ANALYTICAL INSTITUTIONS.

ANALYTICAL INSTITUTIONS,

IN FOUR BOOKS:

ORIGINALLY WRITTEN IN ITALIAN,

BY

DONNA MARIA GAETANA AGNESI,

PROFESSOR OF THE MATHEMATICKS AND PHILOSOPHY IN THE UNIVERSITY OF BOLOGNA.

TRANSLATED INTO ENGLISH

BY THE LATE

REV. JOHN COLSON, M. A. F. R. S.

AND LUCASIAN PROFESSOR OF THE MATHEMATICKS IN THE UNIVERSITY OF CAMBRIDGE.

NOW FIRST PRINTED, FROM THE TRANSLATOR'S MANUSCRIPT,

UNDER THE INSPECTION OF THE

REV. JOHN HELLINS, B. D. F. R. S.

AND VICAR OF POTTER'S-PURY, IN NORTHAMPTONSHIRE.

VOLUME THE FIRST,

CONTAINING THE FIRST BOOK.

To which is prefixed,

AN INTRODUCTION BY THE TRANSLATOR.

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1801.

E. H.

ADVERTISEMENT

BY

THE EDITOR.

THE *Analytical Inſtitutions* of the very learned Italian Lady, *Maria Gaetana Agneſi*, Profeſſor of the Mathematicks and Philoſophy in the Univerſity of *Bologna*, which were publiſhed in two Volumes, Quarto, in the year 1748, are well known and juſtly valued on the Continent; and there cannot perhaps be a better recommendation of them in this Iſland, than that they were tranſlated into Engliſh by that eminent judge of Mathematical Learning, the late Reverend *John Colſon*, M. A. F. R. S. and Lucaſian Profeſſor of the Mathematicks in the Univerſity of *Cambridge*. That learned and ingenious man, who had obliged his Country with an Engliſh Tranſlation of Sir ISAAC NEWTON's Fluxions, together with a Comment on that profound work, in the year 1736,—and was well acquainted with what appeared on the ſame ſubject, in the courſe of fourteen years afterward, in the writings of thoſe very ingenious men, *Emerſon*, *Mac Laurin*, and *Simpſon*,—found, after all, the *Analytical Inſtitutions* of *Agneſi* to be ſo excellent, that he was at the pains of learning the Italian Language, at an advanced age, for the ſole purpoſe of tranſlating that work into Engliſh; that the Britiſh Youth might have the benefit of it as well as the Youth of Italy.

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This great design he lived to accomplish; and had actually transcribed a fair copy of his Translation for the press, and begun to draw up proposals for printing it by subscription. And, in order to render it more easy and useful to the Ladies of this Country, (if indeed they can be prevailed upon by his persuasion and encouragement, to show to the world, as they easily might, that they are not to be excelled by any foreign Ladies whatever, in any valuable accomplishment,) he had designed and begun a popular account of this work, under the title of *The Plan of the Lady's System of Analyticks;* explaining, article by article, what was contained in it. But this he did not live long enough to finish, nor indeed to give more than a rough draught of it so far as article 256 of the first Book.

In this state the Manuscript remained many years; and, considering the great expense which, in the present times, attends the printing of such a work, probably might have remained many more, had it not been for the active and liberal spirit of Mr. BARON MASERES; who, whether we consider his own ingenious and extensive labours in the Mathematicks, or the encouragement which he gives to others who employ their talents in that way, well deserves what Sir ISAAC NEWTON said of Mr. *Collins*, the great encourager of Mathematical Learning in his time —*Vir in Rem Mathematicam promovendam natus* *. But this commendation is far short of the deserts of the Patron of this Work. While he sets a due value upon *Arts* and *Sciences*, he is highly sensible of the much greater importance of REVEALED RELIGION, and *well-constituted Government*, to the happiness of mankind; and is no less pious and loyal than he is learned and liberal. To the truth of these assertions every one who is acquainted with him will readily bear testimony; and they might be supported likewise by passages from various Books which

* See Comm. Epistol. Edit. 1722, p. 148.

are

are well known to be productions of his pen, although ſome of them bear not his name. But I forbear quotations from his works in this place, that I may not, on the one hand, hurt the modeſty of a Friend, nor, on the other, give occaſion to the captious and malevolent to ſay I offer incenſe to my Patron.

When the BARON had reſolved to bear the whole of the expenſe of a handſome Edition of theſe *Inſtitutions*, he was pleaſed to deſire me to ſuperintend the printing of them: to which I readily conſented, in conſequence of favours received from him, and with the hope that I might render ſome little ſervice to the readers of this work, by taking care that it ſhould be correctly printed, which is a matter that requires more time and attention than moſt are aware of, who have not experienced it.

But, beſides correcting the errors of the preſs, it was neceſſary to correct many little ſlips of the pen, and inaccuracies, which I found in the Copy. For, notwithſtanding it was fairly tranſcribed for the preſs in Mr. *Colſon's* own hand-writing, it had evidently been written in haſte, and wanted reviſion; and undoubtedly would have received it from him, if he had lived to ſuperintend the printing of it himſelf. Of theſe inaccuracies, a few were in the language, but more in the mathematical part, where, although I ſeldom found any wrong concluſion, I found many miſtakes in the ſigns and exponents of quantities, as well as omiſſions of numbers and quantities, and ſometimes of whole clauſes. Some of theſe miſtakes I was enabled to correct by means of the foul ſheets on which the Tranſlation was firſt written; but finding errors in them alſo, (ſome of which, I doubt not, were occaſioned by preſs errors in the original, a copy of which I could never obtain),

obtain *,) I ſaw no way of ſatisfying myſelf, but to undertake the labour, great as it was, of examining and recomputing every operation in which I ſuſpected or diſcovered any error: and this was frequently the caſe in the ſecond Volume. In ſhort, my endeavour has been to preſent this Tranſlation to the Public faithfully as the worthy old Profeſſor made it, and would have rendered it, if he had lived to publiſh it; altering nothing in it but the miſtakes before mentioned, nor inſerting any thing of my own but what is included within theſe marks [ ].

With reſpect to the ſtyle of this Tranſlation, ſome of the ſentences, no doubt, might have been better turned; yet the meaning is, in general, plain enough, which is all that is requiſite in books of this kind.

It has been mentioned above, that the Introduction was left unfiniſhed by Profeſſor *Colſon:* I have continued it to the end of the firſt Volume; diſtinguiſhing what I have written from what was found in his Manuſcript by putting it in brackets.

It appears by a paſſage in the Manuſcript of the Introduction, that Mr. *Colſon* intended to make ſome additions to this Work; but what theſe additions were to be is not mentioned. Yet I conjecture that they were to be ſome eaſy pleaſant Queſtions, with their Solutions, in the manner which he has ſhown in Sect. VI. of his Comment on Sir ISAAC NEWTON's Fluxions; merely to exerciſe the learner in the rules given in theſe Inſtitutions, and not to contain any new rules, or additional matter; for he has called this Work of *Agneſi*, *A Complete Syſtem of Analyticks* †. And finding a ſhort Paper of this kind in his handwriting, I have inſerted it at the end of the ſecond Volume.

* In the year 1799, I employed two days in making inquiries amongſt the bookſellers of London, from one end of the city to the other, for a Copy of the Original, without ſucceſs.

† See the Introduction, p. i.

That

That these *Institutions*, considering the great quantity of valuable matter contained in them, the judicious manner in which it is arranged, and the perspicuity with which it is explained, will be esteemed, by all candid judges, as the most valuable work of the kind that has appeared in our language, need not be doubted. Instances of the superiour skill of the Author may be found in various parts of her Work, more especially in the Fourth Book, where it appears in the construction of some fluxionary equations without a separation of the variable quantities, —in the separation of the variable quantities in others,---and in the reduction of others in which there are second and third fluxions to equations having first fluxions only. A single instance of her great skill may serve to gratify the reader, and, for the sake of brevity, is all that I shall produce in this place. It is taken from the beginning of the fifth Article of the first Section of the Fourth Book, where she shows that the equation of the fluents of $y^r\dot{y} = x^n\dot{y} + yx^{n-1}\dot{x}$ is $fny^{r+n-1}\dot{y} = x^ny^n \pm b$; which, by only writing x for y and y for x, is the solution of the equation $y^n\dot{x} + xy^{n-1}\dot{y} = x^r\dot{x}$; from which the solution of the equation $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m\dot{x}}{ay^n}$ is most easily obtained. This equation is taken from page 289 of the second Volume of *Simpson's* Fluxions, (published in the year 1750,) who has there expressed his opinion, That the only case in which this equation admits of a solution "by multiplying, or dividing it, by some power or product of the quantities concerned," is, when $n = 1$: whereas *Agnesi* has given a general solution by that method *. What is here said is only to

* I am aware that a solution of this equation has, of late, been given by several ingenious persons of this Country; which, however, some of them may see reason to revise.

prove the great ſkill of *Signora Agneſi*, and not with any intent to leſſen the reputation of Mr. *Simpſon*; for whoſe memory and abilities I have the higheſt reſpect, eſteeming him as one of the greateſt Mathematical Geniuses that this Country has produced ſince the time of Sir ISAAC NEWTON.

It may perhaps be objected to theſe *Inſtitutions*, That there are a number of Mechanical and Phyſical Problems to be met with, in ſome Treatiſes of Fluxions in our language, which are not found here. The anſwer is, That ſuch Problems are properly placed in Treatiſes of *Mathematical Philoſophy*; but, as the ſolutions of them require a knowledge of Mechanicks, and Natural Philoſophy, they could not, with any more propriety, be admitted into an Elementary Treatiſe of Fluxions, than the Problems of meaſuring Land, or of taking Heights and Diſtances, could be admitted into *Euclid's* Elements of Geometry.

But here I would not be underſtood to inſinuate that theſe *Inſtitutions* are ſo perfect as to admit of neither improvement nor addition: on the contrary, I have obſerved that ſome of the inveſtigations might be made in a ſimpler manner; and that the *Methods of finding the Roots of numerical Equations by Approximation*,—*Of ſolving literal and fluxionary Equations by infinite Series*,---and *Of comparing together homogeneous Fluents*, are wanting in them; all which might be contained in a few ſheets, and which, if added to this Work, would ſave the learner the expenſe of money and time in procuring and reading a number of books on theſe ſubjects. Theſe Methods therefore, together with Notes on ſeveral parts of the Work, I purpoſe to draw up, under the title of

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A Supplement to Maria Agnesi's Analytical Institutions; to be printed with the same type, and on the same kind of paper, as this Work; if health and leisure should permit, and if it should appear to be desired by Mathematical Readers.

The wonderful sagacity which appears in these *Institutions*, and the singular circumstance that so large a work of this kind was performed by a Lady, raised in me a wish to obtain a particular account of the Author; but the confusion and misery which have been brought upon a great part of Europe, and particularly upon Italy, by the French Revolution, have deprived me of the means of getting authentic information respecting this *Phænomenon* of Literature from the University of *Bologna*, of which she was once so bright an ornament. All the information I have been able to get of her, (besides what appears in her excellent Work, and some just encomiums on her skill which I have seen in foreign books,) I have inserted in the following pages; supposing that the reader would be no less desirous than myself of any authentic information respecting so amiable and so extraordinary a person. The account comes, indeed, by way of France; yet, as there is no visible motive for the writers of it to deviate from truth in what they have related of her, I see no reason for disbelieving it.

I have also inserted the Testimony given by Dr. *Saunderson* to the great genius and skill of Mr. *Colson;* conceiving that it might prove useful information to the junior readers of these *Institutions*.

I have only to request of the candid reader that, if, notwithstanding

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the care I have taken in correcting the preſs for this Work, any errors have eſcaped me, (and in printing a work of this kind it is hardly poſſible but ſome will eſcape unnoticed,) he will correct them himſelf, and kindly excuſe the omiſſion.

JOHN HELLINS.

Potter's-Pury,
September 29th, 1801.

SOME

SOME ACCOUNT OF *MARIA AGNESI*,

THE AUTHOR OF THESE *ANALYTICAL INSTITUTIONS*.

IN the Appendix to the XXXIIId Volume of the Monthly Review, pages 516 and 517, is an Account of *Maria Agneſi*, taken from one of M. *De Broſſes'* Letters on Italy, which is nearly the ſame in ſubſtance, but not in perſpicuity, with what is here printed.

'Letter X.—The account given by Monſieur De Broſſes, in the 10th Letter, of a kind of literary phænomenon that he met with in this journey, is ſo remarkable that we cannot avoid tranſcribing it. This was a young lady of *Milan*, about eighteen or twenty years of age, named *la Signorina Agneſi*, whom he calls *a walking Polyglott*, and who, not content with knowing all the oriental languages, undertook to maintain a *Theſis* in any of the ſciences againſt any one who ſhould chooſe to diſpute upon it with her. At a *Converſatione* to which our traveller [Monſieur De Broſſes] and his nephew were invited, they found about thirty perſons, of ſeveral different nations of Europe, ſitting in a circle, and *la Signorina Agneſi*, with her little ſiſter, ſeated under a canopy. She could hardly be reckoned handſome; but ſhe had a fine complexion, and an air of great ſimplicity, ſoftneſs, and feminine delicacy.'

"I had conceived (ſays the Preſident *,) when I went to this converſation-party, that it was only to converſe with this young lady in the uſual way, though on learned ſubjects; but, inſtead of this, Count *Belloni* (who had introduced me to it,) made a fine harangue to the lady in *Latin*, with the formality of a college-declamation. She anſwered with great readineſs and ability in the ſame language; and they then entered into a diſputation, ſtill in the ſame language, on the origin of fountains and on the cauſes of the ebbing and flowing which is obſerved in ſome of them, like the tides in the ſea. She ſpoke like an angel on this ſubject; and I never heard it treated in a manner that gave me more ſatisfaction. Count *Belloni* then deſired me to enter with her on the diſcuſſion of any other ſubject I ſhould chooſe to pitch upon, provided that it related to Mathematicks or Natural Philoſophy. This propoſal alarmed me a good deal,

* M. *De Broſſes* was firſt Preſident of the Parliament of Dijon, and Member of the *Royal Academy of Inſcriptions and Belles Lettres* of Paris. According to the Monthly Reviewer, he travelled in Italy about the year 1740: from which it follows that *Agneſi* was about 28 years of age when her *Analytical Inſtitutions* were publiſhed.

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as I found it was expected that I ſhould hold a converſation in the Latin language, with which I had no longer that familiar acquaintance and readineſs at ſpeaking it, which in the days of my youthful ſtudies I had formerly poſſeſſed. However, I made the lady the beſt excuſes I could for my want of ſufficient ſkill in the Latin language to make me worthy of converſing in it with her, and hoped ſhe would over-look the incorrect expreſſions I might happen to make uſe of in the courſe of the diſcuſſion; and we then entered, firſt, into an inquiry concerning the manner in which the ſoul receives impreſſions from corporeal objects, and in which thoſe impreſſions are communicated from the eyes, and ears, and other parts of the body on which they are firſt made, to the organs of the brain, which is the general *ſenſorium*, or place in which the ſoul receives them; and we afterwards diſputed on the propagation of light and the priſmatick colours. *Loppin* then diſcourſed with her on *tranſparent bodies*, and on *curvilinear figures* in Geometry, of which laſt ſubject I did not underſtand a word. Loppin ſpoke in French; and the lady begged to be permitted to anſwer him in Latin, fearing that ſhe ſhould not be able to recollect the proper French technical names of the ſeveral ſubjects which they ſhould have occaſion to conſider.

" She ſpoke wonderfully well on all theſe ſubjects, though ſhe could not have been prepared before-hand to ſpeak upon them, any more than we were. She is much attached to the Philoſophy of Sir Isaac Newton: and it is marvellous to ſee a perſon of her age ſo converſant with ſuch abſtruſe ſubjects. Yet, however much I may have been ſurprized at the extent and depth of her knowledge, I have been much more amazed to hear her ſpeak Latin (a language which ſhe certainly could not often have occaſion to make uſe of,) with ſuch purity, eaſe, and accuracy that I do not recollect to have ever read any book in modern Latin that was written in ſo claſſical a ſtyle as that in which ſhe pronounced theſe diſcourſes. After ſhe had replied to *Loppin*, the converſation became general, every one ſpeaking to her in the language of his own country, and ſhe anſwering him in the ſame language: for her knowledge of languages is prodigious. She then told me that ſhe was ſorry that the converſation at this viſit had taken that formal turn of an *Academical Diſputation*, declaring that ſhe very much diſliked ſpeaking on ſuch ſubjects in numerous companies; where, for one perſon who received amuſement from the diſcuſſion of them, there were often twenty who were tired to death by it; and that therefore ſuch ſubjects were only fit to be entered-upon in ſmall companies of two or three perſons, who had all

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the ſame taſte for diſcuſſing them. This obſervation, I thought, was very juſt, and was a proof of the ſame good ſenſe and diſcernment which had appeared in her former learned diſcourſes. I was ſorry to hear that ſhe was determined to go into a Convent, and take the veil: which was not from want of fortune, (for ſhe is rich,) but from a religious and devout turn of mind, which diſpoſes her to ſhun the pleaſures and vanities of the world. After the converſation was finiſhed, her little ſiſter played on the harpſichord, with the ſkill of a Rameau, firſt, ſome of Rameau's pieces of muſic, and then ſome pieces of her own compoſition, and concluded by ſinging ſome airs and accompanying her voice on the inſtrument."

M. *Montucla* ſpeaks of *Maria Agneſi*, and of her *Analytical Inſtitutions*, to the following effect, in his *Hiſtoire des Mathématiques*, Volume II, page 171.

"Beſides the foregoing Authors I ought to mention on this occaſion, with much commendation, the *Analytical Inſtitutions* of a learned Italian lady of the name of *Maria Gaetana Agneſi*, which is a work of ſuch merit that ſome female mathematician of France (for we alſo have ſome ladies of that deſcription among us,) would have done well to give us a French tranſlation of it. We cannot behold without the greateſt aſtoniſhment a perſon of a ſex that ſeems ſo little fitted to tread the thorny paths of theſe abſtract ſciences, penetrate ſo deeply as ſhe has done into all the branches of Algebra, both the common and the tranſcendental, or infiniteſimal. She has ſince retired to a cloiſter: and, though we do not preſume to cenſure her conduct in this ſtep, (which we muſt ſuppoſe to proceed from the pureſt and ſincereſt piety,) we cannot but lament that ſhe ſhould have thus deprived the learned world of the uſeful improvements in Literature which her genius and knowledge would have enabled her to communicate to it, not only on ſubjects of a mathematical nature, but on many others of a different kind, in which ſhe had become eminent."

In the Index to the Volume above mentioned, M. *Montucla*, at the name *Agneſi*, refers alſo to the third Volume of his work, which is not yet publiſhed.

Maria Agneſi and her *Analytical Inſtitutions* are mentioned alſo in a note in page 179 of a work intitled "*An Eſſay on the Learning, Genius, and Abilities of the Fair-Sex: proving them Not Inferior to Man, from a Variety of Examples, extracted from Antient and Modern Hiſtory. Tranſ-*

lated from the Spaniſh of ' *El Theatro Critico.*' London 1774." What is there ſaid of her is to the effect following:

" A learned Italian lady of our own times is *Signora Agneſi* *, daughter of a creditable tradeſman in Milan, famed throughout all Europe for her knowledge of the learned languages and for being the author of a profound treatiſe of Algebra, intitled *Analytical Inſtitutions*, which, beſides many eulogiums beſtowed on her by ſeveral Scientifical Societies, has gained her a Profeſſorſhip of Mathematicks in the Univerſity of Bologna. Neither her inclination to theſe favourite intellectual purſuits, nor a deſire of preſerving and increaſing the fame ſhe had acquired by her attainments in them, nor the intreaties of her father have been able to ſtifle the call from heaven which ſhe conceives herſelf to have felt in her child-hood to dedicate herſelf to a monaſtick life amongſt the nuns known by the name of *The Blue Nuns*, than which there are few orders in the Church of Rome ſubject to rules of greater ſeverity. Since her father's death ſhe has given herſelf up to the moſt ſublime devotion, and has ſacrificed to chriſtian ſelf-denial all thoſe enjoyments in the ſociety of the world to which her fine qualities and literary attainments had already introduced her amongſt the moſt reſpectable part of mankind."

DR. SAUNDERSON'S TESTIMONY OF THE GENIUS OF MR. *COLSON*.

DR. NICHOLAS SAUNDERSON, *Lucaſian* Profeſſor of the Mathematicks in the Univerſity of Cambridge, and Fellow of the Royal Society, ſpeaking of Mr. *Colſon* in his *Algebra*, Vol. II. p. 720, has theſe words:

—" The learned Mr. *John Colſon*, a gentleman whoſe great genius and known abilities in theſe ſciences I ſhall always have in the higheſt admiration and eſteem."

Mr. *De Moivre* alſo has, on ſeveral occaſions, ſpoken of the great ſkill of Mr. *Colſon*; but, for want of books, I cannot quote his words. However, Dr. *Saunderſon's* Teſtimony, and the office which Mr. *Colſon* afterward held in the Univerſity of *Cambridge*, are ſufficient vouchers of his ability.

* In the Note above referred to, which ſeems to be a bad tranſlation of a paſſage in a book intitled ' *Obſervations ſur l'Italie*, &c,' her name is erroneouſly printed *Angleſe*. I have therefore given the ſame Account in better Engliſh, as it was communicated to me by Mr. *Baron Maſeres*; to whom alſo I am obliged for all the reſt that is here printed concerning this very extraordinary perſon. J. H.

THE

THE AUTHOR'S DEDICATION.

TO

HER SACRED IMPERIAL MAJESTY,

MARIA TERESA OF AUSTRIA,

EMPRESS OF GERMANY, QUEEN OF HUNGARY, BOHEMIA, &c. &c.

AMONG the various arguments I revolved in my mind, inducing me to hope, that Your Sacred Majeſty, according to your great condeſcenſion, would vouchſafe to receive favourably this Work of mine, which is proud to ſhelter itſelf under your auguſt name, and humbly to crave your gracious patronage and protection; among all theſe arguments, I ſay, none has encouraged me ſo much as the conſideration of your ſex, to which Your Majeſty is ſo great an ornament, and which, by good fortune, happens to be mine alſo. It is this conſideration chiefly that has ſupported me in all my labours, and made me inſenſible to the dangers that attended ſo hardy an enterpriſe. For, if at any time there can be an excuſe for the raſhneſs of a Woman, who ventures to aſpire to the ſublimities of a ſcience, which knows no bounds, not even thoſe of infinity itſelf, it certainly ſhould be at this glorious period, in which a Woman reigns, and reigns with univerſal applauſe and admiration. Indeed, I am fully convinced, that in this age, an age which, from your reign, will be diſtinguiſhed to lateſt poſterity, every Woman ought to exert herſelf, and endeavour to promote the glory of her ſex, and to contribute her utmoſt to increaſe that luſtre, which it happily receives from Your Majeſty; who, having diffuſed, on all ſides, the fame and admiration of your actions, have obliged Mankind to apply to you, with much greater reaſon, what has been ſaid of ſome of the antient Cæſars;—that, by the juſtice and clemency of your Government, you are an honour to human nature, and a near reſemblance of the divine. To thoſe who, zealous for the glory of our ſex, ſhall faithfully tranſmit to poſterity the memory of your deeds; to thoſe (I ſay) I muſt leave to commemorate, how each accompliſhment of the mind is united in Your Majeſty with the moſt engaging gracefulneſs of perſon; to thoſe I ſhall leave the arduous taſk to deſcribe, the ſtrength of your underſtanding, the extenſiveneſs of your genius, but, above all, that ſignal fortitude, that invincible courage and conſtancy of mind, by which you derived freſh vigour, as it were, from your perils and perſecutions themſelves; and, after having been ſo ſeverely tried by the hand of Providence at the beginning of your reign, gave at laſt ſo happy a reverſe to your affairs. Neither will they fail to celebrate the engaging ſweetneſs of your temper, your humane and compaſſionate diſpoſition, nor that generous condeſcenſion with which, amidſt the hurry and tumult of

 arms,

arms, you cherifh and protect the arts and fciences; being duly fenfible how greatly thefe redound to the public welfare; and that by thefe the minds of men are forcibly excited to the purfuit and practice of every focial virtue. Hence it was, that the Sciences fo early took poffeffion of your mind, and that you became well acquainted with the whole circle of them. And though the bufy cares and interruptions of Empire may have withdrawn you from your more ftudious applications, (Heaven having thought it too fmall a commendation for you, to be called the moft knowing and learned Woman of your age,) yet ftill your love of truth is not the lefs fervent; fo that whoever employ themfelves in the fearch of it, are fure to meet with diftinguifhing marks of your approbation.

Vouchfafe, therefore, Madam, to caft a favourable eye on this Performance of mine, not only as a Work which comprehends the higheft attempts of the human underftanding, but alfo as the greateft tribute it was in my power to offer, to the glory of your aufpicious reign; a reign which feems to revive the memory of former heroines, only to render your magnanimity, prudence, and good fortune, the more eminently confpicuous by the comparifon. And if the Volume of Mufic, which my Sifter has had the honour of prefenting to Your Majefty, has been fo fortunate as to excite your voice to melodious accents; let this be fo happy as to have the defired effect, of employing fometimes the fagacity and penetration of your underftanding. As nothing more remains, but to implore of Heaven a long and happy continuance of your glorious reign, for the felicity of the many nations fubject to your command; I therefore proſtrate myſelf, with all humility, at the foot of your Throne, and am

Your Majeſty's

moſt humble,

moſt obedient,

and faithful ſervant,

MARIA GAETANA AGNESI.

THE

THE AUTHOR'S PREFACE

TO

THE READER.

THERE are few ſo unacquainted with Mathematical Learning, but are ſenſible the Study of Analyticks is very neceſſary, eſpecially in our days; they cannot but be apprized what improvements have already been made by it's means, what are ſtill making every day, and what may be yet expected in time to come. For which reaſon I ſhall not amuſe myſelf with making unneceſſary encomiums on this ſcience, which ſtands in no need of any ſuch recommendations, and much leſs of mine. But, notwithſtanding the neceſſity of this ſcience appears ſo evident as to excite our youth to the earneſt ſtudy of it; yet great are the difficulties to be overcome in the attainment of it. For it is very well known, that perſons able and willing to teach it are not to be found in every city, at leaſt not in our Italy; and every one that would be glad to learn has not the means of travelling into diſtant countries, in queſt of proper maſters. This I know by my own experience, as I muſt ingenuouſly confeſs; for, notwithſtanding the ſtrong inclination I had to this ſcience, and the great application I made uſe of to acquire it; I might ſtill have been loſt in a maze of inextricable difficulties, had I not been aſſiſted by the ſecure guidance and ſage direction of the very learned Father *Don Ramiro Rampinelli*, Monk of the Olivetan Order, and now Profeſſor of

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the Mathematicks in the Royal Univerſity of *Pavia*; to whom I acknowledge myſelf indebted for what little progreſs I may poſſibly have made in this kind of ſtudy; on whoſe deſerved praiſe I ſhall forbear to inſiſt, it being unneceſſary to a perſon of his fame and merit, and offenſive to his known, but perhaps too rigid, modeſty. True it is, the aforeſaid inconvenience may, in ſome meaſure, be removed, by having recourſe to good books, written with perſpicuity, and (what is above all) in a proper method. But though what relates to the ſubject of Analyticks may have already been treated of, and is to be found in print; yet as theſe pieces are ſcattered and diſperſed in the works of various authors, and particularly in the *Leipſic* Acts, the Memoirs of the Royal Academy of Sciences at *Paris*, and in other foreign Journals; ſo that it is impoſſible for a beginner to methodize the ſeveral parts, even though he were furniſhed with all the books neceſſary for his purpoſe: this conſideration induced the celebrated Father *Renau* to publiſh that moſt uſeful Work, intitled *L'Analyſe demontrée*, a work deſerving the higheſt commendation. After which, I am very ſenſible, that theſe Inſtitutions of mine may ſeem, at firſt ſight, to be needleſs, ſo many learned Men having thus amply provided for the occaſions of the Public. But, as to this point, I deſire the candid reader to conſider, that, as the Sciences are daily improving, and, ſince the publication of the aforementioned book, many important and uſeful diſcoveries have been made by many ingenious writers; as had happened likewiſe to thoſe who had written before them: Therefore, to ſave ſtudents the trouble of ſeeking for theſe improvements, and newly-invented methods, in their ſeveral authors, I was perſuaded that a new Digeſt of Analytical Principles might be uſeful and acceptable. The late diſcoveries have obliged me to follow a new arrangement of the ſeveral parts; and whoever has attempted any thing of this kind muſt be convinced, how difficult it is to hit upon ſuch a method as ſhall have a ſufficient degree of perſpicuity, and ſimplicity, omitting every thing ſuperfluous, and yet retaining all that is uſeful and neceſſary; ſuch, in ſhort, as ſhall proceed in that natural order, in which

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confifts the clofeft connexion, the ftrongeft conviction, and the eafieft inftruction. This natural order I have always had in view; but whether I have always been fo happy as to attain it, muft be left to the judgment of others.

In the management of various methods, I think I may venture to fay, that I have made fome improvements in feveral of them, which I believe will not be quite devoid of novelty and invention. To thefe the judicious Reader may give what weight he pleafes. It was never my defign to court applaufe, being fatisfied with having indulged myfelf in a real and innocent pleafure; and, at the fame time, with having endeavoured to be ufeful to the Public.

In the Second Volume, in which I treat of the Integral Calculus, or what is alfo called the Inverfe Method of Fluxions, the Reader will meet with a fpeculation entirely new *, and no where before publifhed, concerning *Multinomials*. For this I am indebted to the celebrated Count *James Riccati*, a gentleman who has greatly deferved of every branch of literature, and whofe merit is well known to the learned world. He was pleafed to communicate this to me, which I take as a favour beyond my deferts; and for which both the Public and myfelf are bound to give him our thanks.

To conclude: As it was not my intention, at firft, that the following Work fhould ever appear in public; a work begun and continued in the *Italian* tongue, purely for my own private amufement, or, at moft, for the inftruction of one of my younger Brothers, who poffibly might have a tafte for mathematical ftudies; and as I had not determined to fend it abroad till after it was pretty far advanced, and had grown to the fize

* It does not appear to me, that any thing can be done by this new method, which may not be done as well, or better, without it. J. H.

of

of a juſt volume; then I thought I might be excuſed the trouble of tranſlating it into Latin, (a language which ſome may imagine is more ſuitable to works of this nature,) eſpecially as I had the example of ſo many famous Mathematicians, as well Italians as others, who have publiſhed their Mathematical Works in their own mother-tongues. Nor could I eaſily overcome my natural indolence, in ſubmitting to the drudgery of tranſlating that into Latin which I had already compoſed in *Italian*. Far am I therefore from laying the leaſt claim to any merit ariſing from that purity and elegance of ſtyle, which in ſubjects of a different nature may be laudably attempted; being fully ſatisfied if I have always expreſſed myſelf, as I ſincerely endeavoured, in a plain, but clear and intelligible manner.

A TABLE

A TABLE

OF THE

CONTENTS OF THE WHOLE WORK.

VOLUME THE FIRST.

INTRODUCTION.

BOOK I.

The Analysis of Finite Quantities.

VOLUME THE SECOND.

BOOK II.

The Analysis of Quantities Infinitely Small.

BOOK III.

Of the Integral Calculus.

BOOK IV.

The Inverse Method of Tangents.

THE PLAN

OF

THE LADY'S SYSTEM OF ANALYTICKS.

INTRODUCTION.

THAT we ſhould receive from Italy, the Mother of Arts, a complete Syſtem of Analyticks, is not ſo much to be wondered at; knowing we have often had from that quarter very excellent productions in the ſublimer Mathematicks. But, that we ſhould receive ſuch a preſent from the hands of a Lady; from that ſex which, however capable, yet hardly ever amuſe themſelves with theſe ſeverer ſtudies; is, indeed, very wonderful and ſurpriſing. Yet ſo it is in fact: a very learned, ingenious, and celebrated Lady of *Milan*, by name *Donna Maria Gaetana Agneſi*, a member of the Univerſity of *Bolonia*, and lately advanced by the Pope to a Profeſſorſhip in Mathematicks and Philoſophy in the ſame Univerſity, has publiſhed a Treatiſe in Italian, in two volumes *quarto*, which ſhe calls *Analytical Inſtitutions for the Uſe of the Youth of Italy*; of which ſhe was pleaſed to preſent a Copy to the Royal Society of *London*. This Copy I had the curioſity to inſpect, and thought it might be a proper way of returning the Author's compliment, to have an Account of the work drawn up and read to the Society, and perhaps printed in the Philoſophical Tranſactions, as has often been the practice on ſuch occaſions. This Account, therefore, I undertook to draw up, having the conſent and approbation of our worthy Preſident. But when I came to look into the work more cloſely, I ſoon enlarged my ſcheme;

and, inftead of barely taking the Plan, or giving an Account of it, I thought it highly deferved to be tranflated into our own language, that the Youth of *England* might likewife enjoy the benefit of it. This determined me then to attempt it's tranflation, though I well knew how unequal I was to the tafk. I confefs I alfo entertained fome diftant hopes, that it might excite the curiofity of fome of our *Englifh* Ladies; that it might raife an emulation in them, a laudable ambition to promote the glory of their country, with a generous refolution not to be outdone by any foreign ladies whatever. They want no genius or capacity for the fciences, and have undoubtedly as good abilities as the Ladies of *Italy*. They feem only to want to be properly introduced into thefe ftudies, to be convinced of their ufefulnefs and agreeablenefs, and to prevail on themfelves to ufe the neceffary application and perfeverance. They have here a noble inftance before them, of what the fex is capable to perform, when their faculties are exerted the right way. And they may be fully perfuaded, that what one lady is able to write, other ladies are able to imitate, or, at leaft, to read and underftand. With not much more pains and induftry than what they muft be at, to be expert at Whift or Quadrille, they may become miftreffes of this fcience; which they will find to be much more innocent, more diverting and agreeable, and to have infinitely more amufing variety than thofe, or any other games whatever. Indeed, this is rather to be efteemed a game, or a diverfion, than a ftudy; but then it is a game of fkill, without any mixture of chance, like Chefs and fome other ingenious games: and parties of two, or more, may play at it together, by propofing curious queftions to one another alternately, to their great diverfion and improvement. The games of Whift, Quadrille, Back-gammon, &c. and all other games in which chance predominates, but fkill is alfo required to convert the events of chance to the beft advantage; thefe are only particular cafes of this general game or art, and ought always to be regulated by it. For, in all inftances, Analyticks may be ufed to difcover the odds, or degrees of probability, which are for, or againft, the happening of any particular event, and fo the chance may be made equal on all fides, notwithftanding a fuperiority of fkill on one fide. And thus all games of chance may be made fair and equal; and the well-meaning gamefter will not be impofed on by fharpers, who, by much obfervation, rather than by fkill in Analyticks, always know what they call the beft of the lay, or always have the odds on their fide.

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But this is the leaſt recommendation of this ſcience. The improvement of their minds and underſtandings, which will neceſſarily ariſe from hence, is of much greater importance. They will be inured to think clearly, cloſely, and juſtly; to reaſon and argue conſequentially, to inveſtigate and purſue truths which are certain and demonſtrative, and to ſtrengthen and improve their rational faculties. Now that theſe, and all other readers, may attain theſe advantages with as little trouble as poſſible, I ſhall endeavour to draw out the Plan of this Work at full length, and in a popular manner, inſerting ſome uſeful Obſervations to explain the Art itſelf; ſo that the Work, when publiſhed, may be eaſily read and apprehended, by ſuch as will peruſe it with the neceſſary diligence and attention.

The ſubject of the Work is Analyticks, or the general Science of Computation or Calculation. That is, the Art of reſolving all kinds of Mathematical Queſtions, by finding or computing unknown numbers, or quantities, by the means of others that are known or given. Theſe computations are performed either by common numbers, and then the ſcience is called *Arithmetick:* or by general numbers or arbitrary ſymbols of quantities, which are commonly the letters of the alphabet, and then it is uſually called *Algebra:* or by lines and geometrical figures, which are likewiſe the ſymbols of quantities, and then it is called *Geometry:* or, laſtly, by all theſe conjunctly and indifferently, and then it will properly be called *Analyticks.* All theſe ſciences our Author teaches and explains promiſcuouſly, but in good order and method, at leaſt the higher and more difficult parts of them; for ſhe requires, as very reaſonably ſhe may, that the learner ſhould come prepared with a pretty good ſtock of common Arithmetick, with a competent knowledge of the firſt elements of Geometry, and with ſome inſight into the ſimpler properties of the Conic Sections. Theſe are acquiſitions with which they may be eaſily furniſhed out of the common mathematical books on theſe ſubjects; which will then prepare the way for an eaſy acceſs to her ſublimer ſpeculations. Now, to enter upon our intended Plan. The Author divides her ſubject into two Tomes, or Volumes; in the firſt of which ſhe treats of the common, ordinary, and finite quantities, and their repreſentatives, whether numbers, general ſymbols, or lines. In the ſecond Volume ſhe explains the nature of what ſhe calls *Infiniteſimals,* or infinitely ſmall Quantities; proves their comparative exiſtence, and ſhows their uſe and

 application.

application. This is the grand divifion of the whole Work, which is again divided into four *Books*, and every Book is fubdivided into it's number of *Sections*, according to the nature of the feveral fubjects they treat of. Laftly, there is a further fubdivifion of the Sections into *Articles*, which are numbered without interruption from the beginning to the end of each Book, and which we fhall alfo obferve and enumerate in our explications of them.

PLAN.

THE firft Section of the firft Book is concerning the primary Notions and Operations of the Analyfis of finite Quantities; in which are contained the following Articles. After a fhort Preface concerning the nature of Analyfis, the Author obferves,

1. That it's operations are the fame as thofe of common Arithmetick; this operating with numbers, and that with fpecies, that is, with fymbolical numbers or quantities. By which means Algebra has great advantages over Arithmetick; for, in this, the fteps of the operations will be confounded and loft by the fubfequent ones, but in Algebra they may be preferved, as they are often not actually performed, but only infinuated by proper fymbols; it is alfo more univerfal, and works indifferently with known or unknown quantities.

2. Here the diftinction of pofitive and negative numbers, or quantities, is explained. Negative quantities are not in nature, but depend only on the manner of conceiving them. They are merely artificial, and introduced to fave needlefs repetitions and diftinctions, by which we can confider the oppofite operations of Addition and Subtraction under one general view and comprehenfive idea. In Geometry, they are reprefented by lines drawn oppofite ways. If pofitive lines proceed to the right-hand, then negative ones will be to the left, with the fame direction; or if pofitive ones are upwards, then negative will be downwards.

Then

3. Then different affections of quantities are distinguished, or denoted, by the signs + or −, *plus* or *minus*, placed before them; whether the quantities are represented arithmetically, or by common numbers; or else algebraically, by representative numbers, that is, by the letters of the alphabet: *plus* being the mark of Addition, and *minus* of Subtraction. And the sign $\pm$ and $\mp$ are ambiguous, but contrary to each other. The equality of quantities is denoted by the mark $=$, and majority or minority by the marks $>$ or $<$. Proportion, or equality of ratios, by $::$, and infinitely great by ∞.

4. Quantities are *simple* that are not connected by the signs + or −, and *compound* when they are: of which examples are proposed by the Author.

5. Then is taught the addition of simple quantities being integers, and explained by a sufficient number of examples: also, the use of numeral co-efficients is shown.

6. Likewise, the subtraction of simple integral quantities is taught, in which it is shown that the sign of the quantity to be subtracted must always be changed, and the reason of it, together with examples.

7. Next the Author proceeds to the multiplication of simple quantities, being integers, whether they are positive or negative. Then the product will be represented by the connection of the several factors, and their co-efficients without any sign between them. And if the factors are positive and negative promiscuously, like signs will always produce +, and unlike signs −. This she demonstrates from the nature of proportion.

8. And whereas raising of *powers* is a case of multiplication; she shows how simple powers are formed, and conveniently expressed by their *indices*, or *exponents*, annexed to the roots.

9. These powers are distributed into *squares*, *cubes*, *biquadrates*, &c.; that is, into second, third, fourth, &c. powers, of which the given number, or *root*, is always the first power; and they are marked by the exponents 1, 2, 3, 4, &c. respectively. Their signs are always known by the general rule aforegoing.

10. Then

10. Then comes divifion of fimple quantities, being integers, which is juft the reverfe of multiplication, and refolves, or decompounds, that which the other had compounded; as by the examples.

11. When common letters or quantities are rejected, and the divifion can proceed no further, it muft be infinuated, by making a fraction of what fhall remain.

12. When the figns of the dividend and divifor are the fame, the fign of the quotient muft be pofitive; but when thofe figns are different, the fign of the quotient muft be negative. This proved from the nature of proportion.

13. Whence, in fractions, it is indifferent how the figns are changed in the numerator and denominator, provided the fign of each is changed into it's contrary.

14. The roots of fimple quantities will be extracted, by dividing their exponents by the number which denominates the root to be extracted. As, by 2 for the fquare-root, by 3 for the cube-root, and fo on.

15. If any even root is to be extracted, the fign of that root will be ambiguous; but if an odd root is to be extracted, the fign of that root will be the fame as of the given power.

