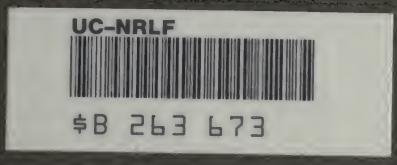

SUGGESTIONS
ON THE
TEACHING of ALGEBRA

With Especial Reference to the Use of DURELL and ARNOLD'S ALGEBRA

BY

FLETCHER DURELL, Ph.D.

HEAD OF MATHEMATICAL DEPARTMENT, THE LAWRENCEVILLE SCHOOL; AUTHOR OF THE DURELL MATHEMATICS SERIES AND JOINT AUTHOR OF THE DURELL-ARNOLD MATHEMATICS SERIES

CHARLES E. MERRILL COMPANY
NEW YORK CHICAGO

CONTENTS

SUGGESTIONS ON THE TEACHING OF ALGEBRA *

WITH ESPECIAL REFERENCE TO THE USE OF

DURELL AND ARNOLD'S ALGEBRAS

1. Introductory Remarks.—The teaching of the early part of algebra, that is, the transition from arithmetic to algebra, has always been the most difficult part of elementary mathematical instruction. This difficulty has been increased in certain ways in recent years by the fact that pupils entering the high school are more and more immature, and by the many new and attractive appeals to their attention in other departments of study as well as in the outside world. Also the recent somewhat radical changes made in the subject matter of first year algebra call for some modifications in methods of instruction in order to meet the situation in the most effective way.

This pamphlet is written in order to give suggestions or illustrations to such teachers as desire all the help they can get, from whatever quarter, in order to get the maximum of results when working under the new conditions.

In the present situation more than ever, the important thing to do is to utilize to the utmost the natural growth processes of the child's mind; for a half year at least to feed young pupils much extremely simple and easily appreciated material so that in time and often without serious effort they unconsciously grow into the power of doing much harder work, and indeed develop an appetite and demand for such work.

Any suggestions from teachers with a view to the betterment of the methods here presented will be welcomed.

In what follows, we discuss first the form or organization of the recitation best suited to meet new conditions and afterward the methods of treating the different kinds of subject matter.

2. The Parts of a Typical Recitation in Algebra are four:

I. Return and discussion of corrected written work (if any) done in the last fifteen minutes of the preceding recitation.

II. Discussion of the advance work for the current recitation.

III. Assignment and explanation of the next lesson in advance.

IV. Written work during the final ten or fifteen minutes of recitation.

The first five or ten minutes of a recitation in algebra may well be occupied by a discussion of the written test work done by the pupils in the last few minutes of the preceding recitation. Special stress is laid on these papers, because the work has all been done under the teacher's eye without aid from any outside source. Each of these papers has been carefully corrected in red pencil by the teacher, and at the opening of the recitation is returned to the pupil.

Any one method of calling attention to the errors, or to specially meritorious points on the papers, if used continuously soon loses its force. Hence in discussing these papers, in order to keep the interest of pupils fresh and active, as well as to use methods which fit the peculiarities of each set of papers, different methods may be used from day to day. Among those available are the following:

(1) Pupils who have correct solutions, or specially good solutions, may be asked to copy these on the blackboard from their papers, after which the teacher may comment

upon them, or other pupils may ask questions concerning them;

(2) The teacher may solve the problems in whole or in part on the blackboard (often time may be saved by putting part of the work on the blackboard before the recitation begins);

(3) In case an error, the correction of which should be emphasized, has been committed by a number of pupils, the teacher after correcting this error may fasten the matter more firmly in the minds of the class by reading a list of those who have made the error, or by stating the number of those who have done so, or by asking all those who have solved the problem correctly to raise their hands. The same applies in presenting any particularly good method of solution found on the papers;

(4) If a given pupil has made a mistake like cancelling the a's in $\frac{ab}{a+x}$ the undivided attention of the class to this error can be obtained by writing the fraction on the board without mentioning the fact that an error has been made, and then calling upon the pupil who has made the mistake to come forward and simplify the fraction as he has done on his paper;

(5) If the class has shown anything like a general weakness in working the assigned problems, it may be well, instead of reviewing the marked papers in any of the above ways, to send the whole class to the blackboard for ten minutes' drill on examples of the given type, including perhaps one or more of the actual examples used in the test under discussion.

