1} \frac{\beta_{1}x'}{1-1} \,, \quad , \frac{\beta_{r-1}x'}{1} \,, \label{eq:Gaussian_state}$$

In order that this may be identical with the value given by (7), it is necessary and sufficient that G(r+1, x'), as given by (6), should become ∞, that is, it is necessary and sufficient that the residual fraction

$$\frac{\beta_{r+2}x'}{1-}\frac{\beta_{r+3}x'}{1-}\dots$$
 ad  $\infty$ 

should converge to 1; but this condition will in general be satisfied if the relation (4) subsist for all values of n, and the condition (9) be also satisfied when  $n \triangleleft r + 2$ .

§ 21.1 As an example of the process of last paragraph, let

$$F(n, x) = 1 + \frac{x}{1!(\gamma + n)} + \frac{x^2}{2!(\gamma + n)(\gamma + n + 1)} + \dots$$
 (11).

Then

$$F(n+1, x) - F(n, x) = -\frac{x}{(\gamma+n)(\gamma+n+1)}F(n+2, x)$$
 (2);

$$G(n, x) = 1 / \left\{ 1 + \frac{x}{(\gamma + n)(\gamma + n + 1)} G(n + 1, x) \right\}$$
 (4'),

where G(n, x) = F'(n + 1, x)/F'(n, x).

Hence

$$G\left(0,\,x\right) = \frac{1}{1} + \frac{x/\gamma\left(\gamma+1\right)\,x/(\gamma+1)\,\left(\gamma+2\right)}{1+} + \cdots + \frac{x'\left(\gamma+n-1\right)\left(\gamma+n\right)}{1-\left\{1-1\right\}G\left(n,\,x\right\}\right\}}$$
(5');

and

$$1 = \frac{1}{G(n, x)} = -\frac{x'(\gamma + n)(\gamma + n + 1)}{1 + \cdots} \cdot \cdot \cdot \cdot \frac{x'(\gamma + n + m - 1)(\gamma + n + m)}{1 - (1 - 1)G(n + m, x)} \cdot \cdot (6').$$

The series (11) will be convergent for all finite values of x, and for all positive integral values of n, including 0, provided y be not 0 or a negative integer. Hence we have obviously, for all finite values of x, LG(n+m, x) - 1 when  $m = \infty$ .

Let us suppose that x is positive. Then the residual continued fraction

$$\frac{x_{j}'(\gamma+n)(\gamma+n+1)}{1+} \frac{x_{j}'(\gamma+n+1)(\gamma+n-2)}{1+} \cdot \cdot \cdot \frac{x_{j}'(\gamma+n+m-1)(\gamma+n+m)}{1+} \cdot \cdot \cdot (S')$$

is (by the criterion of § 14) evidently convergent. Hence the factor  $p_m/q_m - p_{m-1}/q_{m-1}$  in the expression (9) vanishes when  $m = \alpha$ ,

Also, since the q's continually increase,  $Lq_m/q_{m-1} < 1$ .

Therefore we may continue the fraction to infinity when x is positive.

Next suppose x negative, = -y say; we then have

$$G(0, -y) = \frac{1}{1 - y} \frac{y'(\gamma + 1)}{1 - y} \frac{y_{\gamma}(\gamma + 1)(\gamma + 2)}{1 - y'(\gamma + n - 1)(\gamma + n)} \frac{y'(\gamma + n - 1)(\gamma + n)}{1 - \{1 - 1, 1 \in (n, -y)\}} (5'');$$

and

$$1 - \frac{1}{G\left(n, -y\right)} = \frac{y/(\gamma + n)\left(\gamma + n + 1\right)}{1 - \frac{y/(\gamma + n + m - 1)\left(\gamma + n + m\right)}{1 - \left(1 - 1\right)G\left(n + m, y\right)}} \qquad (6'')$$

The fraction (8) in this case is "equivalent" to

$$\frac{1}{\gamma+n}\left\{\frac{y}{\gamma+n+1}-\frac{y}{\gamma+n+2}-\dots,\frac{y}{\gamma+n+m-1}\dots\right\} \quad (8''),$$

which is obviously convergent (by § 16), if y have any finite value whatever. Hence the factor  $p_m/q_m - p_{m-1}/q_{m-1}$  belonging to the equivalent fraction (8) must vanish.

Again, by § 2 (6),

$$q_{m_{-1}}$$

$$= 1 - \frac{y/(\gamma + n + m - 1)(\gamma + n + m)y'(\gamma + n + m - 2)(\gamma + n + m - 1)}{1 - 1}$$

$$= 1 - \frac{y/(\gamma + n + m - 1)(\gamma + n + m)y'(\gamma + n + m - 1)}{1 - 1}$$

$$\cdots \frac{y_{j}(\gamma+n)(\gamma+n+1)}{1}$$

$$=1-\frac{1}{\gamma+n+m}\left\{\frac{y}{\gamma+n+m-1}-\frac{y}{\gamma+n+m-2}-\cdots,\frac{y}{\gamma+n}\right\} (12).$$

If only n be taken large enough, the fraction inside the brackets satisfies the condition of § 16 throughout: its value is therefore < 1, however great m may be; and it follows from (12) that  $L_{Im} \circ I_{m+1} = 1$  when  $m = \infty$ .

Since LG(n+m, -y) = 1 when  $m = \infty$ , it follows that all the requisite conditions are fulfilled in the present case also,

We have thus shown that

$$\begin{array}{c} F(1,x) = \frac{1}{1+} \frac{x/\gamma(\gamma+1)}{1+} \frac{x/(\gamma+1)(\gamma+2)}{1+} \\ F(0,x) = \frac{1}{1+} \frac{x/(\gamma+n-1)(\gamma+n)}{1+} \dots \text{ ad } \infty \end{array} \tag{13},$$

whence, by an obvious reduction,

$$\frac{F(1, x)}{F(0, x)} = \frac{\gamma}{\gamma + \frac{x}{\gamma + 1}} \frac{x}{\gamma + 2} \cdots \frac{x}{\gamma + n} \dots (14),$$

a result which holds for all finite real values of x, except such as render F(0, x) zero\*, and for all values of y, except zero and negative integers.

If we put  $\pm x^2/4$  in place of x in the functions F(0, x) and F(1, x), and at the same time put  $\gamma = \frac{1}{2}$ , we get

$$F(0, -x^2/4) = \cos x, \quad F(1, -x^2/4) = \sin x/x;$$

$$F(0, x^2/4) = \cosh x, \quad F(1, x^2/4) = \sinh x/x.$$

Cor. 1. Hence, from (14), we get at once

$$\tan x = \frac{x}{1 - 3} - \frac{x^2}{3 - 5} - \dots - \frac{x^2}{2n + 1} - \dots$$
 (15);

$$\tanh x = \frac{x}{1+3+5+} \cdot \dots \cdot \frac{x^2}{2n+1+} \dots$$
 (16).

Cor. 2. The numerical constants π and π² are incommensurable, For, if π were commensurable, π/4 would be commensurable, say λ/μ. Hence we should have, by (15),

$$\frac{1}{\gamma}\left\{\gamma+\frac{x}{\gamma+1+}\frac{x}{\gamma+2+}\ldots\right\}$$

which represents F(0, x)/F(1, x) will converge to 0. Of course, two consecutive functions F(n, x), F(n+1, x) cannot vanish for the same value of x; otherwise we should have F(x, x) = 0, which is impossible, since F(x, x) = 1.

<sup>.</sup> In a sense it will hold even then, for the fraction

$$1 = \frac{\lambda/\mu}{1 - 3} \frac{\lambda^2/\mu^2}{5} \frac{\lambda^2/\mu^2}{2n + 1} \cdot \cdot \cdot \cdot \cdot \cdot$$

$$= \frac{\lambda}{\mu} - \frac{\lambda^2}{3\mu} \frac{\lambda^2}{5\mu} \cdot \cdot \cdot \cdot \cdot \frac{\lambda^2}{(2n + 1)\mu} \cdot \cdot \cdot \cdot \cdot \cdot (17).$$

Now, since  $\lambda$  and  $\mu$  are fixed finite integers, if we take n large enough we shall have  $(2n+1)\mu > \lambda^2 + 1$ . Hence, by § 17, the fraction in (17) converges to an incommensurable limit, which is impossible since 1 is commensurable.

That  $\pi^2$  is also incommensurable follows in like manner very readily from (15).

By using (16) in a similar way we can easily show that

Cor. 3. Any commensurable power of e is incommensurable\*. § 22.] The development of last paragraph is in reality a particular case of the following general theorem regarding the hypergeometric series, given by Gauss in his classical memoir on that subject (1812) †:—

If

$$F(a, \beta, \gamma, x) = 1 + \frac{a\beta}{1 \cdot \gamma} x + \frac{a(a+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \dots,$$

and

$$G(\alpha, \beta, \gamma, x) = F(\alpha, \beta + 1, \gamma + 1, x)/F(\alpha, \beta, \gamma, x),$$

then  $G(\alpha, \beta, \gamma, x) = \frac{1}{1-1} \frac{\beta_1 x}{1-1} \frac{\beta_2 x}{1-1} \cdot \cdot \cdot \frac{\beta_{2n} x}{1/G(\alpha + n, \beta + n, \gamma + 2n)}$ (18)

where

$$\begin{split} \beta_1 &= \frac{\alpha \left( \gamma - \beta \right)}{\gamma \left( \gamma + 1 \right)}, & \beta_2 &= \frac{\left( \beta + 1 \right) \left( \gamma + 1 - \alpha \right)}{\left( \gamma + 1 \right) \left( \gamma + 1 - \beta \right)}, \\ \beta_3 &= \frac{\left( \alpha + 1 \right) \left( \gamma + 1 - \beta \right)}{\left( \gamma + 2 \right) \left( \gamma + 3 \right)}, & \beta_4 &= \frac{\left( \beta + 2 \right) \left( \gamma + 2 - \alpha \right)}{\left( \gamma + 3 \right) \left( \gamma + 4 \right)}, \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{2n-1} &= \frac{\left( \alpha + n - 1 \right) \left( \gamma + n - 1 - \beta \right)}{\left( \gamma + 2 n - 1 \right) \left( \gamma + 2 n \right)}, & \beta_{2n} &= \frac{\left( \beta + 1 \right) \left( \gamma + 2 - \alpha \right)}{\left( \gamma + 2 n - 1 \right) \left( \gamma + 2 n \right)} \end{split}$$

<sup>•</sup> The results of this paragraph were first given by Lambert in a memoir which is very important in the history of continued fractions (Hist. d. Pdc. Roy. d. Berlin, 1761). The arrangement of the analysis is taken from Legendre (Lc.), the general idea of the discussion of the convergence of the fraction from Schlömlich.
• Weeke, Bd. m., p. 136.

After what has been done, the proof of this theorem should present no difficulty.

The discussion of the question of convergence is also comparatively simple when x is positive; but presents some difficulty in the case where x is negative. In fact, we are not aware that any complete elementary discussion of this latter point has been given.

Cor. If in (18) we put  $\beta = 0$ , and write  $\gamma - 1$  in place of  $\gamma$ , we get the transformation

$$1 - \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha - 1)}{\gamma(\gamma + 1)} x^{2} + \frac{\alpha(\alpha + 1)}{\gamma(\gamma - 1)} \frac{(\alpha - 2)}{(\gamma - 2)} x^{2} + \dots$$

$$= \frac{1}{1 - \frac{\beta_{1} x}{1 - \frac{\beta_{2} x}{1 - 1}}} \dots (19),$$

where

$$\begin{split} \beta_1 &= \frac{\alpha}{\gamma}, & \beta_2 &= \frac{\gamma - \alpha}{\gamma (\gamma - 1)}, \\ \beta_3 &= \frac{(x - 1)\gamma}{(\gamma - 1)(\gamma - 2)}, & \beta_4 &= \frac{2(\gamma - 1 - \alpha)}{(\gamma - 2)(\gamma + 3)}, \end{split}$$

$$\boldsymbol{\beta}_{2n-1} = \frac{(\alpha+n-1)\cdot \gamma + n - 2}{\left(\gamma - 2n - 3\right)\left(\gamma - 2n - 2\right)}, \quad \boldsymbol{\beta}_{2n} = \frac{n\left(\gamma + n - 1 - \alpha\right)}{\left(\gamma - 2n - 2\right)\left(\gamma + 2n - 1\right)}.$$

Gauss's Theorem is a very general one; for the hypergeometric series includes nearly all the ordinary elementary series.

Thus, for example, we have, as the reader may easily verify,

$$\begin{split} &(1+x)^n=F(-m,\beta,\beta,-x)\,;\\ &\log(1+x)=xF(1,1,2,-x)\,;\\ &\sin x=x\sum_{k=n}L_kF(k,K,\frac{n}{2},x^2|4kk)\,;\\ &\sin x=x\sum_{k=n}L_kF(k,K,\frac{n}{2},-x^2|4kk)\,;\\ &\sin^{-1}x=xF(\frac{1}{2},\frac{1}{2},\frac{n}{2},x^2)\,;\\ &x_N(1-x^2)F(1,1,\frac{n}{2},x^2)\,;\\ &\tan^{-1}x=xF(\frac{1}{2},\frac{1}{2},\frac{n}{2},x^2)\,;\\ &\tan^{-1}x=xF(\frac{1}{2},\frac{1}{2},\frac{n}{2},x^2)\,; \end{split}$$

Examine the converse of the first

(1.)  $1 + \frac{1}{1^2 + 2^$ 

(3.)  $1 + \frac{1}{1} + \frac{2}{1} + \frac{3}{2} + \frac{3}{2}$ (5.)  $1 - \frac{1}{1} + \frac{1}{$ 

 $(9.) \ \frac{2}{1} \frac{1^3}{1^3} \frac{3}{1^3} \frac{2^5}{1^3} \frac{4}{1^3} \frac{1^3}{1^3} \frac{1^3$ 

(11.) Show that the fraction from 1.

verges to a position must if, for a value of a

descending series for a series to a series of the first second class, provided and all and the contract and all are more aim of these condition -.

(14.) Show that

(15.) Evaluate

and

where m is any int - c.

Show that

(16.)  $1 - \frac{1}{b} + \frac{1}{2} = \frac{1}$ 

(17.)  $\sin x = \frac{2}{1 - 2} = \frac{2}{4} = \frac{4}{1 - 2} = \frac{4}{1$ 

(18.) log 1+r 1 1 2 x

(19.) 
$$1 = \frac{1^2}{2^2} \cdot \frac{2^2}{5^2} \cdot \frac{3^2}{7^2} \cdot \dots$$

(20.) 
$$\log \frac{1+x}{x} = \frac{x}{1+} \frac{1^2x}{2x-1+} \frac{2^3x}{3x-2+} \frac{3^2x}{4x-3+} \cdots$$

(21.) 
$$\log y = \frac{y^2 - 1}{1 - 4(y^2 + y + 1) - 4(2y^2 + y + 2) - 4(2y^2 + y + 2) - \dots}$$

$$\frac{(2n-1)^2(y^2-1)^3}{4(ny^2+y+n)} = .$$

(22.) 
$$e^{x} = \frac{1}{1-1} \frac{x}{1+x-2} \frac{x}{2+x-3+x-4} \frac{3x}{4+x-4} \cdots$$

Evaluate the following: -

(23.) 
$$1 + \frac{1}{1-} \frac{1}{3-} \frac{2}{1-} \frac{3}{5-} \frac{4}{5-} \cdots$$
 = e (21.)  $\frac{1}{1+} \frac{2}{2+} \frac{3}{3+} \frac{4}{4+} \cdots$ 

(25.) 
$$\frac{1}{1+1} \frac{1^2}{1+1} \frac{2^2}{1+1} \frac{3^2}{1+} \dots = \log 2$$
 (26.)  $\frac{1}{2+3} \frac{2}{3+4} \frac{3}{4+5} \frac{4}{5+} \dots = \frac{3-2}{2+3}$ 

(27.) Show that tan x and tanh x are incommensurable if x be commensurable.

Establish the following transformations:-

(24.) 
$$e^x = \frac{1}{1-1} \frac{x}{1+2} \frac{x}{2-3} \frac{x}{3+2} \frac{x}{2-5+2} \frac{x}{2-7+\cdots$$

(29.) 
$$\log(1+x) = \frac{x}{1+} \frac{1^2x}{2+} \frac{1^2x}{3+} \frac{2^2x}{4+} \frac{2^2x}{5+} \frac{3^2x}{6+} \frac{3^2x}{7+} \dots$$

(30.) 
$$\tan^{-1} x = \frac{x}{1+} \frac{1^2 x^2}{3+} \frac{2^2 x^2}{5+} \frac{3^2 x^2}{7+} \dots$$

$$tanh^{-1}x = 1 - 3 - 5 - 7 - \cdots$$

(31.) 
$$\tan nx = \frac{n \tan x}{1 - \frac{1}{2}} \frac{(n^2 - 1^2) \tan^2 x}{3 - \frac{5}{2}} \cdot \frac{(n^2 - 2^2) \tan^2 x}{5 - \frac{5}{2}} \cdot \frac{1}{(\text{Euler, Mem. Acad. Pet., 1813.})}$$

$$\frac{\sin{(n+1)}x}{\sin{nx}} = 2\cos{x} - \frac{1}{2\cos{x}} - \frac{1}{2\cos{x}} - \frac{1}{\cos{x}} - \cdots,$$

where there are n partial quotients.

(32.1 (33.) If

$$\begin{aligned} \phi\left(\mathbf{a},\,\beta,\,\gamma,\,x\right) &= 1 + \frac{(q^{\mathbf{a}}-1)\left(q^{\mathbf{\beta}}-1\right)}{(q-1)\left(q^{\mathbf{y}}-1\right)} \frac{x}{q} + \frac{\left(q^{\mathbf{a}}-1\right)\left(q^{\mathbf{a}+1}-1\right)\left(q^{\mathbf{\beta}}-1\right)\left(q^{\mathbf{\beta}+1}-1\right)}{(q-1)\left(q^{\mathbf{y}}-1\right)\left(q^{\mathbf{y}}-1\right)\left(q^{\mathbf{y}}+1-1\right)} \frac{x^{\mathbf{a}}}{q} + \dots \end{aligned}$$

then

$$\frac{\phi\left(a,\,\beta+1,\,\gamma+1,\,x\right)}{\phi\left(a,\,\beta,\,\gamma,\,x\right)} = \frac{1}{1-} \frac{\beta_{1}x}{1-} \frac{\beta_{2}x}{1-} \cdot \cdot \cdot \cdot \cdot$$

(Heine, Crelle's Jour., xxx:1.)

where

$$\begin{split} \beta_{2r} &= \frac{(q^{\beta+r}-1) \left( q^{\gamma+r-a}-1 \right)}{\left( q^{\gamma+r-1}-1 \right) \cdot q^{\gamma+r}-1 \right)} q^{a+r-1}, \\ \beta_{2r+1} &= \frac{(q^{a+r}-1) \left( q^{\gamma+r-\beta}-1 \right)}{\left( q^{\gamma+r}-1 \right) \left( q^{\gamma+r-\beta}-1 \right)} q^{\beta+r}. \end{split}$$

(34.) Show that

now that 
$$a^2 = \left\{ a - 1 + \frac{1}{2(a-1)} + \frac{3^2}{2(a-1)} + \frac{5^2}{2(a-1)} + \cdots \right\}$$

$$\times \left\{ a + 1 + \frac{1}{2(a+1)} + \frac{3^2}{2(a+1)} + \frac{5^2}{2(a+1)} + \cdots \right\}$$

$$\text{Wall is (see Multr, It hil. Mag., 1877)}.$$

## CHAPTER XXXV.

# General Properties of Integral Number.

## NUMBERS WHICH ARE CONGRUENT WITH RESPECT TO A GIVEN MODULUS.

§ 1.] If m be any positive integer whatever, which we call the modulus, two integers, M and N, which leave the same remainder when divided by m are said to be congruent with respect to the modulus m\*.

In other words, if M = pm + r, and N = qm + r, M and N are said to be congruent with respect to the modulus m. Gauss, who made the notion of congruence the fundamental idea in his faunous Disquisitiones Arithmeticae, uses for this relation between M and N the symbolism

 $M \equiv N \pmod{m}$ ; or simply  $M \equiv N$ .

if there is no doubt about the modulus, and no danger of confusion with the use of \(\exists \text{to denote algebraical identity}\).

Cot. 1. If two numbers M and N be congruent with respect to modulus m, then they differ by a multiple of m; so that we have, san, M = N + pm.

Cot 2. If either M or N have any factor in common with m, than the other must also have that factor; and if either be prime to m, the other must be prime to m also.

In the present chapter we shall use only the most elementary consequences of the theory of congruent numbers.

To save repetition, let't be understood, when nothing else is indicated, throughout this chapter every letter stands for a positive or negative integer.

Our object here is simply to give the reader a conspectus of the more elementary methods of demonstration which are employed in establishing properties of integral numbers; and to illustrate these methods by proving some of the elementary theorems which he is likely to meet with in an ordinary course of mathematical study. Further developments must be sought for in special treatises on the theory of numbers.

§ 2.] If we select any "modulus" m, then it follows, from chap, II., § 11, that all integral numbers can be arranged into successive groups of m, such that each of the integers in one of these groups is congruent with one and with one only of the set

0, 1, 2, . . ., 
$$(m-2)$$
,  $(m-1)$  (A),

or, if we choose, of the set

$$0, 1, 2, \ldots, -2, -1$$
 (B),

where there are m integers.

Another way of expressing the above is to say that, if we take any m consecutive integers whatever, and divide them by m, their remainders taken in order will be a cyclical permutation of the integers (A).

Example. If we take m=5, the set (A) is 0, 1, 2, 3, 4. Now if we take the 5 consecutive integers 63, 64, 65, 66, 67 and divide them by 5, the remainders are 3, 4, 0, 1, 2, which is a cyclical permutation of 0, 1, 2, 3, 4.

§ 3.] A large number of curious properties of integral numbers can be directly deduced from the simple principle of classification just explained.

Example 1. Every integer which is a perfect cube is of the form  $7p \pm 1$ . Bearing in mind that every integer N has one or other of the forms 7m,  $7m \pm 1$ ,  $7m \pm 2$ ,  $7m \pm 3$ .

also that . (7m

$$(7m \pm r)^3 = (7m)^3 + 3(7m)^2 r + 3(7m)^2 \pm r^3,$$
  
=  $(7^2m^3 \pm 24m^2r + 3mr^2) 7 \pm r^3,$   
=  $M7 \pm r^3,$ 

we see that in the four possible cases we have

$$N^{\omega} = (7m)^{\pm} = (7^{2}m^{2}) 7$$
;  
 $N^{2} = (7m \pm 1)^{2} = M7 \pm 1$ ;  
 $N^{3} = (7m \pm 2)^{3}$ ,  
 $= M7 \pm 8 = (M \pm 1) 7 \pm 1$ ;  
 $N^{3} = (7m \pm 3)^{3} = (M \pm 4) 7 \mp 1$ .

C. IT.

In every case, therefore, the cube has one or other of the forms Tp or Tp +1.

Example 2. Prove that  $3^{2n+1} + 2^{n+2}$  is divisible by 7 (Wolstenholme). We have  $3^{2n+1} + 2^{n+2} = (7 - 4)^{2n+1} + 2^{n+2}$ .

Now (see above, Example 1, or below, & 4)

 $(7-4)^{2n+1} = M7-4^{2n+1},$  $3^{2n+1} + 2^{n+2} = M7-4^{2n+1} + 2^{n+2},$ 

Hence  $3^{2n+1} + 2^{n+2} = M7 - 4^{2n+1} + 2^{n+2},$ =  $M7 - 2^{n+2} (2^{2n} - 1).$ 

First  $2^{n}-1$  is divisible by  $2^{n}-1$  (see chap. v., § 17), that is, by 7. Hence  $2^{n+2}(2^{2n}-1)=N7$ .

Finally, therefore,  $3^{2n+1} + 2^{n+2} = (M-N) 7$ , which proves the theorem.

Example 3. The product of 3 successive integers is always divisible by 1-2.3.

Let the product in question be m (m+1) (m+2). Then, since m must have on other of the three forms, 3m, 3m+1, 3m-1, we have the following cases to consider:

$$3m (3m + 1) (3m + 2)$$
 (1);  
 $(3m + 1) (3m + 2) (3m + 3)$  (2);

$$(3m+1)(3m+2)(3m+3)$$
 (2),  
 $(3m-1)(3m+1)$  (3).

In (1) the proposition is at once evident; for 3m is divisible by 3, and (3m+1)(3m+2) by 2. The same is true in (2).

In case (3) we have to show that (3m-1)m(3m+1) is divisible by 2. Now this must be so, because, if m is even, m is divisible by 2; and if m be odd, both 3m-1 and 3m+1 are even; that is, both 3m-1 and 3m+1 are divisible by 2.

In all cases, therefore, the theorem holds.

Example 1. To show that the product of p successive integers is always divisible by  $1, 2, 3, \ldots, p$ .

Let us suppose t', it it has been shown, 1st, that the product of any p-1 so sive into rs whatever is divisible by  $1, 2, 3, \dots, p-1$ ; 2nd, that the pool let of p successive intepers beginning with any integer up to x is divisible by  $1, 2, 3, \dots, p-1$ ; p.

Consider the product of p successive integers beginning with x+1. We have

(x+1)(x+2)...(x+p-1)(x+p)

$$= p(x+1)(x-2) \dots (x+p-1) + x(x+1)(x+2) \dots (x+p-1) \dots (1).$$

Now, by our first supp sition,  $(x+1)(x+2) \dots (x+p-1)$  is divisible by  $1, 2, \dots, p-1$ ; and, by our second,  $x(x+1)(x+2) \dots (x+p-1)$  is divisible by  $1, 2, 3, \dots, p$ .

Hence each member on the right of (1) is divisible by 1.2.3...p.

It follows, therefore, that, if our two suppositions be right, then the product of p successive integers beginning with x+1 is divisible by  $1,2,3,\ldots p$ .

But we have shown in Example 3 that the product of 3 consecutive integers is always divisible by 1, 2, 3; and it is self-evident that the product of 4 consecutive integers beginning with 1 is divisible by 1, 2, 3, 4. It follows, therefore, that the product of 4 consecutive integers beginning with 2 is divisible by 1, 2, 3, 4. Using Example 3 acid, and the risult just established, we prove that 4 consecutive integers beginning with 3 is divisible by 1, 2, 3, 4; and thus we finally establish that the product of any 4 consecutive integers whatever is divisible by 1, 2, 3, 4.

Proceeding in exactly the same way, we next show that our theorem holds when p=5; and so on. Hence it holds generally.

This demonstration is a good example of "mathematical induction."

**Example 5.** If a, b, c be three integers such that  $a^2 + b^2 = c^2$ , then they are represented in the most general way possible by the forms

$$a = \lambda (m^2 - n^2), \quad b = 2 \lambda mn, \quad c = \lambda (m^2 + n^2).$$

First of all, it is obvious, on account of the relation  $a^2 + b^2 = c^2$ , that, if any two of the numbers have a common factor  $\lambda$ , then that factor must occur in the other also; so that we may write  $a = \lambda a'$ ,  $b = \lambda b'$ ,  $c = \lambda c'$ , where a', b', c' are prime to each other, and we have

$$a'^2 + b'^2 = c'^2$$
 (1),

No two of the three, a', b', c, therefore, can be even; also both a' and b' cannot be odd, for then  $a'' + b''^2$  would be of the form 4n + 2, which is an impossible form for the number c''.

