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ALGEBRA.

ALGEBRA

For the Use of Colleges and Schools.

WITH NUMEROUS EXAMPLES.

BY

I. TODHUNTER, M.A., F.R.S.

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PREFACE.

THIS work contains all the propositions which are usually included in elementary treatises on algebra, and a large number of examples for exercise.

My chief object has been to render the work easily intelligible. Students should be encouraged to examine carefully the language of the book they are using, so that they may ascertain its meaning or be able to point out exactly where their difficulties arise. The language, therefore, ought to be simple and precise; and it is essential that apparent conciseness should not be gained at the expense of clearness.

In attempting, however, to render the work easily intelligible, I trust I have neither impaired the accuracy of the demonstrations nor contracted the limits of the subject; on the contrary, I think it will be found that in both these respects I have advanced beyond the line traced out by previous elementary writers.

The present treatise is divided into a large number of chapters, each chapter being, as far as possible, complete in itself. Thus the student is not perplexed by attempting to master too much at once; and if he should not succeed in fully comprehending any chapter, he will not be precluded from going on to the next, reserving the difficulties for future consideration: the latter point is of especial importance to those students who are without the aid of a teacher.

The order of succession of the several chapters is to some extent arbitrary, because the position which any one of them should occupy must depend partly upon its difficulty and partly upon its importance. But, since each chapter is nearly independent, it will be in the power of the teacher to abandon the order laid down in the book and to adopt another at his discretion.

The examples have been selected with a view to illustrate every part of the subject, and, as the number of them is more than two thousand, I trust they will supply ample exercise for the student. Complicated and difficult problems have been excluded, because they consume time and energy which may be spent more profitably on other branches of mathematics. Each set of examples has been carefully arranged, commencing with some which are very simple and proceeding gradually to others which are less obvious; those sets which are entitled *Miscellaneous Examples*, together with a few in each of the other sets, may be omitted by the student who is reading the subject for the first time. The answers to the examples, with hints for the solution of some in which assistance may be needed, are given at the end of the book.

I will now give some account of the sources from which the present treatise has been derived.

Dr Wood's Algebra has been so long current that it has become public property, and it is so well known to teachers that an elementary writer would naturally desire to make use of it to some extent. The first edition of that work appeared in 1795, and the tenth in 1835; the tenth edition was the last issued in Dr Wood's life-time. The chapters on Surds, Ratio, and Proportion, in my Algebra are almost entirely taken from Dr Wood's Algebra. I have also frequently used Dr Wood's examples either in my text or in my collections of examples. Moreover, in the statement of rules in the elementary part of my book I have often followed Dr Wood, as, for example, in the

Rule for Long Division; the statement of such rules must be almost identical in all works on Algebra. I should have been glad to have had the advantage of Dr Wood's authority to a greater extent, but the requirements of the present state of mathematical instruction rendered this impossible. The tenth edition of Dr Wood's Algebra contains less than half the matter of the present work, and half of it is devoted to subjects which are now usually studied in distinct treatises, namely, Arithmetic, the Theory of Equations, the application of Algebra to Geometry, and portions of the Summation of Series; the larger part of the remainder, from its brevity and incompleteness, is now unsuitable to the wants of students. Thus, on the whole, a very small number of pages comprises all that I have been able to retain of Dr Wood's Algebra.

For additional matter I have chiefly had recourse to the Treatise on Arithmetic and Algebra in the Library of Useful Knowledge, and the works of Bourdon, Lefebure de Fourcy, Mayer and Choquet, and Schlömilch; I have also studied with great advantage the Algebra of Professor De Morgan and other works of the same author which bear upon the subject of Algebra.

I have also occasionally consulted the edition of Wood's Algebra published by Mr Lund in 1841, Hind's Algebra, 1841, Colenso's Algebra, 1849, and Goodwin's Elementary Course of Mathematics, 1853.

Although I have not hesitated to use the materials which were available in preceding authors, yet much of the present work is peculiar to it; and I believe it will be found that my Algebra contains more that is new to elementary works, and more that is original, than any of the popular English works of similar plan. Originality however in an elementary work is rarely an advantage; and in publishing the first edition of my Algebra I felt some apprehension that I had deviated too far from the ordinary methods. I have had great satisfaction in receiving from eminent teachers favourable opinions of the work generally and also of those parts which are peculiar to it.

The present edition has been carefully revised, and two new chapters have been added. Three hundred miscellaneous examples have also been supplied; these are arranged in sets, each set containing ten examples; the first hundred relate to the first twenty chapters of the book, the second hundred extend to the end of the fortieth chapter, and the last hundred relate to the whole book.

I have to return my thanks to many able mathematicians who have favoured me with suggestions which have been of service to me; the improvements which have been effected in the work will, I trust, render it still more useful in education, and still more worthy of the approbation which it has received.

I have drawn up a treatise on the *Theory of Equations* to form a sequel to the *Algebra*; and the student is referred to that treatise as a suitable continuation of the present work.

I. TODHUNTER.

ST JOHN'S COLLEGE,
October, 1870.

CONTENTS.

ALGEBRA.

I. DEFINITIONS AND EXPLANATIONS OF SIGNS.

1. THE method of reasoning about numbers, by means of letters which are employed to represent the numbers, and signs which are employed to represent their relations, is called *Algebra.*

2. Letters of the alphabet are used to represent numbers, which may be either *known* numbers, or numbers which have to be found and which are therefore called *unknown* numbers. It is usual to represent *known* numbers by the first letters of the alphabet, as a, b, c, and *unknown* numbers by the last letters, as x, y, z; this is not however a necessary rule, and so need not be strictly obeyed.

Numbers may be either whole or fractional. The word *quantity* is frequently used as synonymous with *number.* The word *integer* is often used instead of *whole* number.

3. The sign + signifies that the number to which it is prefixed is to be *added.* Thus $a + b$ signifies that the number represented by b is to be added to the number represented by a. If a represent 9, and b represent 3, then $a + b$ represents 12. The sign + is called the *plus sign,* and $a + b$ is read thus "a *plus* b."

Similarly $a + b + c$ signifies that we are to add b to a, and then add c to the result.

4. The sign − signifies that the number to which it is prefixed is to be *subtracted.* Thus $a - b$ signifies that the number represented by b is to be subtracted from the number represented by a. If a represent 9, and b represent 3, then $a - b$ represents 6. The sign − is called the *minus sign,* and $a - b$ is read thus "a *minus* b."

Similarly $a - b - c$ signifies that we are to subtract b from a, and then subtract c from the result; $a + b - c$ signifies that we are to add b to a, and then subtract c from the result; $a - b + c$ signifies that we are to subtract b from a and then add c to the result.

5. The sign $\times$ signifies that the numbers between which it stands are to be *multiplied* together. Thus $a \times b$ signifies that the number represented by a is to be multiplied by the number represented by b. If a represent 9, and b represent 3, then $a \times b$ represents 27. The sign $\times$ is called the ***sign of multiplication***, and $a \times b$ is read thus "a *into* b." Similarly $a \times b \times c$ denotes the product of the numbers represented by a, b and c.

It should be observed that the sign of multiplication is often omitted for the sake of brevity; thus ab is used instead of $a \times b$, and has the same meaning; so abc is used for $a \times b \times c$. Sometimes a point is used instead of the sign $\times$; thus $a \,.\, b$ is used for $a \times b$ or ab. But the point is here superfluous, because, as we have said, ab is used instead of $a \times b$. Nor is the point, nor the sign $\times$, necessary between a number expressed in the ordinary way by a figure and a number represented by a letter; so that, for example, $3a$ is used instead of $3 \times a$, and has the same meaning.

The sign of multiplication must not be omitted when numbers are expressed by figures in the ordinary way. Thus 45 cannot be used to express the product of 4 and 5, because a different meaning has already been appropriated to 45, namely *forty-five*. We must therefore express the product of 4 and 5 thus 4×5, or thus $4 \,.\, 5$. To prevent any confusion between the point thus used as a sign of multiplication and the point as used in the notation for decimal fractions, it is advisable to write the latter higher up; thus $4{\cdot}5$ may be kept to denote $4 + \frac{5}{10}$.

6. The sign $\div$ signifies that the number which precedes it is to be *divided* by the number which follows it. Thus $a \div b$ signifies that the number represented by a is to be divided by the number represented by b. If a represent 9, and b represent 3, then $a \div b$ represents 3. The sign $\div$ is called the ***sign of division***, and $a \div b$ is read thus "a *by* b." There is also another way of

denoting that one number is to be divided by another; the dividend is placed over the divisor with a line between them. Thus $\frac{a}{b}$ is used instead of $a \div b$ and has the same meaning.

7. The sign $=$ signifies that the numbers between which it is placed are *equal.* Thus $a = b$ signifies that the number represented by a is equal to the number represented by b, that is, a and b represent the same number. The sign $=$ is called the ***sign of equality***, and $a = b$ is read thus "a *equals* b" or "a *is equal to* b."

8. The *difference* of two numbers is sometimes denoted by the sign $\sim$; thus $a \sim b$ denotes the difference of the numbers denoted by a and b, and is equal to $a - b$ or to $b - a$, according as a is greater than b or less than b.

9. The sign $>$ denotes *greater than,* and the sign $<$ denotes *less than;* thus $a > b$ denotes that the number represented by a is greater than the number represented by b, and $b < a$ denotes that the number represented by b is less than the number represented by a. Thus in both signs the opening of the angle is turned towards the greater number.

10. The sign $\therefore$ denotes *then* or *therefore;* the sign $\because$ denotes *since* or *because.*

11. When several numbers are to be taken collectively they are enclosed by *brackets.* Thus $(a - b + c) \times (d + e)$ signifies that the number represented by $a - b + c$ is to be multiplied by the number represented by $d + e$. This may also be written thus $(a - b + c)(d + e)$. The use of the brackets will be seen by comparing what we have just given with $(a - b + c)\, d + e$; the latter denotes that the number represented by $a - b + c$ is to be multiplied by d and then e is to be added to the product.

Sometimes instead of using brackets a line called a *vinculum* is drawn over the numbers which are to be taken collectively. Thus $\overline{a - b + c} \times \overline{d + e}$ is used with the same meaning as $(a - b + c) \times (d + e)$.

12. The letters of the alphabet, and the signs or marks which

we have already introduced and explained, together with those which may occur hereafter, are called *algebraical symbols*, since they are used to *represent* the things about which we may be reasoning. Any collection of algebraical symbols is called an *algebraical expression*, or briefly, an *expression*, or a *formula*. An algebraical expression is sometimes called an *algebraical quantity*, or briefly, a *quantity*.

13. Those parts of an expression which are connected by the signs + or − are called its *terms*. When an expression consists of *two* terms it is called a *binomial expression*; when it consists of *three* terms it is called a *trinomial expression*; any expression consisting of several terms may be called a *multinomial expression* or a *polynomial expression*. When an expression does not contain parts connected by the sign + or the sign − it may be called a *simple expression*, or it may be said to contain only *one term*.

Thus abc is a *simple expression*; $abc + x$ is a *binomial expression*, of which abc is one term, and x is the other; $ab + ac - bc$ is a *trinomial expression*, of which ab, ac, and bc are the *terms*.

14. When one number consists of the product of two or more numbers, each of the latter is called a *factor* of the product. Thus a, b and c are *factors* of the *product* abc.

15. A product may consist of one factor which is a number represented *arithmetically*, and of another factor which is a number represented *algebraically*, that is, by a letter or letters; in this case the former factor is said to be the *coefficient* of the latter. Thus in the product $7abc$ the factor 7 is called the coefficient of the factor abc. Where there is no arithmetical factor, we may supply unity; thus we may say that, in the product abc, the coefficient is unity.

And when a product is represented entirely algebraically, any one factor may be called the coefficient of the product of the remaining factors. Thus, in the product abc, we may call a the coefficient of bc, or b the coefficient of ac, or c the coefficient of ab. If it be necessary to distinguish this use of the word *coefficient* from the former, we may call the latter coefficients *literal coefficients*, and the former coefficients *numerical coefficients*.

16. If a number be multiplied by itself any number of times, the product is called a *power* of that number. Thus $a \times a$ is called the *second power* of a; also $a \times a \times a$ is called the *third power* of a; and $a \times a \times a \times a$ is called the *fourth power* of a; and so on. The number a itself is often called the *first power* of a.

17. Any power of a quantity is usually expressed by placing above the quantity the number which represents how often it is repeated in the product. Thus a^2 is used to express $a \times a$; also a^3 is used to express $a \times a \times a$; and a^4 is used to express $a \times a \times a \times a$; and so on. And a^1 may be used to denote the first power of a or a itself; that is, a^1 has the same meaning as a.

Numbers placed above a quantity to express the powers of that quantity are called *indices of the powers*, or *exponents of the powers*; or more briefly *indices* or *exponents*.

18. Hence we may sum up the two preceding Articles thus: the product of n factors each equal to a is expressed by a^n, and n is called the index or exponent of a^n, where n may denote any whole number.

19. The second power of a or a^2 is often called the *square* of a, and the third power of a or a^3 is often called the *cube* of a. The symbol a^4 is read thus "a *to the fourth power*" or briefly "a *to the fourth*;" and a^n is read thus "a *to the* n^{th}."

20. The *square root* of any assigned number is that number which has the assigned number for its *square* or *second power*. The *cube root* of any assigned number is that number which has the assigned number for its *cube* or *third power*. The *fourth root* of any assigned number is that number which has the assigned number for its *fourth* power. And so on.

21. The square root of a number a is denoted thus $\sqrt[2]{a}$, or simply thus $\sqrt{a}$. The cube root of a is denoted thus $\sqrt[3]{a}$. The fourth root of a is denoted thus $\sqrt[4]{a}$. And so on.

The sign $\sqrt{}$ is said to be a corruption of the initial letter of the word *radix*. This sign is sometimes called the *radical sign*.

22. Terms are said to be *like* or *similar* when they do not differ at all, or differ only in their numerical coefficients; otherwise they are said to be *unlike*. Thus $4a$, $6ab$, $9a^2$ and $3a^2bc$ are respectively similar to $15a$, $3ab$, $12a^2$ and $15a^2bc$. And ab, a^2b, ab^2 and abc are all unlike.

23. Each of the letters which occur in an algebraical product is called a *dimension* of the product, and the number of the letters is the *degree* of the product. Thus a^2b^3c or $a \times a \times b \times b \times b \times c$ is said to be of six dimensions or of the sixth degree. A numerical coefficient is not counted; thus $9a^3b^4$ and a^3b^4 are of the same dimensions, namely of seven dimensions. Thus the *degree* of a term or the *number of dimensions* of a term is the *sum* of the *exponents*, provided we remember that if no exponent is expressed the exponent 1 must be understood as indicated in Art. 17.

24. An algebraical expression is said to be *homogeneous* when all its terms are of the same dimensions. Thus $7a^3 + 3a^2b + 4abc$ is homogeneous, for each term is of three dimensions.

The following examples will serve for an exercise in the preceding definitions.

EXAMPLES.

If $a=1$, $b=3$, $c=4$, $d=6$, $e=2$ and $f=0$, find the numerical values of the following twelve algebraical expressions:

1. $a + 2b + 4c$.
2. $3b + 5d - 2e$.
3. $ab + 2bc + 3ed$.
4. $ac + 4cd - 2eb$.
5. $abc + 4bd + ec - fd$.
6. $a^2 + b^2 + c^2 + f^2$.
7. $\dfrac{cd}{b} + \dfrac{4be}{3a} - \dfrac{cd}{24}$.
8. $c^4 - 4c^3 + 3c - 6$.
9. $\dfrac{b^2 + c^2}{2c - 3a}$.
10. $\dfrac{d^3 - c^3}{d^2 + dc + c^2}$.
11. $\sqrt{(27b)} - \sqrt[3]{(2c)} + \sqrt{(2e)}$.
12. $\sqrt{(3bc)} + \sqrt[3]{(9cd)} - \sqrt[3]{(2e^2)}$.

13. Find the value of $(9-y)(x+1)+(x+5)(y+7)-112$, when $x=3$ and $y=5$.

14. Find the value of $x\sqrt{(x^2-8y)}+y\sqrt{(x^2+8y)}$, when $x=5$ and $y=3$.

X 15. Find the value of $a\sqrt{(x^2-3a)}+x\sqrt{(x^2+3a)}$, when $x=5$ and $a=8$.

16. Find the value of $a+b\sqrt{(x+y)}-(a-b)\sqrt[3]{(x-y)}$, when $a=10$, $b=8$, $x=12$, and $y=4$.

17. If $a=16$, $b=10$, $x=5$ and $y=1$, find the value of

$$(b-x)(\sqrt{a}+b)+\sqrt{\{(a-b)(x+y)\}};$$

and of

$$(a-y)\{\sqrt{(2bx)}+x^2\}+\sqrt{\{(a-x)(b+y)\}}.$$

18. If $a=2$, $b=3$, $x=6$ and $y=5$, find the value of

$$\sqrt[3]{\{(a+b)^2y\}}+\sqrt[3]{\{(a+x)(y-2a)\}}+\sqrt[3]{\{(y-b)^2a\}}.$$

II. CHANGE OF THE ORDER OF TERMS. REDUCTION OF LIKE TERMS. ADDITION, SUBTRACTION, USE OF BRACKETS.

25. When the terms of an expression are connected by the sign + it is indifferent in what *order* they are written; thus $a+b$ and $b+a$ give the same result, namely the sum of the numbers which are denoted by a and b. We may express this fact algebraically thus:

$$a+b=b+a.$$

Similarly

$$a+b+c=a+c+b=b+a+c=b+c+a=c+a+b=c+b+a.$$

26. When an expression consists of some terms preceded by the sign + and some terms preceded by the sign −, we may write the former terms first in any order we please, and the latter terms after them in any order we please. This appears from the same considerations as before. Thus, for example,

$$a+b-c-e=a+b-e-c=b+a-c-e=b+a-e-c.$$

27. In some cases it is obvious that we may vary the order of terms still further, by mixing up the terms preceded by the sign $-$ with those preceded by the sign $+$. Thus, for example, if a represent 10, b represent 6, and c represent 5, then

$$a+b-c=a-c+b=b-c+a.$$

If however a represent 2, b represent 6, and c represent 5, then the expression $a-c+b$ presents a difficulty because we are thus apparently required to take a greater number from a less, namely 5 from 2. It will be convenient to agree that such an expression as $a-c+b$ when c is greater than a shall *be understood to mean the same thing as* $a+b-c$. At present we shall never use such an expression except when c is *less* than $a+b$, so that $a+b-c$ presents no difficulty. Similarly we shall consider $-b+a$ to mean the same thing as $a-b$. We shall recur to this point hereafter in Chapter V.

28. Thus the numerical value of an expression remains the same whatever may be the *order* of the terms which compose it. This, as we have seen, follows, partly from our notions of addition and subtraction, and partly from an *agreement* as to the meaning we ascribe to an expression when our ordinary arithmetical notions are not strictly applicable. Such an agreement is called in Algebra a *convention*, and *conventional* is the corresponding adjective.

29. We shall frequently, as in Article 26, have to distinguish the terms of an expression which are preceded by the sign $+$ from the terms which are preceded by the sign $-$, and thus the following definition is adopted: The terms in an expression which are preceded by no sign or which are preceded by the sign $+$ are called *positive* terms; the terms which are preceded by the sign $-$ are called *negative* terms. This definition is introduced merely for the sake of brevity, and no meaning is to be given to the words *positive* and *negative* beyond what is expressed in the definition. The student will notice that terms preceded by no sign are treated as if they were preceded by the sign $+$.

30. Sometimes an expression includes several *like* terms; in this case the expression admits of simplification. For example, consider the expression $4a^2b - 3a^2c + 9ac^2 - 2a^2b + 7a^2c - 6b^2$; this may be written $4a^2b - 2a^2b + 7a^2c - 3a^2c + 9ac^2 - 6b^2$ (Art. 28). Now $4a^2b - 2a^2b$ is the same thing as $2a^2b$, and $7a^2c - 3a^2c$ is the same thing as $4a^2c$. Thus the expression may be put in the simpler form $2a^2b + 4a^2c + 9ac^2 - 6b^2$.

ADDITION.

31. *The addition of algebraical expressions is performed by writing the terms in succession each preceded by its proper sign.*

For suppose we have to add $c - d + e$ to $a - b$; this is the same thing as adding $c + e - d$ to $a - b$ (Art. 28). Now if we add $c + e$ to $a - b$ we obtain $a - b + c + e$; we have however thus added d too much, and must consequently subtract d. Hence we obtain $a - b + c + e - d$, which is the same as $a - b + c - d + e$; thus the result agrees with the rule above given. The result is called the *sum*.

We may write our result thus:

$$a - b + (c - d + e) = a - b + c - d + e.$$

32. When the terms of the expressions which are to be added are all *unlike*, the sum obtained by the rule does not admit of simplification. But when *like* terms occur in the expressions, we may simplify as in Art. 30. Hence we have the following rules:

When like terms have the same sign their sum is found by taking the sum of the coefficients with that sign and annexing the common letters.

Example; add $5a - 3b$ and $4a - 7b$; the sum is $9a - 10b$. For the $5a$ and the $4a$ together make $9a$, and the $3b$ and $7b$ together make $10b$.

Again; add $4a^2c - 10bde$, $6a^2c - 9bde$ and $11a^2c - 3bde$. The sum is $21a^2c - 22bde$.

When like terms occur with different signs their sum is found by taking the difference of the sum of the positive and the sum of the negative coefficients with the sign of the greater sum and annexing the common letters as before.

Example; add $7a-9b$ and $5b-4a$. The sum is $3a-4b$.

Again; add together $3a^2+4bc-e^2+10$, $5a^2+6bc+2e^2-15$ and $4a^2-9bc-10e^2+21$. The sum is $12a^2+bc-9e^2+16$.

SUBTRACTION.

33. Suppose we have to take $b+c$ from a. Then as each of the numbers b and c is to be taken from a the result is denoted by $a-b-c$. That is

$$a-(b+c)=a-b-c.$$

We enclose the term $b+c$ in brackets, because *both* the numbers b and c are to be taken from a.

Similarly $a+d-(b+c+e)=a+d-b-c-e.$

Next suppose we have to take $b-c$ from a. If we take b from a we obtain $a-b$; but we have thus taken too much from a, for we are required to take, not b but, b diminished by c. Hence we must increase the result by c; thus

$$a-(b-c)=a-b+c.$$

Similarly, suppose we have to take $b-c-d+e$ from a. This is the same thing as taking $b+e-c-d$ from a. Take away $b+e$ from a and the result is $a-b-e$; then *add* $c+d$, because we were to take away, not $b+e$ but, $b+e$ diminished by $c+d$; thus

$$\begin{aligned} a-(b-c-d+e) &= a-b-e+c+d \\ &= a-b+c+d-e. \end{aligned}$$

34. From considering these cases we arrive at the following rule for subtraction: *Change the sign of every term in the expression to be subtracted, and then add it to the other expression.* Here as before, we suppose for shortness, that where there is no sign before a term, + is to be understood.

For example; take $a-b$ from $3a+b$.

$$3a+b-(a-b)=3a+b-a+b=2a+2b.$$

Again; tako $5a^2+4ab-6xy$ from $11a^2+3ab-4xy$.

$$11a^2+3ab-4xy-(5a^2+4ab-6xy)$$
$$=11a^2+3ab-4xy-5a^2-4ab+6xy=6a^2-ab+2xy.$$

BRACKETS.

35. On account of the frequent occurrence of brackets in algebraical investigations, it is advisable to call the attention of the student *explicitly* to the laws respecting their use. These laws have already been established, and we have only to give them a verbal enunciation.

When an expression within brackets is preceded by the sign + the brackets may be removed.

Thus $a-b+(c-d+e)=a-b+c-d+e$, (Art. 31).

And consequently any number of terms in an expression may be enclosed by brackets, and the sign + placed before the whole.

Thus $a-b+c-d+e$ may be written in the following ways:

$$a-b+c+(-d+e),\quad a-d+(c+e-b),\quad a+(-d+c+e-b),$$

and so on.

When an expression within brackets is preceded by the sign − the brackets may be removed if the sign of every term within the brackets be changed, namely + to − and − to +.

Thus $a-(b-c-d+e)=a-b+c+d-e$, (Art. 34).

And consequently any number of terms in an expression may be enclosed by brackets and the sign − placed before the whole, provided the sign of every term within the brackets be changed.

Thus $a-b+c+d-e$ may be written in the following ways:

$$a-b+c-(-d+e),\quad a-(b-c-d+e),\quad a+c-(b-d+e),$$

and so on.

36. Expressions may occur with more than one pair of brackets; these may be removed in succession by the preceding rules *beginning with the inside pair.* Thus, for example,

$$a+\{b+(c-d)\}=a+\{b+c-d\}=a+b+c-d,$$

$$a+\{b-(c-d)\}=a+\{b-c+d\}=a+b-c+d,$$

$$a-\{b+(c-d)\}=a-\{b+c-d\}=a-b-c+d,$$

$$a-\{b-(c-d)\}=a-\{b-c+d\}=a-b+c-d.$$

Similarly,

$$a-[b-\{c-(d-e)\}]=a-[b-\{c-d+e\}]$$
$$=a-[b-c+d-e]=a-b+c-d+e.$$

It will be seen in these examples that, to prevent confusion between various pairs of brackets, we use brackets of different *shapes;* we might distinguish by using brackets of the same shape but of different *sizes.*

A vinculum is equivalent to a bracket; see Art. 11. Thus, for example,

$$a-[b-\{c-(d-\overline{e-f})\}]=a-[b-\{c-(d-e+f)\}]$$
$$=a-[b-\{c-d+e-f\}]=a-[b-c+d-e+f]$$
$$=a-b+c-d+e-f.$$

In like manner more than one pair of brackets may be introduced. Thus, for example,

$$a-b+c-d+e=a-\{b-c+d-e\}=a-\{b-(c-d+e)\}.$$

37. The beginner is recommended always to remove brackets in the order shewn in the preceding Article; namely, by removing first the innermost pair, next the innermost pair of all which remain, and so on. We may however vary the order; but if we remove a pair of brackets including another bracketed expression within it, we must *make no change in the sign of the included expression.* In fact such an included expression counts as a single term. Thus, for example,

$$a+\{b+(c-d)\}=a+b+(c-d)=a+b+c-d,$$
$$a+\{b-(c-d)\}=a+b-(c-d)=a+b-c+d,$$
$$a-\{b+(c-d)\}=a-b-(c-d)=a-b-c+d,$$
$$a-\{b-(c-d)\}=a-b+(c-d)=a-b+c-d.$$

Also, $$a-[b-\{c-(d-e)\}]=a-b+\{c-(d-e)\}$$
$$=a-b+c-(d-e)=a-b+c-d+e.$$

And in like manner, $a-[b-\{c-(d-\overline{e-f})\}]$
$$=a-b+\{c-(d-\overline{e-f})\}=a-b+c-(d-\overline{e-f})$$
$$=a-b+c-d+\overline{e-f}=a-b+c-d+e-f.$$

EXAMPLES.

1. Add together $4a-5b+3c-2d$, $a+b-4c+5d$, $3a-7b+6c+4d$ and $a+4b-c-7d$.
2. Add together x^3+2x^2-3x+1, $2x^3-3x^2+4x-2$, $3x^3+4x^2+5$ and $4x^3-3x^2-5x+9$.
3. Add together $x^2-3xy+y^2+x+y-1$, $2x^2+4xy-3y^2-2x-2y+3$, $3x^2-5xy-4y^2+3x+4y-2$ and $6x^2+10xy+5y^2+x+y$.
4. Add together $x^3-2ax^2+a^2x$, x^3+3ax^2 and $2x^3-ax^2$.
5. Add together $4ab-x^2$, $3x^2-2ab$ and $2ax+2bx$.
6. From $5a-3b+4c-7d$ take $2a-2b+3c-d$.
7. From $x^4+4x^3-2x^2+7x-1$ take $x^4+2x^3-2x^2+6x-1$.
8. Subtract a^2-ax+x^2 from $3a^2-2ax+x^2$.
9. Subtract $a-b-2(c-d)$ from $2(a-b)-c+d$.
10. Subtract $(a-b)x-(b-c)y$ from $(a+b)x+(b+c)y$.
11. Remove the brackets from $a-\{b-(c-d)\}$.
12. Remove the brackets from $a-\{(b-c)-d\}$.
13. Remove the brackets from $a+2b-6a-\{3b-(6a-6b)\}$
14. Remove the brackets from $7a-\{3a-[4a-(5a-2a)]\}$.

15. Also from $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$.

16. Also from $2x-[3y-\{4x-(5y-6x)\}]$.

17. Also from $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$.

18. Also from $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

19. If $a=2$, $b=3$, $x=6$ and $y=5$, find the value of

$$a+2x-\{b+y-[a-x-(b-2y)]\}.$$

20. Simplify

$$4x^3-2x^2+x+1-(3x^3-x^2-x-7)-(x^3-4x^2+2x+8).$$

III. MULTIPLICATION.

38. We have already stated that the *product* of the numbers denoted by any letters may be denoted by writing those letters in succession without any sign between them; thus $abcd$ denotes the *product* of the numbers denoted by a, b, c and d. We suppose the student to know from Arithmetic, that the product of any number of factors is the same in whatever *order* the factors may be taken; thus $abc=acb=bca$, and so on.

39. Suppose we have to form the product of $4a$, $5b$, and $3c$; this product may be written at full thus: $4\times a\times 5\times b\times 3\times c$, or $4\times 5\times 3\times abc$, that is $60abc$. And thus we may deduce the following rule for the multiplication of simple terms: *multiply together the numerical coefficients and put the letters after the product.*

40. The notation adopted to represent the powers of a number, (Art. 17), will enable us to prove the following rule: *the powers of a number are multiplied by adding the exponents*, for $a^3\times a^2=a\times a\times a\times a\times a=a^5=a^{3+2}$; and similarly any other case may be established.

Thus if m and n are any whole numbers, $a^m\times a^n=a^{m+n}$.

41. We may if we please indicate the product of the *same* powers of different letters by writing the letters within brackets, and placing the index over the whole. Thus $a^2 \times b^2 = (ab)^2$; this is obvious since $(ab)^2 = ab \times ab = a \times a \times b \times b$. Similarly,

$$a^3 \times b^3 \times c^3 = (abc)^3.$$

Thus $a^n \times b^n = (ab)^n$; $a^n \times b^n \times c^n = (abc)^n$; and so on for any number of factors.

42. Suppose it required to multiply $a+b$ by c. The product of a and c is denoted by ac, and the product of b and c is denoted by bc; hence the product of $a+b$ and c is denoted by $ac+bc$. For it follows, as in Arithmetic, from our notion of multiplication, that to multiply any quantity by a number we have only to multiply all the parts of that quantity by the number and add the results. Thus

$$(a+b)\,c = ac + bc.$$

43. Suppose it required to multiply $a-b$ by c. Here the product of a and c must be *diminished* by the product of b and c. Thus

$$(a-b)\,c = ac - bc.$$

44. Suppose it required to multiply $a+b$ by $c+d$. It follows, as in Arithmetic, from our notions of multiplication, that if a quantity is to be multiplied by any number, we may separate the multiplier into parts the sum of which is equal to the multiplier, and take the product of the quantity by each part, and add these partial products to form the complete product.

Thus $\quad (a+b)(c+d) = (a+b)\,c + (a+b)\,d$;

also $\quad (a+b)\,c = ac + bc$, and $(a+b)\,d = ad + bd$;

thus $\quad (a+b)(c+d) = ac + bc + ad + bd.$

45. Suppose it required to multiply $a-b$ by $c+d$. Here the product of a and $c+d$ must be *diminished* by the product of b and $c+d$. Thus

$$\begin{aligned}(a-b)(c+d) &= a\,(c+d) - b\,(c+d)\\ &= ac + ad - (bc + bd) = ac + ad - bc - bd.\end{aligned}$$

46. Suppose it required to multiply $a+b$ by $c-d$. Here the product of $a+b$ and c must be *diminished* by the product of $a+b$ and d. Thus

$$(a+b)(c-d) = (a+b)c - (a+b)d$$
$$= ac + bc - (ad + bd) = ac + bc - ad - bd.$$

47. Suppose it required to multiply $a-b$ by $c-d$. Here the product of $a-b$ and c must be *diminished* by the product of $a-b$ and d. Thus

$$(a-b)(c-d) = (a-b)c - (a-b)d$$
$$= ac - bc - (ad - bd) = ac - bc - ad + bd.$$

48. From considering the above cases we arrive at the following rule for multiplying two binomial expressions: *Multiply each term of the multiplicand by each term of the multiplier; if the terms have the same sign, prefix the sign + to their product, if they have different signs prefix the sign −; then collect these partial products to form the complete product.*

The rules with respect to the sign of each partial product are often enunciated thus for shortness: *like signs produce +, and unlike signs produce −.*

49. It appears from the preceding Articles, that corresponding to the terms $-b$ and c which occur in two binomial factors, there is a term $-bc$ in the product of the factors. Hence it is often stated as an independent truth that $-b \times c = -bc$.

Similarly, we observe, that corresponding to the terms $-b$ and $-c$ which occur in two binomial factors, there is a term bc in the product of the factors; hence it is often stated as an independent truth, that $-b \times -c = bc$. These statements will be examined and explained in Chapter V.

50. The rule given in Article 48 will hold for the multiplication of any expressions. This will appear from considering a few examples. Suppose, for instance, we have to multiply

$4a^2 - 5ab + 6b^2$ by $2a^2 - 3ab + 4b^2$. The required product here is

$$2a^2(4a^2 - 5ab + 6b^2) - 3ab(4a^2 - 5ab + 6b^2) + 4b^2(4a^2 - 5ab + 6b^2);$$

thus we obtain

$$(8a^4 - 10a^3b + 12a^2b^2) - (12a^3b - 15a^2b^2 + 18ab^3) + (16a^2b^2 - 20ab^3 + 24b^4),$$

that is,

$$8a^4 - 10a^3b + 12a^2b^2 - 12a^3b + 15a^2b^2 - 18ab^3 + 16a^2b^2 - 20ab^3 + 24b^4.$$

This result agrees with the rule. If we simplify the result by collecting the like terms we obtain

$$8a^4 - 22a^3b + 43a^2b^2 - 38ab^3 + 24b^4.$$

The whole operation may be conveniently arranged thus:

$$\begin{array}{lllll}
4a^2 - 5ab + 6b^2 & & & & \\
2a^2 - 3ab + 4b^2 & & & & \\
\hline
8a^4 & - 10a^3b & + 12a^2b^2 & & \\
 & - 12a^3b & + 15a^2b^2 & - 18ab^3 & \\
 & & + 16a^2b^2 & - 20ab^3 & + 24b^4 \\
\hline
8a^4 & - 22a^3b & + 43a^2b^2 & - 38ab^3 & + 24b^4
\end{array}$$

51. The student should carefully notice the arrangement of the above operation. The expressions which we wish to multiply are here said to be *arranged according to descending powers of a*; for in the expression $4a^2 - 5ab + 6b^2$ the term which contains the *highest power* of a is $4a^2$, and this is placed first; next we place $-5ab$ which contains a, and last we place the term $+6b^2$, which does not contain a at all. Similarly the other factor $2a^2 - 3ab + 4b^2$ is arranged. The partial products which arise are so arranged that like terms occur in the same column, and thus we collect them more easily.

The factors might also have been arranged thus $6b^2 - 5ab + 4a^2$ and $4b^2 - 3ab + 2a^2$; they are then said to be arranged according to *ascending* powers of a. It is of no consequence which order we adopt, but we should take the *same* order for the multiplicand and the multiplier.

52. Again; multiply $2x^2+3x+4$ by $2x^2-3x+4$. The operation may be arranged thus:

$$\begin{array}{l} 2x^2+3x+4 \\ 2x^2-3x+4 \\ \hline 4x^4+6x^3+8x^2 \\ \quad\ -6x^3-9x^2-12x \\ \qquad\qquad +8x^2+12x+16 \\ \hline 4x^4 \qquad\quad +7x^2 \qquad\quad +16 \end{array}$$

Thus the product is $4x^4+7x^2+16$.

53. The following three examples deserve special notice,

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \end{array} \qquad \begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2 \qquad -b^2 \end{array}$$

The first example gives the value of $(a+b)(a+b)$, that is, of $(a+b)^2$; we thus find

$$(a+b)^2=a^2+2ab+b^2.$$

Thus *the square of the sum of two numbers is equal to the sum of the squares of the two numbers increased by twice their product.*

Again we have

$$(a-b)^2=a^2-2ab+b^2.$$

Thus *the square of the difference of two numbers is equal to the sum of the squares of the two numbers diminished by twice their product.*

Also we have

$$(a+b)(a-b)=a^2-b^2.$$

Thus *the product of the sum and the difference of two numbers is equal to the difference of their squares.*

54. We may here indicate the meaning of the sign $\pm$ which is sometimes used, and which is called the *double sign.*

Since $(a+b)^2 = a^2 + 2ab + b^2$,

and $(a-b)^2 = a^2 - 2ab + b^2$,

we may write $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

Thus $\pm$ indicates that we may take either the sign + or the sign $-$; $a \pm b$ is read thus, "a *plus or minus* b."

55. The results given in Art. 53 furnish a simple example of the use of Algebra; we may say that Algebra enables us to *prove general theorems respecting numbers,* and also to *express those theorems briefly.* For example, the result $(a+b)(a-b) = a^2 - b^2$ is proved to be true, and is stated thus by symbols more compactly than by words.

There are other results in multiplication which are of less importance than the three formulæ given in Art. 53, but which are deserving of attention. We place them here in order that the student may be able to refer to them when they are wanted; they can be easily verified by actual multiplication.

$$(a+b)(a^2 - ab + b^2) = a^3 + b^3,$$

$$(a-b)(a^2 + ab + b^2) = a^3 - b^3,$$

$$(a+b)^3 = (a+b)(a^2 + 2ab + b^2) = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a-b)^3 = (a-b)(a^2 - 2ab + b^2) = a^3 - 3a^2b + 3ab^2 - b^3,$$

$$(b+c)(c+a)(a+b) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 2abc,$$

$$(b-c)(c-a)(a-b) = a^2(c-b) + b^2(a-c) + c^2(b-a),$$

$$(a+b+c)(bc+ca+ab) = a^2(b+c) + b^2(c+a) + c^2(a+b) + 3abc,$$

$$(a+b+c)(a^2+b^2+c^2-bc-ca-ab) = a^3+b^3+c^3-3abc,$$

$$(b+c-a)(c+a-b)(a+b-c) = a^2(b+c) + b^2(c+a) + c^2(a+b) - a^3 - b^3 - c^3 - 2abc,$$

$$(a+b+c)^3 = a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3$$
$$= a^3 + 3a^2(b+c) + 3a(b^2+2bc+c^2) + b^3 + 3b^2c + 3bc^2 + c^3$$
$$= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) + 3c^2(a+b) + 6abc.$$

56. By using the *formulæ* given in Art. 53, the process of multiplication may be often simplified. Thus suppose we have to multiply $a+b+c+d$ by $a+b-c-d$. This is the same thing as multiplying $(a+b)+(c+d)$ by $(a+b)-(c+d)$. Then by the third formula we have

$$\{(a+b)+(c+d)\}\{(a+b)-(c+d)\}=(a+b)^2-(c+d)^2.$$

Next we can express $(a+b)^2$ and $(c+d)^2$ by means of the first formula; thus finally

$$(a+b+c+d)(a+b-c-d)=a^2+b^2+2ab-c^2-d^2-2cd.$$

57. From an examination of the examples here given, and those which are left to be worked, the student will recognise the truth of the following laws with respect to the result of multiplying algebraical expressions.

The number of terms in the product of two algebraical expressions is never *greater* than the product of the numbers of the terms in the two expressions, but may be *less*, owing to the simplification produced by collecting like terms.

When the multiplicand and multiplier are both arranged in the same way according to the powers of some common letter, the first and last terms of the product are *unlike* any other terms. For instance, in the example of Art. 50, the multiplicand and multiplier are arranged according to powers of a; the *first* term of the product is $8a^4$ and the *last* term is $24b^4$, and there are no other terms which are *like* these; in fact, the other terms contain a raised to some power *less* than the fourth power, and thus they differ from $8a^4$; and they all contain a to *some* power, and thus they differ from $24b^4$.

When the multiplicand and multiplier are both *homogeneous* the product is *homogeneous*, and the number of the dimensions of the product is the *sum* of the numbers which express the dimensions of the multiplicand and multiplier. Thus in the example of Art. 50, the multiplicand is homogeneous and of two dimensions, and the multiplier is homogeneous and of two dimensions; the product is homogeneous and of four dimensions. In the example of Art. 56 the multiplicand and the multiplier are both homogeneous and of one dimension; the product is homogeneous and of two dimensions. The law here stated and exemplified is of great importance as it serves to test the accuracy of algebraical work; and accordingly the student is recommended to pay great attention to the dimensions of the terms in the results which he obtains.

There is another law which is often useful in testing the accuracy of algebraical work, which we may call the law of *symmetry*. Suppose we require the product

$$(x+a+b)(x+b+c)(x+c+a).$$

Here a, b, and c occur *symmetrically*. If we put a instead of c, and c instead of a, we shall only change the order of the factors; and this will produce no change in the result. Similarly a and b may be interchanged, or b and c may be interchanged, without changing the value of the result. We may expect then that the result will be *symmetrical* with respect to a, b, and c; and we shall find this to be the case. The result is

$$x^3+2x^2(a+b+c)+x\{a^2+b^2+c^2+3(ab+bc+ca)\}$$
$$+a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc.$$

It will be seen that this expression is symmetrical with respect to a, b, and c. Take, for example, the coefficient of x^2; this is $2(a+b+c)$, that is, $2a+2b+2c$: if then a student had obtained an *unsymmetrical* result, suppose $2a+2b+c$, it would be obvious to a person acquainted with the subject that there must be an error in the work.

The law of symmetry is one with which the student will gradually become familiar; for the further he proceeds in Algebra, the more frequently will the law be of service.

EXAMPLES OF MULTIPLICATION.

1. Multiply $2p-q$ by $2q+p$.
2. Multiply $a^2+3ab+2b^2$ by $7a-5b$.
3. Multiply a^2-ab+b^2 by a^2+ab-b^2.
4. Multiply $a^2-ab+2b^2$ by $a^2+ab-2b^2$.
5. Multiply $a^2+2ax+x^2$ by $a^2+2ax-x^2$.

6. Multiply $a^2+4ax+4x^2$ by $a^2-4ax+4x^2$.

7. Multiply $a^2-2ax+bx-x^2$ by $b+x$.

8. Multiply $15x^2+18ax-14a^2$ by $4x^2-2ax-a^2$.

9. Multiply $2x^3+4x^2+8x+16$ by $3x-6$.

10. Multiply $2x^2-8xy+9y^2$ by $2x-3y$.

11. Multiply $4x^2-3xy-y^2$ by $3x-2y$.

12. Multiply $x^5-x^4y+xy^4-y^5$ by $x+y$.

13. Multiply $x+2y-3z$ by $x-2y+3z$.

14. Multiply $2x^2+3xy+4y^2$ by $3x^2-4xy+y^2$.

15. Multiply x^2+xy+y^2 by x^2+xz+z^2.

16. Multiply $a^2+b^2+c^2+bc+ca-ab$ by $a+b-c$.

17. Multiply $x^2-xy+y^2+x+y+1$ by $x+y-1$.

18. Multiply $x^3+4x^2+5x-24$ by $x^2-4x+11$.

19. Multiply $x^3-4x^2+11x-24$ by x^2+4x+5.

20. Multiply x^3-2x^2+3x-4 by $4x^3+3x^2+2x+1$.

21. Multiply $x^4+2x^3+x^2-4x-11$ by x^2-2x+3.

22. Multiply $x^5-5x^4+13x^3-x^2-x+2$ by x^2-2x-2.

23. Multiply $a^4-2a^3+3a^2-2a+1$ by $a^4+2a^3+3a^2+2a+1$.

24. Multiply together $a-x$, $a+x$, and a^2+x^2.

25. Multiply together $x-3$, $x-1$, $x+1$, and $x+3$.

26. Multiply together x^2-x+1, x^2+x+1, and x^4-x^2+1.

27. Multiply $x^4-ax^3+bx^2-cx+d$ by $x^4+ax^3-bx^2+cx-d$.

28. Shew that $(x+a)^4=x^4+4x^3a+6x^2a^2+4xa^3+a^4$.

29. Shew that $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

30. Multiply together $a+x$, $b+x$, and $c+x$.

31. Multiply together $x-a$, $x-b$, $x-c$, and $x-d$.

32. Multiply together $a+b-c$, $a+c-b$, $b+c-a$, and $a+b+c$.

33. Simplify $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

34. Simplify $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2$
$+(a+b-c-d)^2$.

35. Prove that $(x+y+z)^3-(x^3+y^3+z^3)=3(y+z)(z+x)(x+y)$.

36. Simplify $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$.

37. Simplify $(x-y)^3+(x+y)^3+3(x-y)^2(x+y)+3(x+y)^2(x-y)$.

38. Simplify $(a^2+b^2+c^2)^2-(a+b+c)(a+b-c)(a+c-b)(b+c-a)$.

39. Simplify $(a^2+b^2+c^2)^2+(a+b+c)(a+b-c)(a+c-b)(b+c-a)$.

40. Prove that $x^8+y^8+(x+y)^8=2(x^2+xy+y^2)^4$
$$+8x^2y^2(x+y)^2(x^2+xy+y^2).$$

41. Prove that $4xy(x^2+y^2)=(x^2+xy+y^2)^2-(x^2-xy+y^2)^2$.

42. Prove that $4xy(x^2-y^2)=(x^2+xy-y^2)^2-(x^2-xy-y^2)^2$.

43. Multiply together $(x^2-3x+2)^2$ and x^2+6x+1.

44. Multiply $x^6+a^6-ax(x^3+a^3)$ by $x^3+a^3-ax(x+a)$.

45. Multiply $(a+b)^3$ by $(a-b)^3$.

46. If $s=a+b+c$, prove that

$$s(s-2b)(s-2c)+s(s-2c)(s-2a)+s(s-2a)(s-2b)$$
$$=(s-2a)(s-2b)(s-2c)+8abc.$$

IV. DIVISION.

58. Division, as in Arithmetic, is the inverse of Multiplication. In Multiplication we determine the product arising from two given factors; in Division we have the product and one of the factors given, and our object is to determine the other factor. The factor to be determined is called the *quotient*.

59. Since the product of the numbers denoted by a and b is denoted by ab, the quotient of ab divided by a is b; thus $ab \div a = b$; and also $ab \div b = a$. Similarly, we have $abc \div a = bc$, $abc \div b = ac$, $abc \div c = ab$; and also $abc \div bc = a$, $abc \div ac = b$, $abc \div ab = c$. These results may also be written thus:

$$\frac{abc}{a}=bc, \qquad \frac{abc}{b}=ac, \qquad \frac{abc}{c}=ab;$$

$$\frac{abc}{bc}=a, \qquad \frac{abc}{ac}=b, \qquad \frac{abc}{ab}=c.$$

60. Suppose we require the quotient of $60abc$ divided by $3c$. Since $60abc = 20ab \times 3c$ we have $60abc \div 3c = 20ab$. Similarly, $60abc \div 4a = 15bc$; $60abc \div 5ab = 12c$; and so on. Thus we may deduce the following rule for dividing one *simple term* by another: *If the numerical coefficient and the literal product of the divisor be found in the dividend, the other part of the dividend is the quotient.*

61. If the numerical coefficient and the literal product of the divisor be *not* found in the dividend, we can only indicate the division by the notation we have appropriated for that purpose. Thus if $5a$ is to be divided by $2c$, the quotient can only be indicated by $5a \div 2c$, or by $\frac{5a}{2c}$. In some cases we may however simplify the expression for the quotient by a principle already used in arithmetic. Thus if $15a^2b$ is to be divided by $6bc$, the quotient is denoted by $\frac{15a^2b}{6bc}$. Here the dividend $= 3b \times 5a^2$, and the divisor $= 3b \times 2c$; thus in the same way as in Arithmetic we may remove the factor $3b$, which occurs in both dividend and divisor, and denote the quotient by $\frac{5a^2}{2c}$.

62. *One power of any number is divided by another power of the same number by subtracting the index of the latter power from the index of the former.*

Thus $a^5 \div a^2 = a \times a \times a \times a \times a \div a \times a = a \times a \times a = a^3 = a^{5-2}$. Similarly any other case may be established.

Hence if m and n be any whole numbers, and m greater than n, we have $a^m \div a^n$ or $\frac{a^m}{a^n} = a^{m-n}$.

63. Again, suppose we have such an expression as $\frac{a^2}{a^5}$. We may write it thus $\frac{a^2 \times 1}{a^2 \times a^3}$; then, as in Art. 61, we may remove

the common factor a^2. Thus we obtain $\frac{a^2}{a^5} = \frac{1}{a^3}$. Similarly any other case may be established.

Hence if m and n be any whole numbers, and m less than n, we have $a^m \div a^n$ or $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$.

64. Suppose such an expression as $\frac{a^2}{b^2}$ to occur; this may be written thus $\left(\frac{a}{b}\right)^2$. For $\left(\frac{a}{b}\right)^2$ means $\frac{a}{b} \times \frac{a}{b}$, and $\frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}$, as we know from Arithmetic, and as will be shewn in Chapter VIII. Similarly any other case may be established.

Hence if n be any whole number $\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$.

65. *When the dividend contains more than one term, and the divisor contains only one term, we must divide each term of the dividend by the divisor, and then collect the partial quotients to obtain the complete quotient.*

Thus, $\frac{ab - cb}{b} = a - c$; for $(a - c)\,b = ab - cb$.

$\frac{ab^2 - abc + abd}{ab} = b - c + d$; for $(b - c + d)\,ab = ab^2 - abc + abd$.

In the first example we see that corresponding to the term ab in the dividend and to the divisor b there is the term a in the quotient; and corresponding to the term $-cb$ in the dividend and to the divisor b there is the term $-c$ in the quotient.

We have already stated in Art. 49, that the following results are admitted for the present, subject to future explanation:

$$b \times -c = -bc, \qquad -b \times -c = bc.$$

Similarly, the following results may be admitted:

$$\frac{-bc}{-c} = b, \qquad \frac{bc}{-c} = -b.$$

Thus in Division as in Multiplication, the *sign* of the quotient is deduced from the signs of the dividend and divisor by the rule, *like signs produce* +, *and unlike signs produce* −.

66. When the divisor as well as the dividend contains more than one term, we must perform the operation of algebraical division in the same way as the operation called Long Division in Arithmetic. The following rule may be given:

Arrange both dividend and divisor according to the powers of some common letter, either both according to ascending powers, or both according to descending powers. Find how often the first term of the divisor is contained in the first term of the dividend, and write down this result for the first term of the quotient; multiply the whole divisor by this term, and subtract the product from the dividend. Bring down as many terms of the dividend as the case may require, and repeat the operation till all the terms are brought down.

Example. Divide $a^2 - 2ab + b^2$ by $a - b$.

The operation may be arranged thus:

$$
\begin{array}{r|l}
a - b \,) & a^2 - 2ab + b^2 \;(a - b \\
 & \underline{a^2 - ab \qquad\;\;} \\
 & \quad - ab + b^2 \\
 & \quad \underline{- ab + b^2}
\end{array}
$$

The reason for the rule is, that the whole dividend may be divided into as many parts as may be convenient, and the complete quotient is found by taking the sum of all the partial quotients. Thus, in the example, $a^2 - 2ab + b^2$ is really divided by the process into two parts, namely, $a^2 - ab$ and $-ab + b^2$, and each of these parts is divided by $a - b$; thus we obtain the complete quotient $a - b$.

67. It may happen, as in Arithmetic, that the division *cannot be exactly performed.* Thus, for example, if we divide $a^2 - 2ab + 2b^2$ by $a - b$, we shall obtain as before $a - b$ in the

quotient, and there will *then be a remainder* b^2. This result is expressed in a manner similar to that used in Arithmetic; we say $\frac{a^2 - 2ab + 2b^2}{a-b} = a - b + \frac{b^2}{a-b}$; that is, there is a complete quotient $a - b$ and a *fractional part* $\frac{b^2}{a-b}$. To the consideration of algebraical fractions we shall return in Chapter VIII.

68. The following examples are important:

$$\begin{array}{l} x-a)\,x^3 - a^3\,(x^2 + xa + a^2 \\ \quad x^3 - x^2a \\ \hline \qquad x^2a - a^3 \\ \qquad x^2a - xa^2 \\ \hline \qquad\quad xa^2 - a^3 \\ \qquad\quad xa^2 - a^3 \\ \hline \end{array}$$

$$\begin{array}{l} x-a)\,x^4 - a^4\,(x^3 + x^2a + xa^2 + a^3 \\ \quad x^4 - x^3a \\ \hline \qquad x^3a - a^4 \\ \qquad x^3a - x^2a^2 \\ \hline \qquad\quad x^2a^2 - a^4 \\ \qquad\quad x^2a^2 - xa^3 \\ \hline \qquad\qquad xa^3 - a^4 \\ \qquad\qquad xa^3 - a^4 \\ \hline \end{array}$$

The student may also easily verify the following statements:

$$\frac{x^2-a^2}{x+a} = x - a; \qquad \frac{x^4-a^4}{x+a} = x^3 - x^2a + xa^2 - a^3;$$

$$\frac{x^3+a^3}{x+a} = x^2 - xa + a^2; \qquad \frac{x^5+a^5}{x+a} = x^4 - x^3a + x^2a^2 - xa^3 + a^4.$$

Each of these examples of division furnishes an example of multiplication, as the product of the divisor and quotient must be equal to the dividend. Thus we have the following results which are worthy of notice:

$$x^2 - a^2 = (x+a)(x-a),$$

$$x^3 - a^3 = (x-a)(x^2 + xa + a^2),$$

$$x^3 + a^3 = (x+a)(x^2 - xa + a^2),$$

$$x^4 - a^4 = (x-a)(x^3 + x^2a + xa^2 + a^3),$$

$$x^4 - a^4 = (x+a)(x^3 - x^2a + xa^2 - a^3),$$

$$x^5 + a^5 = (x+a)(x^4 - x^3a + x^2a^2 - xa^3 + a^4).$$

69. It will be useful for the student to notice the following facts:

$x^n - a^n$ is always divisible by $x - a$ whether the index n be an *odd* or *even* whole number.

$x^n - a^n$ is divisible by $x + a$ if the index n be an *even* whole number.

$x^n + a^n$ is divisible by $x + a$ if the index n be an *odd* whole number.

It will be easy for the student to verify these statements in any particular case, and hereafter we shall give a general proof of them. See Chapter XXXIII.

70. By means of the results which have been obtained in the preceding Articles we may often resolve algebraical expressions into factors. Thus whatever A and B denote we have

$$A^2 - B^2 = (A + B)(A - B),$$

and the student will frequently have occasion to use this general result with various forms of A and B. For example, suppose $A = a^2$, and $B = b^2$, so that $A^2 = a^4$, and $B^2 = b^4$; then we have

$$a^4 - b^4 = (a^2 + b^2)(a^2 - b^2),$$

and as $$a^2 - b^2 = (a + b)(a - b),$$

we obtain $$a^4 - b^4 = (a^2 + b^2)(a + b)(a - b).$$

Again, suppose $A = a^3$, and $B = b^3$, so that $A^2 = a^6$, and $B^2 = b^6$; then we have

$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3);$$

and, as in Art. 68,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

so that

$$a^6 - b^6 = (a + b)(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2).$$

Again, suppose $A = a^4$ and $B = b^4$, so that $A^2 = a^8$, and $B^2 = b^8$; then we have

$$a^8 - b^8 = (a^4 + b^4)(a^4 - b^4)$$
$$= (a^4 + b^4)(a^2 + b^2)(a + b)(a - b).$$

Again, take the general result

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2),$$

and suppose $A = a^2$, and $B = b^2$; thus we obtain

$$a^6 - b^6 = (a^2 - b^2)(a^4 + a^2b^2 + b^4);$$

and by comparing this with the result just proved,

$$a^6 - b^6 = (a + b)(a - b)(a^2 + ab + b^2)(a^2 - ab + b^2),$$

we infer that

$$(a^2 + ab + b^2)(a^2 - ab + b^2) = a^4 + a^2b^2 + b^4.$$

This can be easily verified by the method of Art. 56.

For $(a^2 + ab + b^2)(a^2 - ab + b^2) = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$

$$= (a^2 + b^2)^2 - a^2b^2$$
$$= a^4 + 2a^2b^2 + b^4 - a^2b^2$$
$$= a^4 + a^2b^2 + b^4.$$

We may also in some cases obtain useful arithmetical applications of our formulæ. For example,

$$(127)^2 - (123)^2 = (127 + 123)(127 - 123)$$
$$= 250 \times 4 = 1000;$$

thus the value of $(127)^2 - (123)^2$ is obtained more easily than it would be by squaring 127 and 123, and subtracting the second result from the first.

The following additional examples are deserving of notice:

$$(a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2) = (a^2 + b^2)^2 - (ab\sqrt{2})^2$$
$$= a^4 + 2a^2b^2 + b^4 - 2a^2b^2$$
$$= a^4 + b^4.$$

$$(a^2+ab\sqrt{3}+b^2)(a^2-ab\sqrt{3}+b^2)=(a^2+b^2)^2-(ab\sqrt{3})^2$$
$$=a^4+2a^2b^2+b^4-3a^2b^2$$
$$=a^4-a^2b^2+b^4.$$

$$a^6+b^6=(a^2+b^2)(a^4-a^2b^2+b^4)$$
$$=(a^2+b^2)(a^2+ab\sqrt{3}+b^2)(a^2-ab\sqrt{3}+b^2).$$

71. The following are additional examples of Division.

Divide $8a^4-22a^3b+43a^2b^2-38ab^3+24b^4$ by $2a^2-3ab+4b^2$.

$$
\begin{array}{r|l}
2a^2-3ab+4b^2 \;) & 8a^4-22a^3b+43a^2b^2-38ab^3+24b^4 \;(4a^2-5ab+6b^2 \\
 & 8a^4-12a^3b+16a^2b^2 \\
\hline
 & -10a^3b+27a^2b^2-38ab^3 \\
 & -10a^3b+15a^2b^2-20ab^3 \\
\hline
 & 12a^2b^2-18ab^3+24b^4 \\
 & 12a^2b^2-18ab^3+24b^4 \\
\hline
\end{array}
$$

The quotient is $4a^2-5ab+6b^2$.

Divide $x^3-(a+b+c)x^2+(ab+bc+ac)x-abc$ by $x-a$.

$$
\begin{array}{r|l}
x-a \;) & x^3-(a+b+c)x^2+(ab+bc+ac)x-abc \;(x^2-(b+c)x+bc \\
 & x^3-ax^2 \\
\hline
 & -(b+c)x^2+(ab+bc+ac)x \\
 & -(b+c)x^2+\quad(ab+ac)x \\
\hline
 & bcx-abc \\
 & bcx-abc \\
\hline
\end{array}
$$

The quotient is $x^2-(b+c)x+bc$.

These two examples suggest the following statement: When the dividend and the divisor are *homogeneous* so also is the quotient; the number of the dimensions of the quotient is equal to the excess of the number which expresses the dimensions of the dividend over the number which expresses the dimensions of the divisor. See Art. 57.

EXAMPLES OF DIVISION.

1. Divide x^3+1 by $x+1$.

2. Divide $27x^3+8y^3$ by $3x+2y$.

3. Divide $a^3-2ab^2+b^3$ by $a-b$.

4. Divide $a^3-2a^2b-3ab^2$ by $a+b$.

5. Divide $64x^6-y^6$ by $2x-y$.

6. Divide a^5+b^5 by $a+b$.

7. Divide $x^3-x^2y+xy^2-y^3$ by $x-y$.

8. Divide x^3-7x-6 by $x-3$.

9. Divide $32x^5+y^5$ by $2x+y$.

10. Divide $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$ by x^3-y^3.

11. Divide $x^4+x^3-4x^2+5x-3$ by x^2+2x-3.

12. Divide $a^4+2a^2b^2+9b^4$ by $a^2+2ab+3b^2$.

13. Divide a^6-b^6 by $a^3+2a^2b+2ab^2+b^3$.

14. Divide $32a^4+54ab^3-81b^4$ by $2a+3b$.

15. Divide x^6-2x^3+1 by x^2-2x+1.

16. Divide $x^6-6x^4+9x^2-4$ by x^2-1.

17. Divide $a^4+a^3b-8a^2b^2+19ab^3-15b^4$ by $a^2+3ab-5b^2$.

18. Divide the product of $x^3-12x+16$ and $x^3-12x-16$ by x^2-16.

19. Divide the product of x^3-2x+1 and x^3-3x+2 by x^3-3x^2+3x-1.

20. Divide the product of x^2-x-1, $2x^2+3$, x^2+x-1, and $x-4$ by x^4-3x^2+1.

21. Divide the product of a^2+ax+x^2 and a^3+x^3 by $a^4+a^2x^2+x^4$.

22. Divide the product of $x^4-4x^3a+6x^2a^2-4xa^3+a^4$ and $x^2+2xa+a^2$ by $x^4-2x^3a+2xa^3-a^4$.

23. Divide $a^3+a^2b+a^2c-abc-b^2c-bc^2$ by a^2-bc.

24. Divide $3x^3 + 4abx^2 - 6a^2b^2x - 4a^3b^3$ by $x + 2ab$.

25. Divide the product of $x^3 - 3x^2 + 3x - 1$, $x^2 - 2x + 1$ and $x - 1$ by $x^4 - 4x^3 + 6x^2 - 4x + 1$.

26. Divide $6a^4 - a^3b + 2a^2b^2 + 13ab^3 + 4b^4$ by $2a^2 - 3ab + 4b^2$.

27. Divide $x^3 + y^3 + 3xy - 1$ by $x + y - 1$.

28. Divide $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.

29. Divide $2a^7b - 5a^6b^2 - 11a^5b^3 + 5a^4b^4 - 26a^3b^5 + 7a^2b^6 - 12ab$ by $a^4 - 4a^3b + a^2b^2 - 3ab^3$.

30. Divide $a^2b^2 + 2abc^2 - a^2c^2 - b^2c^2$ by $ab + ac - bc$.

31. Divide the product of $a + b - c$, $a - b + c$, and $b + c - a$ by $a^2 - b^2 - c^2 + 2bc$.

32. Divide $(a + b + c)(ab + bc + ca) - abc$ by $a + b$.

33. Divide $(a^2 - bc)^3 + 8b^3c^3$ by $a^2 + bc$.

34. Divide $b(x^3 - a^3) + ax(x^2 - a^2) + a^3(x - a)$ by $(a + b)(x - a)$.

35. Divide $xy^3 + 2y^3z - xy^2z + xyz^2 - x^3y - 2yz^3 + x^3z - xz^3$ by $y + z - x$.

36. Divide $a^2(b + c) - b^2(a + c) + c^2(a + b) + abc$ by $a - b + c$.

37. Divide $(a - b)x^3 + (b^3 - a^3)x + ab(a^2 - b^2)$ by $(a - b)x + a^2 - b^2$.

38. Divide $ax^2 - ab^2 + b^2x - x^3$ by $(x + b)(a - x)$.

39. Divide $(b - c)a^3 + (c - a)b^3 + (a - b)c^3$ by $a^2 - ab - ac + bc$.

40. Divide $(ax + by)^2 + (ay - bx)^2 + c^2x^2 + c^2y^2$ by $x^2 + y^2$.

41. Divide $a^2b - bx^2 + a^2x - x^3$ by $(x + b)(a - x)$.

42. Resolve $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$ into two factors.

43. Divide $b(x^3 + a^3) + ax(x^2 - a^2) + a^3(x + a)$ by $(a + b)(x + a)$.

44. Shew that $(x^2 - xy + y^2)^3 + (x^2 + xy + y^2)^3$ is divisible by $2x^2 + 2y^2$.

45. Shew that $(x + y)^7 - x^7 - y^7$ is divisible by $(x^2 + xy + y^2)^2$.

46. If $A = bc - p^2$, $B = ca - q^2$, $C = ab - r^2$, $P = qr - ap$, $Q = rp - bq$, and $R = pq - cr$, find the value of $\dfrac{BC - P^2}{a}$, $\dfrac{CA - Q^2}{b}$, $\dfrac{AB - R^2}{c}$, $\dfrac{QR - AP}{p}$, $\dfrac{RP - BQ}{q}$, and $\dfrac{PQ - CR}{r}$.

47. Resolve $a^{16} - x^{16}$ into five factors.

48. Resolve $4a^2b^2 - (a^2 + b^2 - c^2)^2$ into four factors.

49. Resolve $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$ into four factors.

50. Shew that $(ay - bx)^2 + (bz - cy)^2 + (cx - az)^2 + (ax + by + cz)^2$ is divisible by $a^2 + b^2 + c^2$ and by $x^2 + y^2 + z^2$.

V. NEGATIVE QUANTITIES.

72. In Algebra we are sometimes led to a subtraction which cannot be performed because the number which should be subtracted is greater than that from which it is required to be subtracted. For instance, we have the following relation: $a - (b + c) = a - b - c$; suppose that $a = 7$, $b = 7$ and $c = 3$ so that $b + c = 10$. Now the relation $a - (b + c) = a - b - c$ tacitly supposes $b + c$ to be less than a; if we were to neglect this supposition for a moment we should have $7 - 10 = 7 - 7 - 3$; and as $7 - 7$ is zero we might finally write $7 - 10 = -3$.

73. In writing such an equation as $7 - 10 = -3$ we may be understood to make the following statement: "it is impossible to take 10 from 7, but if 7 be taken from 10 the remainder is 3."

74. It might at first sight seem to the student unlikely that such an expression as $7 - 10$ should occur in practice; or that if it did occur it would only arise either from a mistake which could be instantly corrected, or from an operation being proposed which it was obviously impossible to perform, and which must therefore be abandoned. As he proceeds in the subject the student will find however that such expressions occur frequently; it might happen that $a - b$ appeared at the commencement of a long investigation, and that it was not easy to decide at once whether a were greater or less than b. Now the object of the present Chapter is to shew that in such a case we may proceed on the supposition that a is greater than b, and that if it should finally appear that a is less than b we shall still be able to make use of our investigation.

75. Let us consider an illustration. Suppose a merchant to gain in one year a certain number of pounds and to lose a certain number of pounds in the following year, what change has taken place in his capital? Let a denote the number of pounds gained in the first year, and b the number of pounds lost in the second. Then if a is greater than b the capital of the merchant has been *increased* by $a-b$ pounds. If however b is greater than a the capital has been *diminished* by $b-a$ pounds. In this latter case $a-b$ is the indication of what would be pronounced in Arithmetic to be an impossible subtraction; but yet in Algebra it is found convenient to retain $a-b$ as indicating the change of the capital, which we may do by means of an appropriate system of *interpretation*. Thus, for example, if $a=400$ and $b=500$ the merchant's capital has suffered a diminution of 100 pounds; the algebraist indicates this in symbols, thus

$$400-500=-100,$$

and he may turn his symbols into words by saying that the merchant's capital has been increased by -100 pounds. This language is indeed far removed from the language of ordinary life, but if the algebraist understands it and uses it consistently and logically his deductions from it will be sound.

76. There are numerous instances like the preceding in which it is convenient for us to be able to represent not only the *magnitude* but also what may be called the *quality* or *affection* of the things about which we may be reasoning. In the preceding case a sum of money may be *gained* or it may be *lost*; in a question of chronology we may have to distinguish a date *before* a given epoch from a date *after* that epoch; in a question of position we may have to distinguish a distance measured to the *north* of a certain starting-point from a distance measured to the *south* of it; and so on. These pairs of related magnitudes the algebraist distinguishes by means of the signs + and −. Thus if, as in the preceding Article, the things to be distinguished are gain and loss, he may denote by 100 or by $+100$ a *gain*, and then he will denote by -100 a *loss* of the same extent. Or he may denote a loss by 100

or by + 100, and then he will denote by − 100 a gain of the same extent. There are two points to be noticed; *first*, that when no sign is used + is to be understood; *secondly*, the sign + may be ascribed to either of the two related magnitudes, and then the sign − will throughout the investigation in hand belong to the other magnitude.

77. In Arithmetic then we are concerned only with the numbers represented by the symbols 1, 2, 3, &c., and intermediate fractions. In Algebra, besides these, we consider another set of symbols −1, −2, −3, &c., and intermediate fractions. Symbols preceded by the sign − are called *negative quantities*, and symbols preceded by the sign + are called *positive quantities*. Symbols without a sign prefixed are considered to have + prefixed.

The *absolute value* of any quantity is the number represented by this quantity taken independently of the sign which precedes the number.

78. In the preceding Chapters we have given rules for the Addition, Subtraction, Multiplication, and Division of algebraical expressions. Those rules were based on arithmetical notions and were shewn to be true so long as the expressions represented such things as Arithmetic considers, that is *positive quantities*. Thus, when we introduced such an expression as $a - b$ we supposed both a and b to be *positive quantities* and a to be greater than b. But as we wish hereafter to include *negative quantities* among the objects of our reasoning it becomes necessary to recur to the consideration of these primary operations. Now it is found convenient that the laws of the fundamental operations should be the same whether the symbols denote *positive* or *negative* quantities, and we shall therefore secure this convenience by means of suitable *definitions*. For it must be observed that we have a power over the definitions; for example, *multiplication* of positive quantities is defined in Arithmetic, and we should naturally retain that definition; but *multiplication* of *negative quantities*, or of *a positive and a negative quantity* has not hitherto been defined; the terms are

at present destitute of meaning. It is therefore in our power to define them as we please provided we always adhere to our definition.

79. The student will remember that he is not in a position to judge of the *convenience* which we have intimated will follow from our keeping the fundamental laws of algebraical operation permanent, and giving a wider meaning to such common words as addition and multiplication in order to insure this permanence. He must at present confine himself to watching the accuracy of the deductions drawn from the definitions. As he proceeds he will see that Algebra gains largely in power and utility by the introduction of *negative quantities* and by the extension of the meaning of the fundamental operations. And he will find that although the symbols + and − are used apparently for two purposes, namely, according to the definitions in Arts. 3 and 4, and according to the convention in Art. 76, no contradiction nor confusion will ultimately arise from this circumstance.

80. Two quantities are said to be *equal* and may be connected by the sign = when they have the same numerical value and have the same sign. Thus they may have the same absolute value and yet not be equal; for example, 7 and −7 are of the same *absolute value* but they are not to be called equal.

81. In Arithmetic the object of *addition* is to find a number which alone is equal to the units and fractions contained in certain other numbers. This notion is not applicable to negative quantities; that is, we have as yet no *meaning* for the phrase "add −3 to 5," or "add −3 to −5." We shall therefore give a meaning to the word *add* in such cases, and the meaning we propose is determined by the following rules: *To add two quantities of the same sign add the absolute values of the quantities and place the sign of the quantities before the sum. To add two quantities of different signs, subtract the less absolute value from the greater, and place before the remainder the sign of that quantity which has the greater absolute value.*

Thus, by the first rule, if we add 3 to 5 we obtain 8; if we add -3 to -5 we obtain -8. By the second rule, if we add 3 to -5 we obtain -2; if we add -3 to 5 we obtain 2.

82. It will be seen that the rules above given leave to the word *add* its common arithmetical meaning so long as the things which are to be added are such as Arithmetic considers, namely, *positive quantities*, and merely assign a meaning to the word in those cases when as yet it had no meaning. The reader may perhaps object that no *verbal definition* is given of the word *add* but merely a rule for *adding* two quantities. We may reply that the practical use of a *definition* is to enable us to know that we use a word correctly and consistently when we do use it, and the rules above given will ensure this end in the present case.

83. The rules are not altogether arbitrary: that is, the student may easily see even at this stage of his progress that they are likely to be advantageous. Thus, to take the numerical example given above, suppose a man to be entitled to *receive* 3 shillings from one person and 5 shillings from another, then he may be considered to possess 8 shillings. But suppose him to *owe* 3 shillings to one person and 5 shillings to another; then he *owes* altogether 8 shillings; this may be considered to be an interpretation of the -8 which arises from adding -3 to -5. Next, suppose that he has to receive 3 shillings and to pay 5 shillings; then he *owes* altogether 2 shillings; this may be considered to be an interpretation of the -2 which arises from adding 3 to -5. Lastly, suppose that he has to receive 5 shillings and to pay 3 shillings, then he may be considered to possess 2 shillings; this may be considered to be an interpretation of the 2 which arises from adding -3 to 5.

84. Thus in Algebra *addition* does not necessarily imply augmentation in an arithmetical sense; nevertheless the word *sum* is used to denote the result. Sometimes when there might be an uncertainty on the point, the term *algebraical sum* is used to distinguish such a result from the *arithmetical sum*, which would

be obtained by the arithmetical addition of the *absolute values* of the terms considered.

85. Suppose now we have to add the five quantities -2, $+5$, -13, -4 and $+8$. The sum of -2 and $+5$ is $+3$; the sum of $+3$ and -13 is -10; the sum of -10 and -4 is -14; the sum of -14 and $+8$ is -6. Thus -6 is the *sum* required. Or we may first calculate the sum of the negative quantities -2, -13 and -4, and we thus get -19; then calculate the sum of the positive quantities $+5$ and $+8$, and we thus get $+13$. Thus the proposed sum becomes $+13-19$, that is, -6 as before. It will be easily seen on trial that the same result is obtained whatever be the order in which the terms are taken. That is, for example, $-2-13+5+8-4$, $8-13-2-4+5$, and so on, all give -6.

86. Next suppose we have to *add* two or more algebraical expressions; for example, $2a-3b+4c$ and $-a-2b+c+2d$. We have for the *sum*

$$2a-3b+4c-a-2b+c+2d.$$

Then the like terms may be collected; thus

$$2a-a=a, \quad -3b-2b=-5b, \quad 4c+c=5c;$$

and the *sum* becomes

$$a-5b+5c+2d.$$

Thus we may give the following rule for algebraical addition: *Write the terms in the same line preceded by their proper signs; collect like terms into one, and arrange the terms of the result in any order.*

87. In arithmetical subtraction we have to take away one number, which is called the *subtrahend*, from another which is called the *minuend*, and the result is called the *remainder*. The remainder then may be defined as that number which must be added to the subtrahend to produce the minuend, and the object of subtraction is to find this *remainder*.

We shall use the same definition in algebraical subtraction, that is, we say that in subtraction we have to find the quantity which must be added to the subtrahend to produce the minuend. From this definition we obtain the rule: *Change the sign of every term in the subtrahend and add the result so obtained to the minuend, and the result will be the remainder required.*

For it is obvious, that if to the expression thus formed we add the subtrahend, giving to each term its proper sign, all the terms of the subtrahend will disappear and leave the minuend; which was required.

88. We have still another point to notice. According to what has been laid down, the *sum* of $+a$ and $-b$ is denoted by $a-b$; if we take $-b$ from a, the result is $a+b$; and the sum of $-a$, $+b$, and $-c$ is $-a+b-c$; and so on. But we have as yet supposed that the letters themselves stand for *positive numbers;* for example, when we say that the sum of $+a$ and $-b$ is $a-b$, a may be 6, and b may be 10; but suppose that a is -6, and b is -10, do the rules adopted apply here? Since b is -10, $-b$ or $-(-10)$ will naturally be taken to mean 10, and $+a$ or $+(-6)$ will be taken to mean -6; and the sum of 10 and -6 is 4.

89. Thus if a be itself a *negative quantity*, we have assigned a meaning to $+a$ and to $-a$; and the meanings are these: let $a = -\alpha$, so that α is a positive quantity, then $+a$ or $+(-\alpha) = -\alpha$, and $-a$ or $-(-\alpha) = \alpha$. We said in the preceding Article that these meanings followed *naturally* from what had preceded; it is however of little consequence whether we consider these meanings to follow thus, or whether we look upon them as new interpretations; the important point is to use them uniformly and consistently when once adopted.

Since $+(-\alpha) = -\alpha$, and $-(-\alpha) = \alpha$, that is, $+\alpha$, we may enunciate the same rule as formerly, namely, that *like signs produce + and unlike signs −.*

90. There are four cases to consider in multiplication. Let

a and b denote any two numbers, then we have to consider

$$+a \times +b, \quad -a \times +b, \quad +a \times -b, \quad -a \times -b.$$

The first case is that of common Arithmetic and needs no remark. The ordinary definition of multiplication may also be applied to the second case; for suppose, for example, that $b = 3$, then $-a \times 3$ indicates that $-a$ is to be repeated *three* times, that is, we have $-a-a-a$ or $-3a$ as the result. Thus

$$-a \times +b = -ab.$$

In the other two cases the multiplier is a *negative quantity*, and thus the common arithmetical notion of multiplication is not applicable; we may therefore give by definition a meaning to the term in this case. Now we observe that when the multiplier is positive, the sign of the multiplicand is preserved in the product; thus we are led to adopt the following convention: *When the multiplier is negative, perform the multiplication as if the multiplier were positive, and change the sign of the product.* Hence we conclude immediately that

$$+a \times -b = -ab \text{ and } -a \times -b = +ab.$$

91. Thus we have the following rule: *To multiply two quantities whatever be their signs, multiply them without considering the signs, and put + or − before the product according as the two factors have the same sign or different signs.* As before remarked, the rule for the sign of the product is abbreviated thus: *Like signs give + and unlike signs give −.*

92. In the preceding Articles we supposed a and b themselves to denote arithmetical numbers; it is important however to observe that if they denote any quantities, positive or negative, the four results obtained are true; that is,

$$+a \times +b = +ab, \quad -a \times +b = -ab, \quad +a \times -b = -ab, \quad -a \times -b = +ab.$$

Take, for example, the last of these, and suppose that a is a negative quantity, and so may be denoted by $-\alpha$; then $-a$ is a positive quantity, and $= \alpha$. (Art. 89.) Hence $-a \times -b = \alpha \times -b$; and this by the third case $= -\alpha b$. And $\alpha b = -a \times b = -ab$ by the second case.

Thus the result $-a \times -b = ab$ holds when a is a negative quantity. Similarly any other case may be established.

93. We must now shew that the rule for multiplying binomial and polynomial expressions given in Art. 48 is true, whatever the symbols denote. Take, for example, the case

$$(a-b)\,c = ac - bc.$$

When this was proved, we supposed c a positive quantity; we will now suppose that c is a negative quantity, namely $-\gamma$. By virtue of the convention in Art. 90, to find the product of $a-b$ and $-\gamma$ we must multiply $a-b$ by γ and then change the sign of each term in the result. Now,

$$(a-b)\,\gamma = a\gamma - b\gamma;$$

thus $$(a-b)\,(-\gamma) = -a\gamma + b\gamma.$$

But since $c = -\gamma$, we have

$$ac - bc = -a\gamma + b\gamma;$$

thus the relation $$(a-b)\,c = ac - bc$$

holds whatever c may be, positive or negative. Similarly, any other case may be established.

94. The ordinary definition of division will be universally applicable; we suppose a product and one factor given, and we have to determine the other factor.

Hence if we perform the division without regarding the signs we obtain the quotient apart from its sign. It remains then to determine the sign, for which we may give the following rule:

When the dividend and divisor have the same sign, the quotient must have the sign +; when the dividend and divisor have different signs, the quotient must have the sign −.

This rule follows from the fact that the product of the divisor and quotient must be equal to the dividend. The rule for the sign of the quotient may as before be abbreviated thus: *Like signs give + and unlike signs give −.*

95. The words *greater* and *less* are often used in Algebra in an extended sense. We say that *a is greater than b* or that *b is less than a* if *$a-b$ is a positive quantity.* This is consistent with ordinary language when a and b are themselves both positive, and it is found convenient to extend the meaning of the words *greater* and *less* so that this definition may also hold when a or b is negative, or when both are negative. Thus, for example, in *algebraical language* 1 is greater than -2 and -2 is greater than -3.

96. Before leaving this part of the subject we may make a few general remarks. The subject of Algebra has been divided by some modern writers into two parts, which they have called *Arithmetical Algebra* and *Symbolical Algebra.* In Arithmetical Algebra symbols are used to denote the numbers and the operations which occur in Arithmetic. Here, as shewn in the preceding Chapters of the present work, we begin by defining our symbols, and then arrive at certain results, as for example, at the result $(a+b)(a-b)=a^2-b^2$. In Symbolical Algebra we assume that the rules of Arithmetical Algebra hold universally, and then determine what must be denoted by the symbols and the operations, in order to ensure this result. Thus we may consider, that in the present Chapter we have been examining what meanings must be given to the symbols to make the results of the previous Chapters hold universally. And we have thus been led to the theory of *negative* quantities, and to an extension of the meaning of the words addition, subtraction, multiplication and division.

97. In some of the older works on Algebra, scarcely any reference is made to the extensions of meaning which we have given to some simple arithmetical terms. In such works the proofs and investigations are valid only so long as the symbols have purely arithmetical meanings; and the proofs and investigations are really *assumed* without demonstration to hold when the symbols have not purely arithmetical meanings. In recent works, as in the present, an attempt is made to establish the proofs completely. It must not however be denied that this branch of

the subject presents considerable difficulty to the beginner, and it will probably only be after repeated examination that a conviction will be obtained of the universal truth of the fundamental theorems.

The student is recommended to proceed onwards as far as the Chapter on Equations; he will there see some further remarks on negative quantities, and he may afterwards read the present Chapter again. It would be inconsistent with the plan of this work to enter very largely on this branch of Algebra; but the present Chapter may furnish an outline which the student can fill up by his future reading and reflection.

We shall require in the course of the work certain propositions which are obvious axioms in Arithmetic, and which are also true when we give to the terms and symbols their extended meanings.

98. If equal quantities be added to equal quantities, the sums will be equal.

99. If equal quantities be taken from equal quantities, the remainders will be equal.

Thus, for example, if $A = pB + C$, then by taking C from these equal quantities we have $A - C = pB$.

100. If equal quantities be multiplied by the same or by equal quantities, the products will be equal.

Thus too if $a = b$ then $a^n = b^n$ and $\sqrt[n]{a} = \sqrt[n]{b}$.

101. If equal quantities be divided by the same or by equal quantities, the quotients will be equal.

102. If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

103. If a quantity be both multiplied and divided by another, its value will not be altered.

104. It is important to draw the attention of the reader to the fact, that these propositions are still true whether the quantities spoken of are positive or negative, and when the terms addition, subtraction, multiplication, and division have their extended meanings. For example, if $a=b$, and $c=d$, then $ac=bd$; this is obvious if all the letters denote positive quantities. Suppose however that c is a negative quantity, so that we may represent it by $-\gamma$; then d must be a negative quantity, and if we denote it by $-\delta$, we have $\gamma=\delta$; therefore $a\gamma=b\delta$; therefore $-a\gamma=-b\delta$; and thus $ac=bd$.

MISCELLANEOUS EXAMPLES.

1. Shew that $x^2+y^2+4z^2+2xy+8xz$ and $4(x+z)^2$ become identical when x and y each $=a$.

2. If $a=1$, $b=\frac{2}{3}$, $x=7$ and $y=8$, find the value of

$$5(a-b)\sqrt[3]{\{(a+x)y^2\}}-b\sqrt{\{(a+x)y\}}+a.$$

3. If $a=\frac{5}{7}$, $b=\frac{1}{2}$, $x=5$ and $y=\frac{9}{2}$, find the value of

$$(10a+20b)\sqrt{\{(x-b)y\}}-3a\sqrt[3]{\{y^2(x-b)\}}+5b.$$

4. If $a=\frac{4}{5}$, $b=2$, $x=\frac{10}{3}$ and $y=\frac{4}{3}$, find the value of

$$(a+b)\sqrt[3]{\{(x-b)y^2\}}-a\sqrt{\{y(x-b)\}}+x.$$

5. Substitute $y+3$ for x in $x^4-x^3+2x^2-3$ and arrange the result.

6. Shew that

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}^2=2\{(a-b)^4+(b-c)^4+(c-a)^4\}.$$

7. If $2s=a+b+c$, shew that

$$(s-a)^2+(s-b)^2+(s-c)^2+s^2=a^2+b^2+c^2.$$

8. If $2s=a+b+c$, shew that

$$2(s-a)(s-b)+2(s-b)(s-c)+2(s-c)(s-a)=2s^2-a^2-b^2-c^2.$$

9. If $2s=a+b+c$, shew that

$$2(s-a)(s-b)(s-c)+a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)=abc.$$

10. Shew that

$$(a+b+c)^3-(b+c)^3-(c+a)^3-(a+b)^3+a^3+b^3+c^3=6abc.$$

11. Shew that if $a_1+a_2+\dots+a_n=\frac{n}{2}s$, then

$$(s-a_1)^2+(s-a_2)^2+\dots+(s-a_n)^2=a_1^2+a_2^2+\dots+a_n^2.$$

12. If $2s=a+b+c$ and $2\sigma^2=a^2+b^2+c^2$, shew that

$$(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+(\sigma^2-c^2)(\sigma^2-a^2)$$
$$=4s(s-a)(s-b)(s-c).$$

VI. GREATEST COMMON MEASURE.

105. In Arithmetic the *greatest common measure* of two or more whole numbers is the greatest number which will divide each of them without remainder. The term is also used in Algebra, and its meaning in this subject will be understood from the following definition of the *greatest common measure* of two or more algebraical expressions: Let two or more algebraical expressions be arranged according to descending powers of some common letter; then the factor of highest dimensions in that letter which divides each of these expressions without remainder is called their greatest common measure.

106. The term *greatest common measure* is not very appropriate in Algebra, because the words *greater* and *less* are seldom applicable to algebraical expressions in which specific numerical values have not been assigned to the various letters which occur. It would be better to speak of the *highest common divisor* or of the *highest common measure;* but in conformity with established usage we retain the term *greatest common measure.* The letters G. C. M. will often be used for shortness instead of this term.

When one expression divides two or more expressions without remainders we shall say that it is a *common measure* of them, or more briefly, that it is a *measure* of them.

107. The following is the rule for finding the G. C. M. of two algebraical expressions:

Let A and B denote the two expressions; let them be arranged according to descending powers of some common letter, and suppose the index of the highest power of that letter in A not less than the index of the highest power of that letter in B. Divide A by B; then make the remainder a divisor and B the dividend. Again, make the new remainder a divisor and the preceding divisor the dividend. Proceed in this way until there is no remainder; then the last divisor is the G. C. M. required.

108. Example: find the G. C. M. of

$$x^2-6x+8 \text{ and } 4x^3-21x^2+15x+20.$$

$$\begin{array}{r|l} x^2-6x+8 \,) & 4x^3-21x^2+15x+20 \,(4x+3 \\ & \underline{4x^3-24x^2+32x} \\ & 3x^2-17x+20 \\ & \underline{3x^2-18x+24} \\ & x-4 \end{array}$$

$$\begin{array}{r|l} x-4 \,) & x^2-6x+8 \,(x-2 \\ & \underline{x^2-4x} \\ & -2x+8 \\ & \underline{-2x+8} \end{array}$$

Thus $x-4$ is the G. C. M. required.

109. The truth of the rule given in Art. 107 depends upon the following principles:

(1) If P divide A, then it will divide mA. For since P divides A, we may suppose $A=aP$, then $mA=maP$, thus P divides mA.

(2) If P divide A and B, then it will divide $mA \pm nB$. For since P divides A and B, we may suppose $A=aP$, and $B=bP$, then $mA \pm nB=(ma \pm nb)P$; thus P divides $mA \pm nB$.

We can now prove the rule given in Art. 107.

110. Let A and B denote the two expressions; let them be arranged according to descending powers of some common letter, and suppose the index of the highest power of that letter in A not less than the index of the highest power of that letter in B. Divide A by B; let p denote the quotient, and C the remainder. Divide B by C; let q denote the quotient, and D the remainder. Divide C by D, and suppose that there is no remainder, and let r denote the quotient. Thus we have the following results:

$$
\begin{array}{l}
B)\,A\,(p \\
\quad pB \\
\quad \overline{C)\,B\,(q} \\
\qquad\quad qC \\
\qquad\quad \overline{D)\,C\,(r} \\
\qquad\qquad\quad rD \\
\qquad\qquad\quad \overline{}
\end{array}
$$

$$A = pB + C; \qquad B = qC + D; \qquad C = rD.$$

We shall first shew that D is *a* common measure of A and B.

D divides C, since $C = rD$; hence (Art. 109) D divides qC and also $qC + D$; that is, D divides B. Again, since D divides B and C, it divides $pB + C$; that is, D divides A. Hence D divides A and B.

We have thus shewn that D is *a* common measure of A and B; we shall next shew that it is their *greatest* common measure.

By Art. 109 every expression which divides A and B divides $A - pB$, that is, C; thus every expression which is a measure of A and B is a measure of B and C. Similarly every expression which is a measure of B and C is a measure of C and D. Thus every expression which is a measure of A and B divides D. But no expression higher than D can divide D. Thus D is the G. C. M. required.

111. In the same manner as it is shewn in the preceding Article that D measures A and B, it may be shewn that every *expression which divides* D also measures A and B. And it is shewn in the preceding Article that every expression which measures A and B divides D. Thus every measure of A and B divides their G. C. M.; and every divisor of their G. C. M. measures A and B.

112. As an example of the process in Art. 110, suppose we have to find the G. C. M. of $x^2 + 5x + 4$ and $x^3 + 4x^2 + 5x + 2$.

$$
\begin{array}{r|l}
x^2 + 5x + 4 \,) & x^3 + 4x^2 + 5x + 2 \,(x - 1 \\
& x^3 + 5x^2 + 4x \\ \hline
& -x^2 + x + 2 \\
& -x^2 - 5x - 4 \\ \hline
& 6x + 6
\end{array}
$$

$$
\begin{array}{r|l}
6x + 6 \,) & x^2 + 5x + 4 \,(\frac{x}{6} + \frac{4}{6} \\
& x^2 + x \\ \hline
& 4x + 4 \\
& 4x + 4 \\ \hline
\end{array}
$$

This example introduces a new point for consideration. The last divisor here is $6x + 6$; this, according to the rule, must be the G. C. M. required. We see from the above process that when $x^2 + 5x + 4$ is divided by $6x + 6$ the quotient is $\frac{x}{6} + \frac{4}{6}$. If the other given expression, namely $x^3 + 4x^2 + 5x + 2$, be divided by $6x + 6$, it will be found that the quotient is $\frac{x^2}{6} + \frac{x}{2} + \frac{1}{3}$. It may at first appear to the student that $6x + 6$ cannot be a measure of the two given expressions, since the so-called quotients really contain fractions. But we see that in these quotients the letter of reference x does not appear in the denominator of any fraction although the coefficients of the powers of x are fractions. Such expressions as $\frac{x}{6} + \frac{2}{3}$ and $\frac{x^2}{6} + \frac{x}{2} + \frac{1}{3}$, therefore, may be said to be *integral expressions so far as relates to* x.

Thus, in the example, when we say that $6x + 6$ is the G. C. M. of the two given expressions, we merely mean that no measure can be found which contains *higher powers* of x than $6x + 6$, Other measures may be found which differ from this so far as respects numerical coefficients only. Thus $3x+3$ and $2x+2$ will be found to be measures; these are respectively the *half* and the *third* of $6x+6$, and the corresponding quotients when we divide the given expressions by these measures will be respectively *twice* and *three* times what they were before. Again, $x+1$ is also a measure, and the corresponding quotients are $x+4$ and x^2+3x+2; we may then conveniently take $x+1$ as *the* greatest common measure, since the quotients are free from fractional coefficients.

113. In order to avoid *fractional coefficients* in the quotients it is usual in performing the operations for finding the G. C. M. to *reject* certain factors which do not form part of the G. C. M. required.

Suppose we have to find the G. C. M. of A and B; and at any stage of the process suppose we have the expressions K and R, one of which is to be a dividend and the other a divisor. Let $R=mS$, where m has no factor which K has; then m may be rejected: that is, instead of continuing the process with K and R we may continue it with K and S.

For by what has been already shewn we know that A and B have just the same common measures as K and R have.

Now any common measure of K and S is a common measure of K and R, and is therefore a common measure of A and B.

And any common measure of K and R is a common measure of K and mS. But m has no factor which K has. Therefore any common measure of K and R is a common measure of K and S. Hence any common measure of A and B is a common measure of K and S.

Thus we see that A and B have just the same common measures as K and S have; and this is what we had to shew.

114. A factor of a certain kind may also be *introduced* at any stage of the process.

Suppose we have to find the G. C. M. of A and B; and at any stage of the process suppose we have the expressions K and R, one of which is to be a dividend and the other a divisor. Let $L = nK$, where n has no factor which R has; then n may be introduced: that is, instead of continuing the process with K and R we may continue it with L and R.

For by what has been already shewn we know that A and B have just the same common measures as K and R have.

Now any common measure of K and R is a common measure of L and R; so that any common measure of A and B is a common measure of L and R.

And any common measure of L and R is a common measure of nK and R. But n has no factor that R has. Therefore any common measure of L and R is a common measure of K and R, and is therefore a common measure of A and B.

Thus we see that A and B have just the same common measures as L and R have; and this is what we had to shew.

115. We see then that certain factors may be removed from *either* a dividend or a divisor, or introduced into *either*: in practice we usually remove factors from divisors, and introduce factors into dividends; and such factors are generally *numerical* factors. The reasoning of Arts. 113 and 114 shews that these operations may be performed at any stage of the process, for example at the beginning if we please. By means of such modifications of the process for finding the G. C. M., we may avoid the introduction of fractional coefficients. The following example will guide the student. Required the G. C. M. of

$$3x^5 - 10x^3 + 15x + 8 \text{ and } x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6.$$

$$\begin{array}{l|l} x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6 \,) & 3x^5 \qquad\quad - 10x^3 \qquad\quad + 15x + 8 \;(3 \\ & 3x^5 - 6x^4 - 18x^3 + 12x^2 + 39x + 18 \\ \hline & \qquad 6x^4 + \;\; 8x^3 - 12x^2 - 24x - 10 \end{array}$$

Before proceeding to the next division we may strike out the factor 2 from every term of the new divisor, and multiply every term of the new dividend by 3. Then continue the operation thus:

$$3x^4+4x^3-6x^2-12x-5)\ 3x^5-\ 6x^4-18x^3+12x^2+39x+18\ (x$$
$$3x^5+\ 4x^4-\ 6x^3-12x^2-5x$$
$$\overline{\quad -10x^4-12x^3+24x^2+44x+18}$$

Remove the factor 2 from every term of the last expression, and then multiply every term by 3. Thus we have

$$-15x^4-18x^3+36x^2+66x+27.$$

Proceed with the division

$$3x^4+4x^3-6x^2-12x-5)\ -15x^4-18x^3+36x^2+66x+27\ (-5$$
$$-15x^4-20x^3+30x^2+60x+25$$
$$\overline{\quad 2x^3+\ 6x^2+\ 6x+\ 2}$$

Remove the factor 2 and then continue the operation thus:

$$x^3+3x^2+3x+1)\ 3x^4+4x^3-\ 6x^2-12x-5\ (3x-5$$
$$3x^4+9x^3+\ 9x^2+\ 3x$$
$$\overline{\quad -5x^3-15x^2-15x-5}$$
$$-5x^3-15x^2-15x-5$$

Thus x^3+3x^2+3x+1 is the G. C. M. required.

116. Suppose the original expressions A and B to contain a common factor F, which is obvious on inspection; let $A=aF$, and $B=bF$. Then F will be a factor of the G. C. M.; as is shewn in Art. 111. We may then find the G. C. M. of a and b, and multiply it by F, and the product will be the G. C. M. of A and B.

117. Similarly, if at any stage of the operation we perceive that a certain factor is common to the dividend and divisor, we may strike it out, and continue the operation with the remaining factors. The factor omitted must then be multiplied by the last divisor which is obtained by continuing the operation, and the product will be the required G. C. M.

118. Suppose, for example, that we require the G. C. M. of $(x-1)^2(x-2)(x-3)$ and $(x-1)^3(x-4)(x-5)$. Here the factor $(x-1)^2$ is common to both the proposed expressions, and is therefore a factor of the G. C. M. Moreover in this example $(x-1)^2$ forms the entire G. C. M.; for no common measure can be found, except unity, of $(x-2)(x-3)$ and $(x-1)(x-4)(x-5)$ which are the

remaining factors of the proposed expressions. The last statement can be verified by trial, but when the student is acquainted with the subject of the resolution of algebraical expressions into factors it will be obvious on inspection. The resolution of algebraical expressions into factors is discussed in the *Theory of Equations.*

119. Next suppose we require the G. C. M. of *three* algebraical expressions A, B, C. Find the G. C. M. of two of them, say A and B; let D denote this G. C. M.; then the G. C. M. of D and C is the required G. C. M. of A, B and C.

For by Art. 111 every measure of D and C is a measure of A, B and C; and also every measure of A, B and C is a measure of D and C. Thus the G. C. M. of D and C is the G. C. M. of A, B and C.

120. In a similar manner we may find the G. C. M. of *four* algebraical expressions. Or we may find the G. C. M. of two of the given expressions and also the G. C. M. of the other two; then the G. C. M. of the two expressions thus found will be the G. C. M. of the four given expressions.

121. The definition and operations of the preceding Articles of this Chapter relate to *polynomial* expressions. The meaning of the term *greatest common measure* in the case of *simple* expressions will be seen from the following example:

Required the G. C. M. of $432a^4b^2xy$, $270a^2b^3x^2z$ and $90a^3bx^3$.

We find by Arithmetic the G. C. M. of the numerical coefficients 432, 270, and 90; it is 18. After this number we write every letter which is common to the simple expressions, and we give to each letter respectively the *least* exponent which it has in the simple expressions. Thus we obtain $18a^2bx$, which will divide all the given simple expressions, and is called their greatest common measure.

EXAMPLES OF THE GREATEST COMMON MEASURE.

Find the G. C. M. in the following examples:

1. $x^2 - 3x + 2$ and $x^2 - x - 2$.
2. $x^3 + 3x^2 + 4x + 12$ and $x^3 + 4x^2 + 4x + 3$.
3. $x^3 + x^2 + x - 3$ and $x^3 + 3x^2 + 5x + 3$.
4. $x^3 + 1$ and $x^3 + mx^2 + mx + 1$.
5. $6x^3 - 7ax^2 - 20a^2x$ and $3x^2 + ax - 4a^2$.
6. $x^5 - y^5$ and $x^2 - y^2$.
7. $3x^3 - 13x^2 + 23x - 21$ and $6x^3 + x^2 - 44x + 21$.
8. $x^4 - 3x^3 + 2x^2 + x - 1$ and $x^3 - x^2 - 2x + 2$.
9. $x^4 - 7x^3 + 8x^2 + 28x - 48$ and $x^3 - 8x^2 + 19x - 14$.
10. $x^4 - x^3 + 2x^2 + x + 3$ and $x^4 + 2x^3 - x - 2$.
11. $4x^4 + 9x^3 + 2x^2 - 2x - 4$ and $3x^3 + 5x^2 - x + 2$.
12. $2x^4 - 12x^3 + 19x^2 - 6x + 9$ and $4x^3 - 18x^2 + 19x - 3$.
13. $6x^4 + x^3 - x$ and $4x^3 - 6x - 4x + 3$.
14. $12x^2 - 15yx + 3y^2$ and $6x^3 - 6yx^2 + 2y^2x - 2y^3$.
15. $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$.
16. $2a^4 + 3a^3x - 9a^2x^2$ and $6a^4x - 17a^3x^2 + 14a^2x^3 - 3ax^4$.
17. $2x^3 + (2a - 9)x^2 - (9a + 6)x + 27$ and $2x^2 - 13x + 18$.
18. $a^3x^3 - a^2bx^2y + ab^2xy^2 - b^3y^3$ and $2a^2bx^2y - ab^2xy^2 - b^3y^3$.
19. $x^3 + ax^2 - axy - y^3$ and $x^4 + 2x^3y - a^2x^2 + x^2y^2 - 2axy^2 - y^4$.
20. $x^5 + 3x^4 - 8x^2 - 9x - 3$ and $x^5 - 2x^4 - 6x^3 + 4x^2 + 13x + 6$.
21. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1$ and $4x^4 + 2x^3 - 18x^2 + 3x - 5$.
22. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ and $3x^3 - 7ax^2 + 3a^2x - 2a^3$.
23. $x^3 - 9x^2 + 26x - 24$, $x^3 - 10x^2 + 31x - 30$ and
    $x^3 - 11x^2 + 38x - 40$.
24. $x^4 - 10x^2 + 9$, $x^4 + 10x^3 + 20x^2 - 10x - 21$ and
    $x^4 + 4x^3 - 22x^2 - 4x + 21$.

VII. LEAST COMMON MULTIPLE.

122. In Arithmetic the *least common multiple* of two or more whole numbers is the least number which contains each of them exactly. The term is also used in Algebra, and its meaning in this subject will be understood from the following definition of the *least common multiple* of two or more algebraical expressions: Let two or more algebraical expressions be arranged according to descending powers of some common letter; then the expression of lowest dimensions in that letter which is divisible by each of these expressions is their least common multiple.

123. The letters L. C. M. will often be used for shortness instead of the term *least common multiple;* the term itself is not very appropriate for the reason already given in Art. 106.

Any expression which is divisible by another may be said to be a multiple of it.

124. We shall now shew how to find the L. C. M. of two algebraical expressions. Let A and B denote the two expressions, and D their greatest common measure. Suppose $A = aD$ and $B = bD$. Then from the nature of the greatest common measure, a and b have no common factor, and therefore their least common multiple is ab. Hence the expression of lowest dimensions which is divisible by aD and bD is abD.

And $abD = Ab = Ba = \frac{AB}{D}$.

Hence we have the following rule for finding the L. C. M. of two algebraical expressions: find their G. C. M.; divide either expression by this G. C. M., and multiply the quotient by the other expression. Or thus: divide the product of the expressions by their G. C. M.

125. If M be the least common multiple of A and B, it is obvious that every multiple of M is a common multiple of A and B.

126. ***Every common multiple of two algebraical expressions is a multiple of their least common multiple.***

Let A and B denote the two expressions, M their L. C. M.; and let N denote any other common multiple. Suppose, if possible, that when N is divided by M there is a remainder R; let q denote the quotient. Thus $R = N - qM$. Now A and B measure M and N, and therefore (Art. 109) they measure R. But R is of *lower* dimensions than M; thus there is a common multiple of A and B of lower dimensions than their L. C. M. This is absurd; hence there can be no remainder R; that is, N is a multiple of M.

127. Next suppose we require the L. C. M. of *three* algebraical expressions A, B, C. Find the L. C. M. of two of them, say A and B; let M denote this L. C. M.; then the L. C. M. of M and C is the required L. C. M. of A, B and C.

For every common multiple of M and C is a common multiple of A, B and C (Art. 125). And every common multiple of A and B is a multiple of M (Art. 126); thus every common multiple of A, B and C is a common multiple of M and C. Therefore the L. C. M. of M and C is the L. C. M. of A, B and C.

128. By resolving algebraical expressions into their component factors, we may sometimes facilitate the process of determining their G. C. M. or L. C. M. For example, required the L. C. M. of $x^2 - a^2$ and $x^3 - a^3$. Since

$$x^2 - a^2 = (x - a)(x + a) \text{ and } x^3 - a^3 = (x - a)(x^2 + ax + a^2),$$

we infer that $x - a$ is the G. C. M. of the two expressions; consequently their L. C. M. is $(x + a)(x^3 - a^3)$, that is,

$$x^4 + ax^3 - a^3x - a^4.$$

129. The preceding articles of this Chapter relate to *polynomial* expressions. The meaning of the term *least common multiple* in the case of simple expressions will be seen from the following example:

Required the L. C. M. of $432a^4b^2xy$, $270a^2b^3x^2z$ and $90a^3bx^3$.

We find by Arithmetic the L. C. M. of the numerical coefficients 432, 270 and 90; it is 2160. After this number we write every letter which occurs in the simple expressions, and we give to each letter respectively the *greatest* exponent which it has in the simple expressions. Thus we obtain $2160a^4b^3x^3yz$, which is divisible by all the given simple expressions, and is called their least common multiple.

130. The theories of the greatest common measure and of the least common multiple are not necessary for the subsequent Chapters of the present work, and any difficulties which the student may find in them may be postponed until he has read the Theory of Equations. The examples however attached to the preceding Chapter and to the present Chapter should be carefully worked, on account of the exercise which they afford in all the fundamental processes of Algebra.

EXAMPLES OF THE LEAST COMMON MULTIPLE.

Find the L. C. M. in the following examples:

1. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.

2. $x^3 - 1$ and $x^2 + x - 2$.

3. $x^3 - 9x^2 + 23x - 15$ and $x^2 - 8x + 7$.

4. $3x^2 - 5x + 2$ and $4x^3 - 4x^2 - x + 1$.

5. $(x + 1)(x^2 - 1)$ and $x^3 - 1$.

6. $x^3 + 2x^2y - xy^2 - 2y^3$ and $x^3 - 2x^2y - xy^2 + 2y^3$.

7. $2x - 1$, $4x^2 - 1$ and $4x^2 + 1$.

8. $x^3 - x$, $x^3 - 1$ and $x^3 + 1$.

9. $x^2 - 4a^2$, $(x + 2a)^2$ and $(x - 2a)^2$.

10. $x^3-6x^2+11x-6$, $x^3-9x^2+26x-24$ and $x^3-8x^2+19x-12$.

11. $x^3-9x^2+26x-24$, $x^3-10x^2+31x-30$ and $x^3-11x^2+38x-40$.

12. x^4-10x^2+9, $x^4+10x^3+20x^2-10x-21$ and $x^4+4x^3-22x^2-4x+21$.

13. x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$ and $x^3-2ax^2+4a^2x-8a^3$.

14. $x^2-(a+b)x+ab$, $x^2-(b+c)x+bc$ and $x^2-(c+a)x+ca$.

15. $2x^3+(2a-3b)x^2-(2b^2+3ab)x+3b^3$ and $2x^2-(3b-2c)x-3bc$.

16. $6(a^3-b^3)(a-b)^3$, $9(a^4-b^4)(a-b)^2$ and $12(a^2-b^2)^3$.

VIII. FRACTIONS.

131. We propose to recall to the student's attention some propositions respecting fractions which he has already found in Arithmetic, and then to shew that these propositions hold universally in Algebra. In the following Articles the letters represent *whole* numbers, unless it is stated otherwise.

132. By the expression $\frac{a}{b}$ we indicate that a unit has been divided into b equal parts, and that a of such parts are taken. Here $\frac{a}{b}$ is called a *fraction*; a is the *numerator* and b the *denominator*, so that the denominator indicates into how many equal parts the unit is to be divided, and the numerator indicates how many of those parts are to be taken.

Every integer may be considered as a fraction with unity for its denominator; that is, $p=\frac{p}{1}$.

133. Rule for multiplying a fraction by an integer. *Either multiply the numerator by that integer, or divide the denominator by that integer.*

Let $\frac{a}{b}$ denote any fraction, and c any integer; then will $\frac{a}{b} \times c = \frac{ac}{b}$. For in each of the fractions $\frac{a}{b}$ and $\frac{ac}{b}$ the unit is divided into b equal parts, and c times as many parts are taken in $\frac{ac}{b}$ as in $\frac{a}{b}$; hence $\frac{ac}{b}$ is c times $\frac{a}{b}$.

This demonstrates the first form of the Rule.

Again; let $\frac{a}{bc}$ denote any fraction, and c any integer; then will $\frac{a}{bc} \times c = \frac{a}{b}$. For in each of the fractions $\frac{a}{bc}$ and $\frac{a}{b}$ the same number of parts is taken, but each part in $\frac{a}{b}$ is c times as large as each part in $\frac{a}{bc}$, because in $\frac{a}{bc}$ the unit is divided into c times as many parts as in $\frac{a}{b}$; hence $\frac{a}{b}$ is c times $\frac{a}{bc}$.

This demonstrates the second form of the Rule.

134. Rule for dividing a fraction by an integer. *Either multiply the denominator by that integer, or divide the numerator by that integer.*

Let $\frac{a}{b}$ denote any fraction, and c any integer; then will $\frac{a}{b} \div c = \frac{a}{bc}$. For $\frac{a}{b}$ is c times $\frac{a}{bc}$, by Art. 133; and therefore $\frac{a}{bc}$ is $\frac{1}{c}$ th of $\frac{a}{b}$.

This demonstrates the first form of the Rule.

Again; let $\frac{ac}{b}$ denote any fraction, and c any integer; then will $\frac{ac}{b} \div c = \frac{a}{b}$. For $\frac{ac}{b}$ is c times $\frac{a}{b}$, by Art. 133; and therefore $\frac{a}{b}$ is $\frac{1}{c}$ th of $\frac{ac}{b}$.

This demonstrates the second form of the Rule.

135. If any quantity be both multiplied and divided by the same number its value is not altered. Hence if the numerator and denominator of a fraction be *multiplied* by the same number the value of the fraction is not altered. For the fraction is multiplied by any number by multiplying its numerator by that number, and is divided by the same number by multiplying its denominator by that number. (Arts. 133 and 134.) Thus $\frac{a}{b} = \frac{ac}{bc}$. And so also if the numerator and denominator of a fraction be *divided* by the same number the value of the fraction is not altered.

136. Hence, an algebraical fraction may be reduced to another of equal value by dividing both numerator and denominator by any common measure; when both numerator and denominator are divided by their G.C.M. the fraction is said to be reduced to *its lowest terms*. For example, consider the fraction $\frac{6x^2 - 7x - 20}{4x^3 - 27x + 5}$. Here the G.C.M. of the numerator and denominator will be found to be $2x - 5$; hence, dividing both numerator and denominator by this we obtain

$$\frac{6x^2 - 7x - 20}{4x^3 - 27x + 5} = \frac{3x + 4}{2x^2 + 5x - 1}.$$

137. Since $\frac{a}{b} = \frac{-a}{-b}$ (Art. 94) it is obvious that we may change the signs of the numerator and denominator of a fraction without altering the value of the fraction.

138. To reduce fractions to a common denominator: *multiply the numerator of each fraction by all the denominators except its own for the numerator corresponding to that fraction, and multiply all the denominators together for the common denominator.*

Thus, suppose $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ to be the proposed fractions; then, by Art. 135, $\frac{a}{b} = \frac{adf}{bdf}$, $\frac{c}{d} = \frac{cbf}{bdf}$, and $\frac{e}{f} = \frac{ebd}{bdf}$; thus $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, and

$\frac{ebd}{bdf}$ are fractions of the same value respectively as the proposed fractions, and having the common denominator bdf.

139. If the denominators have any factors in common, we may proceed thus: *find the* L.C.M. *of the denominators and use this as the common denominator; then for the new numerator corresponding to each of the proposed fractions, multiply the numerator of that fraction by the quotient which is obtained by dividing the* L.C.M. *by the denominator of that fraction.*

Thus suppose, for example, that the proposed fractions are $\frac{a}{mx}$, $\frac{b}{my}$, and $\frac{c}{mz}$. Here the L.C.M. of the denominators is $mxyz$; and $\frac{a}{mx}=\frac{ayz}{mxyz}$, $\frac{b}{my}=\frac{bxz}{mxyz}$, and $\frac{c}{mz}=\frac{cxy}{mxyz}$.

140. To add or subtract fractions, *reduce them to a common denominator, then add or subtract the numerators and retain the common denominator.*

For example, $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$; this follows immediately from the meaning of a fraction.

$$\text{So } \frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{cb}{bd}=\frac{ad+cb}{bd};$$

$$\frac{1}{a+b}+\frac{1}{a-b}=\frac{a-b}{a^2-b^2}+\frac{a+b}{a^2-b^2}=\frac{2a}{a^2-b^2};$$

$$a+\frac{b}{c}=\frac{a}{1}+\frac{b}{c}=\frac{ac}{c}+\frac{b}{c}=\frac{ac+b}{c};$$

$$2+\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{2(a^2-b^2)}{a^2-b^2}+\frac{(a+b)^2}{a^2-b^2}+\frac{(a-b)^2}{a^2-b^2}$$

$$=\frac{2a^2-2b^2+a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}=\frac{4a^2}{a^2-b^2};$$

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b};$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd};$$

$$\frac{a}{b}-\frac{c+d}{c-d}=\frac{a(c-d)}{b(c-d)}-\frac{b(c+d)}{b(c-d)}=\frac{ac-ad-(bc+bd)}{b(c-d)}$$

$$=\frac{ac-ad-bc-bd}{b(c-d)};$$

$$\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{(a+b)^2}{a^2-b^2}-\frac{(a-b)^2}{a^2-b^2}=\frac{(a+b)^2-(a-b)^2}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-(a^2-2ab+b^2)}{a^2-b^2}$$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}.$$

141. The rule for the multiplication of two fractions is, *multiply the numerators for a new numerator, and the denominators for a new denominator.*

The following is usually given for a proof. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two fractions which are to be multiplied together; put $\frac{a}{b}=x$, and $\frac{c}{d}=y$; therefore

$$a=bx, \text{ and } c=dy,$$

$$\text{therefore } ac=bdxy;$$

$$\text{divide by } bd; \text{ thus } \frac{ac}{bd}=xy.$$

This process is satisfactory when x and y are *really integers*, though under a fractional form, because then the word *multiplication* has its common meaning. It is also satisfactory when *one* of the two, x and y, is an integer, because we can speak of multiplying a fraction by an integer, as in Art. 133. But when both x and y are fractions we cannot speak of multiplying them together without defining what we mean by the term *multiplication*, for, according to the ordinary meaning of this term, the *multiplier* must be a whole number.

In fact the so-called *rule* for the multiplication of fractions is really a *definition* of what we find it convenient to understand by the multiplication of fractions. And this definition is so chosen that when one of the fractions we wish to multiply together is an integer in a fractional form, or when both are such, the result of the definition coincides with the consequences drawn from the ordinary use of the word *multiplication*.

142. The following verbal definitions may shew more clearly the connection between the meaning of the word multiplication when applied to integers, and its meaning when applied to fractions. When we multiply one integer a by another b, we may describe the operation thus: ***what we did with unity to obtain*** b ***we must now do with*** a ***to obtain*** b ***times*** a. To obtain b from unity *the unit* is repeated b times; therefore to obtain b times a *the number* a is repeated b times. Now let it be required to multiply the fraction $\frac{a}{b}$ by $\frac{c}{d}$; adopting the same definition as above, we may say that, *what we did with unity to obtain* $\frac{c}{d}$ *we must now do with* $\frac{a}{b}$ *to obtain* $\frac{c}{d}$ *times* $\frac{a}{b}$. To obtain $\frac{c}{d}$ from unity the unit is divided into d equal parts, and c of such parts are taken; therefore, to obtain $\frac{c}{d}$ times $\frac{a}{b}$, the fraction $\frac{a}{b}$ is divided into d equal parts, and c such parts are taken. Now, by Art. 134, if $\frac{a}{b}$ be divided into d equal parts, each of them is $\frac{a}{bd}$, and if c such parts be taken the result is $\frac{ac}{bd}$.

The definition then of multiplication may be given thus: to obtain the product of the multiplier and multiplicand we treat the multiplicand in the same way as unity was treated to obtain the multiplier.

143. To multiply three or more fractions together, *multiply all the numerators for the new numerator, and all the denominators for the new denominator.*

144. Suppose we have to *divide* $\frac{a}{b}$ by $\frac{c}{d}$. Here, by the nature of division, we have to find a quantity such that if it be multiplied by $\frac{c}{d}$ the product shall be $\frac{a}{b}$. This is the *meaning* of division applied to integers, and we shall give the same meaning to division applied to fractions, an operation which hitherto has not been defined.

Let $\frac{a}{b} \div \frac{c}{d} = x$; then $\frac{a}{b} = x \times \frac{c}{d} = \frac{xc}{d}$; therefore $\frac{ad}{b} = xc$, and $\frac{ad}{bc} = x$. Thus we obtain the rule for dividing one fraction by another; *invert the divisor, and proceed as in multiplication.*

145. Hitherto we have supposed, in the present Chapter, that the letters represented *whole numbers;* and have thus only recalled rules and proofs which are familiar to the student in Arithmetic. But in virtue of our extended definitions it may be proved that all the rules and formulæ given are true when the letters denote any numbers *whole or fractional.* Take, for example, the formula $\frac{a}{b} = \frac{ac}{bc}$, and suppose we wish to shew that this is true when

$$a = \frac{m}{n}, \; b = \frac{p}{q}, \text{ and } c = \frac{r}{s}.$$

$$\text{Here } \frac{a}{b} = \frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} = \frac{mq}{np};$$

$$\text{also } ac = \frac{mr}{ns}, \text{ and } bc = \frac{pr}{qs};$$

$$\text{thus } \frac{ac}{bc} = \frac{mr}{ns} \div \frac{pr}{qs} = \frac{mr}{ns} \times \frac{qs}{pr} = \frac{mrqs}{nspr} = \frac{mq}{np}.$$

Thus the formula is shewn to be true.

Moreover these formulæ and rules hold when the letters denote *negative quantities* by virtue of the remarks already made in Chapter v.

146. By means of the foregoing rules and formulæ we can simplify algebraical fractions, in which the numerator and denominator are themselves fractional expressions. For example,

$$\frac{\frac{a}{b}+\frac{b}{a+b}}{\frac{a}{a-b}-\frac{b}{a}}=\frac{\frac{a(a+b)+b^2}{b(a+b)}}{\frac{a^2-b(a-b)}{a(a-b)}}=\frac{a^2+ab+b^2}{b(a+b)}\times\frac{a(a-b)}{a^2-ab+b^2}=\frac{a(a^3-b^3)}{b(a^3+b^3)}.$$

147. The beginner requires to be warned that in reducing fractional expressions he should keep the simplest forms which are admissible, in order to avoid unnecessary labour. For example, suppose we have to reduce the following expression to a single fraction,

$$\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}.$$

We might take the product of all the denominators for a common denominator and transform the three fractions accordingly; but a little consideration will shew that there is a much simpler common denominator which we may put in the following symmetrical form,

$$(a-b)(b-c)(c-a)(x-a)(x-b)(x-c).$$

We may write the proposed expression thus,

$$-\frac{a}{(a-b)(c-a)(x-a)}-\frac{b}{(a-b)(b-c)(x-b)}-\frac{c}{(c-a)(b-c)(x-c)};$$

then by reducing to the common denominator we find

$$-\frac{a(b-c)(x-b)(x-c)+b(c-a)(x-a)(x-c)+c(a-b)(x-a)(x-b)}{(a-b)(b-c)(c-a)(x-a)(x-b)(x-c)}.$$

On working out the numerator we find that it reduces to

$$x\{a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)\},$$

and we shall also find that

$$-\{a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)\}=(a-b)(b-c)(c-a).$$

Thus the proposed expression becomes

$$\frac{x}{(x-a)(x-b)(x-c)}.$$

As another example it may be shewn that

$$\frac{a^2}{(a-b)(a-c)(x-a)}+\frac{b^2}{(b-a)(b-c)(x-b)}+\frac{c^2}{(c-a)(c-b)(x-c)}$$
$$=\frac{x^2}{(x-a)(x-b)(x-c)}.$$

EXAMPLES OF FRACTIONS.

Simplify the following fractions:

1. $\dfrac{x^2+2x-3}{x^2+6x-7}$.

2. $\dfrac{x^2-3x-4}{x^2-4x-5}$.

3. $\dfrac{x^3-6x^2+11x-6}{x^2-3x+2}$.

4. $\dfrac{a^3+3a^2b+3ab^2+b^3}{a^2+2ab+b^2}$.

5. $\dfrac{x^4+10x^3+35x^2+50x+24}{x^3+9x^2+26x+24}$.

6. $\dfrac{3x^3-16x^2+23x-6}{2x^3-11x^2+17x-6}$.

7. $\dfrac{6x^3-5x^2+4}{2x^4-x^3-x+2}$.

8. $\dfrac{2x^3+9x^2+7x-3}{3x^3+5x^2-15x+4}$.

9. $\dfrac{3x^3+12x+9}{x^5+5x^3+6}$.

10. $\dfrac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$.

11. $\dfrac{x^4+2x^2+9}{x^4-4x^3+4x^2-9}$.

12. $\dfrac{x^3+2x^2+2x}{x^5+4x}$.

13. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

14. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

15. $\dfrac{bx+2}{2b+(b^2-4)x-2bx^2}$.

16. $\dfrac{(x+y)^7-x^7-y^7}{(x+y)^5-x^5-y^5}$.

Perform the additions and subtractions indicated in the following examples from 17 to 37:

17. $\frac{a}{a+b}+\frac{b}{a-b}$.

18. $\frac{a}{2a-2b}+\frac{b}{2b-2a}$.

19. $\frac{2}{x}-\frac{3}{2x-1}-\frac{2x-3}{4x^2-1}$.

20. $\left(\frac{1}{m}+\frac{1}{n}\right)(a+b)-\left(\frac{a+b}{m}-\frac{a-b}{n}\right)$.

21. $\frac{1}{x-1}-\frac{1}{x+2}-\frac{3}{(x+2)^2}$.

22. $\frac{5}{2(x+1)}-\frac{1}{10(x-1)}-\frac{24}{5(2x+3)}$.

23. $\frac{b-a}{x-b}-\frac{a-2b}{x+b}+\frac{3x(a-b)}{x^2-b^2}$.

24. $\frac{3+2x}{2-x}-\frac{2-3x}{2+x}+\frac{16x-x^2}{x^2-4}$.

25. $\frac{3}{1-2x}-\frac{7}{1+2x}-\frac{4-20x}{4x^2-1}$.

26. $\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}$.

27. $\frac{1}{x^2-y^2}+\frac{1}{(x+y)^2}-\frac{1}{(x-y)^2}$.

28. $\frac{(a^2+b^2)^2}{ab(a-b)^2}-\frac{a}{b}-\frac{b}{a}-2$.

29. $\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}$.

30. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a-4b}{21}$.

31. $\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}$.

32. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}$.

33. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ca}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}$.

34. $\dfrac{bc}{(c-a)(a-b)}+\dfrac{ca}{(a-b)(b-c)}+\dfrac{ab}{(b-c)(c-a)}$.

35. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

36. $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

37. $\dfrac{2}{a-b}+\dfrac{2}{b-c}+\dfrac{2}{c-a}+\dfrac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}$.

38. Multiply $\dfrac{(a-b)^2}{b+a}$ by $\dfrac{b}{x(a-b)}$.

39. Multiply $\dfrac{x^2+xy}{x^2+y^2}$ by $\dfrac{x^3-y^3}{xy(x+y)}$.

40. Multiply together $\dfrac{3ax}{4by}$, $\dfrac{a^2-x^2}{c^2-x^2}$, $\dfrac{bc+bx}{a^2+ax}$ and $\dfrac{c-x}{a-x}$.

41. Prove that

$$\left(\frac{b}{c}+\frac{c}{b}\right)^2+\left(\frac{c}{a}+\frac{a}{c}\right)^2+\left(\frac{a}{b}+\frac{b}{a}\right)^2=4+\left(\frac{b}{c}+\frac{c}{b}\right)\left(\frac{a}{c}+\frac{c}{a}\right)\left(\frac{a}{b}+\frac{b}{a}\right).$$

42. Multiply together $\dfrac{1-x^2}{1+y}$, $\dfrac{1-y^2}{x+x^2}$ and $1+\dfrac{x}{1-x}$.

43. Multiply $\dfrac{x(a-x)}{a^2+2ax+x^2}$ by $\dfrac{a(a+x)}{a^2-2ax+x^2}$.

44. Simplify $\dfrac{a^4-b^4}{a^2-2ab+b^2}\times\dfrac{a-b}{a^2+ab}$.

45. Simplify $\left(\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}\right)\dfrac{x+y}{2y}$.

46. Simplify $\dfrac{a^3-b^3}{a^3+b^3}\cdot\dfrac{a+b}{a-b}\cdot\left(\dfrac{a^2-ab+b^2}{a^2+ab+b^2}\right)^2$.

47. Multiply $\frac{x^2}{a^2} - \frac{x}{a} + 1$ by $\frac{x^2}{a^2} + \frac{x}{a} + 1$.

48. Multiply $x^2 - x + 1$ by $\frac{1}{x^2} + \frac{1}{x} + 1$.

49. Simplify $\frac{x^2 + x(a+b) + ab}{x^2 - x(a+b) + ab} \times \frac{x^2 - a^2}{x^2 - b^2}$.

50. Divide $\frac{ax - x^2}{(a+x)^2}$ by $\frac{x^2}{a^2 - x^2}$.

51. Divide $\frac{4(a^2 - ab)}{b(a+b)^2}$ by $\frac{6ab}{a^2 - b^2}$.

52. Divide $\frac{2y^3}{x^3 + y^3}$ by $\frac{y}{y+x}$.

53. Divide $\frac{2x+y}{x+y} + \frac{2y-x}{x-y} - \frac{x^2}{x^2-y^2}$ by $\frac{x^2+y^2}{x^2-y^2}$.

54. Simplify $\left(\frac{x^2}{y^3} + \frac{1}{x}\right) \div \left(\frac{x}{y^2} - \frac{1}{y} + \frac{1}{x}\right)$.

55. Simplify $\left(\frac{a}{a+b} + \frac{b}{a-b}\right) \div \left(\frac{a}{a-b} - \frac{b}{a+b}\right)$.

56. Simplify $\left(\frac{x+2y}{x+y} + \frac{x}{y}\right) \div \left(\frac{x+2y}{y} - \frac{x}{x+y}\right)$.

57. Divide $x^4 - \frac{1}{x^4}$ by $x + \frac{1}{x}$.

58. Divide $x^2 + \frac{1}{x^2} + 2$ by $x + \frac{1}{x}$.

59. Divide $x^2 + 1 + \frac{1}{x^2}$ by $\frac{1}{x} - 1 + x$.

60. Divide $a^2 - b^2 - c^2 + 2bc$ by $\frac{a+b-c}{a+b+c}$.

61. Divide $\frac{a^3 + 3a^2x + 3ax^2 + x^3}{x^3 - y^3}$ by $\frac{(a+x)^2}{x^2 + xy + y^2}$.

62. Divide $a^2 - b^2 - c^2 - 2bc$ by $\dfrac{a+b+c}{a+b-c}$.

63. Divide $x^2 - 3ax - 2a^2 + \dfrac{12a^3}{x+3a}$ by $3x - 6a - \dfrac{2x^2}{x+3a}$.

64. Divide $\dfrac{x^2}{2a^2} - 4 + \dfrac{6a^2}{x^2}$ by $\dfrac{x}{2a} - \dfrac{3a}{x}$.

65. Simplify $\dfrac{\dfrac{a+b}{c+d} + \dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d} + \dfrac{a-b}{c+d}}$.

66. Simplify $\dfrac{\dfrac{a+x}{a-x} + \dfrac{a-x}{a+x}}{\dfrac{a+x}{a-x} - \dfrac{a-x}{a+x}}$.

67. Simplify $\dfrac{3abc}{bc+ca-ab} - \dfrac{\dfrac{a-1}{a} + \dfrac{b-1}{b} + \dfrac{c-1}{c}}{\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{1}{c}}$.

68. Simplify $\left(\dfrac{a+b}{a-b} + \dfrac{a^2+b^2}{a^2-b^2}\right) \div \left(\dfrac{a-b}{a+b} - \dfrac{a^3-b^3}{a^3+b^3}\right)$.

69. Simplify $\left(\dfrac{c-b}{c+b} - \dfrac{c^3-b^3}{c^3+b^3}\right) \div \left(\dfrac{c+b}{c-b} + \dfrac{c^2+b^2}{c^2-b^2}\right)$.

70. Simplify $\left(\dfrac{x^2+y^2}{x^2-y^2} - \dfrac{x^2-y^2}{x^2+y^2}\right) \div \left(\dfrac{x+y}{x-y} - \dfrac{x-y}{x+y}\right)$.

71. Simplify $\left(\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b}\right) \div \left(\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a^2-b^2}{a^2+b^2}\right)$.

72. Simplify $\dfrac{\dfrac{m^2+n^2}{n} - m}{\dfrac{1}{n} - \dfrac{1}{m}} \times \dfrac{m^2-n^2}{m^3+n^3}$.

73. Simplify $\dfrac{x}{x-a} - \dfrac{x}{x+a} - \dfrac{\dfrac{x+a}{x-a} - \dfrac{x-a}{x+a}}{\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a}}$.

74. Simplify $\dfrac{\dfrac{1}{a} + \dfrac{1}{b+c}}{\dfrac{1}{a} - \dfrac{1}{b+c}} \left\{1 + \dfrac{b^2+c^2-a^2}{2bc}\right\}$.

75. Simplify $\dfrac{1}{x + \dfrac{1}{1 + \dfrac{x+1}{3-x}}}$.

76. Simplify $\dfrac{a}{b + \dfrac{c}{d + \dfrac{e}{f}}}$.

IX. EQUATIONS OF THE FIRST DEGREE.

148. Any collection of algebraical symbols is called an *expression*. When two expressions are connected by the sign of equality the whole is called an *equation*. The expressions thus connected are called *sides* of the equation, or *members* of the equation. The expression to the left of the sign of equality is called the *first* side, and the expression to the right the *second* side.

149. An *identical equation* is one in which the two sides are equal whatever numbers the letters stand for; for example,

$$(x+b)(x-b) = x^2 - b^2$$

is an identical equation. An identical equation is called briefly an *identity*.

Up to the present point the student has been almost entirely occupied with identities. Thus the results given in Arts. 55 and 68 are identically true; and so also are those which will be obtained by solving the examples to Chapters III and IV.

150. An *equation of condition* is one which is not true for every value of the letters, but only for a certain number of values; for example,

$$x + 1 = 7$$

cannot be true unless $x = 6$. An equation of condition is called briefly an *equation.*

151. A letter to which a particular value or values must be given in order that the statement contained in an equation may be true is called an *unknown quantity.* Such particular value of the unknown quantity is said to *satisfy the equation,* and is called a *root of the equation.* To *solve* an equation is to find the particular value or values.

152. An equation involving one unknown quantity is said to be of as many dimensions as is denoted by the index of the highest power of the unknown quantity. Thus, if x denote the unknown quantity, the equation is said to be of *one* dimension when x occurs only in the *first* power; such an equation is also called a *simple equation,* or an equation of the *first degree.* If x^2 occurs, and no power of x higher than x^2 occurs, the equation is said to be of *two* dimensions; such an equation is also called a *quadratic equation,* or an equation of the *second degree.* If x^3 occurs, and no power of x higher than x^3 occurs, the equation is said to be of *three* dimensions; such an equation is also called a *cubic equation,* or an equation of the *third degree.* And so on.

It must be observed that these definitions suppose both members of the equation to be *integral expressions so far as relates to x,* and not to contain x under the *radical sign.*

153. We shall now indicate some operations which may be performed on an equation without destroying the equality which it expresses. It will be seen afterwards that these operations are useful when we have to solve equations.

154. *If every term on each side of an equation be multiplied or divided by the same quantity the results are equal.* This follows from Arts. 100, 101.

155. The principal use of the preceding Article is to *clear an equation of fractions;* this is effected by multiplying every term by the product of all the denominators of the fractions, or, if we please, by the *least common multiple* of those denominators. Suppose, for example,

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13.$$

Multiply every term by $2 \times 3 \times 4$; thus,

$$3 \times 4 \times x + 2 \times 4 \times x + 2 \times 3 \times x = 13 \times 2 \times 3 \times 4;$$

that is, $$12x + 8x + 6x = 312.$$

Divide every term by 2; thus,

$$6x + 4x + 3x = 156.$$

Instead of multiplying every term by $2 \times 3 \times 4$ we may multiply by 12, which is the L. C. M. of 2, 3 and 4. Thus we obtain at once

$$6x + 4x + 3x = 156.$$

156. *Any quantity may be transposed from one side of an equation to the other side by changing its sign.*

Thus suppose $$x - a = b - y.$$

Add a to each side (Art. 98); then

$$x - a + a = b - y + a,$$

that is, $$x = b + a - y.$$

Now subtract b from each side; thus,

$$x - b = b + a - y - b = a - y.$$

Here we see that $-a$ has been removed from one side of the equation, and appears as $+a$ on the other side; and $+b$ has been removed from one side and appears as $-b$ on the other side.

157. *If the sign of every term in an equation be changed the equality still holds.*

This follows from the preceding Article by transposing every term. Thus suppose

$$x-a=b-y.$$

By transposition, $$y-b=a-x,$$

that is, $$a-x=y-b;$$

this result is what we shall obtain if we change the sign of every term in the original equation.

158. We can now give a rule for the solution of any simple equation with one unknown quantity.

Let the equation first be cleared of fractions; then transpose all the terms which involve the unknown quantity to one side of the equation, and the known quantities to the other; divide both sides by the coefficient or the sum of the coefficients of the unknown quantity, and the value required is obtained.

The truth of the rule will be obvious from the principles of the preceding Articles, and we shall now apply it to some examples; in these examples the *unknown quantity* will be denoted by x, and when other letters occur, they are supposed to represent *known* quantities.

159. Solve $$3x-4=24-x.$$

By transposition, $$3x+x=24+4;$$

thus, $$4x=28;$$

by division, $$x=\frac{28}{4}=7.$$

We may *verify* the result by putting 7 for x in the original equation. The first side becomes $3\times 7-4$, that is, $21-4$, that is, 17; the second side becomes $24-7$, that is, 17.

160. Solve $$\frac{5x}{2} - \frac{4x}{3} - 13 = \frac{5}{8} + \frac{x}{32}.$$

Multiply by 96, which is the L. C. M. of the denominators; thus,
$$5 \times 48 \times x - 4 \times 32 \times x - 13 \times 96 = 5 \times 12 + 3x;$$
that is,
$$240x - 128x - 1248 = 60 + 3x;$$
by transposition,
$$240x - 128x - 3x = 1248 + 60;$$
thus,
$$109x = 1308;$$
by division,
$$x = \frac{1308}{109} = 12.$$

We may verify the result by putting 12 for x in the original equation; it will be found that each side of the equation then becomes 1.

161. Sometimes it is convenient to clear of fractions *partially,* and then to effect some reductions before getting rid of the remaining fractional coefficients. For example, solve
$$\frac{x+7}{11} - \frac{2x-16}{3} + \frac{2x+5}{4} = 5\tfrac{1}{3} + \frac{3x+7}{12}.$$

Here we may conveniently multiply by 12; thus,
$$\frac{12(x+7)}{11} - 4(2x-16) + 3(2x+5) = 16 \times 4 + 3x + 7;$$
that is,
$$\frac{12(x+7)}{11} - 8x + 64 + 6x + 15 = 64 + 3x + 7.$$

By transposition and reduction,
$$\frac{12(x+7)}{11} + 8 = 5x.$$

Multiply by 11; thus,
$$12x + 84 + 88 = 55x;$$
by transposition,
$$172 = 43x;$$
by division,
$$x = \frac{172}{43} = 4.$$

We may verify this result as before.

The student should notice one point in this example very carefully. The fraction $\frac{2x-16}{3}$ is equivalent to $\frac{1}{3}(2x-16)$. This fraction is preceded by the sign $-$; and when we multiply by 12 and remove the brackets we obtain $-8x+64$. Thus when we clear of fractions we must regulate the signs of the terms which stood in any numerator in the same way as if they had been between brackets.

162. Solve $\frac{5}{2x+1}=\frac{2}{5x-8}$.

Multiply by $(2x+1)(5x-8)$; thus,

$$5(5x-8)=2(2x+1);$$

that is, $$25x-40=4x+2;$$

by transposition, $$21x=42;$$

by division, $$x=\frac{42}{21}=2.$$

We may verify this result as before.

163. Solve $\frac{2x-3}{3x-4}=\frac{4x-5}{6x-7}$.

Multiply by $(3x-4)(6x-7)$; thus,

$$(2x-3)(6x-7)=(4x-5)(3x-4);$$

that is, $$12x^2-32x+21=12x^2-31x+20.$$

Take away $12x^2$ from both sides; thus,

$$21-32x=20-31x;$$

by transposition, $$21-20=32x-31x;$$

thus, $$x=1.$$

We may verify this result as before.

164. Solve $\frac{x}{2}-8=\frac{10x}{3}-\frac{7}{3}$.

Multiply by 6; thus,

$$3x-48=20x-14;$$

by transposition, $17x = -34$;

by division, $x = -\frac{34}{17} = -2.$

We may verify this result; each side of the equation will be found to become -9.

165. Solve $ax + b = cx + d$.

By transposition, $ax - cx = d - b$;

that is, $(a - c)\,x = d - b$;

by division, $x = \frac{d - b}{a - c}.$

Verification; put this value for x in the original equation; then the first side becomes $\frac{a(d-b)}{a-c} + b$, that is, $\frac{a(d-b)}{a-c} + \frac{b(a-c)}{a-c}$, that is, $\frac{ad - bc}{a - c}$. And the second side becomes $\frac{c(d-b)}{a-c} + d$, that is, $\frac{c(d-b)}{a-c} + \frac{d(a-c)}{a-c}$, that is, $\frac{da - cb}{a - c}$.

166. *An equation of the first degree cannot have more than one root.*

For any equation of the first degree will take the form $ax = b$ if the unknown quantity is brought to one side of the equation, and the known quantities to the other, and to make this true x must be equal to $\frac{b}{a}$, and to nothing else.

The result is sometimes obtained thus. Suppose, if possible, that this equation has two different roots α and β; then by supposition,

$$a\alpha = b, \qquad a\beta = b;$$

therefore, by subtraction,

$$a(\alpha - \beta) = 0;$$

but this is impossible, since by supposition $\alpha - \beta$ is not zero, and a is not zero. Thus an equation of the first degree cannot have more than one root.

EXAMPLES OF EQUATIONS OF THE FIRST DEGREE.

1. $\frac{2x+1}{2}=\frac{7x+5}{8}$.

2. $\frac{x}{2}-2=\frac{x}{4}+\frac{x}{5}-1$.

3. $\frac{x+1}{2}+\frac{3x-4}{5}+\frac{1}{8}=\frac{6x+7}{8}$.

4. $\frac{5x-11}{4}-\frac{x-1}{10}=\frac{11x-1}{12}$.

5. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{1}{2}$.

6. $\frac{x+1}{2}+\frac{x+2}{3}=16-\frac{x+3}{4}$.

7. $x+\frac{11-x}{3}=\frac{26-x}{2}$.

8. $19x+\frac{1}{2}(7x-2)=4x+\frac{35}{2}$.

9. $\frac{x-3}{4}+\frac{x-4}{3}=\frac{x-5}{2}+\frac{x+1}{8}$.

10. $\frac{5x-7}{2}-\frac{2x+7}{3}=3x-14$.

11. $\frac{x-3}{4}-\frac{2x-5}{6}=\frac{41}{60}+\frac{3x-8}{5}-\frac{5x+6}{15}$.

12. $\frac{5x+3}{3}-\frac{3x-7}{2}=5x-10$.

13. $\frac{1}{6}(8-x)+x-1\frac{2}{3}=\frac{x+6}{2}-\frac{x}{3}$.

14. $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$.

15. $\frac{3x-1}{5}-\frac{13-x}{2}=\frac{7x}{3}-\frac{11(x+3)}{6}$.

16. $\frac{5x-3}{7}-\frac{9-x}{3}=\frac{5x}{2}+\frac{19}{6}(x-4)$.

17. $\frac{5x-1}{7}+\frac{9x-5}{11}=\frac{9x-7}{5}$.

