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AN ALGEBRAIC ARITHMETIC

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AN ALGEBRAIC ARITHMETIC

BEING

AN EXPOSITION OF THE THEORY
AND PRACTICE OF

ADVANCED ARITHMETIC

BASED ON THE ALGEBRAIC EQUATION

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PREFACE

THIS arithmetic is not offered to the public as a refinement or super-refinement of the methods of existing text-books on the subject. It is a new departure.

For a number of years arithmetics have been undergoing a progressive change. Voluminous works, in which the isolated treatment of related topics and the multiplicity of detail relating to business arithmetic completely obscured the unity of the science, have by degrees given place to more compendious works. The change has, however, been little more than a process of successive elimination. To the former plethora has succeeded an ever-increasing leanness, until at last the *skeleton* of the subject stands revealed indeed, for it alone remains. The arithmetic of to-day is merely a compilation of examples, classified and miscellaneous, with illustrative solutions accompanied by brief explanatory notes and a few definitions. The task of infusing a living, rational principle into these dry bones is left entirely to the teacher.

A second change in the mathematics of the grammar school, contemporaneous with that above mentioned, has been the introduction of elementary geometry and, more recently, of elementary algebra. The situation is best described by saying that the latter subjects have partially superseded arithmetic, since the whole time devoted to mathematics has remained substantially the same.

The reason for these changes is not far to seek. The mathematics are chiefly valuable as a factor in education in that they afford a means of developing the reasoning powers of the child; and as arithmetic, in spite of the numerous attempts to improve the text-books and the methods of teaching, persisted in remaining little more than a collection of rule-of-thumb methods for turning out "answers," the progressive teacher naturally turned to other branches of mathematics which embodied a logically coherent science not yet perverted by "practical applications."

Much was gained by so doing. Geometry, which in its complete and rigidly demonstrative form is a fairly difficult subject for the high school, was found to contain a large number of facts that could be established by simple yet fairly conclusive reasoning. And not only is the method by which these facts are acquired *in their logical relation* of the highest value in developing the reasoning powers of the child, but the facts themselves possess a value far higher than that of mere utility. For example, it may possibly prove of use to some member of a large grammar-school class to know the empirical rule by which the contents of a cask or the number of feet of lumber in a round log are determined; but it is important that all should know, and should be able to give some simple explanation of the fact, that similar surfaces are to each other as the squares and similar solids as the cubes of their like dimensions. These are universal truths, depending on the nature of space, by which all physical existence is conditioned; and a knowledge of them is therefore an essential part of a complete education; but this, from its very nature, can be true of no empirical rule.

The introduction of elementary algebra into the schools is of more doubtful value for two principal reasons. The conceptions of geometry can be represented by figures and objects, and are therefore readily grasped by the child, but those of algebra can be so represented only to a very limited degree; and, for the most part, are abstractions the nature of which the child comprehends with difficulty. Moreover, the time that can be allotted to the subject in the grammar school is barely sufficient to carry the pupil over the essentially uninteresting details of algebraic manipulation that necessarily precede any but the simplest applications of the science.

But while the wisdom of introducing elementary algebra into the schools may, for these and minor reasons, be seriously questioned, the experience of several years as a teacher has led the author to the conclusion that the application of certain algebraic conceptions to arithmetic would contribute largely toward the rational presentation of the subject, thus increasing its disciplinary value, and at the same time preparing the way for a natural transition to the algebra of the high school.

These conceptions are the use of letters as the general representatives of (positive) numbers and of the equation to express their relations.

Both ideas are introduced into the first chapter, and developed, so far as the purpose demands, in the second. The comparatively large amount of space given to problems is due to the fact that they afford at once the most interesting introduction to the subject and the best means of explaining the significance of the equation and its transformation.*

Among the numerous and important applications of these ideas throughout the book, may be noted the following:

The cases of percentage are reduced to three (Art. 32), and all are shown to be contained in the single equation $p = br$.

All the applications of percentage not involving time are shown to be merely special applications of the three percentage formulas. (For example, see table under Profit and Loss, Art. 33.)

All the formulas of simple interest are derived from the two: $i = prt$ and $a = p + i$; these being obtained directly from the definitions.

The interest formulas are shown to be special developments of the earlier percentage formulas (Art 48).

The principles of proportion are rigidly demonstrated, affording a simple and elegant illustration of the method of algebraic proof, freely used in the chapter on mensuration.

* The attention of teachers is called to the fact that an equation is not a quantity, but an expression of relation, and therefore cannot be operated upon, in the usual sense of the word. Operations are not performed upon an equation, but upon its members. The abbreviated and, to beginners, highly misleading forms of statement, "Multiply the equation by 3," "Subtract 10 from the equation," and the like, should be studiously avoided. For a similar reason, the word "transpose" should not be used. A term is transposed by addition or subtraction, and the specific operation should always be named.

The algebraic and geometrical explanation of evolution are combined in one, the algebraic symbols being the natural method of expressing the geometrical relations. Thus each set of ideas confirms the other.

The chapter on mensuration embodies the principles enunciated in the early part of this preface as fully as the limited time generally allotted to this subject permits. In schools where a more extended course in elementary geometry is given, it will afford a convenient opportunity for a review of the most important results of the course. The author takes pleasure in acknowledging his indebtedness to Hill's Lessons in Geometry for many valuable suggestions in the preparation of this chapter, and recommends the book as being admirably adapted to the needs of grammar schools.

Among the features of the book not resulting from the algebraic method of treatment, the author would call attention to the use of the article to mark the logical divisions of the subject; to the rational explanation of the application of simple and compound proportion to the solution of problems;* and to the treatment of partial payments. The latter subject is placed after compound interest, where the effect of the two usual methods of applying partial payments can be intelligibly discussed, and the manner in which interest is compounded by the United States Rule is fully explained.

It is believed that the examples, which for the most part have been compiled from various sources, present a

* The author has never seen anything on this point in any text-book, except variations of the rule of thumb: More requires more, and less requires less.

fairly extensive and varied application of the subject matter.

The author will receive with pleasure any suggestions or criticisms that will be of assistance in the improvement of later editions of this work.

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CHAPTER I

INTRODUCTION

1. The Use of Letters to denote Numbers. It is often necessary to speak of something which is true not only of one number or set of numbers, but of all numbers or of all similar sets of numbers. For example, the sum of 9 and 6 is 15, and their difference is 3. If we add this sum and difference, we get 18, which is twice the greater of the two numbers. If we subtract the difference from the sum, we get 12, which is twice the smaller of the two numbers. The same relation is true of any two numbers, and may be expressed in the general statement:

If the sum and difference of two numbers be added, the result is twice the greater of the numbers; if the difference be taken from the sum, the result is twice the smaller of the numbers.

This may be expressed more briefly by the aid of signs; thus:

sum of two numbers + difference of the numbers
= twice the larger number.

sum of two numbers − difference of the numbers
= twice the smaller number.

The statement may be still further shortened by using letters to represent the numbers. Thus, let g stand for the greater number, l for the less, s for their sum, and d for their difference. We then have

$$s + d = 2g,$$

and $$s - d = 2l;$$

in which, it must be remembered, a and b denote any two numbers, provided only that a is greater than b.

From the above equations it is clear that

$$g = \frac{s+d}{2}, \tag{1}$$

and $$l = \frac{s-d}{2}. \tag{2}$$

These equations express the fact that the larger of two numbers is equal to one-half the sum found by adding the sum and the difference of the numbers, and the smaller number is equal to one-half the remainder found by subtracting the difference of the numbers from their sum. These equations are, in fact, a very convenient statement of the rule for finding any two numbers when their sum and difference are given.

NOTE. Rules stated in the form of equations are called **formulas.**

The following chapter will afford numerous illustrations of the advantage of using letters to denote numbers when we are studying their general properties.

2. Operations to be performed with numbers denoted by letters are indicated by the usual signs of arithmetic, in the same manner as when the numbers are expressed with figures; with the exception that the product of two numbers is indicated by writing the letters together, the sign of multiplication being omitted.

Thus the sum of any two numbers a and b is indicated by $a+b$, their difference by $a-b$, their product by ab, and their quotient by $a \div b$, or by $\frac{a}{b}$.

In each case the numerical value of the result can be found only when the values of a and b are given.

NOTE 1. The word **numerical** relates to *particular* numbers, that is, to numbers expressed by figures; the word **literal**, to numbers expressed by letters. Thus 12 is a numerical quantity; a, b, c, etc., are literal quantities.

NOTE 2. The sign of multiplication cannot be omitted between numerical factors, but is omitted between a numerical and a literal factor. Thus $5 \times a \times b$ is written $5\,ab$. (Omit "times" in reading.)

EXAMPLES 1

If $a=4$, $b=1$, $c=3$, and $d=2$, find the numerical values of

1. $a+b$.
2. $a-b+c$.
3. $5\,a-2\,d$.
4. $12\,b-2\,a$.
5. $a+8\,b-6\,d$.
6. $ac-2\,bd$.
7. $4\,cd+ab$.
8. $3\,a-5\,b+2\,acd$.
9. $abc+abd+bcd$.
10. $2\,a \div d+4\,c$.

If $a=6$, $b=5$, $c=2$, and $d=0$, find the values of

11. $2\,ab+b-cd$.
12. $a \div c \times b-ac$.
13. $6\,ab \div 5\,c$.
14. $\frac{a}{c}+\frac{d}{b}$.
15. $\frac{2\,ac-b+3\,d}{7\,a+abc}$.

3. The Equation. The statement that two numbers or two sets of numbers are equal is called an **equation.**

Equations are used in Art. 1, and the values of the letters in the examples of Art. 2 are given by means of equations.

The part of an equation on the left of the sign of equality is called the **left side, left member,** or **first member** of the equation; that on the right, the **right side, right member,** or **second member.**

Many problems can be most easily solved by the use of letters to denote the numbers to be found, and equations to express the relations that exist between these numbers and the given numbers of the problem. How this is done will be shown by the following examples:

Ex. 1. If 5 be added to 3 times a certain number, the result is 29. Find the number.

The problem may be stated more briefly thus:

$$3 \text{ times a certain number } + 5 = 29;$$

or, if we let a stand for the number, it may be stated still more briefly by the equation

$$3a + 5 = 29. \qquad (1)$$

Now if from this equation we can find the value of a, that is, the number that a represents, this value will be the answer to the problem. Let us try to do this.

Subtracting 5 from both members of the equation, we get

$$3a = 24. \qquad (2)$$

Dividing the sides of this equation by 3, we have

$$a = 8. \qquad (3)$$

To prove the result, replace a in equation (1) by its value. This gives

$$3 \times 8 + 5 = 29,$$

$$29 = 29.$$

The equation is said to be *satisfied* by $a = 8$; which means that when 8 is substituted for a, the equation is true. It is evident that it would not be satisfied by any other value of a.

Ex. 2. The sum of two numbers is 38, and their difference is 8. What are the numbers?

The answer can be written down at once by substituting $s = 38$, and $d = 8$ in formulas (1) and (2), Art. 1. The pupil should carefully compare this method of solution with the following:

Let us denote the greater of the numbers by x; then, since the smaller number is 8 less than the larger, it will be denoted by $x - 8$.

The problem states that

the larger number + the smaller number = 38.

Hence $$x + (x - 8) = 38, \tag{1}$$

or $$x + x - 8 = 38,$$

or $$2x - 8 = 38.$$

Add 8 to both sides of the equation; then

$$2x = 46.$$

Divide both sides by 2; then

$$x = 23 = \text{larger number}.$$

Hence $$x - 8 = 15 = \text{smaller number}.$$

PROOF. Substituting these values in equation (1), we have

$$23 + 15 = 38,$$
$$38 = 38.$$

NOTE. It is necessary to notice the punctuation after the equations in the solution of a problem; for the equations always occur as parts of sentences, and the punctuation helps to make the meaning clear, just as in the case of any other kind of sentence.

4. Definitions. The figures, letters, and signs used in arithmetic are called **symbols.**

Any combination of symbols denoting a number is called an **expression.** If it contains letters, it is called an **algebraic expression.**

The members of an equation are expressions.

The parts of an expression which are separated from each other by the signs of addition or subtraction are called the **terms** of the expression.

Thus the expression $2ab - c + 5$ has three terms; $5axy$ has one term.

A term may consist of two or more **factors.** Thus the term $5axy$ contains four factors.

If the factors of a product are separated into groups in any way, either group of factors is called the **coefficient** of the other group.

Thus in the term $5axy$, 5 is the coefficient of axy, $5a$ is the coefficient of xy, $5ay$ is the coefficient of x, etc. If a term has a numerical factor, it is generally spoken of as the coefficient of the term.

Terms containing the same literal factors are called **like** terms.

Thus $5\,abx$ and $9\,abx$ are like terms; $3\,ab$ and $7\,cdx$ are unlike terms.

5. Axioms. We have seen that some problems can be stated in the form of equations in which a letter stands for the answer; and that the value of the letter which satisfies the equation is the answer to the problem. In solving such equations, frequent use is made of the following simple truths, or axioms:

Ax. 1. If equal numbers are added to equal numbers, the sums are equal.

Ax. 2. If equal numbers are subtracted from equal numbers, the remainders are equal.

Ax. 3. If equal numbers are multiplied by equal numbers, the products are equal.

Ax. 4. If equal numbers are divided by equal numbers, the quotients are equal.

Thus, if $a = b$ and $c = d$,

then $$a + c = b + d \qquad \text{by Ax. 1.}$$

$$a - c = b - d \qquad \text{by Ax. 2.}$$

$$ac = bd \qquad \text{by Ax. 3.}$$

and $$a \div c = b \div d \qquad \text{by Ax. 4.}$$

The four axioms may be summed up in the statement: *Equal numbers will still remain equal numbers after they have been increased, diminished, multiplied, or divided by equal numbers.*

6. The Solution of Equations. Let us now look again at the solution of Ex. 1, Art. 3. The algebraic (or symbolical) statement of the problem is $3a + 5 = 29$, in which a stands for the answer.

Since the members of this equation are equal numbers, if we subtract 5 from each of them, the remainders will be equal by Ax. 2. This gives

$$3a + 5 - 5 = 29 - 5,$$

or $$3a = 24;$$

that is, we form an equation out of the equal remainders. This could not have been done if we had subtracted more from one member than from the other, for in that case the remainders would have been unequal, and the equation would have been destroyed.

We wish to obtain a alone in the left member of the equation. We can now do this by dividing that side by the coefficient of a; but since we must preserve the *equality* of the members, we divide *both* by 3, and obtain $a = 8$. In this operation we use Ax. 4.

Since, in solving the original equation, we have made use of only those operations which do not destroy the equality of its members, we know that the last equation is true. It therefore gives us the required value of a.

Exercise. Find what axioms have been used in the solution of Ex. 2, Art. 3.

Ex. 1. Find the value of x if

$$2x + 5 = 15 - x.$$

Since we wish to obtain x alone in the left side, and only numerical quantities in the other, we must get rid of the x in the right side and the 5 in the left. The x

will disappear from the right side if we add x to it, since $15 - x + x = 15$. Hence, adding x to both members to preserve their equality, we have

$$2x + 5 + x = 15 - x + x \qquad \text{by Ax. 1.}$$

or

$$3x + 5 = 15.$$

Subtract 5 from both sides; then

$$3x = 10 \qquad \text{by Ax. 2.}$$

Divide both sides by 3; then

$$x = \tfrac{10}{3} = 3\tfrac{1}{3} \qquad \text{by Ax. 4.}$$

NOTE. This value of x satisfies not only the given equation, but also all the equations derived from it; that is, x has the same value throughout the solution, which must be the case in the solution of any equation.

EXAMPLES 2

Solve the following equations:

1. $3x + 4 = x + 10.$
2. $4x + 4 = x + 7.$
3. $5x - 5 = 20 - 2x.$
4. $x + 4 = 2(5 - x).$
5. $14 = \dfrac{1}{x}.$
6. $\dfrac{1}{8} = \dfrac{7}{x}.$
7. $x + \tfrac{2}{3}x = 10.$
8. $\dfrac{x}{5} + \dfrac{x}{4} = 1.$

7. The Solution of Problems.

Ex. 1. What number is that whose half added to 16 gives 25?

Let x denote the number.

Then $\dfrac{x}{2}$ will denote half the number, and $\dfrac{x}{2} + 16$ will denote the half added to 16.

But the problem states that this is 25; hence

$$\frac{x}{2}+16=25.$$

Subtract 16 from both sides; then

$$\frac{x}{2}=9 \qquad \text{by Ax. 2.}$$

Multiply both sides by 2; then

$$x=18 \qquad \text{by Ax. 3.}$$

PROOF: $\frac{18}{2}+16=25,$

$$25=25.$$

Ex. 2. A man having \$92 spent a part of it, and then had 3 times as much as he had spent. How much did he spend?

Let x be the number of dollars he spent.

Then $92-x$ will be the number of dollars he had left.

But the problem tells us that this is 3 times as much as he spent. Hence we have the equation

$$3x=92-x. \qquad (1)$$

Add x to both sides; then

$$4x=92 \qquad \text{by Ax. 1.}$$

Divide both sides by 4; then

$$x=23 \qquad \text{by Ax. 4.}$$

Hence the man spent \$23.

REMARKS. In problems involving concrete numbers, like the last, it is not necessary to express the kind of unit in the equation. Thus, in this problem, we do not write

$$\$3x=\$92-\$x;$$

for, though the statement is correct, it is not so simple as when made without the sign.

The members of the equation are to be regarded as abstract numbers, denoting the *number of times* the concrete unit is contained in the quantities to be compared. Thus the members of (1) denote the number of times \$1 is contained in the sum of money the man had left.

The quantities to be compared must be of the same kind, and must be measured by the same unit. For example, we cannot compare a sum of money with a distance, nor can we compare two sums of money when one is measured in dollars and the other in cents or in dimes.

Such statements as

$$100¢ = \$1,$$

$$16 \text{ oz.} = 1 \text{ lb.},$$

are not equations at all in the sense in which we shall use the word in this book. The first of these statements means that the two sums are equal in *value;* the second, that the two *weights* are equal; but in neither of them are the two *numbers* equal. Equations, as we shall use them, will always mean that the two members are *equal numbers.*

Ex. 3. A can do a piece of work in 10 days, but A and B working together can do it in 6 days. In how many days can B do it alone?

Let $x =$ the number of days it would take B to do the work alone.

Then $\frac{1}{x} =$ the part of the work he can do in one day.

From the problem we know that A can do $\frac{1}{10}$ of the work in one day; and A and B together, $\frac{1}{6}$ of the work in one day.

Hence $$\frac{1}{6}=\frac{1}{x}+\frac{1}{10}. \tag{1}$$

Multiply both sides by $30x$, the L. C. M. of the denominators; then

$$\frac{30x}{6}=\frac{30x}{x}+\frac{30x}{10} \qquad \text{by Ax. 3.}$$

or $$5x=30+3x.$$

Subtract $3x$ from both sides; then

$$2x=30 \qquad \text{by Ax. 2.}$$

Divide both sides by the coefficient of x; then

$$x=15 \qquad \text{by Ax. 4.}$$

Hence B can do the work in 15 days.

Proof by substitution: Replace x in (1) by 15; then

$$\tfrac{1}{6}=\tfrac{1}{15}+\tfrac{1}{10},$$
$$\tfrac{1}{6}=\tfrac{1}{6}.$$

Proof by analysis: Since A can do the work in 10 days, in one day he can do $\frac{1}{10}$ of it; since B can do the work in 15 days, in one day he can do $\frac{1}{15}$ of it. Hence A and B working together can do $\frac{1}{10}+\frac{1}{15}$, or $\frac{1}{6}$, of it in one day, or the whole piece of work in 6 days.

8. From the examples of Art. 3 and Art. 7, the following directions for the solution of similar problems may be deduced:

I. Denote the required number by some letter (it is customary to use x). This is called the **unknown quantity.**

II. If there are other numbers in the problem that depend on the unknown quantity, find expressions for them in terms* of the unknown quantity.

III. Write these expressions in the form of an equation which expresses in symbolic form the conditions of the problem.

IV. Clear the equation of fractions, if there are any, by multiplying both members by the ***l. c. m.*** of the denominators.

V. By addition or subtraction remove all terms containing the unknown quantity to one side of the equation, and all other terms to the other side.

VI. After adding together the terms containing the unknown quantity, divide the members of the equation by its coefficient. This gives the answer.

EXAMPLES 3

1. John is 3 times as old as James, and the sum of their ages is 16 years. What is the age of each?

2. A boy bought a top and a ball for 24 cents, paying 5 times as much for the ball as for the top. What did he pay for each?

3. Ida's sister gave her some money, and her brother gave her twice as much. After spending 12 cents, she had 18 cents left. How much was given her by each?

* A number is said to be expressed in terms of another number when the expression for it contains the letter that represents the other number. Thus in Ex. 2, Art. 3, the smaller number, $x-8$, is expressed in terms of the larger number, x; and in Ex. 2, Art. 7, the number of dollars the man had left, $3x$, is expressed in terms of the number of dollars he spent, x.

4. The sum of two numbers is 50, and their difference is 18. Find them.

5. The sum of three numbers is 126. The second is twice the first, and the third is equal to the sum of the other two. What are the numbers?

6. A boy, after spending half his money, earned 14 cents, and then had 30 cents. How much had he at first?

7. A and B together can do a piece of work in 8 da., and A working alone can do it in 20 da. In what time can B do it?

8. Fred has 3 times as many marbles as Harry, lacking 2; and both together have 26. How many has each?

9. The sum of two numbers is 62, and the greater is 3 less than 4 times the smaller. Find the numbers.

10. A father is 6 years more than 4 times as old as his son, and the sum of their ages is 71 years. Find the age of each.

11. If $\frac{1}{2}$ of a certain number be subtracted from $\frac{2}{3}$ of it, the remainder will be 8. What is the number?

12. Divide 42 into two parts, such that one part shall be $\frac{3}{4}$ of the other.

13. One of two apple trees bore $\frac{3}{7}$ as many apples as the other, and both yielded 21 bu. How many bushels did each yield?

14. A lad having 45 cents bought an equal number of pears, oranges, and bananas; the pears being 3 cents each, the oranges 4 cents, and the bananas 2 cents. How many of each did he buy?

CHAPTER II

ADDITION, SUBTRACTION, MULTIPLICATION, DIVISION

9. We shall now study a little more fully the way in which operations are performed upon numbers denoted by letters.

There are four fundamental operations, or processes, by certain combinations of which all the problems of arithmetic are solved. These are addition, subtraction, multiplication, and division. You have already learned that subtraction is the *inverse* of addition, and division the inverse of multiplication; by which is meant that subtraction *un*-does what addition does, and division un-does what multiplication does.

Hence, if to any number I add any other number, and afterwards subtract the same number, I shall have left the first number unchanged, since the two operations exactly cancel each other.

Thus, if a and b are any two numbers,

$$a + b - b = a. \tag{1}$$

Again, if I multiply any number by any other number, then divide the product by the same number, the quotient will be the first number. If the division be performed first, then the multiplication, the result will still be the first number.

Thus, $$a \times b \div b = a \div b \times b = a, \qquad (2)$$

or $$\frac{ab}{b} = \frac{a}{b} \times b = a.$$

On account of this inverse relation, every fact in addition gives one or more corresponding facts in subtraction, and similarly for multiplication and division.

Thus, since $5 + 7 = 12$, it follows that

$$(7 + 5) - 5 = 7, \text{ or } 12 - 5 = 7,$$

and that $$(5 + 7) - 7 = 5, \text{ or } 12 - 7 = 5;$$

and since $6 \times 8 = 48$, it follows that

$$(6 \times 8) \div 8 = 6, \text{ or } 48 \div 8 = 6,$$

and that $$(8 \times 6) \div 6 = 8, \text{ or } 48 \div 6 = 8.$$

ADDITION

10. If we have a group of a things and a second group of b things, and if we form a single group from these by putting the two groups together, we shall have as many things in the single group thus formed as there were at first in both the groups. It is clear that this will be true whether we put the first group with the second or the second with the first; that is, the *sum* of the things is not changed by the way in which they are brought together.

This fact or *law* is expressed symbolically thus:

$$a + b = b + a; \qquad (1)$$

and briefly in words thus:

Additions may be performed in any order.

Thus, for any three numbers a, b, and c,

$$a+b+c=a+c+b=b+a+c=a+(b+c)=(b+a)+c, \text{ etc.}$$

11. If I take a marbles from a box twice, and afterwards take the same number of marbles 3 times, I have in all taken a marbles $(2+3)$ times, or 5 times; which makes $5a$ marbles.

Thus the number I take the first time is

$$a+a, \text{ or } 2a,$$

and the second time, $\qquad a+a+a, \text{ or } 3a.$

Hence I take in all, $a+a+a+a+a$, or $5a$.

Hence $\qquad 2a+3a=(2+3)a=5a.$

If I take a things m times, and again n times, I take in all a things $(m+n)$ times, or $(m+n)a$ things.

Hence $$\boldsymbol{ma+na=(m+n)a}. \tag{1}$$

From the formula we have the rule: *To add terms having a common factor, write the common factor with a coefficient equal to the sum of the coefficients of the terms added.*

Ex. 1. $3ab+5ab+ab=9ab$.

Ex. 2. $4ac+6bc=(4a+6b)c$.

Exercise. Show that the above equations are true when $a=1,\ b=2,\ c=3$; when $a=4,\ b=3,\ c=2$; when $a=b=c=5$.

Are they true for *all* values of the letters?

SUBTRACTION

12. If there are a apples in one basket and b in another, and I take away c of them, the number remaining will be $a+b-c$. This result does not show whether I take the apples partly from each basket or all from one. It merely indicates that the whole number of apples, $a + b$, has been diminished by c.

If I take them all from the first basket, the number remaining in it will be $a - c$, and the whole number of apples remaining will be $a - c + b$. Similarly, if I take them all from the second basket, the whole number remaining will be $a + (b - c)$, or $b - c + a$.

