# ALGEBRA
## FIRST COURSE

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CORRELATED MATHEMATICS FOR SECONDARY SCHOOLS

# ALGEBRA

## FIRST COURSE

BY

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# PREFACE

In the preparation of this text the authors have had in view two primary aims, first to emphasize and vivify the treatment of Algebra by a systematic correlation with Geometry, and secondly to present the subject-matter in a style sufficiently simple to be easily grasped by students of high school age.

The first object has been accomplished by a free introduction of constructive exercises from Geometry, including theorems and problems of sufficient range to give to the student a fair working knowledge of the elementary properties of important geometric figures. No attempt has here been made at formal demonstration, that being reserved for the book on Geometry proper; the sole aim has been to bring out the fundamental facts regarding such figures as angles, intersecting lines, polygons, circles, and some simple solids. These facts are brought out by drawing, paper cutting, and super-position, and they are then made the basis of further algebraic work.

In bringing out the laws which govern the four fundamental operations, systematic use is made of graphic representation of these operations on a number scale. Experience has shown that this is a simple and effective way to fix these laws so firmly in the mind of the student that they are not easily forgotten.

A lengthy discussion of the contents of the book would be superfluous here. Its scope, aims, and methods are indicated by the table of contents, and can be more fully appreciated only by reading the pages of the text.

The second object, namely to produce a simple treatment, is accomplished largely by the use of a more narrative style than is usual in mathematical text-books. Without this the book could have been made considerably smaller, but it was felt that the addition to the number of pages would be more than compensated by the gain in clearness and fullness of explanation. The time and effort required to cover the year's work will be less than it would be with a more compact and concentrated form of presentation.

The exercises and problems are drawn largely from the student's own experience, and include many that are in the nature of experiments, to be performed by the student individually, or with the coöperation of the class. It is urged that most of these experiments be actually performed, because of the discussion and interest which comes from such concrete applications.

In using the book for the first time there will be perhaps some tendency on the part of the teacher to omit features which at first sight may seem unusual. Such omissions might easily detract materially from the force and usefulness of the text, and it is strongly urged that they be reduced to a minimum.

For those who wish to make a still closer correlation with Geometry, references to that subject (Part II of this series) are given, which outline in a general way a two years' course in Mathematics. For this purpose, if the teacher desires it, the two parts, Algebra and Geometry, may be had bound in one volume. This will be especially advantageous in schools where free text-books are furnished, because the work of the first year is then always at hand for reference during the second year.

E. Long,
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Lincoln, Nebraska,
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## CHAPTER I

### INTRODUCTION

**1. The Operations of Arithmetic.** In arithmetic we deal with the numbers 1, 2, 3, . . ., and with combinations of these numbers by use of the four fundamental operations, namely, addition, subtraction, multiplication, and division. Let us briefly review the meaning of these processes.

**2. The Whole Numbers, or Integers.** We first recall that the numbers 1, 2, 3, . . ., 10, 11, 12, called whole numbers or integers, are merely used as counters, to indicate how many objects are contained in a given group. The same thing could be done in many other ways, as, for example, we might make strokes, one for each object in a group. Thus we have the Roman numerals I, II, III, IV.

**3. Addition.** If we have two groups of objects, say one containing 3 and the other 4 separate objects, their combination into a single group will contain how many objects? Having counted the objects in one group, say 3, we must now count forward 4 more, and so we arrive at the number 7.

We write this result in the form

$$3 + 4 = 7.$$

This is read: "Three plus four equals seven."

This means that when three things of any sort are united into a single group with four others, the result is a group containing seven distinct objects. The process is called *addition.*

The numbers 3 and 4 are called *addends*, and the number 7 their sum.

Evidently we might reverse the order of our counting, and count first the four objects and then the three more to complete the group. This gives us

$$4 + 3 = 7.$$

That is, two numbers may be added in either order; the result is the same.

In like manner we interpret the equation

$$4 + 3 + 5 = 12$$

as the union of three groups into one. In all cases the order of writing the numbers to be added together is immaterial; the result is the same. For example, we may add a column of figures either up or down; we may first rearrange the numbers in the column and then add.

Now in applying the operations of arithmetic to actual problems, we usually deal with objects of a definite sort. We say 3 apples, 3 seconds, 3 feet, 3 quarts, and so on. For example, if we put 3 oranges and 4 apples into a basket, the basket contains 7 objects, but neither 7 oranges nor 7 apples. That is, if the number which results from the addition of several others is to represent so many objects of a certain kind, then all the numbers which are added must represent objects of the same kind.

**4. Subtraction.** The expression $7 - 3$ (seven minus three) means that number which when added to 3 will give 7. From arithmetic we know that this is 4, so we have $7 - 3 = 4$. Subtraction means to find the difference of two numbers, and to do this we find what number must be added to the subtrahend to produce the minuend.

Thus at a store when you buy goods whose value amounts to \$1.37, and you pay with a five-dollar bill, the salesman hands you the purchased article and then *counts on* the change. In other words, in order to subtract \$1.37 from \$5, he determines the amount which must be added to \$1.37 to make \$5.

**5. Multiplication.** The number 3 can be considered as $3 \times 1$, that is, as a unit taken three times to measure the quantity considered. Thus 3 hours is $3 \times 1$ hour; 3 feet is $3 \times 1$ foot, and so on.

To multiply 3 by 4 means to take 3 units 4 times, giving 12 units; that is, $4 \times 3 = 12$.

Instead of the cross, $\times$, a dot is often used to indicate multiplication; thus $4 \cdot 3 = 12$.

From arithmetic we know that the product is the same when the order of the numbers to be multiplied together is reversed; that is, $3 \cdot 4 = 12$.

**6. Division.** To divide one number by another means to find a third number which when multiplied by the divisor will give the dividend.

Thus $12 \div 3$ means find a number which when multiplied by 3 will give 12. Our knowledge of multiplication enables us to guess the right number in many cases. When we cannot easily guess the number, certain rules are used for getting it. We shall later see the reason for these rules, when we study the subject of division further.

The common symbols for division are

$$12 \div 3; \quad \frac{12}{3}.$$

**7. Subject Matter of Algebra.** In the subject of algebra, we continue the study of these four fundamental operations as applied to numbers, and take up other operations. Also we freely use letters to designate numbers and it is chiefly in its free use of symbols that algebra differs from arithmetic.

This can best be brought out by some examples.

*Example* 1 (*a*). If a train travels 5 hours at the rate of 30 miles an hour, what is the distance passed over?

*Solution:* $5 \cdot 30$ miles $= 150$ miles.

(*b*) If a train travels $t$ hours at the rate of $v$ miles an hour, what is the distance passed over?

*Solution:* $t \cdot v$ miles $= d$ miles,

where $t$ stands for the number of hours (time), $v$ for the rate per hour (velocity), and $d$ for the total distance.

Under (*b*) we have the solution of every possible problem of the sort in (*a*).

The result is usually written thus

$$tv = d,$$

the multiplication sign being omitted.

In algebra it is understood that when two letters are written side by side without any sign between them, the numbers for which these letters stand are to be multiplied together.

Thus: $a \cdot b \cdot c = abc.$

*Example* 2 (*a*). If a rectangle is 3 feet wide and 2 feet high, what is its area?

*Solution:* $3 \cdot 2 = 6$, number of square feet in the area.

(*b*) If a rectangle is $b$ feet long and $h$ feet high, what is its area?

*Solution:* $b \cdot h = a$, number of square feet in the area.

Here again the equation under (*b*) contains all possible equations like those under (*a*).

*Example* 3 (*a*). A rectangular tank, whose base is 6 feet by 4 feet, and which is 3 feet high, is filled with water weighing $62\frac{1}{2}$ pounds per cubic foot. What weight of water is contained in the tank?

*Solution:* $6 \cdot 4 \cdot 3 = 72$, number of cubic feet in the contents of the tank.

$72 \cdot 62\frac{1}{2} = 4500$, number of pounds in the tank.

(*b*) A rectangular tank whose base is a rectangle $a$ feet by $b$ feet, and whose height is $h$ feet, is filled with a liquid which weighs $w$ pounds per cubic foot. What is the weight of the liquid in the tank?

*Solution:* $a \cdot b \cdot h = abh$, number of cubic feet in the contents of the tank.

$abh \cdot w = abhw$, number of pounds in the tank.

If we let $W$ stand for the number of pounds in the tank, we have the formula

$$W = abhw.$$

Note that if $w$ stands for the number of pounds per cubic inch, $abh$ must be reduced to cubic inches.

*Example* 4 (*a*). What is the amount of a principal of 300 dollars placed at simple interest for 6 years at 5%?

*Solution:* $\frac{5}{100}$ = number of dollars interest on \$1 in 1 year.

$300 \cdot \frac{5}{100} = 15$ = number of dollars interest on \$300 in 1 year.

$6 \cdot 15 = 90$ = number of dollars interest on \$300 in 6 years.

$300 + 90 = 390$ = number of dollars, amount of \$300 in 6 years.

(*b*) What is the amount of $p$ dollars placed at simple interest for $n$ years at $r$%?

*Solution:* $\frac{r}{100}$ = number of dollars interest on one dollar in one year.

$p \cdot \frac{r}{100} = \frac{pr}{100}$ = number of dollars interest on $p$ dollars in one year.

$n \cdot \frac{pr}{100} = \frac{npr}{100}$ = number of dollars interest on $p$ dollars in $n$ years.

$p + \frac{npr}{100}$ = number of dollars in the amount of $p$ dollars in $n$ years.

If we let $a$ stand for the number of dollars in the amount,

$$a = p + \frac{npr}{100}.$$

In this formula replace $p$ by 300, $n$ by 6, and $r$ by 5; what do you get for the value of $a$?

This result is usually written thus:

$$a = p\left(1 + \frac{nr}{100}\right),$$

where the right-hand member of the equation is to be understood as follows: Multiply the number standing before the parentheses, namely, $p$, into each of the numbers in the parentheses, and add the results.

Thus:

$$5(4 + 3 + 2) = 5 \cdot 4 + 5 \cdot 3 + 5 \cdot 2$$
$$= 45;$$
$$a(b + c + d) = ab + ac + ad.$$

We may first add together the numbers within the parentheses, and then multiply their sum by the number outside.

Thus: $$5(4 + 3 + 2) = 5 \cdot 9$$
$$= 45.$$

In this way we may look upon $b + c + d$ as a single number which is to be multiplied by $a$. So we may look upon $1 + \frac{nr}{100}$ as a single number to be multiplied by $p$.

*Example* 5. What principal must be put at simple interest at 6% so that it shall amount to $480 in 10 years?

According to Example 4 we always have

$$a = p\left(1 + \frac{nr}{100}\right).$$

Then we have

$$p = \frac{a}{1 + \frac{nr}{100}}.$$

But $$a = 480,\ n = 10,\ r = 6.$$

Therefore, $$p = \frac{480}{1 + \frac{10 \cdot 6}{100}}$$

Multiplying numerator and denominator by 100, we have

$$p = \frac{48{,}000}{100 + 60}$$
$$= \frac{48{,}000}{160}$$
$$= \$300.$$

We here use the fact that

if $$a = b \cdot c, \quad \text{then} \quad \frac{a}{b} = c.$$

No matter what numbers are used for $a$, $b$, $c$, the second equation is always true if the first is true.*

We of course use this rule as a check on division in arithmetic. For example, dividing 357 by 17 gives 21. Check the correctness of this answer by showing that 21 times 17 gives 357. That is, since $357 = 21 \cdot 17$, then $\frac{357}{17} = 21$.

* There is only one exception to this rule, namely, when $b$ is zero. Division becomes meaningless when the divisor is zero.

**8. Algebraic Expressions — Their Forms and Values.** In the preceding examples we have given some indication of the way in which letters are used to stand for numbers and have made use of some of the notation which is common in Algebra. We shall now give a list, although not a complete one, of the various forms used in the symbolic arithmetic which we call Algebra. Much of this is repetition of what has been said in the examples above.

In what follows all the letters and their combinations are supposed to stand for ordinary arithmetic numbers.

(1) *Addition.*

$a + b$ means to add the number $b$ to the number $a$; we read it "$a$ plus $b$" just as in arithmetic.

$b + a$ means to add $a$ to $b$; we read it "$b$ plus $a$."

Then we always have

$$a + b = b + a.$$

That is, the sum is the same in whatever order we add.

Likewise, $a + b + c$ means the sum of $a$ and $b$ and $c$. We read it "$a$ plus $b$ plus $c$."

The sum is the same in whatever way the letters may be written. The same is true of any number of letters.

(2) The expression $(a + b)$ means that we are considering the sum of the numbers $a$ and $b$ and are regarding it as a single number.

For example, the expression

$$(a + b) + (c + d)$$

means "the sum of $a$ and $b$ plus the sum of $c$ and $d$. But this is exactly the same as adding together the four numbers $a, b, c, d$. That is

$$(a + b) + (c + d) = a + b + c + d.$$

In the same way $(a + b + c)$ means that we are to regard the sum of the numbers $a, b, c$ as a single number, and so on. Then we would have for example

$$(a + b + c) + (d + e) = a + b + c + d + e.$$

In place of parentheses, ( ), we often use brackets, [ ], or braces, { }, or the vinculum, $\overline{\phantom{xxx}}$

Thus $(a+b)$, $[a+b]$, $\{a+b\}$, $\overline{a+b}$; each means that we are to regard the sum of the numbers $a$ and $b$ as a single number. These marks are called **signs of aggregation.**

**Exercises.** By giving arithmetic values to the letters in the following, show that the equations given are true.

**1.** $(a+b)+c=a+b+c$. (Replace $a$ by 5, $b$ by 2, $c$ by 3.)

**2.** $a+(b+c)=a+b+c$. (Replace $a$ by $\frac{1}{2}$, $b$ by $\frac{1}{3}$, $c$ by $\frac{1}{4}$.)

**3.** $(a+c)+b=a+b+c$. (Replace $a$ by $6\frac{1}{3}$, $b$ by $8\frac{1}{2}$, $c$ by $5\frac{1}{6}$.)

**4.** $c+(a+b)=a+b+c$. (Replace $a$ by $3\frac{2}{5}$, $b$ by $5\frac{1}{4}$, $c$ by $1\frac{2}{3}$.)

**5.** $(a+c)+(b+d)=a+b+c+d$.

**6.** $(a+d)+(c+b)=a+b+c+d$.

**7.** $(a+b+c)+(d+e+f)=a+b+c+d+e+f$.

**8.** $(a+b)+(c+d)+(e+f)=a+b+c+d+e+f$.

**9.** $[m+n]+[k+l]=m+n+k+l$.

**10.** $[s+t+u]+(v+w)=s+t+u+v+w$.

**11.** $[s+v+t]+[w+u]=s+t+u+v+w$.

**12.** $\overline{h+k}+[p+q+r]+\{x+y\}=h+k+p+q+r+x+y$.

We may now state our first rule.

*Rule I. In forming a sum the signs of aggregation,* ( ), [ ], { }, $\overline{\phantom{xxx}}$, *may be omitted.*

(3) *Subtraction.*

$a-b$ means the number which must be added to $b$ to give $a$.

$b-a$ means the number which must be added to $a$ to give $b$, as in arithmetic.

We read these as "$a$ minus $b$" and "$b$ minus $a$" respectively.

Likewise: $a - b + c$ means "start with the number $a$, subtract the number $b$, and to the result add the number $c$." Evidently this is the same as $a + c - b$.

The signs of aggregation are used as before.

Thus: $(a + b) + (c - d)$ means "to the sum of $a$ plus $b$ add the difference of $c$ minus $d$." This is the same as $a + b + c - d$, that is we may omit the parentheses.

$(a + b) - (c - d)$ means "from the sum of $a$ and $b$ subtract the difference of $c$ minus $d$." But we cannot remove the parentheses and say our answer is the same as $a + b - c - d$, as a simple example will show.

We have $\qquad (10 + 7) - (12 - 4) = 17 - 8$

$$= 9.$$

But $\qquad 10 + 7 - 12 - 4 = 1.$

The two results are not the same, so that in subtraction we cannot drop the parentheses as in addition. We shall soon have a rule to cover this case.

**Exercises.** Remove as many as possible of the signs of aggregation in the following exercises.

| | |
|---|---|
| **1.** $(a - b) + (c - d) = ?$ | **2.** $(b - c) + (d - a) = ?$ |
| **3.** $(a + b - c) - d = ?$ | **4.** $d + (b - c - d) = ?$ |
| **5.** $[a - d + c] + [d - e] = ?$ | **6.** $[a + b] + [d - e - c] = ?$ |
| **7.** $\{h - k\} + [m - n] = ?$ | **8.** $\overline{s + t} + \overline{u - v} + \overline{p - q} = ?$ |
| **9.** $(f + g) - (h - j) + k = ?$ | **10.** $(u + v) - (x + y) - (u - z) = ?$ |
| **11.** $n - (p + q) - (s - t) = ?$ | **12.** $(a - b) - \overline{c + d} - [e + f] = ?$ |

Show by inserting arithmetic numbers for letters that the expressions in the last four exercises change their values when the signs of aggregation are omitted.

(4) *Multiplication.*

$a \times b$ means to multiply the number $b$ by the number $a$. Since this is the same as multiplying $a$ by $b$, we have

$$a \times b = b \times a.$$

Using the dot to indicate multiplication,

$$a \times b = a \cdot b = b \cdot a.$$

Usually we do not use either the cross or the dot; we simply write $ab$, thus:

$$a \times b = a \cdot b = ab = ba.$$

Notice that this is contrary to the rule in arithmetic:

$$2\tfrac{1}{2} = 2 + \tfrac{1}{2}, \quad \text{not } 2 \times \tfrac{1}{2}.$$

The numbers $a$ and $b$ are called **factors**, and $ab$ is their **product.**

Similarly we have

$$a \times b \times c = a \cdot b \cdot c = abc,$$

and the "factors" $a$, $b$, $c$ may be taken in any order. Thus:

$$abc = acb = bca, \text{ and so on.}$$

This applies to the product of any number of factors.

We also may use the signs of aggregation:

$$\begin{aligned} a \times (b + c) &= a\,(b + c) \\ &= ab + ac. \\ a \times [b + c - d] &= a\,[b + c - d] \\ &= ab + ac - ad. \end{aligned}$$

(5) *Division.* The symbol $\frac{a}{b}$ means the quotient of $a$ divided by $b$, this quotient being a number which when multiplied by $b$ will give $a$.

In place of $\frac{a}{b}$ we may use $a \div b$.

If we call the quotient $q$, then since the quotient multiplied by the divisor equals the dividend, we have,

$$a = bq \text{ equivalent to } \frac{a}{b} = q.$$

Let the student carefully note our definition:

*The operation of dividing a by b consists in finding a number q which if multiplied by b gives a. That is,* $\frac{a}{b}$ *or* $a \div b$, *each stands for that number q, such that* $bq = a$.

The expression $\frac{a}{b}$ is called a **fraction,** $a$ being the **numerator** and $b$ the **denominator.** The numbers $a$ and $b$ are called the **terms** of the fraction.

Then, as in arithmetic, we have

$$\frac{2}{3} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{n \cdot 2}{n \cdot 3};$$

so we may have $a$ in place of 2 and $b$ in place of 3, and may write

$$\frac{a}{b} = \frac{3\,a}{3\,b} = \frac{na}{nb}.$$

*Rule. The two terms of a fraction may be multiplied by the same number without changing the value of the fraction.* That is,

$$\text{if } \frac{a}{b} = q, \quad \text{then} \quad \frac{na}{nb} = q.$$

(6) *Cancellation.* If both terms of a fraction contain the same factor, this common factor may be cancelled out. Thus,

$$\frac{\not{n}a}{\not{n}b} = \frac{a}{b}.$$

(7) *Division of Products.* When a product of numbers is to be divided by a number, or by a product of numbers, common factors should first be removed by cancellation.

$$\frac{144 \cdot 56 \cdot 18}{36 \cdot 12 \cdot 14} = \frac{\not{3} \cdot \not{2} \cdot \not{2} \cdot \not{3} \cdot \not{2} \cdot \not{2} \cdot \not{7} \cdot \not{2} \cdot 2 \cdot 2 \cdot \not{3} \cdot 3 \cdot 2}{\not{3} \cdot \not{3} \cdot \not{2} \cdot \not{2} \cdot \not{3} \cdot \not{2} \cdot \not{2} \cdot \not{7} \cdot \not{2}} = 24.$$

Likewise

$$\frac{24\,amnr}{8\,bn} = \frac{3 \cdot \not{2} \cdot \not{2} \cdot \not{2}\, a\, m\, \not{n}\, r}{\not{2} \cdot \not{2} \cdot \not{2}\, b\, \not{n}}$$
$$= \frac{3\,a\,m\,r}{b}.$$

Especial attention is called to the fact that if you are called upon to divide a given number by a certain number and then to multiply by that same number, the result will be the given number. Thus, if you are told to divide 629 by 250 and then to multiply by 250 the result is 629. Or, in arithmetic form,

$$250 \cdot \frac{629}{250} = 629.$$

In all cases,

$$a \cdot \frac{b}{a} = b,$$

no matter what the values of $a$ and $b$.

The rule that you can multiply the numerator and denominator by the same number and not change the value of the fraction, gives us the simplest method of reducing a complex fraction to a simple one. You have but to find the least common denominator of all the fractions found in both numerator and denominator, and multiply both terms of the fraction by it. Thus,

given $$\frac{3\frac{2}{3}}{2\frac{7}{15}}.$$

The least common denominator of the fractions is 15. Multiply the numerator and denominator by 15:

$$\frac{3\frac{2}{3}}{2\frac{7}{15}} = \frac{55}{37}.$$

A little practice, to make the student familiar with this method, is worth while.

**Exercises.** Simplify the following fractions:

1. $\dfrac{1\frac{1}{2}}{3}$.  2. $\dfrac{\frac{3}{5}}{\frac{7}{10}}$.  3. $\dfrac{\frac{1}{2}+\frac{1}{4}}{1+\frac{2}{3}}$.

4. $\dfrac{75\frac{1}{3}}{81\frac{5}{6}}$.  5. $\dfrac{\frac{1}{5}+\frac{1}{3}+\frac{1}{15}}{\frac{2}{3}-\frac{1}{15}}$.  6. $\dfrac{728\frac{9}{11}}{824}$.

**Exercises for Review.** Find the numerical value of each of the following expressions when the letters are replaced by the given numbers.

1. $a + b + c$; $a = 3,\ b = 4,\ c = 5$.
2. $a - b + c$; $a = 10,\ b = 3,\ c = 4$.
3. $(a + b) + (c - d)$; $a = 21,\ b = 15,\ c = 40,\ d = 28$.
4. $(a - b) - (c + d)$; $a = 93,\ b = 22,\ c = 12,\ d = 18$.
5. $p\,(q + r - s)$; $p = 15,\ q = 65,\ r = 32,\ s = 12$.
6. $(p + q)\,(p - q)$; $p = \frac{37}{12},\ q = \frac{35}{12}$.
7. $\dfrac{h + k}{h - k}$; $h = \dfrac{1}{5},\ k = \dfrac{1}{6}$.
8. $\dfrac{mn}{m + n}$; $m = \dfrac{3}{7},\ n = \dfrac{15}{8}$.
9. $\dfrac{x + (y - z)}{x - (y + z)}$; $x = \dfrac{3}{2},\ y = \dfrac{2}{3},\ z = \dfrac{1}{6}$.
10. $\dfrac{pqr}{p + q + r}$; $p = 56,\ q = 112,\ r = 28$.
11. $\dfrac{1 + \dfrac{a}{b}}{1 - \dfrac{a}{b}}$; $a = 4,\ b = 5$.
12. $[r + (s - t)] \cdot (s + t)$; $r = \frac{7}{6},\ s = \frac{1}{2},\ t = \frac{1}{3}$.
13. $\dfrac{a + (b - c)}{a - (b + c)}$; $a = 2,\ b = \dfrac{1}{3},\ c = \dfrac{1}{4}$.
14. $\dfrac{\dfrac{m}{n} - \dfrac{n}{m}}{mn}$; $m = \dfrac{3}{4},\ n = \dfrac{2}{3}$.

# CHAPTER II

## MEASUREMENTS — LENGTHS, ANGLES, AREAS, VOLUMES

**9. English Units of Length.** You are familiar with the yardstick, the foot rule, and the inch. To tell some one how long a certain line is you might say that it measures so many yards; the yard is then your unit of length. To state the length of a shorter line you might tell the number of feet which it contains; the foot is then your unit of length; for a still shorter line, such as you might draw on a sheet of paper, the inch would be a convenient unit. Quite long lines, as the length of fence around a farm, are measured with the rod as the unit of length; for distances between cities we use the mile as the unit.

So we have the following common English units of length:

inch, foot, yard, rod, mile.

**Exercise.** How many feet in 3 yards? In $5\frac{1}{2}$ yards? In $n$ yards? How many inches in each of these? How many rods in 1 mile? In $2\frac{1}{4}$ miles? In $m$ miles?

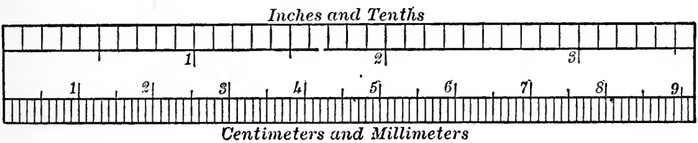
**10. Metric (or French) Units of Length.** The French have chosen a different set of units to measure lengths of lines. This is based on the decimal system and is commonly used in scientific work. In this system there is a unit corresponding nearly to the English yard, and called a *meter*. It is a little longer than the yard and represents one ten-millionth part of the distance from the earth's equator to the pole, measured along a meridian. One tenth of a meter, about four inches, is called a *decimeter;* one hundredth of a meter is a *centimeter,* a little less than half an inch; one thousandth of a meter is a *millimeter*. For long lines, where we would use the mile, the French unit is the kilometer, or one thousand meters, a little more than half a mile.

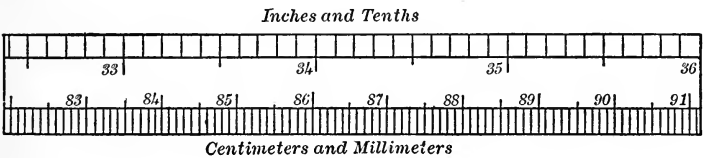
Thus the common metric units of length are:

millimeter, centimeter, decimeter, meter, kilometer.

**Exercise.** How many decimeters in half a meter? In 2.3 meters? In $m$ meters? How many centimeters in each of these?

**11. Comparison of English and Metric Units of Length.** The figures below show parts of a ruler, one edge of which is divided into inches and tenths of inches, the other into centimeters and millimeters. Such a ruler will be found inside the back cover of this book.




**Exercise 1.** From the first figure read off, as exactly as possible, the number of inches in a centimeter; the number of centimeters in an inch.

**Exercise 2.** Draw a line several inches long. Measure it with the edge of the ruler marked in common units; measure it again in metric units. From these measurements find the number of inches there are in a decimeter. Solve as in the following example.

*Example.* Suppose the line measures 4.5 inches and also 1.14 decimeters. To find the number of inches in one decimeter:

Let $i$ = the number of inches in one decimeter.
Then $1.14\,i$ = the number of inches in 1.14 decimeters.
But $4.5$ = the number of inches in 1.14 decimeters;
therefore, $1.14\,i = 4.5$, since both stand for the measurement of the same line.
Therefore $i = 3.95$. (Why?)

So we have 3.95 inches in a decimeter, as nearly as we can tell from the given measurements.

Repeat this experiment, using different lengths of lines.

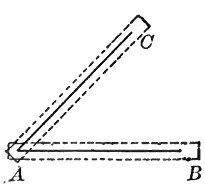
**Exercise 3.** From the second figure above, how many inches in 9 decimeters? Hence how many inches in one decimeter?

**Exercise 4.** Draw a line 2.75 inches long. Measure it in centimeters. Now solve as above to find how many centimeters in an inch. Let $c$ stand for the required number.

**Exercise 5.** Draw a line 0.75 inch long. Measure it in millimeters. Solve as above, to find the number of millimeters in an inch. How does this answer compare with the answer to Exercise 4?

**Exercise 6.** A line measures $x$ inches. How many decimeters does it contain?

**Exercise 7.** A line measures $y$ centimeters. How many feet does it contain?




**12.* Angles — Definitions and Notation.** Suppose that we partially open a fan, or two of the arms of a folding ruler. Suppose also that a line is drawn on each arm of the ruler, starting from the pivot and following the middle of the arm.

*Definitions.* The figure so formed by two straight lines starting out from the same point is called an **angle**. The

* At this point Chapter I of Geometry may be taken up if closer correlation is desired.

point marked $A$ in the figure (the pivot) is called the **vertex** of the angle. The lines $AB$ and $AC$ are called the **arms** of the angle.

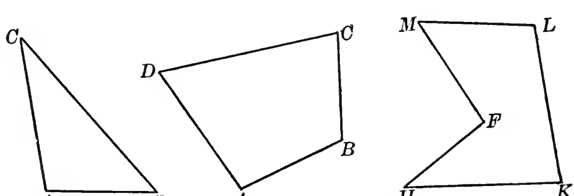
*Notation.* We shall often use a single letter, usually a capital, placed near a point in a figure to designate that point. If we designate a certain point by $A$, and another point by $B$, the straight line through these two points is called *the line AB* or *the line BA*. We would say "the line $AB$" when the line is drawn from $A$ to $B$; we would say "the line $BA$" when the line is drawn from $B$ to $A$. Often it makes no difference which we use; in other cases a distinction is necessary.

To designate the angle formed by the lines $AB$ and $AC$, we say "the angle $BAC$," or, "the angle $CAB$." The first means that in opening up the angle we regard $AB$ as a fixed arm and $AC$ as revolving; the second means that $AC$ is the fixed arm and that $AB$ is revolving. Often it makes no difference which notation is used. In either case, to designate an angle, first name a point on the fixed arm, then name the vertex, and finally name a point on the movable arm.

In fixed figures, angles may usually be read either way.

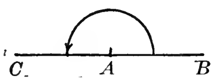
The symbol for the word "angle" is $\angle$; so $\angle BAC$ means "angle $BAC$."

**Exercise.** In each of the figures below read off the points, lines, and angles marked in it.

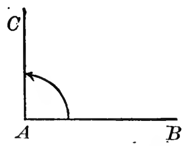



**13. Classification of Angles.** Angles are classified according to the amount of turning done in separating the arms.

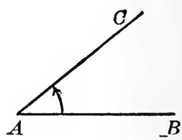
In the figure on p. 16 suppose $AB$ to be fixed and $AC$ to be revolving; when $AC$ has revolved half way around, so that it lies just opposite to $AB$ and forms one straight line with it, the angle $BAC$ is called a **straight angle**; half of a straight angle is a **right angle**; if $AC$ turns through less than a right angle it forms with $AB$ an **acute angle**; if $AC$ turns through more than a right angle but less than a straight angle, it forms with $AB$ an **obtuse angle**; more than a straight angle is called a **reflex angle**; a complete turn is called a **perigon.**



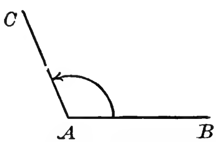
*Straight Angle B A C.*



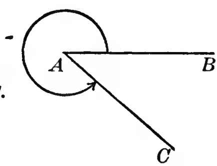
*Right Angle B A C.*



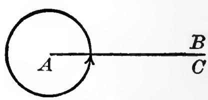
*Acute Angle B A C.*



*Obtuse Angle B A C.*



*Reflex Angle B A C.*



*Perigon B A C.*

**Exercise.** Classify each of the angles in the figures on p. 17.

**14. Measurement of Angles.** To state the size of an angle we adopt some standard angle as a unit, and say how many of these units are needed to fill the given angle. Two different units are in common use, one called the **degree** the other the **radian.**

*Degree Measure of Angles.* When a perigon is divided into 360 equal parts, each such part is called a **degree.**

So we have

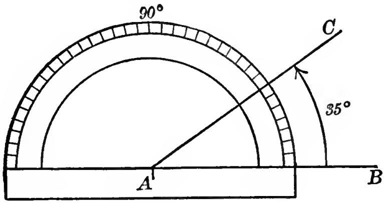
360 degrees = a perigon.

Then 180 degrees = a straight angle

and 90 degrees = a right angle.

The symbol for degrees is °, so that 10° means "ten degrees."

Angles are usually measured with a protractor (see inside

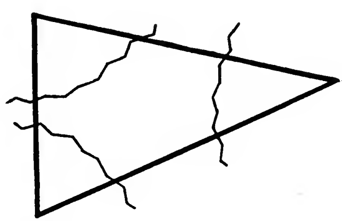


*Protractor*

of back cover) as shown in the above figure. To measure a reflex angle, measure its excess over a straight angle, or measure what is lacking to make a perigon.

**Exercise 1.** Draw ten different angles, some acute, some obtuse, and some reflex. Measure each and write its value on your figure.

**Exercise 2.** Draw a triangle. Measure each angle. What is their sum? Repeat this with another triangle of different shape.



**Exercise 3.** Draw a triangle and tear apart as in the above figure. Place the three angles with their vertices

together, one angle next to the other, without overlapping. What is the sum of the angles of the triangle?

**Exercise 4.** Repeat Exercise 2, using a quadrilateral, that is, a figure bounded by four straight lines. Do not draw a square or a rectangle, but rather a figure whose sides and angles are quite unequal.]

**Exercise 5.** Repeat Exercise 3, using a quadrilateral.

**Exercise 6.** Repeat Exercises 2 and 3, using a pentagon, that is, a figure bounded by five straight lines.

*Radian Measure of Angles.* In this system the unit of measure is a **radian,** instead of a degree as in the system just considered. You can easily make a protractor graduated in radians.

**Exercise 1.** On stiff paper, or, better, light cardboard, draw a circle with a radius of, say, two inches. Carefully cut it out and mark a point on the circumference or rim. On a good-sized sheet of paper draw a straight line and mark off on it parts, each equal to the radius of the circle. Roll the circle carefully along this line, starting with the marked point on the rim placed at the beginning of the first division on the line. Each time that a point on the rim of the rolling circle reaches a division point on the line, mark that point on the rim. Now draw lines from the center of the circle to the points marked on the rim. You then have a series of equal angles, each of which is one radian.

**Exercise 2.** Define a radian.

**Exercise 3.** By rolling the circle so that it makes just one complete turn, find approximately how many radians there are in a perigon. You will find a little more than six radians. Estimate the decimal part as well as you can.

**15. The Number $\pi$.** The number of radian units in a perigon is not a whole number, as you found in the last exercise. Nor can this number be expressed either by a fraction or by a terminating decimal. It is a so-called *incommen-*

*surable number* and by general agreement is always indicated by $2\pi$, $\pi$ being a Greek letter called "pi."

We therefore have

$$2\pi \text{ radians} = \text{a perigon} = 360 \text{ degrees.}$$
$$\pi \text{ radians} = \text{a straight angle} = 180 \text{ degrees.}$$

Then $$1 \text{ radian} = \frac{180}{\pi} \text{ degrees.} \quad (\text{About } 57^\circ.3.)$$

The number for which $\pi$ stands, to four decimal places, is 3.1416; less exactly it is $\frac{22}{7}$. It should be remembered that both of these values are only approximate.

**Exercises.**

1. From Exercise 3 above, find as nearly as you can the number of radii that must be taken to equal the length of the circumference of a circle. The number of diameters. (A diameter of a circle is a line through the center and terminated each way by the circumference.)

2. If $c$ denotes the circumference of a circle, $d$ the diameter, and $r$ the radius, show that

$$c = 2\pi r,$$

and that

$$c = \pi d.$$

State these equations in words and learn them.

3. Using the radian scale on your protractor, repeat Exercise 2 (p. 19).

4. Using the radian scale on your protractor, repeat Exercise 4 (p. 20).

5. Using $\frac{22}{7}$ as the value of $\pi$, find the number of degrees in one radian. Repeat, taking $\pi$ as 3.1416. In each case carry the result to three decimal places.

6. Draw an acute angle. Measure it with the degree scale on your protractor, then with the radian scale. Solve as in the example worked out on p. 16 to find the number of degrees in a radian.

7. Draw an obtuse angle and do the work as instructed in Exercise 6.

8. Draw a reflex angle and repeat work of Exercise 6.

How do the answers to Exercises 6, 7, 8 compare? How do they compare with results of previous work?

9. How many radians are there in the sum of the angles of a pentagon?

**16. Measurement of Areas.** If a unit of length is chosen, then a square whose side is that unit of length is our unit of area. To measure an area we must find how many such square units are required to cover it.

**Exercise 1.** On cross-section paper draw a rectangle of any convenient base and height. How many linear units are there in the base? In the height? How many square units in the area? To answer the last question count the squares. Do your numbers fit the formula

Area of rectangle = base times height?

In letters, this formula may be written

$$a = bh.$$

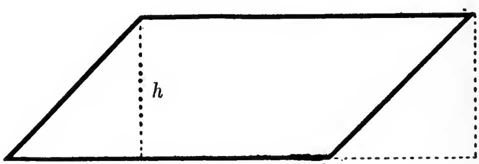
Verify this formula by drawing several other rectangles such as: $a = 3, b = 4$; $a = 5, b = 2$; $a = \frac{1}{2}, b = \frac{1}{3}$.

When the rectangle is a square whose side is $b$, its area is

$$a = bb.$$

In place of $bb$ we usually write $b^2$, so that

$$a = b^2. \quad (a \text{ equals the square on } b.)$$




b

**Exercise 2.** Draw a parallelogram, that is, a quadrilateral whose opposite sides are parallel. Cut this out from your paper. Now cut off a triangle from one side and fit it on the other. Is the area of your parallelogram equal to the area of a rectangle of the same base and height?

Repeat Exercise 1, using a parallelogram. So verify the statement:

Area of parallelogram = base times height.

Using letters: $a = bh,$

as for the rectangle.

**Exercise 3.** On cross-section paper draw a triangle, taking one of its sides along one of the ruled lines. Call this side the base. Draw a line at right angles to the base and leading to the opposite corner of the triangle. Call this line the altitude. How many units are there in the length of the base? In the altitude? Find as well as you can the number of square units in the area of the triangle, adding parts of squares to make whole squares. Do your numbers fit the formula

Area of triangle = $\frac{1}{2}$ (base times height)?

Draw several other triangles and in each case verify the formula

$$a = \tfrac{1}{2}\, bh.$$

**Exercise 4.** Draw a circle on your paper; also draw a triangle with its base equal in length to the circumference of the circle and its height equal to the radius of the circle. Count the number of squares in each and see how they compare, approximately.

How would you find the area of a circle? Letting $a$ stand for the number of square units in the area, $c$ for the number of linear units in the circumference, and $r$ for the number of linear units in the radius, we have the statement

$$a = \tfrac{1}{2}\, cr;$$

but we have shown that $c = 2\,\pi r;$

therefore $a = \tfrac{1}{2} \cdot 2\,\pi r \cdot r$

$$= \pi r^2.$$

**Exercise 5.** On cross-section paper draw a circle with any convenient radius, such as five times the side of one square. Find as nearly as possible the number of square units in the area of the circle. Also find the area of a square whose side equals the radius of the circle. Do your numbers fit the formula

Area of circle = $\pi$ times square on radius?

Draw several other circles of various sizes and in each case verify the formula

$$a = \pi r^2.$$

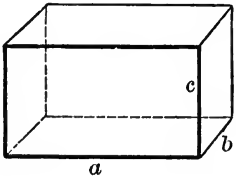
**17. Measurement of Volume.** To measure the cubical content of a solid body we choose as our unit of measure a cube whose edge is one linear unit. The number of such cubic units contained in the body to be measured is called its volume.

For example, a pint jar will contain about 27 cubic inches of water; that is, its volume is 27 cu. in.

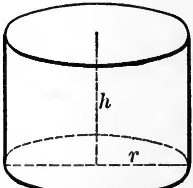
The volumes of a few regular bodies are given below, each in terms of the dimensions of the body. The dimensions in each formula should be expressed in the same unit of length. State each formula in words.

**Exercises.**

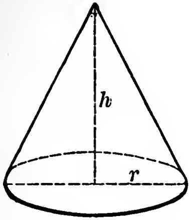
1. Measure the dimensions of a rectangular parallelopiped. Find its volume by immersing it in water in a graduated beaker and noticing how much the water rises. Do your numbers fit the formula?



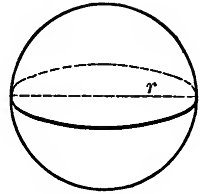

*Rectangular Parallelopiped*
*Volume = abc.*




*Cylinder*
*Volume = $\pi r^2 h$*




Cone
Volume = $\frac{1}{3}\pi r^2 h$.




Sphere
Volume = $\frac{4}{3}\pi r^3$.
($r^3 = r \cdot r \cdot r$)

**2.** Repeat Exercise 1 with a cylinder.

**3.** Repeat Exercise 1 with a cone.

**4.** Repeat Exercise 1 with a sphere.

**5.** Calculate the volume when the dimensions are as below. (Take $\pi$ as $\frac{22}{7}$ or $3\frac{1}{7}$.)

(*a*) Cone.

$r = 2, h = 4$; $r = 1\frac{1}{2}, h = 2\frac{1}{2}$; $r = 3\frac{1}{3}, h = \frac{2}{3}$.

(*b*) Cylinder.

$r = 3, h = 2$; $r = \frac{2}{3}$, $h = 1\frac{1}{4}$; $r = 4\frac{1}{5}, h = 2\frac{1}{3}$.

(*c*) Sphere.

$r = 5$; $r = 3\frac{1}{2}$; $r = 2\frac{2}{3}$.

**6.** Divide the volume of a cylinder by the volume of a cone having the same base and height. What is the quotient?

**7.** If a cylindrical glass is filled with water and then a solid cone of the same base and height is placed in the glass, what part of the water will be forced out?

**8.** A cone-shaped funnel is 8 inches across the top and 6 inches deep. How much water will it hold?

**9.** Measure the dimensions (height and radius) of a pint jar, using the inch as the unit of length. How many cubic inches of water will the jar hold? Is "a pint a pound"?

**10.** A sphere and a cylinder have the same radius. How high should the cylinder be to have the same volume as the sphere? First take the radius, say, 3 inches; then solve again taking the radius equal to $r$.

**Summary.**

### Measurement of Length

| English Units | | Metric Units | |
|---|---|---|---|
| Inch | (in.) | Millimeter | (mm.) |
| Foot | (ft.) | Centimeter | (cm.) |
| Yard | (yd.) | Decimeter | (dm.) |
| Rod | (rd.) | Meter | (m.) |
| Mile | (mi.) | Kilometer | (km.) |

1 rod = $16\frac{1}{2}$ ft. 1 mile = 320 rods = 5280 ft.
1 meter = 10 decimeters = 100 cm. = 1000 mm.
1 km. = 1000 meters.

Angles: straight angle, right angle, acute angle, obtuse angle, reflex angle, perigon.

Angle measure: 360 degrees make a perigon.

$2\pi$ radians make a perigon.

The letter $\pi$ denotes the number of times the diameter of a circle is contained in the circumference; $2\pi$ is the number of times that the radius is contained in the circumference.

$$c = \pi d = 2\pi r.$$
$$\pi = 3.14159 + \cdots = \tfrac{22}{7}, \text{ approximately.}$$

One straight angle = 180 degrees = $\pi$ radians.

One right angle = 90 degrees = $\frac{\pi}{2}$ radians.

The sum of the angles of a triangle is two right angles.

The sum of the angles of a quadrilateral is four right angles.

### Areas

| Rectangle or Parallelogram | Triangle | Circle |
|---|---|---|
| $a = bh.$ | $a = \frac{1}{2}bh.$ | $a = \pi r^2.$ |
| $b = \frac{a}{h}.$ | $b = \frac{2a}{h}.$ | $a = \frac{1}{2}cr.$ |
| $h = \frac{a}{b}.$ | $h = \frac{2a}{b}.$ | |

## Volumes

Rectangular Parallelopiped

Volume $= abc.$

Cylinder

Volume $= \pi r^2 h.$

Cone

Volume $= \frac{1}{3}\pi r^2 h.$

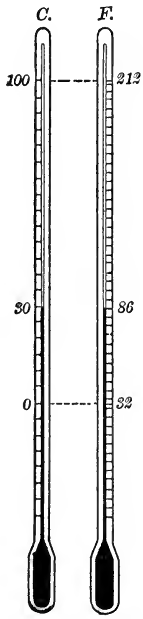
Sphere

Volume $= \frac{4}{3}\pi r^3.$

# CHAPTER III

## MEASUREMENTS CONTINUED — TEMPERATURE, WEIGHT AND DENSITY, FORCE

**18. Measurement of Temperature.** A *thermometer* is an instrument for measuring temperature. It consists of a small bulb or reservoir filled with mercury and connecting with a very narrow vertical tube. When the mercury in the bulb is heated it expands and rises into the tube. So the height of the mercury in the tube indicates the degree of heat to which the bulb is exposed.




*The Centigrade Scale of Temperature.* Immerse a thermometer in melting ice and mark the point on the tube where the mercury stands zero. This is called the freezing point. Next immerse in boiling water and mark the new point 100. This is called the boiling point. Divide the space between zero and 100 into 100 equal parts. Each such part is called *one degree Centigrade.* The graduations are extended below the zero point to indicate temperatures below zero.

*The Fahrenheit Scale of Temperature.* Proceed as above, except that the freezing point is marked 32 and the boiling point 212. Divide the space between these into 180 equal parts. Each such part is called *one degree Fahrenheit.*

The adjacent figure shows two thermometers, one graduated Centigrade, the other Fahrenheit. Both scales may be placed on the same thermometer.

**Exercises.**

**1.** Immerse two thermometers, one graduated C. and the other F., or one thermometer with both scales on it, in a cup of cold water and record the reading on each scale. Next immerse in a cup of hot water and record the readings on each. From these readings find how many degrees Fahrenheit equal one degree Centigrade. Write out each step in full as in the example worked out on p. 16.

**2.** How many degrees Fahrenheit are equal to 100 degrees Centigrade? From this find how many degrees Fahrenheit equal one degree Centigrade.

**3.** When the Fahrenheit scale reads 70°, what is the Centigrade reading ?

**4.** When the Centigrade scale reads 35°, what is the Fahrenheit reading?

**5.** Let $F$ denote the Fahrenheit reading, and $C$ denote the corresponding Centigrade reading. Find $C$ when $F =$ 86°; 68°; 50°; 41°; 14°. Find $F$ when $C =$ 10°; 20°; 30°; 70°; 10° below zero.

**6.** Show that for all temperatures above freezing

$$F = 32 + \tfrac{9}{5}C;$$

and that

$$C = \tfrac{5}{9}(F - 32).$$

**7.** At what temperature Centigrade is the Fahrenheit reading equal to three times the Centigrade reading?

**19. English Units of Weight.** The *weight* of a body is measured by the pull of the earth upon that body. When we say that a body weighs five pounds, we mean that the earth pulls on it with five times the pull on a one-pound weight.

Now what is a one-pound weight? It is the weight of a certain piece of platinum which is carefully preserved by the British government, and used to test other pound weights.

When this weight is hung on a spring balance, the place to which the pointer moves is marked 1. Any other weight which pulls the pointer to the same place is then also a pound. By hanging on two such weights, then three, and so on, each time marking the place where the pointer stops, we get a spring balance graduated in pounds. To weigh a body we need only hang it on the balance and notice where the pointer stops.

Based on the pound we have the English units of weight as follows:

1 pound (Avoirdupois) = 16 ounces = 7000 grains.
100 pounds = 1 hundredweight.
2000 pounds = 1 ton.
2240 pounds = 1 long ton.

There is another pound, called the Troy pound, in less common use. It is divided into 12 ounces and 5760 grains.

We shall deal only with the pound avoirdupois.

**Exercises.**

1. How many ounces in 2 lb.? In $3\frac{1}{2}$ lb.? In $4\frac{1}{3}$ lb.? In $n$ lb.? In $(a + b)$ lb.? How many grains in each of these?

2. How many pounds in 40 oz.? In 75 oz.? In $m$ oz.? In $(h + k)$ oz.? How many grains in each of these?

3. How many pounds in 10,000 gr.? In 1400 gr.? In $r$ gr.? In $(c - d)$ gr.?

4. How many grains in $a$ lb. $+ b$ oz.? How many ounces in $r$ lb. $+ t$ gr.?

**20. Metric (or French) Units of Weight.** In France the unit of weight is the weight of a cubic centimeter of water at a temperature of 4 degrees Centigrade. It is called a **gram.** This is a rather small unit, for it takes nearly 500 grams to make a pound; therefore a larger unit, namely the **kilogram** or 1000 grams, is more commonly used. A kilogram equals about 2.2046 pounds. A half kilo would then be a little more than one pound, and this is the official unit which takes the place of the English pound.

For delicate weighings, such as are needed in physics and chemistry, the gram is divided into smaller units thus:

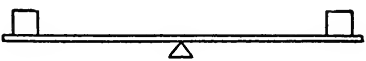
one tenth of a gram = a decigram;
one hundredth of a gram = a centigram;
one thousandth of a gram = a milligram.

**Exercises.**

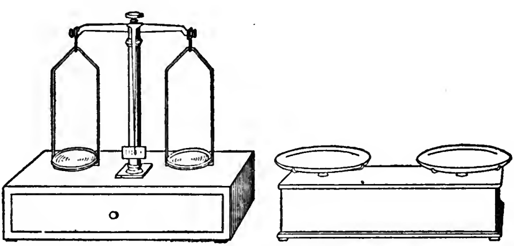
1. How many decigrams in 10 grams? In 1.5 grams? In $g$ grams? In $(m + n)$ grams? How many centigrams?

2. How many milligrams in 5 decigrams? In 3 centigrams? In $k$ decigrams? In $k$ decigrams $+ l$ milligrams? How many grams in each of these?

**21. The Beam Balance.** For accurate weighing the spring balance is replaced by the *beam balance.* This is constructed on the following principle.



When a beam is supported at its middle on a knife edge and equal weights are placed at equal distances from the knife edge, the beam will balance.



For convenience a scalepan is usually fastened to the beam at each end; in the druggists' scales the pans rest on top of the beam; in the chemical balance the pans are hung from the beam by wires.

**Exercises.**

1. In one pan of a beam balance place a weight of one pound; how many grams must be placed in the other pan to balance? How many grams in a pound?

2. Place a piece of iron or other substance in one scale-pan; balance it first by English units, then by metric units, noting the amount of each. From your notes calculate the number of grams in an ounce. Write out the solution in full as in the example worked out on p. 16.

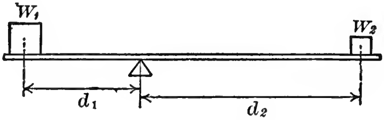
Repeat this experiment several times with different amounts of material.

3. A certain weight is balanced by $q$ lb. and $r$ oz. How many grams does it weigh?

**22. The Lever.** A beam placed on a knife edge is usually called a *lever*. The ordinary "see-saw" is a rough sort of lever. The knife edge is called the *fulcrum;* the parts of the beam on each side of the fulcrum are called the arms of the *lever*.

If two boys, one considerably heavier than the other, were to play see-saw, would they sit at equal distances from the fulcrum? Why not? Which one would sit farther out?

This illustrates the following rule, called the *principle of the lever:*




*Rule.* In order that a weight $W_1$, placed at a distance $d_1$ from the fulcrum, shall balance a weight $W_2$ placed at a distance $d_2$ from the fulcrum, we must have

$$W_1 \times d_1 = W_2 \times d_2.$$

**Exercises.**

**1.** State the last equation in words.

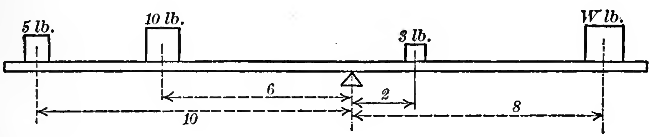
**2.** If a boy weighing 75 lb. sits 6 feet from the fulcrum, where should a boy weighing 100 lb. sit to balance the beam? Give neat solution, letting $d$ stand for the number of feet in the required distance.

**3.** How far from the fulcrum must a 12-lb. weight be placed to balance a 30-lb. weight placed 5 feet from the fulcrum?

**4.** A weight of 200 grams is placed 25 centimeters from the fulcrum. How far from the fulcrum must a weight of half a kilogram be placed to balance?

When several weights are placed at various distances to one side of the fulcrum, and other weights at various distances to the other side of the fulcrum, the beam will balance when *the sum of the products formed by multiplying each weight on one side by its distance from the fulcrum is equal to the sum of such products from the other side.*

*Example.* Weights of 5, 10 and 3 lb. respectively are placed at the distances shown in the figure. What weight $W$ placed 8 feet from the fulcrum will balance the beam?




*Solution:* According to the last rule we must have

$$5 \times 10 + 10 \times 6 = 3 \times 2 + W \times 8.$$

That is, $110 = 6 + 8W,$

or $104 = 8W.$

Therefore, $W = 13$ lbs.

Check this answer.

**Exercises.** Find what is necessary for balance in each of the following cases.

| | | To left of fulcrum | | | To right of fulcrum | | |
|---|---|---|---|---|---|---|---|
| **1.** | $W =$ | 3 lb. | 7 lb.; | | 5 lb. | ? lb. | |
| | $d =$ | 4 ft. | 6 ft.; | | 8 ft. | 2 ft. | |
| **2.** | $W =$ | 8 oz. | 5 oz.; | | ? oz. | 10 oz. | |
| | $d =$ | 7 in. | 4 in.; | | 13 in. | 5 in. | |
| **3.** | $W =$ | 100 gm. | 75 gm.; | | 60 gm. | 20 gm. | |
| | $d =$ | 5 cm. | 8 cm.; | | ? cm. | 10 cm. | |
| **4.** | $W =$ | 2 kgm. | 5 kgm. | 4 kgm.; | 8 kgm. | | |
| | $d =$ | 25 cm. | 15 cm. | ? cm.; | 30 cm. | | |
| **5.** | $W =$ | $2\frac{1}{2}$ lb.; | | | $3\frac{1}{4}$ lb. | 1.4 lb. | 2 lb. |
| | $d =$ | 25 in.; | | | 6 in. | ? in. | $9\frac{1}{2}$ in. |

### 23. Density — Specific Gravity.

**Exercise 1.** Determine the weight of a solid rubber ball in grams. Determine the volume of the ball by dropping it into water in a graduated beaker and noticing the amount the water rises. Express this volume in cubic centimeters. How many grams does this amount of water weigh? Divide the weight of the ball by the weight of the same volume of water. The quotient is called the *density* of the ball.

*Definition.* The density of any substance, when water is taken as the standard, is the quotient obtained by dividing the weight of a given volume of that substance by the weight of an equal volume of water. The density of a liquid is also often called "*specific gravity.*"

**Exercise 2.** Determine the density of iron, wood, stone, and glass by repeating Exercise 1 with pieces of each of these substances. Make a table of your results.

**Other Exercises.**

**3.** Find the density of a solution made by dissolving 10 gm. common salt in 20 gm. water. (Divide the weight of the solution by the weight of an equal volume of water.)

**4.** Find the density of a solution made by dissolving $w$ grams of salt in $n$ grams of water.

**5.** How many grams of salt should be dissolved in 25 cc. water to make a solution whose density is 1.4?

**6.** To 10 cc. of a liquid of sp. gr. 1.2 are added 20 cc. of a liquid of sp. gr. 0.8. What is the sp. gr. of the mixture?

### Table of Densities

| | | | |
|---|---|---|---|
| Aluminum | 2.6 | Lead | 11.4 |
| Brass | 8.4 | Marble | 2.6 |
| Charcoal | 1.6 | Mercury | 13.6 |
| Copper | 8.8 | Nickel | 8.7 |
| Cork | .14–.24 | Paraffin | 0.89 |
| Diamond | 3.53 | Platinum | 21.5 |
| Glass | 2.4–4.5 | Silver | 10.5 |
| Gold | 19.3 | Tin | 7.3 |
| Gutta Percha | 0.97 | Zinc | 7.2 |
| Iron | 7.8 | Water | 1.0 |
| Ivory | 1.8 | | |

### Liquids

| | | | |
|---|---|---|---|
| Alcohol | 0.81 | Glycerine | 1.27 |
| Benzine | 0.90 | Turpentine | 0.88 |
| Ether | 0.73 | Water | 1.00 |

**24. Summary.**

Temperature:
- Centigrade scale: freezing = 0 degrees; boiling = 100 degrees.
- Fahrenheit scale: freezing = 32 degrees; boiling = 212 degrees.

### Units of Weight

| English | | Metrics | |
|---|---|---|---|
| Grain | (gr.) | Milligram | (mgm.) |
| Ounce | (oz.) | Centigram | (cgm.) |
| Pound | (lb.) | Decigram | (dgm.) |
| Hundredweight | (cwt.) | Gram | (gm.) |
| Ton | | Kilogram | (kgm.) |

The Lever. To balance a lever: Multiply each weight on one side of the fulcrum by its distance from the fulcrum and form the sum of these products; do the same with the weights on the other side and form the sum of these products; the two sums must be equal.

The density, or specific gravity, of a substance is the quotient obtained by dividing the weight of a given volume of that substance by the weight of an equal volume of water.

**25. Questions and Problems for Review.**

1. What are the units used to measure angles? Describe each.

2. What are the common English units used to measure lengths, areas, volumes, and weights?

3. What are the French or metric units?

4. Define density.

5. Define the meaning of the words "lever" and "fulcrum."

6. What is the sum of the angles of a triangle in degrees? In radians? What is the sum of the angles of a quadrilateral?

7. On cross-section paper draw a rectangle of any convenient width and height. Draw a triangle of the same width and height as the rectangle. By counting squares verify the formula

$$\text{Area of triangle} = \tfrac{1}{2}\,(\text{area of rectangle}).$$

Express this formula in letters, letting $t$ stand for the area of the triangle and $r$ for the area of the rectangle.

8. Draw a circle; also draw a triangle whose base equals the perimeter of the circle and whose height equals the radius of the circle. By counting squares compare the areas of the two figures as well as possible. If $a$ stands for the number of square units in the area of the circle, $c$ for the number of linear units in the circumference, and $r$ for the number of linear units in the radius, do your numbers fit the equation

$$a = \tfrac{1}{2}\,cr?$$

We have also found experimentally that

$$c = 2\pi r.$$

Using these two equations, show that

$$a = \pi r^2 \quad \text{(p. 23).}$$

9. If one angle is 2 times the size of another angle, and the sum of the two angles is $\pi$ radians, what is the number of radians in each angle? Draw these angles, after you have found the size of each.

*Solution:*

Let $a$ = the number of radians in the first angle.
Then $2a$ = the number of radians in the second angle.
and $3a$ = the number of radians in the sum of the two angles.
But $\pi$ = the number of radians in both angles.
Therefore $3a = \pi$, since $3a$ and $\pi$ stand for the same number of radians.

Then $a = \frac{\pi}{3}$, the number of radians in the first angle,

and $2a = \frac{2\pi}{3}$, the number of radians in the second angle.

Therefore the first angle contains $\frac{\pi}{3}$ radians, and the second angle contains $\frac{2\pi}{3}$ radians.

*Check:* The next step is to check the correctness of our answers. To do this we go back to the original statement of the problem and see whether every statement is fulfilled. To begin with, the one angle must be twice as large as the other.

$$\frac{2\pi}{3} \text{ radians} = 2 \times \frac{\pi}{3} \text{ radians.}$$

Also, the sum of the two angles must be $\pi$ radians.

$$\frac{\pi}{3} \text{ radians} + \frac{2\pi}{3} \text{ radians} = \pi \text{ radians.}$$

Therefore our answers are correct.

10. If one angle is 2 times the size of another angle, and the sum of the two angles is 180 degrees, what is the number of degrees in each angle? Draw.

**11.** Take your answers to exercise 9 and solve to find the number of degrees in each angle. Make use of letters in your solution. How do the answers of exercises 10 and 11 compare?

**12.** The sum of two angles is $\frac{2}{3}\pi$ radians, and one of them is 3 times the size of the other. What is the number of radians in each? Draw.

**13.** State exercise 12 expressing the sum in degrees. Solve and show agreement of answers.

**14.** The sum of three angles is $\frac{7}{4}$ radians. The first is $\frac{1}{3}$ of the size of the second, and the third is $\frac{1}{2}$ of the size of the first. What is the number of radians in each angle? Draw these angles.

**15.** State exercise 14, expressing the sum in degrees and compare, as in exercise 13.

**16.** The length of a rectangle is 3 times its width. The length of the line bounding it (called the perimeter) is 372 mm. What is the length of the rectangle? The width? Draw the rectangle.

**17.** State exercise 16, using the inch unit to express the combined length of the three lines. Solve, draw, and compare.

**18.** Of five lines the first is $2\frac{1}{2}$ times the second, the third is $\frac{1}{2}$ as long as the first, the fourth is as long as the sum of the first and third, and the fifth is $\frac{1}{2}$ as long as the sum of the first and fourth. What is the length of each line if the combined length is 27 millimeters?

**19.** If one of the angles of a triangle is $\frac{1}{3}$ as large as another, and the third is $\frac{1}{4}$ of the sum of the first and second, what is the number of radians in each angle? Draw such a triangle.

**20.** If the first of three angles of a triangle is $\frac{3}{7}$ as large as the second and the third is $\frac{2}{3}$ as large as the first, what is the number of radians in each angle? Draw such a triangle.

**21.** The first of the three angles of a triangle is half as large as the second, and the third is equal to the sum of the first and second. What is the number of degrees in each angle? Draw.