18. $\dfrac{3x+5}{7} - \dfrac{2x+7}{3} + 10 - \dfrac{3x}{5} = 0.$

19. $\dfrac{x}{4} - \dfrac{5x+8}{6} = \dfrac{2x-9}{3}.$

20. $2x - \dfrac{19-2x}{2} = \dfrac{2x-11}{3}.$

21. $\dfrac{7x+9}{4} - \left(x - \dfrac{2x-1}{9}\right) = 7.$

22. $\dfrac{7+9x}{4} - \left(1 - \dfrac{2-x}{9}\right) = 7x.$

23. $\dfrac{x+1}{2} - \dfrac{5-x}{4} = 14 - \dfrac{x+2}{3}.$

24. $\dfrac{7x-8}{11} + \dfrac{15x+8}{13} = 3x - \dfrac{31-x}{2}.$

25. $\dfrac{3x-11}{4} - \dfrac{28-9x}{8} = 4x - 14\tfrac{3}{4}.$

26. $\dfrac{2x-1}{3} - \dfrac{3x-2}{4} = \dfrac{5x-4}{6} - \dfrac{7x+6}{12}.$

27. $\dfrac{2x-9}{27} + \dfrac{x}{18} - \dfrac{x-3}{4} = 8\tfrac{1}{3} - x.$

28. $\dfrac{x-1}{3} + \dfrac{4x-\frac{3}{4}}{5} - \dfrac{7x-6}{8} = 2 + \dfrac{x-2}{2} + \dfrac{3x-9}{10}.$

29. $\dfrac{2x-6}{5} - \dfrac{x-4}{9} - \dfrac{3x}{13} = 0.$

30. $x = 3x - \dfrac{1}{2}(4-x) + \dfrac{1}{3}.$

31. $\dfrac{3x-7}{5} + \dfrac{25-4x}{9} = \dfrac{5x-14}{3}.$

32. $\dfrac{2x+5}{13} + \dfrac{40-x}{8} = \dfrac{10x-427}{19}.$

33. $\dfrac{x}{7} - \dfrac{x-5}{11} + 5 = x - \left(\dfrac{2x}{77} + 1\right).$

34. $\dfrac{x-1}{2} + \dfrac{x-2}{3} = \dfrac{x+3}{4} + \dfrac{x+4}{6} + 1.$

35. $\dfrac{x-1}{x-2} - \dfrac{x-2}{x-3} = \dfrac{x-5}{x-6} - \dfrac{x-6}{x-7}.$

36. $(x-5)(x-2) - (x-5)(2x-5) + (x+7)(x-2) = 0.$

37. $3 - x - 2(x-1)(x+2) = (x-3)(5-2x).$

38. $x-3-(3-x)(x+1)=(x-3)(1+x)+3-x.$

39. $\frac{x+10}{3}-\frac{3}{5}(3x-4)+\frac{(3x-2)(2x-3)}{6}=x^2-\frac{8}{15}.$

40. $\left(x+\frac{5}{2}\right)\left(x-\frac{3}{2}\right)-(x+5)(x-3)+\frac{3}{4}=0.$

41. $\left(x-\frac{5}{2}\right)\left(x+\frac{3}{2}\right)-(x-5)(x+3)-\frac{93}{4}=0.$

42. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{10\frac{1}{4}}{14}.$

43. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}.$ 44. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

45. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}.$

46. $(x+1)^2=\{6-(1-x)\}x-2.$

47. $\frac{1}{x-2}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-8}.$

48. $\frac{2}{2x-5}+\frac{1}{x-3}=\frac{6}{3x-1}.$

49. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=\frac{23}{x+1}+5.$

50. $\frac{1}{2}\left(x-\frac{a}{3}\right)-\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0.$

51. $(a+x)(b+x)=(c+x)(d+x).$

52. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}.$ 53. $ax+b=\frac{x}{a}+\frac{1}{b}.$

54. $\frac{x-a}{b}+\frac{x-b}{c}+\frac{x-c}{a}=\frac{x-(a+b+c)}{abc}.$

55. $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2.$

56. $\frac{a+b}{x-c}=\frac{a}{x-a}+\frac{b}{x-b}.$ 57. $\frac{ax^2+bx+c}{px^2+qx+r}=\frac{ax+b}{px+q}.$

58. $\frac{3abc}{a+b}+\frac{a^2b^2}{(a+b)^3}+\frac{(2a+b)b^2x}{a(a+b)^2}=3cx+\frac{bx}{a}.$

59. $\frac{m(x+a)}{x+b}+\frac{n(x+b)}{x+a}=m+n.$ 60. $\left(\frac{x-a}{x+b}\right)^2=\frac{x-2a-b}{x+a+2b}.$

61. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c).$

62. $\cdot 15x+1\cdot 575-\cdot 875x=\cdot 0625x.$

63. $1\cdot 2x-\frac{\cdot 18x-\cdot 05}{\cdot 5}=\cdot 4x+8\cdot 9.$

64. $4\cdot 8x-\frac{\cdot 72x-\cdot 05}{\cdot 5}=1\cdot 6x+8\cdot 9.$

X. PROBLEMS WHICH LEAD TO SIMPLE EQUATIONS WITH ONE UNKNOWN QUANTITY.

167. We shall now apply the methods already given to the solution of some problems, and thus exhibit to the student specimens of the use of Algebra. In a problem certain quantities are given, and certain others, which have some assigned relations to them, are to be found. The relations are usually expressed in ordinary language in the enunciation of the problem, and the method of solving the problem may be thus described in general terms: *denote the unknown quantities by letters, and express in algebraical language the relations which hold between the unknown quantities and the given quantities; we shall thus obtain equations from which the values of the unknown quantities may be derived.*

We shall now give some examples. In the present Chapter we confine ourselves to problems which may be solved by using only *one* unknown quantity.

168. The sum of two numbers is 89 and their difference is 31: find the numbers.

Let x denote the less number, then the greater number is $31+x$; thus since their sum is 89, we have

$$31+x+x=89,$$

that is,

$$31+2x=89;$$

by transposition, $2x = 89 - 31 = 58$;

by division, $x = \frac{58}{2} = 29.$

Thus the less number is 29, and the greater number is $29 + 31$, that is, 60.

169. A bankrupt owes B twice as much as he owes A, and C as much as he owes A and B together: out of £300 which is to be divided among them, what should each receive?

Let x denote the number of pounds which A should receive; then $2x$ is the number of pounds B should receive; and $x + 2x$, that is $3x$, is the number of pounds C should receive. The whole sum they receive is £300; thus,

$$x + 2x + 3x = 300;$$

that is, $6x = 300$;

and $x = \frac{300}{6} = 50$;

therefore A should receive £50, B £100, and C £150.

170. Divide a line 21 inches long into two parts, such that one may be three-fourths of the other.

Let x denote the number of inches in one part, then $\frac{3x}{4}$ denotes the number of inches in the other part; thus,

$$x + \frac{3x}{4} = 21;$$

clear of fractions; thus,

$$4x + 3x = 84;$$

that is, $7x = 84$;

therefore, $x = \frac{84}{7} = 12.$

Thus one part is 12 inches long and the other part 9 inches.

171. If A can perform a piece of work in 8 days, and B in 10 days, in what time will they perform it together?

Let x denote the number of days required. In one day A can perform $\frac{1}{8}$th of the work, therefore in x days he can perform $\frac{x}{8}$ths of the work. In one day B can perform $\frac{1}{10}$th of the work, therefore in x days he can perform $\frac{x}{10}$ths of the work. Hence since A and B together perform the whole work in x days, we have

$$\frac{x}{8}+\frac{x}{10}=1;$$

clear of fractions by multiplying by 40; thus,

$$5x+4x=40,$$

that is, $$9x=40;$$

therefore, $$x=\frac{40}{9}=4\tfrac{4}{9}.$$

172. A workman was employed for 60 days, on condition that for every day he worked he should receive 15 pence, and for every day he was absent he should forfeit 5 pence; at the end of the time he had 20 shillings to receive: required the number of days he worked.

Let x denote the number of days he worked, then he was absent $60-x$ days; then $15x$ denotes his pay in pence, and $5(60-x)$ denotes the sum he forfeited. Thus,

$$15x-5(60-x)=240;$$

that is, $$15x-300+5x=240;$$

therefore, $$20x=240+300=540;$$

therefore, $$x=\frac{540}{20}=27.$$

Thus he worked 27 days and was absent $60-27$ days, that is, 33 days.

173. How much rye at four shillings and sixpence a bushel must be mixed with fifty bushels of wheat at six shillings a bushel, that the mixture may be worth five shillings a bushel?

Let x denote the number of bushels required; then $9x$ is the value of the rye in sixpences, and 600 is the value of the wheat. The value of the mixture is $10\,(50+x)$. Thus,

$$10\,(50+x) = 9x + 600\,;$$

that is, $$10x + 500 = 9x + 600\,;$$

and $$x = 100.$$

174. A smuggler had a quantity of brandy which he expected would produce £9. 18*s.*; after he had sold 10 gallons a revenue officer seized one-third of the remainder, in consequence of which the smuggler makes only £8. 2*s.*: required the number of gallons he had and the price per gallon.

Let x denote the number of gallons; then $\frac{198}{x}$ is the value of a gallon in shillings. The quantity seized is $\frac{x-10}{3}$ gallons, and the value of this is $\frac{x-10}{3} \times \frac{198}{x}$ shillings; thus,

$$\frac{x-10}{3} \times \frac{198}{x} = 198 - 162 = 36.$$

Multiply by $3x$; thus,

$$198\,(x-10) = 3x \times 36 = 108x\,;$$

therefore, $$198x - 108x = 1980\,;$$

that is, $$90x = 1980,$$

and $$x = \frac{1980}{90} = 22.$$

Thus 22 is the number of gallons, and the price of each gallon is $\frac{198}{22}$ shillings, that is, 9 shillings.

175. The student may now exercise himself in the solution of the following problems. We may remark that in these cases the only difficulty consists in *translating ordinary verbal statements into Algebraical language*, and the student should not be discouraged if at first he is sometimes a little perplexed, since nothing but practice can give him readiness and certainty in this process.

EXAMPLES OF PROBLEMS.

1. The property of two persons amounts to £3870, and one of them is twice as rich as the other; find the property of each.

2. Divide £420 among two persons so that for every shilling one receives the other may receive half-a-crown.

3. How much money is there in a purse when the fourth part and the fifth part together amount to £2. 5*s.*?

4. After paying the seventh part of a bill and the fifth part, £92 is still due; what was the amount of the bill?

5. Divide 46 into two parts, such that if one part be divided by 7 and the other by 3, the sum of the quotients shall be 10.

6. A company of 266 persons consists of men, women and children; there are four times as many men as children, and twice as many women as children. How many of each are there?

7. A person expends one-third of his income in board and lodging, one-eighth in clothing, and one-tenth in charity, and saves £318. What is his income?

8. Three towns, A, B, C, raise a sum of £594; for every pound which B contributes, A contributes twelve shillings, and C seventeen shillings and sixpence. What does each contribute?

9. Divide £1520 among A, B, and C, so that B shall have £100 more than A, and C £270 more than B.

10. A certain sum is to be divided among A, B, and C. A is to have £30 less than the half, B is to have £10 less than the third part, and C is to have £8 more than the fourth part. What does each receive?

11. The sum of two numbers is 5760, and their difference is equal to one-third of the greater: find the numbers.

12. Two casks contain equal quantities of beer; from the first 34 quarts are drawn, and from the second 80; the quantity remaining in one cask is now twice that in the other. How much did each cask originally contain?

13. A person bought a print at a certain price, and paid the same price for a frame; if the frame had cost £1 less and the print 15*s*. more, the price of the frame would have been only half that of the print. Find the cost of the print.

14. Two shepherds owning a flock of sheep agree to divide its value; *A* takes 72 sheep, and *B* takes 92 sheep and pays *A* £35. Required the value of a sheep.

15. A house and garden cost £850, and five times the price of the house was equal to twelve times the price of the garden: find the price of each.

16. One-tenth of a rod is coloured red, one-twentieth orange, one-thirtieth yellow, one-fortieth green, one-fiftieth blue, one-sixtieth indigo, and the remainder, which is 302 inches long, violet. Find the length of the rod.

17. Two-thirds of a certain number of persons received eighteenpence each, and one-third received half-a-crown each. The whole sum spent was £2. 15*s*. How many persons were there?

18. Find that number the third part of which added to its seventh part makes 20.

19. The difference of the squares of two consecutive numbers is 15. Find the numbers.

20. Of a certain dynasty one-third of the kings were of the same name, one-fourth of another, one-eighth of another, one-twelfth of a fourth, and there were five besides. How many kings were there of each name?

21. A crew which can pull at the rate of nine miles an hour, finds that it takes twice as long to come up a river as to go down; at what number of miles an hour does the river flow?

22. *A* and *B* play at a game, agreeing that the loser shall always pay to the winner one shilling more than half the money the loser has; they commence with equal quantities of money, but after *B* has lost the first game and won the second, he has twice as much as *A*: how much had each at the commencement?

23. A person who possesses £12000 employs a portion of the money in building a house. One-third of the money which remains he invests at 4 per cent., and the other two-thirds at 5 per cent., and from these investments he obtains an income of £392. What was the cost of the house?

24. A farmer has oxen worth £12. 10*s.* each, and sheep worth £2. 5*s.* each; the number of oxen and sheep being 35, and their value £191. 10*s.* Find the number he had of each.

25. *A* and *B* find a purse with shillings in it. *A* takes out two shillings and one-sixth of what remains; then *B* takes out three shillings and one-sixth of what remains; and then they find that they have taken out equal shares. How many shillings were in the purse, and how many did each take?

26. A hare is eighty of her own leaps before a greyhound; she takes three leaps for every two that he takes, but he covers as much ground in one leap as she does in two. How many leaps will the hare have taken before she is caught?

27. The length of a field is twice its breadth; another field which is 50 yards longer and 10 yards broader, contains 6800 square yards more than the former; find the size of each.

28. A vessel can be emptied by three taps; by the first alone it could be emptied in 80 minutes, by the second alone in 200 minutes, and by the third alone in 5 hours. In what time will the vessel be emptied if all the taps are opened?

29. If an income tax of 7*d.* in the pound on all incomes below £100 a year, and of 1*s.* in the pound on all incomes above £100 a year realise £18750 on £500000, how much is raised on incomes below £100 a year?

30. A person buys some tea at 3 shillings a pound, and some at 5 shillings a pound; he wishes to mix them so that by selling the mixture at 3*s.* 8*d.* a pound he may gain 10 per cent. on each pound sold: find how many pounds of the inferior tea he must mix with each pound of the superior.

31. A fruiterer sold for 19*s.* 6*d.* a certain number of oranges and apples, of which the latter exceeded the former by 180. He sells the apples at the rate of 5 for 3*d.*, and 15 oranges bring him in $1\frac{1}{2}d.$ more than 35 apples. How many are there of each sort?

32. A cask *A* contains 12 gallons of wine and 18 gallons of water; and another cask *B* contains 9 gallons of wine and 3 gallons of water; how many gallons must be drawn from each cask so as to produce by their mixture 7 gallons of wine and 7 gallons of water?

33. *A* can dig a trench in one-half the time that *B* can; *B* can dig it in two-thirds of the time that *C* can; all together they can dig it in 6 days; find the time it would take each of them alone.

34. A person after paying sevenpence in the pound for Income Tax has £408. 4*s.* $8\frac{1}{2}d.$ left. What had he at first?

35. At what time between one o'clock and two o'clock is the long hand of a clock exactly one minute in advance of the short hand?

36. A person has just a hours at his disposal; how far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

37. A certain article of consumption is subject to a duty of 6 shillings per cwt.; in consequence of a reduction in the duty the consumption increases one-half, but the revenue falls one-third. Find the duty per cwt. after the reduction.

38. A ship sails with a supply of biscuit for 60 days, at a daily allowance of a pound a head; after being at sea 20 days she encounters a storm in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's daily allowance must be reduced to five-sevenths of a pound. Find the original number of the crew.

XI. SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH TWO UNKNOWN QUANTITIES.

176. Suppose we have an equation containing two unknown quantities x and y, for example $5x - 2y = 4$. For every value which we please to ascribe to one of the unknown quantities we can determine the corresponding value of the other, and thus find as many pairs of values as we please which satisfy the given equation. Thus, for example, if $y = 1$ we find $x = \frac{6}{5}$; if $y = 2$ we find $x = \frac{8}{5}$; and so on.

Also, suppose that there is another equation of the same kind, as for example, $4x + 3y = 17$. We can also find as many pairs of values as we please which satisfy this equation.

But suppose we ask for values of x and y which satisfy *both* equations; we shall find then that there is only one value of x and one value of y. For multiply the first equation by 3; thus,

$$15x - 6y = 12;$$

multiply the second equation by 2; thus,

$$8x + 6y = 34.$$

Therefore, by addition,

$$15x - 6y + 8x + 6y = 12 + 34;$$

that is, $$23x = 46,$$

and, $$x = 2.$$

Thus if *both* equations are to be satisfied x *must* equal 2; put this value of x in either of the two given equations; for example, in the second equation; thus we obtain

$$8 + 3y = 17;$$

therefore, $$3y = 17 - 8,$$

and, $$y = 3.$$

177. Two or more equations which are to be satisfied by the *same values* of the unknown quantities are called *simultaneous equations*. We are now about to treat of simultaneous equations involving two unknown quantities where each unknown quantity occurs only in the first degree, and the product of the unknown quantities does not occur.

178. There are three methods which are usually given for solving these equations. The object of all these methods is the same, namely, to obtain from the *two* given equations which contain *two* unknown quantities a single equation containing only *one* of the unknown quantities. By this process we are said to *eliminate* the unknown quantity which does not appear in the single equation.

179. *First method.* The first method is that which we adopted in the example of Art. 176; it may be thus described: *multiply the equations by such numbers as will make the coefficient of one of the unknown quantities the same in the two resulting equations; then by addition or subtraction we can form an equation containing only the other unknown quantity.*

Example. $4x + 3y = 22$; $5x - 7y = 6$.

If we wish to eliminate y we multiply the *first* equation by 7, which is the coefficient of y in the second, and the *second* equation by 3, which is the coefficient of y in the first equation. Thus we obtain

$$28x + 21y = 154; \quad 15x - 21y = 18.$$

Then by *addition*,

$$28x + 15x = 154 + 18;$$

that is, $$43x = 172,$$

and, $$x = \frac{172}{43} = 4.$$

Then put this value of x in either of the given equations, in the first for example; thus,

$$16 + 3y = 22;$$

therefore, $3y = 6,$

and, $y = 2.$

If we wish to solve this example by eliminating x we multiply the first of the given equations by 5, and the second by 4; thus,

$$20x + 15y = 110; \quad 20x - 28y = 24.$$

Then by *subtraction*,

$$20x + 15y - (20x - 28y) = 110 - 24;$$

thus, $43y = 86,$

and, $y = 2.$

180. *Second method. Express one of the unknown quantities in terms of the other from either equation, and substitute this value in the other equation.*

Thus, taking the same example, we have from the first equation

$$4x = 22 - 3y;$$

divide by 4, $x = \dfrac{22 - 3y}{4};$

substitute this value of x in the second equation and we obtain

$$\frac{5(22 - 3y)}{4} - 7y = 6;$$

multiply by 4, $5(22 - 3y) - 28y = 24;$

that is, $110 - 15y - 28y = 24;$

by transposition, $43y = 86,$

and, $y = 2.$

Then substitute this value of y in either of the given equations and we shall obtain $x = 4$.

Or thus; from the first equation we have

$$3y = 22 - 4x;$$

divide by 3, $y = \dfrac{22 - 4x}{3};$

substitute this value of y in the second equation and we obtain

$$5x - \frac{7(22-4x)}{3} = 6;$$

multiply by 3, $\quad 15x - 7(22-4x) = 18;$

that is, $\quad 15x - 154 + 28x = 18;$

that is, $\quad 43x = 172,$

and, $\quad x = 4.$

Then substitute this value of x in either of the given equations and we shall obtain $y = 2$.

181. *Third method. Express the same unknown quantity in terms of the other from each equation and equate the expressions thus obtained.*

Thus, taking the same example, from the first equation $x = \frac{22-3y}{4}$, and from the second equation $x = \frac{6+7y}{5}$;

thus, $\quad \frac{22-3y}{4} = \frac{6+7y}{5};$

clear of fractions, $\quad 5(22-3y) = 4(6+7y);$

that is, $\quad 110 - 15y = 24 + 28y;$

by transposition, $\quad 43y = 86,$

and, $\quad y = 2.$

Hence, as before, we deduce $x = 4$.

Or thus; from the first equation we obtain $y = \frac{22-4x}{3}$,

and from the second equation $y = \frac{5x-6}{7}$; thus,

$$\frac{22-4x}{3} = \frac{5x-6}{7}.$$

Hence as before we shall obtain $x = 4$ and then deduce $y = 2$.

EXAMPLES OF SIMULTANEOUS SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.

1. $x + y = 15, \quad x - y = 7.$
2. $3x - 2y = 1, \quad 3y - 4x = 1.$
3. $3x - 5y = 13, \quad 2x + 7y = 81.$
4. $2x + 3y = 43, \quad 10x - y = 7.$
5. $5x - 7y = 33, \quad 11x + 12y = 100.$
6. $3y - 7x = 4, \quad 2y + 5x = 22.$
7. $21y + 20x = 165, \quad 77y - 30x = 295.$
8. $5x + 7y = 43, \quad 11x + 9y = 69.$
9. $8x - 21y = 33, \quad 6x + 35y = 177.$
10. $11x - 10y = 14, \quad 5x + 7y = 41.$
11. $16x + 17y = 500, \quad 17x - 3y = 110.$
12. $\frac{x}{5} + \frac{y}{6} = 18, \quad \frac{x}{2} - \frac{y}{4} = 21.$
13. $\frac{x}{3} + \frac{y}{4} = 9, \quad \frac{x}{4} + \frac{y}{5} = 7.$
14. $\frac{x}{2} + \frac{y}{3} = 1, \quad \frac{x}{3} + \frac{y}{4} = 1.$
15. $\frac{x+y}{2} - \frac{x-y}{3} = 8, \quad \frac{x+y}{3} + \frac{x-y}{4} = 11.$
16. $\frac{11x - 5y}{11} = \frac{3x + y}{16}, \quad 8x - 5y = 1.$
17. $\frac{2x}{3} - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12}, \quad \frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 6.$
18. $4x + 8y = 2{\cdot}4, \quad 10{\cdot}2x - 6y = 3{\cdot}48.$
19. $x = 4y, \quad \frac{1}{5}(2x + 7y) - 1 = \frac{2}{3}(2x - 6y + 1).$
20. $x + \frac{1}{2}(3x - y - 1) = \frac{1}{4} + \frac{3}{4}(y - 1), \quad \frac{1}{5}(4x + 3y) = \frac{7y}{10} + 2.$

21. $\frac{3x-5y}{2}+3=\frac{2x+y}{5}$, $\qquad 8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3}$.

22. $\frac{3x}{10}-\frac{y}{15}-\frac{4}{9}=\frac{x}{12}-\frac{y}{18}$, $\qquad 2x-\frac{8}{3}=\frac{x}{12}-\frac{y}{15}+\frac{11}{10}$.

23. $\frac{4x-3y-7}{5}=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6}$,

$\frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}=\frac{y-x}{15}+\frac{x}{6}+\frac{11}{10}$.

24. $\dfrac{\frac{2x}{3}-\frac{5y}{12}}{\frac{7}{4}}-\dfrac{\frac{3x}{2}-\frac{y}{3}}{\frac{23}{2}}=2$, $\qquad \frac{x-y}{x+y}=\frac{1}{5}$.

25. $\frac{3x-2y}{3}+1+\frac{11y-10}{8}=\frac{4x-3y+5}{7}+\frac{45-x}{5}$,

$45-\frac{4x-2}{3}=\frac{55x+71y+1}{18}$.

26. $2{\cdot}4x+{\cdot}32y-\frac{{\cdot}36x-{\cdot}05}{{\cdot}5}={\cdot}8x+\frac{2{\cdot}6+{\cdot}005y}{{\cdot}25}$,

$\frac{{\cdot}04y+{\cdot}1}{{\cdot}3}=\frac{{\cdot}07x-{\cdot}1}{{\cdot}6}$.

27. $13x+11y=4a$, $\qquad 12x-6y=a$.

28. $\frac{m}{x}+\frac{n}{y}=1$, $\qquad \frac{n}{x}+\frac{m}{y}=1$.

29. $\frac{x}{a}+\frac{y}{b}=1$, $\qquad \frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}$.

30. $ax+by=c$, $\qquad mx-ny=d$.

31. $\frac{x}{b+c}+\frac{y}{a+c}=2$, $\qquad \frac{ax-by}{(a-b)c}=1$.

32. $\frac{x}{a+b}+\frac{y}{a-b}=2a$, $\qquad \frac{x-y}{4ab}=1$.

XII. SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH MORE THAN TWO UNKNOWN QUANTITIES.

182. If there be three simple equations and three unknown quantities, deduce from two of the equations an equation containing only two of the unknown quantities by the rules of the preceding Chapter; then deduce from the third equation and either of the former two, another equation containing the same two unknown quantities; and from the two equations thus obtained the unknown quantities which they involve may be found. The third quantity may be found by substituting the above values in any of the proposed equations.

Example, suppose,

$$2x + 3y + 4z = 16 \text{ ....(1)},$$

$$3x + 2y - 5z = 8 \text{ ....(2)},$$

$$5x - 6y + 3z = 6 \text{ ....(3)}.$$

For convenience of reference the equations are numbered (1), (2), and (3), and this numbering is continued as we proceed with the solution.

Multiply (1) by 3, and (2) by 2; thus,

$$6x + 9y + 12z = 48,$$

$$6x + 4y - 10z = 16;$$

by subtraction,

$$5y + 22z = 32 \text{ ....(4)}.$$

Multiply (1) by 5, and (3) by (2); thus,

$$10x + 15y + 20z = 80,$$

$$10x - 12y + 6z = 12;$$

by subtraction,

$$27y + 14z = 68 \text{ ....(5)}.$$

Multiply (4) by 27, and (5) by 5; thus,

$$135y + 594z = 864,$$
$$135y + \ \ 70z = 340;$$

by subtraction, $524z = 524,$

therefore, $z = 1.$

Substitute the value of z in (4); thus,

$$5y + 22 = 32;$$

therefore, $y = 2.$

Substitute the values of y and z in (1); thus,

$$2x + 6 + 4 = 16;$$

therefore, $x = 3.$

Sometimes it is convenient to use the following rule: from two of the equations express the values of two of the unknown quantities in terms of the third, and substitute these values in the third equation; hence the third unknown quantity can be found, and then the other two.

Example, suppose

$$3x + 4y - 16z = 0 \ldots\ldots (1),$$
$$5x - 8y + 10z = 0 \ldots\ldots (2),$$
$$2x + 6y + \ \ 7z = 52 \ldots\ldots (3).$$

Multiply (1) by 2, and add to (2); thus

$$11x - 22z = 0; \text{ therefore } x = 2z.$$

Multiply (1) by 5, and (2) by 3, and subtract; thus

$$44y - 110z = 0; \text{ therefore } y = \frac{5z}{2}.$$

Substitute in (3); thus

$$4z + 15z + 7z = 52; \text{ that is } 26z = 52;$$

therefore $z = 2$; and $x = 2z = 4$, $y = \frac{5z}{2} = 5$.

The same methods may be applied when the number of simple equations and of unknown quantities exceeds three.

EXAMPLES OF SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE WITH MORE THAN TWO UNKNOWN QUANTITIES.

1. $3x + 2y - 4z = 15,\ 5x - 3y + 2z = 28,\ 3y + 4z - x = 24.$
2. $x + y - z = 1,\quad 8x + 3y - 6z = 1,\quad 3z - 4x - y = 1.$
3. $2x - 7y + 4z = 0,\quad 3x - 3y + z = 0,\quad 9x + 5y + 3z = 28.$
4. $4x - 3y + 2z = 9,\quad 2x + 5y - 3z = 4,\quad 5x + 6y - 2z = 18.$
5. $2x - 4y + 9z = 28,\quad 7x + 3y - 5z = 3,\quad 9x + 10y - 11z = 4.$
6. $x - 2y + 3z = 6,\quad 2x + 3y - 4z = 20,\quad 3x - 2y + 5z = 26.$
7. $4x - 3y + 2z = 40,\quad 5x + 9y - 7z = 47,\quad 9x + 8y - 3z = 97.$
8. $3x + 2y + z = 23,\quad 5x + 2y + 4z = 46,\quad 10x + 5y + 4z = 75.$
9. $5x - 6y + 4z = 15,\quad 7x + 4y - 3z = 19,\quad 2x + y + 6z = 46.$
10. $\frac{1}{x} + \frac{1}{y} = 1,\quad \frac{1}{x} + \frac{1}{z} = 2,\quad \frac{1}{y} + \frac{1}{z} = \frac{3}{2}.$
11. $\frac{2}{x} + \frac{1}{y} = \frac{3}{z},\quad \frac{3}{z} - \frac{2}{y} = 2,\quad \frac{1}{x} + \frac{1}{z} = \frac{4}{3}.$
12. $\frac{3}{x} - \frac{4}{5y} + \frac{1}{z} = \frac{38}{5},\quad \frac{1}{3x} + \frac{1}{2y} + \frac{2}{z} = \frac{61}{6},\quad \frac{4}{5x} - \frac{1}{2y} + \frac{4}{z} = \frac{161}{10}.$
13. $\frac{3y-1}{4} = \frac{6z}{5} - \frac{x}{2} + \frac{9}{5},$

    $\frac{5x}{4} + \frac{4z}{3} = y + \frac{5}{6},$

    $\frac{3x+1}{7} - \frac{z}{14} + \frac{1}{6} = \frac{2z}{21} + \frac{y}{3}.$
14. $\frac{10x + 4y - 5z}{5} = \frac{4x + 6y - 3z}{9},$

    $10x + 4y - 5z = 4x + 6y - 3z - 8,$

    $\frac{10x + 4y - 5z}{10} + \frac{4x + 6y - 3z}{3} = \frac{x + y + z}{4}.$

15. $7x-3y=1,$
$11z-7u=1,$
$4z-7y=1,$
$19x-3u=1.$

16. $3u-2y=2,$
$5x-7z=11,$
$2x+3y=39,$
$4y+3z=41.$

17. $2x-3y+2z=13,$
$4y+2z=14,$
$4u-2x=30,$
$5y+3u=32.$

18. $7u-13z=87,$
$10y-3x=11,$
$3u+14x=57,$
$2x-11z=50.$

19. $7x-2z+3u=17,$
$4y-2z+v=11,$
$5y-3x-2u=8,$
$4y-3u+2v=9,$
$3z+8u=33.$

20. $3x-4y+3z+3v-6u=11,$
$3x-5y+2z-4u=11,$
$10y-3z+3u-2v=2,$
$5z+4u+2v-2x=3,$
$6u-3v+4x-2y=6.$

21. $\frac{x}{a}+\frac{y}{b}=1, \quad \frac{x}{a}+\frac{z}{c}=1, \quad \frac{y}{b}+\frac{z}{c}=1.$

22. $ay+bx=c, \quad cx+az=b, \quad bz+cy=a.$

23. $\frac{a}{x}+\frac{b}{y}=1, \quad \frac{b}{y}+\frac{c}{z}=1, \quad \frac{c}{z}+\frac{a}{x}=1.$

24. $x+y+z=0,$
$(b+c)x+(c+a)y+(a+b)z=0,$
$bcx+cay+abz=1.$

25. $ax+by+cz=A,$
$a^2x+b^2y+c^2z=A^2,$
$a^3x+b^3y+c^3z=A^3.$

26. $xyz=a(yz-zx-xy)=b(zx-xy-yz)=c(xy-yz-zx).$

27. $x+y+z=a+b+c,$
$bx+cy+az=cx+ay+bz=a^2+b^2+c^2.$

28. $x-ay+a^2z=a^3,$
$x-by+b^2z=b^3,$
$x-cy+c^2z=c^3.$

XIII. PROBLEMS WHICH LEAD TO SIMPLE EQUATIONS WITH MORE THAN ONE UNKNOWN QUANTITY.

183. We shall now give some examples of problems which lead to simple equations with more than one unknown quantity.

A and *B* engage in play; in the first game *A* wins as much as he had and four shillings more, and finds he has twice as much as *B*; in the second game *B* wins half as much as he had at first and one shilling more, and then it appears he has three times as much as *A*: what sum had each at first?

Let x be the number of shillings which *A* had, and y the number of shillings which *B* had; then after the first game *A* has $2x+4$ shillings and *B* has $y-x-4$ shillings. Thus by the question,

$$2x+4=2(y-x-4)=2y-2x-8;$$

therefore, $$2y-4x=12;$$

therefore, $$y-2x=6.$$

Also after the second game *A* has $2x+4-\frac{y}{2}-1$ shillings, and *B* has $y-x-4+\frac{y}{2}+1$ shillings. Thus by the question,

$$y-x-4+\frac{y}{2}+1=3\left(2x+4-\frac{y}{2}-1\right)=6x+12-\frac{3y}{2}-3;$$

therefore, $$2y-2x-8+y+2=12x+24-3y-6;$$

therefore, $$6y-14x=24,$$

and, $$3y-7x=12.$$

And from the former equation,

$$3y-6x=18;$$

hence by subtraction, $$x=6;$$

therefore, $$y=18.$$

184. A sum of money was divided equally among a certain number of persons; had there been three more, each would have received one shilling less, and had there been two fewer, each would have received one shilling more than he did: required the number of persons, and what each received.

Let x denote the number of persons, y the number of shillings which each received. Then xy shillings is the sum divided; thus by the question,

$$(x+3)(y-1)=xy,$$

and also,

$$(x-2)(y+1)=xy.$$

The first equation gives

$$xy+3y-x-3=xy;$$

thus,

$$3y-x=3.$$

The second equation gives

$$xy-2y+x-2=xy;$$

thus,

$$x-2y=2.$$

By addition,

$$3y-x+x-2y=5;$$

that is,

$$y=5.$$

Hence,

$$x=2y+2=12.$$

185. What fraction is that which becomes equal to $\frac{3}{4}$ when its numerator is increased by 6, and equal to $\frac{1}{2}$ when its denominator is diminished by 2?

Let x denote the numerator and y the denominator of the fraction; then by the question,

$$\frac{x+6}{y}=\frac{3}{4},$$

and,

$$\frac{x}{y-2}=\frac{1}{2}.$$

Clear the first equation of fractions by multiplying by $4y$; thus,

$$4(x+6)=3y;$$

therefore,

$$3y-4x=24.$$

Clear the second equation of fractions by multiplying by $2(y-2)$; thus,

$$2x = y - 2;$$

therefore, $$y - 2x = 2,$$

and, $$3y - 6x = 6.$$

By subtraction,

$$3y - 4x - (3y - 6x) = 24 - 6;$$

that is, $$2x = 18,$$

and, $$x = 9.$$

Hence, $$y = 2 + 2x = 20.$$

Thus the required fraction is $\frac{9}{20}$.

EXAMPLES OF PROBLEMS.

1. A certain fraction becomes 1 when 3 is added to its numerator, and $\frac{1}{2}$ when 2 is added to its denominator. What fraction is it?

2. *A* and *B* together possess £570. If *A*'s money were three times what it really is, and *B*'s five times what it really is, the sum would be £2350. What is the money of each?

3. If the numerator of a certain fraction is increased by one the value of the fraction becomes $\frac{1}{3}$; if the denominator is increased by one the value of the fraction becomes $\frac{1}{4}$. What is the fraction?

4. Find two numbers such that if the first be added to four times the second, the sum is 29; and if the second be added to six times the first the sum is 36.

5. If *A*'s money were increased by 36*s.* he would have three times as much as *B*; but if *B*'s money were diminished by 5*s.* he would have half as much as *A*. Find the sum possessed by each.

6. *A* and *B* lay a wager of 10*s.*; if *A* loses he will have twenty-five shillings less than twice as much as *B* will then have;

but if B loses he will have five-seventeenths of what A will then have: find how much money each of them has.

7. Find two numbers, such that twice the first plus the second is equal to 17, and twice the second plus the first is equal to 19.

8. Find two numbers, such that one-half the first and three-fourths of the second together may be equal to the excess of three times the first over the second, and this excess equal to 11.

9. For five guineas can be obtained either 32 pounds of tea and 15 pounds of coffee, or 36 pounds of tea and 9 pounds of coffee: find the price of a pound of each.

10. Determine three numbers such that their sum is 9; the sum of the first, twice the second, and three times the third, 22; and the sum of the first, four times the second, and nine times the third, 58.

11. A pound of tea and three pounds of sugar cost six shillings, but if sugar were to rise 50 per cent. and tea 10 per cent. they would cost 7 shillings. Find the price of tea and sugar.

12. A person has £2550 to invest. The three per cent. consols are at 81, and certain guaranteed railway shares which pay a half-yearly dividend of 10*s*. on each original share of £25 are at £24. Find how many shares he must buy that he may obtain the same income from the railway shares as from the rest of his money invested in the consols.

13. A person possesses a certain capital which is invested at a certain rate per cent. A second person has £1000 more capital than the first person and invests it at one per cent. more; thus his income exceeds that of the first person by £80. A third person has £1500 more capital than the first and invests it at two per cent. more; thus his income exceeds that of the first person by £150. Find the capital of each person and the rate at which it is invested.

14. A sum of money is divided equally among a certain number of persons; if there had been four more each would have

received a shilling less than he did; if there had been five fewer each would have received two shillings more than he did: find the number of persons and what each received.

15. Two plugs are opened in the bottom of a cistern containing 192 gallons of water; after three hours one of the plugs becomes stopped, and the cistern is emptied by the other in eleven more hours; had six hours occurred before the stoppage, it would have required only six hours more to empty the cistern. How many gallons will each plug hole discharge in an hour, supposing the discharge uniform?

16. A person after paying a poor-rate and also the income-tax of 7*d.* in the pound, has £486 remaining; the poor-rate amounts to £22. 10*s.* more than the income-tax: find the original income and the number of pence per pound in the poor-rate.

17. A certain number of persons were divided into three classes, such that the majority of the first and second together over the third was 10 less than four times the majority of the second and third together over the first; but if the first had 30 more, and the second and third together 29 less, the first would have outnumbered the last two by one. Find the number in each class when the whole number was 34 more than eight times the majority of the third over the second.

18. A farmer would spend all his money by buying 4 oxen and 32 lambs; instead of doing this he bought the same number of oxen and half as many lambs, and had a surplus of £9 after paying for them and for their conveyance by railway at an average cost of six shillings per head. Each ox cost as many pounds as its carriage by railway was shillings, and the lambs altogether cost three times as many pounds as the carriage of each was shillings. How much money had the farmer to begin with?

19. *A* and *B* play at bowls, and *A* bets *B* three shillings to two upon every game; after a certain number of games it appears that *A* has won three shillings; but if *A* had bet five shillings to two and lost one game more out of the same number, he would have lost thirty shillings. How many games did each win?

20. Five persons, A, B, C, D, E play at cards; after A has won half of B's money, B one-third of C's, C one-fourth of D's, D one-sixth of E's, they have each £1. 10*s*. Find how much each had to begin with.

21. If there were no accidents it would take half as long to travel the distance from A to B by railroad as by coach; but three hours being allowed for accidental stoppages by the former, the coach will travel the distance all but fifteen miles in the same time; if the distance were two-thirds as great as it is, and the same time allowed for railway stoppages, the coach would take exactly the same time: required the distance.

22. A and B are set to a piece of work which they can finish in thirty days working together, and for which they are to receive £7. 10*s*. When the work is half finished A intermits working eight days and B four days, in consequence of which the work occupies five and a half days more than it would otherwise have done. How much ought A and B respectively to receive?

23. A and B run a mile. First A gives B a start of 44 yards and beats him by 51 seconds; at the second heat A gives B a start of 1 minute 15 seconds, and is beaten by 88 yards. Find the times in which A and B can run a mile separately.

24. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half an hour before B, but missing his way goes a mile and back again needlessly, during which he walks at twice his former pace, and reaches the top six minutes before B. C starts twenty minutes after A and B and walking at the rate of two and one-seventh miles per hour, arrives at the summit ten minutes after B. Find the rates of walking of A and B, and the distance from the foot to the summit of the mountain.

25. A railway train after travelling for one hour meets with an accident which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on,

the train would have arrived 1 hour 20 minutes sooner. Required the length of the line, and the original rate of the train.

26. A, B, and C sit down to play, every one with a certain number of shillings. A loses to B and to C as many shillings as each of them has. Next B loses to A and to C as many as each of them now has. Lastly C loses to A and to B as many as each of them now has. After all every one of them has sixteen shillings. How much had each originally?

27. Two persons A and B could finish a work in m days; they worked together n days when A was called off and B finished it in p days. In what time could each do it?

28. A railway train running from London to Cambridge meets on the way with an accident, which causes it to diminish its speed to $\frac{1}{n}$th of what it was before, and it is in consequence a hours late. If the accident had happened b miles nearer Cambridge, the train would have been c hours late. Find the rate of the train before the accident occurred.

29. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; if the circumference of the fore-wheel be increased by one-fourth of its present size, and the circumference of the hind-wheel by one-fifth of its present size, the six will be changed to four. Required the circumference of each wheel.

30. There is a number consisting of two digits; the number is equal to three times the sum of its digits, and if 45 be added to the number the digits interchange their places: find the number.

31. There is a number consisting of two digits; the number is equal to seven times the sum of its digits, and if 27 be subtracted from the number the digits interchange their places: find the number.

32. A person proposes to travel from A to B, either direct by coach, or by rail to C, and thence by another train to B. The trains travel three times as fast as the coach, and should there be

no delay, the person starting at the same hour could get to *B* 20 minutes earlier by coach than by train. But should the train be late at *C*, he would have to wait there for a train as long as it would take to travel from *C* to *B*, and his journey would in that case take twice as long as by coach. Should the coach however be delayed an hour on the way, and the train be in time at *C*, he would get by rail to *B* and half way back to *C*, while he would be going by coach to *B*. The length of the whole circuit *ABCA* is $76\frac{2}{3}$ miles. Required the rate at which the coach travels.

33. *A* offers to run three times round a course while *B* runs twice round, but *A* only gets 150 yards of his third round finished when *B* wins. *A* then offers to run four times round for *B*'s thrice, and now quickens his pace so that he runs 4 yards in the time he formerly ran 3 yards. *B* also quickens his so that he runs 9 yards in the time he formerly ran 8 yards, but in the second round falls off to his original pace in the first race, and in the third round only goes 9 yards for 10 he went in the first race, and accordingly this time *A* wins by 180 yards. Determine the length of the course.

34. A man starts p hours before a coach, and both travel uniformly; the latter passes the former after a certain number of hours. From this point the coach increases its speed to six-fifths of its former rate, while the man increases his to five-fourths of his former rate, and they continue at these increased rates for q hours longer than it took the coach to overtake the man. They are then 92 miles apart; but had they continued for the same length of time at their original rates they would have been only 80 miles apart. Shew that the original rate of the coach is twice that of the man. Also if $p+q=16$, shew that the original rate of the coach was 10 miles per hour, and that of the man 5 miles per hour.

XIV. DISCUSSION OF SOME PROBLEMS WHICH LEAD TO SIMPLE EQUATIONS.

186. We propose now to solve some problems which lead to Simple Equations, and to examine certain peculiarities which present themselves in the solutions. We begin with the following problem: What number must be added to a number a in order that the sum may be b? Let x denote this number; then,

$$a + x = b;$$

therefore, $$x = b - a.$$

This formula gives the value of x corresponding to any assigned values of a and b. Thus, for example, if $a = 12$ and $b = 25$, we have $x = 25 - 12 = 13$. But suppose that $a = 30$ and $b = 24$; then $x = 24 - 30 = -6$, and we naturally ask what is the meaning of this negative result? If we recur to the enunciation of the problem we see that it now reads thus: What number must be added to 30 in order that the sum may be 24? It is obvious then, that if the word *added* and the word *sum* are to retain their arithmetical meanings, the proposed problem is impossible. But we see at the same time that the following problem can be solved: What number must be *taken from* 30 in order that the *difference* may be 24? and 6 is the answer to this question. And the second enunciation differs from the first in these respects; the words *added to* are replaced by *taken from*, and the word *sum* by *difference*.

187. Thus we may say that, in this example, the *negative* result indicates that the problem in a strictly Arithmetical sense is impossible; but that a new problem can be formed by appropriate changes in the original enunciation to which the *absolute value* of the negative result will be the correct answer.

188. This indicates the convenience of using the word *add* in Algebra in a more extensive sense than it has in Arithmetic. Let x denote a quantity which is to be *added algebraically* to a;

then the Algebraical sum is $a+x$, whether x itself be positive or negative. Thus the equation $a+x=b$ will be possible algebraically whether a be greater or less than b.

We proceed to another problem.

189. *A*'s age is a years, and *B*'s age is b years; when will *A* be twice as old as *B*? Supposed the required epoch to be x years from the present time; then by the question,

$$a+x=2(b+x);$$

hence, $$x=a-2b.$$

Thus, for example, if $a=40$ and $b=15$, then $x=10$. But suppose $a=35$ and $b=20$, then $x=-5$; here, as in the preceding problem, we are led to inquire into the meaning of the negative result. Now with the assigned values of a and b the equation which we have to solve becomes

$$35+x=40+2x,$$

and it is obvious that if a strictly arithmetical meaning is to be given to the symbols x and $+$, this equation is impossible, for 40 is greater than 35, and $2x$ is greater than x, so that the two members cannot be equal. But let us change the enunciation to the following: *A*'s age is 35 years, and *B*'s age is 20 years, when *was* *A* twice as old as *B*? Let the required epoch be x years from the present time, then by the question,

$$35-x=2(20-x)=40-2x;$$

thus, $$x=5.$$

Here again we may say the *negative* result indicates that the problem in a strictly Arithmetical sense is impossible, but that a new problem can be formed by appropriate changes in the original enunciation, to which the *absolute value* of the negative result will be the correct answer.

We may observe that the equation corresponding to the new enunciation may be obtained from the original equation by changing x into $-x$.

190. Suppose that the problem had been originally enunciated thus: *A*'s age is a years, and *B*'s age is b years; find the

epoch at which A's age is twice that of B. These words do not intimate whether the required epoch is before or after the present date. If we suppose it *after* we obtain, as in Art. 189, for the required number of years $x = a - 2b$. If we suppose the required epoch to be x years *before* the present date we obtain $x = 2b - a$. If $2b$ is *less* than a, the first supposition is correct, and leads to an arithmetical value for x; the second supposition is incorrect, and leads to a negative value for x. If $2b$ is *greater* than a, the second supposition is correct, and leads to an arithmetical value for x; the first supposition is incorrect and leads to a negative value for x. Here we may say then that a negative result indicates that we made the wrong choice out of two possible suppositions which the problem allowed. But it is important to notice, that when we discover that we have made the wrong choice, it is not necessary to go through the *whole investigation* again, for we can make use of the *result* obtained on the wrong supposition. We have only to take the absolute value of the negative result and place the epoch *before* the present date if we had supposed it after, and *after* the present date if we had supposed it before.

191. One other case may be noticed. Suppose the enunciation to be like that in the latter part of Art. 189; A's age is a years, and B's age is b years, when was A twice as old as B? Let x denote the required number of years; then

$$a - x = 2(b - x),$$

hence,

$$x = 2b - a.$$

Now let us *verify* this solution. Put this value for x; then $a - x$ becomes $a - (2b - a)$, that is, $2a - 2b$; and $2(b - x)$ becomes $2(b - 2b + a)$, that is, $2a - 2b$. If b is less than a, these results are positive, and there is no Arithmetical difficulty. But if b is greater than a, although the two members are algebraically equal, yet since they are both *negative* quantities, we cannot say that we have arithmetically verified the solution. And when we recur to the problem we see that it is impossible if a is less than b; because if at a given date A's age is less than B's, then A's age never was twice B's and never will be. Or without proceeding to verify the result, we may observe that if b is greater than a, then x is also greater than a, which is inadmissible. Thus it appears that a problem may be really absurd, and yet the result may not immediately present any difficulty, though when we proceed to examine or verify this result we may discover an intimation of the absurdity.

192. The equation $a+x=2(b+x)$ may be considered as the symbolical expression of the following verbal enunciation: Suppose a and b to be two quantities, what quantity must be added to each so that the first sum may be twice the second? Here the words *quantity*, *sum*, and *added* may all be understood in Algebraical senses, so that x, a, and b may be positive or negative. This Algebraical statement includes among its admissible senses the Arithmetical question about the ages of A and B. It appears then that when we translate a problem into an equation, the same equation *may be* the symbolical expression of a more comprehensive problem than that from which it was obtained.

We will now examine another problem.

193. A and B travel in the same direction at the rate of a and b miles respectively per hour. A arrives at a certain place P at a certain time, and at the end of n hours from that time B arrives at a certain place Q. Find when A and B meet.

Let c denote the distance PQ; suppose A and B to travel in the direction from P towards Q, and to meet at R at the end of x hours from the time when A was at P; then since A travels at the rate of a miles per hour, the distance PR is ax miles. Also B goes over the distance QR in $x-n$ hours, so that QR is $b(x-n)$ miles. And PR is equal to the sum of PQ and QR; thus,

$$ax=c+b(x-n)=c+bx-bn;$$

therefore, $$x=\frac{c-bn}{a-b}.$$

We shall now examine this result on different suppositions as to the values of the given quantities.

I. Suppose a greater than b, and c greater than bn; then the value of x is positive, and the travellers *will* meet, as we have supposed, *after* A arrives at P. For when A is at P, the space which B has to travel before he reaches Q is bn miles, and since bn is less than c, it follows that when A is at P he is *behind* B; and A travels more rapidly than B, since a is greater than b. Hence A must at the end of some time overtake B.

The distance $PR = ax = \dfrac{a(c - bn)}{a - b}$. Thus,

$$QR = \frac{a(c - bn)}{a - b} - c = \frac{a(c - bn) - c(a - b)}{a - b} = \frac{cb - abn}{a - b} = \frac{b(c - an)}{a - b}.$$

Now if c be *greater* than an, this expression is a positive quantity, so that R falls, as we have supposed, *beyond* Q; we see that this must be the case, for since c is *greater* than an, it will take A more than n hours to go from P to Q, so that he cannot overtake B until after passing Q. If, however, c be *less* than an, the expression for QR is a *negative* quantity, and this leads us to suppose that some modification is required in our view of the problem. In fact A now takes *less* than n hours to go from P to Q, so that he will overtake B *before* arriving at Q. Hence the figure should now stand thus:

P R Q

And now, since $PR = PQ - RQ$, the equation for determining x would naturally be written

$$ax = c - b(n - x) = c - bn + bx.$$

This, however, we see is really the same equation as before.

Again, if c be *equal* to an the value of RQ is zero. Thus R now coincides with Q; and

$$x = \frac{c - bn}{a - b} = \frac{an - bn}{a - b} = n.$$

Hence A and B meet at Q at the end of n hours after A was at P.

II. Next suppose that a is greater than b, and c less than bn. The value of x is now *negative*, and we may conjecture from what we have hitherto observed respecting negative quantities that A and B instead of meeting $\frac{c-bn}{a-b}$ hours *after* A was at P, will now really have met $\frac{bn-c}{a-b}$ hours *before* A was at P. And in fact, since c is less than bn it follows that B was behind A when A was at P, so that A must have passed B before arriving at P. Hence the correct solution of the problem would now be as follows:

Suppose that A and B meet x hours *before* A arrives at P; let R be the point where they meet. Then $RP = ax$, and $RQ = b(x+n)$. Also $RP = RQ - PQ$; thus,

$$ax = b(x+n) - c;$$

therefore, $$x = \frac{bn-c}{a-b}.$$

III. Next suppose that a is less than b, and c greater than bn. In this case also the expression originally obtained for x is *negative*, and we shall accordingly find that A and B met *before* A was at P. For B now travels more rapidly than A, and is *before* A when A is at P; so that B must have passed A before A was at P. The result now is, as in the second case, that A and B met $\frac{c-bn}{b-a}$ hours *before* A was at P.

IV. Last suppose that a is less than b, and c less than bn. Here the expression originally obtained for x is a *positive* quantity, for it may be written thus, $\frac{bn-c}{b-a}$. Now B travels more rapidly than A and is *behind* A when A is at P; thus B must at some time overtake A. If we suppose A and B to meet *after* A is at Q, the figure will stand thus:

Here we should naturally write the equation thus,

$$ax = c + b(x-n) = c + bx - bn,$$

If we suppose A and B to meet *before* A is at Q, the figure will stand thus:

P R Q

Here we should naturally write the equation thus,

$$ax = c - b(n - x) = c - bn + bx.$$

In the two cases we have, however, really the same equation, and we obtain $x = \frac{bn - c}{b - a}$.

194. The preceding problem may be variously modified; for instance, instead of supposing that A and B travel in the same direction, we may suppose that A travels as before, but that B travels in the opposite direction. In this case, if we suppose, as before, that A and B meet x hours after A arrived at P, we shall find that $x = \frac{c + bn}{a + b}$. Thus the time of meeting will necessarily be *after* A leaves P, and the travellers meet at some point to the right of P. The student should notice that the value of x in the present case coincides with the result obtained by writing $-b$ for b in the original value of x in Art. 193.

195. Or instead of supposing that the arrival of B at Q occurs n hours *after* the arrival of A at P, we may suppose it to occur n hours *before*; and we suppose A and B to travel in the *same* direction. In this case if x have the same meaning as before, we shall find that $x = \frac{c + bn}{a - b}$. This is a *positive* quantity if a is greater than b, and the travellers then really meet *after* the arrival of A at P. If, however, a is less than b, the value of x is a *negative* quantity; this suggests that the travellers now meet $\frac{c + bn}{b - a}$ hours *before* the arrival of A at P, and on examination this will be found correct. The student should notice that the value of x in the present case coincides with the result obtained by writing $-n$ for n in the original value of x in Art. 193.

196. Again, let us suppose that A and B travel in *opposite* directions, and that the arrival of A at P occurs n hours before that of B at Q; and suppose the positions of P and Q in the former figures to be interchanged, so that now A reaches Q before he reaches P, and B reaches P before he reaches Q. If x have the same meaning as before, we shall now find that $x = \frac{bn - c}{a + b}$. If then bn is greater than c, the value of x is a *positive* quantity, and the travellers meet, as we have supposed, *after* the arrival of A at P. If however bn is less than c, the value of x is a *negative* quantity, and it will be found that the travellers meet $\frac{c - bn}{a + b}$ hours *before* the arrival of A at P. The student should notice that the value of x in the present case coincides with the result obtained by writing $-c$ for c in the value of x in Art. 194; it also coincides with the result obtained by writing $-b$ for b, and $-c$ for c in the original value of x in Art. 193.

197. From a consideration of the problems discussed in the present Chapter, and of similar problems, the student will acquire confidence and accuracy in dealing with negative quantities. We will lay down some general principles which have been illustrated in the preceding Articles, and the truth of which the student will find confirmed as he advances in the subject.

(1) A negative result may arise from the fact that the enunciation of a problem involves a condition which cannot be satisfied; in this case we may attribute to the unknown quantity a *quality directly opposite* to that which had been attributed to it, and may thus form a possible problem analogous to that which involved the impossibility.

(2) A negative result may arise from the fact that a wrong supposition respecting the *quality* of some quantity was made when the problem was translated from words into Algebraical symbols; in this case we may correct our supposition by attributing the opposite quality to such quantity, and thus obtain a positive result.

(3) When we wish to alter the suppositions we have made

respecting the *quality* of the known or unknown quantities of a problem, and to attribute an opposite quality to them, it is not necessary to form a *new* equation; it is sufficient to change in the *old* equation the sign of the symbol representing each quantity which is to have its quality changed.

198. We do not assert that the above general principles have been *demonstrated;* they have been suggested by observation of particular examples, and are left to the student to be verified in the same manner. Thus when a negative result occurs in the solution of a problem the student should endeavour to *interpret* that result, and these general principles will serve to guide him. When a problem leads to a negative result, and he wishes to form an analogous problem that shall lead to the corresponding positive result, he may proceed thus: change x into $-x$ in the equation that has been obtained, and then, if possible, modify the verbal statement of the problem, so as to make it coincident with the new equation. We say, *if possible,* because in some cases no such verbal modification seems attainable, and the problem may then be regarded as altogether impossible.

199. We will now leave the consideration of negative quantities, and examine two other singularities that may occur in results.

In Art. 193 we found this result, $x=\frac{c-bn}{a-b}$. Suppose that $a=b$, then the denominator in the value of x is zero; thus, denoting the numerator by N, we have $x=\frac{N}{0}$, and we may ask what is the meaning of this result? Since A and B now travel with equal speed, they must always preserve the same distance; so that they *never* meet. But instead of supposing that a is exactly equal to b, let us suppose that a is very nearly equal to b; then $\frac{N}{a-b}$ may be a very large quantity, since if $a-b$ is very small compared with N, it will be contained a large number of times in N; and the smaller $a-b$ is, the larger will $\frac{N}{a-b}$ be. This is *abbreviated* into the phrase "$\frac{N}{0}$ is infinite," and it is written thus, $\frac{N}{0} = \infty$. But the student must remember that the phrase is *only an abbreviation*, and no absolute meaning can be attached to it.

200. The student should examine every problem, the result of which appears under the form $\frac{N}{0}$, and endeavour to *interpret* that result. He may expect to find in such a case that the problem is impossible, but that by suitable modifications a new problem can be formed which has a *very great number* for its result, and that this result becomes greater the more closely the new problem approaches to the old problem.

201. Again, let us suppose that in Art. 193 we have $a = b$, *and also* $c = bn$; then the value of x takes the form $\frac{0}{0}$. On examining the problem we see that, in consequence of the suppositions just made, A and B are together at P, and are travelling with equal speed, so that they are *always* together. The question, when are A and B together, is in this case said to be *indeterminate*, since it does not admit of a single answer, or of a finite number of answers.

202. The student should also examine every problem in which the result appears under the form $\frac{0}{0}$, and endeavour to interpret that result. In some cases he will find, as in the example considered above, that the problem is not restricted to a finite number of solutions, but admits of as many as he pleases. We do not assert here, or in Art. 200, that the interpretation of the singularities $\frac{N}{0}$ and $\frac{0}{0}$ will *always* coincide with those given in the simple cases we have considered; the student must therefore consider separately each distinct class of examples that may occur.

MISCELLANEOUS EXAMPLES. CHAPTER XIV.

1. Simplify the expression

$$3a - [b + \{2a - (b - c)\}] + \tfrac{1}{2} + \frac{2c^2 - \frac{1}{2}}{2c + 1}.$$

2. Reduce to its lowest terms the expression

$$\frac{6x^4 + 10x^3 + 2x^2 - 20x - 28}{3x^3 + 14x^2 + 22x + 21}.$$

3. Find the value of $\dfrac{x-a}{b} - \dfrac{x-b}{a}$ when $x = \dfrac{a^2}{a-b}$.

4. Simplify $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$.

5. Shew that $\dfrac{d^m(a-b)(b-c) + b^m(a-d)(c-d)}{c^m(a-b)(a-d) + a^m(b-c)(c-d)} = \dfrac{b-d}{a-c}$ when $m = 1$, or 2.

6. Reduce to its simplest form $\dfrac{a^3 + b^3 + c^3 - 3abc}{(a-b)^2 + (b-c)^2 + (c-a)^2}$.

7. If $xy + yz + zx = 1$, shew that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

8. Solve the equation

$$(x - 2a)^3 + (x - 2b)^3 = 2(x - a - b)^3.$$

9. Solve the simultaneous equations

$$x + y + z = a + b + c,$$

$$bx + cy + az = cx + ay + bz = ab + bc + ca.$$

10. Find the least common multiple of

$x^3 + 6x^2 + 11x + 6$, $x^3 + 7x^2 + 14x + 8$,

$x^3 + 8x^2 + 19x + 12$, and $x^3 + 9x^2 + 26x + 24$.

XV. ANOMALOUS FORMS WHICH OCCUR IN THE SOLUTION OF SIMPLE EQUATIONS.

203. We have in the preceding Chapter referred to the forms $\frac{N}{0}$ and $\frac{0}{0}$ which may occur in the solution of an equation of the first degree. We shall now examine the meaning of these forms when they occur in the solution of *simultaneous equations* of the first degree. We will first recall the results already obtained.

204. Every equation of the first degree with one unknown quantity may be reduced to the form $ax = b$. Now from this we obtain $x = \frac{b}{a}$. If $a = 0$ the value of x takes the form $\frac{b}{0}$; in this case no finite value of x can satisfy the equation, for whatever finite value be assigned to x, since $ax = 0$, we have $0 = b$, which is impossible. If $a = 0$ and $b = 0$, the value of x takes the form $\frac{0}{0}$; in this case every finite value of x may be said to satisfy the equation, since whatever finite value be given to x we have $0 = 0$. If $b = 0$ and a is not $= 0$, then of course $x = 0$; this case calls for no remark.

205. Suppose now we have two equations with two unknown quantities; let them be

$$ax + by = c \text{ and } a'x + b'y = c'.$$

We will first make a remark on the notation we have here adopted. We use *certain letters* to denote the known quantities in the first equation, and then we use *corresponding letters with accents* to denote corresponding quantities in the second equation; here a and a' have no necessary connexion as to value, although they have this common point, namely, that each is a coefficient of x, one in the first equation and the other in the second equation. Experience will establish the advantage of this notation.

Instead of accents subscript numbers are sometimes used; thus a_1 and a_2 might be used instead of a and a' respectively.

By solving the given equations we obtain

$$x=\frac{b'c-bc'}{b'a-ba'}, \qquad y=\frac{a'c-ac'}{a'b-ab'}.$$

I. Suppose that $b'a-ba'=0$; then the values of x and y take the forms $\frac{A}{0}$ and $\frac{B}{0}$; we should therefore recur to the given equations to discover the meaning of these results. From the relation $b'a-ba'=0$ we obtain $\frac{a'}{a}=\frac{b'}{b}=k$ suppose; thus $a'=ka$ and $b'=kb$. By substituting these values of a' and b' we find that the second of the given equations may be written thus:

$$kax+kby=c',$$

whence,

$$ax+by=\frac{c'}{k}.$$

Now if $\frac{c'}{k}$ be different from c, the last equation is *inconsistent* with the first of the given equations, because $ax+by$ cannot be equal to two different quantities. We may therefore conclude that the appearance of the results under the forms $\frac{A}{0}$ and $\frac{B}{0}$ indicates that the given equations are inconsistent, and therefore *cannot be solved.*

II. Next suppose that $b'a-ba'=0$, so that $\frac{a'}{a}=\frac{b'}{b}$, and *also* that $\frac{c'}{c}=\frac{a'}{a}$, and therefore of course $=\frac{b'}{b}$. In this case the numerators in the values of x and y become zero as well as the denominators, so that the values of x and y take the form $\frac{0}{0}$. Now by what we have shewn above, the second of the given equations may be written

$$ax+by=\frac{c'}{k}.$$

But now $\frac{c'}{k}=c$, so that the second given equation is only a repetition of the first; we have thus really only one equation involving two unknown quantities. We cannot then *determine* x and y, because we can find as many values as we please which will satisfy *one* equation involving two unknown quantities. In this case we say that the given equations are *not independent*, and that the values of x and y are *indeterminate*.

206. We have hitherto supposed that none of the quantities a, b, c, a', b', c' can be zero; and thus if the value of one of the unknown quantities takes the form $\frac{0}{0}$ or $\frac{A}{0}$ the value of the other takes the *same* form. But if some of the above quantities are zero, the values of the two unknown quantities do not necessarily take the *same* form. For example, suppose a and a' to be zero; then the value of x takes the form $\frac{A}{0}$, and the value of y takes the form $\frac{0}{0}$. Now in this case the given equations reduce to

$$by = c, \quad \text{and } b'y = c';$$

these lead to

$$y = \frac{c}{b}, \text{ and } \quad y = \frac{c'}{b'}.$$

Thus we have two cases. First, if $\frac{c}{b}$ is *not* equal to $\frac{c'}{b'}$, the two equations are inconsistent. Secondly, if $\frac{c}{b}$ is equal to $\frac{c'}{b'}$, the two equations are equivalent to one only. In the second case, since the relation $\frac{c}{b} = \frac{c'}{b'}$, makes the numerator of x also vanish, the values of both x and y take the form $\frac{0}{0}$; in this case x is indeterminate but y is not, for it is really equal to $\frac{c}{b}$.

207. Before we consider the peculiarities which may occur in the solution of three simultaneous simple equations involving three unknown quantities, we will indicate another method of solving such equations.

Let the equations be

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d''.$$

Let l and m denote two quantities, the values of which are at present undetermined; multiply the second of the given equations by l, and the third by m; then, by addition, we have

$$ax + by + cz + l(a'x + b'y + c'z) + m(a''x + b''y + c''z) = d + ld' + md'',$$

that is,

$$x(a + la' + ma'') + y(b + lb' + mb'') + z(c + lc' + mc'') = d + ld' + md''.$$

Now let such values be given to l and m as will make the coefficients of y and z in the last equation to be zero; that is, let

$$b + lb' + mb'' = 0, \qquad c + lc' + mc'' = 0.$$

Thus the equation reduces to

$$x(a + la' + ma'') = d + ld' + md'';$$

therefore,
$$x = \frac{d + ld' + md''}{a + la' + ma''}.$$

We must now find the values of l and m, and substitute them in this expression for x, and then the value of x will be known. We have

$$b + lb' + mb'' = 0, \qquad c + lc' + mc'' = 0;$$

from these we shall obtain

$$l = \frac{b''c - bc''}{b'c'' - b''c'}, \qquad m = \frac{bc' - b'c}{b'c'' - b''c'};$$

substitute these values in the expression for x, and after simplification we obtain

$$x = \frac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}.$$

By a similar method the values of y and z may also be obtained.

208. The above method of solution is called the method of *indeterminate multipliers*, because we make use of multipliers which we do not determine beforehand, but to which a convenient value is assigned in the course of the investigation. The multipliers are not finally *indeterminate;* they are merely at first *undetermined*, and if it were possible to alter established language,

the word *undetermined* might here with propriety be substituted for *indeterminate.*

209. We now proceed to our observations on the values of x, y, and z which are obtained from the equations

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d''.$$

The value of x has been given in Art. 207; if the student investigates the value of y he will find that the denominator of it is the *same as that which occurs in the value of* x, or can be made to be the same by changing the sign of every term in the numerator and denominator. The same remark holds with respect to the denominator in the value of z.

210. We may however obtain the values of y and z from the expression found for the value of x. For the original equations might have been written thus:

$$by + ax + cz = d, \quad b'y + a'x + c'z = d', \quad b''y + a''x + c''z = d'';$$

we may say then that the equations in this form differ from those in the original form only in the following particulars; x and y are interchanged, a and b are interchanged, a' and b' are interchanged, and a'' and b'' are interchanged. We may therefore deduce the value of y from that of x by the following rule: for a, a', and a'' write b, b', and b'' respectively, and conversely. Thus, from

$$x = \frac{d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c)}{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)}$$

we may deduce that

$$y = \frac{d(a'c'' - a''c') + d'(a''c - ac'') + d''(ac' - a'c)}{b(a'c'' - a''c') + b'(a''c - ac'') + b''(ac' - a'c)}.$$

It will be found on comparison that the denominator of the value of y is the same as that of the value of x with the sign of every term changed.

Similarly by interchanging a, a', and a'' with c, c', and c'' respectively, we may deduce the value of z from that of x; or by interchanging b, b', and b'' with c, c', and c'' respectively, we may deduce the value of z from that of y.

211. There is another system of interchanges by which the values of y and z may be deduced from that of x. The given equations are

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d'';$$

they may also be written thus,

$$by + cz + ax = d, \quad b'y + c'z + a'x = d', \quad b''y + c''z + a''x = d''.$$

We may say then that the second form differs from the first only in the following particulars; x is changed into y, y into z, z into x, a into b, b into c, c into a, a' into b', and so on. We may therefore deduce the value of y from that of x by this rule: change a into b, b into c, c into a, and make similar changes in the letters with one accent, and in those with two accents. The value of z may be deduced from that of y by again using the *same rule.*

212. These methods of deducing the values of y and z from that of x by interchanging the letters may perhaps appear difficult to the student at first, but they deserve careful consideration, especially that which is given in Art. 211.

We shall now proceed to examine the peculiarities which may occur in the values of the unknown quantities deduced from the equations

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d''.$$

213. The most important case is that in which d, d', and d'' are all zero. The given equations then become

$$ax + by + cz = 0, \quad a'x + b'y + c'z = 0, \quad a''x + b''y + c''z = 0.$$

It is obvious that $x = 0$, $y = 0$, $z = 0$ satisfy these equations; and from the values found in Art. 210 it follows that these are the *only* values which will satisfy the equations *unless* the denominator there given vanishes, that is, unless

$$a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c) = 0.$$

If this relation holds among the coefficients, the values found for x, y, and z take the form $\frac{0}{0}$, and we must recur to the given equations for further information.

We observe that when this relation holds the equations are not independent; from any two of them the third can be deduced. For multiply the first of the given equations by $b''c' - b'c''$, the second by $bc'' - b''c$, and the third by $b'c - bc'$, and then add the results. It will be found that by virtue of the given relation we arrive at the identity $0 = 0$; thus, in fact, if the first equation be multiplied by $b''c' - b'c''$, and the second equation by $bc'' - b''c$, and the two added, the result is equivalent to the third equation, for it may be obtained by multiplying that equation by $bc' - b'c$.

Suppose then that this relation holds; we may confine ourselves to the first two of the given equations, for values of x, y, and z which satisfy these will necessarily satisfy the third equation. Divide these equations by x; thus

$$\frac{by}{x} + \frac{cz}{x} + a = 0, \qquad \frac{b'y}{x} + \frac{c'z}{x} + a' = 0;$$

hence

$$\frac{y}{x} = \frac{ca' - c'a}{bc' - b'c}, \qquad \frac{z}{x} = \frac{ab' - a'b}{bc' - b'c}.$$

We may therefore ascribe *any value we please* to x, and deduce corresponding values of y and z. Or we may put our result more symmetrically thus; let p denote any quantity whatever, then the given equations will be satisfied by

$$x = p(bc' - b'c), \quad y = p(ca' - c'a), \quad z = p(ab' - a'b).$$

We might in the same way have used the second and third of the given equations, and have omitted the first; we should thus have deduced solutions of the form

$$x = q(b'c'' - b''c'), \quad y = q(c'a'' - c''a'), \quad z = q(a'b'' - a''b'),$$

where q is any quantity. These values however are substantially equivalent to the former; for it will be found that by virtue of the supposed relation among the coefficients,

$$\frac{p(bc' - b'c)}{q(b'c'' - b''c')} = \frac{p(ca' - c'a)}{q(c'a'' - c''a')} = \frac{p(ab' - a'b)}{q(a'b'' - a''b')}.$$

214. We shall now consider the peculiarities which may occur when d, d', and d'' are not all zero.

We shall first shew that if the value of any one of the unknown quantities takes the form $\frac{N}{0}$, the given equations are *inconsistent*. Suppose, for instance, that the value of x takes this form, that is, suppose that

$$a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)$$

is zero. Of course if the given equations were consistent, any equation legitimately deduced from them would also be true. Now multiply the first of the given equations by $b''c' - b'c''$, the second by $bc'' - b''c$, and the third by $b'c - bc'$ and add. It will be found that the coefficients of y and z in the resulting equation vanish; and the coefficient of x is zero by supposition. Thus the first member of the resulting equation vanishes, but the second member does not; hence the resulting equation is impossible, and therefore those from which it was obtained cannot have been consistent.

215. We cannot however affirm certainly, that if the value of *one* of the unknown quantities takes the form $\frac{0}{0}$, the equations are consistent, but not independent. For it is possible that the value of one of the unknown quantities should take this form, while the value of another takes the form $\frac{N}{0}$; and, as we have shewn in the preceding Article, the occurrence of the form $\frac{N}{0}$ is an indication that the given equations are *inconsistent*. For example, suppose the equations to be

$$ax + by + cz = d, \quad a'x + by + cz = d', \quad a''x + by + cz = d''.$$

Here it will be found that the values of y and z take the form $\frac{N}{0}$, and that of x takes the form $\frac{0}{0}$.

Moreover, if the values of *all* the unknown quantities take the form $\frac{0}{0}$, we cannot affirm certainly that the given equations are consistent, but not independent. For example, suppose the equations to be

$$ax + by + cz = d, \quad ax + by + cz = d', \quad ax + by + cz = d'';$$

here it will be found that the values of all the unknown quantities take the form $\frac{0}{0}$, but the equations themselves are obviously inconsistent, unless d, d', and d'' are all equal.

216. We may shew that if the numerators in the values of x, y, and z, all vanish, the denominator will also vanish, assuming that d, d', and d'' are not all zero.

For supposing these numerators to vanish we have

$$d(b'c'' - b''c') + d'(b''c - bc'') + d''(bc' - b'c) = 0,$$
$$d(c'a'' - c''a') + d'(c''a - ca'') + d''(ca' - c'a) = 0,$$
$$d(a'b'' - a''b') + d'(a''b - ab'') + d''(ab' - a'b) = 0.$$

Let us denote these relations for shortness thus,

$$Ad + Bd' + Cd'' = 0, \quad A'd + B'd' + C'd'' = 0, \quad A''d + B''d' + C''d'' = 0.$$

By Art. 213, since d, d' and d'' are not all zero the following relation must also hold,

$$A(B'C'' - B''C') + A'(B''C - BC'') + A''(BC' - B'C) = 0.$$

It will be found that

$$B'C'' - B''C' = a\{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)\};$$

and $B''C - BC''$ and $BC' - B'C$ may be similarly expressed, so that finally the relation becomes

$$\{a(b'c'' - b''c') + a'(b''c - bc'') + a''(bc' - b'c)\}^2 = 0.$$

This establishes the required result.

217. If we adopt the method of *indeterminate multipliers* given in Art. 207, it may happen that the two equations for finding l and m are *inconsistent;* we will examine this case. Suppose then $b''c' - b'c'' = 0$, so that these two equations are inconsistent (Art. 205). In this case the value of x may be obtained from the

second and third of the given equations, without using the first. For multiply the second of the given equations by c'', and the third by c', and subtract; thus the coefficients of y and z vanish, and we have an equation for determining x. For example, suppose the equations to be

$$4x + 2y + 3z = 19, \quad x + y + 4z = 9, \quad x + 2y + 8z = 15.$$

Here the value of x may be found from the second and third equations; we shall obtain $x = 3$; substitute this value of x in the three given equations; from the first we have $2y + 3z = 7$, and from the second or third $y + 4z = 6$; hence $y = 2$ and $z = 1$.

Again, the values of l and m may take the form $\frac{0}{0}$, so that the two equations for finding them are not independent; we will examine this case. Here we have $b''c' - b'c'' = 0$, $bc'' - b''c = 0$, and $b'c - bc' = 0$; these suppositions are equivalent to the two relations $\frac{b'}{b} = \frac{c'}{c}$ and $\frac{b''}{b} = \frac{c''}{c}$. Suppose then that $b' = pb$, and therefore $c' = pc$, and that $b'' = qb$, and therefore $c'' = qc$. Thus the given equations are

$$ax + by + cz = d, \quad a'x + pby + pcz = d', \quad a''x + qby + qcz = d'',$$

and they may be written thus,

$$ax + by + cz = d, \quad \frac{a'}{p}x + by + cz = \frac{d'}{p}, \quad \frac{a''}{q}x + by + cz = \frac{d''}{q}.$$

Here x may be found from any two of the equations; if we do *not* obtain the same value from each pair, the given equations are of course *inconsistent;* if we *do* obtain the same value for x, then the given equations are not independent; and in fact we shall in the latter case have only *one* equation for finding $by + cz$, so that the values of y and z are *indeterminate.* For example, suppose the given equations to be

$$x + 2y + 3z = 10, \quad 3x + 4y + 6z = 23, \quad x + 6y + 9z = 24.$$

From any two of these equations we can find $x = 3$; then substituting this value of x in any one of the three equations we obtain $2y + 3z = 7$, and thus y and z are *indeterminate.* If, however, the right-hand member of one of the given equations be altered, we shall not obtain the same value of x from each pair of the equations, and thus the given equations will be inconsistent.

218. In the preceding Articles we have supposed the given equations to be solved, and from the peculiar forms of the solutions have drawn inferences as to the nature of the given equations. We will now take one example of investigating a relation between the equations without solving them. Suppose, as before, that the equations are

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d'';$$

and let us find the relations which must exist among the known quantities, in order that the third equation may be deducible from the other two by multiplication by suitable quantities and addition. Suppose then that by multiplying the first equation by λ, and the second by μ, and adding, we obtain a result which is coincident with the third equation. Thus,

$$(\lambda a + \mu a')x + (\lambda b + \mu b')y + (\lambda c + \mu c')z = \lambda d + \mu d'$$

is equivalent to $\qquad a''x + b''y + c''z = d'';$

that is, we suppose that

$$\frac{\lambda a + \mu a'}{\lambda d + \mu d'} = \frac{a''}{d''}, \quad \frac{\lambda b + \mu b'}{\lambda d + \mu d'} = \frac{b''}{d''}, \quad \frac{\lambda c + \mu c'}{\lambda d + \mu d'} = \frac{c''}{d''}.$$

From the last three equations we deduce

$$\frac{\lambda}{\mu} = \frac{a''d' - a'd''}{ad'' - a''d}, \quad \frac{\lambda}{\mu} = \frac{b''d' - b'd''}{bd'' - b''d}, \quad \frac{\lambda}{\mu} = \frac{c''d' - c'd''}{cd'' - c''d}.$$

Hence in order that the third equation may be deducible from the other two in the manner proposed, we must have the following relations among the known quantities,

$$\frac{a''d' - a'd''}{ad'' - a''d} = \frac{b''d' - b'd''}{bd'' - b''d} = \frac{c''d' - c'd''}{cd'' - c''d}.$$

It is easy to shew that if these relations hold, the values of x, y, and z take the form $\frac{0}{0}$. For by multiplying up we obtain results which shew that the numerators in the values of x, y, and z vanish; and then by Art. 216 the denominator will also vanish.

MISCELLANEOUS EXAMPLES. CHAPTER XV.

1. Reduce $\dfrac{x^4+3x^3-7x^2-21x-36}{x^4+2x^3-10x^2-11x-12}$ to its simplest form.

2. Shew that

$$(a+b+c)(a^3+b^3+c^3+abc)-(ab+bc+ca)(a^2+b^2+c^2)=a^4+b^4+c^4.$$

3. If $t=\dfrac{2}{2-w}$, $w=\dfrac{2}{2-z}$, $z=\dfrac{2}{2-y}$, $y=\dfrac{2}{2-x}$, find the relation between t and x.

4. If $2s=a+b+c$, shew that

$$\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$$

5. Shew that the G.C.M. of two quantities is the L.C.M. of their common measures.

6. Solve the equation

$$(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10).$$

7. Solve the simultaneous equations

$$x+y+z=0, \qquad ax+by+cz=0,$$
$$bcx+cay+abz+(a-b)(b-c)(c-a)=0.$$

8. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, shew that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{2n+1}=\frac{1}{a^{2n+1}+b^{2n+1}+c^{2n+1}}.$$

9. A person leaves £12670 to be divided among his five children and three brothers, so that after the legacy duty has been paid, each child's share shall be twice as great as each brother's. The legacy duty on a child's share being one per cent. and on a brother's share three per cent., find what amounts they respectively receive.

10. Solve the equation

$$\frac{1}{x+6a}+\frac{2}{x-3a}+\frac{3}{x+2a}=\frac{6}{x+a}.$$

XVI. INVOLUTION.

219. If a quantity be continually multiplied by itself, it is said to be *involved* or raised, and the power to which it is raised is expressed by the number of times the quantity has been employed in the multiplication. The operation is called *Involution.*

Thus, as we have stated (Art. 16), $a \times a$ or a^2 is called the second power of a; $a \times a \times a$ or a^3 is called the third power of a; and so on.

220. If the quantity to be involved have a negative sign prefixed, the sign of the *even* powers will be positive, and the sign of the *odd* powers will be negative.

For, $-a \times -a = a^2$, $-a \times -a \times -a = a^2 \times -a = -a^3$,

$$-a \times -a \times -a \times -a = -a^3 \times -a = a^4,$$

and so on.

221. A *simple* quantity is raised to any power by multiplying the index of every factor in the quantity by the exponent of that power, and prefixing the proper sign determined by the preceding Article.

Thus a^m raised to the n^{th} power is a^{mn}; for if we form the product of n factors, each of which is a^m, the result by the rule of multiplication is a^{mn}. Also $(ab)^n = ab \times ab \times ab \ldots$ to n factors, that is, $a \times a \times a \ldots$ to n factors $\times b \times b \times b \ldots$ to n factors, that is, $a^n \times b^n$. Similarly, a^2b^3c raised to the fifth power is $a^{10}b^{15}c^5$. Also $-a^m$ raised to the n^{th} power is $\pm a^{mn}$, where the positive or negative sign is to be prefixed according as n is an even or odd number. Or as $-a^m = -1 \times a^m$, the n^{th} power of $-a^m$ may be written thus $(-1)^n \times a^{mn}$ or $(-1)^n a^{mn}$.

222. If the quantity which is to be involved be a fraction, both its numerator and denominator must be raised to the proposed power. (Art. 142.)

223. If the quantity which is to be involved be *compound*, the involution may either be represented by the proper index, or may actually be performed.

Let $a+b$ be the quantity which is to be raised to any power,

$$\begin{array}{lll}
a+b & a^2+2ab+b^2 & a^3+3a^2b+3ab^2+b^3 \\
a+b & a+b & a+b \\
\hline
a^2+ab & a^3+2a^2b+ab^2 & a^4+3a^3b+3a^2b^2+ab^3 \\
\quad +ab+b^2 & \quad +a^2b+2ab^2+b^3 & \quad +a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
a^2+2ab+b^2 & a^3+3a^2b+3ab^2+b^3 & a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{array}$$

Thus the square or second power of $a+b$ is $a^2+2ab+b^2$, the cube or third power of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the fourth power of $a+b$ is $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, and so on.

Similarly, the second, third, and fourth powers of $a-b$ will be found to be respectively $a^2-2ab+b^2$, $a^3-3a^2b+3ab^2-b^3$, and $a^4-4a^3b+6a^2b^2-4ab^3+b^4$; that is, wherever an *odd* power of b occurs, the negative sign is prefixed.

We shall hereafter give a theorem, called the Binomial Theorem, which will enable us to obtain any power of a binomial expression without the labour of actual multiplication.

224. It is obvious that the n^{th} power of a^m is the same as the m^{th} power of a^n, for each is a^{mn}; and thus we may arrive at the same result by different processes of involution. We may, for example, find the sixth power of $a+b$ by repeated multiplication by $a+b$; or we may first find the cube of $a+b$, and then the square of this result, since the square of $(a+b)^3$ is $(a+b)^6$; or we may first find the square of $a+b$ and then the cube of this result, since the cube of $(a+b)^2$ is $(a+b)^6$.

225. It may be shewn by actual multiplication that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac,$$

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd.$$

The following rule may be observed to hold good in the above and similar examples: *the square of any multinomial consists of* *the square of each term, together with twice the product of every pair of terms.*

Another form may also be given to these results,

$$(a+b+c)^2 = a^2 + 2a(b+c) + b^2 + 2bc + c^2,$$

$$(a+b+c+d)^2 = a^2 + 2a(b+c+d) + b^2 + 2b(c+d) + c^2 + 2cd + d^2.$$

The following rule may be observed to hold good in the above and similar examples: *the square of any multinomial consists of the square of each term, together with twice the product of each term by the sum of all the terms which follow it.*

These rules may be strictly demonstrated by the process of mathematical induction, which will be explained hereafter.

226. The following are additional examples in which we employ the first of the two rules given in the preceding Article.

$$(a-b+c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ac,$$

$$\begin{aligned}(1-2x+3x^2)^2 &= 1 + 4x^2 + 9x^4 - 4x - 12x^3 + 6x^2 \\ &= 1 - 4x + 10x^2 - 12x^3 + 9x^4,\end{aligned}$$

$$\begin{aligned}(1+x+x^2+x^3)^2 &= 1 + x^2 + x^4 + x^6 + 2x + 2x^2 + 2x^3 + 2x^3 + 2x^4 + 2x^5 \\ &= 1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6.\end{aligned}$$

227. The results given in Art. 55 for the cube of $a+b$, the cube of $a-b$, and the cube of $a+b+c$ should be carefully noticed. The following may also be verified.

$$\begin{aligned}(a+b+c+d)^3 &= a^3 + b^3 + c^3 + d^3 \\ &+ 3a^2(b+c+d) + 3b^2(a+c+d) + 3c^2(a+b+d) + 3d^2(a+b+c) \\ &+ 6bcd + 6acd + 6abd + 6abc.\end{aligned}$$

EXAMPLES OF INVOLUTION.

1. Find $(1+2x+3x^2)^2$.
2. Find $(1-x+x^2-x^3)^2$.
3. Find $(a+b-c)^3$.
4. Find $(1+2x+x^2)^3$.
5. Find $(1+3x+3x^2+x^3)^2 + (1-3x+3x^2-x^3)^2$.
6. Shew that $\dfrac{(27a^4-18a^2b^2-b^4)^2}{64a^2b^4} + \dfrac{(9a^2-b^2)^3(b^2-a^2)}{64a^2b^4} = b^2$.

7. Shew that $(ax^2+2bxy+cy^2)(aX^2+2bXY+cY^2)$
$$=\{axX+cyY+b(xY+yX)\}^2+(ac-b^2)(xY-yX)^2.$$

8. Shew that $(x^2+pxy+qy^2)(X^2+pXY+qY^2)$
$$=(xX+pyX+qyY)^2+p(xX+pyX+qyY)(xY-yX)+q(xY-yX)^2$$
and also
$$=(xX+pxY+qyY)^2+p(xX+pxY+qyY)(yX-xY)+q(xY-yX)^2.$$

9. Simplify
$$\frac{(1-10x^2+5x^4)(5-30x^2+5x^4)+(5x-10x^3+x^5)(20x-20x^3)}{(5x-10x^3+x^5)^2+(1-10x^2+5x^4)^2}.$$

10. Shew that $(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)$
$$=(ap-bq+cr-ds)^2+(aq+bp-cs-dr)^2$$
$$+(ar-bs-cp+dq)^2+(as+br+cq+dp)^2.$$

XVII. EVOLUTION.

228. Evolution, or the extraction of roots, is the method of determining a quantity, which when raised to a proposed power will produce a given quantity.

229. Since the n^{th} power of a^m is a^{mn}, an n^{th} root of a^{mn} must be a^m; that is, to extract any root of a simple quantity, we divide the index of that quantity by the index of the root required.

230. If the root to be extracted be expressed by an odd number, the sign of the root will be the same as the sign of the proposed quantity, as appears by Art. 220. Thus,
$$\sqrt[3]{(-a^3)}=-a.$$

231. If the root to be extracted be expressed by an even number, and the quantity proposed be positive, the root may be either positive or negative; because either a positive or negative quantity raised to an even power is positive by Art. 220. Thus,
$$\sqrt{(a^2)}=\pm a.$$

232. If the root proposed to be extracted be expressed by an even number and the sign of the proposed quantity be negative,

the root cannot be extracted; because no quantity raised to an even power can produce a negative result. Such roots are called *impossible.*

233. A root of a fraction may be found by taking that root of both the numerator and denominator. Thus,

$$\sqrt[3]{\left(\frac{a^3}{b^3}\right)}=\frac{a}{b} \text{ and } \sqrt[3]{\left(-\frac{a^3}{b^3}\right)}=-\frac{a}{b}.$$

234. We will now investigate the method of extracting the square root of a compound quantity.

Since the square root of $a^2+2ab+b^2$ is $a+b$, we may be led to a general rule for the extraction of the square root of an algebraical expression by observing in what manner a and b may be derived from $a^2+2ab+b^2$,

$$\begin{array}{rl} & a^2+2ab+b^2\ (a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline \end{array}$$

Arrange the terms according to the dimensions of one letter a, then the first term is a^2, and its square root is a, which is the first term of the required root. Subtract its square, that is a^2, from the whole expression, and bring down the remainder $2ab+b^2$. Divide $2ab$ by $2a$ and the quotient is b, which is the other term of the required root. Multiply the sum of twice the first term and the second term, that is $2a+b$, by the second term, that is b, and subtract the product, that is $2ab+b^2$, from the remainder. This finishes the operation in the present case. If there were more terms we should proceed with $a+b$ as we did formerly with a; its square, that is $a^2+2ab+b^2$, has already been subtracted from the proposed expression, so we should divide the remainder by the double of $a+b$ for a new term in the root, and then for a new subtrahend we should multiply this term by the sum of twice the former terms and this term. The process must be continued until the required root is found.

235. For example, required the square root of the expression $4x^4 - 12x^3 + 5x^2 + 6x + 1$.

$$
\begin{array}{rl}
 & 4x^4 - 12x^3 + 5x^2 + 6x + 1 \; (2x^2 - 3x - 1 \\
 & 4x^4 \\
\hline
4x^2 - 3x \,) & -12x^3 + 5x^2 + 6x + 1 \\
 & -12x^3 + 9x^2 \\
\hline
4x^2 - 6x - 1 \,) & -4x^2 + 6x + 1 \\
 & -4x^2 + 6x + 1 \\
\hline
\end{array}
$$

Here the square root of $4x^4$ is $2x^2$, which is the first term of the required root. Subtract its square, that is $4x^4$, from the whole expression, and the remainder is $-12x^3 + 5x^2 + 6x + 1$. Divide $-12x^3$ by twice $2x^2$, that is by $4x^2$, the quotient is $-3x$, which will be the next term of the required root; then multiply $4x^2 - 3x$ by $-3x$ and subtract, so that the remainder is $-4x^2 + 6x + 1$. Divide by twice the portion of the root already found, that is by $4x^2 - 6x$; this leads to -1; the product of $4x^2 - 6x - 1$ and -1 is $-4x^2 + 6x + 1$, and when this is subtracted there is no remainder, and thus the required root is $2x^2 - 3x - 1$.

For another example, required the square root of the expression $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$. The operation may be arranged as before,

$$
\begin{array}{rl}
 & x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \; (x^3 - 3ax^2 + 3a^2x - a^3 \\
 & x^6 \\
\hline
2x^3 - 3ax^2 \,) & -6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \\
 & -6ax^5 + 9a^2x^4 \\
\hline
2x^3 - 6ax^2 + 3a^2x \,) & 6a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6 \\
 & 6a^2x^4 - 18a^3x^3 + 9a^4x^2 \\
\hline
2x^3 - 6ax^2 + 6a^2x - a^3 \,) & -2a^3x^3 + 6a^4x^2 - 6a^5x + a^6 \\
 & -2a^3x^3 + 6a^4x^2 - 6a^5x + a^6 \\
\hline
\end{array}
$$

236. It has been already remarked, that all *even* roots admit of a double sign. (Art. 231.) Thus in the first example of Art. 235, the expression $2x^2 - 3x - 1$ is found to be a square root of the expression there given, and $-2x^2 + 3x + 1$ will also be a square root, as may be verified. In fact, the process commenced by the extraction of the square root of $4x^4$, and this might be taken as $2x^2$ or as $-2x^2$; if we adopt the latter and continue the operation in the same manner as before, we shall arrive at the result $-2x^2 + 3x + 1$. Similarly in the second example of Art. 235 we see that $-x^3 + 3ax^2 - 3a^2x + a^3$ will also be a square root.

237. The *fourth* root of an expression may be found by extracting the square root of the square root. Similarly the *eighth* root may be found by three successive extractions of the square root, and the *sixteenth* root by four successive extractions of the square root, and so on.

For example, required the fourth root of the expression

$$81x^4 - 432x^3 + 864x^2 - 768x + 256.$$

Proceed as in Art. 235, and we shall find that the square root of the proposed expression is $9x^2 - 24x + 16$; and the square root of this is $3x^2 - 4$, which is therefore the fourth root of the proposed expression.

238. The preceding investigation of the square root of an Algebraical expression will enable us to prove the rule for the extraction of the square root of a number, which is given in Arithmetic.

The square root of 100 is 10, of 10000 is 100, of 1000000 is 1000, and so on; hence it will follow that the square root of a number less than 100 must consist of only one figure, of a number between 100 and 10000 of two places of figures, of a number between 10000 and 1000000 of three places of figures, and so on. If then a point be placed over every *second* figure in any number beginning with the units, the number of points will shew the number of figures in the square root. Thus the square root of $4\dot{3}5\dot{6}$ consists of two figures, the square root of $\dot{6}1\dot{5}2\dot{4}$ of three figures, and so on.

239. Suppose the square root of 4356 required.

$$\begin{array}{r|l} & 4\dot{3}5\dot{6}\ (60+6 \\ & 3600 \\ 120+6) & 756 \\ & 756 \end{array}$$

Point the number according to the rule; thus it appears that the root consists of two places of figures. Let $a+b$ denote the root, where a is the value of the figure in the tens' place, and b the figure in the units' place. Then a must be the greatest multiple of ten which has its square less than 4300; this is found to be 60. Subtract a^2, that is the square of 60, from the given number, and the remainder is 756. Divide this remainder by $2a$, that is by 120, and the quotient is 6, which is the value of b. Then $(2a+b)\,b$, that is 126×6 or 756, is the quantity to be subtracted; and as there is now no remainder, we conclude that $60+6$ or 66 is the required square root.

It is stated above that a is the greatest multiple of ten which has its square less than 4300. For a evidently cannot be a *greater* multiple of ten. If possible suppose it to be some multiple of ten *less* than this, say x; then since x is in the tens' place, and b in the units' place, $x+b$ is less than a; therefore the square of $x+b$ is less than a^2, and consequently $x+b$ is less than the true root.

If the root consist of three places of figures, let a represent the hundreds and b the tens; then having obtained a and b as before, let the hundreds and tens together be considered as a new value of a, and find a new value of b for the units.

The cyphers may be omitted for the sake of brevity, and the following rule may be obtained from the process.

$$\begin{array}{r|l} & 4\dot{3}5\dot{6}\ (66 \\ & 36 \\ 126) & 756 \\ & 756 \end{array}$$

Point every *second* figure beginning with the units' place, and thus divide the whole number into several periods. Find the greatest number whose square is contained in the first period; this is the first figure in the root; subtract its square from the first period, and to the remainder bring down the next period. Divide this quantity, omitting the last figure, by twice the part of the root already found, and annex the result to the root and also to the divisor, then multiply the divisor as it now stands by the part of the root last obtained for the subtrahend. If there be more periods to be brought down the operation must be repeated.

240. Extract the square root of 611524; also of 10246401.

$$\begin{array}{rl} & 6\dot{1}1\dot{5}2\dot{4}\,(782 \\ & 49 \\ \hline 148) & 1215 \\ & 1184 \\ \hline 1562) & 3124 \\ & 3124 \\ \hline \end{array} \qquad \begin{array}{rl} & 1\dot{0}2\dot{4}6\dot{4}0\dot{1}\,(3201 \\ & 9 \\ \hline 62) & 124 \\ & 124 \\ \hline 6401) & 6401 \\ & 6401 \\ \hline \end{array}$$

In the second example the student should observe the occurrence of the cypher in the root.

241. The rule for extracting the square root of a *decimal* follows from the preceding rule. We must observe, however, that if any decimal be squared there will be an *even* number of decimal places in the result, and therefore there cannot be an exact square root of any decimal which in its simplest state has an *odd* number of decimal places.

The square root of 21·76 is one-tenth of the square root of $100 \times 21{\cdot}76$, that is of 2176. So also the square root of ·0361 is one-hundredth of that of $10000 \times {\cdot}0361$, that is of 361. Thus we may deduce this rule for extracting the square root of a decimal: put a point over every *second* figure beginning at the units' place, and continuing both to the right and left of it; then proceed as in the extraction of the square root of integers, and mark off as many decimal places in the result as the number of periods in the decimal part of the proposed number.

242. The student will probably soon acquire the conviction that many integers have strictly speaking no square root. Take for example the integer 7. It is obvious that 7 can have no integer for its square root; for the square of 2 is less than 7, and the square of 3 is greater than 7. Nor can 7 have any fraction as its square root. For take any fraction which is strictly a fraction and not an integer in a fractional form, and multiply this fraction by itself; then the product will be a fraction: this statement can be verified to any extent by trial, and may be demonstrated by the principles of Chapter LII. Thus 7 has no square root, either integral or fractional. In like manner no integer can have a square root unless that integer be one of the set of numbers 1, 4, 9, 16, ... which are the squares of the natural numbers 1, 2, 3, 4, ..., and are called *square numbers*.

243. In the extraction of the square root of an integer, if there is still a remainder after we have arrived at the figure in the units' place of the root, it indicates that the proposed number has not an exact square root. We may if we please proceed with the approximation to any desired extent by supposing a decimal point at the end of the proposed number, and annexing any even number of cyphers and continuing the operation. We thus obtain a decimal part to be added to the integral part already found.

It may be observed that in such a case by continuing the process we shall not arrive at figures in the root which *circulate* or *recur*. For a recurring decimal can be reduced to a fraction by a rule given in books on Arithmetic, and which will be demonstrated in Chapter XXXI; and therefore, if the square root were a recurring decimal it could be expressed as a fraction, and so there would be an exact square root, which is contrary to the supposition.

Similarly, if a decimal number has no exact square root, we may annex cyphers and proceed with the approximation to any desired extent.

244. The following is the extraction of the square root of twelve to seven decimal places.

$$
\begin{array}{rl}
 & 1\dot{2}\cdot 0\dot{0}0\dot{0}\ldots(3\cdot 4641016 \\
 & 9 \\
64) & 300 \\
 & 256 \\
686) & 4400 \\
 & 4116 \\
6924) & 28400 \\
 & 27696 \\
69281) & 70400 \\
 & 69281 \\
6928201) & 11190000 \\
 & 6928201 \\
69282026) & 426179900 \\
 & 415692156 \\
 & 10487744
\end{array}
$$

Thus we see in what sense we can be said to approximate to the square root of 12: the square of 3·4641016 is less than 12, and the square of 3·4641017 is greater than 12; the former square differs from 12 by the fraction which has 10487744 for numerator and 10^{14} for denominator.

245. It can be demonstrated by the principles of Chapter LII. that no fraction can have a square root unless the numerator and denominator are both square numbers when the fraction is in its lowest terms. But we may approximate to any desired extent to the square root of a fraction.

Suppose for example we require the square root of $\frac{3}{7}$. We might proceed thus: $\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}}$; then approximate to the square root of 3 and to the square root of 7, and divide the former result by the latter. But the following methods are preferable.

Convert $\frac{3}{7}$ into a decimal to any required degree of approximation; and approximate to the square root of this decimal.

Or proceed thus: $\sqrt{\frac{3}{7}} = \sqrt{\frac{3 \times 7}{7 \times 7}} = \frac{\sqrt{(3 \times 7)}}{\sqrt{(7 \times 7)}} = \frac{\sqrt{(21)}}{7}$; then approximate to the square root of 21 and divide the result by 7.

246. *When* n + 1 *figures of a square root have been obtained by the ordinary method,* n *more may be obtained by division only, supposing* 2n + 1 *to be the whole number.*

Let N represent the number whose square root is required, a the part of the root already obtained, x the part which remains to be found; then

$$\sqrt{N} = a + x,$$

so that

$$N = a^2 + 2ax + x^2,$$

therefore,

$$N - a^2 = 2ax + x^2,$$

and

$$\frac{N - a^2}{2a} = x + \frac{x^2}{2a}.$$

Thus $N - a^2$ divided by $2a$ will give the rest of the square root required, or x, *increased by* $\frac{x^2}{2a}$; and we shall shew that $\frac{x^2}{2a}$ is a *proper fraction*, so that by neglecting the remainder arising from the division we obtain the part required. For x by supposition contains n digits, so that x^2 cannot contain more than $2n$ digits; but a contains $2n + 1$ digits, and thus $\frac{x^2}{2a}$ is a proper fraction.

The above demonstration implies that N is an integer with an exact square root: but we may easily extend the result to other cases. For example, suppose we require the square root of 12 to 4 places of decimals. We have in fact to seek the square root of 1200000000, and to divide the result by 10000. Now the process in Art. 244 shews that $1200000000 - 1119 = (34641)^2$. Here N may stand for $1200000000 - 1119$; and then a may stand for 34600 and b for 41. Thus the demonstration assures us that we can obtain 41 by dividing 2840000 by 69200, that is by dividing 28400 by 692: and this coincides with the rule given in books on Arithmetic.

In like manner if we require the square root of 12 to 6 places of decimals, the last three figures, namely 101, can be obtained by dividing 704000 by 6928.

247. We will now investigate the method of extracting the cube root of a compound quantity.

The cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$, and to obtain this we may proceed thus: Arrange the terms according to the dimensions of one letter a, then the first term is a^3, and its cube root is a, which is the first term of the required root. Subtract its cube, that is a^3, from the whole expression, and bring down the remainder $3a^2b + 3ab^2 + b^3$. Divide the first term of the remainder by $3a^2$, and the quotient is b, which is the other term of the required root; then subtract $3a^2b + 3ab^2 + b^3$ from the remainder, and the whole cube of $a + b$ has been subtracted. This finishes the operation in the present case. If there were more terms we should proceed with $a + b$ as we formerly did with a; its cube, that is $a^3 + 3a^2b + 3ab^2 + b^3$, has already been subtracted from the proposed expression, so we should divide the remainder by $3(a+b)^2$ for a new term in the root; and so on.

$$
\begin{array}{l}
a^3 + 3a^2b + 3ab^2 + b^3 \ (a + b \\
a^3 \\
\hline
3a^2)\ 3a^2b + 3ab^2 + b^3 \\
\qquad\ \ 3a^2b + 3ab^2 + b^3 \\
\hline
\end{array}
$$

248. It will be convenient in extracting the cube root of more complex algebraical expressions, and of numbers, to arrange the process of the preceding Article in three columns, as follows:

$$
\begin{array}{lll}
3a+b & 3a^2 & a^3+3a^2b+3ab^2+b^3\ (a+b \\
 & (3a+b)\,b & a^3 \\
\hline
 & 3a^2+3ab+b^2 & \quad 3a^2b+3ab^2+b^3 \\
 & & \quad 3a^2b+3ab^2+b^3 \\
\hline
\end{array}
$$

Find the first term of the root, that is a; put a^3 under the given expression in the third column and subtract it. Put $3a$ in the first column, and $3a^2$ in the second column; divide $3a^2b$ by $3a^2$, and thus obtain the quotient b; add b to the quantity in the first column; multiply the expression now in the first column by b, and place the product in the second column and add it to the quantity already there; thus we obtain $3a^2+3ab+b^2$; multiply this by b and we obtain $3a^2b+3ab^2+b^3$, which is to be placed in the third column and subtracted. We have thus completed the process of subtracting $(a+b)^3$ from the original expression. If there were more terms the process would have to be continued.

249. In continuing the operation we must add such a quantity to the first column as to obtain there *three times the part of the root already found.* This is conveniently effected thus: we have already in the first column $3a+b$; place $2b$ under the b and add; so we obtain $3a+3b$, which is three times $a+b$, that is, three times the part of the root already found.

$$
\begin{array}{r}
3a+\ b \\
2b \\
\hline
3a+3b
\end{array}
$$

Moreover, we must add such a quantity to the second column as to obtain there *three times the square of the part of the root already found.* This is conveniently effected thus: we have already in the second column $(3a+b)\,b$, and below that $3a^2+3ab+b^2$; place b^2 below and *add the expressions in the three lines;* so we obtain $3a^2+6ab+3b^2$, which is three times

$$
\begin{array}{r}
(3a+b)\,b \\
3a^2+3ab+b^2 \\
b^2 \\
\hline
3a^2+6ab+3b^2
\end{array}
$$

$(a+b)^2$, that is, three times the square of the part of the root already found.

250. Example; extract the cube root of

$$8x^6 - 36cx^5 + 66c^2x^4 - 63c^3x^3 + 33c^4x^2 - 9c^5x + c^6.$$

$$\begin{array}{ll}
6x^2 - 3cx & 12x^4 \\
\quad - 6cx & - 3cx(6x^2 - 3cx) \\
\hline
6x^2 - 9cx + c^2 & 12x^4 - 18cx^3 + 9c^2x^2 \\
 & \qquad\qquad + 9c^2x^2 \\
\hline
 & 12x^4 - 36cx^3 + 27c^2x^2 \\
 & \qquad\qquad + c^2(6x^2 - 9cx + c^2) \\
\hline
 & 12x^4 - 36cx^3 + 33c^2x^2 - 9c^3x + c^4
\end{array}$$

$$\begin{array}{l}
8x^6 - 36cx^5 + 66c^2x^4 - 63c^3x^3 + 33c^4x^2 - 9c^5x + c^6 \;(\; 2x^2 - 3cx + c^2 \\
8x^6 \\
\hline
\quad - 36cx^5 + 66c^2x^4 - 63c^3x^3 + 33c^4x^2 - 9c^5x + c^6 \\
\quad - 36cx^5 + 54c^2x^4 - 27c^3x^3 \\
\hline
\qquad\qquad 12c^2x^4 - 36c^3x^3 + 33c^4x^2 - 9c^5x + c^6 \\
\qquad\qquad 12c^2x^4 - 36c^3x^3 + 33c^4x^2 - 9c^5x + c^6 \\
\hline
\end{array}$$

The cube root of $8x^6$ is $2x^2$ which will be the first term of the root; put $8x^6$ under the given expression in the third column and subtract it. Put three times $2x^2$ in the first column, and three times the square of $2x^2$ in the second column; that is, put $6x^2$ in the first column, and $12x^4$ in the second column. Divide $-36cx^5$ by $12x^4$, and thus obtain the quotient $-3cx$, which will be the second term of the root; place this term in the first column, and multiply the expression now in the first column, that is, $6x^2 - 3cx$ by $-3cx$; place the product under the quantity in the second column and add it to that quantity; thus we obtain $12x^4 - 18cx^3 + 9c^2x^2$; multiply this by $-3cx$, and place the product in the third column and subtract. Thus we have a remainder in the third column, and the part of the root already found is $2x^2 - 3cx$.

We must now adjust the first and second columns in the manner explained in Art. 249. We put twice $-3cx$, that is, $-6cx$, under the quantity in the first column, and add the two lines; so we obtain $6x^2-9cx$, which is three times the part of the root already found. We put the square of $-3cx$, that is, $9c^2x^2$, under the quantity in the second column, and add the last three lines in this column; so we obtain $12x^4-36cx^3+27c^2x^2$, which is three times the square of the part of the root already found.

Now divide the remainder in the third column by the expression just obtained, and we arrive at c^2 for the last term of the root; proceed as before and the operation closes.

251. The preceding investigation of the cube root of an Algebraical expression will enable us to deduce a rule for the extraction of the cube root of any number.

The cube root of 1000 is 10, of 1000000 is 100, and so on; hence it will follow that the cube root of a number less than 1000 must consist of only one figure, of a number between 1000 and 1000000 of two places of figures, and so on. If then a point be placed over every *third* figure in any number beginning with the units, the number of points will shew the number of figures in the cube root.

252. Suppose the cube root of 405224 required.

$210+4$	14700	$40\dot{5}22\dot{4}\,(\,70+4$
	856	343000
	15556	62224
		62224

Point the number according to the rule; thus it appears that the root consists of two places of figures. Let $a+b$ denote the root, where a is the value of the figure in the tens' place, and b the figure in the units' place. Then a must be the greatest multiple of ten which has its cube less than 405000; that is, a must be

70. Place the cube of 70, that is 343000, in the third column under the given number and subtract. Place three times 70, that is 210, in the first column, and three times the square of 70, that is 14700, in the second column. Divide the remainder in the third column by the number in the second column, that is, divide 62224 by 14700; we thus obtain 4, which is the value of *b*. Add 4 to the first column; multiply the sum thus formed by 4, that is, multiply 214 by 4; we thus obtain 856; place this in the second column and add it to the number already there. Thus we obtain 15556; multiply this by 4, place the product in the third column and subtract. The remainder is zero, and therefore 74 is the required root. The cyphers may be omitted for brevity, and the process will stand thus:

214	147	$40\dot{5}22\dot{4}$ (74
	856	343
	15556	62224
		62224

253. Example; extract the cube root of 12812904.

63	12	$1\dot{2}81\dot{2}90\dot{4}$ (234
6	189	8
694	1389	4812
	9	4167
	1587	645904
	2776	645904
	161476	

After obtaining the first two figures of the root 23, we adjust the first and second columns in the manner explained in Art. 249. We place twice 3 under the first column and add the two lines giving 69, and we place the square of 3 under the second column and add the last three lines giving 1587. Then the operation is continued as before. The cube root is 234.

254. Example; extract the cube root of 144182818617453.

152	75	144182818617453 (52437
4	304	125
1564	7804	19182
8	4	15608
15723	8112	3574818
6	6256	3269824
157297	817456	304994617
	16	247259907
	823728	5773471 0453
	47169	5773471 0453
	82419969	
	9	
	82467147	
	1101079	
	8247815779	

The cube root is 52437.

255. If the root have any number of decimal places the cube will have thrice as many; and therefore the number of decimal places in a decimal number, which is a perfect cube, and in its simplest state, will necessarily be a multiple of *three*, and the number of decimal places in the root will be a third of that number. Hence if the given cube number be a decimal, we place a point over the units' figure, and over every *third* figure to the right and left of it; then the number of points in the decimal part of the proposed number will indicate the number of decimal places in the cube root.

If a number have no exact cube root we may, as in the extraction of the square root, proceed with the approximation to any desired extent. See Art. 243.

256. Required the cube root of 1481·544.

3 1	3	$\dot{1}48\dot{1}\cdot54\dot{4}$ (11·4
2	3 1	1
3 3 4	3 3 1	4 8 1
	1	3 3 1
	3 6 3	1 5 0 5 4 4
	1 3 3 6	1 5 0 5 4 4
	3 7 6 3 6	

The cube root is 11·4.

257. *When* n + 2 *figures of a cube root have been obtained by the ordinary method,* n *more may be obtained by division only, supposing* 2n + 2 *to be the whole number.*

Let N represent the number whose cube root is required, a the part of the root already obtained, x the part which remains to be found; then

$$\sqrt[3]{N} = a + x,$$

so that $$N = a^3 + 3a^2x + 3ax^2 + x^3;$$

therefore, $$N - a^3 = 3a^2x + 3ax^2 + x^3,$$

and $$\frac{N - a^3}{3a^2} = x + \frac{x^2}{a} + \frac{x^3}{3a^2}.$$

Thus $N - a^3$ divided by $3a^2$ will give the rest of the cube root required, or x, *increased by* $\frac{x^2}{a} + \frac{x^3}{3a^2}$; and we shall shew that the latter expression is a *proper fraction,* so that by neglecting the remainder arising from the division, we obtain the part required. For by supposition, x is less than 10^n, and a is not less than 10^{2n+1}; thus $\frac{x^2}{a}$ is less than $\frac{10^{2n}}{10^{2n+1}}$, that is, less than $\frac{1}{10}$. And $\frac{x^3}{3a^2}$ is less than $\frac{10^{3n}}{3 \times 10^{4n+2}}$, that is, less than $\frac{1}{3 \times 10^{n+2}}$. Hence $\frac{x^2}{a} + \frac{x^3}{3a^2}$ is less than $\frac{1}{10} + \frac{1}{3 \times 10^{n+2}}$, and is thus less than unity.

Remarks similar to those in the latter part of Art. 246 apply here.

EXAMPLES OF EVOLUTION.

Extract the square roots of the expressions contained in the following examples from 1 to 15 inclusive:

1. $x^4 - 2x^3 + 3x^2 - 2x + 1$.
2. $x^4 - 4x^3 + 8x + 4$.
3. $4x^4 + 12x^3 + 5x^2 - 6x + 1$.
4. $4x^4 - 4x^3 + 5x^2 - 2x + 1$.
5. $4x^4 - 12ax^3 + 25a^2x^2 - 24a^3x + 16a^4$.
6. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.
7. $x^6 - 6ax^5 + 15a^2x^4 - 20a^3x^3 + 15a^4x^2 - 6a^5x + a^6$.
8. $(a-b)^4 - 2(a^2+b^2)(a-b)^2 + 2(a^4+b^4)$.
9. $4\{(a^2-b^2)cd + ab(c^2-d^2)\}^2 + \{(a^2-b^2)(c^2-d^2) - 4abcd\}^2$.
10. $a^4 + b^4 + c^4 + d^4 - 2a^2(b^2+d^2) - 2b^2(c^2-d^2) + 2c^2(a^2-d^2)$.
11. $\left(x+\frac{1}{x}\right)^2 - 4\left(x-\frac{1}{x}\right)$.
12. $x^4 - x^3 + \frac{x^2}{4} + 4x - 2 + \frac{4}{x^2}$.
13. $\frac{a^4}{4} + \frac{a^3}{x} + \frac{a^2}{x^2} - ax - 2 + \frac{x^2}{a^2}$.
14. $a^4 + 2(2b-c)a^3 + (4b^2 - 4bc + 3c^2)a^2 + 2c^2(2b-c)a + c^4$.
15. $(a-2b)^2x^4 - 2a(a-2b)x^3 + (a^2 + 4ab - 6a - 8b^2 + 12b)x^2$
$- (4ab - 6a)x + 4b^2 - 12b + 9$.
16. Find the square root of the sum of the squares of ·2, ·4, ·6, ·86.

Extract the cube root of the expressions and numbers in the following examples from 17 to 23 inclusive:

17. $x^6 - 9x^5 + 33x^4 - 63x^3 + 66x^2 - 36x + 8$.
18. $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6$.
19. $8x^6 - 36cx^5 + 102c^2x^4 - 171c^3x^3 + 204c^4x^2 - 144c^5x + 64c^6$.
20. 167·284151.
21. 731189187729.
22. 10970·645048.
23. 1371742108367626890260631.
24. Extract the *fourth* root of $\left(x^2+\frac{1}{x^2}\right)^2 - 4\left(x+\frac{1}{x}\right)^2 + 12$.

25. If a number contain n digits, its square root contains $\frac{1}{4}\{2n+1-(-1)^n\}$ digits.

26. Shew that the following expression is an exact square:

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy).$$

XVIII. THEORY OF INDICES.

258. We have defined a^m, where m is a positive integer, as the product of m factors each equal to a, and we have shewn that $a^m \times a^n = a^{m+n}$, and that $\frac{a^m}{a^n} = a^{m-n}$ or $\frac{1}{a^{n-m}}$ according as m is greater or less than n. Hitherto then an exponent has always been a *positive integer;* it is however found convenient to use exponents which are *not* positive integers, and we shall now explain the meaning of such exponents.

259. As fractional indices and negative indices have not yet been defined, we are at liberty to give what definitions we please to them; and it is found convenient to give such definitions to them as will make the important relation $a^m \times a^n = a^{m+n}$ *always true, whatever* m *and* n *may be.*

For example; required the meaning of $a^{\frac{1}{2}}$.

By supposition we are to have $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^1 = a$. Thus $a^{\frac{1}{2}}$ must be such a number that if it be multiplied by itself the result is a; and the *square root* of a is by definition such a number; therefore $a^{\frac{1}{2}}$ must be equivalent to the square root of a, that is, $a^{\frac{1}{2}} = \sqrt{a}$.

Again; required the meaning of $a^{\frac{1}{3}}$.

By supposition we are to have $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1 = a$.

Hence, as before, $a^{\frac{1}{3}}$ must be equivalent to the cube root of a, that is $a^{\frac{1}{3}} = \sqrt[3]{a}$.

Again; required the meaning of $a^{\frac{3}{4}}$.

By supposition, $a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} \times a^{\frac{3}{4}} = a^3$;

therefore $$a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

These examples would enable the student to understand what is meant by any fractional exponent; but we will give the definition in general symbols in the next two Articles.

260. *Required the meaning of* $a^{\frac{1}{n}}$ *where* n *is any positive whole number.*

By supposition,

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \ldots \text{ to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots \text{ to } n \text{ terms}} = a^1 = a;$$

therefore $a^{\frac{1}{n}}$ must be equivalent to the n^{th} root of a,

that is, $$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

261. *Required the meaning of* $a^{\frac{m}{n}}$ *where* m *and* n *are any positive whole numbers.*

By supposition,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \ldots \text{ to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\ldots \text{ to } n \text{ terms}} = a^m;$$

therefore $a^{\frac{m}{n}}$ must be equivalent to the n^{th} root of a^m,

that is, $$a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

Hence $a^{\frac{m}{n}}$ means the n^{th} root of the m^{th} power of a; that is, in a fractional index the numerator denotes a power and the denominator a root.

262. We have thus assigned a meaning to any positive index, whether whole or fractional; it remains to assign a meaning to negative indices.

For example, required the meaning of a^{-2}.

By supposition, $a^3 \times a^{-2} = a^{3-2} = a^1 = a$,

therefore $$a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

We will now give the definition in general symbols.

263. *Required the meaning of* a^{-n}; *where* n *is any positive number whole or fractional.*

By supposition, whatever m may be, we are to have

$$a^m \times a^{-n} = a^{m-n}.$$

Now we may suppose m positive and greater than n, and then, by what has gone before, we have

$$a^{m-n} \times a^n = a^m; \qquad \text{and therefore} \quad a^{m-n} = \frac{a^m}{a^n}.$$

Therefore $$a^m \times a^{-n} = \frac{a^m}{a^n};$$

therefore $$a^{-n} = \frac{1}{a^n}.$$

In order to express this in words we will define the word *reciprocal*. One quantity is said to be the *reciprocal* of another when the product of the two is equal to unity; thus, for example, x is the *reciprocal* of $\frac{1}{x}$.

Hence a^{-n} is the reciprocal of a^n; or we may put this result symbolically in any of the following ways,

$$a^{-n} = \frac{1}{a^n}, \qquad a^n = \frac{1}{a^{-n}}, \qquad a^n \times a^{-n} = 1.$$

264. It will follow from the meaning which has been given to a negative index that $a^m \div a^n = a^{m-n}$ when m is less than n, as well as when m is greater than n. For suppose m less than n; we have

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{-(n-m)} = a^{m-n}.$$

Suppose $m = n$; then $a^m \div a^n$ is obviously $= 1$; and $a^{m-n} = a^0$. The last symbol has not hitherto received a meaning, so that we are at liberty to give it the meaning which naturally presents itself; hence we may say that $a^0 = 1$.

265. Thus, for example, according to these definitions,

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}, \qquad a^{\frac{3}{2}} = \sqrt{a^3}, \qquad a^{\frac{4}{2}} = \sqrt{a^4} = a^2,$$

$$a^{-3} = \frac{1}{a^3}, \qquad a^{-\frac{1}{2}} = \frac{1}{a^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}, \qquad a^{-\frac{4}{2}} = \frac{1}{a^{\frac{4}{2}}} = \frac{1}{a^2}.$$

Thus it will appear that it is not *absolutely necessary* to introduce fractional and negative exponents into Algebra, since they merely supply us with a new notation for quantities which we had already the means of representing. It is, as we have said, a *convenient* notation, which the student will learn to appreciate as he proceeds.

The notation which we have explained will now be used in establishing some propositions relating to roots and powers.

266. To shew that $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$.

Let $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = x$; therefore

$$x^n = \left(a^{\frac{1}{n}} \times b^{\frac{1}{n}}\right)^n = \left(a^{\frac{1}{n}}\right)^n \times \left(b^{\frac{1}{n}}\right)^n, \text{ (by Art. 41)}, = a \times b.$$

Thus $x^n = ab$, therefore $x = (ab)^{\frac{1}{n}}$, which was to be proved.

In the same manner we can prove that

$$a^{\frac{1}{n}} \div b^{\frac{1}{n}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}.$$

267. As an example of the preceding proposition we have $\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$. Now, as we have seen in Art. 236, a square root admits of a *double* sign; hence strictly speaking our result should be stated thus: the product of one of the square roots of a into one of the square roots of b is equal to one of the square roots of ab. A similar remark applies to other propositions of the present Chapter. In the higher parts of mathematics the matter here noticed is discussed in more detail: see *Theory of Equations*, Chapter XI.

268. Hence $a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (ab)^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$.

And by proceeding in this way we can prove that

$$a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} \times \ldots\ldots \times k^{\frac{1}{n}} = (abc \ldots . k)^{\frac{1}{n}}.$$

Suppose now that there are m of these quantities $a, b, c, \ldots k$, and that each of them is equal to a; then we obtain

$$\left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}.$$

But $\left(a^m\right)^{\frac{1}{n}}$ is, by Arts. 260, 261, $a^{\frac{m}{n}}$; thus

$$\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}.$$

Hence comparing this with Art. 261 we see that the n^{th} root of the m^{th} power of a is equivalent to the m^{th} power of the n^{th} root of a.

269. To shew that $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}}$.

Let $x = \left(a^{\frac{1}{m}}\right)^{\frac{1}{n}}$; therefore $x^n = a^{\frac{1}{m}}$; therefore $x^{mn} = a$; therefore $x = a^{\frac{1}{mn}}$. Thus $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}}$, which was to be proved.

270. To shew that $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$.

Let $x = a^{\frac{m}{n}}$; therefore $x^n = a^m$; therefore $x^{np} = a^{mp}$; therefore $x = a^{\frac{mp}{np}}$. Thus $a^{\frac{m}{n}} = a^{\frac{mp}{np}}$, which was to be proved.

271. The student may infer from what we have said in Art. 265, that the propositions just established may also be established *without using fractional exponents.* Take for example that in Art. 266; here we have to shew that

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{(ab)}.$$

Proceed as before; let $x = \sqrt[n]{a} \times \sqrt[n]{b}$; therefore

$$x^n = (\sqrt[n]{a} \times \sqrt[n]{b})^n = (\sqrt[n]{a})^n \times (\sqrt[n]{b})^n, \text{ (by Art. 41)}, = a \times b.$$

Thus $x^n = ab$, therefore $x = \sqrt[n]{(ab)}$, which was to be proved.

272. We have been led to the definitions of Arts. 260...265 as consequences of considering the relations $a^m \times a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$ to be universally true, whatever m and n may be; we shall now proceed to shew conversely that if we adopt these definitions the relations $a^m \times a^n = a^{m+n}$ and $(a^m)^n = a^{mn}$ are universally true, whatever m and n may be.

273. To shew that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$.

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}, \text{ by Art. 270},$$

$$= \left(a^{ps}\right)^{\frac{1}{qs}} \times \left(a^{qr}\right)^{\frac{1}{qs}}, \text{ by definition},$$

$$= \left(a^{ps} \times a^{qr}\right)^{\frac{1}{qs}}, \text{ by Art. 266},$$

$$= \left(a^{ps+qr}\right)^{\frac{1}{qs}} = a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

274. In the same way we can shew that

$$a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}.$$

275. Thus the relation $a^m \times a^n = a^{m+n}$ is shewn to be true when m and n are positive fractions, so that it is true when m and n are any positive quantities. It remains to shew that it is also true when either of them is a negative quantity, and when both are negative quantities.

(1) Suppose *one* to be a negative quantity, say n; let

$$n = -\nu.$$

Then $a^m \times a^n = a^m \times a^{-\nu} = a^m \times \dfrac{1}{a^\nu} = \dfrac{a^m}{a^\nu} = a^{m-\nu}$, (by Art. 274),

$$= a^{m+n}.$$

(2) Suppose *both* to be negative quantities; let

$$m = -\mu \text{ and } n = -\nu.$$

Then

$$a^m \times a^n = a^{-\mu} \times a^{-\nu} = \frac{1}{a^\mu} \times \frac{1}{a^\nu} = \frac{1}{a^\mu \times a^\nu} = \frac{1}{a^{\mu+\nu}}, \text{ (by Art. 273)},$$

$$= a^{-\mu-\nu} = a^{m+n}.$$

276. Similarly $a^m \times a^n \times a^p = a^{m+n} \times a^p = a^{m+n+p}$; and so on.

Thus if we suppose there to be r quantities $m, n, p, \ldots$, and that each of the others is equal to m, we obtain

$$(a^m)^r = a^{mr},$$

whatever m may be.

277. To shew that $\left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let $x = \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}}$; therefore $x^s = \left(a^{\frac{p}{q}}\right)^r = a^{\frac{pr}{q}}$, by Art. 276; therefore $x^{qs} = a^{pr}$; therefore $x = a^{\frac{pr}{qs}}$, which was to be proved.

278. To shew that $(a^m)^n = a^{mn}$ universally.

By the preceding Article this is true when m and n are any positive quantities; it remains to shew that it is true when either of them is a negative quantity, and when both are negative quantities.

(1) Suppose n to be a negative quantity, and let it $= -\nu$.

Then $$(a^m)^n = (a^m)^{-\nu} = \frac{1}{(a^m)^\nu} = \frac{1}{a^{m\nu}} = a^{-m\nu} = a^{mn},$$

(2) Suppose m to be a negative quantity, and let it $= -\mu$.

Then $$(a^m)^n = (a^{-\mu})^n = \left(\frac{1}{a^\mu}\right)^n = \frac{1}{a^{\mu n}} = a^{-\mu n} = a^{mn}.$$

(3) Suppose both m and n to be negative quantities; let

$$m = -\mu \text{ and } n = -\nu.$$

Then $$(a^m)^n = (a^{-\mu})^{-\nu} = \frac{1}{(a^{-\mu})^\nu} = \frac{1}{a^{-\mu\nu}} = a^{\mu\nu} = a^{mn}.$$

EXAMPLES OF INDICES.

1. Simplify $(x^{\frac{2}{3}} \times x^{\frac{4}{7}})^{\frac{14}{13}}$.

2. Find the product of $a^{\frac{1}{2}}$, $a^{-\frac{1}{3}}$, $a^{-\frac{1}{4}}$, and $a^{-\frac{1}{6}}$.

3. Find the product of $\left(\frac{ay}{x}\right)^{\frac{1}{2}}$, $\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}$ and $\left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

4. Simplify the product of

$$a^{\frac{1}{3}},\ a^{-\frac{3}{4}},\ \sqrt[3]{a^4},\ a^{\frac{1}{12}},\ \sqrt[8]{a^{\frac{25}{3}}},\ \text{and}\ (a^{-\frac{7}{4}})^{\frac{7}{6}}.$$

5. Simplify $\dfrac{\left\{\left(a^m\right)^{\frac{1}{r}}\left(a^q\right)^{\frac{1}{n}}\right\}^{nr}}{\left\{\sqrt[q]{b^n}\left(\sqrt[m]{b}\right)^r\right\}^{mq}} \div \left\{\left(\frac{a}{b}\right)^q\right\}^r$.

6. Multiply $a^{\frac{1}{2}} + b^{\frac{1}{2}} + a^{-\frac{1}{2}}b$ by $ab^{-\frac{1}{2}} - a^{\frac{1}{2}} + b^{\frac{1}{2}}$.

7. Multiply $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

8. Multiply $a^{\frac{7}{2}} - a^3 + a^{\frac{5}{2}} - a^2 + a^{\frac{3}{2}} - a + a^{\frac{1}{2}} - 1$ by $a^{\frac{1}{2}} + 1$.

9. Multiply $a^{\frac{2}{3}} - a^{\frac{1}{3}} + 1 - a^{-\frac{1}{3}} + a^{-\frac{2}{3}}$ by $a^{\frac{1}{3}} + 1 + a^{-\frac{1}{3}}$.

10. Multiply $-3a^{-5} + 2a^{-4}b^{-1}$ by $-2a^{-3} - 3a^{-4}b$.

11. Divide $x^{\frac{3}{2}} - xy^{\frac{1}{2}} + x^{\frac{1}{2}}y - y^{\frac{3}{2}}$ by $x^{\frac{1}{2}} - y^{\frac{1}{2}}$.

12. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.

13. Divide $a^{\frac{3n}{2}} - a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}} - a^{-\frac{n}{2}}$.

14. Divide $2x^5y^{-3} - 5x^4y^{-2} + 7x^3y^{-1} - 5x^2 + 2xy$
by $x^3y^{-3} - x^2y^{-2} + xy^{-1}$.

15. Divide $a^{\frac{5}{2}} - a^{\frac{3}{2}}b + ab^{\frac{3}{2}} - 2a^{\frac{1}{2}}b^2 + b^{\frac{5}{2}}$ by $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$.

16. Simplify $\dfrac{a^{\frac{3}{2}} - ax^{\frac{1}{2}} + a^{\frac{1}{2}}x - x^{\frac{3}{2}}}{a^{\frac{5}{2}} - a^2x^{\frac{1}{2}} + 3a^{\frac{3}{2}}x - 3ax^{\frac{3}{2}} + a^{\frac{1}{2}}x^2 - x^{\frac{5}{2}}}$.

17. Extract the square root of $\frac{y^2}{x} + \frac{x^2}{4y} + \frac{2y^{\frac{3}{2}} - x^{\frac{3}{2}}}{(xy)^{\frac{1}{4}}}$.

18. Extract the square root of

$$4a - 12a^{\frac{1}{2}}b^{\frac{1}{3}} + 9b^{\frac{2}{3}} + 16a^{\frac{1}{2}}c^{\frac{1}{4}} - 24b^{\frac{1}{3}}c^{\frac{1}{4}} + 16c^{\frac{1}{2}}.$$

19. Extract the square root of

$$256x^{\frac{4}{3}} - 512x + 640x^{\frac{2}{3}} - 512x^{\frac{1}{3}} + 304 - 128x^{-\frac{1}{3}} + 40x^{-\frac{2}{3}} - 8x^{-1} + x^{-\frac{4}{3}}.$$

20. If $a^b = b^a$, shew that $\left(\frac{a}{b}\right)^{\frac{a}{b}} = a^{\frac{a}{b}-1}$; and if $a = 2b$, shew that $b = 2$.

XIX. SURDS.

279. When a root of an Algebraical quantity which is required, cannot be exactly obtained, it is called an *irrational* or *surd* quantity. Thus $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ is called a surd. But $\sqrt[3]{a^6}$ or $a^{\frac{6}{3}}$, though apparently in a surd form, can be expressed by a^2, and so is not called a surd.

The rules for operations with surds follow from the propositions established in the preceding Chapter, as will now be seen.

280. *A rational quantity may be expressed in the form of a given surd, by raising it to the power whose root the surd expresses, and affixing the radical sign.*

Thus $a = \sqrt{a^2} = \sqrt[3]{a^3}$, &c.; and $a + x = (a + x)^{\frac{n}{n}}$. In the same manner the form of any surd may be altered; thus

$$(a + x)^{\frac{1}{2}} = (a + x)^{\frac{2}{4}} = (a + x)^{\frac{3}{6}} \ldots\ldots$$

The quantities are here raised to certain powers, and the roots of those powers are again taken, so that the values of the quantities are not changed.

281. *The coefficient of a surd may be introduced under the radical sign, by first reducing it to the form of the surd and then multiplying according to Art.* 271.

For example,

$$a\sqrt{x}=\sqrt{a^2}\times\sqrt{x}=\sqrt{(a^2x)}\,;\qquad ay^{\frac{3}{2}}=(a^2y^3)^{\frac{1}{2}}\,;$$

$$x\sqrt{(2a-x)}=\sqrt{(2ax^2-x^3)}\,;\qquad a\times(a-x)^{\frac{2}{3}}=\{a^3(a-x)^2\}^{\frac{1}{3}}\,;$$

$$4\sqrt{2}=\sqrt{(16\times 2)}=\sqrt{32}.$$

282. *Conversely, any quantity may be made the coefficient of a surd, if every part under the sign be divided by the quantity raised to the power whose root the sign expresses.*

Thus $\sqrt{(a^2-ax)}=a^{\frac{1}{2}}\times\sqrt{(a-x)}\,;\quad \sqrt{(a^3-a^2x)}=a\sqrt{(a-x)}\,;$

$$(a^2-x^2)^{\frac{1}{n}}=a^{\frac{2}{n}}\times\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{n}};\qquad \sqrt{60}=\sqrt{(4\times 15)}=2\sqrt{15}\,;$$

$$\left(\frac{1}{b^2}-\frac{1}{x^2}\right)^{\frac{1}{2}}=\frac{1}{b}\left(1-\frac{b^2}{x^2}\right)^{\frac{1}{2}}=\frac{1}{x}\left(\frac{x^2}{b^2}-1\right)^{\frac{1}{2}}=\frac{(x^2-b^2)^{\frac{1}{2}}}{xb}.$$

283. *When surds have the same irrational part, their sum or difference is found by affixing to that irrational part the sum or difference of their coefficients.*

Thus $a\sqrt{x}\pm b\sqrt{x}=(a\pm b)\sqrt{x}\,;$

$$\sqrt{300}\pm 5\sqrt{3}=10\sqrt{3}\pm 5\sqrt{3}=15\sqrt{3}\text{ or }5\sqrt{3}\,;$$

$$\sqrt{(3a^2b)}+\sqrt{(3x^2b)}=a\sqrt{(3b)}+x\sqrt{(3b)}=(a+x)\sqrt{(3b)}.$$

284. *If two surds have the same index, their product is found by taking the product of the quantities under the signs and retaining the common index.*

Thus $a^{\frac{1}{n}}\times b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}$, (Art. 266); $\quad\sqrt{2}\times\sqrt{3}=\sqrt{6}\,;$

$$(a+b)^{\frac{1}{3}}\times(a-b)^{\frac{1}{3}}=(a^2-b^2)^{\frac{1}{3}}.$$

285. *If the surds have coefficients, the product of these coefficients must be prefixed.*

Thus $a\sqrt{x}\times b\sqrt{y}=ab\sqrt{(xy)}\,;\quad 3\sqrt{8}\times 5\sqrt{2}=15\sqrt{16}=60.$

286. *If the indices of two surds have a common denominator, let the quantities be raised to the powers expressed by their respective numerators, and their product may be found as before.*

Thus $$2^{\frac{3}{2}} \times 3^{\frac{1}{2}} = 8^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (24)^{\frac{1}{2}};$$
$$(a+x)^{\frac{1}{2}} \times (a-x)^{\frac{3}{2}} = \{(a+x)(a-x)^3\}^{\frac{1}{2}}.$$

287. *If the indices have not a common denominator, they may be transformed to others of the same value with a common denominator, and their product found as in Art.* 286.

Thus $$(a^2-x^2)^{\frac{1}{4}} \times (a-x)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{4}} \times (a-x)^{\frac{2}{4}} = \{(a^2-x^2)(a-x)^2\}^{\frac{1}{4}};$$
$$2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = 8^{\frac{1}{6}} \times 9^{\frac{1}{6}} = (72)^{\frac{1}{6}}.$$

288. *If two surds have the same rational quantity under the radical signs, their product is found by making the sum of the indices the index of that quantity.*

Thus $$a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{1}{n}+\frac{1}{m}}, \text{ (Art. 273)};$$
$$\sqrt{2} \times \sqrt[3]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{3}} = 2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}}.$$

289. *If the indices of two surds have a common denominator, the quotient of one surd divided by the other is obtained by raising them respectively to the powers expressed by the numerators of their indices, and extracting that root of the quotient which is expressed by the common denominator.*

Thus, $$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\frac{a}{b}\right)^{\frac{1}{n}}, \text{ (Art. 266)}; \quad \frac{a^{\frac{m}{n}}}{b^{\frac{p}{n}}} = \left(\frac{a^m}{b^p}\right)^{\frac{1}{n}};$$

$$4^{\frac{1}{2}} \div 2^{\frac{3}{2}} = \left(\frac{4}{2^3}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}; \quad \left(\frac{p}{q}\right)^{\frac{1}{m}} \div \left(\frac{r}{s}\right)^{\frac{2}{m}} = \left(\frac{ps^2}{qr^2}\right)^{\frac{1}{m}}.$$

290. *If the indices have not a common denominator, reduce them to others of the same value with a common denominator, and proceed as before.*

Thus $$(a^2-x^2)^{\frac{1}{2}} \div (a^3-x^3)^{\frac{1}{3}} = (a^2-x^2)^{\frac{3}{6}} \div (a^3-x^3)^{\frac{2}{6}} = \left\{\frac{(a^2-x^2)^3}{(a^3-x^3)^2}\right\}^{\frac{1}{6}}.$$

291. *If the surds have the same rational quantity under the radical signs, their quotient is obtained by making the difference of the indices the index of that quantity.*

Thus, $$a^{\frac{1}{m}} \div a^{\frac{1}{n}} = a^{\frac{1}{m}-\frac{1}{n}}, \text{ (Art. 274)};$$
$$\sqrt{2} \div \sqrt[3]{2} = 2^{\frac{1}{2}} \div 2^{\frac{1}{3}} = 2^{\frac{1}{2}-\frac{1}{3}} = 2^{\frac{1}{6}}.$$

292. It is sometimes useful to put a fraction which has a simple surd in its denominator into another form, by multiplying both numerator and denominator by a factor which will render the denominator *rational*. Thus, for example,

$$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

If we wish to calculate numerically the approximate value of $\frac{2}{\sqrt{3}}$ it will be found less laborious to use the equivalent form $\frac{2\sqrt{3}}{3}$. Similarly, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$.

293. It is also easy to rationalise the denominator of a fraction when that denominator consists of *two* quadratic surds.

For $$\frac{a}{\sqrt{b} \pm \sqrt{c}} = \frac{a(\sqrt{b} \mp \sqrt{c})}{(\sqrt{b} \pm \sqrt{c})(\sqrt{b} \mp \sqrt{c})} = \frac{a(\sqrt{b} \mp \sqrt{c})}{b - c}.$$

So also $$\frac{a}{b \pm \sqrt{c}} = \frac{a(b \mp \sqrt{c})}{(b \pm \sqrt{c})(b \mp \sqrt{c})} = \frac{a(b \mp \sqrt{c})}{b^2 - c}.$$

Similarly $$\frac{3+\sqrt{5}}{3-\sqrt{5}} = \frac{(3+\sqrt{5})(3+\sqrt{5})}{(3-\sqrt{5})(3-\sqrt{5})} = \frac{14+6\sqrt{5}}{9-5} = \frac{7+3\sqrt{5}}{2}.$$

294. By two operations we may rationalise the denominator of a fraction when that denominator consists of *three* quadratic surds. For suppose the denominator to be $\sqrt{a}+\sqrt{b}+\sqrt{c}$; first multiply both numerator and denominator by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, thus the denominator becomes $a+b-c+2\sqrt{(ab)}$; then multiply both numerator and denominator by $a+b-c-2\sqrt{(ab)}$, and we obtain a *rational* denominator, namely $(a+b-c)^2-4ab$, that is, $a^2+b^2+c^2-2ab-2bc-2ca$.

295. *A factor may be found which will rationalise any binomial.*

(1) Suppose the binomial $a^{\frac{1}{p}} + b^{\frac{1}{q}}$. Put $x = a^{\frac{1}{p}}$, $y = b^{\frac{1}{q}}$; let n be the least common multiple of p and q; then x^n and y^n are both rational. Now

$$(x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots \pm y^{n-1}) = x^n \pm y^n,$$

where the upper or lower sign must be taken according as n is odd or even. Thus

$$x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots\ldots \pm y^{n-1}$$

is a factor which will rationalise $x + y$.

(2) Suppose the binomial $a^{\frac{1}{p}} - b^{\frac{1}{q}}$. Take x, y, and n as before. Now

$$(x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}) = x^n - y^n.$$

Thus $$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}$$

is a factor which will rationalise $x - y$.

Take, for example, $a^{\frac{1}{2}} + b^{\frac{1}{3}}$; here $n = 6$. Thus we have as a rationalising factor

$$x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5,$$

that is, $$a^{\frac{5}{2}} - a^{\frac{4}{2}}b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - a^{\frac{2}{2}}b^{\frac{3}{3}} + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}},$$

that is, $$a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}.$$

The rational product is $x^6 - y^6$, that is, $a^{\frac{6}{2}} - b^{\frac{6}{3}}$, that is, $a^3 - b^2$.

296. *The square root of a rational quantity cannot be partly rational and partly a quadratic surd.*

If possible let $\sqrt{n} = a + \sqrt{m}$; then by squaring these equal quantities we have $n = a^2 + 2a\sqrt{m} + m$; thus $2a\sqrt{m} = n - a^2 - m$, and $\sqrt{m} = \dfrac{n - a^2 - m}{2a}$, a rational quantity, which is contrary to the supposition. See Art. 242.

297. *If two quadratic surds cannot be reduced to others which have the same irrational part, their product is irrational.*

Let $\sqrt{x}$ and $\sqrt{y}$ be the two quadratic surds, and if possible let $\sqrt{(xy)} = rx$, where r is a whole number or a fraction. Then $xy = r^2x^2$, and $y = r^2x$, therefore $\sqrt{y} = r\sqrt{x}$, that is, $\sqrt{y}$ and $\sqrt{x}$ may be so reduced as to have the same irrational part, which is contrary to the supposition.

298. *One quadratic surd cannot be made up of two others which have not the same irrational part.*

If possible let $\sqrt{x} = \sqrt{m} + \sqrt{n}$; then, by squaring, we have $x = m + n + 2\sqrt{(mn)}$, and $\sqrt{(mn)} = \frac{1}{2}(x - m - n)$, a rational quantity, which is absurd. See Art. 242.

299. *In any equation* $x + \sqrt{y} = a + \sqrt{b}$ *which involves rational quantities and quadratic surds, the rational parts on each side are equal, and also the irrational parts.*

For if x be not equal to a, suppose $x = a + m$; then

$$a + m + \sqrt{y} = a + \sqrt{b},$$

so that $m + \sqrt{y} = \sqrt{b}$; thus $\sqrt{b}$ is partly rational and partly a quadratic surd, which is impossible by Art. 296. Therefore $x = a$, and consequently $\sqrt{y} = \sqrt{b}$.

300. If $\sqrt{(a + \sqrt{b})} = x + \sqrt{y}$, then $\sqrt{(a - \sqrt{b})} = x - \sqrt{y}$.

For since $\sqrt{(a + \sqrt{b})} = x + \sqrt{y}$, we have by squaring

$$a + \sqrt{b} = x^2 + 2x\sqrt{y} + y;$$

therefore $\quad a = x^2 + y$, and $\sqrt{b} = 2x\sqrt{y}$, (Art. 299).

Hence $$a - \sqrt{b} = x^2 - 2x\sqrt{y} + y,$$

and $$\sqrt{(a - \sqrt{b})} = x - \sqrt{y}.$$

Similarly we may shew that if

$$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y},$$

then $$\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

301. *The square root of a binomial, one of whose terms is a quadratic surd and the other rational, may sometimes be expressed by a binomial, one or each of whose terms is a quadratic surd.*

Let $a + \sqrt{b}$ be the given binomial, and suppose

$$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}.$$

By Art. 300, $$\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$$

By multiplication, $$\sqrt{(a^2 - b)} = x - y.$$

By squaring both sides of the first equation,

$$a + \sqrt{b} = x + 2\sqrt{(xy)} + y;$$

therefore $$a = x + y.$$

Hence, by addition and subtraction,

$$a + \sqrt{(a^2 - b)} = 2x, \qquad a - \sqrt{(a^2 - b)} = 2y;$$

therefore $$x = \tfrac{1}{2}\{a + \sqrt{(a^2 - b)}\}, \qquad y = \tfrac{1}{2}\{a - \sqrt{(a^2 - b)}\}.$$

Thus x and y are known, and therefore $\sqrt{(a + \sqrt{b})}$, which is $\sqrt{x} + \sqrt{y}$.

Also $\sqrt{(a - \sqrt{b})}$ is known, for it is $\sqrt{x} - \sqrt{y}$.