The whole number remaining will be the same whichever way I take the c apples; hence

$$\boldsymbol{a + b - c = a - c + b = b - c + a.} \qquad (1)$$

From this we have the law: *Subtractions may be performed in any order.*

NOTE. This law is limited to the case where the minuend is at least as large as the subtrahend.

13. From articles 10 and 12 it is clear that

$$\boldsymbol{a + (b + c) = a + b + c,} \qquad (1)$$

$$\boldsymbol{a + (b - c) = a + b - c.} \qquad (2)$$

Hence *a parenthesis which is preceded by the sign of addition may be removed from an expression without affecting its value.*

14. The expression $a - (b + c)$ means that from a we are to subtract the *sum* of b and c. We shall evidently obtain the same result by first subtracting b from a, then subtracting c from the remainder; hence

$$\boldsymbol{a - (b + c) = a - b - c.} \qquad (1)$$

The expression $a - (b - c)$ means that from a we are to subtract the *difference* between b and c; hence if we subtract b from a, we subtract c units too much. Hence, to obtain the correct result, we must add c to the remainder. That is,

$$\boldsymbol{a-(b-c)=a-b+c}. \qquad (2)$$

From (1) and (2) it follows that *a parenthesis which is preceded by the sign of subtraction may be removed from an expression if all the + signs within the parenthesis be changed to − signs, and all the − signs to + signs.*

Ex. $a-(c-2b+5)=a-c+2b-5$.

EXAMPLES 4

If $a=6$, $b=5$, $c=4$, $d=2$, and $e=1$, find the values of the following expressions (1) by substituting in the given expressions, then performing the indicated operations, (2) by removing parentheses, combining like terms, then substituting the values of the letters. The results should agree.

1. $3ad+(2c-ae)$.
2. $bde-(a-2d)$.
3. $\frac{5a}{d}-\left(10-\frac{2a}{d}+e\right)$.
4. $7cd+(8a-5cd+9)$.
5. $5ab+[bc-(3c-5e)]$.
6. $\frac{a-(b-c)}{d}$.
7. $2abc-\frac{2b-d}{c}$.
8. $3a-(2b-a+5)$.
9. $3ad-(ad-b)+2b-c$.
10. $3a-\left(a+\frac{cd}{b}-2d\right)-4d$.
11. $20e-[3a-(2b+a)]$.
12. $2bc-(ab-bc)+2ab$.
13. $\frac{a+(2c-a)}{c}-\frac{e}{a}$.
14. $2\frac{c}{b}+\left(3-\frac{c}{b}\right)-\frac{a}{cd}$.

MULTIPLICATION

15. For the case where the multiplier is an integer, **multiplication** is defined as the process of taking one number as many times as there are units in another number.

Thus $$3 \times 5 = 5 + 5 + 5$$

(the 5 being taken as many times as there are units in the multiplier 3),

and $$4a = a + a + a + a.$$

This definition fails when the multiplier is a mixed number or a fraction, for we cannot take anything a *fraction of a time.* A fractional multiplier does not indicate *how many times* the multiplicand is to be taken, but *what part* of it.

Thus $\frac{3}{4} \times a$ means that 3 of the 4 equal parts of a are to be taken. The multiplier itself is 3 of the 4 equal parts of *unity.* Hence we have the following definition of multiplication, which holds for *any* value of the multiplier:

Multiplication is the process of doing to the multiplicand what was done to unity to obtain the multiplier.

Numerical examples will make the meaning clear.

Multiply 10 by $\frac{2}{5}$.

To obtain the multiplier, 1 was divided into 5 equal parts, and two of these parts were taken. Hence divide 10 into 5 equal parts and take two of them. The result is 4.

Multiply 5 by 3.

The multiplier is three ones $(1 + 1 + 1)$.

Hence the product is three fives $(5 + 5 + 5)$.

16. The number of dots in the figure, counted by *rows*, is three fives; counted by *columns*, it is five threes. *The number of dots must be the same whichever way they are counted;* hence

$$3 \times 5 = 5 \times 3.$$

The same reasoning holds for the product of any two *integers.* Thus if there are a rows of b dots each, the whole number of dots, counted by *rows,* will be ab; and if counted by *columns* it will be ba.

Hence $$\boldsymbol{ab = ba}. \qquad (1)$$

This is a *law* of multiplication, which, expressed in words, is: The order in which factors are taken does not affect the value of the product; or, more briefly:

The factors of a product may be taken in any order.

NOTE. It should be observed that, in the proof of this law, we have assumed that a and b are integers. The law holds good when either or both the factors are fractions; but the proof is different and rather more difficult.

The law holds for any number of factors; thus for three factors a, b, and c,

$$abc = acb = cba = a(bc), \text{ etc.} \qquad (2)$$

17. It follows directly from the law stated in the preceding article that *a number is multiplied by multiplying any one of its factors.*

Thus, if we wish to multiply the number abc by any number x, we know from this law that

$$\boldsymbol{(abc)\,x = (ax)\,bc = a\,(bx)\,c = ab\,(cx)}. \qquad (1)$$

EXERCISE. Show that 30 is multiplied by 7 by multiplying any one of its prime factors by 7.

18. To multiply 24 by 2 we multiply the 4 units by 2 and the 2 tens by 2; that is,

$$2 \times 24 = 2(20 + 4) = 2 \times 20 + 2 \times 4.$$

We may separate the multiplicand into parts in any way, and multiply it by multiplying each of those parts. For example,

$$2 \times 24 = 2(12 + 7 + 5) = 2 \times 12 + 2 \times 7 + 2 \times 5.$$

The same fact is true of any number.

Thus, $$a(m + n) = am + an. \qquad (1)$$

Hence the law: *A number is multiplied by multiplying each of its parts (terms).*

NOTE. The *parts* of a number are not its factors. A number is produced from its parts by addition, not by multiplication.

Ex. 1. $5(3ab + 2) = 15ab + 10.$

Ex. 2. $2a(b + cd) = 2ab + 2acd.$

Ex. 3. $3b(cd + 3\frac{a}{b} + \frac{2}{3}e) = 3bcd + 9a + 2e.$

EXERCISE. Show that these equations are true when $a = 1$, $b = 2$, $c = 5$, and $d = 3$; also when $a = 6$, $b = 1$, $c = 2$, and $d = 3$. Are they true for other values of the letters?

19. Not only may the multiplicand be separated into parts, but the multiplier may be also. This is done in finding a numerical product when the multiplier consists of more than one figure.

Thus the operation of finding 42×35, when expressed fully, is

$$\begin{array}{rr} 35 = & 30 + 5 \\ 42 = & 40 + 2 \\ \hline 2 \times 5 = & 10 \\ 2 \times 30 = & 60 \\ 40 \times 5 = & 200 \\ 40 \times 30 = & 1200 \\ \hline 42 \times 35 = & 1470 \end{array}$$

If we should separate the factors into parts in any way, and should multiply each part of the multiplicand by each part of the multiplier, the sum of these partial products would be the product of the factors.

Exercise. Find the product of 35 and 42 after separating the factors into the parts: $35 = 12 + 20 + 3$, and $42 = 20 + 22$.

Note. The sign of multiplication is omitted between parentheses, and between a parenthesis and a factor.

Ex. 1. $(a + b)(c + d) = a(c + d) + b(c + d)$
$= ac + ad + bc + bd.$

Ex. 2. $2a(b + 4c) = 2ab + 8ac.$

Ex. 3. $(3a + b)(c + 5e) = 3a(c + 5e) + b(c + 5e)$
$= 3ac + 15ae + bc + 5be.$

Exercise. Show that the above results are true when $a = 1$, $b = c = 2$, $d = 4$, and $e = 3$.

Give the letters a different set of values, and show that the results are true for those values.

20. The expression $4(8-3)$ means that the difference between 8 and 3 is to be taken 4 times.

Hence $$4(8-3)=4\times 5=20.$$

The result can be found differently as follows: If we take 4×8, every time we have taken 8 instead of 5 we have taken 3 too many. Hence we have taken 3 too many 4 times, or 4×3 too many in all; and the result will be correct if we subtract that number.

Hence $$4(8-3)=4\times 8-4\times 3.$$

The same reasoning holds for any numbers; hence, in general,

$$\boldsymbol{a(b-c)=ab-ac}. \qquad (1)$$

Hence the law given in Art. 18 may be extended so as to read: *An expression is multiplied by multiplying each of its terms, whether they are to be added or subtracted.*

21.* To multiply $(c-d)$ by $(a-b)$, first take $(c-d)$ a times, which gives $a(c-d)$, or $ac-ad$. This result is too large, for in taking the multiplicand a times instead of $(a-b)$ times, we have taken it b times too many. Hence we must subtract $b(c-d)$, or $(bc-bd)$.

Hence $$\begin{aligned}(a-b)(c-d)&=a(c-d)-b(c-d)\\&=a(c-d)-(bc-bd)\\&=ac-ad-bc+bd.\end{aligned}$$

From this example we may deduce the following rule for the multiplication of algebraic quantities: *Multiply each term of the multiplicand by each term of the multiplier.*

* This article may be omitted at the discretion of the teacher.

When the two terms of a product have both + or both − before them, put + before their product; when one has + and the other −, put − before their product. In using the first terms of the expressions, which have no sign, apply the rule as if they had the + sign.

The rule for the signs may be briefly stated: *Like signs give + and unlike signs give −.*

Ex.
$$\begin{aligned}(2a-b)(3a-2b) &= 2a(3a-2b)-b(3a-2b)\\ &= 6aa-4ab-3ab+2bb\\ &= 6aa-7ab+2bb.\end{aligned}$$

EXERCISE. Show that this result is true when $a=3$ and $b=2$; when $a=6$ and $b=5$.

22. When a factor is to be taken more than once in a product, instead of repeating the factor the required number of times, it is written only once with a small figure to the right and a little above it. This figure shows how many times the factor is to be repeated, and is called an **exponent**.

Thus, $5^2=5\times 5$, $2^3=2\times 2\times 2$. The answer to the example at the end of the last article would be written

$$6a^2-7ab+2b^2.$$

It should be noticed that a coefficient and an exponent have very different meanings.

Thus
$$\begin{aligned}3\times 6 &= 6+6+6=18;\\ 6^3 &= 6\times 6\times 6=216.\\ 4a &= a+a+a+a;\\ a^4 &= a\times a\times a\times a=aaaa.\end{aligned}$$

Ex. 1. $2a \times 5a^2 = 2 \times 5aaa = 10a^3$.

Ex. 2. $3a(a - 4b) = 3a^2 - 12ab$.

Ex. 3. $(a+b)^2 = (a+b)(a+b) = a(a+b) + b(a+b)$
$$= a^2 + ab + ab + b^2 = a^2 + 2ab + b^2.$$

Exercise. Show that the result of Ex. 3 is true when $a = 20$ and $b = 4$; when $a = 40$ and $b = 3$; when $a = 4$ and $b = 3$.

EXAMPLES 5

Multiply:

1. $2a$ by $4a$.
2. $3a$ by $5cd$.
3. $6a^2$ by ab.
4. $2ab^2$ by $10a^2b$.
5. $4ab^2c$ by $2cx$.
6. $7ce$ by $5x^2y$.
7. $a + b$ by 3.
8. $3a - b$ by 5.
9. $a^2 + a$ by a.
10. $2a^2 - a$ by a^3.
11. $a^2 + 2a - 2$ by $3a$.
12. $bc + ca + ab$ by abc.

If $a = 3$, $b = 2$, $c = 1$, and $d = 5$, find the numerical values of:

13. $2a^2c$.
14. $5ab^2d$.
15. $a^2b^2c^2d^2$.
16. $6ab^2 + 3c^4$.
17. $5a^2c - b^4$.
18. $3(a^2 + b)$.
19. $a(d - c - b)$.
20. $3b^2(a^2 - ac)$.
21. $(a + b)(d - c)$.
22. $(a^2 + 2b)(a - b)$.
23. $(a + b)^2$.
24. $(2b + d)^3$.

Simplify the following by removing parentheses and combining like terms:

25. $2(a-b)+3(a+b)$.

26. $3a(b+c)-(ab+2ac)$.

27. $c(a+b)-c(a-b)$.

28. $7a(b-c)-2b(a-c)$.

29. $\frac{1}{2}(b-2c)-\frac{3}{4}(c-2b)$.

30. $2[3ab-4a(c-2b)]$.

Find:

31. $(x+y)^2$.

32. $(2a+b)^2$.

33. $(a^2+c)^2$.

34. $(a+b)(3a+2b)$.

35. $(a-b)(2a+5b)$.

36. $(a+b)(a-2b)$.

37.* $(a-b)^2$.

38. $(2x-3y)^2$.

39. $(a^3-c^2)^2$.

40. $(a-b)(2a-b)$.

41. $(2a-b^2)(5a-2b^2)$.

42. $(a+b)^3$.

DIVISION

23. We have already referred to division as the inverse of multiplication (Art. 9). It is the process by which, when the product of two factors (the **dividend**) and one of the factors (the **divisor**) are given, the other factor (the **quotient**) is found.

In consequence of this relation between the two processes, it is easy to derive the laws and rules of division from the corresponding laws and rules of multiplication. We shall proceed to do this.

24. Since a number is multiplied by multiplying any one of its factors (Art. 17), it follows that *a number is divided by dividing any one of its factors.*

Thus $$(abc)\div d=\frac{a}{d}\times bc=a\times\frac{b}{d}\times c=ab\times\frac{c}{d}. \qquad (1)$$

* This and the following are to be taken or omitted with Art. 21.

It will be seen from this that the result is the same whether the division is performed before or after any or all of the multiplications; hence:

Divisions may be performed in any order.

Ex. **1**. Show that 336 is divided by 2 by dividing any one of its factors 4, 6, and 14, by 2.

Ex. **2**. Show that 42 is divided by 3 by dividing any one of its prime factors by 3.

25. Since a number is multiplied by multiplying each of its parts (Art. 18), it follows that *a number is divided by dividing each of its parts.*

Thus $$(b + c) \div m = \frac{b}{m} + \frac{c}{m}. \qquad (1)$$

This law is employed in every numerical example in division. Thus the steps in the process of dividing 762 by 3 are as follows:

$$\begin{aligned} 762 &= 600 + 150 + 12, \\ 3\underline{)762} &= \tfrac{600}{3} + \tfrac{150}{3} + \tfrac{12}{3} \\ &= 200 + 50 + 4 \\ &= 254. \end{aligned}$$

If the method of solution seems unfamiliar, it is because we are accustomed to perform the separate steps mentally, and to put down only the result.

26. It follows from Art. 20, that *an expression is divided by dividing each of its terms, whether they are to be added or subtracted.*

Thus $$(a - b) \div c = \frac{a}{c} - \frac{b}{c}. \qquad (1)$$

EXAMPLES 6

Divide:

1. $15a$ by $5a$.
2. $12a^2$ by $4a$.
3. $8ab$ by $2b$.
4. $6x^2y^3$ by $2xy^2$.
5. $30a^2bc$ by $3ac$.
6. $a^3b^2cd^4$ by b^2d.
7. $12a^3+15a^2b$ by $3a$.
8. $18a^2b-12ac$ by $6a$.
9. $5x^2y-xy^3$ by $7xy$.
10. $a^2b+3bcd$ by $3a$.
11. $(a+b)^3$ by $a+b$.
12. $15a^3b^2+5a^2b^3-3a^4b$ by $5a^2b$.

PROBLEMS

NOTE. See Art. 8 for directions.

13. Three boys, counting their money, found they had 190 cents. The second had twice as many cents as the first, and the third as many as both the others, plus 4 cents. How many cents had each?

14. A cistern filled with water has two faucets, one of which will empty it in 5 hr., the other in 20 hr. How long will it take both to empty it?

15. If 12 be added to the half of a certain number, the sum will be 20. Find the number.

16. A farmer divided 52 apples among 3 boys in such a manner that B had $\frac{1}{2}$ as many as A, and C had 2 less than $\frac{3}{4}$ as many as A. How many had each?

17. The whole number of hands employed in a factory is 1000. There are twice as many boys as men, and 11 times as many women as boys. How many of each are there?

18. A and B invest equal amounts in trade. A gains \$1260, and B loses \$870; A's money is now double B's. What sum did each invest?

19. Divide 100 into two parts such that twice one part is equal to 3 times the other.

20. The sum of two numbers is 36, and their difference is half the greater. Find them.

21. A man of 40 has a son 10 yr. old. In how many years will the father be 3 times as old as the son?

22. A father's age is 3 times that of his son, and in 10 yr. it will be twice as great. How old are they?

23. A has \$15 more than B, B has \$5 less than C, and they have \$65 in all. How much has each?

24. In a regiment containing 1200 men, there were 3 times as many cavalry as artillery less 20, and 92 more infantry than cavalry. How many of each?

25. What are the ages of three brothers, whose united ages are 48 years, and their birthdays 2 years apart?

26. The difference of the squares of two consecutive numbers is 15. What are the numbers?

27. At the time of marriage, a man was twice as old as his wife; but 18 years later his age was $\frac{3}{2}$ times hers. Required their ages on the wedding day.

CHAPTER III

PERCENTAGE AND ITS APPLICATIONS

PERCENTAGE

27. Three closely related operations are frequently employed in commercial, or business arithmetic; namely:

I. To find a certain part of a number.

II. To find what part one number is of another.

III. To find a number when a certain part of it is given.

Ex. 1. What is $\frac{3}{5}$ of 50?

What part of 50 is 30?

What is the number of which 30 is $\frac{3}{5}$?

Ex. 2. A man had 75 sheep, and he sold $\frac{2}{3}$ of them. How many did he sell?

A man had 75 sheep, and he sold 50 of them. What part of his sheep did he sell?

A man sold 50 sheep, which was $\frac{2}{3}$ of all he had. How many had he at first?

It will now be seen what is meant by saying that these operations are *closely related.* The three questions in each of the examples involve the same three numbers, of which two are given and the third required; and any one of the three can be found if the other two are given.

28. It is customary in business to express the fraction that one number is of another in *hundredths,* even when the fraction can be readily reduced to lower terms.

Thus 4 is $\frac{50}{100}$ of 8; 3 is $\frac{10}{100}$ of 30; 1 is $\frac{2\frac{1}{2}}{100}$ of 40.

In stating problems, the denominator 100 is omitted, and the phrase *per cent,* which means *hundredths,* is used instead.

The sign % means *per cent.*

The following expressions exhibit the different ways of denoting a fractional part:

$$\frac{1}{2} = \frac{50}{100} = .50 \ = 50 \text{ per cent} = 50\%.$$

$$\frac{1}{8} = \frac{12\frac{1}{2}}{100} = .125 = 12\tfrac{1}{2} \text{ per cent} = 12\tfrac{1}{2}\%.$$

$$\frac{1}{200} = \frac{\frac{1}{2}}{100} = .005 = \tfrac{1}{2} \text{ per cent} = \tfrac{1}{2}\%.$$

$$\frac{5}{4} = \frac{125}{100} = 1.25 = 125 \text{ per cent} = 125\%.$$

$$1 = \frac{100}{100} = 1.00 = 100 \text{ per cent} = 100\%.$$

Note. It should be remembered that 100% of a number is *once* the number, or the number itself.

EXAMPLES 7 (Oral)

Name the corresponding fractions in lowest terms:

2%	15%	40%	80%	100%	$\frac{1}{2}$%
4%	20%	45%	85%	120%	$\frac{1}{6}$%
5%	25%	60%	90%	125%	$\frac{3}{10}$%
10%	35%	75%	95%	175%	$12\frac{1}{2}$%

EXAMPLES 8

Express as fractions in the lowest terms. The results should be memorized:

$6\frac{1}{4}\%$	$16\frac{2}{3}\%$	$62\frac{1}{2}\%$	$87\frac{1}{2}\%$
$8\frac{1}{3}\%$	$33\frac{1}{3}\%$	$66\frac{2}{3}\%$	$2\frac{1}{2}\%$
$12\frac{1}{2}\%$	$37\frac{1}{2}\%$	$83\frac{1}{3}\%$	$\frac{1}{2}\%$

EXAMPLES 9

Express as per cent. Memorize the first four columns:

$\frac{1}{3}$	$\frac{5}{6}$	$\frac{5}{8}$	$\frac{1}{200}$	$\frac{2}{7}$	$\frac{7}{20}$	$\frac{9}{80}$
$\frac{2}{3}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{300}$	$\frac{5}{19}$	$\frac{13}{200}$	$\frac{7}{45}$
$\frac{1}{6}$	$\frac{7}{8}$	$\frac{1}{16}$	$\frac{1}{150}$	$\frac{5}{18}$	$\frac{12}{61}$	$\frac{11}{900}$

29. In computations, per cent is expressed as a common fraction (in lowest terms) or as a decimal, according as the one form or the other is the more convenient.

Ex. 1. What is 20% of 85? (Case I, Art. 27.)
20% of a number is $\frac{1}{5}$ of it; and $\frac{1}{5}$ of 85 is 17.

Ex. 2. What per cent of 30 is 18? (Case II.)
18 is $\frac{18}{30}$ of 30; and $\frac{18}{30}$ is $\frac{3}{5}$, or 60%.

Ex. 3. 8 is 48% of what number? (Case III.)
Since 48 is 8% of the number, 1% of it is $\frac{1}{8}$ of 48, or 6; and 100% of the number is 100×6, or 600. Or,
Since 8%, or $\frac{2}{25}$, of the number is 48, $\frac{1}{25}$ of it is $\frac{1}{2}$ of 48, or 24; and $\frac{25}{25}$ of the number is 25×24, or 600.

Ex. 4. What number diminished by 5% of itself is 38? (Case III.)
$100\% - 5\%$, or 95%, of the number is 38. $95\% = \frac{19}{20}$. Hence the number is 20 times $\frac{1}{19}$ of 38, or 40.

EXAMPLES 10 (Oral)

Find

1. 5% of 120.
2. $12\frac{1}{2}\%$ of 72.
3. 25% of 96.
4. $33\frac{1}{3}\%$ of 66.
5. 8% of 300 sheep.
6. $6\frac{1}{4}\%$ of \$320.
7. $8\frac{1}{3}\%$ of 24 men.
8. 75% of 300 bu.

What per cent of

9. 40 is 15?
10. $12\frac{1}{2}$ is $2\frac{1}{2}$?
11. 1 da. are 16 hr.?
12. 1 lb. are 2 oz.?
13. 72 rd. are 18 rd.?
14. 1 T. are 2 cwt.?
15. 1 gal. are 3 pt.?
16. \$480 are \$24?

What is the number of which

17. 30 is 20%?
18. 84 is 12%?
19. 5 is $\frac{1}{2}\%$?
20. 16 is 32%?
21. 96 is $133\frac{1}{3}\%$?
22. 55 is 125%?
23. 72 is $66\frac{2}{3}\%$?
24. 15 is $16\frac{2}{3}\%$?

25. A farmer had 150 sheep, and sold $16\frac{2}{3}\%$ of them. How many did he sell?

26. A boy increasing his money by 25% of itself has \$1. What had he at first?

27. A grocer bought 150 dozen eggs, and found 20% of them bad or broken. How many were salable?

28. What number increased by $8\frac{1}{3}\%$ of itself is 130?

29. A clerk has his salary increased $12\frac{1}{2}\%$, and he then gets \$18 per week. What was his salary before the increase?

30. A man sold a horse for $100 at 20% above cost. Find the cost.

31. In a school of 75 pupils 3 were absent. What per cent was absent?

32. John has 36 cents, which is $37\frac{1}{2}\%$ of what his brother has. How much has his brother?

33. A clerk spends 88% of his salary and saves $144. What is his salary?

34. $37\frac{1}{2}\%$ of a stock of goods valued at $1200 was destroyed by fire. What was the loss?

30. **Percentage** includes all operations in which a per cent of a number is given or required.

The number of which the per cent is taken is called the **base.**

The per cent, when expressed decimally or as a common fraction, is usually called the *rate* per cent, or simply the **rate.**

The **percentage** is the result obtained by taking a certain per cent of the base. It is, therefore, a *product*, of which the factors are the *base* and the *rate.*

NOTE. It will be seen, from the two definitions of percentage, that the word is used (1) to name a class of *operations*, (2) to name the *result* of an operation.

31. If we use the initial letters of the words *base*, *rate*, and *percentage* to denote the *numbers* called by these names, we can easily express algebraically the relations that these numbers bear to one another. The equations expressing these relations are **percentage formulas.**

From the definition of percentage, we know that

$$p = br. \tag{1}$$

Ex. 1. A man invests \$1265, and gains 12% on his investment. How much does he gain?

$$b = \$1265,\ \ r = .12,\ \ p = ?$$

$$\begin{array}{r} \$1265 \\ .12 \\ \hline \$151.80 = p. \end{array}$$

Divide the members of (1) by b; then

$$\frac{p}{b} = r, \text{ or } r = \frac{p}{b}. \tag{2}$$

Here we have given a product (percentage) and one of the factors (base) to find the other factor (rate per cent).

Ex. 2. A merchant owes \$15,120, of which he can pay only \$9828. What per cent of his debts can he pay?

$$p = \$9828,\ \ b = \$15120,\ \ r = ?$$

$$r = \frac{\$9828}{\$15120} = .65 = 65\%.$$

Divide the members of (1) by r; then

$$b = \frac{p}{r}. \tag{3}$$

Ex. 3. What number increased by 18% of itself is 2950?

2950 is 100% + 18% or 118% of the number; hence

$$p = 2950,\ \ r = 1.18,\ \ b = ?$$

$$b = \frac{2950}{1.18} = 2500.$$

NOTE. In some problems, as in this one, one of the given numbers is not directly stated; but must be found from the conditions of the problem before the formula can be applied.

32. Percentage Formulas. The three cases of percentage and their formulas are:

CASE I. *To find a given per cent of a number.*

$$p = br. \tag{1}$$

CASE II. *To find what per cent one number is of another.*

$$r = \frac{p}{b}. \tag{2}$$

CASE III. *To find a number when a certain per cent of it is given.*

$$b = \frac{p}{r}. \tag{3}$$

EXAMPLES 11

1. A merchant failing was able to pay his creditors but 40%. He owes A \$ 3500, B \$ 1200, C \$ 1134, D \$ 650. What will each receive?