3. In the Second Part of the Typical Recitation the advance lesson which the pupils have prepared since the last recitation is next considered. Individual pupils are now called upon to put on the board solutions of the problems in the

lesson and the proof of principles. While this is being done by part of the class, various methods of instructing the rest of the class may be followed by the teacher. For instance, if some of the solutions being put on the board are difficult and involve details which require close attention, it is often well for the teacher to point out these features while the solutions are being written, and to encourage such pupils as are at their seats to ask questions concerning any difficulties which they have had with these problems.

If the work does not require such close attention as this, some one pupil may be sent to the board to work some newly assigned example involving typical principles of the advance lesson.

Or the time may be spent in oral or sight drills or reviews.

Oral drills may be conducted in various ways, as

(1) By having the pupils who are at their seats open their books at a specified place and solve certain simple examples orally;

(2) By having the teacher write on the blackboard simple improvised problems, which pupils solve orally. One of the advantages of this method is that the attention of the class is held more closely owing to the fact that pupils are curious to know what kind of example will come next. Another advantage is that when any weakness on any point is discerned the teacher can at once follow it up by devising a line of examples adapted to remedy it;

(3) The drill may be a review of definitions, or in having pupils invent and put on the board expressions illustrating the definitions (see § 11), as

Ex. 1. Write two simultaneous equations where p and q are the unknowns and whose solution will give the results $p=1$ and $q=2$.

Ex. 2. Write an algebraic expression of three terms, each of which contains both x and y, one of the terms being of the fourth, another of the third, and another of the second degree.

(4) Oral drill on verbal problems. (See § 8.)

4. The Third Period of the Typical or Standard Recitation consists of an explanation of the principles or processes involved in the next lesson and in the assignment of the

lesson. In explaining the next lesson the exercise of great discretion is necessary on the part of the teacher. If the processes involved in the advance work, are such as the pupil should be able to analyze and grasp, without other aid than that given in the textbook, no explanation at all should be given by the teacher. If an explanation is deemed advisable, the development of the new algebraic process should be made as far as possible by the method of question and answer, and the pupils be expected to supply all possible details of the work, with the reasons for the same.

In all instruction in mathematics, one of the most difficult matters to determine, if not the most difficult, is how much and what kind of help to give pupils. To supply too much help pauperizes them and renders their minds inert; to give too little help often discourages them. A good rule in this matter is to give much help, especially to young and immature pupils, in the early stages of a subject, and then gradually to diminish the amount of aid.

If it is found that pupils are forming the habit of relying too much on help given by the teacher, and are not reading or studying the explanations given in the textbook, a partial remedy for this is for the teacher, instead of giving an explanation, to have some pupil read aloud the explanation given in the textbook and at the same time write the steps of the accompanying process on the board, and answer any questions which the teacher may ask concerning the same. Or all the members of the class may be required to read silently the statements in the textbook, after which the teacher may ask them questions concerning what they have read.

After a method like that of solving a quadratic equation by completing the square has been explained or studied, new interest may be aroused and the matter fixed in the minds of pupils by asking some one pupil to come to the blackboard and work an example by the given method.

His work is followed with the closest attention by every pupil, any mistakes which he may make are quickly noted, and other pupils often ask for the privilege of showing whether they cannot solve a like example without making an error. Also the fact that after an explana-

tion of a new process, pupils may be called upon at once to show whether they understand it, naturally tends to keep the attention of the class more alert during all explanations.

5. The Fourth and Last Part of the Standard Recitation, as has already been stated, usually consists of written work on paper by the members of the class at their seats. Some of this may be extra credit work (see § 12), and in it all every effort should be made to stimulate pupils to form the habit of checking their work (see § 13). In tests of this sort it is also well usually to make one or two of the assigned examples review work.