302. For example, find the square root of $3 + 2\sqrt{2}$.

Here $a = 3, \qquad \sqrt{b} = 2\sqrt{2}, \qquad a^2 - b = 9 - 8 = 1;$

therefore $$x = \tfrac{1}{2}(3 + 1) = 2, \qquad y = \tfrac{1}{2}(3 - 1) = 1.$$

Thus $$\sqrt{(3 + 2\sqrt{2})} = \sqrt{2} + \sqrt{1} = \sqrt{2} + 1.$$

303. Again; find the square root of $7 - 2\sqrt{10}$.

Instead of using the result of Art. 301 we may go through the whole operation as follows:

Suppose $$\sqrt{(7 - 2\sqrt{10})} = \sqrt{x} - \sqrt{y};$$

then, by squaring, $$7 - 2\sqrt{10} = x - 2\sqrt{(xy)} + y;$$

hence $$x + y = 7 \qquad \text{.........................(1)},$$

and $$2\sqrt{(xy)} = 2\sqrt{10};$$

therefore $$(x+y)^2 - 4xy = 49 - (2\sqrt{10})^2,$$

that is, $$(x-y)^2 = 49 - 40 = 9,$$

and $$x - y = 3 \quad \text{......................} (2);$$

therefore, from (1) and (2), $x = 5$, and $y = 2$.

Thus $$\sqrt{(7-2\sqrt{10})} = \sqrt{5} - \sqrt{2}.$$

304. It appears from Art. 301 that

$$\sqrt{x} = \sqrt{\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}}, \qquad \sqrt{y} = \sqrt{\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}};$$

hence, unless $a^2 - b$ be a *perfect square*, the values of $\sqrt{x}$ and $\sqrt{y}$ will be complex surds, and the expression $\sqrt{x} + \sqrt{y}$ will not be so simple as $\sqrt{(a+\sqrt{b})}$ itself.

305. A binomial surd of the form $\sqrt{(a^2c)} + \sqrt{b}$ may be written thus, $\sqrt{c}\left(a + \sqrt{\frac{b}{c}}\right)$. If then $a^2 - \frac{b}{c}$ be a *perfect square*, the square root of $a + \sqrt{\frac{b}{c}}$ may be expressed in the form $\sqrt{x} + \sqrt{y}$; and therefore the square root of $\sqrt{(a^2c)} + \sqrt{b}$ will be $\sqrt[4]{c}\,(\sqrt{x} + \sqrt{y})$.

306. For example, find the square root of $\sqrt{32} + \sqrt{30}$.

Here $$\sqrt{32} + \sqrt{30} = \sqrt{2}\,(4 + \sqrt{15});$$

thus $$\sqrt{(\sqrt{32} + \sqrt{30})} = \sqrt[4]{2} \times \sqrt{(4 + \sqrt{15})};$$

and it may be shewn that

$$\sqrt{(4+\sqrt{15})} = \sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}}.$$

Hence $$\sqrt{(\sqrt{32} + \sqrt{30})} = \sqrt[4]{2}\left(\sqrt{\frac{5}{2}} + \sqrt{\frac{3}{2}}\right) = \frac{1}{\sqrt[4]{2}}(\sqrt{5} + \sqrt{3}).$$

307. Sometimes we may extract the square root of a quantity of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ by assuming

$$\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}=\sqrt{x}+\sqrt{y}+\sqrt{z};$$

then $a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{(xy)}+2\sqrt{(yz)}+2\sqrt{(zx)}$;

we may then put

$$2\sqrt{(xy)}=\sqrt{b}, \qquad 2\sqrt{(yz)}=\sqrt{c}, \qquad 2\sqrt{(zx)}=\sqrt{d},$$

and if the values of x, y, *and* z, *found from these, also satisfy* $x+y+z=a$, we shall have the required square root.

308. For example, find the square root of

$$8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}.$$

Assume $\sqrt{(8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$; then $8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}=x+y+z+2\sqrt{(xy)}+2\sqrt{(yz)}+2\sqrt{(zx)}$.

Put $2\sqrt{(xy)}=2\sqrt{2}, \qquad 2\sqrt{(yz)}=2\sqrt{5}, \qquad 2\sqrt{(zx)}=2\sqrt{10}$;

hence, by multiplication, $\sqrt{(xy)}\times\sqrt{(yz)}=\sqrt{10}$,

and $\sqrt{(zx)}=\sqrt{10}$,

therefore, by division, $y=1$;

hence $x=2$, and $z=5$.

These values satisfy the equation $x+y+z=8$.

Thus the required square root is $\sqrt{2}+\sqrt{1}+\sqrt{5}$,

that is, $1+\sqrt{2}+\sqrt{5}$.

309. If $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$, then $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

For suppose $\sqrt[3]{(a+\sqrt{b})}=x+\sqrt{y}$;

then, by cubing, $a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$;

therefore $a=x^3+3xy, \qquad \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y}$, (Art. 299);

hence $a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}$,

and $\sqrt[3]{(a-\sqrt{b})}=x-\sqrt{y}$.

310. *The cube root of a binomial* $a \pm \sqrt{b}$ *may be sometimes found.*

Assume $\sqrt[3]{(a + \sqrt{b})} = x + \sqrt{y}$,

then $\sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}$.

By multiplication, $\sqrt[3]{(a^2 - b)} = x^2 - y$.

Suppose now that $a^2 - b$ is a perfect cube, and denote it by c^3,

thus $c = x^2 - y$;

and, as in Art. 309, $a = x^3 + 3xy$.

Substitute the value of y;

thus $a = x^3 + 3x(x^2 - c)$;

therefore $4x^3 - 3cx = a$.

From this equation x must be found *by trial*, and then y is known from the equation $y = x^2 - c$.

Thus it appears that the method is inapplicable unless $a^2 - b$ be a *perfect cube;* and then it is imperfect since it leads to an equation which we have not at present any method of solving except by trial. The proposition, however, is of no practical importance.

311. For example, find the cube root of $10 + \sqrt{108}$.

Assume $\sqrt[3]{(10 + \sqrt{108})} = x + \sqrt{y}$, then $\sqrt[3]{(10 - \sqrt{108})} = x - \sqrt{y}$.

By multiplication, $\sqrt[3]{(100 - 108)} = x^2 - y$, that is, $-2 = x^2 - y$. Also $10 = x^3 + 3xy = x^3 + 3x(x^2 + 2)$; therefore $4x^3 + 6x = 10$.

We see that this equation is satisfied by $x = 1$; hence $y = 3$, and the required cube root is $1 + \sqrt{3}$.

Again; find the cube root of $18\sqrt{3} + 14\sqrt{5}$.

$$18\sqrt{3} + 14\sqrt{5} = 3\sqrt{3}\left(6 + \frac{14}{3}\sqrt{\frac{5}{3}}\right).$$

The cube root of $3\sqrt{3}$ is $\sqrt{3}$; and the cube root of $6 + \frac{14}{3}\sqrt{\frac{5}{3}}$, can be found. For here $a^2 - b = 36 - \frac{196}{9} \times \frac{5}{3} = -\frac{8}{27}$; so that $c = -\frac{2}{3}$. Hence we have the equation $4x^3 + 2x = 6$, which we see is satisfied by $x = 1$. Thus the required cube root is $\sqrt{3}\left(1 + \sqrt{\frac{5}{3}}\right)$, that is $\sqrt{3} + \sqrt{5}$.

312. We will now solve an equation involving surds which will serve as a model for similar examples: the equation resembles those already solved in the circumstance that we obtain only a *single* value of the unknown quantity.

Solve $\sqrt{(x+2)} + \sqrt{(x-14)} = 8.$

By transposition, $\sqrt{(x+2)} = 8 - \sqrt{(x-14)}$;

square both sides, $x + 2 = 64 - 16\sqrt{(x-14)} + x - 14$;

transpose, $16\sqrt{(x-14)} = 48$;

divide by 16, $\sqrt{(x-14)} = 3$;

square both sides, $x - 14 = 9$;

therefore $x = 23$.

EXAMPLES OF SURDS.

1. Find a factor which will rationalise $a^{\frac{1}{3}} - b^{\frac{2}{3}}$.

2. Find a factor which will rationalise $\sqrt{2} - \sqrt[3]{3}$.

3. Find a factor which will rationalise $\sqrt{3} + \sqrt[4]{5}$.

4. Given $\sqrt{3} = 1\cdot7320508$, find the value of $\dfrac{1}{2+\sqrt{3}}$.

5. Shew that $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(1+\sqrt{3})} = \dfrac{1}{5}\sqrt{15}$.

6. Shew that $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}} = \sqrt{5}(1+\sqrt{2})$.

7. Extract the square root of

$$9\frac{x}{y} - 24\sqrt{\frac{x}{y}} + 34 - 24\sqrt{\frac{y}{x}} + 9\frac{y}{x}.$$

8. Extract the square root of $(a+b)^2 - 4(a-b)\sqrt{(ab)}$.

Extract the square root of the expressions in the following examples from 9 to 18 inclusive:

9. $4 + 2\sqrt{3}$.

10. $7 - 4\sqrt{3}$.

11. $7 + 2\sqrt{10}$.

12. $18 + 8\sqrt{5}$.

13. $75 - 12\sqrt{21}$.

14. $16 + 5\sqrt{7}$.

15. $ab + c^2 + \sqrt{\{(a^2 - c^2)(b^2 - c^2)\}}$.

16. $\sqrt{27} + \sqrt{15}$.

17. $-9 + 6\sqrt{3}$.

18. $1 + (1 - c^2)^{-\frac{1}{2}}$.

19. Find the value of

$$\frac{1+x}{1+\sqrt{(1+x)}} + \frac{1-x}{1+\sqrt{(1-x)}} \text{ when } x = \frac{\sqrt{3}}{2}.$$

20. Find the value of

$$\frac{1+x}{1+\sqrt{(1+x)}} + \frac{1-x}{1-\sqrt{(1-x)}} \text{ when } x = \frac{\sqrt{3}}{2}.$$

21. Extract the square root of $6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}$.

22. Extract the square root of $5 + \sqrt{10} - \sqrt{6} - \sqrt{15}$.

23. Extract the square root of

$$15 - 2\sqrt{3} - 2\sqrt{15} + 6\sqrt{2} - 2\sqrt{6} + 2\sqrt{5} - 2\sqrt{30}.$$

24. Extract the cube root of $7 + 5\sqrt{2}$.

25. Extract the cube root of $16 + 8\sqrt{5}$.

26. Extract the cube root of $9\sqrt{3} - 11\sqrt{2}$.

27. Extract the cube root of $21\sqrt{6} - 23\sqrt{5}$.

28. Shew that $\sqrt[3]{(\sqrt{5} + 2)} - \sqrt[3]{(\sqrt{5} - 2)} = 1$.

29. Solve the equation $\sqrt{(x + 11)} - \sqrt{x} = 1$.

30. Solve the equation $\sqrt{(3x + 4)} + \sqrt{(3x - 5)} = 9$.

31. Solve the equation $a\sqrt{(b - x)} = b\sqrt{(a - x)}$.

32. Solve the equation $\sqrt{(x + a)} + \sqrt{(x + b)} = \sqrt{c}$.

XX. QUADRATIC EQUATIONS.

313. When an equation contains only the square of the unknown quantity the value of this square can be found by the rules for solving a simple equation; then by extracting the square root the values of the unknown quantity are found. For example, suppose

$$8x^2 - 72 + 10x^2 = 7 - 24x^2 + 89:$$

by transposition, $42x^2 = 168$;

by division, $x^2 = 4$;

therefore $x = \sqrt{4} = \pm 2.$

The double sign is used because the square root of a quantity may be either positive or negative. (Art. 231.)

It might at first appear that from $x^2 = 4$ we ought to infer, not that $x = \pm 2$, but that $\pm x = \pm 2$. It will however be found that the second form is really coincident with the first. For $\pm x = \pm 2$ gives either $+x = +2$, or $+x = -2$, or $-x = +2$, or $-x = -2$; that is, on the whole, either $x = 2$, or $x = -2$. Hence it follows, that when we extract the square root of the two members of an equation it is sufficient to put the double sign before the square root of *one* of the members.

314. Quadratic equations which contain only the square of the unknown quantity are called *pure* quadratics. Quadratic equations which contain the first power of the unknown quantity as well as the square are called *adfected* quadratics. We proceed now to the solution of the latter.

315. We shall first shew that every quadratic equation may be reduced to the form $x^2 + px = q$, where p and q are positive or negative. For we can reduce any quadratic equation to this form by the following steps: bring the terms which contain the unknown quantity to the left-hand side of the equation, and the known quantities to the right-hand side; if the coefficient of x^2 be negative, change the sign of every term of the equation; then divide

every term by the coefficient of x^2. Thus we may represent any quadratic equation by

$$x^2 + px = q.$$

To solve this equation we add $\frac{1}{4}p^2$ to both sides; thus

$$x^2 + px + \frac{p^2}{4} = \frac{p^2}{4} + q.$$

The left-hand member is now a *complete square;* extract the square root of each member; thus

$$x + \frac{p}{2} = \pm\sqrt{\left(\frac{p^2}{4} + q\right)};$$

transpose the term $\frac{p}{2}$, and we obtain

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p^2}{4} + q\right)}.$$

316. For example, suppose

$$-3x^2 + 36x - 105 = 0;$$

transpose, $\quad -3x^2 + 36x = 105;$

change the signs, $\quad 3x^2 - 36x = -105;$

divide by 3, $\quad x^2 - 12x = -35;$

add to both sides $\left(\frac{12}{2}\right)^2$, that is, 36; thus

$$x^2 - 12x + 36 = 36 - 35 = 1;$$

extract the square root of both members; thus

$$x - 6 = \pm 1.$$

Therefore $x = 6 \pm 1$; that is, $x = 7$, or 5. If either of these values be substituted for x in the expression $-3x^2 + 36x - 105$, the result is zero.

317. Hence the following rule may be given for the solution of a quadratic equation:

By transposition and reduction arrange the equation so that the terms involving the unknown quantity are alone on one side, *and the coefficient of* x^2 *is* $+1$; *add to both sides of the equation the square of half the coefficient of* x, *and extract the square root of both sides.*

318. As another example we will take

$$ax^2 + bx + c = 0;$$

transpose, $$ax^2 + bx = -c;$$

divide by a, $$x^2 + \frac{bx}{a} = -\frac{c}{a};$$

add $\left(\frac{b}{2a}\right)^2$, $$x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2};$$

extract the square root, $$x + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a};$$

transpose, $$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

The particular case in which $c = 0$ should be noted. Then, taking the upper sign we have $x = 0$; and taking the lower sign we have $x = -\frac{b}{a}$. In fact in this case the equation reduces to $ax^2 + bx = 0$, or $x(ax + b) = 0$: and it is plain that this is satisfied, either when $x = 0$; or when $ax + b = 0$, that is when $x = -\frac{b}{a}$.

319. When an example is proposed for solution instead of going through the process indicated in Art. 317, we may make use of the *formula* in Art. 318. Thus, take the example in Art. 316, namely, $-3x^2 + 36x - 105 = 0$, and by comparing it with the formula in Art. 318 we see that we may suppose $a = -3$, $b = 36$, $c = -105$. Hence if we put these values for a, b, and c in the result of Art. 318, we shall obtain the value of x. Here

$$b^2 - 4ac = (36)^2 - 12 \times 105 = 36;$$

therefore $$x = \frac{-36 \pm 6}{-6} = 7, \text{ or } 5.$$

320. For another example take the equation

$$x^2 - 6x = -2;$$

add $\left(\frac{6}{2}\right)^2$, $\quad x^2 - 6x + 9 = 9 - 2 = 7;$

extract the square root, $\quad x - 3 = \pm\sqrt{7},$

transpose, $\quad x = 3 \pm \sqrt{7}.$

Here $\sqrt{7}$ cannot be found exactly; but we can find an approximate value of it to any assigned degree of accuracy, and thus obtain the value of x to any assigned degree of accuracy.

321. In the examples hitherto considered we have found *two different* roots of a quadratic equation; in some cases however we shall find really only one root. Take for example the equation $x^2 - 12x + 36 = 0$; by extracting the square root we have $x - 6 = 0$, and therefore $x = 6$. It is however convenient in this case to say that the quadratic equation has *two equal roots.*

322. If the quadratic equation be represented by

$$ax^2 + bx + c = 0,$$

we know from Art. 318 that the two roots are respectively

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}.$$

Now these will be different unless $b^2 - 4ac = 0$, and then each of them is $-\frac{b}{2a}$. This relation $b^2 - 4ac = 0$ is then the condition that must hold in order that the two roots of the quadratic equation may be equal.

323. Consider next the example $x^2 - 10x + 32 = 0$.

By transposition, $\quad x^2 - 10x = -32;$

by addition, $\quad x^2 - 10x + 25 = 25 - 32 = -7.$

If we proceed to extract the square root we have

$$x - 5 = \pm\sqrt{-7}.$$

But the negative quantity -7 has no square root either exact or approximate (Art. 232); thus no real value of x can be found to satisfy the proposed equation. In such a case the quadratic equation has no real roots; this is sometimes expressed by saying that the roots are *imaginary* or *impossible*. We shall return to this point in Chapter XXV.

324. If the quadratic equation be represented by

$$ax^2 + bx + c = 0,$$

we see from Art. 318 that the roots are *real* if $b^2 - 4ac$ is *positive*, that is, if b^2 is algebraically greater than $4ac$, and that the roots are impossible if $b^2 - 4ac$ is *negative*, that is, if b^2 is algebraically less than $4ac$.

EXAMPLES OF QUADRATICS.

1. $x^2 - 4x + 3 = 0$.
2. $x^2 - 5x + 4 = 0$.
3. $6x^2 - 13x + 6 = 0$.
4. $3x^2 - 7x = 20$.
5. $2x^2 - 7x + 3 = 0$.
6. $3x^2 - 53x + 34 = 0$.
7. $x^2 + 10x + 24 = 0$.
8. $7x^2 - 3x = 160$.
9. $14x - x^2 = 33$.
10. $2x^2 - 2x - \frac{3}{2} = 0$.
11. $x^2 - 3 = \frac{1}{6}(x - 3)$.
12. $4(x^2 - 1) = 4x - 1$.
13. $110x^2 - 21x + 1 = 0$.
14. $780x^2 - 73x + 1 = 0$.
15. $(x - 1)(x - 2) = 6$.
16. $(3x - 2)(x - 1) = 14$.
17. $(3x - 5)(2x - 5) = (x + 3)(x - 1)$.
18. $(2x + 1)(x + 2) = 3x^2 - 4$.
19. $(x + 1)(2x + 3) = 4x^2 - 22$.
20. $(x - 1)(x - 2) + (x - 2)(x - 4) = 6(2x - 5)$.
21. $(2x - 3)^2 = 8x$.
22. $(5x - 3)^2 - 7 = 44x + 5$.
23. $(x - 7)(x - 4) + (2x - 3)(x - 5) = 103$.

24. $\frac{5}{7}x^2 + \frac{7}{5}x + \frac{73}{140} = 0.$

25. $\left(x - \frac{1}{2}\right)\left(x - \frac{1}{3}\right) + \left(x - \frac{1}{3}\right)\left(x - \frac{1}{4}\right) = \left(x - \frac{1}{4}\right)\left(x - \frac{1}{5}\right).$

26. $\frac{x}{2} + \frac{2}{x} = \frac{x}{3} + \frac{3}{x}.$

27. $\frac{5x}{21}(x + 1) - \frac{1}{7}(2x^2 + x - 1) = \frac{4}{35}(x + 1).$

28. $8x + 11 + \frac{7}{x} = \frac{21 + 65x}{7}.$

29. $\frac{6}{x} + \frac{x}{6} = \frac{5(x - 1)}{4}.$

30. $\frac{x}{7} + \frac{21}{x + 5} = \frac{23}{7}.$

31. $\frac{21}{5 - x} - \frac{x}{7} = \frac{23}{7}.$

32. $\frac{1}{2(x - 1)} + \frac{3}{x^2 - 1} = \frac{1}{4}.$

33. $\frac{3}{2(x^2 - 1)} + \frac{x}{4(x + 1)} = \frac{3}{8}.$

34. $\frac{x}{15} + \frac{40}{3(10 - x)} = \frac{3(10 + x)}{95}.$

35. $\frac{2x}{15} + \frac{3x - 50}{3(10 + x)} = \frac{12x + 70}{190}.$

36. $\frac{x^2 - 5x}{x + 3} = x - 3 + \frac{1}{x}.$

37. $\frac{x + 2}{x - 1} - \frac{4 - x}{2x} = \frac{7}{3}.$

38. $\frac{x}{x - 1} = \frac{3}{2} + \frac{x - 1}{x}.$

39. $\frac{x + 4}{x - 4} + \frac{x - 4}{x + 4} = \frac{10}{3}.$

40. $\frac{x + 2}{x - 2} - \frac{x - 2}{x + 2} = \frac{5}{6}.$

41. $\frac{x}{x + 1} + \frac{x + 1}{x} = \frac{13}{6}.$

42. $\frac{x - 6}{x - 12} - \frac{x - 12}{x - 6} = \frac{5}{6}.$

43. $\frac{1}{x - 2} - \frac{2}{x + 2} = \frac{3}{5}.$

44. $\frac{4}{x + 1} + \frac{5}{x + 2} = \frac{12}{x + 3}.$

45. $\frac{5}{x + 2} + \frac{3}{x} = \frac{14}{x + 4}.$

46. $\frac{2x - 3}{3x - 5} + \frac{3x - 5}{2x - 3} = \frac{5}{2}.$

47. $\frac{3x - 2}{2x - 5} - \frac{2x - 5}{3x - 2} = \frac{8}{3}.$

48. $\frac{x + 3}{x + 2} + \frac{x - 3}{x - 2} = \frac{2x - 3}{x - 1}.$

49. $\frac{x - 2}{x + 2} + \frac{x + 2}{x - 2} = \frac{2(x + 3)}{x - 3}.$

50. $10(2x+3)(x-3)+(7x+3)^2=20(x+3)(x-1)$.

51. $(7-4\sqrt{3})x^2+(2-\sqrt{3})x=2$.

52. $x^2-2ax+a^2-b^2=0$.

53. $x^2-2ax+b^2=0$.

54. $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1)$.

55. $\dfrac{1}{x-a}+\dfrac{1}{x-b}+\dfrac{1}{x-c}=0$.

56. $\dfrac{1}{(x-b)(x-c)}+\dfrac{1}{(a+c)(a+b)}=\dfrac{1}{(a+c)(x-c)}+\dfrac{1}{(a+b)(x-b)}$.

57. $\dfrac{1}{a+b+x}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{x}$.

58. $(ax-b)(bx-a)=c^2$.

59. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{2c}{x-c}$.

60. $abx^2+\dfrac{3a^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{b^2x}{c}$.

61. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$.

62. $\dfrac{a+c(a+x)}{a+c(a-x)}+\dfrac{a+x}{x}=\dfrac{a}{a-2cx}$.

XXI. EQUATIONS WHICH MAY BE SOLVED LIKE QUADRATICS.

325. There are many equations which, though not really quadratics, may be solved by processes similar to those given in the preceding Chapter. For example, suppose

$$x^4-9x^2+20=0.$$

Transpose, $\qquad x^4-9x^2=-20;$

by addition, $$x^4 - 9x^2 + \left(\frac{9}{2}\right)^2 = \left(\frac{9}{2}\right)^2 - 20 = \frac{1}{4};$$

extract the square root, $$x^2 - \frac{9}{2} = \pm\frac{1}{2};$$

therefore $$x^2 = \frac{9}{2} \pm \frac{1}{2} = 5, \text{ or } 4;$$

therefore $$x = \pm\sqrt{5}, \text{ or } \pm 2.$$

326. Similarly we may solve any equation of the form

$$ax^{2n} + bx^n + c = 0.$$

Transpose, $$ax^{2n} + bx^n = -c;$$

divide by a, $$x^{2n} + \frac{bx^n}{a} = -\frac{c}{a};$$

by addition, $$x^{2n} + \frac{bx^n}{a} + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2};$$

extract the square root, $$x^n + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a};$$

therefore $$x^n = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Hence by extracting the n^{th} root the value of x is known.

327. Suppose, for example,

$$x + 4\sqrt{x} = 21;$$

therefore $$x + 4\sqrt{x} + 4 = 25;$$

therefore $$\sqrt{x} + 2 = \pm 5;$$

therefore $$\sqrt{x} = -2 \pm 5 = 3, \text{ or } -7;$$

therefore $$x = 9, \text{ or } 49.$$

328. Again, suppose

$$x^{-1} + x^{-\frac{1}{2}} = 6;$$

therefore $$x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = \frac{25}{4};$$

therefore $$x^{-\frac{1}{2}} + \frac{1}{2} = \frac{\pm 5}{2};$$

therefore $x^{-\frac{1}{2}} = -\frac{1}{2} \pm \frac{5}{2} = 2, \text{ or } -3;$

therefore $x^{-1} = 4, \text{ or } 9,$

and $x = \frac{1}{4}, \text{ or } \frac{1}{9}.$

329. Suppose we require the solutions of the equation

$$x + \sqrt{(5x + 10)} = 8.$$

By transposition, $\sqrt{(5x + 10)} = 8 - x;$

square both sides; thus

$$5x + 10 = 64 - 16x + x^2;$$

therefore $x^2 - 21x = -54;$

therefore $x^2 - 21x + \left(\frac{21}{2}\right)^2 = \left(\frac{21}{2}\right)^2 - 54 = \frac{225}{4};$

therefore $x - \frac{21}{2} = \pm \frac{15}{2};$

therefore $x = \frac{21}{2} \pm \frac{15}{2} = 18, \text{ or } 3.$

Substitute these values of x in the left-hand side of the given equation; it will be found that 3 satisfies the equation but that 18 does not; we shall find however that 18 does satisfy the equation

$$x - \sqrt{(5x + 10)} = 8.$$

In fact the equation $5x + 10 = 64 - 16x + x^2$ which we obtained from the given equation by transposing and squaring might have arisen also from $x - \sqrt{(5x + 10)} = 8$. Hence we are not sure that the values of x which are finally obtained will satisfy the proposed equation; they *may* satisfy the other form.

330. Again, consider the example

$$x - 2\sqrt{(x^2 + x + 5)} - 14 = 0.$$

By transposition, $x - 14 = 2\sqrt{(x^2 + x + 5)};$

by squaring, $x^2 - 28x + 196 = 4x^2 + 4x + 20$;

therefore $3x^2 + 32x = 176$.

From the last equation we shall obtain $x = 4$, or $\frac{-44}{3}$. It will, however, be found on trial that neither of these values satisfies the proposed equation; each of them however satisfies the equation

$$x + 2\sqrt{(x^2 + x + 5)} - 14 = 0.$$

From this and the preceding example we see that when an equation has been reduced to a rational form by squaring, it will be necessary to examine whether the roots which are finally obtained satisfy the equation in the form originally given. This remark applies for instance to equations like those solved in Arts. 312, 327, and 328.

331. Suppose that all the terms of an equation are brought to one side and the expression thus obtained can be represented as the product of simple or quadratic factors, then the equation can be solved by methods already given. For example, suppose

$$(x - c)(x^2 - 3ax + 2a^2) = 0.$$

The left-hand member is zero either when $x - c = 0$, or when $x^2 - 3ax + 2a^2 = 0$; and in no other case. But if $x - c = 0$, we have $x = c$; and if $x^2 - 3ax + 2a^2 = 0$, we shall find that $x = a$, or $2a$. Hence the proposed equation is satisfied by $x = c$, or a, or $2a$; and by no other values.

332. Facility in separating expressions into factors will be acquired by experience; some assistance however will be furnished by a principle which we will here exemplify. Consider the example

$$x(x - c)^2 = a(a - c)^2.$$

Here it is obvious that $x = a$ satisfies the equation; and we shall find that if we bring all the terms to one side $x - a$ will be a factor of the whole expression. For the equation may be written

$$x^3 - a^3 - 2c(x^2 - a^2) + c^2(x - a) = 0;$$

that is, $(x - a)\{x^2 + ax + a^2 - 2c(x + a) + c^2\} = 0.$

Hence the other roots besides a will be found by solving the quadratic

$$x^2 + ax + a^2 - 2c(x + a) + c^2 = 0.$$

In this manner when one root is obvious on inspection, we may succeed in arranging the equation in the manner indicated in Art. 331.

333. We will now add some miscellaneous examples of equations reducible to quadratics.

(1) Suppose

$$x^2 - 7x + \sqrt{(x^2 - 7x + 18)} = 24.$$

Add 18 to both sides; thus

$$x^2 - 7x + 18 + \sqrt{(x^2 - 7x + 18)} = 42;$$

complete the square; thus

$$x^2 - 7x + 18 + \sqrt{(x^2 - 7x + 18)} + \frac{1}{4} = 42\tfrac{1}{4} = \frac{169}{4};$$

therefore $$\sqrt{(x^2 - 7x + 18)} + \frac{1}{2} = \pm\frac{13}{2};$$

therefore $$\sqrt{(x^2 - 7x + 18)} = 6, \text{ or } -7;$$

therefore $$x^2 - 7x + 18 = 36, \text{ or } 49.$$

Hence we have now two ordinary quadratic equations to solve. We shall obtain from the first $x = 9$, or -2, and from the second $x = \frac{1}{2}(7 \pm \sqrt{173})$. It will be found on trial that the first two only are solutions of the proposed equation; the others apply to the equation

$$x^2 - 7x - \sqrt{(x^2 - 7x + 18)} = 24.$$

(2) Suppose

$$x^4 + x^3 - 4x^2 + x + 1 = 0.$$

Divide by x^2; thus

$$x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0;$$

or $$x^2 + \frac{1}{x^2} + x + \frac{1}{x} - 4 = 0;$$

therefore $$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) - 6 = 0;$$

therefore $$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6,$$

and $$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) + \frac{1}{4} = 6\tfrac{1}{4} = \frac{25}{4};$$

therefore $$x + \frac{1}{x} + \frac{1}{2} = \pm\frac{5}{2};$$

therefore $$x + \frac{1}{x} = 2, \text{ or } -3.$$

First suppose $$x + \frac{1}{x} = 2;$$

therefore $$x^2 - 2x + 1 = 0;$$

therefore $$x = 1.$$

Next suppose $$x + \frac{1}{x} = -3;$$

therefore $$x^2 + 3x = -1;$$

therefore $$x^2 + 3x + \frac{9}{4} = \frac{9}{4} - 1 = \frac{5}{4};$$

therefore $$x + \frac{3}{2} = \pm\frac{\sqrt{5}}{2}, \text{ and } x = \frac{-3 \pm \sqrt{5}}{2}.$$

(3) Suppose

$$x^4 + 3x + 1 = 3x^3 + \frac{4}{9}x^2.$$

Transpose $$x^4 - 3x^3 + 3x + 1 = \frac{4x^2}{9};$$

therefore $$\left(x^2 - \frac{3x}{2}\right)^2 - \frac{9x^2}{4} + 3x + 1 = \frac{4x^2}{9};$$

therefore $$\left(x^2 - \frac{3x}{2}\right)^2 - 2\left(x^2 - \frac{3x}{2}\right) - \frac{x^2}{4} + 1 = \frac{4x^2}{9};$$

therefore $\left(x^2-\frac{3x}{2}\right)^2-2\left(x^2-\frac{3x}{2}\right)+1=\frac{x^2}{4}+\frac{4x^2}{9}=\frac{25x^2}{36}$.

Extract the square root, then

$$x^2-\frac{3x}{2}-1=\pm\frac{5x}{6}.$$

We have now ordinary quadratics, namely, $x^2-\frac{3x}{2}-1=\frac{5x}{6}$, and $x^2-\frac{3x}{2}-1=-\frac{5x}{6}$. From the former we shall obtain $x=\frac{1}{6}(7\pm\sqrt{85})$, and from the latter $x=\frac{1}{3}(1\pm\sqrt{10})$.

(4) Suppose

$$6x\sqrt{x}-11x+6\sqrt{x}-1=0.$$

We may write the equation in the form

$$(x-3\sqrt{x})^2+2(x-3\sqrt{x})+1=x^2.$$

Hence $$x-3\sqrt{x}+1=\pm x.$$

Take the upper sign; thus

$$x-3\sqrt{x}+1=x;$$

therefore $$\sqrt{x}=\frac{1}{3}, \text{ and } x=\frac{1}{9}.$$

Take the lower sign; thus

$$x-3\sqrt{x}+1=-x;$$

therefore $$2x-3\sqrt{x}+1=0.$$

From this we obtain $\sqrt{x}=1$, or $\frac{1}{2}$, and therefore $x=1$, or $\frac{1}{4}$.

(5) Suppose

$$\frac{x+c+\sqrt{(x^2-c^2)}}{x+c-\sqrt{(x^2-c^2)}}=\frac{9(x+c)}{8c}\text{..............(1)}.$$

In solving this equation we shall employ a principle which often abbreviates algebraical work.

Suppose that
$$\frac{a}{b}=\frac{p}{q},$$
then will
$$\frac{a+b}{b}=\frac{p+q}{q}, \quad \frac{a-b}{b}=\frac{p-q}{q}, \quad \frac{a+b}{a-b}=\frac{p+q}{p-q}.$$

For the first of these three results is obtained by adding unity to each of the given equal quantities, the second is obtained by subtracting unity from each of the given equal quantities, and the third result is obtained by dividing the first by the second. Each result is sometimes serviceable. For the present example we employ the third. Thus from (1) we deduce
$$\frac{2(x+c)}{2\sqrt{(x^2-c^2)}}=\frac{9x+17c}{9x+c}.$$

Square both sides, and simplify the left-hand member; thus
$$\frac{x+c}{x-c}=\frac{(9x+17c)^2}{(9x+c)^2}\text{.....................}(2).$$

Again, by employing the third of the above results we deduce from (2)
$$\frac{x}{c}=\frac{(9x+17c)^2+(9x+c)^2}{(9x+17c)^2-(9x+c)^2}=\frac{(9x+17c)^2+(9x+c)^2}{16c(18x+18c)}.$$

By reducing, we obtain
$$63x^2-18xc-145c^2=0,$$
and from this, $x=\frac{5c}{3}$, or $x=-\frac{29c}{21}$.

(6) Suppose
$$\sqrt{\left(\frac{3a}{4}-x\right)}+\sqrt{(3ax-x)}=\frac{3a}{2}\sqrt{(1-4x)}.$$

Transpose; thus
$$\frac{3a}{2}\sqrt{(1-4x)}-\sqrt{\left(\frac{3a}{4}-x\right)}=\sqrt{(3ax-x)}.$$

By squaring, $\frac{9a^2}{4}(1-4x) - 3a\sqrt{(1-4x)}\sqrt{\left(\frac{3a}{4}-x\right)} = 3ax - \frac{3a}{4}$

$$= -\frac{3a}{4}(1-4x).$$

Divide by $\sqrt{(1-4x)}$; thus

$$\frac{9a^2+3a}{4}\sqrt{(1-4x)} = 3a\sqrt{\left(\frac{3a}{4}-x\right)}.$$

By squaring, $\qquad (1+3a)^2(1-4x) = 16\left(\frac{3a}{4}-x\right);$

therefore $\qquad 4x\{(1+3a)^2-4\} = (1+3a)^2 - 12a = (1-3a)^2;$

therefore $\qquad 4x(3a+3)(3a-1) = (3a-1)^2;$

therefore $\qquad x = \frac{3a-1}{12(a+1)}.$

Also corresponding to the factor $\sqrt{(1-4x)}$, which was removed, we have the root $x = \frac{1}{4}$.

This example is introduced in order to draw the attention of the student to the circumstance that when both sides of an equation are to be squared, an advantageous arrangement of the terms on opposite sides of the equation should be made before squaring. If in this example as it originally stands we square both sides, no terms will disappear; but by transposing before squaring we obtain a result in which $-x$ occurs on both sides, and may therefore be cancelled.

(7) Suppose

$$\sqrt{(x^2+9)} + \sqrt{(x^2-9)} = \sqrt{(34)} + 4.$$

We have identically

$$x^2 + 9 - (x^2 - 9) = 18 = 34 - 16.$$

Hence, dividing the members of this identity by the corresponding members of the proposed equation, we obtain

$$\sqrt{(x^2+9)} - \sqrt{(x^2-9)} = \sqrt{(34)} - 4.$$

Therefore, by addition, $\sqrt{(x^2+9)} = \sqrt{(34)}$;

therefore $x^2 = 25$, and $x = \pm 5$.

This equation is introduced for the sake of illustrating the artifice employed in the solution. This artifice may often be employed with advantage; for instance, example (6) may be solved in this way.

(8) $$\sqrt{(2x+4)} - 2\sqrt{(2-x)} = \frac{12x-8}{\sqrt{(9x^2+16)}}.$$

We may write this equation thus,

$$\sqrt{(2x+4)} - 2\sqrt{(2-x)} = \frac{2\{2(x+2) - 4(2-x)\}}{\sqrt{(9x^2+16)}}.$$

The factor $\sqrt{(2x+4)} - 2\sqrt{(2-x)}$ can now be removed from both sides; thus we obtain

$$\sqrt{(9x^2+16)} = 2\{\sqrt{(2x+4)} + 2\sqrt{(2-x)}\}.$$

By squaring, $9x^2 + 16 = 4\{12 - 2x + 4\sqrt{(8-2x^2)}\}$;

therefore $x^2 + 8x = 4(8-2x^2) + 16\sqrt{(8-2x^2)}$;

therefore $x^2 + 8x + 16 = 4(8-2x^2) + 16\sqrt{(8-2x^2)} + 16$.

Extract the square root; thus

$$\pm(x+4) = 2\sqrt{(8-2x^2)} + 4.$$

The solution can now be completed; we shall obtain

$$x = \pm\frac{4\sqrt{2}}{3},$$

and also a pair of imaginary values.

Also, by equating to zero the factor $\sqrt{(2x+4)} - 2\sqrt{(2-x)}$, which was removed, we shall obtain $x = \frac{2}{3}$.

It will be seen that very artificial methods are adopted in some of these examples; the student can acquire dexterity in using such transformations only by practice. More examples will be found in Chapter LIV.

EXAMPLES OF EQUATIONS REDUCIBLE TO QUADRATICS.

1. $3x + 2\sqrt{x} - 1 = 0.$

2. $x^{10} + 31x^5 = 32.$

3. $3x^3 + 42x^{\frac{3}{2}} = 3321.$

4. $x^{\frac{1}{n}} - 13x^{\frac{1}{2n}} = 14.$

5. $x^6 - 35x^3 + 216 = 0.$

6. $x^{\frac{1}{n}} - x^{\frac{2}{n}} + 2 = 0.$

7. $x + 2\sqrt{(ax)} + c = 0.$

8. $3x^4 - 7x^2 = 43076.$

9. $x^4 - 14x^2 + 40 = 0.$

10. $x^{\frac{1}{3}} + \dfrac{5}{2x^{\frac{1}{3}}} = 3\frac{1}{4}.$

11. $\sqrt{(2x)} - 7x = -52.$

12. $3x^n \sqrt[3]{x^n} + \dfrac{2x^n}{\sqrt[3]{x^n}} = 16.$

13. $x + 5 - \sqrt{(x+5)} = 6.$

14. $2\sqrt{x} + \dfrac{2}{\sqrt{x}} = 5.$

15. $x^{\frac{1}{4}} + 5x^{\frac{1}{2}} - 22 = 0.$

16. $3x^{\frac{3}{2}} - 4x^{\frac{3}{4}} = 7.$

17. $2x + \sqrt{(4x+8)} = \dfrac{7}{2}.$

18. $2\left(x^{\frac{1}{n}} + x^{-\frac{1}{n}}\right) = 5.$

19. $\sqrt{(2x+7)} + \sqrt{(3x-18)} = \sqrt{(7x+1)}.$

20. $\dfrac{\sqrt{(x^2-16)}}{\sqrt{(x-3)}} + \sqrt{(x+3)} = \dfrac{7}{\sqrt{(x-3)}}.$

21. $\sqrt{(a+x)} + \sqrt{(a-x)} = \sqrt{b}.$

22. $\sqrt{(x+9)} = 2\sqrt{x} - 3.$

23. $x + \sqrt{(5x+10)} = 8.$

24. $2^{x+1} + 4^x = 80.$

25. $\dfrac{x^3 - 4x}{x-2} + \dfrac{x^2-1}{x+1} = 39.$

26. $\dfrac{\sqrt{(a+x)}}{\sqrt{a} + \sqrt{(a+x)}} = \dfrac{\sqrt{(a-x)}}{\sqrt{a} - \sqrt{(a-x)}}.$

27. $\left(\dfrac{x}{x-1}\right)^2 + \left(\dfrac{x}{x+1}\right)^2 = n(n-1).$

28. $(a+b)\sqrt{(a^2+b^2+x^2)} - (a-b)\sqrt{(a^2+b^2-x^2)} = a^2 + b^2.$

29. $x + \sqrt{x} + \sqrt{(x+2)} + \sqrt{(x^2+2x)} = a.$

30. $2x + \sqrt{(2 + 2x)} = c(1 - x).$

31. $\dfrac{a - x}{\sqrt{a} + \sqrt{(a - x)}} + \dfrac{a + x}{\sqrt{a} + \sqrt{(a + x)}} = \sqrt{a}.$

32. $\dfrac{\sqrt{(x + 2a)} - \sqrt{(x - 2a)}}{\sqrt{(x - 2a)} + \sqrt{(x + 2a)}} = \dfrac{x}{2a}.$

33. $\sqrt{(x + 8)} - \sqrt{(x + 3)} = \sqrt{x}.$

34. $\sqrt{(x + 3)} + \sqrt{(x + 8)} = 5\sqrt{x}.$

35. $\dfrac{x^2 - a^2}{x^2 + a^2} + \dfrac{x^2 + a^2}{x^2 - a^2} = \dfrac{34}{15}.$

36. $\sqrt{(a + bx^n)} - \sqrt{a} = c\sqrt{(bx^n)}.$

37. $\sqrt{(x + 4)} - \sqrt{x} = \sqrt{\left(x + \dfrac{3}{2}\right)}.$

38. $x^2 + \dfrac{1}{x^2} - a^2 - \dfrac{1}{a^2} = 0.$

39. $\dfrac{850}{931} = \dfrac{x(x^4 - a^4)}{x^6 - a^6}.$

40. $\dfrac{\sqrt{(x^2 + 1)} + \sqrt{(x^2 - 1)}}{\sqrt{(x^2 + 1)} - \sqrt{(x^2 - 1)}} + \dfrac{\sqrt{(x^2 + 1)} - \sqrt{(x^2 - 1)}}{\sqrt{(x^2 + 1)} + \sqrt{(x^2 - 1)}} = 4\sqrt{(x^2 - 1)}.$

41. $(a^{\frac{1}{2}} + x^{\frac{1}{2}})^{\frac{1}{3}} = (a^{\frac{1}{3}} + x^{\frac{1}{3}})^{\frac{1}{2}}.$

42. $\dfrac{a^2 + x^2}{a + x} + \dfrac{a^2 - x^2}{a - x} = 4a.$

43. $\sqrt{(1 - x + x^2)} - \sqrt{(1 + x + x^2)} = m.$

44. $\dfrac{x + \sqrt{(x^2 - 1)}}{x - \sqrt{(x^2 - 1)}} + \dfrac{x - \sqrt{(x^2 - 1)}}{x + \sqrt{(x^2 - 1)}} = 34.$

45. $\sqrt{(x^2 - 3ax + a^2)} + \sqrt{(x^2 + 3ax + a^2)} = \sqrt{(2a^2 + 2b^2)}.$

46. $x\sqrt{\left(\dfrac{6}{x} - x\right)} = \dfrac{1 + x^2}{\sqrt{x}}.$

47. $\sqrt[2pq]{(x^{p+q})} - \dfrac{1}{2c}(\sqrt[p]{x} + \sqrt[q]{x}) = 0.$

48. $\sqrt{x} + \sqrt{\{x - \sqrt{(1 - x)}\}} = 1.$

49. $(x + a)^5 - (x - a)^5 = 242a^5.$

50. $\dfrac{x^2 + 1}{x^2 - 1} = x + \sqrt{\dfrac{6}{x}}.$

51. $\sqrt{(x^2+ax+b^2)}+\sqrt{(x^2+bx+a^2)}=a+b$.

52. $\dfrac{25x^2-16}{10x-8}=\dfrac{3(x^2-4)x}{2x-4}$.

53. $\sqrt{(2x+9)}+\sqrt{(3x-15)}=\sqrt{(7x+8)}$.

54. $\sqrt{\dfrac{x}{a}}+\sqrt{\left\{\dfrac{(b-c)(ac-bx)}{abc}\right\}}=1$.

55. $\sqrt{(x^2+2x-1)}+\sqrt{(x^2+x+1)}=\sqrt{2}+\sqrt{3}$.

56. $\sqrt{(x^2+ax-1)}+\sqrt{(x^2+bx-1)}=\sqrt{a}+\sqrt{b}$.

57. $(x^2+1)(x+2)=2$.

58. $(x^2+a)(x+b)=ab$.

59. $(x-a)(x-b)(x-c)+abc=0$.

60. $\dfrac{1}{1-x}-\dfrac{1}{1+x}=\dfrac{4x}{1+x^2}$.

61. $\dfrac{1}{x+a+b}+\dfrac{1}{x-a+b}+\dfrac{1}{x+a-b}+\dfrac{1}{x-a-b}=0$.

62. $\dfrac{(a-x)(x+m)}{x+n}=\dfrac{(a+x)(x-m)}{x-n}$.

63. $\left(\dfrac{a+x}{a-x}\right)^2=1+\dfrac{cx}{ab}$.

64. $2x+1+x\sqrt{(x^2+2)}+(x+1)\sqrt{(x^2+2x+3)}=0$.

65. $x^2+3=2\sqrt{(x^2-2x+2)}+2x$.

66. $x^2+5x+4=5\sqrt{(x^2+5x+28)}$.

67. $\sqrt{(x^2-2x+9)}-\dfrac{x^2}{2}=3-x$.

68. $3x^2+15x-2\sqrt{(x^2+5x+1)}=2$.

69. $(x+5)(x-2)+3\sqrt{\{x(x+3)\}}=0$.

70. $x^2+3-\sqrt{(2x^2-3x+2)}=\dfrac{3}{2}(x+1)$.

71. $x(x+1)+3\sqrt{(2x^2+6x+5)}=25-2x$.

72. $x^2-2\sqrt{(3x^2-2ax+4)}+4=\dfrac{2a}{3}\left(x+\dfrac{a}{2}+1\right)$.

73. $x^2 - x + 3\sqrt{(2x^2 - 3x + 2)} = \frac{x}{2} + 7.$

74. $\frac{9}{1 + x + x^2} = 5 - x - x^2.$

75. $(x+a)(x+2a)(x+3a)(x+4a) = c^4.$

76. $16x(x+1)(x+2)(x+3) = 9.$

77. $\frac{a^2 + ax + x^2}{a^2 - ax + x^2} = \frac{a^2}{x^2}.$

78. $a = x^4 + (1-x)^4.$

79. $x^4 - 2x^3 + x = a.$

80. $x^4 - 2x^3 + x = 132.$

81. $\sqrt{x} + \sqrt{(x+7)} + 2\sqrt{(x^2 + 7x)} = 35 - 2x.$

82. $x^2 - 8(x+1)\sqrt{x} + 18x + 1 = 0.$

83. $2(x^2 + ax)^{\frac{1}{2}} + \sqrt{x} + \sqrt{(a+x)} = b - 2x.$

84. $x^4 + 2x^3 - 11x^2 + 4x + 4 = 0.$

85. $x^4 + 4a^3x = a^4.$

86. $x^4 + ax^3 + bx^2 + cx + \frac{c^2}{a^2} = 0.$

87. $1 + \sqrt{\left(1 - \frac{a}{x}\right)} = \sqrt{\left(1 + \frac{x}{a}\right)}.$

88. $x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) = \frac{142}{9}.$

89. $\sqrt{\left(x - \frac{1}{x}\right)} - \sqrt{\left(1 - \frac{1}{x}\right)} = \frac{x-1}{x}.$

90. $\frac{x^4 + 1}{(x+1)^4} = \frac{1}{2}.$

91. $x^3 + 1 = 0.$

92. $nx^3 + x + n + 1 = 0.$

93. $(x-2)(x-3)(x-4) = 1 \,.\, 2 \,.\, 3.$

94. $(x-1)(x-2)(x-3) - (6-1)(6-2)(6-3) = 0.$

95. $(x-1)(x-2)(x-3) = 24.$

96. $6x^3 - 5x^2 + x = 0.$

97. $x^3 + x^2 - 4x - 4 = 0.$

98. $\frac{x}{a} + \frac{b}{x} + \frac{b^2}{x^2} = 1 + \frac{b}{a} + \frac{b^2}{a^2}.$

99. $8x^2 + 16x = 9.$

100. $x^2 - \frac{2}{3x} = 1\frac{4}{9}.$

101. $3x^6 + 8x^4 - 8x^2 = 3.$

102. $x(x^2-2)=m(x^2+2mx+2)$.

103. $(x^2-a^2)(x+a)b+(a^2-b^2)(a+b)x+(b^2-x^2)(b+x)a=0$.

104. $x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1=0$.

105. $(p-1)^2x^3+px^2+\left(p-1+\frac{1}{p-1}\right)x+1=0$.

XXII. THEORY OF QUADRATIC EQUATIONS AND QUADRATIC EXPRESSIONS.

334. *A quadratic equation cannot have more than two roots.*

For any quadratic equation will take the form $ax^2+bx+c=0$ if all the terms are brought to one side of the equation; and then by Art. 318 the value of x *must be* either

$$\frac{-b+\sqrt{(b^2-4ac)}}{2a} \text{ or } \frac{-b-\sqrt{(b^2-4ac)}}{2a},$$

that is the value of x must be one or the other of *two* quantities.

The result is sometimes obtained thus. If possible let three different quantities α, β, γ be roots of the quadratic equation $ax^2+bx+c=0$; then, by supposition,

$$a\alpha^2+b\alpha+c=0, \quad a\beta^2+b\beta+c=0, \quad a\gamma^2+b\gamma+c=0.$$

By subtraction,

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0;$$

divide by $\alpha-\beta$ which is, by supposition, not zero; thus

$$a(\alpha+\beta)+b=0.$$

Similarly we have $\quad a(\alpha+\gamma)+b=0.$

By subtraction, $\quad a(\beta-\gamma)=0;$

this however is impossible, since by supposition a is not zero, and $\beta-\gamma$ is not zero. Hence there cannot be three different roots to a quadratic equation.

335. *In a quadratic equation where the coefficient of the first term is unity and the terms are all on one side, the sum of the roots is equal to the coefficient of the second term with its sign changed, and the product of the roots is equal to the last term.*

For the roots of $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \frac{-b - \sqrt{(b^2 - 4ac)}}{2a};$$

hence the sum of the roots is $-\frac{b}{a}$, and the product of the roots is $\frac{b^2 - (b^2 - 4ac)}{4a^2}$, that is, $\frac{c}{a}$. And by dividing by a the equation may be written $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$; and thus the proposition is established.

336. Let α and β denote the roots of the equation

$$ax^2 + bx + c = 0;$$

then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. These relations are useful in finding the values of expressions in which α and β occur in a symmetrical manner. For example,

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a};$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{b^2 - 4ac}{a^2};$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = -\frac{b}{a} \div \frac{c}{a} = -\frac{b}{c}.$$

The relations demonstrated in Art. 335 are useful in verifying the solution of a quadratic equation; of course if the roots obtained do not satisfy these relations we are certain that there is some error in the work.

When we know one root of a quadratic equation we can deduce the other root by the aid of either of these relations. Take for example the equation

$$\frac{a + c}{x + a} + \frac{b + c}{x + b} = \frac{2(a + b + c)}{x + a + b}.$$

Here $x = c$ obviously satisfies the equation; clearing of fractions we obtain

$$(a+b)x^2 + \{a^2 + b^2 - c(a+b)\}x - c(a^2+b^2) = 0.$$

Thus the product of the roots is $-\frac{c(a^2+b^2)}{a+b}$; and as one root is c the other must be $-\frac{a^2+b^2}{a+b}$.

337. We have

$$ax^2 + bx + c = a\left\{x^2 + \frac{bx}{a} + \frac{c}{a}\right\};$$

now put for $\frac{b}{a}$ and $\frac{c}{a}$ their values in terms of α and β; thus

$$ax^2 + bx + c = a\{x^2 - (\alpha+\beta)x + \alpha\beta\} = a(x-\alpha)(x-\beta).$$

Thus the *expression* $ax^2 + bx + c$ is identical with the *expression* $a(x-\alpha)(x-\beta)$; that is, the two expressions are equal for all values of x.

Hence we can prove the statement of Art. 334 in another manner. For no other value of x besides α and β can make $(x-\alpha)(x-\beta)$ vanish; since the product of two quantities cannot vanish if neither of the quantities vanishes.

The student may naturally ask if the identity

$$ax^2 + bx + c = a(x-\alpha)(x-\beta)$$

holds in those cases alluded to in Art. 323, where the roots of $ax^2 + bx + c = 0$ are *impossible;* we shall return to this point in Chapter XXV.

338. The student must be careful to distinguish between a *quadratic equation* and a *quadratic expression.* In the quadratic *equation* $ax^2 + bx + c = 0$ we must suppose x to have one of two definite values, but when we speak of the quadratic *expression* $ax^2 + bx + c$, without saying that it is to be equal to zero, we may suppose x to have any value we please.

339. We have

$$ax^2 + bx + c = a\left\{x^2 + \frac{bx}{a} + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right\} = a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}.$$

Now first suppose that $b^2 - 4ac$ is *negative;* then $\dfrac{b^2 - 4ac}{4a^2}$ is also negative; hence $\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2 - 4ac}{4a^2}$ is necessarily positive for all real values of x. In this case, $ax^2 + bx + c$ being equal to the product of a into some positive quantity must have the same sign as a. Thus if $b^2 - 4ac$ be negative, $ax^2 + bx + c$ has the same sign as a for all real values of x.

Next suppose that $b^2 - 4ac$ is *zero;* then

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2.$$

Here, as before, $ax^2 + bx + c$ has the same sign as a; in this case the expression $ax^2 + bx + c$ is a *perfect square* with respect to x, and its square root is

$$\pm\sqrt{a}\left(x + \frac{b}{2a}\right).$$

Last, suppose that $b^2 - 4ac$ is positive; then

$$ax^2 + bx + c = a\left\{x + \frac{b}{2a} - \frac{\sqrt{(b^2 - 4ac)}}{2a}\right\}\left\{x + \frac{b}{2a} + \frac{\sqrt{(b^2 - 4ac)}}{2a}\right\}$$

$$= a(x - \alpha)(x - \beta),$$

where α and β are both real quantities, namely,

$$\alpha = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a} \text{ and } \beta = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a}.$$

The expression $a(x - \alpha)(x - \beta)$ must have the same sign as a except when one of the factors $x - \alpha$ and $x - \beta$ is positive, and the other is negative; and we shall now shew that this can only be the case when x lies in value between α and β. Of the two quantities $\alpha - \beta$ and $\beta - \alpha$ one must be positive; suppose the former, so that α is *algebraically greater* than β. Now if x is *algebraically greater* than α, then $x - \alpha$ is positive, and therefore also $x - \beta$ is positive, and if x is *algebraically less* than β, then $x - \beta$ is negative, and therefore also $x - \alpha$ is negative. But if x lies between α and β, then $x - \alpha$ is negative, and $x - \beta$ is positive.

For such a value of x the sign of the expression ax^2+bx+c is the contrary to the sign of a.

The conclusion of the investigation of the three cases is this: whatever real value x may have ax^2+bx+c and a *never* differ in sign, except when the roots of $ax^2+bx+c=0$ are possible and different, and x is taken so as to lie between them.

340. The roots of

$$ax^2+bx+c=0 \text{ are } \frac{-b\pm\sqrt{(b^2-4ac)}}{2a},$$

and the roots of

$$ax^2-bx+c=0 \text{ are } \frac{b\pm\sqrt{(b^2-4ac)}}{2a}.$$

It is obvious that the latter roots are the same as the former with their signs changed. Hence if two quadratic equations differ only in the sign of the second term, the roots of one may be obtained by changing the signs of the roots of the other.

341. Suppose we want to divide ax^2+bx+c by $x-h$. The first term of the quotient is ax, and the next term $ah+b$, and there is a remainder ah^2+bh+c. If this remainder vanish, so that $ah^2+bh+c=0$, then h is a *root* of the *equation* $ax^2+bx+c=0$. Thus the *expression* ax^2+bx+c is divisible by $x-h$ only when h is a root of the *equation* $ax^2+bx+c=0$.

342. Some particular cases of the equation $ax^2+bx+c=0$ may now be investigated. The roots of the equation are

$$\frac{-b+\sqrt{(b^2-4ac)}}{2a} \text{ and } \frac{-b-\sqrt{(b^2-4ac)}}{2a};$$

we will first examine the results of supposing $a=0$.

The numerator of the first root becomes $-b+b$, that is, 0; thus this root takes the form $\frac{0}{0}$. The numerator of the second root becomes $-2b$; thus this root takes the form $\frac{-2b}{0}$. If in the original equation we put $a=0$, it becomes $bx+c=0$, so that

$x = -\frac{c}{b}$; and we may arrive at this result from the expression which takes the form $\frac{0}{0}$ by a suitable transformation. For multiply both numerator and denominator of $\frac{-b+\sqrt{(b^2-4ac)}}{2a}$ by $b+\sqrt{(b^2-4ac)}$; thus we obtain $\frac{-2c}{b+\sqrt{(b^2-4ac)}}$, and if we now put $a=0$, we obtain $\frac{-2c}{2b}$, that is, $\frac{-c}{b}$. If the root $\frac{-b-\sqrt{(b^2-4ac)}}{2a}$ be transformed by multiplying its numerator and denominator by $b-\sqrt{(b^2-4ac)}$ it becomes $\frac{-2c}{b-\sqrt{(b^2-4ac)}}$, and the smaller a is the smaller is the denominator of this fraction, and the greater the fraction itself: an equivalent result may obviously be obtained without effecting any transformation of the root. Thus we may enunciate our results as follows: in the equation $ax^2+bx+c=0$, if a be very small compared with b and c, one root is very large and the other root is nearly equal to $-\frac{c}{b}$, and the smaller a is, the larger one root becomes, and the nearer the other root approaches to $-\frac{c}{b}$.

343. Next suppose both a and b to be zero; then the ordinary expressions for both roots take the form $\frac{0}{0}$. By transforming the roots as in the preceding Article, we shall see that when a and b are both small compared with c, both roots are very large, and become greater the smaller a and b are.

344. Last, suppose a, b and c to be zero; then the roots take the form $\frac{0}{0}$. In this case, if we transform the roots as in Art. 342, we shall still obtain the form $\frac{0}{0}$; we may say here that the value of x is really indeterminate.

345. We will give an example of the application of the results of Art. 339.

Let it be required to ascertain if the fraction $\frac{x^2-2x+21}{6x-14}$ can assume any value we please by suitably choosing the value of x.

Put $$\frac{x^2-2x+21}{6x-14}=y;$$

therefore $$x^2-2x+21=y(6x-14);$$

therefore $$x^2-2(1+3y)x+21+14y=0.$$

By solving the quadratic we obtain

$$x=1+3y\pm\sqrt{(9y^2-8y-20)}.$$

Hence if x is to be real the quantity $9y^2-8y-20$ must be positive; that is, $9(y-2)\left(y+\frac{10}{9}\right)$ must be positive. Therefore y cannot lie between 2 and $-\frac{10}{9}$, but may have any other value. We conclude then that by suitably choosing the value of x, the fraction $\frac{x^2-2x+21}{6x-14}$ may have any value we please, except values between 2 and $-\frac{10}{9}$.

EXAMPLES ON THE THEORY OF QUADRATIC EQUATIONS AND QUADRATIC EXPRESSIONS.

Resolve the following four quadratic expressions into the product of simple factors:

1. $3x^2-10x-25.$
2. $x^2+73x+780.$
3. $2x^2+x-6.$
4. $x^2-88x+1612.$

5. Form the quadratic equation whose roots are 6 and 8.

6. Form the quadratic equation whose roots are 4 and 5.

7. Form the quadratic equation whose roots are 1 and -2.

8. Form the quadratic equation whose roots are $1 \pm \sqrt{5}$.

9. Find the sum, difference, and product of the roots of

$$x^2 - 42x + 117 = 0.$$

10. For what value of m will the equation $2x^2 + 8x + m = 0$ have equal roots?

11. If α and β be the roots of $x^2 - px + q = 0$, find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ and of $\alpha^3 + \beta^3$.

12. If α and β be the roots of $ax^2 + bx + c = 0$, construct the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

13. Shew that the roots of $x^2 + px + q = 0$ will be rational if $p = k + \frac{q}{k}$, where p, q, k are any rational quantities.

14. Shew that if $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have a common root, then $(a'c - ac')^2 = (a'b - ab')(b'c - c'b)$.

15. If x be real, prove that $\frac{2x - 7}{2x^2 - 2x - 5}$ can have no real value between $\frac{1}{11}$ and 1.

16. If p be greater than unity, then for all real values of x the expression $\frac{x^2 - 2x + p^2}{x^2 + 2x + p^2}$ lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$.

XXIII. SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

346. We will now give some examples of simultaneous equations where one or more of the equations may be of a degree higher than the first; various artifices are employed, the proper application of which must be learned by experience.

(1) Suppose $x^2 - 2y^2 = 71, \quad x + y = 20.$

From the second equation $y = 20 - x$; substitute in the first, thus

$$x^2 - 2(20 - x)^2 = 71;$$

therefore $$-x^2 + 80x - 800 = 71,$$

therefore $$x^2 - 80x = -871.$$

From this quadratic we shall obtain $x = 13$ or 67; then from the equation $y = 20 - x$ we obtain the corresponding values of y, namely, $y = 7$ or -47.

(2) Suppose $x^2 + y^2 = 25, \quad xy = 12.$

Here $$x^2 + y^2 = 25,$$
$$2xy = 24;$$

therefore, by addition,

$$x^2 + 2xy + y^2 = 25 + 24 = 49;$$

that is, $$(x + y)^2 = 49;$$

therefore $$x + y = \pm 7.$$

Similarly, by subtraction,

$$(x - y)^2 = 25 - 24 = 1;$$

therefore $$x - y = \pm 1.$$

We have now four cases to consider; namely,

$$x + y = 7, \quad x - y = 1; \qquad x + y = -7, \quad x - y = 1;$$
$$x + y = 7, \quad x - y = -1; \qquad x + y = -7, \quad x - y = -1.$$

By solving these simple equations we obtain finally

$$x = \pm 3, \quad y = \pm 4; \quad \text{or } x = \pm 4, \quad y = \pm 3.$$

(3) Suppose $2y^2 - 4xy + 3x^2 = 17$, $y^2 - x^2 = 16$.

Let $y = vx$, and substitute in both equations; thus

$$x^2 (2v^2 - 4v + 3) = 17, \qquad x^2 (v^2 - 1) = 16;$$

therefore, by division,

$$\frac{2v^2 - 4v + 3}{v^2 - 1} = \frac{17}{16};$$

therefore $$32v^2 - 64v + 48 = 17v^2 - 17;$$

therefore $$15v^2 - 64v + 65 = 0.$$

From this quadratic we shall obtain $v = \frac{5}{3}$ or $\frac{13}{5}$. Take the former value of v; then $x^2 = \frac{16}{v^2 - 1} = 9$; therefore $x = \pm 3$; and $y = vx = \pm 5$. Again, taking the second value of v we have $x^2 = \frac{25}{9}$; therefore, $x = \pm \frac{5}{3}$; and $y = \pm \frac{13}{3}$.

The artifice here used may be adopted conveniently when the terms involving the unknown quantities in each equation constitute an expression which is homogeneous and of the second degree; see Art. 24.

(4) Suppose $x^2 + xy - 6y^2 = 24$, $x^2 + 3xy - 10y^2 = 32$.

Let $y = vx$; substitute in both equations, and divide; thus

$$\frac{1 + 3v - 10v^2}{1 + v - 6v^2} = \frac{32}{24} = \frac{4}{3};$$

therefore $$6v^2 - 5v + 1 = 0.$$

From this quadratic we shall obtain $v = \frac{1}{2}$ or $\frac{1}{3}$. The value $v = \frac{1}{2}$ we shall find to be inapplicable; for it leads to the inadmissible result $x^2 \times 0 = 24$. In fact the equations from which the values of v were obtained may be written thus,

$$x^2 (1 - 2v)(1 + 3v) = 24, \quad x^2 (1 - 2v)(1 + 5v) = 32;$$

and hence we see that the value of v found from $1-2v=0$ is inapplicable, and that we can only have $\frac{1+3v}{1+5v}=\frac{24}{32}$, which gives $v=\frac{1}{3}$.

Then $$x^2\left(1-\frac{2}{3}\right)(1+1)=24;$$

therefore $x^2=36$; therefore $x=\pm 6$; and $y=\pm 2$.

(5) Suppose $$x+y=a, \quad x^5+y^5=b^5.$$

By division, $$\frac{x^5+y^5}{x+y}=\frac{b^5}{a};$$

that is, $$x^4-x^3y+x^2y^2-xy^3+y^4=\frac{b^5}{a};$$

or $$x^4+y^4-xy(x^2+y^2)+x^2y^2=\frac{b^5}{a}.$$

Now since $$x+y=a,$$
$$x^2+y^2=a^2-2xy;$$

therefore $x^4+y^4+2x^2y^2=(a^2-2xy)^2=a^4-4a^2xy+4x^2y^2$;

therefore $$x^4+y^4=a^4-4a^2xy+2x^2y^2.$$

By substituting the values of x^4+y^4 and x^2+y^2 we obtain
$$a^4-4a^2xy+2x^2y^2-xy(a^2-2xy)+x^2y^2=\frac{b^5}{a},$$

that is, $$5x^2y^2-5a^2xy=\frac{b^5}{a}-a^4.$$

We may obtain this result also in another way. It may be shewn that
$$a^5=x^5+y^5+5xy(x^3+y^3)+10x^2y^2(x+y);$$

thus $$a^5-b^5=5xy(x^3+y^3)+10ax^2y^2;$$

and $$a^3=x^3+y^3+3xy(x+y)$$
$$=x^3+y^3+3axy:$$

therefore $$a^5-b^5=5xy(a^3-3axy)+10ax^2y^2,$$

or $$5ax^2y^2-5a^3xy=b^5-a^5.$$

From this quadratic we can find two values of xy; let c denote one of these values, then we have

$$x+y=a, \qquad xy=c;$$

thus $$(x+y)^2-4xy=a^2-4c,$$

that is, $$(x-y)^2=a^2-4c;$$

therefore $$x-y=\pm\sqrt{(a^2-4c)}.$$

Thus since $x+y$ and $x-y$ are known, we can find immediately the values of x and y.

Or we may proceed thus. Assume $x-y=z$, then since $x+y=a$, we obtain

$$x=\frac{1}{2}(a+z), \qquad y=\frac{1}{2}(a-z).$$

Substitute in the second of the given equations; thus

$$(a+z)^5+(a-z)^5=32b^5,$$

therefore $$5az^4+10a^3z^2=16b^5-a^5.$$

From this quadratic we may find z^2, and hence z, that is, $x-y$; and hence finally x and y.

More examples will be found in Chapter LIV.

EXAMPLES OF SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

1. $4x^2+7y^2=148, \qquad 3x^2-y^2=11.$

2. $x+y=100, \qquad xy=2400.$

3. $x+y=4, \qquad \dfrac{1}{x}+\dfrac{1}{y}=1.$

4. $x+y=7, \qquad x^2+2y^2=34.$

5. $x-y=12, \qquad x^2+y^2=74.$

6. $x-\dfrac{x-y}{2}=4, \qquad y-\dfrac{x+3y}{x+2}=1.$

7. $x^2+y^2=65, \qquad xy=28.$

8. $xy=1, \quad 3x-5y=2.$

9. $\dfrac{1}{x}+\dfrac{1}{y}=2, \qquad x+y=2.$

10. $x^2+xy+2y^2=74, \qquad 2x^2+2xy+y^2=73.$

11. $2x+3y=37, \qquad \frac{1}{x}+\frac{1}{y}=\frac{14}{45}.$

12. $x^2+3xy=54, \qquad xy+4y^2=115.$

13. $x^2+xy=15, \qquad xy-y^2=2.$

14. $x^2+xy+4y^2=6, \qquad 3x^2+8y^2=14.$

15. $x^2+xy=12, \qquad xy-2y^2=1.$

16. $x^2-xy+y^2=21, \qquad y^2-2xy+15=0.$

17. $x^2-4y^2=9, \qquad xy+2y^2=3.$

18. $7x^2-8xy=159, \qquad 5x+2y=7.$

19. $x^2-2xy-y^2=1, \qquad x+y=2.$

20. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{10}{3}, \qquad x^2+y^2=45.$

21. $\frac{x+y}{x-y}+\frac{x-y}{x+y}=\frac{5}{2}, \qquad x^2+y^2=20.$

22. $\cdot 1y+\cdot 125x=y-x, \qquad y-\cdot 5x=\cdot 75xy-3x.$

23. $\cdot 3x+\cdot 125y=3x-y, \qquad 3x-\cdot 5y=2\cdot 25xy+3y.$

24. $\left.\begin{array}{r} y^2-4xy+20x^2+3y-264x=0, \\ 5y^2-38xy+x^2-12y+1056x=0. \end{array}\right\}$

25. $x+y=x^2, \qquad 3y-x=y^2.$

26. $x^2+y^2=\frac{5}{2}xy, \qquad x-y=\frac{1}{4}xy.$

27. $x+2y+\frac{3x}{y}=16, \qquad 3x+y+\frac{3x}{y}=23.$

28. $4(x+y)=3xy, \qquad x+y+x^2+y^2=26.$

29. $x-y=2, \qquad x^3-y^3=8.$

30. $x+y=5, \qquad x^3+y^3=65.$

31. $x+y=11, \qquad x^3+y^3=1001.$

32. $xy(x+y)=30, \qquad x^3+y^3=35.$

33. $\frac{x^2}{y}+\frac{y^2}{x}=18, \qquad x+y=12.$

34. $x+y=18, \qquad x^3+y^3=4914.$

35. $\frac{x^2}{y}+\frac{y^2}{x}=9, \qquad \frac{1}{x}+\frac{1}{y}=\frac{3}{4}.$

36. $x^2(x+y)=80, \qquad x^2(2x-3y)=80.$

37. $x^2y+y^2x=20, \qquad \frac{1}{x}+\frac{1}{y}=\frac{5}{4}.$

38. $x^2+y^2=7+xy, \qquad x^3+y^3=6xy-1.$

39. $x^2+y^2=8, \qquad \frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{2}.$

40. $x+y=4, \qquad x^4+y^4=82.$

41. $x^5-y^5=3093, \qquad x-y=3.$

42. $\left(3-\frac{6y}{x+y}\right)^2+\left(3+\frac{6y}{x-y}\right)^2=82, \qquad xy=2.$

43. $x^2-x^2y^2+y^2=19, \qquad x-xy+y=4.$

44. $x^2-xy+y^2=7, \qquad x^4+x^2y^2+y^4=133.$

45. $x^2+xy+y^2=49, \qquad x^4+x^2y^2+y^4=931.$

46. $x^4-x^2+y^4-y^2=84, \qquad x^2+x^2y^2+y^2=49.$

47. $x(12-xy)=y(xy-3), \qquad xy(y+4x-xy)=12(x+y-3).$

48. $x+y+\sqrt{(xy)}=14, \qquad x^2+y^2+xy=84.$

49. $x+y-\sqrt{(xy)}=7, \qquad x^2+y^2+xy=133.$

50. $x+y=72, \qquad \sqrt[3]{x}+\sqrt[3]{y}=6.$

51. $x+\sqrt{(x^2-y^2)}=8, \qquad x-y=1.$

52. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{7}{\sqrt{(xy)}}+1, \qquad \sqrt{(x^3y)}+\sqrt{(y^3x)}=78.$

53. $x+y=10, \qquad \sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}.$

54. $\sqrt{x} - \sqrt{y} = 2\sqrt{(xy)}, \qquad x + y = 20.$

55. $\sqrt{(x+y)} + 2\sqrt{(x-y)} = \dfrac{2(x-1)}{\sqrt{(x-y)}}, \qquad \dfrac{x^2+y^2}{xy} = \dfrac{34}{15}.$

56. $\sqrt{(3+x^2)} + 2y = 8, \qquad 2x^2 + \sqrt{(5y^2+4x^4)} = 9.$

57. $\dfrac{x}{a} + \dfrac{y}{b} = 1, \qquad \dfrac{a}{x} + \dfrac{b}{y} = 4.$

58. $x^2 - y^2 = a^2, \qquad xy = b^2.$

59. $x + y = a, \qquad x^4 + y^4 = b^4.$

60. $x^4 + y^4 = 14x^2y^2, \qquad x + y = a.$

61. $\dfrac{a}{a+x} + \dfrac{b}{b+y} = 1, \qquad x + y = a + b.$

62. $\dfrac{bx}{y+b} + \dfrac{ay}{x+a} = \dfrac{a+b}{2}, \qquad \dfrac{x}{a} + \dfrac{y}{b} = 2.$

63. $x - y = a, \qquad x^5 - y^5 = b^5.$

64. $\sqrt{(x^2+y^2)} + \sqrt{(x^2-y^2)} = 2y, \qquad x^4 - y^4 = a^4.$

65. $2ab(a+b)x + y^2 = abx^2 + 2aby, \qquad abx + (a+b)y = xy.$

66. $2\sqrt{(x^2-y^2)} + xy = 1, \qquad \dfrac{x}{y} - \dfrac{y}{x} = a.$

67. $x + y = a\sqrt{(xy)}, \qquad x - y = c\sqrt{\dfrac{x}{y}}.$

68. $\sqrt{(x+y)} + \sqrt{(x-y)} = \sqrt{a}, \qquad \sqrt{(x^2+y^2)} + \sqrt{(x^2-y^2)} = b.$

69. $\left(\dfrac{a^2-x^2}{y^2-b^2} + \dfrac{y^2-b^2}{a^2-x^2}\right)^{\frac{1}{2}} + \left(\dfrac{a^2+x^2}{y^2+b^2} + \dfrac{y^2+b^2}{a^2+x^2}\right)^{\frac{1}{2}} = 4, \qquad xy = ab.$

70. $x^2 + y^2 - (x+y) = a, \qquad x^4 + y^4 + x + y - 2(x^3+y^3) = b.$

71. $yz = bc, \qquad \dfrac{x}{a} + \dfrac{y}{b} = 1, \qquad \dfrac{x}{a} + \dfrac{z}{c} = 1.$

72. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 9, \qquad \dfrac{2}{x} + \dfrac{3}{y} = 13, \qquad 8x + 3y = 5.$

73. $y+z=\frac{1}{x}, \qquad z+x=\frac{1}{y}, \qquad x+y=\frac{1}{z}.$

74. $xyz=a^2(y+z)=b^2(z+x)=c^2(x+y).$

75. $x^2+yz=y^2+zx=c, \qquad z^2+xy=a.$

76. $\frac{1}{29}\left(x+\frac{y}{z}\right)=\frac{1}{34}\left(y+\frac{x}{z}\right)=\frac{1}{6}, \qquad x+y+z=15.$

77. $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{2}, \qquad xyz=1.$

78. $x^3+y^3+z^3=x^2+y^2+z^2=x+y+z=1.$

79. $x(x+y+z)=a^2, \qquad y(x+y+z)=b^2, \qquad z(x+y+z)=c^2.$

80. $$\left.\begin{aligned} &xy+xz+yz=26,\\ &xy(x+y)+yz(y+z)+zx(z+x)=162,\\ &xy(x^2+y^2)+yz(y^2+z^2)+xz(x^2+z^2)=538. \end{aligned}\right\}$$

XXIV. PROBLEMS WHICH LEAD TO QUADRATIC EQUATIONS.

347. We shall now solve and discuss some problems which lead to quadratic equations.

A man buys a horse which he sells again for £24; he finds that he thus loses as much per cent. as the horse cost; required the price of the horse.

Let x denote the price in pounds; then the man loses x per cent. and thus his total loss is $\frac{x}{100}\times x$, that is, $\frac{x^2}{100}$; but this loss is also $x-24$; thus

$$\frac{x^2}{100}=x-24;$$

therefore $$x^2 - 100x = -2400,$$

and $$x^2 - 100x + (50)^2 = 2500 - 2400 = 100;$$

hence $$x - 50 = \pm 10,$$

and $$x = 60 \text{ or } 40.$$

Thus all we can infer is, that the price was either £60 or £40, for each of these values satisfies all the conditions of the problem.

348. Divide the number 10 into two parts, such that their product shall be 24.

Let x denote one part, and therefore $10 - x$ the other part; then

$$x(10 - x) = 24;$$

therefore $$x^2 - 10x = -24,$$

and $$x^2 - 10x + 5^2 = 25 - 24 = 1;$$

hence $$x - 5 = \pm 1,$$

and $$x = 4 \text{ or } 6.$$

Here although x may have either of two values, yet there is only one mode of dividing 10, so that the product of the two parts shall be 24; one part must be 4 and the other 6.

349. A person bought a certain number of oxen for £80; if he had bought 4 more for the same sum each ox would have cost £1 less; find the number of oxen and the price of each.

Let x denote the number of oxen, then $\frac{80}{x}$ is the price of each in pounds; if the person had bought 4 more, the price of each in pounds would have been $\frac{80}{x+4}$: thus, by supposition,

$$\frac{80}{x+4} = \frac{80}{x} - 1;$$

therefore $$80x = 80(x + 4) - x^2 - 4x,$$

therefore $x^2 + 4x = 320,$

and $x^2 + 4x + 2^2 = 320 + 4 = 324;$

hence $x + 2 = \pm 18,$

and $x = 16 \text{ or } -20.$

Only the positive value of x is admissible, and thus the number of oxen is 16, and the price of each ox is £5.

In solving problems, as in the proposed example, results will sometimes be obtained which do not apply to the question actually proposed. The reason appears to be that the algebraical mode of expression is more general than ordinary language, and thus the equation, which is a proper representation of the conditions of the problem, will also apply to other conditions. Experience will convince the student that he will always be able to select the result which belongs to the problem he is solving, and that it will be sometimes possible, by suitable changes in the enunciation of the original problem, to form a new problem, corresponding to any result which was inapplicable to the original problem. Thus in the present case we may propose the following modification of the original problem: a person *sold* a certain number of oxen for £80; if he had *sold* 4 *fewer* for the same sum, the price of each ox would have been £1 *more;* find the number of oxen and the price of each.

Let x represent the number; then by the question we shall have

$$\frac{80}{x-4} = \frac{80}{x} + 1.$$

The roots of this quadratic will be found to be 20 and -16; thus the number 20 which appeared with a negative sign as a result in the former case, and was then inapplicable, is here the admissible result.

350. Find a number such that twice its square increased by three times the number itself may amount to 65.

Let x denote the number; then, by the question,

$$2x^2 + 3x = 65.$$

The roots of this quadratic will be found to be 5 and $-\frac{13}{2}$; the first value satisfies the conditions of the question. In order to interpret the second value, we observe, that if we write $-x$ for x in the equation, it becomes

$$2x^2 - 3x = 65;$$

and the roots of the latter equation are $\frac{13}{2}$ and -5, as will be found on trial, or may be known from Art. 340. Hence $\frac{13}{2}$ is the answer to a new question, namely: find a number such that twice its square *diminished* by three times the number itself may amount to 65.

351. Divide a given line into two parts, such that twice the square on one part may be equal to the rectangle contained by the whole line and the other part.

Let a denote the length of the line, and x the length of one part, then $a - x$ is the length of the other part; thus, by the question,

$$2x^2 = a(a - x);$$

therefore $$2x^2 + ax = a^2,$$

and $$x^2 + \frac{ax}{2} = \frac{a^2}{2},$$

and $$x^2 + \frac{ax}{2} + \left(\frac{a}{4}\right)^2 = \frac{a^2}{2} + \frac{a^2}{16} = \frac{9a^2}{16};$$

hence $$x + \frac{a}{4} = \pm \frac{3a}{4},$$

and $$x = \frac{a}{2} \text{ or } -a.$$

Here $\frac{a}{2}$ is the required length. The negative answer suggests the following problem: produce a given line, so that twice the square on the part produced may be equal to the rectangle contained by the given line, and the line made up of the given line and the part produced; the result is, that the part produced must be equal to the given line.

352. In the examples hitherto given, both roots of the quadratic equation have applied to the actual problem, or to an allied problem which was easily formed. Frequently, however, it will be found that only one root applies to the problem proposed, and that no obvious interpretation occurs for the other.

353. Problems may be proposed which involve more than one unknown quantity, and thus lead to *simultaneous equations*; we will give an example.

Two men A and B sell a quantity of wheat for £28. 8*s*. B sells four quarters more than A, and if he had sold the quantity A sold, would have received £10 for it; while A would have received 16 guineas for what B sold. Find the quantity sold by each, and the rates at which they sold it.

Let x denote the number of quarters which A sold, and therefore $x+4$ the number which B sold; and suppose that A sold his wheat at y shillings per quarter, and that B sold his at z shillings per quarter. Then since the value of the wheat sold is 568 shillings, we have

$$xy + (x+4)z = 568 \qquad (1).$$

If B had sold the quantity A sold, he would have received 200 shillings; thus

$$xz = 200 \qquad (2).$$

Similarly,

$$(x+4)y = 336 \qquad (3).$$

From (3) we have $xy = 336 - 4y$; by substitution in (1) we have

$$336 - 4y + 200 + 4z = 568;$$

therefore

$$4(z-y) = 32,$$

and

$$z - y = 8 \qquad (4).$$

From (2) we have

$$x = \frac{200}{z},$$

and from (3) we have

$$x = \frac{336}{y} - 4;$$

thus $$\frac{200}{z} = \frac{336}{y} - 4,$$

and $$\frac{50}{z} = \frac{84}{y} - 1 \text{ ....(5).}$$

We may now find y and z from (4) and (5). Substitute in (5) the value of z from (4); thus

$$\frac{50}{y+8} = \frac{84}{y} - 1;$$

therefore $$50y = 84(y+8) - (y^2 + 8y),$$

hence $$y^2 - 26y - 672 = 0.$$

From this quadratic we shall find $y = 42$ or -16. The former is the only admissible result; thus $z = 50$; and $x = 4$.

EXAMPLES OF PROBLEMS.

1. Find two numbers such that their sum may be 39, and the sum of their cubes 17199.

2. A certain number is formed by the product of three consecutive numbers, and if it be divided by each of them in turn, the sum of the quotients is 47. Find the number.

3. The length of a rectangular field exceeds the breadth by one yard, and the area is three acres: find the length of the sides.

4. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour and 40 minutes: supposing the river to have a current of 2 miles per hour, find the rate at which the crew would row in still water.

5. A farmer wishes to enclose a rectangular piece of land to contain 1 acre 32 perches with 176 hurdles, each two yards long; how many hurdles must he place in each side of the rectangle?

6. A person rents a certain number of acres of land for £84; he cultivates 4 acres himself, and letting the rest for 10*s.* an acre more than he pays for it, receives for this portion the whole rent, £84. Find the number of acres.

7. A person purchased a certain number of sheep for £35: after losing two of them he sold the rest at 10 shillings a head more than he gave for them, and by so doing gained £1 by the transaction. Find the number of sheep he purchased.

8. A line of given length is bisected and produced: find the length of the produced part so that the rectangle contained by half the line and the line made up of the half and the produced part may be equal to the square on the produced part.

9. The product of two numbers is 750, and the quotient when one is divided by the other is $3\frac{1}{3}$: find the numbers.

10. A gentleman sends a lad into the market to buy a shilling's worth of oranges. The lad having eaten a couple, the gentleman pays at the rate of a penny for fifteen more than the market-price; how many did the gentleman get for his shilling?

11. What are eggs a dozen when two more in a shilling's worth lowers the price one penny per dozen?

12. A shilling's worth of Bavarian kreuzers is more numerous by 6 than a shilling's worth of Austrian kreuzers; and 15 Austrian kreuzers are worth 1*d.* more than 15 Bavarian kreuzers. How many Austrian and Bavarian kreuzers respectively make a shilling?

13. Find two numbers whose sum is nine times their difference, and whose product diminished by the greater number is equal to twelve times the greater number divided by the less.

14. Two workmen were employed at different wages, and paid at the end of a certain time. The first received £4. 16*s.*,

and the second, who had worked for 6 days less, received £2. 14*s.* If the second had worked all the time and the first had omitted 6 days, they would have received the same sum. How many days did each work, and what were the wages of each?

15. A party at a tavern spent a certain sum of money. If there had been five more in the party, and each person had spent a shilling more, the bill would have been £6. If there had been three less in the party, and each person had spent eightpence less, the bill would have been £2. 12*s.* Of how many did the party consist, and what did each person spend?

16. A person bought a number of £20 railway shares when they were at a certain rate per cent. discount for £1500; and afterwards when they were at the same rate per cent. premium sold them all but 60 for £1000. How many did he buy, and what did he give for each of them?

17. Find that number whose square added to its cube is nine times the next higher number.

18. A person has £1300, which he divides into two portions and lends at different *rates* of interest, so that the two portions produce equal returns. If the first portion had been lent at the second rate of interest it would have produced £36; and if the second portion had been lent at the first rate of interest it would have produced £49. Find the rates of interest.

19. A person having travelled 56 miles on a railroad and the rest of his journey by a coach, observed that in the train he had performed a quarter of his whole journey in the time the coach took to go 5 miles, and that at the instant he arrives at home the train must have reached a point 35 miles further than he was from the station at which it left him. Compare the rates of the coach and the train, and find the number of miles in the rest of the journey.

20. *A* sets off from London to York, and *B* at the same time from York to London, and they travel uniformly; *A* reaches York 16 hours, and *B* reaches London 36 hours, after they have

met on the road. Find in what time each has performed the journey.

21. A courier proceeds from one place P to another place Q in 14 hours; a second courier starts at the same time as the first from a place 10 miles behind P, and arrives at Q at the same time as the first courier. The second courier finds that he takes half an hour less than the first to accomplish 20 miles. Find the distance of Q from P.

22. Two travellers A and B set out at the same time from two places P and Q respectively, and travel so as to meet. When they meet it is found that A has travelled 30 miles more than B, and that A will reach Q in 4 days, and B will reach P in 9 days, after they meet. Find the distance between P and Q.

23. A vessel can be filled with water by two pipes; by one of these pipes alone the vessel would be filled 2 hours sooner than by the other; also the vessel can be filled by both pipes together in $1\frac{7}{8}$ hours. Find the time which each pipe alone would take to fill the vessel.

24. A vessel is to be filled with water by two pipes. The first pipe is kept open during three-fifths of the time which the second would take to fill the vessel; then the first pipe is closed and the second is opened. If the two pipes had both been kept open together the vessel would have been filled 6 hours sooner, and the first pipe would have brought in two-thirds of the quantity of water which the second pipe really brought in. How long would each pipe alone take to fill the vessel?

25. A certain number of workmen can move a heap of stones in 8 hours from one place to another. If there had been 8 more workmen, and each workman had carried 5 lbs. less at a time, the whole work would have occupied 7 hours. If however there had been 8 fewer workmen, and each workman had carried 11 lbs. more at a time, the work would have occupied 9 hours. Find the number of workmen and the weight which each carried at a time.

XXV. IMAGINARY EXPRESSIONS.

354. Although the square root of a negative quantity is the symbol of an impossible operation, yet these square roots are frequently of use in Mathematical investigations in consequence of a few conventions which we shall now explain.

355. Let a denote any real quantity; then the square roots of the negative quantity $-a^2$ are expressed in ordinary notation by $\pm\sqrt{(-a^2)}$. Now $-a^2$ may be considered as the product of a^2 and -1; so if we suppose that the square roots of this product can be formed, in the same manner as if both factors were positive, by multiplying together the square roots of the factors, the square roots of $-a^2$ will be expressed by $\pm a\sqrt{(-1)}$. We may therefore *agree* that the expressions $\pm\sqrt{(-a^2)}$ and $\pm a\sqrt{(-1)}$ shall be considered equivalent. Thus we shall only have to use one imaginary expression in our investigations, namely, $\sqrt{(-1)}$.

356. Suppose we have such an expression as $\alpha+\beta\sqrt{(-1)}$, where α and β are real quantities. This expression may be said to consist of a real part α and an imaginary part $\beta\sqrt{(-1)}$; or on account of the presence of the latter term we may speak of the whole expression as imaginary. When β is zero, the term $\beta\sqrt{(-1)}$ is *considered to vanish*; this may be regarded then as another convention. If α and β are both zero, the whole expression vanishes, and not otherwise.

357. By means of the conventions already made, and the additional convention that such terms as $\beta\sqrt{(-1)}$ shall be subject to the ordinary rules which hold in Algebraical transformations, we may establish some propositions, as will now be seen.

358. *In order that two imaginary expressions may be equal, it is necessary and sufficient that the real parts should be equal, and that the coefficients of* $\sqrt{(-1)}$ *should be equal.*

For suppose $\alpha + \beta\sqrt{(-1)} = \gamma + \delta\sqrt{(-1)}$;
then, by transposition, $\alpha - \gamma + (\beta - \delta)\sqrt{(-1)} = 0$;
thus, by Art. 356, $\alpha - \gamma = 0$, and $\beta - \delta = 0$;
that is, $\alpha = \gamma$, and $\beta = \delta$.

Thus the equation

$$\alpha + \beta\sqrt{(-1)} = \gamma + \delta\sqrt{(-1)}$$

may be considered as a symbolical mode of asserting the *two* equalities $\alpha = \gamma$ and $\beta = \delta$ in *one* statement.

359. Take now two imaginary expressions $\alpha + \beta\sqrt{(-1)}$ and $\gamma + \delta\sqrt{(-1)}$, and form their sum, difference, product, and quotient.

Their sum is

$$\alpha + \gamma + (\beta + \delta)\sqrt{(-1)}.$$

If the second expression be taken from the first, the remainder is

$$\alpha - \gamma + (\beta - \delta)\sqrt{(-1)}.$$

Their product is

$$\{\alpha + \beta\sqrt{(-1)}\}\{\gamma + \delta\sqrt{(-1)}\} = \alpha\gamma - \beta\delta + (\alpha\delta + \beta\gamma)\sqrt{(-1)};$$

for $\sqrt{(-1)} \times \sqrt{(-1)}$ is, by supposition, -1.

The quotient obtained by dividing the first expression by the second is

$$\frac{\alpha + \beta\sqrt{(-1)}}{\gamma + \delta\sqrt{(-1)}}.$$

This may be put in another form by multiplying both numerator and denominator by $\gamma - \delta\sqrt{(-1)}$. The new numerator is thus

$$\alpha\gamma + \beta\delta + (\beta\gamma - \alpha\delta)\sqrt{(-1)};$$

and the new denominator is $\gamma^2 + \delta^2$; therefore

$$\frac{\alpha + \beta\sqrt{(-1)}}{\gamma + \delta\sqrt{(-1)}} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}\sqrt{(-1)}.$$

360. We will now give an example of the way in which imaginary expressions occur in Algebra. Suppose we have to solve the equation $x^3 = 1$. We may write the equation thus,

$$x^3 - 1 = 0;$$

or in factors, $(x - 1)(x^2 + x + 1) = 0$.

Thus we satisfy the proposed equation either by putting $x-1=0$, or by putting $x^2+x+1=0$. The first gives $x=1$; the second may be written

$$x^2+x=-1,$$

therefore $$x^2+x+\left(\frac{1}{2}\right)^2=\frac{1}{4}-1=-\frac{3}{4};$$

therefore $$x+\frac{1}{2}=\pm\sqrt{\left(-\frac{3}{4}\right)}=\pm\frac{\sqrt{3}}{2}\sqrt{(-1)};$$

and $$x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\sqrt{(-1)}.$$

Thus we conclude that if either of the imaginary expressions last written be cubed, the result will be unity. This we may verify; take the upper sign for example, then

$$\left\{-\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^3=\left(-\frac{1}{2}\right)^3+3\left(-\frac{1}{2}\right)^2\frac{\sqrt{3}}{2}\sqrt{(-1)}$$
$$+3\left(-\frac{1}{2}\right)\left\{\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^2+\left\{\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^3.$$

Now $$\left(-\frac{1}{2}\right)^3=-\frac{1}{8},$$

$$3\left(-\frac{1}{2}\right)^2\frac{\sqrt{3}}{2}\sqrt{(-1)}=\frac{3}{4}\,\frac{\sqrt{3}}{2}\sqrt{(-1)}=\frac{3\sqrt{3}}{8}\sqrt{(-1)},$$

$$3\left(-\frac{1}{2}\right)\left\{\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^2=\left(-\frac{3}{2}\right)\left(-\frac{3}{4}\right)=\frac{9}{8},$$

$$\left\{\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^3=\left\{\frac{\sqrt{3}}{2}\sqrt{(-1)}\right\}^2\frac{\sqrt{3}}{2}\sqrt{(-1)}$$
$$=-\frac{3}{4}\times\frac{\sqrt{3}}{2}\sqrt{(-1)}=-\frac{3\sqrt{3}}{8}\sqrt{(-1)}.$$

Thus the result is unity.

If $x^3=1$, we have $x=(1)^{\frac{1}{3}}$; it appears then that there are three cube roots of unity, namely, 1 and $-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\sqrt{(-1)}$.

361. We have seen in Art. 337, that the quadratic *expression* ax^2+bx+c is always identical with $a(x-p)(x-q)$, where p and q are the roots of the *equation* $ax^2+bx+c=0$. If the roots are imaginary, p and q will be of the forms $\alpha \pm \beta\sqrt{(-1)}$; thus we have then

$$ax^2+bx+c=a\{x-\alpha-\beta\sqrt{(-1)}\}\{x-\alpha+\beta\sqrt{(-1)}\}.$$

This will present no difficulty when we remember the convention that the usual algebraical operations are to be applicable to the term $\beta\sqrt{(-1)}$. For the second side of the asserted identity is

$$a\{(x-\alpha)^2+\beta^2\}, \qquad \text{that is, } a\{x^2-2\alpha x+\alpha^2+\beta^2\},$$

and from the values of α and β we have

$$2\alpha=-\frac{b}{a}, \qquad \text{and } \alpha^2+\beta^2=\frac{c}{a};$$

thus the second side coincides with the first.

362. Two imaginary expressions are said to be *conjugate* when they differ only in the sign of the coefficient of $\sqrt{(-1)}$. Thus $\alpha+\beta\sqrt{(-1)}$ and $\alpha-\beta\sqrt{(-1)}$ are *conjugate*.

Hence the *sum* of two conjugate imaginary expressions is real, and so also is their product. In the above example the sum is 2α, and the product is $\alpha^2+\beta^2$.

363. The positive value of the square root of $\alpha^2+\beta^2$ is called the *modulus* of each of the expressions

$$\alpha+\beta\sqrt{(-1)} \text{ and } \alpha-\beta\sqrt{(-1)}.$$

From this definition it follows that the modulus of a real quantity is the numerical value of that quantity taken positively.

In order that the modulus $\sqrt{(\alpha^2+\beta^2)}$ may vanish, it is necessary that $\alpha=0$ and $\beta=0$; in this case the expressions

$$\alpha+\beta\sqrt{(-1)} \text{ and } \alpha-\beta\sqrt{(-1)}$$

vanish. And conversely, if these expressions vanish, then $\alpha=0$ and $\beta=0$, and thus the modulus vanishes.

364. If two imaginary expressions are equal, their *moduli* are equal. It is not however necessarily true, that the expressions are equal if the moduli are equal.

365. The modulus of the product of $\alpha + \beta \sqrt{(-1)}$ and $\gamma + \delta \sqrt{(-1)}$ is

$$\sqrt{\{(\alpha\gamma - \beta\delta)^2 + (\beta\gamma + \alpha\delta)^2\}}\text{; (see Art. 359).}$$

But $$(\alpha\gamma - \beta\delta)^2 + (\beta\gamma + \alpha\delta)^2 = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2);$$

thus the modulus is

$$\sqrt{(\alpha^2 + \beta^2)} \times \sqrt{(\gamma^2 + \delta^2)}.$$

Hence *the modulus of the product of two imaginary expressions is equal to the product of their moduli.*

Therefore the *product* of two imaginary expressions cannot vanish if neither factor vanishes.

It will follow from this that the modulus of the quotient of two imaginary expressions is the quotient of their moduli. This can also be shewn by forming the modulus of the expression for the quotient given in Art. 359.

366. It is often necessary to consider the powers of $\sqrt{(-1)}$. We may form them by successive multiplication; thus,

$$\{\sqrt{(-1)}\}^1 = \sqrt{(-1)}, \qquad \{\sqrt{(-1)}\}^2 = -1,$$

$$\{\sqrt{(-1)}\}^3 = \{\sqrt{(-1)}\}^2 \times \sqrt{(-1)} = -\sqrt{(-1)}, \qquad \{\sqrt{(-1)}\}^4 = 1.$$

If we proceed to obtain higher powers we shall have a recurrence of the results $\sqrt{(-1)}$, -1, $-\sqrt{(-1)}$, 1. We may then express all the powers by four formulæ. For every whole number must be of one of the four forms $4n$, $4n+1$, $4n+2$, $4n+3$, according as it is exactly divisible by 4, or leaves, when divided by 4, a remainder 1, 2, 3, respectively. And

$$\{\sqrt{(-1)}\}^{4n} = 1, \qquad \{\sqrt{(-1)}\}^{4n+1} = \sqrt{(-1)},$$

$$\{\sqrt{(-1)}\}^{4n+2} = -1, \qquad \{\sqrt{(-1)}\}^{4n+3} = -\sqrt{(-1)}.$$

367. *The square root of an imaginary expression of the form* $\alpha + \beta\sqrt{(-1)}$ *may be expressed in a similar form.*

For suppose $\sqrt{\{\alpha + \beta\sqrt{(-1)}\}} = x + y\sqrt{(-1)}$;

then $\alpha + \beta\sqrt{(-1)} = \{x + y\sqrt{(-1)}\}^2 = x^2 - y^2 + 2xy\sqrt{(-1)}$.

Hence, by Art. 358,

$$x^2 - y^2 = \alpha \text{..............................} (1),$$

$$2xy = \beta \text{..............................} (2);$$

therefore from (1) and (2)

$$(x^2 + y^2)^2 = \alpha^2 + \beta^2,$$

thus

$$x^2 + y^2 = \sqrt{(\alpha^2 + \beta^2)} \text{ ....} (3).$$

From (1) and (3) we obtain

$$x^2 = \tfrac{1}{2}\{\sqrt{(\alpha^2 + \beta^2)} + \alpha\}, \qquad y^2 = \tfrac{1}{2}\{\sqrt{(\alpha^2 + \beta^2)} - \alpha\};$$

hence

$$x = \pm\left\{\frac{\sqrt{(\alpha^2 + \beta^2)} + \alpha}{2}\right\}^{\frac{1}{2}}, \qquad y = \pm\left\{\frac{\sqrt{(\alpha^2 + \beta^2)} - \alpha}{2}\right\}^{\frac{1}{2}}.$$

Since the values of x and y are supposed real, $x^2 + y^2$ is positive, and thus the positive sign must be ascribed to the quantity $\sqrt{(\alpha^2 + \beta^2)}$. And since the values of x and y must satisfy the equation $2xy = \beta$, they must have the *same* sign if β be *positive*, and *different* signs if β be *negative*. On account of the double sign in the values of x and y, we see that $\alpha + \beta\sqrt{(-1)}$ has two square roots which differ only in sign.

368. We may obtain the square roots of $\pm\sqrt{(-1)}$ by supposing that $\alpha = 0$ and $\beta = \pm 1$ in the results of the preceding Article. Thus we shall obtain

$$\sqrt{\{+\sqrt{(-1)}\}} = \pm\frac{1 + \sqrt{(-1)}}{\sqrt{2}}, \qquad \sqrt{\{-\sqrt{(-1)}\}} = \pm\frac{1 - \sqrt{(-1)}}{\sqrt{2}}.$$

If we suppose that $z^4 = -1$, we deduce $z^2 = \pm\sqrt{(-1)}$; thus $z = \pm\sqrt{\{\pm\sqrt{(-1)}\}}$. And since $z^4 = -1$, we have $z = (-1)^{\frac{1}{4}}$. Thus there are four fourth roots of -1, namely, the four expressions contained in $\pm\dfrac{1\pm\sqrt{(-1)}}{\sqrt{2}}$. There are also four fourth roots of 1, since if we put $z^4=1$, we find $z^2=\pm 1$, and $z=\pm\sqrt{1}$ or $z=\pm\sqrt{(-1)}$. Similarly there are eight eighth roots of 1 or -1, and so on.

MISCELLANEOUS EXAMPLES.

1. Simplify $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-c)(b-a)}+\dfrac{c^2}{(c-a)(c-b)}$.

2. If $\dfrac{a-b}{1+ab}+\dfrac{c-d}{1+cd}=0$, shew that

$$\frac{a-d}{1+ad}=\frac{b-c}{1+bc} \text{ and } \frac{a+c}{1-ac}=\frac{b+d}{1-bd}.$$

3. Shew that

$$a^3+b^3+c^3-3abc=$$
$$\tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}\{a+b+c\},$$

$$a^3+b^3+c^3+24abc=$$
$$(a+b+c)^3-3\{a(b-c)^2+b(c-a)^2+c(a-b)^2\},$$

$$(a+b+c)^3-27abc=$$
$$\tfrac{1}{2}\{(a+b+7c)(a-b)^2+(b+c+7a)(b-c)^2+(c+a+7b)(c-a)^2\},$$

$$9(a^3+b^3+c^3)-(a+b+c)^3=$$
$$(4a+4b+c)(a-b)^2+(4b+4c+a)(b-c)^2+(4c+4a+b)(c-a)^2.$$

4. Shew that if $a+b+c$ is zero the following expression is also zero,

$$\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ca}+\frac{c^2}{2c^2+ab}-1.$$

5. If the square root of the product of two quantities is rational, shew that the square root of the quotient obtained by dividing one by the other is also rational.

6. Extract the square root of $\{1+x\}\{1+x^2+2(1-x^2)\sqrt{x}\}$.

7. Express in the form of the sum of two simple surds the roots of the equation $x^4-2ax^2+b^2=0$.

8. Express in the form of the sum of two simple surds the roots of the equation $4x^4-4(1+n^2)a^2x^2+n^2a^4=0$.

9. By performing the operation for extracting the square root, find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square.

10. Shew that if $x^4+ax^3+bx^2+cx+d$ be a perfect square, the coefficients satisfy the relations

$$8c=a(4b-a^2) \text{ and } (4b-a^2)^2=64d.$$

11. If the values of x, y, x', y' be all possible, and

$$1+xx'+yy'=\sqrt{(1+x^2+y^2)}\sqrt{(1+x'^2+y'^2)},$$

shew that $$x=x' \text{ and } y=y'.$$

12. Shew that the equation

$$a^2b^4(x-x')^2+a^4b^2(y-y')^2+(b^2x^2+a^2y^2-a^2b^2)(b^2x'^2+a^2y'^2-a^2b^2)=0$$

is equivalent to the two $a^2b^2-a^2yy'-b^2xx'=0$ and $xy'-x'y=0$.

13. A man sells a horse for £24. 12*s*., and loses 18 per cent. on what the horse cost him: find the original cost.

14. Divide the number 16 into three such parts that the difference of the two less shall be the square root of the greatest, and the difference of the two greater shall be the square of the least.

15. Shew that

$$\left\{\frac{-1+\sqrt{(-3)}}{2}\right\}^n+\left\{\frac{-1-\sqrt{(-3)}}{2}\right\}^n$$

is equal to 2 if n be a multiple of 3, and equal to -1 if n be any other integer.

Solve the following equations:

16. $\dfrac{x+1}{x-1}+\dfrac{x+2}{x-2}=2\dfrac{x+3}{x-3}$.

17. $\dfrac{4}{x^2-2x}=\dfrac{2}{x^2-x}+x^2-x$.

18. $\left(x-\dfrac{1}{x}\right)\left(x-\dfrac{4}{x}\right)\left(x-\dfrac{9}{x}\right)=(x-1)(x-2)(x-3)$.

19. $x^4-8x^3+12x^2+16x-16=0$.

20. $\sqrt{(2x-1)}+\sqrt{(3x-2)}=\sqrt{(4x-3)}+\sqrt{(5x-4)}$.

21. $2b\{\sqrt{(x+a)}-b\}+2c\{\sqrt{(x-a)}+c\}=a$.

22. $\{\sqrt{(a+x)}-\sqrt{a}\}\{\sqrt{(a-x)}+\sqrt{a}\}=nx$.

23. $x+y=a+b,\qquad \dfrac{a}{x}+\dfrac{b}{y}=2$.

24. $\dfrac{ax}{a+x}+\dfrac{by}{b+y}=\dfrac{(a+b)c}{a+b+c},\qquad x+y=c$.

25. $6\left(\dfrac{x}{y}-\dfrac{y}{x}\right)=5=6\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$.

26. $x(bc-xy)=y(xy-ac),\ xy(ay+bx-xy)=abc(x+y-c)$.

27. $\left(x-3y+\dfrac{1}{z}\right)(x+z)=6,\quad \left(x+\dfrac{1}{z}\right)\dfrac{1}{y}=9,\quad \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{9}{2}$.

28.
$$(v+x)(y+z)=b+c-a,$$
$$(v+y)(z+x)=c+a-b,$$
$$(v+z)(x+y)=a+b-c,$$
$$v^2+x^2+y^2+z^2=3(a+b+c).$$

XXVI. RATIO.

369. Ratio is the relation which one quantity bears to another with respect to magnitude, the comparison being made by considering what multiple, part, or parts, the first quantity is of the second.

Thus in comparing 6 with 3, we observe that 6 has a certain magnitude with respect to 3, which it contains twice; again, in comparing 6 with 2, we see that 6 has now a different *relative* magnitude, for it contains 2 three times; or 6 is greater when compared with 2 than it is when compared with 3.

370. The ratio of a to b is usually expressed by two points placed between them, thus, $a : b$; and a is called the *antecedent* of the ratio, and b the *consequent* of the ratio.

371. A ratio is measured by the fraction which has for its numerator the antecedent of the ratio, and for its denominator the consequent of the ratio. Thus the ratio of a to b is measured by $\frac{a}{b}$; then for shortness we may say that *the ratio of* a *to* b *is equal to* $\frac{\mathrm{a}}{\mathrm{b}}$, *or is* $\frac{\mathrm{a}}{\mathrm{b}}$.

372. Hence we may say that the ratio of a to b is equal to the ratio of c to d, when $\frac{a}{b} = \frac{c}{d}$.

373. *If the terms of a ratio be multiplied or divided by the same quantity the ratio is not altered.*

For $$\frac{a}{b} = \frac{ma}{mb}, \text{ (Art. 135)}.$$

374. We may *compare* two or more ratios by reducing the fractions which measure these ratios to a common denominator.

Thus suppose one ratio to be that of a to b, and another ratio to be that of c to d; then the first ratio $\frac{a}{b} = \frac{ad}{bd}$, and the second ratio $\frac{c}{d} = \frac{bc}{bd}$. Hence the first ratio is greater than, equal to, or less than, the second ratio, according as ad is greater than, equal to, or less than bc.

375. A ratio is called a ratio of *greater inequality*, of *less inequality*, or of *equality*, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

376. *A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding any quantity to both terms of the ratio.*

Let the ratio be $\frac{a}{b}$, and let a new ratio be formed by adding x to both terms of the original ratio: then $\frac{a+x}{b+x}$ is greater or less than $\frac{a}{b}$, according as $\frac{b(a+x)}{b(b+x)}$ is greater or less than $\frac{a(b+x)}{b(b+x)}$; that is, according as $b(a+x)$ is greater or less than $a(b+x)$; that is, according as xb is greater or less than xa; that is, according as b is greater or less than a.

377. *A ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking from both terms of the ratio any quantity which is less than each of those terms.*

Let the ratio be $\frac{a}{b}$, and let a new ratio be formed by taking x from both terms of the original ratio: then $\frac{a-x}{b-x}$ is greater or less than $\frac{a}{b}$, according as $\frac{b(a-x)}{b(b-x)}$ is greater or less than $\frac{a(b-x)}{b(b-x)}$; that is, according as $b(a-x)$ is greater or less than $a(b-x)$; that is, according as bx is less or greater than ax; that is, according as b is less or greater than a.

378. If the antecedents of any ratios be multiplied together and also the consequents, a new ratio is obtained, which is said to be *compounded* of the former ratios. Thus the ratio $ac : bd$ is said to be *compounded* of the two ratios $a : b$ and $c : d$.

379. The ratio compounded of two ratios has sometimes been called the *sum* of those two ratios. When the ratio $a : b$ is compounded with itself, the resulting ratio $a^2 : b^2$ is sometimes called the *double* of the ratio $a : b$. Also the ratio $a^3 : b^3$ is called the *triple* of the ratio $a : b$. Similarly, the ratio $a : b$ is sometimes said to be *half* of the ratio $a^2 : b^2$, and the ratio $a^{\frac{1}{n}} : b^{\frac{1}{n}}$ is sometimes said to be $\frac{1}{n}$*th* of the ratio $a : b$.

This language, however, is now not used; the following terms are in conformity with it, and some of them are still retained. The ratio $a^2 : b^2$ is said to be the *duplicate* ratio of $a : b$, and the ratio $a^3 : b^3$ the *triplicate* ratio of $a : b$. Similarly, the ratio $\sqrt{a} : \sqrt{b}$ is called the *subduplicate* ratio of $a : b$, and the ratio $\sqrt[3]{a} : \sqrt[3]{b}$ the *subtriplicate* ratio of $a : b$. And the ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the *sesquiplicate* ratio of $a : b$.

380. *If the consequent of the preceding ratio be the antecedent of the succeeding ratio, and any number of such ratios be taken, the ratio which arises from their composition is that of the first antecedent to the last consequent.*

Let there be three ratios, namely $a : b$, $b : c$, $c : d$; then the compound ratio is $a \times b \times c : b \times c \times d$ (Art. 378), that is, $a : d$. Similarly, the proposition may be established whatever be the number of ratios.

381. *A ratio of greater inequality compounded with another increases it, and a ratio of less inequality compounded with another diminishes it.*

Let the ratio $x : y$ be compounded with the ratio $a : b$; the compound ratio is $xa : yb$, and this is greater or less than the ratio $a : b$, according as $\frac{xa}{yb}$ is greater or less than $\frac{a}{b}$, that is, according as x is greater or less than y.

382. *If the difference between the antecedent and the consequent of a ratio be small compared with either of them, the ratio of their squares is nearly obtained by doubling this difference.*

Let the proposed ratio be $a + x : a$, where x is small compared with a; then $a^2 + 2ax + x^2 : a^2$ is the ratio of the squares of the antecedent and consequent. But x is small compared with a, and therefore x^2 or $x \times x$ is small compared with $2a \times x$, and much smaller than $a \times a$. Hence $a^2 + 2ax : a^2$, that is, $a + 2x : a$, will nearly express the ratio $(a + x)^2 : a^2$.

Thus the ratio of the square of 1001 to the square of 1000 is nearly 1002 : 1000. The real ratio is 1002·001 : 1000, in which the antecedent differs from its approximate value 1002 only by one-thousandth part of unity.

383. Hence we may infer that the ratio of the square root of $a + 2x$ to the square root of a is the ratio $a + x : a$ nearly, when x is small compared with a. That is; *if the difference of two quantities be small compared with either of them, the ratio of their square roots is nearly obtained by halving this difference.*

In the same manner as in Art. 382 it may be shewn when x is small compared with a, that $a + 3x : a$ is nearly equal to the ratio $(a + x)^3 : a^3$, and $a + 4x : a$ is nearly equal to the ratio $(a + x)^4 : a^4$.

These results may be generalised by the student when he is acquainted with the Binomial Theorem.

384. We will place here a theorem respecting ratios which is often of use.

Suppose that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then each of these ratios is equal to $\left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}$, where p, q, r, n are any quantities whatever.

For let $k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$; then

$$kb = a, \quad kd = c, \quad kf = e;$$

therefore $\quad p(kb)^n + q(kd)^n + r(kf)^n = pa^n + qc^n + re^n$;

therefore $k^n = \frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}$, and $k = \left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}}$.

The same mode of demonstration may be applied, and a similar result obtained, when there are more than *three* ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ given equal. It may be observed that p, q, r, n are not necessarily *positive* quantities.

As a particular example we may suppose $n = 1$, then we see that if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ each of these ratios is equal to $\frac{pa + qc + re}{pb + qd + rf}$; and then as a special case we may suppose $p = q = r$, so that each of the given equal ratios is equal to $\frac{a + c + e}{b + d + f}$.

385. Suppose that we have *three* unknown quantities x, y, z connected by the *two* equations

$$ax + by + cz = 0, \quad a'x + b'y + c'z = 0;$$

these equations are not sufficient to determine the unknown quantities, but they will determine the ratios subsisting between them. For multiply the first equation by c', and the second by c, and subtract: thus

$$(ac' - a'c)x + (bc' - b'c)y = 0;$$

therefore
$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a}.$$

Again, multiply the first equation by b', and the second by b, and subtract: thus we shall obtain

$$\frac{x}{bc' - b'c} = \frac{z}{ab' - a'b}.$$

Hence we may write the results in this form:

$$\frac{x}{bc' - b'c} = \frac{y}{ca' - c'a} = \frac{z}{ab' - a'b}.$$

These results are very important, and should be carefully remembered; the second denominator may be derived from the first, and the third from the second, in the manner explained in Art. 211.

Denote the common value of these fractions by k; then

$$x = k(bc' - b'c), \quad y = k(ca' - c'a), \quad z = k(ab' - a'b).$$

Now suppose that we have also a third equation connecting the unknown quantities x, y, z; then by substituting in it for x, y, z the expressions just given, we shall obtain an equation which will determine k: thus the values of x, y, z become known.

Suppose, for example, the third equation is

$$lx^2 + my^2 + nz^2 = 1,$$

then k is determined by

$$k^2\{l(bc' - b'c)^2 + m(ca' - c'a)^2 + n(ab' - a'b)^2\} = 1.$$

EXAMPLES OF RATIO.

1. Write down the duplicate ratio of $2 : 3$, and the subduplicate ratio of $100 : 144$.

2. Write down the ratio which is compounded of the ratios $3 : 5$ and $7 : 9$.

3. Two numbers are in the ratio of 2 to 3, and if 9 be added to each they are in the ratio of 3 to 4. Find the numbers.

4. Shew that the ratio $a : b$ is the duplicate of the ratio $a + c : b + c$ if $c^2 = ab$.

5. There are two roads from A to B, one of them 14 miles longer than the other, and two roads from B to C, one of them 8 miles longer than the other. The distances from A to B and from B to C along the shorter roads are in the ratio of 1 to 2, and the distances along the longer roads are in the ratio of 2 to 3. Determine the distances.

6. Solve the equations

$$\frac{ax+by}{cz}=\frac{cz+ax}{by}=\frac{by+cz}{ax}=x+y+z.$$

7. Prove that if $\frac{a_1+a_2x}{a_2+a_3y}=\frac{a_2+a_3x}{a_3+a_1y}=\frac{a_3+a_1x}{a_1+a_2y}$, each of these ratios is equal to $\frac{1+x}{1+y}$, supposing $a_1+a_2+a_3$ not to be zero.

8. If $\frac{a-b}{ay+bx}=\frac{b-c}{bz+cx}=\frac{c-a}{cy+az}=\frac{a+b+c}{ax+by+cz}$, then each of these ratios $=\frac{1}{x+y+z}$, supposing $a+b+c$ not to be zero.

9. Shew that if $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$, then $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

10. If $\frac{a-a'}{a'-a''}=\frac{b-b'}{b'-b''}=\frac{c-c'}{c'-c''}$, then each of these ratios

$$=\frac{ab'-a'b}{a'b''-a''b'}=\frac{bc'-b'c}{b'c''-b''c'}=\frac{ca'-c'a}{c'a''-c''a'}=\frac{a+b+c-(a'+b'+c')}{a'+b'+c'-(a''+b''+c'')}.$$

11. Solve the equations

$$2x+y-2z=0,\quad 7x+6y-9z=0,\quad x^3+y^3+z^3=1728.$$

12. Solve the equations

$$ax+by+cz=0,\quad bcx+cay+abz=0,\quad xyz+abc(a^3x+b^3y+c^3z)=0.$$

XXVII. PROPORTION.

386. Four quantities are said to be proportionals when the first is the same multiple, part, or parts, of the second, as the third is of the fourth; that is, when $\frac{a}{b}=\frac{c}{d}$, the four quantities a, b, c, d, are called proportionals. This is usually expressed by saying, a is to b as c is to d, and is represented thus, $a:b::c:d$, or thus, $a:b=c:d$.

The terms a and d are called the *extremes*, and the terms b and c are called the *means*.

387. *When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be the four quantities; then since they are proportionals $\frac{a}{b}=\frac{c}{d}$ (Art. 386); and by multiplying both sides of the equation by bd, we have $ad=bc$.

Hence if the first be to the second as the second is to the third, the product of the extremes is equal to the square of the mean.

388. If any three terms in a proportion are given, the fourth may be determined from the equation $ad=bc$.

389. *If the product of two quantities be equal to the product of two others, the four are proportionals; the terms of either product being taken for the means, and the terms of the other product for the extremes.*

Let $xy=ab$; divide by ay, thus, $\frac{x}{a}=\frac{b}{y}$;

or $$x:a::b:y \text{ (Art. 386).}$$

390. If $a:b::c:d$, and $c:d::e:f$, then

$$a:b::e:f.$$

Because $\frac{a}{b}=\frac{c}{d}$ and $\frac{c}{d}=\frac{e}{f}$, therefore $\frac{a}{b}=\frac{e}{f}$;

or $$a:b::e:f.$$

391. *If four quantities be proportionals, they are proportionals when taken* inversely.

If $a:b::c:d$, then $b:a::d:c$.

For $\frac{a}{b}=\frac{c}{d}$; divide unity by each of these equal quantities, thus $\frac{b}{a}=\frac{d}{c}$; or $b:a::d:c$.

392. *If four quantities be proportionals, they are proportionals when taken* alternately.

If $a : b :: c : d$, then $a : c :: b : d$.

For $\frac{a}{b} = \frac{c}{d}$; multiply by $\frac{b}{c}$; thus $\frac{a}{c} = \frac{b}{d}$;

or $$a : c :: b : d.$$

Unless the four quantities are of the *same* kind the alternation cannot take place; because this operation supposes the first to be some multiple, part, or parts, of the third. One line may have to another line the same ratio as one weight has to another weight, but there is no relation, with respect to magnitude, between a line and a weight. In such cases, however, if the four quantities be *represented by numbers*, or by other quantities which are all of the same kind, the alternation may take place.

393. *When four quantities are proportionals, the first together with the second is to the second as the third together with the fourth is to the fourth.*

If $a : b :: c : d$, then $a + b : b :: c + d : d$.

For $\frac{a}{b} = \frac{c}{d}$; add unity to both sides; thus

$$\frac{a}{b} + 1 = \frac{c}{d} + 1; \text{ that is, } \frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b :: c + d : d.$$

This operation is called *componendo.*

394. *Also the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth.*

For $\frac{a}{b} = \frac{c}{d}$; subtract unity from both sides; thus

$$\frac{a}{b} - 1 = \frac{c}{d} - 1; \text{ that is, } \frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b :: c - d : d.$$

This operation is called *dividendo.*

395. *Also the first is to the excess of the first above the second as the third is to the excess of the third above the fourth.*

By the last Article, $\frac{a-b}{b} = \frac{c-d}{d}$;

also $\frac{b}{a} = \frac{d}{c}$;

therefore $\frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}$, or $\frac{a-b}{a} = \frac{c-d}{c}$;

or $a-b : a :: c-d : c$,

and inversely, $a : a-b :: c : c-d$.

This operation is called *convertendo.*

396. *When four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.*

If $a : b :: c : d$, then $a+b : a-b :: c+d : c-d$.

By Art. 393, $\frac{a+b}{b} = \frac{c+d}{d}$,

and by Art. 394, $\frac{a-b}{b} = \frac{c-d}{d}$;

therefore $\frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d}$;

that is, $\frac{a+b}{a-b} = \frac{c+d}{c-d}$;

or $a+b : a-b :: c+d : c-d$.

397. *When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.*

Let $a : b :: c : d :: e : f$;

then $a : b :: a+c+e : b+d+f$.

For $ad = bc$, and $af = be$, (Art. 386),

also $ab = ba$; hence $ab + ad + af = ba + bc + be$;

that is, $a(b + d + f) = b(a + c + e)$.

Hence, by Art. 389, $a : b :: a + c + e : b + d + f$.

Similarly the proposition may be established when more quantities are taken.

398. *When four quantities are proportionals, if the first and second be multiplied, or divided, by any quantity, as also the third and fourth, the resulting quantities will be proportionals.*

Let $a : b :: c : d$, then $ma : mb :: nc : nd$.

For $$\frac{a}{b} = \frac{c}{d}, \text{ therefore } \frac{ma}{mb} = \frac{nc}{nd};$$

or $$ma : mb :: nc : nd.$$

399. *If the first and third be multiplied, or divided, by any quantity, and also the second and fourth, the resulting quantities will be proportionals.*

Let $a : b :: c : d$, then $ma : nb :: mc : nd$.

For $$\frac{a}{b} = \frac{c}{d}; \text{ therefore } \frac{ma}{b} = \frac{mc}{d}, \text{ and } \frac{ma}{nb} = \frac{mc}{nd};$$

or $$ma : nb :: mc : nd.$$

400. *In two ranks of proportionals, if the corresponding terms be multiplied together, the products will be proportionals.*

Let $$a : b :: c : d,$$

and $$e : f :: g : h,$$

then $$ae : bf :: cg : dh.$$

For $$\frac{a}{b} = \frac{c}{d} \text{ and } \frac{e}{f} = \frac{g}{h}; \text{ therefore } \frac{ae}{bf} = \frac{cg}{dh};$$

or $$ae : bf :: cg : dh.$$

This is called *compounding* the proportions. The proposition is true if applied to any number of proportions.

401. *If four quantities be proportionals, the like powers, or roots, of these quantities will be proportionals.*

Let $a : b :: c : d$, then $a^n : b^n :: c^n : d^n$.

For $\frac{a}{b} = \frac{c}{d}$, therefore $\frac{a^n}{b^n} = \frac{c^n}{d^n}$, where n may be whole or fractional; thus

$$a^n : b^n :: c^n : d^n.$$

402. If $a : b :: b : c$, then $a : c :: a^2 : b^2$.

For $\frac{a}{b} = \frac{b}{c}$; multiply by $\frac{a}{b}$, thus $\frac{a}{b} \times \frac{a}{b} = \frac{a}{b} \times \frac{b}{c}$,

that is, $$\frac{a^2}{b^2} = \frac{a}{c};$$

or $$a : c :: a^2 : b^2.$$

The three quantities a, b, c are in this case said to be in *continued proportion;* and b is said to be a *mean proportional* between a and c.

403. Similarly we may shew that if $a : b :: b : c :: c : d$, then $a : d :: a^3 : b^3$. Here the four quantities a, b, c, d are said to be in *continued proportion.*

404. It is obvious from the preceding Articles, that if four quantities are proportionals, we may derive from them many other proportions. We will give another example.

If $a : b :: c : d$, then

$$ma + nb : pa + qb :: mc + nd : pc + qd.$$

For $$\frac{a}{b} = \frac{c}{d}, \text{ therefore } \frac{ma}{b} = \frac{mc}{d};$$

add n to both sides; thus

$$\frac{ma+nb}{b}=\frac{mc+nd}{d}.$$

Similarly
$$\frac{pa+qb}{b}=\frac{pc+qd}{d}.$$

Hence
$$\frac{ma+nb}{b}\div\frac{pa+qb}{b}=\frac{mc+nd}{d}\div\frac{pc+qd}{d};$$

that is,
$$\frac{ma+nb}{pa+qb}=\frac{mc+nd}{pc+qd};$$

or
$$ma+nb : pa+qb :: mc+nd : pc+qd.$$

405. In the definition of Proportion it is supposed that one quantity is some determinate multiple, part, or parts, of another; or that the fraction formed by taking one of the quantities as a numerator, and the other as a denominator, is a determinate fraction. This will be the case whenever the two quantities have any common measure whatever. For let x be a common measure of a and b, and let $a=mx$ and $b=nx$; then

$$\frac{a}{b}=\frac{mx}{nx}=\frac{m}{n},$$

where m and n are whole numbers.

406. But it sometimes happens that quantities are *incommensurable*, that is, admit of no common measure whatever. If, for example, one line is the side of a square, and another line is the diagonal of the same square, these lines are *incommensurable*. In such cases the value of $\frac{a}{b}$ cannot be expressed by any fraction $\frac{m}{n}$ where m and n are whole numbers; yet a fraction of this kind may be found which will express the value of $\frac{a}{b}$ to any *required degree of accuracy*.

For let $b = nx$, where n is an integer; also let a be greater than mx but less than $(m+1)x$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$, but less than $\frac{m+1}{n}$. Thus the difference between $\frac{a}{b}$ and $\frac{m}{n}$ is less than $\frac{1}{n}$. And since $nx = b$, when x is diminished n is increased and $\frac{1}{n}$ is diminished. Hence by taking x small enough, $\frac{1}{n}$ can be made less than any assigned fraction, and therefore the difference between $\frac{m}{n}$ and $\frac{a}{b}$ can be made less than any assigned fraction.

407. If c and d as well as a and b are incommensurable, and if when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, then $\frac{c}{d}$ also lies between $\frac{m}{n}$ and $\frac{m+1}{n}$ however the numbers m and n are increased, $\frac{a}{b}$ is equal to $\frac{c}{d}$.

For if $\frac{a}{b}$ and $\frac{c}{d}$ are not equal, they must have some assignable difference, and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference must be less than $\frac{1}{n}$. But since n may, by supposition, be increased without limit, $\frac{1}{n}$ may be diminished without limit; that is, it may be made *less* than any assigned magnitude; therefore $\frac{a}{b}$ and $\frac{c}{d}$ have no assignable difference, so that we may say that $\frac{a}{b} = \frac{c}{d}$. Hence all the propositions respecting proportionals are true of the four quantities a, b, c, d.

408. It will be useful to compare the definition of proportion which has been given in this Chapter with that which is given in the fifth book of Euclid. The latter definition may be stated thus; four quantities are proportionals when if any equimultiples be taken of the first and third, and also any equimultiples of the second and fourth, the multiple of the third is greater than, equal to, or less than, the multiple of the fourth, according as the multiple of the first is greater than, equal to, or less than, the multiple of the second.

We will first shew that the property involved in Euclid's definition follows from the algebraical definition.

For suppose $a : b :: c : d$; then $\frac{a}{b} = \frac{c}{d}$, therefore $\frac{pa}{qb} = \frac{pc}{qd}$. Hence pc is greater than, equal to, or less than qd, according as pa is greater than, equal to, or less than qb.

409. Next we will shew that the property involved in the algebraical definition follows from Euclid's. Let a, b, c, d be four quantities which are proportional according to Euclid's definition: then shall $\frac{a}{b} = \frac{c}{d}$. For if $\frac{a}{b}$ be not equal to $\frac{c}{d}$, one must be greater than the other, and it will be possible to find some fraction which lies between them. Suppose $\frac{a}{b}$ greater than $\frac{c}{d}$; and let $\frac{p}{q}$ lie between them. Then $\frac{a}{b}$ is greater than $\frac{p}{q}$; therefore qa is *greater* than pb: and $\frac{c}{d}$ is less than $\frac{p}{q}$; therefore qc is *less* than pd. Thus a, b, c, d are *not* proportionals according to Euclid's definition; which is contrary to the supposition. Therefore $\frac{a}{b}$ and $\frac{c}{d}$ cannot be unequal; that is they are equal.

410. It is usually stated that the common algebraical definition of proportion cannot be used in Geometry, because there is no method of representing geometrically the result of the operation

of division. Straight lines can be represented geometrically, but not the abstract number which expresses how often one straight line is contained in another. But it should also be noticed that Euclid's definition is *rigorous* and can be applied to *incommensurable* as well as to *commensurable* quantities, while the algebraical definition is, strictly speaking, confined to the latter quantities. Hence this consideration alone would furnish a sufficient reason for the definition adopted by Euclid.

EXAMPLES OF PROPORTION.

1. The last three terms of a proportion being 4, 6, 8, what is the first term?

2. Find a third proportional to 25 and 400.

3. If 3, x, 1083 are in continued proportion, find x.

4. If 2 men working 8 hours a day can copy a manuscript in 32 days, in how many days can x men working y hours a day copy it?

5. If x and y be unequal and x have to y the duplicate ratio of $x+z$ to $y+z$, prove that z is a mean proportional between x and y.

6. If $a : b :: p : q$, then $a^2+b^2 : \dfrac{a^3}{a+b} :: p^2+q^2 : \dfrac{p^3}{p+q}$.

7. If four quantities are proportionals, and the second is a mean proportional between the third and fourth, the third will be a mean proportional between the first and second.

8. If

$$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d),$$

prove that a, b, c, d are proportionals.

9. Shew that when four quantities of the same kind are proportional, the greatest and least of them together are greater than the other two together.

10. Each of two vessels contains a mixture of wine and water; a mixture consisting of equal measures from the two vessels contains as much wine as water, and another mixture consisting of four measures from the first vessel and one from the second is composed of wine and water in the ratio of 2 : 3. Find the proportion of wine and water in each of the vessels.

11. A and B have made a bet; the money which A stakes bears the same proportion to all the money A has as the money which B stakes bears to all the money B has. If A wins he will have double what B will have, but if he loses, B will have three times what A will have. All the money between them being £168, determine the circumstances.

12. If the increase in the number of male and female criminals be 1·8 per cent., while the decrease in the number of males alone is 4·6 per cent., and the increase in the number of females is 9·8; compare the number of male and female criminals respectively.

XXVIII. VARIATION.

411. The present Chapter consists of a series of propositions connected with the definitions of ratio and proportion stated in a new phraseology, which is convenient for some purposes.

412. One quantity is said to *vary* directly as another when the two quantities depend upon each other, and in such a manner that if one be changed the other is changed in the same proportion.

Sometimes for shortness we omit the word *directly*, and say simply that one quantity varies as another.

413. Thus, for example, if the altitude of a triangle be invariable, the area varies as the base; for if the base be increased or diminished, we know from Euclid that the area is increased or

diminished in the same proportion. We may express this result by Algebraical symbols thus: let A and a be *numbers* which represent the areas of two triangles having a common altitude, and let B and b be *numbers* which represent the bases of these triangles respectively; then $\frac{A}{a} = \frac{B}{b}$. And from this we deduce $\frac{A}{B} = \frac{a}{b}$, (Art. 392). If there be a third triangle having the same altitude as the two already considered, then the ratio of the number which represents its area to the number which represents its base will also be equal to $\frac{a}{b}$. Put $\frac{a}{b} = m$, then $\frac{A}{B} = m$ and $A = mB$. Here A may represent the area of *any* one of a series of triangles which have a common altitude, and B the corresponding base, and m remains constant. Hence the statement that the area varies as the base may also be expressed thus: the area has a constant ratio to the base; by which we mean, in accordance with Article 392, that the *number* which represents the area bears a constant ratio to the *number* which represents the base.

We have made these remarks for the purpose of explaining the *notation* and *language* which will be used in the present Chapter. When we say that A varies as B, we mean that A represents the numerical value of any one of a certain series of quantities, and B the numerical value of the corresponding quantity in a certain other series, and that $A = mB$, where m is some number which remains constant for every corresponding pair of quantities.

We will give a formal proof of the equation $A = mB$ deduced from the definition of Art. 412.

414. *If* A *vary as* B, *then* A *is equal to* B *multiplied by some constant quantity.*

Let a and b denote one pair of corresponding values of the two quantities, and let A and B denote any other pair; then $\frac{A}{a} = \frac{B}{b}$

by definition. Hence $A = \frac{a}{b} B = mB$, where m is equal to the constant $\frac{a}{b}$.

415. The symbol $\propto$ is used to express variation; thus $A \propto B$ stands for A varies as B.

416. One quantity is said to vary *inversely* as another when the first varies as the *reciprocal* of the second; see Art. 263.

Or if $A = \frac{m}{B}$, where m is constant, A is said to vary inversely as B.

417. One quantity is said to vary as two others jointly when, if the former is changed in any manner, the product of the other two is changed in the same proportion.

Or if $A = mBC$, where m is constant, A is said to vary jointly as B and C.

418. One quantity is said to vary directly as a second and inversely as a third, when it varies jointly as the second and the reciprocal of the third.

Or if $A = \frac{mB}{C}$, where m is constant, A is said to vary directly as B and inversely as C.

419. *If* A $\propto$ B, *and* B $\propto$ C, *then* A $\propto$ C.

For let $A = mB$, and $B = nC$, where m and n are constants; then $A = mnC$; and, as mn is constant, $A \propto C$.

420. *If* A $\propto$ C, *and* B $\propto$ C, *then* A $\pm$ B $\propto$ C, *and* $\sqrt{(AB)} \propto$ C.

For let $A = mC$, and $B = nC$, where m and n are constants; then $A + B = (m + n) C$, and $A - B = (m - n) C$; therefore $A \pm B \propto C$. Also $\sqrt{(AB)} = \sqrt{(mnC^2)} = C\sqrt{(mn)}$; therefore $\sqrt{(AB)} \propto C$.

421. *If* A $\propto$ BC, *then* B $\propto \frac{A}{C}$, *and* C $\propto \frac{A}{B}$.

For let $A = mBC$, then $B = \frac{1}{m}\frac{A}{C}$; therefore $B \propto \frac{A}{C}$. Similarly $C \propto \frac{A}{B}$.

422. *If* A ∝ B, *and* C ∝ D, *then* AC ∝ BD.

For let $A = mB$, and $C = nD$, then $AC = mnBD$; therefore $AC \propto BD$.

423. *If* A ∝ B, *then* $A^n \propto B^n$.

For let $A = mB$, then $A^n = m^n B^n$; therefore $A^n \propto B^n$.

424. *If* A ∝ B, *then* AP ∝ BP, *where* P *is any quantity variable or invariable.*

For let $A = mB$, then $AP = mBP$; therefore $AP \propto BP$.

425. *If* A ∝ B *when* C *is invariable, and* A ∝ C *when* B *is invariable, then will* A ∝ BC *when both* B *and* C *are variable.*

The variation of A depends upon the variations of the two quantities B and C; let the variations of the latter quantities take place separately, and when B is changed to b, let A be changed to a'; then, by supposition, $\frac{A}{a'} = \frac{B}{b}$. Now let C be changed to c, and in consequence let a' be changed to a; then, by supposition, $\frac{a'}{a} = \frac{C}{c}$. Thus

$$\frac{A}{a'} \times \frac{a'}{a} = \frac{BC}{bc};$$

that is,

$$\frac{A}{a} = \frac{BC}{bc};$$

therefore $A \propto BC$.

A very good example of this proposition is furnished in Geometry. It can be proved that the area of a triangle varies as the base when the height is invariable, and that the area varies as the height when the base is invariable. Hence when both the base and the height vary, the area varies as the product of the numbers which express the base and the height.

426. In the same manner if there be any number of quantities B, C, D, &c. each of which varies as another A when the rest are constant; when they are all variable, A varies as their product.

Take for example three quantities B, C, D. When B alone varies A varies as B; when C alone varies A varies as C: thus, by Art. 425, when B and C both vary A varies as BC. Again when D alone varies A varies as D, and when BC varies A varies as BC: thus, by Art. 425, when D and BC both vary A varies as BCD.

EXAMPLES ON VARIATION.

1. Given that y varies as x, and that $y=3$ when $x=1$, find the value of y when $x=3$.

2. If a varies as b and $a=15$ when $b=3$, find the equation between a and b.

3. Given that z varies jointly as x and y, and that $z=1$ when $x=1$ and $y=1$, find the value of z when $x=2$ and $y=2$.

4. If z varies as $px+y$, and if $z=3$ when $x=1$ and $y=2$, and $z=5$ when $x=2$ and $y=3$, find p.

5. If x varies directly as y when z is constant, and inversely as z when y is constant, then if y and z both vary, x will vary as $\frac{y}{z}$.

6. If 3, 2, 1, be simultaneous values of x, y, z in the preceding Example, determine the value of x when $y=2$ and $z=4$.

7. The wages of 5 men for 6 weeks being £14. 5*s*., how many weeks will 4 men work for £19? (Apply Example 5.)

8. If the square of x vary as the cube of y, and $x=2$ when $y=3$, find the equation between x and y.

9. Given that y varies as the sum of two quantities, one of which varies as x directly, the other as x inversely, and that $y=4$ when $x=1$, and $y=5$ when $x=2$, find the equation between x and y.

10. If one quantity vary directly as another, and the former be $\frac{3}{4}$ when the latter is $\frac{4}{3}$, find what the latter will be when the former is 9.

11. If one quantity vary as the sum of two others when their difference is constant, and also vary as their difference when their sum is constant, shew that when these two quantities vary independently, the first quantity will vary as the difference of their squares.

12. Given that the volume of a sphere varies as the cube of its radius, prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are 3, 4, 5 inches.

13. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate one inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

14. There are two globes of gold whose radii are r and r'; they are melted and formed into a single globe. Find its radius.

15. If x, y, z be variable quantities such that $y+z-x$ is constant, and that $(x+y-z)(x+z-y)$ varies as yz, prove that $x+y+z$ varies as yz.

16. A point moves with a speed which is different in different miles, but invariable in the same mile, and its speed in any mile varies inversely as the number of miles travelled before it commences this mile. If the second mile be described in 2 hours, find the time occupied in describing the n^{th} mile.

17. Suppose that y varies as a quantity which is the sum of three quantities, the first of which is constant, the second varies

as x, and the third as x^2. And suppose that when $x=a$, $y=0$, when $x=2a$, $y=a$, and when $x=3a$, $y=4a$. Shew that when $x=na$, $y=(n-1)^2 a$.

18. Assuming that the quantity of work done varies as the cube root of the number of agents when the time is the same, and varies as the square root of the time when the number of agents is the same; find how long 3 men would take to do one-fifth of the work which 24 men can do in 25 hours. (See Art. 425.)

XXIX. SCALES OF NOTATION.

427. The student will of course have learned from Arithmetic that in the ordinary method of expressing whole numbers by figures, the number represented by each particular figure is always *some multiple of some power of ten*. Thus in 347 the 3 represents 3 hundreds, that is, 3 times 10^2; the 4 represents 4 tens, that is, 4 times 10^1; and the 7 which represents 7 units, may be said to represent 7 times 10^0.

This mode of representing numbers is called the *common scale of notation*, and 10 is said to be the *base* or *radix* of the common scale.

428. We shall now prove that any positive integer greater than unity may be used instead of 10 for the *radix*, and shall shew how to express a number in any proposed scale. We shall then add some miscellaneous propositions connected with this subject.

The figures by means of which a number is expressed are called *digits*.

When we speak in future of *any radix* we shall always mean that this radix is some positive integer greater than unity.

429. *To shew that any whole number may be expressed in terms of any radix.*

Let N denote the whole number, r the radix. Suppose that r^n is the highest power of r which is not greater than N; divide

N by r^n, and let p_n be the quotient and N_1 the remainder; thus

$$N = p_n r^n + N_1.$$

Here, by supposition, p_n is less than r; also N_1 is less than r^n. Next divide N_1 by r^{n-1}, and let p_{n-1} be the quotient and N_2 the remainder; thus

$$N_1 = p_{n-1} r^{n-1} + N_2.$$

Proceed in this way until the remainder is less than r; thus we find N expressed in the manner indicated by the equation

$$N = p_n r^n + p_{n-1} r^{n-1} + \ldots\ldots + p_2 r^2 + p_1 r + p_0.$$

Each of the *digits* p_n, p_{n-1}, ... p_1, p_0 is less than r, and any one or more of them after the first may be zero.

The best practical mode of determining the digits is given in the next Article.

430. *To express a given whole number in any proposed scale.*

By a *given* whole number we mean a whole number expressed in words or else expressed by digits in some assigned scale. If no scale is mentioned, we understand the common scale to be intended.

Let N be the given number, r the radix of the scale in which it is to be expressed. Suppose p_0, p_1, ... p_n to be the required digits by which N is expressed in the new scale, beginning with that on the right hand; then

$$N = p_n r^n + p_{n-1} r^{n-1} + \ldots\ldots + p_2 r^2 + p_1 r + p_0;$$

we have now to find the value of each digit.

Divide N by r, and let Q_1 denote the quotient; then it is obvious that

$$Q_1 = p_n r^{n-1} + p_{n-1} r^{n-2} + \ldots\ldots + p_2 r + p_1,$$

and that the *remainder* is p_0. Hence p_0 is found by this rule; *divide the given number by the proposed radix, and the remainder is the first of the required digits.*

Again, divide Q_1 by r, and let Q_2 denote the quotient; then it is obvious that

$$Q_2 = p_n r^{n-2} + p_{n-1} r^{n-3} + \ldots\ldots + p_2,$$

and that the *remainder* is p_1. Hence the second of the required digits is found.

By proceeding in this way we shall find in succession all the required digits.