2. A person whose annual income is \$ 450 pays \$ 125 for board, \$ 140 for clothing, \$ 25 for books, and \$ 30 for sundries. What per cent of his income is each item, and what per cent remains?

3. The deaths in a certain city, during the year, are 980, which is $3\frac{1}{2}$% of the population. What is the population?

4. Sold cloth for \$ 3.50 per yard, which was 70% of its cost. What was the cost per yard?

5. A merchant failing owes \$ 3500; his property is valued at \$ 2100. What per cent of his indebtedness can he pay?

6. A shepherd lost 12% of a flock of sheep by disease, and then had 2200. How many were in the flock at first?

7. Sold a house and lot, which cost me \$ 1450.75, at a gain of 15%. What was the gain?

8. A man spent in one year \$ 2150, which was $5\frac{3}{8}$% of what he had. How much had he?

9. A man having \$ 5800 worth of hay lost \$ 870 worth by fire. What per cent of the whole was the part lost?

10. A tailor, after using 75% of a piece of cloth, had $9\frac{3}{4}$ yards left. How many yards were in the whole piece?

11. A man drew 25% of his bank deposits, and spent $33\frac{1}{3}$% of the money thus drawn in the purchase of a horse worth \$ 250. How much money had he in the bank at first?

12. A man owning $\frac{4}{5}$ of a cotton-mill, sold 35% of his share for \$ 24,640. What part of the whole mill did he still own, and what was its value?

PROFIT AND LOSS

33. *Gains, losses,* and *selling price* (S. P.) are always reckoned as a per cent of the cost; in other words, they are *percentages* computed on the cost as *base.*

The following table shows what quantities are denoted by the letters of the percentage formulas (Art. 32) in the various problems that occur in Profit and Loss:

TABLE

$$b = \text{cost}, \begin{cases} r = \text{rate of gain}, \ p = \text{profit}. & (1) \\ r = \text{rate of loss}, \ p = \text{loss}. & (2) \\ \left.\begin{array}{l} r = 1 + \text{rate of gain}, \\ r = 1 - \text{rate of loss}, \end{array}\right\} p = \text{S. P.} & \begin{array}{l}(3) \\ (4)\end{array} \end{cases}$$

34. Ex. 1. A man sells a farm for $2081.25, gaining 11%. What did the farm cost him?

(Case III, Art. 32 and (3) of Table.)

$$p = \text{S. P.} = \$2081.25 \ (= 111\% \text{ of cost}).$$

$$r = 1.11, \quad b = \text{cost} = ?$$

$$b = \$2081.25 \div 1.11 = \$1875. \ \textit{Ans.}$$

Ex. 2. At what price must goods that cost $3.50 per yard be sold to lose 20%?

(Case I, and (4) of Table.)

The S. P. will be 100% – 20%, or 80% of the cost.

$$b = \$3.50, \ r = 1 - .2 = .8, \ p = \text{S. P.} = ?$$

$$p = \$3.50 \times .8 = \$2.80. \ \textit{Ans.}$$

Ex. 3. Find the gain per cent on a horse sold for $72 at a gain of $9.50. (Case II, and (1) of Table.)

$$b = \$72 - \$9.50 = \$62.50 = \text{cost}.$$

$$p = \$9.50, \qquad r = ?$$

$$r = \$9.50 \div \$62.50 = .152 = 15\tfrac{1}{5}\%. \ \textit{Ans.}$$

EXAMPLES 12 (Oral)

1. Bought a cow for $ 40, and sold her for 20% above cost. What did I receive for her?

2. A watch that cost $ 25 was sold at a loss of 10%. What was the loss, and the selling price?

3. A tailor bought cloth at $ 6 a yard, and wished to sell it at a gain of $16\frac{2}{3}$%. At what price must he sell it?

4. A merchant sells silk at a profit of $ 1.50 per yard, which is $37\frac{1}{2}$% gain. What did it cost, and what is the selling price?

5. A watch was sold for $ 34, at a gain of $6\frac{1}{4}$%. What was the cost?

6. A dealer lost $12\frac{1}{2}$% on a reaper by selling it for $ 56. For what should he have sold it to gain $12\frac{1}{2}$% ?

7. Sold melons for $ .40 that cost $ .30. What was the gain per cent?

8. What per cent is gained on an article bought for $ 3 and sold for $ 5?

9. If corn selling for 21 cents a bushel yields a profit of 50%, what did it cost?

EXAMPLES 13

4. A man offers a farm, for which he gave $ 3450, for 20% less than its cost. What is his price?

Note. Computations are simplified by expressing the per cent as a common fraction when it is an aliquot part of 100.

5. For how much per barrel must I sell flour costing $ 4.50 per barrel to gain $16\frac{2}{3}$%?

Suggestion. To $ 4.50 add $\frac{1}{6}$ of it.

6. Sold a cargo of wheat for \$16,000, at a profit of 25%. What was the cost of the cargo?

7. A merchant made a profit of \$156 by selling a quantity of silks at a gain of 12%. What was the cost of the silks, and for how much were they sold?

8. A merchant marked a piece of carpeting 25% more than it cost him, but, anxious to effect a sale, and supposing he would still gain 5%, sold it at a discount of 20% from his marked price. Did he gain or lose?

SUGGESTION. S. P. = 80 % of marked price; marked price = 125 % of cost. S. P. = ? % of cost?

9. Sold a lot of books for \$480, and lost 20%. For what should I have sold them to gain 20%?

10. A man bought a pair of horses for \$450, which was 25% less than their real value, and sold them for 25% more than their real value. What was his gain?

SUGGESTION. The real value is the base in both operations.

11. A merchant pays \$6840 for a stock of goods, and sells them at an advance of $26\frac{1}{2}$% on the purchase price. After deducting \$375 for expenses, what is his gain?

12. A dealer bought 108 bbl. of apples at \$$4.62\frac{1}{2}$, and sold them so as to gain \$$114.88\frac{1}{2}$. What was his gain per cent?

13. My goods are marked to sell at retail at 40% above cost. I furnish my wholesale customers at 12% discount from the retail price. What per cent profit do I make on goods sold at wholesale?

SUGGESTION. 88 % of 140 % of cost, or $\frac{88}{100} \times 140$ % of cost = wholesale S. P.

14. At what price must shovels that cost \$1.12 each be marked in order to abate 5% (of marked price), and yet make 25% profit?

15. By selling coffee at 18 cents per pound, I make a profit of 20%. For how much must I sell it to make a profit of $16\frac{2}{3}$%?

16. Bought land at \$60 an acre. How much must I ask an acre, that I may deduct 25% from my asking price, and still make 20% on the cost?

17. Find the loss per cent on goods sold for \$425.98, at a loss of \$134.52.

18. Sold goods for \$3.50 less than cost, and lost 14%. What per cent should I have gained by selling for \$2.75 above cost?

19. Two sets of furniture were sold for \$35 each. On one there was a gain of $16\frac{2}{3}$%; on the other a loss of $16\frac{2}{3}$%. Was there a gain or a loss on both, and how much per cent?

20. A hardware merchant bought three dozen agate basins at the rate of 3 for \$5, and sold them at a gain of \$10 on the whole. What was the average selling price of each, and what was the gain per cent?

21. I bought a horse of Mr. A. for 15% less than it cost him, and sold it for 30% more than I paid for it. I gained \$15 in the transaction. How much did the horse cost me? How much did it cost Mr. A.? For what did I sell it?

22. If tea, when sold at a loss of 25%, brings \$1.25 per pound, what would be the gain or loss per cent if sold for \$1.60 per pound?

COMMISSION AND BROKERAGE

35. A person who buys and sells goods or lands, collects debts, or transacts other business of like nature for another person is called a **commission merchant** or **agent.**

The pay received for such services is called **commission.** It is usually a percentage on the *money paid* for property bought; on the *money received* for property sold; on the *money collected.*

A **broker** is a person who buys and sells stocks, bonds, bills of exchange, etc., for a commission, which is called **brokerage.**

The money that remains from a sale after the commission and other expenses have been paid is called the **net proceeds.**

TABLE

$$b = \left\{\begin{array}{l}\text{am't of sale,}\\ \text{money collected,}\end{array}\right\} \left\{\begin{array}{ll} r = \phantom{1 -{}} \text{rate of com.,} & p = \text{com.} \qquad (1)\\ r = 1 - \text{rate of com.,} & p = \text{proceeds.} \qquad (2)\end{array}\right.$$

$$b = \text{am't of purchase,} \quad r = 1 + \text{rate of com.,} \quad p = \text{remit.} \qquad (3)$$

36. Ex. 1. Find the commission on the sale at auction of a house and the furniture for \$9346.80 at $6\frac{1}{4}\%$.

$$b = \$9346.80,\ r = .06\tfrac{1}{4},\ p = ?$$
$$p = \$9346.80 \times .0625 = \$584.175 \text{ com.}$$

Ex. 2. Find the net proceeds of the above sale.

$$b = \$9346.80,\ r = 1 - .06\tfrac{1}{4} = .93\tfrac{3}{4},\ p = ?$$
$$p = \$9346.80 \times .93\tfrac{3}{4} = \$8762.625 \text{ proceeds.}$$

Or, from Ex. 1,

$$\$9346.80 - \$584.175 = \$8762.625 \text{ proceeds.}$$

Ex. 3. I send \$3120 to a commission merchant to buy flour at 4% commission. Find cost of flour and commission.

The remittance includes the investment + 4% of it; hence is 104% of the investment.

$$p = \$3120,\ r = 1 + .04 = 1.04,\ b = ?$$

$$b = \$3120 \div 1.04 = \$3000.\ \textit{Ans.}$$

EXAMPLES 14

4. Find the commission on the sale of a farm for \$13,750, at $2\frac{3}{4}$%.

5. A commission merchant sells 225 bbl. of potatoes at \$3.25 per bbl., and 316 bbl. of apples at \$4.50 per bbl. What is his commission at $4\frac{1}{2}$%?

6. A dealer sends his agent in Havana \$6720.80, with which to purchase fruits, after deducting his commission of 5%. What sum did the agent invest, and what was his commission?

7. If \$63 is paid for collecting a debt of \$1260, what is the rate of commission?

8. An architect charges $\frac{3}{8}$% for plans and specifications, and $1\frac{1}{2}$% for superintending the construction of a building which cost \$32,000. What is his fee?

9. My agent has purchased goods for me to the amount of \$12.50, for which he charges a commission of $1\frac{3}{4}$%. What sum must I remit to pay for goods and commission?

10. Sent to my agent in Cincinnati \$765 to purchase bacon, after deducting his commission of 2%. What is his commission, and what does he expend for bacon?

11. A grocer sends $2490 to a commission merchant to buy sugar at $3\frac{3}{4}\%$ commission. If he pays 8 cents a pound for the sugar, for what must the grocer sell the whole to gain $16\frac{2}{3}\%$ on the whole cost, and at how much per pound?

12. A collector collected rents at 3% commission and received $87.60 for his services. What sum of money did he collect?

13. I pay $275 for a lot and build on it a house costing $1720, which my agent rents for $25 a month, charging 5% commission. What per cent do I make a year on the money invested?

14. Find the commission on the sale of 100 bales of cotton, averaging 480 lb. to the bale, at $18 per cwt., the commission being 5%.

15. An agent sells 450 tons of hay at $13 a ton, commission 5%, and with the proceeds buys wool at $22\frac{1}{2}$ cents per pound, commission 4%. What is his whole commission, and how many pounds of wool does he buy?

16. An agent in Boston received 28,000 lb. of cotton, which he sold at $\$.12\frac{1}{2}$ per lb. He paid $45.86 freight and cartage, and after retaining his commission, he remits $3252.89 as the net proceeds of the sale. What was the rate of his commission?

17. A collector remits $1890 to his principal after deducting his commission of 10%. What was the amount collected?

18. A farm was sold for $9384 at a commission of $\frac{7}{8}\%$. Find commission and proceeds of sale.

19. Remitted to a stockbroker \$10,650, to be invested in stocks, after deducting $\frac{1}{4}\%$ brokerage. What amount of stock did he purchase?

20. A broker received \$45,337 to invest in bonds, after deducting a commission of $\frac{1}{2}\%$. What amount did he invest, and what was his commission?

COMMERCIAL DISCOUNT

37. Manufacturers and wholesale dealers avoid the inconvenience and expense of issuing price-lists of their goods with every change in their market value by determining upon a fixed **list price** for every article (largely in excess of its true value), from which they give their customers certain **discounts**, determined by current market prices.

Goods are frequently subject to two or more discounts (the last generally being for cash payment); and in such cases each discount is reckoned by itself on the sum remaining after subtracting the preceding discounts.

In stating commercial discount, the sign % is usually omitted.

38. Find the cost of a bill of goods amounting to \$800 at 20 and 5 off, and 5 off for cash.

$$\begin{array}{rl}
5)\underline{\$\,800} & = \text{list price of goods.} \\
\underline{160} & = 20\% \text{ discount.} \\
20)\underline{\$\,640} & \\
\underline{32} & = 5\% \text{ discount.} \\
20)\underline{\$\,608} & \\
\underline{30.40} & = 5\% \text{ off for cash.} \\
\$\,577.60 & = \text{cost of the goods.}
\end{array}$$

Or as follows:

$$\frac{\$\,\overset{2}{\cancel{8}\cancel{0}\cancel{0}} \times 4 \times 19 \times 19}{5 \times \cancel{20} \times \cancel{20}} = \$\,577.60. \quad \textit{Ans.}$$

EXAMPLES 15

1. Bought goods to the amount of \$ 650 at 10 off, and 5 off for cash. What was the cost?

2. Find the cost of a bill of goods marked at \$ 450 at 40% off, and 5% off for cash.

3. By getting a discount of 20, and 10 off for cash, I pay \$ 1080 for a bill of goods. What was the list price? What single discount would give the same reduction?

4. For what must I sell goods which were sold me for \$ 830, list price, at 30, 10, and 5 off, to gain 20%?

5. Find the amount of a bill of \$ 1560, discounts being 40, 25, and 5. Find the single equivalent discount.

6. Sold a bill of goods marked at \$ 250 for 30, and 5 off. How much more did I receive than if I had given a discount of 35%?

7. Paid \$ 655 for a bill of goods after a discount of $16\frac{2}{3}$%. What was the invoice price?

8. Find the cash value of a bill of cloth amounting to \$ 425.50 at a discount of 10%, and 5% off for cash. Find the equivalent single discount.

9. Find the cost of a stove listed at \$ 25, discounts being 10 and $7\frac{1}{2}$.

10. I paid \$ 1.50 for a book after a discount of 25%, and $16\frac{2}{3}$ off. What was its marked price?

INSURANCE

39. Insurance is a guaranty to pay a certain sum of money in case of loss or damage. It is classed as insurance on *property* and insurance on *life.*

That on property is called **fire insurance,** if against loss by fire; **marine insurance,** if against loss at sea; **stock insurance,** if against the loss of cattle, horses, etc.

The sum paid for obtaining the insurance is called the **premium,** and the written contract is called the **policy.**

The premium is a certain per cent of the sum insured, and is paid in advance. In life insurance it is generally paid annually.

Fire-insurance companies rarely insure property for more than two-thirds of its value, and in no case pay for more than the value of the property destroyed, whatever may be the face of the policy.

EXAMPLES 16

1. What is the premium for insuring goods for $14,500, at $1\frac{1}{2}\%$?

2. A house worth $15,000 is insured for $\frac{2}{3}$ of its value, at $\frac{3}{8}\%$. What is the premium?

3. A ship valued at $40,000 is insured for $\frac{3}{4}$ of its value, at $1\frac{1}{2}\%$, and its cargo, valued at $36,000, at $\frac{4}{5}\%$. What is the cost of insurance?

4. A merchant paid $1450 premium for the insurance of a cargo of cotton, the rate of insurance being $2\frac{1}{2}\%$. For what sum was the cargo insured?

5. If it cost $93.50 to insure a store for $\frac{1}{2}$ of its value, at $1\frac{3}{8}\%$, what was the store worth?

6. A merchant pays \$50 for an insurance of \$32,500 on a shipment of goods from New York to St. Louis. What is the rate of insurance?

7. A house valued at \$1200 had been insured for $\frac{2}{3}$ of its value for 3 years, at 1% per annum. During the third year it was destroyed by fire. What was the actual loss to the owner, no allowance being made for interest?

SUGGESTION. The difference between the amount of the insurance and the premium for the three years is what he gets from the insurance.

8. A merchant has his store and goods insured for \$5500 at $\frac{4}{5}$% premium. What is the cost to him? If the store and goods are destroyed, what sum does the insurance company lose?

9. An insurance company loses \$3528 by the wreck of a carload of flour which it had insured for \$3600. What was the rate of insurance?

10. A merchant insures a cargo of goods for \$81,800, which sum includes the value of the goods and the premium at $2\frac{1}{4}$%. What is the premium, and the value of the goods?

SUGGESTION. The premium is always computed on the *amount of insurance;* hence in this case the *base* is \$81,800.

11. A merchant ships \$31,360 worth of wheat from Chicago to Buffalo. For what must he get it insured at 2% so as to cover both the value of the wheat and the premium paid for its insurance?

SUGGESTION. b = amount of insurance = \$31,360 + premium. Hence \$31,360 is what per cent of b?

12. A merchant shipped a cargo of flour worth $3597 from New York to Liverpool. For what must he insure it, at $3\frac{1}{4}\%$, to cover the value of the flour and the premium?

13. I insure my life for $8000, paying $19.80 per $1000 per year. What do I pay the company if I live 20 years after insurance?

14. The annual premium on a life insurance at $2\frac{1}{4}\%$ is $126. What is the amount of the insurance?

TAXES

40. A **tax** is a sum of money assessed upon the inhabitants of a town, district, county, or state, or upon their property, to meet some public expense, such as the support of the schools, or of the government, or the building of public works.

A tax assessed, without regard to property, upon every male citizen within certain age limits (fixed by law) is called a **poll tax,** or **capitation tax.** A person so assessed is called a **poll.**

A **property tax** is assessed at a certain per cent on the estimated, or assessed, value of taxable property.

Taxable property is of two kinds: (1) **Real estate,** or fixed property; as houses and lands; (2) **Personal,** or movable property; as furniture, merchandise, cattle, money, etc.

EXAMPLES 17

1. What sum must be assessed to raise $83,600 net, after deducting the cost of collection at 5%?

REMARK. The cost of collection is 5% *of the amount collected.* See Art. 35, second paragraph, and (2) of the Table.

2. In a certain district, a school-house is to be built at a cost of \$18,500. What amount must be assessed to cover this and the collector's fees at 3%?

3. A county builds a bridge for \$4410. The property is valued at \$1,000,000. What is the tax per \$100, including the cost of collection at 2%?

4. In a certain town a tax of \$5000 is to be assessed. There are 500 polls, each assessed \$.75, and the valuation of the taxable property is \$370,000. What will be the rate of property tax, and how much will be A's tax, whose property is valued at \$7500, and who pays for 2 polls?

SUGGESTION. Subtract the amount to be raised by poll tax from the whole sum to be assessed; and find the per cent that the remainder is of the value of the taxable property. This is the rate of taxation.

5. A tax of \$11,384, besides cost of collection at $3\frac{1}{4}$%, is to be raised in a certain town. There are 760 polls assessed at \$1.25 each, and the personal property is valued at \$124,000, and the real estate at \$350,000. Find the tax rate, and find a person's tax whose real estate is valued at \$6750 and personal property at \$2500, and who pays for 3 polls.

6. In the above town, how much is B's tax on \$15,000 real estate, \$2750 personal property, and 2 polls?

7. What is C's tax on \$9786 and 1 poll?

8. How much taxes will a person pay whose property is assessed at \$7500, if he pays $\frac{3}{4}$% town tax, $\frac{1}{2}$% state tax, $1\frac{1}{4}$ mills on a dollar school tax?

9. I buy a lot for \$400 and build a house on it for \$2000. I pay an insurance on the house of $\frac{4}{5}\%$ on $\frac{3}{4}$ of its value, and a tax on the whole of 14 mills on a dollar, the property valuation being $\frac{2}{3}$ of the cost. For how much must I rent the house per month to realize 10% a year on my money?

10. A tax of \$56,000, including cost of collecting, is to be raised in a city on a property valuation of \$22,400,000. Assuming that the uncollectible tax will be 10% of the tax assessed, what will be the tax rate expressed in mills on a dollar?

11. In the above city, how much is A's tax on \$27,500?

DUTIES

41. The taxes levied on imported goods are called **customs** or **duties.**

Duties are of two kinds: specific and ad valorem.

A specific duty is a tax on goods according to weight, number, or measure, without regard to value.

An **ad valorem duty** is a percentage of the cost of goods in the country from which they are imported.

Many articles are subject to both kinds of duty.

Gross weight is the weight of goods including the boxes or other packing material.

Net weight is the weight after deducting the weight of the packing material.

Specific duties are calculated on the net weight of goods. All custom-house weights are *long-ton weights.*

The following list is taken from two successive tariffs of the United States. The new superseded the old July 24, 1897:

Articles	Old Rate of Duty	New Rate of Duty
Alcoholic perfumery	$ 2 per gal. and 50%	$ .60 per lb. and 45%
Earthen and crockery ware	30%	55%
Glass, cut, engraved, or painted . . .	35%	60%
Tin plate	$1\frac{1}{5}$ ct. per lb.	$1\frac{1}{2}$ ct. per lb.
Machinery	35%	45%
Cigars	$ 4 per lb. and 25%	$ 4.50 per lb. and 25%
Horses valued at $ 150 or less . .	20%	$ 30 per head
Wheat	20%	25 ct. per bu.
Cotton clothing, ready-made . . .	40%	50%
Cotton hosiery, valued at not more than $1 per doz. pairs	50%	$ .50 per doz. and 15%
Shirts and drawers valued at not more than $ 1.50 per doz.	50%	$ .60 per doz. and 15%
Collars and cuffs of linen	30 ct. per doz. and 3%	40 ct. per doz. and 20%
Laces and embroideries of linen . .	50%	60%
Silk velvets . . .	$ 1.50 per lb.	$ 1.50 per lb. and 15%
Lead pencils . . .	50%	45 ct. per gross and 25%

EXAMPLES 18 (Oral)

1. Which is the higher duty on horses valued at $ 150? Less than $ 150?

2. For what value of wheat are the two duties on that article equal? Which is the higher, and by how much, when wheat is worth $ .60?

3. What is the difference between the new and old duties on $1000 worth of cut glass? On $25,000 worth of machinery?

4. What is the cost per gross of lead pencils on which the two rates of duty are equal? Which is the greater for pencils worth more than that? For pencils worth less?

5. What was the old duty on $5000 worth of ready-made clothing? What is the new?

6. What is the new duty on 100 lb. of perfumery worth $2 per pound?

7. What is the new duty on a dozen collars valued at $1.20 per doz.?

8. What is the difference between the old duty and the new on a ton of tin plate?

EXAMPLES 19

Find the (new) duty:

1. On 1000 boxes of cigars, each containing 100 cigars, invoiced at $7.25 per box. Net weight 12 lb. per 1000.

2. On 12 gross lead pencils at $1.00 per gross.
   20 gross lead pencils at $2.25 per gross.
   5 gross lead pencils at $5.00 per gross.

3. On machinery invoiced at $26,500.

4. On 150 yd. silk velvet at $1.75 per yd. Net weight 75 lb.

5. On 15 doz. shirts at $1.50 per doz.
   20 doz. linen collars at $1.10 per doz.
   50 yd. linen lace at 5¢ per yd.
   7 doz. cotton hose at $.90 per doz.

EXAMPLES 20 (Miscellaneous)

1. A man had $5420 in bank. He drew out 15% of it, then 20% of the remainder, and afterwards deposited $12\frac{1}{2}\%$ of what he had drawn. How much had he then in bank?

2. If a man owning 45% of a steamboat sells $16\frac{2}{3}\%$ of his share for $5860, what is the value of the whole boat?

3. A man sold two houses at $2500 each; on one he gained 20%, on the other he lost 20%. What was his loss on the two sales?

4. A man bought a piece of property which afterwards increased in value each year at the rate of 25% on the value of the previous year, for 4 years; and was then worth $16,000. What did it cost?

5. After deducting $6\frac{1}{4}\%$ commission and $132 for storage, my agent sends me $23,654.25 as the net proceeds of a consignment of pork and flour. What was the amount of the sale?

6. After taking out 15% of the grain in a bin, there remained 40 bu. $3\frac{1}{5}$ pk. How many bushels were there at first?

7. The profits of a farm in 2 years were $3485, and the profits of the second year were 5% greater than those of the first year. What were the profits of each year?

8. If $\frac{4}{9}$ of a farm is sold for what $\frac{5}{8}$ of it cost, what is the gain per cent?

9. What is the cost of goods sold for $47,649, at a profit of $16\frac{2}{3}\%$?

10. A broker receives \$7125 to invest in cotton, after deducting his commission of $2\frac{1}{2}\%$. How many pounds of cotton can he buy at $11\frac{1}{4}$ cents a pound?

11. Sold a farm for \$14,700, and lost 12%. What per cent should I have gained by selling it for \$21,000?

12. I buy a house for \$6500 and spend \$500 for repairs. I rent it for \$77.50 a month, out of which I pay a yearly insurance of $\frac{3}{4}\%$ on $\frac{5}{7}$ of its whole cost, including repairs, and a yearly tax of 1% on $\frac{3}{4}$ of the same. What per cent of income a year do I realize on the whole cost?

13. For what sum must a policy be made out to cover the insurance on property worth \$2100, at $\frac{4}{5}\%$?

14. I bought a lot of coffee at 12¢ per pound. Allowing that the coffee will fall 5% short in weighing it out, and that 10% of the sales will be in bad debts, for how much per pound must I sell it to make a clear gain of 14% on the cost?

15. An agent sells for Johnson & Co. 3500 lb. of butter at 20¢ per pound, and 2580 lb. of cheese at 9¢ per pound, at a commission of 5%. He invests the balance in dry goods, after deducting his commission of $2\frac{1}{2}\%$ for purchasing. How many dollars' worth of goods do Johnson & Co. receive? What is the entire commission of the agent?