In this connection again the question naturally arises as to whether the teacher should give pupils any help in this final period of work, and if so, how much. In the early part of the year's work, it is the writer's habit to allow pupils, when they are in difficulty in this test work, to raise the hand and obtain permission to come to the desk. If he finds that their difficulties are such as they cannot be expected to overcome, using a red pencil he makes such suggestions on their papers as will make it possible for pupils to continue their solutions, the red pencil marks rendering it easy for him afterward to make equitable deductions in grading their papers. Giving help in this way also aids the teacher in gaining knowledge of the mental peculiarities and weaknesses of individual pupils. As the class progresses, however, less and less help of this kind is given and toward the end of the year and in all review tests none whatever is supplied till all the papers have been handed in, corrected, and returned to the class.

As has already been indicated, special stress is laid on this written work which has been done in the presence of the teacher, and of which he knows exactly how much is the pupil's own. Hence especial care is taken in correcting and discussing it, in the manner already described (§ 2).

While the class is doing this written work at their seats, an opportunity is afforded the teacher of grouping and making a rapid appraisement of the papers handed in at the beginning of the recitation which contain the work done

outside the class in preparation for the current recitation. If any member of the class has failed to do this work properly, he is at once called to the desk and asked to state the reason for this failure. If his failure is due either to neglect, or to lack of grasp of the subject matter, he is at once assigned to deficiency study (§ 6).

As with other parts of the recitation, pains should be taken to introduce variety into this last period and thus prevent it from becoming monotonous and ineffective. Thus on some days instead of having pupils do written work at their seats, it is well to send the entire class to the blackboard and drill them there in some way. Thus a special group of examples may be assigned to each pupil. Or a set of examples may be written or indicated on the blackboard to be worked alike by all the pupils. Any tendency to copy each other's work will be diminished by the fact that the abler pupils will quickly distance the weaker ones; but at times in this work a certain amount of co-operation, in which the abler pupils aid weaker ones in overcoming their difficulties is desirable, since a pupil often has a clearer appreciation of the troubles of a fellow pupil than the teacher has, and the progress of the class is much facilitated by such co-operation.

One way of obtaining the co-operation spoken of is the following: If one or more of the pupils finish all of the assigned examples before the other members of the class, ask each of these more successful students to help someone who is having unusual difficulty.

The drill during the final period of the recitation may also sometimes be effectively varied by stopping the work at the blackboard at a certain point and sending all the pupils to their seats and having each pupil work on paper, without aid from anyone, the same examples which have just been worked at the board.

Variety and fresh stimulus may also be introduced into the work by using this last or fourth period of the recitation in some other quite distinct way as in a competitive game or drill of some sort between two halves or different groups into which the class has been divided.

6. Deficiency Study.—In many schools the final hour or period of the day's work is employed in giving extra instruction and drill to those pupils who have neglected to do the assigned work, and also those, who, while working well, are naturally slow and have difficulty in mastering the subject. If the spirit of the class is good, pupils often voluntarily go to this period of extra drill, called deficiency study.

A good form of carrying on the drill in deficiency study is to have all of the pupils go to the blackboard and work a set of examples which have been written on the board or listed there from the textbook. The teacher watches and corrects the work, carrying the answers to the examples on a piece of paper in the hand. Solutions by the pupils are erased as soon as they have been pronounced correct by the teacher.

After a pupil has completed the solutions, if he has made few errors he is allowed to go, or is asked to aid other pupils. If he has shown weakness in his work, he is required to work the same examples again, either at the board, or at his seat on paper.

Not only current work, but also back topics in either algebra or arithmetic, in which members of the class have shown weakness, may be reviewed in deficiency study.

7. The First Lessons in Algebra.—As has been stated in § 1 of this pamphlet, it is increasingly important that, in their first study of algebra, young pupils be fed with much easy work which appeals to them, so that in time they will grow by natural processes into the power to do more diffi-

cult work. The teacher who wishes to carry this plan into practice will find certain features in Durell and Arnold's First Book in Algebra a distinct help in so doing. Some of these features may be followed with little change; others of them need to be modified under certain circumstances and with some classes in order to obtain maximum results.

Each chapter in this book is divided into two parts. In Part I of each chapter only the simplest cases and applications of a principle are given, formal definitions, abstract theory, and complicated applications being placed in the corresponding Part II. None of the Parts II are to be studied till the second half year, after all the Parts I have been gone over.