431. For example, transform 43751 into the scale of which 6 is the radix. The division may be performed and the remainders noted thus:

$$\begin{array}{rl}
6\,)\,4\,3\,7\,5\,1 & \\
6\,)\,7\,2\,9\,1 & \ldots\ldots 5 \\
6\,)\,1\,2\,1\,5 & \ldots\ldots 1 \\
6\,)\,2\,0\,2 & \ldots\ldots 3 \\
6\,)\,3\,3 & \ldots\ldots 4 \\
5 & \ldots\ldots 3
\end{array}$$

Thus $\qquad 43751 = 5.6^5 + 3.6^4 + 4.6^3 + 3.6^2 + 1.6 + 5,$

so that the number is expressed in the new scale thus, 534315.

432. Again, transform 43751 into the scale of which 12 is the radix.

$$\begin{array}{rl}
12\,)\,4\,3\,7\,5\,1 & \\
12\,)\,3\,6\,4\,5 & \ldots\ldots 11 \\
12\,)\,3\,0\,3 & \ldots\ldots\ 9 \\
12\,)\,2\,5 & \ldots\ldots\ 3 \\
2 & \ldots\ldots\ 1
\end{array}$$

Thus $\qquad 43751 = 2.12^4 + 1.12^3 + 3.12^2 + 9.12 + 11.$

In expressing the number in the new scale we shall require a single symbol for *eleven*; let it be e; then the number is expressed in the new scale thus, $2139e$.

We cannot of course use 11 to express *eleven* in the new scale, because 11 now represents $1.12+1$, that is, *thirteen*.

433. We will now consider an example in which a number is given, not in the common scale.

A number is denoted by $t347e$ in the scale of which twelve is the radix, it is required to express it in the scale of which eleven is the radix.

Here t stands for *ten*, and e for *eleven*.

$$\begin{array}{r} e\,)\,t\;3\;4\;7\;e \\ \hline e\;2\;7\;3 \ldots\ldots 2 \end{array}$$

The process of division by eleven is performed thus. First e is not contained in t, for eleven is not contained in ten, so we ask how often is e contained in $t3$? here t stands for ten times twelve, that is one hundred and twenty, so that the question is, how often is eleven contained in one hundred and twenty-three? the answer is eleven times, with two over. Next we ask how often is e contained in 24; that is, how often is eleven contained in twenty-eight? the answer is twice, with six over. Then how often is e contained in 67; that is, how often is eleven contained in seventy-nine? the answer is seven times, with two over. Last, how often is e contained in $2e$; that is, how often is eleven contained in thirty-five? the answer is three times, with two over.

Hence 2 is the first of the required digits.

The remainder of the process we will indicate; the student should carefully work it for himself, and then compare his result with that here given.

$$\begin{array}{r} e\,)\,e\;2\;7\;3 \\ \hline e\,)\,1\;0\;2\;t \ldots\ldots 1 \\ \hline e\,)\,1\;1\;4 \ldots\ldots 2 \\ \hline e\,)\,1\;2 \ldots\ldots 6 \\ \hline 1 \ldots\ldots 3 \\ \hline \end{array}$$

Hence the given number is equal to

$$1 . e^5 + 3 . e^4 + 6 . e^3 + 2 . e^2 + 1 . e + 2;$$

that is, it is expressed in the scale with radix eleven thus, 136212.

434. The process of transforming from one scale to another may be effected also in another manner. Suppose for example that we have to transform to the common scale 24613 which is in the scale of seven. We have in fact to calculate the value of

$$2x^4 + 4x^3 + 6x^2 + x + 3,$$

when $x = 7$. We may adopt the method which is explained in the *Theory of Equations*, Art. 11.

$$\begin{array}{r} 2 + \quad 4 + \quad 6 + \quad 1 + \quad 3 \\ 14 + 126 + 924 + 6475 \\ \hline 18 + 132 + 925 + 6478 \end{array}$$

The result is 6478. This method is advantageous when we have to transform from any other scale into the common scale.

435. It will be easy to form an unlimited number of self-verifying examples. Thus, take two numbers expressed in the common scale and obtain their product, then transform this product into any proposed scale; next transform the two numbers into the proposed scale, and obtain their product in this scale; the result should of course agree with that already obtained. Or, take any number, square it, transform this square into any proposed scale, and extract the square root in this scale; then transform the last result back to the original scale.

436. Next let it be required to transform a given *fraction* from one scale to another. This may be effected by transforming separately the numerator and denominator of the given fraction by the method of Art. 430. Thus we obtain a fraction identical with the proposed fraction, having its numerator and denominator expressed in the new scale.

437. We stated in Art. 427, that in the common scale of notation, each digit which occurs in the expression of any *integer*

by figures represents *some multiple of some power of ten.* This statement may be extended, and we may assert that if a number be expressed in the common scale, and the number be an *integer*, or a *decimal fraction*, or *partly an integer and partly a decimal fraction*, then each digit represents *some multiple of some power of ten.* Thus in 347·958 the 3, the 4, and the 7, have the values assigned to them in Art. 427; the 9 represents $\frac{9}{10}$, that is, 9 times 10^{-1}; the 5 represents $\frac{5}{100}$, that is, 5 times 10^{-2}; and the 8 represents $\frac{8}{1000}$; that is, 8 times 10^{-3}.

It may therefore naturally occur to us to consider the following problem: required to express a given fraction by a series of fractions in any proposed scale analogous to *decimal fractions* in the common scale. We will speak of such fractions as *radix-fractions.*

438. *Required to express a given fraction by a series of radix-fractions in any proposed scale.*

By a *given* fraction we mean a fraction expressed in words or expressed by figures in any given scale.

Let F denote the given fraction, r the radix of the proposed scale. Suppose t_1, t_2, ... the numerators of the required *radix-fractions* beginning from the *left hand;* thus

$$F = \frac{t_1}{r} + \frac{t_2}{r^2} + \frac{t_3}{r^3} + \ldots\ldots,$$

where t_1, t_2, t_3, ... are to be found.

Multiply both members of the equation by r; thus

$$Fr = t_1 + \frac{t_2}{r} + \frac{t_3}{r^2} + \ldots\ldots$$

The right-hand member consists of an integer t_1 and an additional fractional part. Let I_1 denote the integral part of Fr, and F_1 the fractional remainder; then we must have

$$I_1 = t_1, \quad F_1 = \frac{t_2}{r} + \frac{t_3}{r^2} + \ldots\ldots$$

Thus, to obtain the first numerator, t_1, of the series of radix-fractions, we have this rule; *multiply the given fraction by the proposed radix; then the greatest integer in the product is the first of the required numerators.*

Again, multiply F_1 by r; let I_2 be the integral part of the product, and F_2 the fractional remainder; then

$$I_2 = t_2, \quad F_2 = \frac{t_3}{r} + \frac{t_4}{r^2} + \ldots\ldots$$

Hence t_2, the second of the required numerators, is ascertained. By proceeding in this way we shall determine the required numerators in succession. If one of the products which occur on the left-hand side of the equations be an exact integer, the process then terminates, and the proposed fraction is expressed by a finite series of radix-fractions. If no integral product occur, the process never terminates, and the proposed fraction can only be expressed by an infinite series of the required radix-fractions; the numerators of the radix-fractions will *recur* like a recurring decimal.

439. For example, express $\frac{123}{128}$ by a series of radix-fractions in the scale 8.

Multiply $\frac{123}{128}$ by 8; thus we obtain $\frac{123}{16}$, that is $7 + \frac{11}{16}$.

Multiply $\frac{11}{16}$ by 8; thus we obtain $\frac{11}{2}$, that is $5 + \frac{1}{2}$.

Multiply $\frac{1}{2}$ by 8; thus we obtain 4.

Hence $\frac{123}{128} = \frac{7}{8} + \frac{5}{8^2} + \frac{4}{8^3}$.

440. We may remark that the radix ten is not only the base of the common mode of expressing numbers by figures, but is in fact assumed as the base of our *language* for numbers. This will be seen by observing at what stage in counting upwards from unity new *words* are introduced. For example, all numbers between twenty-one and twenty-nine, both inclusive, are expressed

by means of words that have already occurred in counting up to twenty; then a new word occurs, namely *thirty*, and we can count on without an additional new word as far as thirty-nine; and so on.

The number *ten* has only two divisors different from itself and unity, namely 2 and 5; the number *twelve* has four divisors, namely 2, 3, 4, and 6. On this account twelve would have been more convenient than ten as a radix. This may be illustrated by reference to the case of a shilling; since a shilling is equivalent to *twelve* pence, the half, the third, the fourth, and the sixth of a shilling, each contains an exact number of pence; if the shilling were equivalent to ten pence, the half and fifth of a shilling would be the only submultiples of a shilling containing an exact number of pence. Similarly, the mode of measuring lengths by feet and inches may be noticed.

441. We may observe that if *two* be adopted as the radix of a scale, the operations of Arithmetic are in some respects much simplified. In this scale the only *figures* which occur are 0 and 1, so that each separate step of a series of arithmetical operations would be an addition of 1, or a subtraction of 1, or a multiplication by 1, or a division by 1. The simplicity of each operation is however counterbalanced by the disadvantage arising from the increased number of such operations.

We give in the following two Articles two problems connected with the present subject.

442. Determine which of the series of weights 1 lb., 2 lbs., 2^2 lbs., 2^3 lbs., 2^4 lbs.,...... must be used to balance a given weight of N lbs., not more than one weight of each kind being used.

It is obvious that this question is the same as the following; express the number N in the scale of which the radix is 2. Hence it follows from Art. 429 that the problem can always be solved.

443. Suppose it required to determine which of the weights 1 lb., 3 lbs., 3^2 lbs., 3^3 lbs.,... must be selected to weigh N lbs., not

more than one of each kind being used, but *in either scale* that may be necessary.

Divide N by 3, then the remainder must be zero, or one, or two. Let N_1 denote the quotient; then in the first case we have $N = 3N_1$, in the second case $N = 3N_1 + 1$, and in the third case $N = 3N_1 + 2$. In the first or second case divide N_1 by 3; in the third case we may write $N = 3(N_1 + 1) - 1$, then we should divide $N_1 + 1$ by 3. Proceed thus, and we shall finally have a result of the following form,

$$N = q_n 3^n + q_{n-1} 3^{n-1} + \ldots\ldots + q_1 3 + q_0,$$

where each of the quantities $q_0, q_1, \ldots\ldots q_n$ is either zero, or $+1$, or -1. Thus the problem is solved.

444. *In a scale of notation of which the radix is* r, *the sum of the digits of any whole number divided by* r − 1, *or by any factor of* r − 1, *will leave the same remainder respectively as the whole number divided by* r − 1 *or by the factor of* r − 1.

Let N denote the whole number, $p_0, p_1, \ldots\ldots p_n$ the digits beginning with that in the units' place; then

$$\begin{aligned} N &= p_0 + p_1 r + \ldots\ldots + p_n r^n \\ &= p_0 + p_1 + p_2 + \ldots\ldots + p_n \\ &\quad + p(r-1) + p_2(r^2 - 1) + \ldots\ldots + p_n(r^n - 1); \end{aligned}$$

therefore

$$\begin{aligned} \frac{N}{r-1} &= \frac{p_0 + p_1 + p_2 + \ldots\ldots + p_n}{r-1} \\ &\quad + p_1 + p_2(r+1) + \ldots\ldots + p_n \frac{r^n - 1}{r-1}. \end{aligned}$$

But $\dfrac{r^n - 1}{r-1}$ is an integer whatever positive integer n may be; thus $\dfrac{N}{r-1} = \text{some integer} + \dfrac{p_0 + p_1 + \ldots\ldots + p_n}{r-1}$.

Next let p be a factor of $r - 1$, say that $r - 1 = pq$. Then multiplying the last result by q we have

$$\frac{N}{p} = \text{some integer} + \frac{p_0 + p_1 + \ldots\ldots + p_n}{p}.$$

This establishes the proposition.

445. *In a scale of notation in which the radix is* r *let any whole number be divided by* r + 1; *and let the difference between the sum of the digits in the odd places and the sum of the digits in the even places be divided by* r + 1; *then either the remainders will be equal or their sum will be* r + 1.

With the same notation as in the preceding proposition we have

$$N = p_0 + p_1 r + p_2 r^2 + \ldots\ldots + p_n r^n$$
$$= p_0 - p_1 + p_2 - p_3 + \ldots\ldots + (-1)^n p_n$$
$$+ p_1(r+1) + p_2(r^2-1) + p_3(r^3+1) + \ldots + p_n\{r^n - (-1)^n\}.$$

Thus $\dfrac{N}{r+1} = \text{some integer} + \dfrac{p_0 - p_1 + p_2 - \ldots\ldots + (-1)^n p_n}{r+1}$.

First, suppose $p_0 - p_1 + p_2 - \ldots + (-1)^n p_n$ to be *positive*, and denote it by D; then

$$\frac{N}{r+1} = \text{some integer} + \frac{D}{r+1};$$

thus when N and D are divided by $r+1$ the remainders are equal.

Secondly, suppose $p_0 - p_1 + p_2 - \ldots + (-1)^n p_n$ to be *negative*, and denote it by $-D$; then

$$\frac{N}{r+1} = \text{some integer} - \frac{D}{r+1},$$

that is, $$\frac{N}{r+1} + \frac{D}{r+1} = \text{some integer};$$

thus when N and D are divided by $r+1$ the sum of the remainders must be $r+1$, unless either remainder is zero, and then the other remainder also is zero.

For example, suppose $r = 10$ and $N = 263419$. Here

$$9 - 1 + 4 - 3 + 6 - 2 = 13 = D;$$

and N and D when divided by 11 each leave the remainder 2.

Again, suppose $r = 10$ and $N = 615372$. Here

$$2 - 7 + 3 - 5 + 1 - 6 = -12 = -D;$$

and N and D when divided by 11 leave the remainders 10 and 1 respectively.

446. It appears from Art. 444 that a number is divisible by 9 when the sum of its digits is divisible by 9; and that when any number is divided by 9, the remainder is the same as if the sum of the digits of that number were divided by 9. And as 3 is a factor of 9 a number is divisible by 3 when the sum of its digits is divisible by 3; and when any number is divided by 3, the remainder is the same as if the sum of the digits of that number were divided by 3.

It appears from Art. 445 that a number is divisible by 11 when the difference between the sum of the digits in the odd places and the sum of the digits in the even places is divisible by 11.

447. From the property of the number 9, mentioned in the preceding Article, a rule may be deduced which will sometimes detect an error in the multiplication of two numbers.

Let $9a + x$ denote the multiplicand, and $9b + y$ the multiplier; then the product is $81ab + 9bx + 9ay + xy$. If then the sum of the digits in the multiplicand be divided by 9, the remainder is x; if the sum of the digits in the multiplier be divided by 9, the remainder is y; and if the sum of the digits in the product be divided by 9, the remainder ought to be the same as when xy is divided by 9, and will be if there be no mistake in the operation.

EXAMPLES ON SCALES OF NOTATION.

Transform the following sixteen numbers from the scales in which they are given to the scales in which they are required:

1. 123456 from the scale of ten to the scale of seven.

2. 1357531 from the scale of ten to the scale of five.

3. 357234 from the scale of ten to the scale of seven.

4. 333310 from the scale of ten to the scale of eleven.

5. 545 from the scale of six to the scale of ten.

6. 4444 from the scale of five to the scale of ten.

7. 3413 from the scale of six to the scale of seven.

8. 40234 from the scale of five to the scale of twelve.

9. 64520 from the scale of seven to the scale of eleven.

10. 15951 from the scale of eleven to the scale of ten.

11. 15·75 from the scale of ten to the scale of eight.

12. 31462·125 from the scale of ten to the scale of eight.

13. 221·248 from the scale of ten to the scale of five.

14. 444·44 from the scale of five to the scale of ten.

15. 1845·3125 from the scale of ten to the scale of twelve.

16. 3065·263 from the scale of eight to the scale of ten.

17. Express in the scale of seven the numbers which are expressed in the scale of ten by 231 and 452; multiply the numbers together in the scale of seven, and reduce to the scale of ten.

18. Divide 17832126 by 4685 in the scale of nine.

19. Extract the square root of 33224 in the scale of six.

20. Extract the square root of 123454321 in the scale of six.

21. Extract the square root of 3445·44 in the scale of six, and reduce the result to the scale of three.

22. Subtract 20404020 from 103050301 in the scale of eight, and extract the square root of the result.

23. Extract the square root of 11000000100001 in the binary scale.

24. Extract the square root of 67556t21 in the scale of twelve.

25. Express $\frac{117}{192}$ in a series of radix-fractions in the scale of twelve.

26. Find in what scale 95 is denoted by 137.

27. Find in what scale 2704 is denoted by 20304.

28. Find in what scale 1331 is denoted by 1000.

29. Find in what scale 16000 is denoted by 1003000.

30. A number is represented in the denary scale by $35\frac{5}{8}$ and in another scale by 55·5, find the radix of the latter scale.

31. Find in what scale of notation sixteen hundred and sixty-four ten-thousandths of unity is represented by ·0404.

32. Shew that 12345654321 is divisible by 12321 in any scale; the radix being supposed greater than six.

33. Shew that 144 is a perfect square in any scale; the radix being supposed greater than four.

34. Shew that 1331 is a perfect cube in any scale; the radix being supposed greater than three.

35. Find which of the weights 1, 2, 4, 8,......2^n pounds must be selected to weigh 1719 pounds.

36. Find which of the weights 1 lb., 3 lbs., 3^2 lbs.,...... must be selected to weigh 1027 lbs., not more than one of each kind being used, but in either scale that is necessary.

37. Find which of the same weights must be selected to weigh 716 lbs.

38. Find which of the same weights must be selected to weigh 475 lbs.

39. Find by operation in the scale of twelve what is the height of a parallelepiped which contains 94 cubic feet 235 cubic inches, and whose base is 24 square feet 5 square inches.

40. Express 2 feet $10\frac{1}{4}$ inches linear measure, and 5 feet $79\frac{1}{6}$ inches square measure, in the scale of twelve as feet and duodecimals of a foot; and the latter quantity being the area of a rectangle, one of whose sides is the former, find its other side by dividing in the scale of twelve.

41. If p_0, p_1, p_2,...... be the digits of a number beginning with the units, prove that the number itself is divisible by eight if $p_0 + 2p_1 + 4p_2$ is divisible by eight.

42. Prove that the difference of two numbers consisting of the same figures is divisible by nine.

43. Find the greatest and least numbers with a given number of digits in any proposed scale.

44. Prove that if in any scale of notation the sum of two numbers is a multiple of the radix, then (1) the digits in which the squares of the numbers terminate are the same, and (2) the sum of this digit and of the digit in which the product of the numbers terminates is equal to the radix.

45. A certain number when represented in the scale two has each of its last three digits (counting from left to right) zero, and the next digit different from zero; when represented in either of the scales three, five, the last digit is zero, and the last but one different from zero; and in every other scale (twelve scales excepted) the last digit is different from zero. What are these twelve scales, and what is the number?

XXX. ARITHMETICAL PROGRESSION.

448. Quantities are said to be in Arithmetical Progression when they increase or decrease by a common difference.

Thus the following series are in Arithmetical Progression:

$$1,\ 3,\ 5,\ 7,\ 9,\ \ldots\ldots$$

$$40,\ 36,\ 32,\ 28,\ 24,\ \ldots\ldots$$

$$a,\ a+b,\ a+2b,\ a+3b,\ \ldots\ldots$$

$$a,\ a-b,\ a-2b,\ a-3b,\ \ldots\ldots$$

In the first example the common difference is 2, in the second -4, in the third b, in the fourth $-b$.

449. Let a denote the first term of an Arithmetical Progression, b the *common difference;* then the second term is $a+b$, the third term is $a+2b$, the fourth term is $a+3b$, and so on. Thus the n^{th} term is $a+(n-1)b$.

450. *To find the sum of a given number of quantities in Arithmetical Progression, the first term and the common difference being supposed known.*

Let a denote the first term, b the common difference, n the number of terms, l the last term, s the sum of the terms. Then

$$s = a + (a + b) + (a + 2b) + \ldots\ldots + l.$$

And, by writing the series in the reverse order, we have also

$$s = l + (l - b) + (l - 2b) + \ldots\ldots + a.$$

Therefore, by addition,

$$2s = (l + a) + (l + a) + \ldots\ldots \text{ to } n \text{ terms}$$

$$= n(l + a);$$

therefore $$s = \frac{n}{2}(l + a) \qquad (1).$$

Also $$l = a + (n - 1)b \qquad (2),$$

thus $$s = \frac{n}{2}\{2a + (n - 1)b\} \qquad (3).$$

The equation (3) gives the value of s in terms of the quantities which were supposed known. Equation (1) also gives a convenient expression for s, and furnishes the following rule: *the sum of any number of terms in Arithmetical Progression is equal to the product of the number of the terms into half the sum of the first and last terms.*

451. *In an Arithmetical Progression the sum of any two terms equidistant from the beginning and the end is equal to the sum of the first and last terms.*

The truth of this has already been seen in the course of the preceding demonstration; it may be shewn formally thus: Let a be the first term, b the common difference, l the last term; then the r^{th} term from the beginning is $a + (r - 1)b$ and the r^{th} term from the end is $l - (r - 1)b$, and the sum of these terms is therefore $l + a$.

452. *To insert a given number of arithmetical means between two given terms.*

Let a and c be the two given terms, n the number of terms to be inserted. Then the meaning of the problem is, that we are to find $n+2$ terms in Arithmetical Progression, a being the first term, and c the last term. Let b denote the common difference; then $c = a + (n+1)b$; therefore $b = \frac{c-a}{n+1}$. This finds b, and the n required terms are

$$a+b, \quad a+2b, \quad a+3b, \ldots \quad \ldots a+nb.$$

453. In Art. 450 we have five quantities occurring, namely, a, b, l, n, s, and these are connected by the equations (1) and (2), or (2) and (3) there established. The student will find that if any three of these five quantities are given, the other two can be found; this will furnish some useful exercises. We give one as an example.

454. *Given the sum of an Arithmetical Progression, the first term, and the common difference; required the number of terms.*

Here $$s = \frac{n}{2}\{2a + (n-1)b\};$$

therefore $$2s = n^2b + (2a-b)n.$$

By solving this quadratic in n we obtain

$$n = \frac{b - 2a \pm \sqrt{\{(2a-b)^2 + 8sb\}}}{2b}.$$

455. It will be seen that *two* values are found for n in the preceding Article; in some cases both values are applicable, as will appear from the following example. Suppose $a = 11$, $b = -2$, $s = 27$; we obtain $n = 3$ or 9. The arithmetical progression is

$$11, \; 9, \; 7, \; 5, \; 3, \; 1, \; -1, \; -3, \; -5, \; \&c.,$$

and it is obvious that the sum of the first three terms is the same as the sum of the first nine terms.

456. Again, suppose $a=4$, $b=2$, $s=18$; we obtain $n=3$ or -6. The sum of three terms beginning with 4 is $4+6+8$ or 18. If we put on terms *before* 4 we obtain the series

$$-2+0+2+4+6+8,$$

and the sum of these six terms is also 18. From this example we may conjecture that when there is a *negative integral* value for the number of terms as well as a positive integral value, the following statement will be true: begin from the last term of the series which is furnished by the positive value, and count backwards for as many terms as the negative value indicates, then the result will be the given sum. The truth of this conjecture may be shewn in the following manner.

The quadratic equation in n obtained in Art. 454 is

$$2s = n^2 b + (2a - b) n \quad \text{.......................(1).}$$

Suppose a series in which the first term is $b-a$, the common difference b, the number of terms m, and the sum s; then

$$2s = m^2 b + (2b - 2a - b) m \quad \text{....................(2).}$$

The roots of (1) and (2) are of equal values but of opposite signs (Art. 340); so that if the roots of (1) are denoted by n_1 and $-n_2$, those of (2) will be n_2 and $-n_1$. Hence n_2 terms of a series which begins with $b-a$ and has the common difference b, will amount to the given sum s. The last term of the series which begins with a and extends to n_1 terms is $a+(n_1-1)b$; we have therefore to shew that if we begin with this term and count backwards for n_2 terms, we arrive at $b-a$. This amounts to shewing that

$$a + (n_1 - 1) b - (n_2 - 1) b = b - a;$$

that is, that

$$a + (n_1 - n_2) b = b - a.$$

Now

$$n_1 - n_2 = -\frac{2a-b}{b}, \quad \text{(Art. 335)};$$

therefore

$$a + (n_1 - n_2) b = a - (2a - b) = b - a.$$

457. Another point may be noticed in connexion with a *negative integral* value of n.

Let $-n_1$ be a negative integral value of n which satisfies the equation

$$s = \frac{n}{2}\{2a + (n-1)b\};$$

then $$s = -\frac{n_1}{2}\{2a - n_1 b - b\}.$$

Therefore $$-s = \frac{n_1}{2}\{2(a-b) + (n_1 - 1)(-b)\}.$$

This shews that if we count *backwards* n_1 terms beginning with $a - b$, the sum so obtained will be $-s$.

For example, taking the case in Art. 456, by beginning at 2 and counting backwards for six terms we obtain

$$2 + 0 - 2 - 4 - 6 - 8,$$

that is, -18.

458. In some cases, however, only one of the values of n found in Art. 454 is an integer. Suppose $a = 11$, $b = -3$, $s = 24$; we obtain $n = 3$ or $5\frac{1}{3}$. The value $5\frac{1}{3}$ suggests to us that of the two numbers 5 and 6, one will correspond to a sum *greater* than 24, and the other to a sum *less* than 24. In fact the sum of 5 terms is 25, and the sum of 6 terms is 21.

We may notice the following point in connexion with a fractional value of n.

Suppose $\frac{p}{q}$ a fractional value of n which satisfies the equation

$$s = \frac{n}{2}\left\{2a + (n-1)b\right\};$$

then $$s = \frac{p}{2q}\left\{2a + \left(\frac{p}{q} - 1\right)b\right\} = \frac{p}{2}\left\{\frac{2a}{q} + \frac{pb}{q^2} - \frac{b}{q}\right\}$$

$$= \frac{p}{2}\left\{\frac{2a}{q} - \frac{b}{q} + \frac{b}{q^2} + (p-1)\frac{b}{q^2}\right\} = \frac{p}{2}\left\{2\left(\frac{a}{q} - \frac{b}{2q} + \frac{b}{2q^2}\right) + (p-1)\frac{b}{q^2}\right\}.$$

This shews that s is equal to the sum of p terms of an Arithmetical Progression in which the first term is $\frac{a}{q} - \frac{b}{2q} + \frac{b}{2q^2}$ and the common difference is $\frac{b}{q^2}$.

In the example given above $\frac{p}{q} = 5\frac{1}{3} = \frac{16}{3}$; so that $p = 16$ and $q = 3$. And

$$\frac{a}{q} - \frac{b}{2q} + \frac{b}{2q^2} = \frac{11}{3} + \frac{1}{2} - \frac{1}{6} = 4; \quad \frac{b}{q^2} = -\frac{1}{3}:$$

thus 24 is the sum of 16 terms of an Arithmetical Progression in which the first term is 4 and the common difference is $-\frac{1}{3}$.

459. The results in the following two simple examples are worthy of notice.

To find the sum of n *terms of the series* 1, 2, 3, 4,...

Here the n^{th} term is n; thus, by Art. 450,

$$s = \frac{n}{2}(n+1).$$

To find the sum of n *terms of the series* 1, 3, 5, 7,...

Here $a = 1$, $b = 2$; thus, by Art. 450,

$$s = \frac{n}{2}\{2 + 2(n-1)\} = \frac{n}{2} \times 2n = n^2.$$

We add two similar questions which lead to important results, although not very closely connected with the present subject.

460. *To find the sum of the squares of the first* n *natural numbers.*

Let s denote the required sum; then

$$s = 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2,$$

and we shall prove that $s = \frac{n(n+1)(2n+1)}{6}$.

We have

$$n^3-(n-1)^3=3n^2-3n+1,$$

$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1,$$

$$(n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1,$$

....

$$3^3-2^3=3\,.\,3^2-3\,.\,3+1,$$

$$2^3-1^3=3\,.\,2^2-3\,.\,2+1,$$

$$1^3-0^3=3\,.\,1^2-3\,.\,1+1.$$

Hence, by addition,

$$n^3=3\{1^2+2^2+\ldots\ldots+n^2\}-3\{1+2+\ldots\ldots+n\}+n,$$

that is, $$n^3=3s-\frac{3n(n+1)}{2}+n.$$

Therefore $$3s=n^3+\frac{3n(n+1)}{2}-n=\frac{n(n+1)(2n+1)}{2},$$

and $$s=\frac{n(n+1)(2n+1)}{2\,.\,3}.$$

461. *To find the sum of the cubes of the first* n *natural numbers.*

Let s denote the required sum; then

$$s=1^3+2^3+3^3+\ldots\ldots+n^3,$$

and we shall prove that $s=\left\{\frac{n(n+1)}{2}\right\}^2$.

We have

$$n^4-(n-1)^4=4n^3-6n^2+4n-1,$$

$$(n-1)^4-(n-2)^4=4(n-1)^3-6(n-1)^2+4(n-1)-1,$$

$$(n-2)^4-(n-3)^4=4(n-2)^3-6(n-2)^2+4(n-2)-1,$$

....

$$3^4-2^4=4\,.\,3^3-6\,.\,3^2+4\,.\,3-1,$$

$$2^4-1^4=4\,.\,2^3-6\,.\,2^2+4\,.\,2-1,$$

$$1^4-0^4=4\,.\,1^3-6\,.\,1^2+4\,.\,1-1.$$

Hence, by addition,

$$n^4 = 4\{1^3 + 2^3 + \dots\dots + n^3\} - 6\{1^2 + 2^2 + \dots\dots + n^2\} + 4\{1 + 2 + \dots\dots + n\} - n;$$

that is, $$n^4 = 4s - n(n+1)(2n+1) + 2n(n+1) - n.$$

Therefore $$4s = n^4 + 2n^3 + n^2,$$

and $$s = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Hence, by Art. 459, we have the following result: *the sum of the cubes of the first* n *natural numbers is equal to the square of the sum of the numbers.*

EXAMPLES OF ARITHMETICAL PROGRESSION.

1. Sum to 20 terms 2, 6, 10, 14, ...

2. Sum to 32 terms 4, $\frac{15}{4}$, $\frac{7}{2}$, $\frac{13}{4}$, ...

3. Sum to 24 terms $\frac{1}{2}$, $-\frac{3}{4}$, -2, ...

4. Sum to 20 terms 5, $\frac{13}{3}$, $\frac{11}{3}$, ...

5. Sum to 10 terms $1\frac{3}{5}$, $1\frac{1}{5}$, $\frac{4}{5}$, ...

6. Sum to 12 terms 1, $1\frac{3}{4}$, $2\frac{1}{2}$, ...

7. Sum to 21 terms $\frac{5}{7}$, $\frac{2}{3}$, $\frac{13}{21}$, ...

8. Sum to 50 terms $\frac{1}{3}$, $\frac{2}{3}$, 1, ...

9. Sum to 30 terms 116, 108, 100, ...

10. Sum to n terms 9, 11, 13, 15, ...

11. Sum to n terms 1, $\frac{5}{6}$, $\frac{2}{3}$, ...

12. Find an A. P. such that the sum of the first five terms is one-fourth the sum of the following five terms, the first term being unity.

13. The first term of a series being 2, and the fifth term being 7, find how many terms must be taken that the sum may be 63.

14. Given $a = 16$, $b = 4$, $s = 88$, find n.

15. If the sum of m terms of an A. P. be always to the sum of n terms in the ratio of m^2 to n^2, and the first term be unity, find the n^{th} term.

16. The sum of a certain number of terms of the series $21 + 19 + 17 + \ldots\ldots$ is 120: find the last term and the number of terms.

17. What is the common difference when the first term is 1, the last 50, and the sum 204?

18. Insert 6 arithmetical means between 1 and 29.

19. If $2n + 1$ terms of the series 1, 3, 5, 7, 9, ... be taken, then the sum of the alternate terms 1, 5, 9, ... will be to the sum of the remaining terms 3, 7, 11, ... as $n + 1$ to n.

20. Find the sum of the first n numbers of the form $4r + 1$.

21. Find how many terms of $1 + 3 + 5 + 7 + \ldots\ldots$ amount to 1234321.

22. Find how many terms of $16 + 24 + 32 + 40 + \ldots\ldots$ amount to 1840.

23. On the ground are placed n stones; the distance between the first and second is one yard, between the second and third three yards, between the third and fourth five yards, and so on. How far will a person have to travel who shall bring them, one by one, to a basket placed at the first stone?

24. The 14th, 134th, and last terms of an A. P. are 66, 666, and 6666 respectively: find the first term and the number of terms.

25. Find a series of arithmetical means between 1 and 21, such that their sum has to the sum of the two greatest of them the ratio of 11 to 4.

26. The sum of the terms of an A.P. is $28\frac{1}{2}$, the first term is -12, the common difference is $\frac{3}{2}$. Find the number of terms.

27. Find how many terms of the series 3, 4, 5, ... must be taken to make 25.

28. Find how many terms of the series 5, 4, 3, ... must be taken to make 14.

29. Shew that a certain number of terms of an A.P. may be found of which the algebraical sum is equal to zero, provided twice the first term be divisible by the common difference, and the series ascending or descending according as the first term is negative or positive.

30. If the m^{th} term of an A.P. be n and the n^{th} term m, of how many terms will the sum be $\frac{1}{2}(m+n)(m+n-1)$, and what will be the last of them?

31. If $s = 72$, $a = 24$, $b = -4$, find n.

32. If $s = pn + qn^2$ whatever be the value of n, find the m^{th} term.

33. If S_a represent the sum of n of the natural numbers beginning with a, prove that $S_{2a+n-1} = 3S_a$.

34. Prove that the squares of $x^2 - 2x - 1$, $x^2 + 1$, and $x^2 + 2x - 1$ are in A.P.

35. The common difference of an A.P. is equal to the difference of the squares of the first and last terms divided by twice the sum of all the terms diminished by the first and last term.

36. The sum of m terms of an A.P. is n, and the sum of n terms with the same first term and the same common difference is m. Shew that the sum of $m+n$ terms is $-(m+n)$ and the sum of $m-n$ terms is $(m-n)\left(1+\frac{2n}{m}\right)$.

37. Find the number of arithmetical means between 1 and 19 when the second mean is to the last as 1 to 6.

38. How many terms of the natural numbers commencing with 4 give a sum of 5350?

39. In a series consisting of an odd number of terms, the sum of the odd terms (the first, third, &c.) is 44, and the sum of the even terms (the second, fourth, &c.) is 33. Find the middle term and the number of terms.

40. If a^2, b^2, c^2, be in A. P., then $\frac{1}{b+c}$, $\frac{1}{c+a}$, $\frac{1}{a+b}$ are in A. P.

41. Sum to n terms the series whose r^{th} term is $2r-1$.

42. Sum to n terms $1-3+5-7+\dots\dots$

43. Sum to n terms $1-2+3-4+\dots\dots$

44. Given the p^{th} term P, and the q^{th} term Q of a series in A. P., express the sum of n terms in terms of P, Q, p, q, n.

45. The p^{th}, q^{th}, and r^{th} terms of an A. P. are x, y, z, respectively; prove that if x, y, z be positive integers, there is an A. P. whose x^{th}, y^{th}, z^{th} terms are p, q, r, respectively; and that the product of the common differences of the progressions is unity.

46. The interior angles of a rectilinear figure are in A. P.; the least angle is 120° and the common difference 5°. Required the number of sides.

47. Find the sum to n terms of $1.2+2.3+3.4+4.5+\dots$

48. If the second term of an A. P. be a mean proportional between the first and the fourth, shew that the sixth term will be a mean proportional between the fourth and the ninth.

49. If $\phi(n)$ be the sum of n terms of an A. P., find $\phi(n)$ in terms of n and the first two terms.

Also shew that $\phi(n+3)-3\phi(n+2)+3\phi(n+1)-\phi(n)=0$.

50. Sum to n terms the series whose m^{th} term $= 5 - \frac{m}{2}$.

51. Divide unity into four parts in A. P. of which the sum of the cubes shall be $\frac{1}{10}$.

52. A servant agrees for certain wages the first month, on the understanding that they are to be raised a shilling every subsequent month until they reach £3 a month. At the end of the first of the months for which he receives £3, he finds that his wages during his time of service have averaged 48 shillings a month. How long has he served?

53. A sets out from a place and travels 5 miles an hour. B sets out $4\frac{1}{2}$ hours after A, and travels in the same direction 3 miles the first hour, $3\frac{1}{2}$ miles the second hour, 4 miles the third hour, and so on. Find in how many hours B will overtake A.

54. A number of persons were engaged to do a piece of work, which would have occupied them m hours if they had commenced at the same time; but instead of doing so they commenced at equal intervals, and then continued to work till the whole was finished: the payment being proportional to the work done by each, the first comer received r times as much as the last. Find the time occupied.

55. A number of three digits is equal to 26 times the sum of its digits; the digits are in arithmetical progression; if 396 be added to the number the digits are reversed: find the number.

56. Shew that the sum of any $2n+1$ consecutive integers is divisible by $2n+1$.

XXXI. GEOMETRICAL PROGRESSION.

462. Quantities are said to be in Geometrical Progression when each is equal to the product of the preceding and some constant factor. The constant factor is called the *common ratio* of the series, or more shortly, the *ratio*. Thus the following series

are in Geometrical Progression:

$$1,\quad 2,\quad 4,\quad 8,\quad 16, \ldots\ldots$$

$$1,\quad \frac{1}{3},\quad \frac{1}{9},\quad \frac{1}{27},\quad \frac{1}{81}, \ldots\ldots$$

$$a,\quad ar,\quad ar^2,\quad ar^3,\quad ar^4, \ldots\ldots$$

In the first example the common ratio is 2, in the second $\frac{1}{3}$, in the third r.

463. Let a denote the first term of a Geometrical Progression, r the common ratio, then the second term is ar, the third term is ar^2, the fourth term is ar^3, and so on. Thus the n^{th} term is ar^{n-1}.

464. *To find the sum of a given number of quantities in Geometrical Progression, the first term and the common ratio being supposed known.*

Let a denote the first term, r the common ratio, n the number of terms, s the sum of the terms. Then

$$s = a + ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1};$$

therefore $$sr = \quad ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$sr - s = ar^n - a;$$

therefore $$s = \frac{a(r^n - 1)}{r - 1} \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If l denote the last term, we have

$$l = ar^{n-1} \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

hence $$s = \frac{rl - a}{r - 1} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Equation (1) gives the value of s in terms of the quantities which are supposed known. Equation (3) is sometimes a convenient form.

465. We may write the value of s thus,

$$s = \frac{a(1-r^n)}{1-r}.$$

Now suppose r less than unity; then the larger n is the smaller will r^n be, and by taking n large enough r^n can be made as small as we please. If then n be taken so large that r^n may be neglected in comparison with unity, the value of s reduces to $\frac{a}{1-r}$. We may enunciate the result thus: *by taking* n *large enough, the sum of* n *terms of the Geometrical Progression can be made to differ as little as we please from* $\frac{a}{1-r}$. This statement is sometimes abbreviated into the following: *the sum of an infinite number of terms of the Geometrical Progression is* $\frac{a}{1-r}$; but it must be remembered that it is to be considered as nothing more than an abbreviation of the preceding statement.

The preceding remarks suppose that r is less than unity. In future, both in the text and in the examples, when we speak of an *infinite* Geometrical Progression we shall always suppose that r is less than unity.

We may apply the preceding remarks to an example. Consider the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots$; here $a=1$, $r=\frac{1}{2}$; thus the sum of n terms is $\frac{1}{1-\frac{1}{2}}\left(1-\frac{1}{2^n}\right)$, that is, $2-\frac{1}{2^{n-1}}$. Now by taking n large enough, 2^{n-1} can be made as large as we please, and therefore $\frac{1}{2^{n-1}}$ as small as we please. Hence we may say that *by taking* n *large enough, the sum of* n *terms of the series can be made to differ from* 2 *by as small a quantity as we please.* This is abbreviated into the following: *the sum of an infinite number of terms of this series is* 2.

466. *In a geometrical progression continued to infinity each term bears a constant ratio to the sum of all which follow it; the common ratio being supposed less than unity.*

Let the series be $a + ar + ar^2 + ar^3 + \ldots$; then the n^{th} term is ar^{n-1}; the sum of all the terms which follow this

$$= ar^n(1 + r + r^2 + \ldots) = \frac{ar^n}{1-r}.$$

The ratio of the n^{th} term to the sum of all which follow it is

$$ar^{n-1} \div \frac{ar^n}{1-r},$$

that is $\frac{1-r}{r}$. This is constant whatever n may be.

If we wish to determine r so that this ratio may have a given value p we put $\frac{1-r}{r} = p$; therefore $r = \frac{1}{1+p}$.

467. Recurring decimals are cases of what are called infinite Geometrical Progressions. Thus, for example, ·2343434 ...... denotes $\frac{2}{10} + \frac{34}{10^3} + \frac{34}{10^5} + \frac{34}{10^7} + \ldots\ldots$ Here the terms after $\frac{2}{10}$ constitute a Geometrical Progression, of which the first term is $\frac{34}{10^3}$, and the common ratio is $\frac{1}{10^2}$. Hence we may say that the sum of an infinite number of terms of this series is $\frac{34}{10^3} \div \left\{1 - \frac{1}{10^2}\right\}$, that is, $\frac{34}{990}$. Therefore the value of the decimal is $\frac{2}{10} + \frac{34}{990}$. We will now investigate a general rule for such examples.

468. *To find the value of a recurring decimal.*

Let P denote the figures which do not recur, and suppose them p in number; let Q denote the figures which do recur, and suppose them q in number. Let s denote the value of the recurring decimal; then

$$s = \cdot PQQQ\ldots\ldots,$$

$$10^p s = P\cdot QQQ\ldots\ldots,$$

$$10^{p+q} s = PQ\cdot QQQ\ldots\ldots;$$

by subtraction, $$(10^{p+q} - 10^p)s = PQ - P.$$

Now $10^{p+q} - 10^p = (10^q - 1)\,10^p$; and $10^q - 1$ when expressed by figures in the usual way will consist of q *nines*. Hence we deduce the usual rule for finding the value of a recurring decimal: subtract the integral number consisting of the non-recurring figures from the integral number consisting of the non-recurring and recurring figures, and divide by a number consisting of as many nines as there are recurring figures followed by as many cyphers as there are non-recurring figures.

469. *To insert a given number of Geometrical means between two given terms.*

Let a and c be the two given terms, n the number of terms to be inserted. Then the meaning of the problem is that we are to find $n+2$ terms in Geometrical Progression, a being the first term and c the last. Let r denote the common ratio; then $c = ar^{n+1}$; thus $r = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$. This finds r, and the required terms are ar, ar^2, ar^3,......ar^n.

470. In Art. 464 we have five quantities occurring, namely, a, r, l, n, s; and these are connected by the equations (1) and (2), or (2) and (3), there given. We might therefore propose to find any two of these five quantities when the other three are given; it will however be seen that some of the cases of this problem are too difficult to be solved. The following four cases present no difficulty: (1) given a, r, n; (2) given a, n, l; (3) given r, n, l; (4) given r, n, s.

471. Suppose, however, that a, s, n are given, and therefore r and l are to be found. Then r would have to be found from the equation

$$s(r-1) = a(r^n - 1);$$

we may divide both sides by $r-1$, and then we shall have an equation of the $(n-1)^{\text{th}}$ degree in the unknown quantity r, which therefore cannot be solved by any method yet given, if n be greater than 3. Similar remarks will hold in the case where l, s, n are given, and therefore a and r are to be found.

472. Four cases of the problem remain, namely, those four in which n is one of the quantities to be found. Suppose a, r, l given, and therefore s and n are to be found. Here n would have to be found from the equation $l = ar^{n-1}$, where the unknown quantity n occurs as an exponent; nothing has been said hitherto as to the solution of such an equation.

473. *To find the sum of* n *terms of the following series;*

$$a, \quad \{a+b\}r, \quad \{a+2b\}r^2, \quad \{a+3b\}r^3, \ldots\ldots$$

Let s denote the sum; then

$$s = a + \{a+b\}r + \{a+2b\}r^2 + \ldots\ldots + \{a+(n-1)b\}r^{n-1},$$

$$rs = \qquad ar + \{a+\ b\}r^2 + \ldots\ldots + \{a+(n-2)b\}r^{n-1} + \{a+(n-1)b\}r^n.$$

By subtraction

$$s(1-r) = a + br + br^2 + \ldots\ldots + br^{n-1} - \{a+(n-1)b\}r^n$$

$$= a + \frac{br(1-r^{n-1})}{1-r} - \{a+(n-1)b\}r^n,$$

therefore $$s = \frac{a - \{a+(n-1)b\}r^n}{1-r} + \frac{br(1-r^{n-1})}{(1-r)^2}.$$

EXAMPLES OF GEOMETRICAL PROGRESSION.

1. Sum to six terms $\frac{8}{5} + \frac{8}{3} + \frac{40}{9} + \ldots\ldots$

2. Sum to ten terms $2 - 2^2 + 2^3 - 2^4 + \ldots\ldots$

3. Sum to n terms $3 + 2 + \frac{4}{3} + \ldots\ldots$

4. Sum to n terms $\frac{2}{3} + \frac{1}{2} + \frac{3}{8} + \ldots\ldots$

5. Sum to infinity $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \ldots\ldots$

6. Sum to infinity $\frac{4}{3} + 1 + \frac{3}{4} + \ldots\ldots$

7. Sum to infinity $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\ldots$

8. Sum to infinity $3+2+\frac{4}{3}+\ldots\ldots$

9. Sum to infinity $4+\frac{12}{5}+\frac{36}{25}+\ldots\ldots$

10. Sum to infinity $1+\frac{1}{4}+\frac{1}{16}+\ldots\ldots$

11. Sum to infinity $5-\frac{1}{2}+\frac{1}{20}-\frac{1}{200}+\ldots\ldots$

12. Sum to infinity $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\ldots$

13. Sum to infinity $\frac{3}{2}-\frac{2}{3}+\frac{8}{27}-\ldots\ldots$

14. Sum to infinity $\frac{1}{5}-\frac{1}{25}+\frac{1}{125}-\ldots\ldots$

15. Sum to infinity $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots\ldots$

16. Sum to infinity $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\ldots\ldots$

17. Sum to infinity $\frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots\ldots$

18. Sum to n terms $r+2r^2+3r^3+4r^4+\ldots\ldots$

19. Sum to n terms $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots\ldots$

20. Sum to n terms $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots\ldots$

21. Sum to n terms $1-\frac{3}{2}+\frac{5}{4}-\frac{7}{8}+\ldots\ldots$

22. Find the sum of any number of terms in G.P. whose first and third terms are given.

23. If the common ratio of a G.P. is -3, find the common ratio of the series obtained by taking every fourth term of the original series.

24. The sum of £700 was divided among four persons, whose shares were in G.P.; and the difference between the greatest and least was to the difference between the means as 37 to 12. Find their respective shares.

25. Sum to n terms the series whose m^{th} term is $(-1)^m a^{4m}$.

26. If P be the sum of the series $1 + r^p + r^{2p} + r^{3p} + \ldots\ldots ad\ inf.$, and Q be the sum of the series $1 + r^q + r^{2q} + r^{3q} + \ldots\ldots ad\ inf.$, prove that $P^q(Q-1)^p = Q^p(P-1)^q$.

27. Shew that $\sqrt{(\cdot 444\ldots\ldots)} = \cdot 666\ldots\ldots$

28. A person who saved every year half as much again as he saved the previous year had in seven years saved £102. 19*s.* How much did he save the first year?

29. In a G.P. shew that the product of any two terms equidistant from a given term is always the same.

30. In a G.P. shew that if each term be subtracted from the succeeding, the successive differences are also in G.P.

31. The square of the arithmetical mean of two quantities is equal to the arithmetical mean of the arithmetical and geometrical means of the squares of the same two quantities.

32. Find a G.P. continued to infinity, in which each term is ten times the sum of all the terms which follow it.

33. If S_n represent the sum of n terms of a given G.P., find the sum of $S_1 + S_2 + S_3 + \ldots\ldots + S_n$.

34. If n geometrical means be found between two quantities a and c, their product will be $(ac)^{\frac{n}{2}}$.

35. Let s denote the sum of n terms of the series a, ar, ar^2, ...; let s' denote the sum of n terms of the series a, ar^{-1}, ar^{-2}, ...; and let l denote the last term of the first series; then will $as = ls'$.

36. If a, b, c, d be in G. P.,

$$(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2.$$

37. If a, b, c, d be in G. P.,

$$(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2.$$

38. The sum of the first three terms of a G. P. $=21$, and the sum of the first four terms $=45$: find the series.

39. Sum to n terms $\left(r-\frac{1}{r}\right)^2+\left(r^2-\frac{1}{r^2}\right)^2+\left(r^3-\frac{1}{r^3}\right)^2+\ldots\ldots$

40. Sum to n terms $5+55+555+\ldots\ldots$

41. Prove that the two quantities between which A is the arithmetical and G the geometrical mean, are given by the formula

$$A \pm \sqrt{\{(A+G)(A-G)\}}.$$

42. There are four numbers, the first three of which are in G. P., and the last three in A. P.; the sum of the first and last is 14, and the sum of the second and third is 12: find the numbers.

43. Three numbers whose sum is 15 are in A. P.; if 1, 4, and 19 be added to them respectively the results are in G. P. Determine the numbers.

44. If a, b, c be in A. P. shew that

$$\frac{2}{9}(a+b+c)^3=a^2(b+c)+b^2(c+a)+c^2(a+b);$$

if they be in G. P. shew that

$$a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3.$$

45. Find the sum of the infinite series

$$ar+(a+ab)r^2+(a+ab+ab^2)r^3+\ldots$$

r and br being each less than unity.

XXXII. HARMONICAL PROGRESSION.

474. Three quantities a, b, c, are said to be in Harmonical Progression when $a : c :: a-b : b-c$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive quantities are in Harmonical Progression.

475. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let a, b, c be in Harmonical Progression; then

$$a : c :: a-b : b-c,$$

therefore $$a(b-c) = c(a-b).$$

Divide by abc, thus

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}.$$

This proves the proposition.

476. The definition in Art. 474 is sometimes expressed in words thus: *three quantities are in harmonical progression when the first is to the third as the difference of the first and second is to the difference of the second and third.* But it must be remembered then that the differences are to be formed *in the same order:* that is by subtracting the second from the first, and the third from the second; or by subtracting the first from the second, and the second from the third. It would not be correct to subtract the first from the second, and the third from the second. The definition by the aid of symbols has the advantage in brevity and exactness over the definition in words.

Sometimes the property demonstrated in Art. 475 is taken as the definition of harmonical progression, which is stated thus: *quantities are said to be in harmonical progression when their reciprocals are in arithmetical progression.*

The term *harmonical* is derived from a fact with regard to musical sounds. Let there be a series of strings of the same substance, the lengths of which are proportional to 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$; and suppose these strings stretched tight with equal force. Then if any two of the strings are sounded together the effect is found to be harmonious to the ear.

There is no formula for the sum of any number of quantities in Harmonical Progression; the property established in the preceding Article will however enable us to solve some questions relating to Harmonical Progression.

477. *To insert a given number of harmonical means between two given terms.*

Let a and c be the two given terms, n the number of terms to be inserted. Then the meaning of the problem is that we are to find $n+2$ terms in Harmonical Progression, a being the first term and c the last. Hence the problem is reducible to the following: *to insert* n *arithmetical means between* $\frac{1}{a}$ *and* $\frac{1}{c}$. Let b denote the common difference; then

$$\frac{1}{c} = \frac{1}{a} + (n+1)b,$$

therefore $$b = \frac{a-c}{(n+1)ac}.$$

The Arithmetical Progression is

$$\frac{1}{a},\ \frac{1}{a}+b,\ \frac{1}{a}+2b,\ \ldots\ldots\ \frac{1}{a}+nb,\ \frac{1}{c},$$

that is,

$$\frac{1}{a},\ \frac{c(n+1)+a-c}{ac(n+1)},\ \frac{c(n+1)+2(a-c)}{ac(n+1)},\ \ldots\ldots$$

$$\frac{c(n+1)+n(a-c)}{ac(n+1)},\ \frac{1}{c}.$$

Therefore the Harmonical Progression is

$$a,\quad \frac{ac(n+1)}{c(n+1)+a-c},\quad \frac{ac(n+1)}{c(n+1)+2(a-c)},\ \dots\dots\dots \frac{ac(n+1)}{c(n+1)+n(a-c)},\quad c.$$

478. Let a and c be any two quantities; let A be their arithmetical mean, G their geometrical mean, H their harmonical mean. Then

$$A-a=c-A;\ \text{therefore } A=\tfrac{1}{2}(a+c).$$

$$a:G::G:c;\ \text{therefore } G=\sqrt{(ac)}.$$

$$a:c::a-H:H-c;\ \text{therefore } H=\frac{2ac}{a+c}.$$

It follows that $G^2=AH$; therefore $A:G::G:H$. Thus G lies in magnitude between A and H; and A is greater than H, for

$$A-H=\tfrac{1}{2}(a+c)-\frac{2ac}{a+c}=\frac{(a-c)^2}{2(a+c)},$$

that is, $A-H$ is a positive quantity.

479. We may observe that the three quantities a, b, c, are in Arithmetical, Geometrical, or Harmonical Progression, according as $\frac{a-b}{b-c}=\frac{a}{a}$, or $=\frac{a}{b}$, or $=\frac{a}{c}$, respectively.

For in the first case $\frac{a-b}{b-c}=1$, therefore $b=\frac{1}{2}(a+c)$.

In the second case $b(a-b)=a(b-c)$; therefore $b^2=ac$.

The third case is obvious by definition.

EXAMPLES OF HARMONICAL PROGRESSION.

1. Continue the series $3+\frac{6}{5}+\frac{3}{4}$ for two terms.

2. Insert 18 harmonical means between 1 and $\frac{1}{20}$.

3. Find the n^{th} term of an H. P., of which a, b, are respectively the first and second terms.

4. Find the $(p+q)^{\text{th}}$ term of an H. P., of which P is the p^{th} term, and Q the q^{th} term.

5. Find what quantity must be subtracted from each of three given quantities that the three results may be in H. P.

6. Three quantities are in H. P.; if half the middle term be subtracted from each, shew that the three remainders are in G. P.

7. Shew that b^2 is greater than, equal to, or less than ac, according as a, b, c, are in A. P., G. P., or H. P.

8. The arithmetical mean of two numbers is 3, and the harmonical mean is $\frac{8}{3}$: find the numbers.

9. The geometrical mean of two numbers is also the geometrical mean between the arithmetical mean of the two numbers and their harmonical mean. The arithmetical mean *minus* the harmonical mean is equal to the square of the difference of the two numbers divided by twice their sum.

10. If z is the harmonical mean between a and b,

$$\frac{1}{z-a}+\frac{1}{z-b}=\frac{1}{a}+\frac{1}{b}.$$

11. There are three numbers in H. P., such that the greatest is the product of the other two, and if one be added to each the greatest becomes the sum of the other two. Find the numbers.

12. The sum of two contiguous terms in H. P. is $\frac{29}{104}$, and their product is $\frac{1}{52}$. Find the series.

13. If between two numbers there be inserted two arithmetical means A_1 and A_2, and two harmonical means H_1, H_2; and between A_1 and A_2 there be inserted an harmonical mean, and between H_1 and H_2 an arithmetical mean; then the geometrical mean between these is equal to the geometrical mean between the original quantities.

14. The arithmetical mean of two quantities x and y is A; the geometrical mean is G; the harmonical mean is H. If $A-G=a$ and $A-H=b$, find x and y in terms of a and b.

15. If a, b, c be in A. P.; α, β, γ in H. P.; $a\alpha, b\beta, c\gamma$ in G. P.; then will

$$\frac{\alpha}{\gamma}+\frac{\gamma}{\alpha}=\frac{a}{c}+\frac{c}{a}.$$

16. If a, b, c are in H. P., shew that

$$\frac{1}{a-b}+\frac{1}{b-c}+\frac{4}{c-a}=\frac{1}{c}-\frac{1}{a}.$$

17. If a, b, c are in H. P., shew that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are also in H. P.

18. If n arithmetical and the same number of harmonical means be inserted between two quantities a and b, and a series of n terms be found by dividing each arithmetical by the corresponding harmonical mean, the sum of the series

$$=n\left\{1+\frac{n+2}{n+1}\,\frac{(a-b)^2}{6ab}\right\}.$$

19. Any whole number of the form $3a^2-b^2$, where a is greater than b, may be divided into three others in H. P., of which the sum of the squares shall be $3a^4+b^4$.

20. The first of a series of n quantities in H. P. is unity, and the sum of the products of every $(n-1)$ terms is to the product of all the terms as $2n$ is to 1: find the progression.

XXXIII. MATHEMATICAL INDUCTION.

480. We shall in the subsequent parts of this book have occasion to use a method of proof which is called *mathematical induction* or *demonstrative induction*, and we shall now exemplify the method.

481. Suppose the following assertion made: the sum of n terms of the series 1, 3, 5, 7, ... is n^2. This assertion we can

see to be true in some cases; for example, the sum of two terms is $1+3$ or 4, that is, 2^2; the sum of three terms is $1+3+5$ or 9, that is, 3^2; we wish however to prove the theorem universally.

Suppose the theorem were known to be true for a certain value of n; that is, suppose for this value of n that

$$1+3+5+\ldots\ldots+(2n-1)=n^2;$$

add $2n+1$ to both sides; then

$$1+3+5+\ldots\ldots+(2n-1)+(2n+1)=n^2+2n+1=(n+1)^2.$$

Thus, if the sum of n terms of the series $=n^2$, the sum of $n+1$ terms will $=(n+1)^2$. In other words, if the theorem is true when we take a certain number of terms, whatever that number may be, it is true when we increase that number by one. But we see by trial that the theorem *is* true when 3 terms are taken, it is therefore true when 4 terms are taken, it is therefore true when 5 terms are taken, and so on. Hence the theorem must be universally true.

482. We will now take another example; we propose to establish the truth of the following formula:

$$1^2+2^2+3^2+\ldots\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

We can easily ascertain by trial that this formula holds in simple cases, for example, when $n=1$, or 2, or 3; we wish, however, to establish it universally.

Suppose the theorem were known to be true for a certain value of n; add $(n+1)^2$ to both sides; then

$$1^2+2^2+3^2+\ldots\ldots+n^2+(n+1)^2=\frac{n(n+1)(2n+1)}{6}+(n+1)^2.$$

But $\frac{n(n+1)(2n+1)}{6}+(n+1)^2=(n+1)\left\{\frac{n(2n+1)}{6}+n+1\right\}$

$$=\frac{n+1}{6}\{2n^2+7n+6\}$$

$$=\frac{n+1}{6}(n+2)(2n+3)=\frac{m(m+1)(2m+1)}{6}, \text{ where } m=n+1.$$

Thus we obtain the same formula for the sum of $n+1$ terms of the series 1^2, 2^2, 3^2 ...... as was supposed to hold for n terms. In other words, if the formula holds when we take a certain number of terms, whatever that number may be, it holds when we increase that number by one. But the formula *does* hold when 3 terms are taken, therefore it holds when 4 terms are taken, therefore it holds when 5 terms are taken, and so on. Hence the formula must hold universally.

483. The two theorems which we have proved by the method of *induction* may be established otherwise. The first theorem is an example of an Arithmetical Progression, and the second has been investigated in Art. 460. There are many other theorems which are capable of easy proof by the method of induction; for example, that in Art. 461.

The theorems asserted in Art. 69, respecting the divisibility of $x^n \pm a^n$ by $x \pm a$, may be proved by induction. For

$$\frac{x^n - a^n}{x - a} = x^{n-1} + \frac{a(x^{n-1} - a^{n-1})}{x - a};$$

hence $x^n - a^n$ is divisible by $x - a$ when $x^{n-1} - a^{n-1}$ is so. Now we see that $x - a$ is divisible by $x - a$, therefore $x^2 - a^2$ is divisible by $x - a$, therefore again $x^3 - a^3$ is divisible by $x - a$, and so on; hence $x^n - a^n$ is always divisible by $x - a$ when n is a positive integer. Similarly the other cases may be established. As another example the student may consider the theorems in Art. 225.

484. The method of *mathematical induction* may be thus described: We prove that if a theorem is true in one case, whatever that case may be, it is true in another case which we may call the *next* case; we prove by trial that the theorem *is* true in a certain case; hence it is true in the next case, and hence in the next to that, and so on; hence it must be true in every case after that with which we began.

485. It is possible that this method of proof may be less satisfactory to the student than a more direct proceeding; it may

appear to him that he is rather compelled to believe propositions so proved than shewn *why* they hold. But as in some cases this is the only method of proof which can be used, the student must accustom himself to it, and should not pass over it when it occurs until he is satisfied of its validity.

486. We may remark that the student of natural philosophy will find the word *induction* used in a different sense in that subject; the word is there applied to the assumption or conjecture that some law holds generally which is found to be true in certain cases that have been examined. There, however, we cannot be sure that the law holds for any cases except those which we have examined, and can never arrive at the conclusion that it is a *necessary* truth. In fact, induction, as used in natural philosophy, is never absolutely demonstrative, often far from it; whereas the method of *mathematical induction* is as rigid as any other process in mathematics.

MISCELLANEOUS EXAMPLES.

1. Transform 221·342 from the scale with radix ten to the scale with radix five.

2. If the radix of a scale be $4m+2$ the square of any number whose last digit is $2m+1$ or $2m+2$ will terminate with that digit.

3. A digit is written down once, twice, thrice, ... up to n times respectively, so as to form n numbers consisting of one, two, three, ... n, places of figures respectively. If a be the first and b the last of the numbers, and r the radix of the scale, the sum of the numbers is $\dfrac{rb-na}{r-1}$.

4. If m, n be any two numbers, g their geometrical mean, a_1, h_1 the arithmetical and harmonical means between m and g, and a_2, h_2 the arithmetical and harmonical means between g and n, prove that $a_1h_2 = g^2 = a_2h_1$.

5. If between b and a there be inserted n arithmetical means, and between a and b there be inserted n harmonical means, the sum of the series composed of the products of the corresponding terms of the two series is $(n+2)\,ab$.

6. If n harmonical means are inserted between the two positive quantities a and b, shew that the difference between the first and the last bears to the difference between a and b a less ratio than that of $n-1$ to $n+1$.

7. A sets out from a certain place and travels one mile the first day, two miles the second day, three the third, four the fourth, and so on. B sets out five days after A and travels the same road at the rate of 12 miles a day. How far will A travel before he is overtaken by B?

8. From 256 gallons of wine a certain number are drawn and replaced with water; this is done a second, a third, and a fourth time, and 81 gallons of wine are then left. How much was drawn out each time?

9. A and B have made a bet, the amount of the stakes being £90, and the sum staked by each being inversely proportional to all the money he has. If A wins he will then have five times what B has left; if B wins he will then have double what A has left. What sum of money had each?

10. If $(a+b+c)(a+b+d)=(c+d+a)(c+d+b)$, prove that each of these quantities is equal to

$$\frac{(a-c)(a-d)(b-c)(b-d)}{(a+b-c-d)^2}.$$

11. If the roots of $ax^2+2bx+c=0$ be possible and different, those of $(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$ will be impossible; and *vice versâ*.

12. If $a+b+c=0$, $x+y+z+w=0$, then the two equations $\sqrt{(ax)}+\sqrt{(by)}+\sqrt{(cz)}=0$, $\sqrt{(bx)}-\sqrt{(ay)}+\sqrt{(cw)}=0$, are deducible the one from the other.

XXXIV. PERMUTATIONS AND COMBINATIONS.

487. The different orders in which any things can be arranged are called their *permutations*.

Thus the permutations of the letters a, b, c, taken two at a time are ab, ba, ac, ca, bc, cb.

488. The *combinations* of things are the different collections that can be formed out of them, without regarding the order in which the things are placed.

Thus the *combinations* of the letters a, b, c, taken two at a time are ab, ac, bc; ab and ba though different *permutations* forming the same *combination*.

489. We may observe that a difference of language occurs in books on this subject; what we have called *permutations* are called *variations* or *arrangements* by some writers, and they restrict the word *permutations* to the case in which *all* the things are used at once; thus they speak of the *variations* or *arrangements* of four letters taken two at a time, or three at a time, but of the *permutations* of them taken all together.

490. *To find the number of permutations of* n *things taken* r *at a time.*

Suppose there to be n letters a, b, c, d,......; we shall first find the number of permutations of them taken *two* at a time. Put a before each of the other letters; we thus obtain $n-1$ permutations in which a stands first. Next put b before each of the other letters; we thus obtain $n-1$ permutations in which b stands first. Similarly there are $n-1$ permutations in which c stands first; and so on. Thus, on the whole, there are $n(n-1)$ permutations of n letters taken two at a time.

We shall now find the number of permutations of the n letters taken three at a time. It has just been shewn that out of n letters we can form $n(n-1)$ permutations each of two letters; hence out of the $n-1$ letters $b, c, d, \ldots\ldots$ we can form $(n-1)(n-2)$ permutations each of two letters; put a before each of these and we have $(n-1)(n-2)$ permutations each of three letters in which a stands first. Similarly there are $(n-1)(n-2)$ permutations each of three letters in which b stands first. Similarly there are as many in which c stands first; and so on. Thus on the whole there are $n(n-1)(n-2)$ permutations of n letters taken three at a time.

From these cases it might be *conjectured* that the number of permutations of n letters taken r at a time is

$$n(n-1)(n-2)\ldots\ldots(n-r+1),$$

and we shall prove that this is the case. For suppose it true that the number of permutations of n letters taken $r-1$ at a time is

$$n(n-1)\ldots\ldots\{n-(r-1)+1\},$$

we shall shew that a similar formula will give the number of permutations of the letters taken r at a time. For out of the $n-1$ letters $b, c, d, \ldots\ldots$ we can form

$$(n-1)(n-2)\ldots\ldots\{n-1-(r-1)+1\}$$

permutations each of $r-1$ letters; put a before each of these, and we obtain as many permutations each of r letters in which a stands first. Similarly we have as many in which b stands first, as many in which c stands first, and so on. Thus on the whole there are

$$n(n-1)(n-2)\ldots\ldots(n-r+1)$$

permutations of n letters taken r at a time.

If then the formula holds when the letters are taken $r-1$ at a time, it will hold when they are taken r at a time; but it has been proved to hold when they are taken three at a time, therefore it holds when they are taken four at a time, therefore it holds when they are taken five at a time, and so on; thus it holds universally.

491. Hence the number of permutations of n things taken all together is $n(n-1)(n-2) \ldots\ldots 1$.

For the sake of brevity $n(n-1)(n-2) \ldots\ldots 1$ is often denoted by $\underline{|n}$; thus $\underline{|n}$ denotes the product of the natural numbers from 1 to n inclusive. The symbol $\underline{|n}$ may be read, *factorial n*.

492. The formula for the number of permutations of n things taken r at a time may also be obtained in another manner.

Let P denote the number of permutations of n letters taken $r-1$ at a time. To form the permutations of n letters taken r at a time we may proceed thus: take any one of the P permutations, and place at the end of it any one of the $n-r+1$ letters which it does not involve. Thus the whole number of the permutations of the n letters taken r at a time will be $(n-r+1)P$.

Now the number of the permutations of n letters taken one at a time is n; therefore the number taken two at a time is $n(n-1)$; therefore the number taken three at a time is $n(n-1)(n-2)$; and so on.

493. Any combination of r things will produce $\underline{|r}$ permutations. For, by Article 491, the r things which form the given combination can be arranged in $\underline{|r}$ different ways.

494. *To find the number of combinations of* n *things taken* r *at a time.*

The number of combinations of n things taken r at a time is

$$\frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{\underline{|r}}.$$

For the number of permutations of n things taken r at a time is $n(n-1)(n-2) \ldots\ldots (n-r+1)$, by Art. 490; and each combination produces $\underline{|r}$ permutations, by Art. 493; hence the number of combinations must be

$$\frac{n(n-1)(n-2) \ldots\ldots (n-r+1)}{\underline{|r}}.$$

If we multiply both numerator and denominator of this expression by $\lfloor n-r$ it becomes $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$.

495. *The number of combinations of* n *things taken* r *at a time is the same as the number of them taken* n − r *at a time.*

The number of combinations of n things taken $n-r$ at a time is

$$\frac{n(n-1)(n-2)\ldots\ldots\{n-(n-r)+1\}}{\lfloor n-r},$$

that is,
$$\frac{n(n-1)(n-2)\ldots\ldots(r+1)}{\lfloor n-r}.$$

Multiply both numerator and denominator by $\lfloor r$ and we obtain $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$, which, by Art. 494, is the number of combinations of n things taken r at a time.

The proposition which we have thus demonstrated will be evident too if we observe that for every combination of r things which we take out of n things, we leave one combination of $n-r$ things. Hence every combination of r things corresponds to a combination of $n-r$ things which contains the remaining things. Such combinations are called *complementary*.

496. *To find for what value of* r *the number of combinations of* n *things taken* r *at a time is greatest.*

Let $(n)_r$ denote the number of combinations of n things taken r at a time,

$(n)_{r-1}$ the number of combinations of n things taken $r-1$ at a time,

then
$$(n)_r = \frac{n-r+1}{r}(n)_{r-1}.$$

The factor $\dfrac{n-r+1}{r}$ may be written $\dfrac{n+1}{r}-1$, which shews that it decreases as r increases. By giving to r in succession the values 1, 2, 3, ... the number of combinations is continually increased so long as $\frac{n+1}{r}-1$ is greater than unity.

First suppose n even and $=2m$, then $\frac{2m+1}{r}-1$ is greater than 1 until $r=m$ inclusive, and when $r=m+1$ it is less than 1. Hence the greatest number of combinations is obtained when the things are taken m at a time, that is, $\frac{n}{2}$ at a time.

Next suppose n odd and $=2m+1$, then $\frac{2m+1+1}{r}-1$ is equal to unity when $r=m+1$. Hence the greatest number of combinations is obtained when they are taken m at a time or $m+1$ at a time, the result being the same in these two cases, that is, when they are taken $\frac{n-1}{2}$ at a time, or $\frac{n+1}{2}$ at a time.

497. *To find the number of permutations of* n *things taken all together which are not all different.*

Let there be n letters; and suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike; the number of permutations of them taken all together will be

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

For let N represent the required number of permutations. If in any one of the permutations the p letters a were changed into p new letters different from any of the rest, then without altering the situation of any of the remaining letters, we could from the single permutation produce $\lfloor p$ different permutations; and so if the p letters a were changed into p different letters, the whole number of permutations would be $N \times \lfloor p$. Similarly, if the q letters b were also changed into q new letters different from any of the rest, the whole number of permutations we could now obtain would be $N \times \lfloor p \times \lfloor q$; and if the r letters c were also changed, the whole number would be $N \times \lfloor p \times \lfloor q \times \lfloor r$. But this number must be equal to the number of permutations of n dissimilar things taken all together, that is, to $\lfloor n$.

Thus $$N \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n;$$

therefore $$N = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

And similarly any other case may be treated.

498. There is another mode in which the result of the preceding Article may be obtained which will be instructive for the student. We will explain it for simplicity by the aid of a particular example; but the reasoning is perfectly general in character. Suppose we have 10 letters; suppose 2 of them to be a, 3 of them to be b, and 5 of them to be c: required the number of permutations of the 10 letters taken all together.

We may consider that we have 10 places which are to be occupied by the 10 letters. Choose any 2 of the places and put a in each; this can be done in $\frac{10 \cdot 9}{1 \cdot 2}$ ways. Choose any 3 of the remaining 8 places, and put b in each; this can be done in $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}$ ways. Then put c in each of the remaining 5 places; this can be done in 1 way; and $1 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$. Now the product of the results thus obtained will obviously give the total number of permutations: this number therefore is $\frac{\lfloor 10}{\lfloor 2 \lfloor 3 \lfloor 5}$.

499. If there be n things not all different, and we require the number of permutations or of combinations of them taken r at a time, the operation will be more complex; we will exemplify the method in the following case:

There are n *things of which* p *are alike and the rest unlike; required the number of combinations of them taken* r *at a time.*

We shall suppose r *less* than $n - p$, and put $n - p = q$. Consider first the number of combinations that can be formed without using any of the p like things; this is the number of combinations of q things taken r at a time, that is, $\frac{\lfloor q}{\lfloor r \lfloor q-r}$. Next take *one* of the p things and $r-1$ of the q things; the number of ways in which combinations can thus be formed is the same as the number of combinations of q things taken $r-1$ at a time, that is, $\frac{\lfloor q}{\lfloor r-1 \lfloor q-r+1}$. Next take *two* of the p things and combine them with $r-2$ of the q things; this can be done in $\frac{\lfloor q}{\lfloor r-2 \lfloor q-r+2}$ ways. Proceed thus, and add the number of combinations so obtained together, which will give the whole number of combinations.

If however r is *not less* than q we should consider first the case in which $r-q$ things are taken from the p like things, and q things are taken from the q unlike things; this can be done in only one way. Next take $r-q+1$ things from the p things, and $q-1$ from the q things; this can be done in q ways. And so on.

If the number of *permutations* be required, we have only to observe that each combination of r things in which s are alike and the rest unlike, will produce $\frac{\lfloor r}{\lfloor s}$ permutations (Art. 497), and thus the whole number of permutations may be found.

500. By the following method the formula for the number of combinations of n things taken r at a time may be found without assuming the formula for the number of permutations.

Let $(n)_r$ denote the number of combinations of n things taken r at a time. Suppose n letters $a, b, c, d, \ldots\ldots$; among the combinations of these r at a time, the number of those which contain the letter a is obviously equal to the number of combinations of the remaining $n-1$ letters $r-1$ at a time, that is, to $(n-1)_{r-1}$. The number of combinations which contain the letter b is also $(n-1)_{r-1}$, and so for each of the letters. But if we form, first all the combinations which contain a, then all the combinations which contain b, and so on, each particular combination will appear r times; for if $r = 3$, for example, the combination abc will occur among those containing a, among those containing b, and among those containing c. Hence

$$(n)_r = \frac{n}{r}(n-1)_{r-1}.$$

In this formula change n and r first into $n-1$ and $r-1$ respectively, then into $n-2$ and $r-2$ respectively, and so on; thus

$$(n-1)_{r-1} = \frac{n-1}{r-1}(n-2)_{r-2},$$

$$(n-2)_{r-2} = \frac{n-2}{r-2}(n-3)_{r-3},$$

....

$$(n-r+2)_2 = \frac{n-r+2}{2}(n-r+1)_1.$$

Multiply, and cancel like terms, and we obtain

$$(n)_r = \frac{n(n-1)\ldots\ldots(n-r+2)(n-r+1)}{\underline{|r}},$$

for $(n-r+1)_1 = n-r+1$.

501. *To find the whole number of permutations of* n *things when each may occur once, twice, thrice, ... up to* r *times.*

Let there be n letters $a, b, c, \ldots\ldots$ First take them one at a time; this gives the number n. Next take them two at a time; here a may stand before a, or before any one of the remaining letters; similarly b may stand before b, or before any one of the remaining letters; and so on; thus there are n^2 different permutations of the letters taken two at a time. Similarly by putting successively $a, b, c, \ldots\ldots$ before each of the permutations of the letters taken two at a time, we obtain n^3 permutations of the letters taken three at a time. Thus the whole number of permutations when the letters are taken r at a time will be n^r.

502. Since the number of combinations of n things taken r at a time must be some *integer*, the expression

$$\frac{n(n-1)\ldots\ldots(n-r+1)}{\underline{|r}}$$

must be an integer. Hence we see that the product of any r successive integers must be divisible by $\underline{|r}$. We shall give a more direct proof of this proposition in the Chapter on the *theory of numbers.*

EXAMPLES OF PERMUTATIONS AND COMBINATIONS.

1. How many different permutations may be made of the letters in the word *Caraccas* taken all together?

2. How many of the letters in the word *Heliopolis?*

3. How many of the letters in the word *Ecclesiastical?*

4. How many of the letters in the word *Mississippi?*

5. If the number of permutations of n things taken 4 together is equal to twelve times the number of permutations of n things taken 2 together; find n.

6. In how many ways can 2 sixes, 3 fives, and 5 twos be thrown with 10 dice?

7. If there are twenty pears at three a penny, how many different selections can be made in buying six-pennyworth? In how many of these will a particular pear occur?

8. From a company of soldiers mustering 96, a picket of 10 is to be selected; determine in how many ways it can be done, (1) so as always to include a particular man, (2) so as always to exclude the same man.

9. How many parties of 12 men each can be formed from a company of 60 men?

10. If the number of combinations of n things $r-r'$ together be equal to the number of combinations of n things $r+r'$ together, find n.

11. In how many ways can a party of six take their places at a round table?

12. In how many different ways may n persons form a ring?

13. How many different numbers can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9; each of these digits occurring once and only once in each number? How many with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, on the same supposition?

14. Out of 12 conservatives and 16 reformers how many different committees could be formed each consisting of 3 conservatives and 4 reformers?

15. If there be x things to be given to n persons, shew that n^x will represent the whole number of different ways in which they may be given.

16. Suppose the number of combinations of n things taken r together to be equal to the number taken $r+1$ together, and that each of these equal numbers is to the number of combinations of n things taken $r-1$ together as 5 is to 4, find the value of n.

17. Given m things of one kind, and n things of a second kind, find the number of permutations that can be formed containing r of the first and s of the second.

18. Find how many different rectangular parallelepipeds there are satisfying the conditions that each edge shall be equal to some one of n given straight lines all of different lengths; and that no face of a parallelepiped shall be a square.

19. The ratio of the number of combinations of $4n$ things taken $2n$ together, to that of $2n$ things taken n together is

$$\frac{1 \,.\, 3 \,.\, 5 \,\ldots\ldots\, (4n-1)}{\{1 \,.\, 3 \,.\, 5 \,\ldots\ldots\, (2n-1)\}^2}.$$

20. Out of 17 consonants and 5 vowels, how many words can be formed, each containing two consonants and one vowel?

21. Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels?

22. Find the number of words which can be formed out of 7 letters taken all together, each word being such that 3 given letters are never separated.

23. With 10 flags representing the 10 numerals how many signals can be made, each representing a number and consisting of not more than 4 flags?

24. How many words of two consonants and one vowel can be formed from 6 consonants and 3 vowels, the vowel being the middle letter of each word?

25. How many words of 6 letters may be formed with 3 vowels and 3 consonants, the vowels always having the even places?

26. A boat's crew consists of 8 men, 3 of whom can only row on one side and 2 only on the other. Find the number of ways in which the crew can be arranged.

27. A telegraph has m arms, and each arm is capable of n distinct positions: find the total number of signals which can be made with the telegraph, supposing that all the arms are to be used to form a signal.

28. A pack of cards consists of 52 cards marked differently: in how many different ways can the cards be arranged in four sets, each set containing 13 cards?

29. How many triangles can be formed by joining the angular points of a decagon, that is, each triangle having three of the angular points of the decagon for *its* angular points?

30. There are n points in a plane, no three of which are in the same straight line with the exception of p, which are all in the same straight line: find the number of *straight lines* which result from joining them.

31. Find the number of *triangles* which can be formed by joining the points in the preceding Example.

32. There are n points in space, of which p are in one plane, and there is no other plane which contains more than three of them: how many planes are there, each of which contains three of the points?

33. If n points in a plane be joined in all possible ways by indefinite straight lines, and if no two of the straight lines be coincident or parallel, and no three pass through the same point (with the exception of the n original points), then the number of points of intersection, exclusive of the n points, will be

$$\frac{n(n-1)(n-2)(n-3)}{8}.$$

34. There are fifteen boat-clubs; two of the clubs have each three boats on the river, five others have two, and the remaining eight have one: find an expression for the number of ways in which a list can be formed of the order of the 24 boats, observing that the second boat of a club cannot be above the first.

35. A shelf contains 20 books, of which 4 are single volumes, and the others form sets of 8, 5, and 3 volumes respectively: find in how many ways the books may be arranged on the shelf, the volumes of each set being in their due order.

36. Find the number of the permutations of the letters in the word *examination* taken 4 at a time.

37. Find the number of the combinations of the letters in the word *proportion* taken 6 at a time.

38. There are $n-1$ sets containing $2a$, $3a$, ... na things respectively: shew that the number of combinations which can be formed by taking a out of the first, $2a$ out of the second, and so on for each combination, is $\dfrac{\lfloor na}{\{\lfloor a\}^n}$.

39. Find the sum of all the numbers which can be formed with all the digits 1, 2, 3, 4, 5, in the scale of 10.

40. The sum of all numbers that are expressed by the same digits is divisible by the sum of the digits.

XXXV. BINOMIAL THEOREM. POSITIVE INTEGRAL EXPONENT.

503. We have already seen that $(x+a)^2 = x^2 + 2xa + a^2$, and that $(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$; the object of the present Chapter is to find an expression equal to $(x+a)^n$ where n is any positive integer.

504. By ordinary multiplication we obtain

$$(x+a_1)(x+a_2) = x^2 + (a_1 + a_2)x + a_1a_2,$$

$$(x+a_1)(x+a_2)(x+a_3) = x^3 + (a_1 + a_2 + a_3)x^2 + (a_1a_2 + a_2a_3 + a_3a_1)x + a_1a_2a_3,$$

$$(x+a_1)(x+a_2)(x+a_3)(x+a_4) = x^4 + (a_1 + a_2 + a_3 + a_4)x^3 + (a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)x^2 + (a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4)x + a_1a_2a_3a_4.$$

Now in these results we see that the following laws hold:

I. The number of terms on the right-hand side is one more than the number of the binomial factors which are multiplied together.

II. The exponent of x in the first term is the same as the number of binomial factors, and in the succeeding terms each exponent is less than that of the preceding term by unity.

III. The coefficient of the first term is unity; the coefficient of the second term is the sum of the second terms of the binomial factors; the coefficient of the third term is the sum of the products of the second terms of the binomial factors taken two at a time; the coefficient of the fourth term is the sum of the products of the second terms of the binomial factors taken three at a time; and so on; the last term is the product of all the second terms of the binomial factors.

We shall now prove that these laws always hold whatever be the number of binomial factors. Suppose the laws to hold when $n-1$ factors are multiplied together; that is, suppose

$$(x+a_1)(x+a_2)\ldots(x+a_{n-1}) = x^{n-1} + p_1x^{n-2} + p_2x^{n-3} + p_3x^{n-4} + \ldots + p_{n-1},$$

where p_1 = the sum of the terms $a_1, a_2, \ldots\ldots a_{n-1}$,

p_2 = the sum of the products of these terms taken two at a time,

p_3 = the sum of the products of these terms taken three at a time,

....

p_{n-1} = the product of all these terms.

Multiply both sides of this identity by another factor $x + a_n$; thus

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n) = x^n + (p_1+a_n)x^{n-1} + (p_2+p_1a_n)x^{n-2} + (p_3+p_2a_n)x^{n-3} + \ldots\ldots + p_{n-1}a_n.$$

Now $p_1 + a_n = a_1 + a_2 + \ldots\ldots + a_{n-1} + a_n$

= the sum of all the terms $a_1, a_2, \ldots\ldots a_n$;

$p_2 + p_1a_n = p_2 + a_n(a_1 + a_2 + \ldots\ldots + a_{n-1})$

= the sum of the products taken two at a time of all the terms $a_1, a_2, \ldots\ldots a_n$;

$p_3 + p_2a_n = p_3 + a_n(a_1a_2 + a_2a_3 + a_1a_3 + \ldots)$

= the sum of the products taken three at a time of all the terms $a_1, a_2, \ldots\ldots a_n$.

....

$p_{n-1}a_n$ = the product of all the terms $a_1, a_2, \ldots\ldots a_n$.

Hence if the laws hold when $n-1$ factors are multiplied together, they hold when n factors are multiplied together; but they have been proved to hold when 4 factors are multiplied together, therefore they hold when 5 factors are multiplied together, and so on; thus they hold universally.

We shall write the result for the multiplication of n factors thus for abbreviation,

$$(x+a_1)(x+a_2)\ldots(x+a_n) = x^n + q_1x^{n-1} + q_2x^{n-2} + q_3x^{n-3} + \ldots + q_n.$$

The number of terms in q_1 is obviously n; the number of terms in q_2 is the same as the number of combinations of the

n things $a_1, a_2, \ldots\ldots a_n$, taken two at a time, that is, $\frac{n(n-1)}{1 \cdot 2}$; the number of terms in q_3 is the same as the number of combinations of the n things $a_1, a_2, \ldots\ldots a_n$ taken three at a time, that is $\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$; and so on. Now suppose $a_1, a_2, a_3 \ldots\ldots a_n$ each $= a$; thus q_1 becomes na, and q_2 becomes $\frac{n(n-1)}{1 \cdot 2} a^2$, and so on; and we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1 \cdot 2} a^2 x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^3 x^{n-3} + \ldots\ldots + \frac{n(n-1)}{1 \cdot 2} a^{n-2} x^2 + na^{n-1}x + a^n.$$

This formula is called the *Binomial Theorem;* the series on the right-hand side is called the *expansion* of $(x+a)^n$, and when we put this series in the place of $(x+a)^n$ we are said to *expand* $(x+a)^n$. The theorem was discovered by Newton.

505. For example, take $(x+a)^5$; here $n=5$,

$$\frac{n(n-1)}{1 \cdot 2} = \frac{5 \cdot 4}{1 \cdot 2} = 10, \quad \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} = 10,$$

$$\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} = 5;$$

thus $$(x+a)^5 = x^5 + 5x^4a + 10x^3a^2 + 10x^2a^3 + 5xa^4 + a^5.$$

Again, suppose we require the expansion of $(c^2+yz)^5$; we have only to write c^2 for x and yz for a in the preceding identity; thus

$$(c^2+yz)^5 = (c^2)^5 + 5(c^2)^4 yz + 10(c^2)^3(yz)^2 + 10(c^2)^2(yz)^3 + 5c^2(yz)^4 + (yz)^5$$
$$= c^{10} + 5c^8yz + 10c^6y^2z^2 + 10c^4y^3z^3 + 5c^2y^4z^4 + y^5z^5.$$

Similarly,

$$(c^2+2y^2)^5 = (c^2)^5 + 5(c^2)^4 2y^2 + 10(c^2)^3(2y^2)^2 + 10(c^2)^2(2y^2)^3 + 5c^2(2y^2)^4 + (2y^2)^5$$
$$= c^{10} + 10c^8y^2 + 40c^6y^4 + 80c^4y^6 + 80c^2y^8 + 32y^{10}.$$

506. The Binomial Theorem is so very important that the student should pay close attention to the demonstration of it. Three laws are observed to hold when we multiply together a small number of binomial factors; and it is shewn strictly by induction that these laws will hold whatever be the number of binomial factors multiplied together.

The inductive demonstration depends mainly on the following principle: suppose that we have formed all the combinations of $n-1$ letters taken r at a time, and that a new letter is introduced; the combinations of the n letters taken r at a time consist of the combinations of the $n-1$ letters r at a time, together with the combinations obtained by combining the new letter with all the combinations of the old letters $r-1$ at a time. This principle is applied in succession to the cases $r=1$, $r=2$, $r=3$, ... up to $r=n-1$.

But even without the inductive process the universal truth of the laws will be obvious on due consideration. Suppose we have to multiply together n binomial factors $x+a_1$, $x+a_2$, ..., $x+a_n$; when the multiplication is effected every term in the result is a product formed by taking *one letter out of each binomial factor.* Thus if we require the term which involves x^{n-2} we must multiply together the second letter in *any two* binomial factors and the first letter in the remaining $n-2$ binomial factors; hence the coefficient of x^{n-2} must consist of the sum of the products of *every two* of the letters a_1, a_2, ... a_n; and the number of these products will be the same as the number of combinations of n things taken two at a time. Similarly we may determine the coefficient of any other power of x, as x^{n-4} for example.

The Binomial Theorem may also be demonstrated in the following manner: We can verify by trial that the Theorem holds for small values of n as 2, 3, 4; assume then that

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1.2}a^2x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-3} + \ldots;$$

multiply both sides by $x+a$; thus

$$(x+a)^{n+1} = x^{n+1} + nax^n + \frac{n(n-1)}{1.2}a^2x^{n-1} + \frac{n(n-1)(n-2)}{1.2.3}a^3x^{n-2} + \ldots$$
$$+ ax^n + na^2x^{n-1} + \frac{n(n-1)}{1.2}a^3x^{n-2} + \ldots$$

Hence, by putting together like terms, we have

$$(x+a)^{n+1} = x^{n+1} + (n+1)ax^n + \frac{(n+1)n}{1.2}a^2x^{n-1}$$
$$+ \frac{(n+1)n(n-1)}{1.2.3}a^3x^{n-2} + \ldots;$$

that is, we obtain for $(x+a)^{n+1}$ a series of the same form as that for $(x+a)^n$, having $n+1$ in the place of n. This shews that if the Binomial Theorem is true for any exponent it is also true when that exponent is increased by unity. But the Theorem is true when the exponent is 4; therefore it is true when the exponent is 5; therefore it is true when the exponent is 6; and so on. Thus the Theorem is true for any positive integral exponent.

507. In the expansion of $(x+a)^n$ suppose $x=1$; thus

$$(1+a)^n = 1 + na + \frac{n(n-1)}{1.2}a^2 + \frac{n(n-1)(n-2)}{1.2.3}a^3 + \ldots\ldots + a^n;$$

since this is true whatever a may be, we may write x for a; thus

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots\ldots + x^n.$$

The coefficient of the second term in the expansion of $(1+x)^n$ is n; the coefficient of the third term is $\frac{n(n-1)}{1.2}$; and generally the coefficient of the $(r+1)^{\text{th}}$ term, being the number of combinations of n things taken r at a time is, by Art. 494, equal to $\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{\lfloor r}$; by multiplying both numerator and denominator by $\lfloor n-r$ this becomes $\frac{\lfloor n}{\lfloor r \, \lfloor n-r}$.

508. *In the expansion of* $(1+x)^n$ *the coefficient of the* r^{th} *term from the beginning is equal to the coefficient of the* r^{th} *term from the end.*

The coefficient of the r^{th} term from the beginning is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{\lfloor r-1};$$

by multiplying both numerator and denominator by $\lfloor n-r+1$ this becomes $\dfrac{\lfloor n}{\lfloor r-1 \lfloor n-r+1}$.

The r^{th} term from the end is the $(n-r+2)^{\text{th}}$ from the beginning, and its coefficient is

$$\frac{n(n-1)\ldots\ldots\{n-(n-r+2)+2\}}{\lfloor n-r+1}, \text{ or } \frac{n(n-1)\ldots\ldots r}{\lfloor n-r+1};$$

and this also $= \dfrac{\lfloor n}{\lfloor r-1 \lfloor n-r+1}$.

509. It appears from the preceding Article that the coefficient of the r^{th} term may be written thus, $\dfrac{\lfloor n}{\lfloor r-1 \lfloor n-r+1}$. If we apply this to the last term for which $r=n+1$, this expression takes the form $\dfrac{\lfloor n}{\lfloor n \lfloor 0}$. The symbol $\lfloor 0$ has had no meaning hitherto assigned to it; if we agree to consider it equivalent to 1, then the general expression will hold true for the *last* term.

510. *To find the greatest coefficient in the expansion of* $(1+x)^n$.

This has been investigated in the Chapter on Permutations and Combinations (Art. 496); it is there shewn that when n is even, the greatest coefficient is found by putting $\dfrac{n}{2}$ for r in the expression $\dfrac{\lfloor n}{\lfloor r \lfloor n-r}$; when n is odd the greatest coefficient is found by putting $\dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ for r in the expression, the result being the same in the two cases.

511. *To find the greatest term in the expansion of* $(x+a)^n$.

The r^{th} term of the expansion is $\dfrac{n(n-1)\dots(n-r+2)}{\lfloor r-1}x^{n-r+1}a^{r-1}$; the $(r+1)^{\text{th}}$ term may be obtained by multiplying the r^{th} term by $\dfrac{n-r+1}{r}\cdot\dfrac{a}{x}$, that is, by $\left(\dfrac{n+1}{r}-1\right)\dfrac{a}{x}$. This multiplier diminishes as r increases, and $\left(\dfrac{n+1}{r}-1\right)\dfrac{a}{x}$ is greater than 1 only so long as $\dfrac{n+1}{r}-1$ is greater than $\dfrac{x}{a}$, that is, only so long as $\dfrac{n+1}{r}$ is greater than $\dfrac{x}{a}+1$, that is, only so long as r is less than $\dfrac{n+1}{\frac{x}{a}+1}$.

If $\dfrac{n+1}{\frac{x}{a}+1}$ be an integer, then, denoting this integer by p, the p^{th} term of the expansion is *equal* to the $(p+1)^{\text{th}}$ term, and these terms are greater than any other term; but if $\dfrac{n+1}{\frac{x}{a}+1}$ be *not* an integer, then the greatest term is the $(q+1)^{\text{th}}$, where q is the integral part of $\dfrac{n+1}{\frac{x}{a}+1}$.

512. In the theorem for expanding $(x+a)^n$, as a may have any value, we may suppose it negative if we please; thus put $-c$ for a and we have

$$(x-c)^n = x^n - ncx^{n-1} + \frac{n(n-1)}{1.2}c^2x^{n-2} - \dots\dots + n(-c)^{n-1}x + (-c)^n.$$

We may observe that the expansion of a binomial can always be reduced to the case in which one of the two quantities is unity. For

$$(x+a)^n = x^n\left(1+\frac{a}{x}\right)^n = x^n(1+y)^n, \text{ if } y = \frac{a}{x}.$$

We may then expand $(1+y)^n$ and multiply each term by x^n, and thus obtain the expansion of $(x+a)^n$.

513. *To find the sum of the coefficients of the terms in the expansion of* $(1+x)^n$.

The theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots + nx^{n-1} + x^n$$

is true for all values of x; put $x=1$; thus

$$2^n = 1 + n + \frac{n(n-1)}{1.2} + \ldots\ldots + n + 1.$$

That is, the sum of the coefficients $= 2^n$.

514. *The sum of the coefficients of the odd terms in the expansion of* $(1+x)^n$ *is equal to the sum of the coefficients of the even terms.*

Put $x=-1$ in the expansion of $(1+x)^n$; thus

$$0 = 1 - n + \frac{n(n-1)}{1.2} - \frac{n(n-1)(n-2)}{1.2.3} + \ldots\ldots$$

$=$ sum of the odd coefficients $-$ sum of the even coefficients.

Since then the sums are equal, by the preceding Article each must $=\frac{2^n}{2}$; that is, 2^{n-1}.

515. The result in Art. 513 gives a theorem relating to Combinations. For suppose there are n things; then we can take them singly in n ways, we can take them two at a time in $\frac{n(n-1)}{1.2}$ ways, we can take them three at a time in $\frac{n(n-1)(n-2)}{1.2.3}$ ways, and so on. Hence by Art. 513 the total number of ways of taking n things is 2^n-1. This theorem was obtained by the early writers on Algebra before the Binomial Theorem was known; the proof is a simple example of mathematical induction which is deserving of notice. We have to

shew that if unity be added to the total number of ways of taking n things, the result is 2^n. Suppose we have four letters a, b, c, d; form all the possible selections and prefix unity to them. Thus we have

1,

a, *b*, *c*, *d*,

ab, *ac*, *ad*, *bc*, *bd*, *cd*,

abc, *abd*, *acd*, *bcd*,

abcd.

Here the total number of symbols is 16, that is, 2^4. Now take an additional letter e; the corresponding set of symbols will consist of those already given, and those which can be formed from them by affixing e to each of them. The number will therefore be doubled; that is, it will be 2^5. The mode of reasoning is general, and shews that if the theorem is true for n things, it is true for $n+1$ things.

EXAMPLES OF THE BINOMIAL THEOREM.

1. Write down the 3rd term of $(a+b)^{15}$.

2. Write down the 49th term of $(a-x)^{50}$.

3. Write down the 5th term of $(a^2-b^2)^{12}$.

4. Write down the 2001st term of $(a^{\frac{3}{10}}+x^{\frac{3}{10}})^{2002}$.

5. Write down all the terms of $(5-4x)^4$.

6. Write down the 5th term of $(3x^{\frac{1}{2}}-4y^{\frac{1}{2}})^9$.

7. Write down the 6th term of $(2a^{\frac{1}{2}}-b^{\frac{3}{2}})^{10}$.

8. Write down all the terms of $\left(5-\frac{x}{6}\right)^6$.

9. Write down the middle term of $(a+x)^{10}$.

10. Write down the two middle terms of $(a+x)^9$.

11. Expand $\{a+\sqrt{(a^2-1)}\}^6+\{a-\sqrt{(a^2-1)}\}^6$ in powers of a.

12. Write down the coefficient of y in the expansion of

$$\left(y^2+\frac{c^3}{y}\right)^5.$$

13. If A be the sum of the odd terms and B the sum of the even terms in the expansion of $(x+a)^n$, prove that

$$A^2-B^2=(x^2-a^2)^n.$$

14. Prove that the difference between the coefficients of x^{r+1} and x^r in the expansion of $(1+x)^{n+1}$ is equal to the difference between the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1+x)^n$.

15. Shew that the middle term in the expansion of $(1+x)^{2n}$

$$=\frac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{\lfloor n}\,2^n x^n.$$

16. Find the binomial expansion of which four consecutive terms are 2916, 4860, 4320, 2160.

17. Prove that if the term x^r occurs in the expansion of $\left(x+\frac{1}{x}\right)^n$ the coefficient of the term $=\dfrac{\lfloor n}{\lfloor \frac{1}{2}(n-r)\,\lfloor \frac{1}{2}(n+r)}$.

18. Write down the coefficient of x^{2r+1} in the expansion of

$$\left(x-\frac{1}{x}\right)^{2n+1}.$$

19. Find the r^{th} term from the beginning, the r^{th} term from the end, and the middle term of $\left(x-\frac{1}{x}\right)^{2n}$.

20. If $t_0, t_1, t_2, t_3, \ldots\ldots$ represent the terms of the expansion of $(a+x)^n$, shew that

$$(t_0-t_2+t_4-\ldots\ldots)^2+(t_1-t_3+t_5-\ldots\ldots)^2=(a^2+x^2)^n.$$

XXXVI. BINOMIAL THEOREM. ANY EXPONENT.

516. We have seen that when n is a positive integer

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \ldots\ldots$$

We now proceed to shew that this relation holds when n has any value positive or negative, integral or fractional, that is, we shall prove the Binomial Theorem for *any* exponent. We shall make some observations on the proof after giving it in the usual form.

517. Suppose m and n are *positive integers;* then we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{\underline{|3}} x^3 + \ldots\ldots\ldots \quad (1),$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{\underline{|3}} x^3 + \ldots\ldots\ldots \quad (2).$$

But $$(1+x)^m \times (1+x)^n = (1+x)^{m+n};$$

hence the product of the series which form the right-hand members of (1) and (2) must $= (1+x)^{m+n}$; that is,

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2} x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{\underline{|3}} x^3 + \ldots\ldots$$

$$= \left\{1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{\underline{|3}} x^3 + \ldots\ldots\right\}$$

$$\times \left\{1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{\underline{|3}} x^3 + \ldots\ldots\right\} \ldots\ldots\ldots\ldots (3).$$

Equation (3) has been proved on the supposition that m and n are positive integers; but the product of the two series which occur on the right-hand side of (3) must be of the *same form* whatever

m and n may be; we therefore infer that (3) must be true whatever m and n may be. We shall now use a notation that will enable us to express (3) briefly. Let $f(m)$ denote the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{\underline{|3}}x^3 + \ldots\ldots$$

whatever m may be; then $f(n)$ will denote what the series becomes when n is put for m; and $f(m+n)$ will denote what the series becomes when $m+n$ is put for m. And when m is any positive integer $f(m) = (1+x)^m$; also $f(0) = 1$. Thus (3) may be written

$$f(m+n) = f(m) \times f(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

Similarly, $$f(m+n+p) = f(m+n) \times f(p)$$
$$= f(m) \times f(n) \times f(p).$$

Proceeding in this way we may shew that

$$f(m+n+p+q+\ldots\ldots) = f(m) \times f(n) \times f(p) \times f(q) \times \ldots\ldots \quad (5).$$

Now let $m = n = p = q = \ldots\ldots = \frac{s}{r}$, where s and r are positive integers, and suppose the number of terms to be r; then (5) becomes

$$f(s) = \left\{f\left(\frac{s}{r}\right)\right\}^r;$$

therefore $$\{f(s)\}^{\frac{1}{r}} = f\left(\frac{s}{r}\right).$$

But since s is a positive integer $f(s) = (1+x)^s$, and therefore

$$\{f(s)\}^{\frac{1}{r}} = (1+x)^{\frac{s}{r}};$$

therefore $$(1+x)^{\frac{s}{r}} = f\left(\frac{s}{r}\right) = 1 + \frac{s}{r}x + \frac{\frac{s}{r}\left(\frac{s}{r}-1\right)}{1 \cdot 2}x^2 + \ldots\ldots$$

This proves the Binomial Theorem when the exponent is *any positive quantity.*

Again, in (4) put $-n$ for m; thus

$$f(-n) \times f(n) = f(0) = 1;$$

therefore
$$\frac{1}{f(n)} = f(-n).$$

But if n be any positive quantity, $f(n) = (1+x)^n$; hence

$$\frac{1}{(1+x)^n} = f(-n);$$

that is,
$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \ldots\ldots$$

This proves the Binomial Theorem when the exponent is any *negative* quantity.

518. The proof of the Binomial Theorem for any exponent contained in the preceding Article was first given by Euler; although difficult and not altogether satisfactory, it is a valuable exercise for the student. We shall now offer some remarks upon it.

The first point we have to notice is the mode of proving that $f(m+n) = f(m) \times f(n)$. The student should for an exercise write down three or four terms of the series for $f(m)$, and also of the series for $f(n)$, and multiply them together; if the product be arranged according to powers of x, it will be found that so far as it has been completely formed, it will agree with the series for $f(m+n)$. But from knowing what $f(m)$ and $f(n)$ represent when m and n are positive integers, we infer without the trouble of actual multiplication, that the law which is expressed by $f(m+n) = f(m) \times f(n)$ must hold. The mode of establishing this law in the simple case in which m and n are positive integers is a valuable and important algebraical artifice.

But the way in which we infer that $f(m+n) = f(m) \times f(n)$, *whatever* m *and* n *may be*, is still more important. The principle is merely this: the *form* of any algebraical product is the same whether the factors represent whole numbers or fractions, positive or negative numbers; thus, for example,

$$(a+b)(a+c) = a^2 + (b+c)a + bc$$

is true whatever a, b, and c may be. Hence we infer that $f(m) \times f(n)$ will have the same *form* in all cases, whether m and n be positive integers or not.

The student may also notice the proof of this result which is given in the *Theory of Equations*, Chapter XXIV.

519. The most difficult point however to be considered is the meaning of the sign = in the assertion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots\ldots\ldots\ldots\ldots (1).$$

Suppose, for example, that $n = -1$, then the above becomes

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Now we know that the sum of r terms of the series $1 - x + x^2 - x^3 + \ldots\ldots$ is $\frac{1-(-x)^r}{1+x}$; hence when x is numerically *less* than unity, by taking enough terms of the series, we can obtain a result differing *as little as we please* from $\frac{1}{1+x}$, and thus we can in this case understand the assertion in (2). But when x is numerically *greater* than unity, there is no such numerical approximation to the value of $\frac{1}{1+x}$ obtained by taking a large number of terms of the series $1 - x + x^2 - x^3 + \ldots\ldots$

We shall see in the Chapter on the Convergence of Series, that when x is numerically less than unity, we can form a definite conception of the series on the right of (1) whatever n may be. In this case there is no difficulty in the assertion

$$f(m+n) = f(m) \times f(n);$$

each of the three series which it involves is arithmetically intelligible. But when x is numerically greater than unity, we cannot give an arithmetical meaning to the series or to the assertion; all we ought to say is, that if we form the product of the first r terms of $f(m)$ and the first r terms of $f(n)$, the first r terms of the result will agree with the first r terms of $f(m+n)$; but this will not justify us in writing $f(m+n) = f(m) \times f(n)$. The case in which x is numerically equal to unity would require special investigation which would be out of place here.

On the whole then we may conclude that the Binomial Theorem for the expansion of $(1+x)^n$ gives a result which is arithmetically intelligible and true when x is numerically less than unity; in what sense the result is true when x is numerically greater than unity has not yet been explained in an elementary manner. The subject of the expansion of expressions is however properly a portion of the Differential Calculus, to which the student must be referred for a fuller consideration of the difficulties.

520. *To find the numerically greatest term in the expansion of* $(1+x)^n$.

We consider x as positive.

I. Suppose n a positive integer.

The $(r+1)^{\text{th}}$ term may be formed by multiplying the r^{th} term by $\frac{n-r+1}{r}x$, that is, by $\left(\frac{n+1}{r}-1\right)x$; and this multiplier diminishes as r increases. Put

$$\left(\frac{n+1}{p}-1\right)x = 1, \quad \text{therefore } p = \frac{(n+1)\,x}{x+1}.$$

If p be an integer, two terms of the expansion are equal, namely, the p^{th} and the $(p+1)^{\text{th}}$, and these are greater than any other term. If p be not an integer, suppose q the integral part of p, then the $(q+1)^{\text{th}}$ term is the greatest.

II. Suppose n positive but not integral.

As before, the $(r+1)^{\text{th}}$ term may be formed by multiplying the r^{th} term by $\left(\frac{n+1}{r}-1\right)x$.

If then x be greater than unity, there is no greatest term; for the above multiplier can, by increasing r, be made as near to $-x$ as we please; that is, each term from and after some fixed term can be made as nearly as we please *numerically* x times the preceding term, and thus the terms increase without limit.

But if x be not greater than unity *there will be* a greatest term; for if $p = \frac{(n+1)x}{x+1}$, then as long as r is less than p the multiplier is greater than unity, and the terms go on increasing; but when r is greater than p the multiplier is less than unity, and so long as it continues positive it diminishes as r increases; and when the multiplier becomes negative it is still numerically less than unity; so that each term after r has passed the value p is numerically less than the preceding term. Hence, as in the first case, if p be an integer, the p^{th} term is equal to the $(p+1)^{\text{th}}$ term, and these are greater than any other term; if p be not an integer, suppose q the integral part of p, then the $(q+1)^{\text{th}}$ term is the greatest.

III. Suppose n negative.

Let $m = -n$, so that m is positive. The numerical value of the $(r+1)^{\text{th}}$ term may be obtained by multiplying that of the r^{th} term by $\left(\frac{m+r-1}{r}\right)x$, that is, by $\left(\frac{m-1}{r}+1\right)x$.

If x be greater than unity we may shew, as in the second case, that there is no greatest term.

If x be less than unity, put

$$\left(\frac{m-1}{p}+1\right)x = 1, \quad \text{therefore } p = \frac{(m-1)x}{1-x}.$$

If p be a positive integer, the p^{th} term is equal to the $(p+1)^{\text{th}}$ term, and these are greater than any other term. If p be positive but not an integer, suppose q the integral part of p, then the $(q+1)^{\text{th}}$ term is the greatest. If p be negative, then m is less than unity; in this case each term is less than the preceding, and the first term, that is, unity, is the greatest.

If x be equal to unity, then when m is greater than unity the terms continually increase and there is no greatest term, when m is equal to unity the terms are all equal, and when m is less than unity the terms continually decrease so that the first is the greatest.

We have supposed throughout that x is positive; if x be negative, put $y = -x$, so that y is positive; then find the numerically greatest term of $(1+y)^n$, and this will also be the numerically greatest term of $(1+x)^n$.

521. The first term of the expansion of $(1+x)^n$ is unity; any other term is known since the $(r+1)^{\text{th}}$ term is

$$\frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r}x^r.$$

This expression is called the *general term*, because by putting 1, 2, 3, ... successively for r, it gives us in succession the 2^{nd}, 3^{rd}, 4^{th}, ... terms; that is, we can obtain from it *any* term after the first. The expression for the general term may be modified in particular cases, and sometimes simplified, as will be seen in the following examples:

$(1+x)^{-m}$. Here $n=-m$; the general term becomes

$$\frac{(-m)(-m-1)\ldots\ldots(-m-r+1)}{\lfloor r}x^r,$$

which may be written

$$\frac{m(m+1)\ldots\ldots(m+r-1)}{\lfloor r}(-1)^r x^r.$$

$(1+x)^{\frac{1}{2}}$. Here $n=\frac{1}{2}$; the numerator of the coefficient of x^r is

$$\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\ldots\left(\frac{1}{2}-r+1\right);$$

if r is not less than 2, this may be written

$$\frac{1.3.5.7\ldots\ldots(2r-3)}{2^r}(-1)^{r-1};$$

hence in the expansion of $(1+x)^{\frac{1}{2}}$, the first term is 1, the second is $\frac{1}{2}x$, and *any subsequent* term may be found by putting for the $(r+1)^{\text{th}}$ term

$$\frac{1.3.5.7\ldots\ldots(2r-3)}{2^r\lfloor r}(-1)^{r-1}x^r.$$

$(1+x)^{-2}$. This is a particular case of $(1+x)^{-m}$. The coefficient of x^r is

$$\frac{2.3.4\ldots\ldots(2+r-1)}{\lfloor r}(-1)^r, \text{ that is, } (r+1)(-1)^r.$$

$(1-x)^{-2}$. By the preceding example the $(r+1)^{\text{th}}$ term is $(r+1)(-1)^r(-x)^r$, that is, $(r+1)x^r$.

$(1+x)^{-3}$. This is a particular case of $(1+x)^{-m}$. The coefficient of x^r is

$$\frac{3\,.\,4\,.\,5\ldots\ldots(3+r-1)}{\lfloor r}(-1)^r,\ \text{that is,}\ \frac{(r+1)(r+2)}{2}(-1)^r.$$

$(1-x)^{-3}$. By the preceding example the $(r+1)^{\text{th}}$ term is

$$\frac{(r+1)(r+2)}{2}(-1)^r(-x)^r,\ \text{that is,}\ \frac{(r+1)(r+2)}{2}x^r.$$

If x and n are positive it will be found that in the expansion of $(1+x)^{-n}$ the terms are alternately positive and negative; and in the expansion of $(1-x)^{-n}$ the terms are all positive. If x and n are positive, and n not an integer, it will be found that in the expansion of $(1+x)^n$ the terms begin by being positive, and eventually become alternately positive and negative; and in the expansion of $(1-x)^n$ the terms begin by being alternately positive and negative, and eventually become all of one sign.

522. A Multinomial expression may be raised to any power by repeated use of the Binomial Theorem; thus, for example,

$$\{a+b+c\}^3=\{a+(b+c)\}^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3;$$

if we now expand $(b+c)^2$ and $(b+c)^3$ and put the resulting expansions in the place of these quantities respectively, we shall obtain the expansion of $\{a+b+c\}^3$. Similarly,

$$\{a+b+c+d\}^3=\{a+(b+c+d)\}^3=a^3+3a^2(b+c+d)$$
$$+3a(b+c+d)^2+(b+c+d)^3;$$

the expansion may then be completed by finding the expansion of $(b+c+d)^2$ and of $(b+c+d)^3$ in the manner just exemplified. Or we may proceed thus,

$$\{a+b+c+d\}^3=\{(a+b)+(c+d)\}^3=(a+b)^3$$
$$+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3;$$

the expansion may then be completed by expanding $(a+b)^3$, $(a+b)^2$, $(c+d)^3$, and $(c+d)^2$, and effecting the requisite multiplications.

523. *To find the number of homogeneous products of* r *dimensions that can be formed out of* n *letters* a, b, c, …… *and their powers.*

By common division, or by the Binomial Theorem,

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \ldots$$

$$\frac{1}{1-bx} = 1 + bx + b^2x^2 + b^3x^3 + \ldots$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \ldots$$

....

Thus

$$\frac{1}{1-ax} \cdot \frac{1}{1-bx} \cdot \frac{1}{1-cx} \cdots\cdots = \left\{1 + ax + a^2x^2 + a^3x^3 + \cdots\cdots\right\}$$

$$\times \left\{1 + bx + b^2x^2 + b^3x^3 + \cdots\cdots\right\} \times \left\{1 + cx + c^2x^2 + c^3x^3 + \cdots\cdots\right\} \cdots\cdots$$

$$= 1 + S_1x + S_2x^2 + S_3x^3 + \cdots\cdots \text{ suppose.}$$

Here
$$S_1 = a + b + c + \cdots\cdots,$$

$$S_2 = a^2 + ab + b^2 + ac + \cdots\cdots,$$

$$S_3 = a^3 + a^2b + abc + b^3 + \cdots\cdots,$$

....

that is, S_1 is equal to the sum of the quantities $a, b, c, \ldots\ldots$; S_2 is equal to the sum of all the products, each of two dimensions, that can be formed of $a, b, c, \ldots\ldots$ and their powers; S_3 is equal to the sum of all the products, each of three dimensions, that can be formed; and so on. To find the *number* of products in any one of these sets of products, we put $a, b, c, \ldots\ldots$ each $= 1$; thus

$$\frac{1}{1-ax} \cdot \frac{1}{1-bx} \cdot \frac{1}{1-cx} \cdots\cdots \text{ becomes } \frac{1}{(1-x)^n} \text{ or } (1-x)^{-n}.$$

Hence in this case S_r is the coefficient of x^r in the expansion of $(1-x)^{-n}$; that is,

$$=\frac{n(n+1)\ldots\ldots(n+r-1)}{\lfloor r}.$$

This is therefore the number of homogeneous products of r dimensions that can be formed out of a, b, c, ... and their powers.

524. *To find the number of terms in the expansion of any multinomial, the exponent being a positive integer.*

The number of terms in the expansion of $(a_1+a_2+a_3+\ldots+a_r)^n$ is the same as the number of homogeneous products of n dimensions that can be formed out of a_1, a_2, a_3, ... a_r, and their powers. Hence, by the preceding Article, it is

$$\frac{r(r+1)(r+2)\ldots\ldots(r+n-1)}{\lfloor n}.$$

525. The Binomial Theorem may be applied to extract the roots of numbers approximately. Let N be a number whose n^{th} root is required, and suppose $N=a^n+b$; then

$$N^{\frac{1}{n}}=(a^n+b)^{\frac{1}{n}}=a\left(1+\frac{b}{a^n}\right)^{\frac{1}{n}}=a(1+x)^{\frac{1}{n}},$$

where $x=\frac{b}{a^n}$. If now x be a small fraction, the terms in the expansion of $(1+x)^{\frac{1}{n}}$ diminish rapidly, and we may obtain an approximate value of $(1+x)^{\frac{1}{n}}$, and therefore of $N^{\frac{1}{n}}$, by retaining only a few of these terms. It will therefore be convenient to take a so that a^n may differ as little as possible from N, and thus b may be as small as possible. Sometimes it will be better to suppose $N=a^n-b$.

526. We will close this Chapter with six examples which will illustrate the use of the Binomial Theorem.

(1) The ratio $(a+x)^n : a^n$ is nearly equal to the ratio $a+nx : a$ when nx is small compared with a. This holds whether x be positive or negative, and for values of n integral or fractional, positive or negative. See Art. 383.

(2) Expand $\dfrac{a+bx}{p+qx}$ in a series of ascending powers of x.

$$\frac{a+bx}{p+qx}=\frac{a+bx}{p\left(1+\dfrac{qx}{p}\right)}=\frac{1}{p}(a+bx)\left(1+\frac{qx}{p}\right)^{-1};$$

expand $\left(1+\dfrac{qx}{p}\right)^{-1}$ by the Binomial Theorem; thus we have

$$\frac{a+bx}{p+qx}=\frac{1}{p}(a+bx)\left(1-\frac{qx}{p}+\frac{q^2x^2}{p^2}-\frac{q^3x^3}{p^3}+\ldots\ldots\right)$$

$$=\frac{a}{p}+\frac{x}{p}\left(b-\frac{aq}{p}\right)-\frac{qx^2}{p^2}\left(b-\frac{aq}{p}\right)+\ldots\ldots$$

Or we may proceed thus,

$$\frac{a+bx}{p+qx}=\frac{a+\dfrac{aqx}{p}}{p+qx}+\frac{\left(b-\dfrac{aq}{p}\right)x}{p+qx}=\frac{a}{p}+\frac{x}{p}\left(b-\frac{aq}{p}\right)\left(1+\frac{qx}{p}\right)^{-1}$$

$$=\frac{a}{p}+\frac{x}{p}\left(b-\frac{aq}{p}\right)\left(1-\frac{qx}{p}+\frac{q^2x^2}{p^2}-\frac{q^3x^3}{p^3}+\ldots\ldots\right);$$

and thus we obtain the same result as before.

This example frequently occurs in mathematics, especially in cases where x is so small that its square and higher powers may be neglected; we have then *approximately*

$$\frac{a+bx}{p+qx}=\frac{a}{p}+\frac{x}{p}\left(b-\frac{aq}{p}\right).$$

(3) Required approximate values of the roots of the quadratic equation $ax^2+bx+c=0$, when ac is very small compared with b^2.

The roots are $\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$.

And by the Binomial Theorem, $\sqrt{(b^2-4ac)}=b\left(1-\dfrac{4ac}{b^2}\right)^{\frac{1}{2}}$

$$=b\left\{1-\frac{1}{2}\frac{4ac}{b^2}-\frac{1}{8}\left(\frac{4ac}{b^2}\right)^2-\frac{1}{16}\left(\frac{4ac}{b^2}\right)^3-\ldots\ldots\right\}.$$

Thus for the root with the upper sign we get

$$-\frac{c}{b}-\frac{ac^2}{b^3}-\frac{2a^2c^3}{b^5}-\ldots\ldots$$

and for the root with the lower sign we get

$$-\frac{b}{a}+\frac{c}{b}+\frac{ac^2}{b^3}+\frac{2a^2c^3}{b^5}+\ldots\ldots$$

If a be very small, while b and c are not small, the former root does not differ much from $-\frac{c}{b}$, and the latter root is numerically very large. See Art. 342.

It is deserving of notice that the approximate value of the root in the former case coincides with what we shall obtain in the following way. Write the equation thus,

$$bx+c=-ax^2.$$

For an approximate result neglect the term ax^2 as small; thus we obtain $x=-\frac{c}{b}$. Then substitute this approximate value of x in the term ax^2; thus we obtain

$$bx+c=-\frac{ac^2}{b^2},$$

that is,

$$x=-\frac{c}{b}-\frac{ac^2}{b^3}.$$

Again, substitute this new approximate value of x in the term ax^2, and preserve the terms involving a and a^2; thus we obtain

$$bx+c=-\frac{ac^2}{b^2}-\frac{2a^2c^3}{b^4},$$

that is,

$$x=-\frac{c}{b}-\frac{ac^2}{b^3}-\frac{2a^2c^3}{b^5},$$

and so on.

(4) To prove that if n be any positive integer the *integral* part of $(2+\sqrt{3})^n$ is an *odd* number.

The meaning of this proposition will be easily seen by taking some simple cases; thus $2+\sqrt{3}$ lies between 3 and 4 in value, so that the integral part of it is the *odd* number 3; $(2+\sqrt{3})^2$ will be found to lie between 13 and 14 in value, so that the integral part of it is the *odd* number 13.

Suppose then I to denote the *integral* part of $(2+\sqrt{3})^n$, and $I+F$ its complete value, so that F is a proper fraction. We have by the Binomial Theorem

$$I+F=2^n+n2^{n-1}3^{\frac{1}{2}}+\frac{n(n-1)}{1\,.\,2}2^{n-2}3^{\frac{2}{2}}+\ldots\ldots+3^{\frac{n}{2}}\ldots\ldots(1).$$

Now $2-\sqrt{3}$ is a proper fraction, therefore also so is $(2-\sqrt{3})^n$; denote it by F'; then

$$F'=2^n-n2^{n-1}3^{\frac{1}{2}}+\frac{n(n-1)}{1\,.\,2}2^{n-2}3^{\frac{2}{2}}-\ldots\ldots+(-1)^n3^{\frac{n}{2}}\ldots\ldots(2).$$

Now add (1) and (2); the *irrational* terms on the right disappear, and we have

$$I+F+F'=2\left\{2^n+\frac{n(n-1)}{1\,.\,2}2^{n-2}3^{\frac{2}{2}}\right.$$

$$\left.+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}2^{n-4}3^{\frac{4}{2}}+\ldots\ldots\right\}$$

$$=\text{an } even \text{ integer.}$$

But F and F' are proper fractions: we must therefore have

$$F+F'=1, \text{ and } I=\text{an } odd \text{ integer.}$$

A similar result holds for $(a+\sqrt{b})^n$ if a is the integer next greater than $\sqrt{b}$, so that $a-\sqrt{b}$ is a proper fraction.

(5) Required the sum of the coefficients of the first $r+1$ terms of the expansion of $(1-x)^{-n}$. We have

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{1\,.\,2}x^2+\ldots+\frac{n(n+1)\ldots(n+r-1)}{\underline{|r}}x^r+\ldots$$

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots$$

Therefore $(1-x)^{-(n+1)}$ is equal to the product of the two series. Now if we multiply the series together, we see that *the coefficient of x^r* in the product is

$$1+n+\frac{n(n+1)}{1\,.\,2}+\ldots\ldots+\frac{n(n+1)\ldots\ldots(n+r-1)}{\underline{|r}},$$

we may naturally *assume* then that this must be equal to the coefficient of x^r in the expansion of $(1-x)^{-(n+1)}$; that is, to

$$\frac{(n+1)(n+2) \ldots\ldots (n+r)}{\lfloor r};$$

thus the required summation is effected.

(6) The Binomial Theorem may be applied in the manner just shewn to establish numerous algebraical identities; we will give one more example.

Let $$\phi(m, r) = \frac{m(m-1)(m-2) \ldots (m-r+1)}{\lfloor r};$$

it is required to shew that

$$\phi(n, 0)\,\phi(n, r) - \phi(n, 1)\,\phi(n-1, r-1) + \phi(n, 2)\,\phi(n-2, r-2)$$
$$- \phi(n, 3)\,\phi(n-3, r-3) + \ldots\ldots = 0.$$

The expression here given is the expansion of

$$\frac{n(n-1)(n-2) \ldots (n-r+1)}{\lfloor r}(1-1)^r,$$

which must obviously be zero.

EXAMPLES OF THE BINOMIAL THEOREM.

Expand each of the following twelve expressions to four terms:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{1}{4}}$.
3. $(1+x)^{\frac{2}{3}}$.
4. $(1+x)^{-\frac{1}{2}}$.
5. $(1+x)^{-\frac{1}{4}}$.
6. $(1+x)^{-\frac{2}{3}}$.
7. $(1-x)^{\frac{1}{5}}$.
8. $(1-2x)^{\frac{3}{4}}$.
9. $\sqrt{(a^2-x^2)}$.
10. $(3a-2x)^{\frac{2}{3}}$.
11. $(a^2-bx)^{-\frac{2}{5}}$.
12. $(1+5x)^{\frac{17}{5}}$.

Find the $(r+1)^{\text{th}}$ term in the expansion of the following eight expressions:

13. $(1-x)^{-4}$.
14. $(1-x)^{-5}$.
15. $(1-x)^{\frac{1}{n}}$.
16. $(1-px)^{\frac{1}{p}}$.
17. $\dfrac{1}{\sqrt{(1+x)}}$.
18. $(1-x^2)^{-\frac{2}{3}}$.
19. $(1-2x)^{-\frac{7}{2}}$.
20. $\dfrac{1}{\sqrt[4]{(1-x)}}$.

Calculate the following four roots approximately:

21. $\sqrt{(24)}$. 22. $\sqrt[3]{(999)}$. 23. $\sqrt[5]{(31)}$. 24. $\sqrt[5]{(99000)}$.

25. If x be small compared with unity, shew that

$$\frac{\sqrt{(1+x)}+\sqrt[3]{\{(1-x)^2\}}}{1+x+\sqrt{(1+x)}}=1-\frac{5x}{6}\text{ nearly.}$$

26. Shew that the number of combinations of n things when taken in ones, threes, fives, ... exceeds the number when taken in twos, fours, sixes, ... by unity.

27. Shew that the number of homogeneous products of n things of n dimensions is

$$\frac{|\underline{2n-1}}{|\underline{n}\,|\underline{n-1}}.$$

Find the greatest term in the following four expansions:

28. $(1+x)^n$ when $x=\frac{2}{3}$ and $n=4$.

29. $(1+x)^{-n}$ when $x=\frac{1}{5}$ and $n=12$.

30. $(1+x)^{-n}$ when $x=\frac{5}{7}$ and $n=3$.

31. $(1-x)^{-n}$ when $x=\frac{7}{12}$ and $n=\frac{8}{3}$.

32. Find the greatest term in the expansion of $\left(n-\frac{1}{n}\right)^{2n+1}$, where n is a positive integer.

33. Find the number of terms in the expansion of

$$(a+b+c+d)^{10}.$$

34. Find the first term with a negative coefficient in the expansion of $(1+\frac{1}{2}x)^{\frac{11}{3}}$.

35. If p be greater than n, the coefficient of x^p in the expansion of $\frac{x^n}{(1-x)^{2n}}$ is $\frac{p(p^2-1^2)(p^2-2^2)\ldots\ldots\{p^2-(n-1)^2\}}{|\underline{2n-1}}$.

36. The coefficient of x^{2n} in the expansion of $\dfrac{(1-2x)^3}{(1-3x^2)^4}$ is

$$3^{n-1}\frac{(n+1)(n+2)(5n+3)}{2}.$$

37. Find the coefficient of x^n in the expansion of $\dfrac{(1+x)^2}{(1-x)^4}$.

38. Expand $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$ in ascending powers of x. Write down the coefficient of x^{2r} and of x^{2r+1}.

39. Shew that the n^{th} coefficient in the expansion of $(1-x)^{-n}$ is always the double of the $(n-1)^{\text{th}}$.

40. Shew that if t_r denote the middle term in the expansion of $(1+x)^{2r}$, then $t_0+t_1+t_2+\ldots\ldots=(1-4x)^{-\frac{1}{2}}$.

41. Write down the sum of

$$1+\frac{1}{4}+\frac{1\,.\,3}{4\,.\,8}+\frac{1\,.\,3\,.\,5}{4\,.\,8\,.\,12}+\ldots\ldots \textit{ ad inf.}$$

42. Find the sum of the squares of the coefficients in the expansion of $(1+x)^n$, where n is a positive integer.

43. If $p_r=\dfrac{1\,.\,3\,.\,5\ldots\ldots(2r-1)}{2\,.\,4\,.\,6\ldots\ldots 2r}$, prove that

$$p_{2n+1}+p_1p_{2n}+p_2p_{2n-1}+\ldots\ldots+p_{n-1}p_{n+2}+p_np_{n+1}=\tfrac{1}{2}.$$

44. Shew that if m and n are positive integers the coefficient of x^m in the expansion of $\dfrac{1}{(1-x)^{n+1}}$ is equal to the coefficient of x^n in the expansion of $\dfrac{1}{(1-x)^{m+1}}$.

45. Find the coefficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+\ldots\ldots \textit{ ad inf.})^n.$$

XXXVII. THE MULTINOMIAL THEOREM.

527. We have in the preceding Chapter given some examples of the expansion of a multinomial; we now proceed to consider this point more fully. We propose to find an expression for the *general term* in the expansion of $(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots)^n$. The number of terms in the series a_0, a_1, a_2, ... may be any whatever, and n may be positive or negative, integral or fractional.

Put b_1 for $a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$, then we have to expand $(a_0 + b_1)^n$; the general term of the expansion is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\underline{|\mu}} a_0^{n-\mu} b_1^{\mu},$$

μ being a positive integer. Put b_2 for $a_2x^2 + a_3x^3 + \ldots\ldots$, then $b_1^{\mu} = (a_1x + b_2)^{\mu}$; since μ is a *positive integer* the general term of the expansion of $(a_1x + b_2)^{\mu}$ may be denoted either by

$$\frac{\underline{|\mu}}{\underline{|q}\,\underline{|\mu-q}} (a_1x)^{\mu-q} b_2^{q}, \text{ or by } \frac{\underline{|\mu}}{\underline{|q}\,\underline{|\mu-q}} (a_1x)^{q} b_2^{\mu-q};$$

we will adopt the latter form as more convenient for our purpose.

Combining this with the former result, we see that the general term of the proposed expansion may be written

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\underline{|q}\,\underline{|\mu-q}} a_0^{n-\mu} (a_1x)^{q} b_2^{\mu-q}.$$

Again, put b_3 for $a_3x^3 + a_4x^4 + \ldots\ldots$, then $b_2^{\mu-q} = (a_2x^2 + b_3)^{\mu-q}$, and the general term of the expansion of this will be

$$\frac{\underline{|\mu-q}}{\underline{|r}\,\underline{|\mu-q-r}} (a_2x^2)^{r} b_3^{\mu-q-r}.$$

Hence the general term of the proposed expansion may be written

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\underline{|q}\,\underline{|r}\,\underline{|\mu-q-r}} a_0^{n-\mu} (a_1x)^{q} (a_2x^2)^{r} b_3^{\mu-q-r}.$$

Proceeding in this way we shall obtain for the required general term

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots} a_0^{n-\mu} a_1^q a_2^r a_3^s a_4^t \ldots\ldots x^{q+2r+3s+4t+\ldots}$$

where $$q+r+s+t+\ldots\ldots=\mu.$$

If we suppose $n-\mu=p$, we may write the general term in the form

$$\frac{n(n-1)(n-2)\ldots\ldots(p+1)}{\lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots} a_0^p a_1^q a_2^r a_3^s a_4^t \ldots\ldots x^{q+2r+3s+4t+\ldots}$$

where $$p+q+r+s+t+\ldots\ldots=n.$$

Thus the expansion of the proposed multinomial consists of a series of terms of which that just given may be taken as the general type.

It should be observed that $q, r, s, t, \ldots\ldots$ are always *positive integers*, but p is not a positive integer unless n be a positive integer. When p is a positive integer, we may, by multiplying both numerator and denominator by $\lfloor p$, write the factor

$$\frac{n(n-1)(n-2)\ldots\ldots(p+1)}{\lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots}$$

in the more symmetrical form

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots}.$$

In the above expression for the general term we may regard the multiplier of $x^{q+2r+3s+4t+\ldots}$ as the *coefficient* of the term. Sometimes however the word *coefficient* is applied to the factor $\dfrac{n(n-1)\ldots\ldots(p+1)}{\lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots}$; this is usually the meaning of the word in the cases in which x has been put equal to unity, as in the Examples 25...32 at the end of this Chapter.

528. Suppose we require the coefficient of an assigned power of x in the expansion of $(a_0 + a_1x + a_2x^2 + \ldots\ldots)^n$, for example, that of x^m. We have then

$$q + 2r + 3s + 4t + \ldots\ldots = m,$$

$$p + q + r + s + t + \ldots\ldots = n.$$

We must find by trial all the positive integral values of q, r, s, t, ... which satisfy the first of these equations; then from the second equation p can be found. The required coefficient is then the sum of the corresponding values of the expression

$$\frac{n(n-1)(n-2)\ldots\ldots(p+1)}{\lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots} a_0^p a_1^q a_2^r a_3^s a_4^t \ldots\ldots$$

When n is a positive integer, then p must be so too, and we may use the more symmetrical form

$$\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r \lfloor s \lfloor t \ldots\ldots} a_0^p a_1^q a_2^r a_3^s a_4^t \ldots\ldots$$

529. For example, find the coefficient of x^7 in the expansion of $(1 + 2x + 3x^2 + 4x^3)^4$.

Here
$$q + 2r + 3s = 7,$$
$$p + q + r + s = 4.$$

Begin with the greatest admissible value of s; this is $s = 2$, with which we have $r = 0$, $q = 1$, $p = 1$. Next try $s = 1$; with this we may have $r = 2$, $q = 0$, $p = 1$; also we may have $r = 1$, $q = 2$, $p = 0$. Next try $s = 0$; with this we may have $r = 3$, $q = 1$, $p = 0$. These are all the solutions; they are collected in the annexed table.

p	q	r	s
1	1	0	2
1	0	2	1
0	2	1	1
0	1	3	0

Also $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$. Thus the required coefficient is

$$\frac{\lfloor 4}{\lfloor 2} 2^1 . 4^2 + \frac{\lfloor 4}{\lfloor 2} 3^2 . 4^1 + \frac{\lfloor 4}{\lfloor 2} 2^2 . 3^1 . 4^1 . + \frac{\lfloor 4}{\lfloor 3} 2^1 . 3^3;$$

that is, $384 + 432 + 576 + 216$; that is, 1608.

Again; find the coefficient of x^3 in the expansion of

$$(1 + 2x + 3x^2 + 4x^3 + \ldots\ldots)^{\frac{1}{2}}.$$

Here $$q + 2r + 3s + \ldots\ldots = 3,$$

$$p + q + r + s + \ldots\ldots = \frac{1}{2}.$$

p	q	r	s
$-\frac{1}{2}$	0	0	1
$-\frac{3}{2}$	1	1	0
$-\frac{5}{2}$	3	0	0

All the solutions are collected in the annexed table, and the required coefficient is

$$\left(\frac{1}{2}\right)4^1 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)2^1 . 3^1 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\underline{|3}}\, 2^3;$$

that is, $$2 - \frac{3}{2} + \frac{1}{2}; \text{ that is, } 1.$$

In this case, since

$$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots = (1 - x)^{-2},$$

the proposed expression is $\{(1 - x)^{-2}\}^{\frac{1}{2}}$, that is, $(1 - x)^{-1}$. And

$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots;$$

thus we see that the coefficient of x^3 ought to be 1; and the student may exercise himself by applying the multinomial theorem to find the coefficients of other powers of x: for example, the coefficient of x^4 will be found to be

$$\frac{5}{2} - 2 - \frac{9}{8} + \frac{9}{4} - \frac{5}{8}, \text{ that is } 1.$$

530. The form of the coefficient in the Multinomial Theorem in the case in which the exponent is a *positive integer* might be obtained in another way. Suppose, for example, that we have to expand $(\alpha + \beta + \gamma)^{10}$. When the multiplication is effected every term in the result is a product formed by taking one letter out of each of the 10 trinomial factors. Thus if we require the term which involves $\alpha^2\beta^3\gamma^5$ we must take α out of any *two* of the 10 trinomial factors, β out of any *three* of the remaining 8 trinomial factors, and γ out of the remaining 5 trinomial factors. The num-

ber of ways in which this can be done is $\dfrac{\lfloor 10}{\lfloor 2\,\lfloor 3\,\lfloor 5}$, by Art. 498: thus the required term is $\dfrac{\lfloor 10}{\lfloor 2\,\lfloor 3\,\lfloor 5}\,\alpha^2\beta^3\gamma^5$.

Hence it follows that if we have to expand $(\alpha+\beta x+\gamma x^2)^{10}$ the term which involves $\alpha^2\beta^3\gamma^5$ is

$$\frac{\lfloor 10}{\lfloor 2\,\lfloor 3\,\lfloor 5}\,\alpha^2(\beta x)^3(\gamma x^2)^5, \text{ that is } \frac{\lfloor 10}{\lfloor 2\,\lfloor 3\,\lfloor 5}\,\alpha^2\beta^3\gamma^5 x^{3+10}.$$

Similarly any other case might be treated. Thus we could give the investigation of the Multinomial Theorem in the following manner:

Begin by establishing in the way just exemplified the form of the coefficient in the case in which the exponent is a *positive* integer. Then suppose we have to find the general term in the expansion of $(a_0+a_1x+a_2x^2+a_3x^3+\ldots)^n$, where n is not a positive integer. Put b for $a_1x+a_2x^2+a_3x^3+\ldots$; then we have to expand $(a_0+b)^n$; the general term of this expansion is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\lfloor \mu}\,a_0^{\,n-\mu}b^{\mu}:$$

and as μ is a *positive integer* the general term in the expansion of $(a_1x+a_2x^2+a_3x^3+\ldots\ldots)^{\mu}$ is

$$\frac{\lfloor \mu}{\lfloor q\,\lfloor r\,\lfloor s\,\lfloor t\ldots\ldots}\,a_1^{\,q}a_2^{\,r}a_3^{\,s}a_4^{\,t}\ldots\ldots x^{q+2r+3s+4t+\ldots}$$

Hence the required general term is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\mu+1)}{\lfloor q\,\lfloor r\,\lfloor s\,\lfloor t\ldots\ldots}\,a_0^{\,n-\mu}a_1^{\,q}a_2^{\,r}a_3^{\,s}a_4^{\,t}\ldots\ldots x^{q+2r+3s+4t+\ldots}$$

EXAMPLES OF THE MULTINOMIAL THEOREM.

Find the coefficients of the specified powers of x in the expansions in the following 24 Examples:

1. x^4 in $(1+x+x^2)^3$.
2. x^5 in $(1-x+x^2)^4$.

3. x^8 in $(1-2x+3x^2-4x^3)^4$.

4. x^{14} in $(1+x+x^2+x^3+x^4+x^5)^3$.

5. x^6 in $(2-3x-4x^2)^5$.

6. x^8 in $(1-x+2x^2)^{12}$.

7. x^4 in $(2-5x-7x^2)^5$.

8. x^8 in $(1-2x^2+4x^4)^{-2}$.

9. x^4 in $(1+x+x^2)^{-5}$.

10. x^5 in $(1+2x-x^2)^{-\frac{1}{2}}$.

11. x^8 in $\left(1-\dfrac{x^2}{2}+\dfrac{x^4}{4}\right)^{-2}$.

12. x^4 in $(1+2x-4x^2-2x^3)^{-\frac{1}{2}}$.

13. x^6 in $(1-2x+x^4)^{\frac{1}{4}}$.

14. x^4 in $(1+x^{\frac{1}{2}}+x^{\frac{3}{2}}+x^{\frac{5}{2}}-x^{\frac{7}{2}})^5$.

15. x^4 in $(1+x+x^2)^n$.

16. x^4 in $(1+3x+5x^2+7x^3+9x^4+\ldots\ldots)^7$.

17. x^m in $(1+x+x^2+\ldots\ldots)^2$.

18. x^8 in $(1+2x+3x^2)^n$.

19. x^4 in $(1+2x+3x^2+4x^3+\ldots\ldots)^{-\frac{1}{2}}$.

20. x^{12} in $(1+a_1x+a_2x^2+a_3x^3)^5$.

21. x^5 in $(a_0+a_1x+a_2x^2)^n$.

22. x^8 in $(1-x^2+x^3-x^5)^4$.

23. x^2 in $(1+ax+bx^2)^{-\frac{1}{2}}$.

24. x^3 in $(1+a_1x+a_2x^2+a_3x^3+\ldots\ldots)^m$.

25. Find the coefficient of abc^3 in $(a+b+c)^5$.

26. Find the coefficient of $a^2b^3c^2$ in $(a-b-c)^7$.

27. Find the coefficient of $a^2b^4c^3$ in $(a+b+c+d)^9$.

28. Find the coefficient of $ab^2c^3d^4$ in $(a-b+c-d)^{10}$.

29. Write down all the terms which involve powers of b and c as high as the third power inclusive in the expansion of $(a+b+c)^n$.

30. Write down all the terms which contain d^{n-3} in the expansion of $(a+b+c+d)^n$.

31. Find the greatest coefficient in the expansion of

$$(a+b+c)^{10}.$$

32. Find the greatest coefficient in the expansion of

$$(a+b+c+d)^{14}.$$

33. Shew that the greatest coefficient in the expansion of $(a_1+a_2+\ldots\ldots+a_m)^n$ is $\dfrac{\lfloor n}{\{\lfloor q\}^m (q+1)^r}$, where q is the quotient, and r the remainder when n is divided by m.

34. Shew that in the expansion of $(a_0+a_1x+a_2x^2+\ldots\ldots)^2$ the coefficient of x^{2p+1} is $2(a_0a_{2p+1}+a_1a_{2p}+a_2a_{2p-1}+\ldots\ldots+a_pa_{p+1})$.

35. Expand $(1-2bx+x^2)^{-\frac{1}{2}}$ as far as x^4.

36. Expand $(a+bx+cx^2)^{-1}$ as far as x^4.

37. Expand $(1-x-x^2-x^3)^n$ as far as x^3.

38. In the expansion of $(1+x+x^2+\ldots\ldots+x^r)^n$, where n is a positive integer, shew that

(1) the coefficients of the terms equidistant from the beginning and the end are equal;

(2) the coefficient of the middle term, or of the two middle terms, according as nr is even or odd, is greater than any other coefficient;

(3) the coefficients continually increase from the first up to the greatest.

39. If $a_0, a_1, a_2, a_3, \ldots\ldots$ be the coefficients in order of the expansion of $(1+x+x^2+\ldots\ldots+x^r)^n$, prove that

(1) $a_0+a_1+a_2+\ldots\ldots+a_{nr}=(r+1)^n$;

(2) $a_1+2a_2+3a_3+\ldots\ldots+nra_{nr}=\frac{1}{2}nr(r+1)^n$.

40. If $a_0, a_1, a_2, a_3, \ldots\ldots$ be the coefficients in order of the expansion of $(1+x+x^2)^n$, prove that

$$a_0^2-a_1^2+a_2^2-a_3^2+\ldots\ldots+(-1)^{n-1}a_{n-1}^2+\tfrac{1}{2}(-1)^n a_n^2=\tfrac{1}{2}a_n.$$

XXXVIII. LOGARITHMS.

531. Suppose $a^x = n$, then x is called the *logarithm of* n *to the base* a; thus the logarithm of a number to a given base is the index of the power to which the base must be raised to be equal to the number.