CHAPTER IV

APPLICATIONS OF PERCENTAGE INVOLVING TIME

42. The money paid for the use of money is called **interest.** It is always a percentage of the sum loaned.

The sum loaned is called the **principal.**

The rate per cent of the principal paid for its use for a certain time is called the **rate of interest.** It is understood to be for a year unless otherwise specified.

The sum of the principal and the interest is called the **amount.** It is the sum that the borrower must pay back to cancel his debt.

In computing interest for a fraction of a year, it is customary to reckon each month as $\frac{1}{12}$ of a year, and a day as $\frac{1}{30}$ of a month.

SIMPLE INTEREST

43. Ex. 1. What is the interest on $\$100$ for $2\frac{1}{2}$ yr. at 8%?

The interest for 1 yr. is 8% of $\$100$, or $\$8$, and for $2\frac{1}{2}$ yr. is $2\frac{1}{2} \times \$8$, or $\$20$. Or,

Since the interest is 8% of the principal for 1 yr., for $2\frac{1}{2}$ yr. it will be $2\frac{1}{2} \times 8\%$ or 20% of the principal; and 20% of $\$100 = \$20$.

Ex. 2. Find the interest and amount of $\$200$ for 3 yr. at 5%.

The interest is 15% of the principal, or \$30; the amount is \$200 + \$30 = \$230.

EXAMPLES 21 (Oral)

Find the interest and amount of:

3. \$100, at 6%, for 1 yr.; $2\frac{1}{2}$ yr.; 3 yr. 4 mo.
4. \$500, at 5%, for 6 mo.; 2 yr.; $2\frac{1}{2}$ yr.
5. \$50, at 12%, for 1 mo.; $1\frac{1}{2}$ yr.
6. \$1000, at 3%, for 1 yr.; 2 yr. 4 mo.
7. \$40, at 6%, for 2 mo.; 6 mo.; 10 mo.
8. \$5 at 10%, for 2 yr.; $3\frac{1}{2}$ yr.
9. \$10 at 6%, for 1 mo.; 9 mo.
10. \$300 for 6 mo., at 6%; at 8%.
11. \$60 for 8 mo., at 6%; at 12%.
12. \$200 for 3 mo., at 1% a month.
13. \$250 for $2\frac{1}{2}$ yr., at 4%; at 10%.
14. \$6 for 7 mo., at 1% a month.

44. To compute interest at 6 per cent. Reckoning a month as $\frac{1}{12}$ of a year and a day as $\frac{1}{30}$ of a month, the interest, at 6%,

for 1 yr. = .06 of the principal;
for 2 mo. = .01 of the principal;
for 1 mo. = .005 of the principal;
for 6 da. = .001 of the principal;
for 1 da. = $.000\frac{1}{6}$ of the principal.

Hence to find the decimal fraction of the principal that the interest, at 6%, for any given time is, *take 6 times the number of years and $\frac{1}{2}$ the number of months as hundredths, and $\frac{1}{6}$ the number of days as thousandths.*

Ex. 1. Find the interest of $375.50 for 3 yr. 5 mo. 21 da., at 6%.

The interest for

$$\begin{array}{rlll} 3 \text{ yr.} = & 3 \times .06 & = .18 & \text{of the principal;} \\ 5 \text{ mo.} = & 5 \times .005 & = .025 & \text{of the principal;} \\ 21 \text{ da.} = & 21 \times .000\frac{1}{6} & = \underline{.0035} & \text{of the principal;} \\ & & .2085 & \text{of the principal.} \end{array}$$

Or, following the rule exactly,

$$\begin{array}{rl} (3 \times .06) & = .18 \\ (\frac{5}{2} \times .01) & = .025 \\ (\frac{21}{6} \times .001) & = \underline{.0035} \\ & .2085 \end{array} \qquad \begin{array}{r} \$\,375.5 \\ \underline{.2085} \\ 7510 \\ 300 \\ \underline{19} \\ \$\,78.29 \end{array} \text{ interest.}$$

NOTE. In forming the multiplier, the operations indicated in parentheses should be performed mentally, only the results being set down.

EXAMPLES 22

Find the interest and amount, at 6%, of:

2. $760, for 1 yr. 9 mo. 27 da.
3. $179.50, for 1 yr. 1 mo. 8 da.
4. $325, for 2 yr. 11 mo. 6 da.
5. $758.75, for 3 yr. 2 mo. 16 da.

6. $1024.25, for 2 yr. 3 mo. 22 da.

7. $584.50, for 1 yr. 2 mo. 14 da.

8. $725.84, for 1 yr. 3 mo. 11 da.

9. $387.95, for 3 yr. 7 mo. 24 da.

10. $42.20, for 24 da.

45. To find the years, months, and days between two dates, add mentally to the earlier date first the years, then the months, then the days necessary to obtain the later date, in each case recording only the result.

Ex. Find the time from Jan. 26, 1895, to June 8, 1897.

From Jan. 26, 1895, to Jan. 26, 1897, 2 yr.; to May 26, 4 mo.; to June 8, 12 da. (4 in May and 8 in June). Time: 2 yr. 4 mo. 12 da.

NOTE. Observe that (1) the last day is counted, the first is not; (2) where in counting the days we pass from one month to the next, the whole number of days in the former is taken as *thirty* for any month of the year.

46. For rates other than 6 per cent, *the multiplier* is most readily found by the **six per cent method,** as follows:

First find the multiplier for the given time, at 6%; then

for 3% take $\frac{1}{2}$ of it;	for 7% add $\frac{1}{6}$;
for 4% subtract $\frac{1}{3}$ of it;	for 8% add $\frac{1}{3}$;
for 5% subtract $\frac{1}{6}$;	for 9% add $\frac{1}{2}$, etc.

For rates higher than 10%, it is easier to form a 12% multiplier with the months as hundredths, and $\frac{1}{3}$ the days as thousandths.

EXAMPLES 23

Find the interest on:

1. \$721.56, for 1 yr. 4 mo. 10 da., at 6%.
2. \$54.75, for 3 yr. 24 da., at 5%.
3. \$1000, for 11 mo. 18 da., at 7%.
4. \$3046, for 7 mo. 26 da., at 8%.
5. \$1821.50, from April 1 to Nov. 12, at 6%.
6. \$700, from Jan. 15 to Aug. 1, at 10%.
7. \$316.84, from Oct. 20, 1895, to March 10, 1897, at 7%.
8. \$127.36, from Dec. 12, 1893, to July 3, 1895, at $4\frac{1}{2}$%.

Find the amount of:

9. \$3146, for 2 yr. 3 mo. 10 da., at 7%.
10. \$1008.80, for 10 mo. 16 da., at $6\frac{1}{2}$%.
11. \$2000, for 15 da., at $12\frac{1}{2}$%.
12. \$137.60, for 127 da., at 10%.

NOTE. Count 30 da. to a month.

13. \$1671.64, from June 1, 1894, to April 1, 1896, at 7%.
14. \$250, from June 5, 1896, to Feb. 14, 1897, at 8%.
15. \$340.50, from May 25, 1895, to Sept. 9, 1897, at 9%.
16. \$25, for 93 da., at 12%.
17. \$145.20, for 1 yr. 11 mo. 29 da., at 7%.

18. \$450, for 3 yr. 2 mo. 21 da., at 8%.

19. A man engaged in business was making $12\frac{1}{2}\%$ annually on his capital of \$16,840. He quit his business, and loaned his money at $7\frac{1}{2}\%$. What did he lose in 2 yr. 3 mo. 18 da. by the change?

20. A speculator borrowed \$9675, at 6%, April 15, 1894, with which he purchased flour at \$6.25 per bbl. May 10, 1895, he sold the flour at $\$7\frac{3}{8}$ per bbl., cash. What did he gain by the transaction?

21. A man borrows \$1000 at 10% interest, and with it buys a note for \$1100, maturing in 5 mo., but which not being paid when due, runs 1 yr. 6 mo. beyond maturity, drawing interest at 6% after maturity. How much does he gain?

47. Accurate Interest. The common method of computing interest is accurate for whole years; but is not accurate for months, since no month is exactly $\frac{1}{12}$ of a year; nor for days, since a day is reckoned as $\frac{1}{360}$ of a year.

To compute accurate interest, find the interest for years by the common method; and for any fraction of a year, take as many 365ths of a year's interest as there are days.

NOTE. The number of days in each month can be remembered from the following:

"Thirty days hath September,
April, June, and November:
All the rest have thirty-one,
Except the second month alone,
Which has but twenty-eight, in fine,
Till leap-year gives it twenty-nine."

Ex. 1. Find the accurate interest on \$535 from July 25 to Oct. 3, at 6%.

Time:	July,	6 da.
	Aug.,	31
	Sept.,	30
	Oct.,	3
		70 da.

$$\frac{\$535 \times 6 \times 70}{10 \times 365} = \$6.16. \textit{ Ans.}$$

EXAMPLES 24

2. Find the interest in Ex. 1 by the common method.

3. Find the exact interest on 3 United States bonds, of \$1000 each, at 6%, from May 1 to Oct. 15.

4. What is the exact interest on a \$500 United States bond, at 5%, from Nov. 1 to April 10?

Find the exact interest on:

5. \$375, from June 12, 1896, to Dec. 14, 1897, at 7%.

6. \$5760, from Nov. 8, 1896, to March 1, 1897, at 6%.

7. \$12,085, from Sept. 4, to Dec. 17, at 5%.

8. \$1250, from April 1, to Dec. 7, at 6%.

9. What is the difference between the exact interest for 90 da. on \$1,000,000 of 6% bonds and the interest reckoned on the basis of 360 da. to the year?

PROBLEMS IN INTEREST

48. Interest Formulas. Since the interest and the amount in any problem in interest are percentages of the principal, the relation that these quantities bear to one another can easily be expressed by the formulas of Art. 32.

Thus, applying (1) Art. 32 to Ex. 2, Art. 43, we see that:

To find the interest, $b = \$200$, $r = .15$.

Hence p (the interest) $= \$200 \times 0.15 = \$30$.

To find the amount, $b = \$200$, $r = 1.15$.

Hence p (the amount) $= \$200 \times 1.15 = \$230$.

It will be seen that, in finding the interest, r is the product of two factors; namely, the rate of interest and the time. We shall obtain much more useful formulas by denoting *each* of these factors by a letter, and by always using the same letters to denote the same elements. For this purpose we shall use the initials of the names of the elements.

Thus, $p =$ the principal (*base*),

$r =$ the rate of interest (per annum, unless otherwise specified),

$t =$ the time (expressed in the *same denomination* as that for which the rate is given),

$i =$ the interest,

$a =$ the amount $= p + i$.

NOTE. It should be observed that p is not here a percentage of some number, as in the preceding formulas. It is the *base,* of which i and a are percentages.

49. The formula expressing the rule by which we have computed the interest in all preceding examples is

$$i = prt; \tag{1}$$

in which the product rt is *the multiplier* (Art. 46).

This equation involves four quantities, any one of which can be found if the other three are given. For we have only to solve the equation for the unknown quantity; then substitute the given values of the other quantities, and perform the indicated operations.

Ex. 1. What sum of money will gain \$84 interest in 2 yr., at 7%?

$$i = \$84,\ r = .07,\ t = 2,\ p = ?$$

Solve (1) for p by dividing its sides by rt, and interchange the sides; then

$$\boldsymbol{p = \frac{i}{rt}}. \tag{2}$$

Hence $$p = \frac{\$84}{.14} = \$600. \quad \textit{Ans.}$$

NOTE. The value of p is the answer to the question: 84 is 14% of what number? (Case III. Art. 32.)

Ex. 2. At what rate will \$300 gain \$60 in 4 years?

$$p = \$300,\ i = \$60,\ t = 4,\ r = ?$$

Divide the members of (1) by pt; then

$$\boldsymbol{r = \frac{i}{pt}}. \tag{3}$$

Hence $$r = \frac{\$60}{\$1200} = \frac{1}{20} = 5\%. \qquad \text{(Case II.)}$$

ANALYSIS. The interest on \$300 for 4 yr. is the same as the interest on \$1200 for *one* year, the rate remaining the same. Hence the rate is $\frac{60}{1200}$, or 5%.

NOTE. The t in all the interest formulas is really an *abstract multiplier;* its value is the *number* of years. The product of *dollars* and *years* is, of course, impossible.

Second solution and analysis:

$$r = \frac{\overset{15}{\$\,\cancel{60}}}{\$\,300 \times \cancel{4}} = 5\%.$$

The interest for 1 yr. is $\frac{1}{4}$ the interest for 4 yr., or \$15. Hence the rate is $\frac{15}{300}$, or 5%.

Ex. 3. In what time will \$500 gain \$60, at 4%?

$$p = \$\,500,\ i = \$\,60,\ r = .04,\ t = ?$$

Solving (1) for t, we have

$$t = \frac{i}{pr}. \qquad (4)$$

Hence $$t = \frac{\$\,60}{\$\,500 \times 0.04} = \frac{60}{20} = 3.$$ 3 yr. *Ans.*

ANALYSIS. For 1 yr. the interest is \$500 × 0.04, or \$20. Hence to gain \$60 it will take as many years as \$20 is contained *times* in \$60, or 3 yr.

50. From the definition of *amount* we have the formula

$$a = p + i. \qquad (1)$$

Hence, replacing i by its value prt,

$$a = p + prt, \qquad (2)$$

or $$a = p\,(1 + rt), \qquad (3)$$

Equation (3) expresses the fact that the amount may be obtained directly from the principal by adding 1 to the multiplier by which the interest is obtained.

Ex. 1. Find the amount of \$250 for $3\frac{1}{2}$ yr., at 6%.

The interest is 15% of the principal; hence the amount is 115% of it. The multiplier for the interest is .15; for the amount it is $1 + .15$, or 1.15.

$$a = \$250 \times 1.15 = \$287.5. \qquad \text{(Case I.)}$$

Ex. 2. What principal will amount to $267.90 in 2 yr., at 7%?

$$a = \$267.90, \ r = .07, \ t = 2, \ p = ?$$

Solve (3) for p by dividing its sides by $(1 + rt)$; then

$$p = \frac{a}{1 + rt}. \qquad (4)$$

Hence $$p = \frac{\$267.90}{1 + 0.14} = \frac{\$267.90}{1.14} = \$235. \quad \text{(Case III.)}$$

Subtract p from both sides of (2); then

$$a - p = prt,$$

or $$prt = a - p. \qquad (5)$$

Divide both sides of (5) by pt; then

$$r = \frac{a - p}{pt}. \qquad (6)$$

But $$a - p = i.$$

Hence, substituting, $$r = \frac{i}{pt};$$

which is the same as (3) Art. 49.

Divide the members of (5) by pr; then

$$t = \frac{a - p}{pr}. \qquad (7)$$

Replacing $a-p$ by i,

$$t=\frac{i}{pr};$$

which is the same as (4) Art. 49.

51. Interest Formulas. From the last two articles we obtain the following set of interest formulas. References are given to the corresponding percentage formulas.

Given the Principal, Rate, and Time; to find the Interest.

$$i=prt. \qquad \text{(Case I.)} \quad (1)$$

Given the Principal, Rate, and Time; to find the Amount.

$$\left.\begin{aligned} a&=p+prt,\\ a&=p\,(1+rt). \end{aligned}\right\} \qquad \text{(Case I.)} \quad (2)$$

Given the Interest (or Amount), Principal, and Time; to find the Rate.

$$\left.\begin{aligned} &r=\frac{i}{pt}, & r&=\frac{a-p}{pt};\\ \text{or}\quad &r\%=\frac{i}{p(1\%)t}, & r\%&=\frac{a-p}{p(1\%)t}. \end{aligned}\right\} \qquad \text{(Case II.)} \quad (3)$$

Given the Interest, Rate, and Time; to find the Principal.

$$p=\frac{i}{rt}. \qquad \text{(Case III.)} \quad (4)$$

Given the Amount, Rate, and Time; to find the Principal.

$$p=\frac{a}{1+rt}. \qquad \text{(Case III.)} \quad (5)$$

Given the Interest (or Amount), Principal, and Rate; to find the Time.

$$t=\frac{i}{pr},\quad t=\frac{a-p}{pr}. \qquad \text{(None.)} \quad (6)$$

EXAMPLES 25 (Oral)

Find:

1. Sum that will gain \$20 in 2 yr., at 5%.
2. Sum that will amount to \$228 in 2 yr., at 7%.
3. Rate at which \$400 will gain \$84 in 3 yr.
4. Time in which \$200 will gain \$56 at 7%.
5. Rate at which \$120 will gain \$60 in 10 yr.
6. Sum that will amount to \$350 in 15 yr., at 5%.
7. Time in which \$1000 will gain \$250 at 5%.
8. Sum that will gain \$900 in 3 yr., at 3%.
9. Rate at which \$5 will gain \$1 in 3 yr.
10. Sum that will amount to \$260 in 3 yr. 9 mo., at 8%.
11. Time in which \$100 will gain \$15, at 6%.
12. Rate at which \$50 will gain \$1.50 in 6 mo.
13. Sum that will gain \$40 in 6 mo., at 8%.
14. Rate at which \$200 will gain \$25 in $2\frac{1}{2}$ yr.
15. Time in which \$75 will gain \$5, at 4%.
16. Time in which \$150 will gain \$21, at 8%.
17. Rate at which \$400 will amount to \$460 in $2\frac{1}{2}$ yr.
18. Time it will take \$700 to amount to \$749, at 7%.
19. Rate at which any sum will double itself in 10 yr.; in 8 yr. 4 mo.; in 16 yr.; in 20 yr.
20. Time it will take money to double itself, at 5%; at 6%; at 8%.

EXAMPLES 26

Find:

1. The principal that will gain \$ 213 in 5 yr. 10 mo. 20 da., at 7%.

2. Sum that will amount to \$ 1028 in 4 mo. 24 da., at 7%.

3. Time in which \$ 1301.64 will amount to \$ 1522.92, at 5%.

4. Rate at which \$ 1350 will amount to \$ 1539 in 2 yr. 4 mo.

5. Sum that will amount to \$ 761.44 in 3 yr. 4 mo. 24 da., at 5%.

6. Rate at which \$ 1500 will gain \$ 252 in 2 yr. 4 mo. 24 da.

NOTE. $t = 2\frac{2}{5}$ yr. When the months and days are not expressible as a fraction with small terms, it is simpler to form a 1% multiplier for the given time, and solve by the second set of formulas (3). Solve **Ex.** 6 by both methods.

7. Time in which \$ 175.12 will gain \$ 6.43, at 6%.

8. Rate at which \$ 2085 will gain \$ 68.11 in 5 mo. 18 da.

9. Sum that will gain \$ 173.97 in 4 yr. 4 mo., at 6%.

10. Sum that will amount to \$ 1596 in 2 yr. 6 mo., at $5\frac{1}{2}$%.

PRESENT WORTH AND TRUE DISCOUNT

52. The **present worth** of any debt is the sum or principal which at the current rate of interest will amount to that debt when it becomes due.

The difference between the amount of the debt and its present worth is called the **true discount.**

Problems in *present worth and true discount* are solved by formula (5); in which a is the amount of the debt, and p its present worth (*base*).

Ex. 1. A merchant buys a bill of goods for \$700 on 3 months' time. What is the present worth of the debt, money being worth 6%?

$$p = \frac{a}{1+rt} = \frac{\$700}{1+0.015} = \frac{\$700}{1.015} = \$689.66. \quad \textit{Ans.}$$

EXAMPLES 27

2. What is the present worth and discount of a debt of \$1000 due in 1 yr. 6 mo., the current rate of interest being 6%?

3. A merchant buys goods for \$4200 on 4 mo. credit, but is offered a discount of 3% for cash. If money is worth $\frac{1}{2}$% a month, what is the difference?

4. Bought a house and lot for \$19,500 cash, and sold them for \$22,000, payable $\frac{1}{4}$ in cash and the remainder in 1 yr. 6 mo. How much did I gain, computing discount at 6%?

5. A merchant holds two notes, one for \$356.25, due Dec. 1, 1897, and the other for \$497.50, due Feb. 1, 1898. What would be due him in cash on both notes Sept. 15, 1897, discounting at 6%?

6. Which is the more profitable, to buy coal at \$8.75 per ton on 6 mo. credit, or at \$8.60 on 2 mo. credit, money being worth 7%?

7. What sum must I put at interest at 8% to liquidate a debt of $2500 due 3 yr. hence?

8. Bought a house for $7500, payable in 4 mo., and sold it for $7500 cash. If money is worth 6%, what did I gain?

9. Find the difference between the interest and true discount of $270 for 9 mo., at 8%.

BANK DISCOUNT

53. Promissory Notes.

$150.00. CAMBRIDGE, MASS., July 28, 1897.

Ninety days after date, I promise to pay Charles Bond One Hundred and Fifty Dollars, value received.

JOHN BRAINARD.

$2000.00. BERKELEY, CAL., May 15, 1896.

Sixty days after date, I promise to pay to the order of Frank Barnes Two Thousand Dollars, value received, with interest at 6%.

W. B. SLACK.

The above are examples of **promissory notes**; so called because they contain a promise to pay a certain sum, at a specified time, for value received.

The person who signs a note is called the **maker**; the person to whom or to whose order it is to be paid is called the **payee.** The sum named in a note is called its **face.**

In most states a note is not legally due till three days after the expiration of the time specified in the note.

These are called **days of grace.** They are counted in by bankers in discounting notes. A note is said to **mature** on the last day of grace.

To find the date of maturity when the time is expressed in days, count forward from the date of the note, the specified number of days plus the days of grace, reckoning the actual number of days in the months passed over. When the time is expressed in months, calendar months are always to be understood.

A note, like the second given above, made payable to the *order* of the payee, or one made payable to *bearer*, is a **negotiable** note; that is, it can be bought and sold.

54. Bank Discount. If the holder of a negotiable note sells it to a bank, he will receive the *amount of the note at maturity* minus a percentage of *that sum*, called the **bank discount**, which is computed at a certain per cent per month or per annum.

The sum received for the note is called the **proceeds** or **avails** of the note.

NOTE. Banks reckon 12 mo. of 30 da. each, or 360 da. to a year; and count the actual number of days in a given time.

Ex. 1. Find the discount and proceeds of the first note in the last article, if discounted at a bank Aug. 31, at 1% per month.

The term of discount = 93 da. − (3 da. in July + 31 da. in Aug.) = 59 da.

Discount = interest on \$150 for 59 da. at 1% per mo.
$= \$150 \times 0.019\frac{2}{3} = \$2.95.$
Proceeds $= \$150 - \$2.95 = \$147.05.$

Ex. 2. Find the discount and proceeds of the second note, discounted on the day it was made, at 6%.

Interest on \$2000 for 2 mo. 3 da. $=$ \$2000 $\times$ .0105 $=$ \$21.

Amount of note at maturity $=$ \$2021.

Discount $=$ interest on \$2021 for 2 mo. 3 da. $=$ \$21.22.

Proceeds $=$ \$2021 $-$ \$21.22 $=$ \$1999.78.

NOTE. The present worth of the note at 6% discount is, of course, its face; and the *true discount* on the amount at maturity is \$21. The excess of the bank discount above the true discount is equal to the interest on the true discount for the given time (\$21 $\times$ .0105 $=$ \$.22).

EXAMPLES 28

Find the date of maturity, the term of discount, and the proceeds of the following:

3. \$957$\frac{37}{100}$. BOSTON, July 27, 1897.

Three months after date, I promise to pay to the order of M. Levering Nine Hundred Fifty-seven and $\frac{37}{100}$ Dollars, value received.

Discounted Aug. 10, at 8%. T. J. JENNINGS.

4. \$916$\frac{25}{100}$. GLENDALE, CAL., Feb. 5, 1896.

Two months after date, we jointly and severally promise to pay C. R. Crowley, or order, Nine Hundred Sixteen and $\frac{25}{100}$ Dollars, value received, with interest at 8%.

Discounted Feb. 21, at 10%. JAMES LITTLE.
T. B. LONG.

5. $700.00. NEW YORK, April 10, 1897.

Four months after date, I promise to pay to the order of Edward Brill Seven Hundred Dollars, value received.

Discounted June 10, at 8%. A. B. GORDEN.

Write the following in the form of promissory notes, and find the proceeds:

6. Note of $650, given Jan. 8, 1897, payable 60 da. after date; discounted Feb. 1, at 1% per month.

7. Note of $1840, given July 5, 1897, payable in 30 da.; discounted July 8, at 1% per month.

8. Note of $2550, given May 3, 1897, payable in 3 mo., with interest at 6%; discounted May 3, at 6%.

NOTE. The note bears interest for 3 mo. 3 da.; but the term of discount is 95 da., which is 3 mo. 5 da., as banks reckon time.

9. Note of $56.25, given July 29, 1897, payable in 6 mo., with interest at 10%; discounted Oct. 1, at 1% per month.

ANNUAL INTEREST

55. If a note reads "with interest payable annually," or "with annual interest," the interest is *due* at the end of each year, and thereafter draws simple interest until paid. Interest so computed is called **annual interest**.

Ex. 1.

$1000.00. CHICAGO, Jan. 13, 1897.

Three and one-half years from date, I promise to pay Henry Ames, or bearer, One Thousand Dollars, for value received, with interest at 6%, payable annually.

M. J. CLARKSON.

Find the amount of the note at maturity (not counting days of grace), no interest having been paid.

The simple interest on the principal for $3\frac{1}{2}$ yr. = \$210.
The annual interest on the principal = \$60.

The first year's interest remains unpaid $2\frac{1}{2}$ yr.; the second, $1\frac{1}{2}$ yr.; the third, $\frac{1}{2}$ yr. This is equivalent to the use of \$60 for $(2\frac{1}{2}+1\frac{1}{2}+\frac{1}{2})$ yr., or $4\frac{1}{2}$ yr.

Interest on \$60 for $4\frac{1}{2}$ yr. = \$16.20.
Total interest = \$210 + \$16.20 = \$226.20.
Amount due at maturity = \$1000 + \$226.20 = \$1226.20.