16. When roots are furd, and cannot be extracted, they are to be infinuated by radical figns or characters.

17. From thefe operations belonging to fimple quantities, the Author proceeds to thofe of compound quantities, or fuch as confift of feveral fimple quantities, connected by the figns + and —. Thus, Addition will be performed by fetting down all the given quantities together promifcuoufly, and then abbreviating the fum as much as may be, and expunging equivalents with contrary figns.

18. In Subtraction, all the figns are changed of the quantity to be fubtracted, and the remainder, or difference, fo found is to be abbreviated as much as may be done.

19. Mul-

19. Multiplication of compound quantities, being integers, depends on the multiplication of ſimple quantities; and the proceſs is much like the ſame operation in common Arithmetick, as the examples ſhow.

20. But it is often convenient only to inſinuate this multiplication, without actually performing it. And that is done by drawing a line, or *vinculum*, over the ſeveral factors, and connecting them by putting the mark ×, ſignifying *Multiplied by*, between them.

21. The powers of compound quantities, as well as of ſimple, need not always be actually formed, but may often be conveniently inſinuated, by a *vinculum* placed over the root, and a proper index annexed to it. How theſe powers may be actually formed, when occaſion requires, is here ſhown.

22. The Author preſents us with a general Canon, (being Sir *Iſaac Newton's* Binomial Theorem,) for raiſing any binomial quantity, or even multinomial, to any power required; which ſhe exemplifies by a ſufficient number of examples.

23. The Author proceeds to diviſion of compound quantities, being integers, of which ſhe makes three caſes. The firſt is, when the diviſor is ſimple and the dividend compound, and the ſecond is on the contrary. Theſe are eaſily reduced to the foregoing rules.

24. The third caſe is, when both the dividend and diviſor are multinomials, and therefore requires a more prolix proceſs. In order to which, the terms of each are to be diſpoſed according to the dimenſions (or powers) of ſome particular letter contained in them; that is, they are to form numbers belonging to a ſcale, of which that letter is the root, juſt as we do in our common Arithmetick, the root of which is ten, and the numbers are diſpoſed according to the dimenſions of that root. Then the proceſs of diviſion muſt be performed much after the manner of the like proceſs in numbers, and which is ſufficiently explained by the examples produced. When the dividend cannot be intirely exhauſted, the quotient muſt be completed by adding a fraction to it, as in common Arithmetick.

25. The Author proceeds to the extraction of the roots of compound quantities, being integers, and firſt of the ſquare-root. The terms of the given quantity

quantity are to be difpofed, as before, in Divifion; and the procefs of extraction will be nearly as the fame operation in numbers. Indeed, her procefs is fomething different in form from the common one, but is very intelligible, and comes to the fame thing. Her examples make it very clear. When the root is furd, and therefore cannot be extracted, it muft be infinuated by a quadratick vinculum.

26. The procefs of the extraction of the cube-root is much after the fame manner, only more operofe, as being a more complicate operation. The examples render it as plain as the nature of the thing will admit.

27. The biquadratick, or fourth root, is extracted in the fame manner.

28. The fifth root, and all higher roots, may be extracted, by forming rules for them, which are found by raifing a binomial to the fame power. For the like was done in forming rules, by which the fquare and cube-roots have been extracted.

29. The Author then proceeds to the algorithm of fractions fimple and compound; obferving that any quantity may be converted into a fraction with a denominator given, if it be multiplied into that denominator: of which fhe produces feveral inftances. For this fee the Examples.

30. Then comes the reduction of fractions to more fimple expreffions, when that can be done, which it is not always eafy to perceive. When the numerator and denominator are each multiplied by the fame quantity, whether fimple or compound, they may each be divided by it again, and a new fraction will arife equivalent to the former. And fo *toties quoties*. This will be a very ufeful reduction; for, in all our calculations, we fhould always ftudy to abbreviate as much as poffible. See the Examples. How thefe common divifors may be found we fhall be taught afterwards.

31. Then is taught reduction of fractions to a common denominator, which in two fractions is performed by the crofs multiplication of each numerator into the denominator of the other, as by the examples. And fo two by two, if there are more, till all are reduced.

32. This

32. This prepares the way for the addition and subtraction of fractions; for, if they have not a common denominator, those operations can only be insinuated, by writing them after one another with their proper signs. But, when reduced to a common denominator, their numerators may then be added or subtracted, to compleat these operations; as by the examples.

33. The multiplication of fractions requires no such preparation, but is performed directly, by multiplying the numerators together for a new numerator, and the denominators together for a new denominator. The product, or fraction thence arising, may often be reduced by some of the foregoing methods.

34. Division of fractions is reduced to multiplication, by multiplying the dividend by the reciprocal of the divisor; which reciprocal is, when the numerator and the denominator change places. The quotient thus found will often have occasion for some reduction, as by the examples may be seen.

35. As for the extraction of the roots of fractions, whether it be the square-root, the cube-root, &c. the said roots must be extracted severally out of the numerator and denominator, and the fraction thence arising will be the root of the fraction given. But when such root cannot be extracted, it must be insinuated by placing a radical vinculum before the given fraction, as by the examples.

36. To conclude the Doctrine of Fractions, the Author proceeds to a very curious and useful operation, which is, to find the greatest common divisor of two quantities or formulas given. Where it may be observed, that a formula is a combination of quantities, which may serve as a paradigm, or pattern, for all combinations of the like kind. Then, by a process not unlike that in Arithmetick, which is, by subtracting one from the other continually and interchangeably as often as can be done, the last quantity so found will be the greatest common divisor of the two given quantities. Now, if those two quantities form a fraction, and the numerator and denominator are each divided by the greatest common divisor so found, a fraction will thence arise equal to the other, but reduced to the smallest terms. Of this reduction she gives us the process at large, in three several instances.

 37. The

37. The Author goes on then to the Doctrine of Surds or Radicals, which are ſuch quantities whoſe roots cannot be extracted, yet may often admit of a partial extraction, or may be reduced to ſimpler expreſſions; as by the examples may appear.

38. The reduction of different radicals to radicals with the ſame index, will be performed by finding the leaſt number for a common exponent, by which the given exponents may be divided. Then each radical muſt be raiſed, if neceſſary, till it arrives at that exponent. The examples make it plain.

39. Addition and ſubtraction of radicals is eaſily performed, by writing them one after another with their proper ſighs, and then abbreviating when it may be done.

40. Radical quantities are multiplied by thoſe that are rational, by prefixing the rational to the radical, with ſuch ſign as the Rule of Multiplication requires. And when they are complicate, their product will be found by the ſame rule.

41. Radicals of the ſame denomination, or reduced to ſuch, are multiplied by putting their product under the ſame radical vinculum.

42. If the radicals are affected by rational co-efficients, their product muſt be put before the radical ſo found.

43. When like quadratick radicals are multiplied into each other, the radical ſign will be taken away, and the product will often become rational. Several examples of this are exhibited.

44. A rational co-efficient to a radical may at any time be made to paſs under the radical vinculum.

45. The multiplication of radicals of different kinds may be inſinuated, or they may be reduced to the ſame kind.

46. Diviſion of radicals of the ſame kind is performed by leaving out the radical quantity, and dividing the co-efficients only.

47. If the radicals are of the ſame kind, but not of the ſame quantity, the quantities under the vinculum may be divided, and the quotient put under the ſame vinculum.

48. But

48. But if the radicals are different, they may be reduced to the fame exponent, and then divided as before. And thus complicate quantities may be divided as in common Divifion.

49. Then the Author gives us a Rule for extracting the fquare-root of quantities any how compounded of rational and irrational quantities, and thofe either numeral or algebräical; which fhe applies to feveral examples.

50. In order to the calculation of powers, which are expreffed by integer exponents; from any root fhe forms a geometrical progreffion of it's powers, beginning from unity, and afcending one way by pofitive exponents, and defcending the other way by negative exponents, to fhow the correfpondence there is between the increafing powers and their affirmative exponents, and the decreafing powers and their negative exponents. Then obferves, that when any power is in the denominator of a fraction, it may be made to pafs into the numerator, and *vice verfâ*, by only changing the fign of the index.

51. Then, as fractional powers, or roots, are certain intermediate terms, between the integral powers in the foregoing geometrical progreffion; fo their exponents muft be correfponding intermediate terms in the arithmetical progreffion. And this will obtain in the defcending progreffion as well as in the afcending, and whether the terms are fimple or compound.

52. Hence the multiplication or divifion of powers will eafily be performed by their exponents. For, to multiply them, we muft add their exponents; and to divide them, we fubtract the exponent of the divifor from that of the dividend. This fhe proves from the nature of proportion.

53. Hence the raifing of powers, or extracting the roots of any powers, will eafily be performed by their exponents. For the index of any power muft be multiplied by the index of the power to which it is to be raifed; and the index of the given power is to be divided by the index of the root to be extracted.

54. And this obtains as well in compound quantities as in fimple. For all which reductions fee the Examples.

55. Another ufeful operation follows, which is that of finding all the linear or fimple divifors of any given number or formula; or to refolve a compound

 quantity

quantity into the ſeveral quantities of which it is, or may be, compounded by multiplication. The proceſs is exemplified and illuſtrated both in numbers and ſpecies. Indeed, if this could always be done in numbers, it would amount to a very valuable diſcovery, or deſideratum in Analyticks, which is, a method of reſolving a given compound number into the prime numbers of which it is compounded; but though it is only a tentative method, yet, however, it is very uſeful.

56. This is extended to any compound formula, or to a number expreſſed by an indefinite root in an arithmetical ſcale, which may have been formed by the multiplication of ſeveral binomial factors. By this method ſuch a number may again be reſolved into it's factors, by the help of the foregoing operation. And if the number of trials to be made ſhould happen to be too great, the Author ſhows a method of reducing them to a ſmaller number, which is, by changing the root, and ſo exhibiting the given formula by another ſcale.

57. Now, if the firſt term of the given formula ſhould happen to have a numeral co-efficient, it may be convenient (by ſubſtitution) to change it into another formula, or to expreſs it by an equivalent root of another ſcale, the co-efficient of the firſt term of which ſhall be unity.

BOOK I. SECT. II.

Of Equations, and of Plane Determinate Problems.

58. HAVING explained the firſt principles or operations of Analyticks in the foregoing Section, our Author proceeds to the grand inſtrument of the art of computation, which is equation. This is either when ſome of the terms placed before the mark of equality, are collectively equal to all the terms on the other ſide, called the *homogeneum comparationis*; or when the whole are one ſide, and equal to nothing on the other ſide; inſinuating that the affirmative and negative are equal, and ſo deſtroy one another. She explains likewiſe what is meant by the law of *homogeneity*.

59. She

59. She tells us what a *Problem* is, and what is the diſtinction between the *data* and *quæſita* of a problem.

60. Problems are divided into *determinate* and *indeterminate*, of which ſhe gives inſtances from Geometry. But in this Section ſhe treats only of ſuch as are determinate.

61. Here it is explained how equations are formed, from the dependance of quantities upon one another, whether they are known and given quantities, or unknown and required. The inſtances are taken from the properties of lines and figures.

62. How we are to argue from the given conditions of the queſtion till we come to an equation between the quantities given and required. This is explained geometrically, and by an abſtract arithmetical queſtion.

63. No more given quantities are to be aſſumed than are neceſſary, when they can be expreſſed by the known properties of the figure.

64. It will often happen, that the lines given in a figure are not ſufficient for forming the equations; then ſuch other lines muſt be drawn as may complete the figure, and bring us to a determination. A problem is propoſed to illuſtrate this; and the Propoſitions of *Euclid* are enumerated, which will be of uſe for ſuch purpoſes.

65. Here the Author propoſes and ſolves three or four geometrical problems, to ſhow the method of arguing from one condition to another, in order to obtain a final equation.

66. When the conditions of a problem involve the properties of angles, they muſt ſomehow be reduced to the properties of lines. This is exemplified in the problem of finding an equicrural triangle, in which either of the angles at the baſe is double to the angle at the vertex: which is reduced to the linear problem, of dividing a line in extreme and mean proportion.

67. Having thus ſhown how to find equations from the given circumſtances of a problem, ſhe proceeds to the reſolution of theſe equations, or to the finding the unknown quantity, by means of various reductions. For this end

ſhe

ſhe gives us four axioms. By the firſt, ſhe ſhows the uſe of tranſpoſing quantities at pleaſure from one ſide of an equation to the other; which may always be done without deſtroying the equation, only by changing the ſigns of the terms ſo tranſpoſed.

68. By the ſecond axiom ſhe ſhows how we may take away any fractions that ariſe in an equation, and ſo reduce the whole to integral terms.

69. And how, by the ſame, any term may be freed from it's co-efficient.

70. By the third and fourth axiom ſhe ſhows how equations may be freed from ſurds and radicals; and of all theſe reductions gives us a variety of examples.

71. Equations prepared for ſolution, and diſtributed into their terms.

72. Equations further prepared, by which the unknown quantity will be found equal to a combination of known quantities, and a ſimple equation will be ſolved entirely.

73. If any power of the unknown quantity is found equal to known quantities, then the root may be extracted on both ſides.

74. If the equation is an affected quadratick, it may be ſolved by completing the ſquare on one ſide, and then extracting the ſquare-root on each ſide.

75. In quadratick equations the ambiguity of the ſigns will ſupply two values of the unknown quantity, which may therefore be both poſitive, both negative, or one poſitive and the other negative, or both imaginary, according to the values of the known quantities. What is analogous to this difference of ſigns in geometrical figures, is here ſhown, and all is illuſtrated by examples.

76. The Author ſhows us here the uſe of impoſſible or imaginary roots of equations. For they are a ſure indication, that the queſtion (as now propoſed) is impoſſible, either by chance or deſign. And the ſame thing is to be concluded, when the final equation brings us to any abſurdity or contradiction. This ſhe ſhows in ſeveral inſtances.

77. And ſometimes we may be brought to an identical equation; which only ſhows that the point required may be any where in the given line, as by the example.

78. Equations

78. Equations and problems are diftinguifhed into degrees, according to the dimenfions of the unknown quantity contained in them. Alfo, thofe problems are called *Plane*, the refolution of which requires only the ordinary Elements of Geometry. But if they require the defcription of the Conic Sections, or other curves, they are *Solid Problems.*

79. Equations are not always of that degree which their higher powers feem to infinuate, but may often be brought to a lower degree by an eafy reduction: As by the examples.

80. Sometimes neceffity, and fometimes conveniency, will require, that more than one unknown quantity may be introduced in a problem; in which cafe (if the problem is determinate,) as many equations muft be found as there are unknown quantities affumed. Then thefe are to be eliminated one by one, till we finally arrive at an equation, in which there is only one unknown quantity. The way of doing this fhe fhows by an example.

81. This method of elimination may be made ufe of, not only in fimple equations, but alfo in affected quadraticks.

82. Higher equations may fometimes be reduced, by eliminating their greateft powers. And when thofe powers have not the fame index, they may be reduced to fuch as have. Of both thefe reductions the Author produces feveral examples.

83. If there be feveral fimple equations including as many unknown quantities, they may be feverally eliminated, and reduced to one equation including only one unknown quantity, though the calculation will often be tedious.

84. If there are not as many equations to be found as there are unknown quantities, the problem will become *indeterminate*, and will allow an infinite number of anfwers. Of this fhe produces examples.

85. But if the conditions to be fulfilled, or the equations, are more than neceffary, they may be inconfiftent with each other, and fo the problem will become impoffible; or fome of the conditions may coincide with others, and fo be fuperfluous.

86. Having

86. Having laid this foundation for calculating with arithmetical or algebräical quantities; ſhe now does the ſame for calculating with geometrical quantities, or with lines and figures. She begins with the operations of Multiplication and Diviſion, or, what is the ſame thing, with finding ſuch ſimple proportions, or conſtructing ſuch ſimple equations, as will give the values of the quantities required expreſſed by lines.

87. The operations of addition and ſubtraction of lines, when thus found, will be very eaſy and familiar.

88. Hence, by ſubſtitution, any given letter, or letters, may be introduced; or a plane may be transformed into another with a given ſide, or a ſolid into another with one or two given ſides, &c. by which the conſtruction of ſimple equations will be much facilitated.

89. This reduction is eaſily extended to fractions, the numerators or denominators of which are complicate terms.

90. But, without dividing a fraction into ſeveral fractions, the method of transformation may often be preferable, as is ſhown by a variety of examples.

91. Here it is ſhown how lines may be found, that ſhall expreſs the value of any quadratick radical, by only finding geometrically a mean proportional between two given quantities: excepting the caſe when that value is imaginary or impoſſible.

92. But, to reduce radical quantities to this rule, there will often be occaſion to have recourſe to the method of transformation, as appears by the examples.

93. Any quadratick radicals may be conſtructed by a right-angled triangle, either alone or combined with a circle, without transformation; though ſome transformation will often be found convenient. This illuſtrated by various examples.

94. The foregoing rules may eaſily be applied to the conſtruction of any affected quadratick equation; but they may all be conſtructed after a more general manner. For this purpoſe the Author aſſumes a general affected quadratick equation, which ſhe diſtinguiſhes into four, according to the variety of

of their ſigns. Theſe ſhe conſtructs, one after another, by right-angled triangles and a circle, and exhibits the roots, both affirmative and negative, by right lines.

95. The ſame equations may be otherwiſe, and more eaſily, conſtructed, when the laſt term is not a ſquare, but a rectangle.

96. Hitherto the learned Author has been laying down the principal rules of the Art of Computation, whether arithmetical, algebräical, or geometrical; ſhe now proceeds, as ſhe tells us, to ſhow their uſe in the ſolution of ſome particular Problems, to the number of 15, with which ſhe concludes this Section. The firſt is purely arithmetical, and to be found in moſt Books of Algebra.

97. The ſecond Problem is alſo very common, and is about the motion of two bodies with given velocities, in various circumſtances, general and particular.

98. The next is the famous Problem of King *Hiero's* crown, in which *Archimedes* diſcovered the quantity of baſer metal mixed with the gold, and which gave the occaſion to his celebrated ευρηκα.

99. The next Problem is concerning the relation of two weights to each other, and is purely arithmetical. And theſe Problems hitherto have produced only ſimple equations.

100. Then we have a Geometrical Problem, which amounts only to a ſimple equation, and is therefore eaſily reſolved and conſtructed.

101. The next Problem is geometrical, which ariſes to a ſimple quadratick equation, which is there conſtructed, or reſolved, geometrically.

102. Then a Geometrical Problem, teaching to inſcribe a cube in a given ſphere; which amounts only to a ſimple quadratick equation, and is there conſtructed, and the conſtruction proved by a ſynthetical demonſtration.

103. A Geometrical Problem, or rather Theorem, concerning a ſecant drawn through two concentrical circles, ſo that the parts intercepted by the circumferences ſhall be equal. This being the property of every ſuch ſecant, the

ſolution brings to an identical equation, which is a proper caution how to manage ſuch Problems, and what concluſions we are to derive from them.

104. Another Geometrical, or rather Algebräical, Problem.

105. A Geometrical Problem.

106. A Geometrical Problem, in which the magnitude of angles enters the calculation.

107. A Geometrical Problem, with a ſynthetical demonſtration.

108. The Author gives us here a very notable Geometrical Problem, which is, two contiguous arches of a circle being given, and alſo their tangents, to find the tangent of their ſum. And this ſhe extends very artfully to the ſolution of a much higher and more general Problem, which is, any number of arches and their tangents being given, to find the tangent of their ſum. By the way ſhe gives us a general Theorem, for finding all the poſſible combinations of any number of quantities given. She concludes with giving a general canon, or formula, for finding the tangent of any multiple or ſubmultiple arch; as alſo, ſhows the converſe of this Theorem.

109. Then we have a Geometrical Problem, which is, to find a triangle, the ſides of which and the perpendicular are in continued geometrical proportion. This amounts to a high equation, but is reduced to an affected quadratick: which is geometrically conſtructed.

110. The laſt Problem is that famous geometrical one, of triſecting a given angle. This ſhe divides into three caſes, according as the given angle is right, obtuſe, or acute. The firſt caſe ſhe ſolves by a ſimple quadratick equation, of which ſhe alſo gives us the conſtruction. The ſecond and third caſes ariſe to cubic equations, which ſhe reſerves till ſhe comes to treat of thoſe equations.

BOOK

BOOK I. SECT. III.

Of the Conſtruction of Geometrical Places, and of Indeterminate Problems not exceeding the ſecond Degree.

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111. In this article the Author explains the nature of variable quantities; that there muſt always be two of them, at leaſt, in an indeterminate Problem, which are varied according to a conſtant law, which is expreſſed by a given equation.

112. A *Locus Geometricus* is a right line, or a curve, the *abſciſs* and *ordinate* (or the *co-ordinates*) of which are variable right lines, which in all caſes expreſs the variables of the equation. The abſciſs begins from ſome certain point taken at pleaſure in an indefinite right line, and the ordinate is placed at the end of the abſciſs, at a given angle. When a definite value is aſſigned to one of theſe lines, the curve, or locus, will give the definite and relative value of the other, agreeably to the equation: as by the inſtances may be ſeen.

113. Different equations will require different *loci*, and *vice verſâ*. And as the equations are of different degrees, ſo will the loci be alſo.

114. Of a ſimple equation the locus will always be a right line.

115. When any combination of the variables, in any one term, does not exceed the ſecond degree, the equation will always require a conic ſection for it's locus.

116. Theſe loci are here diſtributed into their ſeveral orders.

117. All equations of the firſt order, or which can belong to a right line, are here conſtructed.

118. In ſimple equations, ſometimes a determinate problem may be propoſed as an indeterminate, in which caſe one of the variables will vaniſh out of the equation, or not at all appear in it. Then the locus of the equation will be a

 right

right line, either perpendicular or parallel to the abſciſs. Of this the Author produces an inſtance or two, with their conſtruction.

119. The Author goes on to the circle, as the ſimpleſt curve, of which ſhe exhibits the firſt and ſimpleſt equations, whether we take the beginning of the abſciſs from the centre, or from the end of the diameter; and ſhows what the radius muſt be, in caſes not ſo ſimple: and tells us likewiſe when the circle will be only imaginary.

120. She proceeds then to the parabola, as the next ſimpleſt curve, of which ſhe exhibits the primary equations, whether the parameter be ſimple or complicate, whether the parabola be internal or external.

121. The next conic ſection is the hyperbola, or rather the two oppoſite hyperbolas, of which ſhe exhibits the ſimpleſt equations, when the ordinates are referred to the axis; whether the abſciſs commences from the centre, or from either of the vertices; or whether the equation is expreſſed by the axes, or by the parameter. She finds the equation when the hyperbola is equilateral; and reduces complicate parameters, or diameters, to ſimple ones.

122. She ſhows likewiſe what will be the ſimpleſt equation belonging to the hyperbola between it's aſymptotes.

123. The ſimpleſt equations are alſo derived for the ellipſis, whatever is the angle of ordination; and whether the abſciſs begins from the centre, or from either of the vertices; or whether the equation is expreſſed by the diameters, or the parameter. And what will be the equation, when the diameters and parameter are equal. In this laſt caſe, if the angle of ordination is a right angle, the ellipſis will degenerate into a circle. Complicate diameters and parameters are reduced to ſimple ones, as before in the hyperbola, from the equations of which thoſe of the ellipſis will differ only in their ſigns; ſo that they will eaſily paſs into each other.

124. When the ſimple equations to the diameters of the hyperbola, or ellipſis, are not given exactly in the terms of the diameters, but rather in diſguiſed terms; the Author ſhows how, by the Rule of Proportion, thoſe diameters may be found. Of which reduction ſhe gives Examples.

125. Or

125. Or when the fame equations are expreffed by parameters, though fomething obfcurely; fhe fhows us how to find thofe parameters, and gives Examples of it.

126. Having thus exhibited the fimpleft equations belonging to the Conic Sections, and fhown how we may find the diameters or parameters when involved, by which thefe fections may be defcribed; the Author proceeds to conftruct any complicate equations that may be given, belonging to thefe fections or curves; in order to which, fhe diftributes all fuch equations into three fpecies or claffes. The firft are thofe that contain the fquare of one of the variables, and the rectangle of the other into a conftant quantity. The fecond fpecies contains the rectangle of the two variables, with other fimple terms. The third contains the rectangle and both the fquares of the variables, with any other fimple terms.

127. She then proceeds to conftruct equations of the firft fpecies, however complicate they may be, and reduces them to a fimple form, by one or two fubftitutions of new variables. And of this fhe gives us two Examples. In the firft, by one fubftitution, fhe reduces the given equation to the fimpleft form belonging to the parabola, which fhe then conftructs. In the fecond, fhe reduces the given equation, by two fubftitutions, to the fimpleft form belonging to the hyperbola between the afymptotes, which fhe then conftructs, and purfues it through all it's varieties. When the conftant quantities are fuch, as not to admit of thefe fubftitutions, fhe changes them, by the tranfmutations fhe had taught before, into fuch as will be fit for thofe fubftitutions.

128. Then fhe reduces equations of the fecond fpecies to the firft, by a method not unlike that of extracting the fquare-root of an affected quadratick equation. By which means, and by a fubftitution, fhe introduces a new variable. Of this fhe gives an Example in an equation to the parabola, which fhe reduces and conftructs. Alfo, another to the hyperbola, reduced by two fubftitutions.

129. Then fhe fhows, by an example, how an equation of the third fpecies may be reduced to the firft, and fo conftructed.

130. Here ſhe propoſes various complicate examples, of which ſome are to the parabola, ſome to the hyperbola, and ſome to the ellipſis, which require ſeveral ſubſtitutions and transformations; but are all reduced to ſimple equations, and conſtructed with great art and ingenuity.

131. All the variety of equations to the hyperbola between the aſymptotes, are reduced to four general equations, which are here conſtructed, by one, two, or more ſubſtitutions, or changing of the variables; and that according to all the variety of their ſigns. To illuſtrate theſe conſtructions, and to ſhow their application in particular caſes, ſhe propoſes and reſolves the ſeveral Problems following.

132. The equation of the firſt Problem is found to belong to the parabola, being the property of the focus of the parabola in reſpect of the directrix, which is therefore eaſily conſtructed by one ſubſtitution.

133. The equation of the next Problem is found to be a locus to the hyperbola between the aſymptotes, and is conſtructed by means of two eaſy ſubſtitutions.

134. This Problem is propoſed concerning the properties of two circles and their tangents, but the general ſolution and conſtruction of the equation require all the three conic ſections, according to the three caſes included in it. Theſe caſes are conſtructed ſeparately, by the help of ſeveral ſubſtitutions and tranſmutations.

135. A Problem to the three Conic Sections, according to it's three different caſes.

136. A general Problem ſolved by a canonical equation, and illuſtrated by three Examples of particular curves, of which the laſt ariſes to a cubical equation, and therefore goes beyond the Conic Sections.

137. A Problem concerning two equal interſecting circles, which ariſes to an equation to an ellipſis, which is here conſtructed by means of one ſubſtitution.

138. A Problem, or rather two Problems to the circle, with ſynthetical demonſtrations of the ſolution.

139. A

139. A Problem of a normal fliding between the fides of a right angle, and with one end defcribing a curve. This curve, by it's equation, is found to be an ellipfis, and is here conftructed.

140. The equation of this Problem is either to the parabola, the hyperbola, or the ellipfis, according to different circumftances, and is refolved by various fubftitutions, or changes of the indeterminate quantities, and is here conftructed.

141. The Method of Majority and Minority is here occafionally explained, which proceeds in the fame manner as the reduction of equations. For, by a feries of comparifons duly made, we may know which of two quantities is the greater or leffer.

142. A Problem producing an equation to the hyperbola between the afymptotes, which is very artfully refolved and conftructed, by three fubftitutions, or changes of the variable quantities.

143. Here the Author concludes her Problems, and recommends the proving the folution, after it is finifhed, by tracing back the feveral fubftitutions, and fo returning to the original equation. Of this fhe gives us two Examples in the foregoing Problems.

BOOK I. SECT. IV.

Of Solid Problems and their Equations.

144. The Author having thus difpatched what are called Plane Problems, or fuch as require only equations of two dimenfions; fhe proceeds to thofe called Solid Problems, which require equations of more than two dimenfions, and therefore higher and more difficult conftructions. She begins by informing us what are the roots of fuch affected equations, or what are the values of the unknown and indeterminate quantities, which are to be extracted out of thefe equations. That they are fuch numbers or quantities, that, if they were to be fubftituted in the equation given, inftead of the root, they would reduce the whole to nothing; which would be a full proof, when the root, or roots, are extracted, that they are the true roots of the equation.

145. Or,

145. Or, in another acceptation, thofe fimple equations are often called the Roots of a compound equation, which, being multiplied into each other continually, will produce the equation given. Confequently that equation may be refolved into it's components by continual divifion. Hence every equation will have fo many roots as it has dimenfions. Of this fhe gives us inftances in equations of two, three, or four dimenfions, or of quadratick, cubick, and biquadratick equations, which are formed by the multiplication of fimple, but general equations, and which therefore will be the roots of the equations fo formed.

146. Hence, when any of the roots of a compound equation happen to be known, we have a method, by divifion, of depreffing that equation, and reducing it to a fimpler, which fhall include only the unknown roots.

147. From this way of raifing compound equations by multiplication, we may know the conftitution of every fingle term, when the whole equation is difpofed in a proper and regular order, and made equal to nothing. For the higheft term muft always be pofitive, and have no other co-efficient but unity, which can always be effected. The co-efficient of the fecond term will be the fum of all the roots, under their proper figns. The co-efficient of the third term will be the fum of the products of every pair of roots, &c. And the laft term will be the product of all the roots, affected by their proper figns.

148. It follows from hence, that, if the fecond term is wanting in any equation, then the fum of the pofitive roots will be equal to the fum of the negative; therefore, when that term is prefent and affirmative, the fum of the pofitive roots will be lefs than the fum of the negative; but the contrary, if that term be negative.

149. When any term is wanting in an equation, it's abfence is commonly indicated by putting an afterifm in it's place.

150. If no imaginary root appears in the equation, yet it may have them, two by two, always in pairs, and with contrary figns. If the degree of the equation is odd, it will have, at leaft, one real root; and if it's degree is even, it may have all it's roots imaginary. The like may be obferved of furd roots.

151. Many

151. Many useful indications, concerning the roots of an affected equation, may be had from the signs of the several terms.

152. A proof that, in cubick and biquadratick equations, if the second term is wanting, and the third term is positive, there will necessarily be imaginary roots.

153. In any equation the affirmative roots may be made negative, and the negative affirmative, only by changing the signs of those terms which are in even places. Here the asterism, or vacant place, must always be reckoned for one. This proved by Examples.

154. The roots of affected equations may be increased or diminished by any quantity at pleasure, by resolving the root into two parts, one unknown, and the other known; and that only by a substitution of equivalents. The new equation so found will have the same roots as the given equation, only they will be increased or diminished by a known quantity. See the Author's Examples.

155. By a like substitution of equivalents, the roots of any equation, though unknown, may be multiplied or divided by a given quantity, and undergo many other changes at pleasure.

156. The reason of these several processes is, that, as equals are always substituted for equals, so the results must always come out equal.

157. The uses of these substitutions are many. One of which is, that, though the roots of an equation are unknown, yet, by such a transformation, they may often become known.

158. Another use is, the freeing an equation from fractions or surds. Of this the Author produces several Examples.

159. Some necessary conditions in the equation, in order to it's being freed from surds or radicals.

160. But the chief use of this transmutation of equations, is intirely to take away the second term from any equation by an easy substitution: of which the Author gives several instances.

161. Or the third term may be taken away, by ſolving a quadratick equation, the fourth by a cubic, &c.; as may appear from the Author's general proceſs.

162. In an equation wanting the ſecond term, the penultimate term may be taken away; but it will be by reſtoring the ſecond term.

163. Thus, in an equation wanting the third term, the ante-penultimate term may be taken away; and ſo on.

164. Or any equation, in which any term or terms are wanting, may be made complete by a new ſubſtitution.

165. If equations have diviſors of one, two, or more dimenſions, they are properly of that order, to which they may be reduced by diviſion.

166. Diviſion ought firſt to be tried by a diviſor of one dimenſion, then by thoſe of two, &c.

167. Equations of the third degree, if reducible, may be reduced by a linear or ſimple diviſor, which is to be found in the manner taught in the 56th Article before. If an equation of the fourth degree cannot be reduced by a diviſor of one dimenſion, to be found in the ſame manner, the reduction muſt be attempted by a diviſor of two dimenſions. To perform which, the Author throws out the ſecond term of the equation, as ſhown before, and then aſſumes two general equations of two dimenſions, and multiplies them together, and compares the terms of the produced equation with thoſe of the equation given. By this compariſon ſhe determines the co-efficients of the aſſumed equations, the laſt compariſon of which amounts to an equation, which in effect is no more than cubical. This cubic equation is reſolved by the Method of Diviſors, and it's roots, being ſubſtituted in the aſſumed equations, will make them become diviſors of the biquadratick equation propoſed. Of this method of ſolution ſhe gives us two Examples.

168. Here is the ſame proceſs as before, but after a more general manner, and applied to a particular biquadratick equation, which is reſolved by it.

169. Sometimes this method will ſucceed only by taking away the ſecond term of the equation, which will depreſs it to a quadratick.

170. The

170. The ſame method is purſued, but without taking away the ſecond term of the given biquadratick equation. Two general quadratick equations are aſſumed, and multiplied together, and the general co-efficients of the product are determined and eliminated, as far as may be, by a compariſon with thoſe of the given equation. The laſt co-efficient in theſe compariſons muſt be determined by the foregoing Method of Diviſors. But this way of reſolution ſeems to be too tentative to be of any general uſe. It is illuſtrated by three Examples.

171. The ſame method is carried on to equations of five dimenſions, in which the two aſſumed general equations are, one of two and another of three dimenſions. When, by compariſon, the general co-efficients are determined, they are ſubſtituted in the ſimpleſt of the aſſumed equations, which then becomes a diviſor of the given equation; as by two Examples.

172. The Author extends this method to equations of ſix dimenſions, which ſhe manages with great ſagacity and ſucceſs, though it muſt be owned to be very tedious, precarious, and tentative; but, however, is the beſt that can be had in theſe high equations. She aſſumes two general and ſubſidiary equations, one of two, and another of four dimenſions, which are multiplied together to produce a general formula for equations of ſix dimenſions, that may be reſolved into two ſuch equations. Then the general co-efficients are determined as before, and ſubſtituted in the ſimpleſt of the aſſumed equations, which will then become a diviſor of the given equation. Of this reduction ſhe gives us an Example.

But an equation of the ſixth degree may poſſibly be reſolved into two cubic equations, and not otherwiſe. She therefore aſſumes two general cubic equations, and multiplies them together, to conſtitute a general formula for theſe equations. Then, a particular equation of ſix dimenſions being given, the general co-efficients are determined by compariſon, as far as that can be done, and their values are finally ſubſtituted in one of the aſſumed equations, in order to form a diviſor to the given equation.

173. The Author aſſures us, that the ſame method might be applied to the ſolution of higher equations, if it was not for the exceſſive tedious ſneſs of the operations. It may very well be ſuppoſed, that the calculation will become

 very

very laborious in thofe equations, by what we fee in thefe of a lower order. And as the method is but tentative at beft, it can hardly deferve to be profecuted any further; efpecially as we have an *exegefis numerofa* to recur to in thefe cafes, which, though only an approximation to the root, yet will anfwer all real occafions that can offer. The Author now proceeds to propofe and refolve fome particular Problems, in order to fhow the ufe and application of what is now delivered.

174. The firft Problem is purely arithmetical, and is elegant enough: *To find four numbers which exceed one another by unity, and their product is* 100. The equation of this Problem amounts to a biquadratick equation with all it's terms; but, by throwing out the fecond term, it is reduced to a quadratick with four roots. Thefe are irrational, of which two only are real, one pofitive, the other negative, either of which will folve the Problem. The firft and leaft of the four numbers required, when reduced to a decimal, will be the negative number.

175. The next is a Geometrical Problem, relating to a right-angled triangle. It's equation is a biquadratick with all it's terms, but when the fecond term is taken away, it degenerates into a quadratick with a plane root, but irrational.

176. A Geometrical Problem producing a biquadratick equation, the four roots of which are irrational, and may be all real, and are exhibited by the figure.

177. An equation may often appear of a higher order than the Problem really requires, if a prudent choice is made of the unknown quantity, by which the Problem is determined. This is illuftrated in feveral appofite Examples.

178. Another artifice that often prevents Problems from rifing to too high equations, is finding two values of the fame unknown quantity, and making them equal. An inftance of this is feen in the next Problem.

179. This Problem is, *in a given circle to infcribe a regular heptagon.* The Author gives feveral folutions of this Problem, which amount to high equations; but, by being compared with each other, are reduced lower. At laft fhe brings it to a cubic equation with a plane root. This is performed by finding two different expreffions for the fame quantity, and comparing them together.

180. When

180. When cubic (or higher) equations cannot be thus reduced, their roots may be found analytically, but involved in ſurds, by what are called *Cardan's* Rules. But the geometrical method will be more univerſal, by conſtructing them, and finding their roots by the interſection of curve-lines.

181. She begins with the analytical ſolution, or with finding *Cardan's* Rules: All cubic equations, that want the ſecond term, are repreſented by four general formulæ, differing only in the ſeveral changes of the ſigns. To reſolve the firſt general formula, ſhe divides the unknown root into two parts, which, after ſubſtitution, gives room for ſplitting the equation into two, ſuch as may eaſily be reſolved ſeparately. This finds commodious values for the two aſſumed parts of the root, and brings us two cubical radicals for the value of the root. See the Philoſophical Tranſactions, Number 309.

182. The ſolution of the ſecond formula does not differ from the firſt, but only in the ſigns.

183. The ſame may be ſaid of the third.

184. And likewiſe of the fourth.

185. All the four formulæ are ſolved ſomething differently, in which the two parts of the root have only one cubic radical; but which coincide with the foregoing ſolution, and are eaſily reduced to it.

186. The limits of theſe roots are here aſſigned, and it is ſhown when they will be real, and when two of them will be only imaginary.

187. When one root is found of a cubic equation, the other two may be found without diviſion. For, as unity itſelf has three cubic roots, ſo any other quantity has the ſame. Therefore, multiplying the root found by the three roots of unity ſucceſſively, we ſhall have the three roots of the given equation. This is proved here ſynthetically, by returning to the original equation. See Phil. Tranſ. No. 309.

188. This method of ſolution is illuſtrated, by applying it to a given cubic equation, of which the three roots are thence found.

189. Or, without recurring to the general ſolution, any particular cubic equation may be ſolved, by purſuing the method of that ſolution. Of theſe here are given ſeveral Examples.

190. The Author proceeds to the ſolution of biquadratick equations, of which ſhe takes a general formula, with the ſecond term abſent. Then aſſumes two general quadratick formulæ, which, multiplied together, produce a general biquadratick equation; and, by compariſon with the firſt general equation, ſhe determines the aſſumed co-efficients. This will bring her to a transformed cubic equation, in the manner taught in Article 167 aforegoing. And thus ſhe proceeds to determine the four roots of the aſſumed biquadratick equation. See Phil. Tranſ.

This ſolution ſhe applies to an Example.

191. From the algebräical reſolution of theſe equations, ſhe proceeds to the more general (as ſhe calls it), or to the geometrical ſolution, which is, by conſtructing the ſeveral *loci geometrici*, or curve-lines, adapted to every equation conſiſting of two indeterminates. Every determinate equation may be reſolved into two indeterminate equations, by introducing a quantity into it at pleaſure. Theſe two equations muſt conſiſt of the ſame two variable quantities, and the ſame conſtant quantities, and may be conſtructed by two curves. If thoſe two curves are combined in ſuch manner, as that they ſhall have a common abſciſs, they will alſo have ſome common ordinates at their common points, that is, their points of interſection. Theſe common ordinates will be the roots of the determinate equation, if the quantity repreſenting thoſe roots is made one of the variable quantities. To exemplify this, ſhe aſſumes a determinate biquadratick equation, and alſo an equation to the parabola. This ſhe introduces, by ſubſtitution, into the given biquadratick, which will then be an indeterminate equation to the hyperbola. She then conſtructs theſe two curves upon a common axis, and draws four ordinates from the four points of interſection of the curves, which will be the roots required.

192. From this conſtruction theſe notable circumſtances will evidently follow; that the poſitive and negative roots will be on different ſides of the common abſciſs; that, when two ordinates become equal, or when the two curves do not

cut

cut but touch each other, two roots of the equation will be equal; or, when the two curves cut each other in the vertex, one of the roots will be equal to nothing; and where the curves neither touch nor cut, the roots will be impoſſible.

193. It is here ſhown, that, as there may be great variety in reducing a determinate equation given, to two indeterminate equations, in order to be conſtructed; ſo ſuch a choice is to be made of the two *loci*, that the conſtruction may be as ſimple as poſſible. According as the equation is in degree, ſo each *locus* ſhould be taken, as together to make up nearly the dimenſions of the given equation.

194. Here it is ſhown, by an Example, how the ſeveral *loci* to the Conic Sections are to be diſtinguiſhed from one another.