It appears, then, that one of the two, a', b', say b' ( $a \ge 3$ ), must be even, and that a' and c' must be odd. Hence (c' + a')/2 and (c' - a')/2 must be integers and these integers must be prime to each other; for, if they had a common factor, it must divide their smu which is c' and their difference which is a'; but c' and a' have by hypothesis no common factor.

Now we have from (1)

whence

$$\left(\frac{c'+a'}{2}\right) \times \left(\frac{c'-a'}{2}\right) = \beta^2$$
 (2)

Therefore, since  $\{c'+a'\}/2$  is prime to (c'-a')/2, each of these must be a perfect square; so that we must have

 $\beta = mn$ 

$$\frac{c'+a'}{2} = m^2 \tag{}$$

$$\frac{e' - a'}{2} = n^2$$
 (4),

where m is prime to n.

From (3) and (4), we have, by subtraction and addition,

 $a' = m^2 - n^2$ ,  $c' = m^2 + n^2$ ;  $b' = 2\beta = 2mn$ ,

and, from (5),  $b' = 2\beta = 2mn$ . Returning, therefore, to our original case, we must have generally

 $a = \lambda (m^2 - n^2), b = 2\lambda mn, c = \lambda (m^2 + n^2).$ 

This is the complete analytical solution of the famous Pythagorean problem—to find a right-angled triangle whose sides shall be commensurable.

§ 4.] The following theorem may be deduced very readily from the principles of § 2. Let f(x) stand for  $p_0 + p_1x + p_2x^2 + \dots + p_nx^n$ , where  $p_0, p_1, \dots, p_n$  are positive or negative integers, and x any positive integer; then, if x be congruent with r with respect to the modulus m, f(x) will be congruent with f(r) with respect to modulus m.

By the binomial expansion, we have

$$\begin{aligned} (qm+r)^n &= (qm)^n + {}_nC_1(qm)^{n-1}r + \dots + {}_nC_{n-1}(qm)r^{n-1} + r^n, \\ &= (q^nm^{n-1} + {}_nC_1q^{n-1}m^{n-2}r + \dots + {}_nC_{n-1}qr^{n-1})m + r^n, \\ &= M_nm + r^n; \end{aligned}$$

where  $M_n$  is some integer, since all the numbers  ${}_nC_1$ ,  ${}_nC_2$ , . . . ,  ${}_nC_{n-1}$  are, by § 3, Example 4, or by their law of formation (see chap. iv., § 14) necessarily integers.

Similarly

$$(qm+r)^{n-1}=M_{n-1}m+r^{n-1},$$

Hence, if x = qm + r,

$$f(x) = p_0 + p_1 r + p_2 r^2 + \dots + p_n r^n + (p_1 M_1 + p_2 M_2 + \dots + p_n M_n) m,$$
  
=  $f(r) + Mm$ .

Hence f(x) is congruent with f(r) with respect to modulus m.

Cor. 1. Since all integers are congruent (with respect to modulus m) with one or other of the series

$$0, 1, 2, \ldots, m-1,$$

it follows that to test the divisibility of f(x) by m for all integral values of x, we need only test the divisibility by m of f(0), f(1), f(2), . . . , f(m-1).

Example 1. Let f(x) = x(x+1)(2x+1); and let it be required to find when f(x) = 0 is divisible by 6. We have f(x) = 0, f(1) = 6, f(2) = 30, f(3) = 81, f(4) = 180, f(5) = 330. Each of these is divisible by 6; and every integer is congruent (mod 6) with one of the rix numbers 0, 1, 2, 3, 4, 5; hence x(x+1)(2x+1) is always divisible by 6.

Cor. 2.  $f\{qf(r) + r\}$  is always divisible by f(r); for  $f\{qf(r) + r\} = Mf(r) + f(r) = (M+1)f(r)$ .

Hence an infinite number of values of x can always be found which will make f(x) a composite number,

This result is sometimes stated by saying that no integral function of x can furnish prime numbers only.

Example 2. Show that  $x^4-1$  is divisible by 5 if x be prime to 5, but not otherwise.

With modulus 5 all integral values of x are congruent with  $0, \pm 1, \pm 2$ . If  $f(x) = x^4 - 1$ , f(0) = -1,  $f(\pm 1) = 0$ ,  $f(\pm 2) = 15$ . Now 0 and 15 are each divisible by 5; but -1 is not divisible by 5. Hence  $x^4 - 1$  is divisible by 5 when x is prime to 5, but not otherwise.

Example 3. To show that  $x^2+x+17$  is not divisible by any number less than 17, and that it is divisible by 17 when and only when x is of the form 17m or 17m-1.

Here

These numbers are all primes, hence no number less than 17 will divide  $x^2+x+17$ , whatever the value of x may be; and 17 will do so only when x=m17 or x=m17-1.

§ 5.] Method of Differences.—There is another method for testing the divisibility of integral functions, which may be given here, although it belongs, strictly speaking, to an order of ideas somewhat different from that which we are now following.

Let  $f_n(x)$  denote an integral function of the nth degree.

$$f_{\mathbf{n}}(x+1) - f_{\mathbf{n}}(x) = p_{\mathbf{0}} + p_{\mathbf{1}}(x+1) + \dots + p_{n-1}(x+1)^{n-1} + p_{\mathbf{n}}(x+1)^{n} - p_{\mathbf{0}} - p_{\mathbf{1}}x - \dots - p_{n-1}x^{n-1} - p_{\mathbf{n}}x^{n} \quad (1).$$

Now on the right-hand side the highest power of x, namely  $x^*$ , disappears; and the whole becomes an integral function of the  $\overline{n-1}$ th degree,  $f_{n-1}(x)$ , say. Thus, if m be the divisor, we have

$$\frac{f_n(x+1) - f_n(x)}{m} = \frac{f_{n-1}(x)}{m}$$
(2)

It may happen that the question of divisibility can be at once settled for the simpler function  $f_{n-1}(x)$ . Suppose, for example, that it turns out that  $f_{n-1}(x)$  is always divisible by m, whatever x may be ; then  $f_n(x+1)-f_n(x)$  is always divisible by m, whatever x may be. Suppose, further, that  $f_n(0)$  is divisible by m; then, since  $f_n(1)-f_n(0)$ , as we have just seen, is divisible by m, it follows that  $f_n(1)$  is divisible by m. Similarly, it may be shown that  $f_n(2)$  is divisible by m; and so on.

If the divisibility or non-divisibility of  $f_{n-1}(x)$  be not at once evident, we may proceed with  $f_{n-1}(x)$  as we did before with  $f_n(x)$ , and make the question depend on a function of still lower degree; and so on.

$$\begin{array}{ll} {\rm Example.} & f_{\theta}(x) = x^{0} - x \text{ is always divisible by 5.} \\ & f_{3}(x+1) - f_{6}(x) = (x+1)^{3} - (x+1) - x^{3} + x, \\ & = 5x^{4} + 15x^{2} + 10x^{2} + 10x^{2} + 6x, \\ {\rm Now} & f_{6}(1) = 0, \\ {\rm therefore} & f_{2}(2) - f_{6}(1) = M.5, \\ {\rm and} & f_{3}(2) = M.5, \\ {\rm Similarly,} & f_{9}(3) - f_{5}(2) = M.5, \\ {\rm therefore} & f_{1}(3) = f_{1}(3) = M.5, \\ {\rm for } (3) = M.5, \\ {\rm for } ($$

and so on.

Thus we prove that  $f_{\delta}(1), f_{\delta}(2), f_{\delta}(3), Ac.,$  are all divisible by 5; in other words, that  $x^2 - x$  is always divisible by 5.

#### EXERCISES XXXV.

- (1.) The sum of two odd squares cannot be a square,
- (2.) Every prime greater than 3 is of the form 6n ± 1.
- (3.) Every prime, except 2, has one or other of the forms 4n ± 1.
- (4.) Every integer of the form 4n-1 which is not prime has an odd number of factors of the form 4n-1.
- (5.) Every prime greater than 5 has the form 30m+n, where n=1, 7, 11, 13, 17, 19, 23, or 29.
   (6.) The source of every trime greater than 3 is of the form 24m+1; and
- the square of every integer which is not divisible by 2 or 3 is of the same form.
  - (7.) If two odd primes differ by a power of 2, their sum is a multiple of 3.
- (4.) The difference of the squares of any two odd primes is divisible by 24.
  (9.) None of the forms (3m+2) n²+3, 4mn m-1, 4mn m-n can repre-
- sent a square integer. (Goldbach and Euler.)

  (10.) The nth power of an odd number greater than unity can be presented
- as the difference of two square numbers in a different ways.

  (11.) If N differ from the two successive squares between which it lies by
- x and y respectively, prove that N-xy is a square.

  (12) The cube of every rational number is the difference of the squares of two rational numbers.
- (13.) Any uneven cube, n<sup>4</sup>, is the sum of n consecutive uneven numbers, of which n<sup>2</sup> is the middle one.
- (11) There can always be found a consecutive integers, each of which is not a prime, however great a may be.

- (15.) In the scale of 7 every square integer must have 0, 1, 2, or 4 for its unit digit.
- unit digit. (16.) The scale in which 31 denotes a square integer has a radix of the form n(3n+4) or (n+2)(3n+2).
- (17.) There cannot in any scale be found three different di its such that the three integers formed by placing each dige differently in each integer shall be in Arithmetical Progression, unless the radix of the scale be of the form 3p+1. If this condition be satisfied, there are 2(p-1) such sets of digits; and the common difference of the AP, is the same in all cases.
  - (18.) If x > 2,  $x^4 4x^3 + 5x^2 2x$  is divisible by 12.
- (19.)  $x^5/5 + x^4/2 + x^3/3 x/30$ , and  $x^6/6 + x^5/2 + 5x^4/12 x^2/12$  are both integral for all integral values of x.
- (20.) If x, y, z be three consecutive integers,  $(\Sigma z)^3 3\Sigma z^3$  is divisible by 108.
  - (21.)  $x^3 x$  is divisible by 6.
  - (22.) Find the form of x in order that x<sup>4</sup>+1 may be decisible by 17.
- (23.) Examine how far the forms x<sup>2</sup>+x+11, 2x<sup>2</sup>+29 represent prime numbers.
  - (24.) Find the least value of x for which 2x 1 is divisible by 47.
    - (25.) Find the least value of x for which 2x-1 is divisible by 23.
  - (26.) Find the values of x and y for which 7x y is divisible by 22.
  - (27.) Show that the remainder of  $2^{2x+p}+1$  with respect to  $2^{-x}+1$  is 2. (28.)  $3^{2x} \sim 2^{2y}$  is divisible by 5, if  $x \sim y = 2$ .
  - (29.) Show that 22x+1 + 1 is always divisible by 3.
  - (30.)  $4^{3x\pm 1} + 2^{3x\pm 1} + 1$  is divisible by 7.
  - (31.) x<sup>im</sup> + x<sup>2m</sup> + 1 never represents a prime unless x 0 or x = 1.
- (32.) If P be prime and = a<sup>2</sup> + b<sup>2</sup>, show that I<sup>n</sup> can be resolved into the source of two squares in ½n ways or ½ (n+1) ways, according a n is even or odd, and give one of these resolutions.
- (33.) If  $x^2 + y^2 = z^2$ , x, y, z being integers, then  $xyz \equiv 0 \pmod{60}$ ; and if x be prime and >3,  $y \equiv 0 \pmod{12}$ . Show also that one of the three numbers  $\equiv 0 \pmod{5}$ .
- (34.) The solution in integers of  $x^2+y^2=2z^2$  can be deduced from that of  $x^2+y^2=z^2$ . Hence, or otherwise, find the two lowest solutions in integers of the first of these equations.
- (35.) If the equation x³+y²=z³ had an integral solution, show that one of the three x, y, z must be of the form 7m, and one of the form 3m.
  (36.) The area of a right-angled triangle with commensur, 3/e sides cannot
- be a square number.
  - (37.) The sum of two integral fourth powers cannot be an integral square.
    (38.) Show that (3+5/5)<sup>x</sup>+(3-5/5)<sup>x</sup> is divisible by 2<sup>x</sup>.
- (39.) If x be any odd integer, not divisible by 3, prove that the integral part of 4x − (2 + √2)x is a multiple of 112.
- (40.) If n be odd, show that  $1+{}_nC_4+{}_nC_8+{}_nc_{16}+\ldots$  is divisible by  $2^{(n-3)/2}$

#### ON THE DIVISORS OF A GIVEN INTEGER.

[6] We have already seen (chap. III., § 7) that every composite integer N can be represented in the form  $a^ab^a\sigma$ , . . ., where a, b, c, . . . are primes. If N be a perfect square, all the indices must be even, and we have  $N = a^{aa}b^{a}\sigma$  . . .; so that  $JN = a^{ab}b^{a}\sigma$  . . . ; so that

In this case N is divisible by  $\sqrt{N}$ .

If N be not a perfect square, then one at least of the indices must be odd; and we have, say,

$$N = a^{2\alpha+1}h^{2\beta}e^{\alpha\gamma} \dots = a^{\alpha}h^{\beta}e^{\gamma} \dots a^{\alpha+1}h^{\beta}e^{\gamma'} \dots$$

so that N is divisible by  $a^a\,b^\beta\,c^\gamma$  . . ., which is obviously less than  $\sqrt{N}$ . Hence

Every composite number has a factor which is not greater than its square root.

This proposition is useful as a guide in finding the least factors of large numbers. This has been done, once for all, in a systematic, but more or less tentative, manner, and the results published for the first nine million integers in the Factor Tables of Burckharl. Dase, and the British Association.

§ 7.] The divisors of any given number  $N = a^{\alpha}b^{\beta}c^{\gamma}$ ... are all of the form  $a^{\alpha}b^{\beta}c^{\gamma}$ ..., where  $a', \beta', \gamma'$ ... may have any values from 0 up to a, from 0 up to  $\beta$ , from 0 up to  $\gamma$ ... respectively. Hence, if we include 1 and N itself among the divisors the divisors of N  $a^{\alpha}b^{\beta}c^{\gamma}$ ... are the various terms of dained by distribution the product

For an interesting account of the construction and use of these tables, see J. W. L. triarsher's Report, Rep. Brit. Assoc. (1877).

Cor. 1. Since

$$1 + a + a^{9} + \dots + a^{n} = \frac{a^{n+1} - 1}{a - 1}$$
$$1 + b + b^{2} + \dots + b^{8} = \frac{b^{8+1} - 1}{a - 1}$$

.

and so on,

It follows that the sum of the divisors of V = a they is

$$\frac{(a^{a+1}-1)(b^{a+1}-1)}{(a-1)(b-1)}.$$

If in (1) we put  $a=1,b-1,c=1,\ldots$ , each divisor, that is, each term of the distributed product, becomes unity; and the sum of the whole is simply the number of the different divisors. Hence, since there are  $\alpha+1$  terms in the first bracket,  $\beta+1$  in the second, and so on, it follows that

Cor. 2. The number of the divisors of  $N = a^{\alpha}b^{\beta}e^{\gamma}$ ... is  $(\alpha+1)(\beta+1)(\gamma+1)$ ...

Cor. 3. The number of ways in which\*  $N = a^a b^a e^{\gamma}$ ... can be resolved into two factors is  $\frac{1}{2}\{1 + (a+1)(\beta+1)(\gamma+1)...\}$ , or  $\frac{1}{2}(a+1)(\beta+1)(\gamma+1)...$ , according as N is or is not a square number.

For every factor has a complementary factor, that is to say, every factorisation corresponds to two divisors; unless N be a square number, and then one factor, namely  $\sqrt{N}$ , has itself for complementary factor, and therefore the factorisation  $N = \sqrt{N} \times \sqrt{N}$  corresponds to only one divisor.

Cor. 4. The number of ways in which  $N = a^a b^a c^{\gamma}$ ... can be resolved into two factors that are prime to each other is  $2^{n-1}$ , n being the number of prime factors  $a^a$ ,  $b^a$ ,  $c^{\gamma}$ , . . . .

For, in this kind of resolution, no single prime factor,  $a^a$  for example, can be split between the two factors. The number of different divisors is therefore the same as if  $\alpha, \beta, \gamma, \dots$ 

<sup>\*</sup> This result is given by Wallis in his Discourse of Combinations, Alternations, and Aliquot Parts (1685), chap. III., § 12. In the same work are given most of the results of §§ 6 and 7 above.

were each equal to unity. Hence the number of ways is  $\frac{1}{2}(1+1)(1+1)(1+1)$ . . .  $(n \text{ factors}) = \frac{1}{2} \cdot 2^n = 2^{n-1}$ .

Example 1 Find the different divisors of 360, their sum, and their number.

We have 360 - 21325.

The divisors are therefore the terms in the distributed product  $(1+2+2^2+2^2)(1+3+3^2)(1+5)$ ; that is to say,

1, 2, 4, 8, 3, 6, 12, 24, 9, 18, 36, 72, 5, 10, 20, 40, 15, 30, 60, 120, 45, 90, 180, 360.

Their sum is  $(2^4 - 1)(3^2 - 1)(5^2 - 1)/(2 - 1)(3 - 1)(5 - 1) = 1170$ .

Their number is (1+3)(1+2)(1+1)=24.

Example 2. Find the least number which has 30 divisors. Let the number be  $N = a^2 b^2 c^3$ . There cannot be more than three prime factors; for  $30 = 2 \times 3 \times 5$ , which has at most three factors, must  $= (a+1)(\beta+1)(\gamma+1)$ . There might of rourse be only two, and then we must have  $30 = (a+1)(\beta+1)$ ; or there  $\min_{k \in \mathbb{N}}$  the only one, and then 30 = a+1.

In the first case  $\alpha=1$ ,  $\beta=2$ ,  $\gamma=1$ . Taking the three least primes, 2, 3, 5, and putting the larger indices to the smaller primes, we have N=2+3+5=720.

In the second case we should get 214.3, 25.34, or 29.32.

In the last case, 229.

It will be found that the least of all these is  $2^4$ ,  $3^2$ , 5; so that 720 is the required number.

Example 3. Show that, if  $2^n - 1$  be a prime number, then  $2^{n-1}(2^n - 1)$  is equal to the sum of its divisors (itself excluded).

Since  $2^{n}-1$  is supposed to be prime, the prime factors of the given number are  $2^{n+1}$  and  $2^{n}-1$ . Hence the sum of its divisors, excluding itself, is, by Cor. 1 above.

$$\begin{array}{l} \left\{2^n-1\right\}\left\{(2^n-1)^2-1\right\} \\ \left\{(2^n-1)\left\{(2^n-1)-1\right\} \\ \left\{(2^n-1)\left\{(2^n-1)-1\right\} \\ = \left(2^n-1\right)\left\{2^n-2^{n-1}\right\}, \\ = 2^{n-1}\left\{2^n-1\right\}\left\{2^n-1\right\}, \\ = 2^{n-1}\left\{2^n-1\right\}, \end{array}$$

as was to be shown.

# ON THE NUMBER OF INTEGERS LESS THAN A GIVEN INTEGER AND PRIME TO IT

§ 8.] If we consider all the integers less than a given one, N, a certain number of these have factors in common with N, and the rest have none. The number of the latter is usually denoted

In the language of the ancients such a number was called a Perfect Number. 6, 28, 496, 8428 are perfect numbers.

by  $\phi(N)$ . Thus  $\phi(N)$  is taken to denote the number of integers (including 1) which are less than N and prime to N.

We have the following important theorem, first given by

Euler:-

If 
$$N = a_1^{a_1} \alpha_2^{a_2} \alpha_3^{a_3} \dots \alpha_n^{a_n}$$
, then

$$\phi(N) = N\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right)\left(1 - \frac{1}{a_3}\right) \cdot \cdot \cdot \left(1 - \frac{1}{a_n}\right)$$
 (1).

The proof of this theorem which we shall give is that which follows most naturally from the principles of § 7.

**Proof.**—Let us find the number of all the integers, not greater than N, which have some factor in common with N. That factor must be a product of powers of one or more of the primes  $a_1, a_2, a_3, \dots, a_n$ .

Now all the multiples of a, which do not exceed N are

 $1a_1$ ,  $2a_1$ ,  $3a_1$ , . . . ,  $(N/a_1)a_1$ ,  $N/a_1$  in number (3); all the multiples of  $a_2$  which do not exceed N are

 $1a_2$ ,  $2a_2$ ,  $3a_2$ , ...,  $(N/a_2)a_2$ ,  $N/a_2$  in number (1); and so on.

All the multiples of  $a_1a_2$  which do not exceed N are

 $1a_1a_2$ ,  $2a_1a_2$ ,  $3a_1a_2$ , . . . ,  $(N/a_1a_2)a_1a_2$ ,

 $N/a_1a_2$  in number (5);

and so on.

Similarly, for  $a_1a_2a_3$  we have

 $1a_1a_2a_3$ ,  $2a_1a_2a_3$ ,  $3a_1a_2a_3$ , . . . ,  $(N/a_1a_2a_3)a_1a_2a_3$ ,  $N/a_1a_2a_3$  in number (6).

Let us now consider the number

$$\frac{X}{a_1} + \frac{X}{a_2} + \frac{X}{a_3} + \cdots \\
-\frac{X}{a_1 a_2} - \frac{X}{a_1 a_3} - \frac{X}{a_1 a_4} - \cdots \\
+\frac{X}{a_1 a_2 a_3} + \frac{X}{a_1 a_2 a_4} + \frac{X}{a_2 a_3 a_4} - \cdots \\
-\frac{X}{a_1 a_2 a_3 a_4} - \cdots$$
(7)

The number of terms in the first line is  ${}_{n}C_{1}$ . The number in the second line is  ${}_{n}C_{2}$ , since every possible group of 2 out of the n letters  $a_{1}a_{2}$ , . . .  $a_{n}$  occurs among the denominators. The number in the third line is  ${}_{n}C_{2}$  for a similar reason. And so on.

Now consider every multiple of the r letters  $a_1a_2a_3 \ldots a_r$  which does not exceed N; in other words, every number, not exceeding N, that has in common with it a factor of the form  $a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_r^{\alpha_r}$ . This multiple will be enumerated in the first line, once as a multiple of  $a_1$ , once as a multiple of  $a_2$ , and so on; that is, once for every letter in it, that is,  $rC_1$  times. In the second line the same multiple will be enumerated once as a multiple of  $a_1a_2$ , and so on; that is, once for every group of two that can be formed out of the r letters  $a_1a_2 \ldots a_r$ , that is,  $rC_2$  times. And so on. Hence, paying attention to the signs, the multiple in question will in the whole expression (7) be enumerated

$$_{r}C_{1} - _{r}C_{2} + _{r}C_{3} - \ldots \pm _{r}C_{r-1} \mp _{r}C_{r} = 1 - (1-1)^{r}$$

times; that is, just once. This proof holds, of course, whatever the r letters in the group may be, and whether there be 1, 2, 3, or any number up to n in the group.

It follows, therefore, that (7) enumerates, without repetition or omission, every integer which has a factor in common with N. But, from formula (1), chap. IV., § 10, we see that (7) is simply

$$N - N\left(1 - \frac{1}{a_0}\right)\left(1 - \frac{1}{a_0}\right) \dots \left(1 - \frac{1}{a_n}\right)$$
 (8).

To obtain the number of integers less than N which are prime to N, we have merely to subtract (8) from N. We thus obtain

$$\phi(N) = N\left(1 - \frac{1}{a_1}\right)\left(1 - \frac{1}{a_2}\right), \quad \left(1 - \frac{1}{a_n}\right),$$

which establishes Euler's formula.

Example.  $N = 100 = 2^2 \cdot 5^2$ ;  $\phi(100) = 2^2 \cdot 5^2(1 - \frac{1}{6})(1 - \frac{1}{6}) = 40$ .

§ 9.] If M = PQ, where P and Q are prime to each other, then

$$\phi(M) = \phi(P) \phi(Q) \tag{1}.$$

For, since P and Q are prime to each other, we must have

$$P = a_1^{a_1}a_2^{a_2} \dots ,$$
  
 $Q = b_1\beta_1b_1\beta_2 \dots .$ 

where none of the prime factors are common; and therefore

$$M=a_1^{\alpha_1}a_2^{\alpha_2}\dots b_1^{\beta_1}b_2^{\beta_2}\dots$$

where  $a_1, a_2, \ldots, b_1, b_2, \ldots$  are all primes.

But, by § 8, we then have

 $\phi(M)$ 

$$= M\left(1 - \frac{1}{a_i}\right)\left(1 - \frac{1}{a_2}\right) \dots \left(1 - \frac{1}{b_1}\right)\left(1 - \frac{1}{b_2}\right) \dots,$$

$$= a_1^{a_1}a_2^{a_2} \dots \left(1 - \frac{1}{a_i}\right)\left(1 - \frac{1}{a_i}\right) \dots b_1^{a_1}b_2^{a_2} \dots \left(1 - \frac{1}{b_i}\right)\left(1 - \frac{1}{b_2}\right) \dots,$$

$$= \phi\left(P\right)\phi\left(O\right).$$

Cor. If PQRS. . , be prime to each other, then

$$\phi(PQRS...) = \phi(P)\phi(Q)\phi(R)\phi(S)...$$
 (2)

For, since P is prime to Q, R, S, . . . , it follows that P is prime to the product QRS . . . Hence, by the above proposition,  $\Phi(PQRS, \ldots) = \Phi(P) \Phi(QRS, \ldots),$ 

Repeating the same reasoning, we have

$$\phi(QRS,...)$$
  $\phi(Q)\phi(RS,...)$ :

and so on.

$$\phi(PQRS,...) = \phi(P)\phi(Q)\phi(R)\phi(S)$$
...

Remark.—There is no difficulty in establishing the theorem  $\phi(PQ) = \phi(P) \phi(Q)$  a priori. This may be done, for example, by means of § 13 below (see fross' Alphra, § 230). The theorem of § 8 above can then be deduced from  $\phi(PQR, ...) = \phi(P) \phi(Q) \phi(R)$ ... The course followed above, though not so neat, is, we think, more instructive for the learner.

Example.

$$56 = 7 \times 8,$$
  
 $\phi(56) = 24,$   
 $\phi(7) = 6,$   
 $\phi(8) = 4;$   
 $\phi(56) = \phi(7) \times \phi(8).$ 

 $\S | 10 \rangle | If | d_1, d_2, d_3, \dots, \&c., denote all the divisors of the integer N, then*$ 

$$\phi(d_1) + \phi(d_2) + \phi(d_3) + \dots = N \dots$$
 (1),  
(Gauss, Disg. Arith., § 39.)

For the divisors,  $d_1$ ,  $d_2$ ,  $d_3$ , . . . , are the terms in the distribution of the product

$$(1 + a_1 + a_2^2 + \dots + a_n^{a_n})(1 + a_n + a_n^2 + \dots + a_n^{a_n})$$
.