EXAMPLES 29

Find the annual interest of:

2. \$8000 for 5 yr., at 6%.
3. \$1500 for 4 yr., at 7%.
4. \$3500 for 10 yr., at 8%.
5. \$575 for $9\frac{1}{2}$ yr., at 8%.
6. \$800 for 4 yr., at 7%.

COMPOUND INTEREST

56. In **compound interest,** the interest, when due, is added to the principal, thus forming a new principal for the next period. The interest may be compounded with the principal annually, semiannually, or quarterly, according to agreement. Annual periods are understood unless otherwise stated.

Ex. 1. What is the compound interest of $750 for 1 yr. 3 mo., at 8% per annum, payable semiannually?

$750	1st principal.
1.04	
$780	amount at end of ½ yr., 2d principal.
1.04	
$811.20	amount at end of 1 yr., 3d principal.
1.02	
$827.42	amount in 1 yr. 3 mo.
750.00	
$77.42	compound interest.

EXAMPLES 30

2. Find the simple interest, annual interest, and compound interest of $2500 for 6 yr., at 6%.

3. Find the amount of $350 in 3 yr., at 7% compound interest.

Find the compound interest on:

4. $1200, for 3 yr., at 5%, payable annually.

5. $864.50, for 4 yr., at 8%, payable annually.

6. $680, for 2 yr., at 7%, payable semiannually.

7. $460, for 1 yr. 5 mo. 18 da., at 6%, payable quarterly.

8. $1250, for 3 yr. 7 mo. 18 da., at 5%, payable semiannually.

9. $790, for 9 mo. 27 da., at 8%, payable quarterly.

10. What sum placed at simple interest for 3 yr. 10 mo. 18 da., at 7 %, will amount to the same as $1500 placed at compound interest for the same time, and at the same rate, payable semiannually?

11. How much must a father, at the birth of his son, set apart for his benefit, so that with the interest at 7 %, compounded semiannually, it may amount to $10,000 when his son shall become 21 years of age?

PARTIAL PAYMENTS

57. When partial payments are made on notes or other obligations bearing interest, they may be applied in either of two ways; namely:

(1) To the debt of principal, leaving the interest unpaid till the time of final settlement.

(2) To the debt of interest first, and the remainder, if any, to the principal.

There are other methods which are formed by various combinations of these two.

It will be seen from the following articles that by the first method the debt draws simple interest; by the second, compound interest.

NOTE. All interest is in effect compounded when it is paid, since it allows the lender to loan it again and so draw interest on interest, while, if not paid, the debtor has the use of the interest money without paying interest on it.

The acknowledgment of a partial payment, stating the time and amount of the same, is written on the back of the note; and is called an **indorsement**.

58. The first method mentioned in the last article is commonly used when partial payments are made on mercantile accounts which are past due, and on notes containing the words "*with interest*" and running for a year or less. It is called the **Merchants' Rule.**

Ex. 1. What is due Oct. 1, 1897, on a note of \$750, with interest at 6%, dated June 1, 1897, and bearing the following indorsements: July 1, \$100; Aug. 19, \$250?

Interest on 1st principal for 1 mo.	$= \$750 \times .005 = \$$	3.75
Interest on 2d principal for 1 mo. 18 da.	$= 650 \times .008 =$	5.20
Interest on 3d principal for 1 mo. 12 da.	$= 400 \times .007 =$	2.80
Total interest		\$11.75

Amount due Oct. 1 = \$400 + \$11.75 = \$411.75.

The same result is obtained by the following method of procedure, which is the usual way of stating

The Merchants' Rule

Find the amount of the note or debt from its date to the time of settlement.

Find the amount of each payment from its date to the time of settlement.

Subtract the sum of the amounts of the payments from the amount of the note or debt.

Thus, in the above example,

Amount of \$750 for 4 mo. is . . .		\$765.00
Amount of \$100 for 3 mo. is . . .	\$101.50	
Amount of \$250 for 1 mo. 12 da. is	251.75	353.25
Amount due		\$411.75

EXAMPLES 31

Write out the following in proper form on paper, placing the indorsements on the back, and solve by the Merchants' Rule:

2. Face, $1500. Date, Jan. 1, 1895. Interest, 6%. Indorsements: Aug. 7, 1895, $500; Dec. 7, 1895, $500. What is due Jan. 1, 1896?

3. Face, $480. Date, March 3, 1894. Interest, 7%. Indorsements: Sept. 3, 1894, $196.80; March 3, 1895, $214. Sept. 3, 1895, paid the amount due. Find it.

4. Face, $1000. Date, July 20, 1894. Interest, 8%. Indorsements: March 5, 1895, $50; July 5, 1895, $450. What was still due on the date of last payment?

5. Face, $1230. Date, Jan. 1, 1896. Interest, $5\frac{1}{2}$%. Indorsements: March 1, 1896, $98; June 7, 1896, $500; Sept. 20, 1896, $290; Dec. 10, 1896, $100. What is due Jan. 1, 1897?

6. Face, $800. Date, March 1, 1896. Interest, 10%. Indorsements: Aug. 10, 1896, $200; Sept. 1, 1896, $50; Jan. 1, 1897, $15. What was due March 1, 1897?

59. The second method of applying partial payments, mentioned in Art. 57, is generally employed in the case of interest-bearing notes that run for more than a year; but is also frequently used when the time is less than a year.

Under the application of this method, three cases may arise; namely, a payment may be (1) equal to, (2) greater than, or (3) less than, the interest accumulated at the time of the payment.

In the first case, the payment just cancels the interest, and the principal, or *interest-bearing debt*, remains unchanged. The debtor, in the end, pays just as much as if such payment had been deferred until he was able to make a payment large enough *to diminish the principal;* and, meanwhile, he loses the use of the payment.

In the second case, the principal is diminished; hence the total interest on the debt is diminished by such payment.

In the third case, if the unpaid balance of the interest were added to the principal, the interest-bearing debt would be increased. This would increase the total interest; and besides losing the use of such payment, the debtor would actually have more to pay, in the end, than if he had kept the money till he was able to make a sufficiently large payment to reduce the principal.

It was to prevent such manifest injustice to the debtor that the Supreme Court of the United States adopted the following rule:

The United States Rule

Find the amount of the principal to the time when the payment, or the sum of the payments, equals or exceeds the interest.

From this amount deduct the payment or sum of the payments.

Consider the remainder as a new principal, and proceed as before.

Ex. 1. A note of $ 500, dated Feb. 1, 1895, and bearing interest at 6%, is indorsed as follows: May 1, 1895,

$40; Nov. 14, 1895, $8; April 1, 1896, $18; May 1, 1896, $30. What was due Sept. 16, 1896?

$500.00	1st principal.
7.50	interest to May 1.
$507.50	
40.00	1st payment.
$467.50	2d principal.
15.04	interest to Nov. 14.
10.67	interest to April 1.
$493.21	
26.00	2d and 3d payments.
$467.21	3d principal.
2.34	interest to May 4.
$469.55	
30.00	4th payment.
$439.55	4th principal.
9.89	interest to Sept. 16.
$449.44	amount due.

NOTE. It will be seen from the note to Art. 57 that by the United States Rule the interest is compounded as often as a payment is made which equals or exceeds the unpaid interest.

EXAMPLES 32

2 to 6 inclusive. Solve Ex. 2 to 6 inclusive of the last article by the United States Rule, and compare the results with those obtained by the Merchants' Rule. Account for the difference in the results.

7. What was due Aug. 5, 1896, on a note for $2500, with interest at 7%, dated Aug. 5, 1895, and bearing the following indorsements: Jan. 1, 1896, \$500; March 10, 1896, \$750?

8. A note for \$16,500, dated May 20, 1896, and bearing interest at 7%, is indorsed as follows: Sept. 1, 1896, \$25; Oct. 14, 1896, \$150; March 20, 1897, \$45; July 5, 1897, \$300. Find the amount due Nov. 11, 1897.

9. Find the amount due Jan. 1, 1897, on a note for \$497.39, with interest at 6%, dated March 15, 1894, and indorsed as follows: Nov. 3, 1894, \$57.50; June 15, 1895, \$22.25; Aug. 1, 1895, \$125; Sept. 15, 1895, \$175.

10. A note for \$10,000 runs 4 yr., at 8% interest, on which were made quarterly payments of \$500. What was the amount due at the time of settlement?

11. On a note for \$1000, at 6% interest, payments were made as follows: in 1 yr., \$50; in 1 yr. 6 mo., \$250; in 2 yr., \$224; in 2 yr. 8 mo., \$20; in 2 yr. 10 mo., \$110. Find the amount due at the end of 4 yr.

CHAPTER V

PROPORTION. PARTNERSHIP. AVERAGE OF PAYMENTS

60. Ratio. The relative magnitude of two numbers, measured by the quotient of the first divided by the second, is called their **ratio.**

Thus the ratio of 12 to 3 is 4; of 9 da. to 4 da. is $2\frac{1}{4}$; of 3 pt. to 1 gal., is $\frac{3}{8}$.

Concrete numbers of different kinds can have no ratio to one another. For example, we cannot compare feet and pounds with respect to their magnitude. Moreover, concrete numbers of the same kind must be expressed in the same unit before their ratio can be taken.

A ratio is always an *abstract number,* and may be expressed as a rate per cent (Case II, Art. 32). Thus, in percentage, the rate is the ratio of the percentage to the base.

The ratio of any two numbers a and b is expressed by the notation $a:b$ or $\frac{a}{b}$; and a is called the **first term** of the ratio, or the **antecedent**; and b, the **second term,** or the **consequent.**

The product of two or more ratios is called a **compound ratio.**

Thus the ratio *compounded* of the ratios $3:4$ and $5:7$ is $15:28$; since $\frac{3}{4}\times\frac{5}{7}=\frac{15}{28}$.

PROPORTION

61. A statement of the equality of two ratios is called a **proportion,** and is expressed in three ways; thus:

$$\tfrac{4}{8} = \tfrac{6}{12},$$

$$4:8 = 6:12,$$

$$4:8::6:12.$$

The last is the usual notation, and is read "4 is to 8 as 6 is to 12."

The four terms of a proportion are said to be **proportional** or **in proportion.**

Thus 4, 8, 6, 12, are proportional.

The first ratio of a proportion is called the **first couplet**; the second ratio, the **second couplet.**

The first and fourth terms of a proportion are called **extremes**; and the second and third terms, **means.**

62. Denote any four proportional numbers by the letters a, b, c, d; then

$$a:b::c:d, \qquad (1)$$

or

$$\frac{a}{b} = \frac{c}{d}. \qquad (2)$$

Multiply the sides of this equation by bd; then

$$ad = bc. \qquad (3)$$

Hence the law:

(i.) *The product of the extremes of any proportion is equal to the product of the means.*

If equation (2) or (3) be solved for each of the quantities in succession, we obtain

$$a = \frac{bc}{d},\ d = \frac{bc}{a}; \tag{4}$$

$$b = \frac{ad}{c},\ c = \frac{ad}{b}. \tag{5}$$

From (4) we have the law:

(ii.) *The product of the means divided by either extreme will give the other extreme.*

From (5) we have the law:

(iii.) *The product of the extremes divided by either mean will give the other mean.*

From (i.) it follows that a proportion is verified, or proved, by showing that the product of the extremes is equal to the product of the means.

From (ii.) and (iii.) it follows that if three terms of a proportion are given, the fourth term can be found.

EXAMPLES 33

Verify the following proportions:

1. $12 : 1728 :: 1 : 144$.
2. $27.03 : 9.01 :: 16.05 : 5.35$.
3. $\frac{2}{3} : \frac{4}{7} :: \frac{3}{8} : \frac{9}{28}$.

Find the value of x in each of the following proportions:

4. $8 : 52 :: 20 : x$.
5. $12 : x :: 1 : 144$.
6. $x : 20 :: 120 : 50$.
7. $80 : 4 :: x : \frac{1}{2}$.

8. $2.5 : 62.5 :: 5 : x$.

9. $175.35 : x :: \frac{1}{8} : \frac{3}{7}$.

10. $4\frac{1}{2} : x :: 9\frac{3}{4} : 27\frac{1}{4}$.

11. $x : 9.01 :: 16.05 : 5.35$.

12. $\frac{3}{5} : x :: \frac{7}{8} : 59.0625$.

13. $\frac{5}{16} : \frac{3}{8} :: x : \frac{2}{5}$.

14. $x : 38\frac{1}{4} :: 8\frac{1}{2} : 76\frac{1}{2}$.

15. $7.5 : 18 :: x : 7\frac{1}{15}$.

63. If four numbers a, b, c, d, are proportional, that is, if

$$a : b :: c : d, \tag{1}$$

then it is also true that

$$b : a :: d : c, \tag{2}$$

$$a : c :: b : d, \tag{3}$$

$$c : a :: d : b. \tag{4}$$

This is easily proved; for, from (1), we know that

$$ad = bc, \qquad \text{(Art. 62.)} \tag{5}$$

or

$$bc = ad. \tag{6}$$

Divide the sides of (6) by ac; then

$$\frac{b}{a} = \frac{d}{c}, \text{ or } b : a :: d : c.$$

Similarly, dividing the sides of (6) by ab gives (4); and dividing the sides of (5) by cd gives (3).

Four other proportions can be obtained from the four given by interchanging the couplets.

Exercise. Express the proportionality of the numbers 5, 15, 7, 21, in as many ways as possible.

64. When any substance is sold at a fixed price per pound, the cost of any amount of it is so related to its weight that when the weight is doubled the cost is also doubled, when the weight is halved the cost is also halved, and so on. This relation is expressed by saying that the cost and the weight are *proportional.*

For example, if coffee is 20 cents per lb., 3 lb. will cost 60 cents, and 5 lb. will cost 100 cents; and

$$\frac{3\text{ lb.}}{5\text{ lb.}}=\frac{20\text{ ct.}}{100\text{ ct.}},$$

or $$3\text{ lb.} : 5\text{ lb.} :: 20\text{ ct.} : 100\text{ ct.}$$

And, in general, if a pounds of the coffee cost $\$ b$ and c pounds of it cost $\$ d$, then

$$a\text{ lb.} : c\text{ lb.} :: \$ b : \$ d;$$

which is the symbolical statement of the fact that the ratio of any two values of the cost is equal to the ratio of the corresponding weights.

Definition. One quantity is said to **be proportional to** another when the two are so related that the ratio of any two values of the one is equal to the ratio of the *corresponding* values of the other.

It should be observed that in the proportion, **corresponding values** *are both antecedents or both consequents.*

65. Definition. The **reciprocal** of a number is 1 divided by that number. Hence the reciprocal of a fraction is the fraction inverted, and the reciprocal of a ratio is the ratio formed by interchanging its antecedent and consequent.

If the number of men to do a given piece of work be doubled, the time required to do the work will be halved;

if 3 times as many men work, the time required will be $\frac{1}{3}$ as long; if only $\frac{1}{4}$ as many men work, it will take 4 times as long; and so on. This relation is expressed by saying that the time required to do the work is *inversely proportional* to the number of men working.

For example, if 1 man can do a piece of work in 48 da., 4 men can do it in $\frac{1}{4}$ of 48 da., or 12 da., and 6 men can do it in $\frac{1}{6}$ of 48 da., or 8 da.

Also, $$\frac{4 \text{ men}}{6 \text{ men}} = \frac{8 \text{ da.}}{12 \text{ da.}},$$

or $$4 \text{ men} : 6 \text{ men} :: 8 \text{ da.} : 12 \text{ da.};$$

from which it will be seen that the ratio of the two numbers of men is equal to the *reciprocal* of the ratio of the corresponding numbers of days.

Definition. One quantity is said to **be inversely proportional to** another when the ratio of any two values of the one is equal to the reciprocal of the ratio of the *corresponding* values of the other.

Observe that in the case of *inverse* proportionality, **corresponding values** *are both extremes or both means.*

PROBLEMS IN SIMPLE PROPORTION

66. A statement of the equality of two simple ratios is called a **simple proportion.**

Problems involving two pairs of quantities, proportional or inversely proportional, three of which quantities are given, can be solved by simple proportion.

Ex. 1. If 20 lb. of sugar cost $1.20, what will 45 lb. cost?

Let $\$x$ denote the cost of 45 lb. of sugar; then, since the cost is proportional to the weight,

$$20 \text{ lb.} : 45 \text{ lb.} :: \$1.20 : \$x.$$

Hence $\quad \$x = \dfrac{45 \text{ lb.}}{20 \text{ lb.}} \times \$1.20 = \$2.70.$ *Ans.*

NOTE. In concrete problems the product of the extremes or of the means will be the product of two concrete numbers, and this has no meaning. *A multiplier is necessarily an abstract number.* Hence in the above example we cannot multiply \$1.20 by 45 lb.; but we *can* multiply it by the ratio of 45 lb. to 20 lb., for all ratios are abstract numbers (Art. 60).

Since the ratio of 45 lb. to 20 lb. is the same as the ratio of 45 to 20, it is unnecessary to retain concrete denominations in the solution, except in the case of the number of the same kind as the answer. Hence we may proceed as follows:

$$\$x = \frac{45 \times \$\overset{.06}{\cancel{1.20}}}{\cancel{20}} = \$2.70.$$

The required quantity may be taken as any one of the four terms of the proportion, but it is customary to write it as the fourth.

SOLUTION BY ANALYSIS. If 20 lb. of sugar cost \$1.20, 1 lb. will cost $\frac{1}{20}$ of \$1.20, or 6¢, and 45 lb. will cost 45 times 6¢, or \$2.70.

Ex. 2. In how many days can 12 men do a piece of work that 60 men can do in 8 da.?

The number of men and the number of days are inversely proportional; hence

$$12 \text{ men} : 60 \text{ men} :: 8 \text{ da.} : x \text{ da.}$$

$$x \text{ da.} = \frac{60 \times 8 \text{ da.}}{12} = 40 \text{ da.}$$

Solution by Analysis. Since it takes 60 men 8 da., it will take 1 man 60 times 8 da., and it will take 12 men $\frac{1}{12}$ of 60×8 da., or $\frac{60 \times 8 \text{ da.}}{12}$.

EXAMPLES 34

Note. All problems in proportion can be solved by analysis. The learner should become familiar with both methods of solution.

3. If 20 yd. of cloth cost $ 180, find the cost of 45 yd.

4. If 18 bu. of wheat make 4 bbl. of flour, how many barrels will 200 bu. make?

5. How many men will be required to build 32 rods of wall in the same time that 5 men can build 10 rods?

6. If 5 sheep can be bought for $ 20.75, how many sheep can be bought for $ 398.40?

7. When 10 bbl. of flour cost $ 112.50, what will be the cost of 476 bbl. of flour?

8. If a train runs 30 mi. in 50 min., in what time will it run 260 mi.?

9. If a horse travels 12 mi. in 1 hr. 36 min., how far at the same rate will he travel in 15 hr.?

10. How many days will 12 men require to do a piece of work that 95 men can do in $7\frac{1}{2}$ da.?

11. If $\frac{3}{8}$ of an acre of land cost $ 60, what will $45\frac{3}{4}$ acres cost?

12. If by selling $ 5000 worth of dry goods a merchant gains $ 456.25, what amount must he sell to gain $ 1000?

13. If a pasture will feed 120 horses 81 da., how many horses will it feed 108 da.?

14. If a business yields \$700 profits in 1 yr. 8 mo., in what time will it yield \$1050 profits at the same rate?

15. If it takes a train $2\frac{1}{3}$ hr. to go a certain distance at the rate of 27 mi. an hour, how long will it take to go the same distance at the rate of 21 mi. an hour?

16. If 15 men can build a wall in 6 da., how many men would be required to build it in $4\frac{1}{2}$ da.?

17. If a quantity of provisions is sufficient to support 225 men 25 da., how many days will it support 75 men?

18. If 12 men earn \$78 in 4 da., how many men will earn \$$58\frac{1}{2}$ in the same time at the same wages?

COMPOUND PROPORTION

67. A statement of the equality of two compound ratios, or of a compound ratio and a simple one, is called a **compound proportion.**

For example, the equation

$$\frac{3}{4} \times \frac{2}{5} = \frac{9}{30}$$

may be expressed as a compound proportion; thus:

$$\left.\begin{matrix} 3:4 \\ 2:5 \end{matrix}\right\} :: 9:30;$$

in which form the product of the ratios written one above the other is understood. By *taking* the product, the proportion is reduced to a simple one. Thus the above becomes

$$6:20::9:30.$$

From the *meaning* of a compound proportion, it follows that the product of the extremes is equal to the product of the means, as in the case of a simple proportion; hence a missing term is found in the same way.

Ex. 1. Find the value of x in the proportion

$$\left.\begin{matrix}5:6\\3:7\end{matrix}\right\}::10:x.$$

The proportion means that

$$5 \times 3 : 6 \times 7 :: 10 : x.$$

Hence $$x = \frac{6 \times 7 \times 10}{5 \times 3} = 28.$$

68. The amount of work done by a *given number of men* is proportional to the time, and the amount of work done in a *given time* is proportional to the number of men. If *both* the time and the number of men vary, the amount of work is proportional to their *product*.

For example, 6 men in 4 da. can do 24 times as much work as 1 man in 1 da.

Hence, if 1 man can dig 2 rd. of ditch in 1 da., 6 men in 4 da. can dig $6 \times 4 \times 2$ rd., or 48 rd., and 5 men in 3 da. can dig $5 \times 3 \times 2$ rd., or 30 rd.

Also, $$\frac{6 \text{ men}}{5 \text{ men}} \times \frac{4 \text{ da.}}{3 \text{ da.}} = \frac{48 \text{ rd.}}{30 \text{ rd.}},$$

or $$\left.\begin{matrix}6 \text{ men} : 5 \text{ men}\\4 \text{ da.} \;\; : 3 \text{ da.}\end{matrix}\right\}::48 \text{ rd.} : 30 \text{ rd.};$$

that is, the ratio of the two amounts of work is equal to the product of the ratios of the corresponding numbers of men and days.

DEFINITION. One quantity is said to be proportional to the product of two or more other quantities when the ratio of any two values of that quantity is equal to the product of the ratios of the corresponding values of the others.

PROBLEMS IN COMPOUND PROPORTION

69. Ex. 1. If 18 men build 126 rd. of wall in 60 da., how many rods will 6 men build in 110 da.?

Symbolical statement:

$$\left.\begin{array}{l}18 \text{ men} : 6 \text{ men} \\ 60 \text{ da.} \;\; : 110 \text{ da.}\end{array}\right\} :: 126 \text{ rd.} : x \text{ rd.}$$

Hence $$x \text{ rd.} = \frac{60 \times 110 \times 126 \text{ rd.}}{18 \times 60} = 77 \text{ rd.}$$

SOLUTION BY ANALYSIS. One man in 60 da. will build $\frac{1}{18}$ of 126 rd., or $\frac{126}{18}$ rd.; 6 men in the same time will build 6 times as many rods, or $\frac{126 \times 6}{18}$ rd.; in 1 da. the 6 men will build $\frac{1}{60}$ of $\frac{126 \times 6}{18}$ rd., or $\frac{126 \times 6}{18 \times 60}$ rd., and in 110 da. they will build 110 times as many rods as in 1 da., or $\frac{126 \times 6 \times 110}{18 \times 60}$ rd.

Ex. 2. If 18 men build 126 rd. of wall in 60 da., how many men will it take to build 77 rd. in 110 da.?

$$\left.\begin{array}{l}126 \text{ rd.} : 77 \text{ rd.} \\ 110 \text{ da.} : 60 \text{ da.}\end{array}\right\} :: 18 \text{ men} : x \text{ men.}$$

Hence $$x \text{ men} = \frac{77 \times 60 \times 18 \text{ men}}{126 \times 110} = 6 \text{ men.}$$

Explanation of the Method. In problems of this class all the numbers occur in like pairs, except one which is of the same kind as the answer. Take this as the third term of the proportion; then the fourth term, when found, is the answer. Consider each of the pairs of numbers separately, forming a first couplet from each, as in simple proportion.

In solving by analysis, begin with the number like the answer, and consider the effect upon it of the given change in each of the other numbers, separately. Thus, in the analysis of Ex. 1, we considered the effect upon the number of rods caused first by the change in the number of men from 18 to 6 (the time remaining *unchanged*), then by the change in the number of days from 60 to 110 (the *number of men* remaining unchanged). In each case we first reason to 1 of the number that is changed.

EXAMPLES 35

3. If 8 men earn $ 320 in 8 da., how much will 12 men earn in 4 da.?

4. If it costs $ 41.25 to pave a sidewalk 5 ft. wide and 75 ft. long, what will it cost to pave a similar walk 8 ft. wide and 566 ft. long?

5. If 16 horses consume 48 bu. of oats in 12 da., how many bushels will 20 horses consume in 8 wk.?

6. What sum of money will gain $ 300 in 8 mo., if $ 800 gain $ 70 in 15 mo.?

7. If 10 men can cut 46 cords of wood in 18 da., working 10 hr. a day, how many cords can 40 men cut in 24 da., working 9 hr. a day?

8. What is the cost of $36\frac{1}{2}$ yd. of cloth $1\frac{1}{2}$ yd. wide, if $2\frac{1}{2}$ yd., $1\frac{2}{5}$ yd. wide, cost $ $3.37\frac{1}{2}$?

9. A contractor employs 45 men to complete a work in 3 mo. What additional number of men must he employ to complete the work in $2\frac{1}{2}$ mo.?

10. How many days will 21 men require to dig a ditch 80 ft. long, 3 ft. wide, and 8 ft. deep, if 7 men can dig a ditch 60 ft. long, 8 ft. wide, and 6 ft. deep in 12 da.?

11. When the shadow of a post 10 ft. 6 in. high is 12 ft. 3 in. long, what is the length of shadow of a post 8 ft. 9 in. high?

12. The shadow of a post 16 ft. 3 in. high is 5 ft. 5 in. long. What height of post will give a shadow 3 ft. 4 in. long?

13. If a vat 16 ft. long, 7 ft. wide, and 15 ft. deep holds 384 bbl., how many barrels will a vat $17\frac{1}{2}$ ft. long, $10\frac{1}{2}$ ft. wide, and 13 ft. deep hold?