195. Other cautions to be obſerved, in adapting the *loci* to their equations.

196. Here follow ſome Examples, to illuſtrate the foregoing doctrine. The firſt is, a determinate cubic equation wanting the ſecond term, which is reduced to a biquadratick, by multiplying the whole by the root, and a ſimple equation to the parabola is aſſumed. This is introduced into the given equation by ſubſtitution, by which it becomes an indeterminate equation to the circle. Then theſe two *loci* are combined, or conſtructed to a common abſciſs, and from their interſection a common ordinate is drawn, which will therefore repreſent the root of the given equation. Their other interſection is at the vertex, and therefore it's root will be nothing, which was introduced into the equation. The truth of this conſtruction is confirmed by a demonſtration.

197. The ſame equation is again conſtructed by combining two parabolas, and the conſtruction demonſtrated.

198. Or, to conſtruct the ſame equation, the equilateral hyperbola might be introduced, only by ſubtracting one of the equations to the parabola from the other.

199. Or, laſtly, by a ſmall alteration, one of the *loci* might have been to the circle, the other to the hyperbola.

200. But,

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200. But, without increafing the dimenfions of the cubic equation, it may be conftructed by an hyperbola between the afymptotes, combined with a parabola; as is here performed, and the conftruction demonftrated. And fo may all other equations be conftructed, that do not exceed the third degree.

In her next Example fhe takes a determinate equation of the fourth degree, which fhe changes into an indeterminate, by the fubftitution of an equation to the parabola. Into this fhe introduces an equation to the circle, and then conftructs it by means of thefe two *loci:* which conftruction fhe then demonftrates.

For another Example fhe takes a determinate cubic equation, into which fhe introduces a known root by multiplication, which raifes it to a biquadratick. Then taking an equation to the parabola, by the fubftitution of this after various manners, fhe produces feveral indeterminate equations; the laft of which, being to the circle, fhe makes choice of for conftructing the biquadratick equation. One of it's roots is the known root that was introduced, two are imaginary, and the fourth is a real but negative root. Then fhe demonftrates the conftruction.

Another Example is, an equation of fix dimenfions, but, being divifible by a divifor of two dimenfions, it is reduced to a biquadratick equation. By various fubftitutions of an equation to the parabola, various *loci* are formed, of which fhe conftructs one, which is to the equilateral hyperbola. But thefe two *loci,* being combined as their equations require, will no where interfect each other, or will have no common ordinates. Which proves, that all the roots of the given equation are imaginary and impoffible.

201. In this Example a biquadratick, or cubic, equation is propofed, to be conftructed by two conical *loci,* not to be found (as before) from the given lines of the equation, but fuch as are already known and defcribed, or otherwife by fuch as fhall be like to thefe. This is performed by deriving the two *loci* in general (as before), and then introducing new quantities, which are to be determined from the known lines of the given *loci,* according to their various circumftances. This equation, therefore, is conftructed by means of a given parabola, combined with a given hyperbola.

It

If it ſhould be required to conſtruct a biquadratick equation with a given parabola, and with an ellipſis that is of the ſame ſpecies with an ellipſis given; here is an inſtance of it, by means of introducing new quantities into the equation; which are afterwards to be determined as occaſion ſhall require. And the truth of the conſtruction is demonſtrated at length.

202. The Author here, by way of anticipation, gives us ſome conſtructions of equations that exceed the fourth degree, though ſhe reſerves the fuller treating of ſuch conſtructions to her next Section. She aſſumes a determinate equation of the fifth degree, and likewiſe an indeterminate equation to the parabola, and, by ſubſtitution, forms an equation, or locus, to a line of the third degree, which, combined with the parabola, will conſtruct the given equation. Or, ſhe ſhows how it may be done with the ſame locus combined with an hyperbola. Or, with an hyperbola, and the firſt cubic parabola. Likewiſe, ſhe conſtructs an equation of the ſixth degree, by a parabola combined with a line, or locus, of the third degree: of which equation ſhe finds two real roots, one affirmative and the other negative, and the other four are imaginary.

203. Then ſhe tells in what order the *loci* muſt riſe, by which we would conſtruct higher equations; and conſtructs (for example) an equation of eight dimenſions by means of a parabola, combined with another locus of four dimenſions.

204. She then obſerves, how equations of the ninth degree (and therefore thoſe of the eighth degree, reduced to the ninth by multiplying by the unknown root,) may be conſtructed by combining two *loci* of the third degree: which rule ſhe makes general.

205. The moſt natural way of conſtructing an equation of any degree, is by a right line for one of the *loci*, and a curve of the ſame degree for the other. As an example of this method, the Author aſſumes a definite equation of the fifth degree, makes one of the diviſors of the laſt term to become indefinite, that is, aſſumes a locus to a right line, and, ſubſtituting it in the given equation, makes it become an indefinite equation of the ſame degree as the equation given. This being conſtructed, and the right line drawn as it ought to be by

the nature of the equation, the common ordinates will determine ſo many abſciſſes, which will repreſent the roots of the given equation. Thoſe roots will be impoſſible, where the right line does not meet the curve.

206. She tells us this method may be of uſe in verifying other conſtructions; then proceeds to particular Problems, with their conſtructions.

207. The firſt is a Geometrical, or rather Analytical Problem; *between two given quantities, to find as many mean proportionals as we pleaſe.* This is applied to finding two mean proportionals, and ariſes to a ſimple cubic equation, which ſhe raiſes to an affected biquadratick, by multiplying it by the unknown root. Then aſſumes a locus to the parabola, and, by ſubſtituting it various ways in the given equation, ſhe forms ſeveral other *loci*, one to a parabola, one to an hyperbola, and one to a circle. This laſt ſhe combines with the aſſumed locus to the parabola, and conſtructs the equation given; finding one real affirmative root, and the root that was introduced which is equal to nothing, and the other two roots will be imaginary.

208. Or, without introducing a new root equal to nothing, ſhe conſtructs it by a parabola, and an hyperbola between the aſymptotes.

209. To find three mean proportionals is a plane Problem.

210. To find four mean proportionals amounts to a ſimple equation of the fifth degree, which ſhe conſtructs by means of a parabola combined with an hyperboloid of the third degree.

211. Or, by the common hyperbola between it's aſymptotes, and the ſecond cubical parabola.

212. To find five mean proportionals amounts only to a cubical equation. Then ſhe obſerves, by what *loci* ſix, ſeven, or any other number of mean proportionals may be found.

213. The next is a Geometrical Problem, *of three contiguous chords being given, terminating at the diameter of a circle, to find that diameter*; which Problem has two caſes. For the middle chord may cut the diameter, either within the circle or (produced) without. The equation that ariſes for the ſolution of this

 Problem

Problem is cubical, which ſhe multiplies by the root to make it a biquadratick. Then, aſſuming a locus to the parabola, by ſubſtitution ſhe finds another locus, which is to the circle; by the combination of which two *loci* ſhe finds the three roots, and then determines which of them will ſolve the preſent Problem. After which ſhe proceeds to the other caſe, which, with little variation, requires the ſame conſtruction.

214. A Geometrical Problem, by which the Problem of § 176 is made more general, the equation aſcending to the fourth degree. It is conſtructed by a parabola combined with an hyperbola.

215. This Problem is, *to triſect a given angle*, (ſee § 110.) and amounts to a cubic equation, which is conſtructed by two *loci*, the parabola, and the hyperbola between it's aſymptotes. The conſtruction is demonſtrated, and extended to all the caſes.

216. A further explanation of the triſection of an angle, ſhowing how the three roots of the equation ſerve for all the three ſeveral caſes, which are implied in the triſection of any angle.

217. The ſame otherwiſe conſtructed, by combining two other *loci*, one to the parabola, and the other to the circle.

218. This Problem of dividing a given arch into any given number of parts, is here extended to five equal parts, and ariſes to an equation of the fifth degree. It is conſtructed by aſſuming a locus to the parabola, and thence forming an indeterminate equation of the third degree, which is conſtructed by a curve proper to it. Theſe two, being combined, give all the five roots of the equation.

219. And this may be extended to the dividing any angle into any greater odd number of equal parts.

BOOK I. SECT. V.

The Construction of Loci exceeding the second Degree.

220. Having difcourfed at large of the ufe of the Conic Sections, as geometrical *loci* for the conftruction of equations; the Author proceeds now to higher curves, and their defcription, as the proper *loci* for conftructing equations of more dimenfions. Thefe curves, fhe fays, may be defcribed in two different manners; one is, by finding as many points as we pleafe in each curve, and tracing regular curves through them. The other is, by taking a curve already defcribed of a lower order, and finding by that the points of the other curve, or locus.

221. In order to defcribe a curve by an infinite number of points, from it's equation we muft derive the value of one of it's unknown quantities, and fuppofe it the ordinate of a curve. Then we muft affume a fucceffion of values of the other unknown quantity, or the abfcifs, and then the correfponding ordinate will become known, and fo give us a fucceffion of points in the curve, through which we may trace a regular curve, which will be one locus. Of this fhe propofes an Example in an equation of three dimenfions.

222. This ordinate may be drawn at any conftant angle to it's refpective abfcifs.

223. As an example of this defcription of a curve by points, the Author affumes the equation to an equilateral hyperbola; and, interpreting the abfcifs by fmall numbers continually, fhe finds the correfponding ordinates, which give fo many points in the curve.

224. And the fame thing will obtain if the abfcifs is interpreted by negative numbers, beginning from the centre of the hyperbola; fo that the fame hyperbola will arife, but only in an inverted pofition.

225. And when the ordinate is made nothing, the value of the abfcifs will fhow when the curve cuts the axis.

226. Alfo,

226. Alſo, intermediate points may be found, by intermediate values of the abſciſs and ordinate.

227. A Rule to find whether a curve has aſymptotes or no, and where they are if it has any.

228. But this Rule holds only when the aſymptotes are parallel to the co-ordinates; for the hyperbola has it's aſymptotes, which may be found from another equation to the ſame curve, and by the ſame rule.

229. The affair, of finding the aſymptotes of curves, properly belongs to the Method of Infiniteſimals, to which therefore it is referred.

230. Other circumſtances of the propoſed curve are here inquired into, as, whether it is convex or concave towards it's axis. This is eaſily determined by the Rule of Proportion. For, if a triangle is inſcribed in the curve, and an ordinate is drawn which is in common both to the curve and the triangle; if the ordinate to the triangle is leſs than that to the curve, the curve will be concave to it's axis; otherwiſe not.

231. But this Rule will not always obtain in all curves; for, in ſome, particular methods muſt be uſed, as will be ſeen hereafter. The Author proceeds to give another Example of deſcribing curves by points, which is the firſt cubical parabola. Of this ſhe determines a ſufficient number of points, to ſhow it's progreſs, that it cuts the axis only in one point, that it goes on *ad infinitum*, that it has no aſymptotes, that it is concave towards it's axis, and that it has a negative branch like the poſitive, but contrarily poſited.

The next Example is of the firſt cubical hyperboloid, the form of which ſhe determines by finding it's points; as alſo it's aſymptotes, and other circumſtances.

She then gives an Example of a curve of the fourth degree, the form of which ſhe determines by finding the ſeveral points.

232. She further proſecutes the ſame equation through all it's varieties, of poſitive, negative, and imaginary roots; ſhowing the different circumſtances of the curve, and of it's ſeveral branches, which reſult from thoſe roots.

Another Example of an equation of three dimenſions, from the roots of which, and finding the moſt material points, the form and other circumſtances of the curve belonging to it are determined: as it's aſymptote, it's conjugate oval, &c.

Another Example of a curve of three dimenſions, in which the principal points are determined by the ſeveral roots of the equation.

233. The ſame equation and the ſame curve is further proſecuted, and other of it's properties diſcovered: as it's two parts extending to infinity, their common aſymptote, the convexity towards it's axis, &c.

234. The ſame method, of deſcribing the curve by points, may be extended to equations in which the indeterminates are involved together, and not eaſily ſeparable. The points required may ſtill be found, though the trouble will be increaſed.

235. The Author makes an apology, for ſeeming to depart from the method ſhe had preſcribed to herſelf, in treating of theſe high equations and their curves; and then illuſtrates what ſhe has delivered, by propoſing and ſolving ſeveral Problems.

236. The conſtruction of the firſt Problem produces a well-known curve called the *Ciſſoid* of *Diocles*, and ariſes to an equation of the third degree. This locus the Author deſcribes, by finding ſeveral of the principal points, and determines it's aſymptote.

237. In this Problem the Author finds another curve by it's points, the equation of which ariſes to four dimenſions.

238. A Problem in which the Author conſtructs a curve, which ſhe calls the *Witch*. It's equation ariſes to three dimenſions, and ſhe determines it's aſymptote and other circumſtances.

239. The curve of the next Problem will be the *Conchoid* of *Nicomedes*, the equation of which ariſes to the fourth degree. This ſhe conſtructs by finding it's principal points, it's two diſtinct parts ſeparated by a common aſymptote, it's concavity and convexity, and that it has points of contrary flexure

flexure and regreſſion. This is in the firſt caſe; for ſhe diſtinguiſhes the Problem into three caſes, which ſhe purſues ſeparately.

240. As the firſt caſe depended upon the equality of two certain lines, ſo this requires that one of them ſhall be bigger than the other, and ſo will produce a different figure with ſomething different properties. The point of regreſſion in the former caſe now becomes a node, where the curve croſſes itſelf, and forms a foliate. The aſymptote remains as before, and the curve will have a like concavity and convexity towards it.

241. The other caſe is, when that line, which before was the greateſt of the two, is now the leaſt. This produces a great alteration in the curve of the former caſe; for now the foliate entirely vaniſhes, and makes the curve have a continued curvature at it's vertex, not much unlike that part on the other ſide of the aſymptote.

242. The Author propoſes a way here, of improving this method of deſcribing curves by points; which is by geometrical conſtruction. In this her firſt Example of it, ſhe reſumes the *Ciſſoid* of *Diocles* and it's equation, § 236, which ſhe conſtructs an eaſier way by geometrical effection.

In her ſecond Example ſhe reſumes the curve of § 237, which ſhe conſtructs after a like manner.

Then ſhe does the ſame by the curve called the *Witch*, § 238.

And by the *Conchoid* of *Nicomedes*, of § 239, which ſhe conſtructs geometrically in all it's varieties.

243. The foregoing conſtructions are eaſily performed by the aſſiſtance of a circle; others may be made by the help of other ſimple curves. As, here an equation of four dimenſions is conſtructed by means of a parabola; but that parabola muſt be varied for every new ordinate. However, every new parabola gives four points in the curve.

244. Parabolas are here enumerated, and diſtributed into orders, according to their dimenſions. There is only one of the firſt order, which is the *Apollonian*, or common parabola: two cubic parabolas, or of the ſecond order; three of the third order, or of four dimenſions; &c.

245. In

245. In these several orders of parabolas, those are called first parabolas in whose equation the abscis ascends no higher than to the root, or first power. She begins with the construction of the first cubic parabola, the equation of which she changes (by substitution) into that of the common parabola, which she constructs; then, by means of this she easily finds the points of the other parabola: and that both for the positive and negative branch.

246. The Author proceeds to construct the first parabola of the fourth degree, by changing it's equation of four dimensions (by substitution) into the equation of the first cubical parabola, which has been constructed. Then, by the help of similar triangles, for every ordinate of the assumed parabola she determines a point of the curve required, in each branch affirmative and negative.

247. By the same method, from the first parabola of the fourth degree the Author constructs the first parabola of the fifth degree, as to both it's branches affirmative and negative.

248. She then shows, in general, that we may always construct a first parabola of any degree, by means of a triangle, and of the first parabola of the next lower degree.

249. The Author then proceeds to construct other parabolas besides the first, and that of any degree, by means of the first, which she supposes already described. As, here she describes the second cubic parabola, by finding it's ordinate from that of the first, being reduced to a common abscis. And, in like manner, she constructs the third parabola of the fourth degree, by reducing the value of one ordinate to that of another.

250. She adds here a useful Remark concerning any of these parabolas, or paraboloids; which is, that the second parabola of the fourth degree is no other than the common parabola, only redoubled on the negative side: and so in all other, in which the index of the power of the ordinate is double to that of the abscis, and both even numbers. But if the index of the power of the abscis is an odd number, the curve will be no other than the common parabola, without such reduplication. And this holds good of all hyperbolas as well as parabolas.

251. She

251. She goes on to the conſtruction of hyperbolas (or hyperboloids) of any degree. There are only two of the third degree; the firſt has it's ordinates reciprocally proportional to the ſquares of the abſciſſes, in the ſecond the ſquare of the ordinate is reciprocally as the abſciſs. The firſt of theſe ſhe conſtructs by the help of a common parabola and hyperbola, by means of which ſhe finds it's points. The other will be the ſame curve in effect, and may be conſtructed the ſame way, only by changing the co-ordinates into each other.

252. The Author proceeds to conſtruct hyperboloids of the fourth degree, or ſuch wherein the ordinate is reciprocally as the cube of the abſciſs; or the ſquare of the ordinate is reciprocally as the ſquare of the abſciſs; or the cube of the ordinate is reciprocally as the abſciſs. The firſt ſhe conſtructs by the help of the common hyperbola and the firſt cubical parabola; the ſecond is no other than the common hyperbola itſelf; and the third is the ſame as the firſt, if the co-ordinates change places.

253. She goes on to conſtruct hyperboloids of the fifth degree; and, firſt, that in which the ordinate is reciprocally proportional to the fourth power of the abſciſs. She finds the points of this, by firſt conſtructing a common hyperbola, and then, in proper circumſtances, a firſt paraboloid of the fourth degree. She alſo conſtructs another hyperboloid of the fifth degree, in which the ſquare of the ordinate is reciprocally as the cube of the abſciſs, by aſſuming two other curves of an inferior degree. In all theſe conſtructions ſhe determines the aſymptotes of the curves, and their other affections. And the ſame method might be purſued in *loci* of higher degrees.

254. She obſerves that all firſt parabolas, deſcribed about the ſame axis, will cut one another in the ſame point. This point will be diſtinct from their common vertex; and, beſides, they muſt all have the ſame parameter.

255. Likewiſe, that theſe firſt parabolas, in tending to this common interſection, the higher their dimenſions are, the nearer they approach to the tangent; and, after they are paſt it, the nearer they approach to the axis. And the firſt hyperboloids have alſo a like property.

256. Having conſtructed theſe paraboloids and hyperboloids, or curves of two terms only; the Author proceeds to ſuch as have ſeveral terms, which ſhe

diſtinguiſhes into three caſes. The firſt caſe is of thoſe curves, or their equations, in which the ordinate is but of one dimenſion only, and is found only in one term. In the ſecond, the ordinate ariſes to any power, but is found in one term only.

In the third, the ordinate is found in more terms than one, and of any number of dimenſions.

[257. She gives here an Example of the firſt caſe. The equation of the curve to be conſtructed is of the fourth degree, and has three terms. By a convenient ſubſtitution this equation is reſolved into two others, one of which contains only conſtant quantities, and the other belongs to a firſt parabola of the fourth degree, which is here conſtructed, and the co-ordinates of the other curve are eaſily derived from it; which curve, it is obſerved, will be a portion of a parabola of the ſame degree.

258. Another Example of the ſame caſe, in which the equation of the curve to be conſtructed has three terms and four dimenſions. Here again, firſt, the equation is reſolved into two others by a ſubſtitution, and then the curve is conſtructed by means of two firſt parabolas, one of three, and the other of four dimenſions.

259. A third Example of the ſame caſe. The equation of the curve is of four dimenſions, and has four terms. This likewiſe is reſolved into two other equations by a ſubſtitution, of which one is ſimilar to that which was conſtructed in the preceeding example, and the other is to the *Apollonian* parabola; and by means of theſe two curves the required one is eaſily conſtructed.

Here the Author remarks that, if an equation ſhould more abound in terms, the ſame artifice might be uſed; and that, although the conſtruction in this caſe might become more compounded and perplexed, yet the ſame method would obtain.

She obſerves alſo that the equation in this example might have been reſolved, by another ſubſtitution, into three equations belonging to as many parabolas of different orders; and then, by means of theſe auxiliary curves, the principal curve might have been conſtructed.

260. It

260. It is here ſhown, that the co-ordinates of theſe curves may make any angle.

261. The Author gives here an Example of the ſecond caſe, in the conſtruction of a general equation of many terms, which, by a convenient ſubſtitution, ſhe reduces to caſe the firſt. See the Example.

262. An Example of conſtructing a curve of the third caſe. The equation here propoſed is general, and is reſolved, by a proper ſubſtitution, into otheis which belong to the firſt caſe; ſo that, by the conſtruction of theſe curves, the co-ordinates of the propoſed curve are obtained.

263. Hitherto the Author has conſidered only thoſe equations which have their indeterminate quantities ſeparate; ſhe here obſerves that, when the indeterminates are involved with each other, the foregoing rules cannot take place, but that a ſeparation of the variable quantities muſt be made, either by common diviſion, or by the extraction of roots, or by a congruous ſubſtitution, or by other expedients. She then gives two examples of the ſeparation of the indeterminate quantities: in the firſt, it is performed by common diviſion; in the ſecond, by a convenient ſubſtitution.

264. Having ſhown how to prepare equations of that kind for conſtruction, ſhe proceeds to the actual conſtruction of them, taking here the firſt equation in the preceding article, and conſtructing it by means of equations which come under caſe the firſt of article 256.

265. The conſtruction of the other equation in § 263: which, it is ſhown, may be performed by caſe the third of § 256.

266. A remark, That a convenient ſubſtitution may be of uſe even in thoſe caſes in which the indeterminate quantities are already ſeparated; and may ſuggeſt a conſtruction which is more eaſy and elegant.

267. An inſtance of the truth of the foregoing remark appears here, where the conſtruction of a curve, the equation of which has four dimenſions, is

 facilitated

facilitated by a ſubſtitution, although the variable quantities in that equation were ſeparate. With this the Author ends her examples of the conſtruction of curves.

BOOK I. SECT. VI.

Of the Method De Maximis et Minimis, *of the Tangents of Curves, of Contrary Flexure and Regreſſion; making uſe only of the* Common Algebra.

268. THE Author here obſerves that, although the Calculus of Infiniteſimals* be the ſimpleſt and ſhorteſt method, and the moſt univerſal, for managing ſuch ſpeculations, yet the ſolution of ſuch queſtions may be performed by common Algebra, in curves that are expreſſed by finite algebräical equations.

She begins with the *Maxima* and *Minima*; that is, to find in geometrical curves the greateſt or the leaſt ordinates; and ſhows that, in either caſe, two ordinates coincide, and conſequently two abſciſſes become equal; and thence two roots of the equation belonging to the curve, taken either in terms of the letter which expreſſes the abſciſs, or of the letter which expreſſes the ordinate, become equal to each other.

Her firſt Example is, To find the greateſt or leaſt ordinate when the curve is an Ellipſis; which ſhe does by forming a quadratick equation which has equal roots, and comparing it, term by term, with the equation of the curve. She then ſhows how to perform the ſame thing when the equation of the curve is of the third, fourth or higher degree; which is, by forming an equation of the ſame degree, that has two equal roots, and comparing it, term by term, with the equation of the curve. See the Examples.

269. A ſhorter and eaſier way of doing the ſame thing; which is, by multiplying the terms of the given equation by the terms of an arithmetical

* Rather *Fluxions*. EDITOR.

progreſſion.

progreſſion. For, if an equation has two equal roots, (which is the caſe of a *maximum* or *minimum*,) one of theſe roots will, of neceſſity, be included in the product of that equation multiplied by the arithmetical progreſſion. This is demonſtrated; and the two preceding examples are reſumed, and the ſame reſults obtained, although different progreſſions are uſed.

270. The Author proceeds to find tangents and perpendiculars to curves by a like method; previouſly ſhowing that the queſtion is reduced to this: To find a circle that ſhall touch the curve in any given point. This alſo is performed by means of equations that have two equal roots: which ſhe explains, and illuſtrates by an example of drawing a tangent to the *Apollonian* parabola. The equation which thus ariſes is ſolved, firſt, by comparing it with another quadratick having two equal roots; ſecondly, by multiplying the terms of it by the terms of the arithmetic progreſſion 3, 2, 1; and, laſtly, by multiplying the terms by the progreſſion 2, 1, 0.

271. Another Example of drawing a tangent to a curve of which the equation is cubical, worked both by comparing it with an equation of the ſame degree which has two equal roots, and by multiplying the terms of it by the arithmetical progreſſion 3, 2, 1, 0.

272. It is obſerved, that, in general, the moſt convenient progreſſion will be that which forms the exponents of the letter according to which the equation is ordered.

273. The Problem of drawing tangents is ſolved in a way ſomewhat different, but more ſimple; and the formulæ here derived are of uſe alſo in finding points of contrary flexure and regreſſion.

274. Points of Contrary Flexure and Regreſſion are here defined; and it is ſhown that, as the nature of *maxima* and *minima*, and of tangents, requires equations that have two equal roots, ſo in contrary flexures and regreſſions three equal roots are required. An example of finding the point of contrary flexure is given, by way of illuſtration.

275. The Author obſerves that the operation is the ſame for finding the points of regreſſion in curves, as for finding points of contrary flexure; ſo

that, to diftinguifh them, there is no other way, but to find, by means of a conftruction, the figure and proceeding of the curve.

She fays that the fame ambiguity arifes in queftions *de maximis et minimis*, which can only be removed by acquiring fome knowledge of the difpofition of the curve. She then obferves that, by the fame condition of three equal roots, we may find the *radii* of curvature; but, intending to treat of thefe things in the fecond Volume, fhe here puts an end to the firft.]

N. B. It being my intention to deliver what I have to offer on the fecond Volume in Notes, as is mentioned in my Advertifement prefixed to this Work, the reader will fee the propriety of my continuing the *Plan of the Lady's Syftem of Analyticks* no further. J. H.

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ANALYTICAL INSTITUTIONS.

BOOK I.

ANALYTICAL INSTITUTIONS.

BOOK I.

THE ANALYSIS OF FINITE QUANTITIES.

THE *Analysis* of Finite Quantities, which is commonly called the *Algebra* of *Cartesius*, is a method of solving Problems by the use and management of finite quantities: that is, from certain quantities and conditions, which are given and known, we may come to the knowledge of others which are unknown and required; and that by means of certain operations and methods, which I propose to explain by degrees in the following Sections. Introduction: What is *Algebra*, or *Analyticks.*

SECT. I.

Of the First Notions and Operations of the Analysis of Finite Quantities.

1. THE primary operations of this Algebra, or Analyticks, are the same as those of common Arithmetick; which are, Addition, Subtraction, Multiplication, Division, and Extraction of Roots. But with this difference, that whereas in Arithmetick those operations are performed with numbers, in Algebra they are performed (or perhaps only insinuated) with species, or the letters of the alphabet; by which quantities are denominated and calculated in the abstract, of whatever kind they may be, whether Geometrical or Physical, as Lines, Surfaces, Solids, Forces, Resistances, Velocities, &c. And therefore this kind The operations of Algebra, what.

 of

of Arithmetick is often called *The Algorithm of Quantities*, or *Specious Arithmetick.* And indeed this is of a much more excellent and general nature than that can be, though all it's operations are the ſame; as well becauſe theſe quantities are not confounded one among another in their operations, as numbers are; as becauſe in this Calculus known and unknown quantities are treated indifferently, and with the ſame facility; and laſtly, becauſe Analytical demonſtrations are general, and therefore applicable alike to all caſes; whereas in Arithmetick they are particular, and in every different caſe require a new determination.

Poſitive and negative quantities diſtinguiſhed.

2. Now of theſe quantities ſome are *poſitive*, or ſaid to be greater than nothing; others are leſs than nothing, and therefore are called *negative*. To explain this by an example. The goods in our own poſſeſſion may be called poſitive, but thoſe which we owe to others are negative, becauſe they muſt be ſubtracted from the poſitive, and therefore will diminiſh their ſum total. Wherefore, as the capitals in our poſſeſſion are poſitive, and are anſwerable for our debts; ſo the debts we owe will be negative quantities. In like manner, if a body or point in motion is directed towards a certain mark, and in it's paſſage deſcribes a ſpace, this ſpace may be called poſitive; but afterwards if it receives an oppoſite direction, it will indeed deſcribe a ſpace, but this ſpace will be negative in reſpect of the mark to which it ought to go. Wherefore, in Geometry, if a line drawn one way is aſſumed as poſitive, (for this is quite arbitrary,) a line drawn the contrary way will be negative.

The ſigns of poſitive and negative quantities, with other marks, explained.

3. Poſitive and negative quantities in Algebra are diſtinguiſhed by means of certain marks, or ſigns, which are prefixed to them. To poſitive quantities the ſign $+$, or *plus*, is prefixed: to negative quantities the ſign $-$, or *minus*. And when a quantity has no ſign prefixed, as when it ſtands alone, or is the firſt among others, it is then always ſuppoſed to be affected by the poſitive ſign. The ſign $\pm$, the contrary of which is $\mp$, is an ambiguous ſign, and ſignifies either *plus* or *minus*. So, for example, $\pm a$ inſinuates, that the quantity or number repreſented by a may be taken either affirmative (that is, poſitive) or negative. The mark $=$ ſignifies equality, and therefore $a = b$ informs us, that the two quantities expreſſed by a and b are equal to each other. So $a > b$ means, that a is greater than b. Alſo, $a < b$ tells us, that a is leſs than b. The equality of ratios, or the geometrical proportion of three or four terms, is thus expreſſed: $a . b :: b . c$, when there are three terms; that is, the ratio of a to b is equal to the ratio of b to c. Alſo, $a . b :: c . d$ means, that a is to b as c is to d. Laſtly, the ſign ∞ denotes infinite, and therefore $a = \infty$ ſignifies, that a is equal to infinite, or is an infinite quantity.

Quantities are divided into ſimple or compound.

4. A quantity is ſimple, incomplex, or of one term only, when it is expreſſed by one or more letters, but thoſe are not ſeparated or diſtinguiſhed from one another by the ſign either of addition or ſubtraction. Such are a, ab, aac, and the like. So, on the contrary, quantities are compound, or of ſeveral terms,

terms, when they are expreffed by feveral letters, feparated from one another by the figns + or —. Such are $a + b$, $aa - ff + bb$, and the like. And therefore $a + b$ will be a quantity of two terms, or a binomial; $aa - ff + bb$ will be one of three terms, or a trinomial, &c.

Addition of Simple Quantities, being Integers.

5. Simple quantities are added to one another by writing one after another, prefixing to each it's proper fign. As if we were to add a to b and c, the fum would be reprefented by $a + b + c$. If we were to add a to $-b$, the fum would be $a - b$. To add a to b to a to b, the fum would be $a + b + a + b$. But here it muft be obferved, that $a + a$ is the fame as $2a$, and $b + b$ is the fame as $2b$; therefore the fum will be $2a + 2b$. Therefore, to add the fame quantities, or fuch as are expreffed by the fame letters, it will fuffice to prefix to the fame letter fuch a number as fhall contain fo many units, as are the times that the letter fhould be repeated. Thus, the fum of ac to ac to ac, that is, $ac + ac + ac$, will make $3ac$, and this number is called *the Numeral Co-efficient* of the quantity. And if the quantities to be added, being denominated by the fame letter, fhall have different co-efficients, thofe co-efficients are to be added by the ordinary rules of Arithmetick. Thus the fum of $2a$ and $5a$, together with b and $4b$, will be $7a + 5b$. And thus the fum of a and $3b$, and $-2c$, and $7c$, and $5a$, will be $a + 3b - 2c + 7c + 5a$. But $a + 5a$ will make $6a$, and $-2c + 7c$ make $5c$. Therefore the fum will be $6a + 3b + 5c$. Addition of fimple quantities.

Subtraction of Simple Quantities, being Integers.

6. To fubtract one quantity from another, the fign muft be changed of that quantity which is to be fubtracted, and then with it's fign fo changed it muft be wrote with the other. Thus to fubtract b from a, we muft write $a - b$; where it may be obferved, that if a is a quantity greater than b, the remainder of the fubtraction, or the difference, will be pofitive. But if b is greater than a, in that cafe the difference will be a negative quantity. To fubtract aff from bbc, it will make $bbc - aff$. To fubtract $2a$ from $5a$, it will make $5a - 2a$; but $5a$ leffened by $2a$ make $3a$, fo that the remainder will be $3a$. And to fubtract $-b$ from a, it muft be written $a + b$. Nor fhould it feem ftrange, that to fubtract the negative quantity $-b$ it muft become pofitive, that the remainder Subtraction of fimple quantities.

may be $a + b$; for as much as to ſubtract one quantity from another is the ſame thing as to find the difference between thoſe quantities. Now the difference between a and $-b$ is $a + b$, juſt in the ſame manner as the difference between a capital of 100 crowns and a debt of 50 is 150 crowns. For from having an hundred and having none, the difference is an hundred; and from having none to having a debt of fifty, the difference is fifty; therefore, from having an hundred to having a debt of fifty, the difference muſt be an hundred and fifty. Thus, for the ſame reaſon, to ſubtract b from $-a$, it muſt be written $-a - b$; and to ſubtract $-b$ from $-a$, it muſt be written $-a + b$.

Multiplication of Simple Quantities, being Integers.

Multiplication of ſimple quantities.

7. Simple quantities are multiplied by writing them one after another, without any ſign between, (unleſs ſometimes the mark ×,) and the reſulting quantity is called the *Product*, as alſo the quantities ſo multiplied are called the *Factors* or *Multipliers*. But as to the ſign which is to be prefixed to theſe products, the general rule is this; that if the quantities to be multiplied are both poſitive or both negative, then the poſitive ſign muſt always be prefixed to the product: but if one of thoſe quantities, whichever it is, is poſitive, and the other negative, then the negative ſign muſt always be prefixed to the product. The reaſon of this is, becauſe multiplication is nothing elſe but a geometrical proportion, of which the firſt term is unity, the ſecond and third terms are the two quantities which are to be multiplied together, and the fourth term is the product. And therefore being placed in a row, unity for the firſt term, one of the multipliers for the ſecond, and the other multiplier for the third; therefore, by the nature of geometrical proportion, the fourth muſt be ſuch a multiple of the third, as the ſecond is a multiple of the firſt. If the ſecond and third terms are poſitive, for example, if it is $1 . a :: b .$ to a fourth; the firſt term or unity being poſitive, the fourth muſt therefore be poſitive. But if the ſecond is negative, and the third poſitive, that is, if $1 . -a :: b .$ to a fourth; whereas this fourth muſt be ſuch a multiple of the third as the ſecond is of the firſt, and the ſecond being negative, therefore the fourth muſt be negative. Let the ſecond be poſitive and the third negative, that is, let it be $1 . a :: -b .$ to a fourth. Now, whereas this fourth muſt be ſuch a multiple of the third, as the ſecond is of the firſt, and the ſecond and firſt being poſitive and the third negative, the fourth cannot be otherwiſe than negative. Laſtly, let both the ſecond and third be negative, that is, let it be $1 . -a :: -b .$ to a fourth. Now the ſecond being here a negative multiple of the firſt, it follows that the fourth muſt be a negative multiple of the third. But the third is already negative, and therefore the fourth muſt be poſitive. Wherefore the product of a into b will be ab. That of a into $-b$ will be $-ab$. That of $-a$ into b

 will

will alſo be $-ab$. That of $-a$ into $-b$ will be ab. That of a into b into c will be abc. That of a into $-b$ into c, will be $-abc$; becauſe a into $-b$ will be $-ab$, and $-ab$ into c will be $-abc$. And the product of $-a$ into $-b$ into c will be abc.

If the quantities to be multiplied ſhall have numeral co-efficients, thoſe co-efficients muſt be multiplied together by the common rules of numbers, and the product muſt be prefixed to that of the letters. Hence the product of $6a$ into $-8bc$ will be $-48abc$. And the product of $2a$ into $-2b$ into $-3c$ will be $12abc$. And the like of all others.

8. Now becauſe the product of a into a is aa, that of a into a into a, or of aa into a, is aaa, that of a into a into a into a, or of aaa into a, is $aaaa$, and ſo on ſucceſſively; to prevent the repetition of the ſame letter ſo often, it is uſual to write a^2 inſtead of aa, a^3 inſtead of aaa, a^4 inſtead of $aaaa$, and ſo of others. That is, we may write a little above the letter ſuch a number as ſhews the number of times the letter ought to be repeated, which number is called an *Index* or *Exponent*. We may write indifferently aa or a^2, but higher products or powers are more commonly expreſſed by their exponents.

Notation of ſimple powers.

9. As the product of a number multiplied by itſelf is called the *Square* of that number, or it's ſecond power; ſo if this product is again multiplied by the ſame number, the new product is called the *Cube* or the third power of the ſame number. And the product of the cube by the ſame number is called the *Biquadrate* or fourth power of the ſame number; and ſo on. Thus the quantity a multiplied by a, or a^2, is called the Square of a, or the ſecond power of a; a^3 is it's cube or third power; a^4 it's fourth power, &c. Therefore $2a$ and a^2 will be very different from each other, the firſt being the ſum of a and a, or $a+a$, the other their product, or aa. The ſame is to be underſtood of $3a$ and a^3, of $4a$ and a^4; and ſo of others. Now as the product of $+$ into $+$, or of $-$ into $-$, is always poſitive; it proceeds from thence, that as well the ſquare of a as of $-a$ will be always poſitive, or aa. So on the other hand the cube of a, or a^3, will always be poſitive, but the cube of $-a$ will always be negative, or $-a^3$. For $-a$ into $-a$ makes aa, and aa into $-a$ makes $-a^3$. Thus the fourth power as well of $-a$ as of a will be poſitive. And in general, when the exponent of the power, to which we would raiſe the given quantity, is an even number, whether the quantity itſelf is poſitive or negative, that which reſults will always be poſitive. And when the exponent is an odd number, if the quantity is poſitive, the reſult will be poſitive; and if it be negative, the reſult or power will be negative.

Names of the powers, and their diſtinction into poſitive and negative.

Diviſion.

Division of Simple Quantities, being Integers.

Divifion, what.

10. Divifion is an operation directly contrary to Multiplication; and what this compounds, that again refolves. Thus, becaufe ab is the product of a into b, therefore if we divide ab by a, we fhall have b for the quotient. And if we divide it by b, the quotient will be a. So dividing abc by c, we fhall have the quotient ab. And fo on. The quantity to be divided is called the *Dividend*, that by which the divifion is performed is called the *Divifor*, and that which refults from the divifion is called the *Quotient*, as in common Arithmetick. Therefore whenever in the dividend and the divifor the fame quantities are found, they may be taken out of both, or as it were cancelled, and what is left will give the quotient. Thus, if we are to divide aa by a, the quotient will be a. If we divide a^3 by a, the quotient will be aa. If we divide a^3b^3 by a^2b^2, the quotient will be ab. If the dividend and divifor fhall have numeral co-efficients befides, they are alfo to be divided by the common rules of Arithmetick, and the refulting numeral co-efficient muft be prefixed to the literal quotient. Thus, if we divide $3a^3b^3$ by $3b^3$, the quotient will be a^3. Dividing $56a^2b^3$ by $8ab$, the quotient will be $7ab^2$. And here it may be obferved, that whenever the quantity to be divided is the fame as the divifor, then the quotient will be unity; as dividing b by b, $7a^3$ by $7a^3$, and fuch like. The reafon of which is plain, becaufe to divide is to find how often the divifor is contained in the dividend, the anfwer of which queftion is the quotient.

When quotients are to be reprefented as fractions.

11. Wherefore, in the divifor and dividend, when no common quantities or letters are found, by means of which the divifion may be performed in the foregoing manner, the quotients will receive the form of fractions. Thus, to divide a by b, a^3 by bc, $5aabb$ by $2cc$, &c. the quotients are to be wrote thus: $\frac{a}{b}$, $\frac{a^3}{bc}$, $\frac{5aabb}{2cc}$, &c: that is, place the dividend above, and the divifor under it, with a little line between them; and it is to be underftood, that a ought to be divided by b, a^3 by bc, &c; and thefe are called *Fractions*, in which the quantity above the line is called the *Numerator*, and that below is the *Denominator*. Thus if any of the letters of the divifor, but not all, fhall be in common with the letters of the dividend, thofe that are common are to be taken away from each, and of thofe that remain a fraction is to be formed. Thus, if we were to divide a^3bb by $5abcc$, the quotient will be $\frac{a^3bb}{5abcc}$, or $\frac{a^2b}{5cc}$. And if we divide $10ab^3$ by $15bcc$, the quotient will be $\frac{2abb}{3cc}$. And fo of all others.

12. Now,

12. Now, becauſe both the dividend and diviſor may be either poſitive or negative, it is neceſſary in every combination of caſes to fix a rule, for the ſign which is to be prefixed to the quotient. This rule is the ſame as that which ſerves for multiplication. That is to ſay, that if the dividend and the diviſor have both the ſame ſign, whether poſitive or negative, the quotient will be always poſitive. But if they have contrary ſigns, the quotient muſt be negative. The demonſtration depends on that of multiplication. For as multiplication is a proportion, of which the firſt term is unity, the ſecond and third are the two multipliers, and the fourth is the product; ſo diviſion is the ſame proportion, but inverted. Of this the firſt term is the dividend, the ſecond the diviſor, the third is the quotient, and the fourth is unity. Let it be required to divide $\pm ab$ by $\pm b$. Then the proportion will be $\pm ab \,.\, \pm b :: * a \,.\, 1$. Here I place the mark * before the third term or quotient, as not yet knowing whether it ought to be poſitive or negative. Now, conſidering this proportion to be like that of multiplication, but the terms placed inverſely, it is known that when the ſecond term b is poſitive, the firſt term ab cannot be poſitive, unleſs the third term a is poſitive alſo; and the ſecond b being negative, the firſt ab cannot be negative, unleſs the third a be poſitive. Wherefore, in diviſion, when the two firſt terms, or the dividend and diviſor, are both poſitive or both negative, the third term, or quotient, muſt neceſſarily be poſitive. In like manner, in this proportion, the ſecond term b cannot be poſitive and the firſt ab negative, or the ſecond b negative and the firſt ab poſitive, unleſs the third a be negative. So that in diviſion, the dividend being poſitive and the diviſor negative, or on the contrary, the quotient of neceſſity muſt be negative.