If we take any one of these terms, say  $d_r = a_1^{-a_1} a_2^{-a_2} \cdots$ , then, by § 9, Cor.,

$$\phi(d_r) = \phi(a_1^{(\mathbf{a}_1)}a_2^{(\mathbf{a}_2)}, ...),$$
  
 $= \phi(a_i^{(\mathbf{a}_1)})\phi(a_i^{(\mathbf{a}_2)})...$ 

since  $a_1, a_2, \dots$  are primes.

It follows that the left-hand side of (1) is the same as

$$\{1 + \phi(a_1) + \phi(a_1^2) + \dots + \phi(a_1^{-k})\}$$
  
 $\Rightarrow \{1 + \phi(a_2) + \phi(a_2^2) + \dots + \phi(a_2^{-k})\}$   
 $\Rightarrow \vdots \qquad \vdots \qquad \vdots \qquad \vdots$   
 $\Rightarrow \phi(a_1^n) = a_1^n (1 - \frac{1}{n}) = a_1^n - a_1^{n-1}$ 
(2)

But

$$\phi(a_1^r) = a_1^r \left(1 - \frac{1}{a_1}\right) = a_1^r - a_1^{r-1}$$

Henco

$$1 \cdot \phi(a_i) + \phi(a_i^2) + \dots + \phi(a_i^{a_i})$$
  
=  $1 + a_i - 1 + a_i^2 - \overline{a_i} + \dots + \overline{a_i^{a_i}} - a_i^{a_i - 1}$ ,  
=  $a_i^{a_i}$ :

and so on.

It appears, therefore, that (2) is equal to  $a_1^{\mathbf{a}_1}a_2^{\mathbf{a}_2}\dots$ , that is, equal to N.

Example.  $N = 315 - 3^{\circ}, 5, 7$ .

The divisors are 1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315, and we have  $\phi(1) + \phi(3) + \phi(5) + \dots + \phi(315)$ 

1 + 2 + 4 + 6 + 6 + 8 + 12 + 24 + 24 + 36 + 48 + 144 = 315.

<sup>\*</sup> Here and in what follows 1 is included among the divisors, and, for convenience,  $\phi(1)$  is taken to stand for 1. Strictly speaking,  $\phi(1)$  has no meaning at all,

### PROPERTIES OF m!

§ 11.] The following theorem enables us to prove some important properties of m!:—

The highest power of the prime p which divides mi exactly is

$$I\binom{m}{p} + I\binom{m}{p^2} + I(\frac{m}{p^3}) + \dots,$$

where  $I\left(\frac{m}{p}\right)$ ,  $I\left(\frac{m}{p^2}\right)$ , . . . denote the integral parts of m/p.  $m/p^2$ , . . . ; and the series is continued until the greatest power of p is reached which does not exceed m.

To prove this, we remark that the numbers in the series

which are divisible by p are evidently

$$1p, 2p, 3p, \ldots, kp,$$

where kp is the greatest multiple of  $p \geqslant m$ . In other words, k = I(m/p). Hence I(m/p) is the number of the factors in m! which are divisible by p.

If to this we add the number of those that are divisible by  $p^p$ , namely  $I(m/p^p)$ , and again the number of those that are divisible by  $p^p$ , namely  $I(m/p^p)$ , and so on, the sum will be the power in which p occurs in m!.

Hence, since p is a prime, the highest power of p that will divide m! exactly is

$$I\left(\frac{m}{\nu}\right) + I\left(\frac{m}{\nu^2}\right) + I\left(\frac{m}{\nu^3}\right) + \dots$$

It is convenient for practical purposes to remark that

$$I\binom{m}{p^{\overline{r}}} = I\left\{I\binom{m}{p^{\overline{r-1}}}/p\right\}.$$

For, if

$$m^{r}p^{r-1} = i + k^{r}p^{r-1} (k < p^{r-1})$$
 (1).

then

$$m/p^r = i/\rho + k/p^r$$
(2),

$$= j + l_{i}p + k_{i}p^{r} (l < p)$$
 (3)

Von

$$\begin{array}{c} l \; p + k \; p^r \! \gg \! (p-1) \; p + (p^{r-1}-1)' p^r, \\ \gg \! (p^{r+1}-p) \; p^{r+1}, \\ < 1. \end{array}$$

Hence, by (3),

$$j = I\left(\frac{m}{n^r}\right)$$
.

But since i p - j + l p,

$$j = I\left(\frac{i}{p}\right) - I\left\{I\left(\frac{m}{p^{r-1}}\right)/p\right\},$$
 by (2).

We may therefore proceed as follows:—Divide m by p; take the integral quotient and divide again by p; and so on; until the integral quotient becomes zero; then add all the integral quotients, and the result is the high st power of p which will divide micracity.

Example 1. To find the highest power of 7 which divides 1000! exactly. In dividing successively by 7 the integral quotients are 142, 20, 2; the sum of these is 164. Hence 7 let is the power of 7 required.

Example 2. To decompose 25! into its prime factors.

Write down all the primes less than 25; write under each the successive specients; and then aid. We thus obtain

2	3	5	7	11	13	17	19	23
12 6 3	8 2	5 1	3	2	1	1	1	1
22	10	-6	3	2	1	1	1	1

Horaco 251 252 37 . 56 . 7 . 112 . 13 . 17 . 19 . 23.

Example 3. Express 39% . I in its simplest form as a product of prime latters.

Result, 2<sup>13</sup>, 3<sup>4</sup>, 5<sup>2</sup>, 7<sup>2</sup>, 11, 13<sup>2</sup>, 17, 19, 29, 31, 37,

Example 1. Find the indirect power of 5 that will divide 27, 28, 29...100 tactly.

Result, 519.

Example 5. If m be expressed in the scale of p, in the form

 $m-p_1+p_1p+p_2^2+\dots+p_np^n$ , the highest power of p that will divide  $m^1$  exactly is the

$$m-p-p_1-p_2-p_3$$

Example 6. If  $m=2^n+2^\beta+2^\gamma+\dots$  (k terms), where  $a<\beta<\gamma<\dots$ , the greatest power of 2 that will divid. m! is the (m-k)th.

§ 12.] If  $f+g+h+\ldots > m$ , then  $m! f! g! h' \ldots$  is an integer\*.

To prove this, it will be sufficient to show that, if any prime factor, p say, appear in  $\mathbb{Z}[g/k]$ , . . . it will appear in at least as high a power in m! In other words (§ 11), we have to show that

Now, if d be any integer whatever, we have

$$\begin{split} f_l d &= f'' + f''/d \quad (f'' \geqslant d-1), \\ g' d &= g' + g' \cdot d \quad (g'' \geqslant d-1), \\ h_i d &= h' + h'_i d \quad (h'' \geqslant d-1), \\ \end{split}$$

and we obtain by addition

$$\frac{f+g+h+\dots}{d}=f''+g'+h'+\dots+\frac{f''+g''+h''-\dots}{d}$$

Hence, if  $f'' + g'' + h'' + \dots < d$ .

$$I\left(\frac{f+g+h+\cdots}{d}\right) = f'+g+h'+\cdots,$$

$$= I\left(\frac{f}{f}\right) + I\left(\frac{g}{f}\right) + I\left(\frac{h}{f}\right) + I\left(\frac{h}{f}\right) + \cdots$$

If, on the other hand,  $f'' + g'' + h'' + \dots > d$ , i then

$$I\left(\frac{f+g+h+\cdots}{d}\right) > f'+g'+h'-\cdots,$$

$$> I\left(\frac{f}{d}\right) + I\left(\frac{g}{d}\right) + I\left(\frac{h}{d}\right) + \cdots (2).$$

<sup>&</sup>quot;This theorem might, of course, be inferred from the that the "the that milfiglith". The represents the number of permutations of mathematical which are alike, a slike, halike, halik

<sup>†</sup> If n be the number of the letters  $f, g, h, \ldots$ , the utmost value of  $f'' + g'' + h'' + \ldots$  is n(d-1). Hence the utmost difference very smaller f is I in (d-1), d.

It appears, therefore, that, even if  $m \mid f + g \cdot h \mid \dots$ ,

$$I\binom{n}{d} \leqslant I\binom{f}{d} + I\binom{g}{d} + \dots$$
 (3).

A first ori is this so if  $m > f \circ g + h + \dots$ .

It now we give d the successive values  $p, p, \dots$ , and comtime by addition the inequalities thus obtained from (3), the

Cot. 1. If  $f \circ g \circ h + \dots > m$ , and none of the numbers t g, h, . . . . . . and to m, the integer m'. f.g.h'. . . . is divisible

Cir. 2. The product of \u03c4 successive integers is exactly

dir St. by r. The proofs of these, so far as they require proof, we leave to the reader. Cor. 2 has already been established by a totally different kind of reasoning in § 3, Example 6.

- (1) What is the least multiplier that will convert 945 into a complete
  - (2) Fin 1th plus for of the div sors of 2160, and their sum.
    - ( ) I and the int ... Is lations of
      - $m_0 = 1000x + 10u + 1$
    - (1) No rece ' roft' form at 1 coupt 5 is price.
  - - on The last refere form 2", 3, a claims prime) which shall be
- The last the place of the form Lad . . . . (a, b, c being unequal
- The sum of its divisors. es to a contract of the two "annial of" randers of the forms 2"p1, 2"r. and the late of the property are at the are when each is the sum of the state of the state of the state of the being reckoned as a
- there are a shall be a square
- was live a right many roth-divisors, and P the product

(11.) The sum and the sum of the squares of all the numbers less than N and prime to it are  $\frac{1}{2}N(a-1)(b-1)(c-1)\dots$  and  $\frac{1}{2}N^3(1-1/a)(1-1/b)$  $\dots + \frac{1}{b}N(1-a)(1-b)\dots$  respectively. (Wolstenholme.)

(12.) If p, q, r, . . . be prime to each other, and d (N) denote the sum of the divisors of N, show that

$$d(pqr,...) = d(p)d(q)d(r)...$$

(13.) If N = abc, where a, b, c are prime to each other, then the product of all the numbers less than N and prime to N is

$$(abc-1)! \text{ H } \{(a-1)!/(bc-1)! \ a^{(b-1)(c-1)}\}.$$
(Gony, and Caius Col)., 1482.)

(14.) The number of integers less than (r<sup>2</sup>+1)<sup>n</sup> which are divisible by r

but not by  $r^2$  is  $(r-1)\{(r^2+1)^n-1\}/r^2$ . (15.) Prove that

$$\phi\left(1\right)\frac{x}{1+x^{2}}-\phi\left(3\right)\frac{x^{3}}{1+x^{6}}+\phi\left(5\right)\frac{x^{5}}{1+x^{10}}-\dots\text{ ad }x=\frac{x\left(1-x^{2}\right)}{\left(1+x^{2}\right)^{2}}.$$

- (16.) In a given set of N consecutive integers beginning with A, find the number of terms not divisible by any one of a given set of relatively prime integers, (Cayley,)
  - (17.) If m-1 be prime to n+1, show that me'n is divisible by n+1. (18.)  $(a+1)(a+2) \dots 2a \times b (b+1) \dots 2b/(a+b)!$  is an integer.
- (19.) The product of any r consecutive terms of the series x 1, x2 1,  $x^3 - 1$  . . . is exactly divisible by the product of the first r terms.
- (20.) If p be prime, the highest power of p which divides n! is the greatest integer in  $\{n - S(n)\}/(p - 1)^m$ , where S(n) is the sum of the digits of n when expressed in the scale of p.

If S(m) have the above meaning, prove that  $S(m-n) \triangleleft S(m) - S(n)$  for any radix. Hence show that  $(n+1)(n+2) \dots (n+m)$  is divisible by m!.

(Camb. Math. Jour. (1839), vol. 1., p. 226.)

(21.) If f(n) denote the sum of the uneven, and F(n) the sum of the even, divisors of n, and 1, 3, 6, 10, . . . be the "triangular numbers," then  $f(n) + f(n-1) + f(n-3) + f(n-6) + \dots$ 

$$= F(n) + F(n-1) + F(n-3) + F(n-6) + \cdots$$

it being understood that f(n-n) = 0, F(n-n) = n.

# ON THE RESIDUES OF A SERIES OF INTEGERS IN ARITHMETICAL PROGRESSION.

§ 13.] The least positive remainders of the series of numbers  $k, k+a, k+2a, \dots, k+(m-1)a$ 

with respect to m, where m is prime to a, are a permutation of the numbers of the series

All the remainders must be different; for, if any two different numbers of the series had the same remainders, then we should have

$$k + ra = \mu m + \rho$$
, and  $k + sa = \mu' m + \rho$ ,

whence

$$(r-s)a = (\mu-\mu')m$$
, and  $(r-s)a'm = \mu-\mu'$ .

Now this is impossible, since u is prime to m, and r and s are each s, m, and therefore r - s < m. Hence, since there are only m possible remainders, namely,  $0, 1, 2, \ldots, (m-1)$ , the proposition follows

Cor. 1. If the remainders of k and a with respect to m be k and a, the revainders will occur as follows:—

$$k', k' + a', k' + 2a', \ldots, k' + ra',$$

until we reach a number that equals or surpasses m; this we must diminish by m, and then proceed to add a' at each step as before.

Thus, if 
$$k = 11$$
,  $a = 25$ ,  $r_1 = 7$ , the series is

We have 
$$C = 1$$
 and  $a' = 4$ , here  $c$  the remainders are  $c = 4$ ,  $1 + 4 = 7 - 1$ ,  $-5$ ,  $5 + 4 - 7 = 2$ , &c.

in fact

Cor. 2 If the propersion of numbers be continued beyond at exist, the reactiveless will repeat in the same order as before; and in this periodic series the number of convinctors intervening between two that if the lancing is always the same.

Cor. 3. There are as many terms in the series

$$k$$
,  $l + a$ ,  $k + 2a$ , . . . ,  $k + (m - 1)a$ 

All kan prime to m, as there are in the seri s

$$0, 1, 2, \ldots, (m-1).$$

That to the result of them is the sector in question which are print to make the many Section.

The two street of the first of that two numbers which are contents the post tool are either both prime or both non-

$$0, 1, 2, \dots, r - 1$$

we select those that are less than in and prime to it, say

 $r_1, r_2, \ldots, r_n$  (the number n being  $\phi(m)$ ), then the numbers

$$k+r_1a$$
,  $k+r_2a$ , . . . ,  $k+r_2a$ .

where k = 0 or a multiple of m, and a prime to m as before, are all prime to m; and their remainders with respect to m are a vermutation of

$$r_1, r_2, \dots, r_n$$

For, as we have seen already, all the n remainders are unlike, and every remainder must be prime to m; for, if we had  $k+r_t \iota = \mu m+\rho$ , where  $\rho$  is not prime to m, then  $r_t \iota \iota = \mu m+\rho \iota \cdot k$  would have a factor in common with m, which is impossible, since  $r_t$  and a are both prime to m.

Hence the remainders must be the numbers  $r_1, r_2, \ldots, r_n$  in some order or other.

§ 14.] If m be not prime to a, but have with it the G.C.M. g, so that a = aa', m = aa', the remainders of the series

$$k, k+a, k+2a, \ldots, k+(m-1)a$$

with respect to m will recur in a shorter cycle of m'.

Consider any two terms of the series out of the first m', say k+ra, k+sa. These two must have different remainders, otherwise (r-s)a would be exactly divisible by m: that is, (r-s)qa in would be an integer: that is, (r-s)a'm' would be an integer; which is impossible, since a' is prime to m' and r-s-m'.

Again, consider any term beyond the m'th, say the (m'+r)th, then, since

$$\begin{aligned} \{k+(m'+r)\alpha\} - \{k+r\alpha\} &= m'\alpha, \\ &= gm'\alpha', \end{aligned}$$

it follows that the (m'+r)th term has the same remainder with respect to m as the rth.

In other words, the first m' remainders are all different, and after that they recur periodically, the increment being qn', where a'' is the remainder of a' with respect to m', subject to diminution by m as in last article.

Example. If k=11, a=25, m=15, we have the serie

and here g:5; a'=5; a'=3; a''=2; k'=11; ga''=10. Hence the remainders are

Cor. If the G.C.M., g. of a and m divide k exactly, and, in particular, if k = 0, the remainders of the series

 $k, \quad k+a, \quad k+2a, \quad \dots$ 

$$0g, 1g, 2g, 3g, \dots, (m'-1)g$$

continually repeated in a certain order.

For, in this case, since  $k = q\kappa$ , we have  $(k + ra)/m = (\kappa + ra')/m'$ , hence the remainders are those of the series

$$\kappa$$
,  $\kappa + a'$ ,  $\kappa + 2a'$ , ...

with respect to m' which is prime to a', each multiplied by g. Hence the result follows by § 13.

Example. Let k=10, a=25, m=15; then the series of numbers is

10, 35, 60, 85, 110, 135, 160, 185, . . . . We have q = 5; a' = 5; a' = 3;  $\kappa = 2$ ; and the remainders are

that is to say.

10, 5, 0, 10, 5, 0, 10, 5, . . .;  

$$2 \times 5$$
,  $1 \times 5$ ,  $0 \times 5$ , . . .

§ 15.] From § 13 we can at once deduce Fermat's Theorem\*, which is one of the corner-stones of the theory of numbers.

If m be a prime number, and a be prime to m,  $a^{m-1}-1$  is divisible by m.

If a be prime to m, then we have

$$1a = \mu_1 m + \rho_1,$$
  
 $2a = \mu_2 m + \rho_2,$   
 $\dots$   
 $(m-1) a - \mu_{m-1} m + \rho_{m-1},$ 

where the numbers  $\rho_1, \ \rho_2, \dots, \ \rho_{m-1}$  are the numbers  $1, 2, \dots, (m-1)$  written in a certain order.

Given historical interest attaches to this theorem. It was, with neveral other studing results in the theory of numbers, published without demonstration among Fernards in the to-an odition of Bachet de Mezirine's Diophantas (1670). For many years no demonstration was found. Finally, Euler (Connect, Avid. Fetter), vim. 1714, and Comment, Nov., Acid. Petter), vim. 1715 paye two proofs. Another, due to Lagrange (Nour. Mem. de l'Ac. de Berlin, 1715), is reprodued in § 18.—The proof given above is akin to Euler's second act to that given by Gauss. Dec., Artich, § 19.

Hence

1.2... 
$$(m-1) a^{m-1} = (\mu_1 m + \rho_1) (\mu_2 m + \rho_2) ... (\mu_{m-1} m + \rho_{m-1}),$$
  
=  $Mm + \rho_1 \rho_2 ... ... \rho_{m-1},$ 

 $=Mm+1,2\;,\;\;\ldots(m-1),$  We therefore have

1.2., 
$$(m-1)(a^{m-1}-1) = Mm$$
.

Now, m being a prime number, all the factors of  $1, 2, \ldots (m-1)$  are prime to it. Hence m must divide  $a^{m-1}-1$ .

It is very easy, by the method of differences, explained in § 5, to establish the following theorem:—

If m be a prime,  $a^m - a$  is exactly divisible by  $m^*$ .

Since  $a^m - a = a$   $(a^{m-1} - 1)$ , if a be prime to m, this is simply Fermat's Theorem in another form.

§ 16.] By using Cor. 4 of § 13 we arrive at the following generalisation of Fermat's Theorem, due to Euler:—

If m be any integer, and a be prime to m, then  $a^{\phi(m)}-1$  is exactly divisible by m.

Here  $\phi(m)$  denotes, as usual, the number of integers which are less than m and prime to it.

For, if  $r_1, r_2, \ldots, r_n$  be the integers less than m and prime to it, we have, by the corollary in question,

$$r_1 \alpha = \mu_1 m + \rho_1,$$
  
 $r_2 \alpha = \mu_2 m + \rho_2,$   
 $r_3 \alpha = \mu_3 m + \rho_3,$ 

where the numbers  $\rho_1, \rho_2, \ldots, \rho_n$  are simply  $r_1, r_2, \ldots, r_n$  written in a certain order.

We have therefore, just as in last paragraph,

$$r_1r_2...r_n(a^n-1)=Mm,$$

whence, since  $r_1, r_2, \ldots, r_n$  are all prime to m, it follows that  $a^n - 1$ , that is,  $a^{\phi(m)} - 1$ , is divisible by m.

 $\S$  17.] The famous theorem of Wilson can also be established by means of the principles of  $\S$  13.

<sup>•</sup> For another proof of this theorem see § 18 below.

All to  $m_{ij}$  with see product has the remainder + 1 with  $m_{ij}$  to given modulus m may be called, after Euler, Allied

Co. to a sur the interers

the Course (prime name of the number of them is of course even W = 6.00 prove that if we except the first and last, they can be extensively arranged in allied pairs

For, take any one of them, say r, then, since r is prime to m,

$$r = 1, \quad r = 2, \quad \dots, \quad r = (m-1)$$

atte to number

1. 2. . . . , 
$$(m-1)$$

in the order. Hence, some one of the series, say rr', must have
the property; then rr' will be allies.

 $T^{*}$  is an number r cannot have two different allies, since all  $t_{*}$  is an object at the result of r are dimerent.

Note on the two, r and r', be equal, unless r-1 or = m-1;

$$r^2 = \mu m + 1$$
.

that r = 1 - pm; that is, (r+1)(r-1) must be divisible by m. Let now m is prime, this involves that either r+1 or r-1 be  $d_{n} = d_{n}$  by  $d_{n}$  and, since r cannot be greater than m, this involves is  $d_{n} = d_{n}$  where  $d_{n} = d_{n}$  in the other that r-1.

f = 1 bits, then, 1 and m = 1, we can arrange the series

$$2, 3, \ldots, (m-2)$$

in 39 d pan. Now every product of two allies is of the form m=1 be see the polart  $2,3,\ldots,(m-2)$  is of the form  $0,\ldots,1$  by m+1, which reduces to the form Mm+1.

2 3 . . . (m 2) Mm + 1:

and only by m 1, we get

What

$$1.2.5...(m-1) + 1 = Nm_*$$

That is, if m be a prime, (m-1)! + 1 is divisible by m, which is Wilson's Theorem\*.

It should be observed that, if m be not a prime, (m-1)! + 1 is not divisible by m.

For, if m be not a prime, its factors occur among the numbers 2, 3, . . . , (m-1), each of which divides (m-1)!, and, therefore, none of which divide (m-1)! + 1.

§ 18.] The following Theorem of Lagrange embraces both Fermat's Theorem and Wilson's Theorem as particular cases:—

If 
$$(x + 1) (x + 2) \dots (x + p - 1)$$
  
=  $x^{p-1} + A_1 x^{p-2} + \dots + A_{p-2} x + A_{p-1}$ ,  
and  $p$  be prime, then  $A_1, A_2, \dots, A_{p-2}$  are all divisible by  $p$ .

we have  $A_1, A_2, \ldots, A_{p-2}$  are all divisible by p.

 $(x+p)\{x^{p-1}+A_1A^{p-2}+\dots+A_{p-2}x+A_{p-1}\}$ 

$$\begin{aligned} & = (x+1) \{ (x+1)^{p-1} + A_1 (x+1)^{p-2} + \dots + A_{p-1} (x+1) + A_{p-1} \}, \end{aligned}$$

Hence

$$\begin{split} &px^{p-1} + p\mathcal{A}_1x^{p-2} + p\mathcal{A}_2x^{p-3} + \dots + p\mathcal{A}_{p-2}x + p\mathcal{A}_{p-1} \\ &= \{(x+1)^p - x^p\} + \mathcal{A}_1((x+1)^{p-1} - x^{p-1}\} + \mathcal{A}_1((x+1)^{p-2} - x^{p-2}) + \dots \end{split}$$

Therefore

$$\begin{split} pA_1 &= pC_2 + p_{-1}C_1A_1, \\ pA_2 &= pC_2 + p_{-1}C_2A_1 + p_{-1}C_1A_2, \\ pA_3 &= pC_4 + p_{-1}C_3A_1 + p_{-2}C_1A_2 + p_{-3}C_1A_3. \end{split}$$

Hence, since  $p_{-1}C_1$ ,  $p_{-2}C_1$ ,  $p_{-2}C_1$ , . . . are not divisible by p if p be prime, we get, by successive steps, the proof that  $A_1$ ,  $A_2$ ,  $A_3$ , . . . are all divisible by p.

<sup>•</sup> This theorem was first published by Waring in his Melitations Algebraica (1770). He there attributes it to Sir John Wilson, but gives no proof. The first demonstration was given by Lagrange (Nour. Mrm. de 13.4 at Berlin, 1771); this is reprodued in § 18. A second proof was given by Euler in his Opuscula Analytica (1783), vol. 1., p. 329, depending on the theory of the residues of powers.

The proof above is that given by Gauss (Diog. Arith., §§ 77, 78), who generalises the theorem as follows:—"The product of all the numbers less than m and prime to it is congruent with -1, if  $m = p^n = p^n$ , where p is any prime but 2, or, again, if m = 1; but is congruent with +1 in every other case." This extension depends on the theory of quadratic residues.

Cor L. Put x 1, and we get

2.3. 
$$p = 1 + (A_1 + A_2 + ... + A_{p-2}) + A_{p-1}$$
.

Therefore  $A_{p+1} + 1$ , that is, (p-1)! + 1, is divisible by  $p_*$ . Cor. 2. Multiplying by x and transposing, we get

 $x^{r} = x - x \ (x+1) \ . \ . \ (x+p-1)$ 

$$-(1+A_{r-1})x-(A_1x^{p-1}+A_2x^{p-2}+...+A_{p-2}x^2).$$

For x(x+1), . . . . (x+p-1), being the product of p consecutive int zers, must be divisible by p. Also, if p be prime,  $1+A_{n-1}$  is divisible by p.

Therefore,  $x^p - x$  is divisible by p if p be prime. From which Form  $x^p$ . Theorem follows at once if x by prime to p.

### EXERCISES XXXVII.

(1.1  $x^{13} - x$  is divisible by 2730.

(2) If x be a prime greater than 13, x<sup>12</sup> - 1 is divisible by 21840.

. If the nth power of every number end with the same digit as the number itself, then n-4p+1.

Give a rule for determining by inspection the cube root of every perfect cida less than a million.

(4) If the radix, r, of the scale of notation be prime, show that the rth

p wer of every hat her has the same final digit as the integer itself, and that the country has been supported by the first power of every integer has for its final digit 1.

(5) If n be prime, and x prime to n, then either  $x^{(n-1)/2} - 1$  or  $x^{(n-1)/2} + 1$  divisible by x

(6) If n be prime, and x prime to n, then either  $x^{n(n-1)/2} - 1$  or  $x^{n(n-1)k^2} + 1$ 

(7) If m and n be primes, then

 $m^{n-1} + n^{m-1} \equiv 1 \pmod{mn}$ .

- (5) If  $a, \beta, \gamma, \ldots$  be primes, and  $N = a\beta\gamma \ldots$ , then  $\Sigma (N, v)^{a} = 21 \text{ (mod. } a\beta\gamma \ldots).$
- (2) If n be an odd prine, show that

 $(a+1)^n - (i^n+1) \equiv 0 \pmod{2n}$ 

Hence he with at, if n be an odd prime and p an integer, then any integer express 1 in the scale of 2n will only in the same digit as its (pn-p+1)th power 1 by the Format - Theorem (Math. Trip., 1879.)