**22.** The first angle of a quadrilateral is $\frac{2}{3}$ as large as the second, the third is equal to the first, and the fourth is equal to the second. What is the number of radians in each angle of the quadrilateral? Draw such a quadrilateral.

**23.** One of the four angles of a quadrilateral is $\frac{1}{5}$ as large as another, a third is equal to the sum of the two, and the fourth is equal to the difference between the two. What is the number of degrees in each angle? Draw such a quadrilateral.

**24.** In the following state the problem in good English, using the word instead of the letter which stands for that word. Solve by substituting the number instead of the letter in the formula on p. 26, and answer the question asked in your problem. Illustrate each problem.

For the rectangle or parallelogram:

(*a*) $b = 7,\quad h = 2,\quad a = ?$ (See formulas on p. 26.)

*Illustration.* If a rug is 7 feet long and 2 feet wide, what is the number of square feet in the rug?

*Solution:* Formula, $a = bh.$

Since $b = 2$ and $h = 7,$

$$a = 2 \cdot 7 = 14.$$

Therefore there are 14 square feet in the area of the rug.

(*b*) $a = 100,\quad b = 8,\quad h = ?$
(*c*) $a = 57,\quad h = 3,\quad b = ?$
(*d*) $a = \frac{7}{15},\quad b = \frac{3}{5},\quad h = ?$
(*e*) $b = 53\frac{2}{3},\quad h = 7\frac{3}{5},\quad a = ?$

For the triangle:

(*f*) $b = 7.5,\quad h = 40,\quad a = ?$
(*g*) $a = 286,\quad b = 459,\quad h = ?$
(*h*) $a = 2\frac{7}{9},\quad h = 1\frac{2}{3},\quad b = ?$

For the circle:

| | | | |
|---|---|---|---|
| (*i*) | $r = 110,$ | $c = ?$ | $a = ?$ |
| (*j*) | $c = 3.5,$ | $r = ?$ | $a = ?$ |
| (*k*) | $d = .42,$ | $c = ?$ | $a = ?$ |
| (*l*) | $a = 49\pi,$ | $r = ?$ | $c = ?$ |

**25.** If a certain sum of money is put at simple interest at a certain rate, for a certain time, you have learned from your arithmetic that

$$\text{interest} = \text{principle} \times \text{rate} \times \text{time}.$$

Using the first letter of each word as we have done in previous examples,

$$i = p \cdot r \cdot t.$$

(*a*) What does $p$ equal in terms of $i$, $r$, $t$? What does $t$ equal in terms of $p$, $i$, $r$? What does $r$ equal in terms of $p$, $i$, $t$?

(*b*) If a sum of 200 dollars is placed at simple interest at 6% for 3 years, what is the interest due? Solve by using formula and answering question asked.

(*c*) The interest on $1200 placed at the rate of 5% was $240. What was the time of the note?

(*d*) At what rate of interest must $1400 be placed in order that it bring $294 interest by the end of three years?

**26.** An ivory ball and an India-rubber ball of the same size together weigh 60 grams. Ivory is twice as heavy as India rubber. What is the weight of each ball?

**27.** A solution of alum and water weighs 456 grams. Alum weighs 1.7 times as much as water. How many grams of alum are there in the solution?

**28.** Marble is 1.5 times as heavy as ivory. 9 marble balls weigh 46.8 grams more than 7 ivory balls. What is the weight of each ball?

**29.** A brass ornament plated with gold weighs 50 grams. There are 5 times as many cubic millimeters of brass as there are of gold, and one cubic millimeter of gold weighs

.0193 grams, and one cubic centimeter of brass weighs 8.4 grams. How many cubic millimeters of each metal are there in the ornament?

**30.** If we know the velocity of a moving body, and the length of time it moves, how may we find the distance it moves?

Let $v$ stand for the velocity, $t$ for the time, and $d$ for the distance; state the answer to the last question in letters.

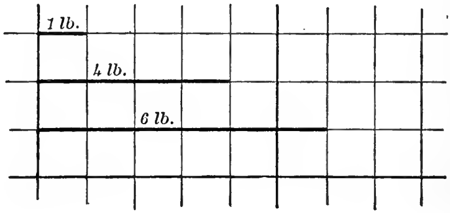
State the formula for $t$ in terms of $d$ and $v$. State the formula for $v$ in terms of $d$ and $t$. Make problems for each of these cases.

## CHAPTER IV

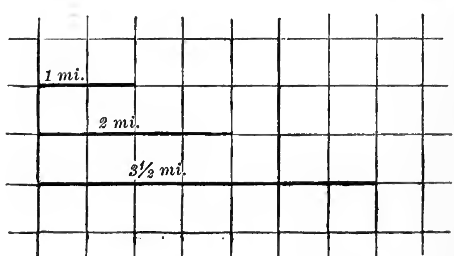
### GRAPHIC REPRESENTATION OF QUANTITY

**26. The Amount of any Quantity Represented by the Length of a Line.** The result of the measurement of any quantity may be expressed by a number which tells us how many units or parts of units of measure are contained in the quantity measured. We may also represent the amount of our quantity by the length of a line; this is especially useful when several quantities are to be compared, because the eye takes in at a glance the various lengths of lines.

*Example* 1. On your section paper let one division of the line represent 1 pound; then four divisions will represent 4 pounds; six divisions will represent 6 pounds; and so on.




*Example* 2. Let two divisions represent a distance of 1 mile. Then 4 divisions will represent 2 miles; 7 divisions will represent $3\frac{1}{2}$ miles.



On the scale of $\frac{1}{2}$ inch to the mile, how wide would a map of the United States be? (The distance east and west is about 3000 miles.)

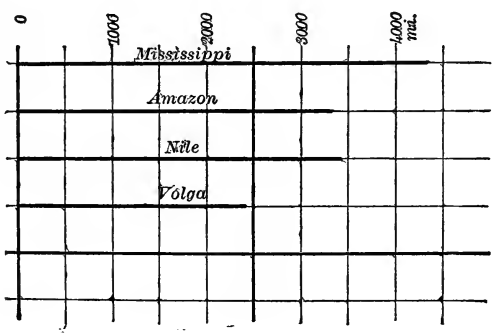
How wide would a map of your state be? How high?

*Example* 3. If a line 1 inch long represents 10,000 people, then a line $2\frac{1}{2}$ inches long represents 25,000 people, a line $\frac{1}{4}$ inch long represents 2500 people.

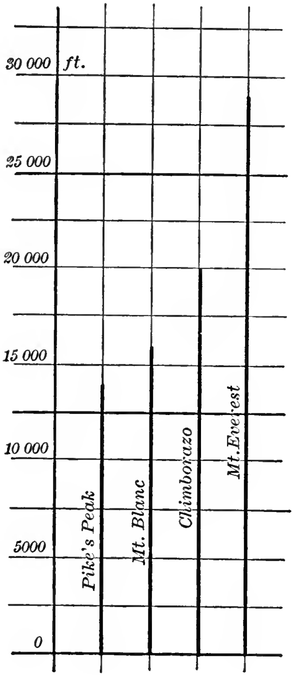

*Example* 4. To compare unequal quantities of the same kind, it is very convenient to represent them by parallel lines drawn under each other. This is illustrated below.

Lengths of rivers.

| | |
|---|---|
| Mississippi | 4300 miles |
| Amazon | 3300 miles |
| Nile | 3400 miles |
| Volga | 2400 miles |




*Example* 5. We may also draw the lines representing the values of quantities perpendicularly. Thus in the following diagram the heavy vertical lines represent the relative heights of four mountain peaks.




Heights of mountains.

| | |
|---|---|
| Pike's Peak...... | 14,000 feet |
| Mt. Blanc....... | 16,000 feet |
| Chimborazo...... | 20,000 feet |
| Mt. Everest...... | 29,000 feet |

**Exercises.** In the following select an appropriate unit and represent by drawings.

**1.** Draw lines whose lengths shall represent 3 lbs.; $2\frac{1}{2}$ lbs.; $2\frac{3}{4}$ lbs.; 20 oz.; 50 oz.

**2.** On the scale of one unit of division on your section paper to the foot, what lengths are represented by the following: 3 divisions; 6 divisions; 12 divisions; 36 divisions; 90 divisions. Draw lines representing 5 feet; $3\frac{1}{2}$ feet; $1\frac{1}{2}$ yards; $\frac{1}{2}$ rod.

Make a diagram of the floor of a room whose dimensions are 10 feet by 15 feet, on a scale of one division to five feet.

**3.** Selecting an appropriate length to represent one day, tell what durations of time the following lengths would represent? 7; 20; $\frac{1}{12}$; $\frac{1}{48}$.

How long a line would be required to represent 10 days? one year? Draw lines to represent 4 hours; 75 minutes; 1 hour and 40 minutes.

**4.** Express by drawings the yearly interest on \$100 at 1%; at 2%; at 4%; at 7%; at 10%.

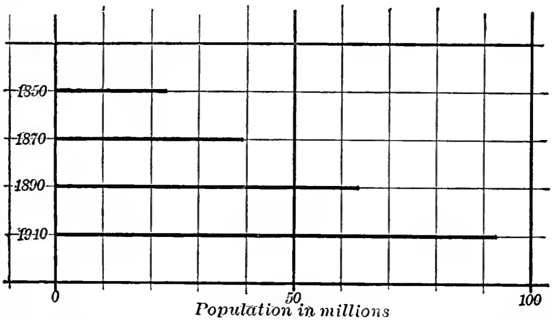
On the same scale represent the yearly interest on \$200 at 1%; on \$400 at 1%; on \$300 at 2%.

**5.** Selecting a length to represent 100,000 people, represent by lengths of lines the population of the following cities:

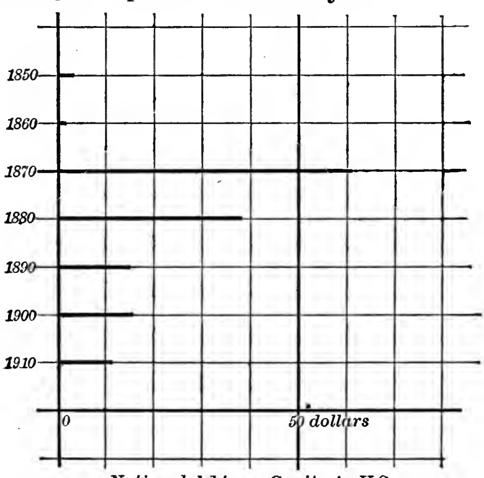
| City: | Milwaukee | Buffalo | Baltimore | Pittsburg | Cleveland |
|---|---|---|---|---|---|
| Population: | 320,000 | 400,000 | 550,000 | 375,000 | 480,000 |

**6.** By measurement on a map, using the scale given on the map, find the distances between various cities. Make a table of the results and then show the distances by lines.

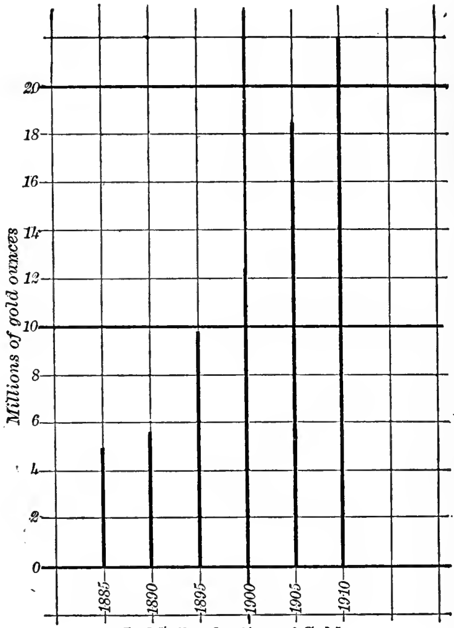
**7.** From the following diagram read off the population of the United States for the various dates.




**8.** From the following diagram read off the amount of national debt per capita in different years.




*National debt per Capita in U.S.*




*World's Production of Gold*

**9.** From the preceding diagram read off the world's production of gold in the various years.

Make diagrams to show the data given in the tables below.

**10.** Distance from New York to Chicago, 910 miles,
to Boston, 160 miles,
to Cleveland, 580 miles,
to Washington, 230 miles.

**11.** Population of countries, 1910:

| | |
|---|---|
| United States, | 93 million people, |
| England, | 45 million people, |
| France, | 39 million people, |

| | |
|---|---|
| Germany, | 63 million people, |
| Italy, | 34 million people, |
| Japan, | 51 million people. |

**12.** Number of people to the square mile, 1910:

| | |
|---|---|
| United States, | 26 persons, |
| England, | 374 persons, |
| France, | 191 persons, |
| Germany, | 311 persons, |
| Italy, | 313 persons, |
| Japan, | 344 persons. |

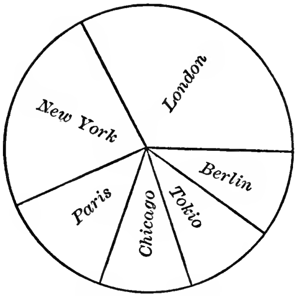
**13.** Distances from the sun to the four inner planets:

| | |
|---|---|
| Sun to Mercury, | 36 million miles, |
| Sun to Venus, | 67 million miles, |
| Sun to Earth, | 93 million miles, |
| Sun to Mars, | 141 million miles. |

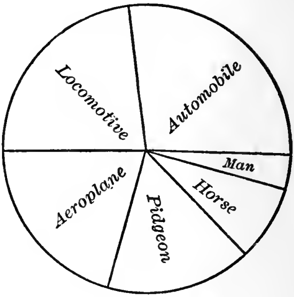
**14.** Make diagrams showing various data. The following are suggested.

Population of your state for every ten years.
Amount of public debt of your state for several years.
Amount of crops — corn, wheat, etc.
Value of crops.
Amount of various manufactures.
Population of leading cities.

The comparison of quantity is also expressed by the comparative sizes of angles. The amount of turning or circular motion is quite as effective in its power to convey comparisons as is the straight line. It is not used so extensively, because a straight line is easier to make. The following pictures illustrate its use.




*Population of Cities*




*Speed Records*

| | | | |
|---|---|---|---|
| London, | 7.2 million, | Automobile, | 142 miles per hour, |
| New York, | 5.2 million, | Locomotive, | 120 miles per hour, |
| Paris, | 2.8 million, | Aeroplane, | 106 miles per hour, |
| Chicago, | 2.2 million, | Pidgeon, | 86 miles per hour, |
| Tokio, | 2.2 million, | Horse, | 43 miles per hour, |
| Berlin, | 2.0 million. | 100 yd. dash, | 21 miles per hour. |

# CHAPTER V

## POSITIVE AND NEGATIVE NUMBERS

### 27. Exercises.

**1.** A boy strikes a ball at exactly opposite points with two mallets at the same time and with the same force. What is the result as to the movement of the ball?

**2.** If he strikes first with the right-hand mallet, and then, after the ball has come to rest, with the left-hand mallet with an equal force, what is the result?

**3.** If he strikes with both mallets at the same time, striking with one mallet with a force which would send the ball 5 feet and with the other mallet with a force which would send the ball 20 feet, what is the result?

**4.** If in Exercise 3 the boy strikes first with one mallet, then, after the ball has come to rest, with the other mallet, what is the result?

**5.** Does it make any difference whether two forces act at the same time or one after the other?

**6.** Two men are driving a stake; one strikes with a force that sends the stake down two inches, the other with a force that sends the stake down three inches. What is the combined effect?

**7.** Three boys are pulling a load on a sled, one with a force of 25 pounds, another with a force of 58 pounds, and the other with a force of 97 pounds. With what force is the load being pulled?

**8.** Two small boys are pulling a small wagon along; one pulls with a force of 25 pounds, and the other pulls with a force of 34 pounds. A boy comes up behind and pulls with a force of 57 pounds in the opposite direction. What is the result?

From the exercises given, and with a little further experiment and thought of your own, you will be thoroughly convinced that forces are constantly acting against each other. An object cannot stop itself when it is once in motion, neither can it change the rate at which it is moving, nor the direction in which it is moving, any more than it can start itself to moving. This truth is known as Newton's first law of motion.

**28. Opposite Qualities.** The exercises given have dealt with forces. Almost all quantities with which we deal may be thought of, and commonly are thought of, as having two opposite qualities which tend to counteract or neutralize each other.

East is opposite to west; if we go east any distance, then west the same distance, we come back to the starting point.

In the same way we have:

North opposite to south;
Up opposite to down;
Right opposite to left;
Forward opposite to backward;
Profit opposite to loss;
Assets opposite to liabilities;
Future time opposite to past time;
Temperature above zero opposite to temperature below zero.

*Example* 1. A man starts from a certain town A and goes 10 miles due east; from the point where he now is he goes 15 miles due west. Where is he then with respect to the original starting point A?

*The answer is:* 5 miles west from A. We must state not only the distance from A, but also the direction.

*Example* 2. A man has $10,000 assets and $15,000 liabilities. What is his financial status?

*The answer is:* He owes $5000. It is not enough to say merely $5000. We must also say whether it is an asset or a liability.

*Example* 3. If at noon the temperature is 20 degrees above zero, and if the temperature rises 10 degrees during the afternoon, then falls 40 degrees during the night, what is the temperature the next morning?

*Answer:* 10 degrees below zero. It is not enough to say 10 degrees. We must also say whether it is above or below zero.

As we have seen, forces, whether acting at the same time or at different times, may act in opposite directions to one another. In mathematics we express the idea that one force acts in the opposite direction to another force by saying that one is the negative of the other, negative meaning opposite.

*Definition.* Whenever a quantity has two opposite qualities, we call one of them **positive** and the other **negative.**

*Notation.* The symbol for positive is $+$, and the symbol for negative is $-$. When no sign is written, the $+$ sign is understood.

*Illustrations.*

If $+$ 100 dollars means gain, then $-$ 100 dollars means loss.

If $+$ 10 feet means 10 feet up, then $-$ 10 feet means 10 feet down.

If $+$ 10 days means 10 days later, then $-$ 10 days means 10 days earlier.

If $+m$ degrees means an angle measured counterclockwise, then $-m$ degrees means an angle measured clockwise.

If $+n$ means count $n$ units forward, then $-n$ means count $n$ units backward.

**Exercise.**

(*a*) If earning money is positive, what is negative?

(*b*) If going west is positive, what is negative?

(*c*) If upstream is positive, what is negative?

(*d*) If the pull of gravity is positive, what is the pull of a balloon?

### 29. Positive and Negative Numbers.

*Definition.* Numbers which express the measurement of positive quantity are called **positive numbers.** Numbers which express the measurement of negative quantity are called **negative numbers.**

*Notation.* A positive number is indicated by the sign + written before the number; a negative number is indicated by the sign − written before it. When no sign is written, the positive sign is understood.

Thus if + 10 means 10 units of quantity of a certain kind, then − 10 means 10 units of quantity of the opposite kind.

Let the student give a number of illustrations.

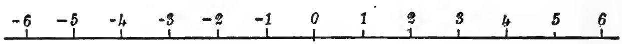
*Absolute Value of a Number.* When we wish to indicate merely the value of a number, without regard to its sign, we use the symbol | |.

Thus $|5| = +5$, and also $|-5| = +5$.

**30. Graphic Representation of Positive and Negative Numbers.** We have already shown how we use lengths of lines to represent numbers, and how the comparative lengths of lines give a clear illustration of comparative values of numbers. We have dealt, however, with numbers of the same quality or sign.

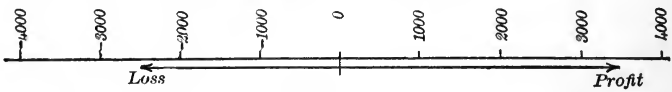
We now show how to represent numbers of opposite quality, so that our diagram will show both value of the number and its quality.

Let us mark off a series of equal distances on an indefinite straight line. Mark one of the points of division zero. This is the point from which we begin to count. Mark the points to the right of this 1, 2, 3 and so on. Mark the points to the left, − 1, − 2, − 3, and so on. Thus



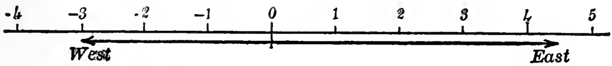

On this scale we can show numbers of opposite quality.

*Example* 1. Represent $3500 profit and $2500 loss.



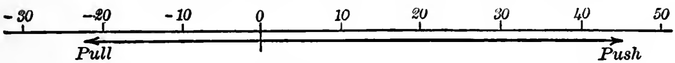

Scale, 1 division to 1000 dollars.

*Example* 2. Represent 4.5 miles east and 3 miles west.




Scale, 1 division to the mile.

*Example* 3. Represent 22 pounds pull and 45 pounds push.




Scale, 1 division to 10 pounds.

Selecting a suitable unit, make diagrams of the following: 4200 births, 2500 deaths.

15 degrees rise, 35 degrees fall.

In all of these diagrams the amount of the quantity is expressed by the number and is represented by a line of the proper length; opposite qualities are expressed by lines drawn in opposite directions. In each case one of the lines represents a positive number and the other line a negative number. That is, positive and negative numbers are represented by oppositely directed lines.

**31. Summary.**

Many quantities, such as force, distance, time, and temperature, admit of the notion of opposite qualities.

In mathematics opposite qualities are usually distinguished by the words *positive* and *negative*.

Measurements of quantities are expressed by numbers; measurement of positive quantities by positive numbers; measurement of negative quantities by negative numbers.

The symbol + indicates positive.

The symbol − indicates negative.

The absolute value of a number is expressed by $| \quad |$.

Geometrically, the measurements of quantities of opposite qualities are expressed by oppositely directed lines.

**Exercises.** Represent by diagrams each of the following pairs of quantities:

**1.** 10 miles north; 15 miles south.

**2.** 15 pounds push; 25 pounds pull.

**3.** 60 degrees above zero; 20 degrees below zero.

**4.** $2000 income; $1500 expenditure.

**5.** If a man earns $150 a month and spends $100 a month, show his total earnings and expenditures in one year.

**6.** If a tank is filled by a pipe which flows 100 gallons a minute and emptied by a pipe which flows 60 gallons a minute, represent the total inflow and outflow in 5 minutes when both pipes are open.

# CHAPTER VI

## ADDITION AND SUBTRACTION

**32. Notation.** The signs of addition, subtraction, multiplication and division are the same as in arithmetic.

To indicate that two numbers $a$ and $b$ are to be added we write $a + b$; this means that $b$ is to be added to $a$. Now the numbers $a$ and $b$ may be both positive; one positive, the other negative; or both negative. Thus:

$(+3) + (+4)$ means add positive 4 to positive 3;
$(+3) + (-4)$ means add negative 4 to positive 3;
$(-3) + (+4)$ means add positive 4 to negative 3;
$(-3) + (-4)$ means add negative 4 to negative 3.

When a number is positive we usually do not write its sign; thus 3 means $+3$, and $3 + 4$ means $(+3) + (+4)$.

We shall now illustrate geometrically what is meant by adding two numbers together, whatever their signs may be. We shall represent each number by a directed line; we then add these directed line segments.

**33. Addition of Directed Line Segments.** Let us call the point from which a line segment starts the *initial point* of the segment, and the point where it ends the *final point* of the segment; the final point will be marked by an arrowhead to show the direction of the line.

Two directed line segments are added by placing the initial point of the line we are adding on at the final point of the line we are adding to, *keeping each line in its original direction.* The line drawn from the initial point of the line we are adding to, to the final point of the line we are adding on, is the sum of the two lines. The direction of the sum is always from the initial point of the line added to, to the final point of the line added on.

This is illustrated in the following figures; here lines extending toward the right are called positive, toward the left negative.

*Illustration I.*

To add the line $RS$ to the line $MN$.


Starting at the initial point, draw the line $MN$. Now add the line $RS$ by placing the point $R$ on the point $N$, taking care that each line is kept in its original direction.

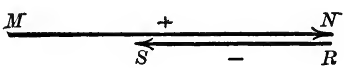

The distance $MS$ is the sum. It is a positive line.

Therefore $MN + RS = MS$.

*Illustration II.*

*Case* 1. To add the line $RS$ to the line $MN$, when $MN$ is longer than $RS$.


As before, starting at the initial point, draw the line $MN$; add the line $RS$ by placing the point $R$ on the point $N$, taking care to keep the lines in their original directions. (See note on p. 57.)

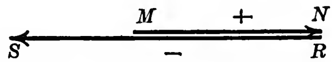



Therefore $MN + RS = MS$. $MS$ is a positive line.

*Case* 2. To add line $RS$ to line $MN$, when $MN$ is shorter than $RS$.


As before, start at the initial point, and draw line $MN$; add the line

*RS* by placing point *R* on point *N*, care being taken to keep the lines in their original directions.

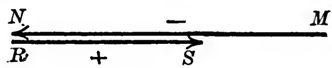



Therefore $MN + RS = MS$. *MS* is a negative line.

*Illustration III.*

*Case* 1. To add line *RS* to line *MN*, when *MN* is longer than *RS*.

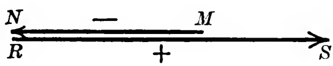

Adding according to the instructions given before, we have




Therefore $MN + RS = MS$. *MS* in this case is negative.

*Case* 2. To add line *RS* to line *MN*, when *MN* is shorter than *RS*.


Adding as before,




Therefore $MN + RS = MS$. *MS* is a positive line.

*Illustration IV.*

To add line *RS* to line *MN*.


Adding as before,


Therefore $MN + RS = MS$. *MS* is a negative line.

*Note.* The student must keep in mind that in Illustrations II and III, the lines *MN* and *RS* coincide — lie one on the other — and that in the picture one is drawn a little below the other to show the directions.

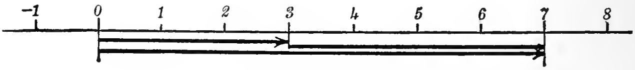
**34. Geometric Addition of Numbers.** We are now ready to illustrate what we mean by adding two *numbers* together, whatever may be their sign. Let the numbers be $a$ and $b$.

*First make a number scale. Starting from zero on this scale, lay off a line segment to represent the number a in amount and direction; using the point that you have now reached as an initial point, lay off a line segment to represent the number b in amount and direction; then the number represented by the line from the origin or zero point to the point last reached is the sum* $a + b$.

As shown above line segments can be added when both are positive, both negative, one positive and one negative; so the numbers which they represent may be added, when both are positive, both negative, one positive and the other negative.

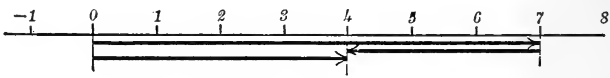
If $a$ and $b$ are both positive, the addition corresponds to the addition of line segments as shown in Illustration I.

Suppose $a$ is 3 and $b$ is 4; we have



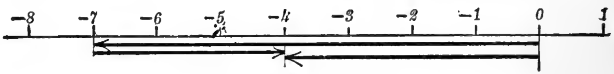

$$3 + 4 = 7.$$

Suppose $a$ is 7 and $b$ is $-3$; this corresponds to Illustration II, Case 1, in the addition of line segments.



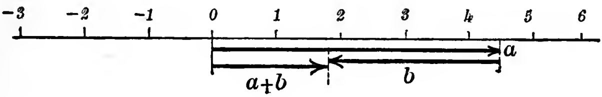

$$7 + (-3) = 4.$$

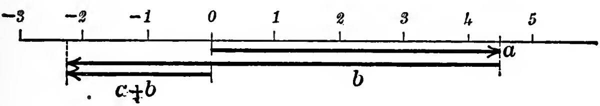
Suppose $a$ is $-7$ and $b$ is 3; this corresponds to Illustration III, Case 1, in the addition of line segments.




$$-7 + 3 = -4.$$

In general, if the two numbers are $a$ and $b$, $a$ being positive and $b$ negative we have the two figures below. The first is for the case where the number of units in $b$ is less than the number of units in $a$, and the second is for the case where the number of units in $b$ is greater than the number of units in $a$.



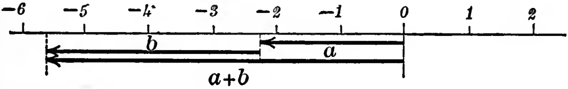

In these figures, as in the case of preceding figures, the fact that $b$ stands for a negative number is shown by the arrow. The sum of $a$ and $b$, $a + b$, is positive in the first figure and negative in the second.

We may now state the following rule:

*If a is positive and b is negative then*

$$\textit{the sum } a + b \textit{ is} \begin{cases} \textit{positive if } |b| \textit{ is less than } a; \\ \textit{negative if } |b| \textit{ is greater than } a. \end{cases}$$

If both numbers are negative, the two directed lines will both extend in the negative direction; hence *the sum of two negative numbers is a negative number whose absolute value equals the sum of the absolute values of the two numbers.*




This corresponds to Illustration IV in the addition of line segments.

Estimate the values of $a$ and $b$ in the above figures.

*Addition of more than two numbers.*

*Example.* Add $4 + (-6) + 8$.

To do this we first add two of the numbers; then to their sum we add the third.

This is shown in the diagram below; $4 + (-6)$ leads to $-2$; adding $+8$ to $-2$ leads to $+6$.


Algebraically expressed this is

$$4 + (-6) + 8 = 6.$$

**35. Exercises and Problems.**

Add the following geometrically, representing the numbers by lengths of lines. The best results will be obtained from the use of squared paper. Care should be taken in each example, especially when fractions occur, to select the unit which may be most conveniently used in the example. Also have care that the arrowheads are always placed to indicate the direction of the line. Write the algebraic solution in each case.

**1.** $6 + 3$.

**2.** $-12 + 2$.

**3.** $2 + (-3) + 14$.

**4.** $\frac{1}{2} + (-\frac{1}{4})$.

**5.** $140 + (-300)$.

**6.** $42 + (-8) + (-3)$.

**7.** $-52 + 40 + (-3)$.

**8.** $-15 + 4 + (-10)$.

**9.** $-3 + (-10) + (-2) + 32$.

**10.** $.7 + (-.3) + (-.05) + 1.4$.

**11.** $3.7 + 5.7 + (-4) + 7$.

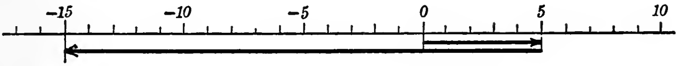
**12.** $\frac{3}{5} + (-\frac{1}{4}) + (-\frac{4}{5})$.

**13.** $-\frac{7}{9} + (-\frac{2}{3}) + \frac{5}{6} + (-\frac{2}{9})$.

In each of the following problems give graphic representation, algebraic expression and result.

**14.** A boy is pulling a small wagon along with a force of 5 lbs.; another boy comes up behind and pulls back with a force of 20 lbs. In what direction and with what force is the wagon pulled?

*Solution:* Let the line – represent a pound force. Then the following will represent the forces and their sum.




The algebraic expression is

$$5 \text{ lb.} + (-20 \text{ lb.}) = -15 \text{ lb.}$$

The negative sign shows that the resulting pull is backward.

**15.** A man earned \$5 on Monday and spent \$3; on Tuesday he earned \$2 and spent \$6; on Wednesday he earned \$7 and spent none; on Thursday he earned \$10 and spent \$4; on Friday he earned \$7 and spent \$7. How much had he at the end of each day? What does a negative answer mean?

**16.** When the mercury in a Fahrenheit thermometer registered 73 degrees, the bulb of the thermometer was in a bottle of ether. It was taken out quickly. As the ether on the bulb evaporated, the mercury fell 2 degrees. The thermometer was then placed in hot water and the mercury rose 56 degrees. What temperature did the thermometer then register?

**17.** A thermometer (at 25 degrees) was placed in finely crushed ice, and the mercury fell 24.6 degrees. The thermometer was then placed in a mixture of salt and ice, and the mercury fell 19 degrees more. When it was finally placed in hot water, the mercury rose 108.2 degrees. What was the temperature of the hot water?

**18.** On consecutive days one winter the following was noticed: On the morning of the first day the thermometer registered zero; by the next morning the mercury had risen 23 degrees; by the following morning it had fallen 9 degrees; by the following morning it had fallen 25 degrees more; and by the next morning it had risen 7 degrees. What did the thermometer then register?

**19.** A boy wished to have the water in his beaker a certain temperature. He tested the hydrant water and found that it registered 34 degrees. He poured in hot water and raised it 23 degrees; he next poured in cold water and lowered it 12 degrees; then hot water and raised it 3 degrees; then cold water, and lowered it 7 degrees; then cold water, and again lowered it 2 degrees, when he found that it was the temperature that he wished it to be. What was the final temperature?

**20.** Make a few experiments using the thermometer, noting changes of temperature. Write problems from these observations.

The rule for adding angles is similar to that for adding lines. Draw the initial arm of the first angle. It is customary to have this extend from left to right. Using your protractor count $a$ angular units, and draw the final arm of your first angle; using this as the initial arm of the second angle count $b$ angular units and draw the final arm of the second angle. The sum is the angle with its initial arm the same as the initial arm of the first angle and its final arm the last line drawn.

In making the drawing lay off a positive angle so that the turning from the initial arm toward the final arm is opposite to the turning of the hands of a clock, that is, counterclockwise; lay off negative angles in the clockwise direction.

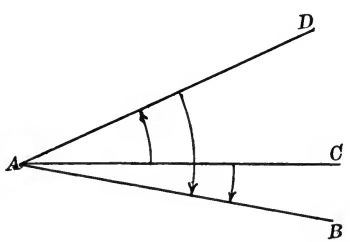



*Illustration.* Add an angle of $-44$ degrees to an angle of 25 degrees.

Reading from the figure we have:

$$\angle CAD + \angle DAB = \angle CAB.$$

The algebraic expression is:

$$25° + (-44°) = -19°.$$

Add the following; give drawing and algebraic equation. Use a protractor, so as to have the work as accurate as possible.

**21.** $25° + (-34°)$.
**22.** $-13° + 56° + (-12°)$.
**23.** $43° + (-156°) + 73°$.
**24.** $43° + (-94°) + 141° + 40°$.
**25.** $\frac{1}{2}$ radian + $\frac{2}{3}$ radian + $(-\frac{1}{4})$ radian.
**26.** $-\frac{1}{3}$ radian + $\frac{1}{4}$ radian + $(-\frac{1}{2})$ radian.
**27.** $\frac{1}{2}\pi$ radian + $(-\frac{2}{3}\pi)$ radian.
**28.** $1\frac{1}{3}\pi$ radian + $(-2\pi)$ radian + $(-\frac{1}{4}\pi)$ radian.

**36. Subtraction.** In the preceding exercises about forces we assumed that we knew the number of pounds and the direction of each force, and we had to find the result of their combined action; that is, the number of pounds and the direction of the force that results from all the forces combined in pulling and pushing the body in one direction or the opposite. It may be just as necessary, in practical work, to be able to tell what force must be combined with another to secure a desired result. For example, what force must be combined with a downward force of 2 pounds to get a resultant upward force of 5 pounds? Plainly this is the inverse of the work that we have been doing. So we determine the answer to this problem by guessing the amount and then adding to test whether we have guessed correctly or not. After a while we become able to guess and test the result very rapidly.

**Definition of Subtraction.** *The process of determining the amount which must be added to one number to produce another is called* **subtraction.**

What we have said about forces will apply to all the other subjects that we treated in addition.

**Exercises.** Give answers to the following by guessing the answer and then adding to test the correctness of the guess.

1. A mass of iron filings lying on the scales weighs 23 grams. What mass of filings must be added in order to make the mass weigh 37 grams?

2. The thermometer registers 69 degrees. How many degrees must be added in order to make it register 40 degrees?

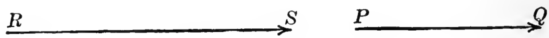
3. The pressure of the atmosphere at one time sustained in a barometer a column of mercury 73.1 centimeters high, and at another time 74.7 centimeters high. How many centimeters have been added by the change of pressure?

4. At what rate can a man row in still water if the rate of a stream is 3 miles an hour, and if he can row at the rate of 3 miles an hour upstream?

5. A boatman rowing upstream finds that at one point he is not moving. If his rate of rowing is 5 miles an hour, what is the rate of the stream at that point?

**Subtraction of Lines.**

*Illustration.* Subtract the line $RS$ from the line $PQ$.




The question is — What length of line must be added to the line $RS$ to get the line $PQ$?

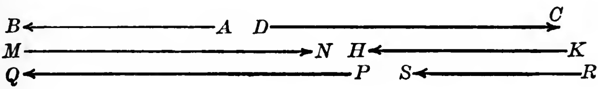
Examining your graphic representation of the addition of two directed lines, you will observe that the line you added to and the line which is the sum of the two lines both have the same initial point. The line added on extends from the final point of the line added to, to the final point of the sum. From this we see that, when we are given a line and wish to determine the length and direction of a line that must be added to it in order to obtain another line, which is in reality the sum of the given line and the one to be found, we place the two given lines with their initial points together, taking care that *each keeps its original direction.* Draw a line from the final point of the line to be added to, to the final point of the line which is the sum. This is the line we seek.

That is place line $RS$ and line $PQ$ with the point $R$ on the point $P$ taking care that each keeps its original direction. Draw a line from point $S$ to point $Q$. The line $SQ$ is the line which we are seeking;

that is, it is the line which must be added to the line $RS$ to obtain the line $PQ$. (See note on p. 57.)


Line $PQ$ − line $RS$ = line $SQ$. $SQ$ is a negative line.
That is, $SQ$ must be added to $RS$ to give $PQ$.

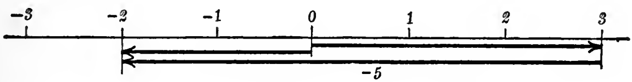
**Exercises.** Subtract the first of the following lines from the second, observing carefully the above remarks.




## Subtraction of Numbers.

*Illustration.* If you have a line 3 units long, what length of line must you add to it in order to have a line − 2 units long?

This does not differ from the preceding work excepting that we use the scale of measurement in order to do the work more quickly. We follow the same rule as above.




Thus we see that we must add a line negative 5 units long to a line 3 units long in order to have a line negative 2 units long.

The algebraic expression for this is

$$(-2) - (+3) = -5.$$

This is read, "Negative 2 minus 3 equals negative 5," or "3 subtracted from negative 2 gives negative 5," and means that which must be added to 3 to get negative 2.

The check to this is: $3 + (-5) = -2$.

**Exercises.** In the following, give graphic representation, algebraic expression, result, and check.

**1.** What length of line must be added to a line negative 3 units long in order to have a line positive 8 units long?

**2.** Subtract a line positive 28 cm. long from a line positive 23 cm. long. We see readily that this would have little meaning if we did not hold to our definition of the word subtract. The question in reality is — What must be added to the line 28 cm. in length (taking into consideration both distance and direction) to have a line 23 cm. long?

**3.** What must be added to a line 11 units long to have a line 2 units long?

**4.** What must be added to a line negative 17 units in length to have a line negative $2\frac{1}{2}$ units in length?

**5.** What must be added to a line 54 mm. long to have a line 0 mm. long?

**6.** What must be added to a line $-7\frac{2}{3}$ cm. long to have a line $-2\frac{3}{4}$ cm. long?

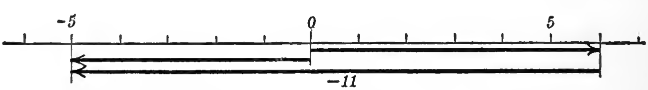
**7.** What must be added to a line $4\frac{6}{7}$ cm. long to have a line $2\frac{3}{7}$ cm. long?

**8.** Repeat the idea of the questions asked in Exercises 3 to 7 using the word subtract.

**9.** If a boy is pulling a block with a force of 6 lb., with how much force and in what direction must another boy pull, if the net result is to be a pull of 5 lb. in the opposite direction? How else may this question be asked?

A graphic representation may be made of this exercise in the same way that we represented the exercises in the addition of forces. That is, we let a certain length of line represent the unit of force, and by taking multiple lengths of this line or fractional parts of it, we can make a line representation of the forces, which lines may be subtracted in the same manner as is given above.

Let one unit of length represent one pound force. Then the graphic representation of the subtraction is as follows.




The algebraic expression for this is

$$(-5) - (+6) = -11 \quad \text{or,} \quad -5 - 6 = -\mathbf{11}.$$

Such a graphic representation may be made of any measurable quantities.

Make graphic representation of the following exercises; give algebraic expression and result as in preceding exercises.

**10.** If a man earns ten dollars, how much must he spend to be in debt four dollars? Since spending is the opposite of earning, this may be asked — How much must a man add to ten dollars in order to have negative four dollars?

**11.** If wheat is $1.05 a bushel on one day, and $.93 the next, what has been added to the price the first day to get that of the second?

**12.** If certain shares of stock sell at 5 cents above par one day, and 2 cents below the next, what has been added to the price of the first day to get that of the next?

**13.** A strip of iron when warm extended past a certain notch 3 cm.; when cooler, the end touched the notch. If we call the effects of expansion positive, how much was added to the length of the strip by contraction? If when cold the strip lacked 2 cm. of reaching to the notch, how much was added to the original length by contraction?

**14.** An iron nail lying in a pan on the scales weighs 10 grams. When a magnet is held over it, the scales register 2 grams. If we regard an upward pull as positive, what is the pull of the magnet upon the nail?

Write problems leading to some of the following algebraic expressions; give graphic representation and answers for all.

**15.** $9 - 2$.

**16.** $-22 - 8$.

**17.** $-22 - (-9)$.

**18.** $0 - (-3)$.

**19.** $0 - 3$.

**20.** $0 - (-15)$.

**21.** $-7 - 0$.

**22.** Subtract 8 from 7.

**23.** Subtract $3\frac{1}{2}$ from $1\frac{5}{6}$.

**24.** Subtract 3.6 from .7.

**25.** Subtract 20.8 from $-.9$.

**26.** Subtract $-.08$ from .25.

**27.** Subtract $-1.7$ from $-.7$.

**28.** Subtract $-\frac{2}{3}$ from $-\frac{3}{5}$.

29. From an angle of 30 degrees subtract an angle of 45 degrees.

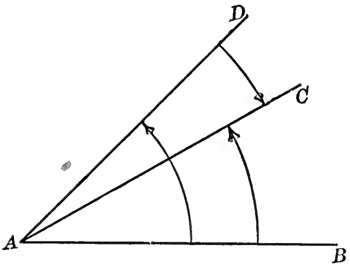



*Illustration:* The question is — What must be added to an angle of 45 degrees to get an angle of 30 degrees?

The symbolic expressions are

$$\angle BAC - \angle BAD = \angle DAC.$$
$$30^\circ - 45^\circ = -15^\circ.$$

*Check:* $\angle BAD + \angle DAC = \angle BAC.$
$$45^\circ + (-15^\circ) = 30^\circ.$$

For the exercises below make geometric picture; give algebraic expression and result.

30. $-\frac{2}{3}\pi$ radians $-\frac{1}{3}\pi$ radians.

31. $\frac{1}{2}\pi$ radians $-\pi$ radians.

32. $27^\circ - 125^\circ$.

33. $-45^\circ - (-60^\circ)$.

34. $-\frac{1}{4}\pi$ radians $-(-\frac{2}{3}\pi)$ radians.

35. What angle must be added to 10 degrees to get a right angle?

36. What angle must be added to 37 degrees to get a right angle?

37. What angle must be added to $\frac{3}{4}\pi$ radians to get a right angle?

Two angles whose sum is a right angle are called *complementary* angles. Either angle is said to be the complement of the other. In the last three exercises you have found the complements of the angles given.

38. What is the complement of 80 degrees? of 53 degrees? of $-\frac{1}{4}\pi$ radians? of $-\pi$ radians? of 175 degrees? of 0 degrees? of 90 degrees?

39. What angle must be added to 10 degrees to get a straight angle?

40. What angle must be added to $\frac{3}{4}\pi$ radians to get a straight angle?

Two angles whose sum is a straight angle are called *supplementary* angles. Either angle is said to be the supplement of the other. In the last two exercises you have found the supplements of the angles given.

What are the supplements of the angles given in Exercise 38?

**Exercises for Quick Oral Drill.** The teacher may read the following or similar exercises, and the student should be able to give the answer as soon as the exercise is read. Read the subtraction exercises in three different ways.

**1.** $2-(-3)$.

**2.** $5-(-4)$.

**3.** $-5-4$.

**4.** $-2-(-6)$.

**5.** $8-9$.

**6.** $3+7+(-4)$.

**7.** $5+(-2)+(-3)+2+7$.

**8.** $3-(-7)$.

**9.** $-3-(-7)$.

**10.** $5-8$.

**11.** $-15-(-3)$.

**12.** $4-12$.

**13.** $3+6+(-15)+9$.

**14.** $0-12$.

**15.** $0-(-12)$.

**16.** $3-21$.

**17.** $21-(-16)$.

**18.** $-13+(-2)+10+3$.

**19.** $-5+7+(-8)+7+8$.

**20.** $-2+(-3)+5$.

**21.** $0-(-4)$.

**22.** $-2-0$.

**23.** $0-2$.

**24.** $-4+5+(-5)+(-3)$.

**25.** $\frac{1}{3}-1-\frac{1}{3}$

**26.** $\frac{2}{5}-2\frac{3}{5}$.

**27.** $-\frac{7}{9}+\frac{2}{9}+(-\frac{3}{9})+\frac{4}{9}$.

**28.** $\frac{1}{15}+(-\frac{4}{15})+(-\frac{3}{15})+\frac{2}{15}$.

**29.** $-\frac{2}{23}+(-\frac{9}{23})+\frac{14}{23}+(-\frac{3}{23})$.

**30.** $\frac{12}{25}-(-\frac{17}{25})$.

**31.** $120-125$.

**32.** $\frac{13}{12}+(-\frac{1}{12})$.

**33.** $\frac{13}{12}-\frac{1}{12}$.

**34.** $120+(-125)$.

**35.** $-120-125$.

**36.** $-120+125$.

**37.** $-120-(-125)$.

**38.** $-15+17+(-1)+(-3)$.

**39.** $22+(-27)+5+3$.

**40.** $-5+(-6)$.

**41.** $-5-6$.

**42.** $5-(-6)$.

**43.** $5+6$.

**44.** $37+(-41)+5$.

**37. Summary.**

*Addition of directed lines.* To add a given directed line to another, place the initial point of the line you are adding, on the final point of the line you are adding to, having care that both lines keep their original direction. The distance and direction of the line extending from the initial point of the line added to, to the final point of the line added on, is the length and direction of the sum.

Comparison of measurable quantities may be expressed by means of lines as well as by numerals and letters.

To express the addition of quantities by the addition of lines, assume a unit length of line to express a unit of quantity. By taking multiples or fractional parts of this unit, draw lines to express the quantities to be added. Add these lines by the above rule.

*Subtraction of directed lines.* Subtraction is the inverse of addition. It consists in finding the quantity which when added to one stated quantity will give another stated quantity.

To subtract one directed line from another directed line, place the two lines with their initial points together having care that the lines keep their original direction. The length and the direction of the line extending from the final point of the line to be added to, to the final point of the sum, is the line required.

To express subtraction of quantities by subtraction of lines, express their values by means of lines as instructed in addition, then subtract these lines by the rule just given.

The terms minuend and subtrahend are used in the same sense as in arithmetic.

# CHAPTER VII

## MULTIPLICATION AND DIVISION

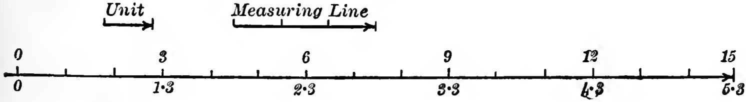
**38. Meaning of Multiplication.** The terms multiplicand and multiplier are used in the same sense as in arithmetic.

The subject of multiplication adds nothing new to the work we have been doing. So far, the numbers that we have been dealing with have told us what to do with a unit of measure if we wished to express the measurement of quantity. For example, when we say that a line is negative 5 cm. in length, we mean to take the unit, turn it over, and use it five times to get the length of line called for. Now instead of a single unit we shall use multiples of or fractional parts of the unit, using them in either the same direction as the unit or in the reverse direction; these numbers are *multiplicands,* and the *multiplier tells us what to do with them* just as *the number* in the example above told us what to do with the *unit* to get it. In other words,

*To multiply a second number by a first, we are to do to the second number the same thing that we did to the unit to get the first.*

**39. Graphic Illustrations of Multiplication.**

*Illustration I.* A line three units long lying in the positive direction must be used five times to measure a line; what is the length of the line?




Our line is 15 units long.

The algebraic expression of this operation is

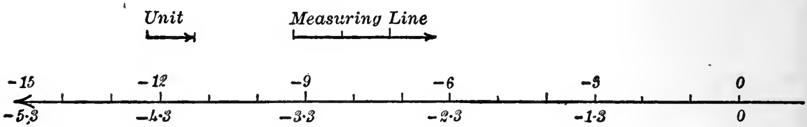
$$5 \cdot 3 = 15.$$

It is read 5 times 3 equals 15.

Did we use the 3, in this case, in the same way that we would have used the unit had we been asked to lay off a line 5 units long? Does our definition of multiplication hold in this case?

*Illustration II.* A line 3 units long is lying in the positive direction. Reverse it and use it 5 times to measure a line. What length of line is measured? What is its direction? What sign is used to indicate "reverse" direction?

The geometric representation of this is




The algebraic expression of this is

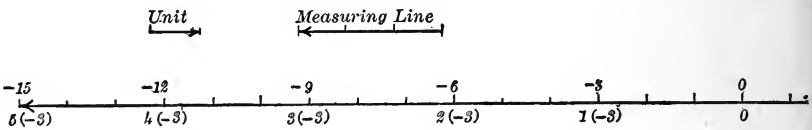
$$-5 \cdot 3 = -15.$$

It is read negative 5 times 3 equals negative 15.

In this case did you do the same thing to 3 that you would have had to do to the unit to get negative 5?

*Illustration III.* A line 3 units long is lying in the negative direction (opposite direction to the unit). Keeping it in this direction lay it off 5 times along the line to be measured. What is the length of line measured? What is its direction? What is the sign used to indicate "keep same direction"?

The geometric representation of this is



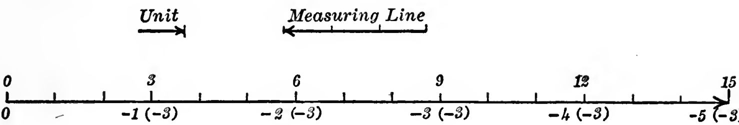

The algebraic expression of this is

$$5 \cdot (-3) = -15.$$

It is read 5 times negative 3 equals negative 15.

*Illustration IV.* A line 3 units long is lying in the negative direction. Reverse it, and lay it off 5 times along the line to be measured. What is the length of line measured? What is its direction? What sign is used to denote reverse direction?

The geometric representation of this is




The algebraic expression of this is

$$-5 \cdot (-3) = 15.$$

It is read, negative 5 times negative 3 equals 15.

**Exercises.**

**1.** Repeat Illustrations I, II, III, IV, using a line $\frac{2}{3}$ of a unit in length as the measuring line. Give geometric representation, and algebraic expression.

**2.** Give geometric representation and result of the following:

$$\tfrac{2}{5} \cdot 4; \quad \tfrac{2}{5} \cdot (-4); \quad -\tfrac{2}{5} \cdot 4; \quad -\tfrac{2}{5} \cdot (-4).$$

**3.** If a piece of iron weighs 25 5-gram weights, and a piece of wood weighs 7 5-gram weights, and a piece of stone weighs 10 5-gram weights, how many 5-gram weights do the pieces together weigh? How many grams do they weigh?

**4.** If three boys, capable of pulling an average of 75 pounds apiece, are pulling a sled, and seven boys, with the same average pulling capacity, come up and pull in the opposite direction, in what direction will the sled be pulled, and with how many times 75 pounds force? With how many pounds force?

**5.** If you go 3 times $3\frac{1}{2}$ units forward, then 6 times $3\frac{1}{2}$ units backward, how many $3\frac{1}{2}$ units are you from the starting point?

**6.** If you go 3 times $-2$ units forward, then 6 times $-2$ units backward, how many times $-2$ units are you from the starting point? How many units?

**7.** If you go 3 times $-3\frac{1}{2}$ units forward and 6 times $-3\frac{1}{2}$ units backward, how many times $-3\frac{1}{2}$ units are you from your starting point?

**8.** If you go 3 times $n$ units forward and 6 times $n$ units backward, how many times $n$ units are you from your starting point?

**9.** A load is being pulled in a certain direction with a force of 3 100-pounds. What force must be added in order that it may be pulled in the opposite direction with a force of 17 100-pounds?

**10.** Two forces are acting in opposite directions with a resultant force of 6.4 5-pounds. If one of the forces is known to be 151.2 5-pounds, what is the amount and direction of the other force?

**11.** If an engine has gone 17 m-meters from the station, how far and in what direction must it go to be $-3$ m-meters from the station.

**12.** If the temperature rises $4\,d$ degrees, then falls $7\,d$ degrees, what is the total change in temperature?

**40. Multiplication of General Numbers.** In the preceding exercises we brought out the following facts:

*If the sign of the multiplier is positive, the sign of the product is the same as the sign of the multiplicand.*

*If the sign of the multiplier is negative, the sign of the product is opposite to the sign of the multiplicand.*

In other words:

A positive number times a positive number gives a positive number.

A positive number times a negative number gives a negative number.

A negative number times a positive number gives a negative number.

A negative number times a negative number gives a positive number.

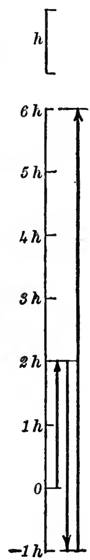
Would these facts have been brought out had we used other numbers in our exercises? Yes, you will say, any numbers might have been used. Let us then use the letter $m$ to stand for the multiplier, and the letter $n$ to stand for the multiplicand. Such letters may then be called *general numbers*, since they may be used to stand for any number we choose.

Our last four rules, in algebraic form, will then be

$$m \cdot n = mn.$$
$$m \cdot (-n) = -mn.$$
$$-m \cdot n = -mn.$$
$$-m \cdot (-n) = mn.$$

**Exercises.**

1. The distance from floor to floor in an office building is on the average $h$ feet. The elevator boy, starting at the main floor, runs the cage up two stories, then down three, then up seven, when he is at the top floor. How many feet is it from the main floor to the top floor?




The geometric solution is shown in the adjacent diagram. The algebraic expression is:

$$2h + (-3h) + 7h = 6h.$$

Furthermore, we could bring out the mathematical thought in a still more general way by having the example read thus:

The distance from floor to floor in an office building is on the average $h$ feet. The elevator boy, starting from the main floor, runs the cage up $a$ stories, then down $b$ stories, then up $c$ stories, when he is at the top floor. How many feet is it from the main floor to the top floor?

The algebraic expression for the second statement, which we might call the general statement, is

$$ah + (-bh) + ch = (a - b + c)h.$$

When we express numbers by means of letters, we have no single symbol as we have in arithmetic to express their sum, so, when necessary to show that a combination of numbers is to be regarded as one quantity we inclose the numbers in parentheses as shown above (§ 8). However, either expression in the second statement may be regarded as the answer to our example.

**2.** A man made it a point to have his average daily expenses the same as the rent from a certain piece of property which he owned. After $t$ days during the month of October the renter of the property moved and the rent stopped. Compute the net amount the man had for the month, if the average rent per day was $r$ dollars.

The work of writing problems leading to given arithmetic or algebraic expressions is of great value. It is the best way of testing a student's insight into the meaning of algebraic symbols.

When a problem is given and the solution is called for, the student arrives at the answer to the question asked by first translating the English statements into algebraic symbol language and then doing the work called for by these symbols.

The writing of a problem is the reverse of this process. The algebraic expression is given and the student is asked to translate its meaning into English.

The correctness of the problem should always be tested by a translation back into the algebraic language; that is, by the solution of the problem.

*Example* 1. Write a problem leading to the expression

$$6 \cdot 40 + 3 \cdot 40.$$

*Solution:* Upon examining this expression we see that there are two terms, each of which is a multiple of 40. So we must write a problem in which 40 is repeated.

Six boys, exerting a force of 40 pounds each, are pulling on a load. Not being able to move it, they call to their assistance 3 other boys each capable of exerting a force of 40 pounds. They succeed in starting the load in motion. What was the amount of force necessary to move the load?

*Check:* Translating into algebraic symbols:

$6 \cdot 40 =$ the number of pounds force the first 6 boys exert;
$3 \cdot 40 =$ the number of pounds force the 3 boys exert;

so that

$6 \cdot 40 + 3 \cdot 40 =$ the number of pounds necessary to move the load.

$6 \cdot 40 + 3 \cdot 40 = 9 \cdot 40$, the number of pounds necessary to move the load.

Give a statement of a problem like this, using general numbers.

*Example* 2. Write a problem leading to the expression

$$6 \cdot (-15) - 6 \cdot 25,$$

or,

$$-6 \cdot 15 - 6 \cdot 25.$$

*Solution:* A train runs due north for 6 hours at the rate of 25 miles an hour. At what rate must it now go for 6 hours so as to arrive at the same place as if it had gone south from the original starting point for 6 hours at 15 miles an hour?

Here we regard north as positive. Check this statement. Give a similar problem involving general numbers.

The best way to learn to write these problems is to study carefully the expressions obtained from problems in the preceding lists. Such lists are given not only for the solution of the problems but to show how such problems arise.

Write problems for some of the following. Give graphic representation and result for all.

**3.** $15 \cdot 4 + 3 \cdot 4$.
**4.** $9 \cdot (-6) - 3 \cdot (-6)$.
**5.** $7 \cdot (-y) - 10 \cdot (-y)$.
**6.** $3 \cdot \frac{2}{5} + (-7) \cdot \frac{2}{5}$.
**7.** $7w + (-3)w + (-4)w$.
**8.** $-f + (-4)f$.
**9.** $-6a + 7a + (-3)a$.
**10.** $-2(-c) + (-5)(-c)$.
**11.** $m(-n) + r(-n)$.
**12.** $-9(-p) - 5(-p)$.
**13.** $a(-b) - (-a)(-b)$.
**14.** $2k - 8k$.

**41. Division.**

*Division of a first number by a second is the process of deter-*

*mining the number by which you must multiply the second number to get the first.*

It is the **inverse** operation of **multiplication,** and is a purely guessing process. You guess the answer and multiply to see if your guess is correct. In this respect it does not differ from the work in arithmetic, for long division is nothing more than a series of guesses and multiplication to test the correctness of the guess.

No new rules need be given for division, as you have but to observe the rules for multiplication when it comes to testing.

*Example.* You are to divide $-ab$ by $b$; that is, you want to know by what you must multiply $b$ to get $-ab$. You have learned that this is $-a$. So $-ab$ divided by $b$ equals $-a$.

The symbols for division are the same as in arithmetic. So symbolically we write

$$-ab \div b; \quad \text{or} \quad \frac{-ab}{b}.$$

Of these two forms the latter is almost always used. We then have

$$\frac{-ab}{b} = -a.$$

Notice that ordinary cancellation, as in arithmetic, can be used here to reduce the given fraction. The only difference is that in arithmetic you used only positive numbers.

**Problems.** In these problems give algebraic expression and result. Multiply to check your answer.

**1.** A chauffeur finds that he has gone $7t$ miles in $t$ hours. What is his speed per hour? What is his speed per hour if he goes $m$ miles in $t$ hours?

**2.** A boy agrees to work for $1.50 per day; how many days must he work to earn $6? How many days must he work to earn $n$ dollars, if he earns $d$ dollars a day?

**3.** A cup made of tin in the form of a cone holds $v$ grams of water, and a cup in the form of a cylinder whose height and circumference around the bottom are the same as that of the cone, holds $3v$ grams. What is the ratio of the volume of the cylinder to that of the cone?

*By* **ratio** *of one number to another we mean the number by which we must multiply the second to get the first.*

**4.** One cup holds $mw$ grams and another holds $w$ grams. What is the ratio of the capacity of the two cups?

**5.** If a boatman rows at the rate of $v$ miles an hour in a stream that flows at the rate of $s$ miles an hour, what is the ratio of his rate of rowing to the rate of the stream?

**6.** If an automobile goes $r$ miles an hour for $t$ hours, and another automobile goes $r$ miles an hour in the opposite direction for $s$ hours, what is the ratio of the distance the first goes to the distance the second goes?

**7.** If a train is going east at the rate of $r$ miles an hour, what is the ratio of its velocity to that of a train going west at the rate of $r$ miles an hour?

**8.** What is the ratio of the length of a line $-2$ cm. long to the length of a line 7 cm. long?

**9.** By what must you multiply a force of 51 lbs. in order to have a force of $\frac{2}{3}$ lb.? Answer this, using the word ratio. Check.

**10.** If 12 men can do a piece of work in a day, how much can 3 men do in a day if the average amount of work done by each is the same?

**11.** If $a$ men can do a piece of work in $m$ days, what part of the work will 1 man do in 1 day, if the average amount of work done by each is the same?

**12.** By what must a line $-5$ units long be multiplied in order to have a line $-12$ units long? Answer this, using the word ratio.

**13.** Divide a line $-1$ cm. long by a line $r$ cm. long. Read this, using the word ratio.

**14.** The same number of boys were trying to keep a door shut as the number that were trying to pull it open. The amount of force exerted was $ma - mb$ pounds. What is the number of boys on each side, if the average number of pounds exerted by each boy on one side is $a$, and the average number of pounds exerted by each on the other side is $b$?

Simplify each of the following expressions as much as possible. State problems leading to some of them.

15. $\frac{6t}{8t}$.

16. $\frac{15a}{5a}$.

17. $\frac{32n^2}{8n}$.

18. $\frac{4n}{6n^2}$.

19. $\frac{2ab}{5a}$.