The ſign of the quotient, what.

13. For this reaſon it will be the ſame thing whether we write (for example) $\frac{a}{-b}$, or $\frac{-a}{b}$; becauſe if a poſitive is to be divided by b negative, or if a negative is to be divided by b poſitive, in both caſes the quotient muſt be negative. Thus it will be the ſame to write $\frac{-a}{-b}$, or $\frac{a}{b}$.

Signs reciprocal in ſimple fractions.

Extraction of the Roots of Simple Quantities, being Integers.

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14. As quantities have their ſeveral powers, the ſquare, the cube, the biquadrate, the fourth power, &c, ſo among the roots of ſuch powers there is the ſquare-root or ſecond root, the cube-root or third root, the fourth root, &c. The denomination of roots is the ſame as that of the exponents of powers. Therefore the index or exponent of the ſquare-root is 2, of the cube-root is 3, &c. And to extract the root of a given quantity, we muſt find ſuch another quantity, as being multiplied into itſelf as many times, all but one, as are the units

Roots of ſimple quantities extracted.

units in the index of the root, ſhall have for the product the quantity whoſe root is propoſed to be extracted. Thus a will be the ſquare-root of aa, the cube-root of a^3, the biquadratick-root of a^4, &c. In the ſame manner the ſquare-root of $aabb$ will be ab, of $16aabbcc$ will be $4abc$; the cube-root of $27a^3x^3$ will be $3ax$; and ſo of others.

Signs of roots. Impoſſible roots.

15. And ſince the product of *minus* into *minus* is always *plus*, as above; thence it follows that the ſquare-root of aa will be either a or $-a$, that is $\pm a$. It is not ſo with the cube-root, which will always be poſitive if the given cube is poſitive, and will be negative if this be negative; for the cube of a will be a^3, and the cube of $-a$ will be $-a^3$. But the fourth root will be either poſitive or negative. And to ſpeak in general, the roots whoſe index is an even number will always be either poſitive or negative; but thoſe whoſe index is an odd number will be poſitive if the power propoſed be poſitive, and negative if that be negative. And hence it is, from the ſame property of the ſigns in multiplication, that no poſitive or negative quantity can ever produce a negative power having an even exponent. So that it is impoſſible to find the root of a negative power with an even exponent. Such roots as theſe, of a negative quantity with an even index, are therefore called *Impoſſible* or *Imaginary*. Thus the ſquare-root of $-aa$ will be imaginary, as alſo the fourth root of $-a^4$, the ſixth root of $-a^6$, &c. But ſuch as theſe will be true and real roots, the cube-root of $-a^3$, the fifth root of $-a^5$, &c.

Roots extracted of imperfect powers.

16. But for the generality the quantities propoſed, of which we are to extract the roots, will not be true ſquares, cubes, or other powers, which are produced by the multiplication of rational quantities into themſelves, but will be the products of another kind; as ab, abc, &c: in which caſe we make uſe of the mark $\surd$, called the *Radical Sign* or *Vinculum*. Hence $\sqrt[2]{ab}$, or ſimply $\sqrt{ab}$, denotes the ſquare-root of ab; $\sqrt[3]{abc}$ denotes the cube-root of abc. And thus $\sqrt[4]{}$ ſtands for the fourth or biquadratick-root, $\sqrt[5]{}$ ſtands for the fifth root, &c. And ſuch quantities as theſe, affected by a radical ſign or vinculum, are called *Surds*, or *Irrational Quantities*.

Addition of Compound Quantities, being Integers.

Compound quantities added.

17. By the addition or ſubtraction of ſimple quantities, compound quantities are produced. Therefore, to add theſe together, it is ſufficient to write them one after another with their proper ſigns. So to add $a + b$ to $c - d$, we may write $a + b + c - d$. To add $2aa - xx$ to $3cc + 2yy$, the ſum will be $2aa - xx + 3cc + 2yy$. To add $aa - xx$ to $bb + xx + yy$, we ſhall have $aa - xx + bb + xx + yy$; but here it is to be obſerved, that $-xx$ and $+xx$ remove

remove or deſtroy each other, and therefore may both be cancelled or omitted, and then the ſum will be $aa + bb + yy$. To add $2aa - 5bb$ to $aa + 2bb + yy$, the ſum will be $2aa - 5bb + aa + 2bb + yy$; but here $2aa + aa$ make $3aa$, and $-5bb + 2bb$ make $-3bb$, and therefore the ſum will be $3aa - 3bb + yy$.

Subtraction of Compound Quantities, being Integers.

18. The ſigns muſt be changed of that quantity which is to be ſubtracted, and then with the ſigns ſo changed it is to be wrote after that, from which the ſubtraction is to be made. Thus to ſubtract $c - d$ from $a + b$, we muſt write them thus, $a + b - c + d$; and the reaſon is plain. For if we were to ſubtract only the quantity c, we ſhould write $a + b - c$. And now having ſubtracted too much, (for we ought to have ſubtracted only $c - d$, or the difference between c and d,) having ſubtracted, I ſay, more than we ought by the quantity d, to make amends we muſt add d, and ſo write the remainder $a + b - c + d$. The ſame is to be done for quantities more compounded. To ſubtract $a + 3b$ from $3a + 2b$, it will be wrote $3a + 2b - a - 3b$; but by a reduction of ſimilar terms, becauſe $3a - a$ is $2a$, and $2b - 3b$ is $-b$, the remainder will become $2a - b$. To ſubtract $3ab - 2bc + 2cd$ from $5ab - 4bc + 2cd$, after a proper reduction the remainder will be $2ab - 2bc$. Compound quantities ſubtracted.

Multiplication of Compound Quantities, being Integers.

19. The rule for the multiplication of ſimple quantities being underſtood, that for compound quantities will be very eaſy. Let one of the factors be wrote under the other, as is uſual in the vulgar Arithmetick, and then all the terms of the multiplicand muſt be multiplied ſucceſſively by every term of the multiplier, according to the rules already given for the multiplication of ſimple quantities; and what reſults, after the uſual reduction of ſimilar terms, will be the product required. Thus if we are to multiply $a + b - c$ by x, let them be wrote as in the margin. Then let every term of the multiplicand, placed above, be multiplied by the multiplier placed under it, and the product will be $ax + bx - cx$, as by the operation. Thus if we were to multiply Compound quantities multiplied.

$$\begin{array}{r} a + b - c \\ x \\ \hline ax + bx - cx \end{array}$$

$$\begin{array}{r} 2a + 3b - c \\ 3x - 2y \\ \hline 6ax + 9bx - 3cx \\ - 4ay - 6by + 2cy \end{array}$$

multiply $2a + 3b - c$ by $3x - 2y$, let them be placed as in the margin. Then multiply all the terms above by $3x$, and do the ſame by the other term $-2y$, and ſo if there were more terms in the multiplier. The product will be as is here to be ſeen. It is no matter whether the operation begins from the right hand or from the left, in regard to either of the factors; or which of them is wrote above, and the other below; or in what order the terms are placed. Suppoſe we were to multiply $aa + xx$ by $aa - xx$; proceed as in the margin, where becauſe $aaxx$ and $-aaxx$ deſtroy each other, the product will be reduced to $a^4 - x^4$.

$$\begin{array}{r} aa + xx \\ aa - xx \\ \hline a^4 + a^2x^2 \\ - a^2x^2 - x^4 \end{array}$$

In long multiplications, in order to reduce ſimilar terms with greater eaſe, it will be convenient to write thoſe ſimilar terms, which will ariſe from the multiplication, one under another as in the foregoing and following example. Let it be propoſed to multiply $4a^3 + 3a^2b - 2ab^2 + b^3$ by $a^2 - 5ab + 6b^2$. The work will ſtand as in the margin. Here it is eaſily perceived, that $+3a^4b - 20a^4b$ make $-17a^4b$. And that $-2a^3b^2 - 15a^3b^2 + 24a^3b^2$ make $7a^3b^2$. And that $+a^2b^3 + 10a^2b^3 + 18a^2b^3$ make $29a^2b^3$. And that $-5ab^4 - 12ab^4$ make $-17ab^4$. Therefore the product is $4a^5 - 17a^4b + 7a^3b^2 + 29a^2b^3 - 17ab^4 + 6b^5$.

$$\begin{array}{l} 4a^3 + 3a^2b - 2ab^2 + b^3 \\ \qquad a^2 - 5ab + 6b^2 \\ \hline 4a^5 + 3a^4b - 2a^3b^2 + a^2b^3 \\ \quad -20a^4b - 15a^3b^2 + 10a^2b^3 - 5ab^4 \\ \qquad\qquad +24a^3b^2 + 18a^2b^3 - 12ab^4 + 6b^5. \end{array}$$

Multiplication how inſinuated.

20. Sometimes it will be unneceſſary actually to perform the multiplication in this manner, but it may be ſufficient to inſinuate it only, which is commonly done by inſerting this mark $\times$, and drawing a line or *vinculum* over each of the multipliers, extended over all the terms which enter the multiplication. Thus $\overline{aa + xx} \times \overline{aa - xx}$ denotes the product of $aa + xx$ by $aa - xx$. But in the quantity $\overline{aa + xx} \times \overline{aa - xx} \pm a^4$, the term $\pm a^4$, not being included in the vinculum, is not intended to be comprehended in the multiplication. So that being wrote in this manner it muſt be underſtood, that to or from the product of $aa + xx$ into $aa - xx$, muſt be further added or ſubtracted the term a^4.

Powers of compound quantities how inſinuated: how actually formed.

21. After the ſame manner that in ſimple quantities the product of a into a is called the ſquare of a, the product of aa into a is called the cube of a, the product of a^3 into a is called the biquadrate of a, &c. So in compound quantities the product of $a + b$ (for example) into $a + b$, or $\overline{a + b} \times \overline{a + b}$, is called the ſquare of $a + b$, which is wrote thus, $\overline{a + b}|^2$, when we would not actually form it by multiplication. In the ſame manner $\overline{a + b}|^2 \times \overline{a + b}$ will be the cube, which may be wrote thus, $\overline{a + b}|^3$; and $\overline{a + b}|^3 \times \overline{a + b}$, or $\overline{a + b}|^2$

$\overline{a+b}|^2 \times \overline{a+b}|^2$, or $\overline{a+b}|^4$ will be the fourth power of $a + b$. And this is to be underſtood of quantities of any number of terms.

Actually to form theſe powers, the quantity given muſt be multiplied into itſelf, and the product by the ſame quantity ſucceſſively, as many times, ſave one, as the exponent of the power required contains unity. But for the ſecond power, or the ſquare, the operation may be thus abbreviated. If the quantity given is a binomial, or conſiſts only of two terms, ſuppoſe $a \pm b$, write down the ſquare of the firſt term, then the two rectangles, or twice the product of the firſt term by the ſecond, with ſuch a ſign as the rule of multiplication requires; and laſtly the ſquare of the ſecond term muſt be added. Thus $\overline{a+b}|^2$ will be $aa + 2ab + bb$; and $\overline{a-b}|^2$ will be $aa - 2ab + bb$. Alſo $\overline{-a-b}|^2$ will be $aa + 2ab + bb$. If the quantity given is a trinomial, or conſiſts of three terms; beſides the ſquare of the two firſt terms found as before, muſt be wrote two rectangles of the firſt into the third, and alſo of the ſecond into the third, (taking care that theſe rectangles may have their proper ſigns, according to the rules of multiplication,) and laſtly the ſquare of the third term. Thus $\overline{a+b-c}|^2$ will be $aa + 2ab + bb - 2ac - 2bc + cc$. If the quantity is a quadrinomial, or of four terms, there muſt be wrote beſides, twice the rectangles of the three firſt terms into the fourth, and alſo the ſquare of the fourth term. And ſo on to other multinomials.

22. But as to all binomial quantities, the following general canon will be of good uſe, not only to raiſe it to the ſquare, but to any power denoted by m, where m ſtands for any number whatever. Therefore let $p + q$ be to be raiſed to the power m; this power will be $p^m + mp^{m-1}q + m \times \frac{m-1}{2}p^{m-2}q^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}p^{m-3}q^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}p^{m-4}q^4$, &c.; which ſeries of terms may be continued as far as we pleaſe, obſerving the ſame law. Powers raiſed by the *Binomial Theorem* of Sir I. N.

From hence let us derive the ſquare of $p + q$. In this caſe m will be 2, and therefore in this canon, by ſubſtituting 2 inſtead of m, the firſt term will be p^2; the ſecond $2p^{2-1}q$, that is $2pq$; the third will be $2 \times \frac{2-1}{2}p^{2-2}q^2$, that is q^2. (Here we do not admit the quantity p, becauſe being raiſed to no power, it is equal to unity, as will be ſhown afterwards. And the fourth term will be $2 \times \frac{2-1}{2} \times \frac{2-2}{3}p^{2-3}q^3$. But $2 - 2$ in the co-efficient is equal to nothing, and therefore this term being multiplied by nothing will be nothing, or will vaniſh. And thus ſince all the following terms are multiplied by nothing, they will all vaniſh, and the canon will terminate after three terms. So then the ſquare required will be $pp + 2pq + qq$.

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If we would have the cube or third power of $p + q$, then $m = 3$; whence the fifth term of the canon, and all the following ones, will be equal to nothing. So that the power required, by fubftituting 3 inftead of m, will be $p^3 + 3p^2q + 3pq^2 + q^3$. If the quantity to be raifed is $p - q$, it will be fufficient to place the fign *minus* before all the terms, in which the index of q is an odd number.

The foregoing canon will not only ferve for the binomial $p \pm q$, but for any other whatever. So that if we would have the third power of $2ax - xx$, we muft fuppofe $p = 2ax$, and $q = - xx$, as alfo $m = 3$. Then in the canon, inftead of p and the powers of p, we muft fubftitute $2ax$ and it's powers; which muft alfo be done by putting $- xx$ inftead of q and it's correfponding powers. Then inftead of m put 3, and the cube will be $8a^3x^3 - 12aax^4 + 6ax^5 - x^6$.

It may likewife ferve for any polynome, or for any quantity confifting of more terms than two. Let there be a trinomial $a + b - c$ to be raifed to the third power, and then it will be $m = 3$. If we make $p = a$ and $q = b - c$, and fubftitute a and it's powers inftead of p and it's powers, and alfo $b - c$ and it's powers inftead of q and it's powers; the cube will be $a^3 + 3aa \times \overline{b-c} + 3a \times \overline{b-c}|^2 + \overline{b-c}|^3$; that is, $a^3 + 3a^2b - 3a^2c + 3ab^2 - 6abc + 3ac^2 + b^3 - 3b^2c + 3bc^2 - c^3$.

Divifion of Compound Quantities, being Integers.

Compound quantities divided.

23. There may be three different cafes, or combinations, in the divifion of compound quantities; the firft is, when the quantity to be divided is compound, and the divifor is fimple; the fecond is on the contrary, when the divifor is compound, and the dividend fimple; the third is when they are both compound quantities. As to the two firft cafes, it will fuffice to make ufe of the rule for fimple quantities. In the firft cafe every term of the quantity propofed is divided by the divifor, and there will arife either integers or fractions, as follows from the nature of divifion of fimple quantities. Thus if we are to divide $aa + ab - ac$ by a, we fhall have for the quotient $a + b - c$. If we are to divide $4ab - 6bc + xx$ by $2b$, we fhall have $2a - 3c + \frac{xx}{2b}$. If we are to divide $4ab - cc + 3xx$ by $3c$, we fhall have $\frac{4ab - cc + 3xx}{3c}$, or elfe $\frac{4ab}{3c} - \frac{c}{3} + \frac{xx}{c}$. In the fecond cafe the divifor is wrote under the dividend, as is ufual in fractions; and if in every term of the numerator and denomi-

nator there ſhall be any common quantity, it may be cancelled; then what remains will always be a fraction. Thus dividing $3a^3b$ by $aa - ax + ab$, the quotient will be $\frac{3aab}{a-x+b}$. And if we divide $6a^4$ by $2aa - 2ax + 2xx$, the quotient will be $\frac{3a^4}{aa - ax + xx}$.

24. In the third caſe it is neceſſary, firſt to put the terms of the dividend in order, and likewiſe of the diviſor, in reſpect to ſome certain letter which ſhall be thought the moſt proper for that purpoſe. This is done by writing that for the firſt term of the dividend, and alſo of the diviſor, in which that letter is found of the higheſt power, or of moſt dimenſions. Then making that the ſecond term, in which that letter is of the next greateſt power. And ſo ſucceſſively till we come to thoſe terms, which are not affected by that letter at all, which therefore muſt be made the laſt. Thus the quantity $a^3 + 2a^2c - a^2b - 3abc + bbc$ will be ordered in reſpect of the letter a, and alſo the diviſor $a - b$. If we would diſpoſe this in order, in reſpect of the letter b, it muſt be done thus; $b^2c - 3abc - a^2b + a^3 + 2a^2c$; and the diviſor thus, $-b + a$. Proceſs of diviſion.

This ſuppoſed, the diviſion muſt be performed after this manner. The firſt term of the dividend muſt be divided by the firſt term of the diviſor, and the quotient muſt be written on one ſide. By this quotient the whole diviſor muſt be multiplied, and the product ſubtracted from the dividend. When the ſubtraction is made, and the terms reduced, in the ſame manner the firſt term of the remainder muſt be divided by the firſt term of the diviſor, and this term of the quotient muſt be wrote after the other, with ſuch ſign as it ought to have. Then the whole diviſor muſt be multiplied by this ſecond quotient, and the product ſubtracted from the dividend, that is from the firſt remainder. And proceeding in this manner, the calculation muſt be repeated, till at laſt there is no remainder. Then the ſum of all theſe quotients, thus found by parts, will be the whole quotient of the diviſion.

Let it be required to divide $a^3 + 2a^2c - a^2b - 3abc + b^2c$ by $a - b$. Let the quantity to be divided be wrote at A, the diviſor at B. Now dividing a^3 by a, the quotient will be a^2, which is written at D. Then finding the product of the quotient into the diviſor, and ſubtracting it from the dividend, there will be left the firſt remainder, as at M. Then dividing the firſt term $2aac$ in this remainder M by the ſaid firſt term of the diviſor a, and writing the quotient $2ac$ after the other at D, we muſt ſubtract the product of $2ac$ into the diviſor B, and we ſhall have the ſecond remainder N. Divide the firſt term $-abc$ of this ſecond remainder by the ſame term a of the diviſor, and write the quotient $-bc$ at D after the other. The product of $-bc$ into the diviſor muſt be ſubtracted from the ſecond remainder, and nothing will now remain. Therefore the compleat quotient will be $aa + 2ac - bc$.

A. a^3

A.	$a^3 + 2a^2c - a^2b - 3abc + b^2c$	B. $a - b$
M.	$2a^2c - 3abc + b^2c$	D. $aa + 2ac - bc.$
N.	$- abc + b^2c$	

Let $a^3 - 3a^2b + 3ab^2 - b^3$ be to be divided by $a - b$. Let the dividend be wrote at A, and the divifor at B. Let the firft term a^3 be divided by a, and the quotient aa be wrote at D. Then finding the product of the quotient into the divifor, and fubtracting it from the dividend, there will be left the firft remainder M. Let the firft term of this remainder, that is $- 2a^2b$, be di-divided by the fame firft term of the divifor a, and let the quotient $- 2ab$ be wrote after the other at D. Then let the product of $- 2ab$ into the divifor be fubtracted from the firft remainder M, and we fhall have the fecond remainder N. If we divide the firft term ab^2 of this fecond remainder by the fame firft term of the divifor a, the quotient bb muft be wrote at D after the other. Then let the product of bb into the divifor B be fubtracted from the fecond remainder N, and nothing will remain; fo that the whole quotient will be $aa - 2ab + bb$.

A.	$a^3 - 3a^2b + 3ab^2 - b^3$	B. $a - b$
M.	$- 2a^2b + 3ab^2 - b^3$	D. $aa - 2ab + bb$
N.	$+ ab^2 - b^3$	

Another Example.

A.	$2aa + 5ab + 2bb - ac - 2bc$	B. $a + 2b$
M.	$+ ab + 2bb - ac - 2bc$	D. $2a + b - c$
N.	$- ac - 2bc$	

Another.

A.	$9d^4 + 12d^3e - 4de^3 - e^4$	B. $3d^2 - e^2$
M.	$12d^3e + 3d^2e^2 - 4de^3 - e^4$	D. $3d^2 + 4de + e^2$
N.	$3d^2e^2 - e^4$	

Another.

A.	$4a^2 + 4ab - 2ac + b^2 - c^2$	B. $2a + b$
M.	$2ab - 2ac + b^2 - c^2$	D. $2a + b - c$
N.	$- 2ac - c^2$	
O.	$bc - c^2$	

Now here it is to be obferved, that the laft remainder at O is not divifible by $2a$, and confequently the operation cannot proceed, but it muft remain as a fraction $\frac{bc - cc}{2a + b}$. That is to fay, that the quantity propofed is not entirely

 divifible

divisible by $2a + b$, but only in part, and therefore the quotient will be partly an integer, and partly a fraction, as $2a + b - c + \frac{bc - cc}{2a+b}$. Or the whole may be wrote as a fraction thus, $\frac{4aa + 4ab - 2ac + bb - cc}{2a + b}$.

Extraction of the Roots of Compound Quantities, being Integers.

25. As in simple quantities, so in compound; the root of any quantity is that, which being multiplied into itself, if once produces the given square, if twice produces the given cube, and so on. Roots how to be extracted; particularly the square-root.

The manner of extracting the square-root in compound quantities is as follows: It being first understood, that the terms must be disposed in order, according to some one of it's letters, agreeably to the caution before given, § 24.

Let the given quantity be $a^2 + 2ab + b^2$, whose root is to be extracted, and let it be wrote down as at A. Extract the square-root of the first term a^2, which will be a, and let it be wrote as at B. The square of this, or a^2, must be subtracted from the quantity proposed, A, and the remainder wrote down at D. Then the quantity a, wrote down at B, must be doubled, and wrote as at M, which will be $2a$. By this quantity $2a$ the first term at D must be divided, and the quotient b wrote at B. Then the divisor $2a$ must be multiplied by the quotient b, and the product subtracted from the quantity D; and moreover the square of b must be subtracted from the same; and as there is no remainder, the root required will be $a + b$.

A.	$a^2 + 2ab + b^2$	B.	$a + b$
D.	$2ab + b^2$	M.	$2a$

Let the quantity given be $a^4 + 6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$; let it be wrote at A, and let the square root of the first term be extracted, which is a^2, and let this root be wrote at B. Let the square of a^2 be subtracted from the quantity A, and there will remain the quantity D. Let a^2 be doubled and wrote at M, and by this double, that is by $2a^2$, let the first term be divided of the first remainder D, and the quotient $3ab$ be wrote at B. Then subtracting the product of $3ab$ into the divisor $2aa$, as also the square of $3ab$, from the first remainder D, there will be left the second remainder H. Let the whole quantity B be doubled, and wrote at G. By it's first term let the first term of H be divided, and the quotient $-2b^2$ be wrote at B. Then subtracting the product of the quotient into the divisor G, and also the square of the same quotient,

quotient, from the quantity H; and, as there is no remainder, the quantity written at B, that is, $aa + 3ab - 2bb$, will be the root required.

A.	$a^4 + 6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$	B.	$a^2 + 3ab - 2b^2$
D.	$6a^3b + 5a^2b^2 - 12ab^3 + 4b^4$	M.	$2a^2$
H.	$-4a^2b^2 - 12ab^3 + 4b^4$	G.	$2a^2 + 6ab$

The Operation of another Example.

A.	$y^4 + 4ay^3 - 8a^3y + 4a^4$	B.	$y^2 + 2ay - 2a^2$
D.	$4ay^3 - 8a^3y + 4a^4$	M.	$2y^2$
H.	$-4a^2y^2 - 8a^3y + 4a^4$	G.	$2y^2 + 4ay$

Another Example.

A.	$16a^4 - 24a^2x^2 - 16a^2b^2 + 12b^2x^2 + 9x^4$	B.	$4a^2 - 3x^2 - 2b^2$
D.	$-24a^2x^2 - 16a^2b^2 + 12b^2x^2 + 9x^4$	M.	$8a^2$
H.	$-16a^2b^2 + 12b^2x^2$	G.	$8a^2 - 6x^2$
K.	$-4b^4$		

In this laſt operation there is a remainder of $-4b^4$, which cannot be divided by $8a^2$, as the method requires, which in this caſe cannot take place. That is to ſay, that the ſquare-root of the propoſed quantity cannot be actually extracted, and therefore we muſt make uſe of the radical ſign, as above at § 16; which expedient muſt alſo be applied in other extractions, as the cube-root, the biquadratick-root, &c. Thus $\sqrt{aa+bb}$ repreſents the ſquare-root of $aa + bb$; and $\sqrt[3]{aab-abb}$ will ſtand for the cubic root of $aab - abb$; and the like for other roots.

The cube-root extracted.

26. As to the cube-root, let it be required to extract the root of the quantity $a^3 + 3a^2b + 3ab^2 + b^3$, as is written below at A. Extract the cube-root of the firſt term a^3, which is a, and is written at B. Let the cube of this, or a^3, be ſubtracted from the given quantity A, and let the remainder be written at D. Then take the triple of the ſquare of a, which is $3aa$, and let it be wrote at M, by which divide the firſt term of the remainder D, and let the quotient b be wrote at B. By this multiply the diviſor $3aa$, and the product, together with the triple of the ſquare of b into a, and the cube of b, muſt be ſubtracted from the remainder D. And as nothing remains, $a + b$ will be the root required.

A.	$a^3 + 3a^2b + 3ab^2 + b^3$	B.	$a + b$
D.	$3a^2b + 3ab^2 + b^3$	M.	$3aa$

Let it be required to extract the cube-root of the quantity $z^6 + 6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$.

Extract

Extract the root of the first term z^6, which is z^2, and let it be wrote at B. Let the cube of B be subtracted from the proposed quantity A, and let the remainder be wrote at D. Take the triple of the square of B, and write it at M, and by that divide the first term of the remainder D, and write the quotient $2bz$ at B. Then subtract the product of $2bz$ into the quantity M, and moreover the triple of the square of $2bz$ multiplied into zz, with the cube of $2bz$, from the remainder D, and write the remainder at H. Then find the triple of the square of B, which write in G, and by the first term divide the first term of the remainder H, and write the quotient $-4bb$ in B. Then multiply this quotient by the quantity G, and the product, together with the triple of the square of $-4bb$ into $2z + 2bz$, and the cube of $-4bb$ must be subtracted from the quantity H, and nothing will remain. Whence the cube-root of the quantity proposed will be the whole quantity B, that is, $zz + 2bz - 4bb$.

A.	$z^6 + 6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$	B.	$z^2 + 2bz - 4b^2$
D.	$6bz^5 - 40b^3z^3 + 96b^5z - 64b^6$	M.	$3z^4$
H.	$-12b^2z^4 - 48b^3z^3 + 96b^5z - 64b^6$	G.	$3z^4 + 12bz^3 + 12b^2z^2$

After the same manner is extracted the cube-root of the following quantity.

A.	$27y^6 - 54cy^5 + 144c^2y^4 - 152c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$
D.	$-54cy^5 + 144c^2y^4 - 152c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$
H.	$108c^2y^4 - 144c^3y^3 + 192c^4y^2 - 96c^5y + 64c^6$
B.	$3y^2 - 2cy + 4c^2$
M.	$27y^4$
G.	$27y^4 - 36cy^3 + 12c^2y^2$

27. For the fourth root. Let the quantity proposed be $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, of which we would extract the biquadratick or fourth root. Let it be wrote at A, and extract the fourth root of the first term, which is a, and write it at B. Subtract the fourth power of B from the quantity A, and write the remainder at D. Then find the quadruple of the cube of a, and write it at M. By this must be divided the first term of the quantity D, and the quotient b must be wrote at B. From the quantity D must be subtracted the product of the quotient b into the divisor $4a^3$, and moreover the sextuple of the square of b into the square of a, and the product of the quadruple of the cube of b into the quantity a, and lastly the biquadrate of b. And as there is no remainder, the root required will be $a + b$.

The fourth root extracted.

A.	$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	B.	$a + b$
D.	$4a^3b + 6a^2b^2 + 4ab^3 + b^4$	M.	$4a^3$

 28. As

The fifth and higher roots extracted.

28. As to the fifth root; in order to difcover in what manner the operations proceed, which are to be made in the extraction, it will be fufficient to form the fifth power of a binomial, fuppofe of $a + b$, which will give a rule here; as the fecond, third, and fourth powers of the fame binomial have fupplied us with rules for the extraction of the fecond, third, and fourth roots. The like obtains in the fixth, feventh, and other roots.

Of Fractions, Simple and Compound.

Notation of fractions.

29. We have feen before, how fractions or broken numbers arife from the divifion of quantities. Therefore a fraction infinuates a divifion that is to be made, of the numerator by the denominator. Whence it proceeds, that if the numerator is the fame as the denominator, as $\frac{a}{a}$, or $\frac{aa - bb}{aa - bb}$, and fuch like, thofe fractions can fignify nothing elfe but unity; becaufe in fact, if we divide a by a, or $aa - bb$ by $aa - bb$, the quotient will be unity. And becaufe multiplication is an operation contrary to divifion, it is plain, that any integer whatever may be reduced to a fraction with what denominator we pleafe, if it is multiplied by the quantity which is to be the denominator, and then divided by it again. Thus to reduce the integer a to a fraction with the denominotor b, we muft write $\frac{ab}{b}$. To reduce $a - b$ to a fraction with the denominator d, we muft write $\frac{ad - bd}{d}$. To reduce $a + b$ to a fraction whofe denominator fhall be $c - d$, we muft write $\frac{\overline{a+b} \times \overline{c-d}}{c-d}$, or $\frac{ac + bc - ad - bd}{c - d}$.

Reduction of Fractions to more fimple Expreffions.

How fractions are to be reduced.

30. When fractions have the fame letter or letters in every term of the numerator and denominator, it will be fufficient to expunge the common letters in both; having regard to their powers, as is faid in Divifion, at § 10. Thus $\frac{a^3b^2}{ac}$ will become $\frac{a^2b^2}{c}$; $\frac{ab^3}{abc}$ will be $\frac{bb}{c}$; $\frac{a^3b - x^3b}{ab - bb}$ will be $\frac{a^3 - x^3}{a - b}$. But though there are not the fame letters in both the numerator and denominator, yet if each of them is multiplied by the fame compound quantity, they may be

 divided

divided by it again, and consequently the fraction may be reduced. Thus $\frac{aac - aad}{cd - dd}$, that is $\frac{aa \times \overline{c-d}}{d \times \overline{c-d}}$, will be reduced to $\frac{aa}{d}$. So $\frac{\overline{aa+2ab}|^2}{aab+2abb}$, that is $\frac{\overline{aa+2ab} \times \overline{aa+2ab}}{b \times \overline{aa+2ab}}$, will be reduced to $\frac{aa + 2ab}{b}$. So $\frac{aac - aad - acd + add}{cd - dd}$, or $\frac{\overline{aa-ad} \times \overline{c-d}}{d \times \overline{c-d}}$, will be reduced to $\frac{aa - ad}{d}$.

Therefore in general, as often as the fraction is such, that it's numerator and denominator are both divisible by one and the same quantity, which in this case is called their common divisor, by actually dividing both, the two quotients will give the fraction reduced. But it must be observed, that, if that common divisor is not the greatest that can be, the fraction indeed will be reduced, but not to the simplest expression. Thus the fraction $\frac{a^3 - abb}{aac + abc}$, that is $\frac{a \times \overline{a+b} \times \overline{a-b}}{a \times c \times \overline{a+b}}$, may be divided, both as to it's numerator and denominator, by a, by $a + b$, and by $aa + ab$, the greatest of which divisors is $aa + ab$. And as the fraction should be reduced to it's least terms, we must divide it by $aa + ab$, and the quotient or fraction reduced will be $\frac{a - b}{c}$. But very often it will be difficult to know if there is a common divisor, and what it is; and therefore we shall give a rule to find it, at § 36. afterwards. At present we shall omit it, that we may not too much discourage young learners, as yet not sufficiently confirmed, and shall proceed to other operations; making use of fractions that are any how reduced to lower and simpler expressions.

Reduction of Fractions to a Common Denominator.

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31. If two fractions are given, let the numerator of the first be multiplied by the denominator of the second, and the numerator of the second be multiplied by the denominator of the first, and each product be divided by the product of the two denominators. Thus $\frac{a}{b} + \frac{x}{y}$ will be $\frac{ay + bx}{by}$; and $\frac{a^3}{y^2} - \frac{2x^2}{3b}$ will be $\frac{3a^3b - 2x^2y^2}{3by^2}$. Also $\frac{aa + xx}{m+n} - \frac{aa}{m}$ will be $\frac{ma^2 + mx^2 - ma^2 - na^2}{mm + mn}$, that is $\frac{mxx - naa}{mm + mn}$. But here we must take notice, that as often as the two denominators of the fractions have a greatest common divisor, in this case the multiplication

Fractions reduced to a common denominator.

plication of the numerators into that common divifor is fuperfluous, and alfo of thofe common divifors into each other, for forming a new denominator; for then it may be neceffary to reduce the fractions to more fimple expreffions. Wherefore the faid numerators fhould be multiplied, not by the denominators, but by the quotients which will refult by dividing the faid denominators by their common divifors: and the denominator will be the product of thofe quotients, and of the faid common divifor. For example, let there be given $\frac{a^3}{mn} + \frac{abb}{mx}$. Being reduced as ufual to a common denominator, it will be $\frac{ma^3x + mnabb}{mmnx}$; that is $\frac{a^3x + nabb}{mnx}$. Therefore it was needlefs to multiply the numerators by m, the common divifor of the denominators, as it was fuperfluous to multiply the denominators together. It was fufficient to multiply a^3 into x, and abb into n, to form the numerators, and to multiply m into n into x, to form the common denominator. Thus to reduce to a common denominator the fractions $\frac{a^3 - b^3}{\overline{a+b}|^2} - \frac{aa}{a+b}$, it will be enough to multiply $-\frac{aa}{a+b}$ into $a + b$, and it will be $\frac{a^3 - b^3 - a^3 - aab}{\overline{a+b}|^2}$, that is $\frac{-b^3 - aab}{\overline{a+b}|^2}$. In like manner to reduce to a common denominator the fractions $\frac{b^4}{a^2c - a^2d} + \frac{a^3 + b^3}{cd - dd}$; becaufe $c - d$ is a common divifor of both the denominators, it will fuffice to multiply b^4 by d, and $a^3 + b^2$ by a^2, as to the numerators; and to multiply a^2 into d into $c - d$, as to the denominator, and therefore it will be $\frac{b^4d + a^5 + a^2b^3}{a^2cd - a^2d^2}$.

If three fractions are to be reduced to a common denominator, let the two firft be reduced, then that which refults from thefe with the third; and fo on fucceffively if there are more. So to reduce thefe to a common denominator, $\frac{a}{b} + \frac{c}{d} - \frac{m}{n}$, let the two firft be reduced, and we fhall have $\frac{ad + bc}{bd}$. Let this be reduced with the third, and we fhall have $\frac{adn + bcn - bdm}{bdn}$. This may alfo be done in refpect to integers; for whereas any integer may be confidered as a fraction, having unity for it's denominator, we may proceed after the fame manner as before. Thus $2aa + \frac{3x^4 - 2y^4}{3x^2 - 8ax}$, that is $\frac{2aa}{1} + \frac{3x^4 - 2y^4}{3x^2 - 8ax}$, will be $\frac{6a^2x^2 - 16a^3x + 3x^4 - 2y^4}{3xx - 8ax}$.

Addition

Addition and Subtraction of Fractions.

32. Fractions are added by writing them one after another with the fame figns. And on the contrary they are fubtracted by changing the figns of the quantities to be fubtracted. And the fame things muft be done, if there are integers with the fractions. Thus to add $\frac{aa}{c}$ to $\frac{bb}{c}$, they are wrote $\frac{aa+bb}{c}$. To add $\frac{aa}{c}$ to $\frac{xx}{m} - y$, it muft be wrote $\frac{aa}{c} + \frac{xx}{m} - y$; which afterwards (if we pleafe) may be reduced to a common denominator, and then it will be $\frac{aam + cxx - cmy}{cm}$. To add $\frac{aab^4}{a^4 - 2a^2b^2 + b^4}$ to $\frac{aabb}{aa-bb}$, the fum will be $\frac{aab^4}{a^4-2a^2b^2+b^4} + \frac{a^2b^2}{aa-bb}$, which if we would further reduce to a common denominator, we may obferve, that the denominator of the firft is the fquare of $aa - bb$; therefore the two denominators have a greateft common divifor $aa - bb$, by which being divided, the quotients will be $aa - bb$ in the firft, and unity in the fecond. Wherefore it will be enough to multiply the numerator of the fecond fraction by $aa - bb$, and to divide the whole by $a^4 - 2a^2b^2 + b^4$, and the fum required will be $\frac{a^2b^4 + a^4bb - a^2b^4}{a^4 - 2aabb + b^4}$, that is $\frac{a^4bb}{\overline{aa-bb)}^2}$. To fubtract $\frac{bb}{c}$ from $\frac{aa}{c}$, it will be wrote $\frac{aa-bb}{c}$. To fubtract $a - \frac{xx}{m}$ from $\frac{yy}{m-n}$, it will be wrote $\frac{yy}{m-n} - a + \frac{xx}{m}$, which being reduced to a common denominator, if we think fit, will be $\frac{myy - amm + amn + mxx - nxx}{mm - mn}$. To fubtract $\frac{b^4}{4a^2c - 4a^2d}$ from $\frac{a^3+b^3}{2cd-2dd}$, it muft be wrote $\frac{a^3+b^3}{2cd-2dd} - \frac{b^4}{4a^2c-4a^2d}$; and to reduce it to a common denominator, we muft multiply $a^3 + b^3$ by $2aa$, and $-b^4$ by d, and the whole muft be divided by $4aacd - 4aadd$; then it will be $\frac{2a^5 + 2aab^3 - b^4d}{4aacd - 4aadd}$.

Fractions how added and fubtracted.

Multiplication of Fractions.

33. The numerators muft be multiplied into one another, and alfo the denominators, and the new fraction will be the product of the fractions to be multiplied. Thus to multiply $\frac{ac}{b}$ into $\frac{bc}{d}$, the product will be $\frac{abcc}{bd}$, which is

Fractions how multiplied.

reduced

reduced to $\frac{acc}{d}$. To multiply $\frac{2ab}{b+c}$ into $\frac{3aa-bb}{5c}$, it will be wrote thus, $\frac{6a^3b-2ab^3}{5bc+5cc}$. The ſame muſt be done if there are integers with them, by conſidering an integer as a fraction, the denominator of which is unity. Thus to multiply $2a$, or $\frac{2a}{1}$, into $\frac{xx-3yy}{3x}$, the product will be $\frac{2axx-6ayy}{3x}$.

Let it be required to multiply $\frac{aa+bb}{a-b}$ into $a-b$. In this and the like caſes, becauſe the quantity which ought to multiply is the ſame as the denominator of the fraction, it will be ſufficient to expunge the denominator, and then the product will be $aa+bb$. If $aa-bb$ is to be multiplied into $\frac{aa-ab}{a+b}$, it may be obſerved, that $aa-bb$ is the ſame as $\overline{a+b}\times\overline{a-b}$, and therefore ſince it would be required to multiply $aa-ab$ into $a+b$ into $a-b$, and afterwards to divide by $a+b$; and becauſe $a+b$ would be a common diviſor both of the numerator and the denominator which would thence ariſe; the multiplication and diviſion by the ſame $a+b$ may be omitted, and it would be ſufficient to multiply the numerator by $a-b$, and the product will be $a^3-2aab+abb$. Thus the product of $\frac{a^3-abb}{xx-yy}$ into $\frac{a^3}{aa-bb}$ will be $\frac{a^4}{xx-yy}$.

Diviſion of Fractions.

Fractions how divided.

34. The Diviſion of Fractions is performed by multiplying croſs-wiſe, that is, by multiplying the numerator of the dividend by the denominator of the diviſor, which product muſt be the numerator of the fraction which is to be the quotient: and then multiplying the denominator of the dividend into the numerator of the diviſor, which product will be the denominator of the quotient. This quotient, if there is occaſion, muſt afterwards be reduced to the moſt ſimple expreſſion. Let it be required to divide $\frac{ab}{c}$ by $\frac{m}{n}$; the quotient will be $\frac{abn}{cm}$. Divide $\frac{ab}{c}$ by $\frac{-m}{n}$; the quotient will be $\frac{abn}{-cm}$, or $\frac{-abn}{cm}$; which is all one by § 13. Let it be required to divide $\frac{a^3-b^3}{a+b}$ by $\frac{aa-ab+bb}{c}$; it will be $\frac{a^3c-b^3c}{a^3+b^3}$.