14. What is the weight of a block of granite 8 ft. long, 4 ft. wide, and 10 in. thick, if a similar block 10 ft. long, 5 ft. wide, and 16 in. thick weighs 5200 lb.?

15. If it costs \$15 to carry 20 tons $1\frac{1}{2}$ mi., what will it cost at the same rate to carry 400 tons $\frac{1}{2}$ mi.?

16. If 6 laborers can dig a ditch 34 yd. long in 10 da., how many days will 20 laborers require to dig a similar ditch 170 yd. long?

17. If a man walk 192 mi. in 6 da., walking 8 hr. a day, how far can he walk in 18 da., walking 6 hr. a day?

PARTNERSHIP

70. The association of two or more persons for the purpose of carrying on business is called **partnership.**

The persons thus associated are called **partners**, and together they form a **company** or **firm.**

The money or property invested is called the **capital** or **stock.**

The money and property of all kinds belonging to a company, including the amounts due it, are called its **resources** or **assets**; its debts are called **liabilities.**

The profits and losses of a company are usually divided among the partners proportionally to the capital of each, if all invest for the same time; and proportionally to the product of capital and time if the times are different.

Problems in partnership are therefore solved by the same methods as other problems in proportion.

EXAMPLES 36

1. A and B form a partnership. A furnishes $400 capital and B $600. They gain $250. What is the profit of each?

SUGGESTION. (1) The whole capital is $1000; hence
$1000 : $400 : : $250 : A's share.
$1000 : $600 : : $250 : B's share.

(2) Each partner receives the same fraction (or per cent) of the whole gain that his capital is of the whole capital.

(3) The gain of each partner is the same fraction (or per cent) of his capital that the whole gain is of the whole capital.

2. A, B, and C traded in company. A put in $8000; B, $4500; and C, $3500. Their profits were $6400. What is each partner's share of the profits?

3. A and B, in trading for 3 yr., make a profit of \$4800. A invested $\frac{3}{5}$ as much stock as B. What is each man's share of the profits?

4. Brooks & Co. fail in business; their liabilities amount to \$22,000; their resources to \$8800. They owe A \$4275, and B \$2175.50. What will each of these creditors receive?

5. Four persons engage in manufacturing, and invest jointly \$22,500. At the end of a certain time A's share of the gain is \$2000; B's, \$2800.75; C's, \$1685.25; and D's, \$1014. How much capital did each put in?

6. Three partners, A, B, and C, furnish capital as follows: A, \$500 for 2 mo.; B, \$400 for 3 mo.; C, \$200 for 4 mo. They gain \$600. What is each partner's share?

SUGGESTION. The use of \$500 for 2 mo. is equivalent to the use of \$1000 for 1 mo.; of \$400 for 3 mo. to \$1200 for 1 mo.; of \$200 for 4 mo. to \$800 for 1 mo. Hence, divide the profits proportionally to 1000, 1200, and 800.

7. A, B, and C gain in trade \$8000. A furnishes \$12,000 for 6 mo.; B, \$10,000 for 8 mo.; and C, \$8000 for 11 mo. Apportion the gain.

8. Jan. 1, 1896, three persons began business with \$1300 capital furnished by A. March 1, B put in \$1000; and Aug. 1, C put in \$900. The profits at the end of the year were \$750. Apportion it.

9. In a certain firm B has 3 times as much capital as A, and C has $\frac{1}{2}$ as much as the other two. What is each one's share in a loss of \$786?

10. In a gain of $600 A received $\frac{1}{2}$; B, $\frac{1}{6}$; and C the remainder. If the whole capital was 12 times A's gain, what was the capital of each?

11. Two men receive $1000 for grading. One furnishes 3 teams 20 da., and the other 5 teams 30 da. If the first receives $100 for overseeing the work, what does each receive?

12. Two men contract to move $5316 cu. yd. of gravel at 25 cents a cu. yd., and agree to share the profits in the ratio of 2 to 3. They employ 5 teams 45 da., at $4 each per day. What did each make?

AVERAGE OF PAYMENTS

71. Ex. 1. A owes B $1200, of which $300 is due in 4 mo., $400 in 6 mo., and $500 in 12 mo. If he wishes to pay the whole debt at one time, when must he do so in order that neither party shall lose?

The loss that is here referred to is the loss of the *use* of money, which is really loss of *interest*.

If A should pay the debt at once, he would lose the use of $300 for 4 mo., $400 for 6 mo., and $500 for 12 mo.; to all of which he is entitled.

The use of	$300 for	4 mo. = the use of	$1 for	1200 mo.
The use of	400 for	6 mo. = the use of	1 for	2400 mo.
The use of	500 for	12 mo. = the use of	1 for	6000 mo.
	$1200			9600 mo.

Hence A is entitled to the use of $1 for 9600 mo., or to the use of the $1200 for $\frac{1}{1200}$ of 9600 mo., or 8 mo. That is, the whole debt will be due in a single payment in 8 mo.

EXAMPLES 37

2. On Dec. 1, 1896, a man gave three notes, the first for \$ 500, payable in 3 mo.; the second for \$ 750, payable in 6 mo.; and the third for \$ 1200, payable in 9 mo. Find the average time of payment.

3. Bought merchandise Jan. 1, 1895, as follows: \$ 350 on 2 mo., \$ 500 on 3 mo., \$ 700 on 6 mo. What is the average time of payment?

4. Find the average date for paying three bills, due as follows: May 31, \$ 100; June 18, \$ 150; July 9, \$ 200. (Compute each from May 31.)

5. If I borrow \$ 250 for 8 mo., how long should I lend \$ 400 to repay me an equal interest?

6. A person owes a debt of \$ 1680, due in 8 mo., of which he pays $\frac{1}{3}$ in 3 mo., $\frac{1}{4}$ in 5 mo., $\frac{1}{5}$ in 6 mo., and $\frac{1}{6}$ in 7 mo. When is the remainder due?

7. On a debt of \$ 2500, due in 8 mo. from Feb. 1, the following payments were made: May 1, \$ 250; July 1, \$ 300; and Sept. 1, \$ 500. When is the balance due?

8. Dec. 1, 1894, purchased goods to the amount of \$ 1200, on the following terms: 25% payable in cash, 30% in 3 mo., 20% in 4 mo., and the balance in 6 mo. Find the average time of payment and the cash value of the goods, computing discount at 7%.

CHAPTER VI

INVOLUTION AND EVOLUTION

72. Involution. *Review Art. 22.*

The product of equal factors is called a **power** of the factor thus repeated.

The factor taken once is called the **first power**; the product of two equal factors is called the **second power**; of three equal factors, the **third power**, and so on.

The second power of a number is also called the **square** of the number, because it is equal to the area of the square the length of whose side is the given number. For a similar reason the third power of a number is called its **cube.**

A number is said to be **squared** when its second power is taken, and to be **cubed** when its third power is taken.

The process of taking any power of a number is called **involution.**

EXAMPLES 38 (Oral)

Find the indicated power:

2^1	6^2	1^2	$\dfrac{1}{2^3}$	100^2	1.2^2
2^2	$(\frac{1}{3})^2$	1.1^2		$.7^2$	90^2
3^3	$\dfrac{1}{3^2}$	$.3^2$	$(\frac{5}{8})^2$	$.1^3$	30^3
2^4	$(\frac{3}{5})^2$	$(\frac{1}{2})^3$	10^2	$.2^3$	$.01^2$

73. To find a Power of a Product. Study carefully the following:

1. $6^2 = (2 \times 3)^2 = (2 \times 3)(2 \times 3) = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2 = 4 \times 9 = 36.$

2. $10^3 = (2 \times 5)^3 = (2 \times 5)(2 \times 5)(2 \times 5) = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2^3 \times 5^3 = 8 \times 125 = 1000.$

3. $(3\,ab^2)^2 = (3\,abb)(3\,abb) = 3 \times 3 \times aabbbb = 9\,a^2b^4.$

4. $(a^3)^2 = a^3 \times a^3 = aaa \times aaa = a^6,$

or $(a^3)^2 = (aaa)^2 = aa \times aa \times aa = a^6.$

A product is raised to any power by raising each of its factors to that power.

EXAMPLES 39 (Oral)

Find:

$(ac)^2$	$(6\,a^2b^3)^2$	$(\frac{1}{2}\,ae)^2$	$(a^2d)^4$
$(2\,ab)^3$	$2\,(ab)^2$	$\frac{1}{2}\,(ae)^2$	$\left(\dfrac{1}{2\,c}\right)^2$
$(5\,ab^2)^2$	$3\,a\,(ab^2)^2$	$\left(\dfrac{ab}{c^2}\right)^2$	$\dfrac{1}{(2\,c)^2}$

74. Evolution. The process of taking one of the equal factors of a number is called **evolution.** It is the inverse of involution.

One of the equal factors of a number is called a **root** of the number. One of the *two* equal factors of a number is called its **square root**; one of the *three* equal factors, its **cube root.**

Thus, since 25 is the square of 5, 5 is the square root of 25; since 27 is the cube of 3, 3 is the cube root of 27.

The square root of a number is indicated by the **radical sign** ($\sqrt{}$) placed before it; the cube root by $\sqrt[3]{}$.

Thus $\sqrt{25} = 5$, $\sqrt[3]{27} = 3$.

The figure placed above the radical sign indicates *what* root is to be taken, and is called the **index** of the root. If no index is written, 2 is understood.

If an expression consists of more than one term or factor, the root of the whole is indicated by placing the radical sign before the expression enclosed in parentheses or placed under a vinculum; otherwise the sign affects only the term or the factor immediately following.

Thus $\sqrt{}16 + 9 = 4 + 9 = 13$; $\sqrt{(16+9)} = \sqrt{16+9} = \sqrt{25} = 5$. $\sqrt{}9\,a^2 = 3\,a^2$; $\sqrt{9\,a^2} = 3\,a$.

EXAMPLES 40 (Oral)

Find:

$\sqrt{49}$	$\sqrt{1.21}$	$\sqrt{2500}$	$\sqrt{\frac{4}{9}}$	$12\,a - 2\sqrt{4\,a^2}$
$\sqrt{400}$	$\sqrt{.04}$	$\sqrt{1600}$	$\sqrt{\frac{25}{49}}$	$\sqrt{25\,a^2b^4} + a\sqrt{9\,b^4}$
$\sqrt[3]{125}$	$\sqrt{.81}$	$\sqrt{1.44}$	$\sqrt[3]{\frac{1}{8}}$	$\sqrt{}a^4b - c$
$\sqrt[3]{64}$	$\sqrt[3]{.001}$	$\sqrt[3]{1}$	$\sqrt[3]{\frac{27}{64}}$	$\sqrt{9\,b^4c^2} + 2\,b^2c$

75. The Square of the Sum of Two Numbers.

Review Arts. 18 and 19.

We have learned (Art. 18) that a number is multiplied by multiplying each of its parts, and that for this purpose it may be separated into parts, or terms, in any way.

If both multiplier and multiplicand consist of more than one term, their product is the sum of the partial products obtained by multiplying each term of the multiplicand by each term of the multiplier (Art. 19).

If multiplier and multiplicand are equal and are separated into parts in the same way, the case is like that of Art. 22, Ex. 3, and the exercise. It was there shown that

$$(a+b)^2 = a^2 + 2ab + b^2. \qquad (1)$$

Since a and b may be any two numbers, we have the law:

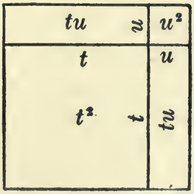
The square of the sum of two numbers equals the square of the first number plus twice the product of the numbers plus the square of the second number.

The operation of squaring a number of two figures is simplest when it is separated into its tens and units. When it is so separated, we shall use t to denote the tens, and u, the units. In this case formula (1) becomes

$$(t+u)^2 = t^2 + 2tu + u^2. \qquad (2)$$

EXERCISE. Express the meaning of (2) in words.

76. Illustration of the Formula. $(t+u)^2$ is the area of the square the length of whose side is $t+u$. The square may be divided into four parts, as shown in the figure. Comparing the right member of the formula with the figure, it will be seen that the first term is the area of the largest part; the second term is the area of the two rectangles; the last term is the area of the small square that fills out the corner, and is always the smallest of the terms.



$t^2 + 2tu + u^2$

$51 = .005.01$

NOTE. The square of any number may be found by the formula. For example, 324 = 32 tens + 4 units; hence $t = 320$, $u = 4$.

EXAMPLES 41

32 u

Find by the formula:

1. 56^2. 2. 73^2. 3. 208^2. 4. 315^2.

SQUARE ROOT

77. Find $\sqrt{784}$.

The problem may be stated thus: Find the side of the square whose area is 784 (sq. in., say).

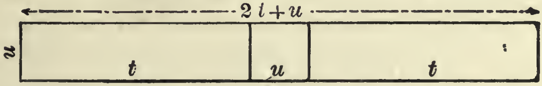
Or thus: Find t and u, when

$$t^2 + 2tu + u^2 = 784.$$

Begin by taking the largest value possible for t. This is easily seen to be 20.

$$\begin{array}{lrl} & t^2 + 2tu + u^2 & = 784 \text{ sq. in.} \\ \text{Subtract} & t^2 & = 400 \text{ sq. in.} \\ \hline \text{Remainder} & = \quad 2tu + u^2 & = 384 \text{ sq. in.} \end{array}$$

Compare with the figure in the last article. What is the remainder the area of? The two rectangles and the



small square have one dimension, u, in common. If placed as in the accompanying figure, they form one long rectangle whose dimensions are $2t + u$ and u, and whose area therefore is $(2t + u)u$.

How is the width of a rectangle found if its length and area are given? To find the width u, we are obliged to use $2t$, or 40, as the length, since the whole length is as yet unknown. This may give too large a value for u; if so, we take one less.

$$384 \div 2t = 384 \div 40 = 9+.$$

This is too large; for it gives

$$(2t + u)u = (40 + 9) \times 9 = 441,$$

and there are only 384 sq. in. Hence take $u = 8$.

This gives $\quad (2t + u)u = (40 + 8) \times 8 = 384.$

Hence $\quad \sqrt{784} = 40 + 8 = 48.$ *Ans.*

FORMULA FOR SQUARE ROOT

$$\begin{array}{l|l}
 & t^2 + 2tu + u^2 \;\underline{|\; t + u} \\
 & t^2 \\
\hline
2t & 2tu + u^2 \\
\underline{\quad + u} & \\
2t + u & \underline{2tu + u^2}
\end{array}$$

OPERATION

$$\begin{array}{r|l}
 & 784 \;\underline{|\; 20 + 8 = 28} \\
20^2 = & \underline{400} \\
2 \times 20 = 40 & 384 \\
\underline{8} & \\
48 & \underline{384}
\end{array}$$

$2t$ is called the **trial divisor.**

$2t + u$ is called the **complete divisor.**

The formula for the square may be written

$$(t + u)^2 = t^2 + (2t + u)u. \tag{1}$$

In this form, *the first term of the coefficient of u is the trial divisor. The whole coefficient is the complete divisor; it is the whole length of the addition to the square of the tens.*

EXAMPLES 42

Solve and prove:

1. $\sqrt{1156}$.
3. $\sqrt{5184}$.
5. $\sqrt{324}$.
2. $\sqrt{4225}$.
4. $\sqrt{841}$.
6. $\sqrt{9604}$.

78. Let any integer of three figures be separated into its hundreds, tens, and units, and denote these parts by the initial letters; then the number will be denoted by the expression $h + t + u$. Let us find its square.

$$\begin{array}{rrrrrr}
 & & & h+ & t & +u \\
 & & & h+ & t & +u \\
\hline
 & & & hu+ & tu & +u^2 \\
 & ht+t^2 & & & tu & \\
h^2+ & ht & + & hu & & \\
\hline
h^2+ & 2ht+t^2 & + & 2hu+ & 2tu & +u^2
\end{array}$$

Hence $(h + t + u)^2 = h^2 + t^2 + u^2 + 2ht + 2hu + 2tu$.

All we need observe here is that the square of the number contains the square of each of the figures plus other terms. This is true of any number.

Thus, the square of 48.7

$$\text{contains}\left\{\begin{array}{lcl} .7^2 & = & .49 \\ 8^2. & = & 64. \\ 4^2\ . & = & 16\ \ . \end{array}\right\}\text{plus other parts;}$$

The square of 12.34

$$\text{contains}\left\{\begin{array}{lcl} .\ 4^2 & = & .\ 16 \\ .3^2 & = & .09 \\ 2^2. & = & 04. \\ 1^2\ . & = & 1\ \ . \end{array}\right\}\text{plus other parts.}$$

It will be seen from this that if the complete square of any number be separated into groups of two figures each, commencing at the decimal point, the number of groups (counting the last figure to the left as a group, if it stands alone) will be equal to the number of figures in the root; and *the square of each figure of the root will lie wholly within the corresponding group.*

79. The square root of any number is found as follows: Separate the number into groups, as above directed, and proceed as in Art. 77, always regarding the part of the root already found as so many tens with respect to the next figure of the root.

Ex. 1. Find the square root of 75076.

FULL OPERATION

$$\begin{array}{r|l}
 & 7'50'76 \quad \underline{|\,274} \\
 & \underline{4} \\
2 \times 20 = 40 & 350 \\
\underline{7} & \\
47 & \underline{329} \\
2 \times 270 = 540 & 2176 \\
\underline{4} & \\
544 & \underline{2176}
\end{array}$$

CONTRACTED

$$\begin{array}{r|l}
 & 7'50'76 \quad \underline{|\,274} \\
 & \underline{4} \\
47 & 350 \\
 & \underline{329} \\
544 & 2176 \\
 & \underline{2176}
\end{array}$$

EXPLANATION. The first trial and complete divisors are obtained from the formula precisely as they would be if the given number were 750. That is $t = 20$ and $u = 7$. For the second divisors $t = 270$ and $u = 4$.

When the cipher is omitted from the trial divisor, as in the contracted operation, omit mentally the right-hand figure of the dividend in finding the figure of the root. Write the latter, when found, in units' place in the trial divisor, thus completing it.

Ex. 2. Extract the square root of 941.578.

$$
\begin{array}{r|l}
 & 9'41'.57'80 \;\underline{|\; 30.685} \\
 & 9 \\
606 & 4157 \\
 & 3636 \\
6128 & 52180 \\
 & 49024 \\
61365 & 315600 \\
 & 306825
\end{array}
$$

Explanation. Complete the last group to the right by the addition of a cipher. Since there is a remainder after using the last group, the root is not exact; but can be found to as many places as desired by annexing groups of ciphers.

The first trial divisor, 6, is contained 0 times in 4. Place 0 in the root; and annex 0 to the trial divisor, and the next group to the dividend.

To find the square root of a fraction, take the square root of its terms separately if they are seen to be perfect squares; otherwise it is best to reduce to a decimal first, as by so doing evolution is performed but once.

EXAMPLES 43

3. $\sqrt{13225}$.	9. $\sqrt{196.1369}$.	15. $\sqrt{\frac{169}{196}}$.
4. $\sqrt{11881}$.	10. $\sqrt{2.251521}$.	16. $\sqrt{\frac{625}{6561}}$.
5. $\sqrt{994009}$.	11. $\sqrt{58.140625}$.	17. $\sqrt{\frac{576}{18225}}$.
6. $\sqrt{20506.24}$.	12. $\sqrt{17.75}$.	18. $\sqrt{\frac{5}{6}}$.
7. $\sqrt{2985.5296}$.	13. $\sqrt{10795.21}$.	19. $\sqrt{30\frac{1}{4}}$.
8. $\sqrt{.001225}$.	14. $\sqrt{2\frac{3}{8}}$.	20. $\sqrt{69\frac{4}{9}}$.

21. A square field contains 1,016,064 sq. ft. What is the length of each side?

22. A square farm contains 361 acres. Find the length of one side.

23. A field is 208 rd. long and 13 rd. wide. What is the length of the side of a square field containing an equal area?

24. If it costs \$312 to enclose a field 216 rd. long and 24 rd. wide, what will it cost to enclose a square field of equal area with the same kind of fence?

25. Find the dimensions of a rectangular field containing 3200 sq. rd., and twice as long as broad.

26. How many rods of fence will enclose a square field of 4 acres?

27. How many rods of fence would be required to enclose a field of 4 acres whose length is twice its width?

28. What is the difference between the areas of two fields, one of which is 14 rd. square and the other 14 sq. rd.?

29. An orchard containing 2401 trees has as many rows as there are trees in a row. How many rows has it?

80. The Cube of the Sum of Two Numbers.

$$\begin{array}{r} (a+b)^2 = a^2 + 2\,ab + b^2 \\ a + b \\ \hline a^2b + 2\,ab^2 + b^3 \\ a^3 + 2\,a^2b + ab^2 \phantom{{} + b^3} \\ \hline a^3 + 3\,a^2b + 3\,ab^2 + b^3 \end{array}$$

Hence $\qquad \boldsymbol{(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3} \qquad (1)$

Exercise 1. State the formula in words, calling a "the first number" and b "the second number."

Solve the following by the formula, and verify by first taking the indicated sum, then cubing:

$$(2+3)^3, \quad (3+5)^3, \quad (12+9)^3, \quad (20+5)^3.$$

The solution is the simplest when the number is separated into its tens and units. For this case we shall write the formula thus:

$$(t+u)^3 = t^3 + 3\,t^2u + 3\,tu^2 + u^3. \qquad (2)$$

EXERCISE 2. Express formula (2) in words.

81. Illustration of the Formula.

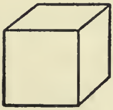


FIG. 1

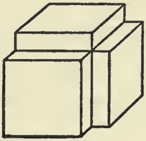


FIG. 2

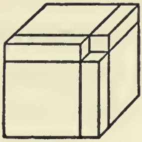


FIG. 3

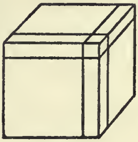


FIG. 4

$(t+u)^3$ is the volume of a cube, the length of whose edge is $t+u$. Such a cube can be formed from 8 solids, as follows: A cube whose edge is t, and whose volume is therefore t^3 (Fig. 1); 3 rectangular solids covering 3 adjacent faces of the cube and of thickness u, the volume of each being t^2u (Fig. 2); 3 rectangular solids filling the edges, the volume of each of which is tu^2 (Fig. 3); a small cube whose edge is u and whose volume is u^3, filling the corner (Fig. 4).

The formula and figures may be applied to any number, if we regard it as being made up of tens and units, as in Art. 76, note.

EXAMPLES 44

Solve by the formula:

1. 15^3. 2. 23^3. 3. 68^3. 4. 127^3.

CUBE ROOT

82. Find $\sqrt[3]{46656}$.

The problem is to find the edge of a cube whose volume is 46656 (cu. in., say), or to find t and u when

$$t^3 + 3t^2u + 3tu^2 + u^3 = 46656 \text{ cu. in.} \tag{1}$$

Begin by taking the largest value possible for t. This is 30; hence $t^3 = 27000$. Subtract from the corresponding members of (1); then

$$3t^2u + 3tu^2 + u^3 = 19656 \text{ cu. in.} \tag{2}$$

What is the remainder the volume of? (See the figures of the last article.) Observe that the seven additions to the cube of the tens have one dimension, u, in common, and that equation (2) may be written

$$(3t^2 + 3tu + u^2)u = 19656 \text{ cu. in.} \tag{3}$$

Suppose the seven solids to be laid side by side, forming one solid. The area of its base would be $3t^2 + 3tu + u^2$, its height would be u, and its volume would be the product of its base and its height, or $(3t^2 + 3tu + u^2)u$.

How is the height of a rectangular solid found when its volume and the area of its base are given? To find the height u we are obliged to use $3t^2$ as the area of the base, since the whole area is not yet known. If we find that this gives too large a value for u, we take one less.

$$19656 \div 3t^2 = 19656 \div (3 \times 30^2) = 7+.$$

By trial we find this too large; hence take $u = 6$.

Then

$$(3t^2+3tu+u^2)u = (3 \times 30^2 + 3 \times 30 \times 6 + 6^2) \times 6 = 19656.$$

Hence $\qquad \sqrt[3]{46656} = 30 + 6 = 36.$ *Ans.*

Formula for Cube Root

$$
\begin{array}{r|l}
 & t^3 + 3\,t^2u + 3\,tu^2 + u^3 \;\underline{|\; t + u} \\
 & t^3 \\ \hline
3\,t^2 & 3\,t^2u + 3\,tu^2 + u^3 \\
+\,3\,tu + u^2 & \\ \hline
3\,t^2 + 3\,tu + u^2 & 3\,t^2u + 3\,tu^2 + u^3 \\ \hline
\end{array}
$$

$3\,t^2$ is called the **trial divisor.**

$3\,t^2 + 3\,tu + u^2$ is called the **complete divisor.**

Solution of $\sqrt[3]{46656}$ by the formula:

$$
\begin{array}{r|l}
 & 46656 \;\underline{|\; 30 + 6 = 36} \\
 & 27000 \\ \hline
3 \times 30^2 = 2700 & 19656 \\
3 \times 30 \times 6 = 540 & \\
6^2 = 36 & \\ \hline
3276 & 19656 \\ \hline
\end{array}
$$

The formula for the cube may be written

$$(t + u)^3 = t^3 + (3\,t^2 + 3\,tu + u^2)\,u. \tag{4}$$

The first term in the parenthesis in the right member is the trial divisor; the whole expression within the parenthesis is the complete divisor.

EXAMPLES 45

1. $\sqrt[3]{15625}$.
2. $\sqrt[3]{166375}$.
3. $\sqrt[3]{10648}$.
4. $\sqrt[3]{912673}$.
5. $\sqrt[3]{42875}$.
6. $\sqrt[3]{474552}$.

83. By multiplying the square of $h + t + u$ (Art. 78) by the first power, the pupil may prove for himself that

$$(h + t + u)^3 = h^3 + t^3 + u^3 + \text{other terms}.$$

That is, the cube of a number of three figures contains, among other parts, the cube of each of the figures. The same is true of any number.