20. $\frac{7a^2b}{14ab^2}$.

21. $\frac{(-a)b}{-b}$.

22. $\frac{b}{-ab}$.

23. $\frac{(-h)(-k)}{-kl}$.

24. $\frac{(-x)(-y)(-z)}{-xy}$.

25. $\frac{(-a)^2m^2}{-am}$.

26. $\frac{x^3(-y)^2}{x^2(-y)^3}$.

27. $\frac{4^2p^3(-r)}{2^3p^2r^2}$.

28. $\frac{\frac{1}{2}a(-b)^3}{\frac{1}{3}a^3(-b)}$.

29. $\frac{(\frac{1}{3})^3m^2(-n)^2}{(\frac{1}{3})^2m^3(-n)}$.

**42. Summary.**

*Definition of multiplication.* To multiply a second number by a first, do to the second number what you must do to the unit to get the first.

*Law of signs in multiplication.* If the sign of the multiplier is positive, the sign of the product is the same as that of the multiplicand.

If the sign of the multiplier is negative, the sign of the product is the opposite to that of the multiplicand.

Literal numbers are used in mathematics when we wish to bring out general truths, that is, truths which do not depend upon the exact amounts, but upon the relation of the quantities under consideration.

*Division.* This is the inverse of multiplication.

# CHAPTER VIII

## ADDITION AND SUBTRACTION OF POLYNOMIALS

**43. Definitions.** An expression consisting of several parts connected by the signs + or − is called a *polynomial.*

For example,

$$a + b, \quad 2h - 3k + l, \quad m + 4n - pq + 3rs$$

are polynomials.

Each part of a polynomial, including its sign, is called a *term* of the polynomial.

Thus $4n$ is a term of the last polynomial written above. Another term is $-pq$.

An expression consisting of a single term is called a *monomial.*

A polynomial consisting of two terms is called a *binomial.*

A polynomial consisting of three terms is called a *trinomial.*

Two terms which differ only by a numerical factor, as $3mn$ and $-5mn$, are called *similar terms.*

**44. Addition and Subtraction of Polynomials.**

*Example* 1. You learned in your arithmetic and reviewed in the chapter on measurement the fact that a quantity may be measured by a unit and subdivisions of that unit.

For example a distance is measured in yards and feet. Say that it measures 9 yards and 2 feet. This may be written

9 yards + 2 feet.

Another distance measures 3 yards and 7 feet. This may be written

3 yards + 7 feet.

Then the sum of these two distances is 12 yards and 9 feet, which may be written

12 yards + 9 feet.

That is, we add the parts of the distances measured by one kind of units together, then add the parts measured by another kind of units together, and combine the results to get the measurement of the sum of the lines.

If we regard the inch as the unit of measure in this exercise, and the yard and the foot as multiples of this unit, our exercise becomes:

Add a line $3 \cdot 36$ inches $+ 7 \cdot 12$ inches long to a line $9 \cdot 36$ inches $+ 2 \cdot 12$ inches long. Adding as before the sum is $12 \cdot 36$ inches $+ 9 \cdot 12$ inches. So that

$$(9 \cdot 36 + 2 \cdot 12) + (3 \cdot 36 + 7 \cdot 12) = 12 \cdot 36 + 9 \cdot 12.$$

Expressing this more generally by letting $y$ stand for the number of inches in a yard and $f$ stand for the number of inches in a foot, we have

$$(9y + 2f) + (3y + 7f) = 12y + 9f.$$

*Example* 2. Suppose that a line measures 9 yards and 1 foot and 4 inches. Let $y$ stand for a length of one yard, $f$ for a length of one foot, and $i$ for a length of one inch, each measured in terms of another unit, as a centimeter. Then the length of the line in centimeters is expressed by

$$9y + f + 4i.$$

This is a trinomial.

Suppose that a second line measures 3 yards and 2 feet and 6 inches. Its length is then expressed by

$$3y + 2f + 6i.$$

Now the sum of the lengths of these lines will be expressed by

$$(9y + f + 4i) + (3y + 2f + 6i),$$

and we know that we can get this sum by adding yards to yards, feet to feet, and inches to inches. So we have

$$(9y + f + 4i) + (3y + 2f + 6i) = 12y + 3f + 10i.$$

In this way any two polynomials are added together, by adding like terms.

The difference of the lengths of our two lines is expressed by

$$(9y + f + 4i) - (3y + 2f + 6i).$$

and again we know that this difference can be found by subtracting terms measured in like units. So we have

$$(9y + f + 4i) - (3y + 2f + 6i) = 6y - f - 2i.$$

Let the student check this by adding.

In the same way we might have such expressions as the following; state each of these as a problem in measurement, and check each result.

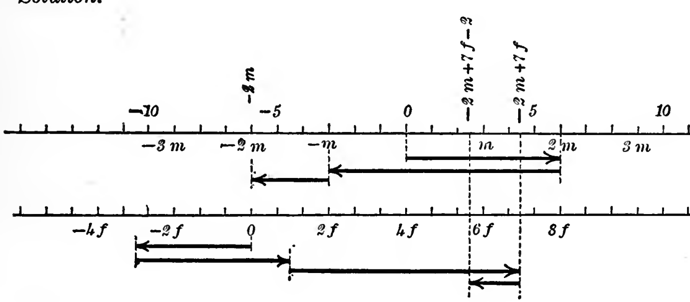
$$(9y + 4i) + (6y + 2f - 8i) = 15y + 2f - 4i.$$
$$(9y + 4i) - (6y + 2f + 6i) = 3y - 2f - 2i.$$
$$(8y - 3f + 5i) - (10y + 2f - 3i) = -2y - 5f + 8i.$$

Make up for yourself several more exercises of this kind.

*Geometric Illustration.*

Add: $(2m - 3f) + (-3m + 4f - 2) + (-m + 6f)$.

*Solution:*




Algebraic expression:

$$(2m - 3f) + (-3m + 4f - 2) + (-m + 6f) = -2m + 7f - 2.$$

In the diagram the measuring line for $f$ might have been laid off on the original scale line, but it is much less confusing to draw a new scale to count by. The student is again cautioned not to forget that all of the segments representing the numbers are supposed to lie on the original scale line. In order not to forget that the thing we are interested in is the distance we are from the point where we began to count, it is well to mark that distance on the scale line, every time we change the measuring line, as shown in the figure.

*Check:* Let $m = 3$, $f = 1\frac{1}{2}$ or $\frac{3}{2}$, as shown in the figure.

Substituting in the given expression,

$$\begin{aligned}(2 \cdot 3 - 3 \cdot \tfrac{3}{2}) &+ (-3 \cdot 3 + 4 \cdot \tfrac{3}{2} - 2) + (-3 + 6 \cdot \tfrac{3}{2})\\ &= (6 - 4\tfrac{1}{2}) + (-9 + 6 - 2) + (-3 + 9)\\ &= 1\tfrac{1}{2} + (-5) + 6\\ &= 2\tfrac{1}{2}.\end{aligned}$$

Substituting in the answer,

$$\begin{aligned}-2 \cdot 3 + 7 \cdot 1\tfrac{1}{2} - 2 &= -6 + 10\tfrac{1}{2} - 2\\ &= 2\tfrac{1}{2}.\end{aligned}$$

Since we get $2\frac{1}{2}$ from each expression, the expressions are equal. This is also the result in the figure.

In the following exercises make diagrams for the addition of some of them; add the others and check all.

It is of great importance that fractional and negative numbers be used in at least part of the checks, since it impresses the fact that the letters given may have either integral or fractional values and that these may be either positive or negative; $-a$ may be either a positive or a negative number according to the value of $a$.

**Exercises.**

1. Add $2r - s$ to $5r - 2s$.
2. Add $6x + 5y - 7$ to $2x - 7y + 5$.
3. Add $3a - 16b + 2$ to $4a + 9b + 4c$.

Add:

4. $(2r - 3p + -6) + (4r + 8p - 2) + (6r + 9)$.
5. $(-6a + 5c - 4b - 6) + (-8a + 6b) + (16a + 9)$.
6. $(3r - 6s + 10 + 2t) + (6r + 7t - 24) + (-5s - 12r)$.
7. $(7s + 5t) + (-4a + 6t) + (7t - 4a + 9s) + (-19t - 4s)$.

In the subtraction of polynomials the same idea enters as in addition. We subtract the terms separately and add the results of the subtractions.

*Example.* Subtract $2r - 3s$ from $r - 4s + 2t$.

Writing in algebraic form:

$$(r - 4s + 2t) - (2r - 3s) = ?$$

The first question is — What must be added to $2r$ to get $r$? The answer is $-r$. The second question is — What must be added to $-3s$ to get $-4s$? The answer is $-s$. The third question is — What must be added to 0 to get $2t$? The answer is $2t$. So we have

$$(r - 4s + 2t) - (2r - 3s) = -r - s + 2t.$$

*Check:* The check is to add the result to the second number to see if you get the first.

$$(2r - 3s) + (-r - s + 2t) = r - 4r + 2t.$$

It would be well to make a drawing of a few of the checks. The student must be careful to do his checking conscientiously; that is, he must actually do the adding or he will be led to make many blunders.

**8.** Subtract $7r + 2s$ from $2r + 6s$.

**9.** Subtract $-3s + 2t$ from $7s - 5t$.

**10.** Subtract $-2a - 3b$ from $a + b$.

In the following do the work indicated.

**11.** $(2c + 7d - e) - (5c - 2e + 7)$.

**12.** $(-5g - 3b - 10) - (-2g - 4b - 19)$.

**13.** $(7b - 4a - 6c + 4) - (-9b - 4a + 6c - 4)$.

**14.** $(2a - 3c) + (5a + 16c) - (3a - 8c)$.

**15.** $(b - 6d - 4a + 9) + (7r + 8b - 17d)$
$- (4r - 7d + 3a - 10)$.

**16.** $(-7f + 8g - 10h - p) + (5f - 8h + h - 10p)$
$- (-5f - 9g - 6)$.

**17.** $(7r - 6t - 9s - 15) + (-4r - 12t + 12s)$
$- (9r + 2s - 17)$.

**18.** Write the supplements of the following angles:
$a$ degrees; $2a - 6$ degrees; $a + 3b$ degrees; $2a - 4b + 3$ degrees; $2r$ radians; $\frac{1}{2}\pi - r$ radians; $1\frac{3}{4}\pi + 2r$ radians.

**19.** Write the complements of the angles of Exercise 18.

**20.** Write the complement of angle $a$, then write the supplement of this complement. Draw this. Do you get the same answer in your drawing as in your algebra?

**21.** Write the complement of the angle $a$, then write the supplement of this complement, then write the complement of this supplement. Draw and see if you get the same result.

**45. Rules for Adding and Subtracting Polynomials.**

(1) *To add two polynomials, add like terms, each term to be taken with the sign before it.*

(2) *To subtract two polynomials, subtract like terms, each term to be taken with the sign before it.*

Also, since subtracting a quantity is equivalent to adding its negative, we may use the following rule for subtracting polynomials.

(3) *To subtract one polynomial from another, change the signs of all the terms of the polynomial to be subtracted and add the result to the other polynomial.*

Try Rule 3 in some of the preceding exercises.

**46. Removal of Parentheses.** From what you have now learned about polynomials you will easily see that such equations as the following are true.

$$+(a-b-c)=a-b-c.$$
$$-(a-b-c)=-a+b+c.$$

*Rule. When a polynomial is inclosed in parentheses preceded by a positive sign, the parentheses may be omitted.*

*When a polynomial is inclosed in parentheses preceded by a negative sign, we may omit the parentheses only if we at the same time change the sign of each term of the polynomial.*

We shall apply this to an example involving several signs of aggregation (§ 8).

*Example.* Remove all signs of aggregation from the expression

$$-\{a-[(b+3d)-(4e-f)]+(2b-7d+3e)\}.$$

Here we may first remove the parentheses inside the brackets [ ]; we then have

$$-\{a-[b+3d-4e+f]+(2b-7d+3e)\}.$$

Now remove brackets after changing the sign of each term inclosed; also omit the parentheses; we then have

$$-\{a-b-3d+4e-f+2b-7d+3e\}.$$

Combining similar terms, we have

$$-\{a+b-10d+7e-f\}.$$

Now remove braces after changing the signs of all terms inclosed; we finally have

$$-a-b+10d-7e+f.$$

In this work it should be noticed that brackets and braces are merely different forms of parentheses.

**Exercises.** Simplify the following, removing all signs of aggregation.

1. $[a+(c-d)-(3a+2d)]$.
2. $[-(2m-n)+(5m+6n)-(8m-3n)+5m]$.
3. $-[2x+(4y-7x)-(8y+3x)-(5x-2y)]$.
4. $-[-(u+2v-3w)+(3u-2v+w)-u+3v-4w]$.
5. $-\{[-(7x-2y)+(8x-4y)]-[2x-5y-(3x-y)]\}$.

N.B. Whenever the expression inclosed in parentheses can be simplified by a reduction or combination of its terms, this should be done first.

**47. Summary.**

The amount of a quantity may be measured by comparing it with different units. The units may be of the standard kind as used in arithmetic, or they may be general units.

When only one length is used this measurement is expressed by means of one term and the expression is called a monomial.

When two different lengths are used in the same measurement, the result is expressed by two terms and the expression is called a binomial.

When three different lengths are used in the same measurement, the result is expressed by means of three terms and the expression is called a trinomial.

In general, when two or more lengths are used in the measurement of a quantity, there are two or more terms in the expression of the measurement and the expression is called a polynomial.

*To add polynomials.* Write the polynomials in parentheses with the plus sign between them. Add the like terms and express the answer with different terms, one following the other, connected by the sign obtained by the addition.

*To subtract polynomials.* Write the polynomials in parentheses, the one to be subtracted following the one from which it is to be subtracted, with the minus sign between them. Subtract the like terms in the manner that you learned to subtract monomials, and write the result with the different terms, one following the other, connecting them by the sign obtained from the subtraction.

*Removal of parentheses.* If a parenthesis is preceded by a plus sign, the parenthesis may be removed without changing the signs of the terms in the parenthesis. If a parenthesis is preceded by a minus sign the parenthesis may be removed, provided the sign of each term in the parenthesis is changed.

**Problems in Addition and Subtraction.**

**1.** On a winter day the temperature at noon was twice the temperature at 6 A.M. and the temperature at 6 P.M. was two-thirds of the temperature at noon. The sum of the three was 65 degrees. Find the temperature at each time.

**2.** The income from a certain business during the second year was double the income during the first year; during the third year it again doubled. The total income was $4500. What was the income each year?

**3.** The volume of a cylindrical cup in cubic inches is 3 times the volume of a conical cup of the same height and base. Using these as measures it is found that 2 measures of the first and 5 of the second just fill a gallon can. What is the volume of each cup?

**4.** If $2a$ dollars and $5b$ dimes are taken from $4a$ dollars and $3b$ dimes the remainder is $4.60. Also $3b$ dollars equal $15b$ dimes. Find the numbers for which $a$ and $b$ stand.

**5.** Show by drawing that

$$(2+3)-(8-5-6)=(2+3)+(-8+5+6).$$

6. Three children are digging dandelions. One digs at the rate of $a$ dozen per hour; the second at the rate of $b$ dozen per hour; the third at the rate of $c$ dozen per hour. The first morning the first works 4 hours, the second 3 hours, and the third 5 hours. The second morning the first works 5 hours, the second 3 hours, and the third 5 hours. How many more dandelions were dug the second day than the first?

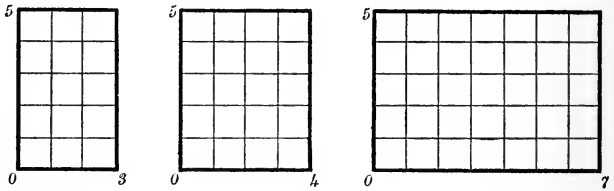
7. There are four sizes of shot. The largest size weighs $s$ grams, the second weighs $r$ grams, the third $p$ grams. On one pan of the scales are 7 of the first kind, 15 of the second, 18 of the third, and 25 of the fourth. On the other pan are 13 of the first, 6 of the second, 30 of the third, and 4 of the fourth. How many grams weight are on the first pan? How many are on the second? How much must be added to the weight on the second pan to have the scales balance?

# CHAPTER IX

## MULTIPLICATION AND DIVISION OF POLYNOMIALS

### 48. Multiplication of a Polynomial by a Monomial.

*Example* 1. The length of a rectangle is 5 inches, its width is 3 inches; the length of a second rectangle is 5 inches and its width is 4 inches. What is the sum of their areas?




Area of the first rectangle is $5 \cdot 3$ square inches.

Area of the second rectangle is $5 \cdot 4$ square inches.

Then the sum of these areas is $(5 \cdot 3 + 5 \cdot 4) = 35$ square inches.

This is equivalent to the area of a rectangle whose dimensions are 5 by 7, as shown in the figures above.

That is, we have

$$\begin{aligned} 5 \cdot 3 + 5 \cdot 4 &= 5\,(3 + 4) \\ &= 5 \cdot 7. \end{aligned}$$

In exactly the same way, if the dimensions of the first rectangle are $a$ by $b$ and the dimensions of the second rectangle are $a$ by $c$, the sum of their areas is

$$a \cdot b + a \cdot c,$$

but this must be equivalent to a single rectangle whose dimensions are $a$ by $(b + c)$. That is, we always have

$$ab + ac = a\,(b + c).$$

*Example* 2. If 3 boys pull a sled forward with a force of 30 pounds and 3 other boys pull the sled forward with a force of 35 pounds each, what is the total pull on the sled?

*Solution:* We can get the result in two different ways.

First: Total pull of the first 3 boys is $3 \cdot 30$ pounds.

Total pull of the other 3 boys is $3 \cdot 35$ pounds.

Therefore the whole is the sum of these,

$$\begin{aligned}(3 \cdot 30 + 3 \cdot 35) \text{ pounds} &= (90 + 105) \text{ pounds}\\ &= 195 \text{ pounds.}\end{aligned}$$

Second: Taking one boy from each set of three, their combined pull is $(30 + 35)$ pounds. Since there are three such pairs of boys, their total pull is

$$\begin{aligned}3\,(30 + 35) \text{ pounds} &= 3 \cdot 65 \text{ pounds}\\ &= 195 \text{ pounds.}\end{aligned}$$

Hence we have

$$3\,(30 + 35) = 3 \cdot 30 + 3 \cdot 35. \quad \text{Why?}$$

This is another illustration of the rule

$$a\,(b + c) = ab + ac.$$

State this rule in words.

It is easy to illustrate this rule when some of the numbers are negative.

*Example* 3. If 3 boys pull a sled forward with a pull of 30 pounds each and if 3 boys pull the sled backward with a force of 35 pounds each, what is the effective pull on the sled?

*Solution:*

Use + for pull forward and − for pull backward.

First: $$\begin{aligned}3 \cdot 30 + 3 \cdot (-35) &= 90 + (-105)\\ &= -15.\end{aligned}$$

That is, the net effect is a backward pull of 15 pounds.

Second: Take the boys in pairs, one pulling forward and the other pulling backward. The net effect of the pulling of each pair is

$$30 + (-35) = -5.$$

Since there are three such pairs, the total effect is

$$\begin{aligned}3\,[30 + (-35)] &= 3 \cdot (-5)\\ &= -15.\end{aligned}$$

Since the expressions in the first and second solutions stand for the same thing, namely total pull, we have

$$3\,[30 + (-35)] = 3 \cdot 30 + 3\,(-35).$$

Using letters, suppose $n$ boys pull the sled forward with a force of

$p$ pounds each, and $n$ other boys pull it backward with a force of $q$ pounds each, then in the first solution we would have

$$n \cdot p + n(-q);$$

In the second solution we would have

$$n[p + (-q)].$$

Here again we must have

$$n[p + (-q)] = n \cdot p + n(-q),$$

which is the same form, except for a difference in the letters, as

$$a(b + c) = ab + ac.$$

If, in the last example, both sets of boys pull backward, show that the total pull may be expressed in either of the forms

$$n[(-p) + (-q)] \text{ or } n(-p) + n(-q),$$

so that we have

$$n[(-p) + (-q)] = n(-p) + n(-q).$$

Another example of this sort is the following.

*Example.* A man who can row at the rate of $v_r$ miles an hour in still water enters a stream which has a velocity of $v_s$ miles an hour. How far can he row against the stream in one hour? How far can he row in three hours?

*Solution:* In this example if we regard the direction in which the man is going as positive, then the direction in which the stream is flowing is negative.

Therefore the distance gone in an hour is

$$v_r + (-v_s) = v_r - v_s.$$

The distance gone in 3 hours is

$$3(v_r - v_s).$$

But the man would be just as far at the end of 3 hours if it were possible for him to stop the stream and he rowed for 3 hours in still water, then he stopped rowing and allowed the stream to flow at its accustomed rate and pull him along for 3 hours. In the first case the two velocities are acting simultaneously, while in the latter case they would be acting consecutively.

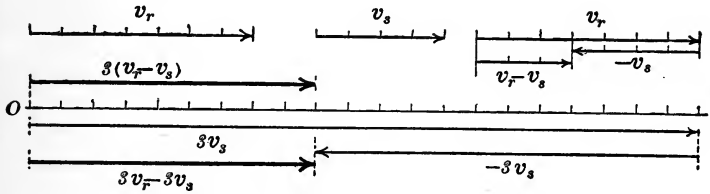
The algebraic expression of the first case is as given above. For the latter case it would be

$$3v_r - 3v_s.$$

Therefore,

$$3\,(v_r - v_s) = 3\,v_r - 3\,v_s.$$

This is illustrated graphically below.




**49. Distributive Law.** The equation $a\,(b + c) = ab + ac$ and the similar equations just considered express what is known as the **distributive law.** This law asserts that the product of a single number by the sum of two numbers is identical with the sum of the products of the first number by the other two numbers taken singly.

As to which of the two ways this shall be written will depend entirely upon the use we are going to make of it in our work.

**Exercises.**

**1.** A girl earned $l$ cents an hour helping in the laboratory, and $c$ cents an hour for correcting papers. She spent 4 hours a day in the laboratory, and 3 hours a day marking papers. How much did she earn during a month of 20 days? First write the algebraic expression for the entire amount earned in one day, then write it for the amount earned in 20 days. Also write the amount earned for laboratory work during the month, then the amount earned for correcting papers during the month, and add the two amounts together, as the total amount earned. These two expressions for the amount earned during the month are equal to each other. Write this equality.

**2.** A teacher prepared for her class $b$ bottles of a solution, each of which contained $w$ grams of water and 15 grams of salt. How many grams of the solution were there? Write the expression for this in two different ways placing the equality sign between them.

**3.** If a student spends $b$ dollars a month for board and $r$ dollars a month for room rent for 6 months, how much does he spend? Write the equality of the two different algebraic expressions for the answer to this.

**4.** From each of $a$ bottles of water, each containing $g$ grams, an average of 3 centigrams evaporated. How many centigrams were left in the bottles?

**5.** If a boy earns $w$ dollars a day and pays $c$ cents a day for his board, how much has he by the end of the month?

**6.** A measuring stick is $3c$ feet and $2b$ inches long. When applied to a line it goes $3\frac{1}{2}$ times. How long is the line?

**7.** A measuring line lacks $l$ feet of being 20 feet long. When applied to a line it goes $f$ times, how long is the line?

**8.** A cubic centimeter of gold weighs $w$ grams and a cubic centimeter of lead weighs $l$ grams less. What is the weight of $l$ cubic centimeters of lead?

Write problems leading to the following expressions; change the letters to any suitable to the problem, and write the two algebraic expressions of the answers equal to each other.

**9.** $3(a+4)$.

**10.** $5(-2a+3)$.

**11.** $7(6a-1)$.

**12.** $5(8a+7b)$.

**13.** $a(a-1)$.

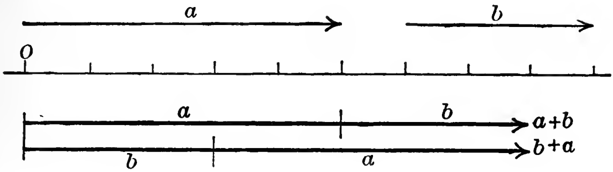
**14.** $a(a+b)$.

**15.** $b(a+2b-c)$.

**50. The Commutative Law of Addition.** This law asserts that the value of the sum of two numbers does not depend upon the order in which the numbers are taken when they are added.

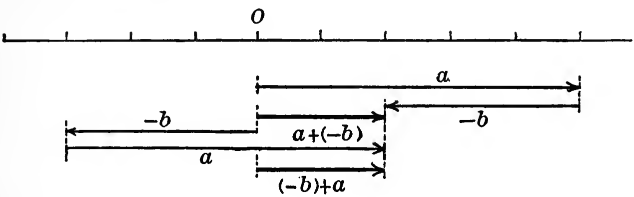
Algebraically expressed, $a+b=b+a$, no matter what the sign of $a$ and $b$.

Geometrically expressed,

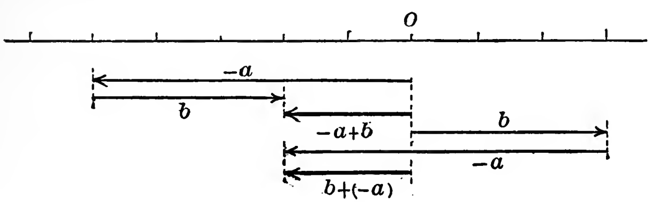



Thus we see that if the two numbers are positive the sums are the same.

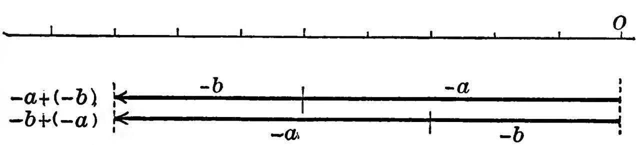
In like manner: $a + (-b) = -b + a$.




In like manner: $-a + b = b + (-a)$.




In like manner: $-a + (-b) = -b + (-a)$.




**51. The Commutative Law of Multiplication.** This law asserts that the value of the product of two numbers does not depend upon the order in which the numbers are taken when they are multiplied.

Algebraically expressed: $ab = ba$, no matter what the sign of $a$ and $b$.

Sylvester's illustration of the commutative law for multiplication. Take $a$ baskets containing $b$ apples each. Now since there are $b$ apples in one basket, in $a$ baskets there are $a$ times $b$ apples or $ab$ apples. Take an apple from each of the baskets and place in a new basket; that basket will have $a$ apples in it. Again take an apple from each of the original baskets, and you will have another new basket with $a$ apples in it. If you keep on this way, you will at last transfer all the apples in the original baskets to $b$ new baskets, and there will be $a$ apples in each basket. Since there are $a$ apples in each basket, in $b$ baskets there will be $b$ times $a$ apples or $ba$ apples.

Therefore, $$ab = ba.$$

**52. Exponents.** The area of a rectangle is the product of its two dimensions. If these dimensions are $a$ and $b$, the area is $ab$. If the rectangle is a square, whose dimension is $a$, then its area is $a \cdot a$. The volume of a cube whose edge is $a$, is $a \cdot a \cdot a$.

To shorten the writing of such products, we use the following symbols.

In place of $a \cdot a$ write $a^2$.
In place of $a \cdot a \cdot a$ write $a^3$.
In place of $a \cdot a \cdot a \cdot a$ write $a^4$.
In place of $a \cdot a \cdot a \cdot a \cdot a$ write $a^5$, and so on.

These are read $a$ square, $a$ cube, $a$ fourth power, $a$ fifth power, and so on.

The little number written above and to the right of another number shows how many times the other number is to be used as a factor; the little number is called the *exponent*, and the other number is called the *base*.

We also write

$$a^2b = a \cdot a \cdot b; \quad a^3b^2 = a \cdot a \cdot a \cdot b \cdot b; \quad \text{and so on.}$$

**Exercises.** As drill to become familiar with this notation, write the following; make use of the exponent. Read what you have written.

**1.** $3\,aaabb.$

**2.** $7 \cdot 7\,mrrmn.$

**3.** $8\,r \cdot 8\,sstttr.$

**4.** $\frac{2}{5} \cdot \frac{2}{5}\,ccdddfd.$

**5.** $4\frac{1}{2}\,hhhhk \cdot 4\frac{1}{2}\,hhkkst.$

**6.** $4 \cdot 4 \cdot 4 \cdot a \cdot 4\,bbba.$

**7.** $2\,r \cdot 2\,r \cdot 5\,r.$

**8.** $-7\,a \cdot (-7\,a)\,ab.$

Write the following without making use of the exponent.

**9.** $9 \cdot 3^5\,m \cdot 3\,r^2m^5n.$

**10.** $\frac{4}{5}\,cd \cdot \frac{15}{16}\,cd^2f.$

**11.** $32\,r^4s^3b^2 \cdot (-3^5r^2s^2b^3).$

**12.** $(-3)^2\,(-3\,a)^3.$

**13.** Write the answers to Exercises 9, 10, 11, 12, making use of the exponent.

**14.** Write as products of prime factors, 20, 72, 36, 75, 98, 312.

**15.** Calculate $2^3 \cdot 3^5 \cdot 5^2$.

*Meaning of word* **coefficient.** In expressions of the kind with which we have just been dealing, we call the figures the numerical factors and the letters the literal factors. The numerical factors are usually written first, that is, preceding the literal factors.

When we wish to refer to one particular factor, we call the other factors coefficients of this one. Thus, in the expression $4\,abc$, if we are interested in $c$, then $4\,ab$ is its coefficient; $4\,bc$ is the coefficient of $a$; $4ac$ is the coefficient of $b$; $4\,b$ is the coefficient of $ac$; 4 is the coefficient of $abc$; $abc$ is the coefficient of 4. Coefficient means co-factor. However, the numerical factor is referred to unless otherwise stated. If you were called upon to give the coefficient in the term $3\,abc$, you would say that it is 3.

## 53. Multiplication of One Polynomial by Another.

*Example* 1. A teacher in a laboratory prepared for a class $w$ bottles, each containing $w$ grams of water and 15 grams of salt. The class used 13 bottles. How many grams of the solution were there left?

*Solution:* We can compute this in two different ways, and give algebraic expressions for the different ways of computing. The easier way to compute it is to subtract the number of bottles used from the number of bottles prepared, and then compute the number of grams of solution in those left. The algebraic expression for this is

$$(w - 13)(w + 15) = \text{the number of grams of the solution left;}$$

or we can compute the number of grams of solution that we had at first and from this subtract the number used, and thus arrive at the number of grams left. The algebraic expression for this is

$$w(w + 15) - 13(w + 15) = \text{the number of grams left;}$$

therefore $(w - 13)(w + 15) = w(w + 15) - 13(w + 15)$; (Why?)
or, writing in the other form, $= (w^2 + 15w) - (13w + 195)$.
Subtracting as indicated, $= w^2 + 2w - 195$, the number of grams left.

Check by letting $w = 60$. Substituting in the original problem:

If a teacher in a laboratory prepared for a class 60 bottles each containing 60 grams of water and 15 grams of salt, there were in each bottle the sum of 60 grams and 15 grams which is 75 grams. There were at first 60 bottles, but the class used 13 bottles; there were then 47 bottles left. Since there were 75 grams in each bottle, in 47 bottles there would be 47 times 75 grams which is 3525 grams. This is the number of grams left after the class used 13 bottles.

Substituting $w = 60$ in the answer, which is supposed to be the number of grams left, we have

$$\begin{aligned} w^2 + 2w - 195 &= 60^2 + 2 \cdot 60 - 195 \\ &= 3600 + 120 - 195 \\ &= 3525. \end{aligned}$$

Since this is the number which we got by our arithmetic solution, $w^2 + 2w - 195$ must be the correct number of grams left after the class used 13 bottles.

*Example* 2. A man who can row at the rate of $v_r$ miles an hour in still water, rows up a stream which flows at the rate of $v_s$ miles an hour. He rows $m$ hours in the morning and $a$ hours in the afternoon. How many miles does he row upstream during the day?

*Solution:* As before, we can compute this in two different ways to arrive at the correct result. First add together the number of hours he rows in the morning and in the afternoon, finding the number of hours he rows during the day; then multiply the number of miles he progresses per hour by this. Second, find the number of miles he rows in the morning and the number of miles he rows in the afternoon; add these, obtaining the number of miles he rows during the day. The algebraic expressions of these two computations are equal to each other. This gives the equation

$$(m + a)(v_r - v_s) = m(v_r - v_s) + a(v_r - v_s);$$

multiplying on the right, $= (mv_r - mv_s) + (av_r - av_s);$

adding as indicated, $= mv_r - mv_s + av_r - av_s,$

the number of miles the man rows up the stream during the day.

Check by letting $m = 3$, $a = 2$, $v_r = 5$, $v_s = 2$. Substituting in the original problem:

If a man who can row at the rate of 5 miles an hour in still water, rows up a stream which flows at the rate of 2 miles an hour, he will go up the stream at the rate of 3 miles an hour. Since he rows 3 hours in the morning and 2 hours in the afternoon, he rows during the day the sum of 3 hours and 2 hours which is 5 hours. Since he advances 3 miles in 1 hour and rows for 5 hours, he will advance 5 times 3 miles, which is 15 miles.

Substituting in the answer:

$$\begin{aligned} mv_r - mv_s + av_r - av_s &= 3 \cdot 5 - 3 \cdot 2 + 2 \cdot 5 - 2 \cdot 2 \\ &= 15 - 6 + 10 - 4 \\ &= 15. \end{aligned}$$

Since this is the number we got from our arithmetic solution $ma_r - mv_s + av_r - av_s$ must be the correct number of miles he rows.

**Exercises.** Solve the following as illustrated above.

**1.** In an alloy of metal there are in each cubic centimeter $g$ grams of one metal and 4 grams of another. In one brick of the metal there are $g$ cubic centimeters, and in another there are 15 cubic centimeters. How many grams do the two bricks together weigh?

**2.** Two boys each agree to work for $w$ dollars a day, out of which 25 cents is taken for dinner. If one works $t$ days and the other 5 days, how much do they together receive?

**3.** How much more does the boy who works $t$ days receive than the boy who works 5 days?

**4.** Two men start out to walk. One walks $t$ days a week for 2 weeks and the other walks $m$ days. If they both walk $d$ miles in the morning and $t$ miles in the afternoon, how much farther does the one go than the other?

**5.** There are two pieces of land. One is square and the other is rectangular. The rectangular piece has a length 6 meters longer than the side of the square, and a width 11 meters shorter than the side of the square. How many more square meters are there in the area of the square than in the area of the rectangle?

**6.** The length of one side of a square is $s$ meters. A rectangle is 1 meter more in length and 1 meter less in width, what is the difference in the area of the two figures? Which is the larger?

**7.** The side of a square is $s$ meters. A rectangle is 3 meters more in length and 3 meters less in width. How much more is the area of the square than that of the rectangle?

**8.** The side of a square is $s$ meters. A rectangle is 5 meters more in length and 5 meters less in width. How much more is the area of the square than the area of the rectangle?

**9.** The side of a square is $s$ meters. A rectangle is 7 meters more in length and 7 meters less in width. How much greater is the area of the square than the area of the rectangle?

**10.** The side of a square is $s$ meters. A rectangle is 12 meters more in length and 12 meters less in width. How much greater is the area of the square than that of the rectangle?

**11.** In Exercises 8, 9, 10, 11 and 12, are the perimeters the same in each case? How do the areas of the different rectangles compare? Draw them, arranging them in the order of their size (use a millimeter instead of meter as the unit). If rectangles have equal perimeters, which is the greatest?

Write problems leading to five of the following. Solve all, expressing the results in the different algebraic forms as in the preceding exercises:

**12.** $(m + 2)(m + 5)$.

**13.** $(t_1 - t_2)(s_1 + s_2)$.

**14.** $(r + 6)(r - 2)$.

**15.** $(p - 8)(p - 7)$.

**16.** $(2k - 7)(3k - 15)$.

**17.** $(5r - h)(3r + 2h)$.

**18.** $(c - 1)(c^2 + c + 1)$.

**19.** $(b^2 + 6ab)(2b^2 - 3ab)$.

**20** $(m + 3rs)(m^2 - 3mrs + 9r^2s^2)$.

**21.** $(a + 3b)(a + 3b)$.

**22.** $(m^2n + 2a^2b)(m^2n + 2a^2b)$.

**23.** $(-4 - 6a^2b)(-4 + 6a^2b)$.

**24.** $(-6x^2y + 2xy^2)(-6x^2y + 2xy^2)$.

**25.** $(-d^2 - 2cd^2)(d^2 - 2cd^2)$.

**26.** $(x^2 + 5)(x^2 + x - 30)$.

## 54. Division of Polynomials.

*Example* 1. A man, who can row at the rate of 5 miles an hour in still water, rows upstream against a 2-mile current. He rows 24 miles the first day and 15 miles the second. How many hours did he row?

*Solution:* He advances in one hour a distance of (5 − 2) miles. He rowed (24 + 15) miles.

Therefore the time he rowed is

$$\frac{24 + 15}{5 - 2} = \frac{39}{3} \quad \text{(Why?)}$$
$$= 13 \text{ hours.}$$

Here we can simplify our first expression by combining the numbers, and afterwards by dividing out.

Suppose the same problems to be stated in a literal form.

A man, who can row at the rate of $r$ miles an hour in still water, rows upstream against the current flowing $c$ miles an hour. He rows $n$ miles the first day and $m$ miles the second. How long did he row?

*Solution:* He advances in one hour a distance $(r - c)$ miles. He rows $(n + m)$ miles. Therefore the time used in rowing is

$$\frac{n + m}{r - c} \text{ hours. (Why?)}$$

We cannot now simplify the result further so we leave it as it stands. It is the indicated quotient of one binomial, namely $(n + m)$, by another binomial, namely $(r - c)$. Such indicated quotients are called *fractions.*

*Example* 2. The cost of seeding a piece of ground was 4 dollars per acre, the cost of cultivating was 2 dollars per acre, and the cost of harvesting was 3 dollars per acre. The total cost was $270. How many acres were there?

*Solution:* The total cost per acre is $(4 + 2 + 3)$ dollars.

Therefore the number of acres is

$$\frac{270}{4+2+3} = \frac{270}{9} = 30.$$

If the cost of seeding is $s$ dollars, the cost of cultivating is $c$ dollars, the cost of harvesting is $h$ dollars, and if the total cost is $a$ dollars, we shall have the number of acres

$$\frac{a}{s+c+h}.$$

We shall study such form farther on in the chapter on factoring.

**Problems in Division of Polynomials.**

**1.** If a man can row at the rate of $r$ miles an hour in still water, and if he rows downstream in a current flowing at the rate of $c$ miles an hour, how many hours will it take him to go $d$ miles? If he rows upstream, how long does it take him to go $d$ miles?

**2.** If it costs $a$ cents to set the type for a printed circular, and if it costs $b$ cents a hundred for the printing, and $c$ cents a hundred for the paper, how many circulars can be made for $n$ dollars?

**3.** To construct an office building costs for each story $f$ dollars for the floor, $w$ dollars for the walls, $p$ dollars for the partitions. The total cost of the building is $c$ dollars, which includes $b$ dollars for the basement and $r$ dollars for the roof. How many stories are there?

**4.** A room is $a$ feet long and $b$ feet wide, and another room is $c$ feet long and $d$ feet wide. It costs $k$ dollars to lay a floor in these two rooms. What is the price of the floor per square foot?

5. If a man loans $p$ dollars to one person and $q$ dollars to each of two others, and receives $i$ dollars interest for the two years, what is the rate of interest?

6. If the temperature at a certain place was $d$, $e$, and $f$ respectively for three consecutive days, what was the average temperature for the time?

7. If three rectangles having the same altitude and bases which are $b$, $c$, and $d$ respectively, are added so that their bases are in a straight line, their area is $a$ square inches. What is their altitude?

8. A cube, a cylinder, a cone, and a sphere of iron weigh $p$, $q$, $r$, $s$ grams respectively, and have a combined volume of $v$ cubic centimeters. What is the density of iron?

9. The momentum of a boy riding his bicycle is $m$ mile-pounds per hour. The weight of the boy was $w_1$ pounds and the weight of the bicycle was $w_2$ pounds. What was the rate per hour at which they were moving? (Momentum in mile-pounds per hour = weight in pounds $\times$ speed in miles per hour.)

**55. Summary.**

*Commutative law of addition.* This law asserts that the value of the sum of two numbers does not depend on the order of summation.

*Commutative law of multiplication.* This law asserts that the value of the product of two numbers does not depend on the order of multiplication.

A positive integral *exponent* of a number is a small number written to the right and a little above the number to show how many times the number is used as a factor. The number to which the exponent is attached is called the *base*.

A *coefficient* of a number is a cofactor with that number.

*To multiply a polynomial by a monomial.* Write the numbers as factors — the monomial followed by the polynomial inclosed in parentheses. Make this equal to the product of each term of the polynomial multiplied by the monomial, the partial products being connected by the proper sign to form the complete product.

*To multiply a polynomial by a polynomial.* Write the two polynomials in parentheses as factors. Multiply the multiplicand by each term of the multiplier, writing the partial products in parentheses connected by the proper sign. Add or subtract as indicated by the sign.

# CHAPTER X

## PROBLEMS LEADING TO SIMPLE EQUATIONS

### 56. Problems.

*Example* 1. Apparatus needed — a pair of balances and some shot (any substance of which you can have a number of the same size and weight will do). Place a quantity of shot in one pan of the balances, and a different quantity in the other together with weights enough to balance. Find the weight of each shot. See that at all times the scales are balanced.

Now the process which you go through in the actual use of the scales in order to find the weight of each shot can be expressed in algebraic symbols.

Suppose you had placed 23 shot in one pan, and 12 in the other, and then found it necessary to place 22 gram weights on the second pan in order to balance the scales. Since you are interested in the weight of one shot only, the thing that you would do in the actual operation of weighing would be to take out of both pans the same number of shot, until you had shot only in one pan. Then you would count the number of shot left in the pan and the number of gram weights left in the other pan, and from this compute the weight of one shot. We shall now represent this in algebraic language.

Let $w$ = the number of grams in the weight of each shot;
then $23w$ = the number of grams in the weight of 23 shot,
and $12w$ = the number of grams in the weight of 12 shot.
Therefore $23w$ = the number of grams in one pan,
while $12w + 22$ = the number of grams in the other pan.

Since these two amounts balance each other, they are equal to each other. Therefore we write

$$23w = 12w + 22.$$

Now, just as we did in the actual weighing, we can take out $12w$ from each pan and have

$$11w = 22.$$

Since $w$ is multiplied by 11 to be equal to 22, and 2 is multiplied by 11 to be equal to 22, therefore,

$$w = 2.$$

Therefore, since $w$ stood for the weight of each shot, each shot must weigh 2 grams.

*Check:* We can test the correctness of this result by computing the weight in each pan.

If 1 shot weighs 2 grams, 23 shot will weigh $23 \cdot 2$ or 46 grams, which is the weight in one pan.

12 shot will weigh $12 \cdot 2$ or 24 grams. To this add the 22 grams, and we have 46 grams as the weight on the other pan.

The two amounts are equal and hence balance each other, therefore our answer must be correct.

**1.** Place different quantities of shot in each pan and then balance the scales by placing gram and fractional gram weights on each pan. Proceed to find the weight of each shot. See that at each moment the scales are kept balanced.

State this experiment in words; represent in algebraic symbols; solve and check.

**2.** A boy found that if he placed 2 iron balls of an equal size on one pan of the scales, he must place 50 gram weights on the other, in order to balance them. What is the weight of each ball?

**3.** In weighing a stone block in the laboratory, a student found that if he placed a 20-gram weight on the side of the balances with the stone, it would balance a 100-gram weight on the other side. What is the weight of the stone? Explain this without using symbols, then solve by means of symbols.

**4.** If 5 marbles of equal size and 5 gram weights balance 2 marbles and 40 gram weights, what is the weight of each marble?

**5.** If your book and 20 grams balance 625 grams, what is the weight of your book?

*Example* 2. If 21 cc. of marble weigh 56.7 grams, what is the density of marble? That is, what is the weight of one cc. of marble?

*Solution:*

| | |
|---|---|
| Let | $d$ = the density of marble. |
| Then | $21\,d$ = the number of grams in the weight of 21 cc. |
| But | $56.7$ = the number of grams in the weight of 21 cc., |
| therefore | $21\,d = 56.7$, |
| and | $d = 2.7$, the density of marble. |

*Check:* If one cubic centimeter of marble weighs 2.7 grams, 21 cc. will weigh 21 times 2.7, which is 56.7. This is the amount that the example states that it should weigh. Therefore the result is correct.

6. How many cubic centimeters are there in a brass ball which weighs 109.2 grams? The density of brass is 8.4.

7. If you place on one pan of the scales a piece of wood and another $\frac{2}{3}$ as large, they together balance 16 lbs. in weights placed on the other pan. What is the weight of each piece of wood?

8. The density of platinum is 2.17 more than the density of gold. Two bars, one containing 20 cc. of platinum and the other containing 20 cc. of gold, together weigh 826 grams. What is the density of platinum? of gold?

9. A cubic centimeter of ebony weighs 1 gram more than a cubic centimeter of cork. A piece of cork containing 16 cc. and a piece of ebony containing 7 cc. placed together on one pan of the scales balance 11.14 gram weights placed on the other pan. What is the density of cork? of ebony? What is the weight of each piece?

10. Iron is 4 times as heavy as ivory. 14 ivory balls and 6 iron balls of the same dimensions are placed on the scales and are found to balance 456 gram weights. What is the weight of each ball? What is the density of ivory if each ball contains $6\frac{2}{3}$ cc.? What is the density of iron?

11. If 20 shot and a 5-gram weight together balance 4 shot and 29 grams, what is the weight of each shot?

12. If 2 iron blocks of the same size, together with 4 oz. weights, just balance a block $\frac{2}{3}$ as large as one of them

together with 12 oz. weights, what is the weight of each block?

**13.** A boy placed in one pan of the scales 110 shot and in the other 50 shot and 25 grams in weights. He then removed from the first pan enough to make the two pans balance. He found afterward by calculation that he had removed shot enough to weigh 15 grams. What was the weight of each shot?

**14.** The weight of one cubic centimeter of copper is 1.1 grams more than that of the same amount of iron. 5 iron balls weigh 278.3 grams more than 3 copper balls of the same size. What is the density of iron, if each ball contains 23 cc.? What is the density of copper?

**15.** The density of lead lacks 4.3 of being 2 times that of iron. 6 iron balls weigh 1.6 grams more than 4 lead balls of the same size. What is the weight of each iron ball if each ball contains 1 cc.? What is the weight of each lead ball?

**16.** An iron pail covered with tin weighs 1.1366 kilograms. There are 125 cc. of iron and 22 cc. of tin. If iron has a density of .5 more than tin, what is the density of iron and of tin?

**17.** If 3 cc. of a substance weigh 9 grams, what does 1 cc. weigh. If 16 cc. of a substance weigh 38 grams, what is the density of the substance? If 84 cc. of a substance weigh 21 grams, what is the density of the substance? If 6 cc. weigh 1 gram, what is the density of the substance? If $v$ cc. weigh $m$ grams, what is the density of the substance?

The answer to this last question is usually expressed in the form

$$d = \frac{m}{v}.$$

This can be used as a formula by which to solve other exercises. Using this as a formula solve the following:

**18.** A block of lead in the shape of a cube whose edge is 10 cm. weighs 11,300 grams. What is the density of lead?

19. A block of marble 6 dm. wide, 6 dm. thick and 1.5 m. long weighs 1,468,800 grams. What is the density of marble?

20. A block of silver 2 cm. long, $c$ cm. wide and $c$ cm. thick weighs 21 grams. What is the density of silver?

21. The mass of a substance is 7 grams and its volume is 10 cc. What is its density?

22. The mass of a substance is $7ab$ grams and its volume is $14\,b$ cc. What is its density?

23. Solve — not using formula:

Alcohol is .8 as heavy as water. A bottle containing 354.3 cc. of alcohol and 118.1 cc. of water weighs 401.54 grams. If the weight of the glass is not included in this weight, what is the density of alcohol? of water?

24. From the Table of Densities on p. 35, write three problems leading to equations and solve.

**57. Formation of Equations.** In all of these exercises you will have noticed that certain amounts are *known* and certain amounts are *unknown*. You will also have noticed that we get the values of those that are unknown through their relations to those that are known. Those quantities that are given by experiment and measurement are called *known quantities*. Those whose values are to be found through their relation to the known quantities are called *unknown quantities*.

Whenever we have enough data on the relations of one unknown quantity to known quantities to be able to form two sets of symbols standing for the same amount, we can place the sets equal to one another, and solve for the unknown quantity as we have done in the preceding exercises.

The expression formed by placing these two sets of symbols equal to one another is called an **equation.**

The truth which we employ in forming the equation is called an **axiom.** It is usually worded:

**Things which are equal to the same thing are equal to each other.**

An axiom is a statement whose truth we assume without proof.

**58. Solution of Equations.** Having formed our equation, we further assume that we may add equal amounts to both sides of the equation without destroying the equality, or that we may subtract equal amounts from both sides of the equation without destroying the equality. Thus we get an equation with the unknown terms on one side of the equation and the known terms on the other. Likewise we assume that both sides of an equation may be multiplied or divided by equal quantities without destroying the equality. This leads to the solution of the equation.

So we have the following five axioms by means of which we solve our equation:

**I. Things equal to the same thing are equal to each other.**
**II. If equals be added to equals, the results are equal.**
**III. If equals be subtracted from equals, the results are equal.**
**IV. If equals be multiplied by equals, the results are equal.**
**V. If equals be divided by equals, the results are equal.**

*Example* 1. A boy riding his wheel starts for his home 20 miles away. He rode 2 miles farther the second hour than he did the first, and 3 miles farther the third hour than he did the first, and reached his destination. How far did he ride each hour?

*Solution:* Let $d$ = the number of miles the boy rode the first hour.
Then $d + 2$ = the number of miles he rode the second hour,
and $d + 3$ = the number of miles he rode the third hour.
Then $3d + 5$ = the number of miles he rode;
but $20$ = the number of miles he rode.
Therefore $3d + 5 = 20$. (Things which are equal to the same thing are equal to each other.)

Now subtract 5 from each side of the equation; we have

$3d = 15.$ (If equals are subtracted from equals, the results are equal.)

Then $d = 5,$ (If equals are divided by equals, the results are equal.)
and $d + 2 = 7,$ (If equals are added to equals, the results are equal.)
and $d + 3 = 8.$ (For the same reason.)

So the boy rode 5 miles the first hour, 7 miles the second hour and 8 miles the third hour.

*Check:* If a boy rode his wheel at the rate of 5 miles an hour the first hour and at the rate of 7 miles an hour the second hour, he rode 2 miles farther the second hour than the first. If he rode 8 miles the third hour, he rode 3 miles farther the third hour than the first. If he rode 5 miles the first hour, 7 miles the second hour and 8 miles the third hour, he rode altogether $(5 + 7 + 8)$ miles, which equals 20 miles, the entire distance. Therefore since these answers check each statement of the original example, they must be correct.

**Problems.**

**1.** Two boys returning from a fishing trip wished to divide the fish caught equally between them. Upon counting them the one who had caught the most said to the other, "If I had 6 more, I should have 2 times as many as you have. I will give you 11, then we shall have an equal number." How many fish did each have?

**2.** A class committee appointed to look up the price of a piece of statuary reported that the money in the treasury was but 50 cents more than half as much as was needed for the purchase, and that it would be necessary for the class to raise 62 dollars more in order to buy the piece selected. How much did the piece of statuary cost?

**3.** Two boys pulled a small wagon in opposite directions, the forward pull being three times the backward pull. The result was a forward pull of 74 lbs. What was the amount and direction of each boy's pull?

**4.** Three men on a building are trying to raise a beam with ropes. One of the men exerts a force of $\frac{2}{3}$ as many pounds as the strongest of the three, and the other a force of $\frac{5}{6}$ as many pounds as the strongest. They succeed in lifting the beam which weighs 470 pounds. What is the number of pounds force of each man's pull?

**5.** A young man wishing to earn money to pay his board while attending school decided to build fires in furnaces at private residences in the neighborhood. He found that the number of engagements that he could make for this kind of work was just $\frac{1}{5}$ of the number of cents that he asked per day at each residence, and that if he would lower his price 5 cents per day at each residence he could get 2 more engagements, and thus earn 15 cents more per day. How much did he ask per residence?

**6.** A lady arranging a flower bed found that she had plants enough to arrange exactly in the form of a square, but if she attempted to arrange them in the form of a bed with 4 more in the length and 2 less in the width, she would lack 16 of having enough plants. How many plants has she?

**7.** Of three lines one is 6 cm. longer than $\frac{3}{5}$ of the length of another, and the third is $7\frac{4}{5}$ cm. less than the sum of the other two, while their combined length is 9 cm. What is the length of each line?

**8.** There are two angles whose sum is 97 degrees. If 13 degrees be subtracted from the larger and added to the smaller, the two angles will be equal. What is the size of each angle?

*Example* 2. A solution of alcohol and water is 95% strong. How much water must be added to have it 75% strong?

Remark. By 95% strong we mean that .95 of any selected amount or part is alcohol, and that 5% is water.

Then in starting an exercise of this kind, unless some definite amount is stated, we can speak of *one part,* a part being any convenient amount.

*Solution:*

Let $w$ = the number of parts of water that must be added to one part of the given solution.

Then $1 + w$ = the number of parts in the new solution.

$.75\,(1 + w)$ = the number of parts of alcohol.

But $.95$ = the number of parts of alcohol, since the amount of alcohol is not changed.

$\therefore \quad .75(1 + w) = .95.$ Why?

$w = \frac{4}{15}$, the number of parts of water that must be added to reduce a 95% solution to a 75% solution.

Check as in previous problems.

9. Upon examining a bottle of rose-water and glycerine which she had just bought, a lady said to the druggist, "There seems to be too much glycerine in this." He answered "I thought you said that you wished equal parts." The lady answered "No, I said that I wished 35% glycerine." How much rose-water must be added if the lady is willing that it be added to the four ounces that are in the bottle?

10. A solution of sulphuric acid and water is $33\frac{1}{3}\%$ sulphuric acid. How much water must be added to have it 10% sulphuric acid?

11. In a solution of iodine and alcohol, there is 6.25% of iodine. How much alcohol must be added in order to have 1% iodine?

12. A 2-ounce bottle of peroxide is 95% strong. This is too strong for general use. How much water must be added to make a 10% solution?

13. A pint of ammonia is purchased for cleaning purposes. It is a 90% solution. This is too strong. How much water must be used if a 50% solution is desired?

14. If sea-water is 12% salt, how much water must be evaporated in order to have it 90% salt?

15. Write three problems relating to solutions with which you are acquainted. Metals in alloy may be used. Solve your problems.

16. If one angle of a triangle is 15 degrees more than another, and a third is 2 times the sum of these two, what is the number of degrees in each angle of the triangle?

17. If one of the angles of a triangle is 15 degrees more than 3 times another, and $\frac{1}{3}$ of the difference between these

two is equal to $\frac{1}{4}$ of the third angle, what is the number of degrees in each angle of the triangle?

**18.** If you subtract 5 degrees from the number of degrees in the first angle of a triangle and multiply the remainder by 3, you will get the number of degrees in the second angle of the triangle. The third angle is 4 times the second. What is the number of degrees in each angle of the triangle?

**19.** The second side of a triangle is 5 times the first, and the third is 5 cm. longer than the first. If the third is subtracted from the second, the difference will lack 2 cm. of being equal to the first. What is the length of each side?

**20.** One side of a triangular lot is 200 feet longer than another, and the third side is 2 times this one. It requires 1100 feet of fence to enclose the lot. What is the length of each side?

**21.** A triangle has two of its sides equal. If 1 were added to the number of centimeters in the length of one of them, the sum would be 3 times the number of centimeters in the length of the third side. If 3 cm. were subtracted from each of the equal sides and added to the third side, the triangle would be equilateral. What is the length of each side of the triangle?

**22.** If one angle of a triangle is $\frac{1}{6}\pi$ radians more than $\frac{3}{4}$ of another, and the third is equal to 4 times the difference between these two, what is the number of radians in each angle of the triangle?

Write problems concerning a triangle leading to the following equations:

**23.** $s + (s + 1) + (2s - 2) = 27.$

**24.** $5a - 30 = 180.$ **25.** $2\frac{1}{2}a + \frac{1}{6}\pi = \pi.$

*Example* 3. If an angle is $2\frac{3}{5}$ times its complement, what is the number of degrees in the angle? in the complement?

*Solution:*

Let $a$ = the number of degrees in the angle.

Then $90 - a$ = the number of degrees in the complement. Why?

$2\frac{2}{3}(90 - a)$ = the number of degrees in the angle. Why?

$2\frac{2}{3}(90 - a) = a$. Why?

$\therefore\ 240 - 2\frac{2}{3}a = a$, doing the multiplication indicated.

$240 = 3\frac{2}{3}a$, adding $2\frac{2}{3}a$ to both members. Why?

$a = \dfrac{240}{3\frac{2}{3}}$. Why?

$= \frac{720}{11}$, multiplying numerator and denominator by 3,

$= 65\frac{5}{11}$, number of degrees in the angle.

$90 - a = 24\frac{6}{11}$, number of degrees in the complement of the angle.

*Check:* Since the angles are complementary their sum should equal 90 degrees. $(65\frac{5}{11} + 24\frac{6}{11})$ degrees is 90 degrees. The angle is as many times its complement as $\dfrac{65\frac{5}{11}}{24\frac{6}{11}}$, which is $\dfrac{720}{270}$, which is $2\frac{2}{3}$. Since the answers obtained satisfy all the conditions of the problem, they must be correct.

**26.** If an angle is 200 degrees more than 2 times its complement, what is the size of the angle and of its complement?

**27.** If $-15$ degrees be added to $\frac{3}{4}$ of the supplement of an angle, the sum will be equal to the angle. What is the number of degrees in the angle? in the supplement of the angle?

**28.** How many radians are there in an angle if its complement is 3 times its supplement?

**29.** Two angles are complementary. If $\frac{1}{18}\pi$ radians be added to one and subtracted from the other, the two angles will be equal. What is the number of degrees in the angles?

**30.** Write three problems concerning complementary angles and supplementary angles. Solve to see that they are correct.

**31.** One acute angle of a right-angled triangle is $1\frac{1}{4}$ times the other. What is the number of radians in each of the angles?

**32.** Of two adjacent sides of a parallelogram one is $\frac{3}{4}$ as long as the other. The perimeter is 49 cm. What is the length of each side?

**33.** Of the six angles of a hexagon, $A$, $B$, $C$, $D$, $E$, $F$, angle $B$ is 5 degrees more than 2 times angle $A$; angle $C$ is 14 degrees more than $\frac{1}{5}$ of angle $B$; angle $D$ is 2 times as large as angle $C$; angle $E$ lacks 10 degrees of equalling $\frac{1}{2}$ angle $A$; angle $F$ equals the difference between angle $A$ and angle $E$. The sum of the angles of a hexagon is 8 right angles. Find the number of degrees in each angle of the hexagon.

**34.** The sum of two angles is 150 degrees; $\frac{3}{4}$ of the smaller equals $\frac{1}{2}$ of the larger. What is the number of degrees in each angle?

**35.** In a triangle whose sides are $a$, $b$, $c$, if the side $c$ contained 1 unit more it would be just $\frac{1}{3}$ of side $a$; if it were 1 unit less it would be $\frac{1}{4}$ the length of side $b$. Furthermore, if one unit be subtracted from side $a$ and added to side $b$, sides $a$ and $b$ would be equal. Find the length of each side of the triangle.

**36.** Three men, $A$, $B$, $C$, on a building, are trying to raise a beam with ropes. $A$ pulls with a force of $\frac{2}{3}$ as many pounds as $C$, and $B$ pulls with a force of $\frac{5}{6}$ as many pounds as $C$. The beam weighs 471 pounds. With what force does each man pull if we regard a downward pull as positive?

**37.** The density of wrought silver is 7.945 more than that of agate. A silver ornament set with agate contains 8 cc. of each, and weighs 110.6 gm. Find the density of silver and of agate.

**38.** A glass bottle containing alcohol is closed with a cork. There are 23 cc. of glass in the bottle and 50 cc. of alcohol; the cork measures 5 cc. The whole weighs 100.12 grams. If a cubic centimeter of alcohol and one of cork together weigh 1.08 grams, and the density of glass is .7 less than 4 times that of alcohol, what is the density of each substance?

**39.** Two men start from a point 200 miles apart and travel toward each other, one at the rate of 5 miles an hour, and the other at the rate of 10 miles an hour. After how many hours will they meet?

**40.** *A*, traveling at the rate of 20 miles a day, has 4 days start of *B*, traveling at the rate of 26 miles a day, in the same direction. After how many days will *B* overtake *A*?

**41.** The circumferences of the front wheel and of the hind wheel of a wagon are 2 and 3 yards respectively. What distance has the wagon moved when the front wheel has made 10 revolutions more than the hind wheel?

Hint. — Let $r$ = number of revolutions the front wheel will make.

**42.** The sum of two digits of a number is 12. If the digits be interchanged, the resulting number will be equal to the original. What is the number?

**43.** The sum of the two digits of a number is 12. If the digits be interchanged, the resulting number exceeds the original number by $\frac{3}{4}$ of the original number. What is the number?

**44.** The sum of two digits of a number is 11. If the digits are interchanged, the resulting number is 45 less than the number. What is the number?

**45.** Write three problems on numbers and their digits and solve.

**46.** One boy says to another: "Think of a number; add 7; double result; take away 8; tell me your answer and I will tell you the number thought of." How can he do it?