(10) If t is tour and -z ther

$$x^{n-1} + x^{n-1} + \dots + x + 1 = 0 \pmod{n}$$
,

(11) If n be an odd trune, the

 $1 + 2(n + 1) + 2^{2}(n + 1) + \dots + n^{-2}(n + 1)^{n-2} = 0 \pmod{n}$ .

(1" It wise it, 1" 2" t . . . . . 11" . 0 (mod. n).

(13.) If n be prime, and p < n,

$$(p-1)!(n-p)! - (-1)^p \equiv 0 \pmod{n}$$

and, in particular,  $[\{\frac{1}{2}(n-1)\}!]^2 + (-1)^{(n-1)/2} \equiv 0 \pmod{n}.$ 

(Waring.)

(14.) Find in what eases one of the two 
$$\{\frac{1}{2}(n-1)\}! \pm 1$$
 is divisible by  $n$ .

What determines which of them is so? (15.) If p be prime, and n not divisible by p-1, then

$$1^n + 2^n + \dots + (p-1)^n \equiv 0 \pmod{p}$$
.

(16.) If p be any odd prime, m any number > 1 which is not divisible by p-1, then

$$1^{2m} + 2^{2m} + \dots + \left(\frac{p-1}{2}\right)^{2m} \equiv 0 \pmod{p}$$
.

(17.) If neither a nor b be divisible by a prime of the form 4n-1, then  $a^{4n-1}-b^{4n-2}$  will not be exactly divisible by a prime of that form.

Hence show that  $a^{4n-2}+b^{4n-2}$  is not divisible by any integer (prime or not) of the form 4n-1.

Also that  $a^2+b^2$  is not divisible by any integer of the form 4n-1 which does not divide both a and b. Also, that any divisor of the sum of two integral squares, which does not divide each of them, is of the form 4n+1.

(Euler.)

(18.) Show, by means of (17), that no square integer can have the form 4mn - m - n<sup>a</sup>, where m, n, a are positive integers. (Euler.)

#### PARTITION OF NUMBERS.

# Euler's Theory of the Enumeration of Partitions.

§ 19.] By the partition of a given integer n is meant the division of the integer into a number of others of which it is the sum. Thus 1+2+2+3+3, 1+3+7, are partitions of 11. There are two main classes of partitions, namely, (I.) those in which the parts may be equal or unequal; (II.) those in which the parts are all unequal. When the worl "Partition" is used without qualification, class (I.) is understood.

We shall use a quadripartite symbol to denote the number of partitions of a given species. Thus  $P(||\cdot|)$  and  $Pu(||\cdot|)$  are used to denote partitions of the classes (L) and (IL) respectively. In the first blank inside the bracket is inserted the number to be partitioned; in the second, an indication of the number of the parts; in the third, an indication of the magnitude or nature of

the pairs. It is always implied, unless the contrary is stated, that the least part admissible is 1; so that  $\geqslant m$  means any integer of the series 1, 2, . . . , m. An asterisk is used to mean any integer of the series 1, 2, . . . , z, or that no restriction is to be put on the number of the parts other than what arises from the nature of the partition otherwise.

Thus P(n|p|q) means the number of partitions of n into p parts the  $2\pi$  west of which is q:P(n|p|>q) the number of a rittions of n into p parts no one of which exceeds q; P(n|\*>p) the number of partitions of n into any number of parts no one of which is to exceed q:Pu(n|>p|\*) the number of partitions of n into p or any less number of unequal parts amestricated in nagnitude: Pu(n|p) odd) the number of partitions of n into p unequal parts cach of which is an odd integer;  $P(n|*1,2,2,2,2,\ldots)$  the number of partitions of n into any number of parts, each part being a number in the series  $1,2,2,2,2,\ldots$ , and so on

The theory of partitions has risen into great importance of hate in connection with the researches of Sylvester and his followers on the theory of invariants. It is also closely connected with the theory of series, as will be seen from Euler's enumeration of certain species of partitions, which we shall now brighty explain.

§ 20.] If we develop the product  $(1+zx)(1+zx^2)\dots$  (1+zx), it is obvious that we get the term  $z^px^n$  in as many different ways as we can produce n by adding together p of the integers  $1, 2, \dots, q$ , can to be taken only once. Hence we have the following counting :=

$$(1+zx)(1+zx)$$
, ...  $(1+zx^{j})\cdot 1+\Sigma Pu(n,p,\gg q)z^{p}x^{n}$  (1).

Again, if to the product on the left of (1) we adjoin the 1-tor  $1 + z - z^2 + z + \dots$  and z (that is,  $1 \cdot (1 - z)$ ), we shall evidently get  $z^2 z^2$  as often as we can produce n by adding tage her any number n at exceeding p of the integers  $1, 2, \dots, q$ . Then fore

$$(1+z)[(1+z)]$$
, ,  $(1+z)((1-z))$   
 $(1+z)[(1+z)]$   
 $(1+z)[(1+z)]$ 

In like manner, we have

$$(1+x)(1+x^2) \dots (1+x^q) = 1 + \sum Pu(n \mid x \mid \ge q) x^n$$
 (3);  
 $(1+zx)(1+zx^2) \dots$  and  $z = 1 + \sum Pu(n \mid p \mid x) z^p x^n$  (4);

$$(1+x)(1+x^2)\dots$$
 and  $x=1+\sum Pu(n+x+x)x^n$  (1),

Also, as will be easily seen, we have

$$1/(1-zx)(1-zx^2)\dots(1-zx^q)=1+\sum P(n|p|>q)z^px^n$$
 (6);

$$1/(1-z)(1-zx)...(1-zx^{q}) = 1 + \sum_{n=0}^{\infty} P(n|p|p|p)z^{p}x^{n}$$
 (7);

$$1/(1-x)(1-x^2)...(1-x^q) = 1 + \Sigma P(n | * | > q) x^n$$
 (8);

$$1/(1-zx)(1-zx^2)\dots \text{ ad } \infty = 1 + \sum_{n=1}^{\infty} P(n|p|*)z^n x^n \qquad (9);$$

$$1/(1-z)(1-zx)(1-zx^2)\dots$$
 ad  $\infty = 1 + \sum_{n=1}^{\infty} P(n) > p + z^n x^n$  (10);

$$1/(1-x)(1-x^2)\dots$$
 and  $\infty = 1 + \sum P(n \mid * \mid *) x^n$  (11);

and so on.

By means of these equations, coupled with the theorems given in chap. XXX., § 2, and Exercises XXI., a considerable number of theorems regarding the enumeration of partitions can be deduced at once.

§ 21.] To find a recurrence-formula for enumerating the partitions of n into any number of parts none of which exceeds q; and thus to calculate a table for P(n|\*|>q).

By (8), we have

$$1/(1-x)(1-x^2)$$
... $(1-x^2)=1+\Sigma P(n|*|>q)x^n$ .

Hence, multiplying on both sides by  $1-x^q$ , and replacing  $1/(1-x)(1-x^2)\dots(1-x^{q-1})$  by its equivalent, we derive

$$1 + \Sigma P(n \mid * \mid > q - 1) x^n = 1 + \Sigma \{P(n \mid * \mid > q) - P(n - q \mid * \mid > q)\} x^n$$
 (12),

where we understand P(0, |\*| > q) to be 1.

Hence, if  $n \not = q$ ,

$$P(n \mid * \mid >q) = P(n \mid * \mid >q-1) + P(n-q \mid * \mid >q)$$
 (13);

and, if n < q,

$$P(n \mid * \mid > q) - P(n \mid * \mid > q - 1)$$
 (14).

By means of (13) and (14) we can readily calculate a table of double entry for  $P(n \mid * \mid \geq q)$ , as follows:—

												73								
4				1		6		8	9	10	11	12	13	11	15	16	17	18	19	20
1	į		1	1	1	1		1	1	1	1	1	ı	1	1	1	1	1	1	1
		-)	2	3	3	1	1	5	5	6	-6	7	7	- 8	8	9	9	10	10	11
4			3	1	5	7	8	10	12	11	16	19	21	21	27	30	33	37	40	4.3
1				j.	6	9	11	15	15	23	27	34	30	47	5.1	61	72	84	94	108
					7	10	1.3	1 -	23	()	37	17	57	70	81	101	110	141	164	192
						11	1.1		26	35	11	58	71	90	110	136	163	199	235	282
							1.5	21	58	.3%	19	6.	45	105	131	164	201	218	300	364
14								22	50	10	52	70	2.1	116	146	1 ~6	230	249	352	431
9									$,  i  \}$	11	51	7.3	94	123	157	201	252	318	393	488
10										42	55	7.5	97	128	164	212	267	340	423	530
11											56	76	99	131	169	219	279	355	415	560
	,											ŀ				_		ļ		

Take a restangle of squared paper BAV; and enter the values of n at the heads of the vertical columns, and the values of q at the ends of the horizontal lines. We remark, first of all, that it (shows from (1)) that all the values in the part of any vertical column below the diagonal AF are the same. We therefore leave all the corresponding spaces blank, the last entry in the claim being understood to be repeated indefinitely.

Next, write the values of  $\vec{P}(1 * > 1)$ , P(2 \* > 1), . . ., that is, 1, 1, . . . , in the row headed 1.

To fill the other rows construct a piece of paper of the form abod. Its use will be understood from the following rule, which is simply a translation of (13):—

To fill the blank immediately after the end of any step, add to the entry above that blank the number which is found at the 1-2 and end of the step.

These to get the number 23, which stands at the end of the stocky 2 on the fourth heri-outal line, we add to 14 the number 9 which lies to the immediate left of ab in the same line as the blank. Asam, in the minth line 157, 146+11; and so not

Ty  $\|\phi\|_{L^{2}(M)}$  by  $\|A\|_{L^{2}(M)}$  by some A(D, w) can all a the table rapidly with little chance of order. We still speak of the table thus constructed as Euler's

Table. It will be found in a considerably extended form in his Introductio, Lib. I., chap. xvi.

A variety of problems in the enumeration of partitions can be solved by means of Euler's Table, as we shall now show.

§ 22.] To find by means of Euler's Table the number of partitions of n into p parts of unrestricted magnitude.

Let us first consider P(n|p|\*). By (9) above, we have

$$1 + \sum P(n \mid p \mid *) x^n z^p = 1/(1 - zx) (1 - zx^2) \dots \text{ad} \quad \infty,$$
  
= 1 + \(\Sigma x^p z^p / (1 - x^2) \dots (1 - x^p), \dots (1 - x^p),

Hence

Therefore,

$$P(n | p | *) = P(n - p | * | > p)$$
 (15).

by Exercises XXI. (18).

Again,

$$1 + \Sigma Pu(n|p|*) x^n z^p = (1 + zx) (1 + zx^2) \dots \text{ad } \infty,$$
  
=  $1 + \Sigma_x t^{1p(p+1)} z^p / (1 - x) (1 - x^2) \dots (1 - x^p),$   
by chap. XXX... § 2. Example 2.

Hence

$$\Sigma Pu(n \mid p \mid *) x^n = x!^{p(p+1)}/(1-x)(1-x^2) \dots (1-x^p),$$
  
=  $\Sigma P(n \mid *| > p) x^{n+\frac{1}{2}p(p+1)}, \text{ by (8)}.$ 

Therefore

$$Pu(n \mid p \mid *) = P(n - \frac{1}{2}p(p+1)] * | > p)$$
 (16).

Example 1.  $P(20 \mid 5 \mid \star) = P(15 \mid \star \mid \succ 5) = 84$ .

Example 2. 
$$Pu(20 \mid 5 \mid *) = P(5 \mid * \mid *5) = 7.$$

 $\S$  23.] If we take any partition of n into p parts in which the largest part is q, and remove that part, we shall leave a partition of n-q into p-1 parts no one of which exceeds q. Hence we have the identity

$$P(n | p | q) = P(n-q | p-1 | p \neq q)$$
 (17)

and, if we make p infinite, as a particular case, we have

$$P(n \mid * \mid q) = P(n - q \mid * \mid > q)$$
 (18).

It will be observed that (18) makes the solution of a certain class of problems depend on Euler's Table. By a mileting (15) and (18), we have the theorem

$$P \circ (s \circ q) \cdot P (s \circ q \circ s),$$

which, however, is only a particular case of a theorem regarding

§ 21 | Theore us requiring emingacy.

$$(L) \qquad P(n \gg p \gg q) \quad P(n \gg q \gg p) \qquad (19)$$

$$H_{eff} = P(n - p \mid q - 1) \ge p) = P(n - q \mid p - 1 \geqslant q)$$
 (20).

$$P(n, p, q) = P(n, q, p) \qquad (21)$$

To prove (L) we observe that by (7), we have

$$1 + \Sigma P(n \geqslant p \geqslant q) z^p x^n - 1 (1 - z x (1 - zx) \dots (1 - zx^q),$$
  
 $-1 + \Sigma z^r \frac{(1 - x^{q+1}) (1 - x^{q+2}) \dots (1 - x^{q+p})}{(1 - x) (1 - x^2) \dots (1 - x^p)}$ 

$$\begin{split} \Sigma P(n \geqslant p \geqslant q) x^{s} &= \frac{(1 - x^{q+1}) (1 - x^{q+2}) \dots (1 - x^{q+p})}{(1 - x) (1 - x^{p}) \dots (1 - x^{p})}, \\ &= \frac{(1 - x) (1 - x^{p}) \dots (1 - x^{q+p})}{(1 - x) (1 - x^{p}) (1 - x^{p}) (1 - x^{p})}. \end{split}$$

Since the function last written is symmetrical as regards p and  $q_i$  it must also be the equivalent of  $\Sigma P(n \gg q \gg p) x^n$ .

The str. (IL) f 'lows from (6) in the same way,

Since, by (17), we have

$$P = a \cdot p \rightarrow P \mid a \cdot q \cdot p = 1 \Rightarrow q),$$
  
 $P \mid a \cdot p \rightarrow P \mid a \cdot q \cdot p = 1 \Rightarrow p);$ 

therefore, by (II ), 
$$P(n,p,q) = P(n,q,p),$$

The fall of particular cases are obtained by making p or

$$P(n + p \rightarrow 1 \mid P, n \rightarrow 3 \cdot p)$$
 (22);

$$P = p - P(n + p)$$
 (23)

§ 25.] The following theorems enable us to solve a number of additional problems by means of Euler's Table:—

$$P(u|p| \geqslant q) = P(u-p|*| \geqslant p) - \Sigma P(u-p_1-p|*| \geqslant p) + \Sigma P(u-p_2-p|*| \geqslant p) - \Sigma P(u-p_2-p|*| \geqslant p) - \Sigma P(u-p_2-p|*| \geqslant p)$$
(24).

Here the summations are with respect to  $\mu_1$ ,  $\mu_2$ , . . . ; and  $\mu_1$  is any one of the numbers  $q, q+1, \ldots, q+p-1, \mu_2$  the sum of any two of them,  $\mu_2$  the sum of any three, and so on. The series of sums is to be continued so long as  $n-\mu_r-p \not \in 0$ . If  $P(n|p|\not = q)$  come out 0 or negative, this indicates that the partition in question is impossible.

$$P(n| \Rightarrow p | \Rightarrow q) = P(n| * | \Rightarrow p) - \exists P(n - r_1 | * | \Rightarrow p)$$

$$+ \exists P(n - r_2 | * | \Rightarrow p)$$

$$- \exists P(n - r_3 | * | \Rightarrow p)$$

$$- \vdots$$
(25).

Here  $\nu_1, \nu_2, \ldots$  have the same meanings with regard to  $q+1, q+2, \ldots, q+p$  as formerly  $\mu_1, \mu_2, \ldots$  with regard to  $q, q+1, \ldots, q+p-1$ .

$$P(n \mid * \mid *)$$
  
=  $P(n-1 \mid * \mid > 1) + P(n-2 \mid * \mid > 2) + ... + P(0 \mid * \mid > n)$  (26).

The demonstrations will present no difficulty after what has already been given above.

## CONSTRUCTIVE THEORY OF PARTITIONS.

§ 26.] Instead of making the theory of partitions depend on series, we might contemplate the various partitions directly, and develop their properties from their inherent character. Sylvester has recently considered the subject from this point of view, and has given what he calls a Constructive Theory of Partitions, which throws a new light on many parts of the subject, and greatly simplifies some of the fundamental demonstrations. Into this

Amer. Jour. Math. (1852).

production our present limits enter; but we desire, the continuous treet, to call the attention of our readers to the article and that of dealing with partitions, which is one of the law that was one of the new theory.

by the simple of a partition is meant a series of rows of the continuous ontaining as many asterisks as there are series as a corresponding port of the partition. Thus

\* \* \* \*

27. ph of the writion 3 + 5 + 3 of the number 11.

that the parts cope in order of magnitude, and all the initial assembles rein one column. Thus the above may be written—

The graph is then said to be regular,

The direct contemplation of the graph at once give us insultive demonstrations of some of the transfer theorems.

Free couple, if we turn the columns of the graph last

about there are as many asterisks as before. The new couple, therefore represents a new partition of 11, which are be said to be conjugate to the former partition-

This to every partition of n into p parts the greatest of set is q, there is a conjugate partition into q parts the greatest of which is p.—Hence

$$P(n \mid p \mid q) = P(n \mid q \mid p),$$

11.5

1 | jet thou if n into p parts no one of which
1 | it is not jugate partition into q or fewer parts
1 | it is not jugate partition into q or fewer parts

$$P = p \simeq_{I} - P (n \geqslant q \mid p) \tag{27},$$

on the second of

way of proving the theorems of

And the second second

§ 27.] The following proof, given by Franklin\*, of Euler's famous theorem that

$$(1-x)(1-x^2)(1-x^3)$$
 . . . ad  $\infty = \sum_{p=0}^{\infty} (-)^p x^{\frac{1}{2}(3p^2 \pm p)}$  (28)†,

is an excellent illustration of the peculiar power of the graphic method.

The coefficient of  $x^n$  in the expansion in question is obviously  $Pu(n \mid \text{even} \mid *) - Pn(n \mid \text{odd} \mid *)$ (29).

Let us arrange the graphs of the partitions (into unequal parts) regularly in descending order. Then the right-hand edge of the graph will form a series of terraces all having slopes of the same angle (this slope may, however, consist of a single asterisk), thus—



We can transform the graph A by removing the top row and placing it along the slope of the last terrace, thus--

We then have a regular graph A'
representing a partition into unequal parts.
This process may be called contraction.

B' We then have a regular graph B' representing a partition into unequal parts.

Every graph can be transformed by con-

traction or by extension, except when the top
row meets the slope of the last terrace; and in
this case also, provided it does not happen that
the number of asterisks in the top row is equal

<sup>·</sup> Comptes Rendus (1980).

<sup>+</sup> Euler originally discovered this theorem by induction from particular cases, and was for long unable to prove it. For other demonstrations, see Legendro, Théorie des Nombres, t. II., § 15, and Sylvester (I.c.).

to the number in the last slope or exceeds it only by one, is, for example, in

Cortication or extension in the first of these would produce an inequality contraction in the second would produce an inequality raph; and extension would produce a graph which corresponds to a partition baying two parts equal. These two cases may be spoken of as morajupate; they can only arise when the results of the partition are

$$p, p+1, p+2, \dots, 2p-1.$$

and the number

$$n = p + (p + 1) + \dots + (2p + 1) - \frac{1}{2}(2p^2 - \frac{1}{2});$$

or when the p parts are

$$p + 1$$
,  $p + 2$ ,  $p + 3$ , ...,  $2p$ ,  $n \cdot (p + 1) + (p + 2) + \dots + 2p - 1 (3p^2 \cdot p)$ ,

and d

Since contraction or extension always converts a partition discoup an even or an odd number of parts into one having an odd or an even number of parts respectively, we see that unless n be a number of the form  $\frac{1}{2}(3p^3 \pm p)$ ,  $P_{n} = (\text{ven} *) \cdot P_{H,n}(n) \cdot (\text{dd} *)$ .

When n has one or other of the forms  $\frac{1}{4}(3p^2+p)$ , there will be one unceresate partition which will be even or odd signature as p is even or +11; all the others will occur in pairs which are conjugate in Franklin's transformation. Hence

$$Pu \mapsto 3p^* \pm p$$
 even  $*i - Pu + [i3p^2 \pm p]_+ \text{ odd}_+ *) = (-1)^p$  (30).

Eu' i's Theorem follows at once.

## EXERCISES XXXXIII

(1) Show I by to evaluate  $I(a|b) > p(|\bullet|)$  by resums of Euler's Table.

(\* / 1 a %)

(3) P(13 = 5 = 8).

(i) I (id . . . .

(5.)  $I_{i,\omega} \circ_i J_i \gg S),$ 

#### Establish the following :-

- (6.) Pu (n | | ) = P (n − ½q (q + 1) | | > q), where ½q (q + 1) just > n.
- (7.)  $Pu(n|p| \bullet) = P(n \frac{1}{2}p(p-1)|p| \bullet).$
- (8.)  $P(n | p | *) = Pu(n + \frac{1}{2}p(p-1) | p | *).$
- (9.)  $Pu(n \mid p \mid p \mid p) = P(n \frac{1}{2}p(p 1) \mid p \mid p \mid p \mid p \mid p)$
- (10.) Is the theorem P(n-p|q-1|\*) = P(n-q|p-1|\*) universally true?
  - (11.) Show how to form a table for the values of P (n | \*| 2, 3, ..., q).
    (See Proc. Edinb. Math. Sec., 1883-4.)
- (12.) Show how to form a table for the number of partitions of n into an indefinite number of odd parts.

## Establish the following:-

- (13.)  $P(n \mid \star \mid 1, 2, 2^2, 2^3, ...) = 1$ .
- (14.)  $Pu(n|p|1, 3, ..., 2q-1) = P(n-p^2+p|p|1, 3, ..., 2q-1).$
- (15.) P(n | p | 2, 4, ..., 2q) = P(n-p | p | 1, 3, ..., 2q-1).
- (16.)  $P(n \mid \star \mid \text{odd}) = Pu(n \mid \star \mid \star)$ .
- (17.)  $P(n \mid p \mid 2, 4, ..., 2q) = P(n \mid p \mid 2, 4, ..., 2p).$
- (18.) P(n+p|p|1, 3, ..., 2q+1) = P(n+q|q|1, 3, ..., 2p+1).
- (19.)  $Pu(n+p^2 | p | 1, 3, ..., 2q+1) = Pu(n+q^2 | q | 1, 3, ..., 2p+1).$
- (20.)  $P(n+2p \mid p \mid 2, 1, \ldots, 2q+2) = P(n+2q \mid q \mid 2, 4, \ldots, 2p+2).$
- (21.) Show that  $P(n \mid p \mid \bullet) = P(n-1 \mid p-1 \mid \bullet) + P(n-p \mid p \mid \bullet)$ ; and hence construct a table for  $P(n \mid p \mid \bullet)$ . (See Whitworth, Choice and Chance, chap. III.)

# CHAPTER XXXVI.

# Probability, or the Theory of Averages.

1.] An elementary account of the Theory of Probability, or as we should prefer to call it, the Theory of Averages, has usually found a place in English text-hooks on algebra. This account is justified by several considerations. The theory in question affords an evellent illustration of the application of the theory of permutations and combinations which is the fundamental part of the algebra of discrete quantity; it forms in its elementary parts an evellent ligical exercise in the accurate use interms and in the nice discrimination of shades of meaning; and, above all, it enters, as we shall see, into the regulation of since of the next inpart in type that concerns of modern life.

The student is probable aware that there are certain occurnation, or classes of events of such a nature that, although we could stain the solution of the certainty assert a particular proposition recording a large number N of them with a decree of certainty which increases (with or without limit,

another care may be) as the number N increases.

For example, if we take any particular man of 20 years of age, to thing could be more uncertain than the statement that he will be a local 25; but, if we consider 1000 such men, we may assert with constant by the could need that 96 per cent of them will live to \$25 - 4; it with a million we might with much greater confidence and the piece of a with even closer accuracy. In so do not have a real ways to be not state the limits both of habit at additional which when the men are to be taken; and, we will a little a limit a consider a little particular and the product of the product of the men are to be taken; and, we will be the state of the product of the prod

the proportion of those who survive for 5 years with absolute accuracy, but be prepared, when we take one million with another, to find occasional small fluctuations about the indicated percentage.

We may, for illustration, indicate the limits just spoken of by saying that "man of 20" is to mean a healthy man or woman living in England in the 18th century. The "event," as it is technically called, here in question is the living for 5 years more of a man of 20; the alternative to this event is not living for 5 years more. The whole, made up of an event and living the result of alternatives, we call its universe. The alternative or alternatives to an event taken collectively we often call the Complementary Event. The living or not living of all the men of 20 in England during the 18th century we may, following Mr Venn\*, call the series of the event. It will be observed that on every occasion embraced by the series the event we are considering is in question; and we express the above result of observation by saying that the probability that a man of 20 living under the assigned conditions reached the age of 25 is '96.

We are thus led to the following abstract definition of the Probability or Chance of an Event:—

If on taking any very large number N out of a series of cases in which an event A is in question, A happens on pN occasions, the probability of the event A is said to be p.

In the framing of this definition we have, as is often done in mathematical theories, substituted an ideal for the actual state of matters usually observed in nature. In practice the number p, which for the purposes of calculation we suppose a definite quantity, would fluctuate to an extent depending on the nature of the series of cases considered and on the number N of specimen cases selected  $\dagger$ . Moreover, the mathematical definition contains no indication of the extent or character of the series of cases.

<sup>\*</sup> Logic of Chance.

<sup>+</sup> We might take more explicit notice of this point by wording the definition thus:— "If, on the arcrage, in N out of a series of cases, &c." But, from the point of view of the ideal or mathematical theory, nothing would thus be gained.

How far the possible fluctuations of p, the extent of the series, and the magnitude of N will affect the bearing of any conclusion on practice must be judged by the light of circumstances. It is obvious, for instance, that it would be unwise to apply to the 14th century the probability of the duration of human life  $\delta$  luced from statistics taken in the 18th. This leads us also to among the total probability is not morely bestorical, as the definition might suggest. Into most of the important practical applications there enters an element of inhoritiny. Thus we do in fact apply in the 19th century a tidde of mortality statistics deduced from observations in the 18th century. The warranty for this extension of the series of cases by induction must be sought in experience, and cannot in most case a be obtained a priori.