It

It is eaſy to perceive, that if the two fractions, the dividend and diviſor, ſhall have the ſame denominator, it would be needleſs to multiply them croſs-wiſe. As if we were to divide $\frac{aa}{m}$ by $\frac{c-d}{m}$, in this caſe it would be enough to divide aa by $c-d$. For by multiplying croſs-wiſe it would be $\frac{aam}{cm-dm}$, and then reducing it to it's leaſt terms, it would be $\frac{aa}{c-d}$. Thus dividing $\frac{a^3-ab^2}{c-d}$ by $\frac{aa+2ab+bb}{c-d}$, the quotient would be $\frac{a^3-abb}{aa+2ab+bb}$; but by reduction, becauſe the numerator is $a\times\overline{a+b}\times\overline{a-b}$, and the denominator is $\overline{a+b}\times\overline{a+b}$, it will become $\frac{aa-ab}{a+b}$. After the ſame manner we muſt proceed when we are to divide an integer by a fraction, or a fraction by an integer; conſidering an integer as a fraction whoſe denominator is unity. Thus dividing the quantity $aa-xx$, or $\frac{aa-xx}{1}$, by $\frac{2yy-3xy}{3a}$, the quotient will be $\frac{3a^3-3axx}{2yy-3xy}$. And ſo of others.

Extraction of the Roots of Fractions.

35. The root of a fraction is extracted by extracting the root of the numerator, and then of the denominator, and the new fraction ariſing ſhall be the root of the fraction propoſed. So the ſquare-root of $\frac{aabb}{cc}$ will be $\frac{ab}{c}$. The ſquare-root of $\frac{a^4-2aabb+b^4}{aa+4ab+4bb}$ will be $\frac{aa-bb}{a+2b}$. The ſquare-root of $4aa+\frac{64xx-160ax}{25}$, that is of $\frac{100aa-160ax+64xx}{25}$, will be $\frac{10a-8x}{5}$. The ſame is to be underſtood of the cube-root, the biquadratick-root, and all others. Roots of fractions how extracted.

But now if the root cannot be extracted out of both the numerator and denominator, yet poſſibly it may be extracted out of one of the two. Let it be extracted out of which of the two it can, and before the other let the radical ſign be placed. Thus the cube-root of $\frac{a^6}{a^3-x^3}$ will be $\frac{aa}{\sqrt[3]{a^3-x^3}}$. The cube-root of $\frac{a^2x-x^3}{a^3b^3}$ will be $\frac{\sqrt[3]{aax-x^3}}{ab}$. And if the root cannot be extracted neither

neither out of the numerator nor denominator, then the whole fraction muſt be included under the radical ſign. Thus the ſquare-root of $\frac{x^4 - a^4}{xx + bx}$ will be $\sqrt{\frac{x^4 - a^4}{xx + bx}}$.

Of the greateſt Common Diviſor of Two Quantities, or Formulas.

Greateſt common diviſor how found.

36. By a Formula I mean any analytical expreſſion whatever, whether complicate or not, the letters of which repreſenting indeterminate quantities, may be what we pleaſe; provided that whatever may be ſaid of that formula is to be underſtood as ſaid of any other, compounded of other letters, but ſimilar to the firſt.

To obtain the greateſt common diviſor of two quantities or formulas; in the firſt place it muſt be obſerved, that if every term of both is multiplied into the ſame quantity or number, in this caſe they muſt be divided by that quantity. Then each of the formulas muſt be ſet in order according to any letter at pleaſure; that is, that muſt be made the firſt term, in which that letter ariſes to the moſt dimenſions, and then the others in order. Let the two given formulas be $18a^3bx - 8a^4b - 3abx^3 - 8a^2bx^2 + bx^4$, and $6a^3b + bx^3 - abx^2 - 8a^2bx$; which becauſe they are diviſible by the letter b, let them be ſo divided, and then ſet in order (if you pleaſe) according to the letter x. They will be thus, $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$, and $x^3 - ax^2 - 8a^2x + 6a^3$. This being done, the firſt term, or that wherein the letter is of moſt dimenſions by which the terms are ſet in order, muſt be divided by the like term in the ſecond, namely x^4 divided by x^3 will give x in the quotient. Then the product of this quotient into the diviſor muſt be ſubtracted from the dividend, and we ſhall have the firſt remainder $-2ax^3 + 12a^3x - 8a^4$, which muſt be reduced to the moſt ſimple expreſſion, (as ought always to be done,) by dividing by $-2a$; then the remainder will be $x^3 - 6a^2x + 4a^3$. And becauſe the dimenſion of x in this remainder is the ſame as in the diviſor, by the ſaid diviſor this remainder muſt be divided; from whence in like manner muſt be ſubtracted the product of the quotient into the diviſor, and we ſhall have a ſecond remainder $ax^2 + 2a^2x - 2a^3$, or dividing by a it will be $x^2 + 2ax - 2a^2$. Now becauſe in this remainder the dimenſion of x is leſs than in the diviſor, the order muſt be inverted, and this remainder muſt be made the diviſor, and the firſt diviſor the dividend. And making the diviſion, the product of the quotient into the ſecond diviſor muſt be ſubtracted from the ſecond dividend, that is from $x^3 - ax^2 - 8a^2x + 6a^3$, and the remainder will be $-3ax^2 - 6a^2x + 6a^3$, which dividing by $-3a$ is $x^2 + 2ax - 2a^2$. Now whereas this laſt remainder is the ſame as the diviſor, it will be the greateſt common diviſor divifor of the two formulas $x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4$, and $x^3 - ax^2 - 8a^2x + 6a^3$; which being multiplied into b, or $bx^2 + 2abx - 2a^2b$, will be the greateft common divifor of the two formulas at firft propofed.

Let the two formulas be $x^4 - 4ax^3 + 11a^2x^2 - 20a^3x + 12a^4$, and $x^4 - 3ax^3 + 12a^2x^2 - 16a^3x + 24a^4$, being ordered according to the letter x. And as this is of the fame dimenfions in both, we are at liberty to take which of them we pleafe for the divifor. Let the firft therefore be divided by the fecond, and fubtracting the product of the quotient into the divifor from the dividend, the firft remainder will be $- ax^3 - a^2x^2 - 4a^3x - 12a^4$, which being divided by $-a$ is $x^3 + ax^2 + 4a^2x + 12a^3$. Here inverting the order, let this remainder be taken for the divifor, and the firft divifor for the dividend. Then making the divifion, and fubtracting the product of the quotient into this fecond divifor from the fecond dividend, the fecond remainder will be $- 4ax^3 + 8a^2x^2 - 28a^3x + 24a^4$, which being divided by $- 4a$ will be $x^3 - 2ax^2 + 7a^2x - 6a^3$. By the fame fecond divifor let the divifion of this fecond remainder be continued, and making the fubtraction as ufual, we fhall have a third remainder $- 3axx + 3a^2x - 18a^3$, or dividing by $- 3a$ it will be $x^2 - ax + 6a^2$. Let the order be again inverted, and let the fecond divifor be divided by this third remainder $x^3 + ax^2 + 4a^2x + 12a^3$, and making the fubtraction as ufual, the remainder will be found to be $2ax^2 - 2a^2x + 12a^3$; or dividing by $2a$, it will be $xx - ax + 6aa$, the fame quantity as that which was a divifor before, and which is therefore the greateft common divifor of the two propofed quantities.

Let the two formulas be $f^4 - aaff - bbff + aabb$, and $f^3 - aff - 2abf + 2a^2b$, which are ordered according to the letter f. Let the firft be divided by the fecond, and the product of the quotient into the divifor being fubtracted from the dividend, will give the firft remainder $af^3 - a^2f^2 + 2abff - bbff - 2a^2bf + a^2b^2$. And if we go on to divide by the fame divifor, and the product of the divifor into the quotient being fubtracted from the dividend, we fhall have a fecond remainder $2abff - b^2ff - 2a^3b + a^2b^2$, or dividing by b it will be $2aff - bff - 2a^3 + a^2b$. Then invert the order, and divide the firft divifor by this fecond remainder, and taking the product of the quotient $\frac{f}{2a-b}$ into the faid remainder, which has now ferved as a divifor, and then making the fubtraction, we fhall have a third remainder $- aff + a^2f - 2abf + 2a^2b$, or dividing by $- a$, it is $ff - af + 2bf - 2ab$. The divifion is to be continued in the fame order, and the product of the quotient $\frac{1}{2a-b}$ into the divifor $2aff - bff + a^2b - 2a^3$ being fubtracted, we fhall have a fourth remainder $- af + 2bf - 2ab + a^2$, by which, inverting the order, the third remainder muft be divided, and the product of the quotient $\frac{f}{2b-a}$ into the divisor being subtracted, we shall have a fifth remainder $2bf - 2ab$, or dividing by $2b$, it is $f - a$. Now if this is divided by the fourth remainder $-af + 2bf - 2ab + a^2$, and the product of the quotient $\frac{1}{2b-a}$ into the divisor is subtracted, nothing will remain. Whence if by the denominator of the last quotient, it being a fraction, the last divisor $-af + 2bf - 2ab + a^2$ shall be divided, the quotient will be $f - a$, the greatest divisor of the two quantities proposed. But because it was at pleasure whether we chose for a divisor that which was made the dividend, or *vice versâ*; that is, we might have divided $-af + 2bf - 2ab + aa$ by $f - a$; let the division be actually made, and the quotient will be $2b - a$ without a remainder; and therefore $f - a$ will be the greatest common divisor, as found above by means of the other division.

Wherefore two formulas may have a greatest common divisor, though being ordered according to some certain letter, it cannot be found in this manner; in which case it must be set in order again, according to some other of it's letters. Now if this be tried by setting it in order according to any other letter, and if it will not then succeed, the quantities proposed will have no greatest common divisor. Thus it would not be found in the last example, by setting them in order according to the letter b; which however is found by ordering them according to the letter f.

Now the fraction $\frac{x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4}{x^3 - ax^2 - 8a^2x + 6a^3}$ being given, if we divide the numerator and denominator by $x^2 + 2ax - 2a^2$, we shall have the fraction $\frac{x^2 - 5ax + 4a^2}{x - 3a}$.

Also the fraction $\frac{x^4 - 4ax^3 + 11a^2x^2 - 20a^3x + 12a^4}{x^4 - 3ax^3 + 12a^2x^2 - 16a^3x + 24a^4}$, by dividing by $x^2 - ax + 6a^2$, will become $\frac{x^2 - 3ax + 2a^2}{x^2 - 2ax + 4a^2}$.

And the fraction $\frac{f^4 - a^2f^2 - b^2f^2 + a^2b^2}{f^3 - af^2 - 2abf + 2a^2b}$, by dividing by $f - a$, will become $\frac{f^3 + af^2 - b^2f - ab^2}{f^2 - 2ab}$.

Thus these fractions are reduced to more simple expressions, as is said above at § 30.

Reduction

Reduction of Irrational Quantities to more ſimple Expreſſions.

37. It has been obſerved already, how irrational quantities ariſe, which are otherwiſe called Surds, or Radicals. For when the root required cannot be actually extracted, then we have recourſe to a *radical vinculum*, which inſinuates it. But it often happens that the quantity under the vinculum is the product of two factors, one of which is a true power of the ſame name as the root required. As if it were $\sqrt{aabc}$, or $\sqrt{a^2b - a^2x}$; the firſt of which is the product of aa into bc, and the other is the product of aa into $b - x$. Thus alſo $\sqrt[3]{a^3x - a^3y}$ is the cube-root of the product of a^3 into $x - y$. In this caſe the root may be extracted out of ſuch of the factors as will admit it, and wrote without the radical ſign, and the other factor may remain under the ſign. And this is called extracting the root in part, or reducing the radical to a more ſimple expreſſion. Thus $\sqrt{aabc}$ will be reduced to $a\sqrt{bc}$. And $\sqrt{aab - aax}$ will be the ſame as $a\sqrt{b - x}$; $\sqrt[3]{a^3x - a^3y}$ will be reduced to $a\sqrt[3]{x - y}$; and ſo of others. In like manner, becauſe $\sqrt{48aabc}$ is the root of the product of $16aa$ into $3bc$, it will be reduced to $4a\sqrt{3bc}$. Thus, becauſe $\sqrt{\frac{a^3b - 4a^2b^2 + 4ab^3}{cc}}$ is the root of the product of $\frac{aa - 4ab + 4bb}{cc}$ into ab, and the root of $\frac{aa - 4ab + 4bb}{cc}$ is $\frac{a - 2b}{c}$; the root reduced will be $\frac{a - 2b}{c}\sqrt{ab}$. Thus the root $\sqrt{\frac{a^2m^2x^2 + 4a^2m^3p}{p^2z^2}}$, when reduced, will be $\frac{am}{pz}\sqrt{x^2 + 4mp}$. And the root $\sqrt[3]{8a^3b + 16a^4}$ will be $2a\sqrt[3]{b + 2a}$. Thus $\sqrt{a^3 - 3a^2b + 3ab^2 - b^3}$, which is the root of the product of $aa - 2ab + bb$ into $a - b$, will be reduced to $\overline{a - b} \times \sqrt{a - b}$. But very often it cannot be known by inſpection only, what are the factors from whence the propoſed radical proceeds. In which caſe we muſt have recourſe to the method of finding all the diviſors, which I ſhall give in it's proper place; and if among theſe ſhall be one, which is exactly a power with the ſame exponent as the radical indicates; the propoſed quantity may then be reduced in the manner now explained.

Surds reduced how.

Reduction of Radicals to the ſame Denomination.

38. Thoſe are called radicals of a different denomination which have a different index or exponent. To reduce them therefore to radicals of the ſame index, we muſt proceed thus. If the index of one of the radicals is an aliquot part of the index of the other, the greater index muſt be divided by the leſſer,

Radicals how reduced to the ſame denomination.

and the quotient ſhows that power, to which the quantities muſt be raiſed which are under the radical of the leſſer index, and to which muſt be prefixed the radical of the greater index. Let it be propoſed to reduce to the ſame index the quantities $\sqrt{\sqrt{ax}}$ and $\sqrt{a}$; or which is the ſame, $\sqrt[4]{ax}$ and $\sqrt[2]{a}$. Becauſe 4 divided by 2 gives 2 for the quotient, therefore the quantity a of the leſſer index muſt be raiſed to it's ſquare, which is aa, and it will be $\sqrt[4]{aa}$, and therefore is reduced to the ſame index or denomination as $\sqrt[4]{ax}$. Thus $\sqrt[6]{a^3b^3 + ab^5}$ and $\sqrt{ab}$ will make $\sqrt[6]{a^3b^3 + ab^5}$ and $\sqrt[6]{a^3b^3}$. But if one of the exponents is not an aliquot part of the other, the leaſt number muſt be found which is diviſible without a fraction by each of the exponents of the given radicals, and this will be the index of the common radical. Then the quantities muſt be raiſed to the next inferior degree of the number, by which the exponents are increaſed of the reſpective radicals, and then to the powers ſo raiſed let the common radical now found be prefixed. Let the two quantities $\sqrt[2]{aq}$ and $\sqrt[3]{aaq}$ be given, to be reduced to a common radical. The leaſt number diviſible by 2 and by 3 will be 6, and therefore $\sqrt[6]{\ }$ will be the common radical. Now, becauſe the index of the ſquare-root is in this caſe increaſed by 4, and that of the cube-root by 3; therefore the firſt will become $\sqrt[6]{a^3q^3}$, and the ſecond will be $\sqrt[6]{a^4qq}$. If the radicals to be reduced are more than two, any two are to be reduced firſt, then the third, and ſo on ſucceſſively.

The manner of reducing rationals to any radical, is plain of itſelf, without the aſſiſtance of rules; by raiſing the rational to any power of the ſame name or index of the radical given, and then prefixing to it the ſame radical.

Addition and Subtraction of Radical Quantities.

Surds how added or ſubtracted.

39. To add them together, the radical quantities are wrote one after another with their proper ſigns. And to ſubtract them, the ſigns of thoſe to be ſubtracted are to be changed, as is done in other quantities. Thus to add $5a\sqrt{bc}$ to $2b\sqrt{bx}$ to $-c\sqrt{zy}$, they muſt be wrote thus, $5a\sqrt{bc} + 2b\sqrt{bx} - c\sqrt{zy}$. To add $5x\sqrt{ab}$ to $3x\sqrt{ab}$ to $y\sqrt{bx}$, they muſt be wrote thus, $5x\sqrt{ab} + 3x\sqrt{ab} + y\sqrt{bx}$; and then reducing like terms, which ought always to be done, they will become $8x\sqrt{ab} + y\sqrt{bx}$. To add $a - b$ to $\sqrt{aa - xx}$, it muſt be wrote $a - b + \sqrt{aa - xx}$. And the ſame is to be done in ſubtraction, having regard to the ſigns.

Multi-

Multiplication of Irrational Quantities.

40. To multiply rational quantities by furds or radicals, the rational is wrote together with the radical, without any fign between, only prefixing to the product fuch fign, whether pofitive or negative, as fhall be required by the common rules of multiplication; and this is to be underftood always to be done. Therefore the product of a into $\sqrt{aa - xx}$ will be $a\sqrt{aa - xx}$. The product of ab into $-\sqrt{ab}$ will be $-ab\sqrt{ab}$. And if the rational quantities or radicals fhall confift of feveral terms, or if they are complicate, every term of one muft be multiplied into every term of the other. Wherefore the product of $aa - xx$ into $\sqrt{xx - yy}$ will be $\overline{aa - xx}\sqrt{xx - yy}$, where it is underftood, that all thofe terms are multiplied into the radical, which are under the *vinculum*. Surds how multiplied.

41. To multiply radicals among themfelves, fuppofing them to be of the fame denomination, or reduced to fuch, the quantities muft be multiplied into each other which are under the radical figns, and to the product muft be put the fame radical vinculum, with fuch a fign, either pofitive or negative, as the common rule requires. Thus to multiply $\sqrt{bc}$ into $\sqrt{xy}$, the product will be $\sqrt{bcxy}$. To multiply $\sqrt{\frac{aa - xx}{x}}$ into $-\sqrt{aa + xx}$, the product will be $-\sqrt{\frac{a^4 - x^4}{x}}$. Surds multiplied by furds.

42. Moreover, if the radicals fhall have rational co-efficients, whether numeral or literal, thofe co-efficients muft be multiplied together, and alfo the radicals together, and the product of the co-efficients muft be put before the radical, without any fign between. Thus $a\sqrt[3]{bbc}$ into $a\sqrt[3]{bxx}$ will be $aa\sqrt[3]{b^3cx^2}$, which reduced is $aab\sqrt[3]{cxx}$. So $2a - \sqrt{aa - xx}$ into $\frac{b}{a}\sqrt{aa + xx}$ will be $2b\sqrt{aa + xx} - \frac{b}{a}\sqrt{a^4 - x^4}$. When they have rational co-efficients.

43. According to this rule, to multiply $m\sqrt{ab}$ into $n\sqrt{ab}$, the product would be $mn\sqrt{aabb}$. But $aabb$ is a fquare whofe root is ab, and therefore the product will be $mnab$. So that, to multiply two like quadratick radicals into each other, it will fuffice to take away the radical vinculum, and the quantities which were under it, multiplied into the product of the co-efficients, will be the total product. Thus $\frac{2b}{a}\sqrt{ax - xx}$ into $-\frac{c}{3}\sqrt{ax - xx}$ will be $-\frac{2bc}{3a} \times \overline{ax - xx}$, that is, $-\frac{2}{3}bcx + \frac{2bcxx}{3a}$. But here it muft be obferved, that if the radicals having Sometimes may become rationals.

having no co-efficients, or unity only, are affected by the ſame ſign, poſitive or negative, the *vinculum* being taken away, the quantities muſt be left with the ſign they have. And if the radicals have contrary ſigns, all the ſigns of the quantity muſt be changed. For example, $\sqrt{\frac{aa-xx}{x}}$ into $\sqrt{\frac{aa-xx}{x}}$, or elſe $-\sqrt{\frac{aa-xx}{x}}$ into $-\sqrt{\frac{aa-xx}{x}}$, will be $\frac{aa-xx}{x}$. Alſo $\sqrt{\frac{aa-xx}{x}}$ into $-\sqrt{\frac{aa-xx}{x}}$ will be $\frac{-aa+xx}{x}$, or $\frac{aa-xx}{-x}$. The reaſon of which is, becauſe $\sqrt{\frac{aa-xx}{x}}$, (and ſo of any other,) is always underſtood to have $+1$ for it's co-efficient, and $-\sqrt{\frac{aa-xx}{x}}$ to have -1. Therefore the product ought to be $1 \times \frac{aa-xx}{x}$ in the firſt caſe, and $-1 \times \frac{aa-xx}{x}$ in the ſecond. Here are other examples of theſe multiplications.

$\sqrt{ab} + \sqrt{aa-xx}$ into $\sqrt{ab} + \sqrt{aa-xx}$ makes the product $ab + \sqrt{a^3b - abx^2} + aa - xx + \sqrt{a^3b - abx^2}$, or $ab + a^2 - x^2 + 2\sqrt{a^3b - abx^2}$.

$x - \sqrt{\frac{\sqrt{4a^4+y^4} - y^2}{2}}$ into $x + \sqrt{\frac{\sqrt{4a^4+y^4} - y^2}{2}}$ makes the product $xx - x\sqrt{\frac{\sqrt{4a^4+y^4} - y^2}{2}} - \frac{\sqrt{4a^4+y^4} - y^2}{2} + x\sqrt{\frac{\sqrt{4a^4+y^4} - y^2}{2}}$, that is, $xx + \frac{1}{2}yy - \frac{\sqrt{4a^4+y^4}}{2}$.

$\sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$ into $\sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$ makes the product $\sqrt[3]{\frac{1}{4}qq + q\sqrt{\frac{1}{4}qq - \frac{1}{27}pp} + \frac{1}{4}qq - \frac{1}{27}pp}$, that is, $\sqrt[3]{\frac{1}{2}qq - \frac{1}{27}pp + q\sqrt{\frac{1}{4}qq - \frac{1}{27}pp}}$.

Rational co-efficients how brought under the vinculum.

44. Becauſe $a\sqrt{ax}$, $\overline{a-b} \times \sqrt{ax-xx}$, and ſuch others, are the products of a rational quantity into a radical, and we already know how to reduce any rational to any radical we pleaſe; we can always make the rational multiplier to paſs under the *vinculum* without any alteration of the quantity. Thus $a\sqrt{a-x}$ will be the ſame as $\sqrt{a^3 - a^2x}$; $\overline{a-b} \times \sqrt{xy}$ will become $\sqrt{a^2xy - 2abxy + b^2xy}$; $ax\sqrt[3]{m-n}$ will be $\sqrt[3]{ma^3x^3 - na^3x^3}$; and ſo of any others.

Different ſurds how multiplied.

45. If the radicals to be multiplied are not of the ſame name, they may be reduced to ſuch, and then the multiplication may be made as before. But very often it will be more commodious to inſinuate it only, without actually performing it, and this by writing one radical after another, without any ſign interpoſed, except the mark of multiplication. Thus $\sqrt{aa-xx} \times \sqrt[3]{xxy}$ will denote the product of theſe two radicals.

Diviſion

Division of Radical Quantities.

46. In every term of the dividend and of the divifor, if the fame radical is found, omitting this, the rational quantities are to be divided as ufual, and what refults will be the quotient. Thus to divide $5a\sqrt{3}$ by $3a\sqrt{3}$, the quotient will be $\frac{5}{3}$. To divide $6\sqrt{a^4 + a^2b^2}$ by $2\sqrt{a^2b^2 + b^4}$, or $6a\sqrt{a^2 + b^2}$ by $2b\sqrt{a^2 + b^2}$, the quotient will be $\frac{3a}{b}$. To divide $aa\sqrt[4]{aa + xx} - 2ax\sqrt[4]{aa+xx} + xx\sqrt[4]{a^2 + x^2}$ by $a\sqrt[4]{aa + xx} - x\sqrt[4]{aa + xx}$, omitting the radical, and dividing $aa - 2ax + xx$ by $a - x$, the quotient will be $a - x$. To divide $aa + bb$ by $\sqrt{aa + bb}$, becaufe the dividend is $\sqrt{aa + bb} \times \sqrt{aa + bb}$, the quotient will be $\sqrt{aa + bb}$. How furds are to be divided.

47. But when the radicals are not the fame, though they have the fame exponent of the root; let the quantities under the *vinculum* be divided by the rational quantities in the ufual manner, and to the quotient prefix the common *vinculum*. Thus to divide $\sqrt[3]{a^3b - ab^3}$ by $\sqrt[3]{aa - bb}$, dividing $a^3b - ab^3$ by $a^2 - b^2$ there arifes ab, and therefore the quotient required is $\sqrt[3]{ab}$. When the index is the fame, but the furds are different.

48. And if the exponents of the roots are different, they may be reduced to the fame, and then the operation will be as before. Thus to divide $\sqrt{a^4 + 2a^3b - 2ab^3 - b^4}$ by $a + b$, the fquare of $a + b$ muft be found, and put under the *vinculum*, which will be then $\sqrt{aa + 2ab + bb}$. Then by the quantity under this *vinculum* the other quantity muft be divided, and the refult will be $aa - bb$. Therefore the quotient required will be $\sqrt{aa - bb}$. When the index alfo is different.

By combining thefe rules with thofe of common divifion, quantities ftill more complicate may be divided. Thus to divide $a^3b - ab^2c - a^2b\sqrt{bc} + b^2c\sqrt{bc}$ by $a - \sqrt{bc}$, it may be performed as is ufual in divifion.

Dividend.	$a^3b - ab^2c - a^2b\sqrt{bc} + b^2c\sqrt{bc}$.	Divifor	$a - \sqrt{bc}$
Rem.	$-ab^2c \qquad - b^2c\sqrt{bc}$	Quotient	$a^2b - b^2c$

Thus dividing $a^3 - abc + a^2\sqrt{bc} - bc\sqrt{bc}$ by $a - \sqrt{bc}$, the quotient will be $aa + bc + 2a\sqrt{bc}$. And when the divifion will not fucceed, the quantities muft be wrote in form of a fraction.

Extraction

Extraction of the Square-Root of Radical Quantities.

The square-root of surds extracted.

49. When quantities any how compounded of rationals and radicals are quadratick radicals, the rule for extracting the square-root will be this. Taking such a part of the quantity proposed as is greater than the remaining part, from the square of this greater part let the square of the lesser part be subtracted, and to the greater part let the square-root of the remainder be added, and likewise be subtracted from it. The square-root of the half of this sum, and of the half of this difference, being taken together, and taking the same sign to this second as belongs to the minor part, will make the square-root of the proposed quantity. Thus let us extract the square-root of the quantity $3 + \sqrt{8}$; subtracting the square of $\sqrt{8}$ from the square of 3, there will remain 1, the root of which is also 1. Adding this therefore to the greater part, or 3, they will make 4, and subtracting it from the same, it will make 2; now the square-root of the half of 4 is $\sqrt{2}$, and the square-root of the half of 2 is 1; therefore $\sqrt{2} + 1$ will be the root required.

If we would have the square-root of $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$; from the square of $6 + \sqrt{8}$ subtracting the square of $-\sqrt{12} - \sqrt{24}$, there remains 8, the root of which $\sqrt{8}$ being added to $6 + \sqrt{8}$, the greater part, will make $6 + 2\sqrt{8}$, and subtracted from the same greater part will make 6. Therefore the first part of the root required will be $\sqrt{\frac{6 + 2\sqrt{8}}{2}}$, that is, $\sqrt{3 + \sqrt{8}}$, and the second part will be $-\sqrt{\frac{6}{2}}$, that is $-\sqrt{3}$, (for the lesser part of the proposed quantity was affected by the negative sign;) whence $\sqrt{3 + \sqrt{8}} - \sqrt{3}$ will be the root required. But by the last example it may be seen, that $\sqrt{3 + \sqrt{8}}$ is the same as $1 + \sqrt{2}$; therefore, lastly, the root of the quantity proposed will be $1 + \sqrt{2} - \sqrt{3}$.

Let us extract the square-root of $aa + 2x\sqrt{aa - xx}$. Taking from the square of aa the square of $2x\sqrt{aa - xx}$, there will remain $a^4 - 4aaxx + 4x^4$, the root of which is $aa - 2xx$. This added to the greater part aa, and taking the half of it, will make $aa - xx$: and subtracted from the same, and taking half the difference, will make xx. Therefore the root required is $\sqrt{aa - xx} + x$.

Let us extract the square-root of the quantity $aa + 5ax - 2a\sqrt{ax + 4xx}$. From the square of $aa + 5ax$, the greater part, subtracting the square of $-2a\sqrt{ax + 4xx}$, there will remain $a^4 + 6a^3x + 9a^2x^2$, the root of which is $aa + 3ax$. This added to the greater part, and taking it's half, it will be $aa + 4ax$; and subtracting and taking the half, it will be ax. Therefore the root required will be $\sqrt{aa + 4ax} - \sqrt{ax}$.

To

To extract the ſquare-root of this quantity $a\sqrt{bc} + d\sqrt{bc} + 2\sqrt{abcd}$. From the ſquare of $a\sqrt{bc} + d\sqrt{bc}$ ſubtracting the ſquare of $2\sqrt{abcd}$, there remains $aabc - 2abcd + bcdd$, the root of which is $a\sqrt{bc} - d\sqrt{bc}$; which being added to the major part, and ſubtracted from the ſame, and taking half of the ſum and difference, the half of the ſum will be $a\sqrt{bc}$, and half of the difference $d\sqrt{bc}$. Therefore the root required is $\sqrt{a\sqrt{bc}} + \sqrt{d\sqrt{bc}}$, that is, $\sqrt{\sqrt{aabc}} + \sqrt{\sqrt{bcdd}}$, or $\sqrt[4]{aabc} + \sqrt[4]{bcdd}$. If the root cannot be extracted, the quantity muſt be put under a radical *vinculum*, as uſual.

The Calculation of Powers.

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50. There is nothing now to be obſerved concerning the Addition or Subtraction of Powers; they are to be written one after another with their proper ſigns in the firſt caſe, and in the ſecond by changing the ſigns of the quantities to be ſubtracted. But as to the other operations which belong to their exponents, it may be firſt obſerved, that, taking unity for the firſt term, and any quantity whatever, as a, for the ſecond, and then ſucceſſively the other powers of the ſame quantity a in order, it is plain we ſhall form an increaſing geometrical progreſſion, $1, a, a^2, a^3, a^4$, &c.; and that the exponents of this progreſſion will form an arithmetical progreſſion increaſing, which will be 0, 1, 2, 3, 4, 5, &c. The firſt term of this is 0, becauſe unity being the firſt term in the geometrical progreſſion, in this the quantity a is raiſed to no power; for $1 = \frac{a}{a} = a^0$. Wherefore, multiplying either $\frac{a}{a}$, or a^0, by a, which does not deſtroy the equality, the product will be $a = a^{0+1}$, which are magnitudes plainly identical. And beſides, if we continue the ſame geometrical progreſſion below unity, it will be $1, \frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4}$, &c. And likewiſe, continuing the arithmetical progreſſion of the exponents, they will become $0, -1, -2, -3, -4$, &c. And therefore the exponents of ſuch powers will be negative. So that $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}$, &c. will be the ſame as a^{-1}, a^{-2}, a^{-3}, &c. And in general, $\frac{1}{a^n}$ will be the ſame as a^{-n}; that is to ſay, we may always make a power to paſs into the numerator of a fraction out of the denominator, and *vice verſâ*, only by changing the ſign of the index.

Powers how calculated when the exponents are integers.

51. Moreover, if we ſhould deſire to introduce new intermediate terms into the geometrical progreſſion, the exponents of theſe would alſo be intermediate terms

When they are fractions.

terms in the arithmetical progreſſion, analogous to the former. So, becauſe $\sqrt{a}$ is a geometrical mean between unity and a, the exponent of this ought to be an arithmetical mean between o and unity, and therefore muſt be $\frac{1}{2}$; ſo that $a^{\frac{1}{2}}$ will be the ſame as $\sqrt{a}$. If two mean proportionals are interpoſed between 1 and a, of which the firſt will be $\sqrt[3]{a}$, and the ſecond $\sqrt[3]{aa}$, there muſt be two arithmetical means between o and 1, which are $\frac{1}{3}$ and $\frac{2}{3}$; ſo that $a^{\frac{1}{3}}$ will be the ſame as $\sqrt[3]{a}$, and $a^{\frac{2}{3}}$ will be the ſame as $\sqrt[3]{aa}$. If three mean proportionals are introduced, they will be $\sqrt[4]{a}$ the firſt, $\sqrt[4]{aa}$ the ſecond, and $\sqrt[4]{aaa}$ the third, and their exponents will be $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$; therefore $\sqrt[4]{a}$ will be the ſame as $a^{\frac{1}{4}}$, and $\sqrt[4]{aa}$ the ſame as $a^{\frac{2}{4}}$, or $a^{\frac{1}{2}}$, and $\sqrt[4]{a^3}$ will be the ſame as $a^{\frac{3}{4}}$. And thus we may proceed to as many mean proportionals as we pleaſe; ſo that, in general, it will be $\sqrt[m]{a^n}$, the ſame as $a^{\frac{n}{m}}$.

The ſame things obtain in reſpect of the progreſſion produced by deſcending below unity. Thus, as $\frac{1}{\sqrt{a}}$ is a mean proportional between unity and $\frac{1}{a}$, or between unity and a^{-1}, ſo it's index ſhould be an arithmetical mean between o and -1, that is $-\frac{1}{2}$; therefore $\frac{1}{\sqrt{a}}$ will be the ſame as $a^{-\frac{1}{2}}$, or $\frac{1}{a^{\frac{1}{2}}}$. Thus likewiſe $\frac{1}{\sqrt[3]{a}}$ and $\frac{1}{a^{\frac{1}{3}}}$ and $a^{-\frac{1}{3}}$ will be the ſame. And $\frac{1}{\sqrt[3]{aa}}$, $\frac{1}{a^{\frac{2}{3}}}$, $a^{-\frac{2}{3}}$ will be the ſame. And ſo, in general, $\frac{1}{\sqrt[m]{a^n}}$, $\frac{1}{a^{\frac{n}{m}}}$, and $a^{-\frac{n}{m}}$ will be the ſame.

And what has been ſaid concerning integral or fractional powers of ſimple quantities, is to be underſtood alſo of compound quantities. Thus, for example, $\frac{1}{\overline{aa+bb}|^n}$ is the ſame as $\overline{aa+bb}|^{-n}$. So $\sqrt[m]{\overline{aa+bb}|^n}$ will be the ſame as $\overline{aa+bb}|^{\frac{n}{m}}$; and the like of others.

Powers how multiplied or divided.

52. From the nature of the two foregoing progreſſions, the geometrical and arithmetical, we obtain a method for the multiplication or diviſion of any two powers of the ſame quantity, whatever they may be; and that is, by adding the exponents together when the powers are to be multiplied, and by ſubtracting

ing the exponent of the divisor from that of the dividend, when the powers are to be divided. For, as to multiplication, as the product is the fourth proportional from unity and the two factors, these four terms will be in a geometrical proportion, and their exponents in an arithmetical progression. Therefore the exponent of the fourth, that is of the product, must be greater than the exponent of the third, by as much as the exponent of the second is greater than the exponent of the first. But the exponent of the second is greater than the exponent of the first, which is o, by it's whole quantity; therefore the exponent of the fourth ought to be greater than the exponent of the third by the whole exponent of the second; that is, it ought to be equal to the sum of the exponents of the second and third. As to division, it has the same proportion as multiplication, but only inverted. It's first term is the dividend, it's second the divisor, the third the quotient, and the fourth is unity. Therefore as much as the exponent of the dividend is greater than the exponent of the divisor, so much the exponent of the quotient ought to be greater than o. Therefore it ought to be exactly the difference of the exponents of the dividend and the divisor.

So that to multiply aa by a, or a^2 by a^1, the product will be a^{2+1} or a^3. To multiply a^3 into a^2, the product is a^{3+2}, or a^5. To multiply a^6 into a^{-2}, the product is a^{6-2}, or a^4. To multiply $a^{\frac{1}{2}}$ into $a^{\frac{1}{3}}$, the product is $a^{\frac{1}{2}+\frac{1}{3}}$, that is $a^{\frac{5}{6}}$. To multiply $a^{-\frac{2}{3}}$ into $a^{\frac{1}{5}}$, the product is $a^{-\frac{2}{3}+\frac{1}{5}}$, that is $a^{-\frac{7}{15}}$. To multiply $a^{\pm\frac{n}{m}}$ into $a^{\pm\frac{r}{t}}$, the product is $a^{\pm\frac{n}{m}\pm\frac{r}{t}}$, or $a^{\frac{\pm nt \pm mr}{mt}}$.

And so to divide a^3 by a^1, the quotient will be a^{3-1}, or a^2. To divide a^5 by a^{-2}, the quotient will be a^{5+2}, or a^7. To divide a^2 by $a^{\frac{1}{2}}$, the quotient will be $a^{2-\frac{1}{2}}$, or $a^{\frac{3}{2}}$. To divide $a^{\frac{2}{3}}$ by $a^{-\frac{1}{2}}$, the quotient will be $a^{\frac{2}{3}+\frac{1}{2}}$, or $a^{\frac{7}{6}}$. To divide $a^{\pm\frac{n}{m}}$ by $a^{\pm\frac{r}{t}}$, the quotient will be $a^{\pm\frac{n}{m}\mp\frac{r}{t}}$, that is, $a^{\frac{\pm nt \mp mr}{mt}}$.

53. And because in the progression before considered, taking any term whatever, the same term with a double exponent will be the square of the term so taken; and a term with a treble exponent will be the cube of the assumed term; and a term with a quadruple exponent will be the fourth power; and so on. And a term with half the exponent will be the square-root of the term assumed; a term with a third part, a fourth part, &c. will be the cube-root, the fourth root, &c. of the term assumed. It follows therefore that, to reduce one power

Powers may be raised, or roots extracted, by the exponents.

power to another, it will be ſufficient to multiply the exponent of the given power by the exponent of that power to which we would raiſe it : and to extract any root, it will be enough to divide it's index by the index of the given root. Thus to raiſe a^2 to it's cube, it will be $a^{2 \times 3}$, or a^6. To raiſe $a^{\frac{2}{3}}$ to the cube, it will be $a^{\frac{2}{3} \times 3}$, or a^2. To raiſe $a^{-\frac{1}{4}}$ to the fifth power, it will be $a^{-\frac{1}{4} \times 5}$, or $a^{-\frac{5}{4}}$. To raiſe $a^{\pm \frac{n}{m}}$ to the power whoſe index is $\pm \frac{r}{t}$, it will be $a^{\pm \frac{nr}{mt}}$. Thus to extract the ſquare-root of a^5, it will be $a^{\frac{5}{2}}$. To extract the cube-root of $a^{\frac{1}{2}}$, it will be $a^{\frac{1}{6}}$. To extract the root r of $a^{\pm \frac{m}{n}}$, it will be $a^{\pm \frac{m}{nr}}$, &c.

Extended to compound quantities.

54. What I have here ſaid concerning the powers or roots of one and the ſame ſimple quantity, may be underſtood in like manner concerning the powers or roots of any compound quantities, as is evident. And by this method the calculus of fractions and radicals will be much facilitated.

Of Linear or Simple Diviſors of any Formula whatever.

Simple diviſors how found; as alſo compound diviſors.

55. Any quantity or formula whatever, whether complicate or not, is ſaid to be *prime* or *ſimple*, when it is not exactly diviſible by any other quantity, except itſelf or unity. And it is called *compound* when it is exactly diviſible by ſome other quantity. Thus, for example, $a + b$, $aa + xx$, $x^3 - aax + aab$, and ſuch others, will be prime or ſimple. But ab is compound, becauſe diviſible by a or b. So $aa - xx$ is compound, becauſe diviſible by $a + x$ or $a - x$. And ſo of others.

Two or more formulas are *relative primes*, when they have no common diviſor, and that the leſſer is not a diviſor of the greater. Such between themſelves will be aa and bb. Alſo $aa + 2ab + bb$ and $aa + bb$, &c. And on the contrary, they are abſolutely and relatively compound, between themſelves, when they have ſome common diviſor, or that one of them can divide the other. Such are aa and ab, which are both diviſible by a; ſuch are $aa - xx$ and $a + x$, which are diviſible by $a + x$, &c.

In order to have all the ſimple diviſors of any quantity, either numeral, or literal, or mixt, it muſt be divided by the leaſt of it's diviſors, and the quotient again by the leaſt of it's diviſors, and ſo on continually till a quotient ariſes, which

which can no longer be divided except by itſelf. The quantities thus ariſing, unity being comprehended among them, will be all the ſimple diviſors. And if they are taken two by two, three by three, and ſo on; according to all the combinations poſſible, they will give likewiſe all the compound diviſors.