**47.** Make up a problem like number 46.

**48.** In sending packages by parcel post for the 5th zone, the amount of increase on each pound is 2 cents less than the cost of the first pound. The cost of 11 pounds is 7 cents more than 8 times the cost of the first pound. What is the cost of each pound in this zone?

**49.** In the 3rd zone the cost for the first pound is 2 cents less than 2 times the additional price per pound. The price for 11 pounds is 4 cents more than 7 times the cost for the first pound. What is the cost for each pound in this zone?

*Note.* — In solving a problem it is not necessary that the letter stand for a number of single units of the thing called for; it may stand for the number of thousand units or the number of million units as the problem may suggest.

**50.** In 1912 the amount of gold produced in Colorado was 1.1 millions less than that produced in California; the amount produced in Alaska was 1.4 millions less than that produced in Colorado; the amount from Nevada was 3.35 millions less than $\frac{1}{2}$ of that of California; the amount from South Dakota was 1.15 millions more than $\frac{1}{2}$ of that from Nevada; from other states it was 1.1 millions less than 2 times that from South Dakota; the total is 2.3 millions less than 5 times that produced by Colorado. What was the amount produced in each state? What was the total gold product of the United States for that year?

**51.** In 1910 the number of people living in rural districts in the United States was 6.7 millions more than the number of people living in cities. The total number of people was 91.9 millions. How many people lived in the rural districts? How many in the cities?

**52.** In 1909 the value of the newspapers and periodicals in this country was placed at 53 millions less than twice the value in 1899. The value in 1899 was 157 millions less than the value in 1904. The total value for the three years was 1684 millions. What was the value placed on them for each of the three years?

**53.** Class should look up statistics and write three problems and solve.

*To form Equations Expressing the Relation between Unknown Numbers.*

Translate the English into algebraic language, then form the equation and reduce it by means of the axioms. Thus we can find the *relation* between numbers in which we are interested, even though we do not know the values of the numbers.

*Example* 1. A certain sum of money placed at simple interest at 6% amounts in a year to the same as another sum placed at interest at 8%. How does the first sum invested compare with the second?

*Solution:*

Let $p$ = the number of dollars invested at 6%

and $q$ = the number of dollars invested at 8%.

Then $1.06\,p$ = the number of dollars in the amount at the end of year of money at 6%

and $1.08\,q$ = the number of dollars in amount at end of year of money at 8%;

therefore $1.06\,p = 1.08\,q$, by statement of the problem.

$$p = \frac{1.08}{1.06}q$$

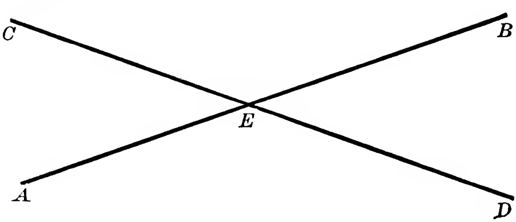
$$= 1\tfrac{1}{53}\,q.$$

That is, the capital invested at 6% is $1\frac{1}{53}$ as much as the capital invested at 8%. We have not found the value of either $p$ or $q$.

*Example* 2. Given two angles that are equal, to find the relation between their complements.

*Solution:* Since the angles are equal, the results obtained by subtracting each of them from a right angle are again equal. (Axiom III.) But these results are the complements of the given angles; therefore the complements of equal angles are also equal angles.

*Example* 3. If two lines are crossed as in the figure, find the relation between angle $DEB$ and angle $CEA$.




*Solution:*

Let $a$ = the number of radians in angle $DEB$,

and $a_1$ = the number of radians in angle $CEA$,

and $b$ = the number of radians in angle $BEC$;

then $a + b = \pi$, Why?

and $b + a_1 = \pi$; Why?
therefore $a + b = b + a_1$. Why?

Subtracting $b$ from each member of the equation,

$$a = a_1.$$

That is, the angle $DEB$ is equal to the angle $CEA$.

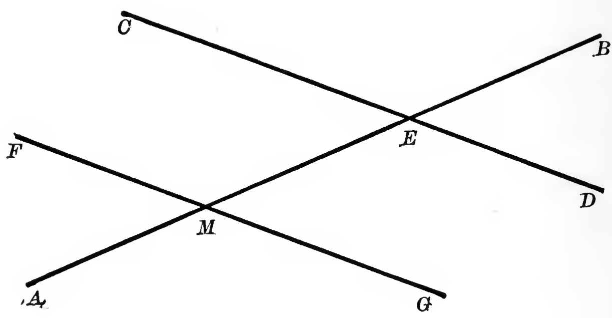
In the figure above the angles $DEB$ and $CEA$ are called **vertical angles,** as are also the angles $BEC$ and $AED$.

**Exercise.** Let the student solve for the relation between angles $BEC$ and $AED$.

These results may be stated thus:

*If two lines intersect, the vertical angles are equal.*

**Exercise.** In the figure of Example 3, draw a line $FG$ parallel to $CD$, making the following figure.

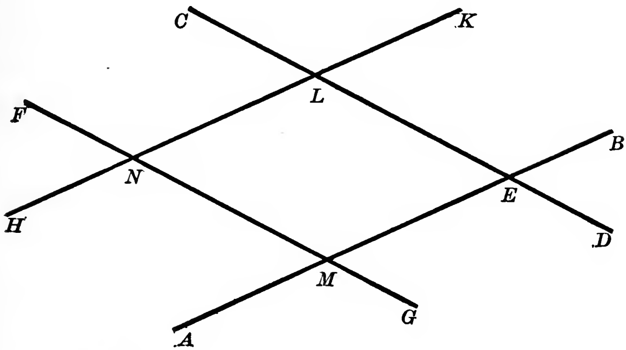



By measurement show that angle $GME$ is equal to angle $CEM$. Assume that this is true.

Then by algebraic solution find the relation between

angle $MED$ and angle $EMF$,
angle $BEC$ and angle $AMG$,
angle $DEB$ and angle $FMA$,
angle $GME$ and angle $MED$.

**Exercise.** In the last figure add a line $HK$, parallel to $AB$, forming the following figure. State relations between the various angles. What can you say about the angles of the parallelogram *ELNM*?




*Solution of More Difficult Equations.*

*Example.* Solve the following equation for $r$:

$$(3r - 2)(\tfrac{3}{2}r + 3) - (2r + 1)(2r - 1) = \tfrac{1}{2}r^2 - 15.$$

Doing the work indicated,

$$[3r(\tfrac{3}{2}r + 3) - 2(\tfrac{3}{2}r + 3)] - [2r(2r - 1) + 1(2r - 1)] = \tfrac{1}{2}r^2 - 15$$
$$[(4\tfrac{1}{2}r^2 + 9r) - (3r + 6)] - [(4r^2 - 2r) + (2r - 1)] = \tfrac{1}{2}r^2 - 15$$
$$(4\tfrac{1}{2}r^2 + 6r - 6) - (4r^2 - 1) = \tfrac{1}{2}r^2 - 15$$
$$\tfrac{1}{2}r^2 + 6r - 5 = \tfrac{1}{2}r^2 - 15$$

Subtracting $\frac{1}{2}r^2$ from both sides of the equation, and also adding 5 to both sides, we have

$$6r = -10. \quad \text{Why?}$$
$$r = -\tfrac{10}{6}$$
$$= -\tfrac{5}{3}.$$

Check by substituting $-\frac{5}{3}$ for $r$ in the first member of the equation.

$$[3(-\tfrac{5}{3}) - 2][\tfrac{3}{2}(-\tfrac{5}{3}) + 3] - [2(-\tfrac{5}{3}) + 1][2(-\tfrac{5}{3}) - 1]$$
$$= (-5 - 2)(-\tfrac{5}{2} + 3) - (-\tfrac{10}{3} + 1)(-\tfrac{10}{3} - 1)$$
$$= -7 \cdot \tfrac{1}{2} - (-\tfrac{7}{3})(-\tfrac{13}{3})$$
$$= -\tfrac{7}{2} - \tfrac{91}{9}$$
$$= -\frac{63 + 182}{18}$$
$$= -\tfrac{245}{18}.$$

Substituting $-\frac{5}{3}$ for $r$ in the second member of the equation,

$$\begin{aligned}\tfrac{1}{2}\left(-\tfrac{5}{3}\right)^2 - 15 &= \tfrac{1}{2}\cdot\tfrac{25}{9} - 15\\ &= \tfrac{25}{18} - 15\\ &= -\tfrac{245}{18}.\end{aligned}$$

Therefore $r = -\frac{5}{3}$ is the value which makes the equation true.

**Exercises.*** Solve:

1. $3r - 2(2 - r) = 21.$
2. $3(s - 1) = 4(s + 1).$
3. $(2 - a)(5 - a) = a^2.$
4. $9t - 3(5t - 6) = -30.$
5. $2(m + 3) - 3(m + 2) = 0.$
6. $p(p^2 + 1) = p(p^2 - 1) + 9.$
7. $3x + 14 - 5(x - 3) = 4(x + 3).$
8. $w(1 + w + w^2) = w^3 + w^2 + 3w - 17\cdot 5.$
9. $(x + 1)(x + 2) - (x + 3)(x + 4) = -50.$

**59. Summary.**

To solve problems leading to equations:

Let any letter, usually the initial letter of the name of the thing whose value you are trying to find, represent the number expressing that value.

Then by analysis find two different algebraic expressions which express the same amount of something spoken of in the problem, or brought out indirectly by the nature of the problem.

Place these two algebraic expressions equal to each other in the form of an equation.

Solve this equation by means of the axioms — bringing the terms which contain the unknown quantity to one side of the equation and the known terms to the other side.

Divide both sides of the equation by the coefficient of the unknown number.

* Chapter II of Geometry may be taken up while the class is solving the exercises and problems which conclude this chapter.

**Miscellaneous Exercises.**

**1.** A man earns $e$ dollars a day and spends $r$ dollars a day for his room and $b$ dollars a day for his board. If he works $e$ days during the month, how much money has he at the end of the month, if his board and room rent are to be paid whether he works or not?

**2.** Write a problem making the above exercise special. After solving, substitute the special numbers used in this exercise in the answers of Exercise 1 and compare answers.

**3.** A measuring line is $30\,c$ feet and $2\,b$ inches long. When applied to a line it goes $3\frac{1}{2}\,c$ times. What is the length of the line measured?

**4.** Simplify $(a + b)(a^2 + 2\,ab + b^2) - 2\,ab\,(a + b)$.

**5.** A man agrees to work for 5 dollars a day for every day that he works, during which time he pays no board or room rent. For the days that he does not work he must pay 1 dollar a day for board and room rent. If he works 20 days during the month, how much money should he receive at the end of the month?

**6.** Make a general problem for the above exercise. Solve and check by placing the special values used in Exercise 5 in the answer of this one. Compare with the answer of Exercise 5.

**7.** Simplify and check $(m - 7)(m + 9) - (m + 3)(m - 2)$.

**8.** If $m$ men do a piece of work in $t$ days, what part of the work will one man do in one day?

**9.** Make the above problem special, and check as in Exercises 1 and 2.

**10.** A train runs at the rate of $r$ miles an hour, stopping $s$ minutes at each station. If it takes it $t$ hours to make the trip and it stops at $n$ stations, what distance has it traveled?

**11.** Make the above exercise special and use as check for Exercise 10.

**12.** $(w + 1)(w + 2) - (w - 3)(w - 4) = 0$. Solve for $w$ and check.

**13.** A man earns $5 a day. If during a certain month he works $21\frac{1}{2}$ days and pays 75 cents a day for board, 50 cents a day for rent, and 75 cents a week for laundry, how much money will he have at the end of the month?

**14.** Make the above exercise general and use as check.

**15.** A man's yearly income is $52\,w$ dollars; his expenses average $e$ dollars a week. He enjoys a vacation of $v$ weeks, during which time his income is the same, but his expenses are increased by 20 dollars a week. How much has he at the end of the year?

**16.** $(t-3)(2t-5)-2t(t-8)=1$. Solve for the value of $t$.

**17.** A cubic centimeter of lead and a cubic centimeter of copper together weigh 20.13 grams. A ball containing 5 cc. of copper and 7 cc. of lead weighs 27.07 grams more than a ball containing 4 cc. of lead and 6 cc. of copper. Find the density of each metal.

**18.** In a parallelogram whose adjacent sides are $a$ and $b$, if 10 units were added to the side $a$ and 3 units subtracted from the side $b$, the sides of the figure would be equal. What is the length of each side?

**19.** If $d$ yards of cloth cost $c$ dollars and $b$ yards are sold for $a$ dollars, what is the gain on each yard sold?

**20.** Make Exercise 19 special. Solve and show correspondence of answers.

*Momentum.* The subject that we shall now look into as to its measurement is the amount of motion in a moving body. The name given to the "amount of motion" is *momentum.* It is measured by the product of the weight by the speed of the moving body.

The amount of motion will be 3 times as much if a ball weighing 6 grams moves at the rate of 100 cc. per second as it will be if a ball weighing 2 grams moves at the rate of 100 cc. per second.

Furthermore, the amount of motion will be 5 times as much if a body weighing 6 grams moves at the rate of 500 cc. per second as it will be if a body weighing 6 grams moves at the rate of 100 cc. per second.

Since velocity is a directed quantity, momentum may be regarded as a directed quantity.

The unit of measure selected for momentum is the amount of motion of 1 unit of matter moving with a velocity of 1 unit of distance during 1 unit of time.

In the centimeter-gram-second system, the unit of momentum is the momentum of 1 gr. of matter moving at the rate of 1 cm. per second.

**21.** What is the momentum of a cannon ball weighing 16 lbs. moving at the rate of 250 ft. per second?

In this exercise let the momentum of 1 lb. moving at the rate of 1 ft. per second be the unit of measure.

Then 16 = the number of units of momentum of 16 lbs. moving at the rate of 1 ft. per second

and 4000 = the number of units of momentum of 16 lbs. moving at the rate of 250 ft. per second.

**22.** Express the momentum of a ball weighing $w$ grams and moving at the rate of $v$ cm. per second. Let $M$ stand for momentum.

**23.** What is the formula for the expression of momentum?

**24.** Compare the momentum of a street car weighing 5 tons going at the rate of 10 miles an hour to that of an engine weighing 20 tons and going at the rate of 40 miles an hour. Solve this exercise, then make it general and solve. Compare answers by substituting the special values given into the general answer obtained.

**25.** If a man weighing $w$ lbs. is riding a wheel weighing $w$ lbs. at the rate of $v$ ft. per second due east, what is the momentum? Would the momentum be the same if he were riding south? How would it compare if he were going west?

**26.** A body composed of two masses, one weighing $a$

pounds and the other weighing $b$ pounds, is moving with a velocity of $(a + b)$ ft. per second. What is the momentum of the body?

**27.** Two balls have equal momenta. The first weighs 100 kilos and moves with a velocity of 20 meters per second. The other moves with a velocity of 500 meters per second. How much does it weigh? Solve this exercise as it stands, then make it general and solve. Compare answers by substituting the special values given into the general answer obtained.

**28.** The velocity of a body is $2v$ meters and the velocity of another is 7 meters. If each weighs $(7v - 4)$ grams, how much more is the momentum of the second body than that of the first?

**29.** A boy weighs 50 pounds more than his bicycle. When he rides at the rate of 10 miles an hour, the momentum acquired is 960 lbs. Find the weight of the boy.

# CHAPTER XI

## SIMPLE AREAS AND THEIR ALGEBRAIC EXPRESSION. ELEMENTS OF FACTORING

**60. Rectangles.** In the chapter on measurement (Chapter II), we brought out the idea that to measure an area we usually take a square unit and find out how many such square units and fractional parts of such square units are required to make the given area.

*Definition.* A four-sided figure all of whose angles are right angles is called a rectangle.

In Chapter II we found, by counting, that if we drew a rectangle with a definite number for a base, and a definite number for its height, the number of square units in its area is equal to the product of these two numbers. Or, expressing this in algebraic language:

If the base of a rectangle consists of $b$ units and the height of $h$ units, the area is

$$b \cdot h = bh \text{ square units.}$$

We may therefore regard the product of any two positive numbers, $m$, $n$, as the area of a rectangle whose dimensions are $m$ and $n$.

This is called the rectangle $m \cdot n$.

A square whose side is $m$ has the area $m \cdot m$ or $m^2$.

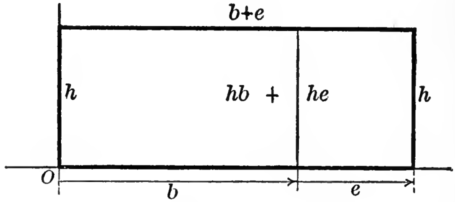
*Definition. Two rectangles are said to be equivalent if they have equal areas.*

**Exercise.** Given that a rectangle whose dimensions are $a \cdot b$ is equivalent to a rectangle whose dimensions are $c \cdot d$. Express this as an equation.

## 61. Geometric Theorems on Factoring.

*Addition of rectangles.* As in Chapter IX, we can express multiplication of polynomials as a combination of rectangles.

Draw the rectangle whose base is $b + e$ and whose height is $h$ units. Thus:




Is the area of the rectangle $h(b + e)$ equal to the sum of the area of rectangles $hb$ and $he$? Write the algebraic equation for this.

We assume here and in what follows that the letters stand for positive numbers.

Do similarly with $m(h + k + l)$.

We can state this as a theorem.

**Theorem 1.** *The rectangle of two given lines is equal to the sum of the rectangles of one of them and the several segments into which the other is divided.*

**Exercises.** Draw the following and write the algebraic expression for the equality of areas.

1. $s(2s + r + t)$.
2. $c(d + e + 3c)$.
3. $m(a + b)$.
4. $am + an$.
5. $bh + br + bd$.
6. $a^2 + ab + ac$.

Express the following algebraically as single rectangles:

7. $bh + bk$.
8. $bh + bk + bl$.
9. $ah + bh + ch + dh$.
10. $m^2 + mn$.
11. $2h^2 + 3hk$.
12. $c^2 + 2c + 3cd$.
13. $5r^2 + 10r$.
14. $2m^2 + 6mn + 2m$.
15. $14s + 28rs + 7r$.
16. $15p^2 + 5p + 5$.
17. $12mn^2 + 6mn$.
18. $4ab^2 + 8ab + 12a^2b$.

*Definition.* When an expression like

$$c^2 + 2c + 3cd$$

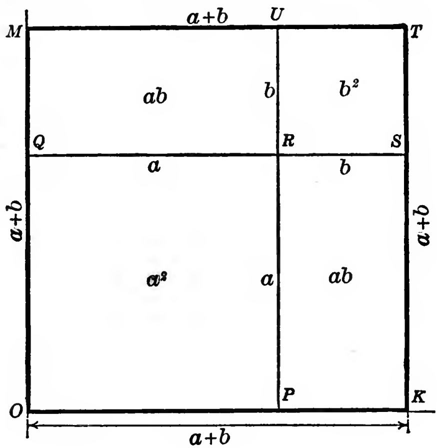
is rewritten in the form of a product,

$$c(c + 2 + 3d),$$

this operation is called **factoring.**

The above exercises are simple exercises in factoring, illustrated geometrically.

**Theorem 2.** *The square on the sum of two lines equals the sum of the squares on those lines plus twice their rectangle.*




We have given the square on the sum of the two lines $a$ and $b$. Call it $OKTM$.

To show that square $OKTM$ is equal to two squares, one of which has a side $a$, the other has a side $b$, and two rectangles of the dimensions $a$ by $b$. In other words we are to show that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Name the parts of which the square $OKTM$ is composed and give the dimensions of each.

Make the algebraic multiplication of $(a+b)(a+b)$. Show the correspondence in the figure of each partial product.

**Exercises.** Make a geometric representation and algebraic expression of the following:

1. $(m+1)(m+1)$.
2. $(2m+n)(2m+n)$.
3. $(r+3s)(r+3s)$.
4. $(2h+3k)(2h+3k)$.
5. $(\frac{1}{2}t+s)(\frac{1}{2}t+s)$.
6. $c^2+2cd+d^2$.
7. $a^2+2a+1$.
8. $m^2+6m+9$.
9. $4s^2+4rs+r^2$.

Give the algebraic expression of the following without drawing:

10. $m^2+4m+4$.
11. $9+6h+h^2$.
12. $9r^2+12rs+4r^2$.
13. $36+2\cdot 30+25$.

On the same principle as above give the geometric picture of the following rectangles, give the algebraic solution, and point out the geometric representation of each step in the solution.

**Exercises.**

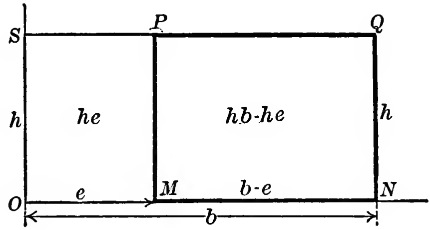
1. $(r+2)(r+3)$.
2. $(b+7)(b+1)$.
3. $(c+8)(c+5)$.
4. $(a+b)(a+c)$.
5. $(p+r)(p+s)$.
6. $c^2+5c+6$.
7. $m^2+7m+10$.
8. $a^2+ac+ab+cb$.

Give the following without drawing:

9. $s^2+3s+2$.
10. $x^2+7x+12$.
11. $y^2+8y+12$.
12. $u^2+13u+12$.
13. $a^2+as+bs+ab$.
14. $k^2+2ak+3bk+6ab$.

*Subtraction of rectangles.* Draw the rectangle $h(b-e)$. Do this by drawing length $b$, then subtract length $e$, $e$ being shorter than $b$. Draw height $h$ and complete the rectangle. Thus:

The rectangle *MNQP* is the rectangle $h(b-e)$. Rectangle *ONQS* is rectangle $hb$; *OMPS* is rectangle $he$; rectangle *OMPS* minus rectangle *ONQS* is rectangle *MNQP*.




Therefore $hb - he = h(b-e)$. Why?

Do the same with $m(h-k+l)$; that is, show that this is equivalent to rectangle $mh$, minus rectangle $mk$, plus rectangle $ml$.

**Exercises.** Resolve the following into simpler rectangles:

**1.** $m(a-b)$. **2.** $5(2s+r-t)$. **3.** $c(d-e+3c)$.

Express the following in single rectangles:

**4.** $c^2+2c-3dc$. **5.** $5r^2-10r$. **6.** $2m^2-6mr+2m$.

**Theorem 3.** *The square on the difference between two lines equals the sum of the squares on the two lines minus twice the rectangle of the lines.*

To draw the square whose side is $(a-b)$, draw length $a$; from it subtract length $b$. The remainder $a-b$ is the base. In like manner draw the altitude $a-b$, so as to form a square. (See figure on next page.)

The resulting figure *PQRS* is the square $(a-b)(a-b)$.

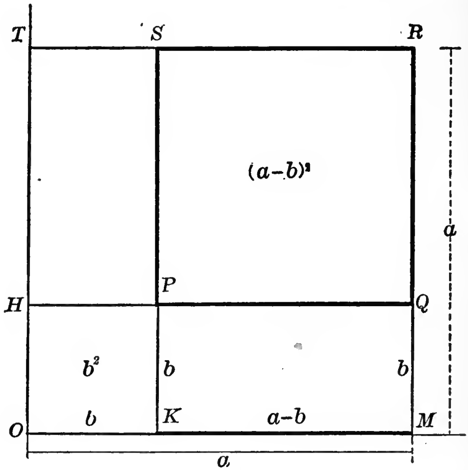
*OMRT* is the square on $a$. Its area is $a^2$.

*OMQH* is the rectangle $ab$.

*OKST* is the rectangle $ab$.

Rectangles *OMQH* and *OKST* are both subtracted from square *OMRT*.

But notice that after subtracting rectangle *OMQH* it will be necessary to add square *OKPH* before we can subtract rectangle *OKST*, so that rectangle *OMRT* minus rectangle




*OMQH* plus rectangle *OKPH* minus rectangle *OKST* gives rectangle *PQRS*.

That is: $(a-b)^2 = a^2 - ab + b^2 - ab,$

or, $$\mathbf{(a-b)^2 = a^2 - 2\,ab + b^2.}$$

**Exercises.** Draw and give algebraic expressions for the following:

**1.** $(c-d)(c-d)$.

**2.** $(m-2n)(m-2n)$.

**3.** $(3-7)(3-7)$.

**4.** $(3r-1)(3r-1)$.

**5.** $b^2 - 2bc + c^2$.

**6.** $4h^2 - 4h + 1$.

Write the algebraic expressions for the following without drawing; in other words, factor the given expressions:

**7.** $d^2 - 2de + e^2$.

**8.** $9 - 6r + r^2$.

**9.** $16 - 8s + s^2$.

**10.** $25t^2 - 30t + 9$.

**11.** $9m^2n^2 - 24mn + 16$.

On the same principle as above, draw and give algebraic expressions for the following rectangles.

**12.** $(r - 2)(r - 5)$.
**13.** $(b - 7)(b - 1)$.
**14.** $(c - 8)(c - 5)$.
**15.** $(a - b)(a - c)$.
**16.** $m^2 - 3m + 2$.
**17.** $p^2 - 8p + 15$.

Give the algebraic expressions for the following without drawing:

**18.** $r^2 - ar - br + ab$.
**19.** $k^2 - 2ak - 3bk + 6ab$.
**20.** $x^2 - 14xy + 48y^2$.
**21.** $d^2 - 15cd + 56c^2$.
**22.** $t^2 - 9ts + 20s^2$.

Draw the following, give algebraic expressions for the multiplication and show correspondence of rectangles to the partial products.

**23.** $(a - 5)(a + 3)$.
**24.** $(c + 7)(c - 2)$.
**25.** $(a - b)(a + c)$.
**26.** $(d - 2e)(d + 3f)$.
**27.** $m^2 - 2m - 15$.
**28.** $d^2 + 2m - 15$.

Write the algebraic expressions for the dimensions of the following rectangles without drawing.

**29.** $c^2 - 7cd - 8d^2$.
**30.** $x^2 + 2xy - 8y^2$.
**31.** $p^2 - mp + np - mn$.
**32.** $k^2 + sk - rk - rs$.
**33.** $t^2 - 2rt + 3st - 6rs$.

Draw the following, give the algebraic expressions for the multiplication, and show correspondence of rectangles to the partial products:

**34.** $(2m + n)(m + 3n)$.
**35.** $(2m - n)(m - 3n)$.
**36.** $(2m - n)(m + 3n)$.
**37.** $6m^2 + mn - 2n^2$.
**38.** $6m^2 - mn - 2n^2$.

Give the algebraic expression for the dimensions of the following rectangles without drawing.

**39.** $10r^2 + 27rs + 5s^2$.
**40.** $6a^2 - 17a + 3$.
**41.** $6a^2 + 17a + 3$.
**42.** $3h^2 - 10h - 8$.
**43.** $15 - x - 6x^2$.

Draw the following and state dimensions:

**44.** $an - bn + ar - br$. **45.** $2pq + 2ps + 7q + 7s$.

**46.** $ac - a - c + 1$.

Without drawing, state dimensions of the following:

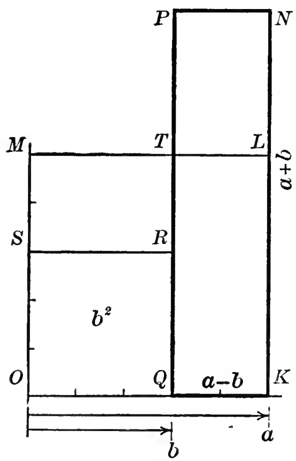
**47.** $dy - cy - b + d$.

**48.** $r(r-1)^2 + r - 1$.

**49.** $(m+n)(a+b) - (m-n)(a+b)$.

**50.** $(s+2)(t-s) + (2-s)(t-s)$.

**Theorem 4.** *The difference of the squares on two lines equals the rectangle of the sum and difference of the two lines.*




We have given the square $OKLM$ on the line $a$, and the square $OQRS$ on the line $b$. We are to show that the difference between these two squares is equal to a rectangle $a + b$ by $a - b$.

The actual difference between these two squares is, by definition, the irregular figure $QKLMSR$. Examine the rectangle $SRTM$. What is the length of its side $SR$? Of its side $SM$? If we take this rectangle and place it with its side $SM$ coinciding with the line $TL$ (why do they coincide?), thus subtracting it from the figure $QKLMSR$ and adding it back in another position, we shall have the rectangle $QKNP$. What are the dimensions of this rectangle? So we have

$$a^2 - b^2 = (a+b)(a-b).$$

**Exercises.** In the following draw the difference of the squares and the rectangle to which this difference is equal; write the algebraic expression for this equality.

1. $c^2 - 4$.
2. $9 - 25$.
3. $m^2 - n^2$.
4. $(a + b)^2 - c^2$.
5. $m^2 - (p + q)^2$.

Factor the following without drawing:

6. $9 - 4h^2$.
7. $25t^2 - 16$.
8. $s^2 - 1$.
9. $1 - 4f^2$.
10. $(a + b)^2 - (a - b)^2$.
11. $(m^2 - 2mn + n^2) - 4$.
12. $9 - (r^2 + 2rs + s^2)$.
13. $q^2 - r^2 - 2rs - s^2$.

*Note.* When it is necessary to inclose terms in parentheses after a minus sign, the sign of each term must be changed. Explain why.

So Exercise 13 would first be written

$$q^2 - (r^2 + 2rs + s^2).$$

Now proceed as in the other exercises.

14. $t^2 - 1 + 8p - 16p^2$.
15. $-r^2 + a^2 - 2ab + b^2$.
16. $49 - m^2 - 2mn + n^2$.
17. $a^2 - 2ab + b^2 - r^2 - 2rs - s^2$.
18. $4 - 4s + s^2 - n^2 - 2nq + q^2$.
19. $1 - 4a^2 + 8ab - 4b^2$.
20. $16 - 8b + b^2 - a^2 + 2ac - c^2$.
21. $a^2 + 2mn + b^2 - 2ab - m^2 - n^2$.
22. $-4r^2 - 8ab + a^2 - 1 - 4r + 16b^2$.

To sum up: Two numbers which multiplied together give a certain number are called the **factors** of that number.

The process of finding the factors of an expression is called **factoring** the expression.

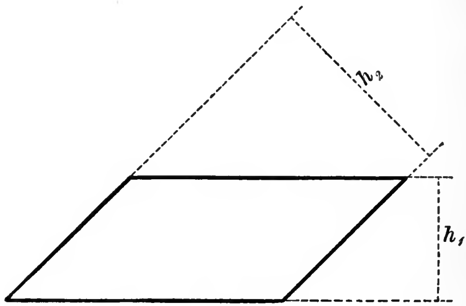
To the geometric process of finding the dimensions of a rectangle of given area, there corresponds the algebraic process of finding the two numbers which multiplied together will give the number which expresses the area. In the preceding work we examined both processes.

Factors are said to be **prime** when they cannot be refactored.

The measurement of an area of any form is obtained essentially through the measurement of an equivalent rectangular area. We shall take up a few simple cases.

**62. Parallelograms.**

*Definitions.* 1. A parallelogram is a four-sided figure whose opposite sides are equal and parallel, that is, the same distance apart throughout their entire length.




2. An altitude or height of a parallelogram is the perpendicular distance between two of its opposite sides. There are two altitudes, $h_1$ and $h_2$, as shown in the figure.

3. When the altitude is drawn, either of the sides to which it is drawn is called the base of the parallelogram.

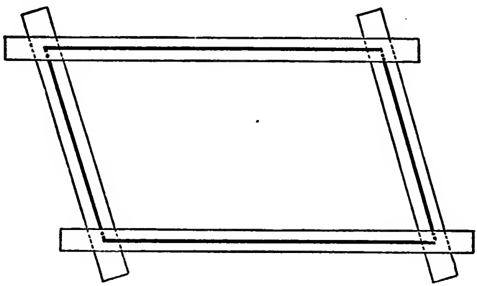
4. A parallelogram with equal sides is called a rhombus.

*Experiment.* In defining a parallelogram we said that the opposite sides are *equal* and *parallel.* Either of these words may be omitted, for if the opposite sides are equal, they will always be parallel; if parallel they will always be equal. Without proving this now we ask the student to verify it experimentally.

Cut four strips of cardboard or stiff paper, two longer and two shorter ones.

Hinge them together by pins or thumbtacks placed at the ends of the lines so drawn that the opposite sides are equal, as shown in the figure.

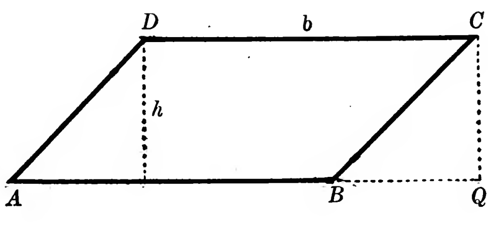
Swing the frame into various positions. See if the opposite sides are always parallel. What about the opposite angles? What have we shown previously that the sum of the angles



of a quadrilateral is equal to? What is the sum of any two consecutive angles of a parallelogram equal to?

How does the area change as the figure is changed from a rectangle to a more oblique form?

**63. Theorem.** *The area of a parallelogram equals the area of a rectangle of the same base and altitude.*



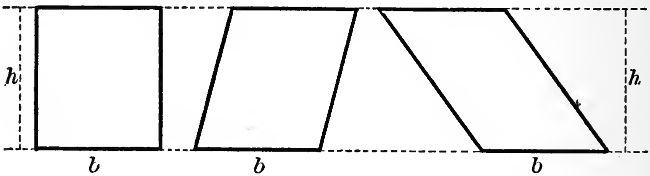

Draw the parallelogram *ABCD*, with base *b* and height *h*. Cut out the figure. Cut off the triangle *APD* and place it in the position of triangle *BQC*. Does it coincide with it — that is, does it fit exactly on it? A rectangle is formed on *b* and *h* as sides and with the same area as the parallelogram *ABCD*.

*Corollary* 1. *The area of a parallelogram is equal to the product of its base by its altitude.*

That is, $a = bh.$

*Corollary* 2. *All parallelograms having equal bases and altitudes are equal in area.*

For they are equivalent to the same rectangle.




**Exercises.** The following are areas of parallelograms; state the lengths of their bases and altitudes.

**1.** $hb - he.$

**2.** $c^2 + 2c + 1.$ What kind of a figure is this parallelogram equal to?

**3.** $m^2 - 4n^2.$ **4.** $r^2 + rs - pr + r.$ **5.** $h^2 + 3h - 4.$

**64. Triangles.**

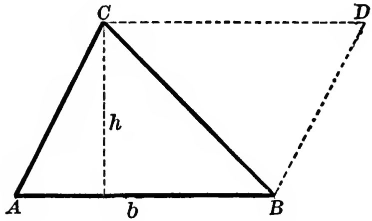
*Definitions.* The three lines which form a triangle are called the sides of a triangle. The three points of intersection of these lines are the vertices. Each is called a vertex. A line drawn from a vertex perpendicular to the opposite side is called an altitude of the triangle. There are three altitudes. The side to which the altitude is drawn is the base of the triangle corresponding to that altitude.

**Theorem.** *The area of a triangle equals half the area of the rectangle having the same base and altitude.*

In our measurement work in Chapter II we showed this by counting the number of square units in each. We shall now show it by another experiment.

We have given the triangle $ABC$ with base $b$ and altitude $h$. Form a parallelogram by drawing $CD$ parallel to $AB$, and

*BD* parallel to *AC*. This is parallelogram *ABCD*. Cut out the figure and cut the paper along the line *BC*. Fit triangles one on the other. How does the triangle *ABC*



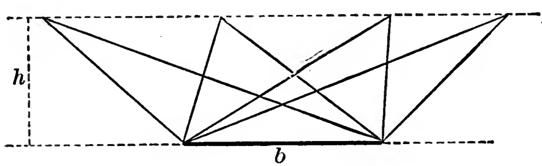

compare with the triangle *BDC*? Do the two triangles coincide? How does the triangle *ABC* compare with the parallelogram *ABCD*? Therefore, how does the triangle *ABC* compare with the rectangle on *b* and *h*?

The algebraic statement, as we have learned before, is

$$a = \frac{1}{2} bh,$$

where *a* stands for the area of the triangle.

*Corollary. All triangles with the same bases and altitudes are equal in area.*




Explain why this is true.

**Exercise.** Take three strips of cardboard all of different lengths. Fasten together with pins as in the experiment with the parallelogram. Try to swing the figure as in the former case. What do you find?

**Exercises.** In the following areas of triangles tell what may be the length of the base and altitude.

Let $a$ = number of square units in the area; $b$ = number of units in the base; $h$ = number of units in the altitude.

| | | |
|---|---|---|
| $a = 24;$ | $b = ?;$ | $h = ?;$ |
| $a = c^2 - d^2;$ | $b = ?;$ | $h = ?;$ |
| $a = m^2 - mn - 2n^2;$ | $b = m - 2n;$ | $h = ?;$ |
| $a = mn - rn + m - r;$ | $b = m + 1;$ | $h = ?.$ |

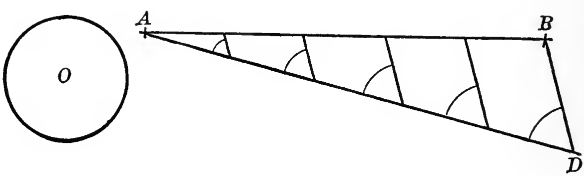
### 65. Regular Polygons.

*Definitions.* A *polygon* is a figure bounded by straight-line segments.

A *regular polygon* is one whose sides are all equal and whose angles are all equal.

*Construction of a regular polygon.* Draw a circle. Divide the circumference into any number of equal parts. Join the successive points of division by straight lines. The resulting figure is a regular polygon.

Plan for dividing the circumference of a circle into equal parts.




Given the circumference whose center is $O$, to divide it into five equal parts.

Draw a straight line. Cut out a circle of same size as circle $O$. Mark a point on its circumference. Starting at this point roll the circle along the straight line until the point again comes to the straight line. You now have the line segment the length of the circumference. Call the line segment $AB$. At the point $A$ draw a line of indefinite length, making an acute angle with line $AB$. On this line lay off a line segment of any convenient length five times. Call the last point $D$. Join $B$ and $D$ by a straight line. Then at each of the other points of division draw an angle equal to angle $BDA$. Extend the arms of these angles until they cut line $AB$, and $AB$ will be divided into five equal parts.

Now roll the circle again marking the points on the circumference.

*Definitions.* The center of the circle is the *center* of the polygon.

The sum of the sides is the *perimeter* of the polygon.

The line from the center perpendicular to one of the sides is called the *apothem* of the polygon.

**66. Theorem.** *The area of a regular polygon equals one-half the product of the apothem by the perimeter.*

If $a$ is the number of units in the length of one side, $n$ the number of sides, and $h$ the number of units in the length of the apothem, we have

$na$ = the number of units in the length of the perimeter.

$\frac{1}{2}nah$ = the number of square units in the area of the polygon.

Let the student verify this by dividing the polygon into equal triangles by lines drawn from the center to each of the vertices.

Show experimentally that these triangles are all equal by drawing a regular polygon on stiff paper and cutting it up. Fit the triangles on one another.

Some simple regular polygons are:

| | |
|---|---|
| Three sides: | equilateral triangle. |
| Four sides: | a square. |
| Five sides: | regular pentagon. |
| Six sides: | regular hexagon. |
| Ten sides: | regular decagon. |

**Exercises.**

1. Draw a regular pentagon whose center call $O$. From $O$ draw lines to each of the vertices. What is the sum of the angles in one triangle? What is the sum of the angles in the five triangles? Since the angles about the point $O$ do not make up the angles of the pentagon, and the other angles of the triangles do make up the angles of the pentagon, if we subtract the sum of the angles about $O$ from the sum of the

angles of the triangles, we will get the sum of the angles of the polygon. Doing this, what do we get for the sum of the angles of the pentagon? What is the value of each angle of a regular pentagon?

2. Draw a regular hexagon and use it in the same way that you did the pentagon.

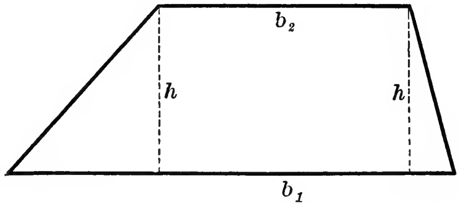
3. Draw a regular decagon and examine in the same manner.

4. From these experiments show that the sum of the angles of these polygons may be expressed by $(n-2)$ straight angles, where $n$ is the number of sides, and that each angle may have its value expressed by $\frac{n-2}{n}$ straight angles.

5. Calculate the value of each angle of a regular seven-sided figure, assuming that the above expressions always hold? Of a nine-sided figure? Of a twenty-sided figure?

**67. Trapezoids.**

*Definition.* A four-sided figure with two of its sides parallel is called a *trapezoid.*



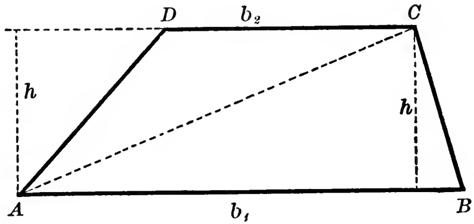

The two parallel sides are called the *bases* of the trapezoid. The perpendicular distance between the two bases is called the *altitude* (height) of the trapezoid.

**Theorem.** *The area of a trapezoid is equal to one-half the product of its altitude by the sum of its parallel sides.*

That is, letting $h$ be the number of units in the altitude, $b_1$ the number of units in the lower base and $b_2$ the number of units in its upper base,

$\frac{1}{2}h(b_1 + b_2)$ = the number of units in the area of the trapezoid.

Let us have given the trapezoid *ABCD*.




Draw the diagonal *AC*. This gives us two triangles *ABC* and *ACD*.

So we have

$\frac{1}{2}hb_1$ = number of square units in area of triangle *ABC*.

$\frac{1}{2}hb_2$ = number of square units in the triangle *ACD*.

Adding:

$\frac{1}{2}hb_1 + \frac{1}{2}hb_2$ = number of square units in the area of the trapezoid.

But

$\frac{1}{2}hb_1 + \frac{1}{2}hb_2 = \frac{1}{2}h(b_1 + b_2)$, by factoring.

This shows our theorem to be true.

**Exercises.**

1. Construct a trapezoid out of strips of cardboard as was done in the parallelogram in § 62. Can you swing the figure into different forms?

2. Draw the trapezoid *ABCD*. Let *E* be the mid-point of side *AD*, and *F* be the mid-point of *BC*. Draw the line *EF*. Cut out the trapezoid, and cut it in two along the line *EF*. Place the two parts together with point *B* on point *C* and *BA* extending in a line with *DC*. What is the shape of the figure you now have? Explain how this may illustrate our theorem about the area of a trapezoid.

**3.** Suppose that we have given that the altitude of a certain trapezoid is 2 units and the sum of the bases is 10 units. Show by drawing that there might be many trapezoids of this description. Will they all have the same area?

**4.** Suppose that we have given that a trapezoid is 2 units high, and that one base is 3 units and the other base 7 units, show by drawing that we may have many trapezoids of this description. Are they all of the same area?

### 68. Circles.

*Definition.* A *circle* is a plane figure bounded by a curved line, all points of which are equally distant from a point within called the *center*.

Circles are easily drawn by the aid of a compass, which the student should secure. The kind that may be slipped on an ordinary lead pencil will do very well.

There is some confusion with regard to the word *circle.* Does it mean the boundary line or does it mean the surface inclosed by this line? It usually means the surface inclosed. The boundary line is called the *perimeter* or the *circumference* of the circle.

The distance from the center to the circumference is called the *radius* of the circle.

A line through the center and terminated at both ends by the circumference is called a *diameter* of the circle.

The diameter is always twice the radius.

The circumference of a circle and the area of a circle may be expressed in terms of the radius as was shown in § 16.

At that time we showed that

$$c = 2\pi r$$

and that

$$a = \pi r^2.$$

As a further experiment to get the approximate area of the circle, draw one with a given radius on cross-section paper. Also draw a square whose side is the given radius. Count the number of squares in the area of the circle, adding together the parts of squares to make the whole squares as well as possible. What is the ratio of the circle to the area of the square? Repeat this for several circles of different sizes, thus showing that

$$a = \pi r^2.$$

**69. Factoring Continued.** In the first part of this chapter we resolved a large number of expressions into factors; we also noted a corresponding geometric process, which consisted in expressing the combined areas of several rectangles as a single rectangle.

To make the geometric picture of the process we supposed the letters contained in the given expressions to stand for positive numbers; we could then suppose these numbers to represent the lengths of the sides of rectangles. But the actual process is just as good when the letters stand for negative numbers.

For example, the equation

$$a^2 - b^2 = (a + b)(a - b)$$

is seen to be true by direct multiplication of the right-hand member; it does not matter whether $a$ and $b$ stand for positive numbers or for negative numbers; or whether $b$ is less than $a$ or greater than $a$.

In this article we merely introduce a little further drill in the factoring of literal expressions, and make some addition to our stock of formulas in factoring.

It is often of great advantage to be able to factor expressions readily. For example, suppose that it is required to find what is the value of the expression

$$a^3 + 3a^2b + 3ab^2 + b^3,$$

when $a = 2$ and $b = 3$.

Substituting we find

$$8 + 3 \cdot 4 \cdot 3 + 3 \cdot 2 \cdot 9 + 27 = 125.$$

But if we had known that the given expression is equal to $(a + b)^3$, that is $(a + b)(a + b)(a + b)$, we would have had its value thus:

$$(2 + 3)^3 = 5^3 = 125.$$

The second calculation is very much simpler.

Again, what is the value of the expression

$$am + bm - an - bn,$$

when $$a = 3,\ b = 4,\ m = 8,\ n = 5?$$

By direct substitution we have

$$3 \cdot 8 + 4 \cdot 8 - 3 \cdot 5 - 4 \cdot 5 = 21.$$

But the given expression is equal to

$$(a + b)(m - n).$$

Substituting in this we have

$$(3 + 4)(8 - 5) = 7 \cdot 3 = 21.$$

Factoring is especially useful when an expression has to be calculated for several values of the letters it contains.

**Exercises.** Factor the following expressions and calculate the value of each when the letters are replaced by the given numbers. Use both the given expression and the factored form.

1. $(a^2 - b^2)$; $a = 100,\ b = 99.$
2. $a^2 + 2ab + b^2$; $a = 5,\ b = 6.$
3. $a^2 - 2ab + b^2$; $a = 56,\ b = 53.$
4. $x^2 + 2x + 1$; $x = 2;\ 3;\ 5;\ -3;\ -5.$
5. $x^2 - 2x + 1$; $x = 3;\ 6;\ -5;\ -7.$
6. $x^2 + 4x + 4$; $x = 1;\ 3;\ 5;\ -1;\ -3;\ -5.$
7. $4m^2 - 9n^2$; $m = 130,\ n = 86;\ m = -5,\ n = -3.$
8. $u^2 + uv - 2v^2$; $u = 72,\ v = 74.$
9. $4x^2 - 12xy + 9y^2$; $x = \frac{5}{3},\ y = \frac{3}{2}.$

**Other useful factors.**

Let the student verify by direct multiplication the following equations; then memorize them.

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$
$$(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2).$$

State these rules in words.

*Example* 1. Let $a = 2\,x$ and $b = 3\,y$; then

$$\begin{aligned}(2\,x+3\,y)^3 &= (2\,x)^3 + 3\,(2\,x)^2\,(3\,y) + 3\,(2\,x)\,(3\,y)^2 + (3\,y)^3\\ &= 8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3.\end{aligned}$$

*Example* 2.

$$\begin{aligned}8\,x^3 - 27\,y^3 &= (2\,x)^3 - (3\,y)^3\\ &= (2\,x - 3\,y)\,[(2\,x)^2 + (2\,x)\,(3\,y) + (3\,y)^2]\\ &= (2\,x - 3\,y)\,(4\,x^2 + 6\,xy + 9\,y^2).\end{aligned}$$

**Exercises.** Write the expanded forms of the following:

1. $(a + 2\,b)^3$.
2. $(a - 2\,b)^3$.
3. $(2\,x + y)^3$.
4. $(3\,h + 4\,k)^3$.
5. $(3\,h - 4\,k)^3$.
6. $(3\,h - 4\,k)^3$.
7. $(5\,u + 3\,v)^3$.
8. $(5\,v - 3\,v)^3$.

Factor the following:

1. $x^3 + 3\,x^2y + 3\,xy^2 + y^3$.
2. $8\,x^3 - 36\,x^2y + 18\,xy^2 - 27\,y^3$.
3. $8\,u^3 + 12\,u^2v + 6\,uv^2 + v^3$.
4. $a^3 + 8\,b^3$.
5. $m^3 - 27\,n^3$.
6. $64\,r^3 + 125\,s^3$.
7. $a^3 + 1$.
8. $1 - x^3$.
9. $1 - x^3y^3$.

*Geometrical picture of the expansion of* $(a+b)^3$. If the dimensions of a rectangular block, for example a brick or a box, are $a$ = number of units in length, $b$ = number of units in width, $c$ = number of units in height,

then, $a \cdot b \cdot c$ = number of cubic units in the contents or volume.

For example a box 4 ft. long, 3 ft. wide and 2 ft. high contains $4 \cdot 3 \cdot 2 = 24$ cubic feet.

If each of the edges of the block has a length $a$ units its cubic contents are

$$a \cdot a \cdot a = a^3 = \text{the number of cubic units.}$$

Such a block is called a cube.

Then $(a + b)^3$ = the number of cubic units in a cubical block each of whose edges is $(a + b)$ units long. (We suppose $a$ and $b$ to stand for positive numbers.)

But we have

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

What volume is represented by each term of the right-hand member of the equation.

Make a paper model showing each of these volumes and fit them together so as to form the cube whose edge is $(a + b)$.

**70. Summary.**

Parallelograms and rectangles are equal

If they have the same or equal bases and lie between the same parallel lines;

If they have equal bases and equal altitudes.

Triangles are equal

If they have the same or equal bases and lie between the same parallel lines;

If they have equal bases and equal altitudes.

A triangle is equal to one-half of a parallelogram having the same base and lying between the same parallel lines, or having the same base and altitude.

The area of a rectangle or a parallelogram is equal to the product of its base by its altitude.

The area of a triangle is equal to one-half of the product of its base by its altitude.

The following equalities have been shown.

$$a^2 + 2ab + b^2 = (a + b)^2.$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$
$$a^2 + (m + n)a + mn = (a + m)(a + n).$$
$$rsa^2 + (rm + sn)a + mn = (sa + m)(ra + n).$$
$$a^2 - b^2 = (a - b)(a + b).$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

# CHAPTER XII

## FRACTIONS

**71. Reduction of Fractions to Lowest Terms — Cancellation.** In arithmetic you learned how to reduce fractions by cancellation, and how to multiply and divide, add and subtract fractions. In algebra these operations are carried on just the same as in arithmetic; we shall give examples of these processes, first using arithmetic fractions and then literal fractions.

*Example* 1. Simplify: $\frac{1575}{2520}$.

Factoring: $$\frac{1575}{2520} = \frac{\overset{5}{\cancel{5^2}} \cdot \cancel{3^2} \cdot \cancel{7}}{2^3 \cdot \cancel{3^2} \cdot \cancel{5} \cdot \cancel{7}}$$

Cancelling: $$= \frac{5}{8}.$$

Why does cancelling factors from the numerator and the denominator of a fraction not change its value?

*Example* 2. Simplify: $\frac{a^2 - b^2}{a^3 + b^3}$.

Factoring: $$\frac{a^2 - b^2}{a^3 + b^3} = \frac{\cancel{(a + b)}(a - b)}{\cancel{(a + b)}(a^2 - ab + b^2)}.$$

Cancelling: $$= \frac{a - b}{a^2 - ab + b^2}.$$

*Example* 3. Simplify: $\frac{12\,m^2q - 27\,n^2q}{60\,m^2q^2 - 6\,mnq^2 - 126\,n^2q^2}$.

Factoring: $$\frac{12\,m^2q - 27\,n^2q}{60\,m^2q^2 - 6\,mnq^2 - 126\,n^2q^2} = \frac{\cancel{3}\cancel{q}\cancel{(2m - 3n)}(2m + 3n)}{2 \cdot \cancel{3}\underset{q}{\cancel{q^2}}\cancel{(2m - 3n)}(5m + 7n)}.$$

Cancelling: $$= \frac{2m + 3n}{2q(5m + 7n)}.$$

## 72. Multiplication of Fractions.

*Example* 1. Simplify: $\frac{3}{10} \cdot \frac{65}{14} \cdot \frac{98}{105}$.

Factoring: $$\frac{3}{10} \cdot \frac{65}{14} \cdot \frac{98}{105} = \frac{\not{3}}{2 \cdot 5} \cdot \frac{\not{5} \cdot 13}{\not{2} \cdot \not{7}} \cdot \frac{\not{2} \cdot \not{7}^2}{\not{3} \cdot \not{5} \cdot \not{7}}.$$

Cancelling: $$= \frac{13}{10}.$$

*Example* 2. $$\frac{b^2 + 3b}{b^2 + 4bc + 4c^2} \cdot \frac{b^2 - 4c^2}{b^2 + b - 6} \cdot \frac{b^2 - 2b + 2bc - 4c}{2b^2 + bc - 10c^2}.$$

Factoring:
$$= \frac{b(b+3)}{(b+2c)(b+2c)} \cdot \frac{(b-2c)(b+2c)}{(b+3)(b-2)} \cdot \frac{(b+2c)(b-2)}{(b-2c)(2b+5c)}.$$

Cancelling: $$= \frac{b}{2b + 5c}.$$

**73. Division of Fractions.** In arithmetic, to divide a number by a fraction is the same as to multiply the number by the fraction inverted.

*Example* 1. $$6 \div \frac{2}{3} = 6 \cdot \frac{3}{2} = 9.$$

Go back to the definition of division and explain why this is so.

This division could have been performed by multiplying the dividend and divisor by a number that would make the divisor an integral number and then dividing. Thus: Multiplying numerator and denominator by 3,

$$\frac{6}{\frac{2}{3}} \cdot \frac{3}{3} = \frac{3 \cdot \not{2} \cdot 3}{\not{2}} = 9.$$

*Example* 2. $$\frac{a^3 - b^3}{a + b} \div \frac{a^2 - 2ab + b^2}{a^2 + 2ab + b^2}$$
$$= \frac{a^3 - b^3}{a + b} \cdot \frac{a^2 + 2ab + b^2}{a^2 - 2ab + b^2}.$$

Factoring: $$= \frac{(a-b)(a^2 + ab + b^2)}{(a+b)} \cdot \frac{(a+b)(a+b)}{(a-b)(a-b)}.$$

Cancelling: $$= \frac{(a+b)(a^2 + ab + b^2)}{(a-b)}.$$

Performing this division by the second plan suggested above:

Factoring:
$$\frac{\dfrac{a^3-b^3}{a+b}}{\dfrac{a^2-2ab+b^2}{a^2+2ab+b^2}}=\frac{\dfrac{(a-b)(a^2+ab+b^2)}{a+b}}{\dfrac{(a-b)(a-b)}{(a+b)(a+b)}}$$

Multiplying both terms by $(a+b)(a+b)$:
$$=\frac{(a-b)(a+b)(a^2+ab+b^2)}{(a-b)(a-b)}.$$

Cancelling:
$$=\frac{(a+b)(a^2+ab+b^2)}{a-b}.$$

**Exercises.**

Simplify the following. Check.

1. $\frac{18}{192}$; $\frac{768}{1440}$; $\frac{441}{4851}$.
2. $\dfrac{3r^2s}{6r^3s}$; $\dfrac{11km^2n^2}{121m^3n}$; $\dfrac{21p^2qr^2}{84pq^3}$.
3. $\dfrac{ma+mb}{a^2-b^2}$; $\dfrac{m^2-n^2}{m^2-2mn+n^2}$.
4. $\frac{2}{15}\cdot\frac{75}{44}\cdot\frac{363}{9}$.
5. $\dfrac{3cd}{5d^3}\cdot\dfrac{2cd^2}{9c^2d}$.
6. $\dfrac{aq-ap}{p}\cdot\dfrac{p^2+pq}{p^2-q^2}$.
7. $\dfrac{a^2-5a+6}{a+2}\cdot\dfrac{a^2+4a+4}{a^2+a-6}\cdot\dfrac{a-2}{a^2-a-6}$.
8. $\dfrac{r^2-4s+4s^2}{r^3+8s^3}\div\dfrac{r-2s}{r^2-2rs+4s^2}$.

**74. Reduction to a Common Denominator.** In arithmetic we learn that to add or subtract fractions requires that they be expressed in the same denominator.

For example:
$$\frac{2}{3}+\frac{4}{5}=\frac{10}{15}+\frac{12}{15}=\frac{22}{15}.$$

Now any set of fractions can be reduced to a common denominator by reducing each to a fraction whose denominator is the product of the denominators of the several denominators; but this is often not the lowest common denominator.

For example: $\frac{2}{9}, \frac{5}{12}, \frac{3}{20}$ may all be reduced to fractions having a common denominator $9\cdot 12\cdot 20=2160$.

For $\frac{2}{9} = \frac{2 \cdot 12 \cdot 20}{9 \cdot 12 \cdot 20}$; $\frac{5}{12} = \frac{5 \cdot 9 \cdot 20}{12 \cdot 9 \cdot 20}$; $\frac{3}{20} = \frac{3 \cdot 9 \cdot 12}{20 \cdot 9 \cdot 12}$.

But if we write the given fractions with their denominators in their factored form, we have

$$\frac{2}{3 \cdot 3}, \quad \frac{5}{3 \cdot 4}, \quad \frac{3}{4 \cdot 5}.$$

Therefore the number $3 \cdot 3 \cdot 4 \cdot 5 = 180$ will contain all the given denominators, and this is the smallest such number. It is called the *least common denominator.* (We shall abbreviate this to *L. C. D.*) We would now have

$$\frac{2}{9} = \frac{2 \cdot 20}{180}; \quad \frac{5}{12} = \frac{5 \cdot 15}{180}; \quad \frac{3}{20} = \frac{3 \cdot 9}{180}.$$

It is better, however, not to multiply out the factors of the denominator until you have added the numerators, for it only makes double work if the same factor is found in the numerator as in the denominator. You would first multiply it into the other factors and then divide it out again. A good rule to follow here and elsewhere is: *Never multiply out until you have to.* It is also shorter to write the denominator but once and write the numerators with the signs of operation above it.

Thus:
$$\frac{2}{9} + \frac{5}{12} - \frac{3}{20} = \frac{40 + 75 - 27}{3 \cdot 3 \cdot 4 \cdot 5}$$
$$= \frac{88}{3 \cdot 3 \cdot 4 \cdot 5}$$
$$= \frac{\not{4} \cdot 22}{3 \cdot 3 \cdot \not{4} \cdot 5}$$
$$= \frac{22}{45}.$$

Literal fractions are handled in exactly the same way.

## 75. Addition and Subtraction of Fractions.

**Rule.** *Reduce the given fractions to their least common denominator; then combine the numerators with their proper sign.*

*Example* 1. $\frac{9}{10}+\frac{14}{75}-\frac{2}{3}=?$

Factoring denominators to find *L. C. D.*:

$$\frac{9}{10}+\frac{14}{75}-\frac{2}{3}=\frac{9}{2\cdot 5}+\frac{14}{3\cdot 5^2}-\frac{2}{3}.$$

Reducing to *L. C. D.*: $=\frac{135+28-100}{2\cdot 3\cdot 5^2}.$

Collecting: $=\frac{63}{2\cdot 3\cdot 5^2}.$

Factoring: $=\frac{3\cdot 21}{2\cdot 3\cdot 25}.$

Cancelling: $=\frac{21}{50}.$

*Example* 2. $\frac{b}{a^3}+\frac{a}{b^2c}-\frac{c}{a^2b}=?$

The least common denominator must contain $a^3$, $b^2$, and $c$ as factors; hence it is $a^3b^2c$. Then, reducing to *L. C. D.*,

$$\frac{b}{a^3}+\frac{a}{b^2c}-\frac{c}{a^2b}=\frac{b\cdot b^2c}{a^3\cdot b^2c}+\frac{a\cdot a^3}{b^2c\cdot a^3}-\frac{c\cdot abc}{a^2b\cdot abc}$$

$$=\frac{b^3c+a^4-abc^2}{a^3b^2c}.$$

*Example* 3. $\frac{n}{m-n}-\frac{2\,mn}{m^2-n^2}-\frac{2\,m}{m+n}.$

Factoring denominator: $=\frac{n}{m-n}-\frac{2\,mn}{(m-n)\,(m+n)}-\frac{2\,m}{m+n}.$

Reducing to *L. C. D.*: $=\frac{n\,(m+n)-2\,mn-2\,m\,(m-n)}{(m-n)\,(m+n)}.$

Multiplying terms of numerator:

$$=\frac{(mn+n^2)-2\,mn-(2\,m^2-2\,mn)}{(m-n)\,(m+n)}.$$

Collecting terms: $=\frac{n^2+mn-2\,m^2}{(m-n)\,(m+n)}.$

Factoring: $=\frac{(n-m)\,(n+2\,m)}{(m-n)\,(m+n)}.$

Cancelling: $$= \frac{-(n+2m)}{m+n}.$$

We might have written the answer in the form $-\frac{n+2m}{m+n}$. Why?

Is $\frac{-10}{5} = \frac{10}{-5} = -2$? Why?

**76. Complex Fractions.** A complex fraction is nothing more nor less than an indicated division in which either one or both of the numbers involved are fractions or mixed numbers. As has been stated the easiest way to handle it is to reduce to a simple fraction by multiplying the numerator and the denominator by the least common denominator of all the fractions, and then to deal with the simple fraction as you have been shown in the preceding examples.