There are, however, some cases where the circumstances are so simple that the probability of the event can be deduced. without elaborate collecting and sifting of observations, merely from our definition of the circumstances under which the event is to take place. The best examples of such cases are games of hazard played with cards, dice, &c. 1f. for example, we assert regarding the tessing of a halfpenny that out of a large number of trials heads will come up nearly as often as tails-in other words, that the probability of heads is ½, what we mean thereby is that all the causes which tend to bring up heads are to neutralise the causes that tend to bring up tails. In every series of cases in question, the assumption, well or ill justified. is made that this counterbalancing of causes takes place. That this is really the r bt point of view will be best brought home to us if we reflect that undoubtedly a machine could be constructed which would infallibly toss a halfpenny so as always to land it he cloup on a thickly sanded floor, provided the coin were always placed the same way into the machine; also, that the coin might have two heads or two tails; and so on,

In cases where the stat ment of probability rests on grounds so simple as this, the difficulty regarding the extension of the series by induction is be—prominent.—The ideal theory in such

<sup>.</sup> It the property is real conse of the work.

cases approximates more closely than usual to the actual circumstances. It is for this reason that the illustrations of the clementary rules of probability are usually drawn from games of hazard. The reader must not on that account suppose that the main importance of the theory lies in its application to such cases; nor must be forget that its other applications, however important, are subject to restrictions and limitations which are not apparent in such physically simple cases as the theory of cards and diec.

Before closing this discussion of the definition of probability as a mathematical quantity, it will be well to warn the learner that probability is not an attribute of any particular event happening on any particular occasion. It can only be predicated of an event happening or conceived to happen on a very large number of "occasions," or, in popular language, of an event "on the average" or in the "long run." Unless an event can happen, or be conceived to happen, a great many times, there is no sense in speaking of its probability, or at least no sense that appears to us to be admissible in the following theory. The idea conveyed by the definition here adopted would be better expressed by substituting the word frequency for the word probability; but, after the above caution, we shall adhere to the accepted term.

§ 2.] The following corollaries and extensions may be added to the definition.

Cor. 1. If the probability of an event be p, then out of N cases in which it is in question it will happen pN times, N being any very large number\*.

This is merely a transposition of the words of the definition.

As an example, let it be required to find the number out of 5000 men of 20 years of age who will on the average live to be 25. The probability of a man of 20 living to be 25 may be taken to be '96; hence the number required is '96 × 5000 = 4800.

Cor. 2. If the probability of an event be p, the probability of its failing is 1-p.

For out of a large number N of cases the event will happen on pN occasions; hence it will fail to happen on N-pN

<sup>\*</sup> It is essential that pN also be a very large number. See Simmons, Proc. L. M. S., xxvi., p. 307 (1895).

(1 + p) N occasions. Hence, by the definition, the probability of the failing of the event is 1 - p.

Cor. 3. If the universe of an event be made up of n alternatives, or, in other words if an event must happen and that in one out of n ways, and if the respective probabilities of its happening in these ways  $h_1, h_2, \dots, h_n$ , then  $h_1 + h_2 + \dots + h_n = 1$ .

For on every one of N occasions the event will happen; and it will happen in the first way on  $p_1N$  occasions, in the second on  $p_2N$  occasions, and so on. Hence  $N = p_1N + p_2N + \ldots + p_NN$ ;

that is,  $1 - p_1 + p_2 + \dots + p_n$ .

Cor. 4. If an event is certain to happen, its probability is 1; if it is certain not to happen, its probability is 0.

For in the former case the event happens on 1. N cases out

of N cases; in the latter on 0. N cases out of N.

The probability of every event is thus a positive number lying between 0 and 1.

Co. 5. If an ecut must happen in one out of n ways all equally probable, or if one out of n events must happen and all are equally probable, then the probability of each way of happening in the first cose, or of each event happening in the second, is 14.

This follows at once from Cor. 3 by making  $p_1 - p_2 - ... = p_n$ . As a particular case, it follows that, if an event be equally likely to happen or to fail, its probability is  $\frac{1}{n}$ .

Definition. The ratio of the probability of the happening of an event to the probability of its failing to happen is called the olds in favour of the event, and the reciprocal of this ratio is called the additional it.

Thus, if the probability of an event be p, the odds in favour are p:1-p; the odds a\_ainst 1-p:p. Also, if the odds in favour be m:n, the probability of the event is m(m+n). If the probability of the event be  $\frac{1}{2}$ , that is, if it be equally likely to be p to fail, the odds in favour are p:1, and are said to be p to p.

Vor. 6. If the universe of an event can be analysed into m+n cases each of which in the long ran will occur equally often \*, and

<sup>.</sup> The is usedly expressed by sayin that all the cases are equally likely,

if in m of these cases the event will happen and in the remaining n fail to happen, the probability of the event is m!(m+u).

After what has been said this will be obvious,

#### DIRECT CALCULATION OF PROBABILITIES,

§ 3.] The following examples of the calculation of probabilities require no special knowledge beyond the definition of probability and the principles of chap. XXIII.

Example 1. There are 5 men in a company of 20 soldiers who have made up their minds to desert to the enemy whenever they are put on outpost duty. If 3 men be taken from the company and sent on outpost duty, what is the probability that all 4 fthem desert?

The 3 men may be chosen from among the 20 in  $_{20}C_2$  ways, all of which are equally likely. Three deserters may be chosen from among the 5 in  $_2C_2$  ways, all equally likely. The probability of the event in question is therefore  $_2C_2C_2C_2=\frac{5.4.3}{1.2.3}\frac{7(9.19.18}{3.1.2.3}=1/111$ .

Example 2. If n people seat themselves at a round table, what is the chance that two named individuals be neighbours?

There are (see chap. xxii., § 4) (n-1)! different ways, all equally likely, in which the people may seat themselves. Among these we may have A and Bor B and A together along with the (n-2)! different arrangements of the rest; that is, we have 2(n-2)! cases favourable to the event and all equally likely. The required chance is therefore  $2(n-2)!(n-1)! \ge 2(n-1)$ .

When n=3, this gives chance =1, as it eught to do. The odds against the vent are in general n=3 to 2; the odds will therefore be even when the number of neonle is 5.

Example 3. If a be a prime integer, and  $n=a^r$ , and if any integer I > n be taken at random, find the chance that I contains a as a factor s times and no more.

The integer I must be of the form  $\Lambda a^*$ , where  $\lambda$  is any integer less than  $a^{r+s}$  and prime to  $a^{r+s}$ . Now, by chap, xxxx,  $\S$  8, the number of integers less than  $a^{r+s}$  and prime to it is  $a^{r+s}(1-1/a)$ . Also the number of integers > n is  $a^r$ . Hence the required chance is  $a^{r+s}(1-1/a)/a^r = a^{r+s}(1-1/a) = 1/a^s - 1/a^{s+s}$ .

Example 4. Find the probability that two men A and B of m and n years of age respectively both survive for p years.

The mortality tables (see § 15 below) give us the numbers out of 100,000 introducing of 10 years of age who complete their mth, mth, mth, nth, n

is in the second. In this way we could select altogether MN pairs of mental area of since or dead after p years have clapsed. The number out of M the M then  $L_{int}$  m(t) = p or  $i \in M_{min}$ ,  $L_{int}$  m(t) = p. On the fitness of the N man is  $Nl_{min}$ ,  $l_m$ . Out of these we could form  $MNl_{min}l_{min}$ ,  $l_m$  pars. This has number will be the number of pairs of arrivors out of the M pairs with which we started. Hence the probability required is  $l_{min}l_{mi$ 

Example 5. A number of balls is to be drawn from an urn, 1, 2, . . . , n being all equally likely. What is the probability that the number drawn be even?

We can sleav 1, 2, ..., n re-pertively in  $a_i^{(k)}$ ,  $a_i^{(k)}$ , ...,  $a_i^{(k)}$ , was a proposed of the universe of the event as consisting of  $a_i^{(k)}$ ,  $a_i^{(k)}$ 

Example 6. A white rook and two black pawns are placed at random on a ches-sheard in any of the positions which they might occupy in an actual game. Find the ratio of the chance that the rook can take one or both of the pawn to the chance that either or both of the pawns can take the rook.

I is so both at the board from the solit of white; and calculate in the first place the whole number of possible arrangements of the pieces. No black pawn can it on any of the front squares; hence we may have the rook on any of these is and the two pawns on any two of the remaining 66; in all, see 2.5.° is 5.66-5.56 arrangements. Apain, we may have the rook on any one of the 56 square and the two pawns on any two of the remaining 55 squares; in all, 56 c. of 54 arrangements. The universe may therefore be supposed to contain 05 c. 55. 55 equally likely cases.

Instal of each study the chance that the rook can take either or both of the pawns, it is at a far, as (for happ as, to calculate the chance of the complainants y war, time by, that the rook can take neither of the pawns. If the rook is on one of the foot row of squares, nither of the pawns can be on the corresponding commit it its the pawns may evenly any two out of 49 equate (the row (s. 1)). Satisfacements. If the rook lies in any one of the rownian 56 in a rise in the roof the pawns must lie in the row of column below as to between the columns (s. 1). The root of the two pawns 42 × 41 position (whe to colors by 12 × 14 trangements. Altogether we have

 $8 \times 49 \times 48 + 56 \times 42 \times 41 = 56 \times 19 \times 42$  arrangements in which the rook can take neither pawn. Hence the chance that the rook can take neither pawn is  $56 \times 49 \times 42/62 \times 56 \times 55 = 1029/1705$ . The chance that the rook can take one or both of the pawns is therefore 1 - 1029/1705 = 676/1705,

Consider now the attack on the rook. If he is on a side square, he can only be attacked by either of the two pawns from one square. For the side squares we have therefore only 24 x 51 arrangements in which the rook can be taken. There remain 36 squares on each of which the rook can be taken from two squares, that is, in 6 ways. For the 36 squares we therefore have  $36 \times 2 + 36 \times 4 \times 53$  arrangements in which the rook can be taken by one or by both the pawns. Altogether there are 9000 arrangements in which the rook may be taken. Hence the chance that he be in dancer is  $9000/32 \times 56 \times 55 =$ The ratio of the two chances is 9464: 1125. 225/4774.

§ 4.] A considerable number of interesting examples can be solved by the method of chap, XXIII., § 15. Let there be r bags. the first of which contains  $a_1, b_1, c_1, \ldots, k_1$  counters, marked with the numbers  $a_1, \beta_1, \gamma_1, \ldots, \kappa_1$ ; the second,  $a_1, b_2, c_2, \ldots, k_n$ marked  $\alpha_2, \beta_2, \gamma_2, \ldots, \kappa_2$ ; and so on. If a counter be drawn from each bag, what is the chance that the sum of the numbers drawn is n?

By chap. XXIII., § 15, the number of ways in which the sum of the drawings can amount to n is the coefficient,  $A_n$  say, of  $x^n$ in the distribution of the product

$$(a_2x^{a_1} + b_1x^{\beta_1} + \dots + k_2x^{s_1})$$
  
  $\times (a_2x^{a_2} + b_2x^{\beta_2} + \dots + k_2x^{s_2})$   
 $\cdot \cdot \cdot \cdot \cdot$   
 $\times (a_rx^{a_r} + b_rx^{\beta_r} + \dots + k_rx^{s_r})$ 

Again, the whole number of drawings possible is the sum of all the coefficients; that is to say,

$$(a_1 + b_1 + \dots + k_1)$$
  
 $\times (a_2 + b_2 + \dots + k_2)$   
 $\cdot \cdot \cdot \cdot \cdot \cdot$   
 $\times (a_r + b_r + \dots + k_r) = D$ , say.

Hence the required chance is  $A_n/D$ .

Example 1. A throw has been made with three dice. The sum is known to be 12; required the probability that the throw was 4, 4, 4.

The number of ways in which 12 can be thrown with three dice is the coefficient of x12 in

$$(x^3 + x^2 + x^3 + x^4 + x^5 + x^7)^{\frac{1}{2}}$$

that is to say, of z? in

$$(1 + x + x^2 + x^3 + x^4 + x^5)^3$$
.

Now the coefficients in  $(1+x+\dots+x^2)^2$  up to the term in  $x^3$  are (see chap,  $w, \S 15$ ) 1+2+3+4+5+6+5+1+3+2. Hence the coefficient of  $x^3$  in the cube of the multinounal is 5+6+5+4+3+2=25. The required probability is therefore  $1,2^5$ .

Example 2. One die has 3 faces marle 11, 2 marked 2, and 1 marked 3; another has 1 face marked 1, 2 marked 2, and 3 marked 3. What is the most probable throw with the two dice, and what the chance of that throw?

The multi-rs of ways in which the sums 2, 3, 4, 5, 6 can be made are the coefficients of x, x', x', x', x', x' in the expansion of  $(3x + 2x^2 + x^2)(x + 2x^2 + 3x^2)$ . Now this product is on all to

 $3x^2 + 8x^3 + 11x^4 + 8x^5 + 3x^6$ .

The sum that will occur oft nest in the long run is therefore 4. The whole number of 4 here at ways in which the different throws may turn out is (3+2+1)(1+2+3)=36. Hence the probability of the sum 4 is 14/36=7/18.

Example 3. An urn contains m counters marked with the numbers  $1, 2, \ldots, m$ . A counter is drawn and replaced r times; what is the chance that the sum of the numbers drawn is  $n \upharpoonright \uparrow$ 

The whole number of possible different drawings is m'.

The number of those which give the sum n is the coefficient of  $x^n$  in  $\{x \mid x^n \mid \dots \mid x^n\}^n$ , that is to say, of  $x^{n+r}$  in  $\{1+x+\dots+x^{m-1}\}^r$ . Now  $1+x+\dots+x^{m-1}=(1-x^m)(1-x)$ . We have therefore to find the coefficient of  $x^n$  in

 $(1-x^n)^r (1-x)^{-r} = \{1-r^r\}^{-r} + r^r\}^{-r^2} x^{2n} - r^r (x^{2n} + \dots)$   $\times \left\{1 + \frac{r}{1}x + \frac{r}{1+2} (r-1) x^2 + \frac{r}{1+2} (r+2) x^3 + \dots\right\} .$ 

The condition in quantum is  $A_{n-r} = \frac{r(r+1)...(n-1)}{(n-r)!} - \frac{r(r+1)...(n-m-1)r}{(n-r-m)!1!}$ 

$$\frac{(n-r-m)!}{r} + \frac{r}{(n-r-2m)!} + \frac{r}{(n-r-2m)!} + \frac{r}{(n-r-2m)!} = \dots$$

The required probability is  $A_{n-r}/m^r$ .

Example 4. If model and never integers (n+m-1) be written down at rate on, show that the chance that no two odd integers are adjacent is  $\pi^*(n+1)!(n-n^*(n-m+1)!)$ .

In order to tad in how many different ways we can write down the interests of that not two old onest come togs they, we may suppose the modd put our written down in any one of the miliposable ways, and consider the r. It spaces between them to other with the two spaces to the right and left of the row. The problem is a site that in how many ways we can fill the

\* We might also have found the coefficient of x<sup>p</sup> by expanding (1 . (1-x) 7, as in Example 4 below.

† This is generally called Primory is Problem. For an interesting account of its history so T. L. rot r, H. t. Pr. 2, 1 p. 53, 85. n even integers into the spaces so that there shall always be one at least in every one of the m-1 spaces. A little consideration will show that the number of ways, irrespective of order, is the coefficient of  $x^n$  in

$$(1+x+x^2+\ldots \text{ ad } x)^2(x+x^2+\ldots \text{ ad } x)^{m-1};$$
  
that is, of  $x^{n-m+1}$  in  $(1+x+x^2+\ldots)^2(1+x+x^2+\ldots)^{m-1};$   
that is, of  $x^{n-m+1}$  in  $(1-x)^{-(m+1)}$ .

This coefficient is

$$\frac{(m+1)(m+2)...(n+1)}{(n-m+1)!} = \frac{(n+1)!}{m!(n-m+1)!}$$

If we remember that every distribution of the n integers among the m+1 spaces can be permutated in n! ways, we now see that the number of ways in which the m+n integers can be arranged as required is

n! m! (n + 1)! / m! (n - m + 1)! = n! (n + 1)! / (n - m + 1)!

The whole number of ways in which the m+n integers can be arranged is (m+n)!, hence the probability required is n!(n+1)!/(n-m+1)!(m+n)!,

#### ADDITION AND MULTIPLICATION OF PROBABILITIES.

§ 5.] In many cases we have to consider the probabilities of a set of events which are of such a nature that the happening of any one of them upon any occasion excludes the happening of any other upon that particular occasion. A set of events so related are said to be mutually exclusive. The set of events considered may be merely different ways of happening of the same event, provided these ways of happening are mutually exclusive.

In such cases the following rule, which we may call the Addition Rule, applies:—

If the probabilities of n mutually exclusive events be  $p_1, p_2, \dots, p_n$ , the chance that one out of those n events happens on any particular occasion on which all of them are in question is  $p_1 + p_2 + \dots + p_n$ .

To prove this rule, consider any large number N of occasions where all the events are in question. Out of these X occasions the n events will happen on  $p_1N, p_2N, \ldots, p_nN$  occasions respectively. There is no cross classification here, since no more than one of the events can happen on any one occasion. Out of N occasions, therefore, one or other of the n events will happen on  $p_1N+p_2N+\ldots+p_nN=(p_1+p_2+\ldots+p_n)N$  occasions. Hence the probability that one out of the n events happens on any one occasion is  $p_1+p_2+\ldots+p_n$ .

It should be observed that the reasoning would lose all force if the events were not unitually exclusive, for then it might be that or the  $p, X \sim cx$  ions on which the first event happens one or more of the officis happen. We shall give the proper formula in this case presently.

As an illustration of the application of this rule, let us suppose that a throw is made with two ordin ay diec, and calculate the probability that the throw does not exceed 8. If one are 7 ways in which the event in question may happ in ramely, to throw may be 2, 3, 4, 5, 6, 7, or 8; and these ways not of course mutually accessive. Now (or 9, 4, Example 1 the probabilities of these 7 throws are 1/3c, 2/3c, 3/3c, 4/3c, 5/3c, 6/3c, 6/3c, 6/3c respectively. If not the probability that a throw with two dies does not exceed 8 is (1+2+3+1+5+6+3) of 2-9/3c, 1/4/3c.

§ 6.] When a set of events is such that the happening of any one of them in no way affects the happening of any other, we say that the events are mutually independent. For such a set of events we have the following Maltiplication Rule:—

Li the respective probabilities of n independent events be  $p_1$ ,  $p_2, \dots, p_n$ , the probability that they all happen on any occasion

in which all of them are in question is  $p_1p_2 \dots p_n$ .

In proof of this rule we may reason as follows:—Out of any large number N of cases where all the events are in question; the first event will happen on  $p_i N$  occasions. Out of these  $p_i N$ occasions the second event will also happen on  $p_i(p_i N) = p_i p_j N$ occasions; so that out of N there are  $p_i p_j N$  occasions on which both the first and second events happen. Continuing in this way, we show that out of N occasions there are  $p_i p_i \dots p_j N$  occasions on which all the n events happen. The probability that all the n events happen on any occasion is therefore  $p_i p_i \dots p_k$ .

It should be noticed that the above reasoning would stand if the events were not independent, provided  $p_2$  denote the probability that  $\phi$  in 2 happen after event 1 has happened,  $p_2$ the probability that 3 happen after 1 and 2 have happened, and

so off.

It must be observed, however, that the probability calculated is then that the events happen in the order 1, 2, 3, ..., n. Hence the following conclusion:

Cor. If the excits 1, 2, . . . , n be interdependent and product the probability of 1, p. the probability that 2 happen aft r 1 has happened, p. the probability that 3 happen aft r 1 and 2 have happened, and so on, then the probability that the exents 1, 2, . . , n happen in the order indicated is p.p. . . p.

As an illustration of the multiplication rule, let us suppose that a die is thrown twice, and calculate the probability that the result is such that the first throw does not exceed 3 and the second by so not exceed 3.

The probability that the first three does not excell 3 is, by the addition rule, 3/6; the probability that the second does not excel 5 is 5.6. The result of the first throw in no way affects the result of the second; hence the probability that the result of the two throws is as indicated is, by the multiplication rule, (3.6) x (5.60 = 5.12).

As an example of the effect of a slight alteration in the working of the question, consider the following:—A die has been thrown twice: what is the probability that one of the throws does not exceed 3 and the other does not exceed 5?

Since the particular throws are now not specified, the event in question happens—lat, if the first throw does not exceed a and the second does not exceed 5; and, if the first throw is 4 or 5 and the second does not case d 3. These cases are mutually exclusive, and the respective probabilities are 5.12 and 1/6. Hence, by the addition rule, the probability of the event in question is 7/12.

§ 7.] The following examples will illustrate the application of the addition and multiplication of probabilities.

Example 1. One turn,  $A_1$  contains m balls, pm bleng white, (1-p)m black; another,  $B_1$  contains n balls, qm white, (1-q)n black. A prison electrone of the two mrns at random, and draws a ball; calculate the charge that it be white; and compare with the change of drawing a white ball when all the m+n balls are in one urn.

There are two ways, mutually exclusive, in which a white 'all may be drawn, namely, from A or from B.

The chance that the drawn selects the urn I is 1/2, and if L is best that unchance of a white ball is g. It have the chance that a white ball is g. It have the chance that a white ball is drawn from I is I is I is drawn from I is I is I. The whole chance of drawing a white ball I is trace I in I is I in I

If all the balls he in one upp, the chance is (pm + qne, n + n).

Now (pm+qn)(m+n) > = <(p+q)/2,according as 2(pm+qn) > = <(p+q)/(n+n),according as (m-n)(p-q) > = <0.

Hence the chance of drawing a white 0.4P will be unaltered by mix 0.14 either the numbers of bubs in A and B be equal, or the 1 reportion of whice balls in each be the same

If the restrict of full is unequal, and the proportions of white be uncred, then the examp of the balls will increase the chance of drawing a size of the number obstains most hills have also the larger proportion of the change of the chance of drawing a white if the number of constant of the proportion of which is a size of the chance of drawing a white if the number proportion of which the size of the change of the

In More of the sense of the additional consolidation of the example to point out the district of the sense of the additional. Let us suppose the two runs to be as for as at A of with 4 bid); B (I who, 3 bid). We might then with a map at algorithm as in the in-The drawer must select either A or B. If the contribution of the sense of white is 30%; if the second by the chance of white is 30%; if the second by the chance of white is 30%; if the second by the sense of white is 30%; if the second by the sense of white is 30%; if the second by the sense of white is 30%; if the second by the sense of white is 30%; if the second by the sense of the sense of the sense of the sense of the white is sense of the sense of th

The new style is the factor 12 will be best seen by reasoning design an the testiment definition. Let us suppose the drawer to make the experiment at a ylaric an other Northmes. In the long run the one urn will be selected as often as the other. Hence out of N times A will be selected by the set of the first selected by the selecte

Extrapl 2. Four earls are drawn from an ordinary pack of 52; what is the corresponding to the value of different state?

We receive the season example of § 6, Cor. The chance that the form with a form of the testing of orderse, I. The chance, after one strict is a received in the testing of orderse, I. The chance, after one strict is an extensive and drawn be of a different suiting of the season of t

3. If we say three new threat he allowed to toss a penny in it is so be 100 to 1 to the sets at least one head?

The complementary event to more head to the complementary event to more head at least is therefore the complementary we must therefore have

hence

$$2^{x} = 101$$
,  
 $x = \log 101/\log 2$ ,  
 $= 2.0043/3010$ ,  
 $= 6.6$ ...

It appears, therefore, that in 6 tosses the odds are less than 100 to 1, and in 7 tosses more.

Example 4. A man tosses 10 pennies, removes all that fall head up; tosses the remainder, and again removes all that fall head up; tosses the remainder, and again removes all that fall head up; and so on. How many times ought he to be allowed to repeat this operation in order that there may be an even chance that before he is done all the pennies have been removed?

Let x be the number of times, then it is clearly necessary and sufficient for his success that each of the 10 pennies shall have turned up head at least once. The chance that each penny come up head at least once in trials is  $1 - (1/2)^2$ . Hence the chance that each of the 10 has turned up heads at least once is  $\{1 - (1/2)^2\}^{10}$ . By the conditions of the problem we must therefore have

$$\{1 - (1/2)^x\}^{10} = 1/2;$$
  
 $(1/2)^x = 1 - (1/2)^{1/10} = 06697;$   
 $x = -\log 06697/\log 2,$   
 $= 3.9 \text{ yery nearly.}$ 

Hence he must have 4 trials to secure an even chance.

Example 5. A man is to gain a shilling on the following conditions. He draws twice (replacing each time) ont of an urn containing one white and one black ball. If the draws white twice he wins. If he falls a black ball is added, he tries twice again, and wins if he draws white twice. If he fails another black ball is added; and so on, ad infinitum. What is his chance of gaining the shilling? (Laurent, Calcul des Probabilities (1873), p. 60.)

The chances of drawing white in the various trials are  $1/2^2$ ,  $1/3^2$ , . . . . The chances of failing in the various trials are  $1-1/2^2$ ,  $1-1/3^2$ , . . ,  $1-1/3^2$ , . . . . Hence the chance of failing in all the trials is  $(1-1/2^2)((1-1/3^2)\dots(1-1/2^2)\dots ad\ z$ .

$$\begin{split} L & \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) & \dots \left(1 - \frac{1}{n^2}\right) \\ & = L & \left\{1 \cdot 3\right\} \left\{2 \cdot 4\right\} & \dots \left\{(n-3)(n-1)\right\} \left\{(n-2)n\right\} \left\{(n-1)(n+1)\right\} \\ & = L & n \cdot (n+1) \\ & = L & 2n^2 & \dots \\ \end{split}$$

The chance of failing to gain the shilling is therefore 1/2 Hence the chance of gaining the shilling is 1/2.

We might have calculated the chance of gaining the shilling directly, by

Above to do the transfer of the following events: 19. the state of the state of the stand coming in 2nd; 3°, failing or many the chance on. In this way the chance

 $1 + (1 - \frac{1}{n}) + \dots + (1 - \frac{1}{n}) / (1 - \frac{1}{n}) \dots + (1 - \frac{1}{n}) \frac{1}{(n-1)^2} + \dots$ the many many must therefore be 1/2. That this is so may

The product of mple among now in which the

I' had A an' B cast along tely with a pair of ordinary dice. A

A sections 6 a to both - 7, and B if he throws 7 before A throws 6. 17.11 - show the black in cof winning; B's=30; 31. (Huyghens.

I the dead to the . ' to sof throwing and of faling to the w 6 at a

in the second twell is resulted to sponting chances for 7.

the are and the file of ways; I, A succeed at 1st throw; 2°, A fail at 1 . Bis and 1 . dec. of at 3rd; also on. His chance is therefore

$$p \cdot {}_{4} 1 + {}_{2} \dots \cdot \dots = {}_{1} 1 + {}_{1} ps - (qs)^{2} + \dots \cdot {}_{n}$$
  
=  $p \cdot (1 - q)$ .

I regard in the f " of everys:-1, A fail at 1st, B succeed at 2nd; the first at 1 to 1 to at 20 to A fail at 3rd, B should at 4th; and so on.

$$qr + q_{-1}r - p_{-1} \cdot p_{-1}r + \cdots = \{r\{1 + (qs) + (q_{-1})^2 + \cdots \}\},$$
  
=  $rr\{1 + q_{-1}\},$ 

Y May ' + ' o + = ' A Te mant r, He t. Prob., p. 21.