For example, let us find all the diviſors, ſimple or compound, of the number 300. Let the given number 300 be wrote at A, and at one ſide, as at B, ſet down it's leaſt diviſor, as 2. Then dividing by 2, write the quotient 150 at A under 300; and again divide this number 150 by 2, and over againſt it at B write the diviſor 2, and the quotient 75 at A under the firſt quotient 150. Now, becauſe 75 is not diviſible by 2, let it be divided by 3, and write the diviſor 3 over againſt it at B, and under it at A the quotient 25. The leaſt diviſor of 25 will be 5, which muſt be wrote over againſt it at B, and the quotient 5 under it at A. The laſt quotient 5 is not diviſible unleſs by itſelf; therefore it muſt be wrote aſide at B, and we ſhall have all the prime diviſors; to which we may add unity, becauſe it is always a diviſor of any quantity. Now to have all the compound diviſors, according to all the combinations, let the firſt and ſecond diviſors be multiplied together, and the product 4 be wrote at B over againſt the ſecond diviſor. By the third diviſor let all above it be multiplied, and let the products 6, 12, be wrote aſide, ſetting down but once thoſe that may chance to be repeated. In like manner, by the fourth let all above it be multiplied, and the products ſet down as before: and ſo on ſucceſſively to the laſt. Now the numbers wrote at B will be all the diviſors of the propoſed number 300.

A.	B.						
	1						
300	2						
150	2	4					
75	3	6	12				
25	5	10	15	20	30	60	
5	5	25	50	75	100	150	300
1							

Let the given formula be $21abb$, of which we are to find all the diviſors. As it is not diviſible by 2, let it be divided by 3, which is to be wrote over againſt it at B, and the quotient $7abb$ under it at A. Let $7abb$ be divided by 7, which is to be wrote over againſt it, and the quotient abb underneath. Let abb be divided by a, which is wrote aſide, and the quotient bb under it. Then divide bb by b, which is wrote aſide, and the quotient b underneath. This is to be divided by b, and wrote over againſt it; and then we ſhall have all the prime diviſors 1, 3, 7, a, b, b, of the propoſed quantity. To have thoſe that are compound we muſt multiply 3 into 7, and the product is 21. Multiply 3, 7, 21 into a, and the products are $3a$, $7a$, $21a$. Multiply all the diviſors 3, 7, 21, a, $3a$, $7a$, $21a$ into b, and there will ariſe $3b$, $7b$, $21b$, ab, $3ab$, $7ab$, $21ab$; and

and ſo proceed. Thus the column B will contain all the diviſors of the quantity propoſed, both ſimple and compound.

A.	B.								
	1								
21*abb*	3								
7*abb*	7	21							
abb	*a*	3*a*	7*a*	21*a*					
bb	*b*	3*b*	7*b*	21*b*	*ab*	3*ab*	7*ab*	21*ab*	
b	*b*	*bb*	3*bb*	7*bb*	21*bb*	*abb*	3*abb*	7*abb*	21*abb*
1									

In like manner, let $2abb - 6aac$ be given. Let it firſt be divided by 2, and the quotient $abb - 3aac$ by a, and the new quotient $bb - 3ac$ by itſelf, as being diviſible by no other quantity. And therefore all the diviſors will be as in the column B.

A.	B.
	1
$2abb - 6aac$	2
$abb - 3aac$	a, $2a$
$bb - 3ac$	$bb - 3ac$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$
1	

Compound formulas how reſolved.

56. But if the laſt quotient, or perhaps the formula itſelf at firſt propoſed, ſhall ſtill be compound, and yet is not diviſible, after the foregoing manner, by any ſimple quantity, ſo that all it's diviſors are compound terms; the way of obtaining them is different, and may be thus. The quantity is to be ſet in order according to ſome one of it's letters, as has been already ſhown at § 24; and if there are fractions, they muſt be reduced to a common denominator. Then all the diviſors of the laſt term muſt be found, compounded of numeral diviſors if there are any, and of the letter of one dimenſion. And if the greateſt term has a numeral co-efficient, it muſt be divided by ſome one of thoſe diviſors, by which that co-efficient of the greateſt term is diviſible. By every one of theſe diviſors, firſt added and then ſubtracted from the letter, by which the formula is ordered, the diviſion muſt be tried; and all thoſe by which it ſucceeds will be ſo many diviſors of the propoſed quantity.

Let the formula $y^3 - 4ay^2 + 5a^2y - 2a^3$ be given. The diviſors of one dimenſion of the laſt term are a and $2a$. Therefore the diviſion muſt be tried by each of theſe added to the letter y, or ſubtracted from it, becauſe the co-efficient of the greateſt term y is unity; that is, by $y \pm a$, or by $y \pm 2a$. Firſt let it be divided by $y - 2a$, and the quotient is $yy - 2ay + aa$, which alſo is diviſible by $y - a$, giving $y - a$ in the quotient. Wherefore the diviſors of the formula propoſed are $y - a$, $y - a$, and $y - 2a$, from the product of which it is derived.

Let

Let the formula be $6y^4 - ay^3 - 21aayy + 3a^3y + 20a^4$. The divisors of one dimension of the last term are a, $2a$, $4a$, $5a$, $10a$, $20a$; and because the first term $6y^4$ is divisible by 1, 2, and 3, we must try the division by $y \pm \frac{1}{2}a$, $y \pm a$, $y \pm 2a$, $y \pm \frac{5}{2}a$, $y \pm 5a$, $y \pm 10a$, $y \pm \frac{1}{3}a$, $y \pm \frac{2}{3}a$, $y \pm \frac{4}{3}a$, $y \pm \frac{5}{3}a$, $y \pm \frac{10}{3}a$, $y \pm \frac{20}{3}a$. But because it would be too tedious and troublesome to try all these divisors; in order to know among so many which are to be selected, we may make $y = z + a$; and substituting this in the place of y, and also it's powers, there will arise another formula, which is this.

$$\begin{array}{rrrrr} 6z^4 + & 24az^3 + & 36aazz + & 24a^3z + & 6a^4 \\ & - az^3 - & 3aazz - & 3a^3z - & a^4 \\ & & - 21aazz - & 42a^3z - & 21a^4 \\ & & & + 3a^3z + & 3a^4 \\ & & & & + 20a^4 \end{array}$$

Which by collecting the terms will be this,

$$6z^4 + 23az^3 + 12a^2z^2 - 18a^3z + 7a^4.$$

Now all the divisors of the last term $7a^4$ of this formula are found to be a and $7a$, which divided by 2 and by 3, the numeral divisors of $6z^4$, will make $\frac{1}{2}a$, $\frac{1}{3}a$, $\frac{7}{2}a$, $\frac{7}{3}a$. And because it was made $y = z + a$, if these divisors can be made use of in the second given formula by z, they will also be useful in the first by y, when they are increased by the quantity a, that is by making them $\frac{3}{2}a$, $\frac{4}{3}a$, $\frac{9}{2}a$, $\frac{10}{3}a$. Therefore let these divisors be compared with the divisors of the first formula, and choose only those which agree with them, that is $\frac{4}{3}a$ and $\frac{10}{3}a$, by which added to and subtracted from y, the division must be tried; which will succeed with $y + \frac{4}{3}a$. But notwithstanding this operation, if there should still remain too many divisors to be selected by this comparison, we may make $y = z - a$, and another formula will arise. From the divisors found by this, the quantity a must be subtracted, and then they are to be compared with those which are selected by means of the second; and by them which agree, which will be fewer in number, the division is to be tried. And proceeding in the same way of operation by new substitutions, making $y = z + 2a$, $y = z - 2a$, &c. the divisors may be reduced to such smaller numbers as will be sufficient.

57. When

How the co-efficient of the first term may be removed.

57. When the proposed formula has it's first or greatest term multiplied by any number, instead of applying the rule aforegoing to this case, it may be more convenient to change the formula into another, the first term of which is multiplied only by unity; and then find the divisors of the same, from which you may afterwards pass to those of the proposed formula.

Let the formula be, for example,

$$\begin{aligned} 3y^3 &+ 9ayy - 12aay - 12aab. \\ &+ 3byy + 9aby \end{aligned}$$

Make $3y = z$, (or, in general, $ny = z$, putting n to represent the numeral co-efficient of the highest power,) and thence $y = \frac{1}{3}z$. This being substituted instead of y, and it's powers expressed in like manner, we shall have the formula $z^3 + 9az^2 + 3bz^2 - 36a^2z + 27abz - 108a^2b$, all divided by 9. Let the divisors of this be found, (at present omitting the denominator 9,) which will be $z + 12a$, $z - 3a$, $z + 3b$; and taking account of the denominator 9, one of these is to be divided by 9, or two of them by 3, and they will be, for example, $z + 12a$, $\frac{z - 3a}{3}$, $\frac{z + 3b}{3}$; but it was made $3y = z$; and substituting this value of z in the divisors, they will become $3y + 12a$. $y - a$, $y + b$, which are the three divisors of the formula proposed.

SECT. II.

Of Equations, and of Plane Determinate Problems.

Equations and their affections what?

58. *Equation* is a relation of equality, which two or more quantities, whether numerical, geometrical, or physical, have with one another when compared together; or which they have with nothing when compared to that. The aggregate of all those terms which are wrote before the mark of equality, is called the First Member of the Equation; and the aggregate of all those which are wrote after it, is called the Second Member, or the *Homogeneum Comparationis*. Those terms of the equation are homogeneous, when each of them is of the same dimension; and therefore in an equation they are said to observe the law of homogeneity, as in this equation $axx - bbx = a^3$. And thus, on the contrary, they are said not to observe the law of homogeneity, when the terms are not such, as in this equation $x^4 - ax^2 = b$.

59. A

59. A *Problem* is a propofition in which it is required to do or to find fome-thing, by means of other things which are known, or of certain conditions which are given, and therefore called the *Data* of the Problem. So thofe things which are required are the *Quæfita* of the Problem.

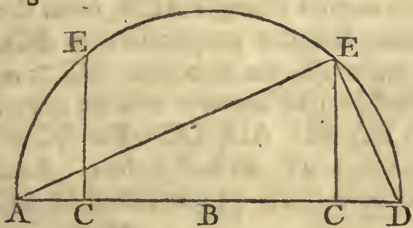
A problem, what.

60. Of Problems fome are *Determinate*, and others *Indeterminate*. The determinate are thofe which have a certain number of folutions, or which can be refolved by one or more determinations, but always in a finite and limited number. Such it would be if we fhould inquire, where we ought to cut the right line AB, fo that the whole line, to it's greater fegment, fhould have the fame ratio, as the greater fegment to the leffer. Becaufe one point only can be affigned in this line, for example C, which will have the property required. The fame thing would be, if in a given circle AED we were to find a point, fuppofe C, in the diameter AD, from whence raifing a perpendicular CE, terminated in the periphery; this perpendicular fhould be juft equal to a third part of the diameter. For there are only two points, each at an equal diftance from the centre, that can fatisfy this demand.

When problems are determinate, —when indeterminate.

Fig. 1.

Fig. 2.



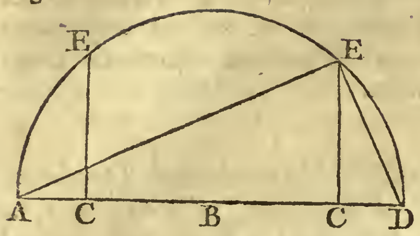
Now if it were propofed to find, out of the right line AD, fuch a point E, fo that drawing from it two right lines EA, ED, to it's extremities A and D, the angle AED fhall be a right angle; it will be found, that there are infinite fuch points as will refolve the problem, or the whole periphery AED, as is known from *Euclid*. In the fame manner, if a point C is required in the diameter AD, from whence raifing the perpendicular EC in the circle, it fhall be a mean proportional between the fegments AC, DC; it will be found, that all the points of the diameter will folve the problem (and therefore fuch points are infinite in number); which is therefore called an *Indeterminate Problem.*

Determinate problems have occafion for one unknown quantity only, but indeterminate ones of two at leaft, though the manner of forming an equation is the fame in both. Of thefe I fhall treat particularly in Sect. III.

61. The given or known quantities are ufed to be denominated by the firft letters of the alphabet, as has been faid already; but the unknown, or fuch as are required, by fome one of the laft letters. And here it may be obferved, that if the quantity fought is a line, it ought always to have it's origin or beginning at fome determinate fixed point. And as that which is required is already fuppofed to be done or known, by calling it, for example, x; fo that from thefe quantities fuppofed as known, others that depend on them come to

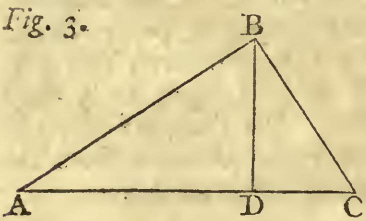
Known and unknown quantities how diftinguifhed.

Fig. 2.



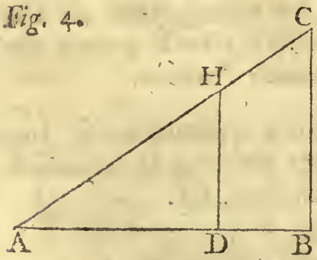
be known or given, as it were by hypothefis. Thus, AD $= a$ being given, and C being fuppofed the point required, and therefore calling AC x, it will be CD $= a - x$; and thus we may argue of feveral others. And further, though many of the quantities are not exprefsly given, like as in the line AD; yet, however, they are given implicitly, and as it were by conftruction. Thus, in the right-angled triangle AED, if the hypothenufe AD $= a$ is given, and the fide ED $= b$; then, by the 47th propofition of the firft Book of *Euclid*, the fide AE $= \sqrt{aa - bb}$ will be therefore given. Thus, in the femicircle AED, the diameter AD $= a$ being given, and the fegment AC $= b$, it will be CD $= a - b$; and therefore, by *Euclid*, vi. 8, it will be CE $= \sqrt{ab - bb}$. Or becaufe AC was called x, it will be CE $= \sqrt{ax - xx}$, which is given both by hypothefis and by conftruction.

Fig. 3.



Thus, in the right-angled triangle ACB, from the right angle B letting fall the perpendicular BD, let be given, for example, the two lines AC $= a$, and AB $= b$; then in like manner will be given all the other lines BC, BD, AD, DC. For BC $= \sqrt{aa - bb}$, by *Euclid*, i. 47, as faid before. And by vi. 8, CD will be a third proportional to AC and CB; wherefore it will be CD $= \frac{aa - bb}{a}$, by the 17th of the fame book. AD will be a third proportional to AC and AB, and therefore AD $= \frac{bb}{a}$. DB will be a mean proportional between AD and DC; or elfe it will be a fourth proportional to AC, CB, AB; and therefore, by 16 of the fame book, it will be DB $= \frac{b\sqrt{aa - bb}}{a}$.

Fig. 4.



Thus, in the right-angled triangle ABC, if DH is parallel to BC, and are given AB $= a$, BC $= b$, AD $= x$; then, by 4 of vi., will be given DH $= \frac{bx}{a}$; AH $= \frac{x\sqrt{aa + bb}}{a}$. And the fame may be obferved of infinite others.

Equations how derived.

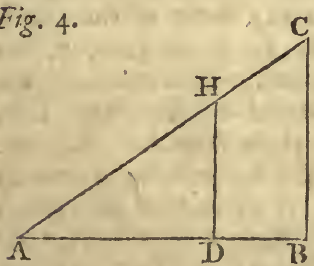
62. Thus, by fuppofing that already done or known, which is to be done or known, and by treating given and fought quantities indifferently, all the conditions may be fulfilled, which are required by the propofition or problem, and we fhall thus arrive at an equation. Let there be a right line AB, which is to be cut in extreme and mean proportion. Let AB $= a$, and let C be the

Fig. 1.

point

point required. Let AC $= x$, and therefore CB $= a - x$. The condition implied is, that it ought to be AB . AC :: AC . CB; that is, $a . x :: x . a - x$. But by the nature of a geometrical proportion, the rectangle of the means muſt be equal to that of the extremes; ſo that $aa - ax = xx$, and thus we are now come to an equation. Again, let there be three numbers given, the firſt is 4, the ſecond is 5, and the third is 10. A fourth number muſt be found, ſuch that, if from the product of this into the third the firſt be ſubtracted, and if the remainder is divided by the firſt, the quotient ſhall be equal to the ſecond number given. Let the number ſought be denoted by x; then the product of this into the third will be $10x$, from which ſubtracting the firſt, the remainder will be $10x - 4$, and dividing this by the firſt, the quotient will be $\frac{10x - 4}{4}$, which by the condition of the problem ſhould be 5, that is $\frac{10x - 4}{4} = 5$, which is the equation required.

Fig. 4.

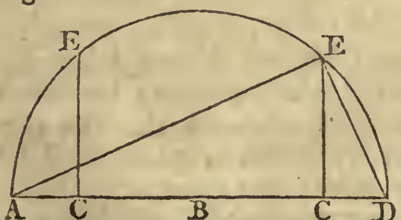


Again, in the triangle ABC, are given the ſides AC $= a$, BC $= b$, and the baſe AB $= c$; we are to find in this ſuch a point D, that drawing DH parallel to BC, the ſquare of DH may be equal to the rectangle AD × DB. Make AD $= x$, whence DB $= c - x$; and becauſe of like triangles ABC, ADH, it will be DH $= \frac{bx}{c}$. Then by completing what the problem requires, we ſhall have the equation $\frac{b^2x^2}{c^2} = cx - xx$.

Some lines to be denominated by inference.

63. If the given triangle ABC is right-angled at B, we ſhall have no need to denominate AC $= a$, but otherwiſe $= \sqrt{bb + cc}$, to expreſs thereby the condition of a right-angled triangle. Thus in the ſemicircle AED is given the diameter AD $= 2a$, and the ſegment AC $= b$; hence conſequently is given the line CE, and therefore it ought not to be expreſſed by a letter at pleaſure, but to be denominated from the property of the circle, by making it $= \sqrt{2ab - bb}$; thereby expreſsly to indicate, that it is an ordinate in the circle at the point C. And in general it is to be underſtood, that the ſame ought to be done in all like caſes.

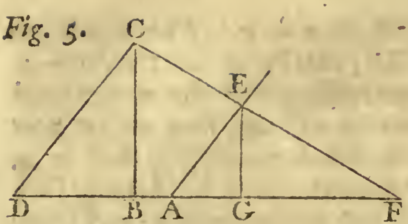
Fig. 2.



New lines to be drawn.

64. But perhaps it may make ſome difficulty, that very often the lines given in a figure, by which the problem is propoſed, are not ſufficient to obtain ſuch quantities or denominations, as are neceſſary to arrive at an equation. Such a caſe

Fig. 5.



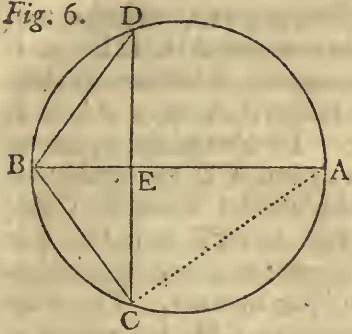
case would be, if two indefinite right lines AE, AF, were given in position, and a point C out of those lines: and if it were proposed to draw a line CF in such a manner from the point C, as that it should include a triangle AEF, equal to a given plane. The expression of the triangle AEF would be half the rectangle of AF into EG, letting fall EG perpendicular to AF. Now make AF $= x$; but yet it will not be possible to determine the value of EG from the lines hitherto described. Upon such occasions it will be necessary to construct or complete the figure, by drawing parallels, raising or letting fall perpendiculars, forming similar triangles, describing circles, or by using the like expedients of the common Geometry; for which it is not possible to give any general rules, as they will depend on the various circumstances of problems, on sagacity, industry, and practice, and often upon chance. But commonly these propositions of the first Book of *Euclid* are used to be of good service, 5, 13, 15, 27, 29, 32, 47; some of the second; these of the third, 20, 21, 22, 27, 31, 35, 36; these of the sixth, 1, 2, 3, 4, 5, 6, 7, 8; and some of the 11th and 12th when solids are concerned. Therefore, in the problem now proposed, from the point C draw CD parallel to EA, and EG, CB, perpendicular to FA produced. Now because the right lines AE, AF, are given in position, and also the point C; the lines AD, CB, will be given in magnitude. Therefore make AD $= a$, CB $= b$, AF $= x$, and let the given plane be $= cc$. And as the triangles FDC, FAE, are similar, as also the triangles DCB, AEG; we shall have the analogies DF . AF :: (DC . AE ::) BC . EG. That is, $a + x \,.\, x :: b \,.\,$ EG. Therefore EG $= \frac{bx}{a+x}$. And because the triangle AEF, that is, half the rectangle of AF into EG, ought to be equal to the given plane cc, we shall at last have the equation $\frac{bxx}{2a + 2x} = cc$.

Equations how formed from different values of the same quantity.

65. The proposing of the problems only, which hitherto I have taken for examples, has brought me immediately and directly to an equation; because it was required that the two quantities so found should be made equal. But this method will not thus succeed, when from certain quantities given, it shall be proposed to find others, without such a condition as will lead us expressly to an equation. Then it may be needful to use a little art to obtain it, and that will be by means of different properties, and compounding the figure if necessary, to find two different expressions of the same quantity, and so to make an equation between them. I said by means of different properties, because the same property, however managed, will always give the same expression. I shall produce three examples of this, which I think may suffice at present.

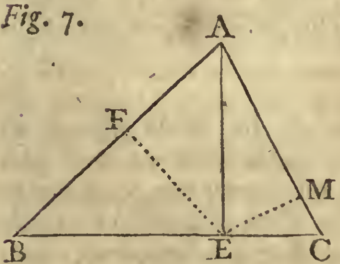
Given

Fig. 6.



Given the isosceles triangle CDB, the diameter AB of the circle CADB is required, in which it may be inscribed. Make CD $= a$, CB $=$ BD $= b$, BA $= x$, which is the diameter required; and draw CA. The two triangles ABC, BCE, will be similar, because the angles BCA and CEB are right ones, and the angle BCE $=$ BDC $=$ BAC. Therefore it will be AB . BC :: BC . BE; that is, $x . b :: b$. BE; whence BE $= \frac{bb}{x}$. Moreover CE is the half of CD, whence CE $= \frac{1}{2}a$. And because of the right-angle CEB, it will be CB$q = \frac{aa}{4} + \frac{b^4}{xx}$. But the square of CB is also $= bb$. Therefore we shall have the equation $bb = \frac{aa}{4} + \frac{b^4}{xx}$.

Fig. 7.

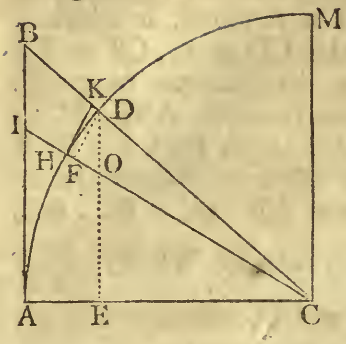


In the triangle ABC the three sides are given, and from the angle A letting fall the perpendicular AE upon BC; the two segments BE, EC are required. Make AB $= a$, AC $= b$, BC $= c$, BE $= x$; then it is EC $= c - x$. By the 47 of the first of *Euclid*, the square of AE will be equal to the square of AB, subtracting the square of BE; that is AEq $=$ ABq $-$ BEq. But by the same it will be also AEq $=$ ACq $-$ ECq. Therefore ABq $-$ BEq $=$ ACq $-$ ECq. And reducing to an algebräick expression, it will be $aa - xx = bb - cc + 2cx - xx$, that is, $aa = bb - cc + 2cx$.

Again another way. Let EF be drawn perpendicular to AB; then, by the 8 of the sixth of *Euclid*, it will be AB . BE :: BE . BF; and therefore BF $= \frac{xx}{a}$. Thence AF $= a - \frac{xx}{a}$. And, by the same proposition, it will be AF . AE :: AE . AB; and therefore AEq $= aa - xx$. From the point E drawing the right line EM perpendicular to AC, by the same way of arguing it will be found, that AEq $= bb - cc + 2cx - xx$; and making a comparison between these two values, we shall have the same equation as before.

The

Fig. 8.



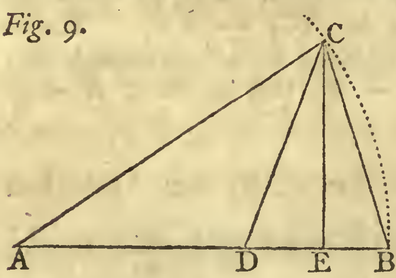
The quadrant AHM being given, and the tangents AI, HK, of the two arches AH, HD; it is required to find AB the tangent of the ſum of theſe two arches. Make the radius CA $= a$, AI $= b$, HK $= c$, and AB $= x$. To obtain an equation, from the point D let be drawn DE perpendicular upon AC. Then by the ſimilar triangles CBA, CDE, we may find the values of CE and DE. Let us examine then if we cannot contrive to denominate the ſame DE in another manner. Therefore drawing DF perpendicular to CH, by means of the ſimilar triangles CAI, CEO, we may have the lines EO, CO; and in like manner, by means of the ſimilar triangles CHK, CFD, we may have the line FD; and from the ſimilar triangles CEO, FOD, we may obtain OD; whence we ſhall finally arrive at OD in another manner, independent on the firſt, and then ED $=$ EO $+$ OD, which will give us an analytical equation.

I have here produced the order of arguing only, which might be uſed to bring us to an equation; omitting the actual operation, becauſe the problem will be completely ſolved in another place.

How we are to proceed when angles are concerned.

66. It will often require ſome particular expedients to be made uſe of, in ſuch problems in which angles are concerned; for by ſome artifice we muſt paſs from the properties of angles to thoſe of lines, which may enter the problem in their ſtead. I will take an example of this from the 10th propoſition of the fourth Book of *Euclid*. Let it be required, upon the given right line AB, to conſtruct an iſoſceles triangle ABC, of which the angle at A ſhall be half of either of the angles ABC, or ACB. Let the triangle ABC be ſuch a triangle, and therefore the two angles ACB, ABC, will be equal to each other, and thence the ſides AC, AB, will alſo be equal. Let the right line CD be drawn in ſuch a manner, that it may biſect the angle ACB. Then the two triangles ACB, CDB, will be ſimilar, from whence we ſhall have this analogy, AB . BC :: BC . BD. But it is BC $=$ DC $=$ AD, and therefore it will be AB . AD :: AD . DB. And now ſee the problem propoſed reduced to another, which is, to divide the given line AB in extreme and mean proportion. Wherefore this ſecond problem being reſolved, the point D will be found, and the problem at firſt propoſed will then be ſolved. For biſecting DB in E, and raiſing the perpendicular EC, it will meet in C an arch BC, which is deſcribed with radius AB from the centre A. Then if from the point found C we draw the lines CA, CB, the triangle ACB ſhall be ſuch as is required.

Fig. 9.



67. Now

67. Now when the equation of a problem is found, all that remains to be done is, to derive the value of the unknown quantity from it; that is, to reduce the unknown quantity to be equal to some known and given quantities, in which consists the solution of the problem. And this is called the *Resolution of the Equation.* Equations how reduced.

For this purpose we must call to our assistance the following Axioms.

1. If to two equal things we shall add equals, or if we shall subtract equals from them, the sums or the remainders will also be equal.

2. If equal things are multiplied or divided by equals, the products or quotients will also be equal.

3. If from equals a root be extracted with an equal index, the roots or quantities resulting will be equal.

4. If equals are raised to a power with an equal index, those powers or resulting quantities will be equal.

From the first of these axioms we learn, that if we should desire that any term of an equation, which is on one side of the mark of equality, should pass to the other side; this may always be done without destroying the equality of the terms. Let the equation be $ax + bb = -xx + cc$; if we add xx to both the members of this equation, it will be $ax + bb + xx = xx - xx + cc$, in which $xx - xx$ expunge one another, and there will remain $ax + bb + xx = cc$, where the term xx has passed into the first member of the equation; from whence if bb is to be taken away, it will be $ax + bb + xx - bb = cc - bb$; but $bb - bb$ expunging one another, the remaining equation will be $ax + xx = cc - bb$, where the term bb has passed into the second member of the equation. Wherefore in general, when we would have any term pass from one side of the equation to the other, it will be enough to expunge it on one side, and write it on the other with it's sign changed. In consequence of this, we may at pleasure make a term positive which in the equation is negative, and so on the contrary; and that will be by writing it on the opposite side, and changing it's sign. Therefore $aa - xx = bb$ will be the same as $aa - bb = xx$, or $xx = aa - bb$. Wherefore if there shall be the same term on each side of the equation, and affected with the same sign, they may both be expunged without injuring the equation. As, if it were $ax - xx = bb - xx$, it would be reduced to $ax = bb$. For, transposing the term $-xx$, it would be then $ax + xx - xx = bb$, where $xx - xx$ destroy each other. The same thing would follow, if, instead of transposing the term which is common to both members, it were added to both if in the equation it were negative, or subtracted from both if affirmative.

68. From the second axiom we learn, that if an equation should have fractions in it, it may always be freed from them without prejudice to the equation; by reducing every term to a common denominator, and then rejecting that Reduced by multiplication.

deno-

denominator: becauſe equal quantities multiplied by equals make equal products. Let the equation be $a - \frac{xx}{b} = b$. Reducing all to a common denominator, it will be $\frac{ab - xx}{b} = \frac{bb}{b}$, and multiplying all by b, or rejecting the common denominator, it will be $ab - xx = bb$. And if beſides we would have the term $- xx$ to be poſitive, it will be $ab = bb + xx$, or otherwiſe $xx = ab - bb$. Let the equation be $\frac{ax}{2} - \frac{bxx}{a} = ab$. Reducing to a common denominator, it will be $\frac{aax - 2bxx}{2a} = \frac{2aab}{2a}$, and multiplying all by $2a$, it will be $aax - 2bxx = 2aab$. And if we deſire beſides, that the term $- 2bxx$ ſhould be poſitive, and moreover that all the terms in which the letter x is concerned ſhould be on one ſide of the equation, make $2bxx - aax = - 2aab$; or reducing the whole equation to one ſide, by which it will be equal to nothing, it will be $2bxx - aax + 2aab = 0$.

Reduced by diviſion.

69. By the ſame axiom we may free any letter, or any power of a letter, in any equation, from it's co-efficient, or from any quantity in which it happens to be multiplied; and that is by dividing every term by that co-efficient. Now let there be $2bxx - aax = - 2aab$, and let it be required to free the term $2bxx$ from it's co-efficient $2b$. Then dividing each member of the equation by the ſame quantity $2b$, the quotients $\frac{2bxx - aax}{2b} = - \frac{2aab}{2b}$ ſhall ſtill be equal, and therefore $xx - \frac{aax}{2b} = - aa$. Again, if the equation is $ax - \frac{a^3}{b} = bb - \frac{3bxx}{2a} - bx$, and if it were deſired that xx ſhould be poſitive, freed from it's fraction and co-efficient, and that all the terms which any how contain the letter x ſhould be on one ſide of the equation, and known terms on the other; write then $\frac{3bxx}{2a} + bx + ax = bb + \frac{a^3}{b}$, multiply all the terms by $2a$, and it will be $3bxx + 2abx + 2aax = 2abb + \frac{2a^4}{b}$; then divide every term by $3b$, and the equation will become $xx + \frac{2}{3} ax + \frac{2aax}{3b} = \frac{2}{3} ab + \frac{2a^4}{3bb}$, which has all the conditions required.

Reduced by raiſing powers.

70. From the fourth axiom we learn, that if an equation contains radicals or ſurds, it may be freed from them, by writing the ſurd term or terms on one ſide of the equation, and the rational quantities on the other, and then ſquaring each member of the equation if the root is quadratick, or cubing if cubick, &c. Thus if we had $\sqrt{aa - xx} + a = x$, we muſt write it thus, $\sqrt{aa - xx} = x - a$, and then ſquaring, $aa - xx = xx - 2ax + aa$, that is $2ax = 2xx$, or $x = a$. Thus

Thus if the equation were $\sqrt[3]{aax - x^3} - a + x = 0$, write it $\sqrt[3]{aax - x^3} = a - x$, and it will be, by cubing, $aax - x^3 = a^3 - 3a^2x + 3ax^2 - x^3$. That is $4a^2x - 3ax^2 - a^3 = 0$, or by dividing by a, $4ax - 3x^2 - a^2 = 0$.

But if the radical terms be two or more, ſo that they will not vaniſh at one operation, it muſt be repeated as often as there is occaſion. Thus $\sqrt{bx} = a + \sqrt{ax}$: write it thus, $\sqrt{bx} - \sqrt{ax} = a$; then ſquaring, it is $bx - 2\sqrt{abxx} + ax = aa$, that is $bx + ax - aa = 2\sqrt{abxx}$. And ſquaring again, $bbxx + aaxx + a^4 + 2abxx - 2aabx - 2a^3x = 4abxx$; that is $b^2x^2 - 2abx^2 + a^2x^2 - 2a^2bx - 2a^3x + a^4 = 0$. Thus $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$ by ſquaring will be $yy = ay + yy - a\sqrt{ay - yy}$, that is $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$. And ſquaring again, $yy = ay - yy$, or $2y = a$.

71. Theſe things being premiſed, the manner of reſolving equations will be eaſy, in order to obtain the value of the unknown quantity, in ſuch terms as are known and given, and which ſerve to the ſolution of the problem. But firſt the equations are ſuppoſed to be freed from all aſymmetry, that is from radicals, if the unknown quantity be under a *vinculum*; and then reduced to the moſt ſimple expreſſion; by expunging ſuperfluous terms, if ſuch there be; by dividing of each member that ſhall be multiplied by the ſame quantity; or by multiplying if ſo divided. As if, for example, we had $\frac{axx - aax + aab}{b} = \frac{a^3 + aab}{b}$, it would be reduced to $xx - ax = aa$. Further, by the firſt term of an equation is meant the aggregate of all thoſe terms, which contain the higheſt power of the unknown quantity. By the ſecond term is meant the aggregate of all thoſe terms which contain the next inferior degree of the ſame quantity, and ſo on. By the known term is meant, the aggregate of all thoſe terms which do not at all contain the unknown quantity. Whence in the equation $axx - bxx - bbx - aax = a^3 - b^3$, or elſe $axx - bxx - bbx - aax - a^3 + b^3 = 0$, the firſt term will be $axx - bxx$, or $\overline{a - b} \times xx$. The ſecond will be $- bbx - aax$, that is $- \overline{aa + bb} \times x$. The known term is $- a^3 + b^3$. In the equation $aaxx - abxx + a^4 - b^4 - a^3b = 0$, the firſt term will be $\overline{aa - ab} \times xx$; the ſecond is wanting, and the known term is $a^4 - b^4 - a^3b$. In the equation $ax^3 + bx^3 - aaxx - a^4 = 0$, the firſt term will be $\overline{a + b} \times x^3$, the ſecond $- a^2x^2$, the third is wanting, and the fourth or known term is $- a^4$. And thus it is to be underſtood in all other equations. Here it may be obſerved, that a term ſuch as $aaxx - bbxx$, (which is likewiſe to be underſtood of any other compound term, having contrary ſigns,) may be either a poſitive or negative quantity; it will be poſitive if a be greater than b, but negative if the contrary. So that when it ſhall be ordered hereafter to make ſuch a term of an equation poſitive, we muſt have regard to this explanation.

How equations are to be reſolved.

Equations further reſolved, and firſt ſimple ones.

72. This being ſuppoſed, in order to reſolve an equation; firſt, if it have a fraction, in the denominator of which the unknown quantity is found, it muſt be reduced to a common denominator. Secondly, the term of the higheſt power of the unknown quantity muſt be made poſitive, and all the terms containing the unknown quantity muſt be wrote in order on one ſide of the equation, and the known terms on the other ſide. And thirdly, if the firſt term, or that which contains the higheſt power of the unknown quantity, ſhould have a denominator, it muſt be freed from it's fraction by what is ſaid, § 68. Laſtly, if it have a co-efficient, or be multiplied into any given quantity, it muſt be freed from this, by what has been taught, § 69.

Hence it is eaſy to perceive, that by proceeding after this manner, if the equation ſhall be ſimple, or have an unknown quantity of one dimenſion only, it will be now intirely reſolved, and that unknown quantity will be found equal to known quantities only, which was the thing propoſed to be done. As if the equation were $aa - ff = \frac{bbx - aax}{2m}$, and aa were greater than bb. Then to make that term poſitive which contains the unknown quantity, write it thus, $\frac{aax - bbx}{2m} = ff - aa$; and freeing it from the denominator, it will be $aax - bbx = 2mff - 2maa$; and then from the co-efficient, it will be $x = \frac{2mff - 2maa}{aa - bb}$, in which the value of x is now intirely known. If aa were leſs than bb, we might then write it thus, $x = \frac{2maa - 2mff}{bb - aa}$, which comes to the ſame without any occaſion of tranſpoſition.

Equations reſolved, having ſimple powers.

73. When the unknown quantity is raiſed to any power, which power is the ſame in all the terms in which it is found; or, which is the ſame thing, if all thoſe terms are conceived to make but one term; then the equation is to be reſolved by the third axiom before, and we ſhall have the unknown quantity equal to known quantities only, by extracting ſuch a root out of both members of the equation, as is denoted by the index of that power. Let the equation be $bb = aa - \frac{axx + bxx}{2c}$. Now to make the term poſitive in which x is found, write $\frac{axx + bxx}{2c} = aa - bb$; and to free it from it's fraction and co-efficient, write it $xx = \frac{2c \times \overline{aa - bb}}{a + b}$, or by diviſion, $xx = 2c \times \overline{a - b}$; and laſtly, by extracting the ſquare-root, $x = \pm \sqrt{2ac - 2bc}$. Here I put the ſign of the root ambiguous, becauſe of what is ſaid at § 15. For the ſame reaſon, if it were $x^3 = a^3 + b^3$, we ſhould have $x = \sqrt[3]{a^3 + b^3}$; and ſo of all others in general.

74. But if the equation contain the unknown quantity raiſed to it's ſquare, together with the rectangle or product of the ſame into known quantities, which is called the ſecond term (and ſuch an equation is called an *Affected Quadratick*, as it is called a *Simple Quadratick* when this ſecond term is wanting); this being prepared as is aforeſaid, to both members of the equation muſt be added the ſquare of half the co-efficient of the ſecond term, (that is to ſay, the ſquare of half that quantity, whether integer or fraction, by which the unknown quantity is multiplied,) and then it is plain that the firſt member will always be a ſquare, the root of which will be the aggregate of the unknown quantity, and of the half co-efficient with it's proper ſign. And then extracting the root, this aggregate ſhall be equal to the ſquare-root of the other member of the equation; and tranſpoſing the half co-efficient as a known quantity, we ſhall finally have the unknown quantity equal to the ſum or difference (according to the nature of the ſigns) of the radical and the ſaid half co-efficient. Thus let the equation be $xx + 2ax = bb$: if we add to each member the ſquare of half the co-efficient of the ſecond term, that is aa, the equation will be $xx + 2ax + aa = aa + bb$, and extracting the ſquare-root, it will be $x + a = \pm \sqrt{aa + bb}$, and by tranſpoſing, it is $x = \pm \sqrt{aa + bb} - a$.

Affected quadraticks reſolved.

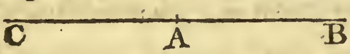
Let the equation be $bbx - aax - mxx + \frac{aabb}{m} = 0$. Making the greateſt term poſitive, and ordering the equation, it will be $mxx + aax - bbx = \frac{aabb}{m}$, and dividing by m, and adding on both ſides the ſquare of half the co-efficient of the ſecond term, it will be $xx + \frac{aa - bb}{m}x + \frac{a^4 - 2aabb + b^4}{4mm} = \frac{a^4 - 2aabb + b^4}{4mm} + \frac{a^2b^2}{m^2}$; and extracting the ſquare-root, it is $x + \frac{aa - bb}{2m} = \pm \sqrt{\frac{a^4 - 2a^2b^2 + b^4}{4m^2} + \frac{a^2b^2}{m^2}}$, and reducing the radical to a common denominator, and tranſpoſing the known term $\frac{aa - bb}{2m}$, it will be $x = \frac{bb - aa}{2m} \pm \sqrt{\frac{a^4 + 2a^2b^2 + b^4}{4m^2}}$. But the root of this radical may be actually extracted, and is either $+ \frac{aa + bb}{2m}$, or $- \frac{aa + bb}{2m}$, becauſe of the ambiguous ſign $\pm$. Therefore there will be two values of x, one is $x = \frac{bb - aa}{2m} + \frac{aa + bb}{2m} = \frac{bb}{m}$, and the other is $x = \frac{bb - aa}{2m} - \frac{aa + bb}{2m} = - \frac{aa}{m}$.