Thus for example in Chapter I, the definition of algebra, of a binomial, etc., are postponed to the second part of the chapter. In teaching this Part I, the teacher is to be at every pains to aid the pupil in realizing that in arithmetic he has already unconsciously learned a considerable amount of algebra in the form of certain symbols and simple formulas, and in learning how to extend this knowledge.

In the case of some classes it will be found advisable after studying the Parts I up to the subject of Simultaneous Equations (that is, through Chapter X), to go back to the beginning of the book and go over the book as a whole, studying all of the Parts I and II in order as they occur.

The teacher may also at times utilize the division of chapters into Parts I and II in another way. Thus, while the class as a whole is going over the Parts I, special problems in, or sections of, the Parts II may be assigned to brighter pupils as extra credit work. In this way some teachers have found it possible to teach the whole book to the brighter part of the class in a half year, this part of the class being able to study some other subject during the remainder of the year while the rest of the class are completing the study of first year algebra.

So much arithmetic is reviewed and covered in the Parts I, that, if pupils are found deficient in any special arithmetical process, it is possible to stress this till it is thoroughly understood.

Still another advantage of the method of studying first year algebra here presented is that if any pupils leave high school at the end of the first half year, they will have studied all of the main principles of a whole year's work in algebra before leaving.

8. Verbal and Written Problems.—It is more and more being recognized that the study of the verbal problem is of the first importance as a means of inculcating the spirit of algebra and enabling pupils to realize its purpose. Hence, such study is far more valuable than practice in intricate manipulations of symbols.

The teacher who wishes to make the utmost use of the verbal problem, will find much material already worked out in Durell and Arnold's Algebra, and in such a form that it may be readily modified or enlarged if this is deemed advisable. In accordance with the general method of the book, many simple verbal problems are given before the more difficult ones are introduced. Though usually presented as oral exercises, these examples may be assigned as written work if the teacher regards this as preferable with any given class.

The study of elementary verbal problems not only cultivates thought power and an appreciation of the spirit of algebra, but also is the best preparation for the more difficult work of solving written problems and of devising and applying formulas. Hence, if pupils have trouble in solving written or formula problems, it is important that the teacher devise and teach many simple verbal problems adapted to prepare for the solution of more difficult ones, as, for instance, Ex. 22, p. 64 prepares for Ex. 16, p. 66; or Ex. 15, p. 88, for Ex. 12, p. 89.

In general, no better sight drill (see § 3) can be given in time available for oral work, than that occupied in answering such questions as the following:

Ex. 1. What is the area in square inches of a rectangle x ft. long and y in. wide?

Ex. 2. What is the interest on y dollars at 6 per cent for t years?

9. Graphs.—It is important in like manner that when the subject of graphs is taken up, the pupil at first be given many simple graphs to construct, till he grasps the essential principles involved and forms a liking for the topic. As his chief difficulty at the outset is that of fixing on a convenient numerical scale on each axis, a completed graph to be used as a model may be given at first. (See p. 27.) In later work, give only a small part of the graph and ask the pupil to complete it. (See Ex. 4, p. 28.) Later still, give no part of the graph, but only the axes marked with their numerical scales and ask the pupil to supply the entire graph. (See pp. 29, 68, 92, 128.)

If the pupil is taught graphing in this simple progressive way during the first half year, he will acquire such confidence in his powers that he will, with comparatively little help and much zest, take up the more difficult cases, where he must determine the numerical scales, draw two graphs on the same diagram with two different scales on the vertical axis, and later determine whether the line, circle, or bar type of graph should be employed in a given case.

Similarly in teaching the interpretation of graphs, it is advantageous to follow the same progressive plan. Simple and easily answered questions like those on page 28 should be asked, till the pupils grow by familiarity and practice into the power of readily seeing a deeper and more comprehensive meaning in graphic forms.