The second of the last played making with the last, m > n, there cannot be more

s ... w that the complication duo . . . . . . . . . . t pr illem. The chances a to the chartes

. a run of m+1 co ecutive 2 V off chances of a run the same and are 1,2 a and and

Hence the chance p of a run of at least m heads is given by

$$p = \frac{n+3}{2^{m+2}} + \frac{n+2}{2^{m+3}} + \dots + \frac{5}{2^{m+n}} + \frac{4}{2^{m+n+1}} + \frac{1}{2^{m+n}}.$$

The summation of the series on the left-hand side is effected (see chap. xx., § 13) by multiplying by  $(1-1/2)^2-1/4$ . We thus find

$$\begin{split} 4p = & \frac{n+3}{2^{m+2}}, & \frac{n+2}{2^{m+2}}, & \frac{n+1}{2^{m+4}} + \dots + \frac{4}{2^{m+n+4}} \\ & - \frac{2(n+3)}{2^{m+3}} - \frac{2(n+2)}{2^{m+4}} - \dots - \frac{2\cdot 5}{2\cdot n \cdot n \cdot 1} - \frac{2\cdot 4}{2^{m+n+2}} \\ & + \frac{n+3}{2^{m+2}} + \dots + \frac{6}{2^{m}} + \frac{1}{2^{m+n+2}} + \frac{1}{2^{m+n+2}} + \frac{1}{2^{m+n+2}}; \\ 4p = & \frac{n+3}{2^{m+2}} - \frac{n+1}{2^{m+2}} - \frac{3}{2^{m+2}} + \frac{2}{2^{m+n+2}} + \frac{1}{2^{m+n+2}}; \\ = & \frac{n+2}{2^{m+2}}. \end{split}$$

Hence  $p = (n + 2)/2^{m+1}$ .

# GENERAL THEOREMS REGARDING THE PROBABILITY OF COMPOUND EVENTS.

§ 8.] The probability that an event, whose probability is p, happen on exactly r out of n occasions in which it is in question is  ${}_{n}C_{r}p^{n-r}$ , where q=1-p is the probability that the exent fail.

The probability that the event happen on r specified occasions and fail on the remaining n-r is by the multiplication rule  $p_0 q p q q$ . . . where there are  $r/p^c$  and  $n-r/q^c$ , that is,  $p^c q^{n-r}$ . Now the occasions are not specified; in other words, the happening, and failing, may occur in any order. There are as many ways of arranging the r happenings and n-r failings as there are permutations of n things r of which are alike and n-r alike, that is to say, n!/r!(n-r)! = s'r. There are therefore n!r mutually exclusive ways in which the event with which we are concerned may happen; and the probability of each of these is  $p^r q^{n-r}$ . Hence, by the addition rule, the probability in question is  $nC_r p^r q^{n-r}$ .

It will be observed that the probabilities that the event happen exactly  $n, n-1, \ldots, 2, 1, 0$  times respectively, are the 1st, 2nd, 3rd, ..., (n+1)th terms of the expansion of  $(p+q)^n$ .

Since, if we make n trials, the event must happen either 0,

or 1, or 2, . . . , or n times, the sum of all these probabilities o(q): to 1 - unity. This is so; for, since p+q-1,  $(p+q)^n=1$ .

It will be seen without further demonstration that the proposition just established is merely a particular case of the

following general theorem :-

If there he meants A. B. C. . . one but not more of which must happen on every wearsion, and if their probabilities he p, q, r, . . . respectively, the probability that on n occasions A happen exactly a times, B exactly \(\beta\) times, \(\Cappa\) exactly \(\gamma\) times, \(\ldots\)

$$n! p^{\alpha} q^{\beta} r^{\gamma}$$
,  $a! \beta! \gamma!$ , ...

where  $a + \beta + \gamma + \dots = n_*$ 

It should be observed that the expression just written is the general term in the expansion of the multinomial  $(n+q+r+\dots)^n$ .

Ex.: pie 1. The faces of a cubical die are marked 1, 2, 2, 4, 4, 6; required the probability that in 8 throws 1, 2, 4 turn up exactly 3, 2, 3 times respectively.

By the general theorem just stated the probability is

$$8! \frac{8!}{3! \; 2! \; 3!} \left(\frac{1}{6}\right)^3 \left(\frac{1}{3}\right)^3 \left(\frac{1}{3}\right)^3 = \frac{7 \cdot 5 \cdot 2}{3^2},$$

$$= \frac{1}{94} \text{ approximately,}$$

It imple 2. Out of n occasions in which an event of probability p is in p > m, on what number of occasions is it most likely to happen?

We have here to determine r so that  ${}_{n}C_{r}p^{r}q^{n-r}$  may be a maximum.

Now  ${}_{n}e_{r}^{r}p^{r}q^{n-r}/{}_{n}C_{r-1}p^{r-1}q^{n-r+1} - (n-r+1)p/rq_{r}$ 

Hence the probability will mercase as r increases, so long as

(n-r+1) p > rq, (n+1) p > r(p+q),

hat is r < (n+1) p

If (n+1)p be an integer, s = s by, then the event will be equally likely to the s = 1 by on s because, s and more likely to happen s = 1 or s times than by other number of times.

If (n+1)p be not an integer, and s be the greatest integer in (n+1)p, then the event is meet likely to happen on s occasions.

When note very late, in the lip differs image exhibit from np. Hence exists a very late tunder notes t, we an event is most likely to happen hap a coson. This, of course, using the fundamental principle of § 2,  $C^{(m)}(t)$ , sarrived at t a circumtous route starting from itself in the first

As a numerical instance, suppose an ordinary die is thrown 20 times, what is the number of aces most likely to appear?

Here n = 20; p = 1/6;  $(n + 1) p = 3\frac{1}{2}$ .

The most likely number of aces is therefore 3.

§ 9.] The probability that an event happen on at least r out of n occasions where it is in question is

$${}_{n}C_{r}p^{r}q^{n-r} + {}_{n}C_{r+1}p^{r+1}q^{n-r-1} + ... + {}_{n}C_{n-1}p^{n-1}q + p^{n}...$$
 (1).

For an event happens at least r times if it happen either exactly r; or exactly r+1; . . .; or exactly n times. Hence the probability that it happens at least r times is the sum of the probabilities that it happens exactly r, exactly r+1, . . ., exactly n times; and this, by § 8, gives the expression (1).

Another expression for the probability just found may be deduced as follows:—Suppose we watch the sequence of the happenings and failings in a series of different cases. After we have observed the event to have happened just r times, we may withdraw our attention and proceed to consider another case; and so on. Looking at the matter in this way, we see that the r happenings may be just made up on the rth, or on the rt 1th, . . ., or on the nth occasion.

If the r happenings have been made up in just s occasions, then the event must have happened on the sth occasion and on any r-1 of the preceding s-1 occasions. The probability of this contingency is

$$p \times_{s-1} C_{r-1} p^{r-1} q^{s-r} = {}_{s-1} C_{s-r} p^r q^{s-r}$$
.

Hence the probability that the event happen at least r times in n trials is

$$p^r + {}_rC_1p^rq + {}_{r+1}C_2p^rq^2 + \dots + {}_{n-1}C_{n-r}p^rq^{n-r}$$
  
=  $p^r\{1 + {}_rC_1q + {}_{r+1}C_2q^2 + \dots + {}_{n-1}C_{n-r}q^{n-r}\}$  (2).

As the two expressions (1) and (2) are outwardly very different, it may be well to show that they are really identical. To do this, we have to prove that  $1 + {}_{r}C_{1}q + {}_{r+1}C_{2}q^{2} + \dots + {}_{n-1}C_{n-r}q^{n-r}$ 

$$\begin{split} &=p^{n-r}\left\{1+{}_{n}C_{1}\left(\frac{q}{p}\right)+{}_{n}C_{2}\left(\frac{q}{p}\right)^{2}+\cdots+{}_{n}C_{n-r}\left(\frac{q}{p}\right)^{n-r}\right\},\\ &=\left(1-q\right)^{n-r}\left\{1+{}_{n}C_{1}\left(\frac{q}{1-q}\right)+{}_{n}C_{2}\left(\frac{q}{1-q}\right)^{2}+\cdots+{}_{n}C_{n-r}\left(\frac{q}{1-q}\right)^{n-r}\right\}\right\}. \end{split}$$

(n-r)th power of q, identical with  $(1-q)^{n-r}(1-q)^n - (1-q)^{-r}$ 

0 = 0 +ij' = 0  $\dots + \frac{1}{n-1}C_{n-r}j^{n-r} + \dots$ 

I William from I hamst be either lost or won: the p, that B gain it  $1 \cdot p = q$ ; what is the I at Besins n? (Pase d's Problem 1º

The state of the diamental at the state of t

 $i_1 + \dots + i_{m+n-1}C_m p^m q^{n-1}$ 

the question given above,

$$(2'),$$

10. The results just arrived at may be considerably Let us conder n independent events A1, A2, 1 to e reme the probabilities are p. 1/2. . . . . pn. Higher place, in contrast to S. 8, 9, let us calculate the

was a state of the property happen. and the count is that none of the n events happen. If  $p = (1-p)(1-p) \dots (1-p_n)$ . Hence the

 $1 - (1 - p - 1 - p - 1 - p - 1 - (1 - p_n) + 2 - p - p - 1 - (1 - p_n - 1) - (1 - p_n - 1)$ 

is the self after probability that one and no more of the n

I dot any particular event, say An and none  $p = (1 - p_1)(1 - p_2) \dots (1 - p_n)$ . Hence

then to . It was first solved for the I deal Toe more general result (1) above . (70 L - ther formula (2) seems to be due

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For the products two and two arise from  $-\sum p_i (p_2 + p_3 + \ldots + p_n)$ , and each pair will come in once for every letter in it. Again, the products three and three arise from  $\sum p_i (p_2 p_3 + p_2 p_4 + \ldots)$ ; hence each triad will come in once for every pair of letters that can be selected from it; and so on.

By precisely similar reasoning, we can show that the probability that r and no more of the n events happen is

$$\Sigma p_1 p_2 \dots p_r (1 - p_{r+1}) (1 - p_{r+2}) \dots (1 - p_n)$$
  
 $= \Sigma p_1 p_2 \dots p_{r-r+1} (r, \Sigma p_1 p_2 \dots p_{r+1} + r_{r+2} (r, \Sigma p_1 p_2 \dots p_{r+2} + r_{r+2} (r, \Sigma p_1 p_2 \dots p_r + r_{r+2} (r, \Sigma p_1 p_2 \dots p_r + r_{r+2} (r, \Sigma p_1 p_2 \dots p_r + r_{r+2} (r, \Sigma p_2 \dots p_r + r_r + r_{r+2} (r, \Sigma p_2 \dots p_r + r_r + r_r + r_r + r_r + r_r + r_r + r_r$ 

We can now calculate the probability that r at least out of the n events happen.

To do so we have merely to sum all the values of (3) obtained by giving r the values r, r+1, r+2, ..., n successively.

In this summation the coefficient of  $\Sigma p_1 p_2 \dots p_{r+s}$  is

$$(-)^s \{r+sC_s-r+sC_{s-1}+r+sC_{s-2}-...(-)^{s-1}r+sC_1+(-1)^s\}$$

Now the expression within the brackets is the coefficient of  $x^*$  in  $(1+x)^{r+s} \times (1+x)^{-1}$ , that is to say, in  $(1+x)^{r+s-1}$ . This coefficient is  $r_{1+s-1}C_s$ . Hence the coefficient of  $\Sigma p_1 p_2 \dots p_{r+s}$  is  $(-)^r p_{r+s-1}C_s$ .

The probability that r at least out of the n events happen is therefore

$$\sum p_r p_2 \dots p_r - C_1 \sum p_r p_2 \dots p_{r+1} + r_{r+1} C_2 \sum p_r p_2 \dots p_{r+2}$$

$$(-)^a_{r+s-1} C_3 \sum p_r p_2 \dots p_{r+s}$$

$$(-)^{n-r_{n-1}} C_{n-r} p_r p_2 \dots p_n \qquad (4).$$

Since the happening of the same event on n different occasions may be regarded as the happening of n different events whose

probabilities are all equal, the formula (3) and (4) above ought, when  $p_1, p_2, \ldots, p_n$  each  $-p_n$  to reduce to  ${}_n\ell^*rp^rq^{n-r}$  and the expression (4) or (2) of § 9 respectively.

If the reader observe that, when  $p_1 - p_2 = \dots = p_n = p$ ,  $\Sigma p, p_1, \dots, p_r = e^r, p^r$ , &c., he will have no difficulty in showing that (3) is actually identical with  ${}_n C_r p^r q^{n-r}$  in the particular case in question.

The particular result derived from (4) is more interesting. We find, for the probability that an event of probability p will happen r times at least out of n occasions, the expression

$${}_{n}C_{r}p^{r} - {}_{r}C_{1:n}C_{r+1}p^{r+1} + \dots + (-)^{s}_{r+s-1}C_{s:n}C_{r+s}p^{r-s} + \dots + (-)^{s-r}_{n-1}C_{n-r}p^{n}$$
 (5).

Here we have yet another expression equivalent to (1) and (2) of § 9. It is not very difficult to transform either of the two expressions of § 9 into the one now found; the details may be left to the reader.

Example. The probabilities of three independent events are  $p,\ q,\ r;$  required the probability of happening—

1st. Of one of the events but not more:

2nd. Of two but not more;

8rd. Of one at least;

4th. Of two at least;

5th. Of one at most; 6th. Of two at most.

The re ults are as follows :--

1-1.  $p + q + r - 2(p_0 + pr + qr) + 3p_0r$ :

2n1. pq + pr + qr 3pqr;

3:3. j+q+r = (pq+pr+qr) + pqr

4th. pq + pr + qr - 2pqr;

5th. 1 - (pq + pr + qr) + 2pqr;

6th, I pyr.

To first four are particular cases of preceding formula; 5 is complementary to 4; and 6 is complementary to "of all three."

§ 11. The Recurrence or Finite Difference Method for solving problems in the theory of probability possesses great historical and practical interest, on account of the use that has been made of it in the solution of some of the most difficult questions in the subject. The sprit of the method may be explained thus. Suppose, for simplicity, that the required probability is a function of one variable x; and let us denote it by  $u_x$ . Reasoning from the data of the problem, we deduce a relation connecting the values of  $u_x$  for a number of successive values of x; say the relation

$$f(u_{x+2}, u_{x+1}, u_x) = 0$$
 (A),

We then discuss the analytical problem of finding a function  $u_{\varepsilon}$  which will satisfy the equation  $(\Lambda)$ .

It is not by any means nece-sary to solve the equation ( $\Lambda$ ) completely. Since we know that our problem is definite, all that we require is a form for  $n_x$  which will satisfy ( $\Lambda$ ) and at the same time agree with the conditions of the problem in certain particular cases. The following examples will sufficiently illustrate the method from an elementary noint of view.

Example 1. A and B play a game in which the probabilities that A and B win are a and B respectively, and the probability that the game be drawn is  $\gamma$ . To start with, A has m and B has n counters. Each time the game is won the winner takes a counter from the loser. If A and B acree to play until one of them loses all his counters, find their respective chances of winning in the end  $^*$ .

Let  $u_x$  and  $v_x$  denote the chances that A and B win in the end when each has zounters. If we put m+n=p, the respective chances at any stage of the game are  $u_x$  and  $v_{p-x}$ .

Consider A's chance when he has x+1 counters. The next round he may, 1st, win; 2nd, lose; 3rd, draw the game. The chances of his ultimately winning on these hypotheses are  $au_{x+2}$ ;  $\beta u_x$ ;  $\gamma u_{x+1}$  respectively. Hence, by the addition rule,

 $u_{x+1} = \alpha u_{x+2} + \beta u_x + \gamma u_{x+1}.$ 

If we notice that  $\alpha+\beta+\gamma=1$  (for the game must be either won, lost, or drawn), we deduce from the equation just written

$$au_{x+2} - (a + \beta) u_{x+1} + \beta u_x = 0$$
 (I).

It is obvious that  $u_x = A\lambda^x$ , where A and  $\lambda$  are constants, will be a solution of (1), provided  $a\lambda^2 - (a + \beta)\lambda + \beta = 0$  (2),

that is, provided  $\lambda = 1$  or  $\lambda = \beta/a$ . Hence  $u_x = A$  and  $u_x = B (\beta/a)^x$  are both solutions of (1); and it is further obvious that  $u_x = A + B (\beta/a)^x$  is a solution

of (1).

We have now the means of solving our problem, for it is clear from (1) that, if we knew two particular values of uz, say uo and u, then all other

<sup>\*</sup> First proposed by Huyghens in a particular case; and solved by James Bernoulli. See Todhunter, Hist. Prob., p. 61.

where the solution of the sol

whence  $A = e^{i} + e^{i} - B^{i}$ ,  $B = a^{i} + B^{i}$ .

Wet or fire has

 $n_x = \alpha^{p-x} \left( \alpha^x - \beta^x \right)' \left( \alpha^p - \beta^p \right)$  :

and, in id. in the  $v_x = J^{-x}\{a^x : J^{-x}\}$ .

The characteristic beams of the same are given by  $u_m = \gamma^n (a^m - \beta^m)^n (a^p - \beta^p),$ 

 $v_n = e^n (\alpha^n + \rho + \alpha^n + \rho^p)$ .  $C(x, A) = If \alpha = \beta$ , then  $s(x) \in \mathbb{N}$  ap.  $\chi_{XY}$ , § 12)

 $n_m = n_c p_c - \mathbf{r}_n = n^t p_s$ 

It must be supposed that when the kill of the placers is unequal this could be compened of by a disparity of consteas. There is, however, a femal, as the following 1, quantum will show, —

Car. 2. The air stal a sty of a a ters cannot reduce the odds in A's

Hence we see that, if A be two as shalful as B (a = 2S), we cannot by copying a group of a and  $a = \{s \mid b \text{ in } a \text{ so } g \text{ which any at all}\}$  make the olds in bit favour less than even.

Law pie 2. A pack of n different earls is hid face downwards. A pit hive a nich, individend and all above it are removed and showed to fit. The time and in r, and so on, until none are left. Required the characteristic transports in he names the top card once at least t.

Let up be the convex of steero har when there are n early; so that  $u_{n+1}$  is that  $u_n \in A$  so  $u_n$  in which there are n-1; and so on. At the first trial this payer is many  $e^{i(t)} = 1$ ; and,  $3, 1, \dots, n$  then the card, the charge of control 1 = 0 and 1 = 1. Now has chances of ultimately store effective M = n + 1, n

 $u_n = 1$   $v_{-1} / n + n$   $n + \dots + u_n / n + u_1 / n$ .

 $n_1 = 1 + n_1 + \dots + n_{n-1}$  (1).

† I . . . I . . . . . . . . I d. I am s, vol. xi ii., p. 63,

<sup>\* 10</sup> s per  $\epsilon$  - for a centre  $\epsilon$  - replace 1 by the considerations of the  $\rho$ 

From (1) we deduce

$$(n-1)u_{n-1} = 1 + v_1 + u_2 + ... + u_{n-3}$$
 (2),

From (1) and (2)

$$nu_n - (n-1)u_{n-1} = u_{n-2}$$

that is,

$$n(u_n - u_{n-1}) = -(u_{n-1} - u_{n-1})$$
 (3),

Hence

$$\begin{aligned} &(n-1)\left(u_{n-1}-u_{n-2}\right) = -\left(u_{n-2}-u_{n-2}\right),\\ &(n-2)\left(u_{n-2}-u_{n-3}\right) = -\left(u_{n-3}-v_{n-1}\right),\\ &3\left(u_{3}-u_{2}\right) = -\left(u_{2}-u_{1}\right). \end{aligned}$$

Hence, multiplying together the last n-2 equations, we deduce  $\frac{1}{2}u!(u_n-u_{n-1})=(-1)^{n-2}(u_n-u_n)$ .

Since  $u_1 = 1$ ,  $u_2 = \frac{1}{2}$ , this gives

bridge 
$$u_1 = 1$$
,  $u_2 = \frac{1}{2}$ , thus gives
$$u_n - u_{n-1} = (-1)^{n-1}/n! \tag{4}.$$

Hence, again,

$$u_{n-1} - u_{n-2} = (-1)^{n-2}/(n-1)!,$$
  
 $u_2 - u_1 = (-1)^1/2!,$   
 $u_3 - 0 = 1.$ 

From the last n equations we derive, by a ldition,

 $u_n = 1 - 1/2! + 1/3! - \dots + (-1)^{n-1}/n!$  (5).

Introducing the sub-factorial notation of chap, xxiii., § 18, we may write the result obtained in (5) in the form  $u_n = 1 - n_1^*/n!$ .

From Whitworth's Table \* we see that the chance when n=8 is 632119. When  $n=\infty$  the chance is  $1-1/\epsilon=632121$ ; so that the chance does not diminish greatly after the number of eards reaches 8.

# EVALUATION OF PROBABILITIES WHERE FACTORIALS OF LARGE NUMBERS ARE INVOLVED.

§ 12.] In many cases, as has been seen, the calculation of probabilities depends on the evaluation of factorial functions. When the numbers involved are large, this evaluation, if pursued directly, would lead to calculations of enormous length \(\text{\text{\text{fine}}}\), and the greater part of this labour would be utterly wasted, since all that is required is usually the first few significant figures of the probability. The difficulty which thus arises is evaded by the use of Stirling's Theorem regarding the approximate value of \(\text{\text{\text{s}}}\).

<sup>.</sup> Choice and Chance, chap, Iv.

t In some cases the process of chap. X: xv., § 11, Lxamples 2 and 3 is useful.

when x is large. In its modern form this theorem may be stated thus--

$$x' = \sqrt{(2\pi x)} x' e^{-x} \left( 1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots \right)$$

(see chap. xxx., § 17).

From this it appears that, if x be a large number, x! may be replaced by  $\sqrt{(2\pi x)} x^x e^{-x}$ , the error thereby committed being of the order 1.12xth of the value of x!.

As an example of the use of Stiftin is Theorem, let us consider the followare problem as A pack of 4n early consists of 4 suits, each consisting of n early. The problem should hand do it out to four players; required the chare that the whole of a particular uit fails to one particular player. The charee in question it cashly found to be waven by

p = (3n)! n!/(4r)!

Hence, by Stirling's Theorem, we have

$$p = \frac{\sqrt{(2\pi \, 3n)\, (3n)^{3n} e^{-3n}} \sqrt{(2\pi n)\, n^n e^{-n}}}{\sqrt{(2\pi \, 1n)\, (4n)^{4n}}, \quad ^{4n}},$$

the error being comparable with 1/11nth of p. Hence, approximately,  $p = \sqrt{(3\pi n/2)(27/256)^n}.$ 

Example. Let In -52, n 13, then

$$n = 1.03 \times 3/1416 \times 13/21/27/256$$

This can be readily evaluated by means of a table of logarithms. We find

 $p = 156/10^{14}$ .

The event in  $\mu r$  -form is therefore not one that would occur often in the experience of one index 1 al.

### Exercises VXXIX

(1.) A starts at half pet one to walk up Princes Strict; what is the present virtual to ment I, who may have started to walk down any time returned mend two leaving Conference in that it takes A 12 minutes to walk up, and E for more to well down.

(2) A base into us 5 what a 4 red, and 5 black balls. Three balls are observed as (100) pc (1.1) yr (1.2) that all three colours: 2nd, that only showed us, i.e., i.e., there is one of (0.1) m, may be a pre-ented.

(\*) A two entring or white and will, "A balls," One is drawn and then a second it; what is the observe of him in our beast one white-list, when the first look is replaced; the drawn it is not replaced.

(i) If n possible most by che six whit is the probability that they all have the sum hathalay, approprietely anth year to be a leap year?

Good consider the post at random on a cliese board, what

- (6.) Three dice are thrown; show that the cast is most likely to be 10 or 11 the probability of each being b.
- (7.) There are three bags, the first of which contains 1, 2, 1 counters, marked 1, 2, 3 respectively; the second 1, 4, 6, 4, 1, marked 1, 2, 3, 4, 5 respectively; the third 1, 6, 15, 20, marked 1, 2, 3, 4 respectively. A counter is drawn from each bag; what is the probability of drawing 6 exactly, and of drawing some number not exceeding 6?
- (8.) Six men are bracketed in an examination, the extreme difference of their marks being 6. Find the chance that their marks are all different.
- (9.) From 2n tickets marked 0, 1, 2, . . . , (2n-1), 2 are drawn; find the probability that the sum of the numbers is 2n.
- (10.) A pack of 4 suits of 13 cards each is dealt to 4 players. Find the chance—1st, that a particular player has no card of a named suit; 2nd, that there is one suit of which he has no card. Show that the odds against the dealer having all the 13 trumps is 158,753,359,899 to 1.
- (11.) If I set down any r-permutation of n letters, what is the chance that two assigned letters be adjacent?
- (12) There are 3 tickets in a bag, marked 1, 2, 3. A ticket is drawn and replaced four times in succession; show that it is 41 to 40 that the sum of the numbers drawn is even.
  - (13.) What is the most likely throw with n dice, when n > 6?
- (14.) Out of a pack of n cards a card is drawn and replaced. The operation is repeated until a card has been drawn twice. On an average how many drawings will there be?
- (15.) Ten different numbers, each > 100, are selected at random and multiplied together; find the chance that the product is divisible by 2. 3, 4, 5, 6, 7, 8, 9, 10 respectively.
- (16.) A undertakes to throw at least one six in a single throw with side; B in the same way to throw at least two six s with twelve dice; and C to throw at least three sixes with eighteen dice. Which has the best chance of succeeding? (Solved by Newton; see Pryps' Dury and Correspondence, ed. by Mynors Bright, vol. v., p. 179.)
- (17) A pitcher is to be taken to the well every day for 4 years. If the odds be 1000: 1 against its being broken on any particular day, show that the chance of its ultimately surviving is rather less than ‡.
- (18.) Five men toss a coin in order till one wins by tossing head; calculate their respective chances of winning.
- (19) A and B, of equal skill, agree to play till one is 5 games ahead. Calculate their respective chances of winning at any stage, supposing that the game cannot be drawn. (Pascal and Fernat.)
- (20.) What are the odds against throwing 7 twice at least in 3 throws with 2 dice?
- (21.) Show that the chance of throwing doublets with 2 dice, 1 of which is loaded and the other true, is the same as if both were true,

- 22.1 A and B that if a larke; A' die is marked 10, 13, 16, 20, 21, 25, and 7, 5, 10, 15, 20, 5, 20. The 1 does throw is to win and equal throws.
- $(2.4) \Delta [past d/2msy (1s, nred, nbb] (1s, i) divided at random into 2 equal part and a could introduce the assist find the chance that the 2 drawn are of the time obtain and counce with the chance of drawing 2 of the same count from the unity <math>l(1)$  at s.
- (21) It is airds, no solar damed sets of m, are distributed into m stacks of 4 c ob face up; find the chance that in no stack is a higher one of any set above one we'r a lower number in the some et.
- (25.) Out of m men in a ring 3 are selected at random; show that the choice that no 2 of them are neighbours is

-(n-1)(m-5) + m - 1 + (m-2)

(2) If m thin is no given to a men and b women, prove that the chance that the northbor is reveal by the group of men is odd is

$$\{\S(b+a)^{-1} - \S(b+a)^{-1}\} (b+a)^{-1}$$
.
(Math. Trip., 18-1.)