75. Therefore the ambiguity of the ſign, which the extraction of the ſquare-root always brings with it, ſupplies two values of the unknown quantity, which may be both poſitive, or both negative, or one poſitive and the other negative; and ſometimes both imaginary, according to the known quantities of which they are compoſed. For example, in the final equation $x = \pm \sqrt{aa + bb} - a$, one value or $\sqrt{aa + bb} - a$ will be poſitive, becauſe, as $\sqrt{aa + bb}$ is greater than a, the

The uſe of the ambiguous ſign.

the difference will be poſitive. The other value $-\sqrt{aa+bb}-a$ will be negative, as is evident. In the equation $x=a\pm\sqrt{aa-bb}$, (ſuppoſing b to be leſs than a,) both the values will be poſitive, becauſe $\sqrt{aa-bb}$ is leſs than a. And for the ſame reaſon, in the equation $x=\pm\sqrt{aa-bb}-a$, both the roots will be negative. Now, if b were greater than a, both would be imaginary, as I have already obſerved at § 15, becauſe then $\sqrt{aa-bb}$ would be the ſquare-root of a negative quantity. In the equation $x^4=a^4-b^4$, which requires twice the extraction of the ſquare-root, that is, $xx=\pm\sqrt{a^4-b^4}$, and thence $x=\pm\sqrt{\pm\sqrt{a^4-b^4}}$, there are four values of x; two real ones, of which one is poſitive and the other negative, that is, $x=\pm\sqrt{+\sqrt{a^4-b^4}}$, ſuppoſing b to be leſs than a; the other two are imaginary, that is, $x=\pm\sqrt{-\sqrt{a^4-b^4}}$; and when b is greater than a, all the four roots will be imaginary: and theſe obſervations may eaſily be applied to all other equations. Theſe negative values or roots, which by ſome authors are called falſe ones, are not leſs real than the poſitive, and have only this difference, that if, in the ſolution of a problem, the poſitive be taken from a fixed point, or beginning of the unknown quantity towards one part, the negative are taken from the ſame point towards the contrary part. Let A be the beginning of the unknown quantity x in a certain problem, and let the final equation (for example) be $x=\pm a$. If we take $AB=a$, and it be determined that the poſitive values ſhall proceed from A towards B; then ſhall $AB=a$ be the poſitive value of x. And conſequently, taking $AC=AB$, but on the contrary part from the point A, we ſhall have $AC=-a$, or the negative value of x. And the problem ſhall have two ſolutions, one at the point B, and the other at the point C. But the practice of all this will be beſt underſtood by the ſolution of the problems which are here to follow.

Fig. 10.



Uſe of imaginary quantities.

76. Therefore, whenever the equation to which we are led by the conditions of the problem ſhall ſupply us with none but imaginary values, this plainly declares, that the problem as now propoſed does not admit of a real ſolution, but is abſolutely impoſſible. The ſame thing is to be concluded, when the final equation brings us to an abſurdity, ſuch as if it ſhould give us a finite quantity equal to nothing, or the whole equal to the part, or ſuch like. We ſhould come to an abſurdity of this kind, if in the right line $AB=a$, it were propoſed to find ſuch a point C, as that the ſquare of the whole line ſhould be equal to the two ſquares of the two ſegments. For, making $AC=x$, it would be $aa=xx+\overline{a-x}|^2=xx+aa-2ax+xx$, that is $2xx=2ax$, or $x=a$; which is as much as to ſay, that the part is equal to the whole. We ſhould likewiſe fall into an inconſiſtency, if, aſſuming

Fig. 1.

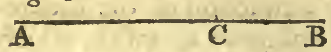
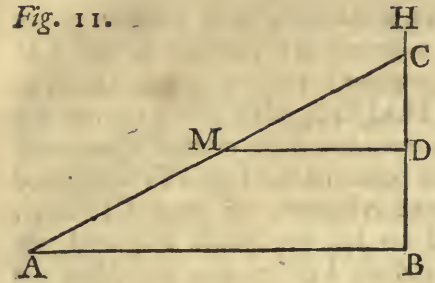


Fig. 11.



assuming a right line, as AB, and raising an indefinite perpendicular upon it BH, we should seek for a point in this, as C, from whence we might draw the right line CA to the given point A, so as that the two lines CB, CA, may be parallel. For, making $BA = a$, $BC = x$, and taking $BD = \frac{1}{2}x$, and drawing DM parallel to BA; because of similar triangles CBA, CDM, it would be $DM = \frac{1}{2}a$. But if CA and CB are parallel, it ought to be $DM = BA$, and therefore $\frac{1}{2}a = a$, which is an impossible equation.

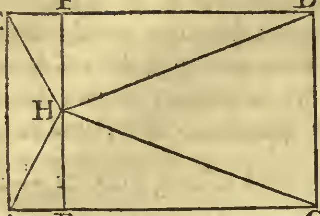
Now if it should be pretended, that the first of the two foregoing equations, or $2xx - 2ax = 0$, is no otherwise absurd, but that it supplies us with two values of x, which, though useless, are however real and consistent; relying upon this argument, that if we divide the equation by $2x - 2a$, there will result $x = 0$, a real value which solves the problem. For taking $x = 0$, or dividing the line AB in the point A, one part of it will be 0, and the other will be a. Therefore the square of the whole line will be equal to the squares of the two segments; that is, $aa = 0 + aa$. Now dividing the same equation by $2x$, there will result $x = a$, which is a real value, and resolves the problem, by dividing the line in the point B. Whoever should argue thus, as I said before, I should not venture to oppose him; but whatever is the true notion of this and such like equations, it is however certain, that they only make us know what we knew before.

For an example of an equation which brings us to an absurd conclusion, I have taken one which gives us a finite quantity equal to nothing, or the whole equal to the part. Yet this must be understood only when the unknown quantity cannot be of an infinite magnitude, and the problem is no more than a determinate problem; for otherwise such equations may be very true, as will be seen hereafter.

77. Sometimes we may meet with equations which contain the same quantities on both sides the mark of equality, and therefore when reduced bring us finally to this conclusion, that $0 = 0$. Such equations as these (which are called *Identical Equations*) inform us only, that the value of the quantity required may be what we please, as it vanishes out of the equation; and that the proposition is rather a theorem than a problem. Here follows an example of this. In

What we learn from identical equations.

Fig. 12.



In the given rectangle ACDE, from a given point B in the side AC is drawn BF parallel to the side AE; in BF is required such a point H, that drawing the lines HA, HC, HD, HE, to the several opposite angles, the sum of the squares of HA, HD, shall be equal to the sum of the squares of HE, HC. Make AB $= a$, BC $= b$, CD $= c$; and supposing H to be the point required, let BH $= x$, and therefore HF $= c - x$. Now the square of HA $= aa + xx$, that of HC $= bb + xx$, that of HD $= bb + cc - 2cx + xx$, and that of HE $= aa + cc - 2cx + xx$. And hence the equation $aa + xx + bb + cc - 2cx + xx = bb + xx + aa + cc - 2cx + xx$. Now as it is an identical equation, the same as $0 = 0$, which is as much as to say, that in the right line BF, wherever we take the point H it will always agree to the property required.

Equations and problems how divided.

78. Equations which reduced contain the unknown quantity of one dimension only, are called *Simple Equations*, or of the *first degree*. Those which contain the unknown quantity raised to the square, whether they are quadraticks simple or affected, are said to be of the *second degree*. Those which contain the unknown quantity raised to the cube, however the other terms may be, are said to be of the *third degree*. And so accordingly are those of the *fourth*, *fifth*, and higher degrees. Moreover, those problems which are expressed by simple equations, as also those of the second degree, are called *Plane Problems*, because they may be constructed by the common Geometry of *Euclid*, or by rules and compasses only. All the others are called *Solid Problems*, because for their construction is required the description of certain curves, which therefore are called *Solid Places*. I shall say nothing here of the Resolution and Construction of Solid Problems, intending to treat of them expresly in Sect. IV.

Equations may sometimes be depressed to a lower degree.

79. There are many equations, which at first sight seem to be of that degree which is intimated by the index of the greatest power of the unknown quantity, which, however, when duly managed may be brought down to an inferiour degree. Of this kind are all those in which, besides the first term, which is that of the highest power of the unknown quantity, and the term which is entirely known, one other term is contained, in which the unknown quantity ascends to a power which is the square-root of the power of the first term. As if the equation were this, $x^4 - 2aaxx = b^4$; which being managed by the Rule of Affected Quadraticks, is reduced to this, $xx = aa \pm \sqrt{a^4 + b^4}$; and therefore $x = \pm \sqrt{aa \pm \sqrt{a^4 + b^4}}$. After the same manner, this equation $x^6 + a^3x^3 - b^6 = 0$, being reduced, becomes $x^3 = \frac{-a^3 \pm \sqrt{a^6 + 4b^6}}{2}$, and

therefore

therefore $x = \sqrt[3]{\frac{-a^3 \pm \sqrt{a^6 + 4b^6}}{2}}$; and infinite others of a like nature. There are others of the ſame kind, which by means of the extraction of a root may be brought down to an inferiour degree. Thus $x^4 - 2ax^3 + aaxx - 2bbxx + 2abbx + b^4 = aabb + b^4$, having it's firſt member a ſquare, the root of which is $xx - ax - bb$, may be reduced to a lower equation, $xx - ax - bb = \pm b\sqrt{aa + bb}$. Thus, in the equation $x^3 + 3axx + 3aax = b^3$, if we add a^3 on both ſides, it will be $x^3 + 3axx + 3aax + a^3 = a^3 + b^3$, of which the firſt member is a cube, whoſe root is $x + a$. Therefore the equation reduced lower will be $x + a = \sqrt[3]{a^3 + b^3}$. But it is not always thus eaſy, to know what quantity may be added or ſubtracted to or from the firſt member of the equation, ſo that it may become a perfect power, nor can any method be aſſigned for it; ſo that the induſtry and practice of the analyſt can only be of ſervice in theſe caſes.

80. But, if the propoſed problem ſhould be of ſuch a nature, that one unknown quantity being aſſumed, would hardly or not at all be ſufficient to have all the denominations that are neceſſary for finding the equation; in this caſe may be taken one, two, three, or as many more unknown quantities as are needful. And if the problem be determinate in it's own nature, it will always ſupply conditions for as many equations as are the unknown quantities aſſumed. Then, by means of each of theſe equations, one of the unknown quantities will be eliminated, or it's value may be found by the remaining and the given quantities; ſo that finally we ſhall arrive at the laſt equation, which will contain one unknown quantity only. The manner of performing theſe operations will be beſt underſtood by the examples.

Problems will often require more unknown quantities than one.

Firſt, let there be two ſimple equations, or of the firſt degree; as, ſuppoſe for example $a + x = b + y$, and $2x + y = 3b$; and let us eliminate y, and retain x. Now, by means of which we pleaſe of the two equations, ſuppoſe of the firſt, by the help of proper tranſpoſitions of the terms, we may find the value of y, which will be $y = a + x - b$. This value may be ſubſtituted inſtead of y in the ſecond, and we ſhall have a new equation $2x + a + x - b = 3b$, that is $x = \frac{4b - a}{3}$. And this value being ſubſtituted inſtead of x in either of the two propoſed equations, we ſhall have the value of $y = \frac{2a + b}{3}$. This may alſo be obtained by deriving two values of y from the two equations, and comparing them together. For from the firſt equation we ſhall have $y = a + x - b$; and from the ſecond, $y = 3b - 2x$; wherefore it will be, by compariſon, $a + x - b = 3b - 2x$, and thence $x = \frac{4b - a}{3}$, as before.

81. After

How they are to be eliminated.

81. After the ſame manner we muſt proceed, when the equations contain the unknown quantity, which is to be eliminated, raiſed to the ſecond dimenſion; if by means of one of the two given equations, or by the tranſpoſition of the terms alone, or by the rule for ſimple or affected quadraticks, we can have a value to be ſubſtituted in the other equation. Let the two equations be $xx + 5ax = 3yy$, and $2xy - 3xx = 4aa$. Now if we would eliminate y, the ſecond equation will give $y = \frac{4aa + 3xx}{2x}$, and therefore $yy = \frac{16a^4 + 24aaxx + 9x^4}{4xx}$. This value being ſubſtituted in the firſt equation, it will be $xx + 5ax = \frac{48a^4 + 72aaxx + 27x^4}{4xx}$; which, by reduction, will be $23x^4 - 20ax^3 + 72aaxx + 48a^4 = 0$. But if we would eliminate x, finding it's value by either of the two equations, for example by the ſecond, we ſhould have $x = \frac{y}{3} \pm \frac{\sqrt{yy - 12aa}}{3}$. This being ſubſtituted in the firſt equation, it will become $\frac{2yy - 12aa \pm 2y\sqrt{yy - 12aa}}{9} + \frac{5ay \pm 5a\sqrt{yy - 12aa}}{3} = 3yy$. This being freed from radicals, and ſet in order, after a long calculation will come out $69y^4 - 90ay^3 + 72aayy + 40a^3y + 316a^4 = 0$.

Quantities to be eliminated by compariſon.

82. Often by two equations, in which the unknown quantity to be eliminated is raiſed in both to the ſame degree, may be found by means of either of them the value of the higheſt power of the unknown quantity; and that is by putting that higheſt power alone on one ſide of the equation, and all the other terms on the other ſide: then theſe two values being compared to each other, will give an equation of a lower degree. The ſame operation may be repeated again, and ſo on, till we have an equation truly ſimple in reſpect of the unknown quantities, and conſequently it's value expreſſed by the other unknown quantity, and by ſuch as are known. Then this value being ſubſtituted in one of the given equations inſtead of the unknown quantity and it's powers, we ſhall have an equation expreſſed by the other unknown quantity only, and ſuch as are known.

Let the two equations be $y^3 + aay = bxx$, and $y^3 - bxx = aax$, out of which y is to be eliminated. Therefore by the firſt it will be $y^3 = bxx - aay$, and by the ſecond, $y^3 = aax + bxx$. Then by compariſon, $bxx - aay = aax + bxx$, or $y = -x$. Then making a due ſubſtitution in either of the two equations, we ſhall have $-x^3 - aax = bxx$, or $x^2 + bx = -aa$. Again, let the two equations be $xx + 5ax = 3yy$, and $2xy - 3xx = 4aa$, from which we are to eliminate x. It will be by the firſt $xx = 3yy - 5ax$, and by the ſecond, $xx = \frac{2xy - 4aa}{3}$. Therefore the equation will be $3yy - 5ax = \frac{2xy - 4aa}{3}$.

From

From hence we ſhall have $x = \frac{9yy + 4aa}{2y+15a}$; and this value being ſubſtituted in one of the propoſed equations, in the firſt for inſtance, it will be as is found above.

But if in the two equations the unknown quantity to be eliminated do not aſcend to the ſame power in the higheſt terms, the equation of the lower degree is to be multiplied by ſuch a power of the ſame quantity, that it may be of the ſame degree as the other; and then you are to proceed as before. Thus, if we have $y^3 = xyy + 3aax$, and $yy = xx - xy - 3aa$, and we are to expunge y; multiply the ſecond equation by y, and it will be $y^3 = xxy - xyy - 3aay$. Therefore $xyy + 3aax = xxy - xyy - 3aay$, which, being compared with the value of yy given by the ſecond propoſed equation $yy = xx - xy - 3aa$, will give $\frac{xxy - 3aay - 3aax}{2x} = xx - xy - 3aa$, or $3xxy - 3aay + 3aax = 2x^3$, and therefore $y = \frac{2x^3 - 3aax}{3xx-3aa}$; which being ſubſtituted in one of the propoſed equations, ſuppoſe in the ſecond, will be $\frac{4x^6 - 12aax^4 + 9a^4xx}{9x^4-18aaxx+9a^4} = xx - 3aa - \frac{2x^4 - 3aaxx}{3xx - 3aa}$; or reducing to the ſame denominator, $x^6 + 18a^2x^4 - 45a^4x^2 + 27a^6 = 0$.

In particular caſes particular expedients may often be uſed, and there may be more expedite methods of coming to a concluſion; but theſe do not fall under any rule. An example may be ſeen of this in theſe two equations, $x + y + \frac{yy}{x} = 20b$, and $xx + yy + \frac{y^4}{xx} = 140bb$. If we would eliminate x we muſt tranſpoſe y in the firſt equation, which will then be $x + \frac{yy}{x} = 20b - y$; and ſquaring both parts, it will be $xx + 2yy + \frac{y^4}{x^2} = 400bb - 40by + yy$, that is $xx + yy + \frac{y^4}{x^2} = 400bb - 40by$. But the firſt member of this equation is the ſame as that of the ſecond propoſed equation, and therefore it will be $400bb - 40by = 140bb$, or $y = \frac{13b}{2}$.

83. By a calculation more laborious and long, but performed after the ſame manner, if there be three, four, or more equations, and as many unknown quantities, we may reduce them to one only. For by means of one equation we may exterminate one unknown quantity, the value of which, expreſſed by the others and known quantities, may be ſubſtituted in every one of the remaining equations. Then by means of another equation we may eliminate another unknown

When there are ſeveral equations.

unknown quantity, and it's value may be ſubſtituted in thoſe that remain; and ſo on to the end. Let there be three equations $x + y = c + z$, $z + x = a + y$, $z + y = b + x$, and we would have only one equation including z. From the firſt equation take the value of y, that is $y = c + z - x$, and ſubſtitute this value in the other two, which are then $z + x = a + c + z - x$, and $z + c + z - x = b + x$, or rather $2x = a + c$, and $2z = b - c + 2x$, which will then be in the place of the ſecond and third. In this laſt, inſtead of $2x$ ſubſtitute it's value from the other, and then it will be $2z = b - c + a + c$, that is $z = \frac{a + b}{2}$. Alſo, the ſame may be done after another manner, thus. From each of the three equations given take the value of y, for example, that is $y = c + z - x$, $y = z + x - a$, $y = b + x - z$. By the compariſon of two and two of theſe equations, which you pleaſe, you will form two equations which have no y. From one of which equations you may take the value of one of the unknown quantities, and ſubſtitute it in the other. Thus, if you make the two equations $c + z - x = z + x - a$, and $c + z - x = b + x - z$, from the firſt take the value of x, or $x = \frac{a + c}{2}$, and ſubſtitute it in the ſecond; then $c + z - \frac{a + c}{2} = b + \frac{a + c}{2} - z$; that is, $z = \frac{a + b}{2}$, as above. In the ſame manner we muſt proceed if the given equations be more in number, and more compounded. The uſe of the rules here taught will be ſeen in the ſolution of the problems.

Sometimes the number of equations may be inſufficient.

84. Whenever the conditions, or the data of the problem, do not ſupply us with as many equations as are the unknown quantities aſſumed, but that two of them will at laſt remain; the problem will always be indeterminate, and we cannot find the value of one of the unknown quantities but on ſuppoſing and determining the value of the other; in which caſe every indeterminate problem becomes determinate. To give ſome idea of theſe indeterminate problems, though by way of anticipation; let it be propoſed to ſeek two numbers, the ſum of which is equal to 30. I call the firſt number x, the ſecond will be $30 - x$ by the condition of the problem, nor ſhall I then have any means of forming another equation. Then I will call the ſecond y, and by the condition of the problem it will be $x + y = 30$. Now becauſe it is not poſſible to find matter for another equation, by which to eliminate one of the two unknown numbers, the problem of it's own nature will be indeterminate. But if I aſſign a determinate value to one of the unknown quantities, and ſuppoſe, for example, that $y = 8$, then it will be $x = 30 - y = 22$. But becauſe we may aſſign infinite values to y ſucceſſively, the values of x will alſo be infinite, and conſequently the problem is capable of an infinite number of ſolutions. I will take another example of this from Geometry. Let it be propoſed to find a rectangle

rectangle equal to a given ſquare. Let y be the baſe of the rectangle required, it's height x, and aa the given ſquare. Then I ſhall have the equation $aa = xy$; and not having matter for another equation, the problem remains indeterminate; there being in fact infinite rectangles equal to the given ſquare, the baſe may be varied infinitely, and the height alſo relatively to it. But if I add this condition to it, that the baſe, for example, ſhall be equal to half the height, or $\frac{1}{2}x$, then it will be $y = \frac{1}{2}x$, and the equation will be $\frac{1}{2}xx = aa$. And thus one of the unknown quantities may be varied an infinite variety of ways, and likewiſe the other, ſo that the problem may have an infinite number of ſolutions.

85. On the contrary, if the conditions of the problem, which are to be fulfilled, ſhall ſupply us with more equations than there are unknown quantities, the problem will be more than determinate, and by that means may become impoſſible. For, in order to be poſſible, the values of the given quantities muſt be reſtrained to a given law, which will often afford innumerable caſes in which the problem will become poſſible. In the foregoing example, of finding two numbers the ſum of which ſhall be 30, when nothing more is required, it will be an indeterminate problem; but if the condition be added, that beſides the difference of the ſquares of thoſe numbers ſhall be given, ſuppoſe for example 60, the problem will then be determinate, we having in this caſe two equations, that is, $x + y = 30$, and $xx - yy = 60$; ſo that, taking from the firſt the value of y, and ſubſtituting it's ſquare in the ſecond, it will be $x = \frac{960}{60}$, or $x = 16$, and conſequently $y = 14$. But beſides, if we ſhould annex a third condition, that the ſum of the ſquares of theſe numbers ſhould be equal to a given number, the problem is more than determinate, and therefore poſſible in one caſe only, in which the number given for the ſum of the ſquares is juſt the ſame as thoſe ſquares, that is 452. Thus, in the other example of a rectangle equal to a given ſquare, if we require that the rectangle ſhould be upon a given baſe, the problem will be determinate; but more than determinate if we ſhould alſo require, that it's ſides ſhould have a given ratio to each other. It will be poſſible only in one caſe, wherein this ratio is exactly the ſame as reſults from the other condition of the given baſe, and from the equality to the given ſquare.

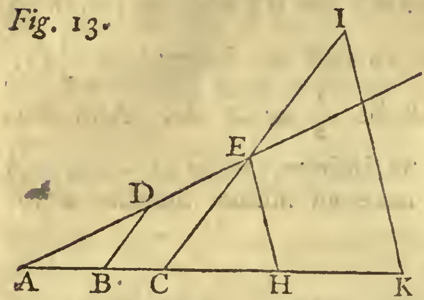
More equations may be given than ſufficient, and the problem become impoſſible.

86. The equations being reſolved, and the values of the unknown quantities being found in geometrical problems, it remains to give the conſtructions of theſe values; that is, from the given lines of the problem we muſt find ſuch, that may exactly repreſent the unknown quantities, which were propoſed to be found. In the firſt place, let the value of the unknown quantity be a ſimple rational

How ſimple equations may be conſtructed geometrically.

rational fraction, such as $x = \frac{ab}{c}$. If we convert this into an analogy, it will be $c . b :: a . x$; so that the fourth proportional required is $\frac{ab}{c}$. Therefore,

Fig. 13.



upon the indefinite right line AC taking AB $= c$, and at any angle drawing BD $= b$, and through the points A, D, drawing the indefinite line AE; if we make AC $= a$, and draw CE parallel to BD, it will be CE $= \frac{ab}{c} = x$. Or else in any angle EAC drawing the indefinite right lines AE, AC, if we take AB $= c$, AD $= b$, AC $= a$, and from the point B to the point D draw the right line BD; from the point C draw CE parallel to BD, it will be AE $= \frac{ab}{c}$. Therefore by these or other theorems or problems of Geometry may be found a fourth proportional to the three given quantities, or a third if only two be given; and we shall have the value of the unknown quantity expressed by lines. If it be $x = \frac{abc}{mn}$, the first analogy is had by taking any one of the letters of the denominator, and two of the numerator; for example, $m . b :: a . \frac{ab}{m}$, which is therefore the fourth. Then let this be found as before, and call it f; therefore it will be $x = \frac{fc}{n}$. The second analogy then will be thus, $n . f :: c . x = \frac{fc}{n}$, which will be the fourth $= \frac{abc}{mn}$. Taking therefore (Fig. 13.) AB $= m$, AC $= a$, BD $= b$, it will be CE $= \frac{ab}{m} = f$; whence producing CE indefinitely, take CH $= n$, CK $= c$, and draw HE; if from the point K the right line KI be drawn parallel to HE, it will be CH . CE :: CK . CI; that is, $n . \frac{ab}{m} :: c . \frac{abc}{mn} = \text{CI} = x$.

If the dimensions in the numerator and denominator shall be more in number, the analogies must also be more, but always in the same order.

Or if they consist of several terms.

87. Whence if the value of the unknown quantity shall be compounded of several simple fractions, or of integers and fractions; find the lines which are equal to each term, and adding or subtracting them according to their signs, they will give the line which expresses the value of the unknown quantity.

88. From

How the terms of an equation may be transformed at pleasure, and so fitted for construction.

88. From this rule we may derive a method of transforming any plane into another with a given side; a solid into another with one or two given sides, &c.; that is, any term of two, three, or more dimensions, into another which shall include any given letter, if it be of two dimensions; or one or two given letters, if it be of three dimensions. Thus let the term be bb which we desire to transform into another, which shall include the letter a. By this letter a let bb be divided, and it will be $\frac{bb}{a}$. By the given rule (Fig. 13.) a line may be found equal to $\frac{bb}{a}$, which call m. Then is $\frac{bb}{a} = m$, and therefore $bb = am$. Let ffc be so transformed as that it may include ab. A line may be found equal to $\frac{ffc}{ab}$, which call n. Then it will be $\frac{ffc}{ab} = n$, or $ffc = abn$. If it had been required that it should only include a, we should have made $\frac{fc}{a} = n$, and therefore $\frac{ffc}{a} = fn$, or $ffc = afn$. This is manifest, and needs no other examples.

How complicate terms may be transformed.

89. This being supposed, let the value of the unknown quantity be a complicate fraction, or more than one, that is, let the denominator have several terms; as $x = \frac{a^3}{bb + cc}$. One of the terms, suppose cc, is to be transformed into another, which shall include the letter b, and let it be bm. Then we shall have $\frac{a^3}{bb + bm}$, which is resolved into these two analogies, $b \,.\, a :: a \,.\, \frac{aa}{b}$, the fourth, and $b + m \,.\, \frac{aa}{b} :: a \,.\, \frac{a^3}{bb + bm}$, the other fourth. And making as usual the construction by the help of similar triangles, we shall have the line which is the value of the unknown quantity x. We might as well have left the term cc in the denominator, and have transformed bb into another, which should have included the letter c, for example cn; then the fraction would have been $\frac{a^3}{cc + cn}$, which is resolved into these analogies, $c \,.\, a :: a \,.\, \frac{aa}{c}$, and $c + n \,.\, \frac{aa}{c} :: a \,.\, \frac{a^3}{cc + cn}$. Let the fraction given be $x = \frac{b^3c}{a^3 + b^3}$; in the denominator the term b^3 may be transformed into aan, and the quantity to be constructed will be $\frac{b^3c}{a^3 + a^2n}$. This may be resolved into three analogies, $a \,.\, b :: b \,.\, \frac{bb}{a}$, and $a \,.\, b :: \frac{bb}{a} \,.\, \frac{b^3}{a^2}$, and $a + n \,.\, c :: \frac{b^3}{a^2} \,.\, \frac{b^3c}{a^3 + a^2n}$. If the denominator should have three terms, then perhaps two of them must be transformed; if it should have

have four, then three are to be transformed, &c. Thus, if there were given $x = \frac{b^3c}{a^3 + b^3 - bcc}$, after having made $b^3 = aan$, and $bcc = aap$, then it would be $x = \frac{b^3c}{a^3 + a^2n - a^2p}$. This, in the ſame manner, is reſolved into three analogies, $a . b :: b . \frac{bb}{a}$, $a . b :: \frac{bb}{a} . \frac{b^3}{a^2}$, and $a + n - p . c :: \frac{b^3}{a^2} . \frac{b^3c}{a^3 + a^2n - a^2p}$ $= \frac{b^3c}{a^3 + b^3 - bcc}$.

It can make no difficulty if the numerator of the fraction ſhould be complicate, or have ſeveral terms; becauſe the fraction will be equivalent to ſo many fractions as are the terms of the numerator. Thus $\frac{aa \pm bb}{a^3 - c^3}$ is the ſame as $\frac{aa}{a^3 - c^3} \pm \frac{bb}{a^3 - c^3}$. Therefore each being reſolved in the manner here explained, the ſum or difference of the lines ſo found, according as their ſigns may require, will give the line which is the value of the unknown quantity required.

Other fractions conſtructed.

90. But without multiplying operations, by reducing a fraction with a complicate numerator to ſeveral fractions, it will be enough to make uſe of a convenient transformation of the terms of the numerator and denominator, after the ſame manner as has already been ſeen for the denominator. Thus let it be $x = \frac{aa + bc}{a + b}$; transform the term bc into am, and the fraction will be $\frac{aa + am}{a + b}$; whence it is $a + b . a + m :: a . \frac{aa + am}{a + b}$. Let it be $\frac{aacc - abcf}{acf + bff}$; make $bf = am$, and the fraction will be $\frac{aacc - aacm}{acf + amf}$, that is $\frac{acc - acm}{cf + mf}$; then $f . a :: c . \frac{ac}{f}$, and $c + m . c - m :: \frac{ac}{f} . \frac{acc - acm}{fc + mf}$.

But if the numerator and denominator of the fraction be ſuch, that without transforming any term they may be reſolved into their linear components; then no uſe is to be made of transformation, which would only multiply operations unneceſſarily. Such will be the fractions $\frac{aab}{aa - cc}$, $\frac{a^3 - ab^2}{ac + cc}$; and ſuch others. The firſt of theſe may be reſolved into theſe two analogies, $a + c . a :: a . \frac{aa}{a+c}$, and $a - c . b :: \frac{aa}{a + c} . \frac{aab}{aa - cc}$. And the ſecond into theſe two, $c . a :: a + b . \frac{aa + ab}{c}$, and $a + c . a - b :: \frac{aa + ab}{c} . \frac{a^3 - abb}{ac + cc}$. Thus very often,

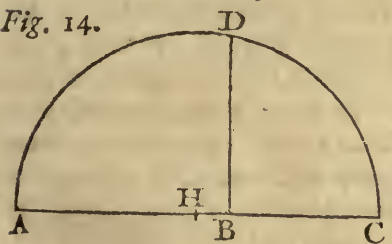
without

without transforming the terms, it will be more convenient to make uſe of the Extraction of Roots, for reſolving a fraction into analogies. Thus the fraction $\frac{aa+bc}{a}$ may be reſolved into this analogy, $a \,.\, \sqrt{aa+bc} :: \sqrt{aa+bc} \,.\, \frac{aa+bc}{a}$; though more ſimply thus, $a + \frac{bc}{a}$. The fraction $\frac{a^3+abb}{aa+cc}$ is reſolved into theſe two analogies, $\sqrt{aa+cc} \,.\, \sqrt{aa+bb} :: \sqrt{aa+bb} \,.\, \frac{aa+bb}{\sqrt{aa+cc}}$, and $\sqrt{aa+cc} \,.\, a :: \frac{aa+bb}{\sqrt{aa+cc}} \,.\, \frac{a^3+abb}{aa+cc}$. Yet ſometimes it may be neceſſary to transform a term; as in the fraction $\frac{a^3+bbc}{aa-cc}$, which cannot be reſolved even by radicals, unleſs one of the terms of the numerator be transformed, ſuppoſe bbc into acm, ſo that it may be $\frac{a^3+acm}{aa-cc}$. For then it may be $a+c \,.\, a :: \sqrt{aa+cm} \,.\, \frac{a\sqrt{a^2+cm}}{a+c}$; and $a-c \,.\, \sqrt{aa+cm} :: \frac{a\sqrt{aa+cm}}{a+c} \,.\, \frac{a^3+acm}{aa-cc} = \frac{a^3+bbc}{aa-cc}$. The ſame obtains in fractions more compounded.

Among the variety of ways here produced, it cannot eaſily be determined which will be beſt in particular caſes; perhaps more than one ſhould be tried, that we may pitch upon that which will furniſh out the ſimpleſt conſtruction of the propoſed problem.

91. As to what concerns the finding ſuch lines as are expreſſed by radicals; in the third place, let the value of the unknown quantity be an integer quadratick radical, ſuppoſe $x = \sqrt{ab}$. That is, x is a mean proportional between a and b. Take AB $= a$, and directly to it BC $= b$, and biſecting the compoſed line AC in H, with radius HC deſcribe the ſemicircle ADC, and from the point B raiſe the perpendicular BD terminated at the circumference. The rectangle of AB into BC will be equal to the ſquare of BD; that is, $ab =$ BDq, and therefore $\sqrt{ab} =$ BD $= x$. Let it be $x = \sqrt{2aa}$; taking AB $= 2a$, and BC $= a$, it will be BD $= \sqrt{2aa}$, &c. Radicals how conſtructed.

Fig. 14.



And if the radical conſiſted of complex quantities, as $x = \sqrt{4aa \pm ab}$, or elſe $x = \sqrt{3aa \pm ab \pm 2ac}$; in the firſt caſe, making AB $= 4a \pm b$; and in the ſecond, AB $= 3a \pm b \pm 2c$, and taking BC $= a$; if a ſemicircle ADC be deſcribed upon the diameter AC, and a perpendicular BD be raiſed, that perpendicular

pendicular in the firſt caſe will be equal to $\sqrt{4aa \pm ab} = x$, and in the ſecond, $\sqrt{3aa \pm ab \pm 2ac} = x$.

And, in general, let the terms under the *vinculum* be as many as you pleaſe, and combined with their ſigns in any manner, it's value may always be conſtructed by means of a ſemicircle, when every term is multiplied into the ſame letter; making, for example, one of the ſegments CB equal to that letter, and the other ſegment BA equal to the ſum or difference of all the terms divided by that letter, and raiſing the perpendicular BD. It is eaſy to perceive, that if the combination of the ſigns ſhould make the ſegment BA a negative quantity, that then the quantity under the *vinculum* would be negative, and therefore that the value of the unknown quantity would be imaginary. Such would be $x = \sqrt{ab - ac}$, ſuppoſing c to be greater than b.

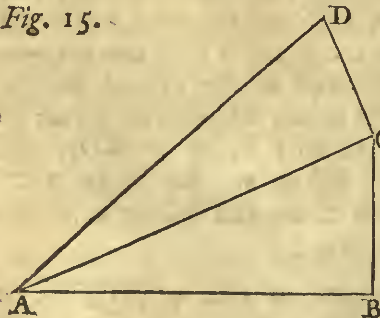
How radicals are to be transformed, in order to conſtruction.

92. Now if every term be not multiplied by the ſame letter, they may become ſuch by transforming thoſe that are not ſo. Thus, if $x = \sqrt{aa \pm bb}$, make $bb = am$, and it will be $x = \sqrt{aa \pm am}$. Then taking AB $= a \pm m$, that is AB $= a \pm \frac{bb}{a}$, and BC $= a$, and deſcribing the ſemicircle, it will be BD $= \sqrt{aa \pm bb} = x$. In like manner, having given $x = \sqrt{aa + bb - cc}$, make $bb = am$, $cc = an$, and it will be $\sqrt{aa + am - an} = x$; and taking AB $= a + m - n$, and BC $= a$, it will be BD $= \sqrt{aa + bb - cc} = x$.

Quadraticks conſtructed without transformation.

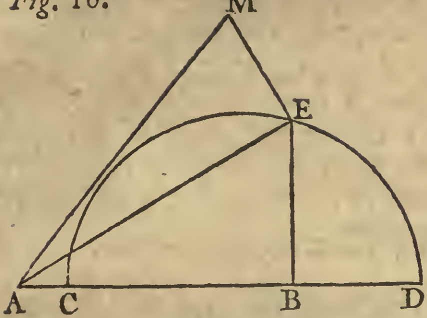
93. But however the terms may be, without making any alteration, quadratick radicals may always be conſtructed, either by a right-angled triangle alone, or by that and a circle together. Let it be $x = \sqrt{aa + bb}$, and take AB $= a$, and BC $= b$ perpendicular to AB, it will be AC $= \sqrt{aa + bb} = x$. If $x = \sqrt{2aa}$, make AB $= a$, and BC $= a$, and it will be AC $= \sqrt{2aa}$. If $x = \sqrt{3aa}$, make, as at firſt, AB $=$ BC $= a$, and upon the right line AC raiſing the perpendicular CD $= a$, it will be AD $= \sqrt{3aa}$. If $x = \sqrt{5aa}$, make AB $= 2a$, BC $= a$, then AC $= \sqrt{5aa}$. If $x = \sqrt{aa + bb + cc}$, make AB $= a$, BC $= b$ and perpendicular to AB, and upon AC raiſe the perpendicular CD $= c$; then the hypothenuſe AD will be $= x = \sqrt{aa + bb + cc}$; and ſo on to quantities more compounded. If $x = \sqrt{aa + bc}$, though the term bc be not

Fig. 15.



transformed,

Fig. 16.



transformed, in the manner ſhown above, taking AB $= a$, BC $= b$, BD $= c$, upon the diameter CD deſcribe the ſemicircle CED, then the ordinate BE will be $= \sqrt{bc}$; and drawing the hypothenuſe AE, it will be $= x = \sqrt{aa + bc}$. If $x = \sqrt{aa + bc + ee}$, upon AE draw the perpendicular EM $= e$, and it will be AM $= x = \sqrt{aa + bc + ee}$. Let $x = \sqrt{aa + bc + cc}$, taking BC $= b + c$, BD $= c$, it will be BE $= \sqrt{bc + cc}$, and AE $= \sqrt{aa + bc + cc}$. If there ſhould be more terms, the operations might increaſe, but not the difficulty. Let $x = \sqrt{aa - bb}$; on the diameter AB $= a$, let the ſemicircle ACB be deſcribed, in which inſcribe the chord AC $= b$; then, becauſe of the right angle ACB, it will be BC $= \sqrt{aa - bb}$. If $x = \sqrt{aa - bb + bb}$, produce AC to M, ſo that it may be CM $= b$; and drawing BM, it will be $= x = \sqrt{aa - bb + bb}$. If $x = \sqrt{aa - bb - bb}$, in the ſemicircle ACB inſcribe the chord AC $= \sqrt{bb + bb}$; then BC $= \sqrt{aa - bb - bb}$. If $x = \sqrt{aa - bc}$, or $x = \sqrt{aa - bc - ce}$; taking AB $= b$ in the firſt caſe, and $= b + e$ in the ſecond, add directly AD $= c$, AH $= a$, if with the diameters BD, AH, be deſcribed the two ſemicircles BCD, AEH; the ordinate AC in the firſt caſe will be $= \sqrt{bc}$, and in the ſecond $= \sqrt{bc + ce}$, and therefore, taking AE $=$ AC, and drawing the chord EH, it will be $\sqrt{aa - bc}$ in the firſt caſe, and $= \sqrt{aa - bc - ce}$ in the ſecond. If it were $x = \sqrt{aa - bc - ee}$, make AB $= b$, AD $= c$, and beſides, taking CF $= e$ perpendicular to AC, it will be AF $= \sqrt{bc + ee}$. Wherefore, making AI $=$ AF, it will be IH $= x = \sqrt{aa - bc - ee}$.

Fig. 17.

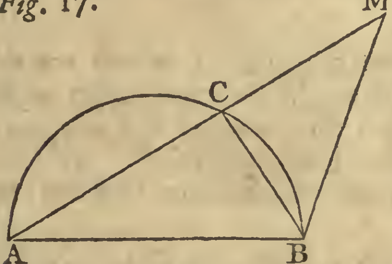
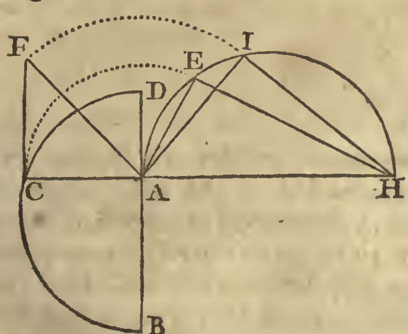


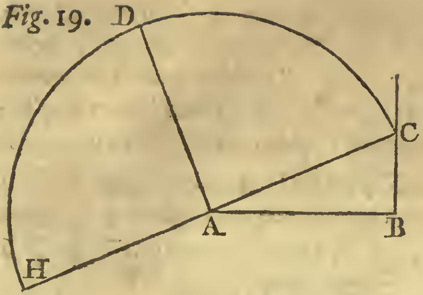
Fig. 18.



If $x = \sqrt[4]{a^4 + b^4}$, that is $x = \sqrt{\sqrt{a^4 + b^4}}$, transform the ſecond term b^4 into $aamm$, and it will be $x = \sqrt{\sqrt{a^4 + a^2m^2}}$; and taking the ſquare aa out of

 the

Fig. 19.



the ſecond radical, it will be $x = \sqrt{a\sqrt{aa + mm}}$. Make AB $= a$, and the normal BC $= m$, it will be AC $= \sqrt{aa + mm}$. Produce CA to H, ſo that it may be AH $=$ AB $= a$. Upon the diameter HC deſcribe the ſemicircle HDC, and from the point A draw AD perpendicular to the diameter; it will be AD $= \sqrt{a\sqrt{aa + mm}} = \sqrt[4]{a^4 + b^4} = x$.

Caſes more compounded are eaſily reducible to theſe here ſpecified. I ſhall add nothing about fractions compounded with rational quantities or radicals, becauſe they require nothing more than applying, or perhaps extending, the rules already given.

Affected quadraticks how conſtructed, independent of their ſolution.

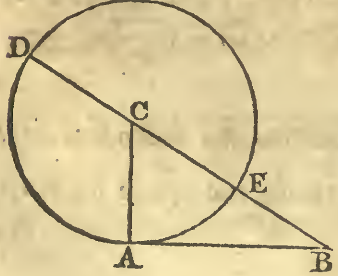
94. As to the conſtruction of affected quadratick equations, which are the higheſt I intend to treat of in this Section, I thought their reſolution to be neceſſary, and have given rules by which to obtain the values of the unknown quantity, and ſo to conſtruct them in the manner juſt now taught. Yet this previous reſolution is not abſolutely neceſſary, and without this they may be conſtructed after the following manner.

All the infinite number of affected quadratick equations may be comprehended and expreſſed by this formula, $xx \pm ax \pm bb = 0$, that is, by theſe four, which ariſe from the four different combinations of their ſigns.

1. $xx + ax - bb = 0$.
2. $xx - ax - bb = 0$.
3. $xx + ax + bb = 0$.
4. $xx - ax + bb = 0$.