10. The Formula owing to its abstract appearance is less suggestive and attractive to the pupil than the graph. Hence, especial pains should be used in following the rule to make the first lessons in its use simple and clear. When first asked to solve a problem by the use of the formula, pupils often say, "I can work this example by arithmetic" and then proceed to do so, neglecting the formula. In this case, in order to emphasize the nature of the formula, it is useful (as when an example like Ex. 3, p. 9, is to be worked) to require the pupil to tabulate the work in some formal way like the following:

Formula	$a=lw.$
Given	$l=32$, and $w=15$,
Find	a
Process	Substituting for the known letters in the formula, we obtain
	$a=32\times15,$
Hence,	$a=480$ $\therefore$ 480 sq. in. *Ans.*

So in teaching the framing of formulas, give much practice at first in simple cases, where all of the needed letters are given. (See Ex. 14, p. 52; Ex. 2, p. 88, etc.) If abundant drill of this kind is given in the first half year, the pupil will grow without much effort into the mastery of the more advanced cases, where he must supply his own letters in formulating a given process, or transform a formula with reference to the different letters in it, or must convert a rule into a formula, or *vice versa*, or eliminate a letter between two formulas, or study the relations between a formula and its graph.

Particular attention is called to Ex. 30, p. 69, Ex. 19, p. 215, and similar examples, by which especially, when treated orally, the pupil is given a large amount of training in the quick transformation of rules

into formulas and the reverse process. These examples, together with much of the drill in oral language work (see, for instance, Exercise 31, p. 63) enable him to acquire the formula habit.

11. Formation of Original Examples.—It was stated in § 8 that one of the best methods of cultivating a pupil's appreciation of the inner meaning and spirit of algebra is drill in the solution of verbal problems. Another important way of developing this appreciation is that of training the pupil to devise an algebraic problem to meet a given set of conditions. Examples of the kind meant are given in § 3, p. 3, of this pamphlet. Other similar examples supplied in the textbook are Ex. 4, p. 12; Ex. 2, p. 18; Exs. 5–7, p. 19; Exs. 32–35, p. 55; Exs. 54–58, p. 147.

Work of this kind, whether given as sight drill or written work, has the double advantage of being both a review of definitions and principles, and a training in thought power.

12. Extra Credit Work.—When most of the members of a class are immature and for a considerable time are receiving such elementary instruction in the early study of algebra as has been described, it is highly desirable to have some means of assigning at times in addition to the regular lesson, a certain amount of more difficult work to be accomplished by the abler members of the class. This additional work should not interfere with the regular plan of instruction, but rather, if possible, should improve it by speeding up and stimulating all members of the class to do harder work and thus outgrow the elementary stage as soon as possible.

One device by which this end may be attained is the utilization of the task and bonus principle now common in the payment of workmen. Thus, if a workman does a normal or average amount of work, he is rated as 100 per cent efficient. If he accomplishes more than this, he is rated as, say, 110 per cent, or 150 per cent efficient.

For instance, if, in shovelling iron ore from cars the task is 40 tons per day, and a workman should succeed in unloading 48 tons, he is regarded as 120 per cent efficient for the day and receives increased pay accordingly.

Similarly, if a pupil does what may fairly be expected of one in his stage of development, we may give him a grade of 100. If he achieves more than this, and does other work called extra credit work, we may give him a grade of, say, 120, or 140.

Thus, we may give a written test or examination in the following form:

Ex. 1. Solve $x^2-x(x+5)=12+x$.

Ex. 2. $5x-2(3x+2)=7$.

Ex. 3. $3(x+1)(x-1)=3x^2-4$.

Ex. 4. $9y=3+2(1+4y)$.

Ex. 5. (*Extra credit*). $24-5(x^2-2)=1-(x-1)(5x-2)$.

The first four of the above examples are intended to represent what the average or normal pupil may be able to solve in the test period, and the correct solution of all of these will entitle him to a grade of 100. If he also succeeds in solving Ex. 5, his grade will be 125.

The same principle is applicable in assigning home work to be done in preparation of a lesson, or the method may be varied slightly by assigning say twelve problems, the solution of any ten of which entitles a pupil to a grade of 100, the grade for the correct solution of all being 120.

The following are among the advantages of the above method of assigning and grading work.

(1) It accelerates the mental growth of the class as a whole and keeps pupils from being content with the more simple work given them at the outset.

(2) It prevents the more gifted pupils from settling down to the

general level of the class, and, in fact, tends to develop them to the utmost.