The logarithm of n to the base a is written $\log_a n$; thus $\log_a n = x$ expresses the same relation as $a^x = n$.

532. For example, $3^4 = 81$; thus 4 is the logarithm of 81 to the base 3.

If we wish to find the logarithms of the numbers 1, 2, 3, ... to a given base 10, for example, we have to solve a series of equations $10^x = 1$, $10^x = 2$, $10^x = 3$, ... We shall see in the next Chapter that this can be done *approximately*, that is, for example, although we cannot find such a value of x as will make $10^x = 2$ *exactly*, yet we can find such a value of x as will make 10^x differ from 2 by as small a quantity as we please.

We shall now prove some of the properties of logarithms.

533. *The logarithm of* 1 *is* 0 *whatever the base may be.*

For $a^x = 1$ when $x = 0$.

534. *The logarithm of the base itself is unity.*

For $a^x = a$ when $x = 1$.

535. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

For let $$x = \log_a m, \quad y = \log_a n;$$

therefore $$m = a^x, \quad n = a^y;$$

therefore $$mn = a^x a^y = a^{x+y};$$

therefore $$\log_a mn = x + y = \log_a m + \log_a n.$$

536. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For let $$x = \log_a m, \quad y = \log_a n;$$

therefore $$m = a^x, \quad n = a^y;$$

therefore $$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y};$$

therefore $$\log_a \frac{m}{n} = x - y = \log_a m - \log_a n.$$

537. *The logarithm of any power, integral or fractional, of a number is equal to the product of the logarithm of the number and the index of the power.*

For let $m = a^x$; therefore $m^r = (a^x)^r = a^{xr}$,

therefore $$\log_a (m^r) = xr = r \log_a m.$$

538. *To find the relation between the logarithms of the same number to different bases.*

Let $$x = \log_a m, \quad y = \log_b m;$$

therefore $$m = a^x \text{ and } = b^y;$$

therefore $$a^x = b^y;$$

therefore $$a^{\frac{x}{y}} = b, \text{ and } b^{\frac{y}{x}} = a;$$

therefore $$\frac{x}{y} = \log_a b, \text{ and } \frac{y}{x} = \log_b a.$$

Hence $$y = x \log_b a, \text{ and } = \frac{x}{\log_a b}.$$

Hence the logarithm of a number to the base b may be found by multiplying the logarithm of the number to the base a by

$$\log_b a, \text{ or by } \frac{1}{\log_a b}.$$

We may notice that $\log_b a \times \log_a b = 1$.

539. In practical calculations the only base that is used is 10; logarithms to the base 10 are called *common* logarithms. We will point out in the next two Articles some peculiarities which constitute the advantage of the base 10. We shall require the following definition: the integral part of any logarithm is called the *characteristic*, and the decimal part the *mantissa*.

540. *In the common system of logarithms, if the logarithm of any number be known we can immediately determine the logarithm of the product or quotient of that number by any power of* 10.

For $\log_{10}(N \times 10^n) = \log_{10} N + \log_{10} 10^n = \log_{10} N + n,$

$$\log_{10} \frac{N}{10^n} = \log_{10} N - \log_{10} 10^n = \log_{10} N - n.$$

That is, if we know the logarithm of any number we can determine the logarithm of any number which has the same figures, but differs merely by the position of the decimal point.

541. *In the common system of logarithms the characteristic of the logarithm of any number can be determined by inspection.*

For suppose the number to be greater than unity and to lie between 10^n and 10^{n+1}; then its logarithm must be greater than n and less than $n+1$: hence the characteristic of the logarithm is n.

Next suppose the number to be less than unity, and to lie between $\frac{1}{10^n}$ and $\frac{1}{10^{n+1}}$, that is, between 10^{-n} and $10^{-(n+1)}$; then its logarithm will be some negative quantity between $-n$ and $-(n+1)$: hence if we agree that the *mantissa shall always be positive,* the characteristic will be $-(n+1)$.

Further information on the practical use of logarithms will be found in works on Trigonometry and in the introductions to Tables of Logarithms.

EXAMPLES OF LOGARITHMS.

1. Find the logarithm of 144 to the base $2\sqrt{3}$.

2. Find the characteristic of the logarithm of 7 to the base 2.

3. Find the characteristic of $\log_3 5$.

4. Find $\log_5 3125$.

5. Give the characteristic of $\log_{10} 1230$, and of $\log_{10} \cdot 0123$.

6. Given $\log 2 = \cdot 301030$ and $\log 3 = \cdot 477121$, find the logarithms of $\cdot 05$ and of $5 \cdot 4$.

7. Given log 2 and log 3 (see Example 6), find the logarithm of $\cdot 006$.

8. Given log 2 and log 3, find the logarithms of 36, 27, and 16.

9. Given $\log 648 = 2 \cdot 81157501$, $\log 864 = 2 \cdot 93651374$, find log 3 and log 5.

10. Given log 2, find $\log \sqrt{(1 \cdot 25)}$.

11. Given log 2, find $\log \cdot 0025$.

12. Given log 2, find $\log \sqrt[3]{(\cdot 0125)}$.

13. Given log 2 and log 3, find log 1080 and $\log (\cdot 0045)^{\frac{1}{5}}$.

14. Given $\log_{10} 2 = \cdot 301030$ and $\log_{10} 7 = \cdot 845098$, find the logarithm of $\left(\frac{4}{343}\right)^{\frac{1}{2}}$ to the base 1000.

15. Find the number of digits in 2^{64}, having given log 2.

16. Given log 2, and $\log 5 \cdot 743491 = \cdot 7591760$, find the fifth root of $\cdot 0625$.

17. If P be the number of the integers whose logarithms have the characteristic p, and Q the number of the integers the logarithms of whose reciprocals have the characteristic $-q$, shew that $\log P - \log Q = p - q + 1$.

18. If $y = 10^{\frac{1}{1-\log x}}$ and $z = 10^{\frac{1}{1-\log y}}$, prove that $x = 10^{\frac{1}{1-\log z}}$.

19. If a, b, c be in G.P., then $\log_a n$, $\log_b n$, $\log_c n$ are in H.P.

20. If the number of persons born in any year be $\frac{1}{45}$th of the whole population at the commencement of the year, and the number of those who die $\frac{1}{60}$th of it, find in how many years the population will be doubled; having given

$$\log 2 = \cdot 301030, \quad \log 180 = 2 \cdot 255272, \quad \log 181 = 2 \cdot 257679.$$

XXXIX. EXPONENTIAL AND LOGARITHMIC SERIES.

542. *To expand* a^x *in a series of ascending powers of* x; *that is, to expand a number in a series of ascending powers of its logarithm to a given base.*

$a^x = \{1 + (a-1)\}^x$; and expanding by the Binomial Theorem we have

$$\{1 + (a-1)\}^x = 1 + x(a-1) + \frac{x(x-1)}{1.2}(a-1)^2$$

$$+ \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3 + \frac{x(x-1)(x-2)(x-3)}{1.2.3.4}(a-1)^4 + \dots$$

$$= 1 + x\{a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \dots\dots\}$$

$$+ \text{ terms involving } x^2,\ x^3, \text{ \&c.}$$

This shews that a^x *can* be expanded in a series beginning with 1 and proceeding in ascending powers of x; we may therefore suppose that

$$a^x = 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots\dots$$

where c_1, c_2, c_3, ... are quantities which do not depend on x, and which therefore remain unchanged however x may be changed; also

$$c_1 = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \dots\dots$$

while c_2, c_3, ... are at present unknown; we proceed to find their values. Changing x into $x+y$ we have

$$a^{x+y} = 1 + c_1(x+y) + c_2(x+y)^2 + c_3(x+y)^3 + \dots\dots;$$

but $\quad a^{x+y} = a^xa^y = a^y\{1 + c_1x + c_2x^2 + c_3x^3 + \dots\dots\}.$

Since the two expressions for a^{x+y} are identically equal, we may *assume* that the coefficients of x in the two expressions are equal, thus

$$c_1 + 2c_2y + 3c_3y^2 + 4c_4y^3 + \dots\dots = c_1a^y$$

$$= c_1\{1 + c_1y + c_2y^2 + c_3y^3 + \dots\dots\}.$$

In this identity we may *assume* that the coefficients of the corresponding powers of y are equal; thus

$$2c_2 = c_1^2; \qquad \text{therefore } c_2 = \frac{c_1^2}{2};$$

$$3c_3 = c_1c_2; \qquad \text{therefore } c_3 = \frac{c_1c_2}{3} = \frac{c_1^3}{1.2.3};$$

$$4c_4 = c_1c_3; \qquad \text{therefore } c_4 = \frac{c_1c_3}{4} = \frac{c_1^4}{1.2.3.4};$$

....

Thus $$a^x = 1 + c_1x + \frac{c_1^2x^2}{\lfloor 2} + \frac{c_1^3x^3}{\lfloor 3} + \frac{c_1^4x^4}{\lfloor 4} + \ldots\ldots$$

Since this result is true for all values of x, take x such that $c_1x = 1$, then $x = \frac{1}{c_1}$, and

$$a^{\frac{1}{c_1}} = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots\ldots;$$

this series is usually denoted by e; thus $a^{\frac{1}{c_1}} = e$, therefore $a = e^{c_1}$ and $c_1 = \log_e a$; hence

$$a^x = 1 + (\log_e a)\, x + \frac{(\log_e a)^2 x^2}{\lfloor 2} + \frac{(\log_e a)^3 x^3}{\lfloor 3} + \ldots\ldots$$

This result is called the *Exponential Theorem.*

Put e for a, then $\log_e a$ becomes $\log_e e$, that is, unity (Art. 534);

thus $$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots\ldots$$

This very important result is true for *all* values of x; and the student should render himself so familiar with it as to be able to apply it to special cases. For example, suppose $x = -1$; thus

$$e^{-1} \text{ or } \frac{1}{e} = \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \frac{1}{\lfloor 5} + \ldots\ldots$$

Or we may put any other symbol for x; thus putting nz for x we have

$$e^{nz} = 1 + nz + \frac{n^2z^2}{\underline{|2}} + \frac{n^3z^3}{\underline{|3}} + \frac{n^4z^4}{\underline{|4}} + \ldots\ldots$$

We shall in Art. 551 make a remark on one part of the preceding investigation, and we shall recur hereafter to the *assumption* which has been made twice in the course of the present Article.

543. By actual calculation we may find approximately the numerical value of the series which we have denoted by e; it is 2·718281828......

544. *To expand* $\log_e(1 + x)$ *in a series of ascending powers of* x.

We have seen in Art. 542, that $c_1 = \log_e a$; that is, by the same Article, $\log_e a = a - 1 - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \frac{1}{4}(a-1)^4 + \ldots\ldots$

For a put $1 + x$; hence $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$

This series may be applied to calculate $\log_e(1+x)$ if x is a proper fraction; but unless x be very small, the terms diminish so slowly that we shall have to retain a large number of them; if x be greater than unity, the series is altogether unsuitable. We shall therefore deduce some more convenient formulæ.

545. We have $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots;$

therefore $$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots,$$

by subtraction we obtain the value of $\log_e(1+x) - \log_e(1-x)$, that is, of $\log_e \frac{1+x}{1-x}$;

therefore $$\log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right\}.$$

In this series write $\frac{m-n}{m+n}$ for x, and therefore $\frac{m}{n}$ for $\frac{1+x}{1-x}$;

thus $$\log \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\ldots\right\} \ldots\ldots(1).$$

Put $n=1$, then

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \ldots\ldots\right\} \ldots\ldots (2).$$

Again, in (1) put $m=n+1$, thus we obtain the value of $\log_e \frac{n+1}{n}$; therefore $\log_e (n+1) - \log_e n$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots\ldots\right\} \ldots\ldots (3).$$

546. The series (2) of the preceding Article will enable us to find $\log_e 2$; put $m=2$, then by calculation we shall find

$$\log_e 2 = \cdot 69314718 \ldots\ldots$$

From the series (3) we can calculate the logarithm of either of two consecutive numbers when we know that of the other. Put $n=2$, and by making use of the known value of $\log_e 2$, we shall obtain

$$\log_e 3 = 1\cdot 09861229 \ldots\ldots$$

Put $n=9$ in (3); then $\log_e n = \log_e 9 = \log_e 3^2 = 2\log_e 3$ and is therefore known; hence we shall find

$$\log_e 10 = 2\cdot 30258509 \ldots\ldots$$

547. Logarithms to the base e are called *Napierian* logarithms, from Napier the inventor of logarithms; they are also called *natural* logarithms, being those which occur first in our investigation of a method of calculating logarithms. We have said that the base 10 is the only base used in the practical application of logarithms, but logarithms to the Napierian base occur frequently in theoretical investigations.

548. From Art. 538 we see that the logarithm of a number to the base 10 can be found by multiplying the Napierian logarithm by $\frac{1}{\log_e 10}$, that is, by $\frac{1}{2\cdot 30258509}$, or by $\cdot 43429448$; this multiplier is called the *modulus* of the common system.

The Napierian base, the modulus of the common system, and the Napierian logarithms of 2, 3, and 5 have all been calculated to upwards of 200 places of decimals. See the *Abstracts of the Papers communicated to the Royal Society*, Vol. VI. p. 397.

The series in Art. 545 may be so adjusted as to give common logarithms; for example, take the series (3), multiply throughout by the modulus which we shall denote by μ; thus

$$\mu \log_e (n+1) - \mu \log_e n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots \right\};$$

that is,

$$\log_{10} (n+1) - \log_{10} n = 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \ldots \right\}.$$

549. By Art. 542 we have

$$(e^x - 1)^n = \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots \right)^n$$

$$= x^n + \text{terms containing higher powers of } x \ldots\ldots\ldots\ldots(1).$$

Again, by the Binomial Theorem,

$$(e^x - 1)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{\underline{|2}} e^{(n-2)x} - \ldots\ldots\ldots\ldots\ldots(2).$$

Expand each of the terms e^{nx}, $e^{(n-1)x}$, ...; thus the coefficient of x^r in (2) will be

$$\frac{n^r}{\underline{|r}} - n\frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}} \frac{(n-2)^r}{\underline{|r}} - \frac{n(n-1)(n-2)}{\underline{|3}} \frac{(n-3)^r}{\underline{|r}} + \ldots\ldots$$

Hence from (1), by the same principle as in Art. 542, we see that

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^r + \ldots\ldots$$

is $= \underline{|n}$ if $r = n$, and is $= 0$ if r be less than n.

It is easy to see that the term on the right-hand side of (1) which involves x^{n+1} is $\frac{n}{2} x^{n+1}$. Thus we get, by the same principle as before,

$$n^{n+1} - n(n-1)^{n+1} + \frac{n(n-1)}{\underline{|2}}(n-2)^{n+1} - \ldots\ldots = \tfrac{1}{2} n \, \underline{|n+1}.$$

550. We will give another method of arriving at the exponential theorem. By the Binomial Theorem

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\frac{1}{n^3}$$
$$+\frac{nx(nx-1)(nx-2)(nx-3)}{\lfloor 4}\frac{1}{n^4}+\ldots\ldots$$

that is,

$$\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}$$
$$+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\left(x-\frac{3}{n}\right)}{\lfloor 4}+\ldots\ldots$$

Put $x=1$, then $\left(1+\frac{1}{n}\right)^n$

$$=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{\lfloor 4}+\ldots\ldots$$

But $$\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x;$$

hence $$1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots$$
$$=\left\{1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\ldots\ldots\right\}^x.$$

Now this being true however large n may be, will be true when n is made infinite; then $\frac{1}{n}$ vanishes and we obtain

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ldots\ldots=\left\{1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\ldots\right\}^x,$$

that is, $=e^x$.

We have thus obtained the expansion of e^x in powers of x; to find the expansion of a^x suppose $a=e^c$ so that $c=\log_e a$, thus

$$a^x=e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\frac{c^4x^4}{\lfloor 4}+\ldots\ldots$$

551. The student will notice that in the preceding Article we have used the Binomial Theorem to expand a power of $1 + \frac{1}{n}$, and if $\frac{1}{n}$ is *less* than unity, we are certain that the expansion gives an *arithmetically* true result (Art. 519). In the proof given of the exponential theorem in the first Article of this Chapter, if $a - 1$ is greater than unity, the expansion by the Binomial Theorem with which the proof commences will not be arithmetically intelligible; and consequently the proof can only be considered sound provided a is less than 2. With this restriction the proof is sound, and x may have any value. In order to complete that proof we have to shew that the theorem is true for any value of a; and as e is greater than 2 we ought not to change a into e until we have removed this restriction as to the value of a. This restriction can be easily removed; for in the theorem

$$a^x = 1 + (\log_e a)\, x + \frac{(\log_e a)^2 x^2}{\underline{|2}} + \frac{(\log_e a)^3 x^3}{\underline{|3}} + \dots\dots$$

put $a = A^y$, and by taking y small enough A may be made as great as we please, while a is less than 2. Then $\log_e a = y \log_e A$;

thus $$A^{yx} = 1 + (\log_e A)\, yx + \frac{(\log_e A)^2 y^2 x^2}{\underline{|2}} + \frac{(\log_e A)^3 y^3 x^3}{\underline{|3}} + \dots\dots;$$

therefore, putting z for yx,

$$A^z = 1 + (\log_e A)\, z + \frac{(\log_e A)^2 z^2}{\underline{|2}} + \frac{(\log_e A)^3 z^3}{\underline{|3}} + \dots\dots;$$

thus the exponential theorem is proved universally.

552. We have found in Art. 550, that when n increases without limit $\left(1 + \frac{1}{n}\right)^{nx}$ ultimately becomes e^x; in the same way we may shew that when n increases without limit $\left(1 + \frac{r}{n}\right)^{nx}$ ultimately becomes e^{rx}.

EXAMPLES OF LOGARITHMIC SERIES.

1. Prove that $\log_e(x+1) = 2\log_e x - \log_e(x-1)$

$$-2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\left(\frac{1}{2x^2-1}\right)^3 + \ldots\ldots\right\}.$$

Given $\log_{10} 3 = \cdot 47712$ and $\dfrac{1}{\log_e 10} = \cdot 43429$, apply the above series to calculate $\log_{10} 11$.

2. Shew that $\log_e(x+2h) = 2\log_e(x+h) - \log_e x$

$$-\left\{\frac{h^2}{(x+h)^2} + \frac{1}{2}\frac{h^4}{(x+h)^4} + \frac{1}{3}\frac{h^6}{(x+h)^6} + \ldots\ldots\right\}.$$

3. If a, b, c be three consecutive numbers,

$\log_e c = 2\log_e b - \log_e a$

$$-2\left\{\frac{1}{2ac+1} + \frac{1}{3(2ac+1)^3} + \frac{1}{5(2ac+1)^5} + \ldots\ldots\right\}.$$

4. If λ and μ be the roots of $ax^2 + bx + c = 0$, shew that

$$\log_e(a - bx + cx^2) = \log_e a + (\lambda + \mu)x - \frac{\lambda^2 + \mu^2}{2}x^2 + \ldots\ldots$$

5. $\text{Log}_e\{1 + 1 + x + (1+x)^2\} = 3\log_e(1+x) - \log_e x$

$$-\left\{\frac{1}{(1+x)^3} + \frac{1}{2}\frac{1}{(1+x)^6} + \frac{1}{3(1+x)^9} + \ldots\ldots\right\}.$$

6. $\text{Log}_e(x+1) = \dfrac{4x}{2x+1}\log_e x - \dfrac{2x-1}{2x+1}\log_e(x-1)$

$$-\frac{2}{2x+1}\left\{\frac{1}{2.3.x^3} + \frac{2}{3.5.x^5} + \frac{3}{4.7.x^7} + \ldots\ldots\right\}.$$

7. $\text{Log}_e\left\{\left(1+x\right)^{\frac{1+x}{x}}\left(1-x\right)^{\frac{1-x}{x}}\right\} = \dfrac{x^2}{1.2} + \dfrac{x^4}{3.4} + \dfrac{x^6}{5.6} + \ldots\ldots$

8. Find the Napierian logarithm of $\frac{501}{499}$. To how many decimal places is your result correct?

9. Assuming the series for $\log_e(1+x)$ and e^x, shew that

$$\left(1+\frac{x}{n}\right)^n = e^x\left(1-\frac{x^2}{2n}\right)$$

nearly when n is large; and find the next term of the series of which the expression on the second side is the commencement.

10. Find the coefficient of x^n in the development of

$$\frac{a+bx+cx^2}{e^x}.$$

11. Shew that $\log_e 4 = 1 + \frac{2}{1.2.3} + \frac{2}{3.4.5} + \frac{2}{5.6.7} + \ldots\ldots$

12. Shew that $n^{n+2} - n(n-1)^{n+2} + \frac{n(n-1)}{\underline{|2}}(n-2)^{n+2} - \ldots\ldots$

$$= \left(\frac{n}{6} + \frac{n(n-1)}{8}\right)\underline{|n+2}.$$

XL. CONVERGENCE AND DIVERGENCE OF SERIES.

553. The expression $u_1 + u_2 + u_3 + u_4 + \ldots\ldots$ in which the successive terms are formed by some regular law, and the number of the terms is unlimited, is called an *infinite series*.

554. An infinite series is said to be *convergent* when the sum of the first n terms cannot numerically exceed some finite quantity however great n may be.

555. An infinite series is said to be *divergent* when the sum of the first n terms can be made numerically greater than any finite quantity, by taking n large enough.

556. Suppose that by adding more and more terms of an infinite series we continually approximate to a certain result, so that the sum of a sufficiently large number of terms will differ from that result by less than any assigned quantity, then that result is called the *sum of the infinite series.*

For example, consider the infinite series

$$1 + x + x^2 + \ldots\ldots,$$

and suppose x a positive quantity.

We know that

$$1 + x + x^2 + \ldots\ldots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

Hence if x be less than 1, however great n may be, the sum of the first n terms of the series is less than $\frac{1}{1-x}$; the series is therefore *convergent.* And, as by taking n large enough the sum of the first n terms can be made to differ from $\frac{1}{1-x}$ by less than any assigned quantity, $\frac{1}{1-x}$ is the *sum* of the infinite series.

If $x = 1$, the series is *divergent;* for the sum of the first n terms is n, and by taking sufficient terms this may be made greater than any finite quantity.

If x is greater than 1, the series is divergent; for the sum of the first n terms is $\frac{x^n - 1}{x - 1}$, which may be made greater than any finite quantity by taking n large enough.

557. *An infinite series in which all the terms are of the same sign is divergent if each term is greater than some assigned finite quantity, however small.*

For if each term is greater than the quantity c, the sum of the first n terms is greater than nc, and this can be made greater than any finite quantity by taking n large enough.

558. *An infinite series of terms, the signs of which are alternately positive and negative, is convergent if each term is numerically less than the preceding term.*

Let the series be $u_1 - u_2 + u_3 - u_4 + \ldots$; this may be written

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots\ldots,$$

and also thus,

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \ldots\ldots$$

From the first mode of writing the series we see that the sum of any number of terms is a positive quantity, and from the second mode of writing the series we see that the sum of any number of terms is less than u_1; hence the series is *convergent.*

It is necessary to shew in this case that the sum of any number of terms is *positive;* because if we only know that the sum is less than u_1, we are not certain that it is not a negative quantity of unlimited magnitude.

An important distinction should be noticed with respect to the series here considered. If the terms $u_1, u_2, u_3, \ldots$ *diminish without limit* the sum of n terms and the sum of $n+1$ terms will differ by an indefinitely small quantity when n is taken large enough. But if the terms $u_1, u_2, u_3, \ldots$ *do not diminish without limit* the sum of n terms and the sum of $n+1$ terms will always differ by a finite quantity. The series continued to an infinite number of terms will have a sum, according to the definition of Art. 556, in the former case, but not in the latter case. In both cases the series is convergent according to our definition. But some writers prefer another definition of convergence; namely, they consider a series convergent only when the sum of an indefinitely large number of terms can be made to differ from one fixed value by less than any assigned quantity: and according to this definition the series is convergent in the first case, but not in the second.

559. *An infinite series is convergent if from and after any fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity.*

Let the series beginning at the fixed term be

$$u_1 + u_2 + u_3 + \ldots\ldots$$

and let S denote the sum of the first n of these terms. Then

$$S = u_1 + u_2 + u_3 + \ldots\ldots + u_n$$
$$= u_1 \left\{1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + \frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1} + \ldots\ldots\right\}.$$

Now first let all the terms be positive, and suppose

$$\frac{u_2}{u_1} \text{ less than } k, \quad \frac{u_3}{u_2} \text{ less than } k, \quad \frac{u_4}{u_3} \text{ less than } k, \ldots\ldots$$

Then S is less than $u_1\{1 + k + k^2 + \ldots\ldots + k^{n-1}\}$; that is, less than $u_1 \dfrac{1-k^n}{1-k}$. Hence if k be less than unity, S is less than $\dfrac{u_1}{1-k}$; thus the sum of as many terms as we please beginning with u_1 is less than a certain finite quantity, and therefore the series beginning with u_1 is convergent.

Secondly, suppose the terms not all positive; then if they are all negative, the numerical value of the sum of any number of them is the same as if they were all positive; if some terms are positive and some negative, the sum is numerically less than if the terms were all positive. Hence the infinite series is still convergent.

Since the infinite series beginning with u_1 is convergent, the infinite series which begins with any fixed term before u_1 will be also convergent; for we shall thus only have to add a *finite* number of *finite* terms to the series beginning with u_1.

560. *An infinite series is divergent if from and after any fixed term the ratio of each term to the preceding term is greater than unity, or equal to unity, and the terms are all of the same sign.*

Let the series beginning at the fixed term be

$$u_1 + u_2 + u_3 + \ldots\ldots,$$

and let S denote the sum of the first n of these terms. Then

$$S = u_1 + u_2 + u_3 + \ldots\ldots + u_n$$

$$= u_1 \left\{1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + \frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1} + \ldots\ldots\right\}.$$

Now, first suppose

$\frac{u_2}{u_1}$ greater than 1, $\frac{u_3}{u_2}$ greater than 1, $\frac{u_4}{u_3}$ greater than 1, ...

Then S is numerically greater than $u_1 \{1 + 1 + \ldots\ldots\ldots\ldots + 1\}$, that is, numerically greater than nu_1. Hence S may be made numerically greater than any finite quantity by taking n large enough, and therefore the series beginning with u_1 is divergent.

Next, suppose the ratio of each term to the preceding to be unity; then $S = nu_1$, and this may be made greater than any finite quantity by taking n large enough.

And if we begin with any fixed term before u_1 the series will obviously still be divergent.

561. The rules in the preceding Articles will determine in many cases whether an infinite series is convergent or divergent. There is one case in which they *do not apply* which it is desirable to notice, namely, when the ratio of each term to the preceding is less than unity, but continually approaching unity, so that we cannot name any finite quantity k which is less than unity, and yet always greater than this ratio. In such a case, as will appear from the example in the following Article, the series *may* be convergent or divergent.

562. Consider the infinite series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots$$

Here the ratio of the n^{th} term to the $(n-1)^{\text{th}}$ term is $\left(\frac{n-1}{n}\right)^p$; if p be positive, this is less than unity, but continually approaches to unity as n increases. This case then cannot be tested by any of the rules already given; we shall however prove that the series is convergent if p be greater than unity, and divergent if p be unity, or less than unity.

I. Suppose p greater than unity.

The first term of the series is 1, the next two terms are together less than $\frac{2}{2^p}$, the following four terms are together less than $\frac{4}{4^p}$, the following eight terms are together less than $\frac{8}{8^p}$, and so on. Hence the whole series is less than

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots\ldots$$

that is, less than

$$1 + x + x^2 + x^3 + \ldots\ldots$$

where $x = \frac{2}{2^p}$. Since p is greater than unity, x is less than unity; hence the series is convergent.

II. Suppose p equal to unity.

The series is now $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots$

The first term is 1, the second term is $\frac{1}{2}$, the next two terms are together greater than $\frac{2}{4}$ or $\frac{1}{2}$, the following four terms are together greater than $\frac{4}{8}$ or $\frac{1}{2}$, and so on. Hence by taking a sufficient number of terms we can obtain a sum greater than any finite multiple of $\frac{1}{2}$; the series is therefore divergent.

III. Suppose p less than unity or negative.

Each term is now greater than the corresponding term in II.; the series is therefore *a fortiori* divergent.

563. We will now give a general theorem which can be proved in the manner exemplified in the preceding Article. If $\phi(x)$ be positive for all positive integral values of x, and continually diminish as x increases, and m be any positive integer, then the two infinite series

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(5) + \ldots\ldots$$

and $$\phi(1) + m\phi(m) + m^2\phi(m^2) + m^3\phi(m^3) + \ldots\ldots$$

are both convergent or both divergent.

Consider all the terms of the first series comprised between $\phi(m^k)$ and $\phi(m^{k+1})$, including the last and excluding the first, k being any positive integer; the number of these terms is $m^{k+1} - m^k$, and their sum is therefore greater than $m^k(m-1)\phi(m^{k+1})$. Thus all the first series beginning with the term $\phi(m^k+1)$ will be greater than $\frac{m-1}{m}$ times the second series beginning with the term $m^{k+1}\phi(m^{k+1})$. Thus if the second series be divergent, so also is the first.

Again, the terms selected from the first series are less than $m^k(m-1)\phi(m^k)$. Thus all the first series beginning with the term $\phi(m^k+1)$ will be less than $m-1$ times the second series beginning with $m^k\phi(m^k)$. Thus if the second series be convergent, so also is the first.

As an example of the use of this theorem we may take the following: *the series of which the general term is* $\frac{1}{n(\log n)^p}$ *is convergent if* p *be greater than unity, and divergent if* p *be equal to unity or less than unity.* By the theorem the proposed series is convergent or divergent according as the series of which the general term is $\frac{m^n}{m^n(\log m^n)^p}$ is convergent or divergent; the latter general term is $\frac{1}{(\log m)^p n^p}$, so that it bears a *constant ratio* to the general term $\frac{1}{n^p}$ for all values of n. Hence the required result follows by Art. 562.

564. *The series obtained by expanding* $(1+x)^n$ *by the Binomial Theorem is convergent if* x *is numerically less than unity.*

For the ratio of the $(r+1)^{\text{th}}$ term to the r^{th} is $\frac{n-r+1}{r}x$. If n is negative and numerically greater than unity the factor $\frac{n-r+1}{r}$ is numerically greater than unity; but it continually approaches unity, and can be made to differ from unity by less than any assigned quantity by taking r large enough. Hence if x is numerically less than unity the product $\frac{n-r+1}{r}x$, when r is large enough, will be numerically less than a quantity which is itself numerically less than unity. Hence the series is convergent. (Art. 559.)

If n is positive the factor $\frac{n-r+1}{r}$ is numerically less than unity when r is greater than n; if n is negative and numerically less than unity this factor is always numerically less than unity; if $n=-1$ this factor is numerically equal to unity: thus in the first case when r is greater than n, and in the other two cases always, if x is numerically less than unity the product $\frac{n-r+1}{r}x$ is numerically less than a quantity which is itself numerically less than unity. Hence the series is convergent. (Art. 559.)

565. *The series obtained by expanding* $\log(1+x)$ *in powers of* x *is convergent if* x *is numerically less than unity.*

For the ratio of the $(r+1)^{\text{th}}$ term to the r^{th} is $-\frac{rx}{r+1}$. If then x be less than unity, this ratio is always numerically less than a quantity which is itself numerically less than unity. Hence the series is convergent. (Art. 559.)

566. *The series obtained by expanding* a^x *in powers of* x *is always convergent.*

For the ratio of the $(r+1)^{\text{th}}$ term to the r^{th} is $\dfrac{x \log_e a}{r}$. Whatever be the value of x, we can take r so large that this ratio shall be less than unity, and the ratio will diminish as r increases. Hence the series is always convergent. (Art. 559.)

EXAMPLES OF CONVERGENCE AND DIVERGENCE OF SERIES.

Examine whether the following ten series are convergent or divergent:

1. $\dfrac{1}{x(x+a)} + \dfrac{1}{(x+2a)(x+3a)} + \dfrac{1}{(x+4a)(x+5a)} + \ldots\ldots$

2. $\dfrac{3}{2}x + \dfrac{5x^2}{5} + \dfrac{7x^3}{10} + \dfrac{9x^4}{17} + \ldots\ldots + \dfrac{2n+1}{n^2+1}x^n + \ldots\ldots$

3. $\dfrac{m+p}{a} + \dfrac{m+2p}{a^2} + \dfrac{m+3p}{a^3} + \ldots\ldots$

4. $(a+1)^2 + (a+2)^2 x + (a+3)^2 x^2 + \ldots\ldots$

5. $1^2 + 2^2 x + 3^2 x^2 + \ldots\ldots$

6. $\dfrac{1}{2} + \dfrac{1}{1+\sqrt{2}} + \dfrac{1}{1+\sqrt{3}} + \dfrac{1}{1+\sqrt{4}} + \ldots\ldots$

7. $\dfrac{x}{1+x^3} + \dfrac{x^2}{1+x^4} + \dfrac{x^3}{1+x^5} + \ldots\ldots$

8. $\dfrac{1}{1^p} + \dfrac{1}{3^p} + \dfrac{1}{5^p} + \dfrac{1}{7^p} + \ldots\ldots$

9. $1^n + 2^n x + 3^n x^2 + \ldots\ldots$

10. $\dfrac{x}{(a+b)^p} + \dfrac{x^2}{(a+2b)^p} + \dfrac{x^3}{(a+3b)^p} + \ldots\ldots$

11. Suppose that in the series $u_0+u_1+u_2+u_3+\ldots\ldots$ each term is less than the preceding; then shew that this series and the series $u_0 + 2u_1 + 2^2u_3 + 2^3u_7 + 2^4u_{15} + \ldots\ldots$ are both convergent or both divergent.

12. Shew that the series $1 + \dfrac{1}{2^n} + \dfrac{2}{3^n} + \dfrac{3}{4^n} + \ldots\ldots$ is convergent if n be greater than 2, and divergent if n be less than 2 or equal to 2.

XLI. INTEREST.

567. *Interest* is money paid for the use of money. The sum lent is called the *Principal.* The *Amount* is the sum of the *Principal* and *Interest* at the end of any time.

568. Interest is of two kinds, *simple* and *compound.* When interest of the Principal alone is taken it is called *simple* interest; but if the interest as soon as it becomes due is added to the principal and interest charged upon the whole, it is called *compound* interest.

569. The *rate* of interest is the money paid for the use of a certain sum for a certain time. In *practice* the sum is usually £100 and the time one year; and when we say that the rate of interest is £4. 6*s.* 8*d.* per cent., we mean that £4. 6*s.* 8*d.*, that is, £4⅓, is due for the use of £100 for one year. In *theory* it is convenient, as we shall see, to use a symbol to denote the interest of *one* pound for one year.

570. *To find the amount of a given sum in any time at simple interest.*

Let P be the principal in pounds, n the number of years for which interest is taken, r the interest of one pound for one year, M the amount.

Since r is the interest of one pound for one year, Pr is the interest of P pounds for one year, and therefore nPr the interest of P pounds for n years;

therefore $$M = P + Pnr.$$

From this equation if any three of the four quantities M, P, n, r are given, the fourth can be found; thus

$$P = \frac{M}{1 + nr}, \qquad n = \frac{M - P}{Pr}, \qquad r = \frac{M - P}{Pn}.$$

571. *To find the amount of a given sum in any time at compound interest.*

Let R denote the amount of one pound in one year, so that $R = 1 + r$, then PR is the amount of P in one year; the amount of PR in one year is PRR or PR^2, which is therefore the amount of P in *two* years at compound interest. Similarly the amount of PR^2 in one year is PR^3, which is therefore the amount of P in *three* years. Proceeding thus we find that the amount of P in n years is PR^n; therefore denoting this amount by M,

$$M = PR^n.$$

Hence $$P = \frac{M}{R^n}, \quad n = \frac{\log M - \log P}{\log R}, \quad R = \left(\frac{M}{P}\right)^{\frac{1}{n}}.$$

The interest gained in n years is $M - P$ or $P(R^n - 1)$.

572. Next suppose interest is due more frequently than *once* a year; for example, suppose interest to be due every quarter, and let $\frac{r}{4}$ be the interest of one pound for one quarter. Then, at compound interest, the amount of P in n years is $P\left(1 + \frac{r}{4}\right)^{4n}$; for the amount is obviously the same as if the number of years were $4n$, and $\frac{r}{4}$ the interest of one pound for one year. Similarly, at compound interest, if interest be due q times a year, and the interest of one pound be $\frac{r}{q}$ for each interval, the amount of P in n years is $P\left(1 + \frac{r}{q}\right)^{qn}$.

At simple interest the amount will be the same in the cases supposed as if the interest were payable yearly, r being the interest of one pound for one year.

573. The formulæ of the preceding Articles have been obtained on the supposition that n is an integer; we may therefore ask whether they are true when n is not an integer. Suppose

$n = m + \frac{1}{\mu}$, where m is an integer and $\frac{1}{\mu}$ a proper fraction. At *simple* interest the interest of P for m years is Pmr; and if the borrower has agreed to pay for *any fraction* of a year the *same fraction* of the annual interest, then $\frac{Pr}{\mu}$ is the interest of P for $\left(\frac{1}{\mu}\right)^{\text{th}}$ of a year; hence the whole interest is $Pmr + \frac{Pr}{\mu}$, that is, Pnr, and the formula for the amount holds when n is not an integer. Next consider the case of *compound* interest; the amount of P in m years will be PR^m; if for the fraction of a year interest is due in the same way as before, the interest of PR^m for $\left(\frac{1}{\mu}\right)^{\text{th}}$ of a year is $\frac{PR^m r}{\mu}$, and the whole amount is $PR^m\left(1 + \frac{r}{\mu}\right)$. On this supposition then the formula is not true when n is not an integer. To make the formula true the agreement must be that the amount of one pound at the end of $\left(\frac{1}{\mu}\right)^{\text{th}}$ of a year shall be $(1 + r)^{\frac{1}{\mu}}$, and therefore the interest for $\left(\frac{1}{\mu}\right)^{\text{th}}$ of a year $(1 + r)^{\frac{1}{\mu}} - 1$. This supposition though not made in practice is often made in theory, in order that the formulæ may hold universally.

Similarly if interest is payable q times a year the amount of P in n years is $P\left(1 + \frac{r}{q}\right)^{qn}$, by Art. 572, if n be an integer; and it is assumed in theory that this result holds if n be not an integer.

574. The amount of P in n years when the interest is paid q times a year is $P\left(1 + \frac{r}{q}\right)^{nq}$, by Art. 572; if we suppose q to increase without limit, this becomes Pe^{nr} (Art. 552), which will therefore be the amount when the interest is due every moment.

575. The *Present value* of an amount due at the end of a given time is that sum which with its interest for the given time

will be equal to the amount. That is, (Art. 567), the Principal is the *present value* of the amount.

576. *Discount* is an allowance made for the payment of a sum of money before it is due.

From the definition of *present value*, it follows that a debt due at some future period is equitably discharged by paying the *present value* at once; hence the *discount* will be equal to the amount due diminished by its present value.

577. *To find the present value of a sum due at the end of a given time and the discount.*

Let P be the present value, M the amount, D the discount, r the interest of one pound for one year, n the number of years, R the amount of one pound in one year.

At simple interest:

$$M = P(1 + nr), \quad \text{(Art. 570)};$$

therefore $$P = \frac{M}{1 + nr}, \qquad D = M - P = \frac{Mnr}{1 + nr}.$$

At compound interest:

$$M = PR^n, \quad \text{(Art. 571)};$$

therefore $$P = \frac{M}{R^n}, \qquad D = M - P = \frac{M(R^n - 1)}{R^n}.$$

578. In practice it is very common to allow the *interest* of a sum of money paid before it is due, instead of the *discount* as here defined. Thus at simple interest, instead of $\frac{Mnr}{1 + nr}$ the payer would be allowed Mnr for immediate payment.

EXAMPLES OF INTEREST.

1. Shew that at simple interest the discount is half the harmonic mean between the sum due and the interest on it.

2. At simple interest the interest on a certain sum of money is £180, and the discount on the same sum for the same time and at the same rate is £150: find the sum.

3. If the interest on £A for a year be equal to the discount on £B for the same time, find the rate of interest.

4. If a sum of money doubles itself in 40 years at simple interest, find the rate of interest.

5. A tradesman marks his goods with two prices, one for ready money, and the other for a credit of 6 months: find what ratio the two prices ought to bear to each other, allowing 5 per cent. simple interest.

6. Find in how many years £100 will become £1050 at 5 per cent. compound interest; having given

$$\log 14 = 1{\cdot}14613, \quad \log 15 = 1{\cdot}17609, \quad \log 16 = 1{\cdot}20412.$$

7. Find how many years will elapse before a sum of money trebles itself at $3\frac{1}{2}$ per cent. compound interest; having given

$$\log 10350 = 4{\cdot}01494, \qquad \log 3 = {\cdot}47712.$$

8. If a sum of money at a given rate of compound interest accumulate to p times its original value in m years, and to q times its original value in n years, prove that $n = m \log_p q$.

XLII. EQUATION OF PAYMENTS.

579. When different sums of money are due from one person to another at different times, we may be required to find the time at which they may all be paid together, so that neither lender nor borrower may lose. The time so found is called the *equated time.*

580. *To find the equated time of payment of two sums due at different times supposing simple interest.*

Let P_1, P_2 be the two sums due at the end of t_1, t_2 years respectively; suppose t_2 greater than t_1; let r be the interest of one pound for one year, x the number of years in the equated time.

The condition of fairness to both parties may be secured by supposing that the discount allowed for the sum paid before it is due is equal to the interest charged on the sum not paid until after it is due.

The discount on P_2 for $t_2 - x$ years is $\dfrac{P_2(t_2 - x)r}{1 + (t_2 - x)r}$;

the interest on P_1 for $x - t_1$ years is $P_1(x - t_1)r$;

therefore $$\frac{P_2(t_2 - x)}{1 + (t_2 - x)r} = P_1(x - t_1).$$

This will give a quadratic equation in x, namely,

$$P_1rx^2 - \{P_1r(t_1 + t_2) + P_1 + P_2\}x + P_1rt_1t_2 + P_1t_1 + P_2t_2 = 0;$$

that root must be taken which lies between t_1 and t_2.

581. Another method of solving the question of the preceding Article is as follows:

The present value of P_1 due at the end of t_1 years is $\dfrac{P_1}{1 + t_1r}$;

the present value of P_2 due at the end of t_2 years is $\dfrac{P_2}{1 + t_2r}$;

the present value of $P_1 + P_2$ due at the end of x years is $\dfrac{P_1 + P_2}{1 + xr}$.

Hence we may propose to find the equated time of payment, x, from the equation

$$\frac{P_1}{1 + t_1r} + \frac{P_2}{1 + t_2r} = \frac{P_1 + P_2}{1 + xr}.$$

582. If such a question did occur in practice however the method would probably be to proceed as in the first solution, with this exception, that the lender would allow *interest* instead of *dis-*

count on the sum paid before it was due; thus we should find x from

$$P_2(t_2-x)r=P_1(x-t_1)r;$$

therefore $$(P_1+P_2)x=P_1t_1+P_2t_2.$$

In this case the interest on P_1+P_2 for x years is equal to the sum of the interests of P_1 and P_2 for the times t_1 and t_2 respectively; this follows if we multiply both sides of the last equation by r. This rule is more advantageous to the borrower than that in Art. 580, for the *interest* on a given amount is greater than the *discount*. See Art. 577.

583. Suppose there are several sums P_1, P_2, P_3, ... due at the end of t_1, t_2, t_3, ... years respectively, and the *equated time* of payment is required.

The first method of solution (Art. 580) becomes very complicated in this case, and we shall therefore omit it.

The second method (Art. 581) gives for determining x, the number of years in the equated time,

$$\frac{P_1}{1+t_1r}+\frac{P_2}{1+t_2r}+\frac{P_3}{1+t_3r}+\ldots\ldots=\frac{P_1+P_2+P_3+\ldots\ldots}{1+xr}.$$

Denote the sum $\frac{P_1}{1+t_1r}+\frac{P_2}{1+t_2r}+\frac{P_3}{1+t_3r}+\ldots\ldots$ by $\Sigma\frac{P}{1+tr}$, and the sum $P_1+P_2+P_3+\ldots\ldots$ by ΣP; then we may write the above result thus,

$$\Sigma\left(\frac{P}{1+tr}\right)=\frac{\Sigma P}{1+xr}.$$

The third method (Art. 582) gives

$$x(P_1+P_2+P_3+\ldots\ldots)=P_1t_1+P_2t_2+P_3t_3+\ldots\ldots;$$

which may be written $x\Sigma P=\Sigma Pt$.

584. *Equation of payments* is a subject of no practical importance, and seems retained in books chiefly on account of the apparent paradox of different methods occurring which may

appear equally fair, but which lead to different results. We refer the student for more information on the question to the article *Discount* in the *English Cyclopædia.* We may observe, however, that the difficulty, if such it be, arises from the fact that *simple* interest is almost a fiction; the moment any sum of money is due, it matters not whether it is called principal or interest, it is of equal value to the owner; and thus if the interest on borrowed money is retained by the borrower, it ought in justice to the lender, to be united to the principal, and charged with interest afterwards.

585. If *compound* interest be allowed, the solutions in Arts. 580 and 581 will give the same result.

For the solution according to Art. 580 will be as follows:

the discount on P_2 for $t_2 - x$ years is $P_2\left(1 - \frac{1}{R^{t_2-x}}\right)$,

the interest on P_1 for $x - t_1$ years is $P_1(R^{x-t_1} - 1)$;

therefore $$P_2\left(1 - \frac{1}{R^{t_2-x}}\right) = P_1(R^{x-t_1} - 1).$$

From this equation x must be found; by transposition we shall see that this is the same equation as would be obtained by the method of Art. 581; for we obtain

$$P_1 + P_2 = \frac{P_2}{R^{t_2-x}} + P_1R^{x-t_1};$$

therefore $$\frac{P_1 + P_2}{R^x} = \frac{P_1}{R^{t_1}} + \frac{P_2}{R^{t_2}},$$

which shews that x is such that the present value of $P_1 + P_2$ due at the end of x years is equal to the sum of the present values of P_1 and P_2 due at the end of t_1 and t_2 years respectively.

586. If there are different sums $P_1, P_2, P_3, \ldots\ldots$ due at the end of $t_1, t_2, t_3, \ldots\ldots$ years respectively, the equated time of payment, x, allowing compound interest, may be found from

$$\frac{P_1 + P_2 + P_3 + \ldots\ldots}{R^x} = \frac{P_1}{R^{t_1}} + \frac{P_2}{R^{t_2}} + \frac{P_3}{R^{t_3}} + \ldots\ldots,$$

which may be written

$$\frac{\Sigma P}{R^z} = \Sigma\left(\frac{P}{R^t}\right).$$

587. We have said in Art. 580, that we must take that root of the quadratic equation which lies between t_1 and t_2; we will now prove that there will in fact be always one root, and only one, between t_1 and t_2.

We have to shew that the *equation*

$$P_1(x-t_1)\{1+(t_2-x)r\}-P_2(t_2-x)=0$$

has one root, and only one, lying between t_1 and t_2.

The *expression* $P_1(x-t_1)\{1+(t_2-x)r\}-P_2(t_2-x)$ is obviously *positive* when $x=t_2$. If this *expression* is arranged in the form ax^2+bx+c, the coefficient a is negative, being $-P_1r$; hence t_2 must lie between the roots of the *equation* by Art. 339; that is, one root is greater than t_2 and one root less than t_2. It is obvious too that no value of x less than t_1 can make the *expression* vanish, so there cannot be a root of the *equation* less than t_1; there must then be *one* root between t_1 and t_2, and *one* root greater than t_2.

It may be remarked that the value $x=t_2+\frac{1}{r}$ also makes the expression *positive*, and so the root which is greater than t_2 must by Art. 339 be greater than $t_2+\frac{1}{r}$.

MISCELLANEOUS EXAMPLES.

1. Find the equated time of payment of two sums, one of £400 due two years hence, the other of £2100 due eight years hence, at 5 per cent. (Art. 580.)

2. Find the equated time of payment of two sums, one of £20 due at the present date, the other of £16. 5*s*. due 270 days hence, the rate of interest being twopence-halfpenny per hundred pounds per day. (Art. 580.)

3. Find the equated time of paying two sums of money due at different epochs, interest being supposed due every moment.

4. A sum of money is left by will to be divided into three parts such that their amounts at compound interest, in a, b, c years respectively, shall be equal: determine the parts.

5. If a and n be positive integers, the integral part of $\{a + \sqrt{(a^2 - 1)}\}^n$ is odd.

6. If a and n be positive integers, the integral part of $\{\sqrt{(a^2 + 1)} + a\}^n$ is odd when n is even, and even when n is odd.

7. Shew that the remainder after n terms of the expansion of $\left(\dfrac{a}{a+x}\right)^2$ in a series of ascending powers of x is

$$\frac{(-1)^n x^n}{a^{n-1}} \cdot \frac{(n+1)a + nx}{(a+x)^2}.$$

8. If $\psi(n, r) = n(n-1)(n-2) \dots (n-r+1)$, shew that

$$\psi(n, r) = \psi(n-2, r) + 2r\psi(n-2, r-1) + r(r-1)\psi(n-2, r-2).$$

9. If $\phi(n, r) = \dfrac{n(n-1)\dots(n-r+1)}{\lfloor r}$, shew that

$$\phi(n, m) = \phi(n-m+1, 1) + \phi(m-1, 1)\phi(n-m+1, 2)$$
$$+ \phi(m-1, 2)\phi(n-m+1, 3) + \dots\dots$$

10. With the same notation shew that

$$a - (a+\beta)\phi(n, 1) + (a+2\beta)\phi(n, 2) - (a+3\beta)\phi(n, 3) + \dots\dots$$
$$\dots\dots + (-1)^n(a+n\beta)\phi(n, n) = 0.$$

11. If s be the sum of n terms of a geometrical progression whose first term is a and common ratio $1 + x$, where x is very small, shew that $n = \dfrac{s}{a}\left\{1 - \dfrac{(s-a)x}{2a}\right\}$ approximately.

12. If a quantity change continuously in value from a to b in a given time t_1, the increase at any instant bearing a constant ratio to its value at that instant, shew that its value at any time t will be $a\left(\dfrac{b}{a}\right)^{\frac{t}{t_1}}$. (Art. 574.)

XLIII. ANNUITIES.

588. *To find the amount of an annuity left unpaid for any number of years, allowing simple interest upon each sum from the time it becomes due.*

Let A be the annuity, n the number of years, r the interest of one pound for one year, M the amount.

At the end of the first year A becomes due, and at the end of the second year the interest of the first annuity is rA; at the end of this year the principal becomes $2A$, therefore the interest due at the end of the third year is $2rA$; in the same way the interest due at the end of the fourth year is $3rA$; and so on; hence the whole interest is $rA + 2rA + 3rA + \dots\dots\dots + (n-1)\,rA$; that is, $\frac{n(n-1)\,rA}{2}$, by Art. 459; and the sum of the annuities is nA:

therefore $$M = nA + \frac{n(n-1)}{2}\,rA.$$

589. *To find the present value of an annuity, to continue for a certain number of years, allowing simple interest.*

Let P denote the present value; then P with its interest for n years should be equal to the amount of the annuity in the same time; that is,

$$P + Pnr = nA + \frac{n(n-1)}{2}\,rA;$$

therefore $$P = \frac{nA + \frac{1}{2}n(n-1)\,rA}{1+nr}.$$

590. Another method has been proposed for solving the question in the preceding Article.

The present value of A due at the end of 1 year is $\frac{A}{1+r}$, (Art. 577); the present value of A due at the end of 2 years is $\frac{A}{1+2r}$; the present value of A due at the end of 3 years is $\frac{A}{1+3r}$, and so on; the present value of the annuity for n years should be equal to the sum of the present values of the different payments: hence

$$P = A\left\{\frac{1}{1+r} + \frac{1}{1+2r} + \frac{1}{1+3r} + \dots\dots + \frac{1}{1+nr}\right\}.$$

591. Some writers on Algebra have adopted the solution given in Art. 589, and others that in Art. 590; we have already intimated in a similar case (Art. 584), that the solution of such questions by *simple* interest must be unsatisfactory. The student may consult on this point Wood's *Algebra*, the *Treatise on Arithmetic and Algebra* in the Library of Useful Knowledge, p. 102; Jones on the *Value of Annuities and Reversionary Payments*, Vol. I. p. 9; and the article *Discount* in the *English Cyclopædia*.

592. The formulæ in Arts. 589 and 590 make the value of a perpetual annuity *infinite*. For the value of P in Art. 589 may be written

$$\frac{A + \frac{1}{2}(n-1)\,rA}{\frac{1}{n} + r};$$

when n is infinite the denominator of this expression becomes r, and the numerator becomes infinite; thus P is infinite. The series given for P in Art. 590 also becomes infinite when n is infinite.

This result is another indication that the value of annuities should be estimated in a different way. We proceed to the supposition of *compound* interest.

593. *To find the amount of an annuity left unpaid for any number of years, allowing compound interest.*

Let A be the annuity, n the number of years, R the amount of one pound in one year, M the required amount.

At the end of the first year A is due; at the end of the second year RA is the amount of the first annuity, hence the whole sum due at the end of the second year is $RA + A$, that is, $(R+1)\,A$; similarly at the end of the third year the whole sum due is $R(R+1)\,A + A$, that is, $(R^2 + R + 1)\,A$; and so on; hence the whole sum due at the end of n years is $(R^{n-1} + R^{n-2} + \ldots\ldots + 1)A$;

thus $$M = \frac{R^n - 1}{R - 1} A.$$

594. *To find the present value of an annuity, to continue for a certain number of years, allowing compound interest.*

Let P denote the present value; then the amount of P in n years should be equal to the amount of the annuity in the same time; that is,

$$PR^n = \frac{R^n - 1}{R - 1} A;$$

therefore $$P = \frac{1 - R^{-n}}{R - 1} A = \frac{1 - (1 + r)^{-n}}{r} A.$$

595. We may also solve the question of the preceding Article by supposing P equal to the sum of the present values of the different payments.

The present value of A due at the end of 1 year is $\frac{A}{R}$,

the present value of A due at the end of 2 years is $\frac{A}{R^2}$;

the present value of A due at the end of 3 years is $\frac{A}{R^3}$;

and so on;

therefore $$P = \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \ldots\ldots + \frac{A}{R^n}$$

$$= \frac{\frac{A}{R}\left(1 - \frac{1}{R^n}\right)}{1 - \frac{1}{R}} = \frac{A(1 - R^{-n})}{R - 1}.$$

If the present value of an annuity A for any number of years be mA, the annuity is said to be worth *m years' purchase.*

596. *To find the present value of a perpetual annuity.*

Suppose n to be infinite in the formula $P = \frac{A(1 - R^{-n})}{R - 1}$,

thus $$P = \frac{A}{R - 1} = \frac{A}{r}.$$

597. *To find the present value of an annuity, to commence at the end of* p *years, and then to continue* q *years.*

The present value of an annuity to commence at the end of p years, and then to continue q years, is found by subtracting the present value of the annuity for p years from the present value of the annuity for $p+q$ years; thus we obtain

$$A\frac{1-R^{-(p+q)}}{R-1}-A\frac{1-R^{-p}}{R-1}, \text{ that is, } \frac{A}{R-1}(R^{-p}-R^{-p-q}).$$

If the annuity is to commence at the end of p years, and then to continue for ever, we must suppose q infinite, and the present value becomes $\frac{AR^{-p}}{R-1}$. This may be obtained directly; for the present value is the sum of the following infinite series,

$$\frac{A}{R^{p+1}}+\frac{A}{R^{p+2}}+\frac{A}{R^{p+3}}+\ldots\ldots$$

598. The preceding Article may be applied to calculate the *fine* which must be paid for the renewal of a lease. Suppose an estate to be worth £A per annum, and that a lease of the estate is granted for $p+q$ years for a certain sum of money paid down; and suppose that when q years have elapsed, the lessee wishes to obtain a new lease for $p+q$ years; he must therefore pay a sum equivalent to the value of an annuity of £A to begin at the end of p years, and to continue for q years. This sum is called the *fine* to be paid for renewing q years of the lease.

599. We have hitherto in the present Chapter confined ourselves to the case in which the interest and the annuity are due only *once* a year. We will now give a more general proposition.

To find the amount of an annuity left unpaid for n *years, at compound interest, supposing interest due* q *times a year, and the annuity payable* m *times a year.*

Let $\frac{r}{q}$ be the interest of one pound for $\left(\frac{1}{q}\right)^{\text{th}}$ of a year; then by Art. 573, the amount of one pound in s years is

$\left(1+\frac{r}{q}\right)^{s}$ whether s be an integer or not; thus the amount of one pound for $\left(\frac{1}{m}\right)^{\text{th}}$ of a year is $\left(1+\frac{r}{q}\right)^{\frac{q}{m}}$; we shall denote this by ρ. Let a be the instalment of the annuity that should be paid each time; then the amount of the annuity at the end of n years is the sum of the following mn terms:

$$a\{\rho^{mn-1}+\rho^{mn-2}+\rho^{mn-3}+\ldots\ldots+\rho+1\},$$

that is, $$a\frac{\rho^{mn}-1}{\rho-1},$$ that is, $$a\frac{\left(1+\frac{r}{q}\right)^{nq}-1}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}.$$

EXAMPLES OF ANNUITIES.

In the examples the interest is supposed *compound* unless otherwise stated.

1. A person borrows £600. 5*s.*: find how much he must pay annually that the whole debt may be discharged in 35 years, allowing simple interest at 4 per cent.

2. Determine what the rate of interest must be in order that the present value of an annuity for a given number of years, at simple interest, may be equal to half the sum of the annuities.

3. A freehold estate of £100 a year is sold for £2500: find at what rate the interest is calculated.

4. The reversion, after 2 years, of a freehold worth £168. 2*s.* a year is to be sold: find its present value, supposing interest at $2\frac{1}{2}$ per cent.

5. If 20 years' purchase must be paid for an annuity to continue a certain number of years, and 26 years' purchase for an annuity to continue twice as long: find the rate per cent.

6. When $3\frac{1}{8}$ per cent. is the rate of interest, find what sum must be paid now to receive a freehold estate of £320 a year 10 years hence; having given

$$\log 1{\cdot}032 = {\cdot}0136797, \qquad \log 7{\cdot}29798 = {\cdot}8632030.$$

7. Supposing an annuity to continue for ever to be worth 25 years' purchase, find the annuity to continue for 3 years which can be purchased for £625.

8. A sum of £1000 is lent to be repaid with interest at 4 per cent. by annual instalments, beginning with £40 at the end of the first year, and increasing 30 per cent. each year on the last preceding instalment. Find when the debt will be paid off; having given

$$\log 2 = \cdot 30103, \qquad \log 3 = \cdot 47712.$$

9. Find the present value of an annuity which is to commence at the end of p years, and to continue for ever, each payment being m times the preceding. What limitation is there as to m?

10. Find what sum will amount to £1 in 20 years, at 5 per cent., the interest being supposed to be payable every instant.

11. If interest be payable every instant, and the interest for one year be $\left(\frac{1}{m}\right)^{\text{th}}$ of the principal, find the amount in n years.

12. A person borrows a sum of money, and pays off at the end of each year as much of the principal as he pays interest for that year: find how much he owes at the end of n years.

13. An estate, the clear annual value of which is £A, is let on a lease of 20 years, renewable every 7 years on payment of a fine: calculate the fine to be paid on renewing, interest being allowed at six per cent.; having given

$$\log 106 = 2\cdot 0253059, \qquad \log 4\cdot 688385 = \cdot 6710233,$$
$$\log 3\cdot 118042 = \cdot 4938820.$$

14. A person with a capital of £a, for which he receives interest at r per cent., spends every year £b, which is more than his original income. Find in how many years he will be ruined.

Ex. If $a = 1000$, $r = 5$, $b = 90$, shew that he will be ruined before the end of the 17th year; having given

$$\log 2 = \cdot 3010300, \qquad \log 3 = \cdot 4771213, \qquad \log 7 = \cdot 8450980.$$

XLIV. CONTINUED FRACTIONS.

600. Every expression of the form $a \pm \cfrac{b}{c \pm \cfrac{d}{e \pm \&c.}}$ is called a *continued fraction.*

We shall confine our attention to continued fractions of the form $a + \cfrac{1}{b + \cfrac{1}{c + \&c.}}$, where $a, b, c, \ldots\ldots$ are all positive integers.

For the sake of abbreviation the continued fraction is sometimes written thus: $a + \frac{1}{b+}\,\frac{1}{c + \&c.}$.

When the number of the terms $a, b, c, \ldots\ldots$ is *finite*, the continued fraction is said to be *terminating;* such a continued fraction may be reduced to an ordinary fraction by effecting the operations indicated.

601. *To convert any given fraction into a continued fraction.*

Let $\frac{m}{n}$ be the given fraction; divide m by n, let a be the quotient and p the remainder; thus $\frac{m}{n} = a + \frac{p}{n} = a + \cfrac{1}{\frac{n}{p}}$. Next divide n by p, let b be the quotient and q the remainder; thus $\frac{n}{p} = b + \frac{q}{p} = b + \cfrac{1}{\frac{p}{q}}$. Similarly, $\frac{p}{q} = c + \frac{r}{q} = c + \cfrac{1}{\frac{q}{r}}$, and so on.

Thus
$$\frac{m}{n} = a + \cfrac{1}{b + \cfrac{1}{c + \&c.}}.$$

If m be less than n, the first quotient a is zero.

We see then that to convert a given fraction into a continued fraction, we have to proceed as if we were finding the greatest common measure of the numerator and denominator; and we must therefore at last arrive at a point where the remainder is zero and the operation terminates: hence every fraction can be converted into a *terminating* continued fraction.

602. The fractions formed by taking one, two, three, ... of the quotients of the continued fraction $a+\frac{1}{b+}\frac{1}{c+\&c.}$ are called *converging fractions* or *convergents*. Thus the first convergent is a; the second convergent is formed from $a+\frac{1}{b}$, it is therefore $\frac{ab+1}{b}$; the third convergent is formed from $a+\cfrac{1}{b+\cfrac{1}{c}}$, that is, from $a+\frac{c}{bc+1}$, it is therefore $\frac{abc+a+c}{bc+1}$; and so on.

603. *The convergents taken in order are alternately less and greater than the continued fraction.*

The first convergent a is too small because the part $\frac{1}{b+\&c.}$ is omitted; $a+\frac{1}{b}$ is too great because the denominator b is too small; $a+\cfrac{1}{b+\cfrac{1}{c}}$ is too small because $b+\frac{1}{c}$ is too great; and so on.

604. *To prove the law of formation of the successive convergents.*

The first three convergents are $\frac{a}{1}$, $\frac{ab+1}{b}$, $\frac{abc+a+c}{bc+1}$; the numerator of the third is $c(ab+1)+a$, that is, it may be formed by multiplying the numerator of the second by the third quotient, and adding the numerator of the first; the denominator of the third convergent may be formed in a similar manner by multiplying the denominator of the second by the third quotient, and adding the denominator of the first. We shall now shew by induction that such a law holds universally.

Let $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$, be three consecutive convergents; m, m', m'', the corresponding quotients; and suppose that

$$p'' = m''p' + p, \qquad q'' = m''q' + q.$$

Let m''' be the next quotient; then the next convergent differs from $\frac{p''}{q''}$ only in taking in the additional quotient m''', so that we have to write $m'' + \frac{1}{m'''}$ instead of m''; thus the next convergent

$$= \frac{\left(m'' + \frac{1}{m'''}\right)p' + p}{\left(m'' + \frac{1}{m'''}\right)q' + q} = \frac{m'''(m''p' + p) + p'}{m'''(m''q' + q) + q'} = \frac{m'''p'' + p'}{m'''q'' + q'}.$$

If therefore we suppose

$$p''' = m'''p'' + p' \text{ and } q''' = m'''q'' + q',$$

the next convergent to $\frac{p''}{q''}$ will be equal to $\frac{p'''}{q'''}$, thus the convergent $\frac{p'''}{q'''}$ may be formed by the same law that was supposed to hold for $\frac{p''}{q''}$; but the law has been *proved* to be applicable for the third convergent, and therefore it is applicable for every subsequent convergent.

We have thus shewn that the successive convergents *may* be formed according to a certain law; as yet we have not proved that when they *are* so formed each convergent is in its lowest terms, but this will be proved in Art. 606.

605. *The difference between any two consecutive convergents is a fraction whose numerator is unity, and whose denominator is the product of the denominators of the convergents.*

This is obvious with respect to the first and second convergents, for $\frac{ab+1}{b} - \frac{a}{1} = \frac{1}{b}$.

Suppose the law to hold for any two consecutive convergents $\frac{p}{q}$, $\frac{p'}{q'}$; that is, suppose $p'q - pq' = \pm 1$, so that

$$\frac{p'}{q'} - \frac{p}{q} = \pm \frac{1}{qq'};$$

then, $p''q' - p'q'' = (m''p' + p)q' - p'(m''q' + q) = pq' - qp' = \mp 1$,

so that $$\frac{p''}{q''} - \frac{p'}{q'} = \mp \frac{1}{q''q'};$$

thus the law holds for the next convergent. Hence it is universally true.

606. *All convergents are in their lowest terms.*

For if the numerator and denominator of $\frac{p}{q}$ had any common measure it would divide $p'q - pq'$ or unity, which is impossible.

607. *Every convergent is nearer to the continued fraction than any of the preceding convergents.*

We shall prove this by shewing that every convergent is nearer to the continued fraction than the preceding convergent.

Let $\frac{p}{q}$, $\frac{p'}{q'}$, $\frac{p''}{q''}$ be consecutive convergents to a continued fraction x; then $\frac{p''}{q''} = \frac{m''p' + p}{m''q' + q}$. Now x differs from $\frac{p''}{q''}$ only in taking instead of m'' the *complete quotient* $m'' + \frac{1}{m''' + \&c.}$; this will be some quantity greater than unity, which we shall denote by μ; thus

$$x = \frac{\mu p' + p}{\mu q' + q};$$

therefore $$\frac{p}{q} - x = \frac{p}{q} - \frac{\mu p' + p}{\mu q' + q} = \frac{\mu(pq' - p'q)}{q(\mu q' + q)} = \frac{\pm \mu}{q(\mu q' + q)},$$

$$x - \frac{p'}{q'} = \frac{\mu p' + p}{\mu q' + q} - \frac{p'}{q'} = \frac{pq' - p'q}{q'(\mu q' + q)} = \frac{\pm 1}{q'(\mu q' + q)}.$$

Now 1 is less than μ and q' is greater than q; hence on both accounts the difference between x and $\frac{p'}{q'}$ is less than the difference between x and $\frac{p}{q}$; that is, $\frac{p'}{q'}$ is nearer to x than $\frac{p}{q}$ is.

608. *To determine limits to the error made in taking any convergent for the continued fraction.*

By the preceding Article the difference between x and $\frac{p}{q}$ is $\frac{\mu}{q(\mu q' + q)}$, or $\frac{1}{q\left(q' + \frac{q}{\mu}\right)}$; this is less than $\frac{1}{qq'}$ and greater than $\frac{1}{q(q'+q)}$. Since q' is greater than q, the error *a fortiori* is less than $\frac{1}{q^2}$ and greater than $\frac{1}{2q'^2}$; these limits are simpler than those first given, though of course not so close.

609. In order that the error made may be less than a given quantity $\frac{1}{k}$, we have therefore only to form the consecutive convergents until we arrive at one $\frac{p}{q}$, such that q^2 is not less than k.

610. *Any convergent is nearer to the continued fraction than any other fraction which has a smaller denominator than the convergent has.*

Let $\frac{p'}{q'}$ be the convergent, and $\frac{r}{s}$ a fraction, such that s is less than q'. Let x be the continued fraction, and $\frac{p}{q}$ the convergent immediately preceding $\frac{p'}{q'}$. Then $\frac{p}{q}$, x, $\frac{p'}{q'}$ are either in ascending or descending order of magnitude by Art. 603. Now $\frac{r}{s}$ cannot lie *between* $\frac{p}{q}$ and $\frac{p'}{q'}$; for then the difference of $\frac{r}{s}$ and $\frac{p}{q}$

would be less than the difference of $\frac{p}{q}$ and $\frac{p'}{q'}$, that is, less than $\frac{1}{qq'}$, and therefore the difference of ps and qr would be less than $\frac{s}{q'}$, that is, an integer less than a proper fraction, which is impossible. Thus either $\frac{p}{q}$, x, $\frac{p'}{q'}$, $\frac{r}{s}$, or $\frac{r}{s}$, $\frac{p}{q}$, x, $\frac{p'}{q'}$ must be in order of magnitude. In the former case $\frac{r}{s}$ differs more from x than $\frac{p'}{q'}$ does; in the latter case $\frac{r}{s}$ differs more from x than $\frac{p}{q}$ does, and therefore *a fortiori* more than $\frac{p'}{q'}$ does.

611. Suppose $\frac{p}{q}$, $\frac{p'}{q'}$ two consecutive convergents to a continued fraction x, then $\frac{pp'}{qq'}$ is greater or less than x^2 according as $\frac{p}{q}$ is greater or less than $\frac{p'}{q'}$. For, as in Art. 607, we have $x = \frac{\mu p' + p}{\mu q' + q}$; therefore $\frac{p}{qx} - \frac{xq'}{p'} = \frac{p(\mu q' + q)}{q(\mu p' + p)} - \frac{q'(\mu p' + p)}{p'(\mu q' + q)}$.

Reduce the fractions on the right-hand side to a common denominator; then the numerator is $pp'(\mu q' + q)^2 - qq'(\mu p' + p)^2$, that is, $\mu^2(pp'q'^2 - qq'p'^2) + pp'q^2 - qq'p^2$, that is, $(\mu^2 p'q' - pq)(pq' - p'q)$.

The factor $\mu^2 p'q' - pq$ is necessarily positive; the factor $pq' - p'q$ is positive or negative, according as $\frac{p}{q}$ is greater or less than $\frac{p'}{q'}$; hence $\frac{p}{qx}$ is greater or less than $\frac{xq'}{p'}$, that is, $\frac{pp'}{qq'}$ is greater or less than x^2, according as $\frac{p}{q}$ is greater or less than $\frac{p'}{q'}$.

EXAMPLES OF CONTINUED FRACTIONS.

Convert the following four fractions into continued fractions:

1. $\frac{1380}{1051}$. 2. $\frac{445}{612}$. 3. $\frac{19763}{44126}$. 4. $\frac{743}{611}$.

5. Find three fractions converging to 3·1416.

6. Find a series of fractions converging to the ratio of 5 hours 48 minutes 51 seconds to 24 hours.

7. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ be three consecutive convergents, shew that $(p_3 - p_1) q_2 = (q_3 - q_1) p_2$.

8. Prove that the numerators of any two consecutive convergents have no common measure greater than unity; and similarly for the denominators.

9. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots$ be successive convergents to a continued fraction greater than unity, shew that $p_n q_{n-1} - p_{n-1} q_n = (-1)^n$.

10. Shew that the difference between the first convergent and the n^{th} convergent is numerically equal to

$$\frac{1}{q_1 q_2} - \frac{1}{q_2 q_3} + \frac{1}{q_3 q_4} - \ldots\ldots + \frac{(-1)^n}{q_{n-1} q_n}.$$

11. Shew that $\left(\frac{p_{n+2}}{p_n} - 1\right)\left(1 - \frac{p_{n-1}}{p_{n+1}}\right) = \left(\frac{q_{n+2}}{q_n} - 1\right)\left(1 - \frac{q_{n-1}}{q_{n+1}}\right)$.

12. If μ_n be the n^{th} quotient in a continued fraction greater than unity, shew that $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-1} \mu_n$.

13. If $\frac{p_{n-2}}{q_{n-2}}, \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \ldots\ldots$ be successive convergents to the continued fraction $\frac{\beta_1}{\alpha_1 +} \frac{\beta_2}{\alpha_2 +} \frac{\beta_3}{\alpha_3 +} \ldots\ldots$ shew that

$$p_n = \alpha_n p_{n-1} + \beta_n p_{n-2}, \qquad q_n = \alpha_n q_{n-1} + \beta_n q_{n-2};$$

and hence that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1} \beta_1 \beta_2 \ldots\ldots \beta_n$.

14. If $\frac{p_n}{q_n}$ denote the n^{th} convergent to a fraction $\frac{P}{Q}$, and R_n denote the n^{th} remainder which occurs in the process of converting the fraction $\frac{P}{Q}$ to a continued fraction, shew that

$$P = p_n R_{n-1} + p_{n-1} R_n, \qquad Q = q_n R_{n-1} + q_{n-1} R_n.$$

15. Shew that the difference of $\frac{P}{Q}$ and $\frac{p_n}{q_n}$ is $\frac{R_n}{Qq_n}$.

16. In converting a fraction in its lowest terms to a continued fraction, shew that any two consecutive remainders have no common measure greater than unity.

XLV. REDUCTION OF A QUADRATIC SURD TO A CONTINUED FRACTION.

612. A quadratic surd cannot be reduced to a *terminating* continued fraction, because the surd would then be equal to a rational fraction, that is, would be commensurable; we shall see, however, that a quadratic surd can be reduced to a continued fraction which does not terminate: we will first give an example, and then the general theory. Take the square root of 6;

$$\sqrt{(6)} = 2 + \sqrt{(6)} - 2 = 2 + \frac{2}{\sqrt{(6)}+2} = 2 + \frac{1}{\frac{\sqrt{(6)}+2}{2}},$$

$$\frac{\sqrt{(6)}+2}{2} = 2 + \frac{\sqrt{(6)}-2}{2} = 2 + \frac{1}{\sqrt{(6)}+2} = 2 + \frac{1}{\frac{\sqrt{(6)}+2}{1}},$$

$$\frac{\sqrt{(6)}+2}{1} = 4 + \frac{\sqrt{(6)}-2}{1} = 4 + \frac{2}{\sqrt{(6)}+2} = 4 + \frac{1}{\frac{\sqrt{(6)}+2}{2}},$$

the steps now recur; thus we have

$$\sqrt{(6)} = 2 + \frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\,\&c.$$

In the above process the expression which occurs at the beginning of any line is separated into two parts, the first part being the *greatest integer* which the expression contains, and the second part the remainder; thus the greatest integer in $\sqrt{6}$ is 2, we therefore write

$$\sqrt{(6)} = 2 + \{\sqrt{(6)} - 2\};$$

again, the greatest integer in $\frac{\sqrt{(6)}+2}{2}$ is 2, we therefore write

$$\frac{\sqrt{(6)}+2}{2} = 2 + \frac{\sqrt{(6)}-2}{2},$$

and so on; the remainder is then made to have its numerator rational, and is expressed as a fraction with unity for numerator; we then begin another line of the process.

We may notice in the example that the quotients begin to recur as soon as we arrive at a quotient which is double of the first. This we shall presently shew is always the case.

613. Let N be any integer which is not an exact square; let a be the greatest integer contained in $\sqrt{N}$; write $\sqrt{N}$ in the form $\frac{\sqrt{(N)}+0}{1}$ for symmetry, and proceed thus:

$$\frac{\sqrt{(N)}+0}{1} = a + \frac{\sqrt{(N)}-a}{1} = a + \frac{r}{\sqrt{(N)}+a}, \text{ if } r = N - a^2;$$

$$\frac{\sqrt{(N)}+a}{r} = b + \frac{\sqrt{(N)}+a-rb}{r} = b + \frac{r'}{\sqrt{(N)}+a'},$$

$$\text{if } a' = rb - a, \text{ and } r' = \frac{N-a'^2}{r};$$

$$\frac{\sqrt{(N)}+a'}{r'} = b' + \frac{\sqrt{(N)}+a'-r'b'}{r'} = b' + \frac{r''}{\sqrt{(N)}+a''},$$

$$\text{if } a'' = r'b' - a', \text{ and } r'' = \frac{N-a''^2}{r'};$$

$$\frac{\sqrt{(N)}+a''}{r''} = b'' + \frac{\sqrt{(N)}+a''-r''b''}{r''} = \&c.$$

In this process we suppose $b, b', b'', \ldots$ to be, like a, the *greatest* integers contained in the expressions from which they respectively spring; hence it follows that $r, r', r'', r''', \ldots$ are all positive. For a^2 is less than N, hence r is positive, and b is the greatest integer in $\frac{\sqrt{(N)}+a}{r}$, so that b is of course less than $\frac{\sqrt{(N)}+a}{r}$; hence a'^2 is less than N, and so r' is positive; and so on. We have noticed this fact, because it follows very obviously from the process; it is, however, included in the proposition of the following Article.

614. In the expressions which occur at the beginning of the lines in Art. 613, we have the following series of quantities:

$$0,\ a,\ a',\ a'',\ a''',\ \&c. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$1;\ r,\ r',\ r'',\ r''',\ \&c. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and the corresponding series of quotients is

$$a,\ b,\ b',\ b'',\ b''',\ \&c. \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We shall now shew that the terms in (1) and (2) are all positive integers; those in (3) are known to be such.

Let a, a', a'' be any three consecutive terms of (1); ρ, ρ', ρ'' the corresponding terms of (2); β, β', β'' those of (3). Let the corresponding convergents to $\sqrt{(N)}$ be $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$, so that $\frac{p''}{q''}=\frac{\beta''p'+p}{\beta''q'+q}$; these convergents can all be formed in the usual way, since all the terms in (3) are positive integers.

Since the complete quotient corresponding to β'' is $\frac{\sqrt{(N)}+a''}{\rho''}$, we have, by Art. 607,

$$\sqrt{(N)}=\frac{\frac{\sqrt{(N)}+a''}{\rho''}p'+p}{\frac{\sqrt{(N)}+a''}{\rho''}q'+q}=\frac{\{\sqrt{(N)}+a''\}p'+\rho''p}{\{\sqrt{(N)}+a''\}q'+\rho''q}.$$

Multiply up, and then equate the rational and irrational parts (Art. 299); thus

$$a''p' + \rho''p = Nq', \qquad a''q' + \rho''q = p';$$

therefore
$$a''(pq' - p'q) = pp' - qq'N,$$
$$\rho''(pq' - p'q) = q'^2N - p'^2.$$

Now $pq' - p'q = \pm 1$, hence a'' and ρ'' are integers. And it is proved in Art. 611 that $pq' - p'q$, $pp' - qq'N$, and $q'^2N - p'^2$ have the *same* sign; hence a'' and ρ'' are *positive* integers.

This investigation may be applied to *any* corresponding pair of quantities in (1) and (2) except the first two pairs; it cannot be applied to these because *two* convergents $\frac{p}{q}$ and $\frac{p'}{q'}$ are assumed to precede the convergent $\frac{p''}{q''}$. But the first two pairs of quantities in (1) and (2), namely 0 and 1, and a and r, are known to be positive integers. Thus *all* the quantities in (1) and (2) are positive integers.

615. The greatest term in (1) is a. For by the mode of formation of the series, $\rho\rho' = N - a'^2$; since ρ and ρ' are positive, a'^2 is less than N, and therefore a' is not greater than a.

616. No term in (2) or (3) can be greater than $2a$. For by the mode of formation of the series, $a' + a'' = \rho'\beta'$; and since neither a' nor a'' can be greater than a, neither ρ' nor β' can be greater than $2a$.

617. If $\rho'' = 1$, then $a'' = a$.

For, by Art. 614, $a'' + \rho''\frac{q}{q'} = \frac{p'}{q'}$, therefore if $\rho'' = 1$ $a'' + \text{a fraction} = \frac{p'}{q'}$. Now $\frac{p'}{q'}$ is a nearer approximation to $\sqrt{N}$ than a is, and a is less than $\sqrt{N}$; therefore $\frac{p'}{q'}$ is greater than a; hence $a'' = a$.

618. If any term in (1), excluding the first, be subtracted from a, the remainder is less than the corresponding term in (2).

For, by Art. 614, $a''q' + \rho''q = p'$; therefore $\frac{q}{q'} = \frac{1}{\rho''}\left(\frac{p'}{q'} - a''\right)$; therefore $\frac{p'}{q'} - a''$ is less than ρ''; therefore, *a fortiori*, $a - a''$ is less than ρ''.

This demonstration will only apply to the *third* or any following term, because in Art. 614 it is supposed that two terms a, a' precede a''. The theorem, however, holds for the second term, as is obvious by inspection, for $a - a$, or zero, is less than r.

619. It is shewn in Arts. 615 and 616 that the values of the terms in (1) and (2) cannot exceed a and $2a$ respectively; hence the same values must recur in the two series simultaneously, and there cannot be *more* than $2a^2$ terms in each series before this takes place.

620. Let the series (1) be denoted by

$$a_1,\ a_2,\ a_3,\ \ldots\ldots\ a_{m-1},\ a_m,\ a_{m+1},\ \ldots\ldots\ a_{n-1},\ a_n,\ a_{n+1},\ \ldots\ldots$$

and let a similar notation be used for (2) and (3). We have proved that a recurrence must take place, suppose then that the terms from the m^{th} to the $(n-1)^{\text{th}}$ inclusive recur, so that

$$a_n = a_m, \qquad a_{n+1} = a_{m+1}, \qquad a_{n+2} = a_{m+2}, \ldots\ldots$$
$$b_n = b_m, \qquad b_{n+1} = b_{m+1}, \qquad b_{n+2} = b_{m+2}, \ldots\ldots$$
$$r_n = r_m, \qquad r_{n+1} = r_{m+1}, \qquad r_{n+2} = r_{m+2}, \ldots\ldots$$

We shall shew that

$$a_{n-1} = a_{m-1}, \qquad b_{n-1} = b_{m-1}, \qquad r_{n-1} = r_{m-1}.$$

We have $r_{m-1}r_m = N - a_m^2$, $\qquad r_{n-1}r_n = N - a_n^2$,

but $r_n = r_m$, and $a_n = a_m$; therefore $r_{n-1} = r_{m-1}$.

Again, $a_{m-1} + a_m = r_{m-1}b_{m-1}$, $\qquad a_{n-1} + a_n = r_{n-1}b_{n-1}$;

therefore $\qquad a_{n-1} - a_{m-1} = (b_{n-1} - b_{m-1})\, r_{m-1}$;

therefore $\frac{a_{n-1} - a_{m-1}}{r_{m-1}} = b_{n-1} - b_{m-1} =$ zero or an integer.

But, by Art. 618, $a - a_{m-1}$ is less than r_{m-1}, and $a - a_{n-1}$ is less than r_{n-1}, so that $a - a_{n-1}$ is less than r_{m-1}; therefore $a_{n-1} - a_{m-1}$ is less than r_{m-1}; therefore $\frac{a_{n-1} - a_{m-1}}{r_{m-1}}$ is less than 1.

Comparing this with the former result, we see that $\frac{a_{n-1} - a_{m-1}}{r_{m-1}}$ must be zero; therefore $a_{n-1} = a_{m-1}$, and $b_{n-1} = b_{m-1}$.

Hence, knowing that the m^{th} term recurs, we can infer that the $(m-1)^{\text{th}}$ term also recurs. This demonstration holds as long as m is not less than 3; for it depends on the theorem established in Art. 618. Hence the terms recur beginning with the complete quotient $\frac{\sqrt{(N)} + a}{r}$.

621. The last quotient will always be $2a$.

For let the last complete quotient be $\frac{\sqrt{(N)} + a_n}{r_n}$, then the next is $\frac{\sqrt{(N)} + a}{r}$; hence $a_n + a = r_n b_n$, $r_n r = N - a^2$; but $r = N - a^2$; therefore $r_n = 1$; therefore, by Art. 617, $a_n = a$; therefore $b_n = 2a$.

622. *Every periodic continued fraction is equal to one of the roots of a quadratic equation with rational coefficients.*

Let $$x = a + \frac{1}{b+} \cdots\cdots \frac{1}{h+} \frac{1}{k+} \frac{1}{y},$$

where $$y = r + \frac{1}{s+} \cdots\cdots \frac{1}{u+} \frac{1}{v+} \frac{1}{y},$$

so that a, b, ... h, k are the quotients which do not recur, and r, s, ... u, v are those which recur perpetually.

Let $\frac{p'}{q'}$ be the convergent formed from the quotients a, b, ... down to k inclusive; and let $\frac{p}{q}$ be the convergent immediately preceding $\frac{p'}{q'}$; then, as in Art. 607,

$$x = \frac{p'y + p}{q'y + q} \qquad (1).$$

Let $\frac{P'}{Q'}$ be the convergent formed from the quotients $r, s, \ldots$ down to v inclusive; and let $\frac{P}{Q}$ be the convergent immediately preceding $\frac{P'}{Q'}$; then

$$y = \frac{P'y + P}{Q'y + Q} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

From (1) and (2) by eliminating y we obtain a quadratic equation in x with rational coefficients. To obtain x we must solve this equation: or we may take the positive value of y found from (2), that is, from $Q'y^2 + (Q - P')\,y - P = 0$, and substitute it in (1).

623. The following theorem in continued fractions may be noticed.

Let $\frac{1}{b+}\,\frac{1}{c+}\cdots\cdots\frac{1}{m+}\,\frac{1}{m'+}\,\frac{1}{m''}$ be the development of a proper fraction $\frac{P}{Q}$; and let the corresponding series of convergents be

$$\frac{1}{b},\ \frac{c}{cb+1},\ \ldots\ldots\ldots\ldots\ldots \frac{p}{q},\ \frac{p'}{q'},\ \frac{p''}{q''},\ \frac{P}{Q}:$$

then the development of $\frac{q''}{Q}$ will be

$$\frac{1}{m''+}\,\frac{1}{m'+}\,\frac{1}{m+}\cdots\cdots\ \frac{1}{c+}\,\frac{1}{b};$$

that is, the same quotients will occur but in the reverse order.

For $Q = m''q'' + q'$, therefore $\frac{q''}{Q} = \cfrac{1}{m'' + \cfrac{q'}{q''}}$;

$q'' = m'q' + q$, therefore $\frac{q'}{q''} = \cfrac{1}{m' + \cfrac{q}{q'}}$;

and so on.

Hence $$\frac{q''}{Q} = \frac{1}{m''+}\,\frac{1}{m'+}\,\frac{1}{m+}\cdots\cdots\cdots\ \frac{1}{c+}\,\frac{1}{b}.$$

624. The preceding theorem will furnish an addition to the results obtained in the present Chapter.

Let $\frac{p}{q}$ and $\frac{p'}{q'}$ be two successive convergents to $\sqrt{N}$, such that $\frac{p'}{q'}$ is the last convergent formed before the quotients recur; therefore by Arts. 614 and 621, $p' = aq' + q$.

Now the development of $\frac{p' - aq'}{q'}$, that is of $\frac{p'}{q'} - a$, will be with the notation of Art. 620

$$\frac{1}{b_2 +} \frac{1}{b_3 +} \frac{1}{b_4 +} \cdots\cdots\cdots\cdots \quad \frac{1}{b_{n-3} +} \frac{1}{b_{n-2} +} \frac{1}{b_{n-1}};$$

and the last convergent will be $\frac{p - aq}{q}$. But we have just seen that $q = p' - aq'$. Hence by Art. 623

$$b_{n-1} = b_2, \quad b_{n-2} = b_3, \quad b_{n-3} = b_4, \ \ldots\ldots\ldots$$

625. There is also a recurrence of the same terms in the reverse order with respect to the second and the third series of Arts. 614 and 620, like that which has just been demonstrated with respect to the first series.

We have universally

$$r_{m-1} r_m = N - a_m^2 \ (1), \qquad a_{m-1} + a_m = r_{m-1} b_{m-1} \ (2).$$

Put in (1) for m successively the values 2 and n; thus

$$r_1 r_2 = N - a_2^2, \qquad r_{n-1} r_n = N - a_n^2;$$

we know that $a_2 = a_n$ for each $= a$, and that $r_1 = r_n$ for each $= 1$: therefore $r_2 = r_{n-1}$.

Put in (2) for m successively the values 3 and n; thus

$$a_2 + a_3 = r_2 b_2, \qquad a_{n-1} + a_n = r_{n-1} b_{n-1};$$

wo know that $a_2 = a_n$, that $r_2 = r_{n-1}$, and that $b_2 = b_{n-1}$: therefore $a_3 = a_{n-1}$.

Again, put in (1) for m successively the values 3 and $n - 1$: hence we obtain $r_3 = r_{n-2}$. Put in (2) for m successively the values 4 and $n - 1$: hence we obtain $a_4 = a_{n-2}$. And so on.

626. The following theorem relating to continued fractions was communicated to the present writer by Mr Rickard of Birmingham. The theorem will furnish high convergents to the square root of a number with little labour.

Let N be a positive integer which is not an exact square, and let the convergents to $\sqrt{N}$ be supposed formed in the usual way; let c be the number of recurring quotients in one complete cycle, or any multiple of that number; let $\frac{p_c}{q_c}$ be the c^{th} convergent, and $\frac{p_{2c}}{q_{2c}}$ the $(2c)^{\text{th}}$ convergent; then will

$$\frac{p_{2c}}{q_{2c}} = \frac{1}{2}\left(\frac{p_c}{q_c} + \frac{Nq_c}{p_c}\right).$$

Let a be the greatest integer in $\sqrt{N}$, and let the quotients obtained by converting $\sqrt{N}$ into a continued fraction in the usual way, be denoted by

$$b_1,\ b_2,\ b_3,\ \ldots\ b_c,\ b_{c+1},\ b_{c+2},\ \ldots\ b_{2c},\ \ldots$$

Then from Arts 620, 621 we have

$$b_2 = b_{c+2},\quad b_3 = b_{c+3},\quad b_4 = b_{c+4}\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

also

$$b_1 = a,\quad b_{c+1} = 2a\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let $\frac{p_{c-1}}{q_{c-1}}$ and $\frac{p_{c+1}}{q_{c+1}}$ be the convergents immediately preceding and following $\frac{p_c}{q_c}$; then $\frac{p_{c+1}}{q_{c+1}} = \frac{b_{c+1}p_c + p_{c-1}}{b_{c+1}q_c + q_{c-1}}$.

Now $\sqrt{N}$ differs from $\frac{p_{c+1}}{q_{c+1}}$ in this respect; instead of using the quotient b_{c+1} we must use the corresponding *complete quotient*, which is $a + \sqrt{N}$, by Art. 621.

Therefore $$\sqrt{N} = \frac{(a+\sqrt{N})\,p_c + p_{c-1}}{(a+\sqrt{N})\,q_c + q_{c-1}};$$

multiply up, and equate the rational and the irrational parts; thus

$$ap_c + p_{c-1} = Nq_c,\quad aq_c + q_{c-1} = p_c\ldots\ldots\ldots\ldots(3).$$

Again, $\frac{p_{2c}}{q_{2c}}$ differs from $\frac{p_{c+1}}{q_{c+1}}$ in this respect; instead of using the quotient b_{c+1} we must use the continued fraction $b_{c+1}+\frac{1}{b_{c+2}+}\cdots\cdots\frac{1}{b_{2c}}$; and this continued fraction by (1) and (2) is equal to $a+b_1+\frac{1}{b_2+}\cdots\cdots\frac{1}{b_c}$, that is, it is equal to $a+\frac{p_c}{q_c}$.

Therefore

$$\frac{p_{2c}}{q_{2c}}=\frac{\left(a+\frac{p_c}{q_c}\right)p_c+p_{c-1}}{\left(a+\frac{p_c}{q_c}\right)q_c+q_{c-1}}=\frac{ap_c+p_{c-1}+\frac{p_c^2}{q_c}}{aq_c+q_{c-1}+p_c}$$

$$=\frac{Nq_c+\frac{p_c^2}{q_c}}{2p_c}\text{, by (3),}\qquad =\tfrac{1}{2}\left(\frac{p_c}{q_c}+\frac{Nq_c}{p_c}\right).$$

We can give an interesting geometrical illustration of the theorem. If N denote the area of a rectangle and $\frac{p_c}{q_c}$ be taken for one side, the other side is $\frac{Nq_c}{p_c}$. Thus $\frac{p_{2c}}{q_{2c}}$ is equal to half the sum of the sides of this rectangle. Let h and k denote the sides of one rectangle; then if $\frac{1}{2}(h+k)$ denote a side of another rectangle of the same area the other side will be $\frac{2hk}{h+k}$; the difference of these two sides will be $\frac{(h-k)^2}{2(h+k)}$, which is *less* than $h-k$. Now in seeking $\sqrt{N}$ we in fact desire the side of a square of which the area is N; and the present theorem may be considered to supply a series of rectangles, in which a side of each rectangle is half the sum of the sides of the preceding rectangle; so that each rectangle is more nearly equilateral than the preceding rectangle: and the rectangles tend to the form of a square. This illustration has been suggested by a paper entitled *The Rectangular Theorem* by Henry Brook.

Suppose for an example that $N=a^2+1$; then the quotients are a, $2a$, $2a$, $2a$, ...; that is, the cycle of recurring quotients reduces to the single quotient $2a$. In this case then c may be any whole number whatever.

Suppose for another example that $N = a^2 - 1$; then the quotients are $a-1$, 1, $2(a-1)$, 1, $2(a-1), \ldots$; thus the cycle of recurring quotients consists of the two quotients 1 and $2(a-1)$. Thus in the above theorem c may be any even whole number. In this case however the theorem will also be true if c be any odd whole number, as we will now shew.

Suppose c any odd whole number. Since the $(c+1)^{\text{th}}$ quotient is unity we have

$$p_{c+1} = p_c + p_{c-1}, \qquad q_{c+1} = q_c + q_{c-1} \ldots\ldots\ldots\ldots\ldots (4).$$

And, in the same manner as equations (3) were proved, we have

$$(a-1)p_{c+1} + p_c = Nq_{c+1}, \qquad (a-1)q_{c+1} + q_c = p_{c+1} \ldots\ldots (5).$$

Now $\frac{p_{2c}}{q_{2c}}$ differs from $\frac{p_{c+1}}{q_{c+1}}$ in this respect; instead of using the quotient unity we must use the continued fraction $1 + \frac{1}{2(a-1)+} \ldots\ldots \frac{1}{1}$; and this continued fraction is equal to $\frac{1}{\frac{p_{c+1}}{q_{c+1}} - (a-1)}$, that is, to $\frac{q_{c+1}}{q_c}$ by the second of equations (5).

Thus $$\frac{p_{2c}}{q_{2c}} = \frac{p_c \frac{q_{c+1}}{q_c} + p_{c-1}}{q_{c+1} + q_{c-1}} = \frac{p_c \frac{q_{c+1}}{q_c} + p_{c+1} - p_c}{2q_{c+1} - q_c}, \text{ by (4).}$$

From equations (5) since $N = a^2 - 1$, it may be deduced that

$$p_{c+1} = \frac{(a-1)p_c + Nq_c}{2(a-1)}, \qquad q_{c+1} = \frac{(a-1)q_c + p_c}{2(a-1)}.$$

Substitute these values in the last expression for $\frac{p_{2c}}{q_{2c}}$ and we obtain $\frac{p_{2c}}{q_{2c}} = \frac{Nq_c + \frac{p_c^2}{q_c}}{2p_c}$.

EXAMPLES OF CONTINUED FRACTIONS FROM QUADRATIC SURDS.

Express the following fourteen surds as continued fractions, and find the first four convergents to each:

1. $\sqrt{8}$.   2. $\sqrt{(10)}$.   3. $\sqrt{(14)}$.   4. $\sqrt{(17)}$.

5. $\sqrt{(19)}$.   6. $\sqrt{(26)}$.   7. $\sqrt{(27)}$.   8. $\sqrt{(46)}$.

9. $\sqrt{(53)}$.   10. $\sqrt{(101)}$.   11. $\sqrt{(a^2+1)}$.

12. $\sqrt{(a^2-1)}$.   13. $\sqrt{(a^2+a)}$.   14. $\sqrt{(a^2-a)}$.

15. Find the 8th convergent to $\sqrt{(13)}$.

16. Find the 8th convergent to $\sqrt{(31)}$.

17. Shew that the 9th convergent to $\sqrt{(33)}$ will give the true value to at least 6 places of decimals.

18. Find limits of the error when $\frac{211}{44}$ is taken for $\sqrt{(23)}$.

19. Shew that $\frac{916}{191}$ differs from $\sqrt{(23)}$ by a quantity less than $\frac{1}{(191)^2}$ and greater than $\frac{1}{2\,(240)^2}$.

20. Find limits of the error when $\frac{1151}{240}$ is taken for $\sqrt{(23)}$.

21. Find limits of the error when the 8th convergent is taken for $\sqrt{(31)}$.

22. Shew that $1+\frac{1}{3+}\frac{1}{2+}\frac{1}{3+}\frac{1}{2+}\dots\dots=\sqrt{\left(\frac{5}{3}\right)}$.

23. Shew that

$$\left(a+\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\dots\dots\right)\left(\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\dots\dots\right)=\frac{a}{b}.$$

24. Shew that

$$2a+\frac{1}{a+}\frac{1}{4a+}\frac{1}{a+}\frac{1}{4a+}\dots\dots=2\sqrt{(1+a^2)};$$

shew that the second convergent differs from the true value by a quantity less than $1\div a\,(4a^2+1)$; and thence by making $a=7$, shew that $\frac{99}{70}$ differs from $\sqrt{2}$ by a quantity less than $\frac{1}{13790}$.

25. Shew that the 3^{rd} convergent to $\sqrt{(a^2+a+1)}$ is $\frac{1}{2}(2a+1)$.

26. Find convergents to $\frac{\sqrt{3}}{4}$; shew that $\frac{13}{30}$ exceeds the true value by a quantity less than $\frac{1}{2910}$.

27. Find the 6^{th} convergent to $\sqrt{\left(\frac{3}{2}\right)}$.

28. Find the 6^{th} convergent to the positive root of

$$2x^2 - 3x - 6 = 0.$$

29. Find the 6^{th} convergent to each root of

$$x^2 - 5x + 3 = 0.$$

30. Find the 7^{th} convergent to the greater root of

$$2x^2 - 7x + 4 = 0.$$

31. Find the 5^{th} convergent to $\frac{1}{\sqrt{(45)}}$.

32. Find the value of $1 + \frac{1}{2+}\frac{1}{2+}\dots\dots$

33. Find the value of $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\dots\dots$

34. Find the value of $1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\dots\dots$

35. Find the value of $\frac{1}{3+}\frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{2+}\frac{1}{1+}\dots\dots$

36. Find the value of $2 + \frac{1}{1+}\frac{1}{3+}\frac{1}{5+}\frac{1}{1+}\frac{1}{5+}\frac{1}{1+}\dots\dots$

XLVI. INDETERMINATE EQUATIONS OF THE FIRST DEGREE.

627. When only one equation is given involving more than one variable, we can generally solve the equation in an infinite number of ways; for example, if $ax + by = c$, we may ascribe any value we please to x, and then determine the corresponding value of y.

Similarly, if there be any number of equations involving more than the same number of variables, there will be an infinite number of systems of solutions. Such equations are called indeterminate equations.

628. In some cases, however, the nature of the problem may be such, that we only want those solutions in which the variables have *positive integral* values. In this case the number of solutions *may* be limited, as we shall see. We shall proceed then to some propositions respecting the solution of indeterminate equations in *positive integers*. The coefficients and constant terms in these equations will be assumed to be integers.

Before we give the general theory we will shew by an example how such equations are often solved in practice.

Required to find corresponding integral values of x and y in the equation $5x + 8y = 37$.

Divide the given equation by 5, the least coefficient: thus $x + y + \frac{3y}{5} = 7 + \frac{2}{5}$, or $x + y - 7 = \frac{2-3y}{5}$. As x and y are to be integers $\frac{2-3y}{5}$ must be an integer; denote it by p so that $2-3y=5p$. Divide by 3: thus $\frac{2}{3} - y = p + \frac{2p}{3}$, or $p + y = \frac{2-2p}{3}$. Hence $\frac{2-2p}{3}$ must be an integer; denote it by q, so that $2 - 2p = 3q$. Divide by 2: thus $1 - p = q + \frac{q}{2}$. Hence $\frac{q}{2}$ must be an integer; denote it by s, so that $q = 2s$. Then $1 - p = 2s + s$, so that $p = 1 - 3s$. Then $2 - 3y = 5p = 5 - 15s$, so that $y = 5s - 1$. Then $5x = 37 - 8y$ $= 45 - 40s$, so that $x = 9 - 8s$.

We have then $y = 5s - 1$ and $x = 9 - 8s$; and if we ascribe *any* integral value to s we shall obtain corresponding integral values of x and y: but the only *positive integral* values of x and y are obtained by putting $s = 1$; then $y = 4$, and $x = 1$.

629. *Neither of the equations* $ax + by = c$, $ax - by = c$ *can be solved in integers if* a *and* b *have a divisor which does not divide* c.

For, if possible, suppose that either of the equations has such a solution; then divide both sides of the equation by the common divisor; thus the left-hand member is integral and the right-hand member fractional, which is impossible.

If a, b, c have any common divisor, it may be removed by division, so that we shall in future suppose that a and b have no common divisor.

630. *Given one solution of* $ax - by = c$ *in positive integers, to find the general solution.*

Suppose $x = \alpha$, $y = \beta$ is *one* solution of $ax - by = c$, so that $a\alpha - b\beta = c$. By subtraction

$$a(x - \alpha) - b(y - \beta) = 0; \text{ therefore } \frac{a}{b} = \frac{y - \beta}{x - \alpha}.$$

Since $\frac{a}{b}$ is in its lowest terms, and x and y are to have integral values, we must have (as will be shewn in the Chapter on the Theory of Numbers),

$$x - \alpha = bt, \qquad y - \beta = at,$$

where t is an integer; therefore

$$x = \alpha + bt, \qquad y = \beta + at.$$

Hence if one solution is known, we may by ascribing to t different positive integral values, obtain as many solutions as we please. We may also give to t such negative integral values as make bt and at numerically less than α and β respectively.

We shall now shew that one solution can always be found.

631. *A solution of the equation* $ax - by = c$ *in positive integers can always be found.*

Let $\frac{a}{b}$ be converted into a continued fraction, and the successive convergents formed; let $\frac{p}{q}$ be the convergent immediately preceding $\frac{a}{b}$; then $aq - bp = \pm 1$.

First suppose $aq - bp = 1$, therefore $aqc - bpc = c$. Hence $x = qc$, $y = pc$ is a solution of $ax - by = c$.

Next suppose $aq - bp = -1$, then $a(b-q) - b(a-p) = 1$; therefore $a(b-q)c - b(a-p)c = c$. Hence $x = (b-q)c$, $y = (a-p)c$ is a solution of $ax - by = c$.

If $a = 1$, the preceding method is inapplicable; in this case the equation becomes $x - by = c$; we can obtain solutions obviously by giving to y any positive integral value, and then making $x = c + by$. Similarly if $b = 1$.

632. *Given one solution of the equation* ax + by = c *in positive integers, to find the general solution.*

Suppose that $x = \alpha$, $y = \beta$ is one solution of $ax + by = c$, so that $a\alpha + b\beta = c$. By subtraction,

$$a(x - \alpha) + b(y - \beta) = 0\,; \text{ therefore } \frac{a}{b} = \frac{\beta - y}{x - \alpha}.$$

Since $\frac{a}{b}$ is in its lowest terms and x and y are to have integral values, we must have

$$x - \alpha = bt, \qquad \beta - y = at,$$

where t is an integer; therefore

$$x = \alpha + bt, \qquad y = \beta - at.$$

633. It may happen that there is no such solution of the equation $ax + by = c$. For example, if c is less than $a + b$, it is impossible that $c = ax + by$ for positive integral values of x and y, excluding *zero* values.

By the following method we can find a solution when one exists. Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding $\frac{a}{b}$; then $aq - bp = \pm 1$.

First suppose $aq - bp = 1$, then $aqc - bpc = c$; combine this with $ax + by = c$; therefore $a(qc - x) - b(pc + y) = 0$; therefore $qc - x = bt$, $pc + y = at$, where t is some integer. Hence

$$x = qc - bt, \qquad y = at - pc.$$

Solutions will be found by giving to t, if possible, positive integral values greater than $\frac{pc}{a}$ and less than $\frac{qc}{b}$.

Next suppose $aq - bp = -1$, then $aqc - bpc = -c$; combine this with $ax + by = c$, therefore $a(x + qc) - b(pc - y) = 0$. Hence

$$x = bt - qc, \qquad y = pc - at.$$

Solutions will be found by giving to t, if possible, positive integral values greater than $\frac{qc}{b}$ and less than $\frac{pc}{a}$.

634. *To find the number of solutions in positive integers of the equation* $ax + by = c$.

Let $\frac{a}{b}$ be converted into a continued fraction, and let $\frac{p}{q}$ be the convergent immediately preceding $\frac{a}{b}$; then $aq - bp = \pm 1$.

Suppose $aq - bp = 1$.

Then by the preceding Article,

$$x = qc - bt, \qquad y = at - pc.$$

I. Suppose $\frac{c}{a}$ and $\frac{c}{b}$ not to be integers.

Let $$\frac{pc}{a} = m + f, \qquad \frac{qc}{b} = n + g,$$

where m and n are integers, and f and g are proper fractions.

Then the least admissible value of t is $m + 1$, and the greatest is n; thus the number of solutions is $n - m$, that is, $\frac{qc}{b} - \frac{pc}{a} + f - g$, that is, $\frac{c}{ab} + f - g$. And as this result must be an integer it must be the nearest integer to $\frac{c}{ab}$, superior or inferior according as f or g is the greater.

II. Suppose $\frac{c}{a}$ an integer.

Then $f=0$; thus when $t=m$ the value of y is *zero*. If we *include* this solution the number of solutions is equal to the greatest integer in $\frac{c}{ab}+1$; if we *exclude* this solution the number of solutions is equal to the greatest integer in $\frac{c}{ab}$.

III. Suppose $\frac{c}{b}$ an integer.

Then $g=0$; thus when $t=n$ the value of x is *zero*. If we *include* this solution the number of solutions is equal to the greatest integer in $\frac{c}{ab}+1$; if we *exclude* this solution the number of solutions is equal to the greatest integer in $\frac{c}{ab}$.

IV. Suppose $\frac{c}{a}$ and $\frac{c}{b}$ to be integers.

Then $f=0$, and $g=0$; thus when $t=m$ the value of y is *zero*, and when $t=n$ the value of x is *zero*. If we *include* these solutions the number of solutions is equal to $\frac{c}{ab}+1$; if we *exclude* these solutions the number of solutions is $\frac{c}{ab}-1$.

Thus the number of solutions is determined in every case.

Similar results will be obtained on the supposition that $aq-bp=-1$.

635. To solve the equation $ax+by+cz=d$ in positive integers we may proceed thus: write it in the form $ax+by=d-cz$, then ascribe to z in succession the values 1, 2, 3, ... and determine in each case the values of x and y by the preceding Articles.

636. Suppose we have the simultaneous equations

$$ax+by+cz=d, \qquad a'x+b'y+c'z=d';$$

eliminate one of the variables, z for example, we thus obtain an equation connecting the other two variables, $Ax + By = C$, suppose. Now if A and B contain no common factors except such as are also contained in C, by proceeding as in the previous Articles, we may obtain

$$x = a + Bt, \qquad y = \beta - At.$$

Substitute these values in one of the given equations, we thus obtain an equation connecting t and z, which we may write $A't + B'z = C'$. From this, if A' and B' contain no common factors except such as are also contained in C', we may obtain

$$t = a' + B't', \qquad z = \beta' - A't'.$$

Substitute the value of t in the expressions found for x and y; thus

$$x = a + (a' + B't')B, \qquad y = \beta - (a' + B't')A,$$

or

$$x = a + Ba' + BB't', \qquad y = \beta - a'A - AB't'.$$

Hence we obtain for each of the variables x, y, an expression of the same form as that already obtained for z.

EXAMPLES OF INDETERMINATE EQUATIONS.

Solve the following six equations in positive integers:

1. $8x + 65y = 81$.
2. $17x + 23y = 183$.
3. $19x + 5y = 119$.
4. $7x + 10y = 297$.
5. $3x + 7y = 250$.
6. $13x + 19y = 1170$.

Find the general integral values in each of the following four equations, and the least values of x and y which satisfy each:

7. $7x - 9y = 29$.
8. $9x - 11y = 8$.
9. $19x - 5y = 119$.
10. $17x - 49y + 8 = 0$.

11. Find in how many ways £500 can be paid in guineas and five-pound notes.

12. Find in how many ways £100 can be paid in guineas and crowns.

13. Find in how many ways £100 can be paid in half-guineas and sovereigns.

14. Find in how many ways 19*s.* 6*d.* can be paid in florins and half-crowns.

15. Find in how many ways £22. 3*s.* 6*d.* can be paid with French five-franc pieces, value 4*s.* each, and Turkish dollars, value 3*s.* 6*d.* each.

16. If there were coins of 7 shillings and of 17 shillings, find in how many ways £30 could be paid by means of them.

17. Find the simplest way for a person who has only guineas to pay 10*s.* 6*d.* to another who has only half-crowns.

18. Supposing a sovereign equal to 25 francs, find how a debt of 44 shillings can be most simply paid by giving sovereigns and receiving francs.

19. Divide 200 into two parts, such that if one of them be divided by 6 and the other by 11, the respective remainders may be 5 and 4.

20. Find how many crowns and half-crowns, whose diameters are respectively ·81 and ·666 of an inch, may be placed in a row together, so as to make a yard in length.

21. Find n positive integers in arithmetical progression whose sum shall be n^2: shew that there are two solutions when n is odd.

22. Find the least number which divided by 28 leaves a remainder 21, and divided by 19 leaves a remainder 17.

23. Find the general form of the numbers which divided by 3, 5, 7, have remainders 2, 4, 6, respectively.

24. Find the least number which being divided by 28, 19, and 15, leaves remainders 13, 2, and 7.

25. Solve in positive integers $17x + 23y + 3z = 200$.

26. Find all the positive integral solutions of the simultaneous equations $5x + 4y + z = 272$, $8x + 9y + 3z = 656$.

27. Find in how many ways a person can pay a sum of £15 in half-crowns, shillings, and sixpences, so that the number of shillings and sixpences together shall equal the number of half-crowns.

28. Find in how many different ways the sum of £4. 16*s*. can be paid in guineas, crowns, and shillings, so that the number of coins used shall be exactly 16.

29. Find how £2. 4*s*. can be paid in crowns, half-crowns, and florins, if there be as many crowns used as half-crowns and florins together.

30. The difference between a certain multiple of ten and the sum of its digits is 99 : find it.

31. The same number is represented in the undenary and septenary scales by the same three digits, the order in the scales being reversed and the middle digit being zero : find the number.

32. A number consists of three digits which together make up 20; if 16 be taken from it and the remainder divided by 2 the digits will be reversed: find the number.

33. Find a number of four digits in the denary scale, such that if the first and last digits be interchanged, the result is the same number expressed in the nonary scale. Shew that there is only one solution.

34. A farmer buys oxen, sheep, and ducks. The whole number bought is 100, and the whole sum paid = £100. Supposing the oxen to cost £5, the sheep £1, and the ducks 1*s*. per head; find what number he bought of each. Of how many solutions does the problem admit?

35. Find three proper fractions in Arithmetical Progression whose denominators shall be 6, 9, 18, and whose sum shall be $2\frac{2}{3}$.

36. Three bells commenced tolling simultaneously, and tolled at intervals of 25, 29, 33 seconds respectively. In less than half an hour the first ceased, and the second and third tolled 18 seconds and 21 seconds respectively after the cessation of the first and then ceased; how many times did each bell toll?

37. Two rods each c inches long, and divided into m, n equal parts respectively, where m and n have no common measure greater than unity, are placed in longitudinal contact with their

ends coincident. Prove that no two divisions are at a less distance than $\frac{c}{mn}$ inches, and that two pairs of divisions are at this distance. If $m = 250$ and $n = 243$, find those divisions which are at the least distance.

38. There are three bookshelves each of which will carry 20 books; when books are composed of 3 sets of 5 volumes each, 6 of 4, and 7 of 3, find how they must be distributed, so that no set is divided.

39. Determine the greatest sum of money that can be paid in 10 different ways and no more, in half-crowns and shillings; allowing a zero number of half-crowns or of shillings.

40. Determine the greatest sum of money that can be paid in 10 different ways and no more, in half-crowns and shillings; excluding a zero number of half-crowns or of shillings.

XLVII. INDETERMINATE EQUATIONS OF A DEGREE HIGHER THAN THE FIRST.

637. The solution in positive integers of indeterminate equations of a degree higher than the first is a subject of some complexity and of little practical importance; we shall therefore only give a few miscellaneous propositions.

638. *To solve in positive integers the equation*

$$mxy + nx^2 + px + qy = r.$$

This equation contains only one of the squares of the variables, and it can always be solved in the manner indicated in the following example. Required to solve in positive integers the equation

$$3xy + 2x^2 = 5y + 4x + 5.$$

Here $y(3x-5) = -2x^2 + 4x + 5$; therefore $y = \frac{-2x^2 + 4x + 5}{3x - 5}$;

let $3x = z$; therefore $9y = \frac{-2z^2 + 12z + 45}{z - 5} = -2z + 2 + \frac{55}{z - 5}$;

therefore $$9y = -6x + 2 + \frac{55}{3x - 5}.$$

Since x and y are to have integral values $3x-5$ must be a divisor of 55, and from this condition we can find by trial the values of x, and then deduce those of y. The only cases for examination are the following:

$$3x-5=\pm 55, \qquad 3x-5=\pm 11,$$
$$3x-5=\pm 5, \qquad 3x-5=\pm 1.$$

Out of these cases only the following give a positive integral value to x:

$$3x-5=55, \text{ therefore } x=20;$$
$$3x-5=1, \text{ therefore } x=2.$$

When $x=20$ we do not obtain a positive integral value for y; when $x=2$ we have $y=5$; this is therefore the only solution of the proposed equation in positive integers.

639. The equation $x^2-Ny^2=1$ can always be solved in integers when N is a whole number and not a perfect square. For in the process of converting $\sqrt{N}$ into a continued fraction we arrive at the following equation (see Art. 614),

$$\rho''(pq'-p'q)=q'^2N-p'^2;$$

and at the end of any complete period of quotients $\rho''=1$ (Art. 621); thus

$$pq'-p'q=q'^2N-p'^2.$$

Suppose now that the number of the recurring quotients is *even*, then $\frac{p'}{q'}$ is always an *even* convergent, and is therefore greater than $\sqrt{N}$, and so greater than $\frac{p}{q}$. Hence $p'q-q'p=1$, and we have $-1=q'^2N-p'^2$; so that $p'^2-Nq'^2=1$. Hence we obtain solutions of the proposed equation by putting $x=p'$ and $y=q'$, where $\frac{p'}{q'}$ is any convergent just preceding that formed with the quotient $2a$.

Next suppose that the number of the recurring quotients is *odd*; then when first $\rho''=1$ the convergent $\frac{p'}{q'}$ is an *odd* convergent, when next $\rho''=1$ the convergent $\frac{p'}{q'}$ is an *even* convergent, and so on. Hence solutions can be obtained by restricting ourselves to *even* convergents occurring just before those formed with the quotient $2a$.

640. If the number of recurring quotients obtained from $\sqrt{N}$ be *odd*, then, as appears in the preceding Article, if $\frac{p'}{q'}$ be any *odd* convergent immediately preceding that formed with the quotient $2a$, we have $pq'-p'q=q'^2N-p'^2$, and $pq'-p'q=1$; thus we obtain in this case solutions in integers of the equation $Ny^2-x^2=1$.

641. The equation $x^2-Ny^2=\pm a^2$ by putting $x=ax'$ and $y=ay'$ becomes $x'^2-Ny'^2=\pm 1$, which we have considered in the preceding Articles.

642. The relation $\rho''(pq'-p'q)=q'^2N-p'^2$, that is, $\pm\rho''=q'^2N-p'^2$, will give solutions of the equation $x^2-Ny^2=\pm c$ in some cases in which c is different from unity. The method will be similar to that given in Arts. 639 and 640.

643. If one solution in integers of the equation $x^2-Ny^2=1$ be known, we may obtain an unlimited number of such solutions. For suppose $x=p$ and $y=q$ to be such a solution, so that $p^2-Nq^2=1$; then $(p-q\sqrt{N})(p+q\sqrt{N})=1$; therefore

$$(p-q\sqrt{N})^n(p+q\sqrt{N})^n=1=(x-y\sqrt{N})(x+y\sqrt{N}),$$

by supposition. Put then

$$x-y\sqrt{N}=(p-q\sqrt{N})^n, \quad x+y\sqrt{N}=(p+q\sqrt{N})^n,$$

thus

$$x=\frac{1}{2}\left\{(p+q\sqrt{N})^n+(p-q\sqrt{N})^n\right\},$$

$$y=\frac{1}{2\sqrt{N}}\left\{(p+q\sqrt{N})^n-(p-q\sqrt{N})^n\right\};$$

it is obvious that if n be any positive integer, these values of x and y will be positive integers.

644. Similarly, if one solution in integers of the equation $x^2 - Ny^2 = -1$ be known, we may obtain an unlimited number of such solutions. For suppose $x = p$ and $y = q$ to be such a solution, then $(p - q\sqrt{N})(p + q\sqrt{N}) = -1$. Now take n any *odd* integer; then

$$(p - q\sqrt{N})^n (p + q\sqrt{N})^n = (-1)^n = -1$$

$$= (x - y\sqrt{N})(x + y\sqrt{N}), \text{ by supposition.}$$

Then we proceed as in Art. 643.

645. If one solution in integers of the equation $x^2 - Ny^2 = a$ be known, we may obtain an unlimited number of such solutions. For suppose $x = p$ and $y = q$ to be such a solution, and let $x = m$ and $y = n$ be a solution of $x^2 - Ny^2 = 1$; then the equation $x^2 - Ny^2 = a$ may be written

$$x^2 - Ny^2 = (p^2 - Nq^2)(m^2 - Nn^2)$$

$$= p^2m^2 + N^2q^2n^2 - N(p^2n^2 + q^2m^2) = (pm \pm Nqn)^2 - N(pn \pm qm)^2;$$

we may therefore take $x = pm \pm Nqn$, $y = pn \pm qm$.

EXAMPLES OF INDETERMINATE EQUATIONS.

1. Solve in positive integers $3xy - 4y + 3x = 14$.

2. Solve in positive integers $xy + x^2 = 2x + 3y + 29$.

3. Find a solution in positive integers of $x^2 - 13y^2 = -1$.

4. Find a solution in positive integers of $x^2 - 101y^2 = -1$.

5. Shew how to find series of numbers which shall be at the same time of the two forms $n^2 - 1$ and $10m^2$, and find the value of the smallest.

6. A gentleman being asked the size of his paddock answered, "between one and two roods; also were it smaller by 3 square yards, it would be a square number of square yards, and if my brother's paddock, which is a square number of square yards, were larger by one square yard, it would be exactly half as large as mine." Find the size of his paddock.

7. Find a whole number which is greater than three times the integral part of its square root by unity: shew that there are two solutions of the problem and no more.

8. Shew that the number of solutions in positive integers of $y^2 + ax^2 = b$ is limited when a is positive.

9. Find all the solutions in positive integers of

$$3y^2 - 2xy + 7x^2 = 27.$$

10. Find all the solutions in positive integers of

$$2x^2 - 9xy + 7y^2 = 38.$$

11. Find a general form for solutions in positive integers of $x^2 - 23y^2 = 1$, having given the solution $x = 24$ and $y = 5$.

12. Find a general form for solutions in positive integers of $x^2 - 2y^2 = 7$, having given the solution $x = 3$ and $y = 1$.

XLVIII. PARTIAL FRACTIONS AND INDETERMINATE COEFFICIENTS.

646. An algebraical fraction may be sometimes decomposed into the sum of two or more simpler fractions; for example,

$$\frac{2x-3}{x^2-3x+2} = \frac{1}{x-1} + \frac{1}{x-2}.$$

The general theory of the decomposition of a fraction into simpler fractions, called *partial fractions*, is given in treatises on the Theory of Equations and on the Integral Calculus. (See *Theory of Equations*, Chap. XXIV., *Integral Calculus*, Chap. II.) We shall here only consider a simple case.

647. Let $\dfrac{ax^2+bx+c}{(x-\alpha)(x-\beta)(x-\gamma)}$ be a fraction, the denominator of which is composed of three different factors of the first degree with respect to x, and the numerator is of a degree not higher than the second with respect to x; this fraction can be decomposed into three simple fractions, which have for their denominators respectively the factors of the denominator of the proposed

fraction, and for their numerators certain quantities independent of x. To prove this, assume

$$\frac{ax^2+bx+c}{(x-a)(x-\beta)(x-\gamma)}=\frac{A}{x-a}+\frac{B}{x-\beta}+\frac{C}{x-\gamma},$$

where A, B, C are at present undetermined; we have then to shew that such constant values can be found for A, B and C, as will make the above equation an *identity*, that is, true whatever may be the value of x. Multiply by $(x-a)(x-\beta)(x-\gamma)$; then all that we require is that the following shall be an *identity*,

$$ax^2+bx+c=A(x-\beta)(x-\gamma)+B(x-a)(x-\gamma)+C(x-a)(x-\beta);$$

this will be secured if we arrange the terms on the right hand according to powers of x, and equate the coefficient of each power to the corresponding coefficient on the left hand; we shall thus obtain three simple equations for determining A, B and C.

648. The method of the preceding Article may be applied to any fraction, the denominator of which is the product of *different* simple factors, and the numerator of lower dimensions than the denominator.

The preceding Article however is not quite satisfactory, because we do not shew that the final equations which we obtain are *independent* and *consistent*. But as we shall only have to apply the method to simple examples, where the results may be easily verified, we shall not devote any more space to the subject, but refer the student to the *Theory of Equations* and the *Integral Calculus*.

649. Suppose we have to develop $\frac{2x-3}{x^2-3x+2}$ in a series proceeding according to ascending powers of x; there are various methods which may be adopted. We may proceed by ordinary algebraical division, writing the divisor in the order $2-3x+x^2$ and the dividend in the order $-3+2x$. Or we may develop $\frac{1}{x^2-3x+2}$ by writing it in the form $(x^2-3x+2)^{-1}$, and finding the coefficients of the successive powers of x by the multinomial

theorem; we must then multiply the result by $2x-3$. It is however more convenient to decompose the fraction into partial fractions and then to develop each of these. Thus

$$\frac{2x-3}{x^2-3x+2}=\frac{1}{x-1}+\frac{1}{x-2}=-\frac{1}{1-x}-\frac{1}{2-x};$$

$$-\frac{1}{1-x}=-(1-x)^{-1}=-\left\{1+x+x^2+x^3+\ldots+x^n+\ldots\right\},$$

$$-\frac{1}{2-x}=-\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}=-\frac{1}{2}\left\{1+\frac{x}{2}+\frac{x^2}{2^2}+\frac{x^3}{2^3}+\ldots\ldots+\frac{x^n}{2^n}+\ldots\ldots\right\}.$$

Hence the required series for $\frac{2x-3}{x^2-3x+2}$ has for its general term $-\left(1+\frac{1}{2^{n+1}}\right)x^n$.

650. Without actually developing such an expression as the above, we may shew that the successive coefficients will be connected by a certain relation; before we can shew this it will be necessary to establish a general property of series.

651. If the series $a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots\ldots$ is always equal to zero whatever may be the value of x, the coefficients $a_0, a_1, a_2, a_3, \ldots\ldots$ must each separately be equal to zero. For since the series is to be zero *whatever may be the value of* x, we may put $x=0$; thus the series reduces to a_0, which must therefore itself be zero. Hence removing this term we have $a_1x+a_2x^2+a_3x^3+\ldots$ always zero; divide by x, then $a_1+a_2x+a_3x^2+\ldots$ is always zero. Hence, as before, we infer that $a_1=0$. Proceeding in this way, the theorem is established.

If the series $\quad a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$

and $\quad A_0+A_1x+A_2x^2+A_3x^3+\ldots\ldots$

are always equal whatever may be the value of x, then

$$a_0-A_0+(a_1-A_1)x+(a_2-A_2)x^2+\ldots\ldots$$

is always zero whatever may be the value of x; hence we infer that

$$a_0-A_0=0, \quad a_1-A_1=0, \quad a_2-A_2=0, \ldots\ldots;$$

that is, the coefficients of like powers of x in the two series are equal.

The theorem here given is sometimes quoted as the *Principle of Indeterminate Coefficients;* we assumed its truth in Arts. 526, 542, and 549.

652. The demonstration of the preceding Article is that which has been usually given in elementary works on Algebra; there is however a difficulty in it which requires examination.

We confine ourselves to the theorem that if the series $a_0 + a_1x + a_2x^2 + \dots$ is always equal to zero, each coefficient must be equal to zero; the theorem in the latter part of the Article follows from this.

When we say that the series is *always* equal to zero we mean that it is equal to zero for all such values of x as make the series *convergent;* for of course a divergent series cannot be said to vanish.

In the demonstration we shew that $a_1x + a_2x^2 + a_3x^3 + \dots$ is always zero; that is xS_1 is always zero, where S_1 stands for $a_1 + a_2x + a_3x^2 + \dots$. Hence if x is not zero S_1 must be zero; but if x is zero xS_1 vanishes whatever finite value S_1 may have: thus in fact we ought not to assume that S_1 is zero when x is zero, and so the result $a_1 = 0$ is not strictly demonstrated. This is the difficulty we have to examine.

We have $S_1 = a_1 + xS_2$ where S_2 stands for $a_2 + a_3x + a_4x^2 + \dots$; and although we are not justified in saying that S_1 is zero when x is zero, yet we may say that S_1 is zero however small x may be. Since the original series is supposed to be convergent S_2 is also a convergent series, and therefore it will not increase beyond some fixed value when x is made small enough; and therefore by making x small enough xS_2 may be made as small as we please: hence a_1 must be zero, for if a_1 were not zero we could not have S_1 zero however small x might be.

Thus the result $a_1 = 0$ follows strictly if S_2 is convergent when x is made as small as we please. In like manner the result $a_2 = 0$ follows strictly if S_3 is convergent when x is made as small as we please, where S_3 stands for $a_3 + a_4x + a_5x^2 + \dots$ And so on.

Since the original series is supposed to be convergent the series S_2, S_3, ... are convergent, when x is made as small as we please; and so the theorem of the preceding Article holds.

653. Suppose that the series $u_0 + u_1x + u_2x^2 + u_3x^3 + \dots\dots$ represents the development of $\dfrac{a+bx}{1-px-qx^2}$; then

$$a + bx = (1 - px - qx^2)(u_0 + u_1x + u_2x^2 + u_3x^3 + \dots\dots).$$

If n be greater than 1, the coefficient of x^n on the right-hand side is $u_n - pu_{n-1} - qu_{n-2}$; hence since there is no power of x higher than the first on the left-hand side, we must have by Art. 651, for every value of n greater than 1,

$$u_n - pu_{n-1} - qu_{n-2} = 0.$$

And by comparing the first and second terms on each side, we have

$$u_0 = a, \qquad u_1 - pu_0 = b;$$

the last two equations determine u_0 and u_1, and then the previous equation will determine u_2, u_3, u_4, ... by making successively $n = 2, 3, 4, \dots\dots$

EXAMPLES OF PARTIAL FRACTIONS AND INDETERMINATE COEFFICIENTS.

Expand each of the following seven expressions in ascending powers of x, and give the general term:

1. $\dfrac{1}{3-2x}$. 2. $\dfrac{5-10x}{2-x-3x^2}$. 3. $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$.

4. $\dfrac{x}{(1-x)(1-px)}$. 5. $\dfrac{1}{1-2x+x^2}$. 6. $\dfrac{5+6x}{(1-3x)^2}$.

7. $\dfrac{1+4x+x^2}{(1-x)^4}$.

Expand each of the following five expressions in ascending powers of x as far as five terms, and write down the relation which connects the coefficients of consecutive terms:

8. $\dfrac{1}{1-x+x^2}$. 9. $\dfrac{1}{1-2x+3x^2}$. 10. $\dfrac{1-x^3}{2-2x-x^2}$.

11. $\dfrac{1}{a^2+ax+x^2}$. 12. $\dfrac{1}{1-px+px^3-x^4}$.

13. Sum the following series to n terms by separating each term into partial fractions:

$$\frac{x}{(1+x)(1+ax)}+\frac{ax}{(1+ax)(1+a^2x)}+\frac{a^2x}{(1+a^2x)(1+a^3x)}+\dots\dots$$

14. Sum in a similar manner the following series to n terms:

$$\frac{x(1-ax)}{(1+x)(1+ax)(1+a^2x)}+\frac{ax(1-a^2x)}{(1+ax)(1+a^2x)(1+a^3x)}+\dots\dots$$

15. Determine $a, b, c, d, e,$ so that the n^{th} term in the expansion of $\dfrac{a+bx+cx^2+dx^3+ex^4}{(1-x)^5}$ may be n^4x^{n-1}.

16. Shew how to decompose $\dfrac{x^p}{(x-a)(x-b)(x-c)\dots}$ into partial fractions, supposing that n is the number of factors in the denominator, and that p is an integer less than n.

If p be less than n, shew that

$$\frac{a^{p-1}}{(a-b)(a-c)\dots}+\frac{b^{p-1}}{(b-a)(b-c)\dots}+\frac{c^{p-1}}{(c-a)(c-b)\dots}+\dots=0.$$

XLIX. RECURRING SERIES.

654. A series is called a *recurring series*, when from and after some fixed term each term is equal to the sum of a fixed number of the preceding terms multiplied respectively by certain *constants*. By *constants* here we mean quantities which remain unchanged whatever term of the series we consider.

655. A geometrical progression is a simple example of a recurring series; for in the series $a+ar+ar^2+ar^3+\dots\dots$ each

term after the first is r times the preceding term. If u_{n-1} and u_n denote respectively the $(n-1)^{\text{th}}$ term and the n^{th} term, then $u_n - ru_{n-1} = 0$; the sum of the coefficients of u_n and u_{n-1} with their proper signs, that is, $1-r$, is called the *scale of relation.*

Again, in the series $2 + 4x + 14x^2 + 46x^3 + 152x^4 + \ldots\ldots\ldots$ the law connecting consecutive terms is $u_n - 3xu_{n-1} - x^2u_{n-2} = 0$; this law holds for values of n greater than 1, so that every term after the second can be obtained from the two terms immediately preceding. The *scale of relation* is $1 - 3x - x^2$.

656. *To find the sum of* n *terms of a recurring series.*

Let the series be $u_0 + u_1x + u_2x^2 + u_3x^3 + \ldots\ldots$, and let the scale of relation be $1 - px - qx^2$, so that for every value of n greater than unity $u_n - pu_{n-1} - qu_{n-2} = 0$. Denote the first n terms of the series by S, then

$$S = u_0 + u_1x + u_2x^2 + u_3x^3 + \ldots\ldots + u_{n-1}x^{n-1},$$

$$pxS = u_0px + u_1px^2 + u_2px^3 + \ldots\ldots + u_{n-2}px^{n-1} + u_{n-1}px^n,$$

$$qx^2S = \qquad u_0qx^2 + u_1qx^3 + \ldots\ldots + u_{n-3}qx^{n-1} + u_{n-2}qx^n + u_{n-1}qx^{n+1};$$

hence

$$S - pxS - qx^2S = u_0 + u_1x - u_0px - u_{n-1}px^n - u_{n-2}qx^n - u_{n-1}qx^{n+1},$$

for all the other terms on the right-hand side disappear by virtue of the relation which holds between any three consecutive terms of the given series; therefore

$$S = \frac{u_0 + x(u_1 - pu_0) - x^n\{pu_{n-1} + qu_{n-2} + qxu_{n-1}\}}{1 - px - qx^2}.$$

If the term $x^n\{pu_{n-1} + qu_{n-2} + qxu_{n-1}\}$ decreases without limit as n increases without limit, we may say that the sum of an infinite number of terms of the recurring series is

$$\frac{u_0 + x(u_1 - pu_0)}{1 - px - qx^2}.$$

It is obvious, that if this expression be developed in a series according to powers of x, we shall recover the given recurring series. (See Art. 653.)

657. If the recurring series be $u_0 + u_1 + u_2 + u_3 + \ldots\ldots$, and the scale of relation $1 - p - q$, we have only to make $x = 1$ in the results of the preceding Article, in order to find the sum of n terms, or of an infinite number of terms.

658. When $1 - px - qx^2$ can be resolved into two *real* factors of the first degree in x, the expression $\dfrac{u_0 + x(u_1 - pu_0)}{1 - px - qx^2}$ may be decomposed into partial fractions, each having for its denominator an expression containing only the first power of x: see Arts. 337 and 647. In this case, since each partial fraction can be developed into a geometrical progression, we can obtain an expression for the general term of the recurring series. We have thus also another method of obtaining the sum of n terms, since the sum of n terms of each of the geometrical progressions is known.

EXAMPLES OF RECURRING SERIES.

Find the expressions from which the following three series are derivable; resolve the expressions into partial fractions, and give the general term of each series:

1. $4 + 9x + 21x^2 + 51x^3 + \ldots\ldots$

2. $1 + 11x + 89x^2 + 659x^3 + \ldots\ldots$

3. $1 + 3x + 11x^2 + 43x^3 + \ldots\ldots$

4. Find how small x must be in order that the series in Example 3 may be convergent.

5. Find the general term of the series $3 + 11 + 32 + 84 + \ldots\ldots$

6. Sum the following series to n terms

$$1 + 5 + 17 + 53 + 161 + 485 + \ldots\ldots$$

7. Find the general term of the series $10 + 14 + 10 + 6 + \ldots$ and the sum to infinity.

8. Find the expression from which the following series is derivable, and obtain the general term

$$2 - x + 2x^2 - 5x^3 + 10x^4 - 17x^5 + \ldots\ldots$$

L. SUMMATION OF SERIES.

659. Series of particular kinds have been summed in the Chapters on Arithmetical Progression, Geometrical Progression, and Recurring Series; we shall here give some miscellaneous examples which do not fall under the preceding Chapters.

660. To find the sum of the series $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2$.

We have already found this sum in Arts. 460, 482; the following method is however usually given. Assume

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = A + Bn + Cn^2 + Dn^3 + En^4 + \ldots\ldots,$$

where A, B, C, D, E, ... are constants at present undetermined. Change n into $n+1$; thus

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 + (n+1)^2 = A + B(n+1) \\ + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots\ldots$$

By subtraction,

$$n^2 + 2n + 1 = B + C(2n+1) + D(3n^2 + 3n + 1) \\ + E(4n^3 + 6n^2 + 4n + 1) + \ldots\ldots$$

Equate the coefficients of the respective powers of n; thus $E = 0$, and so any other term after E would $= 0$;

$$3D = 1; \quad 3D + 2C = 2; \quad D + C + B = 1;$$

hence
$$D = \frac{1}{3}, \quad C = \frac{1}{2}, \quad B = \frac{1}{6}.$$

Thus
$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = A + \frac{n}{6} + \frac{n^2}{2} + \frac{n^3}{3}.$$

To determine A we observe that since this equation is to hold for all positive integral values of n, we may put $n = 1$; thus $A = 0$. Hence the required sum is

$$\frac{1}{6} n(n+1)(2n+1).$$

The same method may be applied to find the sum of the cubes of the first n natural numbers, or the sum of their fourth powers, and so on. See also Art. 671.

661. Suppose the n^{th} term of a series to be

$$\{an+b\}\{a(n+1)+b\}\{a(n+2)+b\}\ldots\ldots\{a(n+m-1)+b\},$$

where m is a fixed positive integer, and a and b known constants; then the sum of the first n terms of this series will be

$$\frac{\{an+b\}\{a(n+1)+b\}\ldots\ldots\{a(n+m-1)+b\}\{a(n+m)+b\}}{(m+1)a}+C,$$

where C is some constant.

Let u_n denote the n^{th} term of the proposed series, S_n the sum of n terms; then we have to prove that

$$S_n=\frac{an+b}{(m+1)a}u_{n+1}+C.$$

Assume that the formula is true for an assigned value of n; add the $(n+1)^{\text{th}}$ term of the series to both sides; then

$$S_n+u_{n+1}=\frac{an+b}{(m+1)a}u_{n+1}+u_{n+1}+C;$$

that is, $$S_{n+1}=\frac{a(n+m+1)+b}{(m+1)a}u_{n+1}+C=\frac{a(n+1)+b}{(m+1)a}u_{n+2}+C;$$

thus the same formula will hold for the sum of $n+1$ terms, which was assumed to hold for the sum of n terms. Hence if the formula be true for any number of terms it is true for the next greater number; and so on. But the formula *will* be true when $n=1$ if we take C such that

$$S_1=\frac{a+b}{(m+1)a}u_2+C, \text{ that is, } u_1=\frac{a+b}{(m+1)a}u_2+C;$$

thus C is determined and the truth of the theorem established.

Since $$u_2=\frac{a(m+1)+b}{a+b}u_1, \text{ we have}$$

$$C=u_1-\frac{a(m+1)+b}{a(m+1)}u_1=-\frac{bu_1}{a(m+1)}.$$

Hence $$S_n=\frac{an+b}{(m+1)a}u_{n+1}-\frac{bu_1}{(m+1)a}.$$

Thus the sum of the first n terms of the proposed series is obtained by subtracting the constant quantity $\frac{bu_1}{(m+1)a}$ from a certain expression which depends on n. This expression is $\frac{an+b}{(m+1)a}u_{n+1}$; we may also put this expression into the equivalent form $\frac{a(n+m)+b}{(m+1)a}u_n$, and to assist the memory we may observe that it can be formed by *introducing an additional factor at the end of u_n, and dividing by the product of the number of factors thus increased and the coefficient of n.*

662. We may obtain the result of the preceding Article in another way. As before, let u_n denote

$$\{an+b\}\{a(n+1)+b\}\{a(n+2)+b\}\ldots\ldots\{a(n+m-1)+b\},$$

and let S_n denote the sum of the first n terms of the series of which u_n is the n^{th} term.

We have

$$u_{n+1} = \frac{a(n+m)+b}{an+b}u_n = u_n + \frac{amu_n}{an+b};$$

let $an+b=p$; thus

$$p(u_{n+1}-u_n) = amu_n;$$

change n into $n-1$, thus

$$\{p-a\}(u_n-u_{n-1}) = amu_{n-1};$$

similarly,

$$\{p-2a\}(u_{n-1}-u_{n-2}) = amu_{n-2},$$

$$\{p-3a\}(u_{n-2}-u_{n-3}) = amu_{n-3},$$

$$\cdots\cdots\cdots$$

$$\{p-(n-1)a\}(u_2-u_1) = amu_1.$$

Hence, by addition,

$$p(u_{n+1}-u_1) - a\{u_n+u_{n-1}+u_{n-2}+\ldots+u_2-(n-1)u_1\} = amS_n;$$

therefore $$p(u_{n+1}-u_1)+nau_1 = amS_n + aS_n;$$

therefore $$S_n = \frac{an+b}{(m+1)a}u_{n+1} - \frac{bu_1}{(m+1)a}.$$

663. Suppose the n^{th} term of a series to be $\frac{1}{u_n}$, where u_n is the same as in the preceding Article; then the sum of the first n terms of this series will be $-\frac{an+b}{(m-1)au_n}+C$.

Assume, as before, $S_n = -\frac{an+b}{(m-1)\,au_n}+C$,

add $\frac{1}{u_{n+1}}$ to both sides, then

$$S_{n+1} = \frac{1}{u_{n+1}} - \frac{an+b}{(m-1)\,au_n} + C$$

$$= \frac{1}{u_{n+1}} - \frac{a(m+n)+b}{(m-1)\,au_{n+1}} + C = -\frac{a(n+1)+b}{(m-1)\,au_{n+1}} + C.$$

Hence, as before, the truth of the theorem is established, provided C be such that $\frac{1}{u_1} = -\frac{a+b}{(m-1)\,au_1} + C$. Thus $C = \frac{am+b}{(m-1)\,au_1}$,

and $S_n = \frac{am+b}{(m-1)\,au_1} - \frac{an+b}{(m-1)\,au_n}$.

This result may also be obtained in the manner of Art. 662.

664. A series may occur which is not directly included in the general form of the preceding Article, but may be decomposed into two or more which are. For example, required the sum of n terms of the series

$$\frac{3}{1.2.4.5} + \frac{4}{2.3.5.6} + \frac{5}{3.4.6.7} + \ldots\ldots$$

Here the n^{th} term

$$= \frac{n+2}{n(n+1)(n+3)(n+4)} = \frac{(n+2)^2}{n(n+1)(n+2)(n+3)(n+4)}.$$

Now $(n+2)^2 = n(n+1) + 3n + 4$; thus the n^{th} term

$$= \frac{n(n+1)+3n+4}{n(n+1)(n+2)(n+3)(n+4)} = \frac{1}{(n+2)(n+3)(n+4)}$$

$$+ \frac{3}{(n+1)(n+2)(n+3)(n+4)} \quad \frac{4}{n(n+1)(n+2)(n+3)(n+4)}.$$

If each term of the proposed series be decomposed in this manner we obtain three series, each of which may be summed by the method of the preceding Article; thus the proposed series can be summed. In the present case the required sum is

$$\frac{1}{24} - \frac{1}{2(n+3)(n+4)} + \frac{1}{24} - \frac{3}{3(n+2)(n+3)(n+4)}$$
$$+ \frac{1}{24} - \frac{4}{4(n+1)(n+2)(n+3)(n+4)}.$$

665. *Polygonal Numbers.* The expression $n + \frac{1}{2}n(n-1)b$ is the sum of n terms of an arithmetical progression, of which the first term is unity and the common difference is b. If we make $b = 0, 1, 2, 3, \ldots$ we obtain expressions which are called the general terms of the 2nd, 3rd, 4th, ... order of *polygonal numbers* respectively. The *first* order is that in which every term is unity. Thus we have

1st order, n^{th} term 1; series 1, 1, 1, ...
2nd order, n^{th} term n; series 1, 2, 3, 4, 5, ...
3rd order, n^{th} term $\frac{1}{2}n(n+1)$; series 1, 3, 6, 10, ...
4th order, n^{th} term n^2; series 1, 4, 9, 16, ...
5th order, n^{th} term $\frac{1}{2}n(3n-1)$; series 1, 5, 12, 22, ...

and so on.