(27) A and I + ch tab 12 counters and play with 3 dice on this condition, II if II is thrown I is so counter to B<sub>i</sub> and if II is thrown B gives no counter to II, and he was the game who first chains all the counters. Show that I's chance is to II in s

244.140.625 : 282.129.536.481.

(11) hars. So Ioth., Hist. Prob., p. 25.)

(28) A and L (d), with 2 dec; if 7 is threwn A wins, if 10 R wins, if a vector is the model from Stow that I's chance of winning stretches 1, 11 (11) mass. Sec. The A B A Pet B 124.

net 1 - romp to have a ptime at epul distances of a yard apart. A mane but 1900 pty and was to any other; prove that the aver of ones the his evolutions can't doing this at random a great

 $(\beta-1)$  ,  $(\beta-1)$  is a sum of faces in n casts with a p+1 -raced draws

$$\int_{\mathbb{T}^{n}} ((p-1)^{n} - \frac{f}{1})^{n} = \frac{f}{1} e^{n} + \frac{f}{1} e^{n} = \frac{1}{1} e^{n} + \frac{$$

(D. m. ivis , Downste of Chances.)

(2+H) and in the correction bar and drawn out successively, the  $(1+e^{-it})$  and  $(1+e^{-it})$  and  $(1+e^{-it})$  and  $(1+e^{-it})$  and its number indicates

(1) 
$$I = I = I = I$$
 (1)  $I = I = I = I$  (1)  $I = I = I = I$ 

(33.) A and B play a game in which their respective chances of winning are a and  $\beta$ . They start with a given number of counters p divided between them; each rives up one to the other when he loses; and they play till one is rained. Show that inequality of counters can be made to compensate for inequality of skill, provided  $a/\beta$  is less than the positive root of the equation  $x^p - 2x^{p-1} + 1 = 0$ . If p be large, show that, to a second approximation, this root is  $2 - \frac{1}{2p-1} - \frac{p-1}{2^2p-1}$ ,

#### MATHEMATICAL MEASURE OF THE VALUE OF AN EXPECTATION.

§ 13.] If a man were asked what he would pay for the privilege of tossing a halfpenny once and no more, with the understanding that he is to receive £50 if the coin turn up head, and nothing if it turn up tail, he might give various estimates, according as his nature were more or less sanguine, of what is sometimes called the value of his expectation of the sum of £50.

It is obvious, however, that in the case where only one trial is to be allowed the expectation has in reality no definite value whatever—the player may get £50 or he may get nothing;

and no more can be said.

If, however, the player be allowed to repeat the game a large number of times on condition of paying the same sum each time for his privilege, then it will be seen that £25 is an equitable payment to request from the player; for it is assumed that the game is to be so conducted that, in the long run, the coin will turn up heads and tails equally often; that is to say, that in a very large number of games the player will win about as often as he loses. With the above understanding, we may speak of £25 as the value of the player's expectation of £50; and it will be observed that the value of the expectation is the sun expected multiplied by the probability of getting it.

This idea of the value of an expectation may be more fully singularized by the case of a lottery. Let us suppose that there are prizes of the value of  $\mathcal{L}a, \mathcal{L}b, \mathcal{L}c_r, \dots$ , the respective probabilities of obtaining which by means of a single ticket are  $p, q, r, \dots$ . If the lottery were held a large number N of times, the holder of a single ticket would get  $\mathcal{L}a$  on pN

coessions  $\mathfrak{L}h$  on qN occasions  $\mathfrak{L}r$  on rN occasions, . . . Hence the hidder of a single ticket in each of the N lotteries would get  $\mathfrak{L}(pNt+qNb+rNr+\ldots)$ . If, therefore, he is to pay the same price  $\mathfrak{L}t$  for his ticket each time, we ought to have, for equity,

$$Nt - pNa + qNh + rNc + \dots$$

that is

$$t = pa + qb + rc + \dots$$

Hence the price of his ticket is made up of parts corresponding to the various prices, namely, pa, qb, rc, . . . These parts are called the cutus of the expectations of the respective prices; and we have the rule that the value of the expectation of a sum of money is that sum multiplied by the chance of pitting it.

The student must however, remember the understanding upon which this definition has been based. It would have no meaning if the lottery were to be held once for all.

Example. A player throws a six-faced die, and is to receive 20s, if he throws are the first throw; half that sum if he throws are the second throw; quarter that sum if he throws are the throw; and so on. Required the value of he extendation.

The player may get 20, 20/2, 20/22, 20/22, ... shillings. His chances of pettin: these sums are 1/6, 5/6, 5/6, 5/6, 5/6, ... Hence the respective values of the core s) adding parts of his expectation are 20/6, 20, 5/6, 2, 20, 5/6, 2, 20, 5/6, 2, 20, 5/6, 2, 20, 5/6, 2, ... shillings. The whole value of his expectation is then fore

$$\frac{20}{6} \left\{ 1 + \frac{5}{12} + \left( \frac{5}{12} \right)^2 + \left( \frac{5}{12} \right)^3 + \dots, \text{ ad } \alpha \right\} = \frac{20}{6} \left\{ \left( 1 - \frac{5}{12} \right) = \frac{40}{7} \text{ shillings,} \right\}$$
The transfer of  $I$ 

§ 11 | It is important to notice that the rule which directs us to add the component parts of an expectation applies whether the separate contingencies be mutually exclusive or not. Thus, if  $p_1, p_2, p_3, \ldots$  be the whole probabilities of obtaining the separate sums  $a_1, a_2, a_3, \ldots$ , then the value of the expectation is  $p_1a_1, p_2a_2, p_3a_3, \ldots$ , even if the expectant may get more than one of the semi-in-question. Observe, however, that  $p_1$  must be the whole probability of getting  $a_1$ , that is, the probability of getting the sum  $a_1$  irrespective of getting or railing to get the other sums

If the expectant may get any number of the sums a1, a2,

...,  $a_n$ , we might calculate his expectation by dividing it into the following mutually exclusive contingencies:- $a_1, a_2, \ldots, a_n$ ;  $a_1 + a_2, a_1 + a_2, \delta c$ ;  $a_1 + a_2 + a_2, \delta c$ ; ...;  $a_1 + a_2 + \ldots + a_n$ . Hence the value of his expectation is

$$\Sigma a_1 p_1 (1 - p_2) (1 - p_3) \dots (1 - p_n)$$
  
 $+ \Sigma (a_1 + a_2) p_1 p_2 (1 - p_3) \dots (1 - p_n)$   
 $+ \Sigma (a_1 + a_2 + a_3) p_4 p_2 p_2 (1 - p_4) \dots (1 - p_n)$   
 $+ (a_1 + a_2 + \dots + a_n) p_4 p_2 p_2 \dots p_n$ 

By the general principle above enunciated the value in question is also \(\Sigma\_{alph}\). The comparison of the values gives a curious algebraic identity, which the student may verify either in general or in particular cases.

Example. A man may get one or other or both of the sums a and b. The chance of getting a is p, and of getting b is q. Required the value of his expectation.

He may get a alone, or b alone, or a+b; and the respective chances are p(1-q), q(1-p), qa. Hence the value of his expectation is ap(1-q) + bq(1-p) + (a+b) pq, which reduces to ap+bq, as it ought to do by the general principle.

N.B.—If the man were to get one or other, but not both of the sums a and b, and his respective chances were p and q, the value of his expectation would still be ap + bq; but p and q would no longer have the same meanings as in last case.

#### LIFE CONTINGENCIES.

§ 15.] The best example of the mathematical theory of the value of expectations is to be found in the valuation of benefits which are contingent upon the duration or termination of one or more human lives. The data required for such calculations are mainly of two kinds—1st, knowledge, or forecast as accurate as may be, of the interest likely to be yielded by investment of capital on good and easily convertible security; 2nd, statistics regarding the average duration of human life, usually embodied in what are called Mortality Tables.

The table printed below illustrates the arrangement of mortality statistics most commonly used in the calculation of life contingencies:—

me and make and the Tool to be following

97	nber II. Z	ta-nt Cx	Age	Living:	Dr. ce troid d <sub>a</sub>	Ag-	Living	ment.
3"	12		-	12	3		- '4	
10	1 0, 100	4 10	40	82.281	549	70	38,124	2371
11	99,510		-11	81.4.0	851	71	35,753	24.7
12	99,113	3.9	42	81,552	865	72	33,320	2497
13	94,741	253	4.3	79,717	8 - 7	73	30,523	2551
11	98,456	271	41	78,830	911	7.1	28,269	2578
15	94,224	2-2	45	77,919	\$650	75	25,691	
16	97,742	314	46	76,949	9.16	70	23,164	2464
17	97,621	37.4	47	75.973	1011	77	20,70 (	2374
13	97.215	4/16	4 <	71,922	10 2	7 4	18,326	2259
19	96,773	556	42	71,870	1101	7.0	16,065	2139
20	96 223	6:0	feet	72,726	1100	5-11	13,930	2 45
21	95,614	643	54	71,500	1193	51	11,915	1853
22	94.971	650	1.2	70,103	12:5	2.2	19,000	1719
23	94,751	6118		61,138	1256	53	8,313	1545
24	90,653	6.22	5.5	67,552	10.3	51	6,768	1046
	90, 401	617		464,513	1399	>5	5,422	1135
26	92,111	618	5.1	65.114	1462	80	4,251	941
27	91,358	6.13	57	61,002	1527	87	1.113	773
24	91,192	654	5.4	62.155	1592	5.9	2,570	615
29	194,533%	673	5.9	60,533	1667	2.0	1,905	495
30	89,865	691	60	58,866	1747	501	1,460	401
31	89,171	7:113	-61	57,119	180	1/1	1,052	329
32	55,475	717	62	55,250	1915	9.2	723	25
33	87,748	727	63	53,374	2001	103	469	198
21	\$7,021	7.50	6.1	51,873	2076	1/4	274	139
35	86,2-1	757	65	49,197	1111	95	135	86
36	85.5.1	779	- 66	47,156	21:3	116	49	40
: 7	81.715	502	67	43,960	2243	97	9	1 1
19	80,943	821	68	42,717	2274	95	0	
.9	8 122	80%	69	40,443	119		-	

In the first column are entered the ages 10, 11, 12, . . . . Oppe ite 10 is entered an arbitrary number 100 000 of children that reach their tenth birthday; opposite 11 the number of these that reach their deve ath birthday; apposite 12 the number that reach their twelfth birthday; and so on. We shall denote these numbers by  $l_{tr}, l_{tr}, l_{tr}, \dots$ . In a third column are entered the differences or "decrements," of the numbers in the second column; these we hall denote by  $d_{tr}, d_{tr}, l_{tr}, \dots$ . It is obvious that  $d_{tr}$  gives the number out of the 100,000 that die between their xth and x+ 10 birthdays. It is impossible here to discuss the methods employed in con tracting a table of mortality, or

to indicate the limits of its use; we merely remark that in applying it in any calculation the assumption made is that the lives dealt with will fall according to the law indicated by the numbers in the table. This law, which we may call the Law of Mortality, is of course only imperfectly indicated by the table itself; for although we are told that  $d_x$  die between the ages of x and x+1, we are not told how these deaths are distributed throughout the intervening year. For rough purposes it is sufficient to assume that the distribution of deaths throughout each year is uniform; although the variation of the decrements from one part of the table to another shows that uniform decrease\* is by no means the general law of mortality.

§ 16.] By means of a Mortality Table a great many interesting problems regarding the duration of life may be solved which do not involve the consideration of money. The following are examples.

Example 1. By the probable duration n of the life of a man of m years of age is meant the number of years which he has an even chance of adding to his life. To find this number.

By hypothesis we have  $l_{m+n}ll_{m}=1/2$ . Hence  $l_{m+n}=l_{m}/2$ ,  $l_{m}/2$  will in general lie between two numbers in the table, say  $l_{p}$  and  $l_{p+1}$ . Hence m+n must lie between p and p+1. We can get a closer approximation by the rule of proportional parts (see chap. xxt, § 13).

Example 2. To find the "mean duration" or "expectancy of life" for a man of m years of age.

By this is meant the average N (arithmetical mean) of the number of additional years of life enjoyed by all men of m years of age.

Let us take as specimen lives the  $l_m$  men of the table who pass their mth birthday; suppose them all living at a particular epoch; and trace their lives till they all die.

In the first year  $l_{m^{-}}l_{m+1}$  die. If we suppose these deaths to be equally distributed through the year, as many of the  $l_{m^{-}}l_{m+1}$  will live any assigned amount over half a year as will live by the same amount under half a year. Hence the  $l_{m^{-}}l_{m+1}$  lives that have failed will contribute  $\frac{1}{2}(l_{m^{-}}l_{m+1})$  years to the united life of the  $l_{m}$  specimen lives. Again, each of the  $l_{m+1}$  who live through the year will contribute one year to the united life. Hence the whole contribution to the united life during the first year is  $\frac{1}{2}(m_{m^{-}}l_{m+1}) + l_{m+1} = \frac{1}{2}(l_{m^{+}}l_{m+1})$ . Similarly, the contribution during the second year is  $\frac{1}{2}(l_{m+1}l_{m+1})$ . Hence the united life is

$$\frac{1}{2}(l_m + l_{m+1}) + \frac{1}{2}(l_{m+1} + l_{m+2}) + \dots = \frac{1}{2}l_m + l_{m+1} + l_{m+2} + \dots$$
 (1),

<sup>.</sup> Demoivre's hypothesis.

the control of the numbers in the table has any significant

If we note that the said his by the number of original lives, we find for the

$$N = \frac{1}{2} + (l_{m+1} + l_{m+2} + ...)/l_m$$
(2)

Owing to our . In restriction the notiform distribution of deaths over the interval between the tabular epoch, it is expression is of course merely an approximation.

Example 3. A and B, where a we are a and b respectively, are both living at a partial or epoch; find the claimer that A survive B.

The corp and event whose clonce is required may be divided into mutually colusive contingencies as to Hower-

and con.

1. 1st containing vir. y be a can divide limto two:—

(c) A and B may both do within the year, B dy ng fir t;

(5) It may die with it the year, and if live beyond the year,

The basis the A and B both die within the first year is  $(l_0 - l_{n+1})$   $A_{n+1} = [l_n]$ . Since the beaths are equally distributed through the year, if  $A_{n+2n} = [l_n]$  and the diament less are one so a likely to survive as the other latter the of an of A satisfaint B on the present hypothesis is  $\frac{1}{2}$ . The channel of the confinency  $(l_0)$  is the order  $(l_0 - l_{n+1})[l_0 - l_{n+1}][2l_0 l_0]$ . The channel of an observal  $v_{n+1}$   $(l_n - l_{n+1})[l_n - l_{n+1}][2l_n l_0]$ .

If  $t = t_1 + t_2 + t_3 + t_4 + t_4 = t_4 + t_4 + t_5 + t_6 + t_6$ 

It was the continue that the chance of the 2nd contingency is

 $10 = \text{the } a = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $10 = \text{the } a = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $11 = \text{the } a = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $12 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $13 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $14 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $15 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $16 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $17 = \frac{(1 + 2) x_0}{(1 + 1)^2}$   $18 = \frac{(1 + 2) x_0}{(1 + 1)^$ 

$$|\cdot|_{0} = \{(l_{1} + l_{1}, + l_{3} + l_{4}) + (-1, 1 + l_{4}, 1)(l_{b+1} - l_{b+2}) + \dots \}/2l_{a}l_{b}$$
 (1).

The last excitations of a last in seeins that (1) may be written in the

$$S_{ab} = \frac{1}{4} + \left(\sum_{r=1}^{r=2} l_{1+r}(l_{-r-1} - l_{b+r+1}) - l_a l_{b+1}\right) + 2l_a l_b$$
 (2),

Let  $\tau$  be the proof stars in the tille for which a significant value of  $t_{\rm c}$  is then

If we have by  $S_{b,a}$  the chance that B survive A, we have, of course,

If 
$$s = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \right) \left( \frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2} \right) = \frac{1}{2}$$

17. Let now consider the following money problem in the section of What should are Insurance Office ask for their theory try or an annew gref 21 to a man of mycars of age. the first payment to be made n + 1 years hence\*, the second n + 2 years hence; and so on, for t wars, if the annuitant live so long.

We suppose that the office makes no charges for the use of the shareholders' capital, for management, and for "margin" to cover the uncertainty of the data of even the best tables of mortality. Allowances on this head are not matters of pure calculation, and differ in different offices, as is well known. We suppose also that the rate of interest on the invested funds of the office is  $\pounds i$  per  $\pounds 1$ , so that the present value, v, of  $\pounds 1$  due one year hence is  $\pounds 1/(1+i)$ . The solution of the problem is then a mere matter of average accounting.

Let  $n|t^{\alpha_m}$  denote the present value of the annuity; and let us suppose that the office sells an amunity of the kind in question to every one of  $l_m$  men of m years of age supposed to be all living at the present date.

The office receives at once  $n_1 \epsilon a_m l_m$  pounds. On the other hand, it will be called upon to pay

$$\mathfrak{L}l_{m+n+1},$$
  $\mathfrak{L}l_{m+n+2},$  . . . ,  $\mathfrak{L}l_{m+n+t},$   $n+1,$   $n+2,$  . . . ,  $n+t$ 

years hence respectively. Reducing all these sums to present value, and balancing outgoings and incomings on account of the  $l_m$  lives, we have, by chap. XXII.,  $\S$  3,

$$a_{n|t}a_{m}l_{m} = v^{n+1}l_{m+n+1} + v^{n+2}l_{m+n+2} + \dots + v^{n+t}l_{m+n+t}$$

Hence

expectation.

$$n_1 t a_m = (v^{n+1} l_{m+n+1} + v^{n+2} l_{m+n+2} + \dots + v^{n+t} l_{m+n+l})/l_{\dots},$$
  
 $= v^n \sum_{r=1}^{n-t} l_{m+n+r} v^r / l_m$  (1).  
The same result might be arrived at by using the theory of

<sup>•</sup> This is what is meant by saying that the annuity begins to run n years

<sup>†</sup> The annity need not necessarily be sold to the person ("nomince") on whose life it is to depend. The life of the nominee merely concerns the definition of the "status" of the annuity, that is, the conditions under which it is to last.

The arguity whose value we have just calculated would be technically described as a deferred temperary annuity.

If the annuity be an immediate temporary annuity, that is, if it commones to run at once, and continue for t years provided the nominee live so long, we must put n=0. Then, using the actuarial notation, we have

$$_{t}u_{n} = \sum_{n=1}^{r-t} l_{m+r}v^{r}/l_{m}$$
 (2).

If the annuity be complete, that is, if it is to run during the  $w^{\dagger}_{0}$  de life of the nominee, the summation must be continued as long as the terms of the series have any significant value; this we may be dicate by putting  $t = \infty$ . Then, according as the annuity is or is not deferred, we have

$$a = e^{a} \sum_{r=1}^{r-x} l_{r-n+r} e^{r/t}, \qquad (3)$$

$$\sigma = \sum_{r=1}^{r=\infty} I_{r-r} v^r I_m \tag{4}.$$

[48.] The function  $a_m$ , which gives the value of an importance on the annuity of a life of a years, is of fundamental injections in the cast lation of contingencies which depend on a short the. Its values have been deduced from various tables of near disc, and takehate b. By means of such tables we can readily solve a variety of problems. Thus, for example,  $a_m a_m = a_{m+1}$ , can all be found from the annuity tables; for we have

$$_{\eta}$$
  $\sigma$  ,  $\tau$   $t_{reg}$   $\sigma$  ...,  $\tau$  , (5);

$${}_{t}a_{ij} - a_{ij} = \tau^{i} I_{ij} a_{ij} I_{ij}$$

$$(6);$$

$$_{n,t}a = (r^n I_{-n} a_{m+n} - e^{n+t} I_{m+n+t} a_{m+n+t}) I_m = (7);$$

as the rest range easily verify by means of formulae (1) to (4).

These results may also be readily established a priori by means of the theory of expectation.

[19] Let us next find a<sub>k,n</sub> the present value of an immediate or it to among quat \(\Lambda\) I on the joint lives of two nominess of 2 and monetrs of a prespectively.

The understanding here is that the annuity is to be paid so

long as both nominees are living and to cease when either of them dies.

The present values of the expectations of the 1st, 2nd, 3rd, . . . instalments are

 $vl_{k+1}l_{m+1}/l_k l_m$ ,  $v^2l_{k+2}l_{m+2}/l_k l_m$ ,  $v^2l_{k+3}l_{m+3}/l_k l_m$ , &c., . . . . . . . . . . . . . . . . .

$$a_{k,m} = (rl_{k+1}l_{m+1} + v^2l_{k+2}l_{m+2} + \dots) l_n l_m,$$

$$= \sum_{k=1}^{r-x} v^r l_{k+r}l_{m+r}/l_k l_m$$
 (1).

Just as in § 18, we obviously have

$$\begin{array}{ll}_{n} \alpha_{k,m} = v^{n} \alpha_{k+n,m+n} l_{k+n} l_{m+n} / l_{k} l_{m}; \\ _{1t} \alpha_{k,m} = \alpha_{k,m} - v^{t} \alpha_{k+t,m+t} l_{k+t} l_{m+t} / l_{k} l_{m}; \\ _{nlt} \alpha_{k,m} = (v^{n} \alpha_{k+n,m+n} l_{k+n} l_{m+n}) \end{array}$$

 $-e^{n-t}u_{t+n+1,m+n+t}l_{t+n+t}l_{u+n+t}/l_t l_n + n + t/l_t l_n + t/l_t l_$ 

number of nominees.

Tables for  $a_{k,m}$  have been calculated; and, by combining them with tables for  $a_{m,n}$  a large number of problems can be solved.

Example 1. To find the present value of an immediate annuity on the last survivor of two lives m and n, usually denoted by  $a_{\overline{m},\overline{n}}$ .

Let  $p_r$ ,  $q_r$  be the probabilities that the nominees are living r years after the present date; then the probability that one at least is living r years hereafter is  $p_r + q_r - p_r q_r$ .

Hence

$$\begin{split} a_{\overline{m,n}} &= \sum_{1}^{\infty} v^r (p_r + q_r - p_r q_r), \\ &= \sum v^r p_r + \sum v^r q_r - \sum v^r p_r q_r, \end{split}$$

 $=a_m+a_n-a_{m-n}$ . This is also obvious from the consideration that, if we paid an annuity on each of the lives, we should pay £1 too much for every year that both lives were in existence.

Example 2. Find the present value  $a_{\overline{k},\overline{m},\overline{n}}$  of an annuity to be paid so long as any one of three nominees shall be alive, the respective ages being k, m, n.

If  $p_s$ ,  $q_s$ ,  $r_s$  be the chances that the respective nominees be alive after s years, then

$$a_{\overline{k},m,n} = \sum r^s \{1 - (1 - p_s)(1 - q_s)(1 - r_s)\},$$
  
 $= \sum r^s \{p_s + q_s + r_s - q_s r_s - r_s p_s - p_s q_s + p_s q_s r_s\},$   
 $= a_k + a_m + a_m - a_{m,m} - a_{m,k} - a_{k,m} + a_{k,m,m}.$ 

The numerical solution of this problem would require a table of annuities on three joint lives, or some other means of calculating  $a_{k_1m_1n}$ .

§ 20.] A contract of life insurance is of the following nature:—A man A agrees to make certain payments to an insurance office, on condition that the office pay at some stated time after his death a certain sum to his heirs. As regards A, he enters into the contract knowing that he may pay less or more than the value of what his heirs ultimately receive according as he lives less or more than the average of human life; his advantage is that he makes the provision for his heirs a certainty, so far as his life is concerned, instead of a contingency. As for a shis life is concerned, instead of a contingency and for A's insurance is such that they shall not ultimately lose if they enter into a large number of contracts of the kind made with A; but, on the contrary, earn a certain percentage to cover expenses of management, interest on shareholders' capital, &c.

The usual form of problem is as follows :-

What annual premium P<sub>m</sub> must a man of m years of age pay (in advance) during all the years of his life, on condition that the office shall pay the sum of £1 to his heirs at the end of the year in which he dies!

 $P_m$  is to be the "net premium," that is, we suppose no abundance made for profit, &c., to the office. Suppose that the office insures  $I_n$  lives of m years, and let us trace the incomings and outgoings on account of these lives alone. The office receives in premiums  $\mathcal{L}P_{mlm}$ ,  $\mathcal{L}P_{mlm+1}$ , . . . at the beginning of the 1-t, 2nd, . . . years respectively. It pays out on lives failed  $\mathcal{L}(I_m + I_{n+1})$ ,  $\mathcal{L}(I_{m+1} - I_{m+1})$ , . . . at the end of the 1st, 2nd, . . . years respectively. Hence, to balance the account, we must have, when all these sums are reduced to present value.

$$P_m(l_n + l_{n+1}v + l_{n+2}v^2 + \dots)$$
  
 $(l_m - l_{m+1})v + (l_{n+1} - l_{n+2})v^2 + (l_{n+2} - l_{n+3})v^3 + \dots$  (1).

the summation to be continued as long as the table gives significant values of  $l_{\star}$ .

Since  $d_m = l_m - l_{m+1}$ , we deduce from (1)

$$P_{m} = \frac{d_{i,r}r + d_{m+1}r^{2} + d_{m+2}r^{3} + \cdots}{l_{i,r} + l_{m+1}r + l_{m+1}r^{2} + \cdots}, \qquad (2).$$

Dividing by  $l_m$ , we deduce from (1)

$$\begin{split} P_m \{ & 1 + (l_{m+1}v + l_{m+2}v^2 + l_{m+3}v^3 + \dots) / . \} \\ & = v + v (l_{m+1}v + l_{m+2}v^2 + \dots) / . \\ & - (l_{m+1}v + l_{m+2}v^2 + \dots) / . . \end{split}$$

Hence

$$P_m(1 + a_m) = v + va_m - a_m,$$
  
 $P_m = v - a_m/(1 + a_m)$  (3).

The last equation shows that the premium for a given life can be deduced from the present value of an immediate complete annuity on the same life. In other words, life insurance premiums can be calculated by means of a table of life annuities,

§ 21.] It is not necessary to enter further here into the details of actuarial calculations; but the mathematical student will find it useful to take a glance at two methods which are in use for calculating annuities and life insurances. They are good specimens of methods for dealing with a mass of statistical information.

Recurrence Method for Calculating Life Annuities.

The reader will have no difficulty in showing, by means of the formulæ of § 17, that

$$\alpha_m = v \left(1 + \alpha_{m+1} \right) l_{m+1} / l_m \qquad (1).$$

From this it follows that we can calculate the present value of an annuity on a life of m years from the present value on a life of m+1 years. We might therefore begin at the bottom of the table of mortality, calculate backwards step by step, and thus gradually construct a life annuity table, without using the complicated formula (4) of  $\S$  17 for each step.

A similar process could be employed to calculate a table for two joint lives differing by a given amount.