(3) It prevents a poor pupil from being discouraged by having a practically unattainable standard of perfection set before him. On the contrary if he has obtained a 100 per cent mark for completing the normal or bogey amount of work, he cheerfully attacks other problems with the stimulating feeling that he can lose nothing, but may gain much by so doing.

(4) The old 100 or absolute system sometimes had the disadvantage that a pupil having gained a mark of 100 or something very close to it, came to feel that he had learned about all that was to be known about a subject. Hence, his progress was checked and perhaps ended. But with possible bonus grades without limit above 100 open to him, endless vistas of achievement are presented and suggested and the pupil is started out upon them.

(5) The method also has certain important broad social and economic educational values. Thus, one obstacle to the satisfactory settlement of certain labor and other problems is the narrow view of efficiency principles held both by certain employers and some workmen, and training by the method here suggested helps broaden all who become familiar with it.

The teacher can carry further this training of the efficiency intelligence and conscience of pupils, by careful instruction in examples like that in § 32, p. 53; or like Ex. 46, p. 57; Ex. 8, p. 202, etc.

In brief the method may be made a stimulus to both weak and able pupils in several ways.

13. Checking Results.—The extra credit principle described in § 12 may be made an aid in overcoming the reluctance which most pupils have to check or prove an answer.

The more complex and strenuous the modern business world and life becomes, the more important it is that every process and detail of work should be tested and proved so that it can be absolutely relied upon, no matter where and how it is used. Hence it is increasingly important that children, as a part of their educational training, should form the habit of checking all answers. Yet children have a marked distaste for this process, and too often merely regard

it as a whimsical requirement on the part of an exacting teacher, especially if the answer is simple and exact and "looks right."

The elimination of about one-third of the more technical and abstruse parts of first year algebra which has been made in some recent syllabi opens the way for giving more attention to the important matter of checking each process and result.

Two devices may be mentioned for overcoming the distaste of pupils for proving answers, and for carrying pupils along till they fully realize the value of doing so, and the process becomes in a measure easy and natural.

The first of these devices is that of making the checking of a problem a separate example and giving the pupil the same credit for checking a process as for the original solution. When this is done a sample test paper would read as follows:

Ex. 1. Solve $x-2=5(x+1)+7$.
Ex. 2. Check the answer obtained in Ex. 1.
Ex. 3. Solve $(x+2)(x-3)=x^2-7$.
Ex. 4. Check the answer to Ex. 3.
And so on alternately.

After pupils have thus been made familiar with the process of proving their answers, they will in time realize its advantage, and they will voluntarily use the method (as in an examination where they are especially anxious to get correct results) after the above artificial stimulus has been removed.

The second method of aiding pupils to form this habit is to begin with it at the outset and apply it to very simple examples. If an example is complicated the proving process usually is long and complex, and hence mistakes are likely to be made by the pupil during its progress. If a discrepancy thus arises, it is a matter of considerable difficulty

for the pupil to determine whether the error has occurred in the process of solving the problem or in that of checking it, and the pupil comes to regard the proving process as merely an added source of perplexity. Hence, the pupil should begin by checking many simple problems till he acquires skill and confidence in the application of the process.

In this connection it may be well to state that some teachers make it a rule never to tell a pupil whether an answer is right or wrong, but require pupils to test their answers so as to make sure for themselves whether these are correct.

14. Self Reliance and Co-operation.—The general plan advocated in the preceding pages, of giving our present immature pupils much simple work at the start and thus putting into action and utilizing the natural growth processes of the child's mind, also has the advantage that when pupils are treated thus they come to work from a higher motive, that is, more to gratify their sense of mastery and expansion and less to obtain good marks. They are less likely to copy each other's work, or, indeed, to get an undue amount of help from any outside source; for the pleasure which comes from personal achievement is so great that they want nothing to interfere with it, and they often desire ever harder work in order to add to this pleasurable sense of achievement.

After this spirit has become general in the class, so that a pupil is not likely to accept aid except when it is really needed, it is possible to allow and even foster a certain amount of co-operation (see § 5) among the members of a class and thus to utilize the remarkable power which some pupils have of realizing the exact nature of the stumbling blocks of other pupils and of aiding their fellow pupils to overcome their difficulties.