Thus the cube of 382.5

$$\text{contains}\left\{\begin{array}{rcl} .5^3 & = & .125 \\ 2^3. & = & 008. \\ 8^3\ . & = & 512\ . \\ 3^3\ \ . & = & 27\ . \end{array}\right\}\text{plus other parts.}$$

It will be seen from this that if the complete cube of any number be separated into groups of three figures each, commencing at the decimal point, the number of groups will be equal to the number of figures in the root; and *the cube of each figure of the root will lie wholly within the corresponding group.*

The last group to the left may contain only one or two figures.

To find the cube root of any number, separate it into groups as above directed, and proceed as in the last article, always regarding the part of the root already found as so many tens with respect to the next figure of the root.

If the last group to the right of the decimal point is incomplete, it must be completed by annexing ciphers. No such number has an exact cube root. Why not?

When a cipher occurs in the root, annex two ciphers to the trial divisor, and another group to the dividend.

If there is a remainder after the root of the last period is found, the result may be found to as many places as desired by annexing groups of ciphers.

Ex. 1. $\sqrt[3]{12812904}$.

$$\begin{array}{r|l}
 & 12'812'904 \;\underline{|\;234} \\
 & 8 \\ \hline
3 \times 20^2 = 1200 & 4812 \\
3 \times 20 \times 3 = 180 & \\
3^2 = 9 & \\ \cline{1-1}
1389 & 4167 \\ \hline
3 \times 230^2 = 158700 & 645904 \\
3 \times 230 \times 4 = 2760 & \\
4^2 = 16 & \\ \cline{1-1}
161476 & 645904 \\ \hline
\end{array}$$

Ex. 2. $\sqrt[3]{8710.37}$.

$$\begin{array}{r|l}
 & 8'710'.370 \;\underline{|\;20.57+} \\
 & 8 \\ \hline
3 \times 20^2 = 1200 & 710 \\
3 \times 200^2 = 120000 & 710370 \\
3 \times 200 \times 5 = 3000 & \\
5^2 = 25 & \\ \cline{1-1}
123025 & 615125 \\ \hline
3 \times 2050^2 = 12607500 & 95245000 \\
3 \times 2050 \times 7 = 43050 & \\
7^2 = 49 & \\ \cline{1-1}
 & 88554195 \\ \hline
\end{array}$$

EXAMPLES 46

3. $\sqrt[3]{1030301}$.
4. $\sqrt[3]{4492125}$.
5. $\sqrt[3]{1045678.375}$.
6. $\sqrt[3]{4080.659192}$.
7. $\sqrt[3]{.091125}$.
8. $\sqrt[3]{.000097336}$.
9. $\sqrt[3]{39.4995}$.
10. $\sqrt[3]{1250.6894}$.
11. $\sqrt[3]{\frac{1000}{1331}}$.
12. $\sqrt[3]{\frac{13824}{15625}}$.
13. $\sqrt[3]{2\frac{7}{8}}$.
14. $\sqrt[3]{\frac{1}{2}}$.

15. What are the dimensions of a cube that has the same volume as a box 2 ft. 8 in. long, 2 ft. 3 in. wide, and 1 ft. 4 in. deep?

16. How many square feet in the surface of a cube whose volume is 91125 cu. ft.?

17. What is the length of the inner edge of a cubical bin that contains 150 bu.? (1 bu. contains 2150.42 cu. in.)

18. What is the depth of a cubical cistern that holds 200 bbl. of water? ($31\frac{1}{2}$ gal. = 1 bbl.; 1 gal. = 231 cu. in.)

19. Find the length of a cubical vessel that will hold 4000 gal. of water.

20. What are the dimensions of a cubical box containing $\frac{1}{2}$ as much as one whose edge is 4 ft.?

CHAPTER VII

MENSURATION

84. The process of measuring lines, surfaces, and solids is called **mensuration.**

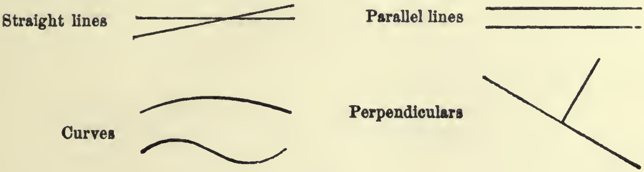
NOTE. All the rules and formulas of mensuration and the facts upon which they depend are proved in geometry. When statements are made without explanation in this chapter, it is not because none can be given, but because they cannot be understood without a knowledge of geometry.

85. Lines. A **straight line** has the same direction at every point.

A **curved line,** or **curve,** changes its direction at every point.

Parallel lines have the same direction; they are everywhere equidistant.

Two straight lines are said to be **perpendicular** to each other when the angle between them is a *right angle* (Art. 86).



86. Angles. An **angle** is the difference in the direction of two straight lines.

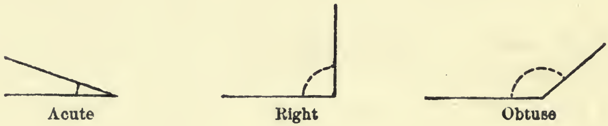
The lines are called the **sides** of the angle; and their point of meeting, its **vertex.**

Angles are measured in degrees, a **degree** being $\frac{1}{360}$ of the whole angular magnitude about a point. Thus the sum of all the angles that can be drawn with a common vertex at a point is 360°.

If two lines intersect so as to form four equal angles, each of these angles is a **right angle.** A right angle is $\frac{1}{4}$ of the angular magnitude about a point, and is therefore equal to 90°.

An **obtuse angle** is greater than a right angle.

An **acute angle** is less than a right angle.



PLANE FIGURES

87. A portion of a plane surface bounded by straight lines or curves is called a **plane figure.** The sum of the lines bounding the figure is called its **perimeter.**

88. Polygons. Any plane figure bounded by straight lines is called a **polygon.** The parts of a polygon are its **sides, angles,** and **vertices.**

A **diagonal** of a polygon is a straight line joining any two vertices not adjacent.

Polygons receive special names according to the number of their sides. A **triangle** has three sides; a **quadrilateral**, four; a **pentagon**, five; a **hexagon**, six.

A **regular polygon** has equal sides and equal angles.

Regular Polygons

Equilateral triangle — Square — Pentagon — Hexagon — Octagon

89. **Quadrilaterals** are classified as follows:

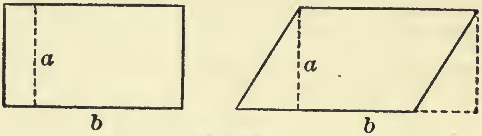
Quadrilaterals (3 classes).

- A **parallelogram** has its opposite sides parallel (4 classes).
  - A **rectangle** has four right angles. 
  - A **square** is a rectangle whose sides are equal. 
  - A **rhomboid** has no right angles. 
  - A **rhombus** has four equal sides and no right angles. 
- A **trapezoid** has only two parallel sides. 
- A **trapezium** has no parallel sides. 

90. Area of Parallelograms.

The **dimensions** of a parallelogram are its **base** (b) and its **altitude** (a).

Any side of a parallelogram may be taken as its base. Its altitude is the perpendicular distance between its base and the opposite side.



$$A = ba$$

The altitude of a rectangle is equal to the side not taken as base.

The area (A) of a parallelogram is equal to the product of its base and its altitude.

Explanation. This is a familiar fact in the case of rectangles; the common form of statement for this case being that the area of a rectangle is equal to the product of its length and width.

From any rhomboid or rhombus, a rectangle *of the same dimensions* can be constructed, by cutting off the triangular portion from one end and fitting it on to the other, as shown in the figure. This change in the *form* of the figure does not change its *area*, since the figure is composed of the same parts as before, only differently placed. It is clear that the area of the figure is now the product of its two dimensions; hence it was equal to the product of these dimensions before its form was changed.

Definition. Figures having the same area are called **equivalent figures**.

It follows from what has been said above that parallelograms having equal bases and equal altitudes are equivalent.

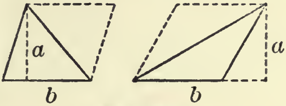
91. Area of Triangles.

The dimensions of a triangle are its **base** and its **altitude.**

Any side of a triangle may be taken as the base. The altitude is the length of the perpendicular from the base to the opposite vertex.

The area of a triangle is equal to one-half the product of its base and altitude.

EXPLANATION. Add to the given triangle an *equal* triangle inverted. The two together form a parallelogram having the same base and altitude as the given triangle. Since the area of the parallelogram is ba, the area of the triangle is $\frac{1}{2}ba$.



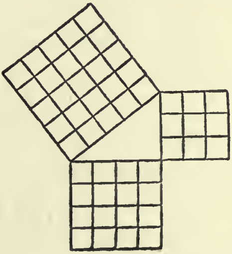
$A = \frac{1}{2}ba$

It follows that triangles having equal bases and equal altitudes are equivalent.

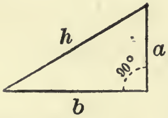
92. A triangle having one right angle is called a **right triangle.**

The side opposite the right angle is called the **hypothenuse,** and the other two sides, the **legs.**

If a right triangle be constructed having legs 3 and 4 units in length respectively, the hypothenuse will be found to be 5 units long; hence the sum of the areas of the squares constructed on the legs will be equal to the area of the square constructed on the hypothenuse. It can be proved that this relation is true of any right triangle. That is:



The square of the hypothenuse of a right triangle is equal to the sum of the squares of the other two sides.



$h^2 = b^2 + a^2$

$b = \sqrt{h^2 - a^2}$

$a = \sqrt{h^2 - b^2}$

This relation is expressed by the formula $h^2 = b^2 + a^2$, where a and b are the legs of the right triangle and h is the hypothenuse. Subtract a^2 from both sides and interchange members; then $b^2 = h^2 - a^2$. Take the square root of both sides, and we have $b = \sqrt{h^2 - a^2}$. In the same way we obtain $a = \sqrt{h^2 - b^2}$.

If any two sides of a right triangle are given, the third side can be found from these formulas.

EXAMPLES 47

1. The base of a rhombus is 10 ft. 6 in., and its altitude 8 ft. What is its area?

2. How many acres in a piece of land in the form of a rhomboid, the base being 8.75 chains, and the altitude 6 chains?

3. Find the area of a triangle whose base is 12 ft. 6 in., and altitude 6 ft. 9 in.

4. What is the cost of a triangular piece of land whose base is 15.48 chains, and altitude 9.67 chains, at \$60 an acre?

5. Find the area of the gable end of a house that is 28 ft. wide, and the ridge of the roof 15 ft. higher than the foot of the rafters.

6. What is the base of a triangle whose area is 189 sq. ft., and altitude 14 ft.?

7. Find the altitude of a triangle whose area is $20\frac{1}{4}$ sq. ft., and base 3 yd.

8. The legs of a right triangle are 12 in. and 16 in. respectively. What is the length of the hypothenuse?

9. The foot of a ladder is 15 ft. from the base of a building, and the top reaches a window 36 ft. above the base. What is the length of the ladder?

10. Find the distance diagonally across a floor 30 by 40 ft.

11. What is the length of a path diagonally across a 10-acre square field?

12. A room is 20 ft. long, 16 ft. wide, and 12 ft. high. What is the distance from one of the lower corners to the opposite upper corner?

13. The hypothenuse of a right angle is 35 ft., and one leg 28 ft. Find the other leg.

14. A ladder 52 feet long stands against the side of a building. How many feet must it be drawn out at the bottom to lower the top 4 ft.?

15. Find the diagonal of a cube containing 729 cu. in.

16. What is the side of a square field whose diagonal is 15 rods? What is its area?

17. A ladder 28 ft. long, placed in a street, reaches the top of a building 18 ft. high on one side, and one 15 ft. high on the other. How wide is the street?

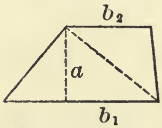
18. Two vessels sail from the same point, one north 58 miles, and the other west 72 miles. How far apart are they?

93. Area of a Trapezoid.

The parallel sides of a trapezoid are called the **bases**; and the distance between them is the **altitude.**

The area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases.

EXPLANATION. Draw a diagonal. This divides the trapezoid into two triangles whose common altitude is the altitude of the trapezoid. The base of one of the triangles is the lower base of the trapezoid, the base of the other is the upper base of the trapezoid.



$A = \frac{1}{2} a(b_1 + b_2)$

The area of the one triangle is $\frac{1}{2} ab_1$, of the other is $\frac{1}{2} ab_2$; the area of the trapezoid is the sum of the areas of the triangles. Hence

$$A = \tfrac{1}{2} ab_1 + \tfrac{1}{2} ab_2 = \tfrac{1}{2} a(b_1 + b_2).$$

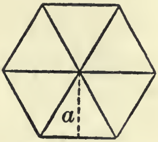
NOTE. The subscripts ($_1$ and $_2$) used here and in subsequent articles have no numerical signification, but are merely used to distinguish between two values of the same letter.

94. The area of a trapezium can be found as the sum of the areas of two triangles, if the length of a diagonal and

of the two perpendiculars from it to the opposite vertices are known.

The area of any polygon can be found by dividing it into triangles, computing their areas separately, and adding the results.

95. Area of Regular Polygons. Every regular polygon can be divided into equal triangles having a common vertex at the centre of the polygon.



$A = \frac{1}{2}ap$

The altitude of these triangles is called the **apothem** of the polygon.

In the figure the dotted line is the apothem.

Since the area of the polygon is the sum of the areas of these equal triangles, it follows that:

The area of a regular polygon is equal to one-half the product of its perimeter (p) *and apothem* (a).

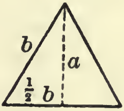
EXAMPLES 48

1. Find the area of a trapezoid whose bases are 23 ft. and 11 ft., and the altitude 9 ft.

2. One side of a quadrilateral field measures 38 rd., the side opposite and parallel to it measures 26 rd., and the distance between the two sides is 10 rd. Find the area.

3. Find the area of a trapezium whose diagonal is 42 ft., and the perpendiculars to this diagonal from the opposite vertices are 16 ft. and 18 ft.

4. Derive the formula $a = \frac{b}{2}\sqrt{3}$ for the altitude (a) of an equilateral triangle, the length of whose sides is b.

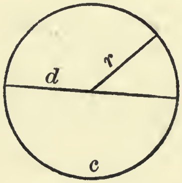


5. Find by the formula of Ex. 4 the altitude of an equilateral triangle whose sides are 9 in.

6. Find the area of a regular hexagon whose sides are 10 in.

SUGGESTION. A regular hexagon is divided into six equilateral triangles by its three diagonals passing through the centre. Hence the apothem can be found by the formula of Ex. 4.

96. The Circle. A **circle** is a plane figure bounded by a curve, called the **circumference**, all points of which are at an equal distance from a point within called the **centre**.



The **radius** (r) of a circle is the distance from its centre to its circumference (c). Its **diameter** (d) is the distance across it measured through the centre. Hence $d = 2r$.

97. The ratio of the circumference of a circle to its diameter *is the same for all circles*, and is generally denoted by the Greek letter π (pronounced *pie*). That is, $\frac{c}{d} = \pi$.

The value of π is a little less than $3\frac{1}{7}$; more accurately, 3.1416. It is not *exactly* expressible by any number; but can be found to as many decimal places as desired.

From the equations $\frac{c}{d} = \pi$ and $d = 2r$ we have

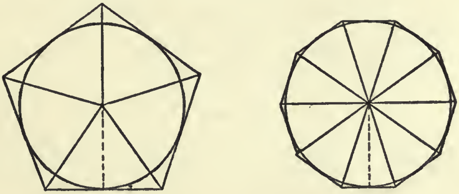
$$c = \pi d = 2\pi r, \tag{1}$$

$$d = \frac{c}{\pi}, \tag{2}$$

$$r = \frac{c}{2\pi}. \tag{3}$$

Hence, if the radius, the diameter, or the circumference of a circle is given, the other dimensions can be computed.

98. The Area of a Circle. If a regular polygon of any number of sides be circumscribed about a circle, its apothem will be the radius of the circle.



Let p denote the perimeter of the polygon, r (radius of circle) its apothem, and A_1 its area; then

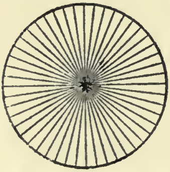
$$A_1 = \tfrac{1}{2} rp. \qquad \text{(Art. 95)}$$

It is evident that A_1 is larger than the area (A) of circle, and that p is larger than the circumference (c) of the circle.

But the greater the number of sides of the polygon, the more nearly will p be equal to c, and also the more nearly will A_1 be equal to A.

If we should go on increasing the number of sides of the polygon, its area would still be found by the formula $A_1 = \frac{1}{2} rp$, and at the same time we could make p as nearly equal to c and A_1 as nearly equal to A as we please.

Hence it follows that it must at least be *very nearly* correct to find the area of the circle by the same formula. It is proved in geometry that it is *exactly* correct.



$A = \frac{1}{2}cr = \pi r^2$

Hence $$A = \tfrac{1}{2}cr. \qquad (1)$$

This amounts to regarding the circle as composed of a very great (infinite) number of triangles, whose common altitude is the radius of the circle, and the sum of whose bases is the circumference.

Since $c = 2\pi r$, (1) may be written

$$A = \pi r^2; \qquad (2)$$

which is the usual formula for finding the area of a circle.

It is sometimes convenient to use the formula

$$A = \tfrac{1}{4}\pi d^2, \qquad (3)$$

which the pupil may derive for himself from (1) and the equation $d = 2r$.

Exercise. Give the meaning of formulas (1), (2), and (3) in words.

99. Let c_1 and c_2 denote the circumferences of two circles; r_1 and r_2 their radii.

Then $c_1 = 2\pi r_1$, $c_2 = 2\pi r_2$. [Art. 97 (1)]

Hence, dividing the members of the first equation by the corresponding members of the second,

$$\frac{c_1}{c_2} = \frac{2\pi r_1}{2\pi r_2} = \frac{r_1}{r_2}; \qquad (1)$$

or, in words: *The ratio of the circumferences of two circles* *is equal to the ratio of their radii.* Or, more briefly: *The circumferences of two circles are to each other as their radii.*

Exercise. Prove that the circumferences of two circles are to each other as their diameters.

100. Let A_1 and A_2 be the areas of two circles; r_1 and r_2 their radii.

Then $$A_1 = \pi r_1^2,\ A_2 = \pi r_2^2. \qquad \text{[Art. 98 (2)]}$$

Hence $$\frac{A_1}{A_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2}. \qquad (1)$$

That is: *The ratio of the areas of two circles is equal to the ratio of the squares of their radii.* Or, *The areas of two circles are to each other as the squares of their radii.*

Taking the square root of the first and last members of (1), and interchanging them, we have

$$\frac{r_1}{r_2} = \frac{\sqrt{A_1}}{\sqrt{A_2}}. \qquad (2)$$

In words: *The radii of two circles are to each other as the square roots of their areas.*

Note. The relation between (1) and (2) is expressed by saying that either is the *converse* of the other.

Exercise. (1) Prove from Art. 98 (3) that the areas of two circles are to each other as the squares of their diameters. Prove the converse.

(2) Prove from Art. 98 (1) that the areas of two circles are to each other as the squares of their circumferences. Prove the converse.

EXAMPLES 49

1. What is the circumference of a circle whose diameter is 20 in.?

2. What is the diameter of a tree whose girt is 18 ft. 6 in.?

3. Find the area of a circle whose diameter is 10 ft.

4. The distance around a circular park is $1\frac{1}{2}$ mi. How many acres does it contain?

5. What is the circumference of a circle whose area is 19.635 sq. ft.?

6. What is the side of a square inscribed in a circle whose diameter is 6 rd.?

7. The area of a circle is 78.54 sq. ft. Find the side of the inscribed square.

8. What is the circumference of a circular pond whose radius is 11 rd.? Its area?

9. What is the radius of a circle equal in area to a triangle whose base is 13 ft. and altitude 10 ft.?

10. A cow is one day tied to the top of a stake 5 ft. high by a rope 20 ft. long. On the next day she is tied to the bottom of the stake by the same rope. Find the difference in the areas over which she can graze.

11. What will it cost at $2 a rod to fence a circular plot of land containing 1 acre?

12. How many times will a carriage wheel 4 ft. in diameter turn round in going 1 mi.?

13. A square field contains 31.5 acres. What is the length of its diagonal? What is the circumference of a circular field of the same area?

101. Similar plane figures are plane figures having the same shape; that is, their corresponding angles are equal and their corresponding lines (like dimensions) are proportional.

Similar figures may be regarded as enlarged or reduced copies of one another.

All circles are similar figures, and all regular polygons of the same number of sides.

It is proved in geometry that:

(i.) *Any corresponding lines of similar plane figures are to each other as their other corresponding lines.*

(ii.) *The areas of similar plane figures are to each other as the squares of their corresponding lines.*

Conversely,

(iii.) *The corresponding lines of similar plane figures are to each other as the square roots of their areas.*

NOTE. These general truths, or *theorems*, were proved in Arts. 99 and 100 for circles. Compare carefully the theorems as given for circles with the more general corresponding theorems of this article.

EXAMPLES 50

1. The length of one side of a triangular field containing 2 A. 80 sq. rd. is 12 chains. Find the area of a field of similar shape whose corresponding side is 48 chains.

SUGGESTION. $12^2 : 48^2 :: 2.5$ A. : x A. [Theorem (ii.)]

2. The side of a square field containing 18 acres is 60 rd. long. Find the side of a square field that contains $\frac{1}{8}$ as many acres.

3. Two circles are to each other as 9 to 16, the diameter of the less being 112 ft. What is the diameter of the greater?

SUGGESTION. $3 : 4 :: 112 : x$. [Theorem (iii.)]

4. A rectangular field contains 720 sq. rd., and its length is to its breadth as 5 to 4. What are its dimensions?

SUGGESTION. Let l = length of field and b = its breadth. The area of a rectangle 5 by 4 is 20. Hence

$$20 : 720 :: 5^2 : l^2; \quad 20 : 720 :: 4^2 : b^2.$$

Solve the proportions for l^2 and b^2, then extract the square roots.

5. It is required to lay out 283 A. 107 sq. rd. of land in the form of a rectangle so that the length shall be 3 times the width. Find the dimensions.

6. A pipe 1.5 in. in diameter fills a cistern in 5 hr. Find the diameter of a pipe that will fill the same cistern in 55.1 min.

7. If it costs $167.70 to enclose a circular field containing 17 A. 110 sq. rd., what will it cost to enclose another $\frac{1}{5}$ as large with the same kind of fence?

8. If 63.39 rd. of fence will enclose a circular field containing 2 A., what length will enclose a circular field of 8 A.?

SOLIDS

102. Prisms and Cylinders. The word **solid** as used in mathematics means a portion of space bounded by surfaces. It has no reference to what the space may contain.

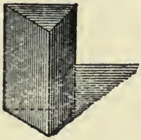
A solid whose ends are equal and parallel polygons and whose sides are rectangles is called a **right prism.**

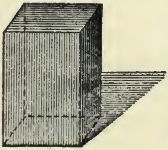
The **height** of a prism is the perpendicular distance between its ends, or bases.

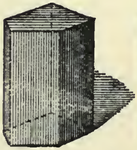
From the form of their bases prisms are called triangular, quadriangular, pentagonal, etc.

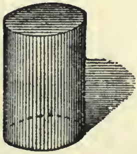
A right prism whose bases are rectangles is called a **quadrangular prism, rectangular solid,** or **parallelopiped.**

A **cube** is a rectangular solid whose faces are all equal squares.


Triangular Prism


Quadrangular Prism


Pentagonal Prism


Cylinder

NOTE. The space passed through by a moving surface is called the solid generated by the surface.

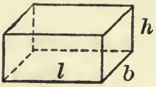
The solid generated by a rectangle rotating about one of its sides is called a **right circular cylinder.**

NOTE. The word *prism* is often used for *right prism*, and *cylinder* for *right circular cylinder*. They are so used in what follows.

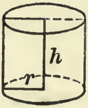
103. The area of the lateral surface (S) **of a prism or a cylinder** *is equal to the product of its height* (h) *and the perimeter* (p) *of its base.*

The volume (V) of a rectangular solid *is equal to the product of its three dimensions.*

The volume of a prism or a cylinder *is equal to the product of its height (h) and the area (A) of its base.*

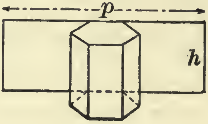


$V = lbh$



$S = 2\pi rh$

$V = \pi r^2 h$



$S = hp$

$V = hA$

EXAMPLES 51

1. Find the area of the lateral surface of a prism whose altitude is 7 in., and its base a pentagon, each side of which is 4 in.

2. What is the entire surface of a cylinder formed by the revolution about one of its sides of a rectangle 6 ft. 6 in. long and 4 ft. wide?

3. Find the solid contents of a cylinder whose altitude is 15 ft., and its radius 1 ft. 3 in.

4. Find the entire surface of a prism whose base is an equilateral triangle, the perimeter being 18 ft., and the height 15 ft.

5. Find the contents of a box whose length, width, and depth are, respectively, 4 ft., 3 ft., and 2 ft.

6. Find its surface.

7. Find the number of square feet necessary to make a stove pipe $2\frac{1}{2}$ ft. long and 5 in. in diameter.

8. Find the amount of tin necessary to make a tin pail cylindrical in form, 6 in. in diameter and 8 in. deep, without a cover.

9. How many quarts will the pail hold?

10. Find the depth of a cylindrical tank that holds 20 gal. and is 18 in. in diameter.

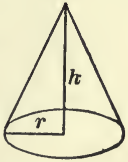
11. A rectangular can is 10 in. square on the bottom and holds 5 gal. How deep is it?

12. What is the difference in the number of square feet of lumber necessary to make the sides of a room 16 ft. long, 12 ft. wide, and 10 ft. high, and one of circular floor containing the same area and of the same height?

104. Pyramids and Cones. A **regular pyramid** is a solid whose base is a regular polygon, and whose sides are equal triangles which terminate in a common vertex.

The common altitude of the triangular sides is called the **slant height** of the pyramid.