*Example* 1. $$\frac{2\frac{5}{14}}{6\frac{2}{7}} = ?$$

Here the *L. C. D.* of numerator and denominator is 14.

Multiplying both terms of the fraction by 14: $$\frac{2\frac{5}{14}}{6\frac{2}{7}} = \frac{33}{88}.$$

Factoring: $$= \frac{3 \cdot \cancel{11}}{8 \cdot \cancel{11}}.$$

Cancelling: $$= \frac{3}{8}.$$

*Example* 2. $$\frac{\frac{a}{b}+1}{1+\frac{c}{b}-\frac{bc+a^2}{b^2}} = ?$$

The *L. C. D.* of numerator and denominator is $b^2$.

Multiplying both terms by *L. C. D.*, namely $b^2$: $$\frac{\frac{a}{b}+1}{1+\frac{c}{b}-\frac{bc+a^2}{b^2}} = \frac{ab+b^2}{b^2+bc-(bc+a^2)}.$$

Collecting and factoring: $$= \frac{b\,\cancel{(a+b)}}{(b-a)\,\cancel{(b+a)}}.$$

Cancelling: $$= \frac{b}{b-a}.$$

Check this by putting numbers for $a$, $b$ and $c$ in the given fraction and in the answer; for example, put $a = 4$, $b = -2$, $c = 3$.

**Exercises.**

Simplify the following and check exercises containing letters.

1. $\frac{4}{45}-\frac{8}{15}+\frac{7}{12}.$

2. $\frac{a}{b}+\frac{b}{a};\quad \frac{ac}{bd^2}-\frac{bc}{ad}.$

3. $\frac{2m}{m^2-n^2}-\frac{n}{m^2-mn}.$

4. $\frac{m+n}{m}+\frac{m-n}{n}.$

5. $\frac{a+b}{a-b}-\frac{4ab}{a^2-b^2}.$

6. $\frac{r-4}{r^2-5r+6}+\frac{r+6}{r^2-6r+9}.$

7. $\frac{a}{2a^2+5ab-3b^2}-\frac{b}{2a^2-ab}-\frac{b}{a^2+3ab}.$

8. $\frac{3a}{a^3-b^3}-\frac{1}{a^2+ab+b^2}+\frac{1}{a^2-ab}.$

9. $\frac{2\frac{2}{7}}{5\frac{1}{21}};\quad \frac{\frac{1}{2}+\frac{2}{3}}{1-\frac{3}{5}};\quad \frac{2\frac{1}{3}+3\frac{1}{5}}{2\frac{1}{3}-3\frac{1}{5}}.$

10. $\frac{a+\frac{1}{b}}{a-\frac{1}{b}};\quad \frac{\frac{3m^2-4mn+n^2}{m^2-n^2}}{\frac{m-n}{m+n}}.$

11. $\frac{\frac{a}{b^2}+\frac{b}{a^2}}{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}};\quad \frac{1-\frac{26r}{7r^2+13r-2}}{1-\frac{6r+1}{7r-1}}.$

**77. Long Division.** If numbers can be factored, the easiest way to divide one by the other is to resolve them into their factors, and then cancel the common factors. However, at times it is necessary to divide one number by another, when neither can be factored by the rules given. In this case we must resort to the use of long division. Long division in algebra is similar to that in arithmetic.

*Example* 1. Divide 1502115 by 6285.

$$\begin{array}{r|l} 1502115 & 6285 \\ 12570\phantom{00} & 239 \\ \hline 24511\phantom{0} & \\ 18855\phantom{0} & \\ \hline 56565 & \\ 56565 & \\ \hline \end{array}$$

So with algebraic expressions.

*Example* 2. Divide $a^5 - 4a^4b + 4a^3b^2 - ab^4$ by $a^2 - 2ab + b^2$.

$$\begin{array}{l|l} a^5 - 4a^4b + 4a^3b^2 - ab^4 & a^2 - 2ab + b^2 \quad \text{(divisor)} \\ a^5 - 2a^4b + a^3b^2 & a^3 - 2a^2b - ab^2, \quad \text{(quotient)} \\ \hline \quad - 2a^4b + 3a^3b^2 + 0 & \\ \quad - 2a^4b + 4a^3b^2 - 2a^2b^3 & \\ \hline \qquad\qquad - a^3b^2 + 2a^2b^3 - ab^4 & \\ \qquad\qquad - a^3b^2 + 2a^2b^3 - ab^4. & \end{array}$$

Therefore $\dfrac{a^5 - 4a^4b + 4a^3b^2 - ab^4}{a^2 - 2ab + b^2} = a^3 - 2a^2b - ab^2.$

Check this by putting $a = 2$ and $b = -3$.

*Rule for dividing one algebraic expression by another.*

Arrange the terms of the dividend and the divisor according to the exponent of some letter ($a$ in our illustration).

Divide the first term of the dividend by the first term of the divisor. The result is the first term of the quotient.

Multiply the entire divisor by this term and subtract.

Divide the first term of the remainder by the first of the divisor. This gives the second term of the quotient.

Multiply the entire divisor by this term of the quotient and subtract.

Continue until there is no remainder or until the exponent of the leading letter of the remainder is less than that of the divisor. In the latter case you do not have an even division. Your answer then is the quotient so far found plus a fraction whose numerator is the last remainder and whose denominator is the divisor.

Check by multiplying the quotient by the divisor. If the work has been correctly done, the result is the dividend.

*Example* 3. $\dfrac{6x^4 - 9x^3 + 11x^2 - 10}{3x^2 - 2} = ?$

$$\begin{array}{l|l} 6x^4 - 9x^3 + 11x^2 - 10 & 3x^2 - 2 \quad \text{(divisor)} \\ 6x^4 \qquad\quad - 4x^2 & 2x^2 - 3x + 5, \quad \text{(quotient)} \\ \hline \quad - 9x^3 + 15x^2 - 10 & \\ \quad - 9x^3 + 6x & \\ \hline \qquad\quad 15x^2 - 6x - 10 & \\ \qquad\quad 15x^2 \qquad - 10 & \\ \hline \qquad\qquad - 6x & \text{(remainder).} \end{array}$$

Therefore $\dfrac{6x^4 - 9x^3 + 11x^2 - 10}{3x^2 - 2} = 2x^2 - 3x + 5 + \dfrac{-6x}{3x^2 - 2}.$

Check this by putting a number in place of $x$, say $x = -2$.

The use of long division is convenient in reducing a fraction to its lowest term, where either the numerator or the denominator can be factored but not both. The plan is to factor the term of the fraction that you can and then divide the other term by one of the factors. If it divides without a remainder, you can easily write the fraction in its lowest terms.

*Example* 4. Simplify: $\dfrac{a^3 - b^3}{a^4 + a^2b^2 + b^4}.$

Factoring: $\dfrac{a^3 - b^3}{a^4 + a^2b^2 + b^4} = \dfrac{(a - b)(a^2 + ab + b^2)}{a^4 + a^2b^2 + b^4}.$

Divide $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$ by long division, and we have the quotient $a^2 - ab + b^2$. Therefore

$$\frac{a^3 - b^3}{a^4 + ab + b^2} = \frac{(a - b)(a^2 + ab + b^2)}{(a^2 + ab + b^2)(a^2 - ab + b^2)}$$
$$= \frac{a - b}{a^2 - ab + b^2}.$$

**Exercises.** Reduce the following, and in each case check by putting numbers for the letters.

Divide:

**1.** $x^5 + y^5$ by $x + y$. **2.** $x^7 + y^7$ by $x + y$.

**3.** $r^4 - s^4$ by $r - s$.

**4.** $m^4 + 4m^2n^2 + 16n^4$ by $m^2 - 2mn + 4n^2$.

**5.** $p^3 + 3p^2q - 4pq^2 + 6$ by $p^2 - pq + 2q$.

Reduce the following fractions to lowest terms:

**6.** $\dfrac{x^3 - 3x^2 - 13x + 15}{x^2 - 2x + 1}.$ **7.** $\dfrac{m^4 + m^2 + 1}{m^3 - 1}.$

**8.** $\dfrac{4a^2 - 9b^2}{4a^3 - 20a^2b - 9ab^2 + 45b^3}.$

**78. Equations involving Fractions.** Fractions often occur in equations drawn from problems, and we may as well examine at this time the method of dealing with such equations.

Suppose that we had formed such an equation as the following:

$$\frac{2t+1}{(t+2)^2}+\frac{2t+1}{t+2}-2=0,$$

where $t$ is the unknown number.

Since this is an equation, we may multiply each term in it by the same number and not destroy the equality. Why? The values of the individual terms will be changed, but we do not care about this fact, so long as we have an equation. So, to get rid of fractions, we shall multiply **each term** of the equation by the number that will multiply out the denominators. The best number to select is the **least common denominator.**

In our equation the least common denominator is

$$(t+2)(t+2).$$

Multiplying each term of the equation by this number, we get

$$(2t+1)+(2t^2+5t+2)-(2t^2+8t+8)=0.$$

Collecting terms: $-t-5=0.$

Therefore $t=-5.$

Check the exercise by substituting this value for $t$ in the equation. We have

$$\frac{-9}{9}+\frac{-9}{-3}-2=-1+3-2=0.$$

Since the equation states that the value should be zero, $-5$ is the correct value of $t$.

**Exercises.**

Solve the following equations and check.

**1.** $\dfrac{2}{m-1}=\dfrac{m}{m^2+3n-4}.$

**2.** $\dfrac{5}{p^2-2p}-\dfrac{7}{p^2-4p+4}=0.$

3. $\dfrac{1}{2r^2-9r+10}-\dfrac{1}{r^2-4}=\dfrac{1}{2r^2-r-10}.$

4. $2+\dfrac{a-1}{3a-2}=\dfrac{7a^2}{3a^2-5a+2}.$

5. $\dfrac{m-1}{m^2-3m}=5-\dfrac{5m^2-m-41}{m^2-9}.$

6. $\dfrac{r}{r-2}-\dfrac{2r}{r^2-4r+4}+\dfrac{1}{r^2-4}=1.$

**Exercises in Fractions.** Reduce the following to lowest terms:

1. $\dfrac{4}{9};\ \dfrac{15}{25};\ \dfrac{64}{56};\ \dfrac{144}{216};\ \dfrac{1442}{3605}.$

2. $\dfrac{ab^2}{ac};\ \dfrac{2d^2}{6dc};\ \dfrac{-12m^2nz^3}{-6mb^2z};\ \dfrac{51a^5b^2cd}{34a^2b^7d}.$

3. $\dfrac{r^2-9}{r^2-5r+6};\ \dfrac{ab+b^2}{ab+a^2};\ \dfrac{r^3+2r^2s+rs^2}{2r^2+2r^2s}.$

4. $\dfrac{2p^3s-4p^2s^2+2ps^3}{2p^2-8ps+6s^2};\ \dfrac{a^3-8d^3}{a^2-4d^2}.$

5. $\dfrac{3-3s^3}{3+3s+3s^2};\ \dfrac{4-(a+b)^2}{6-3a-3b}.$

6. $\dfrac{a^2-b^2}{b^3-a^3};\ \dfrac{2r-rs-2s+s^2}{7r-7s}.$

Multiply the following and see that answers are given in their lowest terms:

7. $\dfrac{2}{5}\cdot\dfrac{35}{15}\cdot\dfrac{225}{14}.$

8. $\dfrac{r}{s}\cdot\dfrac{8ms}{3s}\cdot\dfrac{9s^2}{2m}.$

9. $\dfrac{p^2-2p+1}{p^3+p}\cdot\dfrac{p^4+2p^2+1}{p-1}.$

10. $\dfrac{r}{s-1}\cdot\dfrac{1-s^3}{mr+m}\cdot\dfrac{r-1}{r+rs+rs^2}.$

11. $\dfrac{r(r-1)^2+r-1}{dy-by-b+d}\cdot\dfrac{y^2+y}{r^3+1}.$

12. $\dfrac{6a^2-11a+3}{6a^3}\cdot\dfrac{2a^3-3a^2}{9a^4-a^2}.$

13. $\dfrac{m}{1-m}\cdot\dfrac{1-m^2}{-m}\cdot\dfrac{m^2}{m^3+1}.$

14. $\dfrac{-a^2}{(-a)^2}\cdot\dfrac{a^3}{(-a)^3}\cdot\dfrac{1}{a}.$ 15. $\left(\dfrac{2}{3}\right)^2\cdot\left(\dfrac{9}{2}\right)^5\cdot\dfrac{2^{10}}{3^{10}}.$

16. $\dfrac{(1-n)^2}{1+n}\cdot\dfrac{n^2-1}{2+n-n^2}\cdot\dfrac{2-n}{(n-1)^2}.$

Write the following indicated divisions in simplest form:

17. $\dfrac{\frac{3}{7}}{\frac{1}{2}};\ \dfrac{\frac{8}{21}}{\frac{2}{3}};\ \dfrac{1\frac{1}{2}}{2\frac{1}{5}};\ \dfrac{3\frac{1}{2}}{\frac{1}{15}};\ \dfrac{3-\frac{1}{2}}{\frac{1}{4}+\frac{1}{6}}.$ 18. $\dfrac{\frac{a}{bc}}{\frac{d}{c}};\ \dfrac{\frac{r}{s}}{4};\ \dfrac{\frac{g}{st}}{\frac{g}{3s^2}};\ \dfrac{\frac{m^2}{2n^2}}{\frac{3}{6mn}}.$

19. $\dfrac{a+\frac{1}{a}}{b-a}.$ 21. $\dfrac{\frac{1}{b}-\frac{1}{a}}{\frac{1}{b}+\frac{1}{a}}.$ 23. $\dfrac{\frac{-(a^3+b^3)}{a^2+ab}}{\frac{a^2-b^2}{a}}.$

20. $\dfrac{a+\frac{ab}{a-b}}{b+\frac{ab}{a-b}}.$ 22. $\dfrac{\frac{r^2-2rs+s}{r^2+2rs+s}}{\frac{r-s}{r+s}}.$ 24. $\dfrac{\frac{m^2-n^2-1}{2n}-1}{1+\frac{n^2-m^2+1}{2n}}.$

25. $\dfrac{\frac{r}{r+3}+\frac{r}{r-3}}{\frac{r}{r-3}-\frac{r}{r+3}}.$ 26. $\dfrac{\left[\frac{x}{s+2}+1\right]\left[1-\frac{x^3}{(s+2)^3}\right]}{\left[1-\frac{x^2}{(s+2)^2}\right]\left[\frac{x^2}{(s+2)^2}+1+\frac{x}{s+2}\right]}.$

Perform the indicated operation in the following:

27. $1+\dfrac{1}{2};\ 3-\dfrac{1}{4};\ -7+2\frac{1}{2};\ \dfrac{-7}{15}-1\frac{1}{5}.$

28. $\dfrac{1}{2}-\dfrac{2}{3}+\dfrac{6}{7};\ \dfrac{2}{3}+\dfrac{7}{12}-\dfrac{27}{72}.$

29. $1-\dfrac{1}{n};\ r+\dfrac{1}{s};\ -m+\dfrac{n}{2};\ \dfrac{a}{b}-\dfrac{c}{d}.$

30. $\dfrac{2r}{r-1}-\dfrac{1+r}{r-1}.$ 31. $a-\dfrac{a^2}{a-c}.$

32. $\frac{s}{2r}+\frac{3s}{4r}-\frac{5s}{6r}$.

33. $\frac{1}{m-1}-\frac{m^2+2m}{m^3-1}$.

34. $\frac{1}{n-m}-\frac{3mn}{n^3-m^3}-\frac{m-n}{m^2+mn+n^2}$.

35. $\frac{5x}{9x^2-25y^2}-\frac{2x-3y}{6nx+10ny}-\frac{4x-y}{6nx-10ny}$.

36. $\frac{a}{b^2-ab}+\frac{b}{a^2-ab}$.

Solve the following equations:

37. $\frac{3r}{2}-\frac{r}{3}-\frac{5}{6}=0$. Solve for value of $r$.

38. $\frac{s}{5}-\frac{s-8}{4}=\frac{s}{20}$. Solve for value of $s$.

39. $\frac{3}{m-3}-\frac{5}{m+3}=\frac{2}{m^2-9}$. Solve for value of $m$.

40. $\frac{x-1}{x}-\frac{x+1}{x^2-2x}=\frac{x}{x-2}$. Solve for value of $x$.

41. $\frac{\frac{2x+1}{3}}{\frac{3x-1}{6}}=\frac{3}{2}$. Solve for value of $x$.

42. $\frac{5+\frac{12}{r}}{7-\frac{4}{r}}=\frac{\frac{8}{2r+3}}{\frac{6}{2r+3}}$. Solve for value of $r$.

**Problems leading to Simple Fractional Equations.**

(a) *Formation of the Equation.*

(1) Read the problem with care entirely through. If there is anything not clearly understood, read it through again. Be sure that you understand the English statement before you begin translating into algebraic language.

(2) Decide from the nature of the problem what number is described in two different ways. This is very important for upon this depends your ability to form an equation, which is of the utmost importance to the solution.

(3) If there is but one value called for by the nature of the question let a letter stand for this value. If there are several numbers whose values are called for, let a letter stand for one of them, and form algebraic expressions to stand for the values of the others; these algebraic expressions are merely the translations of the relation existing among the numbers called for, as expressed by the English of the problem.

(4) Translate, clause by clause, as nearly as possible in the order given in the problem, the meaning of the English language into algebraic language. You will always get two algebraic expressions which stand for the same number. This is the number that you have decided upon in instruction (2).

(5) Make these two expressions equal to one another and thus form your equation.

(*b*) *Solution of the Equation.*

(1) If the equation contains fractions, multiply each term of the equation by the least common denominator of the fractions. You now have an equation with no fractions.

(2) If the equation contains parentheses, do the work indicated, collecting similar terms on each side of the equation.

(3) Now add to or subtract from both sides of the equation, bring the unknown numbers to one side of the equation and the known numbers to the other.

(4) Divide both sides of the equation by the coefficient of the unknown number.

(5) The value of the unknown number answers the question asked, if only one number is called for. If several values are called for, find the values of these by substituting in the algebraic expressions which you let stand for these numbers at the opening of the statements.

(6) Check by substituting the value or values of the number called for in the original problem (not the equation formed nor the algebraic expressions written down, for they may be wrong) and ascertain by a process entirely independent of the algebraic solution whether every condition made in the English statements holds. This is usually a pure arithmetic process.

**Problems.**

1. The circumference of the front wheel of a wagon is 2 feet less than that of the hind wheel. In going 14 miles the front wheel makes 1.4 times as many revolutions as the hind wheel. How many feet are there in the circumference of each wheel?

2. The rate a boatman can row in still water is 1 mile more than twice the rate of the stream, and it takes 3 times as long to row 12 miles upstream as it takes to row 10 miles downstream. What is the rate of the rowing and the rate of the stream?

3. A tank has two pipes, one through which water flows in and the other through which water flows out. The volume flowing out per hour when the emptying pipe is open is 11 gallons less than the volume flowing in when the filling pipe is open, so that it takes $4\frac{2}{3}$ times as long to empty the tank as it does to fill it. Find the number of gallons per hour which can flow through each pipe.

4. If $500 yields as much interest in the same length of time as $750 at 2% less rate of interest, what are the rates at which each sum is drawing interest?

5. In planting an orchard, a man found that if he planted 6 more trees in a row, he could plant 1800 in the same number of rows that he had originally planned for 1500 trees. How many rows had he planned for?

6. A man wished to sow a field with wheat, rye and vetch. The price of the wheat was 10 cents more per hundredweight

than the price of the rye, while the price of the vetch was 4 cents more per pound than 12 times the price of the wheat per pound. He arranged to pay for the wheat and rye an average price per hundredweight. He bought as many hundredweight of wheat and rye for $23.75 as he bought pounds of vetch for $4. What was the price of each kind of grain?

7. The number of trees in a row in an apple orchard is 10 more than 3 times the number of rows. The ratio of the number of trees in a row to the number of rows is 5. How many trees are there in a row?

8. There is a number the sum of whose digits is 11. If the digits are interchanged and the original number is divided by the new number thus formed, the quotient is 2 with a remainder of 7. Find the number.

9. Two triangles whose areas are $11\frac{1}{4}$ square inches and 15 square inches respectively have equal altitudes. The length of the base of one is 5 inches less than twice the length of the base of the other. What is the length of the base of each?

10. What number must be subtracted from the denominator of the fraction $\frac{11}{20}$ to increase the value of the fraction by $\frac{11}{5}$?

11. A boy can run two and a half times as fast as he can walk, and he can run 100 yards in 15 seconds less than he takes to walk it. Find his speed in yards per second, when running and when walking.

12. A football player starts with the ball from the middle of the field for a touchdown. An opposing player, who can run 2 yards a second faster, starting 10 yards behind him, catches him on the 20-yard line. How fast did each man run?

13. When 20 grams of a certain liquid are mixed with 5 grams of another liquid whose density is 1.5 greater than that of the first liquid, the density of the mixture equals the product of the densities of the two liquids. What is the density of each liquid?

## CHAPTER XIII

## QUADRATIC EQUATIONS

**79. Definitions.** In Chapter X and in Chapter XII we solved a number of equations for the value of an unknown number. This number was first represented by a letter, then an equation was formed from which the numerical value for which the letter stood could be found. In all cases only the first power of the letter representing the unknown was present in the simplified form of the equation. Our equations could all be reduced finally to the simple form

$$ax + b = 0,$$

where $x$ is the unknown number and $a$ and $b$ are known numbers.

We pass now to equations in which the square of the unknown number also occurs; that is, to equations of the general form

$$ax^2 + bx + c = 0,$$

$x$ being the unknown, and $a$, $b$, $c$ the known numbers. Either $b$ or $c$ may be zero, but $a$ must not be zero. Such an equation is called a *quadratic equation in x;* the expression

$$ax^2 + bx + c$$

is called a *quadratic expression in x.*

Our problem now is, to find the values of $x$, which will reduce this expression $ax^2 + bx + c$ to zero. This is called "solving the quadratic equation $ax^2 + bx + c = 0$ for $x$."

**80. Some Problems Leading to Quadratic Equations.**

*Example* 1. The product of a number by a number 2 units less is 8. What is the number?

*Solution:*

Let $\quad n =$ the number.

Then $\qquad n - 2 =$ the number 2 units less,

and $\qquad n(n-2) =$ the product of the numbers;

but $\qquad 8 =$ the product of the two numbers.

$\therefore \qquad n(n-2) = 8.$ Why?

We are to find the value of $n$ to verify this equation.

Multiplying as indicated

$$n^2 - 2n = 8.$$

Taking 8 from both sides of the equation, we have

$$n^2 - 2n - 8 = 0.$$

This is a quadratic equation in $n$, from which $n$ is to be found.

Factoring, we have

$$(n-4)(n+2) = 0.$$

Here we have the product of two numbers, namely $n - 4$ and $n + 2$, which product is equal to zero. Therefore one or the other of these numbers is zero. Why? Can the product of two numbers be zero without one of the numbers being zero?

Therefore we conclude that

$$\text{either} \quad n - 4 = 0$$
$$\text{or} \quad n + 2 = 0.$$

Let $\qquad n - 4 = 0.$

Then $\qquad n = 4,$ one of the numbers,

$\qquad n - 2 = 2,$ the other number.

*Check:* Since one of the numbers is 4 and the other number is 2, the second is 2 less than the first, so this statement is satisfied by the numbers 4 and 2. The product of the two numbers is $4 \cdot 2$, which is 8. So this statement is satisfied and 4 and 2 are correct answers.

Let the other factor equal zero:

$$n + 2 = 0.$$

Then $\qquad n = -2,$ one of the numbers,

and $\qquad n - 2 = -4,$ the other number.

*Check:* Since one of the numbers is $-2$ and the other is $-4$, the second number is 2 less than the first, and their product is $(-2)(-4)$, which is 8. So $-2$ and $-4$ are correct answers to our problem.

*Example* 2. The height of a rectangle is 2 cm. less than its base. Its area is 8 square centimeters. What are its dimensions?

*Solution:*

Let $\qquad b =$ the number of units in the length of the base;

then $\qquad b - 2 =$ the number of units in the length of the altitude,

so $\qquad b(b-2) =$ the number of square units in the area;

but $\qquad 8 =$ the number of square units in the area.

$\therefore \qquad b(b-2) = 8.$ Why?

This is the same equation as we had in Example 1, so we proceed as before.

Multiplying out:

$$b^2 - 2b = 8.$$

$$b^2 - 2b - 8 = 0, \text{ subtracting 8 from both members.}$$

$$(b - 4)(b + 2) = 0, \text{ factoring.}$$

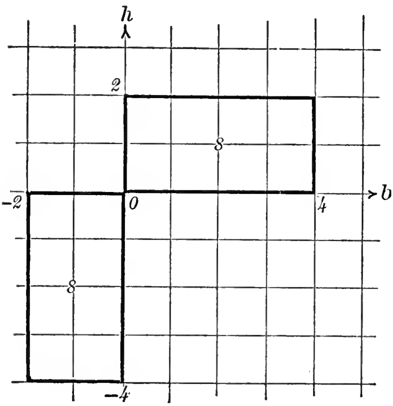
Therefore, either $b - 4 = 0$ or $b + 2 = 0$. Why?

If $b - 4 = 0$,

$b = 4$, the number of cm. in the length of the base.

$b - 2 = 2$, the number of cm. in the length of the altitude.

*Check:* These values satisfy the condition of the problem that the altitude is 2 cm. less than the base and also the condition that the area must be 8 sq. cm., since $2 \cdot 4$ gives 8, which is the area of the rectangle.




But suppose that

$$b + 2 = 0.$$

Then $b = -2$, the number of cm. in the length of the base;

and $b - 2 = -4$, the number of cm. in the length of the altitude.

*Check:* These values satisfy the condition of the problem that the altitude is 2 cm. less than the base, and also the condition that the area is 8 sq. cm., since $-2 \cdot (-4) = 8$.

So we have two rectangles that answer the conditions of the problem.

The graphic representation of this is shown in the figure. The two rectangles differ only in their position with respect to the origin.

**Exercises.**

**1.** A rectangle is 11 cm. longer than it is wide, and contains an area of 26 sq. cm. What are the length and width of the rectangle?

**2.** A rectangle is 5 cm. longer than it is wide and contains

— 6 sq. cm. of area. What is the number of cm. in its width? In its length?

3. A parallelogram has an area of 60 square units. Its base is 11 cm. longer than its corresponding altitude. What is the length of the base? Of the altitude?

4. A square has an area of 625 sq. cm. What is the length of its side? (Be sure that you find two numbers that can be the length of the side of this square.)

5. Find the length and width of a rectangle whose area is 49 sq. in. and whose length is 7 in. more than 6 times its width.

6. Find the length of the base and the altitude of a triangle whose area is 10 sq. in. and the length of whose altitude is 3 less than 2 times its base.

7. If the altitude of a parallelogram be diminished by 4 units the base would be 2 units less than its altitude, and its area would be 24 square units. What is the length of base and altitude of parallelogram?

8. The altitude of a parallelogram is 30 in. less than 9 times its corresponding base. The area is 22 sq. in. What is the length of the base and altitude of the parallelogram?

9. Write three problems about dimensions and areas which will lead to quadratic equations.

**81. Square Roots of Numbers.** Until this time, generally we have made use of numbers expressing measurement of quantities having a common unit of measure. An exception to this was the ratio of the circumference to its diameter. In this case we introduced a new character to express the ratio.

There are other geometric figures which contain lines for which there can be obtained no common measuring unit. Lines which may be measured by a common unit are called commensurable lines (the word means common measure). Lines which cannot be measured by a common unit are called incommensurable lines (the word means no common measure).

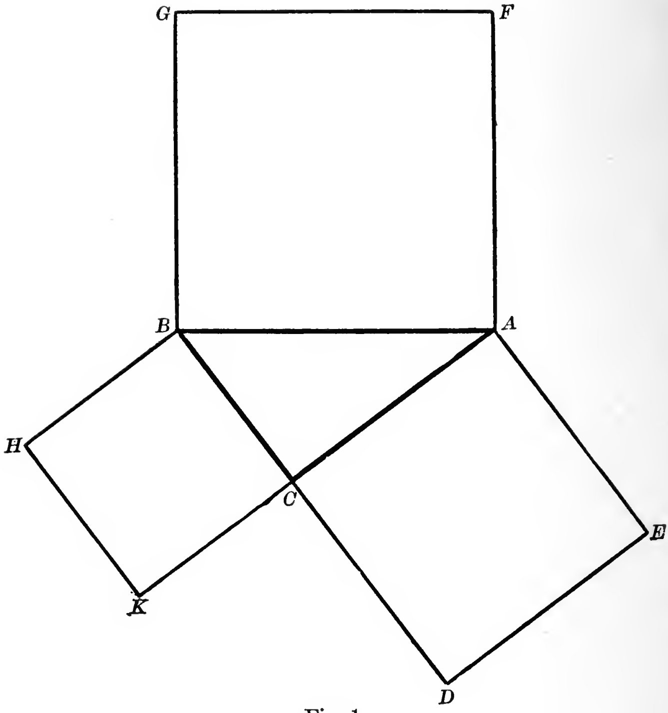



Fig. 1.

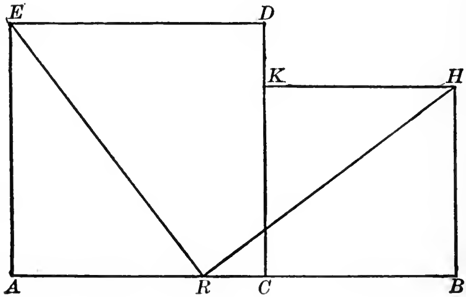



Fig. 2.

We shall investigate such lengths through the following theorem.

**Pythagorean Theorem.*** *The square on the hypotenuse* (the side opposite the right angle) *of a right-angled triangle is equal to the sum of the squares on the other two sides.*

Draw the right-angled triangle *ABC* (p. 170). Draw the squares *BAFG, CBHK, ACDE* on the sides *AB, BC, CA* respectively. We shall, by paper cutting and comparison, show that square *BAFG* = square *CBHK* + square *ACDE.* On stiff paper draw a square equal to the square *ACDE.* Adjoining it, as shown in Fig. 2, draw a square equal to square

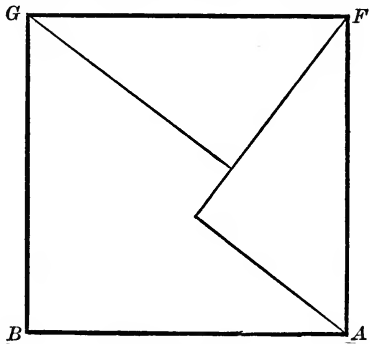



Fig. 3.

*CBHK.* Cut out the figure *ABHKDE.* On the side *AB* lay off a distance *AR* equal to side *BC.* Draw the lines *ER* and *RH.* Cut the figure along these lines. Draw a square equal to the square *BAFG,* and cut it out. On this square arrange the figure *ERHKD* so that *ER* shall fall on *GB,* and *RH* on *BA,* as shown in Fig. 3. Now place triangle *BHR* so that *HR* shall fall on *AF.* Place triangle *ARE* so that *RE* shall fall on *FG.* Can you explain by examining the figures why these parts fit so well together? This shows the correctness of the theorem.

* This theorem takes its name from the Greek mathematician Pythagoras (580–500 B.C.), who first proved the theorem.

Let the letters $a$ and $b$ denote the lengths of the sides of the triangle and let $c$ denote the length of the hypotenuse. Then our theorem, stated as an equation, is

$$c^2 = a^2 + b^2.$$

*Example* 1. If $a = 3$, $b = 4$, $c = ?$

*Solution:* Substituting the values given,

$$c^2 = 3^2 + 4^2$$
$$= 25.$$

From which $c = 5.$

There is a second value, $c = -5$, which we discard, because we are considering the sides of our triangle as measured by positive numbers.

Draw this triangle and check by measurement.

**Exercises.** Similarly solve and check the following:

**1.** If $a = 5$, $b = 12$, $c = ?$

**2.** If $a = 15$, $b = 8$, $c = ?$

**3.** If $a = 16$, $c = 34$, $b = ?$

**4.** If $c = \frac{5}{3}$, $b = 1$, $a = ?$

*Example* 2. If $a = 3$, $b = 5$, $c = ?$

*Solution:* $c^2 = a^2 + b^2.$

Substituting $a = 3,\ b = 5,$

$$c^2 = 3^2 + 5^2$$
$$= 34.$$

Then $c =$ that number whose square is 34.

This number is called the *square root* of 34.

Now there is no number such as we have had up to this time, by which we can express the exact root of 34. In other words, the two lines, namely the hypotenuse and either side of this triangle are incommensurable. It is necessary to introduce a new symbol to express this new number. We write

$$c = 34^{\frac{1}{2}} \text{ or } \sqrt{34}.$$

These two symbols mean the same thing. We may use them interchangeably. The first is read "34, exponent $\frac{1}{2}$," and the second is read, "the square root of 34." There would be no objection to reading the first as "the square root of 34."

Since $c \cdot c = 34$, we must have

$$34^{\frac{1}{2}} \cdot 34^{\frac{1}{2}} = 34;$$

also

$$\sqrt{34} \cdot \sqrt{34} = 34.$$

Such numbers are called *incommensurable* or *irrational numbers* to distinguish them from the numbers which we have been using, which are called *rational* numbers. The expression of the ratio of commensurable lines is a rational number. The expression of the ratio of incommensurable lines is an irrational number.

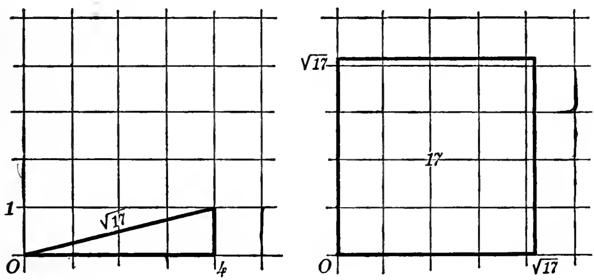
## 82. Geometric Construction of the Square Root of a Number.

*Problem. To draw a square the length of whose side is expressed by an irrational number.*

Although we cannot always express the length of the side of a square in terms of a desired unit, we can always draw the square after finding the length of its side geometrically.

*Example* 1. Draw a square whose area is 17 square units.

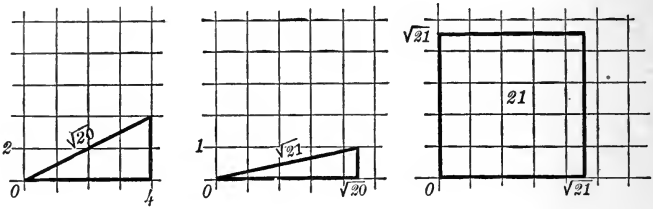
To do this we decide upon two numbers which are squares whose sum is 17. These numbers are 16 and 1. So count 4 to the right and 1 up. Thus:




The line joining the origin to the point last counted is the side of the square called for because the square on it is equal to the sum of the squares on the other two sides of the right-angled triangle one of which is 4 and the other 1.

*Example* 2. Draw a square whose area is 21 square units.

In this case the nearest that we can come to finding two squares whose sum is 21 is 16 and 4, which added give us 20. It will then be necessary first to find the length of the side of a square whose area is 20, and then, using this for one side of our right triangle and 1 as the length of the other side, we shall get the length of the side of the square called for. In taking the length of $20^{\frac{1}{2}}$ from one figure to use in the other it is better to use our compass as we cannot measure it by means of the units on our measuring line. Why?




*Example* 3. Draw the square whose area is 28 square units.

Examining this number you will find that it may be separated into two factors, one of which is a perfect square while the other is not.

$$28 = 4 \cdot 7.$$

Then $$28^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 7^{\frac{1}{2}} = 2 \cdot 7^{\frac{1}{2}}. \quad \text{(Why?)}$$

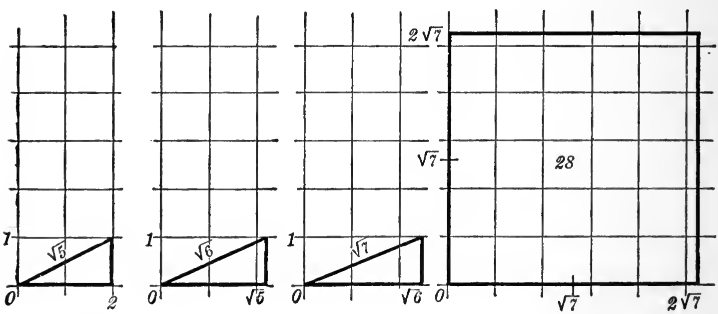
That is, we have the law

$$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}, \text{ or, } \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

Test to see if this is true by letting $a = 9$, $b = 4$.

Does $9^{\frac{1}{2}} \cdot 4^{\frac{1}{2}} = (9 \cdot 4)^{\frac{1}{2}}$?

So in our drawing for the side of the square whose area is 28, we find the length of the side of a square whose area is 7, and take twice this length and draw our square which will be equal to 4 times the square of $7^{\frac{1}{2}}$.




**Exercises.** Draw the squares whose areas are:

13; 27; 37; 48; 23.

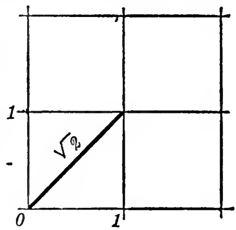
The square roots of numbers will arise often in the solution of quadratic equations. Until this time we have carefully avoided such equations.

*Example* 4. Suppose you were asked to solve the very simple quadratic equation

$$x^2 - 2 = 0.$$

According to the methods previously used, you would factor it into the sum and difference of the roots of $x^2$ and 2. Try it. You are asked to find the square root of 2.

The geometric solution is very easily obtained. You have but to count 1 unit to the right and 1 unit up and join this point to the origin, or, what amounts to the same thing, the diagonal of a square whose side is 1 unit is $2^{\frac{1}{2}}$.




Draw a square whose side is 1 decimeter in length. Draw its diagonal. With your compass lay off on the diagonal a distance equal to the side of the square. How often can you lay it off?

Take one-tenth of the side of the square and lay it off as many times as you can. How often can you lay this off?

Take one hundredth of the side of your square and lay it off. How often can you lay it off?

In the first trial you get 1. In the second trial, 14. In the third trial, 141. If you would use one-thousandth part of the side of the square, you would get 1414. That is, you find that as you diminish your measuring line you come closer to the square root of 2.

$$\begin{aligned} 1^2 &= 1, \\ 1.4^2 &= 1.96, \\ 1.41^2 &= 1.988 +, \\ 1.414^2 &= 1.999 + \end{aligned}$$

and so on.

Since it is true that $(-1)^2 = 1$; $(-1.4)^2 = 1.96$; $(-1.41)^2 = 1.988$; then $-1$, $-1.4$, $-1.41$, $-1.414$ and so on are numbers approaching closer and closer to a number whose square is 2. So there are two numbers whose square is 2.

But we can never arrive at the exact value of these numbers either in the form of a fractional number or as a terminating decimal. We must content ourselves therefore by merely indicating them by the symbol $2^{\frac{1}{2}}$ or by $\sqrt{2}$ for the positive square root and by $-2^{\frac{1}{2}}$ or $-\sqrt{2}$ for the negative square root.

Similarly, the positive number whose square is $a$ is indicated by $\sqrt{a}$ or $a^{\frac{1}{2}}$. The negative number whose square is $a$ is indicated by the symbol $-\sqrt{a}$ or $-(a)^{\frac{1}{2}}$. The symbol $\sqrt{\ }$ is called the *radical sign.* Either one of the two numbers whose square is $a$, that is $\sqrt{a}$ or $-\sqrt{a}$, is called the *square root of a;* they are distinguished as the positive square root and the negative square root respectively.

We can now solve our equation

$$x^2 - 2 = 0.$$

Factoring: $(x - 2^{\frac{1}{2}})(x + 2^{\frac{1}{2}}) = 0.$

Hence $x = 2^{\frac{1}{2}}$ or $\sqrt{2}$,

and $x = -2^{\frac{1}{2}}$ or $-\sqrt{2}$.

These are usually written together:

$$x = \pm\sqrt{2}.$$

Read this "$x$ equals the positive or the negative square root of 2."

**83. Rational and Irrational Numbers.**

*Definitions.* Numbers such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, and so on, whose values can be expressed approximately, but not exactly, by a numerical fraction or a terminating decimal, are called **irrational** numbers.

So far the student has dealt almost entirely with numbers which were either integral, positive or negative, or quotients of such integers, that is, numerical fractions. All such numbers are called **rational numbers.**

**84. Factoring of Quadratic Expressions.** Consider the quadratic expression

$$2a^2 + 7a + 4.$$

No matter how long we try, we cannot find, by means of the processes already given, the factors of this expression. The reason for this is that the factors contain irrational numbers, and the factoring which we have previously considered involves only rational numbers. We shall now investigate the drawings by which we may manage to arrive at the factoring of this expression. However, we shall not start our investigation with this more complicated form, but we shall start with the expression $a^2 + 5a + 6$ (p. 130 Ex. 6), and show that by a longer process we may draw this as a rectangle and find its sides to be the same as we arrived at by the former plan.

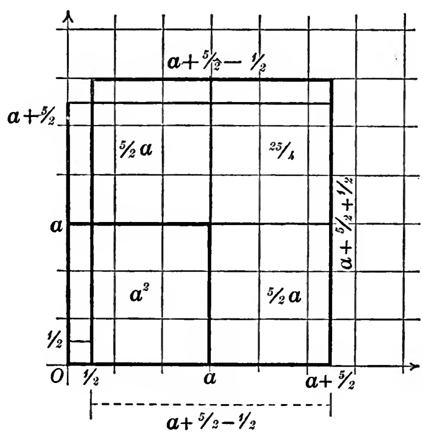
While we factor this simpler expression by this method for the sake of practice, we do not use it in actual work when the factors can be given readily by the shorter plan.

*Example* 1. To factor $a^2 + 5a + 6$;

that is, to find the dimensions of the rectangle of which it expresses the area.

*Note.* See that you follow the directions and answer the questions asked — each one finished before reading the next.

The first term of the expression is $a^2$. What are the dimensions of the square which represents this term? Draw it. The next term is




$5a$. Now, instead of figuring out, as formerly, two different sized rectangles whose sum is $5a$, add to the two sides of the square two rectangles of the same size, whose sum is $5a$. Of course, the area of each is $\frac{5a}{2}$, the dimensions being $a$ and $\frac{5}{2}$. Having drawn this, examine your figure, and determine what area it will be necessary to add in order to make a complete square. What are the dimensions of the area added? What are the dimensions of your figure after adding this area? This process is spoken of as *completing the square*. Is the area of the square you now have the same as that given by the original expression, $a^2 + 5a + 6$? How much more area has been added? What is it necessary to do to bring back the original area?

Since, in order to complete the square we used an area of $6\frac{1}{4}$ square units, and our expressiong ave us an area of only 6 square units it will be necessary to subtract $\frac{1}{4}$ square unit. Subtract a square whose area is $\frac{1}{4}$. We now have the difference of two squares as in the type form $a^2 - b^2$. See p. 134. Proceed as in the drawing for that type form. You produce a rectangle, one side of which is $a + \frac{5}{2} + \frac{1}{2}$ and the other is $a + \frac{5}{2} - \frac{1}{2}$, or, collecting terms, one side is $a + 3$ and the other is $a + 2$.

The algebraic expression for this is:

$$\begin{aligned} a^2 + 5a + 6 &= a^2 + 5a + \tfrac{25}{4} - \tfrac{1}{4} \\ &= (a + 3)(a + 2). \end{aligned}$$

Examine the figure which you drew for the same expression in (Ex.6, p. 130), and satisfy yourself that the resulting rectangles are identical.

**Exercises.** As in the above illustration, draw and write the algebraic expression for the factors of the following. State the values of the leading letter that will make the expression zero. Check the correctness of these values by substituting each in the original expression and doing the work indicated to see that it gives zero.

**1.** $m^2 + 11m + 28$.

**2.** $r^2 + 7r + 6$.

**3.** $s^2 + 10s + 16$.

**4.** $a^2 + 9ab + 14b^2$.

*Example* 2. We are now ready to find the dimensions of the rectangle whose area is given by the expression

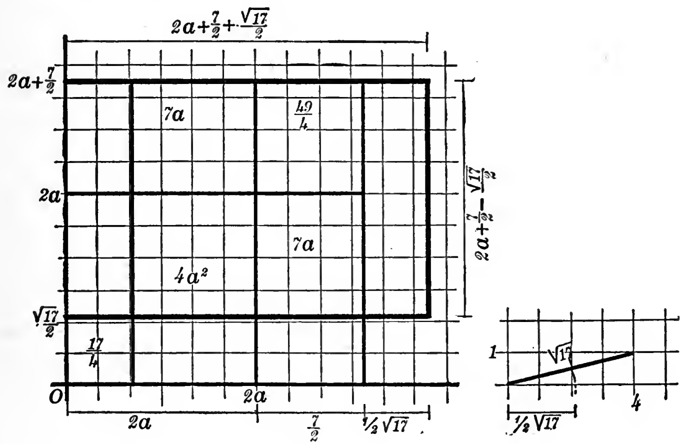
$$2a^2 + 7a + 4.$$

Since the first term is not a perfect square, we make it a square by multiplying the entire expression by 2 (the least number that will make it a square). The rectangle that we draw will have an area of twice the area of the original rectangle. We shall rectify this by taking a rectangle which has one of its dimensions one-half of the one we draw.

$$2a^2 + 7a + 4 = \tfrac{1}{2}(4a^2 + 14a + 8).$$

Proceed as in the preceding example; draw a square whose area is $4a^2$; add two rectangles the sum of whose areas is $14a$, each having one dimension $2a$ and the other $\frac{7}{2}$; complete the square by adding a square whose area is $\frac{49}{4}$. In doing this you have used $\frac{17}{4}$ more than was given in the expression $4a^2 + 14a + 8$, so you must subtract a square whose area is $\frac{17}{4}$. That is, you must subtract a square whose side is $\frac{\sqrt{17}}{2}$, which may be written $\frac{1}{2}\sqrt{17}$.

The drawing is as follows:




The algebraic expressions are:

$$
\begin{aligned}
2a^2 + 7a + 4 &= \tfrac{1}{2}(4a^2 + 14a + 8)\\
&= \tfrac{1}{2}(4a^2 + 14a + \tfrac{49}{4} - \tfrac{17}{4})\\
&= \tfrac{1}{2}(2a + \tfrac{7}{2} - \tfrac{1}{2}\sqrt{17})(2a + \tfrac{7}{2} + \tfrac{1}{2}\sqrt{17}).
\end{aligned}
$$

If we wish to know what values of $a$ will make $2a^2 + 7a + 4$ zero, we make each factor in turn equal to zero and find the value of $a$.

Letting
$$
\begin{aligned}
2a + \tfrac{7}{2} - \tfrac{1}{2}\sqrt{17} &= 0,\\
2a &= -\tfrac{7}{2} + \tfrac{1}{2}\sqrt{17},\\
a &= -\tfrac{7}{4} + \tfrac{1}{4}\sqrt{17}.
\end{aligned}
$$

Substituting this value into the given expression to see if it gives zero, we have

$$
\begin{aligned}
&2(-\tfrac{7}{4} + \tfrac{1}{4}\sqrt{17})^2 + 7(-\tfrac{7}{4} + \tfrac{1}{4}\sqrt{17}) + 4\\
&\quad = 2(\tfrac{49}{16} - \tfrac{7}{8}\sqrt{17} + \tfrac{17}{16}) + (-\tfrac{49}{4} + \tfrac{7}{4}\sqrt{17}) + 4\\
&\quad = (\tfrac{49}{8} - \tfrac{7}{4}\sqrt{17} + \tfrac{17}{8}) + (-\tfrac{49}{4} + \tfrac{7}{4}\sqrt{17}) + 4\\
&\quad = 0.
\end{aligned}
$$

Since we were solving for a number that would make the expression zero, $-\frac{7}{4} + \frac{1}{4}\sqrt{17}$ must be a correct answer.

By making the other factor equal to zero let the student find another value for $a$ that will make the given expression zero, and check.

**Exercises.** By the process just given, factor and find the value of the letter that will make the expression zero.

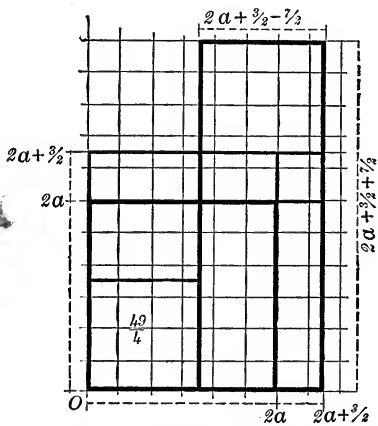
1. $3s^2 + 7s + 2.$
2. $8x^2 + 10x + 1.$
3. $m^2 + 3m + 1.$
4. $5r^2 + 9r + 3.$

*Example* 3. Factor the expression $2a^2 + 3a - 5$.

We write $2a^2 + 3a - 5 = \frac{1}{2}(4a^2 + 6a - 10).$

*Directions:* Draw the square whose area is $4a^2$. Divide the area $6a$ into two equal rectangles, one dimension of each being $2a$. Add the rectangles to the square as in the preceding example. Determine the area necessary to complete the square and add to the figure.

Notice these facts: In the first place you did not have an area equal to $4a^2 + 6a$ and yet you have used that much in your drawing.




You lack 10 square units of having that area, and in addition to this, in order to complete the square, you used $\frac{9}{4}$ square units. All together you used $\frac{49}{4}$ square units more than was given you, so that you must subtract $\frac{49}{4}$ square units. Subtract this in the form of a square and proceed as in the above examples.

By taking a rectangle whose base is the same as this one and whose altitude is $\frac{1}{2}$ of the altitude of this one, we shall have the rectangle called for.

The algebraic expression for this is

$$\begin{aligned} 2a^2 + 3a - 5 &= \tfrac{1}{2}(4a^2 + 6a - 10) \\ &= \tfrac{1}{2}(4a^2 + 6a + \tfrac{9}{4} - \tfrac{49}{4}) \\ &= \tfrac{1}{2}(2a + 5)(2a - 2) \\ &= (2a + 5)(a - 1). \end{aligned}$$

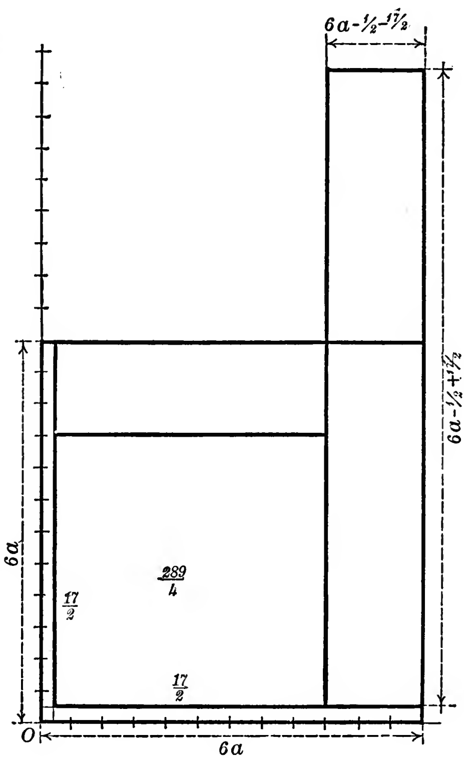
*Example* 4. Factor the expression $6a^2 - a - 12$.

Multiplying the expression by 6 in order to make the first term a square:

$$6a^2 - a - 12 = \tfrac{1}{6}(36a^2 - 6a - 72).$$

As in the preceding example, draw a square whose area is $36a^2$ square units. Subtract two equal rectangles the sum of whose areas is $6a$. See p. 132. Why subtract instead of add?

After subtracting one of these rectangles from your square, you will find that it is necessary to add a square of area $\frac{1}{4}$ before you can subtract the second rectangle, so that when you have subtracted the two rectangles, you have a square which contains $\frac{1}{4}$ more than an area represented by $36a^2 - 6a$. This area is 72 square units more than the area represented by the expression $36a^2 - 6a - 72$. So that, in order to have an area equal to that called for by the expression $36a^2 - 6a - 72$, it will be necessary to subtract a square whose area is equal to the sum of 72 and $\frac{1}{4}$, that is, an area of $\frac{289}{4}$ square units. Subtract this square and then proceed as before.



The algebraic expression for this is

$$\begin{aligned} 6a^2 - a - 12 &= \tfrac{1}{6}(36a^2 - 6a - 72) \\ &= \tfrac{1}{6}(36a^2 - 6a + \tfrac{1}{4} - \tfrac{289}{4}) \\ &= \tfrac{1}{6}(6a - 9)(6a + 8) \\ &= (2a - 3)(3a + 4). \end{aligned}$$

**Exercises.** According to the preceding illustrations draw and write factors for the following; find values of leading letters that will make the expression zero; check.

1. $2r^2 - 3r - 14.$
2. $3h^2 - 10h - 9.$
3. $10r^2 - 27rs + 5s^2.$
4. $6r^2 + 17r - 3.$
5. $4x^2 + 7x - 3.$
6. $3x^2 - 4x - 1.$
7. $3x^2 + 4x - 1.$
8. $3x^2 - 5x + 1.$

**85. Factors of $ax^2 + bx + c$.** We now consider the factoring of the general quadratic expression

$$ax^2 + bx + c,$$

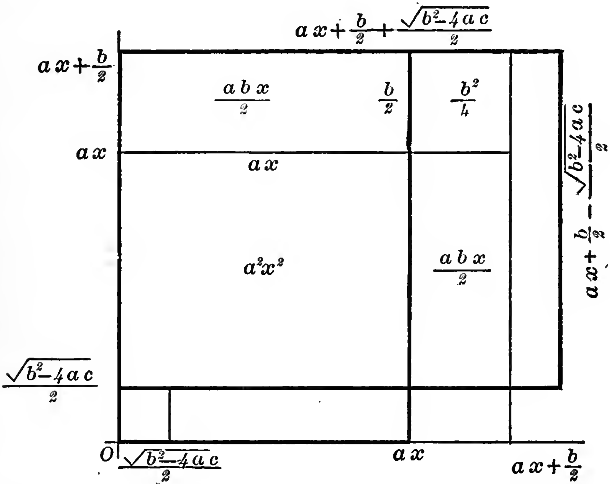
and the finding of the values of $x$ that will make the expression zero. Since this is the general form for all the expressions given above, the treatment is the same.

Make the first term a square by multiplying the expression by $a$:

$$ax^2 + bx + c = \frac{1}{a}(a^2x^2 + abx + ac).$$

The area drawn is $a$ times the correct value; we take one $a$th part of it.

Draw a square whose area is $a^2x^2$. Its side is $ax$. Draw two rectangles the sum of whose areas is $abx$. Each is $\frac{abx}{2}$. Their dimensions are $ax$ and $\frac{b}{2}$. Complete the square by adding a square whose area is $\frac{b^2}{4}$. You had the area $ac$, you used area $\frac{b^2}{4}$, therefore you must subtract a square which is the difference between these two areas, that is, a square whose area is $\frac{b^2 - 4ac}{4}$. The side of this square is $\frac{\sqrt{b^2 - 4ac}}{2}$. Proceeding with the rest of the drawing as in the preceding cases of factoring the difference of two squares, we have the following figure:




The algebraic expressions for the various steps are:

$$ax^2 + bx + c$$
$$= \frac{1}{a}(a^2x^2 + abx + ac)$$
$$= \frac{1}{a}\left(a^2x^2 + abx + \frac{b^2}{4}\right) - \frac{b^2 - 4ac}{4}.$$
$$= \frac{1}{a}\left(ax + \frac{b}{2} - \frac{\sqrt{b^2 - 4ac}}{2}\right)\left(ax + \frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2}\right).$$

Therefore

$$\mathbf{ax^2+bx+c=a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)}.$$

To find the value of $x$ that will make the expression zero, we write each factor in turn equal to zero and solve. Thus

let $$ax + \frac{b}{2} - \frac{\sqrt{b^2 - 4ac}}{2} = 0,$$

then $$ax = -\frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2};$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Let $$ax + \frac{b}{2} + \frac{\sqrt{b^2 - 4ac}}{2} = 0,$$

then $$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The student should convince himself that these values will make the expression $ax^2 + bx + c$ zero, by substituting them in the expression and doing the work indicated. This will be sufficient to show that these values are the correct ones, even though there may be some limitations to our drawings.

These values are generally written together in the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since $a$, $b$, $c$ may take on any values with the limitation that $a$ cannot be zero, we may use this answer as a formula by which we may solve any quadratic equation.

For example, take the expression

$$2x^2 + 7x + 4.$$

To find the value of $x$ that will make the expression zero, or, in other words, solve the equation

$$2x^2 + 7x + 4 = 0.$$

In this equation $a = 2$, $b = 7$, $c = 4$. Substituting these values in the formula, we have

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2}$$

$$= \frac{-7 \pm \sqrt{17}}{4}.$$

This you will recognize as the same exercise used in Example 2. Compare the two processes and convince yourself that they are the same. The check is the same as given in that exercise.

**86. Algebraic Solution of the General Quadratic Equation.** As has been remarked, there are always some limitations to be placed upon drawings. The following solution is purely algebraic.

Consider the equation

$$ax^2 + bx + c = 0,$$

where $a, b, c$ are given numbers. To find $x$ in terms of $a, b, c$.

*First step.* Subtract $c$ from each member of the equation.

$$ax^2 + bx = -c.$$

*Second step.* Divide by $a$,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

*Third step.* Complete the square by adding to each member the square of half the coefficient of $x$, that is

$$\left(\frac{b}{2a}\right)^2 \quad \text{or} \quad \frac{b^2}{4a^2}.$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}.$$

This may be written

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \quad \text{Why?}$$

Hence $$x+\frac{b}{2a}=+\sqrt{\frac{b^2-4ac}{4a^2}} \quad \text{or} \quad -\sqrt{\frac{b^2-4ac}{4a^2}}.$$

Therefore the two values of $x$ are

$$x=-\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{4a^2}},$$

or $$x=-\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{4a^2}}.$$

These values of $x$ may be simplified a little. For we have

$$\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\sqrt{b^2-4ac}}{2a}. \quad \text{Why?}$$

Then the values of $x$ are

$$x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

or $$x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

More compactly,

$$\boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.}$$

Here we use the positive sign before the radical for one value of $x$, the negative sign for the other.

This is the same formula that we developed by our drawing. The student should fix this well in his mind. When the expression cannot be factored readily by inspection, the shortest way to find the values of the unknown quantity in a quadratic equation is by means of this formula.

**Exercises.**

Solve several of the previous exercises by means of this formula and see that you arrive at the same result as before.

Solve by formula and check:

1. $9r^2 - 4r - 6 = 0.$
2. $3s^2 - 7s + 2 = 0.$
3. $12m^2 + 11m + 2 = 0.$
4. $50x^2 - 4x - 6 = 0.$
5. $2m^2 + 26m - 271 = 0.$
6. $2r^2 - 267r + 3240 = 0.$
7. $\frac{m-2}{m+2} - \frac{2m+1}{m-2} = \frac{11m+1}{4-m^2}.$
8. $\frac{4r+7}{19} + \frac{5-r}{3+r} = \frac{4r}{9}.$
9. $x^2 - 2ax = b^2 - a^2.$
10. $ar^2 + 2ar = -r^2 + 1.$
11. $m^2p^2 - 2m^2p + 2mn = 2m^2p + 2mnp^2 - n^2p^2 - m^2 - n^2.$
12. $a^2x^2 + a^2 - b^2x^2 = 2a^2x + b^2 + 2b^2x.$
13. $br^2 - 2br = a - b.$

## 87. Imaginary Roots.

*Example.* Given the equation

$$2x^2 + 2x + 3 = 0.$$

To find the value of $x$ that will satisfy it.

In this equation the special values of $a$, $b$, $c$ are

$$a = 2,\ b = 2,\ c = 3.$$

Using our formula, one of the values of $x$ is

$$\begin{aligned} x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2 + \sqrt{2^2 - 4 \cdot 2 \cdot 3}}{4} \\ &= \frac{-2 + \sqrt{-20}}{4} \\ &= \frac{-2 + 2 \cdot \sqrt{-5}}{4} \\ &= \frac{-1 + \sqrt{-5}}{2}. \end{aligned}$$

Examining this root we find that a new number has arisen which we have not met before in our work. We are called upon to extract the square root of a negative number. But there exists no real number which, when used as a factor twice, will give a negative number. Such an indicated even root of a negative number is called an **imaginary number.**

The symbol $\sqrt{-5}$ we shall regard as equivalent to $\sqrt{5} \cdot \sqrt{-1}$; similarly with regard to the square root of any other negative number.

We can take the square root of 5, either approximately or by drawing. The symbol of interest to us is $\sqrt{-1}$.

There is no other way of expressing this by means of the symbols that we have been using, so mathematicians have agreed to write it and read it $i$. Our expression is, then, $i\sqrt{5}$.

*Definition.* Since the square root of a number when used as a factor twice must give the number, therefore *we define $i$ as that number whose square is* $-1$.

That is:

$$i \cdot i = -1,$$
$$i \cdot i \cdot i = -i.$$
$$i \cdot i \cdot i \cdot i = 1.$$

We now have $i\sqrt{5} \cdot i\sqrt{5} = 5(-1) = -5$, so that $i\sqrt{5}$ is the square root of $-5$. Later we shall discuss this new number more fully.