The numbers in the 2nd, 3rd, 4th, 5th, ... series have been called respectively *linear, triangular, square, pentagonal,* ......

666. The n^{th} term of the r^{th} order of polygonal numbers is

$$n + \tfrac{1}{2}n(n-1)(r-2);$$

the sum of n terms of this series is, by Art. 661,

$$\frac{n(n+1)}{2} + \frac{r-2}{2} \cdot \frac{(n-1)n(n+1)}{3},$$

or $$\tfrac{1}{6}n(n+1)\{(r-2)(n-1)+3\}.$$

Hence for triangular numbers $S_n = \frac{1}{6}n(n+1)(n+2)$, for square numbers $S_n = \frac{1}{6}n(n+1)(2n+1)$, and so on.

667. *To find the number of cannon-balls in a pyramidal pile.*

(1) Suppose the base of the pyramid an equilateral triangle, let there be n balls in a side of the base; then the number of

balls in the lowest layer is $n + (n-1) + (n-2) + \dots\dots + 1$, that is, the *triangular* number $\frac{1}{2}n(n+1)$; the number in the next layer will be found by changing n into $n-1$; and so on. Hence, by Art. 665, the number of all the balls is $\frac{1}{6}n(n+1)(n+2)$.

(2) Suppose the base of the pyramid a square; let there be n balls in a side of the base; then the number of balls in the lowest layer is n^2, in the next layer $(n-1)^2$, and so on. The number of all the balls is $\frac{1}{6}n(n+1)(2n+1)$.

Similarly we may proceed for any other form of pyramid.

We may see from this proposition a reason for the terms *triangular number*, *square number*, ...

If the pile of cannon-balls be *incomplete*, we must first find the number in the pile supposed complete, then the number in the lesser pile which is deficient, and the difference will be the number in the incomplete pile.

668. A question analogous to that in Art. 667 arises when we have to sum the balls in a pile of which the base is *rectangular* but not square. In this case the pile will terminate in a single row at the top; suppose p the number of balls in this row; then the n^{th} layer reckoned from the top has $p+n-1$ balls in its length and n in its breadth, and therefore contains $n(p+n-1)$ balls. Hence the number of balls in n layers is

$$\frac{n(n+1)}{2}p + \frac{(n-1)n(n+1)}{3}, \text{ or } \frac{1}{6}n(n+1)(3p+2n-2).$$

If m be the number in the length of the lowest row, $m = p+n-1$, and the sum may be written $\frac{1}{6}n(n+1)(3m-n+1)$; as n is the number in the breadth of the lowest row, the sum is thus expressed in terms of the numbers in the length and breadth of the base.

669. *Figurate Numbers.* The following series form what are called the different orders of *figurate numbers:*

1st order, 1, 1, 1, 1, 1, ...
2nd order, 1, 2, 3, 4, 5, ...
3rd order, 1, 3, 6, 10, 15, ...
....

the general law is, that the n^{th} term of any order is the sum of n terms of the preceding order. Thus the n^{th} term of the second order is n, of the 3rd order is $\frac{n(n+1)}{1.2}$, of the fourth order is $\frac{n(n+1)(n+2)}{1.2.3}$, and generally the n^{th} term of the r^{th} order is $\frac{n(n+1)\ldots(n+r-2)}{\lfloor r-1}$. This we may prove by induction. For, assuming this expression for the n^{th} term of the r^{th} order, we may find the sum of the first n terms of the r^{th} order by the formula of Art. 661. We have only to put 1 for a, 0 for b, and $r-1$ for m. Hence we obtain for the sum

$$\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{\lfloor r};$$

and then, by definition, this is the expression for the n^{th} term of the $(r+1)^{th}$ order.

670. We have already shewn that the Binomial Theorem may be sometimes applied to find the sum of a series (see Art. 526); we give another example. Find the sum of the series

$$P_1Q_1 + P_2Q_2 + P_3Q_3 + \ldots\ldots + P_{n-1}Q_{n-1},$$

where $\quad Q_r = r(r+1)(r+2)\ldots\ldots(r+q-1)$,

and $\quad P_r = (n-r)(n-r+1)(n-r+2)\ldots\ldots(n-r+p-1)$.

We can see that

$Q_r = \lfloor q \times$ the coefficient of x^{r-1} in the series for $(1-x)^{-(q+1)}$,

and $P_r = \lfloor p \times$ the coefficient of x^{n-r-1} in the series for $(1-x)^{-(p+1)}$.

Hence we have so far as terms not higher than x^{n-2},

$$(1-x)^{-(q+1)} = \frac{1}{\lfloor q}\left\{Q_1 + Q_2x + Q_3x^2 + Q_4x^3 + \ldots\ldots\right\},$$

$$(1-x)^{-(p+1)} = \frac{1}{\lfloor p}\left\{P_{n-1} + P_{n-2}x + P_{n-3}x^2 + P_{n-4}x^3 + \ldots\ldots\right\}.$$

Therefore the series which we have to sum is equal to the product of $\lfloor p \lfloor q$ into the coefficient of x^{n-2} in the expansion of the product of $(1-x)^{-(q+1)}$ and $(1-x)^{-(p+1)}$; that is, the series is equal to the product of $\lfloor p \, \lfloor q$ into the coefficient of x^{n-2} in the expansion of $(1-x)^{-(p+q+2)}$. Hence the series is equal to

$$\frac{\lfloor p \, \lfloor q}{\lfloor p+q+1} \times \frac{\lfloor n-1+p+q}{\lfloor n-2}.$$

671. By the method of Art. 660 we may investigate an expression for the sum $1^r + 2^r + 3^r + \ldots\ldots + n^r$, where r is any positive integer. Denote this sum by S; then it may be shewn, as in Arts. 460 and 461, that S can be put in the form of a series of descending powers of n, beginning with n^{r+1}, and all we have to do is to determine correctly the coefficients of the various powers of n. Assume that $S=$

$$Cn^{r+1} + A_0 n^r + \frac{r}{2} A_1 n^{r-1} + \frac{r(r-1)}{2 \cdot 3} A_2 n^{r-2} + \frac{r(r-1)(r-2)}{2 \cdot 3 \cdot 4} A_3 n^{r-3} + \ldots\ldots$$

It is convenient to represent the coefficients in the manner here exhibited; thus instead of a single letter for the coefficient of n^{r-1} we use the symbol $\frac{r}{2} A_1$, and so on. We shall now proceed to determine the values of $A_0, A_1, A_2, \ldots\ldots$; and it will be found that these quantities are independent of r as well as of n.

In the assumed identity change n into $n+1$; thus

$$S + (n+1)^r = C(n+1)^{r+1} + A_0 (n+1)^r + \frac{r}{2} A_1 (n+1)^{r-1} + \frac{r(r-1)}{2 \cdot 3} A_2 (n+1)^{r-2} + \ldots\ldots$$

Therefore, by subtraction,

$$(n+1)^r = C\{(n+1)^{r+1} - n^{r+1}\} + A_0\{(n+1)^r - n^r\} + \frac{r}{2} A_1 \{(n+1)^{r-1} - n^{r-1}\} + \frac{r(r-1)}{2 \cdot 3} A_2 \{(n+1)^{r-2} - n^{r-2}\} + \ldots\ldots$$

Expand all the expressions $(n+1)^{r+1}$, $(n+1)^r$, $(n+1)^{r-1}$, ... by the Binomial Theorem; and then equate the coefficients of the various powers of n. Thus, by equating the coefficients of n^r, we have $1 = C(r+1)$, then, by equating the coefficients of n^{r-1}, we have $r = \frac{C(r+1)r}{2} + A_0 r$; thus $C = \frac{1}{r+1}$, $A_0 = \frac{1}{2}$.

Equate the coefficients of n^{r-p}, putting for C and A_0 their values; thus we shall obtain generally

$$\frac{1}{p!} = \frac{1}{(p+1)!} + \frac{1}{2\,p!} + \frac{A_1}{2!\,(p-1)!} + \frac{A_2}{3!\,(p-2)!} + \frac{A_3}{4!\,(p-3)!} + \frac{A_4}{5!\,(p-4)!} + \ldots\ldots,$$

where the series on the right-hand side extends as far as the term involving A_{p-1} inclusive; and by putting for p in succession the values 2, 3, 4, ... we determine in succession A_1, A_2, A_3, ...; and we see that these quantities are independent of n and r.

We shall obtain $A_1 = \frac{1}{6}$, $A_2 = 0$, $A_3 = -\frac{1}{30}$, $A_4 = 0$, $A_5 = \frac{1}{42}$, ...

It is remarkable that all the coefficients with *even* suffixes A_2, A_4, A_6, ... are zero; this can be proved as follows:

In the original assumed identity change n into $n-1$, and subtract; thus

$$n^r = C\{n^{r+1} - (n-1)^{r+1}\} + A_0\{n^r - (n-1)^r\} + \frac{r}{2}A_1\{n^{r-1} - (n-1)^{r-1}\} + \frac{r(r-1)}{2\,.\,3}A_2\{n^{r-2} - (n-1)^{r-2}\} + \ldots\ldots$$

Equate the coefficients of n^{r-p}, putting for C and A_0 their values; thus

$$0 = \frac{1}{(p+1)!} - \frac{1}{2\,p!} + \frac{A_1}{2!\,(p-1)!} - \frac{A_2}{3!\,(p-2)!} + \frac{A_3}{4!\,(p-3)!} - \frac{A_4}{5!\,(p-4)!} + \ldots\ldots$$

The result formerly obtained may be expressed thus,

$$0 = \frac{1}{(p+1)!} - \frac{1}{2\,p!} + \frac{A_1}{2!\,(p-1)!} + \frac{A_2}{3!\,(p-2)!} + \frac{A_3}{4!\,(p-3)!} + \frac{A_4}{5!\,(p-4)!} + \ldots\ldots$$

Hence, by subtracting and putting for p in succession the values 3, 5, 7, ... we shew in succession that zero is the value of A_2, A_4, A_6, ...

EXAMPLES OF THE SUMMATION OF SERIES.

1. Shew that the sum of the first n terms of the series of which the n^{th} term is $n(n+1)(n+2)\ldots\ldots(n+m-1)$ is obtained by placing one more factor at the end of this expression, and dividing by the number of factors so increased.

2. Give the formula for summing the series of which the n^{th} term is the *reciprocal* of $n(n+1)(n+2)\ldots\ldots(n+m-1)$.

Sum the following five series to n terms, and also to infinity:

3. $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\dfrac{1}{4.5}+\ldots\ldots$

4. $\dfrac{1}{2.4.6}+\dfrac{1}{4.6.8}+\dfrac{1}{6.8.10}+\dfrac{1}{8.10.12}+\ldots\ldots$

5. $\dfrac{1}{1.4}+\dfrac{1}{2.5}+\dfrac{1}{3.6}+\dfrac{1}{4.7}+\ldots\ldots$

6. $\dfrac{1}{1.3.5}+\dfrac{1}{2.4.6}+\dfrac{1}{3.5.7}+\dfrac{1}{4.6.8}+\ldots\ldots$

7. $\dfrac{4}{2.3.4}+\dfrac{7}{3.4.5}+\dfrac{10}{4.5.6}+\dfrac{13}{5.6.7}+\ldots\ldots$

8. Sum to n terms $1+3+6+10+\ldots\ldots$

9. If n be even, shew that

$$n+2(n-1)+3(n-2)+\ldots\ldots+\frac{n}{2}\left(\frac{n}{2}+1\right)=\frac{n(n+1)(n+2)}{12}.$$

10. Sum to n terms $a^2+(a+1)^2+(a+2)^2+\ldots\ldots$

11. Sum to n terms $1^2+2^2x+3^2x^2+4^2x^3+\ldots\ldots$

12. If the terms of the expansion of $(a+b)^n$ be multiplied respectively by nr, $(n-1)r^2$, $(n-2)r^3$, ..., n being a positive integer, find the sum of the resulting series.

13. Expand $\dfrac{x}{(1-x)^2-cx}$ in a series of ascending powers of x, and shew that the coefficient of x^n is

$$n\left\{1+\frac{n^2-1}{\lfloor 3}c+\frac{(n^2-1)(n^2-4)}{\lfloor 5}c^2+\frac{(n^2-1)(n^2-4)(n^2-9)}{\lfloor 7}c^3+\ldots\ldots\right\}.$$

14. Find the coefficient of $x^m y^n$ in the expansion of

$$\frac{x(1-ax)}{(1-x)(1-ax-by)}.$$

15. Shew that $1+\frac{2n}{3}+\frac{2n(2n+2)}{3.6}+\frac{2n(2n+2)(2n+4)}{3.6.9}+\ldots\ldots$

$$=2^n\left\{1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\ldots\ldots\right\}.$$

16. If p_r denote the coefficient of x^r in the expansion of $(1+x)^n$, where n is a positive integer, shew that

$$\frac{p_1}{p_0}+\frac{2p_2}{p_1}+\frac{3p_3}{p_2}+\ldots\ldots+\frac{np_n}{p_{n-1}}=\frac{n(n+1)}{1.2};$$

$$(p_0+p_1)(p_1+p_2)\ldots\ldots(p_{n-1}+p_n)=\frac{p_1p_2\ldots p_n(n+1)^n}{\lfloor n};$$

$$p_1-\frac{p_2}{2}+\frac{p_3}{3}-\ldots\ldots+\frac{(-1)^{n-1}p_n}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}.$$

17. Shew by developing the identity $\left(\frac{1}{1-x}-1\right)^n=\frac{x^n}{(1-x)^n}$ that

$$\frac{n(n+1)\ldots\ldots(n+p-1)}{\lfloor p}-\frac{n}{1}\cdot\frac{(n-1)\ldots\ldots(n+p-2)}{\lfloor p}$$
$$+\frac{n(n-1)}{1.2}\cdot\frac{(n-2)\ldots\ldots(n+p-3)}{\lfloor p}-\ldots\ldots$$

is zero when n and p are positive integers and n greater than p.

18. If shot be piled on a triangular base, each side of which exhibits 9 shots, find the whole number contained in the pile.

19. Find the number of shot contained in 5 courses of an unfinished triangular pile, the number in one side of the base being 15.

20. The number of balls contained in a truncated pile of which the top and bottom are rectangular, is

$$\frac{p}{6}\{2p^2+3(m+n-1)p+6mn-3m-3n+1\},$$

where m and n represent the number of balls in the two sides of the top, and p the number of balls in each of the slanting edges.

21. Shew that $1^4+2^4+3^4+\dots\dots+n^4$

$$=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}=\frac{n}{30}(n+1)(2n+1)(3n^2+3n-1).$$

22. Shew that

$$(1+xv)(1+x^2v)(1+x^3v)\dots\dots(1+x^pv)$$

$$=1+\frac{1-x^p}{1-x}xv+\frac{(1-x^p)(1-x^{p-1})}{(1-x)(1-x^2)}x^3v^2$$
$$+\frac{(1-x^p)(1-x^{p-1})(1-x^{p-2})}{(1-x)(1-x^2)(1-x^3)}x^6v^3+\dots\dots$$

23. In the expansion of $(1+x)(1+cx)(1+c^2x)(1+c^3x)\dots$ the number of factors being infinite and c less than unity, the coefficient of x^r is

$$\frac{c^{\frac{1}{2}r(r-1)}}{(1-c)(1-c^2)(1-c^3)\dots\dots(1-c^r)}.$$

24. If A_r be the coefficient of x^r in the expansion of

$$(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\left(1+\frac{x}{2^3}\right)^2\dots\dots \textit{ad infinitum},$$

prove that $A_r=\frac{2^2}{2^r-1}(A_{r-1}+A_{r-2})$, and that $A_4=\frac{1072}{315}$.

25. If n be any multiple of 3, shew that

$$1-(n-1)+\frac{(n-2)(n-3)}{1.2}-\frac{(n-3)(n-4)(n-5)}{\underline{|3}}+\dots\dots=(-1)^n.$$

LI. INEQUALITIES.

672. It is often useful to know which is the greater of two given expressions; propositions relating to such questions are usually collected under the head *Inequalities*.

We say that a is *greater* than b when $a-b$ is a *positive* quantity. See Art. 95.

673. *An inequality will still hold after the same quantity has been added to each member or taken from each member.*

For suppose $a > b$, therefore $a - b$ is positive, therefore $a \pm c - (b \pm c)$ is positive, therefore $a \pm c > b \pm c$.

Hence we may infer that a term may be removed from one member of an inequality and affixed to the other *with its sign changed.*

674. *If the signs of all the terms of an inequality be changed the sign of inequality must be reversed.*

For to change all the signs is equivalent to removing each term of the first member to the second, and each term of the second member to the first.

675. *An inequality will still hold after each member has been multiplied or divided by the same positive quantity.*

For suppose $a > b$, therefore $a - b$ is positive, therefore if m be positive $m(a - b)$ is positive, therefore $ma > mb$; and similarly $\frac{1}{m}(a - b)$ is positive, and $\frac{a}{m} > \frac{b}{m}$.

In like manner we can shew that if each member of an inequality be multiplied or divided by the same *negative* quantity, the sign of inequality must be reversed.

676. If $a > b$, $a' > b'$, $a'' > b''$, ... then

$$a + a' + a'' + \ldots\ldots > b + b' + b'' + \ldots\ldots$$

For by supposition, $a - b$, $a' - b'$, $a'' - b''$, ... are all positive; therefore $a - b + a' - b' + a'' - b'' + \ldots\ldots$ is positive; therefore

$$a + a' + a'' + \ldots\ldots > b + b' + b'' + \ldots\ldots$$

677. If $a > b$, $a' > b'$, $a'' > b''$, ... and all the quantities are positive, then it is obvious that $aa'a''\ldots\ldots > bb'b''\ldots\ldots$

678. If $a > b$, and a and b are positive, then $a^n > b^n$, where n is any *positive* quantity.

This follows from the preceding Article if n be an *integer*. If n be fractional suppose it $= \frac{p}{q}$; let $a^{\frac{1}{q}} = h$ and $b^{\frac{1}{q}} = k$; then h is $> k$, and we have to prove that $h^{\frac{1}{q}} > k^{\frac{1}{q}}$; this we can prove indirectly; for if $h^{\frac{1}{q}} = k^{\frac{1}{q}}$, then $h = k$; and if $h^{\frac{1}{q}} < k^{\frac{1}{q}}$, then $h < k$; both of these results are false; hence we must have $h^{\frac{1}{q}} > k^{\frac{1}{q}}$.

If n be a *negative* quantity, let $n = -m$, so that m is positive; then $\frac{1}{a^m} < \frac{1}{b^m}$; that is, $a^n < b^n$.

679. Let $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n}$ be fractions of which the denominators are all of the same sign, then the fraction

$$\frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n}$$

lies in magnitude between the least and the greatest of the fractions

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n}.$$

For suppose $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n}$ to be in *ascending* order of magnitude, and suppose that all the denominators are *positive;* then

$$\frac{a_1}{b_1} = \frac{a_1}{b_1}, \text{ therefore } a_1 = b_1 \times \frac{a_1}{b_1};$$

$$\frac{a_2}{b_2} > \frac{a_1}{b_1}, \text{ therefore } a_2 > b_2 \times \frac{a_1}{b_1};$$

$$\frac{a_3}{b_3} > \frac{a_1}{b_1}, \text{ therefore } a_3 > b_3 \times \frac{a_1}{b_1};$$

and so on;

therefore, by addition,

$$a_1 + a_2 + a_3 + \dots\dots + a_n > (b_1 + b_2 + b_3 + \dots\dots + b_n)\frac{a_1}{b_1};$$

therefore $$\frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b} > \frac{a_1}{b_1}.$$

Similarly we may prove that

$$\frac{a_1 + a_2 + a_3 + \dots\dots + a_n}{b_1 + b_2 + b_3 + \dots\dots + b_n} < \frac{a_n}{b_n}.$$

In like manner the theorem may be established when all the denominators are supposed *negative*.

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \dots$, then each of these fractions is *equal* to the fraction whose numerator is the sum of the numerators and denominator the sum of the denominators. See Art. 384.

680. Since $(x-y)^2$ or $x^2 - 2xy + y^2$ is a positive quantity or zero, according as x and y are unequal or equal, we have

$$\tfrac{1}{2}(x^2 + y^2) > xy,$$

the inequality becoming an equality when $x = y$. Hence

$$\tfrac{1}{2}(a+b) > \sqrt{(ab)};$$

that is, the arithmetic mean of two quantities is greater than the geometric mean, the inequality becoming an equality when the two quantities are equal.

681. Let there be n positive quantities, $a, b, c, \dots k$; then

$$\left(\frac{a+b+c+\dots+k}{n}\right)^n > abc \dots k,$$

unless the n quantities are all equal, and then the inequality becomes an equality.

For $ab < \left(\frac{a+b}{2}\right)^2, \qquad cd < \left(\frac{c+d}{2}\right)^2;$

therefore $$abcd < \left(\frac{a+b}{2} \cdot \frac{c+d}{2}\right)^2;$$

and $$\frac{a+b}{2} \cdot \frac{c+d}{2} < \left\{\frac{\frac{1}{2}(a+b) + \frac{1}{2}(c+d)}{2}\right\}^2;$$

therefore $$abcd < \left(\frac{a+b+c+d}{4}\right)^4.$$

By proceeding in this way we can shew that if p be any positive integral power of 2,

$$abcd \dots (p \text{ factors}) < \left(\frac{a+b+c+d+\dots}{p}\right)^p.$$

Now let $p = n + r$, and let $\frac{a+b+c+d+\dots(n \text{ terms})}{n} = t$, and suppose each of the remaining r quantities out of the p quantities to be equal to t; we have then

$$abcd \ldots (n \text{ factors}) \times t^r < \left(\frac{nt + rt}{n + r}\right)^{n+r}; \text{ that is, } < t^{n+r};$$

therefore $abcd \ldots (n \text{ factors}) < t^n$; that is, $< \left(\frac{a + b + c + d + \ldots}{n}\right)^n$.

Thus the theorem is proved whatever be the number of quantities $a, b, c, d, \ldots$ The inequality becomes an equality when all the n quantities are equal.

We may also write the theorem thus,

$$\frac{a + b + c + d + \ldots\ldots}{n} > (abcd \ldots)^{\frac{1}{n}};$$

by extending the signification of the terms *arithmetical mean* and *geometrical mean*, we may enunciate the theorem thus: *the arithmetical mean of any number of positive quantities is greater than the geometrical mean.*

682. The following proof of the theorem given in the preceding Article will be found an instructive exercise.

Let P denote $(abcd\ldots\ldots k)^{\frac{1}{n}}$, and Q denote $\frac{a+b+c+d+\ldots\ldots+k}{n}$.

Suppose a and b respectively the greatest and least of the n quantities $a, b, c, d, \ldots\ldots\ldots k$; let $a_1 = b_1 = \frac{1}{2}(a + b)$, and let $P_1 = (a_1 b_1 cd \ldots\ldots k)^{\frac{1}{n}}$; then since $a_1 b_1 > ab$, we have $P_1 > P$. Next if the factors in P_1 be not all equal, remove the greatest and least of them, and put in their places two new factors, each equal to half the sum of those removed; let P_2 denote the new geometrical mean; then $P_2 > P_1$. If we proceed in this way, we obtain a series $P, P_1, P_2, P_3, \ldots\ldots P_r$, each term of which is greater than the preceding term; and by taking r large enough, we may have the factors which occur in P_r as nearly equal as we please; thus when r is large enough, we may consider $P_r = Q$; therefore P is less than Q.

683. We will now compare the quantities

$$\frac{a^m + b^m}{2} \text{ and } \left(\frac{a + b}{2}\right)^m.$$

We suppose a and b positive, and a not less than b.

$$a^m + b^m = \left(\frac{a+b}{2} + \frac{a-b}{2}\right)^m + \left(\frac{a+b}{2} - \frac{a-b}{2}\right)^m$$

$$= 2\left\{\left(\frac{a+b}{2}\right)^m + \frac{m(m-1)}{1.2}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2\right.$$

$$\left. + \frac{m(m-1)(m-2)(m-3)}{\underline{|4}}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4 + \ldots\ldots\right\}.$$

Since $\frac{a-b}{2}$ is less than $\frac{a+b}{2}$, the series is *convergent* (Art. 564), so that we have a result which is arithmetically intelligible and true. Hence if m be *negative* or any *positive integer*, it follows that $\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m$. If m be *positive and less than unity*, $\frac{a^m+b^m}{2} < \left(\frac{a+b}{2}\right)^m$. It remains to consider the case in which m is *positive and greater than unity, but not an integer.* Suppose $m = \frac{p}{q}$, where p is $> q$, and let $\alpha = a^{\frac{1}{q}}$, $\beta = b^{\frac{1}{q}}$, $A = \alpha^p$, $B = \beta^p$. Then $\frac{a^{\frac{p}{q}} + b^{\frac{p}{q}}}{2}$ is $>$ or $<$ $\left(\frac{a+b}{2}\right)^{\frac{p}{q}}$, according as $\frac{\alpha^p+\beta^p}{2}$ is $>$ or $<$ $\left(\frac{\alpha^q+\beta^q}{2}\right)^{\frac{p}{q}}$; that is, according as $\left(\frac{\alpha^p+\beta^p}{2}\right)^{\frac{q}{p}}$ is $>$ or $<$ $\frac{\alpha^q+\beta^q}{2}$; that is, according as $\left(\frac{A+B}{2}\right)^{\frac{q}{p}}$ is $>$ or $<$ $\frac{A^{\frac{q}{p}}+B^{\frac{q}{p}}}{2}$. We know by what has already been proved, that the expression on the left-hand side is the greater, since $\frac{q}{p}$ is positive and less than unity; hence $\frac{a^m+b^m}{2}$ is $> \left(\frac{a+b}{2}\right)^m$ when m is positive and greater than unity.

684. Let there be n positive quantities $a, b, c, \ldots\ldots k$; then

$$\frac{a^m + b^m + c^m + \ldots\ldots + k^m}{n} > \left(\frac{a+b+c+\ldots\ldots+k}{n}\right)^m$$

when m is negative, or positive and greater than unity; but the reverse holds when m is positive and less than unity. The inequality becomes an equality when all the n quantities are equal.

This may be proved by a method similar to that used in Art. 681. We will suppose m negative, or positive and greater than unity. Then $a^m + b^m > 2\left(\frac{a+b}{2}\right)^m$, $c^m + d^m > 2\left(\frac{c+d}{2}\right)^m$;

therefore $$a^m + b^m + c^m + d^m > 2\left\{\left(\frac{a+b}{2}\right)^m + \left(\frac{c+d}{2}\right)^m\right\}$$

$$> 2 \,.\, 2\left(\frac{a+b+c+d}{4}\right)^m;$$

therefore $$\frac{a^m + b^m + c^m + d^m}{4} > \left(\frac{a+b+c+d}{4}\right)^m.$$

By proceeding in this way we can establish the theorem in the case where the number of quantities is p, if p be any positive integral power of 2. Now let $p = n + r$, and let the last r of the p quantities be all equal, and each equal to t, say, where

$$t = \frac{a+b+c+ \ldots\ldots (n \text{ terms})}{n};$$

therefore $$\frac{a^m + b^m + c^m + \ldots\ldots}{n+r} > \left(\frac{a+b+c+\ldots\ldots}{n+r}\right)^m,$$

therefore $$a^m + b^m + c^m + \ldots\ldots + rt^m > (n+r)\left(\frac{nt+rt}{n+r}\right)^m;$$

that is, $$> (n+r)\,t^m;$$

therefore $$a^m + b^m + c^m + \ldots\ldots > nt^m;$$

which was to be proved.

In a similar way we may establish the rest of the theorem, namely, that when m is positive and less than unity the reverse holds.

The theorem of this Article may also be established by a method similar to that used in Art. 682.

685. If x and β are positive quantities, and x and βx less than unity, $(1+x)^\beta$ is less than $\frac{1}{1-\beta x}$.

We have in fact to shew that $(1+x)^{-\beta}$ is greater than $1-\beta x$. Now, by the Binomial Theorem,

$$(1+x)^{-\beta}=1-\beta x+\frac{\beta(\beta+1)}{\underline{|2}}x^2-\frac{\beta(\beta+1)(\beta+2)}{\underline{|3}}x^3+\ldots\ldots;$$

each term of this series is greater than the succeeding term, for $\frac{\beta+n}{n+1}x$ is less than unity, since x and βx are each less than unity. Hence, as in Art. 558, the series is greater than $1-\beta x$.

686. Let γ be a positive quantity greater than β; then $1+\gamma x$ is greater than $\frac{1}{1-\beta x}$ provided $(1+\gamma x)(1-\beta x)$ is greater than 1; that is provided $(\gamma-\beta)x$ is greater than $\beta\gamma x^2$, that is provided $\gamma-\beta$ is greater than $\beta\gamma x$. Hence we have the following result: if x, β, and γ are positive, and γ greater than β, then by taking x small enough we can make $(1+x)^\beta$ less than $1+\gamma x$; this holds however small the excess of γ over β may be.

687. If x be positive $\log(1+x)$ is less than x.

For suppose $y=\log(1+x)$, then $1+x=e^y$; and, by Art. 542, $e^y=1+y+\frac{y^2}{\underline{|2}}+\frac{y^3}{\underline{|3}}+\ldots\ldots$, which is greater than $y+1$.

As an example put for x in succession $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \frac{1}{n}$: we have $\log\frac{3}{2}<\frac{1}{2}$, $\log\frac{4}{3}<\frac{1}{3}$, ... $\log\frac{n+1}{n}<\frac{1}{n}$. Hence, by addition, $\log\frac{n+1}{2}<\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}$.

688. If x be positive and less than unity $\log(1+x)$ is greater than $x-\frac{x^2}{2}$.

For $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$; hence, as in Art. 558, we see that $\log(1+x)-\left(x-\frac{x^2}{2}\right)$ is a finite positive quantity.

689. If x be positive and less than unity $\log \frac{1}{1-x}$ is greater than x.

For $\log \frac{1}{1-x} = -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots;$ hence $\log \frac{1}{1-x} - x$ is a finite positive quantity.

690. The following three examples will illustrate the subject of Inequalities, and furnish results of some interest.

I. If $u_n = \frac{1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n}$ and $v_n = \frac{3 \cdot 5 \cdot 7 \ldots\ldots (2n+1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n}$, then when n is infinite u_n is zero, v_n is infinite and $u_n v_n$ is finite.

We have $$u_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \ldots\ldots \frac{2n-1}{2n} \ldots\ldots\ldots\ldots\ldots (1);$$

therefore, by Art. 376, $$u_n < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \ldots\ldots \frac{2n}{2n+1} \ldots\ldots\ldots\ldots\ldots (2).$$

Therefore, by multiplication, $u_n^2 < \frac{1}{2n+1}$.

Hence, by increasing n we can make u_n less than any assigned quantity.

Similarly, $$v_n = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \ldots\ldots \frac{2n+1}{2n} \ldots\ldots\ldots\ldots\ldots (3);$$

therefore, by Art. 376, $$v_n > \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \ldots\ldots \frac{2n+2}{2n+1} \ldots\ldots\ldots\ldots\ldots (4).$$

Therefore, by multiplication, $v_n^2 > \frac{2n+2}{2}$, that is, $> n+1$.

Hence, by increasing n we can make v_n greater than any assigned quantity.

Last, by (1) and (4) we see that

$$u_n v_n > \frac{1}{2} \frac{2n+2}{2n+1}, \text{ that is,} > \frac{n+1}{2n+1}, \text{ and therefore, } a\ fortiori, > \frac{1}{2};$$

and by (2) and (3) we see that $u_n v_n < 1$.

Hence $u_n v_n$ lies between $\frac{1}{2}$ and 1, and is therefore finite.

II. If m, n, a are in descending order of magnitude, then

$$\left(\frac{m+a}{m-a}\right)^m \text{ is } < \left(\frac{n+a}{n-a}\right)^n.$$

For take the logarithms of both sides; thus we have to compare

$$m \log \frac{1+\frac{a}{m}}{1-\frac{a}{m}} \text{ with } n \log\left(\frac{1+\frac{a}{n}}{1-\frac{a}{n}}\right);$$

or $2a\left\{1+\frac{1}{3}\frac{a^2}{m^2}+\frac{1}{5}\frac{a^4}{m^4}+\dots\dots\right\}$ with $2a\left\{1+\frac{1}{3}\frac{a^2}{n^2}+\frac{1}{5}\frac{a^4}{n^4}+\dots\dots\right\}$, and the first of these is less than the second. Hence the required result follows.

III. Let there be n positive quantities a, b, c, ... k; then

$$\left(\frac{a+b+c+\dots\dots+k}{n}\right)^{a+b+c+\dots\dots k} \text{ is } < a^a b^b c^c \dots\dots k^k,$$

unless the n quantities are equal, and then the inequality becomes an equality.

Let there be two unequal quantities a and b; we have to shew that $a^a b^b$ is $>\left(\frac{a+b}{2}\right)^{a+b}$.

Suppose a greater than b; let $a=c+x$, $b=c-x$.

We have to shew that $\left(1+\frac{x}{c}\right)^{c+x}\left(1-\frac{x}{c}\right)^{c-x}$ is >1,

that is, that $$\left(1-\frac{x^2}{c^2}\right)^c\left(\frac{1+\frac{x}{c}}{1-\frac{x}{c}}\right)^x \text{ is } >1,$$

or that $$(1-z^2)\left(\frac{1+z}{1-z}\right)^z \text{ is } >1, \quad \text{where } z=\frac{x}{c}.$$

Now the logarithm of $\left(\frac{1+z}{1-z}\right)^z(1-z^2)$ is

$$2z\left\{z+\frac{1}{3}z^3+\frac{1}{5}z^5+\dots\dots\right\}-\left\{z^2+\frac{1}{2}z^4+\frac{1}{3}z^6+\dots\dots\right\},$$

and this is a positive quantity; and as the logarithm is positive the expression is greater than unity.

The demonstration is then extended to the case of three or more quantities by a method similar to that used in Art. 682.

The problems in the next three Articles are analogous to the subject considered in the present Chapter.

691. Divide a given number $2a$ into two parts, such that their product shall have the greatest possible value.

Let x denote one part and $2a-x$ the other part, and let y denote the product; then we have to determine x so that y may have the greatest possible value. Since $y = x(2a-x)$, we have $x^2 - 2ax + y = 0$; therefore $x = a \pm \sqrt{(a^2 - y)}$. Thus since x must be real y cannot be greater than a^2, and $x = a$, when $y = a^2$.

692. Divide a given number $2a$ into two parts, such that the sum of their square roots shall have the greatest possible value.

Let x denote one part and $2a-x$ the other part, and let y denote the sum of the square roots of the parts; then we have to determine x so that y may have the greatest possible value.

Since $\sqrt{x} + \sqrt{(2a-x)} = y$, $2a - x = (y - \sqrt{x})^2 = y^2 - 2y\sqrt{x} + x$, and $2x - 2y\sqrt{x} + y^2 - 2a = 0$; therefore $\sqrt{x} = \dfrac{y}{2} \pm \dfrac{\sqrt{(4a - y^2)}}{2}$.

Since $\sqrt{x}$ must be real y^2 cannot be greater than $4a$, thus $2\sqrt{a}$ is the greatest value of y, and $x = a$ when $y = 2\sqrt{a}$.

693. Find the least value which $\dfrac{x^2 + a^2}{x}$ can have whatever real value x may have.

Put $\dfrac{x^2 + a^2}{x} = y$, then $x^2 - xy + a^2 = 0$; thus $x = \dfrac{y}{2} \pm \dfrac{\sqrt{(y^2 - 4a^2)}}{2}$.

Hence y^2 cannot be less than $4a^2$; or $2a$ is the least value of y,

Or thus, $\dfrac{x^2 + a^2}{x} = x + \dfrac{a^2}{x}$; suppose x positive, then we can put this expression in the form $\left(\sqrt{x} - \dfrac{a}{\sqrt{x}}\right)^2 + 2a$; and as $2a$ is constant the least value of the whole expression will be obtained when the positive term $\left(\sqrt{x}-\frac{a}{\sqrt{x}}\right)^2$ vanishes, that is, when $x=a$. It is unnecessary to consider negative values of x, because $\frac{x^2+a^2}{x}$ has the same numerical value when x has any *negative* value as when x has the corresponding *positive* value.

EXAMPLES OF INEQUALITIES.

In the following examples the symbols are supposed to denote positive quantities; and the *inequalities* may, in certain cases, become *equalities*, as in some of the Articles of the text.

1. If a, b, c be such that any two of them are greater than the third, $2(ab+bc+ca)>a^2+b^2+c^2$.

2. If $l^2+m^2+n^2=1$, and $l'^2+m'^2+n'^2=1$, then
$$ll'+mm'+nn'<1.$$

3. $(a+b-c)^2+(a+c-b)^2+(b+c-a)^2>ab+bc+ca$.

4. $\left(\frac{a^2}{b}\right)^{\frac{1}{2}}+\left(\frac{b^2}{a}\right)^{\frac{1}{2}}>\sqrt{a}+\sqrt{b}$.

5. $ab(a+b)+bc(b+c)+ca(c+a)>6abc$ and $<2(a^3+b^3+c^3)$.

6. $(a+b)(b+c)(c+a)>8abc$.

7. Shew that $x^2-8x+22$ is never less than 6, whatever may be the value of x.

8. Which is greater, $2x^3$ or $x+1$?

9. If n be >1, then $x+\frac{1}{nx}$ is $>1+\frac{1}{n}$, if x be >1, or $<\frac{1}{n}$.

10. Find the least value of $\frac{(a+x)(b+x)}{x}$.

11. Divide an odd integer into two other integers, of which the product may be the greatest possible.

12. If $a>b$, then $\sqrt{(a^2-b^2)}+\sqrt{(2ab-b^2)}>a$.

13. If a, b, c, d are in harmonical progression, $a+d>b+c$.

14. If a, b, c are in harmonical progression and n a positive integer, $a^n+c^n>2b^n$.

15. If $a>b$, shew that $\frac{x+a}{\sqrt{(x^2+a^2)}}$ is $>$ or $<\frac{x+b}{\sqrt{(x^2+b^2)}}$, according as x is $>$ or $<\sqrt{(ab)}$.

16. If a, b, c, or b, c, a, or c, a, b are in *descending* order of magnitude, $a^2b+b^2c+c^2a>a^2c+b^2a+c^2b$; if they are in *ascending* order of magnitude, $a^2b+b^2c+c^2a<a^2c+b^2a+c^2b$.

17. $(A^2+B^2+C^2+\ldots)(a^2+b^2+c^2+\ldots)>(Aa+Bb+Cc+\ldots)^2$.

18. $3(a^3+b^3+c^3)>(a+b+c)(ab+bc+ca)$.

19. $9abc<(a+b+c)(a^2+b^2+c^2)$.

20. $\frac{n-1}{2}(a_1+a_2+a_3+\ldots+a_n)>\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\sqrt{(a_2a_3)}+\ldots$

21. The difference between the arithmetic and the geometric mean of two quantities is less than one-eighth of the squared difference of the numbers divided by the less number, but greater than one-eighth of such squared difference divided by the greater number.

22. $\lfloor n<\left(\frac{n+1}{2}\right)^n$.

23. $\lfloor n>n^{\frac{n}{2}}$.

24. $1.3.5\ldots(2n-1)<n^n$.

25. $\left(2-\frac{1}{n}\right)\left(2-\frac{3}{n}\right)\ldots\left(2-\frac{2n-1}{n}\right)>\frac{1}{\lfloor n}$.

26. $a^4+b^4+c^4>abc(a+b+c)$.

27. $8(a^3+b^3+c^3)>3(a+b)(b+c)(c+a)$.

28. $\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}>3$.

29. $(a+b+c)^3>27abc$ and $<9(a^3+b^3+c^3)$.

30. If p and q be each less than unity,

$$\frac{\log_a(1-p)}{\log_a(1-q)} \text{ is } <\frac{p}{q(1-p)}, \text{ and } >\frac{p(1-q)}{q}.$$

31. $\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\ldots\ldots+\frac{a_{n-2}}{a_{n-1}}+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}>n$.

32. If a and x both lie between 0 and 1, then $\frac{1-a^x}{1-a}>x$.

LII. THEORY OF NUMBERS.

694. Throughout the present Chapter the word *number* is used as an abbreviation for *positive integer.*

695. A number which can be divided exactly by no number except itself and unity is called a *prime number*, or shortly a *prime.*

696. Two numbers are said to be prime to each other when there is no number, except unity, which will divide each of them exactly. Instead of saying that two numbers are prime to each other, the same thing is expressed by saying that one of them is prime to the other.

697. *If a number* p *divides a product* ab, *and is prime to one factor* a, *it must divide the other factor* b.

Suppose a greater than p; perform the operation of finding the greatest common measure of a and p; let $q, q', q'', \ldots$ be the successive quotients, and $r, r', r'', \ldots$ the corresponding remainders.

Thus $a = pq + r$, $p = rq' + r'$, $r = r'q'' + r''$, ... multiply each member of each of these equations by b, and divide by p; therefore $\frac{ab}{p} = bq + \frac{br}{p}$, $b = \frac{br}{p} \times q' + \frac{br'}{p}$, $\frac{br}{p} = \frac{br'}{p} \times q'' + \frac{br''}{p}$, ...

Since $\frac{ab}{p}$ is an integer, it follows from the first of these equations that $\frac{br}{p}$ is an integer; then from the second of these equations $\frac{br'}{p}$ is an integer; then from the third $\frac{br''}{p}$ is an integer; and so on. But, since a and p are prime to each other, the last of the remainders $r, r', r'', \ldots$ is unity; therefore $\frac{b \times 1}{p}$ is an integer; that is, b is divisible by p.

698. *When the numerator and denominator of a fraction are prime to each other the fraction cannot be reduced to an equivalent fraction in lower terms.*

Suppose that a is prime to b, and, if possible, let $\frac{a}{b}$ be equal to $\frac{a'}{b'}$, a fraction in lower terms. Since $\frac{a}{b} = \frac{a'}{b'}$, we have $a' = \frac{ab'}{b}$; therefore b divides ab'; but b is prime to a, therefore b divides b' (Art. 697); but this is impossible, since b' is less than b by supposition. Hence $\frac{a}{b}$ cannot be reduced to an equivalent fraction in lower terms.

699. *If* a *is prime to* b, *and* $\frac{a}{b} = \frac{a'}{b'}$, *then* a′ *and* b′ *must be the same multiples of* a *and* b *respectively.*

Since $\frac{a'}{b'} = \frac{a}{b}$, we have $a' = \frac{ab'}{b}$; but b is prime to a, therefore b divides b'; hence $b' = nb$, where n is some integer; therefore $a' = na$.

700. *If a prime number* p *divides a product* abcd... *it must divide one of the factors of that product.*

For since p is a prime number, if p does not divide a it is prime to a, and therefore it must divide bcd... (Art. 697). Similarly, if p does not divide b, it is prime to b, and therefore it must divide cd... By proceeding in this way we shall prove that p must divide one of the factors of the product.

701. *If a prime number divides* a^n, *where* n *is any positive integer, it must divide* a.

This follows from the preceding Article by supposing all the factors equal.

702. *If a number* n *is divisible by* p, p′, p″, ... *and each of these divisors is prime to all the others,* n *is also divisible by the product* pp′p″...

For since n is divisible by p, we have $n = pq$, where q is some integer. Since p' divides pq and is prime to p, p' must divide q; hence $q = p'q'$, where q' is some integer; thus $n = pp'q'$, and is therefore divisible by pp'. By proceeding thus we may shew that n is divisible by $pp'p''$...

703. *If* a *and* b *be each of them prime to* c, *then* ab *is prime to* c.

For if ab is not prime to c, suppose $ab = nr$ and $c = ns$, where n, r, and s are integers; then, since a and b are prime to c, they are prime to ns, and therefore to n; but $ab = nr$, therefore $\frac{a}{n} = \frac{r}{b}$; therefore b is a multiple of n (Art. 699). Hence b is both prime to n and a multiple of n, which is impossible. Therefore ab is prime to c.

704. *If* a *and* b *are prime to each other,* a^m *and* b^n *are prime to each other;* m *and* n *being any positive integers.*

For since a is prime to b, it follows that $a \times a$ or a^2 is prime to b (Art. 703); similarly $a^2 \times a$ or a^3 is prime to b; and so on; thus a^m is prime to b. Again, since a^m is prime to b, it follows that a^m is prime to $b \times b$ or b^2; and so on.

This Article establishes the result to which reference was made in Art. 242.

705. *No rational integral algebraical formula can represent prime numbers only.*

For, if possible, suppose that the formula $a + bx + cx^2 + dx^3 + \ldots$ represents prime numbers only; suppose when $x = m$ that the formula takes the value p, so that $p = a + bm + cm^2 + dm^3 + \ldots$ Put for x, in the formula, $m + np$, and suppose the value then to be p'; thus $p' = a + b(m + np) + c(m + np)^2 + d(m + np)^3 + \ldots$ $= a + bm + cm^2 + dm^3 + \ldots\ldots + M(p) = p + M(p)$, where $M(p)$ denotes *some multiple of* p; thus p' is divisible by p, and is therefore *not* a prime.

706. *The number of prime numbers is infinite.*

For if the number of prime numbers be not infinite, suppose p the greatest prime number; the product of all the prime numbers up to p, that is, $2 \,.\, 3 \,.\, 5 \,.\, 7 \,.\, 11 \ldots . p$ is divisible by each of these prime numbers; add unity to this product, and we obtain a number which is not divisible by any of these prime numbers; this number is therefore either itself a prime number, or is divisible by some prime number greater than p; thus p is not the greatest prime number, which is contrary to the supposition. Hence the number of prime numbers is infinite.

707. *If* a *is prime to* b, *and the quantities* a, 2a, 3a, ... (b – 1) a, *are divided by* b, *the remainders will all be different.*

For, if possible, suppose that two of these quantities ma and $m'a$ when divided by b leave the same remainder r, so that

$$ma = nb + r \text{ and } m'a = n'b + r;$$

then $$(m - m')\,a = (n - n')\,b;$$

therefore $$\frac{a}{b} = \frac{n - n'}{m - m'};$$

hence $m - m'$ is a multiple of b (Art. 699); but this is impossible, since m and m' are both less than b.

708. *A number can be resolved into prime factors in only one way.*

Let N denote the number; suppose $N = abcd\,......$, where $a, b, c, d, ...$ are prime numbers equal or unequal. Suppose, if possible, that N also $= \alpha\beta\gamma\delta\,..$, where $\alpha, \beta, \gamma, \delta, ...$ are other prime numbers. Then $abcd\,...... = \alpha\beta\gamma\delta\,......$; hence α must divide $abcd\,......$, and therefore must divide one of the factors of this product; but these factors are all prime numbers; hence α must be equal to one of them, a suppose. Divide by a or α, then $bcd\,...... = \beta\gamma\delta\,......$; from this we can prove that β must be equal to one of the factors in $bcd\,......$; and so on. Thus the factors in $abcd\,......$ cannot be different from those in $\alpha\beta\gamma\delta\,......$

709. *To find the highest power of a prime number* a *which is contained in the product* $\lfloor m$.

Let $I\left(\dfrac{m}{a}\right)$ denote the greatest integer contained in $\dfrac{m}{a}$,

let $I\left(\dfrac{m}{a^2}\right)$ denote the greatest integer contained in $\dfrac{m}{a^2}$,

let $I\left(\dfrac{m}{a^3}\right)$ denote the greatest integer contained in $\dfrac{m}{a^3}$,

and so on;

then the highest power of the prime number a which is contained in $|\underline{m}$ is $I\left(\frac{m}{a}\right) + I\left(\frac{m}{a^2}\right) + I\left(\frac{m}{a^3}\right) + \ldots\ldots$

For among the numbers 1, 2, 3, ... m, there are $I\left(\frac{m}{a}\right)$ which contain a at least once, namely the numbers a, $2a$, $3a$, $4a$, ... Similarly there are $I\left(\frac{m}{a^2}\right)$ which contain a^2 at least once; there are $I\left(\frac{m}{a^3}\right)$ which contain a^3 at least once; and so on. The sum of these expressions is the required highest power.

This proposition will be illustrated by considering a numerical example. Suppose for instance that $m = 14$ and $a = 2$; then we have to find the highest power of 2 which is contained in $|\underline{14}$.

Here $I\left(\frac{m}{a}\right) = 7$, $I\left(\frac{m}{a^2}\right) = 3$, $I\left(\frac{m}{a^3}\right) = 1$; thus the required power is 11. That is, 2^{11} will divide $|\underline{14}$, and no higher power of 2 will divide $|\underline{14}$. Now let us examine in what way this number 11 arises. Of the factors 1, 2, 3, 4, ... 14 there are *seven* which we can divide at once by 2, namely 2, 4, 6, 8, 10, 12, 14. There are *three* factors which can be divided by 2 a second time, namely 4, 8, 12. There is *one* factor which can be divided by 2 a third time, namely 8.

Thus we see the way in which $7 + 3 + 1$, that is 11, arises.

710. *The product of any* n *successive integers is divisible by* $|\underline{n}$.

Let $m + 1$ be the first integer; we have then to shew that $\frac{(m+1)(m+2)\ldots\ldots(m+n)}{|\underline{n}}$ is an integer. Multiply both numerator and denominator of this expression by $|\underline{m}$; it then becomes $\frac{|\underline{m+n}}{|\underline{m}\,|\underline{n}}$, which we shall denote by $\frac{P}{Q}$. Let a be any prime number; let r_1, r_2, r_3, ... denote the greatest integers in $\frac{m+n}{a}$, $\frac{m+n}{a^2}$, $\frac{m+n}{a^3}$, ... respectively; let s_1, s_2, s_3, ...

denote the greatest integers in $\frac{m}{a}, \frac{m}{a^2}, \frac{m}{a^3}, \ldots\ldots$ respectively; and let $t_1, t_2, t_3, \ldots\ldots$ denote the greatest integers in $\frac{n}{a}, \frac{n}{a^2}, \frac{n}{a^3}, \ldots\ldots$ respectively. Then in P the factor a occurs raised to the power $r_1 + r_2 + r_3 + \ldots\ldots$; in Q the factor a occurs raised to the power $s_1 + s_2 + s_3 + \ldots\ldots + t_1 + t_2 + t_3 + \ldots\ldots$ Now it may be easily shewn that r_1 is either equal to $s_1 + t_1$ or to $s_1 + t_1 + 1$, and that r_2 is either equal to $s_2 + t_2$ or to $s_2 + t_2 + 1$, and so on. Thus a occurs in P raised to at least as high a power as in Q. Similarly any prime factor which occurs in Q occurs in P raised to at least as high a power as in Q. Thus P is divisible by Q.

711. *If* n *be a prime number, the coefficient of every term in the expansion of* (a + b)^n, *except the first and last, is divisible by* n.

For the general form of the coefficients excluding the first and last is $\frac{n(n-1)\ldots\ldots(n-r+1)}{\underline{|r}}$, where r may have any value from 1 to $n-1$ inclusive. Now, by Art. 710, this expression is an integer; also since n is a prime number and greater than r, no factor which occurs in $\underline{|r}$ can divide n; therefore $(n-1)(n-2)\ldots\ldots(n-r+1)$ must be divisible by $\underline{|r}$. Hence every coefficient, except the first and last, is divisible by n.

712. *If* n *be a prime number, the coefficient of every term in the expansion of* (a + b + c + d + ...)^n, *except those of* a^n, b^n, c^n, d^n, ..., *is divisible by* n.

Put β for $b + c + d + \ldots\ldots$; then

$$(a + b + c + d + \ldots\ldots)^n = (a + \beta)^n.$$

By Art. 711, every coefficient in the expansion of $(a+\beta)^n$ is divisible by n, except those of a^n and β^n, and the coefficient of each of these terms is unity. Again,

$$\beta^n = (b + c + d + \ldots\ldots)^n = (b + \gamma)^n \text{ suppose};$$

and every coefficient in the expansion of $(b+\gamma)^n$ is divisible by n except those of b^n and γ^n. By proceeding in this way we arrive at the theorem enunciated.

713. *If* n *be a prime number, and* N *prime to* n, *then* $N^{n-1}-1$ *is a multiple of* n. (Fermat's Theorem.)

By the preceding Article,

$$(a+b+c+d+\ldots\ldots+k)^n = a^n+b^n+c^n+d^n+\ldots\ldots+k^n+M(n),$$

where $M(n)$ denotes some multiple of n. Let each of the quantities $a, b, c, d, \ldots\ldots k$ be equal to unity, and suppose there are N of them; thus $N^n = N + M(n)$; therefore $N(N^{n-1}-1) = M(n)$.

Since N is prime to n, it follows that $N^{n-1}-1$ is divisible by n.

We may therefore say that $N^{n-1} = 1 + pn$, where p is some positive integer.

714. Since n is a prime number in the preceding Article, $n-1$ is an *even* number except when $n=2$; hence we may write the theorem thus, $(N^{\frac{n-1}{2}}-1)(N^{\frac{n-1}{2}}+1) = M(n)$; therefore, either $N^{\frac{n-1}{2}}-1$ or $N^{\frac{n-1}{2}}+1$ is divisible by n, so that $N^{\frac{n-1}{2}} = pn+1$, or else $= pn-1$, where p is some positive integer.

715. The following theorem is an extension of Fermat's. Let n be any number; and let $1, a, b, c, \ldots\ldots n-1$, be all the numbers which are less than n and prime to n; suppose there are m of these numbers; then will $x^m - 1 = M(n)$, when for x we substitute any one of the above m numbers, except unity. For multiply all the m numbers by any one of them except unity, and denote the multiplier by x; thus we obtain $1.x, ax, bx, cx, \ldots\ldots (n-1)x$; these products are all different and all prime to n. It may be easily shewn that when these products are divided by n, the remainders are *all different and all prime to* n; thus the remainders must be the original m numbers $1, a, b, c, \ldots\ldots n-1$; they will not necessarily occur in this order, but that is immaterial for the object we have in view. Hence the product of the new series of m numbers $x, ax, bx, cx, \ldots\ldots (n-1)x$, can only differ from the product of the original m numbers *by some multiple of* n; thus

$$x^m abc \ldots\ldots (n-1) = abc \ldots\ldots (n-1) + M(n).$$

Since two of the three terms which enter into this identity are divisible by $abc......(n-1)$, the third term must likewise be so divisible, and as $abc......(n-1)$ is prime to n, the quotient after $M(n)$ is divided by $abc......(n-1)$ must still be some multiple of n, and may be denoted by $M(n)$; thus

$$x^m = 1 + M(n), \text{ and } x^m - 1 = M(n).$$

716. We will now deduce Fermat's theorem from the result of the preceding Article. Suppose n a prime number; then the numbers 1, 2, 3, ... $n-1$, are all prime to n; thus $m = n-1$. Therefore $x^{n-1} - 1 = M(n)$, where x may be any number less than n. Next let y denote any number which is greater than n and prime to n, then we can suppose $y = pn + x$, where p is some integer and x is less than n. Therefore

$$y^{n-1} = (pn + x)^{n-1} = x^{n-1} + (n-1)\,x^{n-2}pn + \ldots\ldots = x^{n-1} + M(n);$$

but we have already shewn that $x^{n-1} = 1 + M(n)$; thus

$$y^{n-1} = 1 + M(n), \text{ and } y^{n-1} - 1 = M(n).$$

Thus Fermat's theorem is established.

717. *If* n *be a prime number,* $1 + \underline{|n-1}$ *is divisible by* n. (Wilson's Theorem.)

By Art. 549 we have

$$\underline{|n-1} = (n-1)^{n-1} - (n-1)(n-2)^{n-1}$$
$$+ \frac{(n-1)(n-2)}{1.2}(n-3)^{n-1} - \frac{(n-1)(n-2)(n-3)}{1.2.3}(n-4)^{n-1} + \ldots\ldots;$$

by Fermat's theorem we have

$$(n-1)^{n-1} = 1 + p_1 n, \quad (n-2)^{n-1} = 1 + p_2 n, \quad (n-3)^{n-1} = 1 + p_3 n, \ldots\ldots$$

where $p_1, p_2, p_3, \ldots\ldots$ are positive integers. Therefore

$$\underline{|n-1} = M(n) + 1 - (n-1)$$
$$+ \frac{(n-1)(n-2)}{1.2} - \frac{(n-1)(n-2)(n-3)}{1.2.3} + \ldots\ldots;$$

the series $1 - (n-1) + \frac{(n-1)(n-2)}{1.2} - \ldots$, of $n-1$ terms, is equal to $(1-1)^{n-1} - (-1)^{n-1}$, that is, to -1, since $n-1$ is an even number. Thus $\underline{|n-1} = M(n) - 1$; therefore $1 + \underline{|n-1}$ is divisible by n.

If n be *not* a prime number, $1 + |\underline{n-1}$ is *not* divisible by n. For suppose p a factor of n; then p is less than $n-1$, and therefore $|\underline{n-1}$ is divisible by p; hence $1 + |\underline{n-1}$ is not divisible by p, and therefore not divisible by n.

718. The following inference may be drawn from Wilson's Theorem: If $2p+1$ be a prime number, $\{|\underline{p}\}^2 + (-1)^p$ is divisible by $2p+1$.

By Wilson's Theorem, since $2p+1$ is a prime number, $1 + |\underline{2p}$ is divisible by $2p+1$. Put n for $2p+1$, then $|\underline{2p}$ may be written thus, $1(n-1)\,2(n-2)\,3(n-3) \ldots\ldots p(n-p)$; if these factors be supposed multiplied out, it is obvious that we shall obtain $(-1)^p 1^2 2^2 3^2 \ldots\ldots p^2$ together with some multiple of n.

Hence $1 + (-1)^p \{|\underline{p}\}^2$ must be divisible by n, and therefore $\{|\underline{p}\}^2 + (-1)^p$ must be divisible by n.

719. Let x denote any positive integer; then the number of positive integers which are less than x and prime to x will be denoted by $L(x)$.

Consider, for example, the positive integer 12; there are 4 positive integers which are less than 12 and prime to 12, namely 11, 7, 5, 1: thus $L(12) = 4$.

720. *If* m *be prime to* n *then* $\mathrm{L(mn) = L(m) \times L(n)}$.

For let $1, a, b, \ldots\ldots m-1$ be the positive integers which are less than m and prime to m; then, r denoting any one of these, the following n positive integers are all less than mn and are all prime to m,

$$r, \quad r+m, \quad r+2m, \ldots\ldots \quad r+(n-1)m.$$

And every positive integer which is less than mn and is prime to m must be of the form $r+pm$, where p is zero or some positive integer less than n, and r is one of the positive integers $1, a, b, \ldots$ $m-1$.

Hence we see that the number of positive integers less than mn and prime to m is $n \times L(m)$.

Out of the positive integers which are less than mn and prime to m we must now determine those which are also prime to n.

Let r have the same meaning as before. If we divide each term of the set

$$r, \quad r+m, \quad r+2m, \ldots\ldots \quad r+(n-1)m$$

by n the remainders will all be different; this is shewn by the method of Art. 707: thus the remainders must be $0, 1, 2, \ldots n-1$; though they will not necessarily occur in this order. If a remainder be prime to n the corresponding dividend is prime to n; and conversely if a dividend is prime to n the corresponding remainder is prime to n. It follows therefore that out of the n positive integers in the above set there are $L(n)$ which are prime to n. And since this holds for each such set of integers as we have considered it follows that $L(mn) = L(m) \times L(n)$.

Hence if l, m, n are all prime to each other, we have

$$L(lmn) = L(lm) \times L(n) = L(l) \times L(m) \times L(n);$$

and a similar result holds for any number of factors which are all prime to each other.

721. *To find the number of positive integers which are less than a given number and prime to it.*

Let N denote the number, and first suppose $N = a^p$, where a is a prime number. The only terms of the series $1, 2, 3, 4, \ldots\ldots N$ which are *not* prime to N are $a, 2a, 3a, 4a, \ldots\ldots \frac{N}{a}a$; and there are $\frac{N}{a}$ of these terms. Hence after rejecting these multiples of a, we have remaining $N - \frac{N}{a}$ terms, that is, $N\left(1-\frac{1}{a}\right)$ terms; thus there are $N\left(1-\frac{1}{a}\right)$ positive integers which are less than N and prime to N.

Next, suppose $N = a^p b^q c^r \ldots\ldots$ where $a, b, c, \ldots\ldots$ are all prime numbers.

Then, by Art. 720,

$$L(N) = L(a^p) \times L(b^q) \times L(c^r) \times \ldots\ldots$$

$$= a^p\left(1-\frac{1}{a}\right) \times b^q\left(1-\frac{1}{b}\right) \times c^r\left(1-\frac{1}{c}\right) \times \ldots\ldots$$

by the first case.

Thus finally if $N = a^p b^q c^r d^s \ldots\ldots$ where $a, b, c, d, \ldots\ldots$ are all prime numbers, the number of positive integers which are less than N and prime to N is

$$N\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\left(1-\frac{1}{d}\right)\ldots\ldots$$

It will be observed that in this theorem *unity* is considered to be one of the positive integers which are less than N and prime to N.

722. *To find the number of divisors of any given number.*

Let N denote the number, and suppose $N = a^p b^q c^r \ldots\ldots$, where $a, b, c, \ldots\ldots$ are prime numbers. It is evident that N will be divisible by any number which is formed by the product of powers of $a, b, c, \ldots\ldots$ provided the exponent of the power of a be comprised between 0 and p, the exponent of the power of b between 0 and q, the exponent of the power of c between 0 and r, and so on; and no other number will divide N. Hence the divisors of N will be the various terms of the product

$$(1 + a + a^2 + \ldots + a^p)(1 + b + b^2 + \ldots + b^q)(1 + c + c^2 + \ldots + c^r)\ldots;$$

the *number* of the divisors will therefore be $(p+1)(q+1)(r+1)\ldots$ This includes among the divisors unity and the number N itself.

723. *To find the number of ways in which a number can be resolved into two factors.*

Let N denote the number, and suppose $N = a^p b^q c^r \ldots\ldots$, where $a, b, c, \ldots\ldots$ are prime numbers. First, suppose N *not* a perfect square; then *one* at least of the exponents $p, q, r, \ldots\ldots$ is an odd number; the required number then is $\frac{1}{2}(p+1)(q+1)(r+1)\ldots\ldots$, because there are *two* divisors of N corresponding to every way in which N can be resolved into two factors. Next suppose N a perfect square, then all the exponents $p, q, r, \ldots\ldots$ are even; the required number is found by increasing $(p+1)(q+1)(r+1)\ldots\ldots$ by unity, and taking half the result; for in this case the square root of N is one of the divisors, and if this be taken as one factor of N, the other factor is equal to it, so that only *one* divisor arises from this mode of resolving N into *two* factors.

It will be observed that in this theorem $N \times 1$ is counted as one of the ways of resolving N into two factors.

724. *To find the sum of the divisors of a number.*

With the notation of Art. 722, we have the *sum* equal to

$$(1+a+a^2+\ldots+a^p)(1+b+b^2+\ldots+b^q)(1+c+c^2+\ldots+c^r)\ldots;$$

that is,

$$\frac{a^{p+1}-1}{a-1}\cdot\frac{b^{q+1}-1}{b-1}\cdot\frac{c^{r+1}-1}{c-1}\ldots\ldots$$

725. *To find the number of ways in which a number can be resolved into two factors which are prime to each other.*

Let the number $N = a^p b^q c^r \ldots.$ as before. Since the two factors are to be prime to each other, we cannot have some power of a in one factor, and some power of a in the other factor, but a^p must occur in one of the factors. Similarly, b^q must occur in one of the factors; and so on. Hence the required number is the same as half the number of divisors of $abc\ldots$, and is therefore 2^{n-1}, where n is the number of *different* prime factors which occur in N.

EXAMPLES OF THE THEORY OF NUMBERS.

1. If p and q are whole numbers, and $p+q$ is an even number, then $p-q$ is also even.

2. Find the least multiplier of 3234 which will make the product a perfect square.

3. Find the least multiplier of 1845 which will make the product a perfect cube.

4. Find the least multiplier of 6480 which will make the product a perfect cube.

5. Find the least multiplier of 13168 which will make the product a perfect cube.

6. If the sum of an odd square number and an even square number is also a square number, then the even square number is divisible by 16.

7. Every square number is of the form $5n$ or $5n \pm 1$.

8. Every cube number is of the form $7n$ or $7n \pm 1$.

9. If a number be both a square and a cube it is of the form $7n$ or $7n+1$.

10. No square number is of the form $3n-1$.

11. No triangular number is of the form $3n-1$.

12. If n be any number whatever, a the difference between n and the next number greater than n which is a square number, and b the difference between n and the next number less than n which is a square number, then $n-ab$ is a square number.

13. If the difference of two numbers which are prime to each other, be an odd number, any power of their sum is prime to every power of their difference.

14. If there be three numbers one of which is the sum of the other two, twice the sum of their fourth powers is a square number.

15. Shew when n is any prime number, that x^n-1 and $(x-1)^n$ will leave the same remainder when divided by n.

16. If $2p+1$ be a prime number and the numbers $1^2, 2^2, \ldots p^2$, be divided by $2p+1$, the remainders are all different.

17. Every even power of every odd number is of the form $8n+1$.

18. Every odd power of 7 is of the form $8n-1$.

19. If n be any integer, n^2-n+1 cannot be a square number.

20. If n be any odd integer, n^2+1 cannot be a square number.

21. If a and x are integers, the greatest value of $ax-2x^2$ is the integer equal to or next less than $\dfrac{a^2}{8}$.

22. Shew that $n(n+1)(2n+1)$ is always divisible by 6.

23. If n be odd, $(n-1)n(n+1)$ is divisible by 24.

24. If n be odd and not divisible by 3, then n^2+5 is divisible by 6.

25. If n be a prime number greater than 5, then n^4-1 is divisible by 240.

26. Shew that $\frac{m^5}{120}-\frac{m^3}{24}+\frac{m}{30}$ is an integer if m be.

27. Shew that n^7-n is always divisible by 42.

28. If n be any prime number and x any integer, prove that x^n and x when divided by n will leave the same remainder.

29. If n be any prime number and N prime to n, then N^m-1 is divisible by n^2, where $m=n(n-1)$.

30. If n be any prime number greater than 2, except 7, then n^6-1 is divisible by 56.

31. If n be any prime number greater than 2 and N any odd number prime to n, then $N^{n-1}-1$ is divisible by $8n$.

32. If n be any prime number greater than 3 and N prime to n, then N^n-N is divisible by $6n$.

33. If n and N be different prime numbers, and each greater than 3, then $N^{n-1}-1$ is divisible by $24n$.

34. Shew that $1^n+2^n+3^n+\ldots+(rn)^n$ is a multiple of n, if n be any prime number greater than 2.

35. Shew that the 10^{th} power of any number is of the form $11n$ or $11n+1$.

36. Shew that the 12^{th} power of any number is of the form $13n$ or $13n+1$.

37. Shew that the 9^{th} power of any number is of the form $19n$ or $19n\pm1$.

38. Shew that the 11^{th} power of any number is of the form $23n$ or $23n\pm1$.

39. Shew that the 20^{th} power of any number is of the form $25n$ or $25n+1$.

40. How many positive integers are less than 140 and prime to 140?

41. How many positive integers are less than 360 and prime to 360?

42. How many positive integers are less than 1000 and prime to 1000?

43. How many positive integers are less than $3^4 \times 7^2 \times 11$ and prime to it?

44. How many positive integers are less than 10^n and prime to it?

45. Find the number of divisors of 140.

46. Find the number of divisors of 1845.

47. Find how many divisors there are of $\lfloor 9$, and the sum of these divisors.

48. Find the number of ways in which 1845 can be resolved into two factors.

49. In how many ways can a line of 100800 inches long be divided into equal parts, each some multiple of an inch?

50. In how many ways can four right angles be divided into equal parts so that each part may be a multiple of the angular unit, (1) when the unit is a degree, (2) when the unit is a grade?

51. How many different positive integral solutions are there of $xy = 10^n$?

52. If N be any number, n the number of its divisors, and P the product of its divisors, shew that $P = N^{\frac{n}{2}}$: shew that N^n is in all cases a complete square.

53. Find the least number which has 30 divisors.

54. Find the least number which has 64 divisors.

55. Suppose a prime to b, and let the series of quantities $a,\ 2a,\ 3a, \dots (b-1)\,a$ be divided by b: prove that the sum of the quotients arising from any two terms equidistant from the be-

ginning and end will be $a-1$, and that the sum of the corresponding remainders will be b.

56. If any number of square numbers be divided by a given number n there cannot be more than $\frac{n}{2}$ different remainders.

57. Express generally the rational values of x and y which satisfy $140x = y^2$.

58. If r, the radix of a scale of notation, be a prime number greater than 2, there are $\frac{r+1}{2}$ different digits in which square numbers terminate in that scale.

59. If any number n can be resolved into the sum of p squares, $2(p-1)n$ can be resolved into the sum of $p(p-1)$ squares.

60. If n be any positive integer $2^{6n} + 15n - 1$ is divisible by 9.

61. If P_r denote the sum of the products of the first n numbers taken r together, $1 + P_1 + P_2 + \ldots + P_{n-1}$ is a multiple of $\lfloor n$.

62. Shew that the 100^{th} power of any number is of the form $125n$ or $125n + 1$.

LIII. PROBABILITY.

726. If an event may happen in a ways and fail in b ways, and all these ways are equally likely to occur, the probability of its happening is $\frac{a}{a+b}$, and the probability of its failing is $\frac{b}{a+b}$. This may be regarded as a definition of the meaning of the word *probability* in mathematical works. The following explanation is sometimes added for the sake of shewing the consistency of the definition with ordinary language: The probability of the happening of the event must, from the nature of the case, be to the probability of its failing as a to b; therefore the probability of its happening is to the sum of the probabilities of its happening and failing as a to $a+b$. But the event must either happen or fail, hence the sum of the probabilities of its happen-

ing and failing is certainty. Therefore the probability of its happening is to certainty as a to $a+b$. So if we represent certainty by unity, the probability of the happening of the event is represented by $\frac{a}{a+b}$.

727. Hence if p be the probability of the happening of an event, $1-p$ is the probability of its failing.

728. The word *chance* is often used in mathematical works as synonymous with *probability*.

729. When the probability of the happening of an event is to the probability of its failing as a to b, the fact is expressed in popular language thus; the *odds* are a to b *for* the event, or b to a *against* the event.

730. Suppose there to be any number of events A, B, C, &c., such that one event must happen and only one can happen; and suppose a, b, c, &c., to be the numbers of ways in which these events can respectively happen, and that all these ways are equally likely to occur, then the probabilities of the events are proportional to a, b, c, &c. respectively. For simplicity let us consider three events, then A can happen in a ways out of $a+b+c$ ways and fail in $b+c$ ways; therefore, by Art. 726, the probability of A's happening is $\frac{a}{a+b+c}$, and the probability of A's failing is $\frac{b+c}{a+b+c}$. Similarly the probability of B's happening is $\frac{b}{a+b+c}$, and the probability of C's happening is $\frac{c}{a+b+c}$.

731. We will now exemplify the mathematical meaning of the word *probability*.

If n balls A, B, C, ..., be thrown promiscuously into a bag and a person draw out one of them, the probability that it will be A is $\frac{1}{n}$; the probability that it will be either A or B is $\frac{2}{n}$.

The same supposition being made, if *two* balls be drawn out the probability that these will be A and B is $\frac{2}{n(n-1)}$. For the number of pairs of balls is the same as the number of combinations of n things taken two at a time, that is, $\frac{1}{2}n(n-1)$; and one pair is as likely to be drawn out as another; therefore the probability of drawing out an assigned pair is $1 \div \frac{1}{2}n(n-1)$, that is, $\frac{2}{n(n-1)}$.

Again, suppose that 3 white balls, 4 black balls, and 5 red balls are thrown promiscuously into a bag, and a person draws out one of them; the probability that this will be a white ball is $\frac{3}{12}$, the probability that it will be a black ball is $\frac{4}{12}$, and the probability that it will be a red ball is $\frac{5}{12}$. But suppose *two* balls to be drawn out: we proceed to estimate the probabilities of the different cases. The number of pairs that can be formed out of 12 things is $\frac{1}{2} \times 12 \times 11$, that is, 66. The number of pairs that can be formed out of the 3 white balls is 3; hence the probability of drawing two white balls is $\frac{3}{66}$. Similarly the probability of drawing two black balls is $\frac{6}{66}$; and the probability of drawing two red balls is $\frac{10}{66}$. Also since each white ball might be associated with each black ball, the number of pairs consisting of one white ball and one black ball is 3×4, that is, 12; hence the probability of drawing a white ball and a black ball is $\frac{12}{66}$. Similarly the probability of drawing a black ball and a red ball is $\frac{20}{66}$; and the probability of drawing a red ball and a white ball is $\frac{15}{66}$. The sum of the six probabilities which we have just found is unity, as, of course, it should be.

We will give one example from a subject which constitutes an important application of the theory of probability. According to the Carlisle Table of Mortality, it appears that out of 6335 persons living at the age of 14 years, only 6047 reach the age of 21 years. As we may suppose that each individual has the same probability of being one of these survivors, we may say that $\frac{6047}{6335}$ is the probability that an individual aged 14 years will reach the age of 21 years: and $\frac{288}{6335}$ is the probability that he will not reach the age of 21 years.

732. Suppose that there are two independent events of which the respective probabilities are known: we proceed to estimate the probability that both will happen.

Let a be the number of ways in which the first event may happen, and b the number of ways in which it may fail, all these ways being equally likely to occur; and let a' be the number of ways in which the second event may happen, and b' the number of ways in which it may fail, all these ways being equally likely to occur. Each case out of the $a+b$ cases may be associated with each case out of the $a'+b'$ cases; thus there are $(a+b)(a'+b')$ compound cases which are equally likely to occur. In aa' of these compound cases both events happen, in bb' of them both events fail, in ab' of them the first event happens and the second fails, and in $a'b$ of them the first event fails and the second happens. Thus

$\frac{aa'}{(a+b)(a'+b')}$ is the probability that both events happen,

$\frac{bb'}{(a+b)(a'+b')}$ is the probability that both events fail,

$\frac{ab'}{(a+b)(a'+b')}$ is the probability that the first event happens and the second event fails,

$\frac{a'b}{(a+b)(a'+b')}$ is the probability that the first event fails and the second event happens.

Thus if p and p' be the respective probabilities of two independent events, pp' is the probability of the happening of both events.

733. The probability of the concurrence of two *dependent* events is the product of the probability of the first into the probability that when that has happened the second will follow. This is only a slight modification of the principle established in the preceding Article, and is proved in the same manner; we have only to suppose that a' is the number of ways in which after the first event has happened the second will follow, and b' the number of ways in which after the first event has happened the second will not follow, all these ways being supposed equally likely to occur.

734. In like manner, if there be any number of *independent* events, the probability that they will all happen is the product of their respective probabilities of happening. Suppose, for example, that there are *three* independent events, and that p, p', p'' are their respective probabilities. By Art. 732, the probability of the concurrence of the first and second events is pp'; then in the same way the probability of the concurrence of the first two events and the third is $pp' \times p''$, that is, $pp'p''$. Similarly the probability that all the events fail is $(1-p)(1-p')(1-p'')$. The probability that the first event happens and the other two events fail is $p(1-p')(1-p'')$; and so on.

735. We will now exemplify the estimation of the probability of compound events.

(1) Required the probability of throwing an ace in the first only of two successive throws with a single die. Here we require a compound event to happen; namely at the first throw the ace is to appear, at the second throw the ace is not to appear. The probability of the first simple event is $\frac{1}{6}$, and of the second simple event $\frac{5}{6}$; hence the required probability is $\frac{5}{36}$.

(2) Suppose 3 white balls, 4 black balls, and 5 red balls to be thrown promiscuously into a bag; required the probability that in two successive trials two red balls will be drawn, *the ball first drawn being replaced before the second trial.* Here the probability of drawing a red ball at the first trial is $\frac{5}{12}$, and the probability is the same of drawing a red ball at the second trial; hence the probability of drawing two red balls is $\left(\frac{5}{12}\right)^2$.

(3) Suppose now that we require the probability of drawing two red balls, *the ball first drawn not being replaced before the second trial.* This will be an example of Art. 733. Here the probability of drawing a red ball at the first trial is $\frac{5}{12}$; if a red ball be drawn at first, out of the eleven balls which remain four are red, and therefore the probability that a second trial will give a red ball is $\frac{4}{11}$; hence the probability of drawing two red balls is $\frac{5}{12} \times \frac{4}{11}$. This is the same result as we found in Art. 731, for the probability of drawing two red balls *simultaneously;* and a little consideration will shew that the results ought to coincide.

(4) Required the probability of throwing an ace with a single die in two trials. The probability of failing the first time is $\frac{5}{6}$, and the probability of failing the second time is also $\frac{5}{6}$; hence the probability of failing twice is $\left(\frac{5}{6}\right)^2$, that is, $\frac{25}{36}$. Hence the probability of *not* failing twice is $1 - \frac{25}{36}$, that is, $\frac{11}{36}$; this is therefore the probability of succeeding.

(5) In how many trials will the probability of throwing an ace with a single die amount to $\frac{1}{2}$? Suppose x the number

of trials; therefore the probability of failing x times in succession is $\left(\frac{5}{6}\right)^x$, by Art. 734. Hence the probability of succeeding is $1-\left(\frac{5}{6}\right)^x$; therefore $1-\left(\frac{5}{6}\right)^x=\frac{1}{2}$; hence $\left(\frac{5}{6}\right)^x=\frac{1}{2}$; hence $x\log\frac{5}{6}=\log\frac{1}{2}$, therefore $x=\frac{\log 2}{\log 6-\log 5}$. By using the values of the logarithms, we find $x=3\cdot8$ nearly. Thus we conclude that in 3 trials the probability of success is less than $\frac{1}{2}$, and that in 4 trials it is greater than $\frac{1}{2}$.

(6) In how many trials is it an even wager to throw sixes with two dice? The probability of sixes at a single throw with two dice is $\frac{1}{6}\times\frac{1}{6}$, that is, $\frac{1}{36}$; hence the probability of not having sixes is $\frac{35}{36}$. Suppose x the number of trials; then we have $1-\left(\frac{35}{36}\right)^x=\frac{1}{2}$; hence $\left(\frac{35}{36}\right)^x=\frac{1}{2}$; therefore $x=\frac{\log 2}{\log 36-\log 35}$. By using the values of the logarithms, we find x lies between 24 and 25, which we interpret as before.

(7) To find the probability that two individuals, A and B, whose ages are known, will be alive at the end of a year. Let p be the probability that A will be alive at the end of a year, p' the probability that B will be alive; then pp' is the probability that both will be alive at the end of a year. The values of p and p' can be found from the Tables of Mortality in the manner exemplified in Art. 731.

(8) To find the probability that one at least of two individuals, A and B, whose ages are known, will be alive at the end of a given number of years. Let p be the probability that A will be alive at the end of the given number of years, p' the probability that B will be alive. Then $1-p$ is the probability that A will be dead, and $1-p'$ is the probability that B will be dead. Hence $(1-p)(1-p')$ is the probability that *both* will be dead. The probability that both will not be dead, that is, that one at least will be alive, is $1-(1-p)(1-p')$, that is, $p+p'-pp'$.

736. If an event may happen in different independent ways, the probability of its happening is the sum of the probabilities of its happening in the different independent ways.

If the independent ways of happening are all equally probable, this proposition is merely a repetition of the definition of probability given in Art. 726; and if they are not all equally probable, the proposition seems to follow so naturally from that definition, that it is often assumed without any remark. The following method of illustrating it is sometimes given: Suppose two urns A and B; let A contain 2 white balls and 3 black balls, and let B contain 3 white balls and 4 black balls; required the probability of obtaining a white ball by a single drawing from one of the urns taken at random. Since each urn is equally likely to be taken, the probability of taking the urn A is $\frac{1}{2}$, and the probability then of drawing a white ball from it is $\frac{2}{5}$; hence the probability of obtaining a white ball so far as it depends on A is $\frac{1}{2} \times \frac{2}{5}$. Similarly, the probability of obtaining a white ball so far as it depends on B is $\frac{1}{2} \times \frac{3}{7}$. Hence the proposition asserts that the probability of obtaining a white ball is $\frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{7}$, that is, $\frac{1}{2}\left(\frac{2}{5} + \frac{3}{7}\right)$. The accuracy of this result may be confirmed by the following steps: First, without affecting the question, we may replace the urn A by an urn A', containing any number of balls we please, *provided the ratio of the white balls to the black balls be that of* 2 *to* 3; and similarly, we may replace the urn B by an urn B', containing any number of balls we please, *provided the ratio of the white balls to the black balls be that of* 3 *to* 4. Let then A' contain 14 white balls and 21 black balls, and let B' contain 15 white balls and 20 black balls; thus A' and B' each contain 35 balls. Secondly, without affecting the question, we may now suppose the balls in A' and B' collected in a single urn; thus there will be

70 balls, of which 29 are white. The probability of drawing a white ball will therefore be $\frac{29}{70}$; that is, $\frac{14+15}{70}$; that is, $\frac{1}{2}\left(\frac{14}{35}+\frac{15}{35}\right)$; that is, $\frac{1}{2}\left(\frac{2}{5}+\frac{3}{7}\right)$.

737. The probability of the happening of one or other of two events which cannot concur is the sum of their separate probabilities. For the complete event we are considering occurs if the first event happens, or if the second event happens; thus the proposition is a case of the preceding proposition.

738. The probability of the happening of an event in one trial being known, required the probability of its happening once, twice, three times, &c., exactly in n trials.

Let p denote the probability of the happening of the event in one trial, and q the probability of its failing, so that $q=1-p$. The probability that in n trials the event will occur in one *assigned trial*, and fail in the other $n-1$ trials is pq^{n-1} (Art. 734); and since there are n trials, the probability of its happening in *some one* of these and failing in the rest is npq^{n-1}. The probability that in n trials the event will occur in two *assigned trials*, and fail in the other $n-2$ trials, is p^2q^{n-2}; and there are $\frac{n(n-1)}{1.2}$ ways in which the event may happen twice and fail $n-2$ times in n trials; therefore the probability that it will happen exactly twice in n trials is $\frac{n(n-1)}{1.2}p^2q^{n-2}$. Similarly the probability that the event will happen exactly three times in n trials is $\frac{n(n-1)(n-2)}{1.2.3}p^3q^{n-3}$; and the probability that it will happen exactly r times in n trials is $\frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r}p^rq^{n-r}$.

Similarly, the probability that the event will fail exactly r times in n trials is $\frac{n(n-1)\ldots\ldots(n-r+1)}{\lfloor r}p^{n-r}q^r$.

739. Thus if $(p+q)^n$ be expanded by the Binomial Theorem

in the series $p^n + np^{n-1}q + \&c.$, the terms will represent respectively the probabilities of the happening of the event exactly n times, $n-1$ times, $n-2$ times, &c., in n trials. Hence we may determine what is the most probable number of successes and failures in n trials; we have only to ascertain the *greatest term* in the above series. Let us suppose, for example, that $p=\frac{a}{a+b}$, $q=\frac{b}{a+b}$, $n=m(a+b)$, where a, b, and m are integers; then, by Art. 511, the most probable case is, that of r failures and $n-r$ successes, where r is the greatest integer contained in $\frac{n+1}{\frac{p}{q}+1}$, that is, in $mb+\frac{b}{a+b}$; so that $r=mb$, and $n-r=ma$. The most probable case therefore is, that in which the numbers of successes and failures are proportional to the probabilities of success and failure respectively in a single trial.

740. The probability of the happening of the event *at least* r times in n trials is

$$p^n + np^{n-1}q + \frac{n(n-1)}{1.2}p^{n-2}q^2 + \ldots\ldots + \frac{n(n-1)(n-2)\ldots\ldots\ldots(r+1)}{\lfloor n-r}p^r q^{n-r};$$

for if the event happen every time, or fail only once, twice, ... $(n-r)$ times, it happens r times; therefore the probability of the happening of the event *at least* r times is the sum of the probabilities of its happening every time, of failing only once, twice, ... $n-r$ times; and the sum of these is the expression given above.

For example; in five throws with a single die what is the probability of throwing *exactly* three aces? and what is the probability of throwing *at least* three aces?

Here $p=\frac{1}{6}$, $q=\frac{5}{6}$, $n=5$, $r=3$; thus the probability of throwing exactly three aces is $\frac{5.4.3}{1.2.3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$, that is, $\frac{250}{7776}$;

and the probability of throwing at least three aces is $\left(\frac{1}{6}\right)^5 + 5\left(\frac{1}{6}\right)^4 \frac{5}{6} + \frac{5 \cdot 4}{1 \cdot 2}\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$; that is, $\frac{276}{7776}$.

The following four Articles contain problems illustrating the subject.

741. *A* and *B* play a set of games, in which *A*'s probability of winning a single game is p, and *B*'s probability is q; required the probability of *A*'s winning m games out of $m+n$.

If *A* wins in *exactly* $m+r$ games he must win the last game and $m-1$ games out of the preceding $m+r-1$ games; the probability of this is $Mp^{m-1}q^r p$, where M is the number of combinations of $m+r-1$ things taken $m-1$ at a time; that is, the probability is

$$\frac{\lfloor m+r-1}{\lfloor m-1 \lfloor r} p^m q^r.$$

Now in order that *A* may win m games out of $m+n$, he must win m games in *exactly* m games, or in *exactly* $m+1$ games, ..., or in *exactly* $m+n$ games. Hence the probability required is the sum of the series obtained by giving to r the values 0, 1, 2, ... n in the expression $\frac{\lfloor m+r-1}{\lfloor m-1 \lfloor r} p^m q^r$, that is, the required probability is

$$p^m \left\{1 + mq + \frac{m(m+1)}{1 \cdot 2} q^2 + \dots\dots + \frac{m(m+1) \dots\dots (m+n-1)}{\lfloor n} q^n\right\}.$$

If *A* in order to win the set must win m games *before* *B* wins n games, *A* must win m games out of $m+n-1$; the probability of this event is given by the preceding expression *with the omission of the last term.* Similarly, the probability of *B*'s winning n games out of $m+n-1$ is

$$q^n \left\{1 + np + \frac{n(n+1)}{1 \cdot 2} p^2 + \dots\dots + \frac{n(n+1) \dots\dots (n+m-2)}{\lfloor m-1} p^{m-1}\right\}.$$

This problem is celebrated in the history of the theory of probabilities, as the first of any difficulty which was discussed; it was proposed to Pascal in 1654, with the simplification however which arises from supposing p and q to be equal.

It appears from the preceding investigation that the probability of A's winning r games out of n is

$$p^r \left\{1 + rq + \frac{r(r+1)}{1.2} q^2 + \ldots\ldots + \frac{r(r+1)\ldots\ldots(n-1)}{\lfloor n-r} q^{n-r}\right\};$$

but this probability must from the nature of the question be the same as the probability of the happening of an event at least r times in n trials when the probability of the event is p. Thus the expression just given must be equivalent to that given in Art. 740; we may verify this as follows: Denote the expression just given by v_n, and that given in Art. 740 by u_n, and let v_{n+1} and u_{n+1} denote respectively what they become when n is changed to $n+1$; then we shall shew that if $u_n = v_n$ when n has any specific value, then also $u_{n+1} = v_{n+1}$.

We have $u_n = u_n(p+q)$; now $u_n(p+q)$ gives two series, and when the like terms in these two series are united we obtain

$$u_n(p+q) = u_{n+1}$$

$$-\frac{(n+1)n\ldots\ldots(r+1)}{\lfloor n+1-r} p^r q^{n+1-r} + \frac{n(n-1)\ldots\ldots(r+1)}{\lfloor n-r} p^r q^{n+1-r};$$

therefore $$u_{n+1} = u_n(p+q) + \frac{n(n-1)\ldots\ldots r}{\lfloor n+1-r} p^r q^{n+1-r};$$

and obviously $$v_{n+1} = v_n + \frac{n(n-1)\ldots\ldots r}{\lfloor n+1-r} p^r q^{n+1-r}.$$

This shews that $u_{n+1} = v_{n+1}$ if $u_n = v_n$. Now obviously u_n is equal to v_n when $n = r$; therefore u_n is equal to v_n for every value of n greater than r.

For some more remarks on this problem the student is referred to the *History of Probability*, page 98.

742. A bag contains $n+1$ tickets which are marked with the numbers 0, 1, 2, ... n, respectively. A ticket is drawn and replaced: required the probability that after r drawings the sum of the numbers drawn is s.

The number of drawings which can occur is $(n+1)^r$, for any one of the tickets may be drawn each time. The number of ways in which the sum of the drawings will amount to s is the coefficient of x^s in the expansion of $(x^0 + x^1 + x^2 + \ldots\ldots + x^n)^r$; because this coefficient arises from the different modes of forming s by the addition of r numbers of the series 0, 1, 2, ... n. Thus the probability required is found by dividing this coefficient by $(n+1)^r$.

The above coefficient may be obtained by the Multinomial Theorem; or we may proceed thus:

$$(x^0 + x^1 + x^2 + \ldots\ldots + x^n)^r = \left\{\frac{1 - x^{n+1}}{1 - x}\right\}^r = (1 - x^{n+1})^r (1 - x)^{-r};$$

and $(1 - x^{n+1})^r = 1 - rx^{n+1} + \frac{r(r-1)}{1.2}x^{2n+2} - \frac{r(r-1)(r-2)}{1.2.3}x^{3n+3} + \ldots\ldots$

$$(1 - x)^{-r} = 1 + rx + \frac{r(r+1)}{1.2}x^2 + \frac{r(r+1)(r+2)}{1.2.3}x^3 + \ldots\ldots$$

We must therefore find the coefficient of x^s in the *product* of these two series; it is

$$\frac{r(r+1)\ldots\ldots(r+s-1)}{\lfloor s} - r.\frac{r(r+1)\ldots\ldots(r+s-n-2)}{\lfloor s-n-1}$$

$$+\frac{r(r-1)}{1.2}.\frac{r(r+1)\ldots\ldots(r+s-2n-3)}{\lfloor s-2n-2} - \&c.;$$

this series is to stop at the $(i+1)^{\text{th}}$ term, where i is the greatest integer contained in $\frac{s}{n+1}$; then the required probability is obtained by dividing this series by $(n+1)^r$.

It is not difficult to determine the probability that after r drawings the sum of the numbers drawn shall not exceed s; see *History of Probability*, page 208.

743. A box has three equal compartments, and four balls are thrown in at random: determine the probability of the different arrangements, assuming that it is equally likely that any ball will fall into any compartment.

Since it is equally likely that a ball will fall into any compartment there are 3 equally likely cases for *each* ball; and on the whole there are 3^4 *equally likely cases.* Now there are four possible arrangements.

I. All the balls may be in one compartment; this can happen in 3 ways.

II. *Any* three of the balls may be in *any* one of the compartments, and the remaining ball in *either* of the remaining compartments; this can happen in 4.3.2 ways.

III. *Any* two of the balls may be in any *one* compartment, and one of the remaining balls in one of the remaining compartments and the other in the other; this can happen in 6.3.2 ways.

IV. *Any* two of the balls may be in any one compartment, and the other two balls in either of the remaining compartments; this can happen in 6.3 ways.

Thus the probabilities of the different arrangements are respectively $\frac{3}{81}, \frac{24}{81}, \frac{36}{81}, \frac{18}{81}$; the sum of these fractions is, of course, unity.

In the preceding solution the point which deserves particular attention is the statement that there are 81 equally likely cases; for when this is admitted all the rest follows necessarily. If this is not admitted and the student substitutes any other statement in the place of it, he will be really taking another problem instead of the one intended. In fact in a problem which relates to permutations, combinations, or probabilities, it is not unfrequently found that different results are obtained because different meanings have been attached to the enunciation; especial care is necessary in these subjects to ensure that whatever meaning is given to the enunciation should be consistently retained throughout the solution.

We will next consider the general problem of which the present is a particular case.

744. A box is divided into m equal compartments. If n balls are thrown in promiscuously, required the probability that there will be a compartments each containing α balls, b compartments each containing β balls, and so on, where $a\alpha + b\beta + c\gamma + \ldots\ldots = n$.

Since *any* ball may fall into *any* compartment, there are m^n cases equally likely to occur. We shall first shew that the number of *different* ways in which the n balls can be divided into $a+b+c+\dots\dots$ parcels containing $\alpha, \beta, \gamma, \dots\dots$ balls respectively is

$$\frac{\lfloor n}{\{\lfloor \alpha\}^a \{\lfloor \beta\}^b \{\lfloor \gamma\}^c \dots\dots \lfloor a \lfloor b \lfloor c \dots\dots} = N \text{ say.}$$

For consider first in how many ways a parcel of α balls can be selected from n balls; the result is $\dfrac{n(n-1)\dots\dots(n-\alpha+1)}{\lfloor \alpha}$ ways. Then consider in how many ways a *second* parcel of α balls can be selected from the *remaining* $n-\alpha$ balls; the result is $\dfrac{(n-\alpha)(n-\alpha-1)\dots\dots(n-2\alpha+1)}{\lfloor \alpha}$. Similarly a *third* parcel of α balls can be selected from the *remaining* $n-2\alpha$ balls in $\dfrac{(n-2\alpha)(n-2\alpha-1)\dots\dots(n-3\alpha+1)}{\lfloor \alpha}$ ways. We might then at first infer that the number of ways in which *three* parcels of α balls each can be selected from n balls is $\dfrac{n(n-1)\dots\dots(n-3\alpha+1)}{\lfloor \alpha \lfloor \alpha \lfloor \alpha}$, and this is correct in a certain sense; but each distinct group of three parcels has in this way occurred $\lfloor 3$ times, and we must therefore divide by $\lfloor 3$ in order to obtain the number of *different* ways in which *three* parcels of α balls each can be selected from n balls. Similarly the number of *different* ways in which a parcels of α balls each can be selected from n balls is $\dfrac{n(n-1)\dots\dots(n-a\alpha+1)}{\{\lfloor \alpha\}^a \lfloor a}$. By proceeding thus we obtain the proposed result.

Now the number of ways in which the parcels can be arranged in the m compartments is $m(m-1)(m-2)\dots\dots(m-s+1)$, where $s=a+b+c+\dots\dots$ Hence, the probability required is

$$\frac{Nm(m-1)(m-2)\dots\dots(m-s+1)}{m^n}.$$

For example, suppose six balls thrown into a box which has three compartments. The seven possible modes of distribution are, 6, 0, 0; 1, 5, 0; 2, 4, 0; 3, 3, 0; 1, 1, 4; 1, 2, 3; 2, 2, 2; and their respective probabilities are fractions whose common denominator is 243, and numerators 1, 12, 30, 20, 30, 120, 30.

745. If p represent a person's chance of success in any transaction, and m the sum of money which he will receive in case of success, then the sum of money denoted by pm is called his *expectation.* This is a *definition* of the meaning we shall attach to the word *expectation*, and might of course be stated arbitrarily without any further remark; it is however usual to illustrate the propriety of the definition as follows. Suppose that there are $m+n$ slips of paper, each having the name of a person written upon it, and no name recurring; let these be placed in a bag, and one slip drawn at random, and suppose that the person whose name is drawn is to receive £a. Now all the expectations must be of equal value, because each person has the *same* chance of obtaining the prize; and the sum of the expectations must be worth £a, because if one person bought up the interests of all the persons named, he would be certain of obtaining £a. Hence, if £x denote the expectation of each person, we have $(m+n)\,x=a$;

thus $$x=\frac{a}{m+n}.$$

Also, it is evident that the value of the expectation of two persons is the sum of the values of their respective expectations; and so for three or more persons. Hence the value of the expectation of m persons is $\frac{ma}{m+n}$. Now suppose that one person has his name on m of the slips; then his expectation is the same as the sum of the expectations of m persons, each of whom has his name on one slip; that is, his expectation is $\frac{ma}{m+n}$. But his chance of winning the prize is $\frac{m}{m+n}$, since he has m cases out of $m+n$ in his favour; thus his expectation is the product of his chance of success into the sum of money which he will receive in case of success.

746. An event has happened which must have arisen from some one of a given number of causes: required the probability of the existence of each of the causes.

Let there be n causes, and suppose that the probability of the existence of these causes was estimated at $P_1, P_2, \dots P_n$ respectively, *before* the event took place. Let p_1 denote the probability of the event on the hypothesis of the existence of the first cause, let p_2 denote the probability of the event on the hypothesis of the existence of the second cause, and so on. Then the probability of the existence of the r^{th} cause, estimated *after* the event, is $\frac{P_r p_r}{\Sigma Pp}$, where ΣPp stands for $P_1 p_1 + P_2 p_2 + \dots + P_n p_n$.

From our first notions of probability we must admit that the probability that the r^{th} cause was the true cause is *proportional* to the antecedent probability that the event would happen from this cause, and may therefore be represented by $CP_r p_r$. And since some one of the causes must be the true cause we have $C\{P_1 p_1 + P_2 p_2 + \dots + P_n p_n\} = 1$, therefore $C = \frac{1}{\Sigma Pp}$; therefore the probability that the r^{th} cause was the true cause is $\frac{P_r p_r}{\Sigma Pp}$.

747. The preceding Article will require some illustration before it will be fully appreciated by the student. Let there be, for example, two urns, one containing 7 white balls and 3 black balls, and the other 5 white balls and 1 black ball; suppose that a white ball has been drawn, and we wish to know what the probability is that it came from the first urn, and what the probability is that it came from the second urn. It must have come from one of the two urns, so that the sum of the required probabilities is unity. Instead of the given urns let us substitute two others which have the whole number of balls the same in each urn, and such that each urn has its white and black balls in the same proportion as the urn which it replaces. Thus we may suppose one urn with 21 white balls and 9 black balls, and the other with 25 white balls and 5 black balls. Each urn now contains 30 balls, and the chance

of each ball being drawn, is the same. Since, by supposition, a white ball *is* drawn we may suppose the black balls to have been removed, and all the white balls put into a new urn. Thus there would be 46 white balls; and the probability that the white ball drawn was one of the 21 is $\frac{21}{46}$, and the probability that it was one of the 25 is $\frac{25}{46}$. Now here $p_1 = \frac{7}{10}$, and $p_2 = \frac{5}{6}$; thus $\frac{p_1}{p_1 + p_2} = \frac{21}{46}$, and $\frac{p_2}{p_1 + p_2} = \frac{25}{46}$. Thus the result agrees with that given by the theorem in Art. 746, supposing that P_1 and P_2 are equal.

Next, suppose that there had been 4 urns, each having 7 white balls and 3 black balls, and 3 urns, each having 5 white balls and 1 black ball. In this case, by proceeding in the manner just shewn, we may deduce that the probability that a white ball which was drawn came from the group of 4 similar urns is $\frac{4 \times 21}{4 \times 21 + 3 \times 25}$; and the probability that it came from the group of 3 similar urns is $\frac{3 \times 25}{4 \times 21 + 3 \times 25}$. Now let us apply the theorem of Art. 746 to estimate the probability that the white ball came from the first group and the probability that it came from the second group. Since there are 7 urns, of which 4 are of the first kind and 3 of the second, we take $P_1 = \frac{4}{7}$, and $P_2 = \frac{3}{7}$; also $p_1 = \frac{7}{10}$, and $p_2 = \frac{5}{6}$. Thus

$$Q_1 = \frac{\frac{4}{7} \times \frac{7}{10}}{\frac{4}{7} \times \frac{7}{10} + \frac{3}{7} \times \frac{5}{6}}, \quad Q_2 = \frac{\frac{3}{7} \times \frac{5}{6}}{\frac{4}{7} \times \frac{7}{10} + \frac{3}{7} \times \frac{5}{6}},$$

and these results agree with those which we have already indicated.

748. It is usual to call the quantities $P_1, P_2, \ldots P_n$ of Art. 746 the *a priori* probabilities of the existence of the respective causes; and $Q_1, Q_2, \ldots Q_n$ the *a posteriori* probabilities. Students

are sometimes perplexed in endeavouring to estimate $P_1, P_2, \ldots P_n$; the safest plan is to observe that the *product* $P_r p_r$ denotes the probability that the event will happen as the result of the r^{th} cause; and the correctness of the *product* is the important part of the solution, because P_r and p_r do not occur *separately* in the results. The whole proposition may be best understood if arranged in the following order. First suppose the different causes all equally probable *before* the observed event; let ϖ_r denote the probability of the occurrence of the event on the hypothesis of the existence of the r^{th} cause; then the probability of the r^{th} cause, estimated *after* the occurrence of the observed event, is $\frac{\varpi_r}{\Sigma\varpi}$. This seems nearly self-evident, and if any doubt remains it may be removed by the mode of illustration given in the first part of Art. 747. Secondly, suppose that the terms in $\Sigma\varpi$ can be arranged in groups; suppose there to be μ_1 terms in the first group, and that each term is equal to p_1, suppose there to be μ_2 terms in the second group, and that each term is equal to p_2, and so on, the last group consisting of μ_n terms, each equal to p_n. Then $\Sigma\varpi$ may be written $\Sigma\mu p$, where the series $\Sigma\mu p$ consists of n terms. Thus the probability of the r^{th} cause is $\frac{\varpi_r}{\Sigma\mu p}$. Also the probability of the first group of causes is the sum of the separate probabilities of the members of that group, that is, $\frac{\mu_1 p_1}{\Sigma\mu p}$. Similar expressions hold for the probabilities of the other groups. Thus we finally arrive at the results given in Art. 746, where, in fact,

$$Q_1 = \frac{\mu_1 p_1}{\Sigma\mu p}, \qquad Q_2 = \frac{\mu_2 p_2}{\Sigma\mu p}, \text{ \&c.}$$

749. When an event has been observed, we may, by Art. 746, estimate the probability of each cause from which that event could have arisen; we may then proceed to estimate the probability that the event will occur again, or that some other event will occur. For by Art. 736 we multiply the probability of each cause by the probability of the happening of the required event on

the hypothesis of the existence of that cause, and the sum of all such products is the probability of the happening of the required event.

For example, a bag contains 3 balls, and it is not known how many of these are white; a white ball has been drawn and replaced, what is the probability that another drawing will give a white ball?

There are three possible hypotheses: (1) all the balls may be white, (2) only two of the balls may be white, (3) only one of the balls may be white. We have first to find the probability of each hypothesis by the method of Art. 746. On the first hypothesis, the observed event is certain, that is, the probability of it is 1; on the second hypothesis, the probability of the observed event is $\frac{2}{3}$; on the third hypothesis, the probability of the observed event is $\frac{1}{3}$. Hence, assuming that before the observed event the three hypotheses were equally probable, we have *after* the observed event,

$$\text{probability of first hypothesis} = 1 \div \left\{1 + \frac{2}{3} + \frac{1}{3}\right\} = \frac{1}{2},$$

$$\text{probability of second hypothesis} = \frac{2}{3} \div \left\{1 + \frac{2}{3} + \frac{1}{3}\right\} = \frac{1}{3},$$

$$\text{probability of third hypothesis} = \frac{1}{3} \div \left\{1 + \frac{2}{3} + \frac{1}{3}\right\} = \frac{1}{6}.$$

The probability that another drawing will give a white ball is $\frac{1}{2} \times 1$, so far as it depends on the first hypothesis; it is $\frac{1}{3} \times \frac{2}{3}$, so far as it depends on the second hypothesis; and it is $\frac{1}{6} \times \frac{1}{3}$, so far as it depends on the third hypothesis. Hence the required probability is $\frac{1}{2} + \frac{2}{9} + \frac{1}{18}$; that is, $\frac{7}{9}$.

Suppose that in the enunciation of this problem instead of the words "it is not known how many of these are white" we had the words "it is known that each ball is either white or black." We may understand the new enunciation as equivalent to the old, and so give the same solution as before. We may however, and perhaps most naturally, understand the new enunciation differently, namely that the probability that each ball is white is to be taken as $\frac{1}{2}$ before the observed event. In this case we cannot assume that the three hypotheses are equally probable before the observed event; the probabilities must be $\frac{1}{8}$, $\frac{3}{8}$, and $\frac{3}{8}$ respectively by Art. 734. Then *after* the observed event we shall obtain $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$ respectively for the probabilities. And the probability that another drawing will give a white ball is $\frac{1}{4}+\frac{1}{3}+\frac{1}{12}$.

750. We give another example. Suppose a bag in which the ratio of the number of white balls to the whole number of balls is unknown, and it is equally probable, *a priori*, that the ratio is any one of the following quantities x, $2x$, $3x$, ... nx; suppose a white ball to be drawn and replaced: required the probability that another drawing will give a white ball.

Here n hypotheses can be formed. On the first hypothesis the probability of the observed event is x, on the second hypothesis it is $2x$, on the third $3x$, and so on. Hence the probability of the first hypothesis is $\frac{x}{x(1+2+\dots\dots+n)}$; that is, $\frac{2}{n(n+1)}$. The probability of the second hypothesis is $\frac{2\times 2}{n(n+1)}$. The probability of the third hypothesis is $\frac{2\times 3}{n(n+1)}$. And so on. Hence the probability that another drawing will give a white ball is $\frac{2x}{n(n+1)}$ on the first hypothesis, $\frac{2x\times 2^2}{n(n+1)}$ on the second hypothesis, $\frac{2x\times 3^2}{n(n+1)}$ on the third, and so on. Hence the required probability is

$$\frac{2x}{n(n+1)}\left\{1^2+2^2+\dots\dots+n^2\right\};$$

that is, $\frac{2x}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6}$; that is, $\frac{x(2n+1)}{3}$.

When n is very great this approximates to $\frac{2nx}{3}$. If the ratio of the number of the white balls to the whole number of balls is equally likely, *a priori*, to have *any* value between zero and unity, then x is indefinitely small and $nx = 1$, so that the required probability is $\frac{2}{3}$.

751. The following problems will illustrate the subject.

(1) A bag contains m white balls and n black balls; if $p + q$ balls are drawn out, what is the probability that there will be p white balls and q black balls occurring *in an assigned order?* We suppose p less than m and q less than n; and the balls are not replaced in the bag after being drawn out.

Suppose, for example, that the first ball is required to be white, the second to be black, the third to be black, the fourth to be white, and so on in any assigned order. Then the required probability is the product of

$$\frac{m}{m+n}, \frac{n}{m+n-1}, \frac{n-1}{m+n-2}, \frac{m-1}{m+n-3}, \ldots\ldots$$

therefore the required probability is

$$\frac{m(m-1)(m-2)\ldots(m-p+1)\,n(n-1)(n-2)\ldots(n-q+1)}{(m+n)(m+n-1)(m+n-2)\ldots(m+n-p-q+1)};$$

and it will be observed that so long as p white balls and q black balls are required, the probability is the same *whatever may be the assigned order* in which they are to occur.

(2) The suppositions being the same as in (1), what is the probability of p white balls and q black balls occurring *in any order whatever?*

Let N represent the number of different orders in which p white balls and q black balls can occur; then the required probability is obtained by multiplying the probability found in (1) by N. And $N = \frac{\underline{|p+q}}{\underline{|p}\,\underline{|q}}$.

The problems (1) and (2) are introductory to one which we shall now consider.

(3) A bag contains m balls which are known to be all either white or black, but how many of each kind is unknown; suppose p white balls and q black balls have been drawn and *not replaced*; find the probability that another drawing will give a white ball.

The observed event here is the drawing of p white balls and q black balls. To render this possible, the original number of white balls may have been any number from $m-q$ to p inclusive, and the number of black balls any number from q to $m-p$ inclusive. Let us denote the hypothesis of $m-q$ white and q black by H_1, and the hypothesis of $m-q-1$ white and $q+1$ black by H_2, and so on. Then H_1 gives for the probability of the observed event

$$M \times \frac{(m-q)(m-q-1)\ldots\ldots(m-q-p+1)\,1\,.\,2\,.\,3\ldots\ldots q}{m(m-1)\ldots\ldots(m-q-p+1)},$$

where M denotes the number of different ways in which p white balls and q black balls can be combined in $p+q$ trials. Put C for

$$\frac{M}{m(m-1)\ldots\ldots(m-q-p+1)};$$

then H_1 gives for the probability of the observed event CP_1Q_1,

where $\quad P_1=(m-q)(m-q-1)\ldots\ldots(m-q-p+1)$,

and $\quad Q_1 = 1\,.\,2\,.\,3\ldots\ldots q.$

Also, H_2 gives for the probability of the observed event CP_2Q_2,

where $\quad P_2=(m-q-1)\ldots\ldots(m-q-p)$,

and $\quad Q_2 = 2\,.\,3\,.\,4\ldots\ldots q\,(q+1).$

Thus, if $n=m-p-q+2$, we find for the probability of H_1,

$$\frac{P_1Q_1}{P_1Q_1+P_2Q_2+\ldots\ldots+P_{n-1}Q_{n-1}}; \text{ this we may denote by } \frac{P_1Q_1}{S}.$$

Similarly the probability of H_2 is $\frac{P_2Q_2}{S}$; and so on. Now the probability of drawing a white ball on another trial

on the hypothesis H_1 is $\frac{P_1Q_1}{S} \times \frac{m-p-q}{m-p-q}$;

on the hypothesis H_2 is $\frac{P_2Q_2}{S} \times \frac{m-p-q-1}{m-p-q}$;

and so on. Thus the whole probability of drawing a white ball is

$$\frac{1}{S.(m-p-q)}\{P_1Q_1(m-p-q)+P_2Q_2(m-p-q-1)+\&c.\}.$$

The series in brackets is of the same kind as S with $p+1$ written instead of p, the number of terms being *one* less than in S.

Now by Art. 670, $S=\dfrac{\lfloor p \lfloor q}{\lfloor p+q+1} \times \dfrac{\lfloor n-1+p+q}{\lfloor n-2}$,

hence the series within brackets is $\dfrac{\lfloor p+1 \lfloor q}{\lfloor p+q+2} \dfrac{\lfloor n-1+p+q}{\lfloor n-3}$;

and the required probability is $\dfrac{p+1}{p+q+2} \times \dfrac{n-2}{m-p-q} = \dfrac{p+1}{p+q+2}$.

For a more general investigation connected with this important problem the student is referred to the *History of Probability*, page 454.

752. The mathematical theory of probability has been applied to estimate the probability of statements which are supported by assertions or by arguments. We will give some examples.

The probability that A speaks truth is p, and the probability that B speaks truth is p'; what is the probability of the truth of an assertion which they agree in making? There are two possible hypotheses; (1) that the assertion is true, (2) that it is not. If it be true, the chance that they both make the assertion is pp'; if it be false, the chance that they both make it is $(1-p)(1-p')$. Hence, by Art. 746, the probabilities of the truth and the falsehood of the assertion are respectively

$$\frac{pp'}{pp'+(1-p)(1-p')} \text{ and } \frac{(1-p)(1-p')}{pp'+(1-p)(1-p')}.$$

Similarly, if the assertion be also made by a third person whose probability of speaking truth is p'', the probabilities of the truth and the falsehood of the assertion are respectively

$$\frac{pp'p''}{pp'p''+(1-p)(1-p')(1-p'')} \text{ and } \frac{(1-p)(1-p')(1-p'')}{pp'p''+(1-p)(1-p')(1-p'')};$$

and so on if more persons join in the assertion.

753. We will make a few remarks on the preceding Article.

When we say that the probability of A's speaking truth is p, we mean that out of a large number of statements made by A, the ratio of the number that are true to the number that are not true is that of p to $1-p$; thus the value of p depends on the correctness of A's judgement as well as on his veracity.

The result in Art. 752 gives the probability of the truth of the assertion, so far as that truth depends solely on the testimony of the witnesses considered; there may be from other sources additional evidence for or against the assertion. Thus the person who is estimating the probability may himself have a conviction more or less decided in favour of the assertion which is independent of the testimony he receives from the witnesses. It has been proposed to combine this conviction with the testimonies which are considered in the problem. Thus, if there be two witnesses with probabilities p and p' respectively of speaking the truth, and a third person estimates the probability of the truth of the assertion at p'' from his own independent sources of belief, then to him the odds in favour of the truth of the assertion are

$$pp'p'' \text{ to } (1-p)(1-p')(1-p'').$$

Still the result is considered unsatisfactory by some writers, who object with great reason to the solution on the ground that it omits all consideration of the circumstance that it is the *same* occurrence to which the several testimonies are offered. In the following problem this circumstance is expressly considered.

754. Two persons, whose probabilities of speaking the truth are p and p' respectively, assert that a *specified* ticket has been drawn out of a bag containing n tickets: required the probability of the truth of the assertion.

The *observed event* here is the coincident testimony of A and B in favour of a specified ticket.

Here $\frac{1}{n}$ is the *a priori* probability that the specified ticket would be drawn. The probability of the observed event on the hypothesis that the specified ticket was drawn is then $\frac{pp'}{n}$. The probability of the observed event on the hypothesis that it was not drawn might at first be supposed to be $(1-p)(1-p')\frac{n-1}{n}$; but if the persons have no inducement to select the specified ticket among those really undrawn, this expression must be multiplied by $\frac{1}{(n-1)^2}$, which is the probability of their selecting the same number among the undrawn numbers. Thus the probability of the observed event on the second hypothesis is $\frac{(1-p)(1-p')}{n(n-1)}$. Thus the odds in favour of the truth of the assertion are

$$\frac{pp'}{n} \text{ to } \frac{(1-p)(1-p')}{n(n-1)}, \text{ or } pp' \text{ to } \frac{(1-p)(1-p')}{n-1}.$$

755. The question in Art. 752 is respecting the truth of *concurrent* testimony; we may now consider the truth of *traditionary* testimony. A says that B says that a certain event took place: required the probability that the event did take place. Let p and p' be the probabilities of speaking the truth of A and B respectively. The event did take place if they both speak truth, or if they both speak falsehood; and the event did not take place if only one of them speaks truth. Thus the odds that the event did take place are

$$pp' + (1-p)(1-p') \text{ to } p(1-p') + p'(1-p).$$

756. If there be n witnesses, each of whom has transmitted a statement of an occurrence to the next, and if p be the probability of speaking the truth of each witness, the probability of the truth of the statement is to the probability of its falsehood as the sum of the odd terms of the expansion of $(p+q)^n$ is to the sum of the even terms, q being put equal to $1-p$ after the expansion has been effected. For the statement is true if all the witnesses speak truth, or if two, or four, or any *even* number speak falsehood.

757. Suppose that certain *arguments* are logically sound, and that the probabilities of the truth of their respective premises

are known: required the probability of the truth of the conclusion. For example, suppose that there are three arguments, and let p, p', p'' denote the respective probabilities of their premises. The conclusion is valid unless *all* the arguments fail. The chance that they all fail is $(1-p)(1-p')(1-p'')$; hence the chance that they do not all fail is $1-(1-p)(1-p')(1-p'')$, which is, therefore, the required probability.

758. Of such an extensive subject as the Theory of Probability only an outline can be given in an elementary work on Algebra. The student who is prepared for further investigation will find a list of the necessary books in the article *Probability* in the *English Cyclopædia;* to that list may be added the work of Professor Boole on the *Laws of Thought.* For a discussion of the first principles of the subject the student may consult De Morgan's *Formal Logic*, Chapters IX. and X., and Venn's *Logic of Chance.* We may also refer to the *History of the Mathematical Theory of Probability, from the time of Pascal to that of Laplace;* this work introduces the reader to almost every process and every species of problem which the literature of the subject can furnish.

EXAMPLES ON PROBABILITY.

1. The odds against a certain event are 3 to 2; and the odds in favour of another event independent of the former are 4 to 3. Find the odds for or against their happening together.

2. Supposing that it is 8 to 7 against a person who is now 30 years of age living till he is 60, and 2 to 1 against a person who is now 40 living till he is 70: find the probability that one at least of these persons will be alive 30 years hence.

3. A party of 23 persons take their seats at a round table: shew that it is 10 to 1 against two specified individuals sitting next to each other.

4. The chance that A can solve a certain problem is $\frac{1}{4}$; the chance that B can solve it is $\frac{2}{3}$: find the chance that the problem will be solved if they both try.

5. Find the chance of drawing two black balls and one red from an urn containing five black, three red, and two white.

6. Find the probability that an ace and only one will be thrown in two trials with one die.

7. Find the probability of throwing one ace at least in two trials with one die.

8. Find the odds against throwing one of the two numbers 7 or 11 in a single throw with two dice.

9. Two purses contain the same number of sovereigns and a different number of shillings; one purse is taken at random and a coin is drawn out: shew that it is more likely to be a sovereign than it would be if all the coins had been in one purse.

10. There are four men, A, B, C, D whose powers of rowing may be represented by the numbers, 6, 7, 8, 9, respectively; two of them are placed by lot in a boat, and the other two in a second boat. Find the chance which each man has of being a winner in a race between the boats.

11. In one throw with a pair of dice find the chance that there is neither an ace nor doublets.

12. If from a lottery of 30 tickets marked 1, 2, 3, ... four tickets be drawn, find the chance that 1 and 2 will be among them.

13. A has 3 shares in a lottery where there are 3 prizes and 6 blanks; B has 1 share in another where there is but 1 prize and 2 blanks. Shew that A has a better chance of getting a prize than B in the ratio of 16 to 7.

14. Two bags contain each 4 black and 3 white balls; a person draws a ball at random from the first bag, and if it be white he puts it into the second bag and then draws a ball from it: find the chance of his drawing two white balls.

15. A coin is thrown up n times in succession: find the chance that the head will present itself an odd number of times.

16. When n coins are tossed up, find the chance that one and only one will turn up head.

17. Supposing the House of Commons to consist of m Tories and n Whigs, find the probability that a committee of $p+q$ selected by lot may consist of p Tories and q Whigs.

18. Find the chance that a person with two dice will throw aces at least four times in six trials.

19. Find the chance of throwing an ace with a single die once at least in six trials.

20. If on an average 9 ships out of 10 return safe to port, find the chance that out of 5 ships expected at least 3 will arrive.

21. In three throws with a pair of dice, find the probability of having doublets one or more times.

22. Find the chance of throwing double sixes once or oftener in three throws with a pair of dice.

23. In a lottery containing a large number of tickets where the prizes are to the blanks as 1 to 6, find the chance of drawing at least 2 prizes in 5 trials.

24. If four cards be drawn from a pack, find the probability that there will be one of each suit.

25. If four cards be drawn from a pack, find the probability that they will be marked one, two, three, four, of the same suit.

26. If A's skill at any game be double that of B, the odds against A's winning 4 games before B wins 2 are 131 to 112.

27. Two persons A and B engage in a game in which A's skill is to B's as 2 to 3. Find the chance of A's winning at least 2 games out of 5.

28. Three white balls and five black are placed in a bag, and three persons draw a ball in succession (the balls not being replaced) until a white ball is drawn. Shew that their respective chances are as 27, 18 and 11.

29. In each game that is played it is 2 to 1 in favour of the winner of the game before. Find the chance that he who wins the first game shall win three or more of the next four.

30. A certain stake is to be won by the first person who throws ace with a die of n faces. If there be p persons, find the chance of the r^{th} person.

31. There are 3 parcels of books in another room and a particular book is in one of them. The odds that it is in one particular parcel are 3 to 2; but if not in that parcel it is equally likely to be in either of the others. If I send for this parcel giving a description of it, and the odds I get the one I describe are 2 to 1, find my chance of getting the book I want.

32. In a purse are ten coins, all shillings except one which is a sovereign; in another are ten coins all shillings. Nine coins are taken from the former purse and put into the latter, and then nine coins are taken from the latter and put into the former. A person is now permitted to take whichever purse he pleases: find which he should choose.

33. One urn contained 5 white balls and 5 black balls; a second urn contained 10 white balls and 10 black balls; a ball, of which colour is not known, was removed from one urn, but which is not known, into the other. A drawing being now made from one of the urns chosen at random, what is the chance that it will give a white ball?

34. Find the chance of throwing 15 in one throw with 3 dice.

35. Find the chance of throwing 17 in one throw with 3 dice.

36. Find the chance of throwing not more than 10 with 3 dice.

37. When $2n$ dice are thrown, prove that the sum of the numbers turned up is more likely to be $7n$ than any other number.

38. When $2n+1$ dice are thrown, prove that the chance that the sum of the numbers turned up is $7n+4$ equals the chance that the sum of the numbers turned up is $7n+3$, and that the chance is greater than the chance that the sum is any other number.

39. Out of a set of cards numbered from 1 to 10 a card is drawn and replaced: after ten such drawings what is the probability that the sum of the numbers drawn is 24?

40. Counters numbered 0, 1, 2, ... n, are placed in a box; after one is drawn it is put back, and the process is repeated. Find the probability that m drawings will give the counter marked s.

41. There are 10 tickets 5 of which are blanks and the others are marked 1, 2, 3, 4, 5: find the probability of drawing 10 in three trials, the tickets being replaced.

42. Find the probability in the preceding Example if the tickets are not replaced.

43. From a bag containing n balls p balls are drawn out and replaced, and then q balls are drawn out. Shew that the probability of exactly r balls being common to the two drawings is

$$\frac{\lfloor p \, \lfloor q \, \lfloor n-p \, \lfloor n-q}{\lfloor n \, \lfloor r \, \lfloor p-r \, \lfloor q-r \, \lfloor n-p-q+r}.$$

44. Eight persons of equal skill at chess draw lots for partners and play four games; the four winners draw lots again for partners and play two games; and the two winners in these play a final game: find the chance that two assigned persons will have played together.

45. In a bag are m white balls and n black balls. Find the chance of drawing first a white, then a black ball, and so on alternately until the balls remaining are all of one colour.

If m balls are drawn at once, find the chance of drawing all the white balls at the first trial.

46. In a bag are n balls of m colours, p_1 being of the first colour, p_2 of the second colour, ... p_m of the m^{th} colour. If the balls be drawn one by one, find the chance that all the balls of the first colour will be first drawn, then all the balls of the second colour, and so on, and lastly all the balls of the m^{th} colour.

47. A bag contains n balls; a person takes out one and puts it in again; he does this n times: find the probability of his having had in his hand every ball in the bag.

48. Two players of equal skill, A and B, are playing a set of games. A wants 2 games to complete the set, and B wants 3 games. Compare the chances of A and B for winning the set.

49. If three persons dine together find in how many different ways they can be seated in a row. When they have dined together exactly so many times, taking their places by chance, find the probability that they will have sat in every possible arrangement.

50. N is a given number; a lower number is selected at random, find the chance that it will divide N.

51. A handful of shot is taken at random out of a bag: find the chance that the number of shot in the handful is prime to the number of shot in the bag. For example, suppose the number of shot in the bag to be 105.

52. If $n = a^r$, and any number not greater than n be taken at random, the chance that it contains a as a factor s times and no more is $\frac{1}{a^s} - \frac{1}{a^{s+1}}$.

53. Two persons play at a game which cannot be drawn, and agree to continue to play until one or other of them wins two games in succession: given the chance that one of them wins a single game, find the chance that he wins the match described. For example, if the odds on a single game be 2 to 1, the odds on the match will be 16 to 5.

54. A person has a pair of dice, one a regular tetrahedron, the other a regular octahedron: find the chance that in a single throw the sum of the marks is greater than 6.

55. There are three independent events of which the probabilities are respectively p_1, p_2, p_3: find the probability of the happening of one of the events at least; also of the happening of two of the events at least.

56. A certain sum of money is to be given to one of three persons A, B, C, who first throws 10 with three dice: supposing them to throw successively in the order named until the event has happened, find their respective chances.

57. The decimal parts of the logarithms of two numbers taken at random are found from a table to 7 places: find the pro-

bability that the second can be subtracted from the first without *borrowing* at all.

58. A undertakes with a pair of dice to throw 6 before B throws 7; they throw alternately, A commencing. Compare their chances.

59. A person is allowed to draw two coins from a bag containing four sovereigns and four shillings: find the value of his expectation.

60. If six guineas, six sovereigns, and six shillings be put into a bag, and three be drawn out at random, find the value of the expectation.

61. Ten Russian ships, twelve French, and fourteen English are expected in port. Find the value of the expectation of a merchant who will gain £2100 if one of the first two which arrive is a Russian and the other a French ship.

62. From a bag containing 3 guineas, 2 sovereigns, and 4 shillings, a person draws 3 coins indiscriminately: find the value of his expectation.

63. Find the worth of a lottery-ticket in a lottery of 100 tickets, having 4 prizes of £100, ten of £50, and twenty of £5.

64. A bag contains 9 coins, 5 are sovereigns, the other four are equal to each other in value: find what this value must be in order that the expectation of receiving two coins out of the bag may be worth 24 shillings.

65. From a bag containing 4 shilling pieces, 3 unknown English silver coins of the same value, and one unknown English gold coin, four are to be drawn. If the value of the drawer's chance be 15 shillings, find what the coins are.

66. A and B subscribe a sum of money for which they toss alternately beginning with A, and the first who throws a head is to win the whole. In what proportion ought they to subscribe? If they subscribe equally, how much should either of them give the other for the first throw?

67. There are a number of counters in a bag of which one is marked 1, two 2, &c. up to r marked r; a person draws a number at random for which he is to receive as many shillings as the number marked on it: find the value of his expectation.

68. A bag contains a number of tickets of which one is marked 1, four marked 2, nine marked 3, ... up to r^2 marked r; a person draws a ticket at random for which he is to receive as many shillings as the number marked on it: find the value of his expectation.

69. A man is to receive a certain number of shillings; he knows that the digits of the number are 1, 2, 3, 4, 5, but he is ignorant of the order in which they stand: find the value of his expectation.

70. From a bag containing a counters some of which are marked with numbers, b counters are to be drawn, and the drawer is to receive a number of shillings equal to the sum of the numbers on the counters which he draws: if the sum of the numbers on all the counters be n, find the value of his expectation.

71. There are two urns; one contains 8 white balls and 4 black balls, and the other contains 12 black balls and 4 white balls; from one of these, but it is not known from which, a ball is taken and is found to be white: find the chance that it was drawn from the urn containing 8 white balls.

72. Five balls are in a bag, and it is not known how many of these are white; two being drawn are both white: find the probability that all are white.

73. A purse contains n coins and it is not known how many of these are sovereigns; a coin drawn is a sovereign: find the probability that this is the only sovereign.

74. A bag contains 4 white and 4 black balls; two are taken out at random, and without being seen are placed in a smaller bag; one is taken out and proves to be white, and replaced in the smaller bag: one is again taken out and proves to be again white, find the probability that both balls in the smaller bag are white.

75. Of two purses one originally contained 25 sovereigns, and the other 10 sovereigns and 15 shillings; one purse is taken by chance and 4 coins drawn out which prove to be all sovereigns: find the probability that this purse contains only sovereigns, and the value of the expectation of the next coin that will be drawn from it.

76. A bag contains three bank notes, and it is known that there is no note which is not either a £5, a £10, or a £20 note; at three successive dips in the bag (the note being replaced after each dip) a £5 note was drawn. Find the probable value of the contents of the bag.

77. It is 3 to 1 that A speaks the truth, 4 to 1 that B does, and 6 to 1 that C does: find the probability that an event took place which A and B assert to have happened and which C denies.

78. A speaks truth 3 times out of 4, B 4 times out of 5; they agree in asserting that from a bag containing 9 balls, all of different colours, a white ball has been drawn: shew that the probability that this is true is $\frac{96}{97}$.

79. Suppose thirteen witnesses, each of whom makes but one false statement in eleven, to assert that a certain event took place; shew that the odds are ten to one in favour of the truth of their statement, even although the *a priori* probability of the event be as small as $\frac{1}{10^{12}+1}$.

80. One of a pack of 52 cards has been removed; from the remainder of the pack two cards are drawn and are found to be spades: find the chance that the missing card is a spade.

81. Two persons walk on the same road in opposite directions during $a+b+c$ minutes, one completing the distance in a minutes and the other in b minutes: find the chance of their meeting.

82. Find how many odd numbers taken at random must be multiplied together, that there may be at least an even chance of the last figure being 5. Given $\log_{10} 2 = \cdot 30103$.

LIV. MISCELLANEOUS EQUATIONS.

759. Equations may be proposed which require peculiar artifices for their solution; in the following collection the student will find ample exercise: he should himself try to solve the equations, and afterwards consult the solution here given.

1. $$\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}.$$

Here $x+1+\dfrac{1}{x+1}+x+4+\dfrac{4}{x+4}=x+2+\dfrac{2}{x+2}+x+3+\dfrac{3}{x+3}$,

so that $$\frac{1}{x+1}+\frac{4}{x+4}=\frac{2}{x+2}+\frac{3}{x+3};$$

therefore $$\frac{1}{x+1}-\frac{2}{x+2}=\frac{3}{x+3}-\frac{4}{x+4},$$

that is $$-\frac{x}{x^2+3x+2}=-\frac{x}{x^2+7x+12};$$

therefore either $x=0$, or $x^2+3x+2=x^2+7x+12$;

from the latter $$4x=-10;$$

therefore $$x=-2\tfrac{1}{2}.$$

2. $$\frac{1}{(x+a)^2-b^2}+\frac{1}{(x+b)^2-a^2}=\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2}.$$

Here $$\frac{1}{x+a+b}\left\{\frac{2x}{x^2-(a-b)^2}\right\}=\frac{1}{x^2-(a+b)^2}+\frac{1}{x^2-(a-b)^2};$$

therefore $$\frac{1}{x+a+b}\,\frac{x-(a+b)}{x^2-(a-b)^2}=\frac{1}{x^2-(a+b)^2};$$

therefore $$\frac{x-(a+b)}{x^2-(a-b)^2}=\frac{1}{x-(a+b)};$$

therefore $$\{x-(a+b)\}^2=x^2-(a-b)^2;$$

therefore $$2x(a+b)=(a+b)^2+(a-b)^2;$$

therefore $$x=\frac{a^2+b^2}{a+b}.$$

3. $$\frac{x^2}{3}+\frac{48}{x^2}=10\left(\frac{x}{3}-\frac{4}{x}\right).$$

Here $$3\left(\frac{x^2}{9}+\frac{16}{x^2}\right)=10\left(\frac{x}{3}-\frac{4}{x}\right);\ \text{call } \frac{x}{3}-\frac{4}{x}=y,$$

then $$3\left(y^2+\frac{8}{3}\right)=10y,\ \text{ so that } y^2-\frac{10y}{3}+\frac{25}{9}=\frac{1}{9};$$

therefore $$y=2 \text{ or } \frac{4}{3};$$

therefore $$x^2-12=6x \text{ or } 4x;$$

therefore $$x=6 \text{ or } -2 \text{ or } 3\pm\sqrt{(21)}.$$

4. $$\frac{(5x^4+10x^2+1)(5a^4+10a^2+1)}{(x^4+10x^2+5)(a^4+10a^2+5)}=ax.$$

Here $$\frac{5x^4+10x^2+1}{x^5+10x^3+5x}=\frac{a^5+10a^3+5a}{5a^4+10a^2+1};$$

adding and subtracting the numerator and denominator of each fraction, we have

$$\left(\frac{x+1}{x-1}\right)^5=\left(\frac{1+a}{1-a}\right)^5;$$

therefore $$\frac{x+1}{x-1}=\frac{1+a}{1-a};\ \text{ therefore } x=\frac{1}{a}.$$

5. $$(x-1)^3+(2x+3)^3=27x^3+8.$$

Since $(x-1)+(2x+3)=3x+2$, divide both sides by $3x+2$, which gives $x=-\frac{2}{3}$ for one value of x; and we obtain

$$(x-1)^2-(x-1)(2x+3)+(2x+3)^2=9x^2-6x+4,$$

that is $$3x^2+9x+13=9x^2-6x+4,$$

therefore $$6x^2-15x=9;$$

therefore $$x^2-\frac{5x}{2}+\frac{25}{16}=\frac{25}{16}+\frac{3}{2}=\frac{49}{16};$$

therefore $$x-\frac{5}{4}=\pm\frac{7}{4};\quad \text{therefore } x=3 \text{ or } -\frac{1}{2}.$$

6. $$31\left\{\frac{24-5x}{x+1}+\frac{5-6x}{x+4}\right\}+370=29\left\{\frac{17-7x}{x+2}+\frac{8x+55}{x+3}\right\}.$$

Here $$31\left\{\frac{24-5x}{x+1}+\frac{5-6x}{x+4}+11\right\}=29\left\{\frac{17-7x}{x+2}+\frac{8x+55}{x+3}-1\right\},$$

or $$31\left\{\frac{24-5x}{x+1}+5+\frac{5-6x}{x+4}+6\right\}=29\left\{\frac{17-7x}{x+2}+7+\frac{8x+55}{x+3}-8\right\},$$

therefore $$31\left\{\frac{29}{x+1}+\frac{29}{x+4}\right\}=29\left\{\frac{31}{x+2}+\frac{31}{x+3}\right\};$$

therefore $$\frac{1}{x+1}+\frac{1}{x+4}=\frac{1}{x+2}+\frac{1}{x+3};$$

therefore $$\frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{x+3}-\frac{1}{x+4};$$

therefore $$(x+1)(x+2)=(x+3)(x+4);$$

therefore $$3x+2=7x+12;$$

therefore $$4x=-10;$$

therefore $$x=-2\tfrac{1}{2}.$$

7. $$\frac{1}{5}\frac{(x+1)(x-3)}{(x+2)(x-4)}+\frac{1}{9}\frac{(x+3)(x-5)}{(x+4)(x-6)}-\frac{2}{13}\frac{(x+5)(x-7)}{(x+6)(x-8)}=\frac{92}{585}.$$

It is clear that the numerator and denominator of each fraction involves the expression x^2-2x, put therefore $(x-1)^2=y$; then the equation becomes

$$\frac{1}{5}\frac{y-4}{y-9}+\frac{1}{9}\frac{y-16}{y-25}-\frac{2}{13}\frac{y-36}{y-49}=\frac{92}{585}.$$

Now $$\frac{1}{5}+\frac{1}{9}-\frac{2}{13}=\frac{92}{585};$$

subtracting corresponding terms, we have

$$\frac{1}{5}\frac{5}{y-9}+\frac{1}{9}\frac{9}{y-25}-\frac{2}{13}\frac{13}{y-49}=0,$$

that is $$\frac{1}{y-9}+\frac{1}{y-25}-\frac{2}{y-49}=0;$$

therefore $$\frac{1}{y-9}-\frac{1}{y-49}=\frac{1}{y-49}-\frac{1}{y-25};$$

that is $$\frac{-40}{y-9}=\frac{24}{y-25};$$

therefore $$3(y-9)+5(y-25)=0,$$

that is $$8y=152;$$

therefore $$y=19 \text{ and } x=1\pm\sqrt{(19)}.$$

8. $$x.\frac{x+3a}{c+3x}=\sqrt{(ac)}\,\frac{a+3x}{x+3c}.$$

Here $\dfrac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\dfrac{x+3a}{a+3x}=\dfrac{c^{\frac{1}{2}}}{x^{\frac{1}{2}}}\dfrac{c+3x}{x+3c}$, that is $\dfrac{x^{\frac{3}{2}}+3ax^{\frac{1}{2}}}{a^{\frac{3}{2}}+3a^{\frac{1}{2}}x}=\dfrac{c^{\frac{3}{2}}+3c^{\frac{1}{2}}x}{x^{\frac{3}{2}}+3cx^{\frac{1}{2}}}$;

adding and subtracting the numerator and denominator of each fraction, we have $\dfrac{(x^{\frac{1}{2}}+a^{\frac{1}{2}})^3}{(x^{\frac{1}{2}}-a^{\frac{1}{2}})^3}=\dfrac{(c^{\frac{1}{2}}+x^{\frac{1}{2}})^3}{(c^{\frac{1}{2}}-x^{\frac{1}{2}})^3}$;

therefore $\dfrac{x^{\frac{1}{2}}+a^{\frac{1}{2}}}{x^{\frac{1}{2}}-a^{\frac{1}{2}}}=\dfrac{c^{\frac{1}{2}}+x^{\frac{1}{2}}}{c^{\frac{1}{2}}-x^{\frac{1}{2}}}$; therefore $\dfrac{x}{a}=\dfrac{c}{x}$;

therefore $$x=\pm\sqrt{(ac)}.$$

9. $$(x+a)\left(1+\frac{1}{x^2+a^2}\right)+\sqrt{(2ax)}\left(1-\frac{1}{x^2+a^2}\right)=2.$$

Here $$\{x+\sqrt{(2ax)}+a\}+\frac{x-\sqrt{(2ax)}+a}{x^2+a^2}=2,$$

therefore $$x+\sqrt{(2ax)}+a+\frac{1}{x+\sqrt{(2ax)}+a}=2;$$

therefore $$\{x+\sqrt{(2ax)}+a\}^2-2\{x+\sqrt{(2ax)}+a\}+1=0;$$

therefore $$x+a+\sqrt{(2ax)}=1;$$

therefore $$(x+a)^2-2(x+a)+1=2ax;$$

therefore $$x^2-2x+1=2a-a^2;$$

therefore $$x=1\pm\sqrt{(2a-a^2)}.$$

10. $$(x+a)(x+2a)(x-3a)(x-4a)=c^4.$$

Here $$(x+a)(x-3a)(x+2a)(x-4a)=c^4,$$

that is $$(x^2-2ax-3a^2)(x^2-2ax-8a^2)=c^4.$$

Let $$x^2-2ax=ya^2,$$

then $$(y-3)(y-8)=\frac{c^4}{a^4};$$

therefore $$y^2-11y+\frac{121}{4}=\frac{c^4}{a^4}+\frac{25}{4};$$

therefore $$y=\frac{11}{2}\pm\frac{(4c^4+25a^4)^{\frac{1}{2}}}{2a^2};$$

therefore $$x^2-2ax=\frac{11a^2}{2}\pm\frac{(4c^4+25a^4)^{\frac{1}{2}}}{2};$$

therefore $$x=a\pm\sqrt{\left\{\frac{13a^2}{2}\pm\frac{(4c^4+25a^4)^{\frac{1}{2}}}{2}\right\}}.$$

11. $$\frac{1}{6x^2-7x+2}+\frac{1}{12x^2-17x+6}=8x^2-6x+1.$$

Here $$\frac{1}{(2x-1)(3x-2)}+\frac{1}{(3x-2)(4x-3)}=(2x-1)(4x-1),$$

or $$\frac{1}{3x-2}\times\frac{6x-4}{(2x-1)(4x-3)}=(2x-1)(4x-1);$$

therefore $$2=(2x-1)^2(4x-1)(4x-3)$$
$$=(2x-1)^2\{2(2x-1)+1\}\{2(2x-1)-1\}.$$

Let $$y=2x-1,$$

then $$y^2(4y^2-1)=2;$$

therefore $$y^4-\frac{y^2}{4}+\frac{1}{64}=\frac{1}{64}+\frac{2}{4}=\frac{33}{64};$$

therefore $$y^2=\frac{1}{8}(1\pm\sqrt{33});$$

therefore $$x=\frac{1+y}{2}=\frac{1}{2}\left[1\pm\sqrt{\left\{\frac{1}{8}(1\pm\sqrt{33})\right\}}\right].$$

12. $$\left(\frac{x+6}{x-6}\right)\left(\frac{x-4}{x+4}\right)^2 + \left(\frac{x-6}{x+6}\right)\left(\frac{x+9}{x-9}\right)^2 = 2\frac{x^2+36}{x^2-36}.$$

Here $$\frac{x+6}{x-6}\left(\frac{x-4}{x+4}\right)^2 + \frac{x-6}{x+6}\left(\frac{x+9}{x-9}\right)^2 = \frac{x+6}{x-6} + \frac{x-6}{x+6};$$

therefore $$\frac{x-6}{x+6}\left\{\left(\frac{x+9}{x-9}\right)^2 - 1\right\} = \frac{x+6}{x-6}\left\{1 - \left(\frac{x-4}{x+4}\right)^2\right\},$$

that is $$\frac{x-6}{x+6} \times \frac{36x}{(x-9)^2} = \frac{x+6}{x-6}\,\frac{16x}{(x+4)^2};$$

therefore $x=0$ is one value; and for the other values we have

$$\left(\frac{x-6}{x+6}\right)^2 = \frac{16}{36}\left(\frac{x-9}{x+4}\right)^2, \quad \text{therefore } \frac{x-6}{x+6} = \pm\frac{2}{3}\,\frac{x-9}{x+4};$$

therefore $$3(x^2-2x-24) = \pm 2(x^2-3x-54);$$

these quadratics can now be solved in the ordinary way.