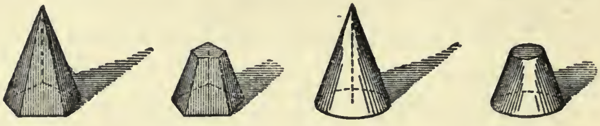
The solid generated by a right triangle rotating about one of its legs is called a **right circular cone.**



The length of the hypothenuse of the generating triangle is the **slant height** of the cone.

The **height** of a pyramid or cone is the perpendicular distance from its vertex to its base.

The words *pyramid* and *cone* are frequently used for *regular pyramid* and *right circular cone* respectively.

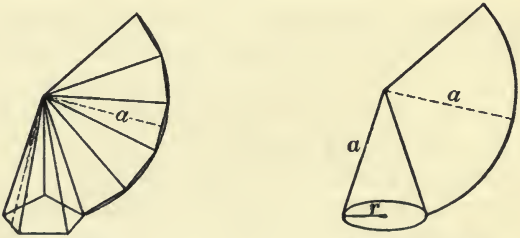


Pyramid Frustum Cone Frustum

The **frustum** of a pyramid or of a cone is the part that remains after cutting off a portion of the top by a plane parallel to the base.

105. The lateral surface (S) **of a pyramid or a cone** *is equal to one-half the product of its slant height* (a) *and the perimeter* (p) *of its base.*

This follows directly from the formula of Art. 95, and formula (1) of Art. 98.

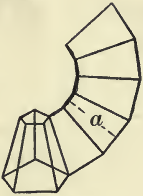


$S = \frac{1}{2}ap$
$V = \frac{1}{3}hA$

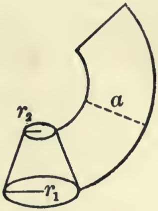
$S = \frac{1}{2}ap = \pi ar$
$V = \frac{1}{3}hA = \frac{1}{3}\pi hr^2$

The lateral surface of a frustum of a pyramid or a cone *is equal to one-half the product of the slant height* (a) *and the sum of the perimeters* (p_1 and p_2) *of its bases.*

This follows for the frustum of a pyramid from Art. 93. The lateral surface of the frustum of a cone may be regarded as made up of a very great (infinite) number of trapezoids.



$S = \frac{1}{2} a(p_1 + p_2)$
$V = \frac{1}{3} h(A_1 + A_2 + \sqrt{A_1A_2})$



$S = \pi a(r_1 + r_2)$
$V = \frac{1}{3} \pi h(r_1^2 + r_2^2 + r_1r_2)$

Let r_1 be the radius of the lower base of the frustum of a cone, and r_2 the radius of the upper base.

Then $$S = \tfrac{1}{2} a(p_1 + p_2) = \tfrac{1}{2} a(2\pi r_1 + 2\pi r_2)$$
$$= \tfrac{1}{2} a \times 2\pi(r_1 + r_2) = \pi a(r_1 + r_2).$$

The volume of a pyramid or a cone *is equal to one-third the product of its height* (h) *and the area* (A) *of its base.* (See note to Art. 84.)

The volume of the frustum of a pyramid or a cone is found as follows: *Add the areas* (A_1 and A_2) *of the bases and the square root of their product, and multiply this sum by one-third of the height.* (See note to Art. 84.)

For the frustum of a cone,

$$\begin{aligned} V &= \tfrac{1}{3} h(A_1 + A_2 + \sqrt{A_1A_2}) \\ &= \tfrac{1}{3} h(\pi r_1^2 + \pi r_2^2 + \sqrt{\pi r_1^2 \pi r_2^2}) \\ &= \tfrac{1}{3} h(\pi r_1^2 + \pi r_2^2 + \pi r_1 r_2) \\ &= \tfrac{1}{3} \pi h(r_1^2 + r_2^2 + r_1 r_2). \end{aligned}$$

EXAMPLES 52

1. Find the lateral surface of a triangular pyramid, the slant height being 16 ft., and each side of the base 5 ft.

2. Find the lateral surface of a cone whose diameter is 17 ft. 6 in., and the slant height 30 ft.

3. Find the entire surface of a square pyramid whose base is 8 ft. 6 in. square, and its slant height 21 ft.

4. How many cubic feet in the mast of a ship, its height being 50 ft., the circumference at one end 5 ft., and at the other 3 ft.?

5. Find how much water can be put into a tin pail 10 in. deep, like a frustum of a cone in form, whose bottom is 8 in. across, and top 12 in. across.

6. How many square feet of tin in the pail described in the last example, without cover?

7. A conical wood pile is 6 ft. high and 12 ft. in diameter at the base. How many cords are in it?

8. How many bushels of oats in a conical pile 2 ft. high and 12 ft. around it at the base?

9. Find the number of cubic feet enclosed by a barn 60 ft. long, 40 ft. wide, and 20 ft. high, with a pyramidal roof 8 ft. high; all inside measurements.

10. How many cubic feet of wood are in a log 20 ft. long and 14 in. in diameter?

11. At 28 cents per cubic foot, what is the cost of a stone wall 28 in. thick at the base and 18 in. at the top, 4 ft. high and 36 rd. long?

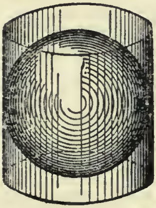
12. How many cubic feet in a regular eight-sided post 10 ft. high, the width of one side being 3 in., and the distance through it 7.24 in.?

106. The Sphere. A **sphere** is a solid bounded by a uniformly curved surface, all points of which are equally distant from a point within called the centre.

A sphere is generated by a semicircle rotating about its diameter. The radius and the diameter of the generating semicircle are the radius and the diameter, respectively, of the sphere.

The section of a sphere made by a plane passing through its centre is called a **great circle** of the sphere.

107. The surface (S) of a sphere *is equal to the lateral surface of the circumscribed cylinder.* (Art. 84, note.)



$S = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$

The diameter and the height of the circumscribed cylinder are each equal to the diameter of the sphere; hence

$$S = 2\pi r \times 2r = 4\pi r^2.$$

The area of the surface of a sphere is equal to four times the area of its great circle.

108. Let r_1 and r_2 be the radii of two spheres, S_1 and S_2 their surfaces.

Then $\quad S_1 = 4\pi r_1^2, \quad S_2 = 4\pi r_2^2.$

Hence $$\frac{S_1}{S_2} = \frac{4\pi r_1^2}{4\pi r_2^2} = \frac{r_1^2}{r_2^2};$$

or $$S_1 : S_2 :: r_1^2 : r_2^2.$$

The surfaces of two spheres are to each other as the squares of their radii.

Conversely, $$\frac{r_1}{r_2}=\frac{\sqrt{S_1}}{\sqrt{S_2}}.$$

The radii of two spheres are to each other as the square roots of their surfaces.

EXERCISE. (1) Prove that the surfaces of two spheres are to each other as the squares of their diameters. Prove the converse.

(2) Prove that the surfaces of two spheres are to each other as the squares of the circumferences of their great circles. Prove the converse.

109. The Volume of a Sphere. We have seen that a circle may be regarded as made up of a very great number of triangles having a common vertex at its centre. Similarly, a sphere may be regarded as made up of a very great (infinite) number of pyramids, having their bases in the surface of the sphere and their common vertex at its centre.

The surface of the sphere is the sum of the bases of these pyramids, and its radius is their height.

Now the volume of a pyramid is the product of its base and one-third its height; hence the volume of a sphere is the product of its surface and one-third its radius.

Hence, since $$S=4\,\pi r^2,$$

$$V=\tfrac{1}{3}\,Sr=\tfrac{4}{3}\,\pi r^3. \qquad (1)$$

110. Let r_1 and r_2 be the radii of two spheres; V_1 and V_2 their volumes.

Then $$V_1 = \tfrac{4}{3}\pi r_1^3,\quad V_2 = \tfrac{4}{3}\pi r_2^3.$$

Hence $$\frac{V_1}{V_2} = \frac{\tfrac{4}{3}\pi r_1^3}{\tfrac{4}{3}\pi r_2^3} = \frac{r_1^3}{r_2^3};$$

and $$\frac{r_1}{r_2} = \frac{\sqrt[3]{V_1}}{\sqrt[3]{V_2}}.$$

The volumes of two spheres are to each other as the cubes of their radii.

Conversely,

The radii of two spheres are to each other as the cube roots of their volumes.

Exercise. (1) Prove that the volumes of two spheres are to each other as the cubes of their diameters; as the cubes of the circumferences of their great circles.

(2) Prove the converse of each of the above.

EXAMPLES 53

1. Find the surface of a sphere whose diameter is 9 in.

2. Find the volume of a sphere whose diameter is 18 in.

3. The glass tank of a lamp is spherical in shape and 4 in. in diameter on the inside. How much oil will it hold?

4. The diameter of the earth is about 8000 mi. Find its surface and volume.

111. Similar solids are solids having the same form. Their corresponding surfaces are similar, their corresponding angles equal, and their corresponding lines proportional.

All spheres are similar solids, and all cubes.

Theorems:

(i.) *Any corresponding lines of similar solids are to each other as their other corresponding lines.*

(ii.) *The surfaces of similar figures* (plane or solid) *are to each other as the squares of their corresponding lines.* Conversely,

(iii.) *The corresponding lines of similar figures are to each other as the square roots of their surfaces.*

(iv.) *The volumes of similar solids are to each other as the cubes of their corresponding lines.* Conversely,

(v.) *The corresponding lines of similar solids are to each other as the cube roots of their volumes.*

NOTE. These theorems were proved in Arts. 108 and 110 for spheres. Compare the theorems given there with the more general ones of this article. Why are these more general?

The *reason* for the truth of these theorems is that all *lines* have but *one* dimension — length; all *surfaces* are proportional to the product of *two* dimensions — length and width; and all *volumes* to the product of *three* dimensions — length, width, and thickness. (Compare with Art. 68.)

This is evident in the case of squares and cubes. If the side of one square is twice that of another, its area is 4 times as great. If the edge of one cube is twice that of another cube, its volume is 8 times as great, etc.

Illustrate the last two statements by drawings.

EXAMPLES 54

1. If a marble column 10 in. in diameter contains 27 cu. ft., what is the diameter of a column of equal length that contains 81 cu. ft.?

2. A ball 4.5 in. in diameter weighs 18 oz. What is

the weight of another ball of the same density, that is 9 in. in diameter?

3. Two vessels have the same shape. One is 12 in. deep and holds 7 gal. The other is 7 in. deep; what does it hold?

4. A tank was made of 20 sq. ft. of sheet iron, and a tank of the same shape of 30 sq. ft. What is the ratio of their capacities?

SUGGESTION. First find the ratio of their linear dimensions.

5. What is the ratio of the corresponding edges of two similar rectangular solids whose volumes are respectively 2.7 cu. ft. and 1.5 cu. ft.?

6. What is the edge of a cube whose entire surface is 1050 sq. ft., and what is its volume?

7. What must be the inner edge of a cubical bin to hold 1250 bu. of wheat?

8. How many gallons will a cistern hold whose depth is 7 ft., the bottom being a circle 7 ft. in diameter and the top 5 ft. in diameter?

9. What is the value of a stick of timber 24 ft. long, the larger end being 15 in. square, and the smaller 6 in., at 28 cents a cubic foot?

10. The surface of a sphere is the same as that of a cube, the edge of which is 12 in. Find the volume of each.

ANSWERS

Examples 1

1. 5.
2. 6.
3. 16.
4. 4.
5. 0.
6. 8.
7. 28.
8. 55.
9. 26.
10. 16.
11. 65.
12. 3.
13. 18.
14. 3.
15. $\frac{19}{102}$.

Examples 2

1. 3.
2. 1.
3. $3\frac{4}{7}$.
4. 2.
5. $\frac{1}{14}$.
6. 56.
7. 6.
8. $\frac{20}{9}$.

Examples 3

1. 4 yr., James; 12 yr., John.
2. 4¢, top; 20¢, ball.
3. 10¢, sister; 20¢, brother.
4. 16 and 34.
5. 21, 42, 63.
6. 32¢.
7. $13\frac{1}{3}$ da.
8. 7 marbles, H; 19 marbles, F.
9. 13 and 49.
10. 13 yr., son; 58 yr., father.
11. 48.
12. 18 and 24.
13. $6\frac{3}{10}$ bu.; $14\frac{7}{10}$ bu.
14. ~~9.~~ 5

Examples 4

1. 38.
2. 8.
3. 10.
4. 73.
5. 163.
6. $2\frac{1}{2}$.
7. 238.
8. 9.
9. 35.
10. $6\frac{2}{5}$.
11. 18.
12. 90.
13. $1\frac{5}{6}$.
14. $3\frac{1}{20}$.

Examples 5

1. $8a^2$.
2. $15acd$.
3. $6a^3b$.
4. $20a^3b^3$.
5. $8ab^2c^2x$.
6. $35cex^2y$.
7. $3a + 3b$.
8. $15a - 5b$.
9. $a^3 + a^2$.
10. $2a^5 - a^4$.
11. $3a^3 + 6a^2 - 6a$.
12. $ab^2c^2 + a^2bc^2 + a^2b^2c$.
13. 18.
14. 300.
15. 900.
16. 75.
17. 29.
18. 33.
19. 6.
20. 72.
21. 20.
22. 13.

23. 25.
24. 729.
25. $5a + b$.
26. $2ab + ac$.
27. $2bc$.
28. $5ab - 7ac + 2bc$.
29. $2b - \frac{7}{4}c$.
30. $22ab - 8ac$.
31. $x^2 + 2xy + y^2$.
32. $4a^2 + 4ab + b^2$.
33. $a^4 + 2a^2c + c^2$.
34. $3a^2 + 5ab + 2b^2$.
35. $2a^2 + 3ab - 5b^2$.
36. $a^2 - ab - 2b^2$.
37. $a^2 - 2ab + b^2$.
38. $4x^2 - 12xy + 9y^2$.
39. $a^6 - 2a^3c^2 + c^4$.
40. $2a^2 - 3ab + b^2$.
41. $10a^2 - 9ab^2 + 2b^4$.
42. $a^3 + 3a^2b + 3ab^2 + b^3$.

Examples 6

1. 3.
2. $3a$.
3. $4a$.
4. $3xy$.
5. $10ab$.
6. a^3cd^3.
7. $4a^2 + 5ab$.
8. $3ab - 2c$.
9. $\frac{5}{7}x - \frac{1}{7}y^2$.
10. $\frac{1}{3}ab + \frac{bcd}{a}$.
11. $a^2 + 2ab + b^2$.
12. $3ab + b^2 - \frac{3}{5}a^2$.
13. 31¢, 1st; 62¢, 2d; 97¢, 3d.
14. 4 hr.
15. 16.
16. A, 24 apples; B, 12 apples; C, 16 apples.
17. 40 men; 80 boys; 880 women.
18. $3000.
19. 40 and 60.
20. 12 and 24.
21. 5 yr.
22. 10 yr., son; 30 yr., father.
23. A, $30; B, $15; C, $20.
24. 164 artillery; 472 cavalry; 564 infantry.
25. 14 yr.; 16 yr.; 18 yr.
26. 7 and 8.
27. 18 yr.; 36 yr.

Examples 11

1. A, $1400; B, $480; C, $453.60 D, $260.
2. $27\frac{7}{9}$%; $31\frac{1}{9}$%; $5\frac{5}{9}$%; $6\frac{2}{3}$%; $28\frac{8}{9}$%.
3. 28000.
4. $5.
5. 60%.
6. 2500 sheep.
7. $217.61.
8. $40000.
9. 15%.
10. 39 yd.
11. $3000.
12. 52%; $45,760.

Examples 13

4. $2760.
5. $5.25.
6. $12800.
7. $1300 cost.
8. Neither.
9. $720.
10. $300.
11. $1437.60.
12. 23%.
13. $23\frac{1}{5}$%.
14. $1.47.
15. $.17½.
16. $96.
17. 24%.
18. 11%.
19. $2\frac{7}{9}$% loss.
20. $1.94⅓; $16\frac{2}{3}$%.

21. $50;
$58.82;
$65.
22. 4% loss.

Examples 14

4. $378.13.
5. $96.90.
6. $6400.76 investment.
7. 5%.
8. $600.
9. $1271.88.
10. $15 com.;
$750 inv.
11. $2905;
$9\frac{41}{60}$¢ per lb.
12. $2920.
13. $14\frac{2}{7}$%.
14. $432.
15. $506.25;
23750 lb.
16. $5\frac{3}{4}$%.
17. $2100.
18. $82.11 com.;
$9301.89 proceeds.
19. $10623.44.
20. $45111.44;
$225.56.

Examples 15

1. $555.75.
2. $256.50.
3. $1080;
28%.
4. $596.11.
5. $666.90;
$57\frac{1}{4}$%.
6. $3.75.
7. $786.
8. $363.80;
$14\frac{1}{2}$%.
9. $20.81.
10. $2.40.

Examples 16

1. $217.50.
2. $37.50.
3. $738.
4. $58000.
5. $13600.
6. $\frac{2}{13}$%.
7. $424.
8. $44 prem.;
$5456 loss.
9. 2%.
10. $1840.50 premium;
$79959.50 value.
11. $32000.
12. $3717.83.
13. $3168.
14. $5600.

Examples 17

1. $88000.
2. $19072.16.
3. $.45.
4. $1\frac{1}{4}$%, rate;
$95.25, A's tax.
5. .0228 tax rate;
$214.65.
6. $407.20.
7. $224.37.
8. $103.13.
9. $21.43.
10. $2\frac{7}{9}$ mills.
11. $76.39.

Examples 19

1. $7212.50.
2. $37.15.
3. $11925.
4. $151.88.
5. $30.72.

Examples 20

1. $3902.40.
2. $78133.33.
3. $208.33.
4. $6553.60.
5. $25372.
6. 48 bu.
7. $1700, 1st yr.;
$1785, 2d yr.
8. $40\frac{5}{8}$%.
9. $40842.
10. 61788.6 lb.
11. $25\frac{5}{7}$% nearly.
12. 12%.
13. $2116.94.
14. 16¢.
15. $863.99.
$68.21.

Examples 22

2. $83.22;
$843.22.
3. $11.91;
$191.41.

4. $57.20 ; $382.20.
5. $146.19 ; $904.94.
6. $142.03 ; $1166.28.
7. $42.28 ; $626.78.
8. $55.77 ; $781.61.
9. $83.02 ; $470.97.
10. $.17; $42.37.

Examples 23

1. $58.93.
2. $8.40.
3. $67.67.
4. $159.75.
5. $67.09.
6. $38.11.
7. $30.81.
8. $8.93.
9. $3647.61.
10. $1066.36.
11. $2010.42.
12. $142.45.
13. $1886.17.
14. $263.83.
15. $410.70.
16. $25.78.
17. $165.50.
18. $410.73.
19. $1936.60.
20. $1120.69.
21. $7.33.

Examples 24

2. $6.06.
3. $82.36.
4. $10.96.
5. $39.55.
6. $106.99.
7. $172.17.
8. $51.37.
9. $205.48.

Examples 26

1. $516.71.
2. $1000.
3. 3 yr. 4 mo. 24 da.
4. 6%.
5. $650.80.
6. 7%.
7. 7 mo. 10 da.
8. 7%.
9. $669.12.
10. $1403.08.

Examples 27

2. $917.43; $82.57.
3. $43.65 in favor of dis.
4. $1137.61.
5. $838.26.
6. The first by 5¢ per ton.
7. $2016.13.
8. $147.06.
9. $.92.

Examples 28

3. Mat. Oct. 30; 81 da. term of dis.; $940.14 proceeds.
4. Apr. 8; 46 da.; $917.21.
5. Aug. 13; 64 da.; $690.04.
6. $641.55.
7. $1821.60.
8. $2548.53.
9. $56.69.

Examples 29

2. $2688.
3. $464.10.
4. $3808.
5. $586.04.
6. $247.52.

Examples 30

2. $900 simple; $1035 annual; $1046.30 comp.
3. $428.76.
4. $189.15.
5. $311.64.
6. $100.32.
7. $41.99.
8. $245.77.
9. $53.38.
10. $1540.79.
11. $2357.79.

Examples 31

2. $576.
3. $98.33.

4. $575.34.
5. $284.79.
6. $601.08.

Examples 32

2. $577.38.
3. $99.88.
4. $576.67.
5. $285.99.
6. $603.49.
7. $1386.78.
8. $1284.11.
9. $162.25.
10. $4408.21.
11. $523.43.

Examples 34

3. $4.05.
4. $44\frac{4}{9}$ bbl.
5. 16 men.
6. 96 sheep.
7. $5355.
8. 7 hr. $13\frac{1}{3}$ min.
9. $112\frac{1}{2}$ mi.
10. $59\frac{3}{8}$ da.
11. $7320.
12. $10958.90.
13. 90 horses.
14. 2 yr. 6 mo.
15. 3 hr.
16. 20 men.
17. 75 da.
18. 9 men.

Examples 35

3. $240.
4. $498.08.
5. 280 bu.
6. $6428.57.
7. $220\frac{4}{5}$ cd.
8. $52.79.
9. 9 men.
10. $2\frac{2}{3}$ da.
11. 10 ft. $2\frac{1}{2}$ in.
12. 10 ft.
13. 546 bbl.
14. 2080 lb.
15. $100.
16. 15 da.
17. 432 mi.

Examples 36

1. A's, $100; B's, $150.
2. A's, $3200; B's, $1800; C's, $1400.
3. A's, $1800; B's, $3000.
4. A, $1710; B, $870.20.
5. A, $6000; B, $8402.25; C, $5055.75; D, $3042.
6. A's, $200; B's, $240; C's, $160.
7. A, $2400; B, $2666.67; C, $2933.33.
8. A, $388.71; B, $249.17; C, $112.12.
9. A, $131; B, $393; C, $262.
10. A, $1800; B, $600; C, $1200.
11. 1st, $$357\frac{1}{7}$; 2d, $$642\frac{6}{7}$.
12. $171.60; $257.40.

Examples 37

2. June 27, 1897.
3. May 5, 1895.
4. June 23.
5. 5 mo.
6. 5 yr. 20 da. from date of last payment.
7. Nov. 26.
8. Mar. 7; $1178.01.

Examples 43

3. 115.
4. 109.
5. 997.
6. 143.2.
7. 54.64.
8. .035.
9. 14.0048+.
10. 1.5005+.
11. 7.625.
12. 4.213+.
13. 103.9.
14. 1.5411+.

15. $\frac{13}{14}$.
16. $\frac{25}{81}$.
17. $\frac{24}{135}$.
18. .91287+.
19. 5.5.
20. $8\frac{1}{3}$.
21. 1008 ft.
22. 240.33 rd.
23. 52 rd.
24. $187.20.
25. 80 × 40 rd.
26. 101.2 rd.
27. 107.33 rd.
28. 182 sq. rd.
29. 49 rows.

Examples 46

3. 101.
4. 165.
5. 101.5.
6. 15.98.
7. .45.
8. .046.
9. 3.4056.
10. 10.77.
11. $\frac{10}{11}$.
12. $\frac{24}{25}$.
13. 1.42+.
14. .7936.
15. 2 ft.
16. 12150 sq. ft.
17. 5 ft. 8+ in.
18. 9 ft. 5.3+ in.
19. 8 ft. 1.4 in.
20. 3.17 ft.

Examples 47

1. 84 sq. ft.
2. $5\frac{1}{4}$ A.
3. $42\frac{3}{16}$ sq. ft.
4. $449.07.
5. 210 sq. ft.
6. 27 ft.
7. $4\frac{1}{2}$ ft.
8. 20 in.
9. 39 ft.
10. 50 ft.
11. 56.57 rd.
12. 28 ft. 3.36 in.
13. 21 ft.
14. 20 ft.
15. 15.59. in.
16. 10.6 rd.; 112.5 sq. rd.
17. 45.08 ft.
18. 92.45 mi.

Examples 48

1. 153 sq. ft.
2. 2 A.
3. 714 sq. ft.
4. Given.
5. 7.794+ in.
6. 259.8+ sq. in.

Examples 49

1. 5 ft. 2.83 in.
2. 5 ft. 10.67 in.
3. 78.54 sq. ft.
4. 114.59 A.
5. 15.708 ft.
6. 4.24 rd.
7. 7.07+ ft.
8. 69.12 rd.; 380.13 sq. rd.
9. 4.55 ft.
10. 78.54 sq. ft.
11. $89.68.
12. 420+ times.
13. 100.399 sq. rd. diagonal; 251.6 rd. circum.

Examples 50

1. 40 A.
2. 34.64+ rd.
3. 149 ft. 4 in.
4. 30 rd.; 20 rd.
5. 369 rd.; 123 rd.
6. 3.5 in.
7. $75.
8. 126.78 rd.

Examples 51

1. 140 sq. in.
2. 263.89 sq. ft.
3. 73.63 cu. ft.
4. 301.177 sq. ft.
5. 24 cu. ft.
6. 52 sq. ft.
7. $3.27\frac{1}{4}$ sq. ft.
8. 179.07 sq. in.
9. 3.91 qt.
10. 18.16 in.
11. 11.55 in.
12. 68.80 sq. ft.

Examples 52

1. 120 sq. ft.
2. 824.67 sq. ft.
3. $429\frac{1}{4}$ sq. ft.
4. 64.99 cu. ft.
5. $795.87\frac{1}{5}$ cu. in.
6. 257 sq. ft.
7. 1 cd. $98\frac{1}{5}$ cu. ft.
8. 6.14 bu.
9. 54400 cu. ft.
10. 21.38 cu. ft.
11. $1275.12.
12. 3+ cu. ft.

Examples 53

1. 254.47 sq. in.
2. 3053.6+ cu. in.
3. .58 qt.
4. 201062400 sq. mi.; 268083200000 cu. mi.

Examples 54

1. 14.42 in.
2. 9 lb.
3. 1.389+ gal.
4. .54433.
5. 1.2164+.
6. 13.228 ft. edge; 2315.03 cu. ft. vol.
7. 11 ft. 7 in.
8. 1494.257 gal.
9. $5.46.
10. 1 cu. ft. vol. of cube; 1 cu. ft. 659 cu. in. vol. of sphere.