We are ready now to check our exercise. Substituting $\dfrac{-1 + i\sqrt{5}}{2}$ in place of $x$ in the first member of the given equation, we have

$$\frac{2(-1 + i\sqrt{5})^2}{4} + \frac{2(-1 + i\sqrt{5})}{2} + 3$$
$$= \frac{1 - 2 \cdot i\sqrt{5} + 5(-1)}{2} + \frac{-2 + 2 \cdot i\sqrt{5}}{2} + 3$$
$$= 0,$$

as the equation states that it should.

Finish this example by substituting

$$a = 2, \quad b = 2, \quad c = 3,$$

in the second value of $x$, that is, in

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Check as illustrated above.

**Exercise.** Find the values of $x$ that will satisfy the following equations:

1. $5x^2 + 3x + 1 = 0.$
2. $x^2 + x + 1 = 0.$
3. $3x^2 - 7x + 5 = 0.$
4. $7x^2 - 2x + 1 = 0.$
5. $4x^2 + x + 2 = 0.$
6. $15x^2 + 7 = 0.$

## 88. To Calculate the Square Root of a Number.

*Example* 1. What is the square root of 55,696?

By inspection we find that it is more than 200 and less than 300. Call it $200 + b$. Then we are to find $b$, so that

*First step:* $$(200 + b)^2 = 55{,}696,$$

or $$200^2 + 2 \cdot 200\,b + b^2 = 55{,}696.$$

Then $$2 \cdot 200\,b + b^2 = 15{,}696.$$

A trial value for $b$ is now obtained by neglecting $b^2$ and taking

$$2 \cdot 200\,b = 15{,}696;$$

$$b = \frac{15{,}696}{400} = 30 +.$$

(The neglecting of $b^2$ may lead to too large a value for $b$; if so, this will be found out at the next step.)

Our approximate square root is now $230 +$.

So we start again.

*Second step:* Find $b$ so that

$$(230 + b)^2 = 55{,}696,$$

or $$52{,}900 + 2 \cdot 230\,b + b^2 = 55{,}696.$$

$$2 \cdot 230\,b + b^2 = 2{,}796.$$

Again neglecting $b^2$, we find the next trial value from

$$2 \cdot 230\,b = 2796,$$

$$b = \frac{2796}{460} = 6 +.$$

Hence our square root is $236 +$.

*Third step:* $$(236 + b)^2 = 55{,}696.$$

We now find that $$236^2 = 55{,}696;$$

hence $b = 0$; that is, 236 is the exact square root.

The numerical work here is usually written as follows:

| | | | |
|---|---|---|---|
| *First step:* | | 5,56,96 | 236 square root |
| | | 4 | |
| *Second step:* | 43 | 156 | |
| | | 129 | |
| *Third step:* | 466 | 2796 | |
| | | 2796 | |
| | | 0 | |

*Example* 2. When the given number is not a perfect square, we get by this process as close an approximation to its square root as we like.

Find to three decimal places the square root of 29.

| | | |
|---|---|---|
| (1) | | 29.00,00,00 \|5.385 square root. |
| | | 25 |
| (2) | 10.3 | 4.00 |
| | | 3.09 |
| (3) | 10.69 | .9100 |
| | | .8552 |
| (4) | 10.765 | .054800 |
| | | .053825 |
| | | .000975 |

*Example* 3. Find the square root of 131.12, to two decimal places.

The required square root lies between 11 and 12. Hence we may neglect the decimal point for the present and proceed as in the preceding examples.

**Exercises.**

Find the square roots of the following:

**1.** (*a*) 2304; (*b*) 1369; (*c*) 8649; (*d*) 5776; (*e*) 331,776; (*f*) 50,625; (*g*) 5,764,801; (*h*) 43,046,721; (*i*) 123,454,321.

**2.** Calculate to two decimal places the value of each of the following:

(*a*) $13^{\frac{1}{2}}$; (*b*) $21^{\frac{1}{2}}$; (*c*) $\sqrt{1.5}$; (*d*) $\sqrt{57.75}$; (*e*) $(2.7)^{\frac{1}{2}}$; (*f*) $\sqrt{.0121}$; (*g*) $\sqrt{.006}$; (*h*) $(.075)^{\frac{1}{2}}$.

**3.** Calculate each of the following to three decimals:

(*a*) $\sqrt{\frac{1}{3}}$.

$\left(\text{Notice that } \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\text{; so calculate } \sqrt{3} \text{ and divide by 3.}\right)$

(*b*) $\sqrt{\frac{2}{3}}$. $\left(\text{Notice that } \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}.\right)$

(*c*) $\frac{2}{\sqrt{5}}$. $\left(\text{Notice that } \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.\right)$

(d) $\frac{1}{\sqrt{10}}$. (e) $\frac{7}{2\sqrt{15}}$.

(f) $\frac{1}{\sqrt{2}+1}$. (Notice that $\frac{1}{\sqrt{2}+1}=\frac{1}{\sqrt{2}+1}\cdot\frac{\sqrt{2}-1}{\sqrt{2}-1}=\frac{\sqrt{2}-1}{2-1}=\sqrt{2}-1.$)

(g) $\frac{3}{\sqrt{7}-2}$.

(Notice that $\frac{3}{\sqrt{7}-2}=\frac{3}{\sqrt{7}-2}\cdot\frac{\sqrt{7}+2}{\sqrt{7}+2}=\frac{3\sqrt{7}+6}{7-4}=\sqrt{7}+2.$)

(h) $\frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}}$.

(Notice that $\frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}}=\frac{\sqrt{2}}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{10}-\sqrt{6}}{2}.$)

(i) $\frac{\sqrt{2}-1}{\sqrt{2}+1}$. (j) $\frac{1+\sqrt{5}}{1-\sqrt{5}}$.

(k) $\frac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}$. (l) $\frac{\sqrt{8}+\sqrt{7}}{\sqrt{7}-\sqrt{2}}$.

**89. Summary.**

An expression of the form of $ax^2 + bx + c$ is called a quadratic expression in $x$.

An equation of the form $ax^2 + bx + c = 0$ is called a quadratic equation in $x$.

*Pythagorean Theorem.* If $a$ and $b$ are the sides of a right-angled triangle, and $c$ is the hypotenuse, then

$$a^2 + b^2 = c^2.$$

This theorem is used in the construction of square roots of numbers.

Arithmetic integers and fractions, positive or negative, form the system of *rational numbers.* All other numbers such as $\pi$ or $\sqrt{2}$ are called *irrational numbers.*

Any rational number can be expressed as the quotient of two integers; irrational numbers cannot be so expressed.

Factors of $ax^2 + bx + c$:

$$ax^2+bx+c=a\left[x-\frac{-b+\sqrt{b^2-4\,ac}}{2\,a}\right]\left[x-\frac{-b-\sqrt{b^2-4\,ac}}{2\,a}\right].$$

This formula is to be used when factoring by inspection cannot be done readily.

Solution of the equation $ax^2 + bx + c = 0$. The two solutions are

$$x = \frac{-b+\sqrt{b^2-4\,ac}}{2\,a} \text{ and } x = \frac{-b-\sqrt{b^2-4\,ac}}{2\,a}.$$

The imaginary unit $i$ or $\sqrt{-1}$ is defined as that number whose square is $-1$.

When $b^2 - 4\,ac$ is negative, the values of $x$ contain this imaginary unit.

**Problems Involving Quadratic Equations.***

In the following problems translate the English into algebraic expression according to instructions given on p. 162. Form the equation and if it is quadratic add and subtract from both members until the right-hand member is zero. Factor the left-hand member by inspection if you can, and solve for the value of the unknown number. If you cannot factor readily, write the values of the unknown by means of the formula.

If a geometric figure is involved, draw a careful figure for each answer if possible. If irrational numbers are involved, draw by making use of the Pythagorean theorem.

Always examine both answers to see if each has a meaning, or if one is to be discarded as meaningless.

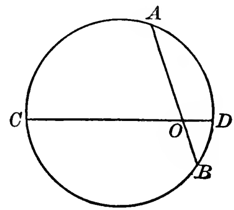
1. The altitude of a parallelogram is 7 units less than twice its base and the area is 10 square units. Find the length of the base and of the altitude of the parallelogram.

2. The upper base of a trapezoid is one unit less than

* While proceeding with the solution of problems in this list, Parts I and II of chapter VI of Geometry may be studied.

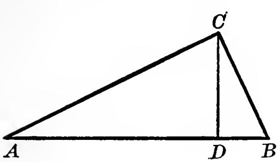
one-half the lower base. The altitude is 5 units less than $1\frac{1}{2}$ times the lower base. The area is 36 square inches. Find the length of each base and of the altitude.

**3.** In the adjacent figure we have a circle with chords intersecting at point $O$. There is a theorem which states that the product of the segments of one of the chords is equal to the product of the segments of the other chord. By measurement verify the truth of this statement.




Supposing that $OD$ is 1 unit less than 2 times $CO$, and that if 2 units be subtracted from $CO$, it will be 3 times as long as $OB$, while $AO$ is 1 unit more than 7 times $OB$, find the length of each of the segments.

**4.** Making use of segments of chords intersecting within a circle write a problem leading to a quadratic equation.

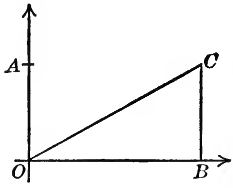



**5.** *Theorem.* If from the vertex of the right angle of a right triangle a line is drawn perpendicular to the hypotenuse, the square of the perpendicular is equal to the product of the segments into which it divides the hypotenuse. Verify this by measurement.

In the right triangle $ABC$ with $CD$ perpendicular to $AB$, suppose that $CD$ is 4 units less than 3 times $AD$, and that $DB$ is 4 units more than the sum of $AD$ and $CD$. Find the lengths of the lines. Then by means of the Pythagorean theorem find the lengths of $AC$ and $BC$ and check by showing that $AC^2 + CB^2 = AB^2$.

**6.** Making use of the theorem just stated write a problem leading to a quadratic equation.

On p. 49, it was brought out that if a ball was hit by two mallets the result of the action would be the same whether the strokes were simultaneous or the second stroke was given when the ball had come to rest after the first stroke. This is true even if the strokes are given at right angles to one another. When you strike the ball $O$ with two mallets at right angles to one another, one having the power to send it the distance $OB$ and the other to send it the distance $OA$, it will stop at the point $C$ no matter whether the action is simultaneous or consecutive; if simultaneous, the ball will go along the path $OC$; if consecutive, it will go along line $OB$ to $B$, then along $BC$ to $C$. So that it is possible to find the resultant force when two forces act at right angles to one another by making use of the Pythagorean theorem. If $OA$ equals 3 and $OB$ equals 4, the result would be equivalent to that of a blow which would send the ball 5 units along $OC$, since $3^2 + 4^2 = 5^2$.




**7.** The resultant of two forces acting at right angles to one another is 10 pounds. One force is 2 pounds less than the other. Find the number of pounds in each force.

*Solution:*

Let $f$ = the number of pounds in one of the forces.

Then $f - 2$ = the number of pounds in the other force.

$(f-2)^2 + f^2$ = the square of the number of pounds in the resultant force. Why?

But $100$ = the square of the number of pounds in the resultant force.

$$\therefore \quad (f-2)^2 + f^2 = 100.$$

$$2f^2 - 4f + 4 = 100.$$

$$2f^2 - 4f - 96 = 0.$$

$$f^2 - 2f - 48 = 0.$$

$$(f+6)(f-8) = 0.$$

If $f + 6 = 0$,

$f = -6$, the number of pounds in one of the forces,

and $f - 2 = -8$, the number of pounds in the other force.

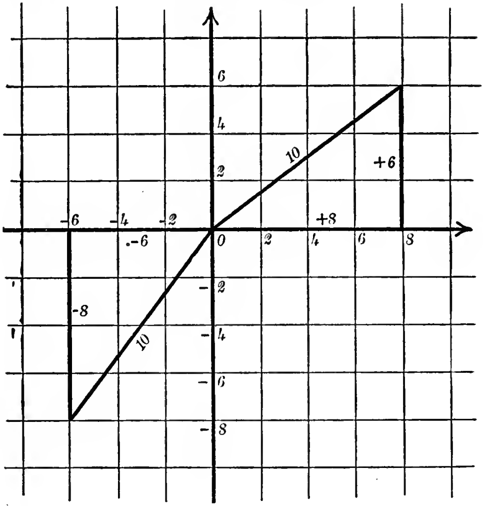
If $f - 8 = 0$,

$f = 8$, the number of pounds in one of the forces,

and $f - 2 = 6$, the number of pounds in the other force.

Check each set of answers.

The drawing for this is shown in the figure.




**8.** If the larger of two forces acting at right angles to one another is 5 lbs. less than 5 times the other, and the resultant of the two forces is 4 less than 5 times the other, what is the magnitude of each force and of their resultant? Make the drawing in each case.

**9.** Two forces act at right angles to one another with a resultant force of 17 kilograms. One of the forces is 1 kilogram less than 2 times the other. What are the forces?

**10.** A car is moving at the rate of 12 miles an hour. What is the rate of motion, with reference to the ground, of a man crossing the car at the rate of 5 miles an hour?

**11.** The speed of a parcel thrown from a train at right angles to it was 17 miles more than the speed of the train. The speed at which it went was 3 miles less than 2 times the speed of the train. What was the speed of the train? Of the parcel?

**12.** * Write three problems on forces or velocities acting at right angles to one another.

**13.** Two men can do a piece of work in 6 days. How long will it take each of them working alone to do the work if it takes one of them 5 days longer than the other?

*Hint.* It will be necessary to find the amount each can do in one day in order to form the equation.

Let $t$ = the number of days that it takes one to do the work.

Then $t + 5$ = the number of days that it takes the other to do the work.

$\frac{1}{t}$ = the part of the work that the first can do in one day.

$\frac{1}{t+5}$ = the part of the work that the second can do in one day.

$\frac{1}{t} + \frac{1}{t+5}$ = the part of the work that both can do in one day.

But $\frac{1}{6}$ = the part of the work that both can do in one day.

$$\therefore \quad \frac{1}{t} + \frac{1}{t+5} = \frac{1}{6}. \quad \text{Why?}$$

Finish the solution and check.

**14.** $A$ can reap one and one-half times as fast as $B$, and $B$ can reap one and two-thirds times as fast as $C$. How many hours will each require to reap a field of grain which all three together reap in 30 hours?

**15.** A vessel can be filled by one of its two pipes in two hours less time than by the other, and by both together in

* To get rational numbers which may be used as the sides of a right triangle substitute special values for $m$ and $n$ in the expressions $m^2 + n^2$, $2mn$, $m^2 - n^2$. Why is this true?

2 hours and 55 minutes. How long will it take each pipe alone to fill it?

**16.** Three men can finish a job in 1 hour and 20 minutes. Working alone, $C$ would take twice as long as $A$ and 2 hours longer than $B$. How long would it take each one to do it alone?

**17.** A steamer on account of poor coal makes 3 miles per hour less speed than usual and requires 165 hours more time to make a trip of 4840 miles. What is the usual speed?

**18.** A vessel steams at the rate of $11\frac{2}{55}$ miles per hour. It takes as long to steam 23 miles up the river as 47 miles down. Find the velocity of the river and time required for given distances.

**19.** $A$ sets out on a journey at the rate of $r_a$ miles per hour, and $m$ hours later $B$ sets out after him at the rate of $r_b$ miles per hour. In how many hours will $B$ overtake $A$ and how many miles will each have walked?

Make a careful drawing in each of the following and compare result with answer found by algebra:

**20.** Of two rectangles $R_1$ and $R_2$ the base of $R_1$ is 2 times that of $R_2$. The altitude of $R_1$ is $b$ units and that of $R_2$ is $a$ units. If $4\,b^2$ square units be subtracted from the area of $R_1$ and $a^2$ square units be subtracted from the area of $R_2$, the product of the areas will be $4\,a^2b^2$. Find the base of each.

**21.** In a right-angled triangle one side is one unit less than twice the other, and the hypotenuse is 17 units. What is the area of the triangle?

**22.** The altitude of a parallelogram is $27\,r$ inches less than 9 times the base, and the area is $22\,r^2$ square units. What are its dimensions?

**23.** A regular octagon (a figure with eight equal sides and eight equal angles) is formed by cutting off the corners of a square whose side is 1 foot. Find the side of the octagon.

**24.** If it takes A 7 days longer than it takes B to do a piece of work, and they both together take $8\frac{2}{5}$ days to do it, how long will it take each working alone to do it?

**25.** If it takes A $a$ days longer than B to do a piece of work, and they both working together take $t$ days to do it, how long will it take each working alone to do it?

**26.** Using the answers to Exercise 25 as formula, solve Exercise 24 by substituting the special values there given in the answers of Exercise 25. Compare results with those obtained when you solved Exercise 24.

**27.** Write a special problem of your own, which may be solved by using the answers to Exercise 25 as formula. Solve.

**28.** Solve your problem of Exercise 27 without using the answers of Exercise 25. Compare results. Compare processes, and explain the advantage of one method over the other.

**29.** If one side of a rectangle is $r$ units longer than $n$ times the other, and the area is $a$ square units, how long and how wide is the rectangle?

**30.** Write a special problem which may be solved by using the answers to Problem 29 as formulæ. Solve.

**31.** Two forces are acting at right angles to one another. The larger is $a$ units more than the smaller, and the resultant force is $b$ units more than the smaller. Find the amount of each force.

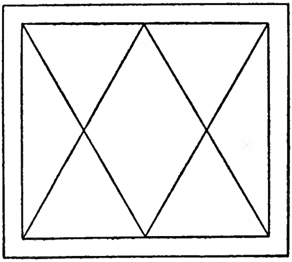
**32.** Write a special problem which may be solved by using answers to Problem 31 as formulæ. Solve.

**33.** In order to cover a table top which contains 972 square inches, it was found that a strip 9 inches wide must be cut from a square bought for the purpose in order that the remainder might just fit the table. How much goods had been purchased?

**34.** In making a centerpiece a lady found that $\frac{1}{2}$ of the length of the side of the linen square out of which she cut the circular center must be 1 inch more than the width of the lace for the border. She had $160\frac{2}{7}$ square inches of lace. What was the side of the square out of which she cut the center?

*Hint.* You may use $\frac{22}{7}$ for $\pi$ in this problem.

**35.** Write a problem about circles which will lead to a quadratic equation.



**36.** How wide would a rug 28 inches long have to be in order to have a border of 2 inches on all sides and a design of two equilateral triangles standing one on each side with its vertex on the opposite side, as shown in the drawing?

**37.** A class decorating committee was instructed to buy material of two colors for a table cover; the center to be of one color and the border of the other. The table was 8 inches longer than wide. They found material for the center that was just as wide as the table was long, so they bought a piece as long as the table was wide. In the other color they found material for the border which when divided into 4 strips would give a width which if the center were 2 inches wider would be $\frac{1}{3}$ the width of the center. They decided to purchase a length of this equal to the length of their cover and to use the extra 384 inches for pennants. The center material was $1 per yard, and the other was 85 cents per yard. What was the cost of the cover? By drawing show the most economical way for cutting the pennants out of the material left.

**38.** Write a problem about a rectangular flower bed with a border, the solution of which will involve a quadratic equation.

**39.** The first of two automobiles weighs $\frac{1}{2}$ ton less than the second. The number of miles per hour that the second is running is 6 more than 12 times the number of tons in the weight of the second, while the number of miles per hour that the first is running is 3 more than 11 times the number of tons in the weight of the second. The momentum of the second is 15 less than twice that of the first. Find the weight and speed of each.

**40.** Two boys starting from camp to fish at a point 7 miles away found themselves short of flies. They decided that one should ride to a store 10 miles distant and from there walk to the point, a distance of 4 miles. In order to reach the point at the same time it was necessary for the boy to ride at a rate 2 miles more per hour than twice the rate that the other boy walked and to walk $\frac{9}{13}$ of a mile per hour more than the other boy walked. What was the rate each walked? At what rate did the boy ride?

**41.** Of two sums of money which were drawing simple interest one was \$200 more than twice the other. The rate of interest on the smaller sum was \$2 more than twice the rate on the larger sum. The interest on the smaller sum was \$40 and the interest on the larger sum was \$24. What were the amounts invested? What was the rate of interest each drew?

**42.** Two spheres were to be decorated, one gilded and the other painted. The cost for gilding a square foot was $\frac{1}{2}$ cent more than 3 times the cost for painting. The number of square feet in the one to be painted was 12.4 more than 11 times the number of square feet in the one to be gilded. The price paid for the work was \$18.60 for painting the larger sphere and \$5.58 for gilding the smaller. What was the price per square foot for each? What was the radius of each? (Here let $\pi$ be 3.1.)

**Exercises and Problems for Review.**

The following exercises are suggested for review, to be assigned one or two a day with the work which follows.

Draw rectangles whose areas shall represent the following algebraic expressions; then write the factors of these expressions.

**1.** $a^2 + 2ab + b^2$.

**2.** $a^2 - 2ab + b^2$.

**3.** $a^2 - b^2$.

**4.** $ab + ac - ad$.

**5.** $a^2 + 5a + 6$.

**6.** $6a^2 - a - 12$.

Factor the following expressions.

7. $15p^2 + 5p - 5$.

8. $y^2 - 8y + 12$.

9. $a^2 + ac + ab + cb$.

10. $k^2 + 2ak + 3bk + 6ab$.

11. $9 - 6r + r^2$.

12. $9m^2n^2 - 24mn + 16$.

13. $t^2 - 9ts + 20s^2$.

14. $m^2 - 2m - 15$.

15. $6m^2 + mn - 2n^2$.

16. $3h^2 - 10h + 3$.

17. $15 - x - 6x^2$.

18. $dy - by - b + d$.

19. $r(r-1)^2 - r + 1$.

20. $ac - a - c + 1$.

21. $9 - 25$.

22. $(a-b)^2 - c^2$.

23. $m^2 - (p-q)^2$.

24. $a^2(b-1) - b^2(b-1)$.

25. $a^3 + b^3$.

26. $a^3 - b^3$.

27. $a^3 - b^3 - a + b$.

28. $8k^3 + h^3 - 2k - h$.

29. $r^2(m+n) - ar(m+n)$.

30. $s^2 - r^2 - s + r$.

31. $(a+b+c)^2 - a^2$.

32. $a^2 + ac + ad + cd$.

33. $a^3 - 3a^2b + 3ab^2 - b^3$.

34. $x^3 - 5x^2 + x - 5$.

35. $x^3 + 6x^2 + 4x + 24$.

36. $a^2 - c^2 - 2cd - d^2$.

37. $a^3 - b^3 - 2a^2 + 2b^2$.

38. $1 - r^2 - 2rs - s^2$.

39. $r - r^3 - 2r^2s - rs^2$.

40. $(a+d)^2 - 4(a+d) + 4$.

41. $a^2 + 2ab + b^2 - m^2 - 2mn - n^2$.

42. $(s-r)(a^2+b^2) - 2ab(s-r)$.

43. $p^2 - q^2 + 2pq - t^2 + 2pt - p^2$.

44. $x^2 - 9b^2 + 9y^2 - a^2 + 6ab - 6xy$.

45. $49x^2 + 4y^2 - 9a^2 - 16b^2 - 28xy - 24ab$.

46. $\frac{9}{5}c^2d - \frac{1}{5}d^3$.

47. $\frac{1}{8}m^3 + 8n^3$.

48. $a^2 - ab + \frac{1}{4}b^2$.

49. $\frac{4}{25}a^2 - \frac{1}{9}b^2$.

50. $\frac{1}{r^3} - \frac{8}{s^3}$.

51. $\frac{27}{m^3} + \frac{64}{n^3}$.

52. $5 - \frac{10}{r} + \frac{20}{r^2}$.

53. $\frac{2}{n^2} + \frac{3}{mn} - \frac{2}{m^2}$.

54. $3 - \frac{17}{a} + \frac{10}{a^2}$.

55. $\frac{9}{x^2} + \frac{30}{x} + 25$.

56. $(a+b)^2 + 5(a+b) + 6$

57. $3u^2 + 8u + 4$.

58. $(a+b)^2 - (a-b)^2$.

59. $(x+y)^2 - (x-y)^3$.

**60.** (*a*) $(a-b)^2-5(a-b)+6$.
(*b*) $3(m+n)^2-3(m+n)-18$.
(*c*) $4(p-q)^2+14s(p-q)+6s^2$.
(*d*) $6s(p-q)^2+21s^2(p-q)+9s^3$.
(*e*) $mnx^2+(mp+nq)x+pq$.

**61.** (*a*) $a^2-c^2-2cd-d^2$.
(*b*) $1-r^2-2rs-s^2$.
(*c*) $r-r^3-2r^2s-rs^2$.
(*d*) $r^2-r^4-2r^3s-r^2s^2$.
(*e*) $r^n-r^{n+2}-2r^{n+1}s-r^ns^2$.
(*f*) $r^{n-1}-r^{n+1}-2r^ns-r^{n-1}s^2$.

**62.** (*a*) $a^{2n}-b^{2m}$.
(*b*) $a^{2n+1}-ab^{2m}$.
(*c*) $a^{2n+2}-a^2b^{2m}$.
(*d*) $a^{2n+m}-a^mb^{2m}$.

**63.** (*a*) $\frac{4}{25}a^{n-2}-\frac{1}{9}a^nb^{2m}$.
(*b*) $\frac{9}{5}c^2d-\frac{1}{5}d^3$.
(*c*) $m^{2r}-6m^r-7$.
(*d*) $p^{2n+s}-9p^{n+s}-10p^s$.
(*e*) $p^{2n+s}q+p^{n+s}q^{m+1}-2p^sq^{2m+1}$.
(*f*) $a^3-b^3-2a^2+2b^2$.

**64.** (*a*) $r^2+5r+6-r-3$.
(*b*) $p^{2n}-4p^n+4-p^n+2$.
(*c*) $p^{2n}-4p^n+4+p^{3n}-8$.
(*d*) $a^{18}-b^{18}$.

**65.** Solve for the value of $p$ that will verify the equation

$$\frac{3p-16}{p}=\frac{5}{3}.$$

**66.** A river flows at the rate of 2 miles an hour, and a fisherman finds that he can row upstream a few miles in 6 hours, but that it takes him only 3 hours to come back. How fast does the fisherman row in still water?

**67.** Two balls, one of lead containing 3 cc. and the other of granite containing 7 cc., are set in motion. One moves at the rate of 14 cm. in 4 sec., and the other at the rate of 27 cm. in 5 sec. The momentum of the lead ball is 19.11 units more than that of the granite ball. If the density of lead is .76 more than 4 times that of granite, what is the density of each? What is the momentum of each?

**68.** A and B start from Lincoln in the direction of Omaha. At noon A has gone $\frac{3}{5}$ of the distance to Omaha, and B has gone $\frac{4}{7}$ of the distance, and they are just $1\frac{5}{7}$ miles apart. What is the distance from Lincoln to Omaha?

**69.** Factor:

$$9m^2 - 36n^2; \qquad 625t^2 - 1;$$
$$81r^2 - 225; \qquad 1 - 100b^2.$$

Simplify: $$\frac{x^2 - x - 20}{x^2 - 25} \cdot \frac{x^2 - x - 2}{2x + 8} \div \frac{x + 1}{x^2 + 5x}.$$

**70.** A man rows 9 miles downstream in 45 min.; he rows back, near the bank, where the current is only half strength, in $1\frac{1}{2}$ hours. What is the speed of the boat and of the stream?

**71.** In a three-digit number the tens digit exceeds the hundreds digit by 3. The units digit is 4 less than 2 times the hundreds digit. Interchanging the units and tens digit decreases the number by 45. What is the number?

**72.** Solve for $p$: $$\frac{1}{2} + \frac{2}{p + 2} = \frac{13}{8} - \frac{5p}{4p + 8}.$$

**73.** We are to construct a square and two rectangles such that the first rectangle has a length 2 units more than and $\frac{1}{6}$ less than the side of the square. The second rectangle has a length 4 units more than and a width $2\frac{2}{3}$ units less than the side of the square. The areas of the two rectangles are the same. Find the dimensions of each of the figures and make the drawing.

**74.** Factor the following:

| | |
|---|---|
| $a^2 + 3a + 2;$ | $4 - a^2;$ |
| $m^2 - 10m + 16;$ | $86^2 - 14^2;$ |
| $4p^2 + 4p + 1;$ | $(m+2)^2 + 2(m+2) + 1;$ |
| $5^2 + 2 \cdot 5 + 1;$ | $z^2 - 3z - 40.$ |

**75.** If $P$ is any point within the parallelogram $ABCD$, prove that the triangle $PAB$ plus the triangle $PCD$ equals one-half of the parallelogram.

**76.** If the velocity of a body flying through space is 144 miles a second, and a particle is thrown from it at right angles to its course, with a velocity of 130 miles a sec., what is the velocity of the particle at the moment it leaves the body?

**77.** A triangle $X$ is equal to a fixed triangle $T$, and has a common base with $T$. On what line or lines must the vertex of $X$ fall?

**78.** Simplify: $\dfrac{m}{2(m-n)} - \dfrac{m}{2(m+n)} + \dfrac{2m^4}{m^2(m^2-n^2)}.$

**79.** Two forces are acting at right angles to one another. The smaller of the two is 1 kilogram less than the larger, and the resultant force is 8 kilograms more than the larger. What is the intensity of the two forces?

**80.** If $P$ is a point on the side $AB$ of the parallelogram $ABCD$, and $Q$ is any point on the side $CD$, prove that triangle $PCD$ equals triangle $QAB$.

**81.** The side of an equilateral triangle is $s$; find its altitude and then its area.

**82.** Use the answer to Exercise 81 as a formula to find the following areas: $s = 8$; $s = 15$; $s = 10$.

**83.** Simplify: $\dfrac{1}{a+1} - \dfrac{a}{a^2-a+1} + \dfrac{a^2-4}{a^3+1}.$

**84.** Of two rectangles having the same width, the length of the first is $a$ units, and of the second is $b$ units. If a square of side $a$ units be subtracted from the first and a square of side $b$ units be subtracted from the second, the areas of the figures left will be equal. Find the width of each rectangle and draw.

**85.** Simplify: $\dfrac{x+y}{x-y}-\dfrac{x^3+y^3}{x^3-y^3}.$

**86.** A train is going at the rate of 56 miles an hour. A parcel is thrown from it at right angles to the direction of the train. The velocity of the parcel at the moment it leaves the train is 65 miles an hour. With what velocity was the parcel thrown?

**87.** Find the value of $r$ that will verify the following equation:

$$\frac{r-2}{r+2}=\frac{r-1}{r+1}+\frac{2r+4}{r^2-1}.$$

**88.** Factor:

$s^3+8r^3$; $27a^3-64b^3$;
$8t^3+27a^3$; $a^3b^3-8c^3$;
$m^3-1$.

**89.** A lead ball moving at the rate of 7 cm. per sec. stops an ivory ball moving at the rate of 8 cm. per sec. The balls measure respectively 13 cc. and 71 cc. If 1 cc. of lead weighs 6.8 grams less than 10 times the same amount of ivory, what is the density of each substance?

**90.** Write the formula for the area of a trapezoid, and solve for $h$. Use this formula to find the altitude of a trapezoid whose bases are 34 units and 45 units respectively, and whose area is 237 sq. units.

**91.** Factor:

$9a^2+30d+25$; $a^3+2a^2d+ad^2$;
$4m^2-12m+9$; $5r^2-10r+20$;
$x^2-x-6$; $2m^2+3mn-2n^2$;
$x^2+7x-30$; $2a^2+a-15$;
$r^3-8s^3$; $3r^2-17r+10$;
$27m^3+64n^3$; $(m+n)^2-10(m+n)-39$;
$6x^2+x-12$; $8t^3-1$.

## CHAPTER XIV

## VARIABLES. CONSTANTS. FUNCTIONS OF VARIABLES. GRAPHIC REPRESENTATION OF FUNCTION OF A SINGLE VARIABLE

**90. Variables.** Up to this time in your work algebra has treated almost entirely of literal arithmetic. The material has been furnished from several sources, principally geometry and physics. You will continue to use quantity furnished by these as a basis for your discussions, but you will now study quantities in their relation to one another. If there is a change in any element of a geometric figure, there are consequent changes in other elements which depend upon it. If there is a change in any magnitude in the physical world, there must be consequent changes in other magnitudes which depend upon it. It is this dependency of quantity upon quantity that we shall now investigate.

*Definition.* A quantity whose magnitude changes is called a *variable quantity,* or simply a *variable.*

Examples of variables are found everywhere about us; we mention a few.

The height of a growing tree.
The weight of a growing apple.
The temperature of outdoor air.
The barometric pressure.
The distance of a moving train from a station.
The speed of a falling body.
The volume of an expanding soap bubble.

Such examples may be multiplied indefinitely.

**91. Constants.** A quantity whose magnitude is fixed is called a *constant quantity*, or simply a *constant*.

In nature almost everything is changing or variable, so that it would be hard to find anything that is absolutely fixed or unchanging. But many things are very nearly so.

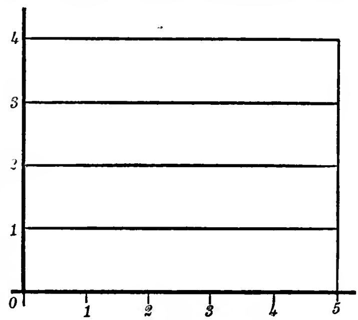
For example pick out any object about you, in the room or out of doors. What can you say as to its size, form, weight, color?

In algebra, then, constants are quantities which for the time being, or in connection with a certain problem, we suppose to be fixed in value.

**92. Functions of Variables.** Many variable quantities depend upon one or more other variable quantities. We give some examples.

*Example* 1. Draw two lines at right angles to each other, and on them mark scales on which to count the bases and altitudes of rectangles. Using these same lines and counting from the origin each time, draw the following rectangles:

Base 5 and altitude 1.
Base 5 and altitude 2.
Base 5 and altitude 3.
Base 5 and altitude 4.




Examine carefully these rectangles. You have kept the base the same and have changed the altitudes. Did anything else about your rectangles change? Using the customary notation to represent the number of units in the area, base and altitude of a rectangle, we have

$$a = bh.$$

We see that as we varied the altitude $h$ and allowed the base $b$ to remain constant, the value of the area, $a$, changed. That is, $a$ takes on a new and definite value to correspond to each new and definite value that we give to $h$. In other words, $a$ depends upon $h$ for its value while $b$ remains constant.

*Definition.* When one variable depends upon another for its value, we say that the first is a **function** of the second. When a quantity depends upon several others, it is said to be a function of these several quantities. In the equation above, to express the thought brought out by our drawing, we say that $a$ is a function of $h$. This is written:

$$a = f(h). \quad (a \text{ is a function of } h.)$$

Both $h$ and $a$ are variables, and to distinguish them, we call $h$ the *independent variable*, and $a$ the *dependent variable.*

Make a drawing similar to the one just made, but considering $h$ the constant, $b$ the independent variable, $a$ the dependent variable. In this case, the equation is the same,

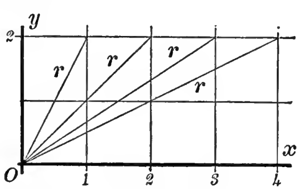
$$a = bh,$$

but

$$a = f(b).$$

Make a drawing changing both the base and altitude at the same time. Still $a = bh$, but now $a = f(b, h)$, that is, $a$ is a function of $b$ and $h$. Here $a$ depends upon both $b$ and $h$ for its value.

Explain how the same idea may be illustrated by a triangle; a trapezoid.




*Example* 2. Draw a right-angled triangle, calling the sides $x$, $y$, $r$. Considering $y$ as constant, using the same origin, draw triangles, letting $x = 1, 2, 3, 4$, etc. See the adjacent figure, which is drawn for $y = 2$.

Examining the drawings what do you find as to the length of $r$? As you have learned $r^2 = x^2 + y^2$; in this example $r^2 = x^2 + 4$. Therefore $r = f(x)$.

Consider $x$ as the constant and $y$ as the independent variable, make a drawing. State the equation of functional relation of the variables.

Make drawings when both $x$ and $y$ vary. Write the equation of functional relation of the variables.

Make drawings to illustrate $x = f(r)$, taking $y = 1$. Again, take $y = 2$. Again, take $y = 3$.

*Example* 3. Draw circles using as radii 1, 2, 3, 4, etc. What is the area of each? If $a$ is the number of square units in the area, and $r$ the number of linear units in the radius, then

$$a = \pi r^2. \quad \text{Here } a = f(r).$$

As usual $\pi$ denotes the numerical constant whose value is about $\frac{22}{7}$; more accurately, $\pi = 3.1416 -$.

*Example* 4. Draw a circle. At the center draw angles of 10°, 20°, 30°, etc. These are called **central angles.** Pick out various magnitudes in the figure which change because the central angle is changed.

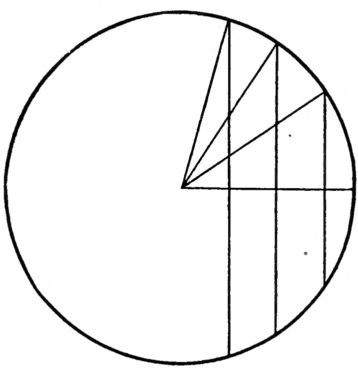
*Definition.* A portion of the circumference of a circle is called an **arc.** The portion of the area of the circle inclosed between the arms of a central angle and the arc is called a **sector.**

If $s$ stands for the number of square units in the area of the sector, and $a$ for the number of degrees in the angle, without writing the equation, we may express the dependency of the magnitudes $s$ and $a$ by writing $s = f(a)$. In this case the radius of the circle is the constant.

If $c$ stands for the number of units in the length of the arc, write the statement of dependency of $c$ on $a$.

Mark arcs of various lengths on the circumference. Join the ends to the center, thus forming angles. In this case what is the independent variable? The dependent variable? State this by means of symbols.

*Example* 5. Draw a circle and draw a line across it terminated at each end by the circumference. Such a line is called a **chord** of the circle. Draw several chords parallel to the one you have drawn, each one nearer to the center. Thus:



The distance from the center to the chord is represented by the perpendicular line drawn from the center to the chord. By testing you will find that this perpendicular bisects the chord.

The area of the portion of the circle bounded by the arc and its chord is called a **segment** of the circle.

In the drawing, what magnitudes change because the chord is drawn nearer to the center? Assign letters to the various magnitudes, and express the dependency by symbol language. Do all magnitudes grow larger because the magnitudes upon which they depend grow larger?

Draw the radii to the ends of the chords. Then we have the equation

$$(\tfrac{1}{2}c)^2 = r^2 - d^2,$$

where $c$ stands for the number of units in the length of the chord, $r$ for the number of units in the radius, and $d$ for the number of units from the center to the chord. Explain how we get this equation.

In our drawing

$$c = f(d).$$

Compare this with Example 2.

Suppose you draw chords such that each one is longer than the one preceding. What can you say about the distances from the center? Express this algebraically.

*Examples of Functional Relations not Geometrical.*

*Example* 1. Let $d$ = the number of units of distance.
$v$ = the number of units of speed.
$t$ = the number of units of time.

Then the relation is

$$d = vt. \qquad \text{Here } d = f(v, t).$$

The last equation is read: "$d$ is a function of $v$ and of $t$."

*Example* 2. Let $R$ denote the reading of a barometer at a height $h$ above sea level. When a barometer is carried up a mountain, it can be seen that the reading changes, but we do not know precisely the law connecting $R$ and $h$. We can say

$$R = f(h). \quad (R \text{ is a function of } h.)$$

But we do not know just what $f(h)$ is. So we merely indicate that $R$ depends on $h$ by writing $R = f(h)$.

*Example* 3. The speed of a falling stone depends on the time since it started to fall.

Let $s$ = number of units in speed at any time,
$t$ = number of units of time since stone was dropped.

Then we know from experiment that

$$s = 16t^2. \qquad \text{Here } s = f(t).$$

This is at least a very close approximation; it is not the exact value of $s$ in terms of $t$, but serves for most purposes.

*Example* 4. The interest on a principal depends on the amount invested, the rate of interest, and on the time during which it is invested.

Let $I$ = the number of dollars interest (simple or compound).
$P$ = the number of dollars principal.
$r$ = the yearly rate of interest.
$t$ = the number of years.

Then $I = Prt$. Here $I = f(P, r, t)$.

That is, $I$ is a function of three variables, $P$, $r$, $t$.

*Example* 5. The temperature of the outer air at a given place on a given day depends on the hour of the day.

Let $T$ = number of degrees of temperature,
$t$ = number representing time of day, counted from any particular moment, as noon, say.

Then $T = f(t)$.

But here we do not know just how to express $T$ in terms of $t$.

**Exercises.***

In the following exercises give the exact form of the function.

1. The distance passed over by a train going 30 miles an hour for $t$ hours. Here $d = f(t)$. What is $f(t)$?

2. The cost of $n$ yards of cloth at \$2 a yard. Here $c = f(n)$. What is $f(n)$?

3. The time required to go one mile at the rate of $v$ feet per second. Here $t = f(v)$.

4. The cost of building a cement walk $l$ feet long and $w$ feet wide at 14 cts. a square foot. Here $c = f(l, w)$.

5. The cost of excavating a cellar $x$ feet long, $y$ feet wide and $z$ feet deep. Here $c = f(x, y, z)$.

6. Cost of making $n$ photographs, if the cost is 30 cts. to make the negative and 10 cts. each to print, mount, and finish the pictures.

In the following first express the required quantity as a function and then give the precise form of the function when possible.

7. The cost of using 3 electric lights at 1 ct. each per hour.

* Chapter IV of "Geometry" may well be introduced while the class is working these exercises.

8. The price of a quantity of lumber at 9 cts. per board foot.

9. The cost of paving a street at $2 a square yard.

10. The amount of lumber needed to make a box.

11. The amount of a given principal which is placed at 5% simple interest.

12. The weight of a solid rectangular block of material. (Depends on volume and density.)

13. The length of an iron rail which expands when heated. (Note the open space between the adjoining rails of a track.)

14. The volume occupied by a cubic foot of water which expands when heated. (Hence the need of an expansion tank in heating a house by hot water. What is the object of such a tank?)

**93. Functions of a Single Variable.** The area of a rectangle is

$$a = bh.$$

Here $a$ is a function of two quantities $b$ and $h$, either of which may vary. Suppose we keep $b$ fixed and let $h$ alone vary. Then $a$ is a function of the single variable $h$. We can write

$$a = f(h),$$

since $b$ has a fixed numerical value.

We shall now consider some functions of a single variable; any letter may be used to indicate the quantity which is varying; then all other letters contained in the expression with which we deal will stand for fixed numbers.

As a general rule we use the first few letters of the alphabet to indicate fixed quantities and the last letters to indicate variables. But quite often the initial letter of a word is used to indicate a constant or a variable without regard to this rule.

**94. The Function Notation.**

*Example.* Consider the quantity $x^2 - 4x + 3$.

It is a function of $x$, so we may write

$$f(x) = x^2 - 4x + 3.$$

Such a function might have come from a variety of sources. It might represent the area of a rectangle whose dimensions are $x - 1$ and $x - 3$. For $(x - 1)(x - 3) = x^2 - 4x + 3$.

It might be the weight of $x - 1$ cubic feet of stone which weighs $x - 3$ pounds per cubic foot. And so on. Let the student give other illustrations.

We shall now use the following notation:

$f(1)$ shall mean the value of $f(x)$ when $x$ is replaced by 1.

$f(-2)$ shall mean the value of $f(x)$ when $x$ is replaced by $-2$ and so on. Thus

$$f(1) = (1)^2 - 4(1) + 3 = 1 - 4 + 3 = 0.$$
$$f(-2) = (-2)^2 - 4(-2) + 3 = 4 + 8 + 3 = 15.$$
$$f(0) = (0)^2 - 4(0) + 3 = 0 - 0 + 3 = 3.$$
$$f(a) = a^2 - 4a + 3.$$
$$f(-b) = (-b)^2 - 4(-b) + 3 = b^2 + 4b + 3.$$

To calculate $f(-1) \cdot f(2)$ we would proceed as follows:

$$f(-1) = (-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$$
$$f(2) = 2^2 - 4 \cdot 2 + 3 = 4 - 8 + 3 = -1.$$
$$f(-1) \cdot f(2) = 8 \cdot (-1) = -8.$$

**Exercises.**

1. $f(x) = 1 - x$. Calculate the numerical value of $f(x)$ when $x = -3, -2, -1, 0, 1, 2, 3$. Arrange your results in tabular form.

Thus: $x = -3,\ -2,\ \ldots\ldots\ldots\ldots,\ 3.$
$f(x) = 4,\ 3,\ \ldots\ldots\ldots\ldots,\ -2.$

2. $f(x) = 3 - 2x$. Calculate $f(x)$ when $x = -2, -1, 0, 1, 2$.

Arrange in tabular form as in Exercise 1.

3. $f(x) = 4 + x$. Calculate $f(x)$ when $x = -4, -2, 0, 2, 4$.
4. $f(x) = 2x - 5$. Calculate $f(4), f(2), f(0), f(-2)$.
5. $f(x) = x^2$. Calculate $f(0), f(1), f(2), f(3), f(1) \cdot f(3), f(6) \div f(2)$.
6. $f(x) = 3x^2 + 2x - 1$. Calculate $f(0), f(2), f(-1), f(-2) \cdot f(3), f(1) \div f(-3)$.
7. $f(x) = x^3 - 3x$. Calculate $f(-1), f(1), f(-3), f(3), 2f(5) \div f(-5)$.

**8.** $f(x) = (x^2 + 1)^2$. Calculate $f(\frac{1}{2})$, $f(-\frac{1}{3})$, $f(1) \cdot f(\frac{1}{4})$, $f(\frac{1}{3}) \div f(\frac{1}{6})$.

Sometimes we have to deal with two or more functions in the same problem. For example $(x^2 - 1)(x^2 + 1)$ can be regarded as the product of the two functions $x^2 - 1$ and $x^2 + 1$.

If we say that $f(x)$ stands for the function $x^2 - 1$, we cannot say $f(x)$ stands for $x^2 + 1$ also. We would say, for example, $F(x)$ stands for $x^2 + 1$, using the capital letter to indicate that $F(x)$ is a different expression from $f(x)$. Often Greek letters are used; thus, to indicate a number of different functions of $x$, we might use the symbols

$$f(x), \quad F(x), \quad \phi(x), \quad \psi(x), \text{ etc.}$$

These are read:
The $f$ function of $x$.
The capital $F$ function of $x$.
The Phi function of $x$.
The Psi function of $x$, etc.

**Exercises.*** (*Note.* The symbol $\equiv$ is often used in place of the words "stands for," or "is identical with.")

**1.** If $f(x) \equiv \dfrac{x^2 + x - 6}{x^2 - x - 20}$, $F(x) \equiv \dfrac{x^2 - 3x - 10}{x^3 + 27}$,

$$\phi(x) \equiv \frac{x^2 - 3x + 9}{x^2 - 4},$$

find the value of $f(x) \cdot F(x) \cdot \phi(x)$.

**2.** If $f(r) \equiv \dfrac{r^2 - 6}{r^3 + 8} + \dfrac{4}{5r^2 - 10r + 20} - \dfrac{1}{r + 2}$,

solve the equation $f(r) = 0$.

* At this point it would be well to take up the study of the Trigonometric Functions in Part III, Chapter VI of Geometry, up to the point there indicated.

**3.** If $f(x, y, z) \equiv \dfrac{x}{z(x-y)}$, $F(x, y, z) \equiv \dfrac{z}{x(y-x)}$,
find the value of $f(x, y, z) - F(x, y, z)$.

**4.** If $f(x, y, z) \equiv \dfrac{xy}{(z-x)(z-y)}$, $F(x, y, z) \equiv \dfrac{yz}{(x-y)(x-z)}$,

$\phi(x, y, z) \equiv \dfrac{zx}{(y-z)(y-x)}$,

find the value of $f(x, y, z) + F(x, y, z) + \phi(x, y, z)$.

**5.** If $f(r, s) \equiv \dfrac{2rs}{r+s} - r$, $F(r, s) \equiv \dfrac{1}{s} - \dfrac{1}{r-2s}$,

find the value of $\dfrac{f(r, s)}{F(r, s)}$.

**6.** If $f(x) \equiv x^2 - 4x + 3$, $F(x) \equiv x^4 - 10x^2 + 9$,

write the value of $\dfrac{f(x)}{F(x)}$.

**7.** If $f(r) \equiv r^3 - 6r^2 + 5r$, $F(r) \equiv r^2 + 2r - 35$,

write the value of $\dfrac{f(r)}{F(r)}$.

**8.** If $f(x) \equiv \dfrac{d^2+x^2}{d^2-x^2}$, $F(x) \equiv \dfrac{d^3-x^3}{d+x}$, $\phi(x) \equiv \dfrac{d^3+x^3}{d-x}$,

find the value of $f(x) + F(x) + \phi(x)$.

**9.** If $f(u) \equiv \dfrac{u^3+8}{u^3-8}$, $F(u) \equiv \dfrac{u^2+2u+4}{u^2-2u+4}$,

find the value of $f(u) \cdot F(u)$.

## 95. Graphic Representation.

*Example.* Given the function $f(x) = 3 - 2x$.

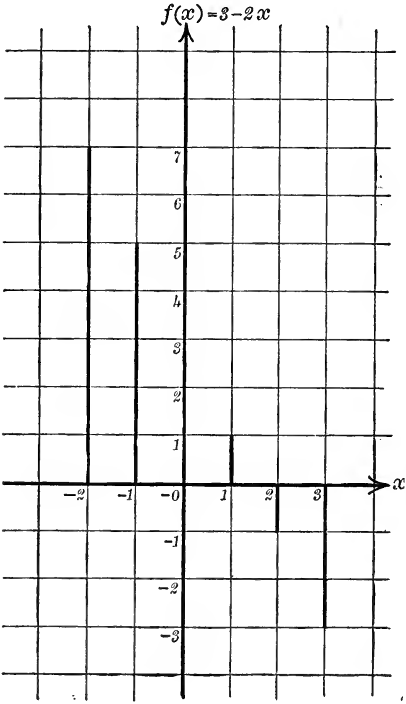
Make a table as in Exercise 1, § 94, showing the values of this function for a number of assumed values of $x$, thus:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x =$ | $-2$, | $-1$, | 0, | 1, | 2, | 3, etc. |
| $f(x) =$ | 7, | 5, | 3, | 1, | $-1$, | $-3$, etc. |

| $x$ | $f(x)$ |
|---|---|
| $-2$ | 7 |
| $-1$ | 5 |
| 0 | 3 |
| $+1$ | 1 |
| $+2$ | $-1$ |
| $+3$ | $-3$ |

Usually it is better to write these in columns, beginning with the negative values, in this case $x = -2$ and going step by step to $x = 3$; this gives the adjacent column of values.

An inspection of these columns shows that as $x$ increases from $-2$ to 3, $f(x)$ decreases from 7 to $-3$.




We now give a scheme for representing these values by a diagram from which we can see at a glance how $f(x)$ changes as $x$ changes.

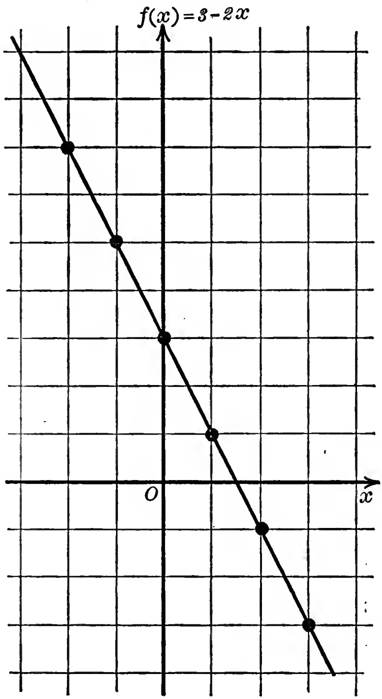
Draw two lines at right angles. On one of them, say the one running from left to right, mark off a number scale to show values of $x$, as in the figure; the zero of the scale is at the intersection of the two lines, and positive numbers are usually marked off to the right.

At $-2$ on the $x$-scale, or on the $x$-axis as it is usually called, lay off a perpendicular upward 7 units long. This indicates the value of $f(x)$ when $x = -2$.

At $-1$ lay off a perpendicular 5 units long, showing the value of $f(x)$ when $x = -1$; at 0 and 1 lay off perpendiculars upward 3 and 1 units long respectively. The perpendiculars are getting shorter, showing that the values of $f(x)$ are decreasing.

At $x = +2$, $f(x) = -1$, from our table. To show this on our diagram at 2 on the $x$-axis draw a perpendicular downward one unit long. We shall understand that when the perpendicular is drawn downward the function has a negative value. Similarly when $x = 3$, $f(x) = -3$; hence at 3 on the $x$-axis draw a perpendicular downward 3 units long.

It is now evident on inspection that the free ends of these perpendiculars all lie on a straight line. In the following figure these free ends are marked by heavy dots and the straight line through them is drawn.




**Exercise.** On a sheet of cross-ruled paper draw accurately the above figure, showing the perpendiculars, the dots and the straight line. Now draw in some more perpendiculars showing the values of $3-2x$ for other values of $x$, such as $x=-\frac{3}{2}, -\frac{1}{4}, \frac{3}{4}, 2\frac{1}{2}$.

Where do the ends of these perpendiculars fall? What does the straight line tell you about the function $3-2x$?

From your diagram read off the values of $3-2x$ when $x=-1\frac{3}{4}, -\frac{1}{3}, \frac{1}{3}, 1\frac{1}{2}, 2\frac{2}{3}$. Check these by direct calculation.

State a rule for finding the value of $3-2x$ for any assumed value of $x$ from the diagram.

*Definition.* The straight line drawn through the dots in the above diagram is called the **graph** of the function $3-2x$.

This graph is a geometric picture which shows us the value of $3-2x$ for the values of $x$ coming within the limit of the diagram. For larger values of $x$, to the left or right, we would have to use a larger paper, or a smaller scale.

**Exercises.** Draw graphs showing values of the following:

1. $3 + 2x$, from $x = -3$ to $x = 2$.
2. $2x - 3$, from $x = -2$ to $x = 4$.
3. $2 - \frac{1}{2}x$, from $x = -2$ to $x = 6$.

*Graph of $f(x)$ when $f(x) = ax + b$.* It will be observed that in all of the above exercises the graph is a *straight line.*

We would infer from this, that *whenever the function $f(x)$ has the form $ax + b$, its graph is a straight line.* For this reason $ax + b$ is called a *linear function* of $x$.

We shall assume the correctness of this rule, without stopping to give a complete proof. Can you give such a proof?

**Exercises.** Using the rule just stated draw the graphs of the following. Notice that you will need only two points to fix the entire graph. Hence only two values of the function need be calculated.

1. $x - 1$.
2. $x + 2$.
3. $2x - 5$.
4. $1 - 2x$.
5. $3 + \frac{1}{2}x$.
6. $2 - 3x$.

In place of the letter $x$ any other letter might be used. Draw the graph of the following functions:

7. $f(t) = t + 3$.
8. $f(r) = 2r - 1$.
9. $f(q) = 8 - 6q$.
10. $f(y) = 5 - 3y$.
11. $f(h) = \frac{1}{3}h + 6$.
12. $F(w) = \frac{1}{2}(w - 2)$.

## 96. Graph of Quadratic Functions.

*Definition.* A function whose form is

$$f(x) = ax^2 + bx + c$$

is called a **quadratic function** of $x$.

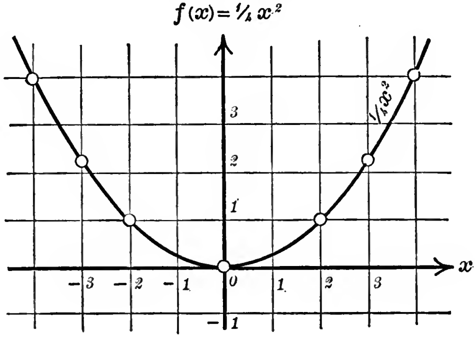
Here $a$, $b$, $c$ are given numbers, and either $b$ or $c$, or both, may be zero; but $a$ must *not* be zero, for then our function would be linear.

The graph of such a function is obtained precisely as in the case of those just studied. It is not a straight line but has nevertheless a very simple form.

*Example* 1. $f(x) = \frac{1}{4}x^2$. Here $a = \frac{1}{4}$, $b = 0$, $c = 0$.

We first make our table of values of $f(x)$ for assumed values of $x$.

| $x$ | $f(x)$ |
|---|---|
| $-4$ | 4 |
| $-3$ | $2\frac{1}{4}$ |
| $-2$ | 1 |
| $-1$ | $\frac{1}{4}$ |
| 0 | 0 |
| 1 | $\frac{1}{4}$ |
| 2 | 1 |
| 3 | $2\frac{1}{4}$ |
| 4 | 4 |




Notice that, on account of the $x^2$, we get the same value for $f(x)$ when $x = -4$ and when $x = +4$; similarly when $x = \pm 3$, $\pm 2$, etc.

We now represent these values of $f(x)$ by perpendiculars, as before. The free ends of these perpendiculars we mark by small circles. Then we draw a smooth curve through the circles and so we have the graph of the function $\frac{1}{4}x^2$.

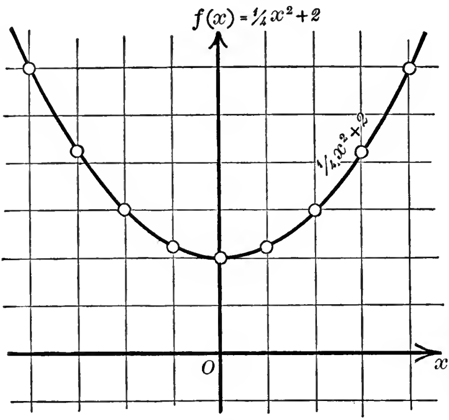
If we take any value of $x$, as $x = 2\frac{1}{2}$, calculate from it the value of $\frac{1}{4}x^2$, and draw a perpendicular of this length at the point $x = 2\frac{1}{2}$ on the $x$-axis, the end of this perpendicular marks another point on the curve. So we might fill in any number of points besides those already shown. As a rule we draw only enough perpendiculars to clearly outline the curve.

**Exercise.** From the diagram, read off the values of $\frac{1}{4}x^2$ when $x = -3\frac{1}{2}$, $-1\frac{1}{4}$, $2\frac{3}{4}$. Check these by direct calculation. What does the curve tell you about the function $\frac{1}{4}x^2$?

From the diagram estimate the values of $x$ for which $\frac{1}{4}x^2 = 2, \frac{1}{2}, 3\frac{1}{2}$.

*Example* 2. $f(x) = \frac{1}{4}x^2 + 2$. Here $a = \frac{1}{4}$, $b = 0$, $c = 2$.

Notice that this function is just two units greater than the function in Example 1.

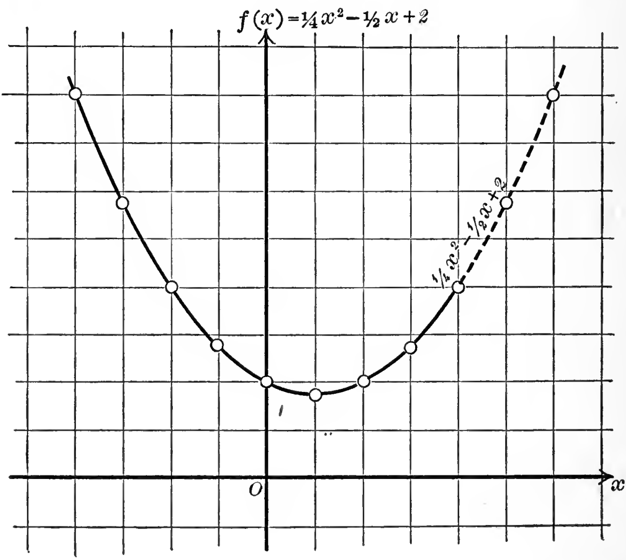



Making the table:

| $x$ | $f(x)$ |
|---|---|
| − 4 | 6 |
| − 3 | $4\frac{1}{4}$ |
| − 2 | 3 |
| − 1 | $2\frac{1}{4}$ |
| 0 | 2 |
| 1 | $2\frac{1}{4}$ |
| 2 | 3 |
| 3 | $4\frac{1}{4}$ |
| 4 | 6 |

Drawing the graph as before we see that we have exactly the same curve, only raised up two units.

*Example* 3. $f(x) = \frac{1}{4}x^2 - \frac{1}{2}x + 2.$ Here $a = \frac{1}{4}$, $b = -\frac{1}{2}$, $c = 2$.




| $x$: | − 4 | − 3 | − 2 | − 1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$: | 8 | $5\frac{3}{4}$ | 4 | $2\frac{3}{4}$ | 2 | $1\frac{3}{4}$ | 2 | $2\frac{3}{4}$ | 4 |

The above figure shows the graph. Suppose we extend the figure farther to the right; $x = 5, f(x) = \frac{23}{4}$; $x = 6, f(x) = 8$.

This is shown by the dotted part of the figure.

**Exercise 1.** We can easily verify now that this is the same curve as before. Draw these three curves accurately on separate sheets of thin paper. Superpose the curves and see if you can make any one of them cover up either of the others.

**Exercise 2.** From the last graph estimate the value of $\frac{1}{4}x^2 - \frac{1}{2}x + 2$, when $x = -3\frac{1}{2}, -1\frac{1}{4}, 4\frac{1}{2}$. Check by direct calculation.