13. $$\frac{x^2+2ax+ac}{x^2+2cx+ac} = \frac{ax}{(x+a)(x+c)}.$$

Let $$(x+a)(x+c) = xy,$$

then $$\frac{x^2+2ax+ac}{x^2+2cx+ac} = \frac{a}{y};$$

therefore $$\frac{2(x^2+ax+cx+ac)}{2x(a-c)} = \frac{a+y}{a-y},$$

or $$\frac{(x+a)(x+c)}{x(a-c)} = \frac{a+y}{a-y},$$

thus $$\frac{y}{a-c} = \frac{a+y}{a-y};$$

therefore $$y^2 - yc = ac - a^2;$$

therefore $$y = \frac{c}{2} \pm \frac{1}{2}\sqrt{(c^2+4ac-4a^2)} = \alpha \text{ suppose};$$

thus $$x^2 + x(a+c) + ac = x\alpha;$$

therefore $$x^2 + x(a+c-\alpha) = -ac;$$

therefore $$x = -\frac{a+c-\alpha}{2} \pm \frac{1}{2}\sqrt{\{(a+c-\alpha)^2 - 4ac\}}.$$

14. $$2(x+a)(x+c)+(a-c)^2=\frac{(x+c)^4}{c(2x+a+c)}.$$

Here $$(x+a)^2+(x+c)^2=\frac{(x+c)^4}{c\{(x+a)+(x+c)\}}.$$

Let $$x+a=y(x+c)\ldots\ldots\ldots\ldots\ldots(\alpha),$$

then $$y^2+1=\frac{x+c}{c(y+1)}\ldots\ldots\ldots\ldots\ldots(\beta).$$

From (α) $$x+c+a-c=y(x+c);$$

therefore $$x+c=\frac{a-c}{y-1};$$

therefore (β) becomes $$y^2+1=\frac{a-c}{c}\,\frac{1}{y^2-1};$$

therefore $$y^4=\frac{a-c}{c}+1=\frac{a}{c};\ \text{therefore}\ y=\left(\frac{a}{c}\right)^{\frac{1}{4}};$$

therefore $$x=\frac{yc-a}{1-y}=\frac{a^{\frac{1}{4}}c-ac^{\frac{1}{4}}}{c^{\frac{1}{4}}-a^{\frac{1}{4}}}=(ac)^{\frac{1}{4}}\,\frac{a^{\frac{3}{4}}-c^{\frac{3}{4}}}{a^{\frac{1}{4}}-c^{\frac{1}{4}}}.$$

15. $$\frac{(x+a+b)^5+(x+c+d)^5}{(x+a+c)^5+(x+b+d)^5}=\frac{m}{n}\ldots\ldots\ldots\ldots(1).$$

Let $\left.\begin{array}{l}a+b=\alpha+\beta\\ c+d=\alpha-\beta\end{array}\right\}$; therefore $\alpha=\frac{1}{2}(a+b+c+d)$, $\beta=\frac{1}{2}(a+b-c-d)$,

let $\left.\begin{array}{l}a+c=\alpha_1+\beta_1\\ b+d=\alpha_1-\beta_1\end{array}\right\}$; therefore $\alpha_1=\frac{1}{2}(a+b+c+d)=\alpha$, $\beta_1=\frac{1}{2}(a-b+c-d)$.

Hence by assuming $x+\alpha=y$, (1) may be put into the shape

$$\frac{(y+\beta)^5+(y-\beta)^5}{(y+\beta_1)^5+(y-\beta_1)^5}=\frac{m}{n},\ \text{or}\ \frac{y^5+10y^3\beta^2+5y\beta^4}{y^5+10y^3\beta_1^{\,2}+5y\beta_1^{\,4}}=\frac{m}{n},$$

or $$y^4(n-m)+10y^2(n\beta^2-m\beta_1^{\,2})=5(m\beta_1^{\,4}-n\beta^4)\ldots\ldots(2),$$

which is a common quadratic equation.

If $$\frac{m}{n}=\frac{\beta^2}{\beta_1^{\,2}}=\frac{(a+b-c-d)^2}{(a-b+c-d)^2},$$

(2) takes the form $y^4=5\beta^2\beta_1^{\,2}$; therefore $y=(5)^{\frac{1}{4}}(\beta\beta_1)^{\frac{1}{2}}$,

or $$x=y-\alpha=\tfrac{1}{2}\left[5^{\frac{1}{4}}\sqrt{\{(a-d)^2-(b-c)^2\}}-(a+b+c+d)\right].$$

16. $$x^2+a^2+y^2+b^2=\sqrt{2}\,\{x(a+y)-b(a-y)\},$$
$$x^2-a^2-y^2+b^2=\sqrt{2}\,\{x(a-y)+b(a+y)\}.$$

Adding and subtracting,

$$x^2+b^2=\sqrt{2}\,(ax+by)\ldots\ldots(a),$$
$$y^2+a^2=\sqrt{2}\,(xy-ab);$$

multiplying together,

$$(x^2+b^2)(y^2+a^2)=2(ax+by)(xy-ab),$$

or $$(ax+by)^2+(xy-ab)^2=2(ax+by)(xy-ab);$$

therefore $ax+by=xy-ab$; therefore $y=a\dfrac{x+b}{x-b}$.

Substituting in (a),

$$x^2+b^2=a\sqrt{2}\left\{x+\frac{bx+b^2}{x-b}\right\}=a\sqrt{2}\,\frac{x^2+b^2}{x-b};$$

therefore, neglecting the impossible root, $x-b=a\sqrt{2}$;

therefore $$x=a\sqrt{2}+b,$$

and $$y=a\frac{x+b}{x-b}=b\sqrt{2}+a.$$

17. $$(x^2+y^2+c^2)^{\frac{1}{2}}+(x-y+c)^{\frac{2}{3}}=2(4xy)^{\frac{1}{3}}\ldots\ldots(1),$$
$$\frac{1}{y}=\frac{1}{x}+\frac{1}{c}\ldots\ldots(2).$$

Since $(x-y+c)^2=x^2+y^2+c^2-2xy+2xc-2yc$,

and from (2) $$xc-xy-yc=0\ldots\ldots(a);$$

therefore $$(x-y+c)^2=x^2+y^2+c^2;$$

therefore (1) becomes $(x-y+c)^2=4xy=4c(x-y)$ from (a);

therefore $$(x-y-c)^2=0;$$

therefore $$y=x-c,$$

but $y=\dfrac{cx}{x+c}$; therefore $x^2-c^2=cx$; therefore $x^2-cx+\dfrac{c^2}{4}=\dfrac{5c^2}{4}$;

therefore $x=\dfrac{c}{2}(1\pm\sqrt{5})$, and $y=\dfrac{c}{2}(-1\pm\sqrt{5})$.

18. $$2(x^2+xy+y^2-a^2)+\sqrt{3}(x^2-y^2)=0 \text{..........(1)},$$
$$2(x^2-xz+z^2-b^2)+\sqrt{3}(x^2-z^2)=0 \text{..........(2)},$$
$$y^3-c^3+3(yz^2-c^3)=0 \text{..........(3)}.$$

Multiplying (1) by 2 it becomes

$$3(x+y)^2+(x-y)^2+2\sqrt{3}(x^2-y^2)=4a^2;$$

therefore $\qquad \sqrt{3}(x+y)+x-y=\pm 2a.$

Similarly from (2) $\qquad \sqrt{3}(x-z)+x+z=\pm 2b.$

By subtraction we obtain on the left-hand side $(\sqrt{3}-1)(y+z)$, and on the right-hand side $\pm 2(a-b)$ or $\pm 2(a+b)$; thus we have four values for $y+z$: choose any one of these and denote it by m.

From (3) $2y^3+6yz^2=8c^3$, that is $(y+z)^3+(y-z)^3=8c^3$;

therefore $\qquad (y-z)^3=8c^3-m^3;$

therefore $\qquad y-z=(8c^3-m^3)^{\frac{1}{3}};$

therefore $\quad y=\frac{1}{2}\{m+(8c^3-m^3)^{\frac{1}{3}}\}$, and $z=\frac{1}{2}\{m-(8c^3-m^3)^{\frac{1}{3}}\}$.

And $\qquad x\{\sqrt{3}+1\}=\pm 2a-y\{\sqrt{3}-1\}$

$$=\pm 2a-\frac{\sqrt{3}-1}{2}\{m+(8c^3-m^3)^{\frac{1}{3}}\};$$

thus x is known.

19. $$3x+3y-z=3 \text{ .... (1)},$$
$$x^2+y^2-z^2=\frac{14-9z}{2} \text{ .... (2)},$$
$$x^3+y^3+z^3=3xyz+\frac{17z+44}{4} \text{ .... (3)}.$$

From (1) $\qquad 3(x+y+z)=4z+3 \text{ .... } (\alpha),$

From (2) $\qquad x^2+y^2+z^2=2z^2+7-\frac{9z}{2} \text{ .... } (\beta),$

From (3) $\qquad 2(x^3+y^3+z^3-3xyz)=\frac{17z+44}{2} \text{ .... } (\gamma);$

then multiplying (α) and (β) together and subtracting (γ), we have

$$x^3 + y^3 + z^3 + 3(x^2y + xy^2 + xz^2 + x^2z + y^2z + yz^2) + 6xyz$$
$$= 8z^3 - 12z^2 + 6z - 1;$$

or $(x + y + z)^3 = (2z - 1)^3$;

therefore $\qquad x + y = z - 1.$

From (1) $\qquad x + y = \frac{z}{3} + 1;$

therefore $\qquad z - 1 = \frac{z}{3} + 1;$ therefore $z = 3$;

therefore $\qquad x + y = 2, \quad x^2 + y^2 = z^2 + \frac{14 - 9z}{2} = \frac{5}{2};$

therefore $2(x^2 + y^2) - (x + y)^2 = 5 - 4 = 1$; therefore $x - y = \pm 1$;

therefore $x = 1\frac{1}{2}$ or $\frac{1}{2}$, and $y = \frac{1}{2}$ or $1\frac{1}{2}$.

20. $$\frac{(ac + 1)(x^2 + 1)}{x + 1} = \frac{(a^2 + 1)(xy + 1)}{y + 1} \quad \text{............(1)},$$

$$\frac{(ac + 1)(y^2 + 1)}{y + 1} = \frac{(c^2 + 1)(xy + 1)}{x + 1} \quad \text{............(2)}.$$

From (1) $$\frac{x^2 + 1}{xy + 1} = \frac{x + 1}{y + 1} \cdot \frac{a^2 + 1}{ac + 1} \quad \text{..............}(\alpha),$$

From (2) $$\frac{y^2 + 1}{xy + 1} = \frac{y + 1}{x + 1} \cdot \frac{c^2 + 1}{ac + 1};$$

therefore $$\frac{(x^2 + 1)(y^2 + 1)}{(xy + 1)^2} = \frac{(a^2 + 1)(c^2 + 1)}{(ac + 1)^2}.$$

Subtracting denominators from numerators, we have

$$\frac{(x - y)^2}{(xy + 1)^2} = \frac{(a - c)^2}{(ac + 1)^2}; \text{ therefore } \frac{x - y}{xy + 1} = \pm \frac{a - c}{ac + 1} \quad \text{........}(\beta);$$

therefore $$x - y = (xy + 1)\frac{a - c}{ac + 1}, \text{ or } (xy + 1)\frac{c - a}{ac + 1};$$

therefore using the first value and calling $\frac{a - c}{ac + 1} = m$,

we have $y(1 + mx) = x - m$; therefore $y = \frac{x - m}{1 + mx}$.

Now from (α) $\dfrac{x^2+1}{x+1}=\dfrac{xy+1}{y+1}\cdot\dfrac{a^2+1}{ac+1}$;

therefore $\dfrac{x^2+1}{x+1}=\dfrac{a^2+1}{ac+1}\cdot\dfrac{\dfrac{x^2-mx}{1+mx}+1}{\dfrac{x-m}{1+mx}+1}=\dfrac{a^2+1}{ac+1}\cdot\dfrac{x^2+1}{1+mx+x-m}$;

therefore $(ac+1)(1+mx+x-m)=(a^2+1)(x+1)$;

or $1+ac+x(a-c)+x(1+ac)-(a-c)=(a^2+1)+x(a^2+1)$;

therefore $x(a-c)-ax(a-c)=a(a-c)+(a-c)$;

therefore $x(1-a)=1+a$; therefore $x=\dfrac{1+a}{1-a}$;

and $$y=\frac{x-m}{1+mx}=\frac{\dfrac{1+a}{1-a}-\dfrac{a-c}{1+ac}}{1+\dfrac{(1+a)(a-c)}{(1-a)(1+ac)}}=\frac{1+c}{1-c}.$$

Similarly, if we use the negative sign in (β), we have $\dfrac{1-a}{1+a}$ and $\dfrac{1-c}{1+c}$ for the corresponding values of x and y.

21. $(2y-1)(x^4+4x+3)^{\frac{1}{2}}-(2x-1)(y^4+4y+3)^{\frac{1}{2}}$
$$=(x-y)(x+y-2xy+4)\ldots\ldots\ldots(1),$$
$$\sqrt{\left(\frac{y+1}{xy-1}\right)}-\sqrt{\left(\frac{2y-1}{2x-1}\right)}=\frac{y+1}{x+1}\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From (1) $(2y-1)(x^4+4x+3)^{\frac{1}{2}}-(2x-1)(y^4+4y+3)^{\frac{1}{2}}$
$$=x^2-y^2-2x^2y+2xy^2+4x-4y$$
$$=y^2(2x-1)-x^2(2y-1)+2(2x-1)-2(2y-1)$$
$$=(y^2+2)(2x-1)-(x^2+2)(2y-1);$$

therefore
$$(2y-1)\{x^2+2+\sqrt{(x^4+4x+3)}\}=(2x-1)\{y^2+2+\sqrt{(y^4+4y+3)}\},$$

or $$\frac{x^2+2+\sqrt{(x^4+4x+3)}}{2x-1}=\frac{y^2+2+\sqrt{(y^4+4y+3)}}{2y-1}\ldots\ldots\ldots(\alpha).$$

Now $x^4+4x+3=(x^2+2x+1)(x^2-2x+3)=uv$

if $u=x^2+2x+1$ and $v=x^2-2x+3$;

therefore $u+v=2(x^2+2)$ and $u-v=2(2x-1)$.

Hence (a) assumes the form

$$\frac{(\sqrt{u}+\sqrt{v})^2}{u-v}=\frac{(\sqrt{u_1}+\sqrt{v_1})^2}{u_1-v_1},$$

where $\qquad u_1=y^2+2y+1$, and $v_1=y^2-2y+3$;

thus $\qquad \dfrac{\sqrt{u}+\sqrt{v}}{\sqrt{u}-\sqrt{v}}=\dfrac{\sqrt{u_1}+\sqrt{v_1}}{\sqrt{u_1}-\sqrt{v_1}}$; therefore $\dfrac{u}{v}=\dfrac{u_1}{v_1}$;

therefore $\qquad \dfrac{x^2+2x+1}{x^2-2x+3}=\dfrac{y^2+2y+1}{y^2-2y+3}$;

adding and subtracting the numerator and denominator of each fraction, we have

$$\frac{x^2+2}{2x-1}=\frac{y^2+2}{2y-1};$$

therefore $\qquad 2yx^2+4y-x^2-2=2xy^2+4x-y^2-2$;

therefore $\qquad 2yx(x-y)-(x^2-y^2)-4(x-y)=0$;

therefore $\quad x=y$; or $2xy=x+y+4$, so that $y=\dfrac{x+4}{2x-1}$.

Substituting the value $y=x$ in (2), we have

$$\sqrt{\left(\frac{x+1}{x^2-1}\right)}=2, \text{ or } \frac{1}{x-1}=4; \text{ therefore } x=1\tfrac{1}{4} \text{ and } y=1\tfrac{1}{4}.$$

Again, if $y=\dfrac{x+4}{2x-1}$, then $\dfrac{y+1}{xy-1}=\dfrac{3(x+1)}{(x+1)^2}=\dfrac{3}{x+1}$,

thus $\qquad \dfrac{2y-1}{2x-1}=\dfrac{9}{(2x-1)^2}$, and $\dfrac{y+1}{x+1}=\dfrac{3}{2x-1}$.

Hence equation (2) becomes

$$\sqrt{\left(\frac{3}{x+1}\right)}-\frac{3}{2x-1}=\frac{3}{2x-1}, \text{ or } \sqrt{\left(\frac{3}{x+1}\right)}=\frac{6}{2x-1};$$

therefore $\qquad \dfrac{1}{x+1}=\dfrac{12}{(2x-1)^2}$, or $4x^2-16x=11$;

therefore $\;4x^2-16x+16=27$; therefore $2x-4=\pm 3\sqrt{3}$;

therefore $x=\frac{1}{2}(4\pm 3\sqrt{3})$; and $y=\dfrac{x+4}{2x-1}=\dfrac{1}{2}\left(\dfrac{4\pm\sqrt{3}}{1\pm\sqrt{3}}\right)$.

MISCELLANEOUS EXAMPLES.

1. Solve $\sqrt{(1+x^2)} - \sqrt{(1-x^2)} = \sqrt{(1-x^4)}$.

2. Solve
$$x^2(b-y) = ay(y-n),$$
$$y^2(a-x) = bx(x-n).$$

3. If
$$x^2+xy+y^2=c^2,$$
$$x^2+xz+z^2=b^2,$$
$$y^2+yz+z^2=a^2,$$
prove that
$$xy+yz+zx=\sqrt{\left\{\frac{1}{3}(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)\right\}};$$
and shew how to solve the equations.

4. Solve
$$\frac{x^2-4x-8}{\sqrt{(x^2+2x+11)}}=2\sqrt{2}.$$

5. Determine c so that $5x+4y=c$ may have *ten* positive integral solutions excluding zero values, and c may be as great as possible.

6. If $\dfrac{x^2-yz}{x(1-yz)}=\dfrac{y^2-xz}{y(1-xz)}$ and x, y, z be unequal, then each member of this equation will be equal to $\dfrac{z^2-xy}{z(1-xy)}$, to $x+y+z$, and to $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}$.

7. Shew that if n and N are very nearly equal,
$$\left(\frac{N}{n}\right)^{\frac{1}{2}}=\frac{N}{N+n}+\frac{n+N}{4n} \text{ very nearly,}$$
and that the error is approximately $\dfrac{(N-n)^4}{8n(N+n)^3}$.

8. A man's income consists partly of a salary of £200 a year, and partly of the interest at 3 per cent. on capital, to which he each year adds his savings; his annual expenditure is less by £95 than five-fourths of his income: shew that whatever be the origi-

nal capital its accumulated value will approximate to £6000. If the original capital be £1000, shew that it will be doubled in about thirty years; having given

$$\log 2 = \cdot 301030, \qquad \log 397 = 2\cdot 598790.$$

9. If n be a positive integer, and $c = \frac{m}{(m+1)^2}$, shew that

$$1 - (n-1)c + \frac{(n-2)(n-3)}{2}c^2 - \frac{(n-3)(n-4)(n-5)}{\lfloor 3}c^3 + \ldots$$
$$= \frac{m^{n+1}-1}{m-1}\,\frac{1}{(m+1)^n}.$$

10. If x be any prime number, except 2, the integral part of $(1+\sqrt{2})^x$, diminished by 2, is divisible by $4x$.

11. If any number of integers taken at random be multiplied together, shew that the chance of the last figure of their product being 5 continually diminishes as the number of integers multiplied together increases.

12. Two purses contain sovereigns and shillings; shew that if either the total numbers of coins in the two purses are equal, or if the number of sovereigns is to the number of shillings in the same ratio in both, then the chance of drawing out a sovereign is the same when one purse is taken at random and a coin drawn out as it is when the coins are all put in one purse and a coin drawn out. If neither of these conditions holds, the chance is in favour of the purse taken at random whenever the purse with the *greater* number of coins has the *smaller* proportion of sovereigns.

LV. MISCELLANEOUS PROBLEMS.

760. We have already given in previous Chapters collections of problems which lead to simple or quadratic equations; we add here a few examples of somewhat greater difficulty with their solutions.

1. Each of three cubical vessels A, B, C, whose capacities are as 1 : 8 : 27 respectively, is partially filled with water, the

quantities of water in them being as 1 : 2 : 3 respectively. So much water is now poured from A into B and so much from B into C as to make the depth of water the same in each vessel. After this $128\frac{4}{7}$ cubic feet of water is poured from C into B, and then so much from B into A as to leave the depth of water in A twice as great as the depth of water in B. The quantity of water in A is now less by 100 cubic feet than it was originally. How much water did each of the vessels originally contain?

Let $x =$ the number of cubic feet in A originally;

therefore $2x =$ the number of cubic feet in B originally;

and $3x =$ the number of cubic feet in C originally.

Now when the depth of the fluid is the same in all, it is clear that the *quantities* vary as the areas of the bases of the vessels, that is, are as 1 : 4 : 9. Therefore, since $6x$ is the total quantity, the quantity in $A = \frac{6x}{9+4+1} = \frac{3x}{7}$, and the quantities in B and C are $\frac{12x}{7}$ and $\frac{27x}{7}$ respectively.

Again, when the depth in A is *twice* that in B, the quantity in A is *half* as much as that in B.

Now A contains $x - 100$; therefore B contains $2(x-100)$, and C contains $\frac{27x}{7} - 128\frac{4}{7}$.

therefore $$3(x-100) + \frac{27x}{7} - 128\frac{4}{7} = 6x;$$

therefore $$\frac{6x}{7} = 300 + 128\frac{4}{7};$$

therefore $$x = 350 + \frac{900}{7} \times \frac{7}{6} = 500;$$

therefore the quantities in A, B, C at first were 500, 1000, 1500 cubic feet respectively.

2. Three horses A, B, C start for a race on a course a mile and a half long. When B has gone half a mile, he is three times

as far ahead of A as he is of C. The horses now going at uniform speeds till B is within a quarter of a mile of the winning post, C is at that time as much behind A as A is behind B, but the distance between A and B is only $\frac{1}{11}^{\text{th}}$ of what it was after B had gone the first half mile. C now increases his pace by $\frac{1}{53}^{\text{rd}}$ of what it was before, and passes B 176 yards from the winning post, the respective speeds of A and B remaining unaltered. What was the distance between A and C at the end of the race?

Let $11x =$ the distance in yards between B and C at the end of the first $\frac{1}{2}$ mile, $33x =$ the distance in yards between B and A at the end of the first $\frac{1}{2}$ mile. When B has gone $1\frac{1}{4}$ miles B is $3x$ ahead of A, and $6x$ ahead of C; therefore while B went $\frac{3}{4}$ mile or 1320 yards, A went $1320 + 30x$ yards, and C went $1320 + 5x$ yards.

Hence, after C increases his pace, the speeds of A, B, C will be proportional to $1320 + 30x$, 1320, and $\frac{54}{53}(1320+5x)$ respectively.

Since C passes B when he is 176 yards from the post; therefore while B was going $440 - 176$ or 264 yards, C went $264 + 6x$;

therefore $$1320 : \frac{54}{53}(1320 + 5x) :: 264 : 264 + 6x,$$

therefore $$1320 + 30x = \frac{54}{53}(1320 + 5x);$$

therefore $$x(1590 - 270) = 1320;$$

therefore $$x = 1;$$

also it will be found that C's increased pace is equal to A's; therefore there will be the same distance between them at the end of the race as there is when B is $\frac{1}{4}$ mile from the winning post, namely $3x$ or 3 yards.

3. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining at the rate of 11 per cent. more on his outlay than he would gain were the balance *true*. If, however, the scale-pans in

which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Determine the legitimate gain per cent. on the article.

Let w and w_1 be the apparent weights of the same article when *bought* and when *sold*.

Let p = the prime cost of a unit of weight,

x = the legitimate gain per cent.;

then an article which cost pw is sold for $w_1\left(p+\frac{px}{100}\right)$;

therefore by the question $w_1\left(p+\frac{px}{100}\right)-wp=\frac{(x+11)pw}{100}$ ......(1).

Again in the supposed case the cost of the article $=pw_1$ and the selling price $=pw\left(1+\frac{x}{100}\right)$;

therefore $$pw_1=pw\left(1+\frac{x}{100}\right)\text{..............................(2).}$$

From (1), $$w_1\left(1+\frac{x}{100}\right)=w\left(1+\frac{x+11}{100}\right);$$

from (2), $$w\left(1+\frac{x}{100}\right)=w_1;$$

therefore $$\left(1+\frac{x}{100}\right)^2=1+\frac{x+11}{100};$$

therefore $x^2+100x=1100$, so that $(x+50)^2=3600$;

therefore $$x+50=\pm 60;$$

therefore $$x=10.$$

4. A person buys a quantity of corn, which he intends to sell at a certain price; after he has sold half his stock the price of corn suddenly falls 20 per cent., and by selling the remainder at this reduced price, his gain on the whole is diminished 30 per cent.; if he had sold $\frac{3}{4}$ths of his stock before the price fell, and the diminution in the price had been in the proportion of £20 on the prime cost of what he before sold for £100, he would have gained by the whole as many shillings as he had bushels of

corn at first. Find what the corn cost him per bushel, and what he hoped to gain per cent.

Let $x=$ the cost price, in pounds, per bushel,

$y=$ the gain per cent. he expected; then

$x\left(1+\frac{y}{100}\right)=$ the price per bushel for which he sold half his corn;

$\frac{4}{5}x\left(1+\frac{y}{100}\right)=$ the price per bushel for which he sold the other half; therefore the average price per bushel $=\frac{9x}{10}\left(1+\frac{y}{100}\right)$;

therefore his gain per bushel $=\frac{9x}{10}\left(1+\frac{y}{100}\right)-x$.

If he had sold the whole as he sold the first half, the gain per bushel would have been $\frac{yx}{100}$;

therefore by the question $\frac{9x}{10}\left(1+\frac{y}{100}\right)-x=\frac{7}{10}\frac{yx}{100}$;

therefore $\frac{y}{500}=\frac{1}{10}$; therefore $y=50$.

Now the prime cost of what he at first sold for 100 is $\dfrac{100}{1+\frac{y}{100}}$, that is $\frac{200}{3}$, and if he were to lose £20 on this, the loss per cent. would be $\dfrac{20\times 100}{\frac{200}{3}}$, that is 30.

Thus in the supposed case the average selling price of a bushel $=\frac{3x}{4}\left(1+\frac{y}{100}\right)+\frac{x}{4}\left(1+\frac{y}{100}\right)\times\frac{7}{10}=\frac{x}{4}\left(\frac{9}{2}+\frac{21}{20}\right)$; therefore the gain on a bushel $=\frac{x}{4}\times\frac{111}{20}-x=\frac{31x}{80}$, and this by the question equals one shilling; therefore $\frac{31x}{80}=\frac{1}{20}$; therefore $x=\frac{4}{31}$.

5. A and B having a single horse travel between two mile-stones, distant an even number of miles, in $2\frac{62}{63}$ hours, riding alternately mile and mile, and each leaving the horse tied to a mile-stone until the other comes up. The horse's rate is twice that of B; B rides first, and they come together to the seventh mile-stone. Finding it necessary to increase their speed, each man after this walks half a mile per hour faster than before, and the horse's rate is now twice that of A, and B again rides first. Find the rates of travelling, and the distance between the extreme mile-stones.

Let $2x =$ the distance they travelled in miles. Now at first A walks 4 miles and rides 3 miles while B walks 3 miles and rides 4 miles, or A walks 4 while B walks 3 and rides 1; that is, since the horse's rate is double of B's, while B walks $3\frac{1}{2}$ miles; therefore A's and B's rates at first may be represented by $8y$ and $7y$ respectively.

Again, A walks $x-3$ and rides $x-4$, while B walks $x-4$ and rides $x-3$; therefore A walks $x-3$ while B walks $x-4$ and rides 1, that is, while B walks $x-4$ and A walks $\frac{1}{2}$; therefore A walks $x-\frac{7}{2}$ while B walks $x-4$;

but A walks $8y+\frac{1}{2}$ while B walks $7y+\frac{1}{2}$;

therefore $$\frac{x-\frac{7}{2}}{x-4} = \frac{8y+\frac{1}{2}}{7y+\frac{1}{2}},\ \text{from which}\ y = \frac{1}{4x-30}.$$

Now the total time A took in hours is

$$\frac{4}{8y} + \frac{3}{14y} + \frac{x-3}{8y+\frac{1}{2}} + \frac{x-4}{2\left(8y+\frac{1}{2}\right)} = 2\tfrac{62}{63},$$

therefore $$\frac{5}{7y} + \frac{3x-10}{16y+1} = 2\tfrac{62}{63};$$

therefore $$\frac{5}{7} + \frac{3x-10}{4x-14} = \frac{188}{63} \times \frac{1}{4x-30};$$

therefore $$\frac{41x-140}{4x-14}=\frac{94}{9}\times\frac{1}{2x-15};$$

therefore $$9(82x^2-895x+2100)=376x-1316;$$

therefore $$738x^2-8431x+20216=0,$$

from which $x=8$; therefore $y=\frac{1}{2}$;

therefore the distance $=16$ miles; the rates of travelling at first $=4$ and $3\frac{1}{2}$ miles per hour respectively.

6. A and B set out to walk together in the same direction round a field, which is a mile in circumference, A walking faster than B. Twelve minutes after A has passed B for the third time, A turns and walks in the opposite direction until six minutes after he has met him for the third time, when he returns to his original direction and overtakes B four times more. The whole time since they started is three hours, and A has walked eight miles more than B. A and B diminish their rates of walking by one mile an hour, at the end of one and two hours respectively. Determine the velocities with which they began to walk.

Let $x=$ the number of miles per hour of A at the first,

$y=$ the number of miles per hour of B at the first.

In 3 hours A has gone $x+2(x-1)$ or $3x-2$ miles,

and B has gone $2y+(y-1)$ or $3y-1$ miles;

therefore by the question $3x-2-(3y-1)=8$; therefore $x-y=3$, that is, the *relative* speed of A and B is 3 miles per hour; therefore A will gain a circumference on B in $\frac{1}{3}$ of an hour, and will therefore be passing B for the third time at the end of the first hour.

Also since the *relative* speed of A and B is the same in the last hour as in the first, and since A passes B for the *fourth* time at the end of the third hour, therefore he will pass him all the *four* times within the last hour; the first time being exactly at the commencement of the third hour.

Now in 12 minutes after the first hour the distance between A and B is $\frac{1}{5}(x-y-1)=\frac{2}{5}$ miles; therefore the time of first meeting $=\frac{2}{5}\div(x+y-1)$; and the time of meeting *twice* more $=2\div(x+y-1)$. In 6 minutes the distance between them $=\frac{1}{10}(x+y-1)$; therefore if A now turns, the time of overtaking B

$$=\frac{\frac{1}{10}(x+y-1)}{x-y-1}=\frac{1}{20}(x+y-1);$$

therefore $$\frac{1}{5}+\frac{\frac{2}{5}}{x+y-1}+\frac{2}{x+y-1}+\frac{1}{10}+\frac{1}{20}(x+y-1)=1,$$

that is, $$\frac{12}{5u}+\frac{u}{20}=\frac{7}{10}, \text{ if } u=x+y-1;$$

therefore $u^2-14u=-48$; therefore $u-7=\pm 1$; therefore $u=8$ or 6;

therefore $$x+y=9 \text{ or } 7; \text{ and } x-y=3;$$

therefore $$x=6 \text{ or } 5, \quad y=3 \text{ or } 2.$$

761. The equations in the preceding Chapter and their solutions, and the solutions in the present Chapter, are due to the Rev. A. Bower, late Fellow of St John's College. Should any student desire more exercises of this kind, he is referred to the collection of algebraical equations and problems edited by Mr W. Rotherham of St John's College.

MISCELLANEOUS EXAMPLES.

1. Exhibit $\{n\sqrt{(a^2+b^2)}-a\sqrt{(m^2+n^2)}\}^2+b^2m^2$ as a square.

2. Extract the square root of $6+\sqrt{6}+\sqrt{14}+\sqrt{21}$.

3. Find the radix of the scale of notation in which the number 16640 of the common scale appears as 40400.

4. Shew that $\frac{3}{4}+\frac{4}{8}+\frac{5}{16}+\frac{6}{32}+\ldots\ldots ad\ inf.=2$.

5. At a contested election the number of candidates was one more than twice the number of persons to be elected, and each elector by voting for one, or two, or three, ... or as many persons as were to be elected, could dispose of his vote in 15 ways: required the number of candidates.

6. In how many ways may the sum of £24. 15*s*. be paid in shillings and francs, supposing 26 francs to be equal to 21 shillings?

7. Find the sum of n terms of the series

$$\frac{1}{1+z}+\frac{z}{(1+z)(1+z^2)}+\frac{z^3}{(1+z)(1+z^2)(1+z^4)}+\frac{z^7}{(1+z)(1+z^2)(1+z^4)(1+z^8)}+\ldots\ldots$$

8. Shew that $1+2x^4$ is never less than x^2+2x^3.

9. If an equal number of arithmetic and geometric means be inserted between any two quantities, shew that the arithmetic mean is always greater than the corresponding geometric mean.

10. If x be any prime number, except 2, the integral part of $(2+\sqrt{3})^x-2^{x+1}+1$ is divisible by $12x$.

11. Shew that if $n=pq$, where p and q are positive integers,

$$\frac{\lfloor n}{\{\lfloor p\}^q \lfloor q} \text{ is an integer.}$$

12. Shew that $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\ldots\ldots+\frac{1}{n}-\log n$ is finite when n is infinite.

13. If p be the probability *à priori* that a theory is true, q the probability that an experiment would turn out as indicated by the theory even if the theory were false, shew that after the experiment has been performed, supposing it to have turned out as expected, the probability of the truth of the theory becomes $\frac{p}{p+q-pq}$.

14. Of two bags one (it is not known which) is known to contain two sovereigns and a shilling, and the other to contain one sovereign and a shilling; a person draws a coin from one of

the bags, and it is a sovereign, which is not replaced. Shew that the chance of now drawing a sovereign from the same bag is half the chance of doing so from the other. Supposing the drawer might keep the coin he draws, find the value of the expectation.

15. All that is known of two bags, one white and one red, is that one of them, but it is not known which, contains one sovereign and four shilling pieces, and that the other contains two sovereigns and three shilling pieces; but a coin being drawn from each, the event is a sovereign out of the white bag and a shilling out of the red bag. These coins are now put back, one into one bag, and the other into the other, but it is not known into which bag the sovereign was put. Shew that the probability of now drawing a sovereign is in favour of the red bag as compared with the white bag in the ratio of 13 to 9.

16. If n be the number of years which any individual wants of 86, find the value of an annuity of £1 to be paid during his life; adopting De Moivre's supposition, that out of 86 persons born, one dies every year until they are all extinct.

LVI. CONVERGENCE AND DIVERGENCE OF SERIES.

762. In Chapter XL. we have discussed the subject of the convergence and divergence of series. The chief general result which has been obtained may be expressed thus: *an infinite series is convergent if from and after any fixed term the ratio of each term to the succeeding term is greater than some quantity which is itself numerically greater than unity; and divergent if this ratio is unity or less than unity, and the terms are all of the same sign.* There is one case to which this result does not apply, which it is desirable to notice, namely the case in which the ratio is greater than unity but continually approaching unity. See Arts. 559, 560 and 561. The statements of those Articles are here reproduced, but in a different form, as for our present purpose it is convenient to regard the ratio of a term to the *succeeding* term instead of to the *preceding* term.

763. We shall now investigate theorems which will supply tests of convergence and divergence for the case to which the former tests do not apply. In the infinite series which we shall consider we shall suppose that all the terms are positive, at least from and after some fixed term if not from the beginning.

764. *A series is convergent if from and after some fixed term the ratio of each term to the succeeding term is never less than the corresponding ratio in a second series which is known to be convergent.*

It is obvious in this case that the proposed series is not greater than a certain convergent series; and is therefore convergent.

765. *A series is divergent if from and after some fixed term the ratio of each term to the succeeding term is never greater than the corresponding ratio in a second series which is known to be divergent.*

It is obvious in this case that the proposed series is not less than a certain divergent series; and is therefore divergent.

766. *Let* u_n *denote the* n^{th} *term of a series; then if from and after some fixed value of* n *the value of* $n\left(\frac{u_n}{u_{n+1}}-1\right)$ *is always greater than some positive quantity which is itself greater than unity, the series is convergent.*

Suppose that from and after some fixed value of n the value of $n\left(\frac{u_n}{u_{n+1}}-1\right)$ is always greater than γ, where γ is positive and greater than unity. Then $\frac{u_n}{u_{n+1}}-1$ is greater than $\frac{\gamma}{n}$; and therefore $\frac{u_n}{u_{n+1}}$ is greater than $1+\frac{\gamma}{n}$.

Now, by Art. 686, a positive quantity p greater than unity can be found, such that when n is large enough $\left(\frac{n+1}{n}\right)^p$ is less than $1+\frac{\gamma}{n}$. Hence, when n is large enough, $\frac{u_n}{u_{n+1}}$ is greater than $\left(\frac{n+1}{n}\right)^p$. But, by Art. 562, the series of which the n^{th} term is $\frac{1}{n^p}$ is convergent when p is positive and greater than unity; hence by Art. 764 the series of which the n^{th} term is u_n is convergent.

767. *Let* u_n *denote the* n^{th} *term of a series; then if from and after some fixed value of* n *the value of* $n\left(\frac{u_n}{u_{n+1}}-1\right)$ *is never positive and greater than unity, the series is divergent.*

For here after some fixed value of n the value of $\frac{u_n}{u_{n+1}}$ is equal to $1+\frac{1}{n}$ or is less than $1+\frac{1}{n}$. But, by Art. 562, the series of which the n^{th} term is $\frac{1}{n}$ is divergent; hence, by Art. 765, the series of which the n^{th} term is u_n is divergent.

768. The rules given in Arts. 766 and 767 will often enable us to decide on the convergence or divergence of series in the case noticed in Art. 762 in which our former rules do not apply. There is one case to which the new rules will not apply, which it is desirable to notice, namely that in which from and after some fixed value of n the value of $n\left(\frac{u_n}{u_{n+1}}-1\right)$ is always positive and greater than unity, but continually approaching unity. We shall proceed to investigate theorems from which we shall deduce tests for this case.

769. It is obvious from the nature of a logarithm that if n increases indefinitely, so also does $\log n$. But it is important to observe that $\log n$ increases far less rapidly than n increases; in fact $\frac{\log n}{n}$ can be made as small as we please by taking n large enough. For suppose $n=e^x$, so that $\log n = x$; then as n increases

indefinitely, so also does x. Now $\frac{\log n}{n} = \frac{x}{e^x} = \frac{x}{1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots}$;

this is less than $\frac{x}{x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots}$, that is less than $\frac{1}{1 + \frac{x}{\lfloor 2} + \frac{x^2}{\lfloor 3} + \dots}$;

and it is obvious that this can be made as small as we please by taking n large enough.

These remarks will be found useful in studying the remainder of the present Chapter. We shall adopt the following notation for abbreviation: let $\log n$ be denoted by $\lambda(n)$; let $\log(\log n)$ be denoted by $\lambda^2(n)$; let $\log\{\log(\log n)\}$ be denoted by $\lambda^3(n)$; and so on.

770. *The series of which the general term is*

$$\frac{1}{n\lambda(n)\,\lambda^2(n)\dots\dots\lambda^r(n)\,\{\lambda^{r+1}(n)\}^p} \quad\dots\dots\dots\dots\dots\dots(1)$$

is convergent if p *be greater than unity, and divergent if* p *be equal to unity or less than unity.*

We suppose n so large that $\lambda^{r+1}(n)$ is possible and positive.

The truth of this theorem when $r = 0$ has been shewn in Art. 563; we shall prove it generally by Induction.

By Art. 563 the series of which (1) is the general term is convergent or divergent simultaneously with the series of which the general term is

$$\frac{m^n}{m^n\lambda(m^n)\,\lambda^2(m^n)\dots\dots\lambda^r(m^n)\,\{\lambda^{r+1}(m^n)\}^p} \quad\dots\dots\dots\dots (2),$$

where m is any positive integer.

I. Suppose p greater than unity. Let m be any positive integer *greater* than the base of the Napierian logarithms; then $\lambda(m^n)$ is *greater* than n. Hence it follows that the general term (2) is *less* than

$$\frac{1}{n\lambda(n)\,\lambda^2(n)\dots\dots\lambda^{r-1}(n)\,\{\lambda^r(n)\}^p} \quad\dots\dots\dots\dots\dots (3);$$

thus by Art. 764 if the series of which (3) is the general term is convergent, so also is that of which (2) is the general term, and so also is that of which (1) is the general term. Therefore if the series of which (3) is the general term is convergent when r has any specific value, it is convergent when r is changed into $r+1$. But since p is greater than unity, by Art. 563 the series of which (3) is the general term *is* convergent when $r=1$, and therefore when $r=2$, and therefore when $r=3$, and so on. Thus the series of which (1) is the general term is convergent.

II. Suppose p equal to unity. Let $m=2$ which is a positive integer *less* than the base of the Napierian logarithms; then $\lambda(m^n)$ is *less* than n. Hence it follows that the general term (2) is *greater* than

$$\frac{1}{n\lambda(n)\,\lambda^2(n)\ldots\ldots\lambda^{r-1}(n)\,\lambda^r(n)}.$$

Hence by proceeding as in I. we can shew that the series of which (1) is the general term is divergent.

III. Suppose p less than unity. Then the general term (1) is greater than it would be if p were equal to unity, at least when n is large enough, and therefore *à fortiori* the series is divergent.

A simple demonstration of this theorem by means of the *Integral Calculus* is given in the *Integral Calculus*, Chapter IV.

771. Let u_n denote the general term of any proposed series. If from and after any value of n the value of

$$u_n n\lambda(n)\,\lambda^2(n)\ldots\ldots\lambda^r(n)\,\{\lambda^{r+1}(n)\}^p$$

is always finite, p being any fixed quantity greater than unity, the proposed series is convergent.

For in this case the terms of the proposed series have a finite ratio to the terms of a series which has been proved to be convergent.

If from and after any value of n the value of

$$u_n n\lambda(n)\,\lambda^2(n)\ldots\ldots\lambda^r(n)\,\lambda^{r+1}(n)$$

is always finite or infinite, the proposed series is divergent.

For in this case the terms of the proposed series have at least a finite ratio to the terms of a series which has been proved to be divergent.

772. The theorem of Art. 771 may be used in cases in which the tests already given of convergence and divergence do not apply; but it will in general be more convenient to use the rules which we shall demonstrate in the next Article.

773. *Let* P_0 *stand for* $n\left(\frac{u_n}{u_{n+1}} - 1\right)$; *then if from and after some fixed value of* n *the value of* $\lambda(n)(P_0 - 1)$ *is always greater than some positive quantity which is itself greater than unity the series of which the* n^{th} *term is* u_n *is convergent; and if from and after some fixed value of* n *the value of* $\lambda(n)(P_0 - 1)$ *is never positive and greater than unity the series is divergent.*

I. Suppose that from and after some fixed value of n the value of $\lambda(n)(P_0 - 1)$ is always greater than γ, where γ is positive and greater than unity. Then $P_0 - 1$ is greater than $\frac{\gamma}{\lambda(n)}$; therefore $\frac{u_n}{u_{n+1}}$ is greater than $1 + \frac{1}{n} + \frac{\gamma}{n\lambda(n)}$.

Let $v_n = \frac{1}{n\{\lambda(n)\}^p}$; then $\frac{v_n}{v_{n+1}} = \frac{n+1}{n}\left\{\frac{\lambda(n+1)}{\lambda(n)}\right\}^p$.

Now $\lambda(n+1) = \lambda(n) + \lambda\left(1 + \frac{1}{n}\right)$; therefore $\lambda(n+1)$ is less than $\lambda(n) + \frac{1}{n}$ by Art. 687; and therefore $\frac{v_n}{v_{n+1}}$ is less than $\left(1 + \frac{1}{n}\right)\left\{1 + \frac{1}{n\lambda(n)}\right\}^p$; and therefore when n is large enough $\frac{v_n}{v_{n+1}}$ is less than $\left(1 + \frac{1}{n}\right)\left\{1 + \frac{q}{n\lambda(n)}\right\}$, provided q be greater than p: see Art. 686. Thus $\frac{v_n}{v_{n+1}}$ is less than $1 + \frac{1}{n} + \frac{q}{n\lambda(n)} + \frac{q}{n^2\lambda(n)}$; and when n is taken large enough the last of the four terms just given is incomparably smaller than the third; and therefore $\frac{v_n}{v_{n+1}}$ is less than $1 + \frac{1}{n} + \frac{r}{n\lambda(n)}$, provided r be greater than q.

This result holds however small may be the excess of q above p, and however small may be the excess of r above q: hence since γ is greater than unity we may suppose that γ is greater than r, and yet have p positive and greater than unity.

Since γ is greater than r we have $\frac{u_n}{u_{n+1}}$ greater than $\frac{v_n}{v_{n+1}}$. But, by Art. 770, the series of which the general term is v_n is convergent when p is positive and greater than unity; hence, by Art. 764, the series of which the n^{th} term is u_n is convergent.

II. Suppose that from and after some fixed value of n the value of $\lambda(n)(P_0 - 1)$ is never positive and greater than unity. Then $P_0 - 1$ is positive and not greater than $\frac{1}{\lambda(n)}$ or is negative. In both cases $\frac{u_{n+1}}{u_{n+2}}$ is less than $1 + \frac{1}{n} + \frac{1}{n\lambda(n)}$.

Let $v_n = \frac{1}{n\lambda(n)}$; then $\frac{v_n}{v_{n+1}} = \frac{n+1}{n}\,\frac{\lambda(n+1)}{\lambda(n)}$.

Now $\lambda(n+1) = \lambda(n) + \lambda\left(1 + \frac{1}{n}\right)$; therefore $\lambda(n+1)$ is greater than $\lambda(n) + \frac{1}{n} - \frac{1}{2n^2}$ by Art. 688; and therefore $\frac{v_n}{v_{n+1}}$ is greater than $\left(1 + \frac{1}{n}\right)\left\{1 + \frac{1}{n\lambda(n)} - \frac{1}{2n^2\lambda(n)}\right\}$; and therefore when n is large enough $\frac{v_n}{v_{n+1}}$ is greater than $1 + \frac{1}{n} + \frac{1}{n\lambda(n)}$.

Thus when n is large enough $\frac{u_{n+1}}{u_{n+2}}$ is less than $\frac{v_n}{v_{n+1}}$. But, by Art. 770, the series of which the general term is v_n is divergent; hence, by Art. 765, the series of which the n^{th} term is u_n is divergent.

774. The theorem of Art. 773 does not apply to the case in which $\lambda(n)(P_0 - 1)$ is always positive and greater than unity, but continually approaching unity; another theorem may then be used which also is inapplicable in a certain case. A series of theorems can thus be obtained each of which may be advanta-

geously tried in succession if all that precede it are inapplicable. The theorems will be found in the *Integral Calculus*, Chapter IV.; they might be demonstrated in the manner of Art. 773, but as they will not be required for elementary purposes we need not consider them here: as an exercise for the student the theorem which is next in order to that of Art. 773 is given as the last Example in the set at the end of the present Chapter.

We shall illustrate the rules which have been demonstrated by applying them in the next three Articles.

775. The name *hypergeometrical* has been given to the series

$$1+\frac{\alpha.\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\dots;$$

we shall now determine when the series is convergent, and when divergent.

Denote the series by $u_0+u_1+u_2+u_3+\dots$; thus

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)(n+\gamma)}{(n+\alpha)(n+\beta)x}=\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)x};$$

thus, by Art. 762, if x is less than unity the series is convergent, and if x is greater than unity the series is divergent. Put $x=1$; then

$$P_0=\frac{1+\gamma-\alpha-\beta+\frac{\gamma-\alpha\beta}{n}}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)};$$

thus if $\gamma-\alpha-\beta$ is positive the series is convergent, and if $\gamma-\alpha-\beta$ is negative the series is divergent: see Arts. 766, 767. If $\gamma-\alpha-\beta$ is zero we must use Art. 773; we have then

$$\lambda(n)(P_0-1)=-\frac{\frac{\alpha\beta}{n}\left(1+\frac{1}{n}\right)\lambda(n)}{\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\beta}{n}\right)};$$

this can be made as small as we please by taking n large enough, and therefore the series is divergent.

776. Suppose that $\frac{u_n}{u_{n+1}} = \frac{n^k + an^{k-1} + bn^{k-2} + cn^{k-3} + \ldots}{n^k + An^{k-1} + Bn^{k-2} + Cn^{k-3} + \ldots}$, where k is a positive integer, and no exponent is negative; and $a, b, c, \ldots$ $A, B, C, \ldots$ are any constant quantities: we shall shew that the series of which the n^{th} term is u_n is convergent, if $a - A - 1$ is positive, and divergent if $a - A - 1$ is negative or zero.

Here $$P_0 = \frac{(a-A)\,n^k + (b-B)\,n^{k-1} + (c-C)\,n^{k-2} + \ldots}{n^k + An^{k-1} + Bn^{k-2} + \ldots};$$

thus if $a - A - 1$ is positive the series is convergent, and if $a - A - 1$ is negative the series is divergent: see Arts. 766, 767. If $a - A - 1$ is zero we have

$$P_0 = \frac{n^k + (b-B)\,n^{k-1} + \ldots}{n^k + An^{k-1} + Bn^{k-2} + \ldots};$$

we may still in some cases determine whether the series is convergent or divergent without using any new rule, for instance if $b - B - A$ is negative the series will be divergent by Art. 767. But it will be more convenient to use Art. 773; we have then

$$\lambda(n)\,(P_0 - 1) = \frac{\lambda(n)\,\{(b-B-A)\,n^{k-1} + (c-C-B)\,n^{k-2} + \ldots\}}{n^k + An^{k-1} + Bn^{k-2} + \ldots};$$

this can be made as small as we please by taking n large enough, and therefore the series is divergent.

777. We shall now examine the expansion of $(1+x)^m$ by the Binomial Theorem and determine whether it is convergent or divergent when $x = 1$ or -1.

Let u_r denote the r^{th} term in the expansion of $(1+x)^m$; then

$$u_{r+1} + u_{r+2} + u_{r+3} + \ldots\ldots$$
$$= u_r \left\{\frac{m-r+1}{r}x + \frac{(m-r+1)(m-r)}{r(r+1)}x^2 + \ldots\ldots\right\}.$$

We must then consider the series included between the brackets.

I. Suppose $x = 1$. Let r be numerically not less than m; then the terms of the series between the brackets are alternately positive and negative.

If m is positive, or negative and numerically less than unity, each term is numerically less than the preceding term and the series is convergent by Art. 558.

If $m=-1$ the series between the brackets takes the form

$$-1+1-1+\ldots\ldots,$$

which is convergent according to the definition of Art. 554.

If m is negative and numerically greater than unity each term of the series between the brackets is numerically greater than the preceding term and the series is divergent.

II. Suppose $x=-1$. Then the series between the brackets is

$$\frac{r-m-1}{r}+\frac{(r-m-1)(r-m)}{r(r+1)}+\frac{(r-m-1)(r-m)(r-m+1)}{r(r+1)(r+2)}+\ldots.$$

Let r be numerically not less than m; then the terms of this series are all of the same sign. In Art. 775 put $\alpha=1$, $\beta=r-m-1$, and $\gamma=r$: hence we find that the series is convergent if m is positive, and divergent if m is negative.

EXAMPLES OF CONVERGENCE AND DIVERGENCE OF SERIES.

1. Shew how to determine whether the product of an infinite number of factors u_1, u_2, u_3, u_4 ... is finite or not.

2. Shew that the value when n is infinite of

$$\frac{\lfloor n \, n^x}{(x+1)(x+2)\ldots(x+n)}$$

is finite except when x is a negative integer.

3. Shew that when x is unity the value of u_n in Art. 775 increases indefinitely with n if $\alpha+\beta-\gamma-1$ is positive.

4. Shew that when x is unity the value of u_n in Art. 775 is finite when n increases indefinitely if $\alpha+\beta-\gamma-1$ is zero.

5. Shew that when x is unity the value of u_n in Art. 775 is indefinitely small when n increases indefinitely if $\alpha+\beta-\gamma-1$ is negative.

6. Determine whether the following series is convergent or divergent, x being positive:

$$ax + \frac{a(a+1)}{\lfloor 2} x^2 + \frac{a(a+1)(a+2)}{\lfloor 3} x^3 + \dots$$

7. If $u_n = \dfrac{1}{n^{\frac{n+1}{n}}}$ shew that the series is divergent.

8. Determine whether the following series is convergent or divergent, x being positive:

$$1 + \frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{3} + \frac{1.3}{2.4} \cdot \frac{x^3}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^4}{7} + \dots$$

9. Determine whether the following series is convergent or divergent, β being a positive proper fraction:

$$1 + \frac{\beta(1-\beta)}{1^2} + \frac{(1+\beta)\beta(1-\beta)(2-\beta)}{1^2.2^2} + \frac{(2+\beta)(1+\beta)\beta(1-\beta)(2-\beta)(3-\beta)}{1^2.2^2.3^2} + \dots$$

10. If $u_n = \dfrac{n^p}{(n-1)^q}$, where p and q are positive, determine whether the series is convergent or divergent.

11. Shew that if from and after some fixed value of n the value of $n \log \dfrac{u_n}{u_{n+1}}$ is always greater than some positive quantity which is itself greater than unity the series is convergent.

12. Shew that if from and after some fixed value of n the value of $n \log \dfrac{u_n}{u_{n+1}}$ is never positive and greater than unity the series is divergent.

13. Determine whether the following series is convergent or divergent, x being positive:

$$\frac{a+x}{1} + \frac{(a+2x)^2}{\lfloor 2} + \frac{(a+3x)^3}{\lfloor 3} + \dots$$

14. Give an investigation of the results of Art. 775 without using Art. 773.

15. Give an investigation of the results of Art. 776 without using Art. 773.

16. If $\dfrac{u_n}{u_{n+1}} = \dfrac{n^\alpha + an^\beta + bn^\gamma + \dots}{n^\alpha + An^\beta + Bn^\gamma + \dots}$ where $\alpha, \beta, \gamma, \dots$ are positive constants in descending order of magnitude, and $a, b, \dots A, B, \dots$ are any constants, determine whether the series of which the n^{th} term is u_n is convergent or divergent.

17. Shew that the two series $u_0 + u_1 + u_2 + u_3 + \dots$

and $$\frac{u_1}{u_0} + \frac{u_2}{u_1 + u_0} + \frac{u_3}{u_2 + u_1 + u_0} + \frac{u_4}{u_3 + u_2 + u_1 + u_0} + \dots$$

are both convergent or both divergent; $u_0, u_1, u_2, \dots$ being all positive quantities.

18. Let P_1 stand for $\lambda(n)(P_0 - 1)$; then if from and after some fixed value of n the value of $\lambda^2(n)(P_1 - 1)$ is always greater than some positive quantity which is itself greater than unity the series of which the n^{th} term is u_n is convergent; and if from and after some fixed value of n the value of $\lambda^2(n)(P_1 - 1)$ is never positive and greater than unity the series is divergent.

LVII. CONTINUED FRACTIONS.

778. The most general form of a continued fraction is

$$\cfrac{b_1}{a_1 \pm \cfrac{b_2}{a_2 \pm \cfrac{b_3}{a_3 \pm \dots}}}.$$

Here $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ may denote any quantities, whole or fractional, positive or negative. The simple fractions $\dfrac{b_1}{a_1}, \dfrac{b_2}{a_2}, \dfrac{b_3}{a_3}, \dots$ may be called *components* of the continued fraction. Either sign might be taken where $\pm$ occurs; but we shall consider only two cases, namely that in which every sign is $+$, and that in which every sign is $-$. We shall thus have two classes of continued fractions, which we shall call the first class and the second class respectively.

In Chapters XLIV. and XLV. we confined ourselves to continued fractions of the first class in which every component had unity for its numerator, and a positive integer for its denominator: but we shall now give some propositions relating to the more general form.

779. The fractions obtained by stopping at the first, second, third,... component are called the first, second, third, ... *convergents.*

Thus the first convergent is $\frac{b_1}{a_1}$; the second convergent is $\cfrac{b_1}{a_1 \pm \cfrac{b_2}{a_2}}$, that is $\frac{a_2 b_1}{a_1 a_2 \pm b_2}$; and so on.

780. In Articles 781...785 we shall treat of continued fractions of the *first* class; in Arts. 786...793 we shall treat of continued fractions of the *second* class: in all these Articles we shall assume that every component has both its numerator and its denominator *positive.*

781. Denote the successive convergents by $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots$ Then we can shew as in Art. 604 that the successive convergents may be obtained by these laws:

$$p_n = a_n p_{n-1} + b_n p_{n-2}, \quad q_n = a_n q_{n-1} + b_n q_{n-2}.$$

Hence $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = -\frac{b_{n+1} q_{n-1}}{q_{n+1}} \left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right)$;

and $\frac{b_{n+1} q_{n-1}}{q_{n+1}} = \frac{b_{n+1} q_{n-1}}{a_{n+1} q_n + b_{n+1} q_{n-1}}$, which is a *proper fraction.* Thus $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}$ is numerically less than $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}$, and is of the contrary sign.

Now $\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{b_1}{a_1} - \frac{a_2 b_1}{a_1 a_2 + b_2} = \frac{b_1 b_2}{q_1 q_2}$; and this is positive. Hence we see that the following series consists of *positive* quantities which are in descending order of magnitude:

$$\frac{p_1}{q_1} - \frac{p_2}{q_2}, \; \frac{p_3}{q_3} - \frac{p_2}{q_2}, \; \frac{p_3}{q_3} - \frac{p_4}{q_4}, \; \frac{p_5}{q_5} - \frac{p_4}{q_4}, \; \frac{p_5}{q_5} - \frac{p_6}{q_6}, \ldots$$

This result involves the following facts for a continued fraction of the first class:

The convergents of an odd order continually decrease, and the convergents of an even order continually increase.

Every convergent of an odd order is greater, and every convergent of an even order is less, than all following convergents.

782. Now suppose the number of components infinite. It may happen that by taking n large enough we can make the difference between the n^{th} convergent and the next convergent less than any assigned quantity; or it may happen that however large n may be the difference between the n^{th} convergent and the next convergent is always greater than some fixed quantity.

In the former case the value towards which the odd convergents continually decrease, and the even convergents continually increase, may be called the value of the infinite continued fraction: and we shall say that the infinite continued fraction is *definite*. In the latter case the infinite continued fraction cannot be said to have a single value; but it may be considered to represent two values, one being that to which the odd convergents tend and the other that to which the even convergents tend.

783. *If from and after some fixed value of* r *the value of* $\frac{a_r a_{r+1}}{b_{r+1}}$ *is greater than some fixed positive quantity, the infinite continued fraction is definite.*

Let γ denote the fixed positive quantity.

By successive applications of the result in Art. 781 we have

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = (-1)^n \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right) \frac{b_3 q_1}{q_3} \frac{b_4 q_2}{q_4} \cdots\cdots \frac{b_{n+1} q_{n-1}}{q_{n+1}}.$$

Now $$\frac{b_{r+1} q_{r-1}}{q_{r+1}} = \frac{b_{r+1} q_{r-1}}{a_{r+1} q_r + b_{r+1} q_{r-1}} = \frac{b_{r+1} q_{r-1}}{a_{r+1}(a_r q_{r-1} + b_r q_{r-2}) + b_{r+1} q_{r-1}}$$

$$= \frac{1}{1 + \frac{a_r a_{r+1}}{b_{r+1}} + \frac{a_{r+1} b_r q_{r-2}}{b_{r+1} q_{r-1}}};$$

and this is less than $\frac{1}{1+\gamma}$ since $\frac{a_r a_{r+1}}{b_{r+1}}$ is greater than γ.

Hence $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}$ is numerically less than $\frac{c}{(1+\gamma)^{n-r+1}}$, where c is some constant; and by taking n large enough this may be made less than any assigned value. Therefore the infinite continued fraction is *definite.*

We shew here that the condition stated is *sufficient* to ensure that the infinite continued fraction is definite; we do not assert that the condition is *necessary.*

784. *An infinite continued fraction of the first class in which every component is a proper fraction with its numerator and its denominator integral must be an incommensurable quantity.*

For if possible suppose the continued fraction commensurable, and denote it by $\frac{B}{A}$, where A and B are positive integers. Thus $\frac{B}{A} = \frac{b_1}{a_1 + \rho_1}$, where ρ_1 denotes the infinite continued fraction beginning with the component $\frac{b_2}{a_2}$. Therefore $\rho_1 = \frac{Ab_1 - Ba_1}{B}$; the numerator of this fraction is an integer, which we will denote by C: and C must be positive for ρ_1 is positive. In like manner, if ρ_2 denote the infinite continued fraction beginning with the component $\frac{b_3}{a_3}$ we find that $\rho_2 = \frac{D}{C}$, where D is also a positive integer. And so on.

Moreover $\frac{B}{A}, \frac{C}{B}, \frac{D}{C}, \ldots$ must all be proper fractions. For $\frac{B}{A}$ is less than $\frac{b_1}{a_1}$, and this is a proper fraction; $\frac{C}{B}$ is less than $\frac{b_2}{a_2}$, and this is a proper fraction; and so on.

Hence $A, B, C, D, \ldots$ form a series of *positive integers,* which are in *descending* order of magnitude, and yet *infinite* in number: this is absurd. Hence the infinite continued fraction cannot be a commensurable quantity.

785. *If some of the components of the infinite continued fraction are not proper fractions, but from and after a certain component*

all the others are proper fractions the infinite continued fraction is incommensurable.

For suppose that $\frac{b_{n+1}}{a_{n+1}}$ and all the subsequent components are proper fractions, then by Art. 784 the infinite continued fraction beginning with $\frac{b_{n+1}}{a_{n+1}}$ is incommensurable; denote it by x. As in Art. 781 we have

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + b_n p_{n-2}}{a_n q_{n-1} + b_n q_{n-2}},$$

and the value of the infinite continued fraction will be obtained by changing a_n into $a_n + x$; so that it is $\frac{(a_n + x) p_{n-1} + b_n p_{n-2}}{(a_n + x) q_{n-1} + b_n q_{n-2}}$, that is, $\frac{p_n + x p_{n-1}}{q_n + x q_{n-1}}$. This cannot be commensurable unless $\frac{p_n}{q_n} = \frac{p_{n-1}}{q_{n-1}}$, and this by aid of the value of $\frac{p_n}{q_n}$ leads to $\frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-2}}{q_{n-2}}$; and so we should arrive at $\frac{p_2}{q_2} = \frac{p_1}{q_1}$. This is impossible, as we cannot have $b_1 = 0$ or $b_2 = 0$.

786. *A continued fraction of the second class in which the denominator of every component exceeds the numerator by unity at least, has all its convergents positive proper fractions which are in ascending order of magnitude.*

The first convergent $\frac{b_1}{a_1}$ is a positive proper fraction by hypothesis. The second convergent is $\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2}}$; and as $\frac{b_2}{a_2}$ is a proper fraction, and a_1 exceeds b_1 by unity at least, $a_1 - \frac{b_2}{a_2}$ is positive and greater than b_1; and thus the second convergent is a positive proper fraction. The third convergent may be denoted by $\cfrac{b_1}{a_1 - \cfrac{\beta_1}{a_1}}$ where $\frac{\beta_1}{a_1}$ stands for $\cfrac{b_2}{a_2 - \cfrac{b_3}{a_3}}$, so that $\frac{\beta_1}{a_1}$ is a positive proper fraction for the same reason that the second convergent is: hence for the same reason the third convergent is a positive proper fraction. The fourth convergent may be denoted by $\dfrac{b_1}{a_1 - \dfrac{\beta_2}{a_2}}$, where $\dfrac{\beta_2}{a_2}$ denotes a fraction of the same form as the third convergent, which is therefore a positive proper fraction: hence the fourth convergent is a positive proper fraction. And so on.

Again; as in Art. 781 we shall find that the successive convergents may be obtained by these laws:

$$p_n = a_n p_{n-1} - b_n p_{n-2}, \qquad q_n = a_n q_{n-1} - b_n q_{n-2}.$$

Hence $$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{b_{n+1}q_{n-1}}{q_{n+1}}\left(\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right);$$

thus $\dfrac{p_{n+1}}{q_{n+1}} - \dfrac{p_n}{q_n}$ is of the same sign as $\dfrac{p_n}{q_n} - \dfrac{p_{n-1}}{q_{n-1}}$.

Now $\dfrac{p_2}{q_2} - \dfrac{p_1}{q_1} = \dfrac{a_2 b_1}{a_1 a_2 - b_2} - \dfrac{b_1}{a_1} = \dfrac{b_1 b_2}{q_1 q_2}$; and this is positive.

Hence it follows that $\dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}, \dfrac{p_3}{q_3}, \ldots\ldots$ form a series of positive proper fractions in ascending order of magnitude.

787. If the number of components is infinite the convergents form an infinite series of proper fractions in ascending order of magnitude; and so the terms will never exceed some fixed value which is unity at most. We may say then that *an infinite continued fraction of the second class in which the denominator of every component exceeds its numerator by unity at least is definite.*

788. We shall now shew that p_n and q_n in Art. 786 increase with n.

For $p_n - p_{n-1} = (a_n - 1)p_{n-1} - b_n p_{n-2}$; now $a_n - 1$ is at least as large as b_n; therefore p_n is greater than p_{n-1} if p_{n-1} is greater than p_{n-2}; and so on: and p_2 is obviously greater than p_1. Thus p_n is greater than p_{n-1}. Similarly q_n is greater than q_{n-1}.

789. *If in an infinite continued fraction of the second class every component has its numerator not less than unity and its denominator greater than its numerator by unity, the value of the infinite continued fraction is unity.*

Here we have always $a_n = b_n + 1$; therefore, by Art. 786,

$$p_n = (b_n + 1) p_{n-1} - b_n p_{n-2};$$

so that $$p_n - p_{n-1} = b_n (p_{n-1} - p_{n-2}).$$

Now $p_1 = b_1$, $p_2 = a_2 b_1 = (b_2 + 1) b_1$; thus $p_2 - p_1 = b_1 b_2$. Hence we obtain in succession

$$p_3 - p_2 = b_1 b_2 b_3, \quad p_4 - p_3 = b_1 b_2 b_3 b_4, \ldots\ldots, \quad p_n - p_{n-1} = b_1 b_2 \ldots b_n.$$

Therefore, by addition,

$$p_n = b_1 + b_1 b_2 + b_1 b_2 b_3 + \ldots\ldots + b_1 b_2 b_3 \ldots b_n.$$

Similarly we have $q_n - q_{n-1} = b_n (q_{n-1} - q_{n-2})$. Now $q_1 = b_1 + 1$, $q_2 = (b_1 + 1)(b_2 + 1) - b_2$; so that $q_2 - q_1 = b_1 b_2$. Hence we obtain in succession

$$q_3 - q_2 = b_1 b_2 b_3, \quad q_4 - q_3 = b_1 b_2 b_3 b_4, \ldots\ldots, \quad q_n - q_{n-1} = b_1 b_2 b_3 \ldots b_n.$$

Therefore, by addition,

$$q_n = 1 + b_1 + b_1 b_2 + b_1 b_2 b_3 + \ldots\ldots + b_1 b_2 b_3 \ldots b_n.$$

Thus, $q_n = p_n + 1$; and $\dfrac{p_n}{q_n} = \dfrac{p_n}{p_n + 1} = \dfrac{1}{1 + \dfrac{1}{p_n}}$. Now p_n by our hypothesis is not less than n, and so may be made as great as we please by taking n large enough; therefore $\dfrac{p_n}{q_n}$ may be made to differ from unity by less than any assigned quantity: and we may therefore say that the value of the infinite continued fraction is unity.

790. It will be seen that the investigation of the preceding Article establishes rather more than is contained in the enunciation which we used for simplicity. The essential conditions are that $a_n = b_n + 1$ for all values of n; and that p_n should increase indefinitely with n. It is *sufficient* for the latter condition that b_n should be never less than unity, but not *necessary*. The necessary and sufficient condition is that the infinite series of which the m^{th} term is $b_1 b_2 b_3 \ldots b_m$ should be divergent; this would be secured for example if $b_m = \dfrac{m}{m+1}$: see Art. 562.

791. *If the denominator of any component exceeds its numerator by more than unity while the denominator of every component exceeds its numerator by unity at least the value of the infinite continued fraction is less than unity.*

Suppose, for example, that $a_2 = b_2 + p$ where p is positive and greater than unity. The infinite continued fraction is equivalent to $\dfrac{b_1}{a_1 - \dfrac{b_2}{b_2 + p - \rho}}$, where ρ is a positive quantity which represents the infinite continued fraction beginning with the component $\dfrac{b_3}{a_3}$.

Now ρ cannot exceed unity by Art. 787; hence $\dfrac{b_2}{b_2 + p - \rho}$ is a positive proper fraction; and therefore as in Art. 786 we see that $\dfrac{b_1}{a_1 - \dfrac{b_2}{b_2 + p - \rho}}$ is a positive proper fraction.

792. *An infinite continued fraction of the second class in which every component is a proper fraction with its numerator and its denominator integral, and in which the value of the infinite continued fraction beginning with any component is less than unity cannot be a commensurable quantity.*

For if possible, suppose the continued fraction commensurable, and denote it by $\dfrac{B}{A}$, where A and B are positive integers. Thus $\dfrac{B}{A} = \dfrac{b_1}{a_1 - \rho_1}$ where ρ_1 denotes the infinite continued fraction beginning with the component $\dfrac{b_2}{a_2}$. Therefore $\rho_1 = \dfrac{a_1 B - b_1 A}{B}$; the numerator of this fraction is an integer, which we will denote by C; and C must be positive for ρ_1 is positive. In like manner, if ρ_2 denote the infinite continued fraction beginning with the component $\dfrac{b_3}{a_3}$ we find that $\rho_2 = \dfrac{D}{C}$, where D is also a positive integer. And so on.

Moreover $\frac{B}{A}, \frac{C}{B}, \frac{D}{C}, \ldots$ must all be proper fractions by hypothesis.

Hence $A, B, C, D, \ldots$ form a series of *positive integers*, which are in *descending* order of magnitude, and yet *infinite* in number: this is absurd. Hence the infinite continued fraction cannot be a commensurable quantity.

Article 785 applies here also, with the condition of the enunciation in Art. 792.

793. We have supposed in the preceding Article that the infinite continued fraction beginning with any component is less than unity. By Arts. 789, 791, this will always be secured except in the case in which from and after some fixed component the denominator of every component exceeds the numerator by unity.

794. For an example of an infinite definite continued fraction of the first class, suppose that every component is $\frac{b}{2a}$, where a and b are positive. Denote the continued fraction by x; then

$$x = \cfrac{b}{2a + \cfrac{b}{2a + \ldots}}; \text{ so that } x = \frac{b}{2a + x};$$

therefore $x^2 + 2ax - b = 0$; therefore $x = -a \pm \sqrt{(a^2 + b)}$: the upper sign must be taken, since the infinite continued fraction is positive. Thus, by transposition, we obtain

$$\sqrt{(a^2 + b)} = a + \cfrac{b}{2a + \cfrac{b}{2a + \ldots}}.$$

This formula gives various modes of expressing a square root in the form of a continued fraction. For example, take $\sqrt{17}$. We may put $17 = 16 + 1$, or $= 9 + 8$; and so on. Thus,

$$\sqrt{17} = 4 + \cfrac{1}{8 + \cfrac{1}{8 + \ldots}} = 3 + \cfrac{8}{6 + \cfrac{8}{6 + \ldots}}.$$

795. For an example of an infinite definite continued fraction of the second class, suppose that every component is $\frac{b}{2a}$, where a and b are positive, and $2a$ exceeds b by unity at least. Denote the continued fraction by x; then

$$x = \cfrac{b}{2a - \cfrac{b}{2a - \dots}}, \text{ so that } x = \frac{b}{2a - x};$$

therefore $x^2 - 2ax + b = 0$; therefore $x = a \pm \sqrt{(a^2 - b)}$. The lower sign must be taken, for with the upper sign we have a result greater than $a + a - b$, that is greater than $2a - b$, that is greater than unity: but the infinite continued fraction cannot be greater than unity, by Art. 787. Thus, by transposition, we obtain

$$\sqrt{(a^2 - b)} = a - \cfrac{b}{2a - \cfrac{b}{2a - \dots}}.$$

796. In Art. 781 we have

$$p_n = a_n p_{n-1} + b_n p_{n-2}, \quad q_n = a_n q_{n-1} + b_n q_{n-2};$$

and in Art. 786 we have similar relations with the sign + changed to −. Now suppose that the values of a_n and b_n are given for all values of n, and that p_1 and p_2 and q_1 and q_2 have been obtained; then from the above general relations we can determine in succession $p_3, p_4, p_5, \dots$ and $q_3, q_4, q_5, \dots$ Sometimes we may by special artifices discover such a law of formation of the successive terms as will enable us to give general expressions for p_n and q_n: an example has already occurred in Art. 789. Or a law may appear by trial to hold, and may be verified by induction. The investigation of the general expressions for p_n and q_n belongs however to a higher branch of mathematics, namely the Calculus of Finite Differences.

A particular case may be noticed. Suppose that a_n and b_n are constant for all values of n; denote the former by a, and the latter by b. Then we have $p_n = ap_{n-1} + bp_{n-2}$; and we see by the aid of Art. 656 that p_n is equal to the coefficient of x^n in the expansion according to ascending powers of x of

$$\frac{p_1 + (p_2 - ap_1)x}{1 - ax - bx^2};$$

also $p_1 = b$, and $p_2 = ab$, so that this expression becomes

$$\frac{b}{1 - ax - bx^2}.$$

Similarly, q_n is equal to the coefficient of x^n in the expansion of

$$\frac{q_1 + (q_2 - aq_1)x}{1 - ax - bx^2};$$

also $q_1 = a$, and $q_2 = a^2 + b$, so that this expression becomes

$$\frac{a + bx}{1 - ax - bx^2}, \text{ that is } \frac{1}{x(1 - ax - bx^2)} - \frac{1}{x}.$$

797. We will now shew how to convert a series having a finite number of terms into a continued fraction.

The series $\frac{1}{u_0} + \frac{x}{u_1} + \frac{x^2}{u_2} + \ldots\ldots + \frac{x^n}{u_n}$ is identically equal to a continued fraction of the second class with $n + 1$ components, in which the first component is $\frac{1}{u_0}$, the second is $\frac{u_0{}^2 x}{u_0 x + u_1}$, and generally the r^{th} is $\frac{u^2_{r-2} x}{u_{r-2} x + u_{r-1}}$.

This may be demonstrated by induction.

It is obvious that $\frac{1}{u_0} = \frac{1}{u_0}$,

and that $$\frac{1}{u_0} + \frac{x}{u_1} = \cfrac{1}{u_0 - \cfrac{u_0{}^2 x}{u_0 x + u_1}};$$

assume that the theorem holds when there are $n + 1$ terms in the series: we will shew then that it will hold when there are $n + 2$ terms.

For change u_n into $u_n - \frac{u_n{}^2 x}{u_n x + u_{n+1}}$; then $\frac{x^n}{u_n}$ is changed into

$$\frac{x^n}{u_n - \dfrac{u_n^2 x}{u_n x + u_{n+1}}}, \text{ that is into } \frac{x^n (u_n x + u_{n+1})}{u_n u_{n+1}}, \text{ that is into } \frac{x^n}{u_n} + \frac{x^{n+1}}{u_{n+1}};$$

so that another term is in fact added to the series. Also if the change of u_n into $u_n - \dfrac{u_n^2 x}{u_n x + u_{n+1}}$ be made in the continued fraction with $n+1$ components we obtain a continued fraction with $n+2$ components formed according to the same law.

Hence if the theorem holds when the series has $n+1$ terms it holds when the series has $n+2$ terms; and it has been shewn to hold when the series has two terms: hence it holds universally.

798. We may deduce the following result from that of Art. 797 by simplifying the fractions which occur; or we may establish it directly in the manner of Art. 797: the series

$$\frac{1}{v_0} + \frac{x}{v_0 v_1} + \frac{x^2}{v_0 v_1 v_2} + \dots + \frac{x^n}{v_0 v_1 v_2 \dots v_n}$$

is identically equal to a continued fraction of the second class with $n+1$ components in which the first component is $\dfrac{1}{v_0}$, the second is $\dfrac{v_0 x}{x + v_1}$, and generally the r^{th} is $\dfrac{v_{r-2} x}{x + v_{r-1}}$.

799. In the identities of Arts. 797 and 798 we may if we please change the sign of x; take for instance the identity of Art. 798; hence we obtain the following result: the series

$$\frac{1}{v_0} - \frac{x}{v_0 v_1} + \frac{x^2}{v_0 v_1 v_2} - \dots + \frac{(-1)^n x^n}{v_0 v_1 v_2 \dots v_n}$$

is identically equal to a continued fraction of the first class with $n+1$ components, in which the first component is $\dfrac{1}{v_0}$, the second is $\dfrac{v_0 x}{v_1 - x}$, and generally the r^{th} is $\dfrac{v_{r-2} x}{v_{r-1} - x}$. This result may also be established directly in the manner of Art. 797.

800. In Arts. 797, 798 and 799 we may suppose n as great as we please provided the series remain convergent; and then we

can transform an infinite convergent series into an infinite continued fraction.

801. A very important formula on this subject is due to Gauss. Denote the *hypergeometrical* infinite series of Art. 775 by $F(a, \beta, \gamma)$; then Gauss has transformed $\frac{F(a, \beta+1, \gamma+1)}{F(a, \beta, \gamma)}$ into an infinite continued fraction: the transformation holds provided $F(a, \beta, \gamma)$ and $F(a, \beta+1, \gamma+1)$ are both convergent.

The essential part of the demonstration consists of the following relation: let z stand for $\frac{F(a, \beta+1, \gamma+1)}{F(a, \beta, \gamma)}$, then

$$z = \cfrac{1}{1 - \cfrac{k_1}{1 - k_2 z_2}},$$

where $k_1 = \frac{a(\gamma-\beta)x}{\gamma(\gamma+1)}$, $k_2 = \frac{(\beta+1)(\gamma+1-a)x}{(\gamma+1)(\gamma+2)}$, and z_2 is what z becomes when in z we change a, β, γ into $a+1, \beta+1, \gamma+2$ respectively. This we shall now shew.

In the series for $F(a, \beta, \gamma)$ change β into $\beta+1$, and γ into $\gamma+1$, and subtract the original value: thus we obtain

$$F(a, \beta+1, \gamma+1) - F(a, \beta, \gamma) = \frac{a(\gamma-\beta)x}{\gamma(\gamma+1)} F(a+1, \beta+1, \gamma+2) \quad \ldots\ldots(1).$$

Similarly we have

$$F(a+1, \beta, \gamma+1) - F(a, \beta, \gamma) = \frac{\beta(\gamma-a)x}{\gamma(\gamma+1)} F(a+1, \beta+1, \gamma+2) \quad \ldots\ldots(2).$$

From (1) by division

$$1 - \frac{1}{z} = k_1 \frac{F(a+1, \beta+1, \gamma+2)}{F(a, \beta+1, \gamma+1)} \quad \ldots\ldots(3).$$

From (2) by division after changing β into $\beta+1$, and γ into $\gamma+1$,

$$1 - \frac{F(a, \beta+1, \gamma+1)}{F(a+1, \beta+1, \gamma+2)} = k_2 z_2 \quad \ldots\ldots(4).$$

From (3) and (4) we obtain the required result.

Then the continued fraction for z may be prolonged by the aid of the relation

$$z_2 = \cfrac{1}{1 - \cfrac{k_3}{1 - k_4 z_4}};$$

and this may be prolonged to any extent, the general terms being

$$k_{2r-1} = \frac{(\alpha + r - 1)(\gamma + r - 1 - \beta)x}{(\gamma\ 2r\ \ - 2)(\gamma + 2r - 1)},$$

$$k_{2r} = \frac{(\beta + r)(\gamma + r - \alpha)x}{(\gamma + 2r - 1)(\gamma + 2r)},$$

$$z_{2r} = \frac{F(\alpha + r,\ \beta + r + 1,\ \gamma + 2r + 1)}{F(\alpha + r,\ \beta + r,\ \gamma + 2r)}.$$

We assume throughout that the infinite series are convergent; as we cannot employ (1) and (2) without this condition; it will be seen from Art. 775 that if the numerator or denominator of z is convergent then all the infinite series which occur are convergent.

When r is indefinitely large z_{2r} will not differ sensibly from unity.

For
$$z_{2r} = \frac{1 + A_1 x + A_2 x^2 + \ldots}{1 + B_1 x + B_2 x^2 + \ldots},$$

where $B_1 = \dfrac{(\alpha + r)(\beta + r)}{1\,(\gamma + 2r)}$, $B_2 = \dfrac{(\alpha + r + 1)(\beta + r + 1)}{2\,(\gamma + 2r + 1)} B_1, \ldots$;

and $A_1, A_2, \ldots$ may be obtained from $B_1, B_2, \ldots$ respectively by changing β into $\beta + 1$ and γ into $\gamma + 1$.

Thus $\dfrac{A_1}{B_1}, \dfrac{A_2}{B_2}, \ldots$ may be considered to be all equal to unity when r is indefinitely great; and so by Art. 679 we may consider z_{2r} to be also unity.

Since z_{2r} may be considered to be unity $k_{2r} z_{2r}$ becomes simply k_{2r}.

Thus z is transformed into an infinite continued fraction.

802. For a particular case put $\dfrac{x^2}{\alpha\beta}$ instead of x; then suppose

that $\beta = a$, and that a increases indefinitely: thus the denominator of z becomes

$$1 + \frac{x^2}{1 \cdot \gamma} + \frac{x^4}{1 \cdot 2 \cdot \gamma(\gamma+1)} + \frac{x^6}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} + \dots$$

which we will denote by $f(\gamma)$; the numerator may be obtained from the denominator by changing γ into $\gamma + 1$.

Also k_{2r-1} becomes $-\dfrac{x^2}{(\gamma+2r-2)(\gamma+2r-1)}$,

and k_{2r} becomes $\quad -\dfrac{x^2}{(\gamma+2r-1)(\gamma+2r)}$.

Thus finally $\dfrac{f(\gamma+1)}{f(\gamma)}$ is transformed into an infinite continued

fraction $\cfrac{1}{1+\cfrac{p_1}{1+\cfrac{p_2}{1+\dots}}}$, where $p_m = \dfrac{x^2}{(\gamma+m-1)(\gamma+m)}$.

This result may be obtained independently in the manner of Art. 801; for we have

$$f(\gamma) - f(\gamma+1) = \frac{x^2}{\gamma(\gamma+1)} f(\gamma+2); \text{ thus}$$

$$\frac{f(\gamma)}{f(\gamma+1)} = 1 + \frac{x^2}{\gamma(\gamma+1)} \frac{f(\gamma+2)}{f(\gamma+1)}; \text{ and so on.}$$

803. In the result of the preceding Article put $\frac{1}{2}$ for γ and $\frac{y}{2}$ for x. Then it will be found that $\dfrac{f(\gamma+1)}{f(\gamma)}$ becomes $\dfrac{e^y - e^{-y}}{y(e^y + e^{-y})}$; and that p_m becomes $\dfrac{y^2}{4m^2-1}$. By multiplying by y and simplifying the fractions we ultimately obtain for $\dfrac{e^y - e^{-y}}{e^y + e^{-y}}$, an infinite continued fraction of the first class in which the first component is $\dfrac{y}{1}$, the second is $\dfrac{y^2}{3}$, and generally the r^{th} is $\dfrac{y^2}{2r-1}$.

For y put $\dfrac{m}{n}$ where m and n are positive integers; then by simplifying the fractions we obtain for $\dfrac{e^{\frac{m}{n}} - e^{-\frac{m}{n}}}{e^{\frac{m}{n}} + e^{-\frac{m}{n}}}$ an infinite continued fraction of the first class in which the first component is $\dfrac{m}{n}$, the second is $\dfrac{m^2}{3n}$, and generally the r^{th} is $\dfrac{m^2}{(2r-1)n}$.

When r is large enough $(2r-1)n$ exceeds m^2; hence by Art. 785 the infinite continued fraction beginning with a suitable component is incommensurable; and therefore the whole continued fraction is incommensurable. Hence $e^{\frac{m}{n}}$ is incommensurable for all integral values of m and n.

EXAMPLES OF CONTINUED FRACTIONS.

1. Find the value of $5 - \cfrac{1}{10 - \cfrac{1}{10 - \dots}}$

2. Shew that $\left\{n + \dfrac{1}{2n+}\,\dfrac{1}{2n+}\dots\right\}^2 - \left\{n - \dfrac{1}{2n-}\,\dfrac{1}{2n-}\dots\right\}^2 = 2.$

3. In a continued fraction of the first class every component is $\dfrac{b}{a}$: shew that $p_{n+1} = bq_n$.

4. In a continued fraction of the first class every component is $\dfrac{b}{a}$: find the values of p_n and q_n.

5. In a continued fraction of the first class if $a_n = b_n = n$, shew that $p_n + q_n = \underline{|n+1}$.

6. In a continued fraction of the first class if $b_{n+1} = 1 + a_n$, shew that $p_n - b_{n+1}p_{n-1} = A(-1)^n$, $q_n - b_{n+1}q_{n-1} = B(-1)^n$; where A and B are constant whatever n may be.

7. In an infinite continued fraction of the first class the n^{th} component is $\dfrac{(n-1)^2+1}{n^2}$; shew that $p_n - (n^2+1)p_{n-1} = (-1)^{n+1}$.

8. Shew that e^{-x} can be transformed into an infinite continued fraction of the first class in which the first component is $\frac{1}{1}$, the second is $\frac{x}{1-x}$, and generally the r^{th} is $\frac{(r-2)x}{r-1-x}$.

9. Shew that $\log 2$ is equal to an infinite continued fraction of the first class in which the first component is $\frac{1}{1}$, the second is $\frac{1}{1}$, and generally the r^{th} is $\frac{(r-1)^2}{1}$.

10. Obtain from Art. 801 an infinite continued fraction of the first class for $\frac{1}{x}\log(1+x)$.

LVIII. MISCELLANEOUS THEOREMS.

804. The present Chapter consists of some miscellaneous theorems on the following subjects: abbreviation of algebraical multiplication and division, vanishing fractions, permutations and combinations, and probability.

805. In multiplying together two algebraical expressions it is sometimes convenient to abridge the written work by expressing only the coefficients. For example, suppose it required to multiply $2x^4+x^2-3x+1$ by x^2+3x-2; we may proceed thus:

$$\begin{array}{l}
2+0+1-3+1 \\
\underline{1+3-2} \\
2+0+1-3+1 \\
\phantom{2+{}}6+0+3-9+3 \\
\underline{\phantom{2+6+{}}-4-0-2+6-2} \\
2+6-3+0-10+9-2
\end{array}$$

Thus the required result is $2x^6+6x^5-3x^4-10x^2+9x-2$.

A similar abridgement of the written work may be made in division.

This mode of operation has been sometimes called the *method of detached coefficients.*

806. *Synthetic Division.* The operation of division may however be still more abridged by a method which is due to the late Mr Horner, and which is called *synthetic division.*

Suppose it required to divide

$$Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} + \dots$$

by

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + a_4x^{n-4} + \dots;$$

let the quotient be denoted by

$$Ax^{m-n} + A_1x^{m-n-1} + A_2x^{m-n-2} + A_3x^{m-n-3} + \dots,$$

then it is our object to shew how A_1, A_2, A_3, ... may be determined.

If we multiply the quotient by the divisor we obtain the dividend; this operation may be indicated as follows, expressing only the coefficients,

$$\begin{array}{rrrrr}
A\,+ & A_1\,+ & A_2\,+ & A_3\,+ & A_4 + \dots \\
1\,+ & a_1\,+ & a_2\,+ & a_3\,+ & a_4 + \dots \\
\hline
A\,+ & A_1\,+ & A_2\,+ & A_3\,+ & A_4 + \dots \\
 & a_1A + & a_1A_1 + & a_1A_2 + & a_1A_3 + \dots \\
 & & a_2A + & a_2A_1 + & a_2A_2 + \dots \\
 & & & a_3A + & a_3A_1 + \dots \\
 & & & & a_4A + \dots \\
 & & & & \dots \\
\hline
A\,+ & B\,+ & C\,+ & D\,+ & E + \dots
\end{array}$$

here the last line is supposed to be obtained in the usual way by adding the vertical columns between the horizontal lines. Now A, B, C, ... are known, and we have to find A_1, A_2, A_3, ...; for this purpose we *reverse* the above operation and perform the following:

$$\begin{array}{r|rrrrr}
 & A\,+ & B\,+ & C\,+ & D\,+ & E + \dots \\
-a_1 & & -a_1A & -a_1A_1 & -a_1A_2 & -a_1A_3 - \dots \\
-a_2 & & & -a_2A & -a_2A_1 & -a_2A_2 - \dots \\
-a_3 & & & & -a_3A & -a_3A_1 - \dots \\
-a_4 & & & & & -a_4A - \dots \\
 & & & & & \dots \\
\hline
 & A\,+ & A_1\,+ & A_2\,+ & A_3\,+ & A_4 + \dots
\end{array}$$

Here each vertical column expresses the same result as the corresponding vertical column of the former operation, but expresses it in a form more convenient for our object. For example, the fourth vertical column of the former operation gave

$$A_3 + a_1A_2 + a_2A_1 + a_3A = D;$$

and the fourth vertical column in the present operation gives

$$D - a_1A_2 - a_2A_1 - a_3A = A_3.$$

The method then may be described as follows:

(1) If the first term of the divisor have a numerical coefficient, divide every coefficient of the dividend and divisor by this coefficient; the resulting coefficients are those intended in the following rules.

(2) Write the coefficients of the dividend in a horizontal line, with their proper signs, putting 0 when any term is wanting. This gives the horizontal row $A + B + C + D + E + \ldots$

(3) Draw a vertical line to the left of this series of coefficients, and write in a vertical column the coefficients of the divisor with their signs changed, putting 0 when any term is wanting. This gives the vertical column $-a_1 - a_2 - a_3 \ldots$ no notice being taken of *unity*, which is the coefficient of the first term of the divisor.

(4) Multiply each term of this vertical column by the first coefficient of the quotient, and arrange the results in the first *oblique* column. This gives the *oblique* column $-a_1A - a_2A - a_3A - \ldots$ the first term of which is to be placed under B.

(5) Add the terms in the second vertical column to the right of the vertical line; this gives the coefficient of the second term of the quotient. That is, $B - a_1A = A_1$.

(6) With the coefficient thus obtained form the next *oblique* column. This gives $-a_1A_1 - a_2A_1 - a_3A_1 - \ldots\ldots$ the first term of which is placed under C.

(7) Add the terms in the third vertical column to the right of the vertical line; this gives the coefficient of the third term of the quotient. That is, $C - a_1A_1 - a_2A = A_2$.

(8) Continue these operations until the work terminates, or as many terms are found as are required.

807. For example, divide $4x^4 + 3x^2 - 3x + 1$ by $x^2 - 2x + 3$;

$$\begin{array}{r|l}
 & 4 + 0 + \ 3 - \ 3 + \ 1 \\
2 & \quad\ 8 + 16 + 14 - 26 - 92 \\
-3 & \qquad\ - 12 - 24 - 21 + 39 + 138 \\
\hline
 & 4 + 8 + \ 7 - 13 - 46 - 53
\end{array}$$

Thus the quotient is $4x^2 + 8x + 7 - 13x^{-1} - 46x^{-2} - 53x^{-3} \ldots\ldots$ Or if we wish to *stop* at $46x^{-2}$, we have

$$\frac{4x^4 + 3x^2 - 3x + 1}{x^2 - 2x + 3} = 4x^2 + 8x + 7 - 13x^{-1} - 46x^{-2} - \frac{53x^{-1} - 138x^{-2}}{x^2 - 2x + 3}.$$

If we wish to *stop* at $-13x^{-1}$, the oblique column $-92 + 138$ must be suppressed, and the result is $4x^2 + 8x + 7 - 13x^{-1} - \dfrac{46 - 39x^{-1}}{x^2 - 2x + 3}$. If we wish to *stop* at 7, the oblique column $-26 + 39$ must also be suppressed, and the result is $4x^2 + 8x + 7 - \dfrac{13x + 20}{x^2 - 2x + 3}$.

808. We may observe that the principle which is exemplified in Art. 332 is often of use in effecting algebraical reductions. For example, suppose it required to prove the following identity:

$$(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4$$
$$= 12abc\,(a + b + c).$$

We see that if $a = 0$, the expression which forms the left-hand member of the proposed identity vanishes; we therefore infer that this expression is divisible by a. In the same manner we infer that the expression is divisible by b and by c. Thus abc is a factor of the expression. And since the expression is of the *fourth* degree, there must be another factor which is of the *first* degree; and since the expression is *symmetrical* with respect to a, b, and c, this factor must be $a + b + c$.

Hence the expression must be equal to $kabc\,(a + b + c)$, where k denotes some numerical coefficient which retains the same value for all values of a, b, and c. To determine k we may ascribe to a, b, and c any values we find convenient; for example, we may suppose $b = a$ and $c = a$, and we find that $k = 12$.

Thus the proposed identity is demonstrated.

The following identity may be demonstrated in the same manner:

$$(a+b+c+d)^4+(a+b-c-d)^4+(a+c-b-d)^4+(a+d-b-c)^4$$
$$-(a+b+c-d)^4-(a+b-c+a)^4-(a-b+c+d)^4-(-a+b+c+d)^4$$
$$=192abcd.$$

809. *Vanishing Fractions.* A fraction in which the numerator and the denominator are both zero on some supposition as to the value of any quantity involved, is then called a *vanishing fraction.* For example, the numerator and the denominator of the fraction $\frac{x^{\frac{1}{3}}-a^{\frac{1}{3}}}{x^{\frac{1}{4}}-a^{\frac{1}{4}}}$ vanish when $x=a$; the fraction *then* takes the form $\frac{0}{0}$, and we cannot strictly say that it has any definite value. But we can find the value of the fraction when x has any value different from a; and we can shew that the more nearly x approaches to a the more nearly does the value of the fraction approach to a certain definite value. For put $x=a+h$; then by the Binomial Theorem the fraction becomes

$$\frac{a^{\frac{1}{3}}+\frac{1}{3}a^{-\frac{2}{3}}h-\frac{1}{9}a^{-\frac{5}{3}}h^2+\ldots-a^{\frac{1}{3}}}{a^{\frac{1}{4}}+\frac{1}{4}a^{-\frac{3}{4}}h-\frac{3}{32}a^{-\frac{7}{4}}h^2+\ldots-a}, \text{ that is, } \frac{\frac{1}{3}a^{-\frac{2}{3}}-\frac{1}{9}a^{-\frac{5}{3}}h+\ldots}{\frac{1}{4}a^{-\frac{3}{4}}-\frac{3}{32}a^{-\frac{7}{4}}h+\ldots}.$$

Now as h diminishes the numerator and the denominator of the last fraction approach to the values $\frac{1}{3}a^{-\frac{2}{3}}$ and $\frac{1}{4}a^{-\frac{3}{4}}$ respectively; and by taking h small enough, the numerator and the denominator may be made to differ from these values by as small a quantity as we please. Thus the fraction can be made to approach as near as we please to $\frac{\frac{1}{3}a^{-\frac{2}{3}}}{\frac{1}{4}a^{-\frac{3}{4}}}$, that is, to $\frac{4}{3}a^{\frac{1}{12}}$. This result is expressed by saying that $\frac{4}{3}a^{\frac{1}{12}}$ is the *limit* to which the fraction approaches as x approaches to a.

We may also arrive at this result without using the Binomial

Theorem. For suppose $x = y^{12}$ and $a = b^{12}$; then the proposed fraction becomes $\frac{y^4 - b^4}{y^3 - b^3}$; so long as y is not absolutely equal to b we may divide both numerator and denominator by $y - b$, and so put the fraction in the form $\frac{y^3 + y^2b + yb^2 + b^3}{y^2 + yb + b^2}$. As y approaches to b this fraction approaches to $\frac{4b}{3}$, and the fraction may be made to differ as little as we please from $\frac{4b}{3}$ by making $y - b$ small enough. Thus the *limit* of the fraction as y approaches to b is $\frac{4b}{3}$; that is, the *limit* of the fraction as x approaches to a is $\frac{4}{3}a^{\frac{1}{12}}$.

Questions respecting *vanishing fractions* and *limits* belong properly to the Differential Calculus, to which the student is therefore referred for more information.

810. We will now give two Articles, which form a supplement to the Chapter on Permutations and Combinations. They are due to H. M. Jeffery, Esq. of Cheltenham.

811. *To find the number of combinations of* n *things taken* 1, 2, 3, ...n *at a time, when there are* p *of one sort,* q *of another,* r *of another, and so on.*

Let there be n letters, and suppose p of them to be a, q of them to be b, r of them to be c, and so on. The product

$$(1 + ax + a^2x^2 + \ldots\ldots + a^px^p)(1 + bx + b^2x^2 + \ldots\ldots + b^qx^q)$$
$$(1 + cx + c^2x^2 + \ldots\ldots + c^rx^r)\ldots\ldots$$

contains the combinations of the n letters taken 1, 2, 3, ... n at a time, namely in the coefficients of x, x^2, x^3, ... x^n respectively. The *number* of the combinations in each case is found by equating a, b, c, ... to unity. Thus the *number* of combinations of the n letters taken k at a time, is the coefficient of x^k in the expansion of

$$(1 + x + x^2 + \ldots + x^p)(1 + x + x^2 + \ldots + x^q)(1 + x + x^2 + \ldots + x^r)\ldots\ldots$$

The number of combinations when the letters are taken k at a time, is the same as the number when they are taken $n-k$ at a time; this may be shewn as in Art. 495.

The total number of combinations is found by equating x to unity in the above expression, and subtracting one from the result, since the first term in the expansion of the expression does not contain x, and therefore does not denote the number of any combination. Thus the total number is $(p+1)(q+1)(r+1)\ldots\ldots-1$.

The expression to be expanded may be written thus,

$$\frac{1-x^{p+1}}{1-x}\cdot\frac{1-x^{q+1}}{1-x}\cdot\frac{1-x^{r+1}}{1-x}\ldots\ldots,$$

that is, $$(1-x^{p+1})(1-x^{q+1})(1-x^{r+1})\ldots\ldots(1-x)^{-\mu},$$

where μ is the number of different sorts of letters.

For example, take the letters in the word *notation*. It will be found that the numbers of the combinations when the letters are taken 1, 2, ... 8 at a time, are respectively 5, 13, 22, 26, 22, 13, 5, 1.

812. *To find the number of permutations of* n *things taken* 1, 2, 3, ... n *at a time, when there are* p *of one sort,* q *of another,* r *of another, and so on.*

Let there be n letters, and suppose p of them to be a, q of them to be b, r of them to be c, and so on.

Form the product of the following series;

$$1+Pax+\frac{P^2a^2x^2}{1.2}+\frac{P^3a^3x^3}{\lfloor 3}+\ldots\ldots+\frac{P^pa^px^p}{\lfloor p},$$

$$1+Pbx+\frac{P^2b^2x^2}{1.2}+\frac{P^3b^3x^3}{\lfloor 3}+\ldots\ldots+\frac{P^qb^qx^q}{\lfloor q},$$

$$1+Pcx+\frac{P^2c^2x^2}{1.2}+\frac{P^3c^3x^3}{\lfloor 3}+\ldots\ldots+\frac{P^rc^rx^r}{\lfloor r},$$

....

After the product has been formed and arranged according to powers of Px, change P into 1, change P^2 into $\lfloor 2$, change P^3 into $\lfloor 3$, and so on; then the coefficient of x^k in the result will consist of the permutations of the n letters taken k at a time. The truth of this statement may be seen by examining the mode of formation of each coefficient in particular cases; for example, suppose $n = 4$, and $p, q, \ldots\ldots$ each $= 1$; or suppose $n = 4$, $p = 2$, $q = 1$, $r = 1$. The *number* of the permutations will be found by making $a, b, c, \ldots\ldots$ each equal to unity; this may be done before the product of the above series is formed.

For example, take the letters in the word *notation*. It will be found that the numbers of the permutations when the letters are taken 1, 2, ... 8 at a time, are respectively, 5, 23, 96, 354, 1110, 2790, 5040, 5040.

813. We will now give some further remarks on the subject of Probability.

It is observed by Dr Wood in his Algebra, that there is no subject in which the learner is so liable to mistake as in the calculation of probabilities. Dr Wood proceeds thus: "A single instance will shew the danger of forming a hasty judgment, even in the most simple case. The probability of throwing an ace with one die is $\frac{1}{6}$, and since there is an equal probability of throwing an ace in the second trial, it might be supposed that the probability of throwing an ace in two trials is $\frac{2}{6}$. This is not a just conclusion; for it would follow by the same mode of reasoning, that in six trials a person could not fail to throw an ace. The error, which is not easily seen, arises from a tacit supposition that there must necessarily be a second trial, which is not the case if an ace be thrown in the first."

The above extract is introduced for the sake of the important remarks which it contains, and also for the purpose of drawing attention to the last sentence, which students have often found difficult. It should be observed, to prevent any ambiguity, that the problem under discussion is the following: Required the probability of throwing one ace at least in two trials with a single die. Dr Wood's last sentence indicates the following as his method of solution. The chance of an ace in the first trial is $\frac{1}{6}$; if an ace is obtained in this trial there will be no need of a second trial. But suppose we fail to throw ace the first time; the chance of this failure is $\frac{5}{6}$, and then the chance of success in the next trial is $\frac{1}{6}$. Thus the chance of obtaining one ace at least in two trials is $\frac{1}{6}+\frac{5}{6}\cdot\frac{1}{6}$; that is, $\frac{11}{36}$. And the error of a person who estimates the chance at $\frac{1}{6}+\frac{1}{6}$ may be ascribed to the circumstance that he changes the $\frac{5}{6}$ in the product $\frac{5}{6}\cdot\frac{1}{6}$ into unity, thus assuming that there will be always a second trial, although the second trial may be rendered unnecessary by reason of the first trial having been successful.

This solution is of course quite correct, but it would probably be considered by the person who estimated the chance at $\frac{1}{6}+\frac{1}{6}$ that it does not shew him his error, but substitutes a different solution altogether; and he might say *there is no uncertainty about the occurrence of the second trial, for two trials are guaranteed in the enunciation of the problem, or at least are allowed to us if we please to make them.*

The error really arises from neglect of the following consideration: when events are *mutually exclusive*, so that the supposition that one takes place is incompatible with the supposition that any other takes place, *then and not otherwise* the chance of one or another of the events is the *sum* of the chances of the separate events.

In the present problem success in the first trial is not incompatible with success in the second trial, and therefore we cannot take the sum of the chances as the chance of success in one or other of the trials.

It is easy to present the correct solution of the problem in different ways. Thus besides Dr Wood's solution, another has been given in Art. 735. We may also proceed thus. The desired event may be considered as one of the following three; success in the first trial and failure in the second, failure in the first trial and success in the second, success in the first trial and success in the second. The chances of these events are respectively $\frac{1}{6}\cdot\frac{5}{6}$, $\frac{5}{6}\cdot\frac{1}{6}$, $\frac{1}{6}\cdot\frac{1}{6}$; and the events are mutually exclusive, so that the chance of obtaining one or another of them is $\frac{5}{36}+\frac{5}{36}+\frac{1}{36}$, that is, $\frac{11}{36}$.

814. This discussion naturally leads us to investigate the probability of the happening of *one or more* events out of events which *are* or which are *not* mutually exclusive. We shall now give some theorems on this subject.

I. Let there be any number of independent events of which the respective probabilities are α, β, γ, ...: required the probability of the happening of one at least.

The probability of all failing is $(1-\alpha)(1-\beta)(1-\gamma)$ ......; therefore the probability of the happening of one at least is $1-(1-\alpha)(1-\beta)(1-\gamma)\ldots$ This may be written $\Sigma\alpha-\Sigma\alpha\beta+\Sigma\alpha\beta\gamma-\ldots$ or $P_1-P_2+P_3-P_4+\ldots$ suppose, where P_1 is the sum of the probabilities of the single events, P_2 is the sum of the probabilities of pairs of events, P_3 the sum of the probabilities of triads of events, and so on.

II. The theorem just proved is true even when the events are *not independent;* that is, the probability of the happening of one at least of the events is $P_1-P_2+P_3-P_4+\ldots\ldots$ where P_1, P_2, P_3, P_4, ... have the meanings already stated.

For consider only two events A and B; let n denote the whole number of equally probable cases, n_α the number in which A occurs, n_β the number in which B occurs, $n_{\alpha\beta}$ the number in which both A and B occur. To find the number of cases in which neither A nor B occurs we proceed thus: from n take away n_α and n_β; we have thus taken away too many cases, because the cases, in number $n_{\alpha\beta}$, in which both A and B occur have been taken away twice; restore then $n_{\alpha\beta}$. Therefore the whole number of cases in which neither A nor B occurs is $n-(n_\alpha+n_\beta)+n_{\alpha\beta}$.

Hence the number of cases in which one at least of the events occurs is $n_\alpha+n_\beta-n_{\alpha\beta}$.

Therefore the probability of the occurrence of one at least

$$=\frac{n_\alpha+n_\beta-n_{\alpha\beta}}{n}=\frac{n_\alpha+n_\beta}{n}-\frac{n_{\alpha\beta}}{n}=P_1-P_2.$$

Similarly any other case may be treated.

III. Supposing that there are n events, required the probability that an assigned m of them will happen, and no more.

Suppose that the events of which the probabilities are $\alpha, \beta, \gamma, \ldots\ldots$ are to happen, and the events of which the probabilities are $\lambda, \mu, \nu, \ldots\ldots$ are not to happen. Then if the events are independent the required probability is

$$\alpha\beta\gamma \ldots\ldots (1-\lambda)(1-\mu)(1-\nu) \ldots\ldots;$$

that is, $\alpha\beta\gamma \ldots\ldots$ to m factors $\left\{1-\Sigma\lambda+\Sigma\lambda\mu-\Sigma\lambda\mu\nu+\ldots\ldots\right\}$.

This we may denote by $Q_m-Q_{m+1}+Q_{m+2}-Q_{m+3}+\ldots\ldots$, where Q_m is the probability of the occurrence of the m assigned events, Q_{m+1} is the sum of the probabilities of the occurrence of every collection of $m+1$ events which includes the m assigned events, Q_{m+2} is the sum of the probabilities of the occurrence of every collection of $m+2$ events which includes the m assigned events, and so on.

IV. As before we may shew that the theorem in III. is true even when the events are not independent.

V. Required the probability of the occurrence of *any* m of the events and no more.

With the previous notation this is

$$\Sigma Q_m-\Sigma Q_{m+1}+\Sigma Q_{m+2}-\Sigma Q_{m+3}+\ldots$$

It may happen that in some cases

$$\Sigma Q_m = \frac{\lfloor n}{\lfloor m \, \lfloor n-m} Q_m, \quad \Sigma Q_{m+1} = \frac{\lfloor n}{\lfloor m+1 \, \lfloor n-m-1} Q_{m+1}, \text{ and so on;}$$

this will be the case when the events are all similar.

VI. In II. we have found the probability that *at least* one event shall happen, and in V. the probability that *just* one event shall happen; by subtracting the second result from the first we obtain the probability that *two events at least* shall happen. Then again we know from V. the probability that *just two* events shall happen; by subtracting this from the probability that *two events at least* shall happen we obtain the probability that *three events at least* shall happen. And so on.

MISCELLANEOUS EXAMPLES.

1. Having given

$$x = by + cz + du, \ y = ax + cz + du, \ z = ax + by + du, \ u = ax + by + cz,$$

shew that
$$1 = \frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d};$$

x, y, z, u being supposed all unequal.

2. If $\frac{x}{y+z} = a$, $\frac{y}{z+x} = b$, and $\frac{z}{x+y} = c$, find the relation between a, b and c; and shew that $\frac{x^2}{a(1-bc)} = \frac{y^2}{b(1-ca)} = \frac{z^2}{c(1-ab)}$.

3. Find the relation between a, b and c, having given

$$\frac{x}{a} + \frac{a}{x} = \frac{y}{b} + \frac{b}{y} = \frac{z}{c} + \frac{c}{z}, \ xyz = abc,$$

and
$$x^2 + y^2 + z^2 + 2(ab + ac + bc) = 0.$$

4. Find the relation between a, b and c, having given

$$\frac{y}{z} + \frac{z}{y} = a, \quad \frac{z}{x} + \frac{x}{z} = b, \quad \frac{x}{y} + \frac{y}{x} = c.$$

5. Eliminate x, y, z between the equations

$$x^2(y+z) = a^3, \ y^2(x+z) = b^3, \ z^2(x+y) = c^3, \ xyz = abc.$$

6. Eliminate a and b from the equations

$$\frac{a^3-x^3}{b^3-y^3}=\frac{2x+3y}{3x+2y},\quad a^3-b^3=(x-y)^3,\quad a^{\frac{2}{3}}+b^{\frac{2}{3}}=z^{\frac{2}{3}}.$$

7. Eliminate x and y from the equations

$$x+y=a,\quad x^3+y^3=b^3,\quad x^5+y^5=c^5.$$

8. Eliminate x from the equations

$$32\frac{c}{a}=\left(\frac{x}{a}\right)^5+10\frac{x}{a}+5\left(\frac{a}{x}\right)^3,\quad 32\frac{a}{c}=\left(\frac{a}{x}\right)^5+10\frac{a}{x}+5\left(\frac{x}{a}\right)^3.$$

9. Eliminate x, y, z from the equations

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\alpha,\quad \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=\beta,$$

$$\left(\frac{x}{y}+\frac{y}{z}\right)\left(\frac{y}{z}+\frac{z}{x}\right)\left(\frac{z}{x}+\frac{x}{y}\right)=\gamma.$$

10. Eliminate x and y from the equations

$$ax+by=0,\quad x+y+xy=0,\quad x^2+y^2-1=0.$$

11. Eliminate x and y from the equations

$$y^2-x^2=\alpha y-\beta x,\quad 4xy=\alpha x+\beta y,\quad x^2+y^2=1.$$

12. If $(x+y)^2=4c^2xy$, $(y+z)^2=4a^2yz$, $(z+x)^2=4b^2zx$, shew that

$$a^2+b^2+c^2\pm 2abc=1.$$

13. Eliminate a from $\dfrac{x}{a^2+x^2}=\dfrac{2y}{a^2+y^2}=\dfrac{4z}{a^2+z^2}$.

14. Eliminate x and y from

$$4(x^2+y^2)=ax+by,\quad 2(x^2-y^2)=ax-by,\quad xy=c^2.$$

15. Shew that unless $abc+2a'b'c'=aa'^2+bb'^2+cc'^2$, the following equations cannot be simultaneously true:

$$a=xx',\ b=yy',\ c=zz',\ 2a'=yz'+zy',\ 2b'=zx'+xz',\ 2c'=xy'+yx'.$$

16. Find the number of permutations which can be formed with the letters composing the word *examination* taken 3 at a time.

17. Find the chance of a one, a two, and a three, of the same suit, lying together in a pack of cards which consists of m suits, and has n cards numbered 1, 2, 3, ... in each suit.

18. A rectangular garden is surrounded by a walk and is divided into mn rectangular beds by $m-1$ walks parallel to two sides and $n-1$ walks parallel to the other two sides. Find the number of ways, no two of which are exactly alike, in which a person can walk from one corner to the opposite corner so as to make the distance equal to half the perimeter of the rectangle.

19. If x be a proper fraction, shew that

$$\frac{x}{1-x^2}-\frac{x^3}{1-x^6}+\frac{x^5}{1-x^{10}}-\ldots\ldots=\frac{x}{1+x^2}+\frac{x^3}{1+x^6}+\frac{x^5}{1+x^{10}}+\ldots\ldots$$

20. If x be a proper fraction, shew that

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}=(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots$$

21. Eliminate x, y, z from the equations

$$(x-y)(y-z)(z-x)=p^3, \qquad (x+y)(y+z)(z+x)=q^3,$$
$$(x^2+y^2)(y^2+z^2)(z^2+x^2)=r^6, \qquad (x^4+y^4)(y^4+z^4)(z^4+x^4)=s^{12}.$$

22. Shew that if $aX+bY+cZ=0$, and $a_1X+b_1Y+c_1Z=0$; where $X=ax+a_1x_1+a_2$, $Y=bx+b_1x_1+b_2$, $Z=cx+c_1x_1+c_2$; then

$$X^2+Y^2+Z^2=\frac{\{a_2(bc_1-b_1c)+b_2(ca_1-c_1a)+c_2(ab_1-a_1b)\}^2}{(bc_1-b_1c)^2+(ca_1-c_1a)^2+(ab_1-a_1b)^2}.$$

23. If $a_1, a_2, \ldots a_n$, and $b_1, b_2, \ldots b_n$ be two series of positive numbers, each arranged in descending order of magnitude, shew that $\frac{a_1}{b_1}+\frac{a_2}{b_2}+\ldots+\frac{a_n}{b_n}$ is less, and $\frac{a_1}{b_n}+\frac{a_2}{b_{n-1}}+\ldots+\frac{a_n}{b_1}$ is greater, than if the denominators $b_1, b_2, \ldots b_n$ were arranged in any other order under the numerators $a_1, a_2, \ldots a_n$

24. If a be less than b, shew that a series of which the general term is $-\left(\frac{2}{n}-\frac{1}{n-1}\right)\frac{(b-a)^n}{b^{n-1}}$ is equal to the logarithm of $\left(\frac{a}{b}\right)^{a+b}$.

25. If a be less than b, shew that $\left(\frac{a}{b}\right)^{a+b}$ is increased by adding the same quantity to a and b.

MISCELLANEOUS EXAMPLES.

1. Simplify $x - [2y + \{3z - 3x - (x + y)\}] + 2x - (y + 3z)$.

2. Multiply $x^2 + 4y^2 + 9z^2 - 2xy - 3xz + 6yz$ by $x - 2y - 3z$; and divide $a^2x^8 + (2ac - b^2)x^4 + c^2$ by $ax^4 - bx^2 + c$.

3. Reduce to its lowest terms $\dfrac{5x^3 + 2x^2 - 15x - 6}{7x^3 - 4x^2 - 21x + 12}$.

4. Add $\dfrac{3x - a}{5x + 3a}$ to $\dfrac{x + 3a}{7x + 9a}$; take $\dfrac{a - x}{2a^2 + 3ax + x^2}$ from $\dfrac{2a + x}{a^2 - x^2}$.

5. Solve $\dfrac{4x + 1}{15} - \dfrac{5x - 1}{3} = x - 2$.

6. Solve $10x - 4y = 11$, $3x + 2y = 14\frac{1}{2}$.

7. A, who travels $3\frac{1}{4}$ miles an hour, starts $2\frac{1}{2}$ hours before B who goes the same road at $4\frac{1}{2}$ miles an hour: find when B overtakes A.

8. A bill of £100 was paid with guineas and half-crowns, and 48 more half-crowns than guineas were used; find how many of each were paid.

9. Find the square root of $a^4 + 2a^3 - a + \dfrac{1}{4}$.

10. Solve $\left(\dfrac{3}{x} - 1\right)(3x - 1) = \dfrac{5}{2}$.

11. If $a = 1$, $b = \dfrac{1}{2}$, $c = 3$, $d = \dfrac{1}{5}$, find the value of

$$a - [2a - 3b - \{4a - 5b - 6c - (7a - 8b - 9c - 10d)\}].$$

12. Multiply $x^2 + (2a + 3b)x + 6ab$ by $x^2 - (2a + 3b)x + 6ab$; and divide $14x^5 - 11x^4y - 66x^3y^2 - 7x^2y^3 + 49xy^4 + 15y^5$ by $2x^2 - 3xy - 5y^2$.

13. Find the L. C. M. of $x^2 + 5x + 6$ and $x^2 + 6x + 8$.

14. Take $\dfrac{2x + 3a}{3x + 4a}$ from $\dfrac{23x^2 + 18ax + 17a^2}{12x^2 + 31ax + 20a^2}$.

15. Solve $\dfrac{1}{x - 1} + \dfrac{2}{x - 2} = \dfrac{3}{x - 3}$.

16. Solve $7x - 9y = 23$, $9x - 7y = 57$.

17. Find the time between 9 and 10 o'clock when the hour-hand and the minute-hand of a watch are together.

18. *A*, after doing three-fifths of a work in 30 days, calls in *B*, and with his help finishes it in 10 days: find in how many days each could do the work alone.

19. Find the square root of $4x^2 - 12xy + 9y^2 + 4xz - 6yz + z^2$.

20. Solve $\frac{3}{x-1} - \frac{x-4}{x-3} = 1$.

21. If $a = 1$, $b = 2$, $c = \frac{1}{6}$, $d = \frac{1}{9\frac{1}{2}}$, find the value of

$$a - [3a - 5b - \{7a - 9b - 11c - (13a - 15b - 17c - 19d)\}].$$

22. Multiply $x^2 + (3a - 2b)x - 6ab$ by $x^2 + (3b - 2a)x - 6ab$; and divide $x^4 - 56x + 15$ by $1 - 4x + x^2$.

23. Find the G.C.M. of $x^2 - 4$, $x^2 + 10x + 16$, and $x^2 - 7x - 18$.

24. Simplify $\frac{2x^2 - x + 2}{4x^3 + 3x + 2} \times \frac{4x^2 - 1}{2x - 1}$.

25. Solve $\frac{x-1}{3} + \frac{11x - 3}{20} - \frac{3x - 9}{10} = 2\frac{1}{6}$.

26. Solve $\frac{x+y}{3} + \frac{x-y}{4} = \frac{11}{12}$, $5x - 3y = 6$.

27. A person starts from Ely to walk to Cambridge, which is distant 16 miles, at the rate of $4\frac{1}{3}$ miles per hour, at the same time that another person leaves Cambridge for Ely, walking at the rate of a mile in 18 minutes: find where they meet.

28. In a concert-room a certain number of persons are seated on benches of equal length; if there were ten more benches one person less might sit on each bench; if there were fifteen fewer benches two persons more must sit on each bench: find the number of benches, and the number of persons seated on each.

29. Find the square root of $x^6 - 4x^5 + 6x^4 - 8x^3 + 9x^2 - 4x + 4$.

30. Solve $11x^2 - 11\frac{1}{4} = 9x$.

31. If $a=1$, $b=2$, $c=3$, $d=4$, find the values of

$$\frac{ab+cd}{bc-ad},\quad c^b-b^c,\quad \sqrt[3]{(d^2+3c+b)}.$$

32. Multiply $\cdot 3+x^2$ by $\cdot 2-x$; and find the value of the product when $x=\cdot 1$.

33. Reduce to its lowest terms $\dfrac{x^4-115x+24}{24x^4-115x^3+1}$.

34. Add together $\dfrac{1+x}{1-x}$, $\dfrac{1-x}{1+x}$, and $\dfrac{1+x^2}{1-x^2}$.

35. Solve $(x-3)^3-3(x-2)^3+3(x-1)^3-x^3=9-x$.

36. Solve $5y-3x=2$, $8y-5x=1$.

37. A farmer bought equal numbers of two kinds of sheep, one at £3 each, the other at £4 each. If he had expended his money equally in the two kinds he would have had 2 sheep more than he did: find how many he bought.

38. The sum of £177 is to be divided among 15 men, 20 women and 30 children, in such a manner that a man and a child may together receive as much as two women, and the women may together receive £60: find what they respectively receive.

39. Find the square root of $\dfrac{x^2}{y^2}+\dfrac{y^2}{4x^2}-\dfrac{x}{y}+\dfrac{y}{2x}-\dfrac{3}{4}$.

40. Solve $\dfrac{2x}{x-4}+\dfrac{4x-3}{x+1}=9$.

41. If $a=1$, $b=3$, $c=5$, $d=7$, find the values of

$$\frac{cd-ab}{bc-ad},\quad b^{c-2}-d^{b-1},\quad \sqrt[3]{(d^2+3c)}.$$

42. Shew that

$$a(a-x)(a-2x)=(a-b)(a-b-x)(a+2b-2x)+b(b-x)(3a-2b-2x).$$

43. Find the G.C.M. and the L.C.M. of $x^5-x^4+x^3-x^2+x-1$ and x^3-1.

44. Simplify $\dfrac{x^2+5x+6}{x^2+5x}\times\dfrac{x^2+6x+5}{x^2+3x}$.

45. Solve $\dfrac{1}{x+1}+\dfrac{4}{2x-1}+\dfrac{9}{3x-1}=\dfrac{36}{6x-1}$.

46. Solve

$$2x + 3y - 8z + 35 = 0, \quad 7x - 4y + z - 8 = 0, \quad 12x - 5y - 3z + 10 = 0.$$

47. Find how many gallons of water must be mixed with 80 gallons of spirit which cost 15 shillings a gallon, so that by selling the mixture at 12 shillings a gallon there may be a gain of 10 per cent. on the outlay.

48. A and B can together do a work in 12 days; A and C in 15 days; B and C in 20 days: find in how many days they will do the work, all working together.

49. Find the square root of $a - c + 2\sqrt{(ab + bc - ca - b^2)}$.

50. Solve $x = \cfrac{3}{4 - \cfrac{3}{4 - \cfrac{3}{4 - x}}}$.

51. Simplify

$$(a+b+c)(x+y+z) + (a+b-c)(x+y-z)$$
$$+ (b+c-a)(y+z-x) + (c+a-b)(z+x-y).$$

52. If $s = \dfrac{a+b+c}{2}$, shew that

$$\{(s-a)+(s-b)\}^3 = (s-a)^3 + (s-b)^3 + 3(s-a)(s-b)c.$$

53. Find the G.C.M. of $x^4 - 2x^3y + 5x^2y^2 - 2xy^3 + 4y^4$ and $x^4 - 3x^3y + 6x^2y^2 - 3xy^3 + 5y^4$.

54. Simplify $\dfrac{x-a}{x-b} + \dfrac{x-b}{x-a} - \dfrac{(a-b)^2}{(x-a)(x-b)}$.

55. Solve $(3x-1)^2 + (4x-2)^2 = (5x-3)^2$.

56. Solve $\dfrac{x+3}{x-3} + \dfrac{y-3}{y+3} = 2, \qquad \dfrac{x-3}{2x+3} + \dfrac{y-3}{2y+3} = 1.$

57. A, B, C are employed on a piece of work. After 3 days A is discharged, one-third of the work being done. After 4 days more B is discharged, another third of the work being done. C then finishes the work in 5 days. Find in how many days each could separately do the work.

58. A person walks from A to B, a distance of $7\frac{1}{2}$ miles, in 2 hours $17\frac{1}{2}$ minutes, and returns in 2 hours 20 minutes. His rates of walking up hill, down hill, and on a level road being 3, $3\frac{1}{2}$, and $3\frac{1}{4}$ miles respectively, find the length of level road between A and B.

59. Find the cube root of

$$8x^9 - 12x^8 + 6x^7 - 37x^6 + 36x^5 - 9x^4 + 54x^3 - 27x^2 - 27.$$

60. Solve $\dfrac{(x+a)(x+mb)}{(x-ma)(x-b)} = \dfrac{(mx+a)(x+b)}{(x-a)(mx-b)}$.

61. Simplify $24\left\{x - \frac{1}{2}(x-1)\right\}\left\{x - \frac{2}{3}(x-2)\right\}\left\{x - \frac{3}{4}(x - 1\frac{1}{3})\right\}$, and subtract the result from $(x+2)(x+3)(x+4)$.

62. Divide $\left(\dfrac{x^2}{a^2} + \dfrac{a^2}{x^2} - 2\right)^2$ by $\dfrac{x}{a} - \dfrac{a}{x}$.

63. Find the G.C.M. of

$$5x^3 - 18x^2y + 11xy^2 - 6y^3 \text{ and } 7x^2 - 23xy + 6y^2.$$

64. Simplify $\dfrac{x^2 - x + 1}{x^2 + x + 1} + \dfrac{2x(x-1)^2}{x^4 + x^2 + 1} + \dfrac{2x^2(x^2-1)^2}{x^8 + x^4 + 1}$.

65. Solve $\dfrac{4}{x-6} - \dfrac{x-2}{x-3} = \dfrac{x+4}{x-5} - 2\dfrac{x-1}{x-4}$.

66. Solve $\dfrac{x-2a}{x-3a} + \dfrac{y-4b}{y-3b} = 2, \quad \dfrac{x+2a}{x+a} = \dfrac{y+5b}{y+3b}$.

67. A man bought a house which cost him 4 per cent. on the purchase money to put it in repair. It then stood empty for a year, during which time he reckoned he was losing 5 per cent. upon his total outlay. He then sold it again for £1192, by which means he gained 10 per cent. on the original purchase money: find what he gave for the house.

68. A certain resolution was carried in a debating society by a majority which was equal to one-third of the number of votes given on the losing side; but if with the same number of votes 10 more votes had been given to the losing side, the resolution would only have been carried by a majority of one: find the number of votes given on each side.

69. Solve $\sqrt{x} - \sqrt{a} + \sqrt{(x + a - b)} = \sqrt{b}$.

70. Solve $(x - 2)(x - 3) = \dfrac{155 \times 78}{77^2}$.

71. If $a = 2$, $b = 3$, $c = 6$, $d = 5$, find the value of

$$\sqrt[3]{\{(a + c - b)^2 d\}} + \sqrt[3]{\{(b + d)(5d - 4c)\}} + \sqrt[3]{\{(c - a)(d - b)\}}.$$

72. Shew that

$$x(y + z)^2 + y(z + x)^2 + z(x + y)^2 - 4xyz = (y + z)(z + x)(x + y).$$

73. Find the G.C.M. of

$$5x^3 - 19x^2 + 55x - 425 \text{ and } 4x^3 - 15x^2 - 38x + 65.$$

74. Simplify $\dfrac{bc(x - a)^2}{(a - b)(a - c)} + \dfrac{ca(x - b)^2}{(b - c)(b - a)} + \dfrac{ab(x - c)^2}{(c - a)(c - b)}$.

75. Solve $\sqrt{\{(x - a)^2 + 2ab + b^2\}} = x - a + b$.

76. Solve $ax + cy + bz = cx + by + az = bx + ay + cz = a^3 + b^3 + c^3 - 3abc$.

77. A and B start together from the same point on a walking match round a circular course. After half an hour A has walked three complete circuits, and B four and a half. Assuming that each walks with uniform speed, find when B next overtakes A.

78. On a certain day mackerel were being sold at a certain price per dozen; on the next day twice as many mackerel could be bought for one shilling as dozens could be bought for a sovereign on the day before: the whole price of 20 mackerel bought 10 on one day and 10 on the other being 2*s.* 2*d.*, determine the price of a mackerel on each day.

79. If $x = \sqrt[3]{(a + \sqrt{a^2 + b^3})} + \sqrt[3]{(a - \sqrt{a^2 + b^3})}$, shew that

$$x^3 + 3bx - 2a = 0.$$

80. Solve $(x^3 + 8x^2 + 16x - 1)^{\frac{1}{3}} - x = 3$.

81. Shew that $(p + q + r)^4 =$

$$4(p^3 + q^3 + r^3 + 3pqr)(p + q + r) + 6q^2r^2 + 6r^2p^2 + 6p^2q^2 - 3p^4 - 3q^4 - 3r^4.$$

82. If $X = ax + cy + bz$, $Y = cx + by + az$, $Z = bx + ay + cz$, shew that $X^2 + Y^2 + Z^2 - YZ - ZX - XY$

$$= (a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy).$$

83. Find the G.C.M. of $7x^4 - 10ax^3 + 3a^2x^2 - 4a^3x + 4a^4$ and $8x^4 - 13ax^3 + 5a^2x^2 - 3a^3x + 3a^4$.

84. Simplify $\frac{1}{1-x} - \frac{1}{1+x} - \frac{2x}{1+x^2} - \frac{4x^3}{1+x^4} - \frac{8x^7}{1+x^8}$.

85. Solve $\frac{4x^3 + 4x^2 + 8x + 1}{2x^2 + 2x + 3} = \frac{2x^2 + 2x + 1}{x + 1}$.

86. Solve

$$x + y + z = a + b + c,$$
$$ax + by + cz = bc + ca + ab,$$
$$(b - c)x + (c - a)y + (a - b)z = 0.$$

87. The present income of a railway company would justify a dividend of 6 per cent., if there were no preference shares. But as £400000 of the stock consists of such shares, which are guaranteed $7\frac{1}{2}$ per cent. per annum, the ordinary shareholders receive only 5 per cent. Find the amount of ordinary stock.

88. The road from a place A to a place B first ascends for five miles, is then level for four miles, and afterwards descends for six miles, the rest of the distance; a man walks from A to B in 3 hours 52 minutes; the next day he walks back to A in 4 hours, and he then walks half way to B and back again in 3 hours 55 minutes: find his rates of walking up hill, on level ground, and down hill.

89. Find the value to five places of decimals of

$$\{161 + \sqrt{19360}\}^{-\frac{1}{4}}.$$

90. Solve $\frac{a}{x + a - c} + \frac{b}{x + b - c} = 2$.

91. Find the value when $x = 5$ of

$$3x - [5y - \{2x - (3z - 3y) + 2z - (x - 2y - z)\}].$$

92. Shew that $(y - z)^4 + (z - x)^4 + (x - y)^4$

$$= 2\{(y - z)^2(z - x)^2 + (z - x)^2(x - y)^2 + (x - y)^2(y - z)^2\}$$
$$= 2(x^2 + y^2 + z^2 - yz - zx - xy)^2.$$

93. Find the G.C.M. of $x^3 + (5m - 3)x^2 + (6m^2 - 15m)x - 18m^2$ and $x^3 + (m - 3)x^2 - (2m^2 + 3m)x + 6m^2$.

94. Shew that $\dfrac{a^2\left(\frac{1}{b}-\frac{1}{c}\right)+b^2\left(\frac{1}{c}-\frac{1}{a}\right)+c^2\left(\frac{1}{a}-\frac{1}{b}\right)}{a\left(\frac{1}{b}-\frac{1}{c}\right)+b\left(\frac{1}{c}-\frac{1}{a}\right)+c\left(\frac{1}{a}-\frac{1}{b}\right)}=a+b+c.$

95. Solve $\left(\dfrac{x^2-11x+19}{x^2+x-11}\right)^2+\dfrac{3(x-2)}{x+2}=0.$

96. Solve $x^3+y^3+z^3=3xyz,\quad x-a=y-b=z-c.$

97. A bag contains sixpences, shillings, and half-crowns; the three sums of money expressed by the different coins are the same: if there are 102 coins in the bag find the number of sixpences, shillings, and half-crowns.

98. A person walks from A to B at the rate of $3\frac{1}{2}$ miles per hour, and from B to C at 4 miles per hour; in returning he calculates that he can complete the distance in the same time by walking uniformly at $3\frac{3}{4}$ miles per hour, but being detained 14 minutes at B he has to walk to A at 4 miles per hour to finish it in the same time: find the distance from A to B, and from B to C.

99. If $X=ax+cy+bz,\quad Y=cx+by+az,\quad Z=bx+ay+cz,$ shew that

$$X^3+Y^3+Z^3-3XYZ=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz).$$

100 Solve $x^2-223x+12432=0.$

101. Solve $(4x+2)^4-(3x-1)^4=(2x+4)^4-(x-3)^4.$

102. Find three consecutive numbers whose product is equal to fifteen times the middle number.

103. Solve $x+y=9,\quad \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2}.$

104. If x varies jointly as y and z; and y varies directly as $x+z$; and if $x=2$ when $z=2$, find the value of z when $x=9$.

105. Sum to 18 terms $1+\dfrac{5}{6}+\dfrac{2}{3}+\ldots$

106. Sum to 6 terms and to infinity $14-7+3\dfrac{1}{2}-\ldots$

107. If the number of combinations of $2n$ things taken $n-1$ together be to the number of combinations of $2(n-1)$ things taken n together as 132 is to 35, find n.

108. Shew that $2^m - \frac{m}{1} 2^{m-1} + \frac{m(m-1)}{\lfloor 2} 2^{m-2} - \ldots + (-1)^m = 1.$

109. In the expansion of $(a_1 + a_2 + \ldots + a_m)^n$ if n is a positive integer, and m greater than n, shew that the coefficient of any term in which none of the quantities $a_1, a_2, \ldots a_m$ appears more than once is $\lfloor n$.

110. Given $\log 2 = \cdot 3010300$ and $\log 3 = \cdot 4771213$, find the integral values between which x must lie in order that the integral part of $(1 \cdot 08)^x$ may contain four digits.

111. Solve $\{a(b+x-a)\}^{\frac{1}{2}} + \{b(a+x-b)\}^{\frac{1}{2}} = \{x(a+b-x)\}^{\frac{1}{2}}$.

112. If α and β be the roots of the equation $ax^2 + bx + c = 0$, form the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

113. Solve $\frac{x}{y} + \frac{y}{x} = \frac{5}{2}, \quad xy = 8.$

114. If $x-4 : x-2 :: x-1 : x+3$, find x.

115. Sum nine terms of an arithmetical progression of which 18 is the middle term.

116. Sum to n terms $\frac{1}{1+\sqrt{2}} + \frac{1}{3+2\sqrt{2}} + \frac{1}{7+5\sqrt{2}} + \ldots$

117. Prove that the number of ways in which p positive signs and n negative signs may be placed in a row so that no two negative signs shall be together is equal to the number of combinations of $p+1$ things taken n together.

118. Determine the coefficient of x^r in the expansion according to ascending powers of x of $\frac{(n-m+1)x(1-x) - x^{m+1} + x^{n+2}}{(1-x)^2}$, where m and n are positive integers of which m is the less.

119. Determine whether the series whose n^{th} term is $\sqrt{(n^2+1)} - n$ is convergent or divergent.

120. Find the value of $\frac{1}{\cdot 05}\left\{\frac{1}{(1\cdot 05)^{13}} - \frac{1}{(1\cdot 05)^{20}}\right\}$.

Given $\log 105 = 2\cdot 0211893$, $\log 5303214 = 6\cdot 7245391$,

$\log 3768894 = 6\cdot 576214$.

121. Solve $(4 + 5x - x^2)^{\frac{1}{2}} = 2^{\frac{3}{2}} x^{\frac{1}{2}} + (x^2 + 3x - 4)^{\frac{1}{2}}$.

122. Find the relation between the coefficients of the equation $ax^2 + bx + c = 0$, that one root may be double of the other.

123. Solve $\frac{1}{x} + \frac{1}{y} = \frac{x+y}{12} = \frac{7}{x+y+5}$.

124. Divide 111 into three parts so that the products of each pair may be in the proportion of 4, 5, and 6.

125. Find the number of terms of an arithmetical progression of which the first term, the sum, and the common difference are given: find the conditions which must hold if there be two such numbers.

126. Find the sum of the reciprocals of n terms of a geometrical progression of which the first term is a and the common ratio r.

127. Shew that the number of ways in which mn things can be divided among m persons so that each shall have n of them is $\frac{\lfloor mn}{\{\lfloor n\}^m}$.

128. Shew that the coefficient of x^{n+r-1} in the expansion of $\frac{(1+x)^n}{(1-x)^3}$ is $2^{n-3}\{(n+2r)(n+2r+2)+n\}$, r being 0 or any positive integer.

129. Find the coefficient of x^4 in the expansion of

$$(1 + 2x - 3x^2 + x^3)^3.$$

130. Shew that $1 + \frac{2^3}{\lfloor 2} + \frac{3^3}{\lfloor 3} + \frac{4^3}{\lfloor 4} + \ldots = 5e$.

131. Solve $\sqrt{(x^2 - 8x + 15)} + \sqrt{(x^2 + 2x - 15)} = \sqrt{(4x^2 - 18x + 18)}$.

132. The numerically greater root of $ax^2 - bx + c = 0$ has the same sign as $\frac{b}{a}$; and the numerically less root the same sign as $\frac{b}{c}$.

133. Solve $x+y+z=a+b+c$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$,

$$x^2+y^2+z^2=a^2+b^2+c^2.$$

134. Two persons A and B divide equally a sum of money consisting of half-crowns, shillings, and sixpences; the values of the several parts being respectively in the proportion of 15, 4, and 1. It is found that each has 60 coins, A having two half-crowns more than B. Determine the sum and the coins each had.

135. The p^{th} term of an arithmetical progression is $\frac{1}{q}$, and the q^{th} term is $\frac{1}{p}$: shew that the sum of pq terms is $\frac{pq+1}{2}$.

136. If a, b, c be in arithmetical progression, and α, β, γ in harmonical progression, and $\frac{a}{\gamma}+\frac{\gamma}{a}=\frac{c}{a}+\frac{a}{c}$, shew that $a\alpha$, $b\beta$, $c\gamma$ are in geometrical progression.

137. Find the number of words beginning and ending with a consonant which can be formed out of the word *equation*.

138. If a_r be the coefficient of x^r in the expansion of $(1+x)^{2n}$, shew that $a_0^2-a_1^2+a_2^2-a_3^2+\ldots=\frac{(-1)^n\,\lfloor 2n}{\lfloor n\,\lfloor n}$.

139. Determine whether the following series is convergent or divergent: $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[4]{4}}+\ldots$

140. If $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots$ shew that $x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\ldots$

141. Solve $(x-3)^2+3x-22=\sqrt{(x^2-3x+7)}$.

142. The number of soldiers present at a review is such that they could all be formed into a solid square, and also could be formed into four hollow squares each four deep and each containing 24 more men in the front rank than when formed into a solid square: find the whole number.

143. Solve $6x^2-xy-12y^2=0$, $x^2+2y^2=\frac{17}{16}$.

144. If the speed on a railway is 20 miles an hour it is found that the expenses are just paid. If the speed is more than 20 miles an hour the increase of the receipts is found to vary as the increase of the velocity, while the increase of the cost of working is found to vary as the square of the increase of the velocity; at the rate of 40 miles per hour the expenses are just paid: find the velocity at which the profits will be greatest.

145. Shew that the number $p_0 + 10p_1 + 10^2p_2 + \dots + 10^np_n$ is divisible by 13 if the following expression is,

$$p_0 - p_3 + p_6 - \dots - 3(p_1 - p_4 + p_7 - \dots) - 4(p_2 - p_5 + p_8 - \dots).$$

146. If s be the sum of an odd number of terms in geometrical progression, and s' the sum of the series when the signs of the even terms are changed, shew that the sum of the squares of the terms will be equal to ss'.

147. If there be n straight lines lying in one plane, no three of which meet at a point, the number of groups of n of their points of intersection in each of which no three points lie in one of the straight lines is $\frac{1}{2}\underline{|n-1}$.

148. Shew that $2^{\frac{1}{4}} . 4^{\frac{1}{8}} . 8^{\frac{1}{16}} . 16^{\frac{1}{32}} \dots = 2$.

149. Find the coefficient of x^4 in the expansion of

$$(1 + x - x^2 - 3x^3 - x^4)^{-3}.$$

150. Shew that if the logarithms of n quantities with respect to n bases in geometrical progression be all equal they will also be equal to the logarithm of the ratio of any one among these quantities to the preceding quantity, with respect to the common ratio of the progression as base.

151. Solve $\dfrac{2(3x-4)}{2x+1} + \dfrac{9(x-1)}{3x+1} = \dfrac{2(3x-2)}{2x+3} + \dfrac{3(x-2)}{x-1}$.

152. Shew that if a quadratic equation be satisfied by more than two values of the unknown quantity the equation is an identity. Apply this property to establish the identity

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

153. Solve $(x^2+y^2)\frac{x}{y}=6$, $(x^2-y^2)\frac{y}{x}=1$.

154. Bronze contains 91 per cent. of copper, 6 of zinc, and 3 of tin. A mass of bell-metal (consisting of copper and tin only) and bronze fused together is found to contain 88 per cent. of copper, 4·875 of zinc, and 7·125 of tin. Find the proportion of copper and tin in bell-metal.

155. Shew that the sum of the products of the first n natural numbers taken two and two together is $\frac{(n-1)n(n+1)(3n+2)}{24}$.

156. Four numbers are taken, the first three in G. P., and the last three in H. P.; again four numbers are taken, the first three in H. P., and the last three in G. P.: shew that if the first two numbers are the same in each set the last of the first set will be less than the last of the second.

157. Find the number of different arrangements that can be made of bars of the seven prismatic colours, so that the blue and the green bars shall never come together.

158. If $(5\sqrt{2}+7)^m=n+a$, where m and n are positive integers and a less than unity, shew that $a(n+a)=1$.

159. Find the coefficient of x^4 in the expansion of

$$(1-2x+3x^2-4x^3+\dots)^{-\frac{1}{2}}.$$

160. If the whole number of persons born in any month be $\frac{1}{480}$ of the whole population at the beginning of the month, and the number of persons who die $\frac{1}{600}$, find the number of months in which the population will be doubled.

Given $\log 2=\cdot 3010300$, $\log 3=\cdot 4771213$, $\log 7=\cdot 8450980$.

161. Solve $x^4+1=2(1+x)^4$.

162. A and B run a race round a two mile course. In the first heat B reaches the winning-post 2 minutes before A. In the second heat A increases his speed 2 miles an hour, and B diminishes his by the same quantity; and A then reaches the winning-post 2 minutes before B. Find at what rate each ran in the first heat.

163. Solve $\dfrac{x+3y+5}{x+y+1}+\dfrac{3x+y+4}{x+y-1}=4,$

$$(x+2y)^2+(y+2x)^2=5(x+y)^2+4y.$$

164. Solve $\dfrac{x}{y+z+1}=\dfrac{y}{z+x}=\dfrac{z}{x+y-1}=x+y+z.$

165. Shew that the number $p_0+10p_1+10^2p_2+\ldots+10^np_n$ is divisible by 101 if the following expression is,

$$p_0+10p_1-(p_2+10p_3)+(p_4+10p_5)-\ldots.$$

166. If a, b, c be three quantities such that a is the arithmetical mean between b and c, and c the harmonical mean between a and b, shew that b is the geometrical mean between a and c: and compare a, b, c.