# Columnar or Commutation Method.

Let us construct a table as follows :-

In the 1st column tabulate  $l_x$ ; , 2nd ,  $d_x$ ; , 3rd ,  $e^{it}$ ,  $D_x$ , say; , 4th ,  $e^{it}$ ,  $e^{it}$ , say,

Next been the 5th column by adding the numbers in the 3rd coler and methods on upwords. In other words, tabulate

$$N = D_{x,z} + D_{x+z} + D_{x,z} + \dots$$
  
In like a onner, in the 6th column tabel ite

 $M = C_r + C_{r+1} + C_{-r,2} + \dots$ 

$$M = C_x + C_{x+1} + C_{-x,2} + \dots$$

All this can be done systematically, the main part of the labour being the multiplications in calculating D and  $C_s$ .

From a table of this kind we can calcul to annuities and life premiums with great case. Referring to the formula above.

$$\sigma = N - D_{eq}$$
 (2)

$$n \in \mathcal{N}_{+n} D_n \qquad ($$

$$\int_{t^{D}}^{h} \frac{(N-N_{-t})D}{(N-N_{-t})D}, \qquad (1);$$

$$\int_{t^{D}}^{h} \frac{(N-N_{-t})D}{(N-N_{-t})}D, \qquad (5);$$

$$P = M - N_{\gamma=1} \tag{6}.$$

22.] In the for aing coptor the object has been to illustrate as many as possible of the elementary mathematical methods that have been used in the Calculus of Probabilities: and at the same time to indicate practical applications of the theory.

All matter of debetable character or of doubtful utility has been excluded. Under this head fall, in our opinion, the theory of a priori or inverse probability, and the applications to the theory of evidence. The very meaning of some of the propositions usually stated in parts of these theories seems to us to be doubtful. Notwithstanding the weighty support of Laplace. Poisson, Do Margail, and others, we think that many of the criticisms of Mr Venn on this part of the doctrine of chances are unan wetable. The mildest judgment we could pronounce w ald be the fellowing words of De Morgan himself, who seems. after a l. to flave " bubted": "My own impression, derived from this a point in the theory of errors and many other circounstances come ted with the malysis of probabilities, is, that

<sup>\*</sup> And see hills and and nation Application to Life Continger and In State Overs (De Mir an), Cabbact Cy 'of rdia, App., P 3 11

The reader who wishes for further information should consult the elementary works of De Morgan (just quoted) and of Whitworth (Choice and Chance); also the following, of a more advanced character:—Laurent, Traité du Calcul des Probabilités (Paris, 1873); Meyer, Vorlesungan über Wahrscheinlichkeitswehnung (Leipzig, 1879); Articles, "Annuities," "Insurance," "Probabilities," Encyclopædia Britannica, 9th edition.

The classical works on the subject are Montmort's Essui d'Analyse sur les Jeux de Huzards, 1708, 1714; James Bernoulli's Ars Conjectuali, 1713; Demoivre's Doctrine of Chances, 1718, 1736; Laplace's Théorie And ytique des Probabilités, 1812, 1820; and Todhunter's History of the Theory of Probability, 1865. The work last mentioned is a mine of information on all parts of the subject; a peru-al of the preface alone will give the reader a better idea of the historical development of the subject than any note that could be in-erted here. Suffice it to say that few branches of mathematics have engaged the attention of so many distinguished cultivators, and few have been so fruitful of novel analytical processes, as the theory of probability.

# Exercises XL.

- (I.) A bag contains 4 shillings and 4 sovereigns. Three coins are drawn; find the value of the expectation.
- (2.) A bag contains 3 sovereigns and 9 shillings. A man has the option, 1st, of drawing 2 coins at once, or, 2nd, of drawing first one coin and afterwards another, provided the first be a shilling. Which had he better do?
- (3.) One bag contains 10 sovereigns, another 10 shillings. One is taken out of each and placed in the other. This is done twice; find the probable value of the contents of each bag thereafter.
- (4.) A player throws n coins and takes all that turn up head; all that do not turn up head is throws up again, and takes all the heads as before; and so on r times. Find the value of his expectation; and the chance that all will have turned up head in r throws at most. (St John's Coll., Camb., 1870)
- (5.) Two men throw for a guinea, equal throws to divide the stake. A uses an ordinary die, but B, when his turn comes, w∈s a die marked 2, 3, 4, 5, 6, 6; show that B thereby increases the value of his expectation by 5/18ths.
  - (6.) The Jeu des Noyaux was played with 8 discs, black on one side and

white on the other. A stake S was named. The discs were tossed up by the player; if the number of blacks turned up was odd the player won S, if all whites he was 2S, otherwise he best S to his opponent. Show that the supertations of the player and opponent are 1318/256 and 1258/256 respectively. (Mortmort. See Todh, Hatt. Prob., p. 95.)

(7.) If promises to give B a shilling if he throws 6 at the first throw with 2 dice, 2 shillings if he throws 6 at the second throw, and so on, until

a 6 is thrown. Calculate the value of B's expectation,

(-.) A man is allowed one throw with 2 ordinary diee and is to gain a number of shillings equal to the greater of the two numbers thrown; what ought he to pay for each throw? Generalise the result by supposing that each the has n faces.

- (9.) A bay contains a crtain number of balls, some of which are white. I am to get a shilling for every ball so long as I continue to draw white only (the balls drawn not being replaced). But an additional ball not white having been introduced, I claim as a compensation to be allowed to replace every white ball I draw. Show that this is fair.
- (10) A per on throws up a coin n times; for every sequence of m(m>n) expected or m tails he is to receive 2<sup>m</sup> − 1 shillings; prove that the value of his expectate in is n(n+3)/4 shillings.
- (11) A manufacturer has n-swing machines, each requiring one worker, and ach yielding everyday it works a times the worker's wages as net profit. The machines are never all in working order at once; and it is equally likely that 1, 2, 3, ..., or any number of them, are out of repair. The worker's wages must be paid whether there is a machine for him or not. Prove that the most position number of workers to engage permanently is the integer next to migraph 1, 3. (Math. Trip. 1875).

(12.) A blackleg bets £5 to £1, £7 to £6, £9 to £5 against horses whose chances of winning are \$, \(\frac{1}{2}\), \(\frac{1}{2}\) respectively. Calculate the most and the

least that he can win, and the value of his expectation.

(13.) The odds against n horses which start for a race are a:1; a+1:1; ..., a+n-1:1. Show that it is possible for a bookmaker, by properly baying bets of different amounts, to make certain to win if  $n > (a+1)(\epsilon+1)$ , and impossible if  $n < a(\epsilon-1)$ , where  $\epsilon$  is the Napierian base.

(11.) If J denote the value of an annuity to last during the joint lives of p persons of the same age, prove that the value of an equal annuity, to continue so lone as there is a survivor out of n persons of that age, may be found by means of the formula.

$$nA_1 = \frac{n(n-1)}{2!}A_2 + \frac{n(n-1)(n-2)}{3!}A_3 - \dots + A_n$$

(15.) M is a number of matried couples, the husbands being m years of a c. What is the number of living pairs, widows, widowers, and dead pairs after two as?

Work out the case where M = 500, m = 10, n = 50.

(io) If Sa, I we the meaning of § 16, show that

$$-2l_a l_b \gamma_{a,b} - 2l_{a+1} l_{b+1} S_{a+1,b+1} = (l_a + l_{a+1}) (l_b - l_{b+1}).$$

(17.) Find the probability that a man of 80 survive one or other of twomen of 90 and 95 respectively.

(18.) If a<sub>km,n</sub> . . . denote the present value of an immediate complete annuity of £1 on the joint lives of a set of men of \( \lloar\), m, m. . . . . years of age respectively, show that the present value of an immediate annuity of £1 which is to continue so long as the re is a survivor out of \( k\) men whose a respect, m, n. repectively is

$$\sum a_{I} - \sum a_{I_{1}m} + \sum a_{I_{1}m,n} = \dots$$

(19.) What annual premium must a married couple of ages m and n respectively pay in order that the survivor of them may enjoy an annuity of £1 when the other dies?

(20.) Calculate the annual premium to insure a sum to be paid n years hence, or on the death of the nominec, if he dies within that time.

(21.) Show how to calculate the annual premium for in-uring a sum which diminishes in arithmetical progression as the life of the nominee lengthens.

(22.) An annuity, payable so long as either A (m years of age) or B (n years of age) survive C (p years of age), is to be divided equally between A and B so long as both are alive, and is to go to the survivor when one of them dies. Show that the present values of the interests of A and B are

$$a_{ni} = \frac{1}{2} a_{mi,n} - a_{mi,p} + \frac{1}{2} a_{mi,n,p}$$
  
 $a_{ni} = \frac{1}{2} a_{mi,n} - a_{ni,p} + \frac{1}{2} a_{mi,n,p}$ 

and respectively.

(23.) If the population increase in a geometrical progression whose ratio is r, show that the proportion of men of n years of age in any large number of the community taken at random is  $(l_n l^n), \sum_{n=1}^{\infty} u_n^l u^n)$ .



# RESULTS OF EXERCISES

# Ĭ.

(1.) 504000. (2.) 1210809600. (3.) 720. (4.) 12. (5.) 6. (3.) 5040; 6164890. (9.) 123520. (10.) 6188;3003;3185. (11.) 408888;18 ways of setting together on the front. 10 ways of setting at equal distances all rounds  $(12.) - (1_{10}C_{11}^{2}C_{12}^{2}C_{13}^{2}C$ 

(12) ∫<sub>1</sub> (1<sub>2</sub>(1) ± (1<sub>4</sub> ± (1<sub>4</sub>) ± (1<sub>2</sub>) ± (1<sub>3</sub> ± (1<sub>3</sub>) ± (1<sub>3</sub>

# Π.

(1)  $418266240x^2$ , (2.) -2093, (3.)  $2^n$ , 1.3 . . . (2n-1)/n!, (4.)  $(-)^{n+r}(2n)l/(n+r)!(n-r)!$ , (5.)  $2^{2n}$ , 1.3 . . . (4n-1)/(2n)!, (6.) If n be even, the middle term is (n)l/2n)!,  $(n^2+1)^2$ ,  $(n^2+1)^2$ , if n be odd, the two middle terms are  $\{nl/2(n-1)l/2(n+1)l$ 

#### HI.

(1.) 944. (2.) 20. (3.) (n+1) (n+2) (n+3) (n+4) (n+5)/5! if the separate numbers thrown be attended to; 5n+1 if the sum of the numbers thrown be alone attended to. (4.) 231. (5),  $p_{14}C_{n}$ . (7.) 62. (8.)  $15s_{n}^{*}C_{n}$ . (11.)  $(2n)!/2^{n}n!$ . (13.) (X+a+b+c-3)!/n!b!c!. (15.) 1 or 0 according as n is even or odd;  $\{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}\}/2^{n+1}\sqrt{5}$ . (17.)  $2_{m-1}C_{r-1}$  n-1  $C_{r-1}$ . (18.) 116250.

#### V.

(1.) x/y must not lie between 1 and b²/a². (2.) x must lie between ½ (7 − √53) and ½ (7 + √53). (3.) x between (dc − b²)/(dd − bc) and (d² − ab)/(ad − bc), and y between (ab − c²)/(ad − bc) and (a² − cd)/(ad − bc).
(15.) Greater. (17.) Less. (39.) 31.7.

#### VI.

(1) Sobe. (2)  $abcB_1(3, -4)$   $bm/3^{m-1}$  is a minimum value if m do not lie between 0 and 1, otherwise a maximum. (5.) Minimum when  $aprF = b_0p^2 = crc^2$ . (7.) There is a maximum or minimum when  $(x+1) \log a = (y+m) \log b \cdot c(x+1) \log c$ , according as  $\log a \log b \log c$  is positive or negative. (8)  $x = (abb/ms)^{-m/2}$ . (9.)  $x = 1, x = 3 \cdot 15$  siye maxima; x = 2, x = 3 minima. (10.)  $\frac{1}{2}abc$ . (11.) Minimum when x = mc/(m-n), y = nc/(m-n). (15.) Minimum 2, (ab)/(a+b).

## VII.

## VIII

(1) Div. (2.) Div. (3.) Cuv. if x be positive. (4) Conv. (5.) Div. (6.) P.v. if m. | x > x; conv. if med x > x. (7.) Conv. if x > 1; div. if x < 4. (8.) Cunv. (9.) Piv. (x < 1). (10.) Conv. (11) Piv. (12.) Conv. if a > 1; div. if x > 1. (13.) Piv. (14.) Div. (15.) Abs. conv. (16.) Piv.

#### IX.

(1)  $(-)^{-2}3,1,1,1,3,\dots(2r-5)[2,4,6,8,\dots,2r,-(2)]$  1.3 ...  $(2r-1)[2,4,\dots,2r,-(3)]$  3.7, 11...  $(4r-1)[1,8,12,\dots,4r,-(4)]$  2.1.4,7... (4r-2)[2,2,4,3,6] 8... (4r-1)[4] 8.12... (4r-4)[4] 2.1.4,7... (4r-4)[4] 8.12... (4r-4)[4] 8.12... (4r-4)[4] 8.12... (4r-4)[4] 8.12... (4r-4)[4] 8.12... (4r-4)[4] 8.13... (4r-4)[4] 8.14... (4r-

#### X

(1.)  $\Sigma 1/a^r(c-a)(a-b)$ . (2.) 0. (3.)  $\Sigma 1/a^{r-a-2}(c-a)(a-b)$ . (4.)  $2r+1+1/2^{r+1}$ . (5.) r, if r be even; r-1, if r-4+1; r+1, if r=1r-1. (6.)  $a^{1}$ ,  $a^{1}$ ,  $a^{2}$ ,  $a^{1}$ ,  $a^{2}$ ,  $a^{1}$ ,  $a^{2}$ ,  $a^{2$ 

#### XI

(2.)  $275/12^3$ , (3.) 869699/256, (4.) 48; 0, (5.) 11989305/2048, (6.)  $(-)^*\{(r-1)+(r+5)/2^{r+2}\}$ , (10.)  $1\cdot0001005084$ ;  $1\cdot0004000805$ , (11.) 2mx, (12.)  $1+2x(1-r^n)/(1-r)$ . (13.)  $1+(-)^{n-1}x/2^n$ ,

#### XII.

(1.) ·367879.
 (2.) ·04165.
 (5.) (1-x)<sup>ex</sup>.
 (6.) 3 (ε-1).
 (7.) ε+1.
 (8.) 1/ε.
 (9.) 15ε.

# XIII.

(4) 917. (5)  $2 \log \{(x-1)/(x+1)\} + \log \{(x+2)/(x-2)\}$ . (6.)  $\log \{(2e), (7.) (1+1/p) \log (1+x) - 1$ . (8.)  $\frac{1}{2}(x-x^{-1}) \log \{(1+x)/(1-x)\} + \frac{1}{2}$ . (9.) When x=1 the sum is  $18 - 21 \log 2$ . (10.)  $\frac{1}{2}$ . (12.)  $\sum \{x^{(n)}/(3n-2) + x^{2n-1}/(3n-1) - 2x^{2n}/(3n)\}$ .

#### VVV

(1.)  $\frac{1}{2}n(n+1) + \frac{1}{2}(r-2)n(n+1)(n-1)$ . (2.)  $\frac{1}{2}n(n+1)(n+4)(n+5)$ . (3.) 3/4 - 1/2n - 1/2(n+1). (4.) 1/15 - 1/5(5n+3). (5.) 1/12 - 1/4(2n+1)(2n+3). (6.) 1/18 - 1/3(n+1)(n+2)(n+3). (7.) a/2 + b/4 - a/(n+2) - b/2(n+1)(n+2). (8.) 1/8 - (4n+3)/8(2n+1)(2n+3). (9.) 7/36 - (3n+7)/(n+1)(n+2)(n+3). (10.) 11/180 - (6n + 11)/12(2n + 1)(2n + 3)(2n + 5). (11.) 3/4 + n - (2n + 3)/12(n+1)(n+2), (12.)  $u_n = (n+1)(n+3)(n+5)/n(n+1)$ , . . . (n+6); apply 8 3. Example 4. (13.)  $\sin \theta \sec (n+1)\theta \sec \theta$ . (14.)  $\cot (\theta/2^n)/2^n - \cot \theta$ . (15.)  $\tan^{-1}na^n$ . (16.)  $\tan^{-1}1 + \tan^{-1}1/2 - \tan^{-1}1/n - \tan^{-1}1/(n+1)$ . (17.) (m+n)!/(m+1)(n-1)!. (18.)  $\{1/(m-1)! - (n+1)!/(m+n-1)!\}/(m-2)$ . (19.)  $(-)^n_{m-1}C_n$ . (21.)  $\{m-1-(n)!/m^{-n-1+}\}/(m-2)$ . (22.)  $(a^{(n+r)}/c^{(n)}$  $a^{(r+1)}/(a-c+r+1)$ . (23.)  $(a^{(n+2)}/c^{(n+r+1)}-a/c^{(r)})/(a-c-r+1)$ . (24.)  $\{(a-1)^{(m-1)}/c^{(m-1)}-(a+n)^{(m-1)}/(c+n+1)^{(m-1)}\}/(m-1)(a-c-1).$  (25.) Deduce from (24). (26.) Deduce from (24). (27.)  $2m\{1-(-)^n 2^n (m-1)\}$ (m-2) , , (m-n)/1 , 3 , , (2n-1)/(2m-1).

#### IVZZ

(1.)  $2^{n+1} + \frac{1}{2}(3^{n+1} - 3)$ . (2.)  $\frac{1}{2}\{1 + (-1)^n\} + 6 - 3\{n^{n+1} + (-1)^{n+1}\} - \frac{1}{2}\{n^n - (-1)^n\}$ . (3.)  $11\{1 - (4x)^{n+1}\}[1 - 1x] - 9\{1 - (3x)^{n+1}\}[1 - 3x^n\}$ ;  $\{2 + 3x\}[1 - 7x + 12x^2\}$ ,  $x + \frac{1}{2}$ . (4.)  $3\{1 - (2x)^{n+1}\}[1 - 2x] + 2\{1 - (3x)^{n+1}\}[1 - 3x] + \frac{1}{2}\{1 - (3x)^{n+1}\}[1 - 5x + 6x^2]$ ,  $x - \frac{1}{2}$ . (6.)  $\frac{1}{2}\{1 - (3x)^{n+1}\}[1 - 3x) + \frac{1}{2}\{1 - (5x)^{n+1}\}[1 - 5x) + (1 - 1x)[1 - 5x + 15x^2]$ ,  $x - \frac{1}{2}$ . (6.)  $3\{1 - (2x)^{n+1}\}[1 - 2x] - 2\{1 - x^{n+1}\}[1 - x\} + (1 + x)[1 - 3x + 2x^2]$ ,  $x - \frac{1}{2}$ .

#### HYXX

(1.)  $(1+2x^3)/(1-x^3)^2$ , (2.)  $-\lceil \log ((1-x))/(1+x+x^2)\rceil = \sqrt{3} \tan^{-1} t \cdot t \cdot 3\pi t$ (2+x)(1/3x);  $\frac{1}{3}\{e^x+2e^{-x/2}\cos(\sqrt{3x/2})\}$ . (4.)  $\frac{1}{3}[e^{-x}+e^{x/2}(\cos(\sqrt{3x/2})+\sqrt{3x/2})]$  $\sin(3\pi/2)$ ]]. (5.)  $\frac{1}{2}(2^m+2\cos(m\pi/3);\frac{1}{2}3^{m/2}\cos(m\pi/6),\frac{1}{2}-1/(n+2)]$ , (7.)  $\{2^{m+3}-1-(rr+3)(m+4)/2\}/(m+1)(m+2)(m+3)$ . (8) 1/(1+x)= $\log (1+x)$ . (9.)  $\frac{1}{2}\cos \theta - \frac{1}{4}\cos 2\theta$ . (10.)  $1 - (2n+3)/(n+2)^2$ . (11.)  $2 - 4\log 2$ . (14.) san majma; cosh ma.

#### XXVIII

The partial quotients are as follows :-

- (1.) 0, 4, 1, 6, 2. (2.) 0, 2, 4, 8, 16. (3.) 1, 15, 1, 1, 1, 3, 1, 14, 1, 1, 5. (4) 31, 1, 1, 1, 1, 1, 1, 1, 1, 3, (5) 2, 1, 2, 1, 1, 4, 1, 1, 6, 3, 12, 3, 5, 1, 2, (6.) 0, 126, 1, 1, 2, 1, 1, 6, (7.) 1, 2, (8.) 2, 4, (9.) 3, 3, 6,
- (10.) 3, 6. (11.) 3, 2, 6. (12.) 1, 4, 2. (13.) 2, 1, 2. (14.) 3, 1, 5, (16.) 0, 2,  $\hat{1}$ ; 0,  $\hat{1}$ . (17.) a,  $\hat{2}$ ,  $\hat{2}a$ ; a = 1,  $\hat{2}$ , 2(a = 1).

# XXIX.

(1.) The 1st, 2nd, 3rd, . . . convergents are 1, 2/3, 9/13, 20/29, 20/12, 78/113. . . .: the errors corresponding less than 1/3, 1/39, 1/377, 1/1218. 1/4746, 1,17515, . . . (2.) 972/1393. (3.) 2177/528. (4.) Transits at the same node will occur 8, 243, . . . years after; after 8 years Venus will be less than 1° 5 from the node. (5.) Transits at the same node will occur 13, 33, . . . vears after.

# TXXX

(6.) 2, 1;

- 0.3 10 20 (2.) 0, 1, 126, 2: 0. 10. 0. 0. 63.63:
  - 64, 63, 1, i
- (3,) 1, 5, 3, 1, 8, 1, 3, 5, 2;

0, 12, 13, 8, 12, 12, 8, 13, 12;

12. 5, 7, 20, 3, 20, 7, 5,

(4) 0, 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 11;

0, 0, 7, 5, 7, 5, 4, 6, 4, 5, 7, 5, 7;

61. 1. 12. 3. 4. 9. 5. 5, 9, 4, 3, 12.

(5) 1. 2. 10. 2. 1:

10, 15, 25, 25, 15, 2. 2:

25 20 5 20 25 2. i.

(7.) 
$$2 + \frac{1}{1+} \frac{1}{1+} \frac{1}{3+}$$
 (8.)  $3 + \frac{1}{6+} ; -\frac{1}{6+} \frac{1}{3+}$ 

(10.)  $a + \frac{1}{2n+}$ ;  $a + (\alpha^{n-1} - \beta^{n-1})/(\alpha^n - \beta^n)$ ,  $\alpha$  and  $\beta$  being the roots of

$$\begin{split} x^2 - 2ux - 1 &= 0. \quad (11) \cdot \frac{1}{2} (a + \langle (a^2 + 4)) : \; (a^{n+1} - \beta^{n+1})/(a^n - \beta^n), \text{where } a \text{ and } \beta \text{ are the roots of } x^2 - ax - 1 &= 0. \quad (12) \cdot \frac{1}{2} (a - \langle (a^2 - 4)) : \; (a^n - \beta^{n+1})/(a^{n+1} - \beta^{n+1}), \text{ where } a \text{ and } \beta \text{ are the roots of } x^2 - ax + 1 &= 0. \quad (13) \cdot \frac{1}{4} = ab + \sqrt{(a^2b^2 + 4ab)} \}^{2a}; \\ \text{if } a, \beta & \text{be the roots of } x^2 - (ab + 2)x + 1 &= 0, \text{ then } p_{2n} = b(a^n - \beta^n)/(a - \beta), \\ q_{2n} = (a^{n+1} - \beta^{n+1} - a^n + \beta^n)/(a - \beta), \text{ and } p_{2n-2} + (b_{2n} - \beta^{n+1})/(a^{n+1} - \beta^{n+1}), \text{ where } a \\ \text{and } \beta \text{ are the roots of } x^2 - x - 1 &= 0. \quad (20) - \frac{1}{2} a + \sqrt{(\frac{1}{4}a^2 + \beta^n - 1)} (a^{n+1} - \beta^{n+1}), \\ (\frac{1}{4}a^2 - \beta^{n-1})/(a^{n+2} - \beta^{n-1}), \text{ where } a \\ \text{and } \beta \text{ are the roots of } x^2 - x - 1 &= 0. \end{cases}$$

#### XXXII.

(1.) 3+7t,  $2\cdot 5t$ , (2.) 17t+7, 16t+5. (3.) 2206-7t. 11t-3300. (4.) 1013t-3021756, 1367t-407774t. (5.) 13. (6.)  $2\times 0$ . (7.) 6. (8.) If  $2\times 0$ , (1.) (9.) Buy 300 of each and spend 1021d. (10.) 69. (12.) 19. (13.) 715. (14.) 697.

#### VIXXIV.

Converges. (2.) Converges. (3.) Oscillates. (4.) Converges. (5.)
 Converges. (6.) Converges. (7.) Converges if k : 2, oscillates it k > 2.
 Converges. (9.) Oscillates. (10.) Oscillates. (15.) Each of the fractions converges to 1. (23.) c. (24.) 1/(1-c). (25.) log<sub>2</sub> 2. (26.) (3-c)/(c-2).

#### XXXIX.

(a) 11/30. (b) 3/11, 29/44, 3/44. (a)  $m(m+2n)/(m+n)^2$ , m(m+2n-1)/(m+n), (m+n-1). (4)  $(365.4^{n}+1)/(1461)^{n}$ . (5) 4/9. (7) 55/972,  $299/26^{n}$ . (8) 1/42. (9) (n-1)/n(2n-1). (10) (395/76).53/4, 4(395/76).53/2. (11) 2(r-1)/n(m-1). (13) 7n/2, or, if this be not integral, the two integers either side of it. (14)  $\frac{n^2}{r^2}/(r-1)n(n-1)$ . .  $(n-r+2)/n^2$ . (18) 16/31. 8/31, 4/31, 2/31, 1/31. (19). The channes in  $A^{n}$ 's favour are 6/10, 7/10, s/10, 9/10, when he is 1, 2, 3, 4 up respectively. (20) 25 to 2. (23) (1-1/n)/2, (1-1/n)/2, (1-1/n)/2,

#### XL

(1.) £1:11:6. (2.) His expectations are 11s. 6d. and 10s.  $4\frac{1}{2}d$ , respectively. (3.) £8:5:9 $\frac{1}{2}$ , £2:4:2 $\frac{1}{2}$ . (4.)  $n(1-1/2^r)$ ,  $(1-1/2^r)^n$ . (7.) 7s.  $2\frac{1}{2}d$ ; (n+1)(1n-1)/6n. (12.) £6, £1, £4:2:2 $\frac{1}{2}$ .

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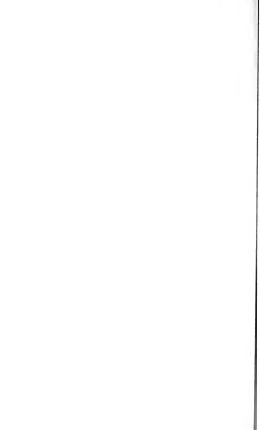
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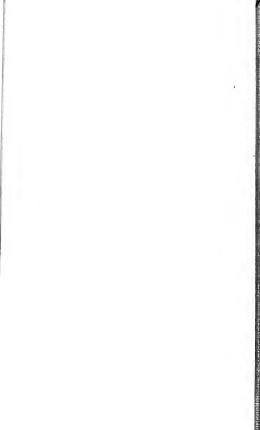
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