Also estimate the value of $x$ for which $\frac{1}{4}x^2 - \frac{1}{2}x + 2$ has the value 7; 5; 3.

*Definition.* The curve in the above diagram is called a **parabola.** Such curves are of frequent occurrence. When a telephone wire sags in the middle, its form is that of a nearly flat parabola. Skyrockets or projectiles from guns follow parabolic curves. Most comets, as they sweep through the solar system, describe huge parabolas.

We shall ask the student to verify, in working the exercises below, that the graph of a quadratic function is always a parabola; that quadratic functions having the same values of $a$ but different values of $b$ and $c$, give the same parabola, but in different positions; that the parabola points downward when $a$ is a positive number; upward, when $a$ is a negative number. The curve is symmetrical with respect to a line parallel to the function axis and distant $-\frac{b}{2a}$ from it.

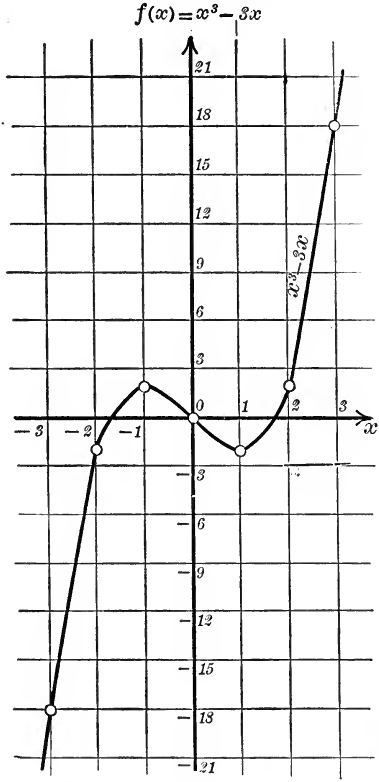
**Exercises.** Draw the graph of the following functions; superpose the curves in each exercise and show that they can be made to coincide.

1. $\frac{1}{2}x^2$; $\frac{1}{2}x^2 + 1$; $\frac{1}{2}x^2 - x + 1$.
2. $-\frac{1}{2}x^2$; $1 - \frac{1}{2}x^2$; $1 - x - \frac{1}{2}x^2$.
3. $2x^2$; $2x^2 - 4x$; $2x^2 - 4x + 2$.
4. $-2x^2$; $-2x^2 + 4x$; $-2x^2 + 4x - 2$.

5. $x^2$; $x^2 - 7x$; $x^2 - 7x + 10$.

6. $-x^2$; $-x^2 + 7x$; $-x^2 + 7x - 10$.

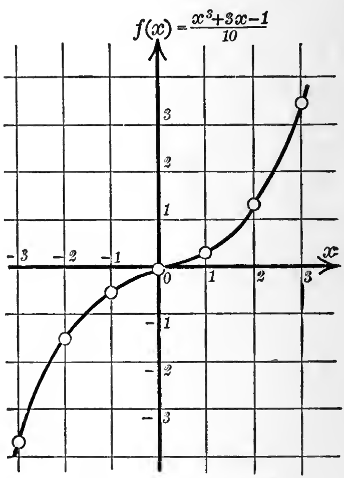
**97. Graphs of Other Functions.** By following step by step the process shown in the preceding examples, we can draw the graphs of many other functions. We shall give two illustrations.




*Example* 1. $f(x) = x^3 - 3x$.

Table of Values.

| $x$ | $f(x)$ |
|---|---|
| $-3$ | $-18$ |
| $-2$ | $-2$ |
| $-1$ | $2$ |
| $0$ | $0$ |
| $1$ | $-2$ |
| $2$ | $2$ |
| $3$ | $18$ |




*Example* 2. $f(x) = \dfrac{x^3 + 3x - 1}{10}$.

Table of Values.

| $x$ | $f(x)$ |
|---|---|
| $-3$ | $-3.7$ |
| $-2$ | $-1.5$ |
| $-1$ | $-0.5$ |
| $0$ | $0.1$ |
| $1$ | $0.3$ |
| $2$ | $1.3$ |
| $3$ | $3.5$ |

**Exercises.**

1. From the figure under Example 1 estimate the following values:

(*a*) What is the value of $x^3 - 3x$ when $x = \frac{1}{2}$? When $x = 1.5$? When $x = 2.75$? When $x = -2.25$?

(*b*) For what values of $x$ is $x^3 - 3x$ equal to zero? Equal to 1? Equal to $-1$? Equal to 10?

2. From the figure under Example 2 estimate the following values:

(*a*) What is the value of $\frac{x^3 + 3x - 1}{10}$ when $x = 1.5$? When $x = 2\frac{1}{3}$? When $x = -2{\cdot}75$?

(*b*) For what values of $x$ is $\frac{x^3 + 3x - 1}{10}$ equal to 1? Equal to $-1$? Equal to 3? Equal to $-3$? Equal to 0?

3. Explain how you would obtain an approximate value of $x$ which satisfies the equation $x^3 + 3x - 1 = 0$.

4. Draw graphs of $x^3$; of $\frac{x^3}{8}$; of $1 - x^3$.

5. Draw the graph of the function

$$f(x) = x^3 - 3x^2 - 2x + 6.$$

From your figure read off as exactly as possible the solutions of the equation

$$x^3 - 3x^2 - 2x + 6 = 0.$$

6. Solve graphically the equation $2x^3 - 5x + 1 = 0$.

**98. Summary.**

*Definition.* A variable quantity is one which, in a given problem, may take on different values.

A constant quantity is one which, in a given problem, is supposed to be fixed in value.

There are two kinds of variables; independent and dependent.

The independent variable may be assigned values at will.

The dependent variable takes its value from the independent variable.

A function of a variable is a quantity whose value depends on the value of the variable. A function is a dependent variable.

To indicate that a quantity depends on a variable, say $x$, that quantity is represented by the symbol $f(x)$, which is read "function of $x$." Then the symbol $f(a)$ means the value of the function when $x$ equals $a$.

A function of a variable may be represented graphically by a diagram in which one scale shows the values of the variable and the other scale shows the corresponding values of the function.

When $f(x)$ has the form of $ax + b$, its graph is a straight line. For this reason $ax + b$ is called a linear expression.

When $f(x)$ has the form $ax^2 + bx + c$, its graph is a parabola.

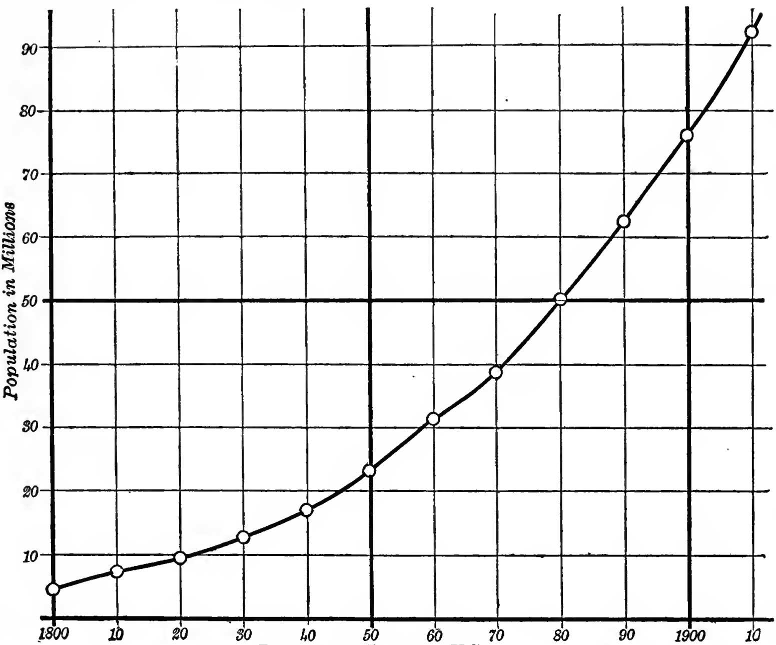
# CHAPTER XV

## USES OF THE GRAPH

**99. Graphic Representation of Measurements of a Variable Quantity.**

*Example* 1. *Population Curve.* From the Census Reports we find the population of the United States as follows:

| Year | Population | Year | Population |
|---|---|---|---|
| 1800 | 4.3 million | 1860 | 31.4 million |
| 1810 | 7.2 million | 1870 | 38.6 million |
| 1820 | 9.6 million | 1880 | 50.2 million |
| 1830 | 12.9 million | 1890 | 62.6 million |
| 1840 | 17.1 million | 1900 | 76.3 million |
| 1850 | 23.2 million | 1910 | 92.2 million |




*Population Curve for U.S.*

By inspection of the figures we see that there has been a steady rise in population. But a graphic representation of the data, as in the above figure, brings this out more clearly.

The height of each dot above the line running from right to left represents the population for the corresponding census year; by drawing a smooth curve through the dots we get the "population curve." In drawing this curve we assume that the population changes gradually during each decade.

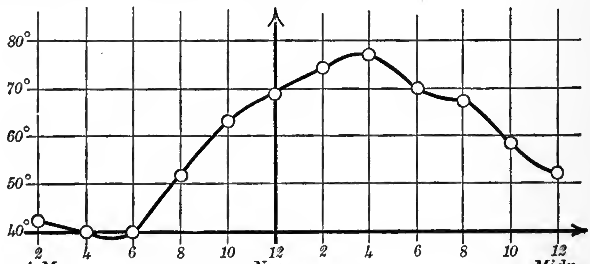
**Exercise.** From this diagram estimate the population in 1845. In 1855. In 1868. In 1883.

Notice that the gain in population during any decade is greater than during the preceding decade. Can you explain why? Was this true from 1860 to 1870? Why not?

*Example* 2. *Temperature Curve.* From the records of the U. S. Weather Bureau at Lincoln, Neb., we take the following temperatures for April 26, 1910.

| Time | Temperature | Time | Temperature |
|---|---|---|---|
| 2 A.M. | + 42° F. | 2 P.M. | + 74° F. |
| 4 A.M. | 40° F. | 4 P.M. | 77° F. |
| 6 A.M. | 40° F. | 6 P.M. | 70° F. |
| 8 A.M. | 52° F. | 8 P.M. | 67° F. |
| 10 A.M. | 63° F. | 10 P.M. | 58° F. |
| Noon | 69° F. | Midnight | 52° F. |

From these readings we construct the following figure. Notice that we have to represent temperatures from 40° to 77°. We therefore start our degree scale at 40° in place of at 0°. In drawing a smooth curve through the dots we assume that the temperature changes gradually from one value to the next.

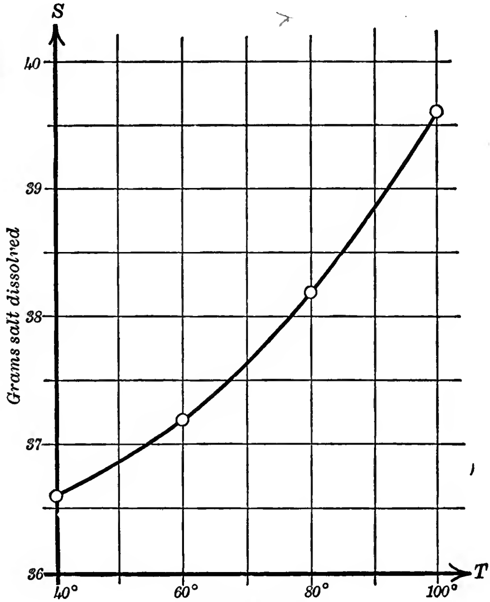



*Temperature Curve*

**Exercise.** From the temperature curve estimate the temperature at 7 A.M. At 9.30 A.M. At 5.30 P.M. At 9 P.M. What were the highest and lowest temperatures during the day? When did these occur?

*Example* 3. *Solubility Curve.* By experiment we can find the number of grams of common salt that can be dissolved in 100 grams of water when the water is heated to different temperatures. So we get the following data, from which the solubility curve is constructed.

| Temperature of Water | Weight of Salt |
|---|---|
| 40° C. | 36.6 gms. |
| 60° C. | 37.2 gms. |
| 80° C. | 38.2 gms. |
| 100° C. | 39.6 gms. |



*Solubility Curve*

**Exercise.** From the diagram read off the number of grams of salt that can be dissolved in 100 gms. water heated to 50° C. To 85° C. To 97° C. Find the amount of salt that can be dissolved in 500 gms. water heated to 65° C. Find the number of ounces of salt that can be dissolved in a pint of water heated to 150° F.

**Exercises.** Draw curves showing the following sets of data.

1. Enrollment in public schools of U. S.

| | | | | |
|---|---|---|---|---|
| Year: | 1880 | 1890 | 1900 | 1910 |
| Pupils: | 9.8 million | 12.7 million | 15.5 million | 17.8 million |

**2.** World's production of gold.

| Year: | 1885 | 1890 | 1895 | 1900 | 1905 | 1910 |
|---|---|---|---|---|---|---|
| Gold ounces: (millions) | 5.0 | 5.7 | 9.8 | 12.3 | 18.9 | 22.0 |

**3.** Yield of wheat per acre.

| Year | Bushels |
|---|---|
| 1890 | 11.1 |
| 1895 | 13.7 |
| 1900 | 12.3 |
| 1905 | 14.5 |
| 1910 | 13.9 |

**4.** Amount of one dollar at 5% compound interest.

| Years | Amount |
|---|---|
| 10 | $1.63 |
| 20 | 2.65 |
| 30 | 4.32 |
| 40 | 7.04 |
| 50 | 11.47 |

**5.** Solubility of potassium carbonate in 100 gms. water.

| Temperature | Salt dissolved |
|---|---|
| 20° C. | 112 gms. |
| 40° C. | 117 gms. |
| 60° C. | 127 gms. |
| 80° C. | 140 gms. |
| 100° C. | 156 gms. |

**6.** Boiling point of water in which salt is dissolved.

| Salt dissolved in 100 gms. water | Boiling point |
|---|---|
| 7 gms. | 101° C. |
| 12 gms. | 102° C. |
| 22 gms. | 104° C. |
| 30 gms. | 106° C. |
| 40 gms. | 109° C. |

**7.** Average height of boys.

| Age | Height |
|---|---|
| 6 years | 44.0 inches |
| 8 years | 47.0 inches |
| 10 years | 51.8 inches |
| 12 years | 55.0 inches |
| 14 years | 59.3 inches |
| 16 years | 64.3 inches |
| 18 years | 67.0 inches |
| 20 years | 67.5 inches |

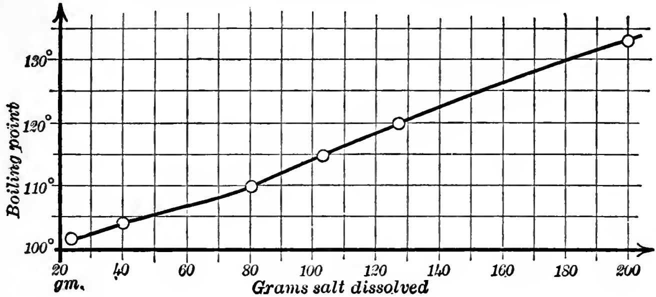
**8.** Number of years that persons of different ages will probably live.

| Age | Expectation of Life |
|---|---|
| 0 years | 41 years |
| 10 years | 47 years |
| 20 years | 40 years |
| 30 years | 33 years |
| 40 years | 27 years |
| 50 years | 20 years |
| 60 years | 14 years |

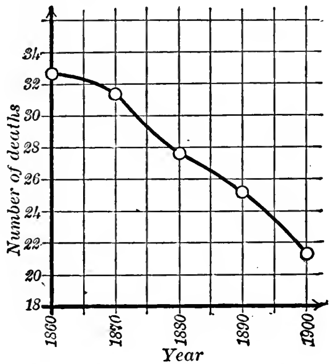
Let the student look up other data from which curves may be constructed.

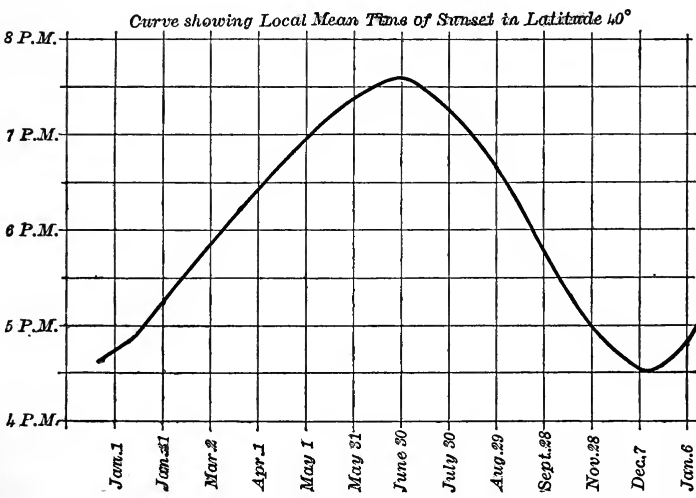
Study the following curves and state what they show.

Curve showing the temperature at which water boils when various amounts of potassium carbonate are dissolved in 100 c.c.



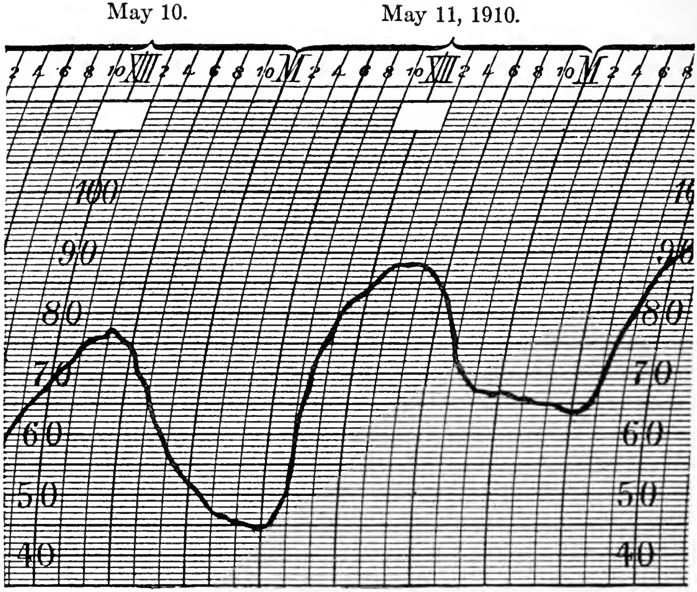

Curve showing the number of deaths from tuberculosis per 10,000 population



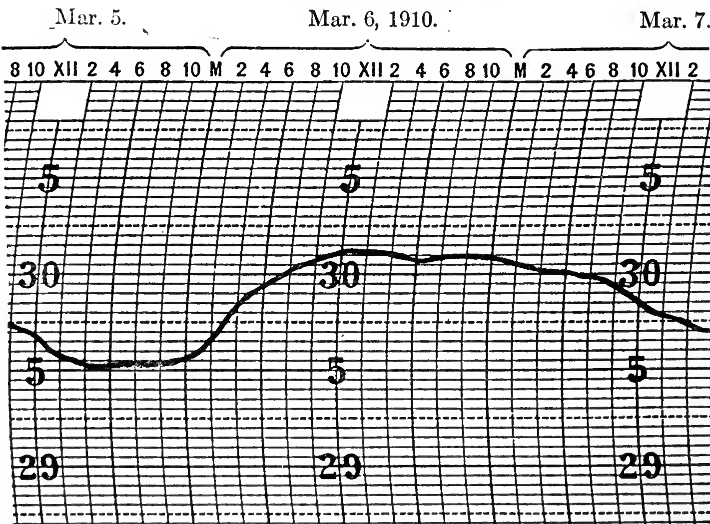
THERMOGRAPH.

Curve showing the temperature in degrees F. (From the records of the U. S. Weather Bureau at Lincoln, Neb.)




BAROGRAPH.

Curve showing barometric pressure in inches. (From records of U. S. Weather Bureau at Lincoln, Neb.)



## 100. Graphic Solution of Equations.

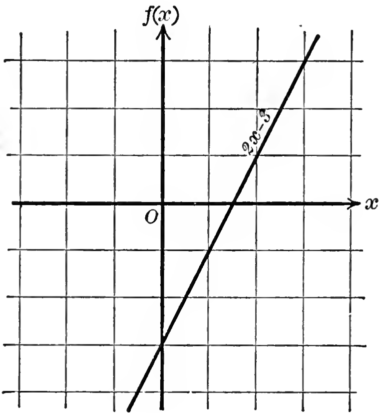
*Example* 1. Suppose $f(x) = 2x - 3$. For what value of $x$ is $f(x)$ equal to zero?

*Graphic solution:* Draw the graph of the function $2x - 3$. It is a straight line, so only two points are needed.

$$x = 0, \quad f(x) = -3;$$
$$x = 3, \quad f(x) = 3.$$

So we get the graph as shown below.

Reading off the value of $x$ where the graph crosses the $x$-axis, we get $x = \frac{3}{2}$. This is the required solution. Explain why.




Check by substituting this value in place of $x$ in $2x - 3$ and see if you get zero. In other words see if $f(\frac{3}{2}) = 0$.

*Algebraic solution:*

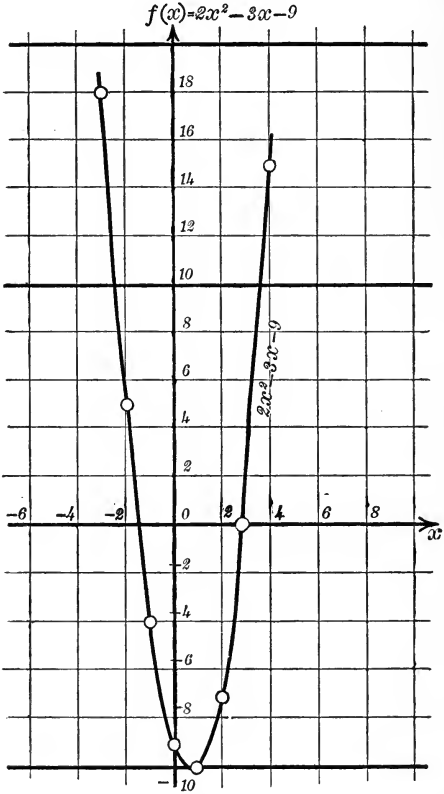
$$2x - 3 = 0,$$
$$2x = 3,$$
$$x = \tfrac{3}{2}.$$

Notice the correspondence of answers in the geometric and the algebraic solutions.

*Example* 2. Let $f(x) = 2x^2 - 3x - 9$. For what value of $x$ is $f(x) = 0$. We are to find $x$ so that $2x^2 - 3x - 9 = 0$.

Draw the graph of the function $2x^2 - 3x - 9$. It will be a parabola.

| $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|---|---|---|---|
| $-3$ | $+18$ | $-2$ | $+5$ | $-1$ | $-4$ | $0$ | $-9$ |
| $+1$ | $-10$ | $+2$ | $-7$ | $+3$ | $0$ | $+4$ | $+15$ |




The curve crosses the $x$-axis where $x = -1.5$ and where $x = 3$. These are the required solutions. Explain why.

Check by showing that $f(-1.5) = 0$. Also that $f(3) = 0$.

*Algebraic solution:* By formula, since $a = 2$, $b = -3$, $c = -9$,

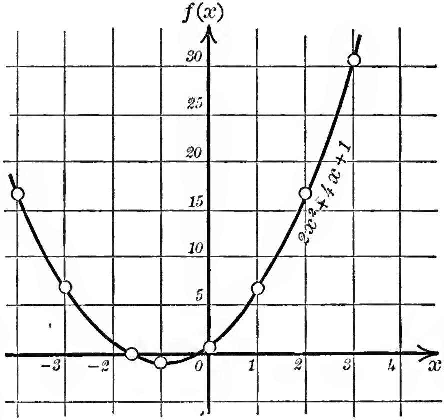
$$x = \frac{3 \pm \sqrt{9 + 72}}{4}$$

$$= 3 \text{ or } -\tfrac{3}{2}.$$

Note the correspondence of the geometric and the algebraic solutions.

*Example* 3. Let $f(x) = 2x^2 + 4x + 1$.

| $x$ | $f(x)$ |
|---|---|
| 3 | 31 |
| 2 | 17 |
| 1 | 7 |
| 0 | 1 |
| − 1 | − 1 |
| − 2 | + 1 |
| − 3 | 7 |




The figure gives $x = -.3$ or $x = -1.7$ approximately.

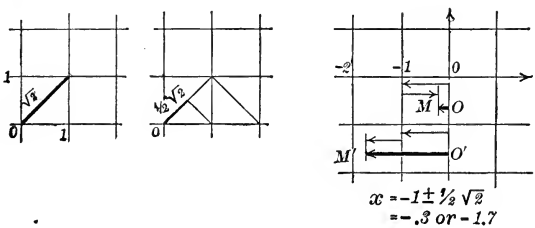
If we wish to find algebraically where the curve crosses the $x$-axis, solve for $x$ as in the preceding exercises. You should obtain the answers

$$x = -1 + \tfrac{1}{2}\sqrt{2},$$

or

$$x = -1 - \tfrac{1}{2}\sqrt{2}.$$

Drawing these we have:




$OM$ is the length as expressed by $-1 + \frac{1}{2}\sqrt{2}$,
$O'M'$ is the length as expressed by $-1 - \frac{1}{2}\sqrt{2}$.

Comparing these with the distances from the origin to where the curve in the figure above crosses the $x$-axis, we find that they are the same.

Check these answers as you have done in preceding exercises.

**Exercises.** Give graphic and algebraic solutions of the following. By counting if possible, by drawing, if not, show that the geometric and algebraic solutions agree.

1. $x^2 - 5x + 6 = 0.$
2. $x^2 + 5x + 6 = 0.$
3. $x^2 - x - 6 = 0.$
4. $x^2 - 4 = 0.$
5. $4x^2 - 14 = 0.$
6. $9x^2 - 20 = 0.$
7. $6x^2 - 7x - 20 = 0.$
8. $7x^2 + 10x + 2 = 0.$
9. $3x^2 - 6x + 2 = 0.$
10. $5x^2 + 2x - 2 = 0.$
11. $3x^2 - 4x - 9 = 0.$
12. $-x^2 + 5x + 1 = 0.$
13. $2x^2 - x = 0.$
14. $7x^2 + 22x = 0.$
15. $-3x^2 - 14x = 0.$
16. $-4x^2 + 20x = 0.$

17. Physics tells us that the distance a body falls during any stated length of time is expressed by the equation

$$d = \tfrac{1}{2} gt^2,$$

where $d$ is in feet, $t$ is in seconds, and $g = 32$ approximately.

Construct a drawing by which you can determine the distance passed over by a body falling from rest during the first 13 seconds; during the first 7 seconds.

How long would it take a stone to fall from the top of the Washington monument to the ground?

*Discussion of the quadratic expression and its graph.* When you made the graphic representation of the quadratic expressions, § 96, your attention was called to the fact that all of the curves have similar forms, but take different positions with reference to the axes and that some are broader than others.

As such expressions differ only in the values of $a$, $b$ and $c$, this must be the cause of the variations in the position curve and the width of the curve.

We shall now make drawings to find out what effect the changing of each in turn has upon the curve.

**Exercise 1.** Using the same axes, give graphic solution of:

(*a*) $x^2 - 2x - 2 = 0.$ (*d*) $x^2 - 2x + 1 = 0.$
(*b*) $x^2 - 2x - 1 = 0.$ (*e*) $x^2 - 2x + 2 = 0.$
(*c*) $x^2 - 2x \quad = 0.$ (*f*) $x^2 - 2x + 3 = 0.$

You have changed the value of $c$. What effect does this have upon the curve?

When $c$ is zero, through what point of interest does the curve pass? Do you think that the graph of $f(x) = ax^2 + bx$ will pass through the origin, no matter what values $a$, $b$, take on? Show this to be true by finding the roots of the equation

$$ax^2 + bx = 0.$$

Through what other point will the curve for $f(x) = ax^2 + bx$ always pass? Fix these truths well in mind.

**Exercise 2.** Make the graph for the solution of the following equations with reference to the same axes. (Not the same as used in Exercise 1.)

(*a*) $x^2 - 2x - 1 = 0.$ (*c*) $x^2 - 1 \quad = 0.$
(*b*) $x^2 - \quad x - 1 = 0.$ (*d*) $x^2 + x - 1 = 0.$
(*e*) $x^2 + 2x - 1 = 0.$

In this exercise you have changed the value of $b$. What effect does this have upon the curve?

When $b$ is zero, what is the fact of interest about the position of the curve?

When $b$ and $c$ are both zero, what is true about the curve?

Write the quadratic equation with $b$ and $c$ both zero, and find by algebra the value of $x$ for the points where the curve crosses the $x$-axis. How many points are there?

**Exercise 3.** With reference to the same axes (not the same as used in Exercises 1 and 2) make graphs for the solution of the following:

(*a*) $2x^2 - 3x - 1 = 0.$ (*c*) $.1x^2 - 3x - 1 = 0.$
(*b*) $x^2 - 3x - 1 = 0.$ (*d*) $.01x^2 - 3x - 1 = 0.$

In this exercise you have changed $a$. What effect does this have upon the curve?

What will be true of the graph if you continue to make $a$ more nearly equal to zero? What would be true if you made $a$ zero?

Suppose $a$, $b$, $c$ were all zero, what would be the graph of the equation?

Make a summary of the truths brought out in the preceding exercises, with reference to the graphs used to solve the following equations.

$$ax^2 + bx = 0.$$
$$ax^2 + c = 0.$$
$$bx + c = 0.$$
$$ax^2 = 0.$$

State these truths in words and fix them in mind.

**101. Nature of the Roots of a Quadratic Equation.** The next thing of interest about the quadratic equation is to be able to tell by examining the equation whether or not the curve will cross the $x$-axis.

You will notice by studying the curves of Exercise 1 that some do and some do not cross the axis.

We shall now examine the algebraic equation and its roots and determine the cause of this.

The roots of

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Examining these roots we see that the only difference in them is the sign of $\sqrt{b^2 - 4ac}$. If this quantity be added to $-b$ and the sum divided by $2a$, we get one root, or one crossing point of the curve with the $x$-axis. If we subtract this quantity from $-b$ and divide by $2a$ we get the other root, or the other crossing point.

Now $b^2 - 4ac$ will be positive, zero, or negative, according to the relative values of $a$, $b$, $c$.

*If* $b^2 - 4ac$ *is greater than zero*, the two roots are unequal and the curve crosses the $x$-axis in two distinct points, whose distances from the origin can be counted, or can be drawn by making use of the Pythagorean theorem. Examples 2 and 3, of § 100, illustrate this. In Example 2 the roots are rational; in Example 3 they are irrational.

See Exercise 1 (*a*), (*b*), (*c*). Examine the graphs and compute the value of $b^2 - 4ac$ to see if the above statements hold.

*If* $b^2 - 4ac$ *is zero*, the roots will be equal, since adding zero to $-b$ gives the same result as subtracting zero from $-b$. In this case the parabola is tangent to the $x$-axis.

Examine Exercise 1 (*d*), to see if the above statements hold.

*If* $b^2 - 4ac$ *is negative*, $\sqrt{b^2 - 4ac}$ is imaginary; this means that the roots are imaginary, and the parabola shows no point in common with the $x$-axis.

See Exercise 1 (*e*) and (*f*). Examine as to the truth of the preceding statements.

To sum up:

**when $b^2 - 4ac$ is positive, the roots are real and unequal;**

**when $b^2 - 4ac$ is zero, the roots are real and equal;**

**when $b^2 - 4ac$ is negative, the roots are imaginary.**

Fix in mind the above facts about the roots of the quadratic equation and the position of the graph with reference to the $x$-axis.

Your attention is called to another feature of interest.

You will notice that if you should draw a line parallel to the function axis, through the vertex of your curve, it will divide it in such a way that if you revolve the curve through a straight angle about this line as an axis, the curve will fall in identically the same place that it was before you revolved it.

The line is called the *axis of symmetry*, and the curve is said to be symmetrical to this line. The equation of this line is

$$x = \frac{-b}{2a}.$$

By drawing in this line you can test to see that you have your curve well drawn, for you should be able to start with any point on your curve, draw a perpendicular to the axis of symmetry, continue in the same direction for the same distance and strike another point on your curve. In other words all lines which cross your curve perpendicular to the axis of symmetry must be bisected by it. Otherwise your curve is not properly drawn.

Another point of less interest, but which will aid in your drawing, is that the curve crosses the function axis at the point $c$ units from zero. Why?

Examine the equations on p. 234, and tell all you can about the position of the graph with reference to the axes, the nature of the roots, and the axis of symmetry.

## 102. Graphic Solution of Equations of Higher Degree.

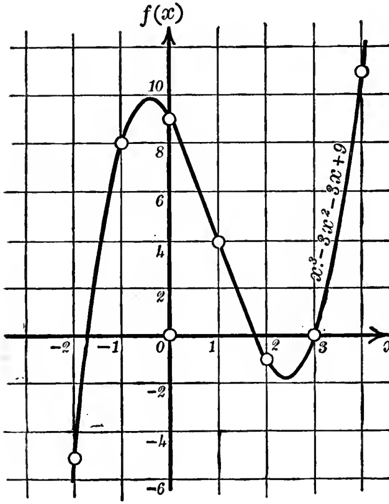
*Example*: Let $f(x) = x^3 - 3x^2 - 3x + 9$. For what values of $x$ does $f(x)$ equal zero?

We are to find $x$ so that

$$x^3 - 3x^2 - 3x + 9 = 0.$$

The algebraic solution of such an equation is rather complicated, unless we can factor $f(x)$. Can you do this? We pass to the graphic solution.

| $x$ | $f(x)$ | $x$ | $f(x)$ |
|---|---|---|---|
| $-2$ | $-5$ | 2 | $-1$ |
| $-1$ | $+8$ | 3 | 0 |
| 0 | $+9$ | 4 | $+11$ |
| 1 | $+4$ | | |




By inspection of the graph we see that $f(x) = 0$ when $x = -1.7$ approximately, when $x = +1.7$ approximately, and when $x = 3$. By greatly enlarging the part of the graph near $x = 1.7$, say from $x = 1.5$ to $x = 2$, can you get the second decimal place in this value of $x$? Do this.

**Exercises.** Obtain graphic solutions of the following equations. Give exact solutions when possible, by factoring.

**1.** $x^3 - 4x = 0$.

**2.** $x^3 - 2x^2 + x = 0$.

**3.** $4x^3 - 9x^2 = 0$.

**4.** $x^3 - x^2 + x - 1 = 0$.

**5.** $x^3 + x^2 - 9x - 9 = 0$.

**6.** $x^3 + x^2 - 9x - 5 = 0$.

**7.** $x^4 - 5x^2 + 4 = 0$.

**8.** $x^4 = 4x + 4$.

**103. Summary.**

The practical use of the graph is to represent tabulated values of a function.

Graphs are also used to solve equations. Draw the graph of the function and measure the values of the variable to the points where the graph crosses the axis of the variable. These values are the solutions. When there are no crossing points the solutions are imaginary.

To solve graphically the linear equation $ax + b = 0$, draw the graph of $ax + b$ and note the point where the straight line crosses the $x$-axis.

To solve the quadratic equation $ax^2 + bx + c = 0$ graphically, draw the graph of the expression $ax^2 + bx + c$ and note the points where it crosses the $x$-axis.

This graph is a parabola whose position depends on the numerical values of $a$, $b$ and $c$.

If $b^2 - 4\,ac$ is positive, the parabola cuts the $x$-axis twice. In this case the quadratic has two real and unequal solutions.

If $b^2 - 4\,ac$ is zero, the parabola just touches the $x$-axis. In this case the quadratic has two equal solutions.

If $b^2 - 4\,ac$ is negative, the parabola does not cross the $x$-axis. In this case the roots are imaginary.

# CHAPTER XVI

## LOCI OF POINTS. SIMULTANEOUS EQUATIONS *

**104. Meaning of the Word Locus.** You cannot locate a place without mentioning its relative position with reference to some other place. And, furthermore, as the following examples will illustrate, you cannot locate a place without giving two descriptions of its location.

*Example* 1. I wish to tell you the location of a house. I say that it is a mile from here. Immediately as you think of houses one mile from here they arrange themselves on the circumferemce of a circle which has our present location for a center and one mile for a radius. Any house nearer or farther as you are able to estimate it, is put out of mind. There may be many houses which answer the description but they all stand on the circumference. The circumference is the *place* of the house. Mathematically I would say that the circumference is the *locus* of the *house* described as being one mile from here. The word *locus* means *place.* As is readily seen, one description is not sufficient to locate the house that I have in mind. I will say further that the house is three blocks from the street that runs in front of the school house. Immediately your mind runs along a street three blocks away and when it reaches what you estimate to be a mile from here you have in mind the house we are speaking of. The street is a second place or locus of the house and where the two loci come together is the place where the house stands. There might be two such houses. Explain how so.

*Example* 2. Locate a place when the description is given as 105° west longitude, and 41° north latitude.

As you learned in your geography, for the sake of locating places on the earth's surface, a line of reference is supposed to be drawn through Greenwich, running north and south from pole to pole. Also a second line of reference, the equator, runs at right angles to this. A place

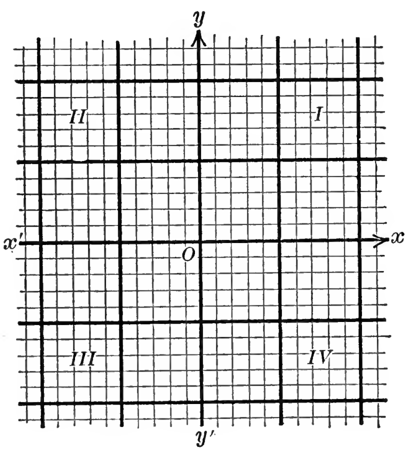
* For closer correlation this chapter may be preceded by Chapter V of Geometry.

whose description is given as 105° west longitude might be anywhere along the meridian 105° west from the line of reference. If we use a mathematical term, the meridian would be its locus. A place described as 41° north latitude might be anywhere along the parallel 41° north of the equator. This parallel would be its locus. Where these two lines cross would be the place sought.

Give other examples of ways for locating places.

**105. Coördinates.** To locate the position of a point on a sheet of paper or any plane surface, we follow the same plan as is used to locate a place on the earth's surface. Draw two lines of reference which cut at right angles; one extending from left to right and the other up and down. (Always use cross-section paper for this work.)

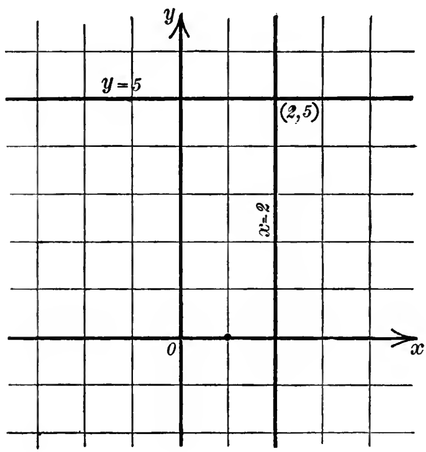
The plane of the paper is now divided into four quarters, called quadrants. They are numbered I, II, III, IV as shown in the figure.




We now locate a point by giving its distance from the two reference lines, stating also whether these distances are measured to the right or left, upward or downward. The line $x'x$, as shown in the figure, is called the *axis of the abscissa.* The line $y'y$ is called the *axis of the ordinate.* Distances upward from the axis of the abscissa are positive; downward, negative. Distances to the right of the axis of the ordinate are positive; to the left, negative. The point of intersection marked zero is called the *origin.* The distance which a point is to the right or left of the axis of the ordinate, measured in the direction of the axis of the abscissa, is called the *abscissa* of the point. The distance which a point is from the axis of the abscissa, measured in the direction of the axis of the ordinate, is called the *ordinate* of the point. The two together are called the *coördinates* of the point which is fixed by them.

We usually represent the abscissa of a point by $x$, and its ordinate by $y$; then the two numbers $x$, $y$ are the coördinates of the point. However, any other letters may be used.

*Example.* Suppose we wish to locate a point for which $x = 2$ and $y = 5$, that is, the abscissa is 2 and the ordinate is 5.




Since the abscissa is 2, the point lies 2 units to the right of the axis of the ordinate. It is therefore on a line drawn 2 units to the right of the axis of the ordinate and parallel to that axis. This line we designate by the equation $x = 2$, because this equation is true for any point on the line, and is true for no other point. This line is the locus of the point whose abscissa is 2.

Since the ordinate of the point is 5, the point must be somewhere on a line drawn parallel to the axis of the abscissa and 5 units above it. This line is designated by the equation $y = 5$. Why? This line is the locus of a point whose ordinate is 5.

The required point must be at the intersection of the loci just drawn. We call it the point (2, 5); here we inclose in parentheses the coördinate of the point, writing first the abscissa, and then the ordinate, with a comma between.

A point whose abscissa is $x$ and whose ordinate is $y$ is designated by the symbol $(x, y)$.

**Exercises.** Draw the lines on which each of the following points must lie. On each line write the equation which describes it. As has just been explained such a line is the locus of the point which lies on it.

1. What is the locus of a point whose abscissa is 3? Whose ordinate is 7? What point is at the intersection of the two loci? Draw a complete figure as shown above, marking in the equations of the line and the symbol for their point of intersection.

2. Proceed as in Exercise 1, when the abscissa is 3 and the ordinate is − 7.

3. Proceed as in Exercise 1, when the abscissa is − 5 and the ordinate is 2.

4. Proceed as in Exercise 1, when the abscissa is − 5 and the ordinate is − 2.

5. Proceed as in Exercise 1, when the abscissa is − 3 and the ordinate is − 3.

6. Make diagrams as in Exercise 1 for each of the following points:
(2, 6); (− 5, 4); (6, − 9); (12, − 7); (− 2, − 3); (0, − 9); (2, 0).

7. What line is described by the equation $x = 0$? What line is described by the equation $y = 0$? What point lies at the intersection of these loci?

8. Locate the points (9, 4); (− 3, − 1); (4, − 3). Join by straight lines. What kind of a figure is formed?

9. Draw the quadrilateral whose vertices are the points (3, 4), (− 1, 4), (− 1, − 2), (3, − 2). How long is the diagonal of this quadrilateral?

**106. Straight Line Loci in General.** We shall study examples of loci which are not parallel to one of the reference lines in the exercises that follow. Use the same sheet of paper for Exercises 1 to 7 inclusive, drawing the axis rather heavy near the middle of the sheet each way.

**Exercises.**

**1.** Bisect the pair of vertical angles made by the axes of the coördinates which form the first and third quadrants. Place five points along this bisector, some on one side of the origin and some on the other. Count the abscissa and the ordinate of each point. How do they compare in each case?

The algebraic statement of this thought is

$$x = y, \text{ or } x - y = 0.$$

Write the equation on the line.

**2.** Draw the line $y = 1$. Bisect the pair of vertical angles formed by the lines $y = 1$ and $x = 0$, corresponding to the pair bisected in Exercise 1. Place 5 points along this line, some on one side of the point of intersection and some on the other. Ascertain by counting the comparative lengths of the abscissa and ordinate of each point. (Remember to count from the axes of the abscissa and ordinate each time.) Explain why you arrive at the result that you do each time.

The algebraic statement of this thought is

$$y = x + 1.$$

**3.** Repeat the work of Exercise 2, bisecting the angle formed by the lines $y = 2$, $x = 0$; $y = 3$, $x = 0$; $y = -1$, $x = 0$; $y = -2$, $x = 0$. On each bisector write the algebraic expression of the truth, found by counting, concerning the relation of the coördinates of each point.

**4.** These bisectors look parallel. We shall assume that they are parallel.

**5.** Bisect the other pair of vertical angles formed by the lines $y = 0$, $x = 0$. By counting establish the relation of the abscissa and ordinate of points on this line.

The algebraic expression for this thought is

$$y = -x, \text{ or } y + x = 0.$$

Write this name on the line.

**6.** Draw the bisectors of the other pairs of vertical angles, and by counting establish the relation of abscissa and ordinate of points on each, express in algebraic language, and write the name on each line.

**7.** To generalize the idea brought out in Exercises 1 to 6, consider $y = b$ to be the line parallel to the axis of the abscissa ($b$ being any distance up or down from the axis of the abscissa). The bisectors of the angles formed by $y = b$ and $x = 0$, will be parallel to the bisectors of the angles formed by $y = 0$ and $x = 0$. This being true, we can state that the general name of these lines is $y = x + b$ for the first set, and $y = - x + b$ for the second. On a new page of graph paper, with axes of reference drawn as before, place Exercises 8, 9 and 10.

**8.** Starting at the origin to count, mark a point whose abscissa is 1 and whose ordinate is 2. Join this point to the origin, extending the line indefinitely in either direction. By counting establish as in previous cases the relation of the abscissa and ordinate of points on this line. You find in each case the ordinate to be 2 times the abscissa.

The algebraic expression for this relation is

$$y = 2x.$$

**9.** Starting from the point of intersection of $y = 1$ and $x = 0$, count 1 unit to the right and 2 units up. Join the point to the point of intersection of $y = 1$ and $x = 0$, extending the line indefinitely.

By counting establish the relation of abscissa and ordinate of points on this line.

The algebraic expression of this relation is

$$y = 2x + 1.$$

In every case be sure to write names on the lines.

**10.** Repeat instructions of Exercise 9, counting from the point of intersection of the following.

$y = 2$ and $x = 0$; $y = 3$ and $x = 0$; $y = - 1$ and $x = 0$; $y = - 2$ and $x = 0$.

By a discussion as in Exercise 7, bring out the statement that the general algebraic expression for the equation of these lines is

$$y = 2x + b,$$

where $b$ is any distance measured up or down from the origin.

On a new page of graph paper place Exercises 11, 12, 13.

**11.** Repeat the instructions of Exercises 8, 9, 10, establishing the line by counting 1 to the right and 3 up from the origin and the points of intersection. You should be able to write the names on these lines without any trouble. Do this work on a fresh sheet of paper.

Write the algebraic expression for the relation of the abscissa and ordinate of the points on each line.

**12.** Repeat Exercises 8 to 11 counting to the left and up, instead of to the right. Write the names on these lines without counting. Then count for one point to see that you have written correctly.

**13.** We shall now try to write an algebraic equation, which will be a general expression of the idea brought out by Exercises 1 to 12. Examine your equations and you will see that the coefficient of $x$ is the expression of the ratio of $y$ to $x$ on the line through the origin. That is, it gives you the *slope* of the line. All other lines which have the same coefficient of $x$ are parallel to it and hence have the same slope. If you count 1 unit to the right and $a$ units up, you have

$$y = ax, \text{ } a \text{ being positive in this case.}$$

If you count 1 unit to the left and $a$ units up, you have

$$y = ax, \text{ } a \text{ being negative in this case.}$$

The line parallel to any one of these lines, and passing through the intersection of $y = b$ and $x = 0$, has for its equation

$$y = ax + b.$$

This equation is very general. You cannot draw any straight line on your paper the relation of whose coördinates this equation will not describe. Moreover you cannot locate a point on your paper that will not fall on one or more of these lines; that is, you cannot locate a point whose coördinate will not satisfy this equation.

Being given a line, we have learned how to write its equation. We shall now investigate the converse of this.

Being given an algebraic expression, to determine the line which is its graph. We shall do this by examining a special case.

*Example* 1. Given the equation $2x + 3y = 0$.

To find the locus of points whose coördinates satisfy this equation. The equation may be written in the form

$$y = -\tfrac{2}{3}x.$$

Comparing this with the equation $y = ax$, $a = -\frac{2}{3}$; so we can locate one point on the line by counting 3 to the left and 2 upward, that is, locate the point $(-3, 2)$. Draw a straight line through this point and the origin, extending it each way indefinitely, and we have the graph of the equation $2x + 3y = 0$.

Test this by selecting any other point on this line, counting its coördinate distances, and see if they satisfy the equation given.

*Example* 2. Given the equation $2x + 3y = 19$.

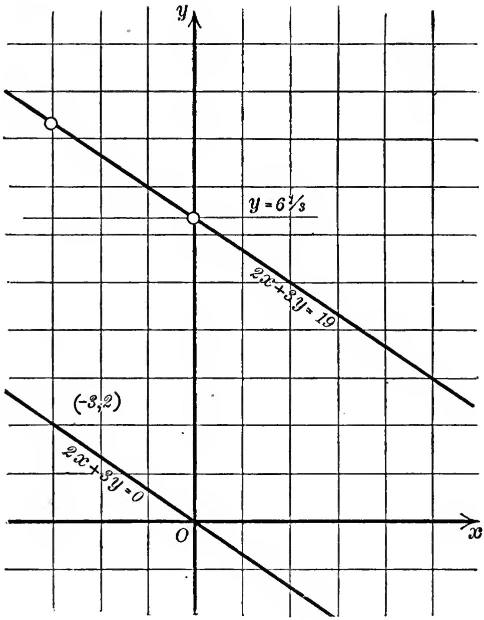
Subtracting $2x$ from both members of the equation, and dividing by 3, this equation becomes,

$$y = -\tfrac{2}{3}x + 6\tfrac{1}{3}.$$

We now have $y$ as a function of $x$, of the form $ax + b$.
Here we have $a = -\frac{2}{3}$, and $b = 6\frac{1}{3}$.

To make the graph locate the point where the line $y = 6\frac{1}{3}$ intersects the line $x = 0$. From this point count 3 units to the left and 2 upward. Join these two points and this line extended indefinitely in either direction is the graph of the given equation. Test this by selecting another point on the line and counting to determine its coordinates. See if they satisfy the equation.

Compare this work with the work in § 95.




**Exercises.** By the process shown above draw the locus for each of the following relations between $x$ and $y$.

1. $x - y = 0.$
2. $x + y = 1.$
3. $6x + 3y = 0.$
4. $6x + 3y = 7.$
5. $4x + 3y = 0.$
6. $4x + 3y = 12.$
7. $3x - 4y = 0.$
8. $3x - 4y = -14.$
9. $2x - y = 3.$
10. $x - 2y = 3.$
11. $x - 2y + 4 = 0.$
12. $3x - 2y = 6.$
13. $2x + 3y + 6 = 0.$
14. $x - y = 2.$

These equations are all of the form

$$ax + by + c = 0,$$

where the letters $a$, $b$ and $c$ stand for given numbers, positive or negative. Such an equation is called an equation of the *first degree* or a *linear equation in two variables.*

From a study of the preceding exercises, what kind of a locus is determined by a linear equation?

Does your work suggest the following conclusions?

(*a*) Any point whose coördinates $x$ and $y$ satisfy the given equation lies on the locus of that equation.

(*b*) Conversely, if we read off the coördinates of any point on the locus, we shall have a pair of values of $x$ and $y$ for which the equation is true.

In other words, the locus of the equation gives us a complete geometric picture of all the pairs of values of $x$ and $y$ which make the equation true.

**107. Simultaneous Equations.** Since, as we have seen, one equation gives us a locus of points whose coördinates satisfy the equation, that is, one equation gives us one relation between $x$ and $y$, if we wish to locate a definite point, it will be necessary to have two equations, so that we may have two relations. The coördinates of the point or points of intersection of the loci will then satisfy both relations.

*Definition.* Two equations which are satisfied by the same pair of values of $x$ and $y$ are called *simultaneous equations.*

*Example.* Given the two equations

(1) $$2x + 3y = 19,$$

(2) $$3x - 4y = -14.$$

To find by graph the point whose coördinates satisfy the two equations.

*Graphic Solution:* Solve each for $y$ in terms of $x$, or writing $y = f(x)$.

(1) $$y = -\tfrac{2}{3}x + 6\tfrac{1}{3},$$

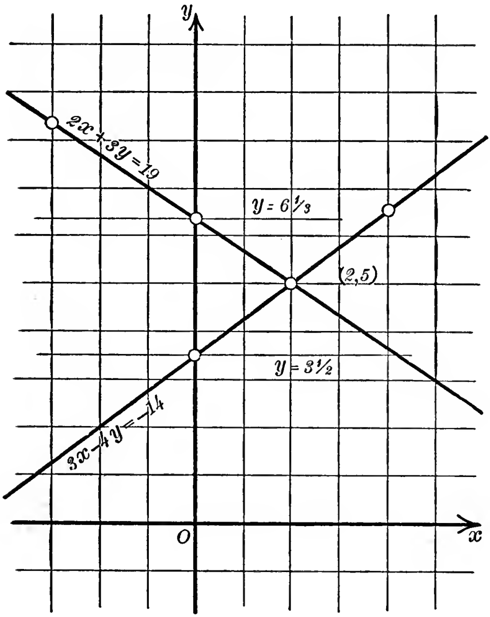
(2) $$y = \tfrac{3}{4}x + 3\tfrac{1}{2}.$$

Drawing the graph of these as shown in the preceding article, we have the figure on the next page.

Check this by substituting the values 2 and 5 for $x$ and $y$ respectively in the *given* equations to see if they verify them.

Thus $2 \cdot 2 + 3 \cdot 5 = 4 + 15 = 19$, as the first equation states.

$3 \cdot 2 - 4 \cdot 5 = 6 - 20 = -14$, as the second equation states.




*Algebraic solutions:* We may obtain the coördinates of the point of intersection by algebra as well as by drawing.

(*a*) *Solution by Comparison.* Observe that for the point of intersection and for no other point on the lines, the value of $y$ is the same for the same value of $x$, whether we regard the point as belonging to one locus or the other.

Therefore, since $\quad y = -\frac{2}{3}x + 6\frac{1}{3},$
and also $\quad y = \frac{3}{4}x + 3\frac{1}{2},$

it follows that for the point of *intersection* and for this point only

$$-\tfrac{2}{3}x + 6\tfrac{1}{3} = \tfrac{3}{4}x + 3\tfrac{1}{2}.$$

Multiplying both members by 12, we have

$$-8x + 76 = 9x + 42,$$
$$17x = 34,$$
$$x = 2.$$

Substituting this value for $x$ in the equation

$$y = -\tfrac{2}{3}x + 6\tfrac{1}{3},$$

we have

$$y = -\tfrac{2}{3}\cdot 2 + 6\tfrac{1}{3}$$
$$= 5.$$

These are the coördinates of the point of intersection as shown in the drawing. Check by substituting these values in the second equation.

**Exercises.** In each of the following exercises make graph; find the point common to the two loci; solve by algebra to find the values of $x$ and $y$ that will satisfy both equations; check to make sure that your algebraic solution is correct. Compare these values with the coördinates of the point of intersection, and make sure they are the same.

**1.** $x - 2y = 6,$
$7x + 5y = 23.$

**2.** $4x + y = -13,$
$8x + 3y = -27.$

**3.** $5x + 4y = +2,$
$3x + y = -3.$

**4.** $3y - 8x = 19,$
$9x - y = 0.$

**5.** $\frac{1}{4}x + \frac{1}{2}y = 12,$
$x - y = 4.$

**6.** $\frac{3}{4}x + \frac{4}{5}y = 21,$
$\frac{2}{3}x + \frac{3}{5}y = 17.$

In the algebraic solution when you find the values of $y$ in terms of $x$ and the constant, and equate these values, you *eliminate* one of the variables, namely $y$, and get an equation containing only one variable.

This special way of eliminating one of the variables is called *elimination by comparison.*

The main idea in the process is to obtain an equation containing only one variable. Any process that will accomplish this result, and not violate any of the laws of mathematics, will answer our purpose.

There are two other methods that may be used, which we shall now investigate.

(*b*) *Solution by Substitution.* Write one of the two given equations in the form of $y = f(x)$, as in the preceding method. Since for the point of intersection, the value of $y$ is the same for each equation, substitute this value of $y$ in place of $y$ in the other equation, thus obtaining an equation containing one variable.

*Example.* Using the same equations that we have been discussing,

$$2x + 3y = 19,$$
$$3x - 4y = -14.$$

Writing the first equation in the form of $y = f(x)$,

$$y = -\tfrac{2}{3}x + 6\tfrac{1}{3}.$$

Substituting this value in the second equation we have

$$3x - 4(-\tfrac{2}{3}x + 6\tfrac{1}{3}) = -14.$$

From which $\frac{17}{3}x = 11\tfrac{1}{3},$

and $x = 2.$

Since $y = -\tfrac{2}{3}x + 6\tfrac{1}{3},$

and $x = 2,$

$$y = -\tfrac{2}{3}\cdot 2 + 6\tfrac{1}{3}$$
$$= 5.$$

(2, 5) is the point of intersection as shown in the graph.

(*c*) *Elimination by Multipliers.* The third method for obtaining an equation containing only one variable is the most *commonly* used. It consists in first multiplying the equations by numbers that will make the coefficients of one of the variables the same in the two equations, then if the terms which have like coefficients have like signs, we will eliminate one of the variables by subtracting one equation from the other, thus obtaining an equation of only one variable. If the terms with like coefficients have unlike signs, we can accomplish the same result by adding the two equations. What is the axiom that is applied in either case?

This process is often called *elimination by addition or subtraction.*

To illustrate this we shall use the same equations which we solved by the other two methods,

$$2x + 3y = 19,$$
$$3x - 4y = -14.$$

Multiplying the first equation by 3 and the second by 2, we have

$$6x + 9y = 57, \qquad \text{Why?}$$
$$6x - 8y = -28. \qquad \text{Why?}$$

Subtracting one equation from the other, we have,

$$17y = 85,$$
$$y = 5.$$

We might have multiplied the first equation by 4 and the second by 3, thus obtaining

$$8x + 12y = 76, \qquad \text{Why?}$$
$$9x - 12y = -42. \qquad \text{Why?}$$

Adding one equation to the other (since the sign of $12y$ is positive in one equation and negative in the other)

$$17x = 34. \qquad \text{Why?}$$
$$x = 2.$$

The check is the same in all three cases.

This method can best be carried out by the following scheme.

| | | |
|---|---|---|
| 4 | 3 | $2x + 3y = 19$ |
| 3 | $-2$ | $3x - 4y = -14$ |

$$17y = 85, \quad y = 5,$$
$$17x = 34, \quad x = 2.$$

**Exercises.** Solve the exercises on p. 252 by the last method. Also solve the following:

**1.** $a_1x + b_1y = c_1,$
$a_2x + b_2y = c_2.$

**2.** $ax - y = 2m,$
$x - by = n.$

**3.** $(c - b)x - y = c,$
$b^2x + (c + b)y = -cb.$

**4.** $x + ay = b,$
$dx - y = c.$

There are sets of simultaneous equations which are not linear, but which may be solved as such.

In the following regard $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns; do not clear fractions, but multiply each equation by a number that will make the coefficients of one of the variables the same, and add or subtract as the case may require, to eliminate one of the variables. Then clear fractions and solve for the other variable. By the same process find the value of the other variable and check as in the former work.

**5.** $\frac{7}{x} + \frac{5}{y} = 19,$
$\frac{8}{x} - \frac{3}{y} = 7.$

**6.** $\frac{3}{x} + \frac{8}{y} = 3,$
$\frac{15}{x} - \frac{4}{y} = 4.$

**7.** $\frac{1}{x} + \frac{1}{y} = \frac{5}{6},$
$\frac{1}{x} - \frac{2}{y} = -\frac{5}{6}.$

**8.** $\frac{1}{x} - \frac{1}{y} = a,$
$\frac{2}{x} - \frac{3}{y} = b.$

**108. Summary and Problems.**

The locus of a point is a line or lines on which a point must fall in order to fulfil a condition imposed upon it. Every point on the locus fulfils the condition and every point that fulfils the condition is on the locus.

A plane is divided into quadrants by means of two straight lines drawn perpendicular to each other. These lines are axes of reference; the one running from left to right is called the *axis* of the *abscissa;* the one running up and down is called the *axis* of the *ordinate.* The point of intersection is called the *origin.*

Every point in the plane may be located with reference to these straight lines; its distance from the axis of the ordinate is called the *abscissa* of the point; its distance from the axis of the abscissa is called the *ordinate* of the point. The two distances are called the *coördinates* of the point.

If a relation is established between the coördinates of a point, and if this relation is expressed in algebraic symbols, the expression is called an *equation in two variables.* If the graph is a straight line, its equation is called the linear equation in two variables.

There are indefinitely many solutions to a linear equation in two variables. The graphs of two linear equations in two variables are necessary to locate a point.

Two linear equations whose graphs intersect in a point are called *simultaneous* linear equations.

The coördinates of the point of intersection of the graphs of simultaneous equations will satisfy each equation.

There are three methods for finding algebraically the values which will satisfy each of two simultaneous equations:

Elimination by addition or subtraction;

Elimination by substitution;

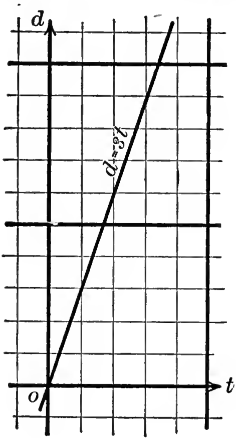
Elimination by comparison.

## Problems in Loci and Simultaneous Equations.

*Example* 1. If a body moves along a straight line 3 ft. per second, how far will it go in 10 seconds?

*Solution:* Draw the standard lines perpendicular to one another. Mark one $t$, on which to count the time; and the other $d$, on which to count the distance.

Construct a line which is the locus of distances passed over in all seconds. By counting we can quickly tell the distance passed over in any special number of seconds called for, as, in this case, 10.




Since nothing is said about it, we shall assume that the body starts from rest.

The ratio of the distance passed over during any number of seconds to the number of seconds is 3. Algebraically expressed

$$d = 3t.$$

Construct a line through the origin such that the ratio of $d$ to $t$ is 3. You can do this by counting to the right 1 and up 3, joining this point to the origin. Count 10 to the right then upward until you come to this line, and you have the distance passed over in 10 seconds. So for any other number of seconds that you choose.