167. In a plane there are m straight lines which all pass through a given point, n others which all pass through another given point, and p others which all pass through a third given point: supposing no other three to intersect at any point find the number of triangles formed by the intersection of the straight lines.

168. If $a_r=r-(r-1)\,n+(r-2)\dfrac{n(n-1)}{\underline{|2}}$

$$-(r-3)\frac{n(n-1)(n-2)}{\underline{|3}}+\ldots\ldots$$

to r terms, shew that $a_r=(-1)^n a_{n-r}$ if r be less than $n-1$, $a_r=0$ if r be greater than $n-1$, and $a_{n-1}=(-1)^n$.

169. Find the coefficient of x^6 in the expansion of

$$(1+2x-3x^2-x^4)^{\frac{7}{3}}.$$

170. Given $\log_{10}2=\cdot30103$, find $\log_{25}50$.

171. Solve $x^{\frac{1}{3}}+x^{-\frac{1}{3}}=\dfrac{4}{13}(x+x^{-1})$.

172. If α and β are the roots of the quadratic $ax^2+bx+c=0$, form the quadratic whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.

173. Solve $8\sqrt{(x^2-y^2)}=x+9y,$

$$x^4+2x^2y+y^2+x=2x^3+2xy+y+506.$$

174. A and B engage to reap a field in 12 days. The times in which they could separately reap an acre are in the proportion of 2 to 3. At the end of 6 days, as they find they cannot finish the work in the stipulated time, they call in C and finish it with his help. The time in which A and C together could have reaped the field is to the time in which B and C together could have reaped it as 7 is to 8. Find in how many days the field would have been reaped if C had worked from the first.

175. A tradesman has eight weights, two of 1 oz. each, two of 5 oz. each, two of 25 oz. each, two of 125 oz. each: shew that he can weigh with a pair of scales any integral number of ounces from 1 up to 312.

176. Find four numbers in geometrical progression so that their sum may be 15, and the sum of their squares 85.

177. Out of $2n$ men who have to sit down, half on each side of a long table, p particular men desire to sit on one side and q on the other: find the number of ways in which this may be done.

178. Shew that the coefficient of x^{3r} in the expansion of $(9a^2+6ax+4x^2)^{-1}$ is $2^{3r}(3a)^{-3r-2}$.

179. Shew that the series $u_1+u_2+\dots+u_n+\dots$ is convergent if from and after a certain term the value of $(u_n)^{\frac{1}{n}}$ is always less than some finite quantity which is itself less than unity, and divergent if the value is unity or greater than unity.

180. Shew that $1-\dfrac{1}{2(n+1)}-\dfrac{1}{2\,.\,3(n+1)^2}-\dfrac{1}{3\,.\,4(n+1)^3}-\dots$ $=\log\left(1+\dfrac{1}{n}\right)^n$. Hence shew that $\left(1+\dfrac{1}{n}\right)^n$ increases with n.

181. Solve $9x^2+4x^3=1+12x^4$.

182. Three persons A, B, C, whose ages are in geometrical progression, divide among them a sum of money in amounts proportional to the ages of each. Five years afterwards when C is double the age of A they similarly divide an equal sum; A now received £17. 10*s.* more than before, and B £2. 10*s.* more than before. Find the sum divided on each occasion.

183. Solve $\left(\frac{x}{y}\right)^{\frac{2}{3}} + \left(\frac{y}{x}\right)^{\frac{2}{3}} - 1 = \frac{61}{(xy)^{\frac{1}{3}}}$, $\quad x^{\frac{3}{4}}y^{\frac{1}{4}} + y^{\frac{3}{4}}x^{\frac{1}{4}} = 78$.

184. If $x = cy + bz$, $y = az + cx$, $z = bx + ay$, shew that

$$\frac{x}{\sqrt{(1-a^2)}} = \frac{y}{\sqrt{(1-b^2)}} = \frac{z}{\sqrt{(1-c^2)}};$$

and find the relation between a, b, and c.

185. Shew that in the scale with radix nine, every number which is a perfect cube must end with 0 or 1 or 8.

186. Find the sum of the products which can be formed by multiplying together any three terms of an infinite G. P.; and shew that if this sum be one-third of the sum of the cubes of the terms the common ratio is $\frac{1}{2}$.

187. A vessel is filled with a gallons of wine, another with b gallons of water; c gallons are taken out of each; that from the first is transferred to the second, and that from the second to the first; this operation is repeated r times: shew that the quantity of wine in the second vessel will be $\frac{ab}{a+b}(1-p^r)$ where $p = 1 - \frac{c}{a} - \frac{c}{b}$.

188. By comparing two expansions of $\frac{1+2x}{1-x^3}$, shew that

$$(-1)^n = 1 - 3n + \frac{(3n-1)(3n-2)}{\lfloor 2} - \frac{(3n-2)(3n-3)(3n-4)}{\lfloor 3} + \frac{(3n-3)(3n-4)(3n-5)(3n-6)}{\lfloor 4} - \dots$$

where n is any positive integer, and the series stops at the first term that vanishes.

189. Determine whether the following series is convergent or divergent: $1 + \frac{1}{2}x + \frac{\lfloor 2}{3^2}x^2 + \frac{\lfloor 3}{4^3}x^3 + \frac{\lfloor 4}{5^4}x^4 + \dots$

190. If $\log \frac{1}{1 - x - x^2 + x^3}$ be expanded in a series of powers of x, shew that the coefficient of x^n is $\frac{3}{n}$ or $\frac{1}{n}$ according as n is even or odd.

191. Solve $(1+x^2)^2 = 2ax(1-x^2)$.

192. Shew that if x, y, z are real quantities

$$x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)$$

cannot be negative.

193. Solve $x^2+y^2+1=m^2xy-x^2y^2, \quad xy(n^2x-y)=x-n^2y$.

194. Shew that the equations $ax+by+cz=0$ and $ax^2+by^2+cz^2=0$ will be satisfied by taking $\dfrac{x}{1-av}=\dfrac{y}{1+bv}=\dfrac{z}{1+abv^2}$; where $a+b+c+abcv^2=0$.

195. In Art. 458 we arrive at an A. P. of which the first term is $\dfrac{a}{q}-\dfrac{b}{2q}+\dfrac{b}{2q^2}$, and the common difference is $\dfrac{b}{q^2}$: shew that if this be arranged in groups of q terms each, the m^{th} group is equal to the m^{th} term of the A. P. of which the first term is a and the common difference is b.

196. The first term of a certain series is a, the second term is b, and each subsequent term is an arithmetic mean between the two preceding terms: shew that the n^{th} term is

$$\frac{2}{3}(b-a)\left\{1-\left(-\frac{1}{2}\right)^{n-1}\right\}+a.$$

197. If all the permutations of n things $a, b, c, \ldots l$ taken all together be formed, and from any permutation as $abc \ldots l$ be formed the fraction $\dfrac{1}{a(a+b)(a+b+c)\ldots(a+b+\ldots l)}$, shew that the sum of all these fractions is $\dfrac{1}{abc\ldots l}$.

198. Shew that

$$1+\frac{n^2x}{1}+\left\{\frac{n(n-1)}{\lfloor 2}\right\}^2x^2+\left\{\frac{n(n-1)(n-2)}{\lfloor 3}\right\}^2x^3+\ldots$$

$$=(1+x)^n\left\{1+\frac{n(n-1)}{1\,.\,1}\,\frac{x}{(1+x)^2}+\frac{n(n-1)(n-2)(n-3)}{\lfloor 2\lfloor 2}\,\frac{x^2}{(1+x)^4}+\ldots\right\}.$$

199. Determine whether the following series is convergent or divergent: $\left(\dfrac{3^2}{2^2}-\dfrac{3}{2}\right)^{-2}+\left(\dfrac{4^3}{3^3}-\dfrac{4}{3}\right)^{-3}+\left(\dfrac{5^4}{4^4}-\dfrac{5}{4}\right)^{-4}+\ldots$

200. If n is any positive integer, find the value of

$$n^{n+3} - n(n-1)^{n+3} + \frac{n(n-1)}{1 \,.\, 2}(n-2)^{n+3} - \dots$$

201. Multiply out $(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)$; and find the form of the series up to x^{12} when the number of factors is infinite.

202. Shew that

$$\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3} = (a+b)(b+c)(c+a).$$

203. Shew that money will increase fifty-fold in a century at 4 per cent. per annum compound interest, having given $\log 2 = \cdot 301030$, $\log 13 = 1 \cdot 113943$.

204. Shew that $\sqrt{p^4+4p} = p + \frac{2}{1+}\,\frac{1}{p+}\,\frac{1}{1+}\,\frac{1}{p+}\,\frac{1}{1+}\dots$

205. Find the number of ways in which a substance of a ton weight may be weighed by weights of 9 lbs. and 14 lbs.

206. If $\frac{1}{(1-2x)(1-2x+x^2)}$ be expanded in ascending powers of x, find the general term.

207. If n is a positive integer, and x a positive proper fraction, shew that $\frac{1-x^{n+1}}{n+1}$ is less than $\frac{1-x^n}{n}$.

208. Shew that $n^4 - 4n^3 + 5n^2 - 2n$ is divisible by 12 for all values of n greater than 2.

209. From a bag containing 10 counters, 3 of which are marked, 5 are to be drawn; and the drawer is to receive a shilling if in his drawing the three marked counters come out together: find the value of his expectation.

210. Determine whether the following series is convergent or divergent: $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \dots$

211. If the square of the sum of n real quantities is equal to $\frac{2n}{n-1}$ times the sum of their products taken two and two together, the n quantities are all equal to one another.

212. Shew that

$$25\{(b-c)^7+(c-a)^7+(a-b)^7\}\{(b-c)^3+(c-a)^3+(a-b)^3\}$$
$$=21\{(b-c)^5+(c-a)^5+(a-b)^5\}^2.$$

213. If a man 48 years old can buy an annuity of £150 a year for £1812. 16*s*., interest being reckoned at 5 per cent., determine what is considered the expectation of life at 48. Having given that $\log 2 = \cdot3010300$, $\log 3 = \cdot4771213$, $\log 7 = \cdot8450980$, $\log 1\cdot1872 = \cdot0745239$.

214. If $\frac{p_r}{q_r}$ denote the r^{th} convergent to $\frac{\sqrt{5}+1}{2}$, shew that

$$p_3+p_5+\ldots+p_{2n-1}=p_{2n}-p_2, \qquad q_3+q_5+\ldots+q_{2n-1}=q_{2n}-q_2.$$

215. Find the proper fractions which satisfy the condition that the sum of five times the numerator and eleven times the denominator shall be 1031.

216. Shew that if n be a positive integer, and x such that no denominator vanishes,

$$\frac{1}{x+1}-\frac{n}{1}\frac{1}{x+2}+\frac{n(n-1)}{1\,.\,2}\frac{1}{x+3}-\ldots+\frac{(-1)^n}{x+n+1}$$
$$=\frac{\lfloor n}{(x+1)(x+2)\ldots(x+n+1)}.$$

217. If p be a positive proper fraction, and a and b positive quantities, shew that $(a+b)^p a^{1-p}$ is less than $a+pb$.

218. If 3, or 5, or 7, or 9 be raised to any power, shew that the digit in the tens' place is always even; if 6 be raised to any power, shew that the digit in the tens' place is always odd.

219. There are three balls in a bag, and it is not known how many of these are black; a person draws a ball from the bag and replaces it; this is done three times: if every drawing gave a black ball find the chance that all the balls are black.

220. If $x=y+\frac{1}{2y+}\frac{1}{2y+}\ldots$ shew that $y=x-\frac{1}{2x-}\frac{1}{2x-}\ldots$

221. If $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}=1$, shew that two of the three fractions on the left-hand side must be equal to 1, and the other to -1.

222. Solve $yz + zx + xy = a^2 - x^2 = b^2 - y^2 = c^2 - z^2$.

223. If p years' purchase must be paid for an annuity to continue a certain number of years, and q years' purchase for an annuity to continue twice as long, find the rate per cent.

224. Convert $\sqrt{\left(a^2 + \frac{2a}{b}\right)}$ into a continued fraction.

225. Resolve $2x^2 - 21xy - 11y^2 - x + 34y - 3$ into rational factors of the first degree.

226. Shew that a recurring series whose scale of relation is $1 - px - qx^2$ is convergent or divergent according as x is numerically less or greater than the numerically least root of the equation $1 - px - qx^2 = 0$; the roots being supposed real.

227. Shew that if all the letters denote positive quantities and $p_1, p_2, p_3 \ldots$ and $a_1, a_2, a_3, \ldots$ are both in ascending or both in descending order of magnitude, $\frac{p_1a_1^2 + p_2a_2^2 + \ldots + p_na_n^2}{p_1 + p_2 + \ldots + p_n}$ is greater than $\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^2$.

228. If $a^2 + b^2 = c^2$, and a, b, c are integers, shew that one of them is divisible by 5.

229. A number, of n digits, is written down at random: shew that whatever be the value of n, provided it be given, the chance that the number is a multiple of 9 is $\frac{1}{9}$.

230. If n be any positive integer, shew that the integer next greater than $(3 + \sqrt{5})^n$ is divisible by 2^n.

231. If the two expressions $x^3 + px^2 + qx + r$ and $x^3 + p'x^2 + q'x + r'$ have the same quadratic factor, then $\frac{r - r'}{p - p'} = \frac{p'r - pr'}{q - q'} = \frac{q'r - qr'}{r - r'}$.

232. Shew in the preceding Example that the third factors are $x + \frac{p - p'}{r - r'}r$ and $x + \frac{p - p'}{r - r'}r'$ respectively; and that the quadratic factor is $x^2 + \frac{q - q'}{p - p'}x + \frac{r - r'}{p - p'}$.

233. The present value of an annuity of £100 on the life of a person aged 21 is by the Carlisle Tables of mortality £2150, interest being at 3 per cent. If out of every 10 children born 6 reach the age of 21, find what sum ought to be paid down immediately on the birth of a child in order to secure it an annuity of £100 on its reaching 21, the deposit being forfeited if the child dies previously. Having given $\log 43 = 1{\cdot}63347$, $\log 2 = {\cdot}30103$, $\log 103 = 2{\cdot}01284$, $\log 1155 = 3{\cdot}0628$.

234. Convert $\sqrt{\left(a^2 - \frac{a}{n}\right)}$ into a continued fraction, n being greater than unity.

235. There is a number which if its digits be reversed becomes less by unity than its half: find the number.

236. Shew that if n be a positive integer, and x such that no denominator vanishes,

$$\frac{1}{x+1} - \frac{n}{(x+1)(x+2)} + \frac{n(n-1)}{(x+1)(x+2)(x+3)} - \dots + \frac{(-1)^n \lfloor n}{(x+1)(x+2)\dots(x+n+1)} = \frac{1}{x+n+1}.$$

237. Shew that $x^n - 1$ is greater than $n\left(x^{\frac{n+1}{2}} - x^{\frac{n-1}{2}}\right)$ if n is any positive integer, and x any positive quantity greater than unity.

238. In the successive powers of 4 shew that the digit in the tens' place is alternately even and odd; in the successive powers of 2 and of 8, shew that there are alternately two even digits and two odd digits.

239. A digit from 2 to 9 inclusive is taken at random, and raised to a high power: shew that the chance that the digit in the tens' place is odd is $\frac{5}{16}$.

240. Determine whether the series whose n^{th} term is $\frac{2n^2+3n+2}{(n+1)(n+2)(n+3)}$ is convergent or divergent.

241. A series $a_1, b_1, a_2, b_2, \dots$ is formed in the following way: a_n is an arithmetical mean between a_1 and b_{n-1}, and b_n is an harmonical mean between b_1 and a_{n-1}. Shew that $a_n b_n = a_1 b_1$.

242. Shew that the following equations are either inconsistent or insufficient for determining the values of x, y, and z:

$$x^2-a^2=zx+xy-yz,\quad y^2-b^2=xy+yz-zx,\quad z^2-c^2=yz+zx-xy.$$

243. A person starts with a certain capital which produces him 4 per cent. per annum compound interest. He spends every year a sum equal to twice the original interest on his capital. Find in how many years he will be ruined. Having given $\log 2 = \cdot 3010300$, $\log 13 = 1\cdot 1139434$.

244. Convert $\sqrt{\left(a^2+\frac{4a+2}{3}\right)}$ into a continued fraction.

245. A farmer laid out £25 in buying sheep at £1. 10*s.* a piece, and bullocks at £5 a piece: find how many sheep and bullocks he bought.

246. By comparing the coefficients of the various powers of x, shew that

$$\frac{1}{m}(1-x)^n+\frac{n}{m(m-1)}(1-x)^{n-1}+\frac{n(n-1)}{m(m-1)(m-2)}(1-x)^{n-2}+\dots$$
$$=\frac{1}{m-n}-\frac{n}{1}\cdot\frac{x}{m-n+1}+\frac{n(n-1)}{1\,.\,2}\,\frac{x^2}{m-n+2}-\dots;$$

n being a positive integer, and m such that no denominator vanishes.

247. If all the n letters $a, b, c \dots k$ denote positive quantities, shew that $n\,(a^{p+q}+b^{p+q}+c^{p+q}+\dots+k^{p+q})$ is greater than $(a^p+b^p+c^p+\dots+k^p)\,(a^q+b^q+c^q+\dots+k^q)$.

248. If n be a prime number, and N not divisible by n, shew that N^m-1 is divisible by n^r; where m stands for n^r-n^{r-1}.

249. A box contains three bank notes, and it is known that there is no note which is not either a £5, a £10, or a £20 note; one is drawn, found to be a £5 note, and replaced: determine the value of another draw.

250. Apply the process of Synthetic Division to divide $x^4+3x^2-12x+4$ by $x^2-4x+12$ as far as the term involving x^{-2}; and give the remainder.

251. Solve $x^2y+x=xy+x^2y^2-4y+4$, $xy+1=3xy^2-x^2y^2$.

252. There are two numbers a and b: it is required to find n intermediate numbers $a_1, a_2, \ldots a_n$, so that $a_1 - a$, $a_2 - a_1$, $a_3 - a_2$, $\ldots b - a_n$, may be in arithmetical progression with the common difference d. Find also the limits between which d must lie.

253. When the 3 per cents. are at 88, the sum of £100 is given for a perpetual annuity of £3 per annum, and an annuity terminable in 30 years: supposing the value of money to be fixed by the price of the 3 per cents., find the amount per annum of the terminable annuity. Having given $\log 1{\cdot}1 = {\cdot}04139$, $\log 1{\cdot}3 = {\cdot}11394$, $\log 2 = {\cdot}30103$, $\log 7 = {\cdot}84510$, $\log 3{\cdot}658 = {\cdot}56320$.

254. If $\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}$ be three successive convergents to $\sqrt{(a^2+1)}$, shew that $2(a^2+1)q_n = p_{n-1} + p_{n+1}$, $2p_n = q_{n-1} + q_{n+1}$.

255. A boy laid out a shilling in buying apples, pears, and peaches; the apples were five for a penny, the pears were one penny each, and the peaches were twopence each, and he got a dozen in all: find how many of each kind of fruit he bought.

256. If $\dfrac{a+bx}{(1-cx)\left(1-\dfrac{x}{c}\right)}$ be expanded in powers of x, shew that the coefficient of x^n is $\dfrac{a+bc-(ac+b)c^{2n+1}}{c^n(1-c^2)}$.

257. Shew that $\{\lfloor n\}^2$ is less than $\left\{\dfrac{(n+1)(2n+1)}{6}\right\}^n$, and that $\{\lfloor n\}^3$ is less than $\left\{\dfrac{n(n+1)^2}{4}\right\}^n$.

258. If n be a prime number, and N not divisible by n, shew that N^m+1 or N^m-1 is divisible by n^2; where m stands for $\dfrac{n(n-1)}{2}$.

259. A number taken at random is squared. Shew that it is an even chance that the digit in the units' place of the result is an even number, that it is 4 to 1 that the digit in the tens' place is an even number, and that it is 59 to 41 that the next higher digit is an even number.

260. In the expansion of $\dfrac{(1+cx)(1+c^2x)(1+c^3x)\ldots}{(1-cx)(1-c^2x)(1-c^3x)\ldots}$, the number of factors being infinite, and c less than unity, shew that the coefficient of x^r is $c^r\dfrac{(1+1)(1+c)(1+c^2)\ldots(1+c^{r-1})}{(1-c)(1-c^2)(1-c^3)\ldots(1-c^r)}$.

261. If α and β are the roots of the equation $ax^2+bx+c=0$, find the value of $\alpha^4+\alpha^2\beta^2+\beta^4$.

262. If the m^{th} term of a series in harmonical progression be n, and the n^{th} term be m, then the r^{th} term will be $\dfrac{mn}{r}$.

263. The first term of a certain series is a, the second term is b, and each subsequent term is a geometrical mean between the two preceding: shew that as n increases the n^{th} term tends to the value $\sqrt[3]{(ab^2)}$.

264. If $\dfrac{a}{b}$ be a proper fraction shew that it may be expressed thus: $\dfrac{a}{b}=\dfrac{1}{q_1}+\dfrac{1}{q_1q_2}+\dfrac{1}{q_1q_2q_3}+\ldots+\dfrac{1}{q_1q_2\ldots q_n}$, where $q_1, q_2, \ldots q_n$ are positive integers. Take for example $\dfrac{5}{7}$.

265. The diameters of two coins are ·81 and ·666 inches respectively: find the smallest number of coins which can be placed in a row of 9 feet long. Find also the smallest sum of money which such a row can be made to represent, supposing that the value of the larger coin is twice that of the smaller.

266. Shew that the difference between any two consecutive odd convergents to $\sqrt{(a^2+1)}$ is a fraction whose numerator is divisible by $2a$.

267. In a geometrical progression of which all the terms are positive the arithmetical mean of the extremes is greater than the arithmetical mean of all the terms.

268. If $a^2+b^2=c^2$, and a, b, c are integers, shew that abc is divisible by 60; and that if a is a prime number greater than 3, then b is divisible by 12.

269. There are n tickets in a bag numbered 1, 2, ... n. A man draws two tickets together at random, and is to receive a number of shillings equal to the product of the numbers he draws: find the value of his expectation.

270. If A be the present value of an annuity of £1 on the life of an individual, shew that in order to receive £P at his death the payment to be made immediately and repeated annually during his life is $\frac{P}{R} - \frac{AP}{1+A}$, where R is the amount of £1 in one year.

271. If $\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z}$,

shew that $$y^z z^y = x^z z^x = x^y y^x.$$

272. Solve $\sqrt{(x^2+a^2)(y^2+b^2)} + \sqrt{(x^2+b^2)(y^2+a^2)} = (a+b)^2$,

$$x + y = a + b.$$

273. Find a series of square numbers which when divided by 7 leave a remainder 4.

274. If $\frac{p_n}{q_n}$ be the n^{th} converging fraction to $\sqrt{(a^2+1)}$, shew

that $$\frac{p_n}{q_n} = \sqrt{(a^2+1)}\,\frac{(a+\sqrt{a^2+1})^n + (a-\sqrt{(a^2+1)})^n}{(a+\sqrt{a^2+1})^n - (a-\sqrt{(a^2+1)})^n}.$$

275. Expand $\frac{1+7x-x^2}{(1+3x)^2(1-10x)}$ in a series of ascending powers of x.

276. Find the scale of relation in each of the following series:

$$1 + 4x + 18x^2 + 80x^3 + 356x^4 + \ldots$$
$$1 + 2x + 3x^2 + 8x^3 + 13x^4 + 30x^5 + 55x^6 + \ldots$$

277. If S be the sum of the m^{th} powers of the n positive quantities $a, b, c, \ldots k$; and P the sum of the products of the quantities m together; shew that $(n-1)!\, S$ is greater than $(n-m)!\, m!\, P$.

278. If n be a prime number greater than 2, shew that any number in the scale whose radix is $2n$ ends with the same digit as its n^{th} power.

279. A bag contains 5 coins, and it is known that they can be nothing but shillings or sovereigns; two shillings are drawn together, and are not replaced: determine the value of another draw of two coins.

280. If n be a positive integer, and m such that no denominator vanishes, shew that

$$\frac{1}{m}(1+x)^n - \frac{n}{m(m+1)}(1+x)^{n-1} + \frac{n(n-1)}{m(m+1)(m+2)}(1+x)^{n-2} - \ldots$$
$$-\left\{\frac{1}{m}(1-x)^n - \frac{n}{m(m+1)}(1-x)^{n-1} + \frac{n(n-1)}{m(m+1)(m+2)}(1-x)^{n-2} - \ldots\right\}$$
$$= 2\left\{\frac{n}{m+n-1}x + \frac{n(n-1)(n-2)}{(m+n-3)\,\underline{|3}}x^3 + \ldots\right\}.$$

281. Determine the limits between which $\dfrac{x^2-2x-3}{2x^2+2x+1}$ lies for all real values of x.

282. Solve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, $(x^2+y^2)^{\frac{1}{2}} + (2xy)^{\frac{1}{2}} = b$.

283. If $\dfrac{p_n}{q_n}$ be the n^{th} convergent to the continued fraction $\dfrac{1}{a+}\,\dfrac{1}{b+}\,\dfrac{1}{a+}\,\dfrac{1}{b+}\ldots$ shew that p_n and q_n are respectively the coefficients of x^{n-1} in the expansions of the expressions $\dfrac{1+bx-x^2}{1-(ab+2)x^2+x^4}$ and $\dfrac{a+(ab+1)x-x^3}{1-(ab+2)x^2+x^4}$.

284. Shew in the preceding Example that if λ and μ are the values of x^2 found from the equation $1-(ab+2)x^2+x^4=0$;

$$ap_{2n} = bq_{2n-1} = \frac{ab(\lambda^n - \mu^n)}{\lambda - \mu}, \quad p_{2n+1} = q_{2n} = \frac{\lambda^{n+1} - \mu^{n+1} - \lambda^n + \mu^n}{\lambda - \mu}.$$

285. Find two numbers such that the first may be equal to the product of the digits of the second, and also less by 100 than twice the second.

286. If A_m denote the value of an annuity to last during the joint lives of m persons of the same given age, shew that the

value of an equal annuity to continue so long as there is a survivor out of n persons of that age may be found by means of tables giving the values of A_m from the formula

$$nA_1 - \frac{n(n-1)}{2} A_2 + \frac{n(n-1)(n-2)}{\underline{|3}} A_3 - \dots \pm A_n.$$

287. If x, y, z be real quantities, shew that

$$a^2(x-y)(x-z) + b^2(y-x)(y-z) + c^2(z-x)(z-y)$$

cannot be negative; provided that any two of the three quantities a, b, c are together greater than the third.

288. Shew that any square number is of one of the forms $5m$ or $5m \pm 1$. Shew that $n^5 - n$ is always divisible by 30; and if n be odd by 240.

289. A bag contains n balls, but nothing is known about their colours. A ball is drawn out and found to be black; it is replaced, and then a second draw is made with the same result: supposing the ball drawn the second time to be replaced, shew that it is $3n+3$ to $n-1$ in favour of a third draw giving a black ball.

290. If x is a proper fraction and p positive, shew that $n^p x^n$ is indefinitely small when n is indefinitely great.

291. If 1, x, x^2 and 1, y^2, y^3 be each in H.P., shew that $-y^2, y, x, x^2$ will be in A.P., and that their sum will be $x^3 + y^3$, supposing $x + y$ not to be zero, and x and y not to be unity.

292. Shew that $1^2r + 3^2r^2 + 5^2r^3 + \dots + (2n-1)^2 r^n$

$$= \frac{r(1+6r+r^2) - \{(2n-1)(1-r)+2\}^2 r^{n+1} - 4r^{n+2}}{(1-r)^3}.$$

293. Shew that if r be less than unity and the series in the preceding Example be continued to infinity it will be convergent: and find the sum to infinity.

294. Find two solutions in positive integers of $x^2 - 7y^2 = 1$.

295. In converting $\sqrt{N}$ into a continued fraction if the first two quotients be each 5, find N.

296. Shew that if x is positive the least value of the fraction $\dfrac{x^3+2a^3}{x}$ is when $x=a$.

297. The amount of fuel consumed by a steamer varies as the cube of the velocity. She consumes 1·5 tons of coal per hour at 18 shillings per ton when her speed is 15 miles per hour. She costs for other expenses 16 shillings per hour. Find the least cost for a voyage of 2000 miles.

298. Shew that if any odd number has an even digit in the tens' place, then all its integral powers must have an even digit in the tens' place.

299. There are three tickets in a bag numbered 1, 2, 3; a ticket is drawn and put back: if this be done four times, shew that it is 41 to 40 that the sum of the numbers drawn is even.

300. Prove that the continued fraction

$$\frac{1}{\frac{2}{1}+}\,\frac{1}{\frac{2}{2}+}\,\frac{1}{\frac{2}{3}+}\cdots\cdots\frac{1}{\frac{2}{n}}=S,$$

where $$S=\frac{1}{1\,.\,2}-\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}-\ldots+\frac{(-1)^{n+1}}{n\,(n+1)}.$$

Hence find the value of the continued fraction when n is infinite.

ANSWERS. I. II. III.

I. 1. 23. 2. 35. 3. 63. 4. 88. 5. 92.
6. 26. 7. 15. 8. 6. 9. 5. 10. 2.
11. 9. 12. 10. 13. 0. 14. 26. 15. 43.
16. 38. 17. 76. 536. 18. 9.

II. 1. $9a-7b+4c$. 2. $10x^2-4x+13$.
3. $12x^2+6xy-y^2+3x+4y$. 4. $4x^3+a^2x$.
5. $2ab+2x^2+2ax+2bx$. 6. $3a-b+c-6d$.
7. $2x^2+x$. 8. $2a^2-ax$. 9. $a-b+c-d$.
10. $2bx+2by$. 11. $a-b+c-d$. 12. $a-b+c+d$.
13. $a-7b$. 14. $5a$. 15. $2a-b-d$. 16. $12x-8y$.
17. $3a$. 18. a. 19. $2a+x-2b+y=9$. 20. $3x^2$.

III. 1. $3pq+2p^2-2q^2$. 2. $7a^3+16a^2b-ab^2-10b^3$.
3. $a^4-a^2b^2+2ab^3-b^4$. 4. $a^4-a^2b^2+4ab^3-4b^4$.
5. $a^4+4a^3x+4a^2x^2-x^4$. 6. $a^4-8a^2x^2+16x^4$.
7. $a^2b+(a-b)^2x-2ax^2-x^3$. 8. $60x^4+42x^3a-107x^2a^2+10xa^3+14a^4$.
9. $6x^4-96$. 10. $4x^3-22x^2y+42xy^2-27y^3$.
11. $12x^3-17x^2y+3xy^2+2y^3$. 12. $x^6-x^4y^2+x^2y^4-y^6$.
13. $x^2-4y^2+12yz-9z^2$. 14. $6x^4+x^3y+2x^2y^2-13xy^3+4y^4$.
15. $x^4+x^3(y+z)+x^2(y^2+yz+z^2)+xyz(y+z)+y^2z^2$.
16. $a^3+b^3-c^3+3abc$. 17. $x^3+y^3+3xy-1$.
18. $x^5+151x-264$. 19. $x^5-41x-120$.
20. $4x^6-5x^5+8x^4-10x^3-8x^2-5x-4$. 21. $x^6+10x-33$.
22. $x^7-7x^6+21x^5-17x^4-25x^3+6x^2-2x-4$.
23. $a^8+2a^6+3a^4+2a^2+1$. 24. a^4-x^4.
25. x^4-10x^2+9. 26. x^8+x^4+1.
27. $x^8-x^6a^2+2x^5ab-(b^2+2ac)x^4+2x^3(bc+ad)-(c^2+2bd)x^2+2xcd-d^2$.
30. $abc+(ab+bc+ca)x+(a+b+c)x^2+x^3$.
31. $x^4-x^3(a+b+c+d)+x^2(ab+ac+ad+bc+bd+cd)$
$-x(bcd+acd+abd+abc)+abcd$.
32. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$. 33. b^2-d^2.

34. $4(a^2+b^2+c^2+d^2)$. 36. $2(a^2+b^2+c^2)$. 37. $8x^3$.
38. $2(a^4+b^4+c^4)$. 39. $4(b^2c^2+c^2a^2+a^2b^2)$.
43. $x^6-22x^4+60x^3-55x^2+12x+4$.
44. $x^8-2x^7a+2x^5a^3-2x^4a^4+2x^3a^5-2xa^7+a^8$.
45. $a^5-a^4b-2a^3b^2+2a^2b^3+ab^4-b^5$.

IV. 1. x^2-x+1. 2. $9x^2-6xy+4y^2$.
3. a^2+ab-b^2. 4. a^2-3ab.
5. $32x^5+16x^4y+8x^3y^2+4x^2y^3+2xy^4+y^5$.
6. $a^4-a^3b+a^2b^2-ab^3+b^4$. 7. x^2+y^2. 8. x^2+3x+2.
9. $16x^4-8x^3y+4x^2y^2-2xy^3+y^4$. 10. x^2-xy+y^2.
11. x^2-x+1. 12. $a^2-2ab+3b^2$. 13. $a^3-2a^2b+2ab^2-b^3$.
14. $16a^3-24a^2b+36ab^2-27b^3$. 15. $x^4+2x^3+3x^2+2x+1$.
16. x^4-5x^2+4. 17. $a^2-2ab+3b^2$. 18. x^4-8x^2+16.
19. x^3+3x^2+x-2. 20. $2x^3-8x^2+3x-12$.
21. $a+x$. 22. x^2-a^2. 23. $a+b+c$.
24. $3x^2-2abx-2a^2b^2$. 25. x^2-2x+1. 26. $3a^2+4ab+b^2$.
27. $x^2-xy+y^2+x+y+1$. 28. $a^2+b^2+c^2+bc+ca-ab$.
29. $b(2a^3+3a^2b-ab^2+4b^3)$. 30. $ab-ac+bc$. 31. $b+c-a$.
32. $(b+c)(c+a)$. 33. $a^4-4a^2bc+7b^2c^2$. 34. a^2+ax+x^2.
35. $(x+2z)y^2+(x^2-2z^2)y-xz(x+z)$. 36. $ab+bc+ca$.
37. $x^2-(a+b)x+ab$. 38. $x-b$. 39. $ab-ac+b^2-c^2$.
40. $a^2+b^2+c^2$. 41. $a+x$. 42. $(a+b-c-d)(a-b+c-d)$.
43. x^2-ax+a^2. 45. The quotient is $7xy(x+y)$.
46. Each is $abc-ap^2-bq^2-cr^2+2pqr$.
47. $(a-x)(a+x)(a^2+x^2)(a^4+x^4)(a^8+x^8)$.
48. $(a+b+c)(b+c-a)(a-b+c)(a+b-c)$.
49. $(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)$.

V. 2. 9. 3. 70. 4. 6. 5. $y^4+11y^3+47y^2+93y+69$.

VI. 1. $x-2$. 2. $x+3$. 3. x^2+2x+3. 4. $x+1$.
5. $3x+4a$. 6. $x-y$. 7. $3x-7$. 8. $x-1$. 9. $x-2$.
10. x^2+x+1. 11. $x+2$. 12. $x-3$. 13. $2x-1$.
14. $x-y$. 15. x^2+2x+3. 16. $a(2a-3x)$. 17. $2x-9$.
18. $ax-by$. 19. $x^2+(a+y)x+y^2$. 20. $(x+1)^2$.
21. $2x^3-4x^2+x-1$. 22. $x-2a$. 23. $x-2$. 24. x^2-1.

VII. 1. $(2x^2+3x-2)(3x+1)$. 2. $(x^2-1)(x+2)$.
3. $(x^3-9x^2+23x-15)(x-7)$. 4. $(3x-2)(4x^3-4x^2-x+1)$.
5. $(x+1)^2(x^2-1)$. 6. $(x^2-y^2)(x^2-4y^2)$. 7. $16x^4-1$.
8. $x(x^6-1)$. 9. $(x^2-4a^2)^2$. 10. $(x-1)(x-2)(x-3)(x-4)$.
11. $(x-2)(x-3)(x-4)(x-5)$. 12. $(x^2-1)(x^2-9)(x+7)$.
13. x^4-16a^4. 14. $(x-a)(x-b)(x-c)$.
15. $(x+c)(2x-3b)(x^2+ax-b^2)$. 16. $36(a^4-b^4)(a^2-b^2)^2(a^3-b^3)$.

VIII. 1. $\frac{x+3}{x+7}$. 2. $\frac{x-4}{x-5}$. 3. $x-3$. 4. $a+b$. 5. $x+1$.
6. $\frac{3x-1}{2x-1}$. 7. $\frac{3x+2}{x+1}$. 8. $\frac{2x+3}{3x-4}$. 9. $\frac{3x+9}{x^4-x^3+6x^2-6x+6}$.
10. $\frac{x-5}{x+5}$. 11. $\frac{x^2+2x+3}{x^2-2x-3}$. 12. $\frac{1}{x^2-2x+2}$. 13. $\frac{(x-1)^2}{x^2-3x+1}$.
14. $\frac{a^2+b^2}{a}$. 15. $\frac{1}{b-2x}$. 16. $\frac{7(x^2+xy+y^2)}{5}$. 17. $\frac{a^2+b^2}{a^2-b^2}$.
18. $\frac{1}{2}$. 19. $\frac{-2}{(4x^2-1)x}$. 20. $\frac{2a}{n}$. 21. $\frac{9}{(x-1)(x+2)^2}$.
22. $\frac{2x-3}{(x^2-1)(2x+3)}$. 23. $\frac{ax-b^2}{x^2-b^2}$. 24. $\frac{1}{x+2}$. 25. 0.
26. $\frac{2ab^2}{a^4-b^4}$. 27. $\frac{x^2-4xy-y^2}{(x^2-y^2)^2}$. 28. $\frac{4ab}{(a-b)^2}$. 29. $\frac{4a}{a+x}$.
30. $\frac{81a-4b}{84}$. 31. 0. 32. 0. 33. 0.
34. $\frac{a^2b+b^2c+c^2a-b^2a-c^2b-a^2c}{(b-c)(c-a)(a-b)}=-1$. 35. $\frac{1}{abc}$. 36. 0. 37. 0.
38. $\frac{(a-b)b}{x(a+b)}$. 39. $\frac{x^2-y^2}{y(x^2+y^2)}$. 40. $\frac{3x}{4y}$. 42. $\frac{1-y}{x}$.
43. $\frac{ax}{a^2-x^2}$. 44. $\frac{a^2+b^2}{a}$. 45. 2. 46. $\frac{a^2-ab+b^2}{a^2+ab+b^2}$.
47. $\frac{x^4}{a^4}+\frac{x^2}{a^2}+1$. 48. $x^2+1+\frac{1}{x^2}$. 49. $\frac{(x+a)^2}{(x-b)^2}$. 50. $\frac{(a-x)^2}{x(a+x)}$.
51. $\frac{2(a-b)^2}{3b^2(a+b)}$. 52. $\frac{2y}{x^2-xy+y^2}$. 53. $\frac{y^2}{x^2+y^2}$. 54. $\frac{x+y}{y}$.
55. 1. 56. 1. 57. $x^3-x+\frac{1}{x}-\frac{1}{x^3}$. 58. $\frac{x^2+1}{x}$.

59. $\frac{x^2+x+1}{x}$. 60. $a^2-b^2+c^2+2ac$. 61. $\frac{a+x}{x-y}$.

62. $a^2-b^2+c^2-2ac$. 63. $\frac{x^2+3ax-2a^2}{x+6a}$. 64. $\frac{x^2-2a^2}{ax}$. 65. $\frac{ac-bd}{ac+bd}$.

66. $\frac{a^2+x^2}{2ax}$. 67. $\frac{bc+ca+ab}{bc+ca-ab}$. 68. $-\frac{a^4+a^2b^2+b^4}{ab(a-b)^2}$.

69. $-\frac{bc(b-c)^2}{b^4+c^4+b^2c^2}$. 70. $\frac{xy}{x^2+y^2}$. 71. $\frac{(a^2+b^2)^2}{2a^2b^2}$. 72. m.

73. $\frac{4a^2x}{x^4-a^4}$. 74. $\frac{(a+b+c)^2}{2bc}$. 75. $\frac{4}{3(x+1)}$. 76. $\frac{adf+ae}{bdf+be+cf}$.

IX. 1. 1. 2. 20. 3. 3. 4. 11. 5. $\frac{6}{7}$. 6. 13.

7. 8. 8. 1. 9. 7. 10. 7. 11. 4. 12. 3.
13. 5. 14. 28. 15. 2. 16. 2. 17. 3. 18. 10.
19. $1\frac{1}{3}$. 20. $2\frac{1}{2}$. 21. 5. 22. $\frac{1}{5}$. 23. 13. 24. 9.
25. 4. 26. 4. 27. 9. 28. $\frac{4}{13}$. 29. 13. 30. $\frac{2}{3}$.
31. 4. 32. 56. 33. 7. 34. $8\frac{3}{5}$. 35. $4\frac{1}{2}$. 36. $2\frac{3}{13}$.
37. $1\frac{4}{7}$. 38. 3. 39. 2. 40. 12. 41. 12. 42. 2.
43. 3. 44. -2. 45. 1. 46. 1. 47. 5. 48. $\frac{79}{29}$.

49. $3\frac{3}{8}$. 50. $\frac{8a}{25}$. 51. $\frac{cd-ab}{a+b-c-d}$. 52. $\frac{a^2(b-a)}{b(b+a)}$.

53. $\frac{a(1-b^2)}{b(a^2-1)}$. 54. $\frac{a^2c+b^2a+c^2b-a-b-c}{ac+bc+ab-1}$. 55. $\frac{ac}{b}$.

56. $\frac{ab(a+b-2c)}{a^2+b^2-ac-bc}$. 57. $\frac{rb-cq}{pc-ra}$. 58. $\frac{ab}{a+b}$. 59. $\frac{nb-ma}{m-n}$.

60. $\frac{a-b}{2}$. 61. $\frac{1}{3}(a+b+c)$. 62. 2. 63. 20. 64. 5.

X. 1. £1290, £2580. 2. £120, £300. 3. £5.
4. £140. 5. 28, 18. 6. 38 children, 76 women, 152 men.
7. £720. 8. £144, £240, £210. 9. £350, £450, £720.

10. A £162, B £118, C £104. 11. 3456, 2304.
12. 126 quarts. 13. £2. 15s. 14. £3. 10s.
15. £600, £250. 16. 400 inches. 17. 30. 18. 42.
19. 7, 8. 20. 8, 6, 3, 2; 24 kings in all. 21. 3.
22. 6 shillings. 23. £3600. 24. 11 oxen, 24 sheep.
25. 5 shillings taken by each; there were 20 shillings in the purse.
26. 240. 27. 90 by 180, and 100 by 230. 28. 48 minutes.
29. £8750. 30. 5. 31. 60 oranges and 240 apples.
32. 10 from A, 4 from B. 33. 11, 22, 33. 34. £420. 10s.
35. $6\frac{6}{11}$ past one. 36. $\frac{abc}{b+c}$. 37. 2s. 8d. 38. 40.

XI. 1. $x=11,\ y=4$. 2. $x=5,\ y=7$. 3. $x=16,\ y=7$.
4. $x=2,\ y=13$. 5. $x=8,\ y=1$. 6. $x=2,\ y=6$.
7. $x=3,\ y=5$. 8. $x=3,\ y=4$. 9. $x=12,\ y=3$.
10. $x=4,\ y=3$. 11. $x=10,\ y=20$. 12. $x=60,\ y=36$.
13. $x=12,\ y=20$. 14. $x=-6,\ y=12$. 15. $x=18,\ y=6$.
16. $x=7,\ y=11$. 17. $x=2,\ y=7$. 18. $x=\cdot 4,\ y=\cdot 1$.
19. $x=4,\ y=1$. 20. $x=\frac{10}{3},\ y=\frac{20}{3}$. 21. $x=12,\ y=6$.
22. $x=2,\ y=-1$. 23. $x=3,\ y=2$. 24. $x=18,\ y=12$.
25. $x=5,\ y=6$. 26. $x=10,\ y=5$. 27. $x=y=\frac{a}{6}$.
28. $x=y=m+n$. 29. $x=3a,\ y=-2b$. 30. $x=\frac{nc+bd}{mb+na},\ y=\frac{mc-nd}{mb+na}$.
31. $x=b+c,\ y=a+c$. 32. $x=(a+b)^2,\ y=(a-b)^2$.

XII. 1. $x=7,\ y=5,\ z=4$. 2. $x=2,\ y=3,\ z=4$.
3. $x=1,\ y=2,\ z=3$. 4. $x=2,\ y=3,\ z=5$.
5. $x=2,\ y=3,\ z=4$. 6. $x=8,\ y=4,\ z=2$.
7. $x=10,\ y=2,\ z=3$. 8. $x=4,\ y=3,\ z=5$.
9. $x=3,\ y=4,\ z=6$. 10. $x=\frac{4}{3},\ y=4,\ z=\frac{4}{5}$.
11. $x=\frac{7}{6},\ y=-\frac{7}{2},\ z=\frac{21}{10}$. 12. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}$.
13. $x=2,\ y=3,\ z=1$. 14. $x=6,\ y=\frac{20}{3},\ z=\frac{46}{3}$.

15. $x=4$, $y=9$, $z=16$, $u=25$. 16. $u=4$, $x=12$, $y=5$, $z=7$.
17. $x=3$, $y=1$, $u=9$, $z=5$. 18. $x=3$, $y=2$, $u=5$, $z=-4$.
19. $x=2, y=4, z=3, u=3, v=1$. 20. $x=2, y=1, z=3, u=-1, v=-2$.
21. $x=\frac{a}{2}, y=\frac{b}{2}, z=\frac{c}{2}$. 22. $x=\frac{b^2+c^2-a^2}{2bc}$. 23. $x=2a, y=2b, z=2c$.
24. $x=\frac{1}{(a-b)(a-c)}$. 25. $x=\frac{A(A-b)(A-c)}{a(a-b)(a-c)}$. 26. $\frac{2}{x}=-\left(\frac{1}{b}+\frac{1}{c}\right)$.
27. $x=b+c-a$. 28. $x=abc$, $y=ab+bc+ca$, $z=a+b+c$.

XIII. 1. $\frac{5}{8}$. 2. 250, 320. 3. $\frac{4}{15}$. 4. 5, 6. 5. 42*s.*, 26*s.*
6. 75*s.* and 35*s.* 7. 5 and 7. 8. 7, 10. 9. 2*s.* 6*d.*, 1*s.* 8*d.*
10. 1, 3, 5. 11. Tea, 5*s.* per pound; sugar 4*d.* 12. 50.
13. £3000, £4000, £4500, at 4, 5, 6 per cent respectively.
14. 20 persons; 6 shillings each. 15. 8 and 12. 16. £540; 17 pence.
17. 300, 140, 218. 18. £70. An ox costs £10 and a lamb 18*s.* 9*d.*
19. *A* wins 21 games, *B* 13 games. 20. *A* 11*s.*, *B* 38*s.*, *C* 33*s.*, *D* 32*s.*, *E* 36*s.* 21. 90 miles. 22. *A* could do the work alone in 80 days, *B* in 48 days; *A* must receive $\frac{11}{32}$ of the money, and *B* $\frac{21}{32}$ of the money. 23. *A* in 5 minutes, *B* in 6 minutes.
24. $2\frac{1}{2}$, 2 miles per hour; distance 5 miles. 25. 100 miles; original rate 25 miles per hour. 26. *A* 26, *B* 14, *C* 8. 27. *A* in $\frac{pm}{p+n-m}$ days, *B* in $\frac{pm}{m-n}$ days. 28. $\frac{b(n-1)}{a-c}$ miles per hour. 29. 4 yards and 5 yards. 30. 27. 31. 63. 32. Coach goes 10 miles an hour; train goes 30 miles. From *A* to *B* is $16\frac{2}{3}$ miles; from *A* to *C* is 20 miles; from *C* to *B* is 40 miles. 33. 600 yards.

XIV. 1. a. 2. $\frac{2x^2-4}{x+3}$. 3. $\frac{b}{a}$. 4. 0. 6. $\frac{a+b+c}{2}$.
8. $x=a+b$. 9. $x=a$, $y=b$, $z=c$. 10. $(x+1)(x+2)(x+3)(x+4)$.

XV. 1. $\frac{x^2+2x+3}{x^2+x+1}$. 3. $t=x$. 6. $\frac{11}{2}$. 7. $x=b-c$, $y=c-a$, $z=a-b$. 8. Clear the given relation of fractions; thus we find $(a+b)(b+c)(c+a)=0$, therefore one of the three factors must vanish; hence the required result follows. 9. Each child obtains £1920. 12*s.*, and each brother £960. 6*s.* 10. $x=-3a$.

XVI. 1. $1+4x+10x^2+12x^3+9x^4$.
2. $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$.
4. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$. 5. $2(1+15x^2+15x^4+x^6)$.
9. The numerator will be found to be equal to $5(1+x^2)^4$ and the denominator to $(1+x^2)^5$, so that the fraction $=\dfrac{5}{1+x^2}$.

XVII. 1. x^2-x+1. 2. x^2-2x-2. 3. $2x^2+3x-1$.
4. $2x^2-x+1$. 5. $2x^2-3ax+4a^2$. 6. $5x^2-3ax+4a^2$.
7. $(x-a)^3$. 8. a^2+b^2. 9. $(a^2+b^2)(c^2+d^2)$. 10. $a^2-b^2+c^2-d^2$.
11. $x-2-\dfrac{1}{x}$. 12. $x^2-\dfrac{x}{2}+\dfrac{2}{x}$. 13. $\dfrac{a^2}{2}+\dfrac{a}{x}-\dfrac{x}{a}$.
14. $a^2+(2b-c)a+c^2$. 15. $(a-2b)x^2-ax+2b-3$. 16. 1·14.
17. x^2-3x+2. 18. $2x^2+4cx-3c^2$. 19. $2x^2-3cx+4c^2$.
20. 5·51. 21. 9009. 22. 22·22.
23. 111111111. 24. $x-\dfrac{1}{x}$. 26. The given expression

$$=(x^2-yz)\{(x^2-yz)^2-(y^2-zx)(z^2-xy)\}+\text{two similar expressions}$$
$$=(x^2-yz)\,x\,\{x^3+y^3+z^3-3xyz\}+\text{two similar expressions}$$
$$=\{x^3+y^3+z^3-3xyz\}^2.$$

XVIII. 1. $x^{\frac{4}{3}}$. 2. $a^{-\frac{17}{60}}$. 3. $\dfrac{y^{\frac{1}{3}}}{(bx)^{\frac{1}{6}}}$. 4. 1. 5. $\left(\dfrac{a}{b}\right)^{mn}$.
6. $a^{\frac{3}{2}}b^{-\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{1}{2}}+a^{-\frac{1}{2}}b^{\frac{3}{2}}$. 7. $x^{\frac{5}{2}}+x^{\frac{3}{2}}y-xy^{\frac{3}{2}}-y^{\frac{5}{2}}$. 8. a^4-1.
9. $a+a^{\frac{1}{3}}-1+a^{-\frac{1}{3}}+a^{-1}$. 10. $-4a^{-7}b^{-1}+9a^{-9}b$. 11. $x+y$.
12. $x^{\frac{2}{3}}-x^{\frac{1}{3}}a^{\frac{1}{3}}+a^{\frac{2}{3}}$. 13. a^n+1+a^{-n}. 14. $2x^2-3xy+2y^2$.
15. $a+a^{\frac{1}{2}}b^{\frac{1}{2}}-b$. 16. $\dfrac{x+a}{x^2+3xa+a^2}$. 17. $\dfrac{y}{x^{\frac{1}{2}}}+y^{\frac{1}{4}}x^{\frac{1}{4}}-\dfrac{x}{2y^{\frac{1}{2}}}$.
18. $2a^{\frac{1}{2}}-3b^{\frac{1}{3}}+4c^{\frac{1}{4}}$. 19. $16x^{\frac{2}{3}}-16x^{\frac{1}{3}}+12-4x^{-\frac{1}{3}}+x^{-\frac{2}{3}}$.

XIX. 1. $a^{\frac{5}{2}}+a^2b^{\frac{2}{3}}+a^{\frac{3}{2}}b^{\frac{4}{3}}+ab^2+a^{\frac{1}{2}}b^{\frac{8}{3}}+b^{\frac{10}{3}}$.
2. $2^{\frac{5}{2}}+2^2.3^{\frac{1}{3}}+2^{\frac{3}{2}}.3^{\frac{2}{3}}+2.3+2^{\frac{1}{2}}3^{\frac{4}{3}}+3^{\frac{5}{3}}$.

3. $3^{\frac{3}{2}} - 3 . 5^{\frac{1}{4}} + 3^{\frac{1}{2}} 5^{\frac{2}{4}} - 5^{\frac{3}{4}}$. 4. $\cdot 2679492$. 7. $3\sqrt{\frac{x}{y}} - 4 + 3\sqrt{\frac{y}{x}}$.

8. $a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} - b$. 9. $1 + \sqrt{3}$. 10. $2 - \sqrt{3}$. 11. $\sqrt{5} + \sqrt{2}$.

12. $\sqrt{10} + 2\sqrt{2}$. 13. $3\sqrt{7} - 2\sqrt{3}$. 14. $\sqrt{\frac{25}{2}} + \sqrt{\frac{7}{2}}$.

15. $\sqrt{\left\{\frac{(a+c)(b+c)}{2}\right\}} + \sqrt{\left\{\frac{(a-c)(b-c)}{2}\right\}}$. 16. $\sqrt[4]{3}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{5}{2}}\right)$.

17. $\sqrt[4]{3}\left(\frac{3}{\sqrt{2}} - \frac{\sqrt{3}}{\sqrt{2}}\right)$. 18. $\frac{1}{\sqrt[4]{(1-c^2)}}\left\{\sqrt{\left(\frac{1+c}{2}\right)} + \sqrt{\left(\frac{1-c}{2}\right)}\right\}$.

19. $\frac{5}{3}\sqrt{3} - 2$. 20. 1. 21. $1 + \sqrt{2} + \sqrt{3}$.

22. $1 + \sqrt{\frac{5}{2}} - \sqrt{\frac{3}{2}}$. 23. $\sqrt{6} + \sqrt{3} - \sqrt{5} - 1$. 24. $1 + \sqrt{2}$.

25. $1 + \sqrt{5}$. 26. $\sqrt{3} - \sqrt{2}$. 27. $\sqrt{6} - \sqrt{5}$. 29. $x = 25$.

30. $x = 7$. 31. $x = \frac{ab}{a+b}$. 32. $x = \frac{a^2 + b^2 + c^2 - 2bc - 2ca - 2ab}{4c}$.

XX. 1. $x = 1, 3$. 2. 1, 4. 3. $\frac{2}{3}, \frac{3}{2}$. 4. $4, -\frac{5}{3}$.

5. $3, \frac{1}{2}$. 6. $17, \frac{2}{3}$. 7. $-4, -6$. 8. $5, -\frac{32}{7}$.

9. 3, 11. 10. $\frac{3}{2}, -\frac{1}{2}$. 11. $\frac{5}{3}, -\frac{3}{2}$. 12. $\frac{3}{2}, -\frac{1}{2}$.

13. $\frac{1}{10}, \frac{1}{11}$. 14. $\frac{1}{13}, \frac{1}{60}$. 15. $4, -1$. 16. $3, -\frac{4}{3}$.

17. $4, \frac{7}{5}$. 18. $6, -1$. 19. $5, -\frac{5}{2}$. 20. $8, \frac{5}{2}$.

21. $\frac{1}{2}, \frac{9}{2}$. 22. $3, -\frac{1}{25}$. 23. $10, -2$. 24. $-\frac{1}{2}, -\frac{73}{50}$.

25. $\frac{2}{3}, \frac{3}{10}$. 26. $\pm\sqrt{6}$. 27. $-1, \frac{3}{5}$. 28. $7, -\frac{7}{9}$.

29. $3, -\frac{24}{13}$. 30. 2, 16. 31. $-2, -16$. 32. $5, -3$.

33. $3, -5$. 34. $29, -10$. 35. $10, -29$. 36. $1, \frac{3}{5}$.

37. $3, -\frac{4}{5}$. 38. $2, \frac{1}{3}$. 39. $8, -8$. 40. $10, -10$.

41. $2, -3.$ 42. $24, \frac{42}{5}.$ 43. $3, -\frac{14}{3}.$ 44. $3, -\frac{5}{3}.$

45. $3, -\frac{4}{3}.$ 46. $\frac{7}{4}, 1.$ 47. $\frac{13}{3}, 1.$ 48. $0, 4.$

49. $0, \frac{4}{3}.$ 50. $1, -\frac{3}{7}.$ 51. $2 + \sqrt{3}$, and $-(2+\sqrt{3})\,2.$

52. $a \pm b.$ 53. $a \pm \sqrt{(a^2 - b^2)}.$ 54. $\frac{a+b}{a-b}, \frac{a-b}{a+b}.$

55. $\frac{1}{3}\{a+b+c \pm \sqrt{(a^2+b^2+c^2-ab-bc-ca)}\}.$ 56. $a+b+c.$

57. $-a, -b.$ 58. $\frac{a^2+b^2 \pm \sqrt{\{(a^2-b^2)^2+4abc^2\}}}{2ab}.$

59. $0, \frac{2ab-ac-bc}{a+b-2c}.$ 60. $\frac{2a-b}{ac}, -\frac{3a+2b}{bc}.$

61. $\frac{1}{a+b+c}[ab+bc+ca \pm \sqrt{\{a^2b^2+b^2c^2+c^2a^2-abc\,(a+b+c)\}}].$

62. $-a, \frac{a\,(1+c)}{c\,(2c+3)}.$

In the following Chapters the irrational roots and the impossible roots have not always been given; and some of the roots given are not applicable; see Arts. 329, 330.

XXI. 1. $1, \frac{1}{9}.$ 2. $1, -2.$ 3. $(-41)^{\frac{2}{3}}, 9.$ 4. $14^{2n}, (-1)^{2n}.$

5. $2, 3.$ 6. $2^n, (-1)^n.$ 7. $\{-\sqrt{a} \pm \sqrt{(a-c)}\}^2.$ 8. $\pm 11.$

9. $\pm 2, \pm\sqrt{10}.$ 10. $8, \frac{125}{64}.$ 11. $8, \left(-\frac{13}{7}\sqrt{2}\right)^2.$ 12. $2^{\frac{3}{2n}}, \left(-\frac{8}{3}\right)^{\frac{3}{2n}}.$

13. $4, -1.$ 14. $4, \frac{1}{4}.$ 15. $16, \left(-\frac{11}{5}\right)^4.$ 16. $(-1)^{\frac{4}{3}}, \left(\frac{7}{3}\right)^{\frac{4}{3}}.$

17. $\frac{17}{4}, \frac{1}{4}.$ 18. $2^n, \frac{1}{2^n}.$ 19. $9, -\frac{18}{5}.$ 20. $\pm 5.$ 21. $\frac{\pm\sqrt{(4ab-b^2)}}{2}.$

22. $16, 0.$ 23. $18, 3.$ 24. $2^x = 8$ or -10; so that $x = 3.$

25. $5, -8.$ 26. $0, \frac{\pm\sqrt{3}}{2}a.$ 27. $x^2 = \frac{n}{n-2}$ or $\frac{n-1}{n+1}.$

28. $x^2 = -ab \pm \frac{1}{2}\sqrt{(3a^4+3b^4-6a^2b^2)}.$

29. $\{\sqrt{(x+2)}+\sqrt{(x^2+2x)}\}^2=(a-x-\sqrt{x})^2$, a quadratic in $\sqrt{x}$, from which $\sqrt{x}=\dfrac{-(2+a)\pm\sqrt{(2a^3+3a^2)}}{2+2a}$. 30. $1, \dfrac{c^2-2}{(c+2)^2}$.

31. Multiply up and arrange $x\{\sqrt{(a-x)}-\sqrt{(a+x)}\}=\sqrt{a}\{\sqrt{(a^2-x^2)}-a\}$, square, &c. $x=0, \pm\dfrac{a\sqrt{3}}{2}$. 32. $2a, -2a$. 33. $1, -\dfrac{25}{3}$. 34. $1, \dfrac{1}{21}$.

35. $\pm 2a, \pm 2a\sqrt{(-1)}$. 36. $x^n=0$ or $\dfrac{4c^2a}{(c^2-1)^2b}$ 37. $\dfrac{1}{2}, -\dfrac{25}{6}$.

38. $\pm a, \pm\dfrac{1}{a}$. 39. $\pm\dfrac{5a}{3}, \pm\dfrac{a\sqrt{(-34)}}{3}$. 40. $\pm\sqrt{2}$. 41. $0, \dfrac{a\{1\pm\sqrt{(-8)}\}^6}{3^6}$.

42. $\dfrac{a}{2}(1\pm\sqrt{5})$. 43. $x^2=\dfrac{m^4-4m^2}{4(m^2-1)}$. 44. $x^2=9$. 45. $x^2=\dfrac{a^4-b^4}{7a^2-2b^2}$.

46. $x^2=\dfrac{2\pm\sqrt{2}}{2}$. 47. $\{c\pm(c^2-1)\}^{\frac{2pq}{p-q}}$. 48. $0, \dfrac{16}{25}$. 49. $\pm 2a, \pm a\sqrt{(-6)}$.

50. $\dfrac{3}{2}, \dfrac{2}{3}$. 51. $0, -\dfrac{4(a+b)(a^2+b^2)}{3a^2+3b^2+10ab}$. 52. $1, -\dfrac{4}{3}$.

53. $8, -\dfrac{23}{5}$. 54. $\dfrac{ac^2}{b^2}$. 55. $1, \dfrac{47-44\sqrt{6}}{23}$.

56. $1, \dfrac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$. 57. $0, -1$. 58. $0, \frac{1}{2}(-b\pm\sqrt{b^2-4a})$.

59. $0, \frac{1}{2}\{a+b+c\pm\sqrt{(a^2+b^2+c^2-2bc-2ca-2ab)}\}$.

60. $0, \dfrac{1}{\pm\sqrt{3}}$. 61. $0, \pm\sqrt{(a^2+b^2)}$. 62. $0, \pm\sqrt{\{mn+a(m-n)\}}$.

63. $0, a\left(1\pm 2\sqrt{\dfrac{b}{c}}\right)$. 64. Transpose and square; we get $2x(2x+1)\sqrt{(x^2+2)}=2(x^2+1)(2x+1)$; it will be found from this that the only solution is $x=-\frac{1}{2}$. 65. 1. 66. $4, -9$. 67. $0, 2$.

68. $0, -5, \dfrac{1}{3}, -\dfrac{16}{3}$. 69. $1, -4, \dfrac{-3\pm\sqrt{109}}{2}$. 70. $1, \dfrac{1}{2}$.

71. $2, -5, \dfrac{1}{2}\{-3\pm\sqrt{241}\}$. 72. $a+2, -\dfrac{a+6}{3}$. 73. $2, -\dfrac{1}{2}$.

74. $1, -2$. 75. $x^2+5ax=-5a^2\pm\sqrt{(a^4+c^4)}$; whence x.

76. $x^2+3x=\dfrac{1}{4}$ or $-\dfrac{9}{4}$; whence x.

77. $\dfrac{a^2+x^2}{ax}=\dfrac{a^2+x^2}{a^2-x^2}$, &c.; $x=\dfrac{a}{2}(-1\pm\sqrt{5})$.

78. $a=\left(x-\frac{1}{2}+\frac{1}{2}\right)^4+\left(x-\frac{1}{2}-\frac{1}{2}\right)^4$. Quadratic in $\left(x-\frac{1}{2}\right)^2$.

79. $(x^2-x)^2-(x^2-x)=a$. 80. 4, -3.

81. $\{\sqrt{x}+\sqrt{(x+7)}\}^2+\sqrt{x}+\sqrt{(x+7)}=42$. $x=9$ or $\left(\frac{29}{12}\right)^2$.

82. $(x-4\sqrt{x})^2+2(x-4\sqrt{x})+1=0$. $x=7\pm4\sqrt{3}$.

83. $\{\sqrt{x}+\sqrt{(a+x)}\}^2+\sqrt{x}+\sqrt{(a+x)}=b+a$; &c.

84. $(x^2+x)^2+4(x^2+x)+4=16x^2$. $x=1$ or 2.

85. $(x^2+a^2)^2=2a^2(x-a)^2$. 86. $\left(x+\frac{c}{ax}\right)^2+a\left(x+\frac{c}{ax}\right)+b=\frac{2c}{a}$.

87. $\left(\frac{x}{a}-\frac{a}{x}\right)^2-2\left(\frac{x}{a}-\frac{a}{x}\right)+1=0$. 88. $x+\frac{1}{x}=\frac{10}{3}$ or $-\frac{16}{3}$.

89. $\left(x-\frac{1}{x}\right)^2-2\left(x-\frac{1}{x}\right)+1=0$ after expunging $\sqrt{(x-1)}$.

90. $1+\sqrt{3}\pm\sqrt{(3+2\sqrt{3})}$, $1-\sqrt{3}\pm\sqrt{(3-2\sqrt{3})}$.

91. $(x+1)(x^2-x+1)=0$. 92. $(x+1)\{1+n(x^2-x+1)\}=0$.

93. $x=5$ is obviously one solution. 94. $x=6$ is obviously one solution. 95. $x=5$ is obviously one solution.

96. $x=0$ is obviously one solution. 97. $(x^2-4)(x+1)=0$.

98. $x=a$ is obviously one solution.

99. $8x^3-1+8(2x-1)=0$; therefore $x=\frac{1}{2}$ is one solution.

100. $x^2-\frac{4}{9}=\frac{1}{x}\left(x+\frac{2}{3}\right)$; therefore $x=-\frac{2}{3}$ is one solution.

101. $x^2=1$ is obviously a solution.

102. $x=-m$ is obviously a solution. 103. $x=a, b$, or $-(a+b)$.

104. $x+p-1$ is a factor. 105. $x(p-1)+1$ is a factor.

XXII. 1. $3(x-5)\left(x+\frac{5}{3}\right)$. 2. $(x+60)(x+13)$.

3. $2(x+2)\left(x-\frac{3}{2}\right)$. 4. $(x-62)(x-26)$. 5. $x^2-14x+48=0$.

6. $x^2-9x+20=0$. 7. $x^2+x-2=0$. 8. $x^2-2x-4=0$.

9. 42, 36, 117. 10. $m=8$. 11. $\frac{p^2-2q}{q}$, $p(p^2-3q)$.

12. $cx^2+bx+a=0$.

XXIII. 1. $x=\pm 3$; $y=\pm 4$. 2. $x=60, 40$; $y=40, 60$.

3. $x=2$; $y=2$. 4. $x=4, \frac{16}{3}$; $y=3, \frac{5}{3}$. 5. $x=7, 5$; $y=-5, -7$.

6. $x=2, 5$; $y=6, 3$. 7. $x=\pm 7, \pm 4$; $y=\pm 4, \pm 7$.

8. $x=-1, \frac{5}{3}$; $y=-1, \frac{3}{5}$. 9. $x=1$; $y=1$.

10. $x=\pm 3, \mp 8$; $y=\pm 5$. 11. $x=5, \frac{333}{28}$; $y=9, \frac{370}{84}$.

12. $x=\pm 3, \pm 36$; $y=\pm 5, \mp \frac{23}{2}$. 13. $x=\pm 3, \pm \frac{5}{\sqrt{2}}$; $y=\pm 2, \pm \frac{1}{\sqrt{2}}$.

14. $x=\pm 2, \pm \sqrt{\frac{2}{5}}$; $y=\pm \frac{1}{2}, \mp 2\sqrt{\frac{2}{5}}$.

15. $x=\pm 3, \pm \frac{8}{\sqrt{6}}$; $y=\pm 1, \pm \frac{1}{\sqrt{6}}$. 16. $x=\pm 4, \pm 3\sqrt{3}$; $y=\pm 5, \pm \sqrt{3}$.

17. $x=\pm \frac{15}{\sqrt{21}}$, $y=\pm \frac{3}{\sqrt{21}}$. 18. $x=3, -\frac{53}{27}$; $y=-4, \frac{227}{27}$.

19. $x=\pm \sqrt{\frac{5}{2}}$; $y=2 \mp \sqrt{\frac{5}{2}}$. 20. $x=\pm 6$, $y=\pm 3, \mp 3$.

21. $x=\pm 3\sqrt{2}$; $y=\pm \sqrt{2}, \mp \sqrt{2}$. 22. $x=0, 4$; $y=0, 5$.

23. $x=0, -1$; $y=0, -\frac{12}{5}$. 24. $x=0, 15$; $y=0, 45$.

25. $x=0, 2, \pm \sqrt{2}$; $y=0, 2, 2 \mp \sqrt{2}$. 26. $x=0, 4, -2$; $y=0, 2, -4$.

27. $x=5, \frac{21}{5}$; $y=3, \frac{7}{5}$. 28. $x=4, 2$; $y=2, 4$.

29. $x=2, 0$; $y=0, -2$. 30. $x=1, 4$; $y=4, 1$.

31. $x=1, 10$; $y=10, 1$. 32. $x=3, 2$; $y=2, 3$.

33. $x=8, 4$; $y=4, 8$. 34. $x=17, 1$; $y=1, 17$.

35. $x=4, 2, -1 \pm \sqrt{\frac{11}{3}}$; $y=2, 4, -1 \mp \sqrt{\frac{11}{3}}$.

36. $x=4$; $y=1$. 37. $x=1, 4$; $y=4, 1$.

38. $x=2, 3$; $y=3, 2$. 39. $x=\pm 2, y=\pm 2$; or $x=\pm 2, y=\mp 2$.

40. $x=3, y=1$; $x=1, y=3$. 41. $x=5, -2$; $y=2, -5$.

42. $x=\pm 2, \pm 1$; $y=\pm 1, \pm 2$. 43. $x=\frac{1}{4}(9 \pm \sqrt{73})$, $y=\frac{1}{4}(9 \mp \sqrt{73})$.

44. $x=\pm 3, \pm 2$; $y=\pm 2, \pm 3$. 45. $x=\pm 5, \pm 3$; $y=\pm 3, \pm 5$.

46. $x = \pm 3, \pm 2$; $y = \pm 2, \pm 3$. 47. The first equation may be written thus, $xy(y+x-3) = 3(4x+y-xy)$; combine this with the second equation $x = \pm\sqrt{(-3)}, \pm\sqrt{3}$; $y = 3 \mp \sqrt{(-3)}, \pm 2\sqrt{3}$.
48. $x = 8, 2$; $y = 2, 8$. 49. $x = 9, 4$; $y = 4, 9$.
50. $x = 8, 64$; $y = 64, 8$. 51. $x = 5, 13$; $y = 4, 12$.
52. $x = 4, 9$; $y = 9, 4$. 53. $x = 2, 8$; $y = 8, 2$.
54. $\sqrt{x} = 2 \pm \sqrt{6}, \frac{1}{2}\{\pm\sqrt{(15)} - 5\}$; $\sqrt{y} = -2 \pm \sqrt{6}, \frac{1}{2}\{\pm\sqrt{(15)} + 5\}$.
55. $x = 5, y = 3$. 56. $x = \pm 1, y = 3$. 57. $x = \frac{a}{2}, y = \frac{b}{2}$.
58. $x^2 = \frac{1}{2}\{a^2 \pm \sqrt{(a^4 + 4b^4)}\}$; $y^2 = \frac{1}{2}\{-a^2 \pm \sqrt{(a^4 + 4b^4)}\}$.
59. $xy = \frac{1}{2}\{2a^2 \pm \sqrt{(2a^4 + 2b^4)}\}$; whence we may proceed.
60. $x = \frac{a}{2}\{1 \pm \sqrt{3}\}, \frac{a}{2}\left\{1 \pm \frac{1}{\sqrt{3}}\right\}$; $y = \frac{a}{2}\{1 \mp \sqrt{3}\}, \frac{a}{2}\left\{1 \mp \frac{1}{\sqrt{3}}\right\}$.
61. $x = a, b$; $y = b, a$. 62. $x = a, \frac{3ab - a^2}{a+b}$; $y = b, \frac{3ab - b^2}{a+b}$.
63. Proceed as on page 199. 64. $x^2 = \pm\frac{5a^2}{3}, \pm a^2$; $y^2 = \frac{4a^2}{3}, 0$.
65. $x = 0, 2(a+b)$; $y = 0, 2ab$. 66. $4axy = (1-xy)^2$; this gives a quadratic in xy. 67. $\frac{x+y}{x-y} = \frac{ay}{c}$, thus $x = \frac{(ay+c)y}{ay-c}$, &c.
68. $x = \frac{a^3 \pm b(2b^2 - a^2)}{4(a^2 - b^2)}$; $y^2 = ax - \frac{a^2}{4}$. 69. $x^2 = b^2\{2 \pm \sqrt{3}\}$; $y^2 = a^2\{2 \mp \sqrt{3}\}$.
70. Add; thus $x^2(x-1)^2 + y^2(y-1)^2 = a + b$; also the first given equation may be written $x(x-1) + y(y-1) = a$; thus we get $x(x-1) = \frac{1}{2}\{a \pm \sqrt{(2a + 2b - a^2)}\}$; $y(y-1) = \frac{1}{2}\{a \mp \sqrt{(2a + 2b - a^2)}\}$.
71. $x = 0, 2a$; $y = b, -b$; $z = c, -c$.
72. $x = \frac{1}{2}, \frac{5}{26}$; $y = \frac{1}{3}, \frac{15}{13}$; $z = \frac{1}{4}, \frac{15}{44}$. 73. Three simple equations for finding xy, yz, zx. 74. Three simple equations for finding $\frac{1}{xy}, \frac{1}{yz}, \frac{1}{zx}$; also x, y, and z may each $= 0$.
75. From the first and second equations by subtraction $x = y$ or $x + y = z$; then use the third equation to complete the solution. We shall thus obtain $x = y = \pm\frac{1}{2}\{2c + a \pm \sqrt{(a^2 + 4ac - 4c^2)}\}^{\frac{1}{2}}$, and $z = \{2c - a \mp \sqrt{(a^2 + 4ac - 4c^2)}\} \div 4x$; or $x\,2\sqrt{2} = \sqrt{(a+c)} + \sqrt{(5c - 3a)}$, $y\,2\sqrt{2} = \sqrt{(a+c)} - \sqrt{(5c - 3a)}$, $z\sqrt{2} = \sqrt{(a+c)}$.

76. Form a quadratic in z; then $z = 6$ or $-\frac{5}{2}$; with the first value we get $x = 4$ and $y = 5$; with the second $x = \frac{355}{42}$, $y = \frac{190}{21}$.

77. By eliminating z we get $x + y + \frac{1}{xy} = \frac{7}{2}$ and $xy + \frac{x+y}{xy} = \frac{7}{2}$; therefore $(x+y)\left(1 - \frac{1}{xy}\right) = \frac{x^2y^2 - 1}{xy}$, &c. 2, 1, $\frac{1}{2}$ are the values of x, y, z; these values may be arranged in six ways.

78. We may deduce $xyz = 0$; thus one or more of the three x, y, z must be zero. The results are 0, 0, 1, which may be arranged in three ways. 79. $x = a^2 \div \pm \sqrt{(a^2 + b^2 + c^2)}$.

80. Form a quadratic in $x + y + z$ which gives 9 for one value, this leads to a cubic in xy, of which the roots may be seen to be 6, 8, 12; hence for the values of x, y, z we get 2, 3, 4, which may be arranged in six ways.

XXIV. 1. 15 and 24. 2. 3.4.5; that is, 60. 3. 120 and 121 yards. 4. Five miles per hour. 5. 66 on one side, 22 on the other. 6. 28 acres. 7. 14. 8. $\frac{a}{4}(1 + \sqrt{5})$ is the produced part; a being the given line. 9. 50 and 15. 10. 18. 11. Ninepence. 12. 30 Austrian; 36 Bavarian. 13. 5 and 4. 14. The first worked 24 days at 4*s.* per day; the second 18 days at 3*s.* per day. 15. 15 persons; each spent 5 shillings. 16. 100 shares at £15 each. 17. $x^2 + x^3 = 9(x+1)$; therefore $x^2 = 9$; the number is 3. 18. 7 per cent. and 6 per cent. 19. Rate of train is $\frac{7}{2}$ that of coach; 14 miles. 20. *A* 40 hours; *B* 60 hours. 21. 70 miles. 22. 150 miles. 23. 5 hours and 3 hours. 24. 15 hours and 10 hours. 25. 36 workmen, and each carried 77 lbs. at a time; or 28 workmen, and each carried 45 lbs. at a time.

XXV. 1. 1. 4. The expression $= \frac{abc(3abc - a^3 - b^3 - c^3)}{(2a^2 + bc)(2b^2 + ca)(2c^2 + ab)}$; then see Art. 55. 6. $1 + x^{\frac{1}{2}} + x^{\frac{3}{2}} - x^2$. 7. $\frac{1}{\sqrt{2}}\{\sqrt{(a+b)} + \sqrt{(a-b)}\}$.

8. $\frac{a}{2}\{\sqrt{(1+n+n^2)}+\sqrt{(1-n+n^2)}\}$. 9. $x=10$.

12. We get by working out $(b^2xx'+a^2yy'-a^2b^2)^2+a^2b^2(xy'-yx')^2=0$.

13. £30. 14. 2, 5, 9. 16. $x=0,\ \frac{5}{3}$. 17. $x=1\pm\sqrt{2},\ 1\pm\sqrt{(-1)}$.

18. $x=1,\ 2,\ 3,\ \frac{1}{12}\{-11\pm\sqrt{(-23)}\}$. 19. $x=3\pm\sqrt{5},\ 1\pm\sqrt{5}$.

20. $\sqrt{(2x-1)}-\sqrt{(5x-4)}=\sqrt{(4x-3)}-\sqrt{(3x-2)}$; then square; $x=1$.

21. $x-a+4c\sqrt{(x-a)}+4c^2=x+a-4b\sqrt{(x+a)}+4b^2$; extract the square root; $x=(c\pm b)^2+\frac{a^2}{4(c\pm b)^2}$. 22. $nx=n(x+a-a)$; divide by $\sqrt{(x+a)}-\sqrt{a}$; $x=0,\ \frac{4an(1-n^2)}{(1+n^2)^2}$. 23. $x=a,\ \frac{1}{2}(a+b)$; $y=b,\ \frac{1}{2}(a+b)$.

24. $x=\frac{ac}{a+b}$; $y=\frac{bc}{a+b}$. 25. $x=3,\ \frac{2}{5}$; $y=2,\ -\frac{3}{5}$.

26. $2x=a+c-b\pm\sqrt{(a^2+b^2+c^2-2bc-2ca-2ab)}$; $x+y=c$. Also $x=\sqrt{(ac)}$, $y=\sqrt{(bc)}$. 27. $x=2,\ y=\frac{1}{3},\ z=1$.

28. Add the four equations; thus $(v+x+y+z)^2=4(a+b+c)$, and from this and the first given equation $(v+x-y-z)^2=8a$;

$$2v=\pm\sqrt{(a+b+c)}\pm\sqrt{(2a)}\pm\sqrt{(2b)}\pm\sqrt{(2c)}.$$

XXVI. 1. 4 : 9; 10 : 12. 2. 7 : 15. 3. 18 and 27.

5. Short road from A to B is 26 miles; from B to C 52 miles.

6. Either $xa=yb=zc=\frac{2abc}{bc+ca+ab}$; or else $xa+yb+zc=0$ and $x+y+z=-1$. 11. $x=6,\ y=8,\ z=10$. 12. $x=\pm a(b^2-c^2)$, $y=\pm b(c^2-a^2)$, $z=\pm c(a^2-b^2)$; also x, y, and z may each $=0$.

XXVII. 1. 3. 2. 6400. 3. 57. 4. $\frac{2\,.\,8\,.\,32}{xy}$. 9. Suppose $ad=bc$; then $a+d-(b+c)=a-b-\left(c-\frac{bc}{a}\right)=\frac{(a-b)(a-c)}{a}$.

10. In the first the wine is $\frac{1}{3}$ of the whole; in the second $\frac{2}{3}$.

11. A has £72 and B has £96; each stakes $\frac{5}{12}$ of his money.

12. Female criminals four-fifths of the male.

XXVIII. 1. 9. 2. $a=5b$. 3. 4. 4. 1. 6. $\frac{3}{4}$. 7. 10. 8. $27x^2=4y^3$. 9. $y=2x+\frac{2}{x}$. 10. 16. 13. 10. 14. $(r^3+r'^3)^{\frac{1}{3}}$.
15. We have $y+z-x=A$, $(x+y-z)(x+z-y)=Byz$; thus $x^2-(y-z)^2=Byz$, therefore $x^2-(y+z)^2=(B-4)yz$, therefore $(x-y-z)(x+y+z)=(B-4)yz$, or $-A(x+y+z)=(B-4)yz$.
16. $2(n-1)$ hours. 18. 4 hours.

XXIX. 1. 1022634. 2. 321420111. 3. 3015333. 4. 20846*t*. 5. 209. 6. 624. 7. 2223. 8. 15*t*1. 9. 1105*t*. 10. 22441. 11. 17·6. 12. 75346·1. 13. 1341·111. 14. 124·96. 15. 1099·39. 16. 1589·349609375. 17. 450, 1214; product 613260. 18. 3483. 19. 152. 20. 11111. 21. 44·4; in scale 3 it is 1001·2. 22. 62444261; square root is 7071. 23. 1101111. 24. 8*te*7. 25. ·739. 26. Eight. 27. Six. 28. Eleven. 29. Five. 30. Six. 31. Five. 35. $2^{10}+2^9+2^7+2^5+2^4+2^2+2+1$. 36. $3^6+3^5+3^4-3^3+1$. 37. 3^6-3^2-3-1. 38. $3^6-3^5-3^2-3+1$. 39. Three feet eleven inches. 40. Twenty-three inches and a third. 43. r^n-1 and r^{n-1}; r being the radix and n the given number. 45. The number is one hundred and twenty.

XXX. 1. 800. 2. 4. 3. -333. 4. $-26\frac{2}{3}$. 5. -2. 6. $61\frac{1}{2}$. 7. 5. 8. 425. 9. 0. 10. $n(8+n)$. 11. $\frac{n(13-n)}{12}$. 12. Common difference -3. 13. 9. 14. 4 or -11. 15. $2n-1$. 16. Number of terms is 10 or 12; last term 3 or -1. 17. Common difference 7. 18. 5, 9, 13, 17, 21, 25. 20. $\frac{n}{2}\{2+4(n-1)\}$ that is $n(2n-1)$. 21. 1111. 22. 20. 23. $\frac{1}{3}(n-1)n(2n-1)$ yards. 24. 1, 1334. 25. Nine means, 3, 5, 7, ... 19. 26. Number of terms 19 or -2. 27. 5 or -10. 28. 4 or 7. 30. The number of terms is $m+n-1$ or $m+n$; in the former case the last term is 1; in the latter case the last term is zero. 31. 4 or 9. 32. $p+q+(m-1)2q$. 37. 17. 38. 100 or -107.

39. Number of terms 7; middle term 11. 41. n^2. 42. $-n(-1)^n$.
43. $\frac{1}{4}\{1-(2n+1)(-1)^n\}$. 46. 9. 47. $\frac{1}{3}n(n+1)(n+2)$.
50. $\frac{n}{4}(19-n)$. 51. $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}$. 52. 25 months.
53. 15. 54. $\frac{2mr}{r+1}$ hours. 55. 468.

XXXI. 1. $\frac{12}{5}\left\{\left(\frac{5}{3}\right)^6-1\right\}$. 2. $-\frac{2}{3}\{2^{10}-1\}$. 3. $9\left\{1-\left(\frac{2}{3}\right)^n\right\}$.
4. $\frac{8}{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}$. 5. 2. 6. $\frac{16}{3}$. 7. 1. 8. 9. 9. 10.
10. $\frac{4}{3}$. 11. $\frac{50}{11}$. 12. $\frac{2}{3}$. 13. $\frac{27}{26}$. 14. $\frac{1}{6}$. 15. $\frac{1}{3}$.
16. $4+3\sqrt{2}$. 17. $\frac{25}{24}\left(\frac{2}{5}+\frac{3}{5^2}\right)$. 18. $\frac{r-nr^{n+1}}{1-r}+\frac{r^2(1-r^{n-1})}{(1-r)^2}$.
19. $4-(n+2)\,2^{-n+1}$. 20. $6-(2n+3)\,2^{-n+1}$.
21. $\frac{1}{9}\left\{2+(-1)^{n-1}\frac{6n+1}{2^{n-1}}\right\}$. 23. 81. 24. £108, £144, £192, £256.
25. $\frac{a^4}{a^4+1}\{a^{4n}(-1)^n-1\}$. 28. £3. 4*s*. 32. Common ratio $\frac{1}{10+1}$.
33. $\frac{ar(r^n-1)}{(r-1)^2}-\frac{na}{r-1}$. 38. $r=2, a=3$; r is found by an easy cubic.
39. $\frac{r^{2n}-1}{r^2-1}\left(r^2+\frac{1}{r^{2n}}\right)-2n$. 40. $\frac{50}{81}(10^n-1)-\frac{5n}{9}$. 42. 2, 4, 8, 12;
or $\frac{25}{2}, \frac{15}{2}, \frac{9}{2}, \frac{3}{2}$. 43. 2, 5, 8. 45. $\frac{ar}{(1-r)(1-br)}$.

XXXII. 1. $\frac{6}{11}, \frac{3}{7}$. 2. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots \frac{1}{19}$. 3. Let p denote it, then $\frac{1}{p}=\frac{1}{a}+(n-1)\left(\frac{1}{b}-\frac{1}{a}\right)$. 4. $\frac{PQ(p-q)}{pQ-qP}$.
8. 2 and 4. 11. 2, 3, 6. 12. The terms are $\frac{2}{13}$ and $\frac{1}{8}$; then the series can be continued. 14. We may shew that $A=\frac{a^2}{2a-b}$ and $G=\frac{ab-a^2}{2a-b}$; as A and G are thus known in terms

of a and b, we can find the two quantities in terms of a and b. 19. a^2+ab, a^2-b^2, a^2-ab. 20. The common difference in the arithmetical progression formed by the reciprocals is $\frac{2}{n-1}$.

XXXIII. 1. 1341·132$\dot{3}$. 7. 36 miles. 8. 64 gallons. 9. A £100; B £80.

XXXIV. 1. 1120. 2. 453600. 3. 454053600.

4. 34650. 5. 6. 6. $\frac{\underline{|10}}{\underline{|2}\,\underline{|3}\,\underline{|5}}$. 7. $\frac{20.19}{1.2}$, $\frac{19.18}{1.2}$.

8. $\frac{\underline{|95}}{\underline{|9}\,\underline{|86}}$, $\frac{\underline{|95}}{\underline{|10}\,\underline{|85}}$. 9. $\frac{\underline{|60}}{\underline{|12}\,\underline{|48}}$. 10. $2r$. 11. $\frac{\underline{|5}}{2}$.

12. Suppose one person to remain fixed, and all possible permutations formed of the other $n-1$ persons. This gives $\underline{|n-1}$ as the number of ways. But this counts as different ways a pair of cases in which each person has the same neighbours, but the right-hand neighbour of one case becomes the left-hand neighbour of the other, and *vice versa*. If such a pair of cases is counted as only one case, we must divide our former result by 2. For example, if there are three persons, there is only one way of arranging them, in the latter view. 13. $\underline{|9}$, $\underline{|10}-\underline{|9}$.

14. $\frac{12.11.10}{\underline{|3}} \times \frac{16.15.14.13}{\underline{|4}}$. 15. If there is only one thing, it may be given away in n ways; then as a second thing may be given away in n ways, there are n^2 ways of giving away two things; and so on. 16. $n=2r+1$; $r=8$.

17. $\frac{\underline{|m}}{\underline{|r}\,\underline{|m-r}} \times \frac{\underline{|n}}{\underline{|s}\,\underline{|n-s}} \times \underline{|s+r}$. Or if the m things are exactly alike, and also the n things, $\frac{\underline{|s+r}}{\underline{|r}\,\underline{|s}}$. 18. $\frac{n(n-1)(n-2)}{\underline{|3}}$.

20. 4080. 21. 86400. 22. $\underline{|5}\times\underline{|3}$; if however the *three* letters are to retain an invariable order, the answer is $\underline{|5}$. 23. 10.9.8.7 − 9.8.7 with 4 flags; 10.9.8 − 9.8 with 3 flags; 10.9 − 9 with 2 flags; 10 with 1 flag. 5265 signals in all.

24. 90. 25. 36. 26. $3 \times \underline{|4} \times \underline{|4}$. 27. n^m. 28. $\dfrac{\underline{|52}}{\{\underline{|13}\}^4}$. 29. 120.

30. $\dfrac{n(n-1)}{1.2} - \dfrac{p(p-1)}{1.2} + 1$. 31. $\dfrac{n(n-1)(n-2)}{\underline{|3}} - \dfrac{p(p-1)(p-2)}{\underline{|3}}$.

32. Increase the preceding result by unity. 34. $\dfrac{\underline{|24}}{\{\underline{|3}\}^2\{\underline{|2}\}^5}$. 35. $\underline{|7}$; if however each set may be in order, either from left to right, or from right to left, the answer is $8 \times \underline{|7}$. 36. I. 8.7.6.5 cases without repetition. II. $\dfrac{7.6}{1.2} \times \dfrac{\underline{|4}}{2}$ cases in which *a* occurs twice; also as many in which *i* occurs twice; and as many in which *n* occurs twice. III. $\dfrac{\underline{|4}}{\underline{|2}\,\underline{|2}}$ cases in which *a* and *i* each occur twice; also as many in which *i* and *n* each occur twice; and as many in which *a* and *n* each occur twice. Total 2454. 37. 53. 39. $\underline{|4} \times 11111 \times 15$.

XXXV. 1. $\dfrac{15.14}{1.2} a^{13} b^2$. 2. $\dfrac{50.49}{1.2} x^{48} a^2$. 3. $\dfrac{12.11.10.9}{\underline{|4}} a^{16} b^8$.

4. $\dfrac{2002.2001}{1.2} a^{\frac{6}{10}} x^{600}$. 5. $625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$.

6. $\dfrac{9.8.7.6}{\underline{|4}} 3^5 x^{\frac{5}{2}} 4^4 y^2$. 7. $-\dfrac{10.9.8.7.6}{\underline{|5}} 2^5 a^{\frac{5}{2}} b^{\frac{15}{2}}$. 9. $\dfrac{\underline{|10}}{\underline{|5}\,\underline{|5}} a^5 x^5$.

10. $\dfrac{\underline{|9}}{\underline{|4}\,\underline{|5}}(a^5x^4 + a^4x^5)$. 11. $64a^6 - 96a^4 + 36a^2 - 2$. 12. $10c^9$.

14. This follows directly; or thus, $(1+x)^{n+1}(1-x) = (1+x)^n(1-x^2)$.

16. From 2nd to 5th terms of $(3+2)^6$. 18. $\dfrac{\underline{|2n+1}}{\underline{|n-r}\,\underline{|n+r+1}}(-1)^{n-r}$.

19. $\dfrac{\underline{|2n}(-1)^{r-1} x^{2n-2r+2}}{\underline{|r-1}\,\underline{|2n-r+1}}$, $\dfrac{\underline{|2n}(-1)^{r-1} x^{2r-2n-2}}{\underline{|r-1}\,\underline{|2n-r+1}}$; $\dfrac{(-1)^n \underline{|2n}}{\underline{|n}\,\underline{|n}}$.

20. $(x^2 + a^2)^n = \{x + a\sqrt{(-1)}\}^n \{x - a\sqrt{(-1)}\}^n$

$$= \{A + B\sqrt{(-1)}\}\{A - B\sqrt{(-1)}\} = A^2 + B^2.$$

XXXVI. 13. $\frac{(r+1)(r+2)(r+3)}{\lfloor 3} x^r$.

14. $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4} x^r$.

15. $-\frac{(n-1)(2n-1)(3n-1)\ldots\ldots\{(r-1)n-1\}}{n^r \lfloor r} x^r$.

16. $-\frac{(p-1)(2p-1)(3p-1)\ldots\ldots\{(r-1)p-1\}}{\lfloor r} x^r$.

17. $\frac{1.3.5\ldots\ldots(2r-1)}{\lfloor r\, 2^r}(-1)^r x^r$. 18. $\frac{2.5.8\ldots\ldots(3r-1)}{\lfloor r\, 3^r} x^{3r}$.

19. $\frac{7.9.11\ldots\ldots(2r+5)}{\lfloor r} x^r$. 20. $\frac{1.5.9\ldots\ldots(4r-3)}{4^r \lfloor r} x^r$.

28. 2nd and 3rd terms $\frac{4}{1}\times\frac{2}{3}=\frac{8}{3}$. 29. 3rd term $=\frac{12.13}{1.2}\frac{1}{5^2}=\frac{78}{25}$.

30. 5th and 6th terms $=\frac{3.4.5.6}{\lfloor 4}\left(\frac{5}{7}\right)^4=\frac{9375}{2401}$.

31. 3rd term $=\frac{8.11}{3.6}\left(\frac{7}{12}\right)^2$. 32. If $n=1$ the 2nd and 3rd terms are the greatest; if $n=2$ the 2nd term is the greatest; and for all other values of n the first term is the greatest.

33. $\frac{11.12.13}{\lfloor 3}$. 34. Sixth term. 37. $\frac{n+1}{3}(2n^2+4n+3)$.

38. Coefficient of x^{2r} is $\frac{1.3.5\ldots\ldots(2r-1)}{2^r a^{2r}\lfloor r}$; coefficient of x^{2r+1} is obtained by dividing this expression by a. 41. $\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}$, that is, $\sqrt{2}$. 42. $\frac{\lfloor 2n}{\lfloor n \lfloor n}$. 45. $\frac{2n(2n+1)\ldots\ldots(2n+r-1)}{\lfloor r}$.

XXXVII. 1. 6. 2. -16. 3. $2^6.3^2+2^7.3+2^5.3^3+3^4=1905$.
4. 3. 5. $-2^9 5+2^6.3^3.5-2^2 3^4 5$.

6. $\lfloor 12\left\{\frac{2^4}{\lfloor 8\lfloor 4}+\frac{2^3}{\lfloor 7\lfloor 2\lfloor 3}+\frac{2^2}{\lfloor 6\lfloor 4\lfloor 2}+\frac{2}{\lfloor 5\lfloor 6}+\frac{1}{\lfloor 4\lfloor 8}\right\}$.

7. $2^4.5.7^2-2^3.3.5^3.7+2.5^5$. 8. -64. 9. -20.

10. $-\frac{15}{8}-\frac{35}{4}-\frac{63}{8}=-\frac{37}{2}$. 11. $-\frac{1}{4}$. 12. $-3+6+15+\frac{35}{8}$.

13. $\left(\frac{3.7}{2^5}-\frac{7.11.19}{2^{10}}\right)$. 14. 50. 15. $\frac{n^4+6n^3-13n^2+6n}{24}$.

16. The expression is $\{(1+x)(1-x)^{-2}\}^7$. Hence the coefficient is

$$\frac{7.6.5.4}{\lfloor 4}+\frac{7.6.5}{\lfloor 3}\cdot\frac{14}{1}+\frac{7.6}{1.2}\cdot\frac{14.15}{1.2}+\frac{7}{1}\cdot\frac{14.15.16}{\lfloor 3}+\frac{14.15.16.17}{\lfloor 4}.$$

17. $m+1$. 18. $\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}3^4+\frac{n(n-1)\ldots(n-4)}{\lfloor 2\lfloor 3}2^2 3^3$

$$+\frac{n(n-1)\ldots(n-5)}{\lfloor 4\lfloor 2}2^4 3^2+\frac{n(n-1)\ldots(n-6)}{\lfloor 6}2^6 3+\frac{n(n-1)\ldots(n-7)}{\lfloor 8}2^8.$$

19. 0. 20. $5a_2^4+20a_1a_2a_3^2+10a_2^2a_3^2$.

21. $\frac{n(n-1)(n-2)}{1.2}a_0^{n-3}a_1a_2^2+\frac{n(n-1)\ldots\ldots(n-3)}{\lfloor 3}a_0^{n-4}a_1^3a_2$

$$+\frac{n(n-1)\ldots\ldots(n-4)}{\lfloor 5}a_0^{n-5}a_1^5.$$ 22. -23. 23. $-\frac{b}{2}+\frac{3a^2}{8}$.

24. $ma_3+m(m-1)a_1a_2+\frac{m(m-1)(m-2)}{\lfloor 3}a_1^3$. 25. 20.

26. -210. 27. 1260. 28. 12600.

29. $a^n+na^{n-1}(b+c)+\frac{n(n-1)}{1.2}a^{n-2}(b+c)^2+\frac{n(n-1)(n-2)}{\lfloor 3}a^{n-3}(b+c)^3$.

30. $\frac{n(n-1)(n-2)}{\lfloor 3}d^{n-3}(a+b+c)^3$. 31. $\frac{\lfloor 10}{\{\lfloor 3\}^3 4}$. 32. $\frac{\lfloor 14}{\{\lfloor 3\}^4 4^2}$.

35. $1+bx-\left(\frac{1}{2}-\frac{3b^2}{2}\right)x^2-\left(\frac{3b}{2}-\frac{5b^3}{2}\right)x^3+\left(\frac{3}{8}-\frac{15b^2}{4}+\frac{35b^4}{8}\right)x^4$.

36. $a^{-1}-a^{-2}bx-(a^{-2}c-a^{-3}b^2)x^2$

$$+(2a^{-3}bc-a^{-4}b^3)x^3+(a^{-3}c^2-3a^{-4}b^2c+a^{-5}b^4)x^4.$$

37. $1-nx+\frac{n(n-3)}{2}x^2-\frac{n(n-2)(n-7)}{6}x^3$.

38. May be proved by Induction. 39. For the first part put $x=1$. For the second part, let S denote the series, so that $S=a_1+2a_2+3a_3+\ldots+nra_{nr}$; and as the coefficients of terms equidistant from the beginning and the end are equal, by Ex. 38, $S=a_{nr-1}+2a_{nr-2}+\ldots+nra_0$. Then, by addition,

$$2S=nr\{a_0+a_1\ldots+a_{nr}\}=nr(r+1)^n.$$

40. $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n-2}x^{2n-2} + a_{2n-1}x^{2n-1} + a_{2n}x^{2n}$; change the sign of x, and, since the coefficients of terms equidistant from the beginning and the end are equal, we have

$$(1-x+x^2)^n = a_{2n} - a_{2n-1}x + a_{2n-2}x^2 - a_{2n-3}x^3 + \ldots$$

Multiply together, and select the coefficient of x^{2n}; this will therefore be equal to the coefficient of x^{2n} in

$$(1+x+x^2)^n(1-x+x^2)^n, \text{ that is, in } (1+x^2+x^4)^n.$$

Then put a_0^2 for a_{2n}^2, a_1^2 for a_{2n-1}^2, ... and divide both sides by 2.

XXXVIII. 1. 4. 2. 2. 3. 1. 4. 5.
5. 3; -2. 6. $\cdot 698970 - 2$; $\cdot 732393$. 7. $\cdot 778151 - 3$.
10. $\frac{1}{2}\{\log 10 - 3\log 2\}$. 15. 20. 20. About 125 years.

XXXIX. 1. This is an example of equation (1), Art. 545, in which $m = (x+1)(x-1)$ and $n = x^2$.

2. $\log(x+2h)x - \log(x+h)^2 = \log\left\{1 - \frac{h^2}{(x+h)^2}\right\}$. 3. See Ex. 1.

5. $\log(3+3x+x^2)x - 3\log(1+x) = \log\left\{1 - \frac{1}{(1+x)^3}\right\}$. 6. We have to find a series for $\log(x+1) - \frac{4x}{2x+1}\log x + \frac{2x-1}{2x+1}\log(x-1)$, that is, for $\log\left(1+\frac{1}{x}\right) + \frac{2x-1}{2x+1}\log\left(1-\frac{1}{x}\right)$, that is, for

$$\frac{2x}{2x+1}\log\left(1-\frac{1}{x^2}\right) + \frac{1}{2x+1}\log\frac{1+\frac{1}{x}}{1-\frac{1}{x}}.$$

9. $\left(1+\frac{x}{n}\right)^n = e^x\left(1 - \frac{x^2}{2n} + \frac{x^3}{3n^2} \ldots\right)$.

XL. 1. Series $= \frac{1}{a}\left\{\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \text{\&c.}\right\}$; convergent by Art. 558. 2. Divergent if $x>1$, convergent if $x<1$. If $x=1$ the general term is $\frac{2n+1}{n^2+1}$, which is $>\frac{1}{n}$, and the series is divergent.
3. Convergent if $a>1$; divergent if $a<1$. If $a=1$ the series is obviously divergent. 4. Divergent if $x>1$, convergent if $x<1$. If $x=1$ the series is obviously divergent. 5. Same result as Ex. 4.
6. Series $>1 + \frac{1}{1+2} + \frac{1}{1+3} + \frac{1}{1+4}$ &c., and therefore divergent.

7. Divergent if $x>1$, convergent if $x<1$; if $x=1$, obviously divergent. 8. Results the same as in Art. 562. 9. Divergent if $x>1$, convergent if $x<1$; if $x=1$ it is a series discussed in Art. 562. 10. Convergent if $x<1$, divergent if $x>1$; if $x=1$ the results are the same as in Art. 562.

XLI. 2. £900. 3. $\frac{B-A}{A}$. 4. $2\frac{1}{2}$. 5. 40 : 41.
6. Between 48 and 49. 7. Nearly 32.

XLII. 1. 7 years. 2. 120 days.
4. $\frac{x}{R^{-a}}=\frac{y}{R^{-b}}=\frac{z}{R^{-c}}=\frac{\text{the given sum}}{R^{-a}+R^{-b}+R^{-c}}$. 8. Equate the coefficients of x^r in $(1+x)^n=(1+x)^s(1+x)^{n-s}$. 9. Equate the coefficients of x^m in $(1+x)^n=(1+x)^{n-m+1}(x+1)^{m-1}$. 10. It will be found that the whole coefficient of a vanishes, and also the whole coefficient of β.

XLIII. 1. £24. 10*s.* 2. Cent. per cent. 3. 4 per cent.
4. £6400. 5. $3\frac{1}{2}$. 6. £7297·98. 7. £$225\frac{425}{1331}$.
8. $\frac{\log 15-\log 2}{\log 5-\log 4}$ = a little more than 9. 9. $\frac{A}{R^{p-1}}\cdot\frac{1}{R-m}$, where A is the first payment; m must be less than R. 10. e^{-1}.
11. $P\left(\frac{m+1}{m}\right)^n$. 12. $P(1-r)^n$. 13. $A\times 2{\cdot}617238$.

XLIV. 1. $1+\frac{1}{3+}\frac{1}{5+}\frac{1}{7+}\frac{1}{9}$. 2. $\frac{1}{1+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{55}$.
3. $\frac{1}{2+}\frac{1}{4+}\frac{1}{3+}\frac{1}{2+}\frac{1}{1+}\frac{1}{2+}\frac{1}{170}$. 4. $1+\frac{1}{4+}\frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{1+}\frac{1}{3}$.
5. $\frac{3}{1}$, $\frac{22}{7}$, $\frac{355}{113}$. 6. $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{39}{161}$, ...

XLV. 1. $\frac{2}{1}$, $\frac{3}{1}$, $\frac{14}{5}$, $\frac{17}{6}$. 2. $\frac{3}{1}$, $\frac{19}{6}$, $\frac{117}{37}$, $\frac{721}{228}$.
3. $\frac{3}{1}$, $\frac{4}{1}$, $\frac{11}{3}$, $\frac{15}{4}$. 4. $\frac{4}{1}$, $\frac{33}{8}$, $\frac{268}{65}$, $\frac{2177}{528}$.
5. $\frac{4}{1}$, $\frac{9}{2}$, $\frac{13}{3}$, $\frac{48}{11}$. 6. $\frac{5}{1}$, $\frac{51}{10}$, $\frac{515}{101}$, $\frac{5201}{1020}$.

7. $\frac{5}{1}, \frac{26}{5}, \frac{265}{51}, \frac{1351}{260}$. 8. $\frac{6}{1}, \frac{7}{1}, \frac{27}{4}, \frac{34}{5}$.

9. $\frac{7}{1}, \frac{22}{3}, \frac{29}{4}, \frac{51}{7}$. 10. $\frac{10}{1}, \frac{201}{20}, \frac{4030}{401}, \frac{80801}{8040}$.

11. $a+\frac{1}{2a+}\frac{1}{2a+}\frac{1}{2a+}\dots \frac{a}{1}, \frac{2a^2+1}{2a}, \frac{4a^3+3a}{4a^2+1}, \frac{8a^4+8a^2+1}{8a^3+4a}$.

12. $a-1+\frac{1}{1+}\frac{1}{2(a-1)+}\frac{1}{1+}\frac{1}{2(a-1)+}\dots \frac{a-1}{1}, \frac{a}{1}, \frac{2a^2-a-1}{2a-1}, \frac{2a^2-1}{2a}$.

13. $a+\frac{1}{2+}\frac{1}{2a+}\frac{1}{2+}\frac{1}{2a}\dots \frac{a}{1}, \frac{2a+1}{2}, \frac{4a^2+3a}{4a+1}, \frac{8a^2+8a+1}{8a+4}$.

14. $a-1+\frac{1}{2+}\frac{1}{2(a-1)+}\frac{1}{2+}\frac{1}{2(a-1)+}\dots$

$\frac{a-1}{1}, \frac{2a-1}{2}, \frac{4a^2-5a+1}{4a-3}, \frac{8a^2-8a+1}{8a-4}$.

[13 and 14 are connected, because $a^2-a=(a-1)^2+a-1$.]

15. $\frac{256}{71}$. 16. $\frac{1520}{273}$. 18. $\frac{1}{(44)^2}$ and $\frac{1}{2(49)^2}$. 20. $\frac{1}{(240)^2}$ and $\frac{1}{2(2111)^2}$.

21. $\frac{1}{(273)^2}$ and $\frac{1}{2(2885)^2}$. 26. $\frac{1}{2}, \frac{3}{7}, \frac{13}{30}, \frac{42}{97}$. 27. $\frac{485}{396}$. 28. $\frac{211}{80}$.

29. $\frac{1549}{360}, \frac{251}{360}$. 30. $\frac{114}{41}$. 31. $\frac{17}{114}$. 32. $\sqrt{2}$.

33. Positive root of $x^2+2x-2=0$. 34. That of $7x^2-8x-3=0$.

35. That of $7x^2+8x-3=0$. 36. That of $59x^2-319x+431=0$.

XLVI. 1. $x=2,\ y=1$. 2. $x=4,\ y=5$.

3. $x=1$ or 6, $y=20$ or 1. 4. $y=1+7t,\ x=41-10t$.

5. $x=25-7t,\ y=25+3t$. 6. $x=90-19t,\ y=13t$.

7. $x=8,\ y=3$. 8. $x=7,\ y=5$. 9. $x=11,\ y=18$.

10. $x=37,\ y=13$. 11. 4 or 5. 12. 19 or 20.

13. 4, or 5. 14. 2. 15. 16. 16. 5.

17. 3 guineas, 21 half-crowns. 18. 3 sovereigns, 20 francs.

19. 185, 15; 119, 81; 53, 147. 20. 28, 20.

21. When n is even, the common difference is 2; when n is odd, the common difference may be 1 or 2. 22. 245.

23. $104+3.5.7.t$. 24. 97. 25. Ascribe to y successively the values 1, 2,...8; and in each case find the correspond-

ing values of x and z. 26. $x = 1 + 3t$, $y = 51 - 7t$, $z = 63 + 13t$. 27. Allowing a zero, there are 15 solutions; excluding it, there are 14. The solutions are found from $100 - t$ half-crowns, $6t$ shillings, and $100 - 7t$ sixpences. 28. Allowing zeros, 4 solutions; excluding them, 2. The solutions are found from $4 - t$ guineas, $5t$ crowns, and $12 - 4t$ shillings. 29. 6 crowns, 4 half-crowns, 2 florins. 30. 100. 31. 205, 502. 32. 974. 33. 5567. 34. 80 ducks, 19 oxen, 1 sheep; or 100 sheep. 35. $\frac{5}{6}, \frac{8}{9}, \frac{17}{18}$. 36. 49, 43, 38. 37. The 107th and 104th divisions reckoned from either of the common ends. 38. We must solve $5x + 4y + 3z = 20$: the accompanying table exhibits the solutions of this equation. Then we can use (1), (4), (5); or (2), (3), (5); or (3), (4), (4).

x	0	0	1	1	2	4
y	2	5	0	3	1	0
z	4	0	5	1	2	0
	(1)	(2)	(3)	(4)	(5)	(6)

39. £2. 11*s*. 6*d*. 40. £2. 15*s*.

XLVII. 1. $x = 2$, $y = 4$; $x = 3$, $y = 1$. 2. $x = 4$, $y = 21$; $x = 5$, $y = 7$. 3. $x = 18$, $y = 5$. 4. $x = 10$, $y = 1$. 5. 360. 6. 1684 square yards. 7. 10 and 7. 9. $x = 0$, $y = 3$; $x = 2$, $y = 1$. 10. $x = 1$, $y = 3$; $x = 53$, $y = 15$.

XLVIII. 1. $\frac{1}{3}\left(\frac{2x}{3}\right)^n$. 2. $\left\{3(-1)^n - \frac{3^n}{2^{n+1}}\right\}x^n$.

3. $-\left\{\frac{1}{2} - \frac{1}{2^{n-1}} + \frac{7}{2 \cdot 3^{n+1}}\right\}x^n$. 4. $\frac{1-p^n}{1-p}x^n$.

5. $(n+1)x^n$. 6. $(7n+5)(3x)^n$. 7. $(n+1)^2 x^n$.

8. $1 + x - x^3 - x^4 \ldots\ldots$ 9. $1 + 2x + x^2 - 4x^3 - 11x^4 \ldots\ldots$

10. $\frac{1}{2} + \frac{x}{2} + \frac{3x^2}{4} + \frac{x^3}{2} + \frac{7x^4}{8} \ldots\ldots$ 11. $\frac{1}{a^2} - \frac{x}{a^3} + \frac{x^2}{a^5} - \frac{x^4}{a^6} \ldots\ldots$

12. $1 + px + p(p-1)x^2 + (p^3 - 2p^2 + 1)x^3 + p(p^3 - 3p^2 + p + 2)x^4 \ldots\ldots$

13. $\frac{1}{a-1}\left(\frac{1}{1+x} - \frac{1}{1+a^n x}\right)$.

14. $-\frac{1}{(1-a)^2}\left(\frac{1}{1+x} - \frac{1}{1+ax} - \frac{1}{1+a^n x} + \frac{1}{1+a^{n+1}x}\right)$.

15. $a = 1$, $b = 11$, $c = 11$, $d = 1$, $e = 0$.

XLIX. 1. $\dfrac{4-11x}{1-5x+6x^2}$; $(3x)^n+3(2x)^n$.

2. $\dfrac{1+x}{1-10x+21x^2}$; $2(7x)^n-(3x)^n$. 3. $\dfrac{1-2x}{1-5x+4x^2}$; $\frac{1}{3}(1+2^{2n+1})x^n$.

4. x less than $\frac{1}{4}$. 5. $2^{n-1}(5n+6)$. 6. 3^n-n-1.

7. $\dfrac{64}{2^n}-\dfrac{54}{3^n}$; 47. 8. $\dfrac{2+5x+5x^2}{(1+x)^3}$; $(-1)^n x^n(n^2-2n+2)$.

L. 3. $1-\dfrac{1}{1+n}$; 1. 4. $\dfrac{1}{8}\left\{\dfrac{1}{4}-\dfrac{1}{2(n+1)(n+2)}\right\}$; $\dfrac{1}{32}$.

5. $\dfrac{1}{3}\left(\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{n+1}-\dfrac{1}{n+2}-\dfrac{1}{n+3}\right)$; $\dfrac{11}{18}$.

6. $\dfrac{11}{96}-\dfrac{1}{2(n+2)(n+3)}-\dfrac{3}{4(n+1)(n+2)(n+3)(n+4)}$; $\dfrac{11}{96}$.

7. $\dfrac{5}{6}-\dfrac{3n+5}{(n+2)(n+3)}$; $\dfrac{5}{6}$. 8. $\dfrac{n(n+1)(n+2)}{6}$.

11. $\dfrac{x^n\{n(x-1)-1\}^2+x^{n+1}-(x+1)}{(x-1)^3}$. 12. $nar(a+br)^{n-1}$.

13. Expand and we get $\dfrac{x}{(1-x)^2}\left\{1+\dfrac{cx}{(1-x)^2}+\dfrac{c^2x^2}{(1-x)^4}+\dots\dots\right\}$.

14. $b^n\left\{1+na+\dfrac{n(n+1)}{1\,.\,2}a^2+\dots\dots+\dfrac{n(n+1)\dots(n+m-2)}{\lfloor m-1}a^{m-1}\right\}$.

15. $\left(1-\dfrac{2}{3}\right)^{-n}=2^n\left(1-\dfrac{1}{3}\right)^{-n}$. 18. 165. 19. 460.

22. Proceed thus; suppose $(1+xv)(1+x^2v)(1+x^3v)\dots\dots(1+x^pv)$ $=1+A_1v+A_2v^2+\dots\dots+A_pv^p$, where $A_1, A_2, \dots\dots A_p$ do not contain v. *Now change v into xv;* thus we can infer that

$$(1+A_1v+A_2v^2+\dots\dots+A_pv^p)(1+x^{p+1}v)$$
$$=(1+A_1xv+A_2x^2v^2+\dots\dots+A_px^pv^p)(1+xv).$$

Now equate the coefficients of the same powers of v on the two sides.

25. $\dfrac{1+x}{1+x^3}=\dfrac{1}{1-x+x^2}$; therefore $(1+x)\{1-x^3+x^6-x^9+\dots\dots\}$

$$=\frac{1}{1-x}\left(1+\frac{x^2}{1-x}\right)^{-1}=\frac{1}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^4}{(1-x)^3}-\frac{x^6}{(1-x)^4}+\dots\dots$$

Expand each term of the last line by the Binomial Theorem and then equate the coefficients of x^n on the two sides.

LI. 8. $2x^3$ is $>$ or $< x+1$ according as x is $>$ or <1. 16. This depends on the sign of $(a-b)(b-c)(c-a)$. 22 and 24 depend on Art. 681. 23. As many of the following inequalities as may be required will be found to hold: $2(n-1) > n$, $3(n-2) > n$, ...; then by multiplication the result is obtained. 25. This may be deduced from Ex. 23. 29. See Ex. 3 of Chapter XXV. 31. Multiply up; then use Art. 681. 32. Put $1-a=b$, and expand $(1-b)^x$ by the Binomial Theorem; the series will be convergent. We shall then have to shew that $1 - \frac{(x-1)b}{\lfloor 2} + \frac{(x-1)(x-2)b^2}{\lfloor 3} - \ldots\ldots > 1$; and this is obvious, since x is <1.

LII. 2. 66. 3. $3.5^2.41^2$. 4. $2^2.3^2.5^2$. 5. $2^2.(823)^2$. 12. Suppose n to lie between m^2 and $(m+1)^2$; then $n-ab=(m^2+m-n)^2$. 19. n^2-n+1 is greater than $(n-1)^2$ and less than n^2. 20. Suppose, if possible, $n^3+1=m^2$; then $n^3=(m-1)(m+1)$. Now no factor, except 2, can divide both $m-1$ and $m+1$, and 2 cannot here divide them, for n is odd. Hence $m-1$ and $m+1$ must both be perfect cubes; but this is impossible; for the difference of two cubes cannot be so small as 2. 35, 36, 37, 38. These all depend on Fermat's Theorem. 40. 48. 41. 96. 42. 400. 43. 22680. 44. $2^{n+1}5^{n-1}$. 45. 12. 46. 12. 47. 160; 1481040. 48. 6. 49. 126. 50. 24; 15. 51. $(n+1)^2$. 53 and 54 must be solved by trial; the answer to 53 is $2^4.3^2.5$, and the answer to 54 is $2^3.3^3.5.7$. 57. $x=2.5^2.7^2.t^3$; $y=2.5.7.t$.

LIII. 1. 27 to 8 against. 2. $\frac{29}{45}$. 4. $\frac{3}{4}$. 5. $\frac{1}{4}$. 6. $\frac{5}{18}$. 7. $\frac{11}{36}$. 8. 7 to 2. 10. A's chance of losing is $\frac{2}{3}$, and of neither winning nor losing is $\frac{1}{3}$; D's chance of winning is $\frac{2}{3}$, and of neither winning nor losing is $\frac{1}{3}$; B and C have each the chance $\frac{1}{3}$ of winning, $\frac{1}{3}$ of losing, $\frac{1}{3}$ of neither. Or more simply, A's chance of winning is $\frac{1}{6}$, B's and C's $\frac{1}{2}$, and D's $\frac{5}{6}$, if we suppose that one of the boats *must* win. 11. $\frac{5}{9}$. 12. $\frac{2}{145}$. 14. $\frac{3}{14}$. 15. $\frac{1}{2}$.

16. $\frac{n}{2^n}$. 18. $\frac{18586}{(36)^3}$. 19. $\frac{31031}{6^6}$. 20. $\frac{12393}{12500}$. 21. $1-\left(\frac{5}{6}\right)^3$.

22. $1-\left(\frac{35}{36}\right)^3$. 23. $\frac{2551}{7^5}$. 24. $\frac{13^3\times 6}{51\times 50\times 49}$. 25. $\frac{\lfloor 4 . 4}{52.51.50.49}$.

27. $\frac{2072}{5^5}$. 29. $\frac{4}{9}$. 30. $\frac{1}{n}\left(\frac{n-1}{n}\right)^{r-1} \div \left\{1-\left(\frac{n-1}{n}\right)^r\right\}$. 31. $\frac{7}{15}$.

32. The chance of the sovereign being in the first purse is to the chance of its being in the second as 10 is to 9. 33. $\frac{1}{2}$.

34. $\frac{10}{6^3}$. 35. $\frac{3}{6^2}$. 36. $\frac{1}{2}$. 39. $\frac{1}{10^{10}\lfloor 9}\left\{\frac{\lfloor 23}{\lfloor 14} - \frac{10\lfloor 13}{\lfloor 4}\right\}$.

40. $1-\left(\frac{n}{n+1}\right)^m$. 41. ·033. 42. $\frac{1}{60}$.

44. $\frac{1}{7}+\frac{6}{7}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{3}+\frac{6}{7}\cdot\frac{2}{3}\left(\frac{1}{2}\right)^4$. 45. $\frac{\lfloor m \lfloor n}{\lfloor m+n}$ in both cases.

46. $\frac{\lfloor p_1 \lfloor p_2 \lfloor p_3 \ldots\ldots}{\lfloor n}$. 47. $\frac{\lfloor n}{n^n}$. 48. 11 to 5. 49. 6; $\frac{\lfloor 6}{6^6}$. 51. $\frac{16}{35}$.

53. Let *A*'s chance of winning a single game be x, and *B*'s chance $1-x$; then *A*'s chance of winning the set is $\frac{x^2(2-x)}{1-x+x^2}$. 54. $\frac{9}{16}$.

55. $p_1+p_2+p_3-p_1p_2-p_2p_3-p_3p_1+p_1p_2p_3$; $p_1p_2+p_2p_3+p_3p_1-2p_1p_2p_3$.

56. $\frac{64}{169}$, $\frac{56}{169}$, $\frac{49}{169}$. 57. $(·55)^7$. 58. $\frac{30}{61}$ and $\frac{31}{61}$.

59. 21 shillings. 60. 42 shillings. 61. £400. 62. 35*s*. 8*d*.

63. £10. 64. A florin. 65. 3 florins, 1 sovereign.

66. 2 to 1; $\frac{1}{3}$ of what each stakes. 67. $\frac{2r+1}{3}$. 68. $\frac{3r(r+1)}{2(2r+1)}$.

69. 33333 shillings. 70. $\frac{b}{a}n$ shillings. 71. $\frac{8}{11}$. 72. $\frac{1}{2}$.

73. $\frac{2}{n(n+1)}$. 74. $\frac{3}{5}$. 75. $\frac{1265}{1286}$; £$\frac{5087}{5144}$. 76. £$\frac{910}{46}$.

77. $\frac{2}{3}$. 80. $\frac{11}{50}$. 81. $\frac{ab+ac+bc}{(a+c)(b+c)}$. 82. 4.

LIV. 1. $\sqrt{(1-x^4)}=-1\pm\sqrt{3}$. 2. Substitute for x^2 from the first equation in the second; thus we obtain either $y^2=bn$ or $x=\dfrac{ay}{y-b}$.
4. Square; and put the equation in the form $(x^2-4x)^2=24(x-1)^2$.
5. $c=220$. 6. Multiply up in the given relation.
7. $\left(\dfrac{N}{n}\right)^{\frac{1}{2}}=\left\{\dfrac{(N+n)^2}{4n^2}-\dfrac{(N-n)^2}{4n^2}\right\}^{\frac{1}{2}}$; and $=\left\{\dfrac{(N+n)^2}{4N^2}-\dfrac{(N-n)^2}{4N^2}\right\}^{-\frac{1}{2}}$.
9. Equate the coefficient of x^n in the expansion of $\dfrac{1}{1-x+cx^2}$, and in the expansion of the *partial fractions* into which this expression may be decomposed.

LV. 1. $\{\sqrt{(m^2+n^2)}\sqrt{(a^2+b^2)}-na\}^2$. 2. $1+\sqrt{\dfrac{3}{2}}+\sqrt{\dfrac{7}{2}}$.
3. 8. 5. 5. 6. $x=26t$; $y=495-21t$. 7. $1-\dfrac{(1-z)z^{p-1}}{1-z^p}$, where $p=2^n$. 8. $(1-x^2)^2+x^2(1-x)^2$ is never negative.
12. $-\log n=\log\dfrac{1}{2}\cdot\dfrac{2}{3}\cdot\dfrac{3}{4}\cdots\cdots\dfrac{n-1}{n}$. Hence we may regard the general term of the series as $\dfrac{1}{n}+\log\left(1-\dfrac{1}{n}\right)$; and by expanding $\log\left(1-\dfrac{1}{n}\right)$ the general term is found to be numerically less than $-\dfrac{1}{n^2}$. Then see Art. 562. 14. If he draws again from the *same* bag, his chance of getting a sovereign is $\frac{2}{7}$, and his chance of getting a shilling is $\frac{5}{7}$; thus his expectation is $\frac{45}{7}$ shillings. If he draws from the *other* bag, his chance of getting a sovereign is $\frac{4}{7}$, and his chance of getting a shilling is $\frac{3}{7}$; thus his expectation is $\frac{83}{7}$ shillings. 16. $\dfrac{(n-1)R-n+R^{1-n}}{n(R-1)^2}$, where R is the amount of one pound in one year.

LVI. 6. Convergent if x is less than unity, divergent if x is greater than unity; if x is equal to unity, convergent if a is negative, divergent if a is positive. 8. Divergent if x is greater than unity, convergent if x is not greater than unity.

9. Divergent. 10. Convergent if $q-p-1$ is positive, divergent if $q-p-1$ is negative or zero. 13. Convergent if x is less than e^{-1}, divergent if x is not less than e^{-1}. 16. I. Suppose $a-A$ positive: the series is convergent if $\beta+1$ is greater than a, divergent if $\beta+1$ is less than a; if $\beta+1=a$ the series is convergent if $a-A$ is greater than unity, divergent if $a-A$ is not greater than unity. II. Suppose $a-A$ negative: the series is divergent. III. Suppose $a-A=0$; then apply Art. 767, and discriminate as in Case I.

LVII. 4. $$p_n = ba^{n-1} + (n-2)b^2a^{n-3} + \frac{(n-3)(n-4)}{\lfloor 2}b^3a^{n-5} + \frac{(n-4)(n-5)(n-6)}{\lfloor 3}b^4a^{n-7} + \dots\dots;$$

then q_n can be obtained by Example 3.

10. Every component has unity for denominator; the numerator of the first component is 1, of the second is $\frac{1}{2}x$, and generally of the $(2r)^{\text{th}}$ is $\frac{r^2x}{(2r-1)2r}$, and of the $(2r+1)^{\text{th}}$ is $\frac{r^2x}{2r(2r+1)}$.

LVIII. 2. $ab+bc+ca+2abc=1$.

3. $(a^2+b^2+c^2)^3 = -8(ab+bc+ca)^3$. 4. $a^2+b^2+c^2-abc=4$.

5. $a^2b^2c^2(a^3+b^3+c^3+2abc)=a^3b^3c^3$. 6. $(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2 = z^{\frac{1}{2}}$.

7. $5(a^3-b^3)(2a^3+b^3)=9a(a^5-c^5)$.

8. $\left(\frac{c^2+a^2}{ac}\right)^{\frac{2}{3}} - \left(\frac{c^2-a^2}{ac}\right)^{\frac{2}{3}} = 1$. 9. $\alpha\beta=1+\gamma$.

10. $(a-b)^2(a^2+b^2)=a^2b^2$. 11. $(\alpha+\beta)^{\frac{2}{3}}+(\alpha-\beta)^{\frac{2}{3}}=2$.

13. $x(y^2-z^2)+2y(z^2-x^2)+4z(x^2-y^2)=0$.

14. $(a+b)^{\frac{2}{3}}-(a-b)^{\frac{2}{3}}=(8c)^{\frac{2}{3}}$. 16. 399.

17. This problem can be solved by the aid of the principles I. and II. of Art. 814. Let p_1 be the probability of a single event with three cards of a selected suit; let p_2 be the probability of a selected pair of events; let p_3 be the probability of a selected triad of events; and so on. Then $P_1 = mp_1$; $P_2 = \frac{m(m-1)}{2}p_2$; $P_3 = \frac{m(m-1)(m-2)}{\lfloor 3}p_3$; ...... We have now to find $p_1, p_2, p_3, \dots\dots$

Imagine three cards fastened together, so as to form one card; we should then have $mn-2$ cards instead of mn. The number of favourable cases would be $|\!\underline{\,mn-2\,}$, and the whole number of cases $|\!\underline{\,mn\,}$; this would give a chance denoted by $\dfrac{|\!\underline{\,mn-2\,}}{|\!\underline{\,mn\,}}$; and to obtain p_1 we must multiply this result by $|\!\underline{\,3\,}$, for the cards imagined to be fastened together could be permuted among themselves in $|\!\underline{\,3\,}$ ways. Thus $p_1 = \dfrac{6}{mn(mn-1)}$. Similarly $p_2 = \dfrac{6^2\,|\!\underline{\,mn-4\,}}{|\!\underline{\,mn\,}}$; and so on. Hence, finally, the required chance is

$$\frac{6m}{mn(mn-1)} - \frac{6^2\dfrac{m(m-1)}{2}}{mn\ldots(mn-3)} + \frac{6^3\dfrac{m(m-1)(m-2)}{|\!\underline{\,3\,}}}{mn\ldots\ldots(mn-5)} - \ldots\ldots$$

18. $\dfrac{|\!\underline{\,m+n\,}}{|\!\underline{\,m\,}\,|\!\underline{\,n\,}}$.   19. The expression $\dfrac{x}{1-x^2} - \dfrac{x^3}{1-x^6} + \dfrac{x^5}{1-x^{10}} - \ldots$ becomes by expansion

$$\begin{array}{l} x + x^3 + x^5 + x^7 + x^9 + \ldots \\ -x^3 - x^9 - x^{15} - x^{21} - x^{27} - \ldots \\ +x^5 + x^{15} + x^{25} + x^{35} + x^{45} + \ldots \\ -x^7 - x^{21} - x^{35} - x^{49} - x^{63} - \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Then, by adding the vertical columns, we obtain

$$\frac{x}{1+x^2} + \frac{x^3}{1+x^6} + \frac{x^5}{1+x^{10}} + \ldots\ldots$$

20. Let

$\alpha = (1-x)(1-x^3)(1-x^5)\ldots$, $\beta = (1+x)(1+x^3)(1+x^5)\ldots$,
$\gamma = (1-x^2)(1-x^4)(1-x^6)\ldots$, $\delta = (1+x^2)(1+x^4)(1+x^6)\ldots$;
then $\alpha\beta = (1-x^2)(1-x^6)(1-x^{10})\ldots$, $\gamma\delta = (1-x^4)(1-x^8)(1-x^{12})\ldots$;
thus $\alpha\beta\gamma\delta = \gamma$; therefore $\alpha\beta\delta = 1$, and therefore $\dfrac{1}{\alpha} = \beta\delta$.

21. $4\{r^6 \pm \sqrt{(2s^{12} - p^6q^6)}\} = \{q^3 \pm \sqrt{(2r^6 - p^6)}\}^2$.

ANSWERS TO MISCELLANEOUS EXAMPLES.

1. $7x-2y-6z$. 2. ax^4+bx^2+c.

3. $\dfrac{5x+2}{7x-4}$. 4. $\dfrac{26x^2+38ax}{(5x+3a)(7x+9a)}$; $\dfrac{3a^2+6ax}{(2a+x)(a^2-x^2)}$.

5. 1. 6. $x=\dfrac{5}{2}$, $y=\dfrac{7}{2}$. 7. B travels $6\frac{1}{2}$ hours before he overtakes A. 8. 80, 128. 9. $a^2+a-\dfrac{1}{2}$. 10. 2, $\dfrac{1}{2}$.

11. 10. 12. $x^4-(4a^2+9b^2)x^2+36a^2b^2$; $7x^3+5x^2y-8xy^2-3y^3$.

13. $(x+2)(x+3)(x+4)$. 14. $\dfrac{15x^2-4ax+2a^2}{(3x+4a)(4x+5a)}$. 15. $\dfrac{3}{2}$.

16. $x=11$, $y=6$. 17. $49\frac{1}{11}$ minutes past 9. 18. Each in 50 days. 19. $2x-3y+z$. 20. 2, 4. 21. 17.

22. $x^4+(a+b)x^3-(6a^2-ab+6b^2)x^2-6ab(a+b)x+36a^2b^2$; $x^2+4x+15$.

23. $x+2$. 24. 1. 25. 3. 26. $x=\dfrac{3}{2}$, $y=\dfrac{1}{2}$.

27. $9\frac{1}{7}$ miles from Ely. 28. 90 benches; 10 persons on each.

29. x^3-2x^2+x-2. 30. $\dfrac{3}{2}$, $-\dfrac{15}{22}$. 31. 7, 1, 3.

32. $\dfrac{6}{100}-\dfrac{3x}{10}+\dfrac{2x^2}{10}-x^3$; ·031. 33. $\dfrac{x^2+5x+24}{24x^2+5x+1}$. 34. $\dfrac{3(1+x^2)}{1-x^2}$.

35. 15. 36. $x=11$, $y=7$. 37. 48 of each kind. 38. A man receives £4. 4*s*., a woman £3, a child £1. 16*s*. 39. $\dfrac{x}{y}-\dfrac{1}{2}-\dfrac{y}{2x}$.

40. 6, $-\dfrac{8}{3}$. 41. 4, 2, 4. 43. The second expression will divide the first; so the second is the G. C. M., and the first is the L. C. M.

44. $\dfrac{(x+1)(x+2)}{x^2}$. 45. $\dfrac{2}{5}$. 46. $x=3$, $y=5$, $z=7$. 47. 30.

48. 10. 49. $\sqrt{(a-b)}+\sqrt{(b-c)}$. 50. 1, 3. 51. $4(ax+by+cz)$.

53. x^2+y^2. 54. 2. 55. $\frac{1}{2}$. 56. $x=\dfrac{3}{2}$, $y=-\dfrac{9}{2}$.

57. A in 36 days, B in 60 days, C in 15 days. 58. $4\frac{1}{8}$ miles.

59. $2x^3-x^2-3$. 60. 0, $\pm\sqrt{(ab)}$. 61. $2(x+4)$. 62. $\dfrac{(x^2-a^2)^2}{x^2a^2}$.

63. $x-3y$. 64. $\frac{x^8-x^4+1}{x^8+x^4+1}$. 65. 2. 66. $x=a$, $y=b$.

67. £1000. 68. 84 for the resolution, and 63 against it. 69. b.

70. $\frac{76}{77}$, $\frac{309}{77}$. 71. 9. 73. $x-5$. 74. x^2. 75. $2a$.

76. $x=y=z=a^2+b^2+c^2-ab-bc-ca$. 77. In 10 more minutes.

78. Twopence on the first day, $\frac{3}{8}$ of a penny on the second day.

80. $-4, -7$. 83. $x^2-2ax+a^2$. 84. $\frac{16x^{15}}{1-x^{16}}$. 85. 2.

86. $x=\frac{1}{2}(b+c)$, $y=\frac{1}{2}(c+a)$, $z=\frac{1}{2}(a+b)$. 87. £600000 of ordinary stock. 88. 3, 4, 5 miles an hour respectively.

89. ·05772. 90. $c, c-\frac{a+b}{2}$. 91. 20. 93. $x^2+(2m-3)x-6m$. 95. $x=1$.

96. $x-a=y-b=z-c=-\frac{1}{3}(a+b+c)$. 97. 60, 30, 12.

98. 14 miles from A to B, 16 from B to C. 100. 111, 112.

101. $\pm 1, \frac{-3\pm\sqrt{5}}{2}$. 102. 3, 4, 5. 103. $x=3, 6$; $y=6, 3$.

104. $3, -12$. 105. $-7\frac{1}{2}$. 106. $\frac{28}{3}\left(1-\frac{1}{2^6}\right)$; $\frac{28}{3}$. 107. 6.

110. Between 90 and 119, both inclusive. 111. $\pm(a+b)$, $\pm(a-b)$.

112. $x^2+\frac{2ac-b^2}{ac}x+1=0$. 113. $x=\pm 2, \pm 4$; $y=\pm 4, \pm 2$.

114. 7. 115. 162. 116. $\frac{1}{\sqrt{2}}\{1-(\sqrt{2}-1)^n\}$.

118. $n-m+1$ if r is not greater than m; $n-r+1$ if r lies between $m+1$ and $n+1$ both inclusive; 0 if r is greater than $n+1$.

119. Divergent. 120. 3·06864. 121. $1, -4, \frac{5\pm\sqrt{41}}{2}$. 122. $2b^2=9ac$.

123. $3, 4, -6\pm 2\sqrt{6}$; $4, 3, -6\mp 2\sqrt{6}$. 124. 30, 36, 45. 125. $\frac{b-2a}{b}$ must be a positive integer, and $\left(\frac{2a}{b}-1\right)^2+\frac{8s}{b}$ must be a perfect square and a positive integer: these two integers must be both even or both odd, and the former integer greater than the square

root of the latter. 126. $\dfrac{1-r^n}{ar^{n-1}(1-r)}$. 129. 3. 131. 3, $\dfrac{17}{3}$.

133. $x-a=\dfrac{k(b-c)}{bc}$, $y-b=\dfrac{k(c-a)}{ca}$, $z-c=\dfrac{k(a-b)}{ab}$, where $k=0$ or $=-2abc\dfrac{a^2(b-c)+b^2(c-a)+c^2(a-b)}{a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2}$. 134. *A* 31 half-crowns, 16 shillings, 13 sixpences; *B* 29 half-crowns, 24 shillings, 7 sixpences. 137. $6\,\underline{|6}$. 139. Divergent. 141. 6, −3. 142. 6400.

143. $x=\pm\dfrac{3}{4}$, $\pm\dfrac{\sqrt{2}}{2}$; $y=\pm\dfrac{1}{2}$, $\mp\dfrac{3\sqrt{2}}{8}$. 144. 30 miles an hour.

149. 18. 151. 5, $-\dfrac{13}{32}$. 153. $x=2^{\frac{3}{4}}$, $\left(\dfrac{27}{4}\right)^{\frac{1}{4}}$; $y=2^{\frac{1}{4}}$, $\left(\dfrac{3}{4}\right)^{\frac{1}{4}}$.

154. 75 per cent. 157. $\underline{|7}-2\,\underline{|6}$. 159. 0. 160. 1666 nearly.

161. $x+\dfrac{1}{x}=-4\pm\sqrt{6}$, whence x may be found. 162. *A* 10 miles an hour, *B* 12 miles an hour. 163. $x=-2$, $-\dfrac{3}{11}$; $y=\dfrac{19}{7}$, 0.

164. $x=0$, $y=0$, $z=0$; or $x=\dfrac{1}{2}$, $y=\dfrac{1}{6}$, $z=-\dfrac{1}{6}$. 166. Either $a=b=c$; or $b=-2a$, and $c=4a$. 169. $-\dfrac{40726}{3^8}$. 170. 1·21534 nearly.

171. ± 8, $\pm\dfrac{1}{8}$. 172. $a^4x^2-2a^2(b^2-2ac)x+b^2(b^2-4ac)=0$.

173. $x=5$, $-\dfrac{23}{5}$, $\dfrac{25\pm\sqrt{19968}}{29}$; $y=3$, $-\dfrac{69}{25}$, $\mp\dfrac{21}{29}\cdot\dfrac{25\pm\sqrt{19968}}{29}$.

174. 9 days. 176. 1, 2, 4, 8. 177. $\dfrac{\underline{|2n-p-q}}{\underline{|n-p}\,\underline{|n-q}}\underline{|n}\,\underline{|n}$.

181. 1, $\dfrac{1}{3}$, $-\dfrac{1}{2}$. 182. £1045. 183. $x=81, 16$; $y=16, 81$.

184. $1=a^2+b^2+c^2+2abc$. 186. $\dfrac{a^2}{6}\left\{\dfrac{1}{(1-r)^3}+\dfrac{2}{1-r^3}-\dfrac{3}{(1-r)(1-r^2)}\right\}$.

189. Convergent if x is less than e; otherwise divergent.

191. $x-\dfrac{1}{x}=-a\pm\sqrt{(a^2-4)}$, whence x may be found.

193. x and y may be found from $x^2+1=\pm\frac{m}{n}x$, $y^2+1=\pm mny$.

199. Convergent. 200. $\underline{|n+3}\left\{\frac{n}{24}+\frac{n(n-1)}{12}+\frac{n(n-1)(n-2)}{48}\right\}$.

201. $1-x-x^2+x^5+x^6+x^7-x^8-x^9-x^{10}+x^{13}+x^{14}-x^{15}$; $1-x-x^2+x^5+x^7-x^{12}$.

205. The solutions are found from $14t$ weights of 9 lbs. and $160-9t$ weights of 14 lbs. 206. $(2^{n+2}-n-3)x^n$.

209. $\frac{1}{12}$ of a shilling. 210. Divergent. 213. 19 years.

215. The solutions are found from taking $204-11t$ for the numerator, and $1+5t$ for the denominator; so that t may have any integral value between 13 and 18 both inclusive. 219. $\frac{3}{4}$.

222. $x^2-a^2=y^2-b^2=z^2-c^2=\frac{b^4c^4+c^4a^4+a^4b^4-2a^2b^2c^2(a^2+b^2+c^2)}{4a^2b^2c^2}$.

223. $rp+\frac{q}{p}=2$. 224. The quotients are $a, b, 2a, b, 2a, \ldots$

225. $(x-11y+1)(2x+y-3)$. 233. £693. 234. The quotients are $a-1, 1, 2(n-1), 1, 2(a-1), \ldots$ 235. 52. 240. Divergent. 243. 18 nearly.

244. The first quotient is a; then we have 1, 2, a, 2, 1, $2a$, which recur. 245. Either 10 sheep, and 2 bullocks; or 5 bullocks.

249. £$7\frac{1}{3}$. 250. $x^2+4x+7-32x^{-1}-208x^{-2}-\frac{448x^{-1}-2496x^{-2}}{x^2-4x+12}$.

251. $x=1, \frac{4}{3}$; $y=1, \frac{3}{2}$. 252. $a_r-a=r(a_1-a)+\frac{r(r-1)}{2}d$; put $n+1$ for r and b for a_{n+1}; thus a_1 becomes known: d must lie between $-\frac{2(b-a)}{n(n+1)}$ and $\frac{2(b-a)}{n(n+1)}$. 253. £645 nearly. 255. Either 5 apples, 3 pears, and 4 peaches; or 12 pears. 261. $\frac{b^4}{a^4}-\frac{4b^2c}{a^3}+\frac{3c^2}{a^2}$.

265. The smallest number of coins consists of 121 of the larger and 15 of the smaller; the smallest sum of money consists of 10 of the larger and 150 of the smaller. 269. $\frac{(n+1)(3n+2)}{12}$ shillings.

272. $x=a, b$; $y=b, a$. 273. $(7t\pm 2)^2$. 275. The coefficient of x^n is $10^n+n(-3)^{n-1}$. 276. $1-4x-2x^2$; $1-3x^2-2x^3$. 279. $11\frac{1}{2}$ shillings.

281. Between 1 and -4. 282. $(xy)^{\frac{1}{2}}=\dfrac{a^2-b^2}{4a-2b\sqrt{2}}$: from this and the first given equation we can find $x^{\frac{1}{2}}$ and $y^{\frac{1}{2}}$. 285. 56, 78.

293. $\dfrac{r(1+6r+r^2)}{(1-r)^3}$. 294. $x=8$, $y=3$; $x=127$, $y=48$.

295. 27. 297. £240. 300. $2\log 2-1$.

CORRECTION.

Page 525. In lines 1 and 6 *for* x^n *read* x^{n-1}.

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