If we regard the body as not having started from rest, but as having been in motion before we became interested in finding out about its movements, as is frequently the case, the question then arises, how far away was the body 8 seconds before, or 20 seconds before, as the case may be. To find this out count to the left 8 and down until you come to the locus and you have the answer to the question.

Check answer by substituting 8 for $t$ in the algebraic equation, and solve for $d$. Do you get the same result?

The body may be a distance away from us when we begin to record its movements, and we may wish to know its distance from us at the end of a certain time.

*Example* 2. Suppose the body that we were speaking of in Example 1 is 4 feet from us when we begin to observe it.

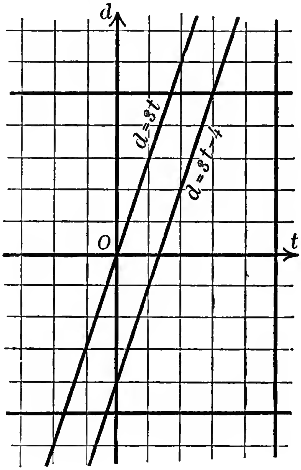
For any particular number of seconds, the distance as recorded by the perpendicular line will be 4 feet less than what it was when we considered it as starting near us.

Therefore the distance passed over during any particular number of seconds is registered by a new line parallel to the former line, every point in the new line being 4 units below the corresponding point in the former line.

The quickest way to construct the graph of problems of this type is to first construct a parallel graph through the origin and from this to construct the required line.

Thus, in Example 2, a body is 4 feet south of us and is moving at the rate of 3 feet per second northward. How far from us will it be at the end of 10 seconds?

The construction would first be made as for the first example, and then as described above for Example 2.




**1.** How far was the body described in Example 2 from us 5 seconds before we began to count.

**2.** The algebraic expression for the thought of Example 2 is

$$d = 3t - 4.$$

Substitute the number of seconds mentioned in Exercises 2 and 3, solve, and compare with answers obtained from graph.

**3.** A body is $7\frac{1}{2}$ feet south of us and is traveling at the rate of $\frac{1}{2}$ of a foot per second. With reference to the same standard lines construct the graph of the distance gone in $t$ seconds.

How long will it be from the time we began to count until the two bodies are together? How far from us will they be?

4. Solve the equations

$$d = 3t - 4,$$
$$d = \frac{1}{2}t - 7\frac{1}{2}.$$

Compare results with those obtained from graph.

5. Construct the graph of

$$c = 2\pi r.$$

For this work you may use $\pi = 3.1$.

From the graph tell the approximate length of the circumference whose radius is 2.5; whose radius is 3.2.

6. Construct a graph which may be used to convert feet into yards. Explain its use.

7. Construct a graph by which you may determine the length of the altitude of an equilateral triangle when you have given the side. Use Pythagorean theorem to construct the length of an irrational quantity.

8. Construct a graph by which you may determine the momentum of different weights moving at the rate of 2 feet per second.

9. According to physics the velocity which a body falling from rest acquires in $t$ seconds is expressed by the equation

$$v = gt. \quad (g = 32 \text{ approximately.})$$

Construct a graph by which you can determine the velocity at any given second. Use a quite small vertical scale.

From the graph determine how fast the body is falling at the beginning of the 5th second; the 8th second.

10. Suppose the body starts with a velocity of 3 feet per second, construct the graph for determining the velocity at the beginning of any given second.

11. The distance passed over in any one second is expressed by the equation

$$d = \frac{1}{2}g(2t - 1).$$

Construct graph and from it state the distance passed over during the 13th second; the 17th second.

**12.** Construct a graph from which, by weighing a mass of irregular shape, you can determine its volume if you know the density of the substance of which it is composed.

**13.** Two ladies arrange to attend a meeting together. One lives 8 blocks north of the hall where the meeting is held, and the other lives 11 blocks south. One called up the other and said, "I am just starting. You start immediately, and we will meet and go to the hall together." One walked at the rate of two blocks in three minutes, and the other at the rate of three blocks in five minutes. Determine by graph the time until they met and the distance and direction they were from the hall.

*Solution.*

Let $t$ = number of minutes until they meet

and $d$ = number of blocks the meeting place was from the hall, positive if north;

then $8 + d$ = number of blocks 1st walks

and $11 - d$ = number of blocks 2nd walks;

but $\frac{2}{3}t$ = number of blocks walked by 1st

and $\frac{3}{5}t$ = number of blocks walked by 2nd.

$\therefore$ $8 + d = \frac{2}{3}t.$ Why?

And $11 - d = \frac{3}{5}t.$ Why?

$$+d - \tfrac{2}{3}t = -8.$$

$$-d - \tfrac{3}{5}t = -11.$$

Adding and solving: $t = 15$, no. of minutes until they meet.

Substituting: $8 + d = \frac{2}{3} \cdot 15.$

$d = 2$, no. of blocks the meeting place is north of hall.

Check as in preceding problems.

Compare answers with solution by graph.

**14.** Two cisterns contain water, the first to the depth of 4 feet and the second to the depth of 7 inches. Water is allowed to run into them at a uniform rate in each case, and at the end of 5 minutes the water in the first is 7 inches, and the other 9 inches. How long will it be before the depth of the water is the same in each cistern, and what is the depth?

**15.** How long will it be before $25 placed at 5% simple interest will amount to the same as $28 at 3% simple interest? What is the amount?

**16.** Solve Exercises 13, 14, 15 by algebra, and compare answers with those obtained by graph.

**17.** A boy starts out on his motorcycle at the rate of 10 miles an hour. An hour later another boy wishing to overtake him rides at the rate of 18 miles an hour. In what time and at what distance will he come up with him? Solve by both graph and algebra. Compare results.

Solve the following problems algebraically.*

**18.** Two boys were in the habit of spending 6 days each spring in clearing out their strawberry patch. This year, however, when it came time to do the work, one of them was ill, so that the other one worked for 4 days alone. Then the first one joined, and it took 3 days longer to complete the work. How long would it have taken each alone to do the work?

**19.** A man planned to plant an apple orchard. He engaged two men who assured him that they could finish the work in 21 days. It turned out, however, that one did not come to work until 3 days after the other had started, so that it took 7 days longer than was planned. How many days would it have taken each to do the work alone?

**20.** A reservoir is filled by means of two pipes. If one is open for 6 hours and the other is open for 7 hours, the reservoir will be filled. If one is open for 2 hours and the other for 12 hours, the reservoir will be filled. How long would it take each pipe alone to fill the reservoir?

**21.** The bases of two rectangles are 7 and 11 feet respectively. The combined area of the two is 109 square feet. The difference between the two areas is 67 square feet. What is the area of each?

* If a further extension of work on loci is desired, the latter portion of Part III of Chapter VI of Geometry may now be studied.

**22.** An aëroplane went $66\frac{2}{3}$ miles in 40 minutes with the wind and $53\frac{1}{3}$ miles in the same time against it. What was the velocity of the wind? What was the velocity of the machine?

**23.** The sum of the two digits of a number is 1 more than 3 times the units digit. If $\frac{1}{2}$ be added to $\frac{1}{2}$ of the number, the digits will be reversed. What is the number?

**24.** A boatman can row 12 miles upstream in 3 hours, and 10 miles downstream in 1 hour. What is the rate of the stream and what is the rate of rowing in still water?

**25.** A number is composed of two digits whose sum is 13. If 27 be added to the number, the sum will be a new number with the digits reversed. What is the number?

**26.** An ornament made of 5 grams of brass and 2 grams of gold weighs 80.6 grams. If it had been made of 2 grams of brass and 5 grams of gold, it would have weighed 113.3 grams. What is the density of brass and of gold?

**27.** A tank is filled by two pipes, when both are running together, in 9 hours and 36 minutes. If the smaller is allowed to run for 3 hours, and then both are allowed to run, it will take 8 hours and 24 minutes after the larger one is opened to fill the tank. How long would it take each running alone to fill the tank?

**28.** A tank is filled by two pipes and emptied by one. The capacity of the smaller filling pipe and the emptying pipe is the same. If the two filling pipes are open and the emptying pipe is closed, the tank is filled in 7 hours and $8\frac{4}{7}$ minutes. If the smaller filling pipe is closed and the larger filling pipe and the emptying pipe are open, the tank is filled in 16 hours and 40 minutes. How long will it take each filling pipe alone to fill the tank?

**29.** The boys of a club were making an assessment among themselves for the expenses of a picnic. After fixing the charge for each boy, 4 said that they could not go, so it was necessary to tax the rest 10 cents more per boy. Later 4 more decided not to go. It was necessary to make another increase in taxes per boy of 15 cents. How many boys were there in the club, and how much was the original assessment per member?

**30.** If a rectangle of given area had 10 feet added to its base and 5 feet subtracted from its altitude its area would not be changed. Again if it had 5 feet subtracted from its base and 4 feet added to its altitude, its area would not be changed. What is the length of the base and the altitude of the rectangles?

**31.** If $19\frac{1}{2}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose one pound when weighed in water, how much silver and how much gold is there in a mass which weighs 20 pounds and loses $1\frac{1}{4}$ pounds when weighed in water?

**32.** There are two angles the sum of whose complements is $75\frac{1}{2}$ degrees and the difference of whose supplements is $28\frac{1}{2}$ degrees. What is the number of degrees in each angle?

**33.** There are two angles such that the supplement of the first is 8 degrees less than three times the complement of the second, and the complement of the first is $20\frac{1}{2}$ degrees more than $\frac{1}{2}$ of the supplement of the second. What is the size of each angle?

## CHAPTER XVII

## EXPONENTS AND RADICALS *

**109. Definitions.** We have already used the following abbreviations.

$a^3$ stands for $a \cdot a \cdot a$; $a^4$ stands for $a \cdot a \cdot a \cdot a$, etc. Likewise:

$a^m$ stands for $a \cdot a \cdot a \cdot \ldots$ to $m$ factors, $m$ being a positive integer;

$a^n$ stands for $a \cdot a \cdot a \cdot \ldots$ to $n$ factors, $n$ being a positive integer.

**110. To multiply $a^m$ by $a^n$.** By direct multiplication we see that:

$$a^3 \times a^4 = a \cdot a \cdot a \times a \cdot a \cdot a \cdot a = a^7 = a^{3+4};$$
$$a^2 \times a^5 = a \cdot a \times a \cdot a \cdot a \cdot a \cdot a = a^7 = a^{2+5};$$

and so in general,

$$\begin{aligned} a^m \times a^n &= a \cdot a \cdot a \cdot \ldots \text{ to } m \text{ factors } \times \\ &\qquad a \cdot a \cdot a \cdot \ldots \text{ to } n \text{ factors} \\ &= a \cdot a \cdot a \cdot a \cdot \ldots \text{ to } m + n \text{ factors} \\ &= a^{m+n}. \end{aligned}$$

So we have our first rule.

*Rule I. To multiply $a^m$ by $a^n$, add the exponents; the product is $a^{m+n}$.*

As an equation:

$$a^m \cdot a^n = a^{m+n}.$$

* For closer correlation it would be well to take up Chapter VII of Geometry before going on with the study of exponents.

**Exercises.** Apply this rule to the following products:

1. $2^2 \cdot 2^3$.
2. $3^4 \cdot 3^2$.
3. $4 \cdot 4^3$.
4. $a^5 \cdot a^4$.
5. $r^3 \cdot r^6$.
6. $t^4 \cdot t^5$.
7. $(-b)^2(-b)^3$.
8. $(2a)^3(2a)^4$.
9. $(-3c)^3(-3c)^6$.
10. $a^p \cdot a^q$.
11. $a^p \cdot a^q \cdot a^r$.
12. $x^h \cdot x^k \cdot x^l$.

**111. To multiply $a^m$ by $b^m$.** By pairing off the $a$'s and $b$'s we can easily see that the following results are correct.

$a^2 \times b^2 = a \cdot a \times b \cdot b = ab \cdot ab = (ab)^2$. Why?
$a^3 \times b^3 = a \cdot a \cdot a \times b \cdot b \cdot b = ab \cdot ab \cdot ab = (ab)^3$. Why?

So in general,

$$a^m \times b^m = ab \cdot ab \cdot ab \cdot \ldots \text{ to } m \text{ factors} = (ab)^m.$$

*Rule II. The product of $a^m$ by $b^m$ is equal to $(ab)^m$.*

That is:
$$a^m \cdot b^m = (ab)^m.$$

In the same way we can show that

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m.$$

Do this, and state the rule in words.

*Example* 1.
$$2^3 \cdot 3^3 = (2 \cdot 3)^3 = 6^3 = 216.$$

*Example* 2.
$$\frac{6^3}{3^3} = \left(\frac{6}{3}\right)^3 = 2^3 = 8.$$

**Exercises.** Reduce the following expressions:

1. $3^3 \cdot 4^3$.
2. $2^4 \cdot 5^4$.
3. $6^3 \cdot 5^3$.
4. $4^6 \cdot 5^6$.
5. $2^5 \cdot 3^5 \cdot 5^5$.
6. $7^3 \cdot 5^3 \cdot 2^3$.
7. $\frac{8^5}{4^5}$.
8. $\frac{12^4}{4^4}$.
9. $\frac{10^3}{2^3}$.
10. $\frac{3^2}{6^2}$.
11. $\frac{15^3}{30^3}$.
12. $\frac{27^4}{81^4}$.

**112. To divide $a^m$ by $a^n$.** By cancellation we get such equations as the following:

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3 = a^{5-2};$$

$$\frac{a^7}{a^3} = a^4 = a^{7-3}; \qquad \frac{a^8}{a^5} = a^3 = a^{8-5}.$$

So in general we have the rule,

$$\frac{a^m}{a^n} = a^{m-n}, \text{ provided } m \text{ is greater than } n.$$

Here we subtract the exponent of the denominator from the exponent of the numerator.

Now what is the result when $m$ is less than $n$, that is, when there are fewer $a$'s in the numerator of the fraction than in the denominator? Let us look at some examples.

$$\frac{a^2}{a^5} = \frac{a \cdot a}{a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a \cdot a \cdot a} = \frac{1}{a^3}.$$

Likewise:

$$\frac{a^3}{a^7} = \frac{1}{a^4}; \qquad \frac{a^5}{a^8} = \frac{1}{a^3}.$$

Now we shall use the symbol $a^{-n}$ to stand for $\frac{1}{a^n}$; for example:

$$a^{-3} = \frac{1}{a^3}; \quad a^{-4} = \frac{1}{a^4}; \text{ and so on.}$$

Then we have

$$\frac{a^2}{a^5} = \frac{1}{a^3} = a^{-3} = a^{2-5}.$$

$$\frac{a^3}{a^7} = \frac{1}{a^4} = a^{-4} = a^{3-7};$$

$$\frac{a^5}{a^8} = \frac{1}{a^3} = a^{-3} = a^{5-8}.$$

We see again that we get the result by subtracting the exponent of the denominator from the exponent of the numerator.

*Rule III. To divide $a^m$ by $a^n$, subtract the exponent of the divisor from the exponent of the dividend; the quotient is $a^{m-n}$, whether $m$ is greater or less than $n$.*

**Exercises.** Apply this rule to the following quotients:

1. $\frac{a^7}{a^2}$. 3. $\frac{x^2}{x^4}$. 5. $\frac{2^4}{2^7}$. 7. $\frac{(2\,a)^8}{(2\,a)^5}$.

2. $\frac{a^6}{a^3}$. 4. $\frac{x^6}{x^9}$. 6. $\frac{3^5}{3^3}$. 8. $\frac{(3\,b)^7}{(3\,b)^9}$.

9. $\frac{a^2 \cdot a^5}{a^3 \cdot a^7}$. (First reduce numerator and denominator by Rule I.)

10. $\frac{a^4 \cdot a^7}{a^3 \cdot a^9}$. 11. $\frac{b^5 \cdot b^7}{b^3 \cdot b^4}$. 12. $\frac{x^4 \cdot x^2 \cdot x^5}{x^8 \cdot x^3 \cdot x^6}$.

**113. The Meaning of $a^0$.** Suppose that in Rule III $n$ is equal to $m$; for example, suppose $m = 3$ and $n = 3$. Following the rule, we have

$$\frac{a^3}{a^3} = a^{3-3} = a^0.$$

But $$\frac{a^3}{a^3} = 1; \text{ therefore } a^0 = 1.$$

In the same way

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But $$\frac{a^m}{a^m} = 1; \text{ therefore } a^0 = 1.$$

That is, we must regard $a^0$ as equal to 1, if Rule III is to apply to cases where $m$ and $n$ are equal. So we have

$$a^0 = 1.$$

**114. The Meaning of $(a^m)^n$.** The expression $(a^m)^n$ stands for the quantity

$$a^m \cdot a^m \cdot a^m \cdot \ldots \text{ to } n \text{ factors.}$$

But $$a^m = a \cdot a \cdot a \cdot \ldots \text{ to } m \text{ factors.}$$

Therefore $(a^m)^n = a \cdot a \cdot a \cdot a \cdot \ldots$ to $m \times n$ factors;
that is, $(a^m)^n = a^{m\ n} = a^{mn}$.

*Rule IV. To raise the expression $a^m$ to the nth power, multiply its exponent by n.*

As an equation,

$$(a^m)^n = a^{mn}.$$

The same rule is true when $n$ is negative, or when $m$ is negative, or when both $m$ and $n$ are negative. This will be seen from the following examples.

1. To show that $(a^4)^{-3} = a^{4\times(-3)} = a^{-12}$.

$$(a^4)^{-3} = \frac{1}{a^4 \cdot a^4 \cdot a^4} = \frac{1}{a^{12}} = a^{-12}.$$

2. To show that $(a^{-4})^3 = a^{-4\times 3} = a^{-12}$.

$$(a^{-4})^3 = a^{-4} \cdot a^{-4} \cdot a^{-4} = \frac{1}{a^4}\cdot\frac{1}{a^4}\cdot\frac{1}{a^4}\cdot = \frac{1}{a^{12}} = a^{-12}.$$

3. To show that $(a^{-4})^{-3} = a^{-4\times(-3)} = a^{12}$.

$$(a^{-4})^{-3} = \frac{1}{(a^{-4})^3} = \frac{1}{a^{-12}} = \frac{1}{\frac{1}{a^{12}}} = a^{12}.$$

**115. Summary of Results.** Collecting our results we have:

**Rule I.** $a^m \cdot a^n = a^{m+n}$.

**Rule II.** $a^m \cdot b^m = (ab)^m$; $\quad \frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$.

**Rule III.** $a^m \div a^n = a^{m-n}$.

**Rule IV.** $(a^m)^n = a^{mn}$.

Also: $a^{-n} = \frac{1}{a^n}$; $\quad a^0 = 1$.

Here $m$ and $n$ may be either positive or negative.

*Examples.*

1. Multiply $2\,a^3b^2c^4$ by $3\,a^2b^5c^5$.

Multiplying together the like factors we have

$$2\,a^3b^2c^4 \times 3\,a^2b^5c^5 = 6\,a^5b^7c^9.$$

Similarly,

$$5\,a^{-4}b^2c^{-3} \times 4\,a^3b^{-5}c^{-2} = 20\,a^{-1}b^{-3}c^{-5} = \frac{20}{ab^3c^5}.$$

2. $\dfrac{6\,p^5q^6r^{-3}}{2\,p^{-3}q^4\,r^{-2}} = 3\,p^{5-(-3)}q^{6-4}r^{-3-(-2)} = 3\,p^8q^2r^{-1} = \dfrac{3\,p^8q^2}{r}.$

3. $(x^{-2}+y^{-2})\,(x^{-2}-y^{-2}) = (x^{-2})^2 - (y^{-2})^2 = x^{-4} - y^{-4}.$

4. $\dfrac{h^{-6}-k^{-6}}{h^{-2}-k^{-2}} = \dfrac{(h^{-2})^3-(k^{-2})^3}{h^{-2}-k^{-2}} = (h^{-2})^2 + h^{-2}k^{-2} + (k^{-2})^2$

$$= h^{-4} + h^{-2}k^{-2} + k^{-4},$$

$$= \frac{1}{h^4} + \frac{1}{h^2k^2} + \frac{1}{k^4},$$

$$= \frac{k^4 + h^2k^2 + h^4}{h^4k^4}.$$

5. Simplify $[(-2\,x^3)^{-4}]^6 \div [64\,(x^2)^4]^{-5}$

Starting with the dividend:

$$(-2\,x^3)^{-4} = (-2)^{-4}\cdot x^{-12}.$$

Then $[(-2\,x^3)^{-4}]^6 = [(-2)^{-4}x^{-12}]^6 = (-2)^{-24}x^{-72} = 2^{-24}x^{-72}.$

Simplifying the divisor:

$$[64\,(x^2)^4]^{-5} = [2^6\,x^8]^{-5} = 2^{-30}\,x^{-40}.$$

Therefore

$$[(-2\,x^3)^{-4}]^6 \div [64\,(x^2)^4]^{-5} = 2^{-24}x^{-72} \div 2^{-30}x^{-40} = 2^6x^{-32} = \frac{64}{x^{32}}.$$

**Exercises.** Simplify the following; write answers with negative exponents, and also without negative exponents.

**1.** $x^2y^3 \cdot x^3y^2.$

**2.** $a^{-3}b^{-2}\cdot a^{-2}b^{-3}.$

**3.** $a^{-4} \div a^{-6}.$

**4.** $5\,m^3 \div 5^2\,m^{-2}.$

**5.** $p^0q^{-2}\cdot 2^{-2}q^3.$

**6.** $c^0d^5 \div 3^{-2}d^3.$

**7.** $2\,rs^2\cdot s^{-2}.$

**8.** $3\,a^{-1} \div 3^{-2}a^{-3}.$

**9.** $(2\,u)^2\cdot(3\,u^3)\cdot u^{-5}.$

**10.** $x^0 + y^0.$

**11.** $x^0 - y^0.$

**12.** $5^3\,(5\,x^{-1})^{-3}.$

**13.** $(2^2)^{-2}.$

**14.** $[(2^2)^{-2}]^2.$

**15.** $4^{-4}\cdot 2^8.$

**16.** $\dfrac{6^5}{2^4\cdot 3^4}.$

**17.** $\dfrac{8^3\cdot 4^{-3}}{2^7\cdot 2^{-5}}.$

**18.** $\dfrac{16\,a^{-2}b^{-3}}{2^5\,a^{-3}b^{-4}}.$

**19.** $(a^2+b^2)^2.$

**20.** $(x^3-y^3)^2.$

**21.** $(u^{-1}+v^{-1})^2.$

**22.** $(a^n+b^n)^2.$

**23.** $(x^m-y^m)^2.$

**24.** $(p^{-n}+q^{-n})^2.$

**25.** $x^{-4}(1+x^2).$

**26.** $x^3(1-x^{-3}).$

**27.** $(r^2s^{-2}-1)^2.$

**28.** $\dfrac{a^{2m-2}}{a^{2m-3}}.$

**29.** $\dfrac{a^{2m}b^{n+1}}{a^mb^{2n-1}}.$

**30.** $\dfrac{u^{1-n}v^{1+n}}{u^{n+1}v^{n-1}}.$

**31.** The edge of one cube is $x$ and of another cube $2x$; what is the ratio of their volumes?

**32.** What is the ratio of the volume of two cubes whose edges are $2x$ and $3x$ respectively?

**33.** What is the ratio of the volume of a sphere whose radius is $r$ to a cube whose edge is $r$?

**34.** What is the ratio of the volume of a sphere of radius $r$ to the volume of a cylinder of radius $r$ and height $h$?

Factor the following:

**35.** $(1 - x^2)$. **38.** $(a^{-2} - b^{-2})$. **41.** $(x^3 - y^3)$.

**36.** $(1 - x^{-2})$. **39.** $(p^4 - q^4)$. **42.** $(x^{-3} - y^{-3})$.

**37.** $(a^2 - b^2)$. **40.** $(p^{-4} - q^{-4})$. **43.** $(x^{-3} + y^{-3})$.

**116. Fractional Exponents.**

*Example* 1. What is the side of a square whose area is 2? We know that the answer must be a number which when multiplied by itself will give 2. We express this number by the symbol $2^{\frac{1}{2}}$ (or by $\sqrt{2}$).

Similarly if the area of a square is $a$, its side is expressed by $a^{\frac{1}{2}}$. Here $a^{\frac{1}{2}}$ is a symbol used to indicate the number whose square is $a$; as an equation,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a.$$

Rule I applies here; for using this rule,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

*Example* 2. What is the edge of a cube whose volume is 2? This must be a number which when multiplied by itself three times will give 2. We express this number by the symbol $2^{\frac{1}{3}}$ (or by $\sqrt[3]{2}$). It is called the cube root of 2.

If the volume of the cube is $a$, its edge is expressed by the symbol $a^{\frac{1}{3}}$; it is called the cube root of $a$. We must then have

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a.$$

Rule I again applies; for it gives

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a.$$

Following out this idea, we define the symbol $a^{\frac{1}{n}}$ as follows, $n$ being a positive integer.

*Definition.* The symbol $a^{\frac{1}{n}}$ (or $\sqrt[n]{a}$) expresses that number which when multiplied by itself $n$ times will give $a$; it is called *the nth root of a.* That is,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot \ . \ . \ . \text{ to } n \text{ factors} = a.$$

This is in accordance with Rule I; for

$$\begin{aligned} a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot \ . \ . \ . & \text{ to } n \text{ factors} \\ &= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+ \ . \ . \ . \text{ to } n \text{ terms}} \\ &= a^{n \times \frac{1}{n}} = a^1 = a. \end{aligned}$$

We next obtain a meaning for a symbol such as $a^{\frac{3}{2}}$. In order to make Rule I apply to such cases we think of $a^{\frac{3}{2}}$ as follows:

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = a^{\frac{3}{2}}.$$

That is, we regard $a^{\frac{3}{2}}$ as the cube of $a^{\frac{1}{2}}$.

In exactly the same way we regard $a^{\frac{m}{n}}$ as the $m$th power of $a^{\frac{1}{n}}$.

We may also, using Rule IV, regard $a^{\frac{3}{2}}$ as follows:

$$(a^3)^{\frac{1}{2}} = a^{3 \times \frac{1}{2}} = a^{\frac{3}{2}}.$$

That is, $a^{\frac{3}{2}}$ is the same as the square root of $a^3$.

Similarly, if Rule IV is used,

$$(a^m)^{\frac{1}{n}} = a^{m \times \frac{1}{n}} = a^{\frac{m}{n}}.$$

So we regard $a^{\frac{m}{n}}$ as the $n$th root of $a^m$.

We may thus think of $a^{\frac{m}{n}}$ as a symbol representing the $m$th power of the $n$th root of $a$, or as representing the $n$th root of the $m$th power of $a$.

In algebraic language this is:

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (a^m)^{\frac{1}{n}}.$$

**Exercises.** Express each of the following in the two forms just given; state each result in words.

| | | | |
|---|---|---|---|
| **1.** $3^{\frac{3}{2}}$. | **5.** $27^{\frac{5}{3}}$. | **9.** $a^{\frac{5}{2}}$. | **13.** $x^{\frac{3}{4}}$. |
| **2.** $2^{\frac{2}{3}}$. | **6.** $81^{\frac{3}{4}}$. | **10.** $a^{\frac{2}{5}}$. | **14.** $y^{\frac{6}{5}}$. |
| **3.** $5^{\frac{3}{4}}$. | **7.** $(\frac{1}{8})^{\frac{2}{3}}$. | **11.** $b^{\frac{7}{2}}$. | **15.** $c^{\frac{5}{7}}$. |
| **4.** $32^{\frac{3}{5}}$. | **8.** $(\frac{1}{16})^{\frac{3}{4}}$. | **12.** $b^{\frac{2}{7}}$. | **16.** $(a+b)^{\frac{3}{5}}$. |

**17.** If the volume of a cube is 8, show that the area of one of its faces is $8^{\frac{2}{3}} = 4$. If the volume is $v$, show that the area of one face is $v^{\frac{2}{3}}$.

**18.** If the area of one face of a cube is 9, show that its volume is $9^{\frac{3}{2}} = 27$. If the area of one face is $a$, show that the volume is $a^{\frac{3}{2}}$.

A negative fractional exponent we define in the same manner as a negative integral exponent; that is,

$$\text{the symbol } a^{-\frac{m}{n}} \text{ means } \frac{1}{a^{\frac{m}{n}}}.$$

Thus: $$a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}; \quad a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}};$$

$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}} = \frac{1}{(16^{\frac{1}{4}})^3} = \frac{1}{2^3} = \frac{1}{8}.$$

It is easy to see that Rule II applies also to the case of fractional exponents. By Rule II we would have

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}, \quad \text{and} \quad \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

But both of these equations are correct; for raising both sides of each equation to the $n$th power, these equations become

$$ab = ab, \text{ and } \frac{a}{b} = \frac{a}{b}.$$

*Example.* $\sqrt[3]{54} = (54)^{\frac{1}{3}} = (27 \cdot 2)^{\frac{1}{3}} = 27^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} = 3\sqrt[3]{2}$.

**Exercises.** Reduce each of the following as in the last example.

1. $\sqrt[3]{16}$.
3. $32^{\frac{1}{4}}$.
5. $\sqrt[3]{40}$.
7. $24^{\frac{1}{2}}$.
2. $\sqrt[3]{81}$.
4. $64^{\frac{1}{4}}$.
6. $\sqrt[4]{80}$.
8. $108^{\frac{1}{3}}$.

Let us now examine a product like $a^{\frac{1}{m}} \cdot a^{\frac{1}{n}}$. We can easily show that (work this through, taking $m = 2$ and $n = 3$.)

$$a^{\frac{1}{m}} \cdot a^{\frac{1}{n}} \cdot = a^{\frac{1}{m}+\frac{1}{n}}.$$

For
$$a^{\frac{1}{m}} = a^{\frac{n}{mn}} = \left(a^{\frac{1}{mn}}\right)^n;$$
$$a^{\frac{1}{n}} = a^{\frac{m}{mn}} = \left(a^{\frac{1}{mn}}\right)^m;$$

also,
$$a^{\frac{1}{m}+\frac{1}{n}} = a^{\frac{m+n}{mn}} = \left(a^{\frac{1}{mn}}\right)^{m+n}.$$

Then
$$a^{\frac{1}{m}} \cdot a^{\frac{1}{n}} = \left(a^{\frac{1}{mn}}\right)^n \cdot \left(a^{\frac{1}{mn}}\right)^m$$
$$= \left(a^{\frac{1}{mn}}\right)^{m+n}$$
$$= a^{\frac{m+n}{mn}}$$
$$= a^{\frac{1}{m}+\frac{1}{n}}.$$

Show in a similar manner that Rule III applies to fractional exponents; that is, that

$$a^{\frac{1}{m}} \div a^{\frac{1}{n}} = a^{\frac{1}{m}-\frac{1}{n}}.$$

Again work out the case where $m = 2$ and $n = 3$.

Finally we can show that Rules I and III apply in general as follows:

*Example.* Show that

$$a^{\frac{2}{3}} \cdot a^{\frac{4}{5}} = a^{\frac{2}{3}+\frac{4}{5}}.$$

*Proof.* The common denominator of the exponents is 15. Raise both sides of the equation to the 15th power. We get

$$(a^{\frac{2}{3}})^{15} \cdot (a^{\frac{4}{5}})^{15} = (a^{\frac{2}{3}+\frac{4}{5}})^{15}.$$

Simplifying we get

$$a^{10} \cdot a^{12} = a^{10+12},$$

which is a true equation.

Prove in the same way that

$$a^{\frac{2}{3}} \div a^{\frac{4}{5}} = a^{\frac{2}{3}-\frac{4}{5}}.$$

Similarly we show that

$$a^{\frac{p}{q}} \cdot a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}};$$

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

Prove these by raising both sides of each equation to the $q \cdot s$th power. Does this proof require that the letters $p$, $q$, $r$, $s$, stand for positive numbers? Try a numerical example.

Finally let us consider Rule IV, when $m$ and $n$ are fractional numbers.

*Example.* Verify that $(a^{\frac{5}{4}})^{\frac{2}{3}} = a^{\frac{5}{4} \times \frac{2}{3}} = a^{\frac{5}{6}}$.

Raising to the 6th power,

$$(a^{\frac{5}{4}})^4 = a^5 \quad \text{(Why?)}$$

Taking the fourth root,

$$a^{\frac{5}{4}} = a^{\frac{5}{4}}. \quad \text{(Why?)}$$

Similarly we can verify Rule IV when $m$ and $n$ are any two fractional numbers.

So we see that the rules on p. 267 apply to fractional as well as to integral exponents.

**Exercises.** Simplify the following expressions, giving the reason for each step.

**1.** $4^{\frac{1}{6}} \cdot 4^{\frac{1}{3}}$.
**4.** $a^{\frac{3}{2}} \cdot a^{\frac{1}{4}}$.
**7.** $s^{-\frac{3}{5}} \div s^{\frac{7}{10}}$.

**2.** $8^{\frac{2}{3}} \cdot 8^{-\frac{1}{3}}$.
**5.** $x^{\frac{2}{5}} \cdot x^{\frac{1}{10}}$.
**8.** $(b^{\frac{5}{3}})^{\frac{3}{5}}$.

**3.** $16^{\frac{3}{4}} \div 8^{\frac{2}{3}}$.
**6.** $c^{\frac{1}{2}} \cdot c^{\frac{1}{3}} \cdot c^{\frac{1}{4}}$.
**9.** $(m^{-\frac{3}{2}})^{\frac{3}{4}}$.

**10.** Show that $a^{\frac{2}{3}} - b^{\frac{2}{3}} = (a^{\frac{1}{3}} + b^{\frac{1}{3}})(a^{\frac{1}{3}} - b^{\frac{1}{3}})$.

**11.** Factor $x^{\frac{4}{3}} - y^{\frac{4}{3}}$.
**12.** Factor $u^{\frac{6}{5}} - v^{\frac{6}{5}}$.

**13.** Multiply $a^{\frac{1}{3}} + b^{\frac{1}{3}}$ by $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

**14.** Multiply $x^{\frac{2}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.

**15.** Divide $x - y$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

**16.** Divide $u^2 - v^2$ by $\sqrt{u} - \sqrt{v}$.

**17.** Solve for $x$: $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$. (Regard $x^{\frac{1}{3}}$ as unknown.)

**18.** Solve for $x$: $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} - 8 = 0$.

**19.** Is $\sqrt{a^2 + b^2}$ equal to $a + b$? Is $\sqrt[3]{a^3 + b^3}$ equal to $a + b$?

### 117. Irrational Numbers or Surds.

*Definitions.* Numbers such as $3\sqrt{2}$, $4\sqrt[3]{5}$, etc., which involve an indicated root that can not be exactly found, are called **irrational numbers** or **surds.** See p. 176.

The number under the radical sign is called the base of the surd, and the number indicating the root to be extracted is called the index of the surd.

Surds expressed by the same base and index are called **similar surds.**

Thus: $4\sqrt[3]{5}$, $7\sqrt[3]{5}$, $-3\sqrt[3]{5}$ are similar surds.

Also: $\sqrt[3]{5}$ and $\sqrt[3]{10}$ are dissimilar surds.

### 118. Examples Involving Surds.

*Example* 1. Reduce $\sqrt{48}$, $\sqrt{75}$, and $\sqrt{27}$ to similar surds.

These can all be expressed as multiples of $\sqrt{3}$.

For: $\sqrt{48} = \sqrt{16 \cdot 3} = 4\sqrt{3}$. (Why?)

$$\sqrt{75} = \sqrt{25 \cdot 3} = 5\sqrt{3};\quad \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3}.$$

*Example* 2. Simplify $\sqrt{48} + \sqrt{75} - \sqrt{27}$.

$$\sqrt{48} + \sqrt{75} - \sqrt{27} = 4\sqrt{3} + 5\sqrt{3} - 3\sqrt{3} = 6\sqrt{3}.$$

*Example* 3. Simplify $16^{\frac{2}{3}} - 54^{\frac{2}{3}} + 128^{\frac{2}{3}}$.

$$16^{\frac{2}{3}} = 8^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} = 4 \cdot 2^{\frac{2}{3}},$$

$$54^{\frac{2}{3}} = 27^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} = 9 \cdot 2^{\frac{2}{3}},$$

$$128^{\frac{2}{3}} = 64^{\frac{2}{3}} \cdot 2^{\frac{2}{3}} = 16 \cdot 2^{\frac{2}{3}}.$$

$$16^{\frac{2}{3}} - 54^{\frac{2}{3}} + 128^{\frac{2}{3}} = 4 \cdot 2^{\frac{2}{3}} - 9 \cdot 2^{\frac{2}{3}} + 16 \cdot 2^{\frac{2}{3}} = 11 \cdot 2^{\frac{2}{3}}.$$

*Rule. To combine surds by addition or subtraction, they must first be reduced to their simplest forms. Then if they are similar they may be combined; not otherwise.*

**Exercises.** Simplify the following.

1. $5\sqrt{8} + 3\sqrt{18}$.
2. $3\sqrt[3]{81} - 2\sqrt[3]{24}$.
3. $\frac{1}{2}\sqrt{125} + \frac{1}{3}\sqrt{20}$.
4. $\sqrt{\frac{1}{3}} + \sqrt{3}$.
5. $2 \cdot 63^{\frac{1}{2}} + 5 \cdot 28^{\frac{1}{2}} - 175^{\frac{1}{2}}$.
6. $3 \cdot 44^{\frac{1}{2}} + \frac{1}{3} \cdot 99^{\frac{1}{2}} + \frac{1}{3} \cdot 275^{\frac{1}{2}}$.
7. $(\frac{1}{3})^{\frac{1}{2}} - (\frac{27}{16})^{\frac{1}{2}} - (\frac{9}{48})^{\frac{1}{2}}$.
8. $a(18a^3b)^{\frac{1}{2}} - (a^2 - b^2)(2ab)^{\frac{1}{2}}$.
9. $3\sqrt[3]{54} - 5\sqrt[3]{16} - 2\sqrt[3]{432}$.
10. $(\frac{3}{16})^{\frac{1}{5}} - (\frac{2}{81})^{\frac{1}{5}}$.
11. $2ab\sqrt[6]{a^7b} - 2\sqrt[6]{a^{13}b^7}$.
12. $5m\sqrt[3]{27m^4} - 3\sqrt[3]{8m^7}$.

*Example* 4. Find the value of $\frac{\sqrt{3}}{2 - \sqrt{3}}$ to two decimals.

Rationalizing the denominator:

$$\frac{\sqrt{3}}{2-\sqrt{3}} = \frac{\sqrt{3}}{2-\sqrt{3}} \cdot \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{2\sqrt{3}+3}{4-3} = 2\sqrt{3} + 3.$$

Now replace $\sqrt{3}$ by its approximate value, 1.73;

$$\frac{\sqrt{3}}{2-\sqrt{3}} = 2 \cdot 1.73 + 3 = 6.46.$$

**Exercises.** Calculate each of the following to two decimal places.

1. $\frac{2}{\sqrt{3}}$.
2. $\frac{1}{\sqrt{5}}$.
3. $\sqrt{\frac{8}{5}}$.
4. $\frac{3}{\sqrt{45}}$.
5. $\frac{3}{7-\sqrt{6}}$.
6. $\frac{2+\sqrt{5}}{2-\sqrt{5}}$.
7. $\frac{2-2\sqrt{3}}{3+\sqrt{3}}$.
8. $\frac{10}{\sqrt{3}+\sqrt{8}}$.
9. $\frac{1}{\sqrt{27}-\sqrt{18}}$.
10. $\frac{\sqrt{11}+\sqrt{7}}{\sqrt{11}-\sqrt{7}}$.

Rationalize the denominators in the following:

**11.** $\dfrac{a}{\sqrt{a}-\sqrt{b}}$. **13.** $\dfrac{b}{1+b^{\frac{1}{2}}}$. **15.** $\dfrac{1+b}{b-b^{\frac{1}{2}}}$.

**12.** $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}$. **14.** $\dfrac{\sqrt{m}+\sqrt{2}}{\sqrt{m}-\sqrt{2}}$. **16.** $\dfrac{\sqrt{x^3}-2}{2+\sqrt{x^3}}$.

*Example* 5. Reduce $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{3}$ to surds with the same index.

Write each surd with a fractional exponent, thus

$$2^{\frac{1}{2}},\quad 5^{\frac{1}{3}},\quad 3^{\frac{1}{4}}.$$

Reduce these exponents to the least common denominator:

$$2^{\frac{1}{2}} = 2^{\frac{6}{12}};\quad 5^{\frac{1}{3}} = 5^{\frac{4}{12}};\quad 3^{\frac{1}{4}} = 3^{\frac{3}{12}}.$$

But
$$2^{\frac{6}{12}} = \sqrt[12]{2^6} = \sqrt[12]{64};$$
$$5^{\frac{4}{12}} = \sqrt[12]{5^4} = \sqrt[12]{625};$$
$$3^{\frac{3}{12}} = \sqrt[12]{3^3} = \sqrt[12]{27}.$$

So the three given surds are equal respectively to

$$\sqrt[12]{64},\quad \sqrt[12]{625},\quad \sqrt[12]{27}.$$

These have the common index 12.

**Exercises.** Reduce the following to surds with the same index.

1. $\sqrt{3}$, $\sqrt[3]{2}$.
2. $\sqrt[3]{4}$, $\sqrt{5}$.
3. $\sqrt[3]{2}$, $\sqrt[4]{3}$.
4. $\sqrt{3}$, $\sqrt[3]{2}$, $\sqrt[4]{5}$.
5. $\sqrt{\frac{1}{2}}$, $\sqrt[4]{\frac{1}{3}}$, $\sqrt[5]{\frac{1}{4}}$.
6. $\sqrt[3]{\frac{2}{5}}$, $\sqrt{\frac{1}{3}}$, $\sqrt[6]{\frac{3}{4}}$.
7. $\sqrt{a^3}$, $\sqrt[4]{a}$, $\sqrt[5]{a^2}$.
8. $\sqrt[3]{a^2b}$, $\sqrt[6]{ab^2}$, $\sqrt[4]{ab}$.

*Example* 6. Multiply $\sqrt[3]{4a^2}$ by $\sqrt[4]{8a^3}$.

First reduce to same index:

$$\sqrt[3]{4a^2} = \sqrt[12]{(4a^2)^4} = \sqrt[12]{4^4a^8} = \sqrt[12]{2^8a^8}.$$
$$\sqrt[4]{8a^3} = \sqrt[12]{(8a^3)^3} = \sqrt[12]{8^3a^9} = \sqrt[12]{2^9\cdot a^9}.$$

Then
$$\sqrt[3]{4a^2}\cdot\sqrt[4]{8a^3} = \sqrt[12]{2^8a^8}\cdot\sqrt[12]{2^9a^9}$$
$$= \sqrt{2^{17}\cdot a^{17}}$$
$$= 2a\sqrt[12]{32a^5}.$$

Similarly in division of surds it is usually best to reduce to the same index.

**Exercises.** Perform the following operations, reducing all results to simplest form.

1. $\sqrt{3} \cdot \sqrt[3]{3}$.
2. $\sqrt[3]{2} \cdot \sqrt[4]{3}$.
3. $\sqrt{\frac{1}{3}} \cdot \sqrt[3]{\frac{1}{2}}$.
4. $\sqrt[3]{a^2} \cdot \sqrt{a^3}$.
5. $\sqrt{m} \cdot \sqrt[3]{m}$.
6. $\sqrt[4]{x^3} \cdot \sqrt[3]{x^4}$.
7. $\sqrt[5]{3x^2} \cdot \sqrt[3]{2x^4}$.
8. $a\sqrt[4]{6a} \cdot \sqrt{3a}$.
9. $\sqrt[3]{5ab} \cdot \sqrt{10a^2b^2}$.
10. $\dfrac{\sqrt{a^5}}{\sqrt[3]{a}}$.
11. $\dfrac{\sqrt[5]{x^2}}{\sqrt{x^5}}$.
12. $\dfrac{\sqrt[3]{b^2}}{\sqrt[5]{b^3}}$.
13. $\dfrac{\sqrt[3]{m^5}}{\sqrt[4]{m^7}}$.
14. $\dfrac{\sqrt[6]{u^3}}{\sqrt[9]{u^8}}$.
15. $\dfrac{\sqrt[7]{8a^2}}{\sqrt[3]{2a}}$.

**119. Equations with Irrational Terms.** Occasionally equations arise which contain irrational terms. Several times our equations have contained one irrational term. You saw readily that these could be easily handled by writing the irrational term as one member of the equation, and the rational terms as the other member; you could then easily rid your equation of the irrational term by squaring both members.

When an equation contains two irrational terms, it is customary to write one of the irrational terms as one member and the other irrational term together with the rational terms as the other member. Squaring both members of the equation will rid the equation of one of the irrational terms. After this the equation can be rid of the other irrational term as above described. The following illustrations will show these processes.

*Example* 1. Given $(x + 3)^{\frac{1}{2}} = 7$.

Squaring: $x + 3 = 49;$

$x = 46.$

Check by substituting in the first member of the equation.

$$(46 + 3)^{\frac{1}{2}} = 49^{\frac{1}{2}} = 7,$$

as the problem states that it should.

*Example* 2. Given

$$(x+3)^{\frac{1}{2}}+(x-6)^{\frac{1}{2}}=9,$$

$$(x+3)^{\frac{1}{2}}=9-(x-6)^{\frac{1}{2}},$$

Squaring:
$$x+3=x+75-18\,(x-6)^{\frac{1}{2}},$$

$$-72=-18\,(x-6)^{\frac{1}{2}},$$

$$4=(x-6)^{\frac{1}{2}},$$

$$16=x-6,$$

$$x=22.$$

Check by substituting in the first member of the equation.

$$(22+3)^{\frac{1}{2}}+(22-6)^{\frac{1}{2}}=5+4=9,$$

as the given equation calls for.

*Example* 3. Given

$$\sqrt{2\,x+2}-\sqrt{4\,x-3}+\sqrt{x-6}=0.$$

$$\sqrt{2\,x+2}=\sqrt{4\,x-3}-\sqrt{x-6}.$$

Squaring:
$$2\,x+2=5\,x-9-2\sqrt{4\,x-3}\,\sqrt{x-6},$$

$$-3\,x+11=-2\sqrt{4\,x-3}\,\sqrt{x-6}.$$

Squaring:
$$9\,x^2-66\,x+121=16\,x^2-108\,x+72.$$

$$7\,x^2-42\,x-49=0,$$

$$x^2-\ \ 6\,x-\ \ 7=0.$$

Solve and finish as in past examples of this kind. Check as in preceding examples.

**Exercises.** Solve the following; check all answers.

**1.** $(2\,x+9)^{\frac{1}{2}}=5.$ **2.** $(5\,x-9)^{\frac{1}{2}}=6.$

**3.** $(2\,x-6)^{\frac{1}{2}}-(5\,x-6)^{\frac{1}{2}}=-3.$

**4.** $\sqrt{3\,x-2}-\sqrt{x-1}-\sqrt{x-8}=0.$

**5.** $\dfrac{(2\,x+1)^{\frac{1}{2}}}{(3\,x+4)^{\frac{1}{2}}}=\dfrac{3}{4}.$ **6.** $\dfrac{5\,x-9}{\sqrt{5\,x}+3}-1=\dfrac{\sqrt{5\,x}-3}{2}.$

**7.** $\dfrac{\sqrt{3\,x+1}+\sqrt{3\,x}}{\sqrt{3\,x+1}-\sqrt{3\,x}}=4.$

## 120. Summary and Exercises for Review.

The symbol $a^m$, when $m$ is a positive integer, means the product $a \cdot a \cdot a \cdot \ldots$ to $m$ factors.

The symbol $a^{\frac{1}{m}}$, when $m$ is a positive integer means that number whose $m$th power is $a$; that is, $a^{\frac{1}{m}} \cdot a^{\frac{1}{m}} \cdot a^{\frac{1}{m}} \cdot \ldots$ to $m$ factors is $a$.

$$a^{-m} \text{ means } \frac{1}{a^m}; \qquad a^{-\frac{1}{m}} \text{ means } \frac{1}{a^{\frac{1}{m}}};$$

$$a^{-\frac{m}{n}} \text{ means } \frac{1}{a^{\frac{m}{n}}}; \qquad \text{in general, } a^{-x} = \frac{1}{a^x}.$$

*Rules.* These rules apply whether the exponents are integral or fractional, positive or negative.

$$a^p \cdot a^q = a^{p+q}.$$
$$a^p \div a^q = a^{p-q}.$$
$$(a^p)^q = a^{pq}.$$
$$a^p b^p = (ab)^p.$$

State these rules in words.

*Surds* (irrational numbers) are numbers involving indicated roots that can not be exactly found. Example, $\sqrt[3]{5}$. Here 5 is called the *base* and 3 the *index*.

Surds expressible by the same base and index are *similar*.

To add or subtract surds, they must be similar.

In working with surds, it is often best to reduce radical forms to fractional exponents. Then apply the rules given above.

### Miscellaneous Exercises for Review.

Simplify the following, and write each result in the form of a single radical.

1. (a) $3^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$; (b) $8^{\frac{2}{3}} \cdot 16^{\frac{1}{3}}$; (c) $a^{\frac{1}{3}}b^{\frac{3}{4}}$; (d) $16^{\frac{3}{4}}\,x^{\frac{2}{5}}$.
2. (a) $3\,(2^{\frac{1}{2}}\,c^{\frac{3}{4}})$; (b) $4\,(a^{\frac{2}{3}}b^{\frac{1}{6}})$; (c) $2^{\frac{1}{8}}\,x^{\frac{1}{2}}y^{\frac{1}{4}}$.
3. (a) $(2\,a^{\frac{1}{2}})^{\frac{3}{4}}$; (b) $5\,(7^{\frac{3}{4}}\,a^{\frac{2}{3}})^{\frac{3}{6}}$; (c) $(m^{\frac{1}{2}}n^{\frac{3}{4}})^{\frac{2}{3}}$.
4. (a) $[(x^{\frac{2}{3}})^{\frac{2}{3}}]^{\frac{2}{3}}$; (b) $[(2\,b)^{\frac{1}{3}}]^{\frac{3}{4}}$; (c) $(c^{\frac{1}{2}}d^{\frac{2}{3}})^{\frac{3}{4}}$.
5. (a) $\dfrac{5\,a^{\frac{1}{3}}}{2\,a^{\frac{1}{2}}}$; (b) $\dfrac{7\,m^{\frac{1}{2}}}{4\,m^{\frac{3}{4}}}$; (c) $\dfrac{2\,a^{\frac{1}{2}}b^{\frac{2}{3}}}{3\,a^{\frac{2}{3}}b^{\frac{3}{4}}}$.

In the following perform indicated operations wherever possible and reduce all results to their simplest forms. State rules and principles used.

6. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$.
7. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$.
8. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})\,(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
9. $(s - t) \div (s^{\frac{1}{2}} - t^{\frac{1}{2}})$.
10. $(a^{2n} - b^{2n}) \div (a^n - b^n)$.
11. $(\sqrt[3]{p^2} - \sqrt{p^3})^2$.
12. $(\sqrt{r} + \sqrt[3]{s})\,(\sqrt{r} - \sqrt[3]{s})$.
13. $(\sqrt{x} + \sqrt{y} + \sqrt{z})^2$.
14. $(m + n)^{\frac{2}{3}}\,(m - n)^{\frac{2}{3}}$.
15. $(m^{\frac{1}{2}} + n^{\frac{1}{2}})\,(m^{-\frac{1}{2}} + n^{-\frac{1}{2}})$.
16. $\sqrt{a^{\frac{2}{3}} - 2\,a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$.
17. $\sqrt{a + b} \cdot \sqrt{a - b}$.
18. $(m + \sqrt{mn} + n)\,(m - \sqrt{mn} + n)$.
19. $(2\,m^{\frac{2}{3}} + 3\,\sqrt{m})\,(3\,m^{\frac{2}{3}} - 2\,\sqrt{m})$.

20. $\dfrac{a^{\frac{2}{3}} - 2\,a^{\frac{1}{3}}b^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{1}{2}} - 2\,b}$.
21. $\dfrac{x^{\frac{1}{3}} - y^{\frac{1}{3}}}{x^{\frac{2}{3}} - y^{\frac{2}{3}}}$.
22. $\dfrac{(\sqrt{m} - \sqrt{n})^2}{m - n}$.
23. $\dfrac{a^{\frac{1}{2}} - b^{\frac{1}{2}}}{a^{\frac{3}{4}} - a^{\frac{1}{4}}b^{\frac{1}{2}}}$.
24. $\dfrac{x - y}{x^{\frac{1}{3}} - y^{\frac{1}{3}}}$.
25. $\dfrac{x + 3\,\sqrt{x} + 2}{x + 6\,\sqrt{x} + 5}$.

*Hint.* $x + 3\,\sqrt{x} + 2 = (\sqrt{x} + 1)\,(\sqrt{x} + 2)$.

26. $\dfrac{y^{\frac{2}{3}} - 4\,y^{\frac{1}{3}} - 4}{y^{\frac{2}{3}} - 5\,y^{\frac{1}{3}} + 6}$.
27. $\dfrac{m^{\frac{4}{5}} - 7\,m^{\frac{2}{5}} + 12}{m^{\frac{4}{5}} - 8\,m^{\frac{2}{5}} + 15}$.
32. $\dfrac{\sqrt[3]{r}}{2\,\sqrt{r}} - \dfrac{\sqrt{r}}{3\sqrt[3]{r}}$.
33. $\dfrac{1}{\sqrt{x} + \sqrt{y}} + \dfrac{1}{\sqrt{x} - \sqrt{y}}$.

28. $\dfrac{r + 10\sqrt{rs} + s}{r - 2\sqrt{rs} - 15s}$.

29. $\dfrac{\sqrt[3]{x^2} - 3\sqrt[3]{x} + 2}{\sqrt[3]{x^2} - 1}$.

30. $\dfrac{p^{\frac{4}{3}} - p^{\frac{2}{3}}q^{\frac{2}{3}} - 20q^{\frac{4}{3}}}{p^{\frac{4}{3}} + p^{\frac{2}{3}}q^{\frac{2}{3}} - 30q^{\frac{4}{3}}}$.

31. $\dfrac{\sqrt{a} + \sqrt{b}}{3\sqrt{a}} - \dfrac{\sqrt{a} - \sqrt{b}}{4\sqrt{a}}$.

34. $\dfrac{2x^{\frac{3}{5}}}{x^{\frac{3}{5}} + x^{\frac{6}{5}}} - 2$.

35. $\dfrac{m^2}{m^2 - m^{-2}} + \dfrac{m^{-2}}{m^2 + m^{-2}}$.

36. $\dfrac{1 + 3s^{-3}}{1 - 3s^{-3}} - \dfrac{1 - 3s^{-3}}{1 + 3s^{-3}}$.

37. $\dfrac{1}{x^{-1} + y^{-1}} + \dfrac{2y^{-1}}{x^{-2} - y^{-2}}$.

38. $\dfrac{a^{-3} - b^{-3}}{2(a^{-3} + b^{-3})} + \dfrac{a^{-6} + b^{-6}}{a^{-6} - b^{-6}}$.

39. $\dfrac{a^{-r}}{x^r(a^{-r} + x^r)} - \dfrac{x^r}{a^{-r}(a^{-r} + x^r)}$.

40. $\dfrac{1}{m^n + 1} - \dfrac{2}{m^n + 2} + \dfrac{1}{m^n + 3}$.

41. $\dfrac{3}{x^3} - \dfrac{5}{2x^3 - 1} - \dfrac{2x^3 - 7}{4x^6 - 1}$.

42. $\dfrac{2}{x^{\frac{1}{3}}} + \dfrac{3}{1 - 2x^{\frac{1}{3}}} - \dfrac{2x^{\frac{1}{3}} - 3}{4x^{\frac{2}{3}} - 1}$.

43. $7\sqrt{2} - \sqrt{18}$.

44. $9\sqrt[3]{5} - 2\sqrt[3]{40}$.

45. $\sqrt{75} + \sqrt{48} - \sqrt{147}$.

46. $5\sqrt[3]{4} + 2\sqrt[3]{32} - \sqrt[3]{108}$.

47. $\sqrt{2} + 3\sqrt{32} + \frac{1}{2}\sqrt{128}$.

48. $3\sqrt[4]{3} - 5\sqrt[4]{48} + \sqrt[4]{243}$.

49. $(\sqrt{2} + \sqrt{7})(\sqrt{2} - \sqrt{7})$.

50. $(1 + \sqrt[3]{2})^3$.

51. $(\sqrt[3]{9} - \sqrt[3]{3})^3$.

52. $\sqrt{4 + \sqrt{7}}\sqrt{4 - \sqrt{7}}$.

53. $\sqrt{6 + \sqrt{11}}\sqrt{6 - \sqrt{11}}$.

54. $(5 + 2\sqrt{3})(3 - 5\sqrt{3})$.

55. $(\sqrt[3]{9} - 2\sqrt[3]{4})(4\sqrt[3]{3} + \sqrt[3]{2})$.

56. $\sqrt{6} \cdot \sqrt{12} \cdot \sqrt{72}$.

57. $\sqrt[3]{3} \cdot \sqrt[3]{18}$.

58. $\sqrt[3]{81} \cdot \sqrt[3]{-45}$.

59. $\sqrt[8]{5^3} \cdot \sqrt[8]{5^7} \cdot \sqrt[8]{5^6}$.

60. $\sqrt[4]{9} \cdot \sqrt[4]{81} \cdot \sqrt[4]{729}$.

61. $\sqrt[5]{3}(\sqrt[5]{81} - \sqrt[5]{243})$.

Solve the following equations:

62. $5x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 2 = 0$.

63. $9y^{\frac{1}{2}} + 15y^{\frac{1}{4}} - 6 = 0$.

64. $3m^{\frac{4}{3}} - 7m^{\frac{2}{3}} - 6 = 0$.

65. $7r^{\frac{2}{n}} - 9r^{\frac{1}{n}} - 10 = 0$.

66. $2u^{-2} - 13u^{-1} + 15 = 0$.

# CHAPTER XVIII

## BINOMIAL THEOREM

**121. The Binomial Theorem.** By multiplying find the values of the following.

$$(a+b)^2;\ (a+b)^3;\ (a+b)^4;\ (a+b)^5.$$

Examine the exponent of the first term in each expansion. Is it the same as the exponent of the binomial? What would be the first term of $(a+b)^7$?

Examine the exponents of the successive term of each of the expansions. What do you find of interest about them? What will be the exponents of $a$ and $b$ in each of the successive terms of $(a+b)^7$?

Examine the coefficient of the first term of each expansion. What is it? What is the coefficient of the first term of $(a+b)^7$?

What do you find of interest about the coefficient of the second term of each expansion? What is the second term of $(a+b)^7$?

Will $\frac{3 \cdot 2}{1 \cdot 2}$ give you the coefficient of the third term of the expansion of $(a+b)^3$? Will $\frac{4 \cdot 3}{1 \cdot 2}$ give the coefficient of the third term of the expansion of $(a+b)^4$? Will $\frac{5 \cdot 4}{1 \cdot 2}$ give the coefficient of the third term of the expansion of $(a+b)^5$? Write by this rule the coefficient of the third term of the expansion of $(a+b)^7$.

Will $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}$ give the coefficient of the fourth term of the expansion of $(a+b)^4$? Will $\frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}$ give the fourth coefficient of $(a+b)^5$? Write the fourth term of $(a+b)^7$.

The laws just brought out may be expressed in algebraic language:

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n.$$

This equation expresses what is known as the **Binomial Theorem.**

Let $n = 7$;

$$(a+b)^7 = a^7 + 7a^6b + \frac{7\cdot 6}{1\cdot 2}a^5b^2 + \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}a^4b^3 + \frac{7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4}a^3b^4 + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{1\cdot 2\cdot 3\cdot 4\cdot 5}a^2b^5 + \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}ab^6 + b^7$$

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7.$$

*Example.* $(a - 2b^2)^5 = ?$

In the formula replace $n$ by 5, leave $a$ unchanged, and replace $b$ by $-2b^2$. Then

$$(a-2b^2)^5 = a^5 + \frac{5}{1}a^4(-2b^2) + \frac{5\cdot 4}{1\cdot 2}a^3(-2b^2)^2 + \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}a^2(-2b^2)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4}a(-2b^2)^4 + (-2b^2)^5$$

$$= a^5 - 10a^4b^2 + 40a^3b^4 - 80a^2b^6 + 80ab^8 - 32b^{10}.$$

**Exercises.** Write expansions of the following:

1. $(2a - b)^4$.
2. $(r^2 - s)^6$.
3. $(\frac{1}{2}a + b^2)^6$.
4. $\left(\frac{1}{2a} - 2a^2\right)^5$.
5. $(a - b)^8$.
6. $(a^{-1} + a)^7$.
7. $\left(3a^{\frac{1}{2}} - \frac{1}{3a}\right)^5$.
8. $.98^4 = (1 - .02)^4$.
9. $1.01^5 = (1 + .01)^5$.
10. $99^6 = (100 - 1)^6$.
11. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^4$.
12. $(\sqrt{2a} - \sqrt[3]{a})^4$.
13. $(u^{\frac{2}{3}} + v^{\frac{2}{3}})^5$.
14. $(x^{\frac{1}{5}}y^{\frac{1}{5}} - 1)^5$.
15. $(2t^3 - \frac{1}{2}t^{\frac{1}{3}})^6$.
16. $(a^{-\frac{2}{5}} - 2\sqrt{a})^5$.

[illegible]