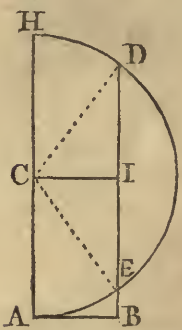
It is to be underſtood, that the letter a repreſents the whole quantity which forms the co-efficient of the ſecond term; and b is the ſquare-root of the aggregate of all the known terms. Now to contract the two firſt, take CA $= \frac{1}{2}a$, AB at right angles to it, and equal to b. With radius CA let a circle AED be deſcribed, and from the point B let the right line BD be drawn, terminating in the periphery at D, and paſſing through the centre C. Then will BE be the poſitive value of the unknown quantity, in the equation $xx + ax - bb = 0$, and BD it's negative value: as on the contrary, in the equation $xx - ax - bb = 0$, BD will be the poſitive value, and BE the negative value. And in effect, by reſolving the two equations, they

Fig. 20.



they are $x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb}$, and $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb}$. And by the conſtruction, it being CA = CD = CE = $\frac{1}{2}a$, and AB = b, it will be CB = $\sqrt{\frac{1}{4}aa + bb}$, and therefore BE = $\sqrt{\frac{1}{4}aa + bb} - \frac{1}{2}a$, which is the poſitive value of the unknown quantity in the firſt equation; and BD taken negatively, $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$, will be the negative value. Thus BD, taken poſitively, $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$, is the poſitive value of the unknown quantity in the ſecond equation; and becauſe of CB greater than CE, EB will be negative, $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa + bb}$, which is the negative value.

Fig. 21.



The third and fourth formulas are thus conſtructed. Taking CA = $\frac{1}{2}a$, and AB at right angles equal to b, as in the foregoing conſtruction; and with radius CA deſcribing a ſemicircle ADH; draw BD parallel to AC. The two right lines BE, BD, will be the two values, or the two negative roots of the equation $xx + ax + bb = 0$; and the two poſitive values in the equation $xx - ax + bb = 0$. Now reſolving the equations, the third will give us $x = -\frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$, and the fourth $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$. Therefore, drawing the right lines CD, CE, and CI perpendicular to BD, it will be ID = IE = $\sqrt{\frac{1}{4}aa - bb}$, and therefore BE negative $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, the negative value of the unknown quantity in the third equation, becauſe BI is greater than IE. And BD taken negative will be $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, the other negative value in the ſame third equation. On the contrary, BD will be poſitive, $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, and BE poſitive, $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$, both being the poſitive values of the unknown quantity in the fourth equation.

Therefore, to conſtruct any affected quadratick equation, it will ſuffice to aſſume the radius CA equal to half the co-efficient of the ſecond term, and the tangent AB equal to the ſquare-root of the laſt term; and the reſt as in one or the other of the two figures, according as the laſt term ſhall be poſitive or negative. Thus, for example, to conſtruct the equation $xx + ax - bx - aa + cc = 0$, make AC $= \frac{a-b}{2}$, and AB $= \sqrt{aa - cc}$ in the firſt of the two figures, if a be greater than c; and AB $= \sqrt{cc - aa}$, in the ſecond, if a be leſs than c. By this example it may be ſeen how we are to proceed in all other caſes.

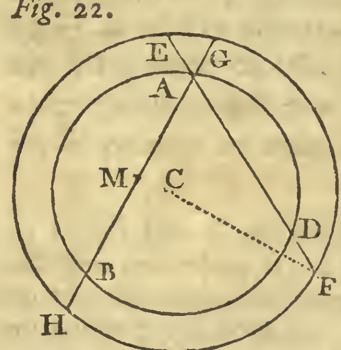
A caſe may happen, that, in the conſtruction of Fig. 21, the right line BD ſhall not cut, but touch the circle ADH; or that it may neither cut nor touch it. It will touch it when it is AC = AB, that is, $\frac{1}{2}a = b$, and the two values of the unknown quantity of the equation, BE, BD, ſhall be equal, one poſitive and the other negative. It will neither touch it nor cut it when BA is greater than AC, that is, b greater than $\frac{1}{2}a$; and the unknown quantities will not have any value at all, but will be impoſſible or imaginary. And this agrees perfectly with the analytical reſolution, becauſe when it is $\frac{1}{2}a = b$, it will be $\frac{1}{4}aa - bb = 0$, and therefore the two values $x = -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, and $x = \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, will be $x = -\frac{1}{2}a$, and $x = \frac{1}{2}a$. And when $\frac{1}{2}a$ is leſs than b, then $\sqrt{\frac{1}{4}aa - bb}$ will be an imaginary quantity, and therefore the two values of the unknown quantity will be imaginary.

Or otherwiſe thus conſtructed.

95. In theſe conſtructions it is neceſſary to find the ſquare-root of the laſt term of the equation, which is to ſupply us with the tangent AB of the circle. If therefore this laſt term is equal to a rectangle, or if we have a mind to make it ſo, which thing is in our own power, the four formulas aforegoing may be thus conſtructed, after another manner.

1. $xx + ax - bc = 0.$
2. $xx - ax - bc = 0.$
3. $xx + ax + bc = 0.$
4. $xx - ax + bc = 0.$

Fig. 22.

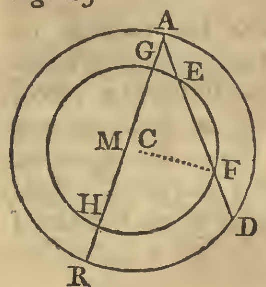


Let the circle BAD be deſcribed with any diameter, provided it be not leſs than either a or $b - c$; where I ſuppoſe b greater than c, or that b is the greater ſide of the rectangle given, and c the leſſer ſide. Now, from any point A in the periphery let the two chords $AB = a$ and $AD = b - c$ be inſcribed in the circle, and let this laſt be produced to F, ſo as that $DF = c$. With centre C of the firſt circle, and with radius CF, let a ſecond circle FGH be deſcribed, which may cut the chords AD, AB produced, in the points F, E, G, H. This being done, AG will be the poſitive value or root, and AH the negative, in the equation $xx + ax - bc = 0$. And on the contrary, AG will be the negative root, and AH the poſitive, in the equation $xx - ax - bc = 0$.

Now, to apprehend the reaſon of this, it is neceſſary to have recourſe to two properties of the circle, which are demonſtrated by geometricians; which are, that the right lines EA, DF, are equal to each other, as alſo the two GA, BH, are equal, and that the rectangles $EA \times AF$, and $GA \times AH$ are alſo equal. Theſe two theorems being ſuppoſed, the line BA is to be biſected in M. Then, by *Euclid*, ii. 6, the ſquare of MG will be equal to the ſquare of MA, together with the rectangle $BG \times GA$, that is $HA \times AG$, that is $FA \times AE$. But the

the ſquare of MA, by the conſtruction, is equal to $\frac{1}{4}aa$, and the rectangle FA × AE is equal to bc. Therefore it will be MG $= \sqrt{\frac{1}{4}aa + bc}$, and thence AG $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, the poſitive value. But AH $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, whence AH negative $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa + bc}$ the other value which is negative; both exactly as they ariſe from the reſolution of the firſt equation. For the ſame reaſon, AG negative will be $= \frac{1}{2}a - \sqrt{\frac{1}{4}aa + bc}$, and AH poſitive $= \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bc}$, which are the values of the unknown quantity in the ſecond equation.

Fig. 23.



As to the third and fourth equation, let any circle RAD be deſcribed with a diameter not leſs than a, or $b + c$. From any point of the periphery A let two chords be inſcribed in it, that is AR $= a$, and AD $= b + c$; and making DF $= c$, with centre C of the firſt circle, and with radius CF, let another circle GHF be deſcribed, which ſhall cut the two chords AR, AD, in the points G, H, F, E. This being done, AG, AH, ſhall be the two negative values in the third equation, and the two poſitive in the fourth. For, biſecting RA in M, it will be, by *Euclid*, ii. 6, the ſquare of MA equal to the rectangle HA × AG, that is RG × GA, that is DE × EA, together with the ſquare of MG. Therefore it will be $\frac{1}{4}aa = bc + \text{MG}q$, or MG $= \sqrt{\frac{1}{4}aa - bc}$. And therefore $-$ MA + MG, that is GA negative, will be $= -\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bc}$. And $-$ MG $-$ MR, that is GR negative, will be $= -\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bc}$, both the negative values of the unknown quantity in the third equation. In like manner, MG + MR, that is $\frac{1}{2}a + \sqrt{\frac{1}{4}aa - bc}$, will be GR poſitive; and MA $-$ MG, that is $\frac{1}{2}a - \sqrt{\frac{1}{4}aa - bc}$, will be AG poſitive, both the poſitive values of the unknown quantity in the fourth equation.

It is plain, both by the conſtruction of Fig. 23, and by the reſolution of the third and fourth equations, that when it is $bc = \frac{1}{4}aa$, the circle HGEF will touch the right line RA, and the two values will be equal. And if bc ſhall be greater than $\frac{1}{4}aa$, it will neither touch it nor cut it, and then the two values will become imaginary.

Having thus laid down the principal rules, I ſhall proceed to ſhow their uſe in the ſolution of ſome particular Problems.

PRO-

PROBLEM I.

An arithmetical problem.

96. Let there be a certain ſum of ſhillings, which is to be diſtributed among ſome poor people; the number of which ſhillings is ſuch, that if 3 were given to each, there would be 8 wanting for that purpoſe; and if 2 were given, there would be an overplus of 3 ſhillings: It is required to know, what was the number of the poor people, and how many ſhillings there were in all.

Let us ſuppoſe the number of poor to be x; then, becauſe the number of ſhillings was ſuch, that, giving to each 3, there would be 8 wanting; the number of ſhillings was therefore $3x - 8$. But, giving them 2 ſhillings a-piece, there would be an overplus of 3; therefore again the number of ſhillings was $2x + 3$. Now, making theſe two values equal, we ſhall have the equation $3x - 8 = 2x + 3$, and therefore $x = 11$ was the number of the poor. And becauſe $3x - 8$, or $2x + 3$, was the number of the ſhillings, if we ſubſtitute 11 inſtead of x, the number of ſhillings will be 25.

PROBLEM II.

A problem of equable motion.

97. The velocities of two bodies being given, their diſtance, and the difference of time in which they begin to move in a right line; the point in that line, and the time is required, in which the bodies will meet.

Fig. 24.

Let the firſt body be at A, the velocity of which is ſuch, that it would deſcribe the ſpace c in the time f. Let B be the ſecond body, with ſuch a velocity, that it would deſcribe the ſpace d in the time g. Let the difference of time in which they begin to move be h, and let their diſtance AB be e. Firſt, let them move the ſame way, and let them come together at the point D. Make $AD = x$, then $BD = x - e$. To obtain an equation it muſt be conſidered, that, having given the difference of time from the beginning of the motion of the body A, and of the body B, the time muſt be found employed by the body A, and alſo by the body B, and to the leſſer of theſe times, or to that of the body which moves laſt, muſt be added the given difference, and then theſe two portions of time ought to be made equal. Therefore, by the rule of proportion, we muſt ſay, if the body A deſcribe the ſpace c in the time f, in what time will it

5 deſcribe

describe the space x? That is, $c \,.\, f :: x \,.\, \frac{fx}{c}$, which is therefore the fourth term. Likewise, if the body B describe the space d in the time g, in what time will it describe the space $x - e$? That is, $d \,.\, g :: x - e \,.\, \frac{gx - ge}{d}$, which is the fourth term. Therefore the time of the body A is $\frac{fx}{c}$, and the time of the body B is $\frac{gx - ge}{d}$, and their difference is b. And if the body A began to move after the body B, it will be $\frac{fx}{c} + b = \frac{gx - ge}{d}$; and reducing to a common denominator, it will be $fdx + cdb = cgx - cge$, that is, $cgx - fdx = cdb + ceg$; and, dividing by $cg - fd$, it is $\frac{cdb + ceg}{cg - df} = x$.

If the body A move before the body B, it will be $\frac{fx}{c} = b + \frac{gx - ge}{d}$; and reducing to a common denominator, it is $dfx = cdb + cgx - ceg$, that is, $cgx - dfx = ceg - cdb$. And, dividing by $cg - df$, it is $x = \frac{ceg - cdb}{cg - fd}$. Now, if instead of x we substitute it's value now found, in the expression of the whole time $\frac{fx}{c} + b$ in the first case, and in $\frac{fx}{c}$ in the second, we shall have the time required.

I shall apply the formula to some examples. Let the body A have such a velocity, as to move 9 miles in 1 hour, and the body B to move 15 miles in 2 hours; and let them be distant from each other 18 miles, and let B begin to move 1 hour before A. Then it will be $b = 1$, $f = 1$, $c = 9$, $g = 2$, $d = 15$, $e = 18$; and therefore $x = \frac{324 + 135}{18 - 15} = 153$. Substitute this value instead of x, and also the others, in the expression of the time $\frac{fx}{c} + b$, and it will be $= 18$. Therefore the two moving bodies will be together at the distance from the point A of 153 miles, after 18 hours from the beginning of the motion.

Let the body A have such a velocity as to move 4 miles in 1 hour, and the body B to move 5 miles in 1 hour, and let them be distant 6 miles, and A begin to move 2 hours before B. Therefore it will be $b = 2$, $f = 1$, $c = 4$, $g = 1$, $d = 5$, $e = 6$. Taking the formula of the second case, it will be $x = \frac{24 - 40}{4 - 5} = 16$. And substituting this value of x with the others in the

expression

expreſſion of the time $\frac{fx}{c}$, it will be $= 4$. Therefore the two bodies A and B will be together at the diſtance of 16 miles from the point A, after 4 hours from the beginning of the motion.

But if the two bodies move contrary ways, or towards each other, let them meet, for example, in the point M; then calling AM $= x$, and retaining the ſame denominations as above, BM only will be changed, which will now be $= e - x$; and conſequently the time of the body B to deſcribe the ſpace BM will be $\frac{ge - gx}{d}$. Wherefore, if A begin it's motion after the body B, it will be $\frac{fx}{c} + h = \frac{ge - gx}{d}$; and if it begin it's motion firſt, it will be $\frac{fx}{c} = h + \frac{ge - gx}{d}$; of which equations the firſt is $fdx + cdh = cge - cgx$, that is $x = \frac{cge - cdh}{cg + fd}$; and the ſecond is $fdx = cdb + cge - cgx$, or $x = \frac{cgc + cdh}{fd + cg}$.

Let the body A have ſuch a velocity, as to deſcribe 7 miles in two hours, and the body B 8 miles in 3 hours, and let them be diſtant 59 miles, and A begin to move 1 hour before B. Therefore it will be $h = 1$, $f = 2$, $c = 7$, $g = 3$, $d = 8$, $e = 59$; and therefore, taking the ſecond formula $x = \frac{cge + cdh}{cg + fd}$, and ſubſtituting theſe values, it will be $x = \frac{1239 + 56}{21 + 16}$, that is $x = 35$. Therefore the two bodies will meet each other at the diſtance of 35 miles from the point A, after 10 hours from the beginning of motion; as will be ſeen by ſubſtituting theſe values in the expreſſion $\frac{fx}{c}$, which is the whole time of motion.

PROBLEM III.

Ευρηκα, a famous problem of Archimedes.

98. Having given the maſs of the crown of King *Hiero*, made up of a mixture of gold and ſilver, and the ſpecifick gravity of gold, of ſilver, and of the crown; it is required to find the quantity of each metal in the crown.

Let the maſs of the crown be repreſented by m, the ſpecifick gravity of gold to ſilver be as 19 to $10\frac{1}{3}$, and to the ſpecifick gravity of the crown as 19 to 17. Make x the quantity of gold in the crown, and therefore $m - x$ will be the quantity of the ſilver. The maſs of a body divided by it's denſity or ſpecifick gravity

gravity is equal to it's volume; therefore the volume of the crown will be $\frac{m}{17}$, that of the gold $\frac{x}{19}$, and that of the silver $\frac{m - x}{10\frac{1}{3}}$. But the volume of the crown is equal to both the volumes of the gold and silver together which compose it. Therefore we shall have the equation $\frac{m}{17} = \frac{x}{19} + \frac{m - x}{10\frac{1}{3}}$, that is, by reducing it to order, $\frac{19 - 10\frac{1}{3}}{19 \times 10\frac{1}{3}} x = \frac{17 - 10\frac{1}{3}}{17 \times 10\frac{1}{3}} m$, and therefore $x = \frac{6\frac{2}{3} \times 19}{8\frac{2}{3} \times 17} m$, or $x = \frac{190}{221} m$. Hence, supposing, for example, the mass of the crown to be 5 pounds, the quantity of the gold in it will be $4\frac{66}{221}$ pounds, and of the silver $\frac{115}{221}$ parts of a pound.

PROBLEM IV.

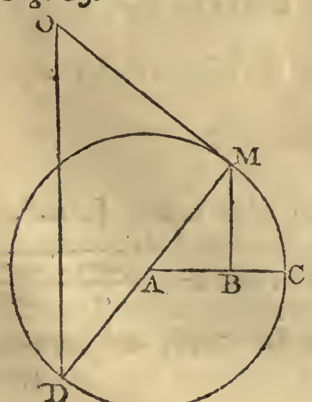
99. Let there be two weights so related, that if we take from the first 1 pound, the remainder shall be equal to the second weight increased by 1 pound. And, adding 1 pound to the first, and taking 1 pound from the second, the sum shall be double to the remainder. The quantity of each weight is required.

An arithmetical problem.

Let us call the first weight x, and the second y. Then it will be $x - 1 = y + 1$ by the first condition, and $\frac{x + 1}{2} = y - 1$ by the second. By the first we obtain this value $y = x - 2$, which, substituted in the second, will give $\frac{x + 1}{2} = x - 3$, and therefore $x + 1 = 2x - 6$; that is, $x = 7$, and consequently $y = 5$.

PROBLEM V.

Fig. 25.



100. In a given circle DCM, a line AB being given, which is intercepted between the centre and the line MB, drawn from the extremity of the diameter DM perpendicular to AC: it is required to find a point O in the tangent MO, from whence the rectangle of OM into MB may be equal to the rectangle of DM into AB.

A geometrical problem.

Make $AB = b$, $AM = a$, $MO = x$; it will be $MB = \sqrt{aa - bb}$, and by the condition of the problem, $x\sqrt{aa - bb} = 2ab$, that is, $x = \frac{2ab}{\sqrt{aa - bb}}$.

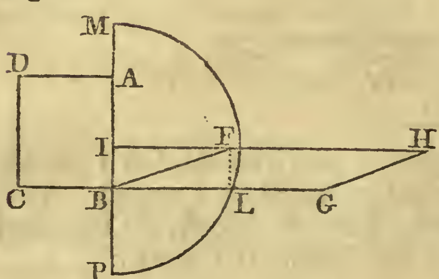
From

From the point D let there be drawn DO parallel to BM; then the triangles MBA, DMO, will be fimilar, and therefore it will be MB . BA :: DM . MO, that is $\sqrt{aa - bb} \,.\, b :: 2a \,.\, MO = \frac{2ab}{\sqrt{aa - bb}} = x$.

PROBLEM VI.

Another.

Fig. 26.



101. A rectangle being given, a parallelogram is required, the fides of which are multiples in a given ratio of the fides of the rectangle, and it's area fubmultiple.

Let ABCD be the given rectangle, AB $= a$, BC $= b$, and therefore the area $= ab$. Let the parallelogram required be BFHG, whofe fide BF fhould be to AB as n to e; and therefore BF $= \frac{an}{e}$. The fide BG fhould be to BC as m to e; and therefore GB $= \frac{bm}{e}$. Laftly, the area BFHG fhould be to the given rectangle ab, as e to r. Make BL $= x$, and therefore, drawing FL perpendicular to BG, it will be FL $= \sqrt{\frac{aann}{ee} - xx}$. Wherefore the parallelogram BFHG, that is FL × BG, will be $\frac{bm}{e}\sqrt{\frac{aann}{ee} - xx}$. And, fince this fhould be to the rectangle ABCD as e to r, we fhall have the analogy $\frac{bm}{e}\sqrt{\frac{aann}{ee} - xx} \,.\, ab :: e \,.\, r$; whence the equation $\frac{bmr}{ee}\sqrt{\frac{aann}{ee} - xx} = ab$. And taking away the radical, it will be $\frac{aann}{ee} - xx = \frac{aae^4}{mmrr}$, that is $xx = \frac{aann}{ee} - \frac{a^2e^4}{m^2r^2}$; and extracting the fquare-root, $x = \pm\sqrt{\frac{aann}{ee} - \frac{aae^4}{mmrr}}$.

In the fide BA take BI $= \frac{aee}{mr}$, and IM $= \frac{an}{e}$; and with centre I, radius IM, defcribe the femicircle MLP. The ordinate will be BL $= \sqrt{\frac{aann}{ee} - \frac{aae^4}{mmrr}} = x$. Then from the point L raifing the perpendicular LF $=$ BI, and drawing

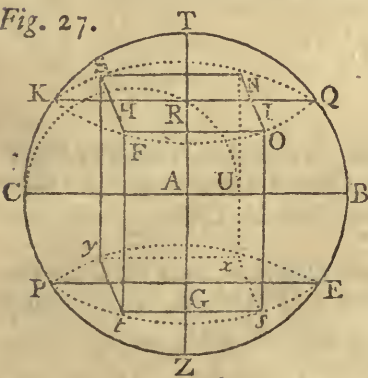
BF, take $BG = \frac{bm}{e}$, and compleating the parallelogram BHFG, it will be $= BG \times FL = \frac{abe}{r}$; that is, it will be to the rectangle $BADC = ab$, as e to r. And the ſide BF will be equal to $\sqrt{BLq + LFq} = \frac{an}{e}$: which was to be conſtructed.

The extraction of the ſquare-root has introduced an ambiguity of ſigns, and therefore two values of the unknown quantity, and conſequently two ſolutions of the problem. But it is eaſy to perceive, that theſe two values are the ſame, and differ from each other only in this, that the ſame conſtruction may be made on the ſide of B towards C.

PROBLEM VII.

102. To inſcribe a cube in a given ſphere.

Fig. 27.



A geometrical problem.

Let KQEP be a great circle of the ſphere, A it's centre, $AT = a$ it's radius, AR half of the height, or of the ſide of the cube to be inſcribed, and therefore make $AR = x$. Through the point R let there be conceived to paſs a plane perpendicular to AT, the common ſection of which, with the ſphere, ſhall be the circle QNSKFO, and the ſquare inſcribed in this circle ſhall be one face, or one plane of the parallelopiped inſcribed in the ſphere. But, becauſe this parallelopiped ought to be a cube, it will therefore follow, that $GR = SN = NO$, or $AR = RI = IO$; and beſides, that the planes which incloſe it ſhould be at right angles. In the circle KPEQ, the ordinate will be $KR = QR = \sqrt{aa - xx}$; and taking $RI = RA = x$, it will be $KI = \sqrt{aa - xx} + x$, and $IQ = \sqrt{aa - xx} - x$. And in the circle NKOQ, the ordinate $IO = \sqrt{KI \times IQ} = \sqrt{aa - 2xx}$. Therefore the equation will be $\sqrt{aa - 2xx} = x$, and thence $aa = 3xx$, or $x = \pm\sqrt{\frac{1}{3}aa}$. Now, taking AU equal to a third part of the radius AB, upon the diameter CU deſcribe the ſemicircle CRU; the point R in which it cuts the radius AT ſhall be the point required. And it will be $AR = \sqrt{\frac{1}{3}aa}$, half the ſide of the cube, taking it's poſitive value on the ſide of T, and the negative towards Z. Whence taking $AG = AR$, and through the points R, G, the ſphere being cut by two planes

perpendicular

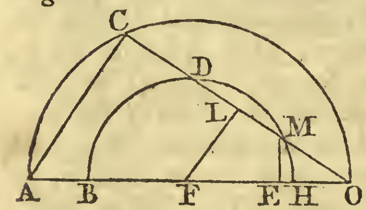
perpendicular to RG; and taking $RH = RI = RA$, and through the points I, H, the ſphere being cut by two other planes perpendicular to HI, and by two others through SN, FO, perpendicular to NO, the cube will be inſcribed. For, becauſe, by the conſtruction, as it plainly appears, the planes are perpendicular to one another, and it being $AR = RI = \sqrt{\frac{1}{3}aa}$, it will be, by the property of the circle KQEP, the ordinate $RQ = \sqrt{\frac{2}{3}aa}$, and therefore $IQ = \sqrt{\frac{2}{3}aa} - \sqrt{\frac{1}{3}aa}$, and $IO = \sqrt{KI \times IQ} = \sqrt{\frac{1}{3}aa}$; and conſequently all the ſides are equal, as was to be demonſtrated.

From the conſtruction of this problem ariſes a pretty ſimple ſynthetical demonſtration. Since AU is a third part of the radius AC, the rectangle CAU, that is the ſquare of AR, will be a third part of the ſquare of the radius, and therefore $AR = RI$. If from the centre A of the ſphere be drawn a right line AI to the point I, the ſquare of AI will be double the ſquare of AR, that is, two third parts of the ſquare of the radius. And if from the ſaid centre A a radius AO be ſuppoſed to be drawn, the ſquare of IO will be equal to the ſquare of AO, leſſened by the ſquare of AI; that is, equal to the ſquare of the radius, leſſened by two third parts of the ſame ſquare, and therefore equal to one third part of the ſquare of the radius, and conſequently IO is equal to AR, &c.

PROBLEM VIII.

Another, producing an identical equation.

Fig. 28.

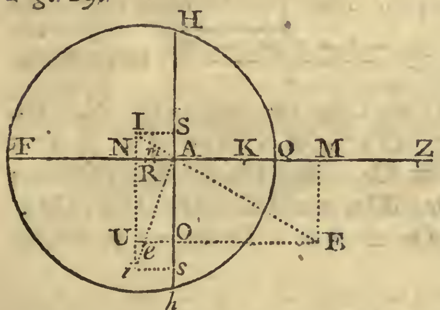


103. Two concentric circles ACO, BDH, being given, from the point O to draw a chord in ſuch manner, that it may be $OM = DC$.

Let OC be the chord required, and let F be the centre. Make $FH = a$, $FO = b$, and letting fall the perpendicular ME to AO, let $FE = x$. Then $EM = \sqrt{aa - xx}$, $EO = b - x$, and therefore $OM = \sqrt{aa - 2bx + bb}$. From the point C draw CA to the extremity of the radius FA. Then the two triangles OEM, OCA, will be ſimilar, and therefore $OM . OE :: OA . OC$. That is, $\sqrt{aa - 2bx + bb} . b - x :: 2b . OC = \frac{2bb - 2bx}{\sqrt{aa - 2bx + bb}}$. But, by *Euclid*, iii. 36, it is $DO \times OM = BO \times OH$; and therefore $DO . BO :: OH . OM$; that is $DO = \frac{\overline{a+b} \times \overline{b-a}}{\sqrt{aa - 2bx + bb}}$. And conſequently $CD = CO - DO = \frac{bb - 2bx + aa}{\sqrt{aa - 2bx + bb}} = \sqrt{bb - 2bx + aa}$. But, by the condition of the problem, it ought to be $OM = CD$. Therefore it it will be $\sqrt{bb - 2bx + aa} = \sqrt{aa - 2bx + bb}$, which is an identical equation. Whence we gather, that, however we may draw the chord OC from the point O, it will always be OM = CD. And this may alſo be known, by drawing from the centre F the perpendicular FL to any chord whatever OC. For F being the centre of both the circles, the right line FL will biſect both DM and CO; and therefore, if from the equals LC, LO, we take the equals LD, LM, there will remain equals CD, MO.

PROBLEM IX.

Fig. 29.



104. The indefinite right line NZ being propoſed, and three points N, A, K, being given in it, a fourth point M is required, ſuch that NM may be a third proportional to NK, AM. A geometrical, or rather arithmetical, problem.

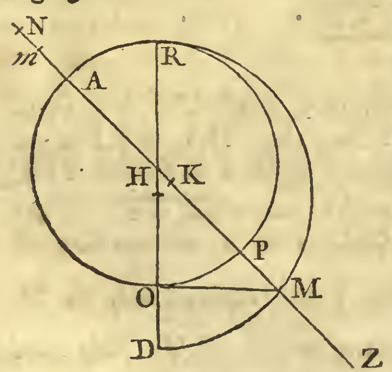
Becauſe the three points N, A, K, are given, make NA $= a$, NK $= b$, AM $= x$, and therefore MN $= a + x$. Then, by the condition of the problem, we ſhall have $b \,.\, x :: x \,.\, a + x$; and, reducing this analogy to an equation, it will be $xx = ab + bx$, or $xx - bx = ab$, which is an affected quadratick. Wherefore, if we add to each ſide the ſquare of half the co-efficient of the ſecond term, that is $\frac{1}{4}bb$, it will be $xx - bx + \frac{1}{4}bb = ab + \frac{1}{4}bb$; and extracting the ſquare-root, it is $x - \frac{1}{2}b = \pm\sqrt{ab + \frac{1}{4}bb}$, that is $x = \frac{b \pm \sqrt{4ab + bb}}{2}$.

On the right line NZ produced both ways indefinitely, take AR, AQ, equal to each other, and each equal to NK $= b$; and RF four times NA, or RF $= 4a$. Then it will be AF $= 4a + b$. With the diameter FQ let a ſemicircle FHQ be deſcribed; at the point A the ordinate will be AH $= \sqrt{4ab + bb}$. Then adding directly AO = NK $= b$, and biſecting OH in S, it will be OS $= \frac{b + \sqrt{4ab + bb}}{2} = x$. Then taking AM = OS, the point required will be M, as to the poſitive root. For, deſcribing the rectangles SN, AU, MO, and drawing the diagonals AI, AE; becauſe it is OS $= \frac{b + \sqrt{4ab + bb}}{2}$, it will be AS $= \frac{\sqrt{4ab + bb} - b}{2}$, and the rectangle OS × SA will be equal to ab, that is, equal to the rectangle OA × AN. Therefore the ſides of theſe rectangles will be be to one another in a reciprocal ratio, that is, OA . OS :: SA . AN, or EM . MA :: IN . NA. Wherefore the two lines IA, AE, will be directly to each other, and consequently the triangles IUE, AOE, will be similar, and therefore it will be AO . OE :: IU . UE; but AO = NK, OE = AM, IU = OS = AM, UE = NM. Wherefore NK . AM :: AM . NM.

The foregoing construction belongs only to the positive value of the unknown quantity, that radical being taken which is affected by the affirmative sign. But, in a like manner, that will be constructed in which the sign is negative. For the other semicircle FbQ being described, and drawing the ordinate Ab, it will be Ob $= b - \sqrt{4ab + bb}$, a negative quantity; and bisecting Ob in S, it will be Os $= \frac{b - \sqrt{4ab + bb}}{2} = x$. So that x is a negative quantity, and therefore, taking Am = Os from A towards F, m will be the other point which solves the problem. For, because it is As = Ab — sb $= \frac{-b - \sqrt{4ab + bb}}{2}$, it is therefore O$s$ × sA $= ab$ = OA × AN; so that, making the rectangle Ns, and drawing the diagonal Ai, because As × sO = OA × AN, and AN = si, it will be As . si :: AO . Oe, and therefore Os = Oe. But Os = Am, therefore Ue = Nm. But, by the similar triangles AOe, iUe, we shall have AO . Oe :: iU . Ue, and it is AO = NK, iU = Os = Oe = Am. Therefore it will be NK . Am :: Am . mN.

Without resolving the equation $xx - bx - ab = 0$, the problem may be constructed independently, by the help of § 94, in the following manner. Take RO = NK = b, and directly to it OD = NA = a. Then with the diameter RD let the semicircle RMD be described; the ordinate will be OM = $\sqrt{ab}$. With the diameter OR let another circle ARPO be described, and from the point M through the centre H let the right line MN be drawn. And taking AN = a, NK = b, AM will be the positive value of the unknown quantity. And taking the part Am = Pm from A towards N, Am will be the negative value. I omit the construction of the same equation by means of § 95, because it is evident enough of itself.

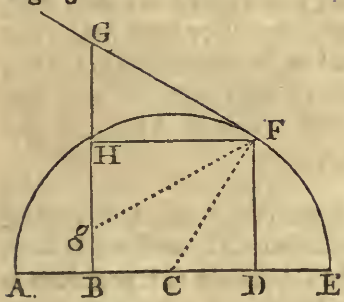
Fig. 30.



PRO-

PROBLEM X.

Fig. 31.



105. The diameter AE of the circle AFE being given, and the two portions CB, CD, from the centre C, and raiſing the perpendiculars DF, BH; in BH produced, ſuch a point G is required, that, drawing the right line GF to the point F, the rectangle GF × FD may be equal to the rectangle AC × BD. A geometrical problem.

Draw FH parallel to AE, and make the radius CA = r, CB = a, CD = b; it will be DF = $\sqrt{rr - bb}$ = BH, and make HG = x. Therefore HF = CB + CD = $a + b$, and GF = $\sqrt{aa + 2ab + bb + xx}$. Then, by the condition of the problem, we ſhall have $\sqrt{aa + 2ab + bb + xx} \times \sqrt{rr - bb} = ar + br$, and, to take away the aſymmetry, it will be $a^2r^2 + 2abr^2 + b^2r^2 = a^2r^2 + 2abr^2 + b^2r^2 + r^2x^2 - a^2b^2 - 2ab^3 - b^4 - b^2x^2$, and, by reducing, $r^2x^2 - b^2x^2 - a^2b^2 - 2ab^3 - b^4 = 0$. That is, $x^2 = \frac{a^2b^2 + 2ab^3 + b^4}{r^2 - b^2}$; and, extracting the ſquare-root, it is $x = \pm \frac{ab + bb}{\sqrt{rr - bb}}$. Therefore x, the quantity ſought, is a fourth proportional to FD, DC, and FH. Now, becauſe the angles in D and H are right, if we make the angles GFH, gFH, each equal to the angle CFD, the triangles GFH, gFH, CFD, will be ſimilar, and the points G, g, (that is G in reſpect to the poſitive value, and g in reſpect of the negative value,) will ſatisfy the queſtion. For it will be FG (or Fg) . FH :: FC . FD. But FH = BD, FC = AC; ſo that it will be GF (gF) . BD :: AC . FD. And therefore GF (gF) × FD = BD × AC.

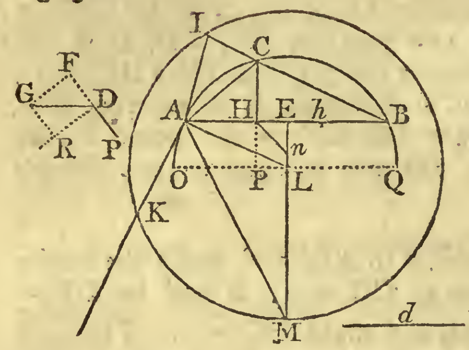
It is eaſy to perceive, that, in reſpect of the poſitive value, it is enough to draw the tangent FG at the point F, becauſe the angles GFC, HFD, are right angles. And taking away the common HFC, the angles GFH and CFD will be equal.

PRO-

PROBLEM XI.

A geometrical problem.

Fig. 32.



106. From the extremities of the given line AB, to draw two right lines AC, BC, in fuch a manner, that they may make the angle ACB equal to the given angle GDP; and that the fum of the fquares of AC and BC may be to the triangle ABC, in the given ratio of $4d$ to a.

Let AB be bifected in E, and letting fall the perpendicular CH, make $EH = x$, $HC = y$. Now, becaufe the problem is determinate, and here are taken two unknown quantities, it will be neceffary to find two equations. Make $EA = a$, then it will be $AH = a - x$, $HB = a + x$; therefore the fquare of AC will be $aa - 2ax + xx + yy$, and the fquare of CB will be $aa + 2ax + xx + yy$, and the triangle $ACB = ay$; but, by the fecond condition of the problem, the fum of thefe fquares fhould be to the triangle ABC in the given ratio of $4d$ to a; therefore we fhall have $2aa + 2xx + 2yy \,.\, ay :: 4d \,.\, a$, and thence the equation $aa + xx + yy = 2dy$. Befides, the angle ACB ought to be equal to the given angle GDP, and therefore, PD being produced, if the angle GDP be obtufe, and taking GD at pleafure, draw GF perpendicular to PF; then the angle GDF will be known, the angle GDP being given. And, becaufe alfo DG is known, which was taken at pleafure, the two lines will be given, DF which make $= b$, and GF, which make $= c$. Then, from the point A draw AI perpendicular to BC produced, the two triangles GDF, ACI, will be fimilar. Now, becaufe of the fimilar triangles BCH, BAI, we fhall have $AI = \frac{2ay}{\sqrt{aa + 2ax + xx + yy}}$, $BI = \frac{2aa + 2ax}{\sqrt{aa + 2ax + xx + yy}}$, and therefore $CI = \frac{aa - xx - yy}{\sqrt{aa + 2ax + xx + yy}}$. And now, becaufe it muft be CI . AI :: DF . FG, we fhall have $\frac{aa - xx - yy}{\sqrt{aa + 2ax + xx + yy}} \,.\, \frac{2ay}{\sqrt{aa + 2ax + xx + yy}} :: b \,.\, c$; and thence the fecond equation $2aby = aac - cxx - cyy$.

To eliminate one of the two unknown quantities; from the two equations (by § 82.) may be deduced the value of xx, that is, from the firft $xx = 2dy$

$- yy$

$-yy-aa$, and from the second, $xx = aa - yy - \frac{2aby}{c}$. Whence the equation $2dy - yy - aa = aa - yy - \frac{2aby}{c}$. That is, $dy = aa - \frac{aby}{c}$, or (making $\frac{ab}{c} = f$) $y = \frac{aa}{d+f}$, which is a value of y expressed by known quantities only. This substituted instead of y in the equation $xx = 2dy - yy - aa$, we shall have at last $xx = \frac{2aad}{d+f} - \frac{a^4}{\overline{d+f}|^2} - aa$, or $xx = \frac{a^2d^2 - a^2f^2 - a^4}{\overline{d+f}|^2}$, and thence $x = \pm \frac{a\sqrt{dd-ff-aa}}{d+f}$, a value expressed by given quantities only.

Draw AK indefinitely, making the angle KAB equal to the given angle GDP; and from the point E let fall the indefinite perpendicular EM, and from the point A the right line AL perpendicular to AK. Then making DR perpendicular to PD, the angle RDG will be equal to the angle DGF. In like manner, the angle LAE will be equal to the same DGF, and besides, the angles at E and F are right ones. Therefore the triangles LAE, GDF, will be similar, and thence $EL = \frac{ab}{c} = f$, and $AL = \sqrt{aa+ff}$. In EL produced take $LM = d$, and with centre L, radius LM, let a circle be described, which shall cut AK in K. And, because the angle KAL is a right one, the ordinate will be $AK = \sqrt{dd-ff-aa}$. Whence, making $En = AK$, and drawing MA, and nH parallel to it from the point n, it will be $ME . EA :: nE . EH$; that is, $d+f . a :: \sqrt{dd-ff-aa} . \frac{a\sqrt{dd-ff-aa}}{d+f} = EH = x$. This being done, with centre L, and radius LA, let a circle OCQ be described, and at the point H raising the perpendicular CH, draw CA, CB, and ACB shall be the triangle required. For, by *Euclid*, iii. 32, the angle ACB is equal to the angle KAE, that is, by the construction, to the angle GDP; and, by the property of the circle, $PC = \sqrt{OP \times PQ} = \frac{df+ff+aa}{d+f}$; and therefore $HC = \frac{aa}{d+f}$. And, by making the calculation, we shall find, that the sum of the squares of AC and CB is to the triangle ACB precisely in the ratio of $4d$ to a.

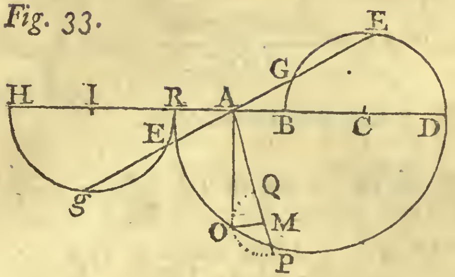
The ambiguous sign of the final equation gives us two equal values of x, one positive, and the other negative. If, therefore, EH taken towards A be considered as positive, then Eh taken towards B, and equal to EH, will be the negative value; which will require the same construction.

It is evident, that the problem will be impossible as often as dd is less than $ff + aa$, that is, LM less than LA; for then the radical will become impossible, or only imaginary.

PROBLEM XII.

Another.

Fig. 33.



107. The femicircle BED being given, and a point A being given in the diameter produced; from that point to draw a fecant AE in fuch manner, that the intercepted part GE may be equal to the radius CB.

Make CB = c, AB = b, AD = a, and AG = x. Therefore, by the condition of the problem, it will be AE = $c + x$. Now, by *Euclid*, iii. 36, the rectangle EAG is equal to the rectangle DAB, and therefore we fhall have the analogy AE . AD :: AB . AG. That is, $c + x \,.\, a :: b \,.\, x$. Whence the equation $xx + cx = ab$; which is an affected quadratick, and, being refolved as ufual, will give us $x = \pm\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c$.

On the right line DA produced, taking AR = AB = b, let the femicircle ROD be defcribed on the diameter RD; and drawing the ordinate AO, it will be $= \sqrt{ab}$. Draw OM $= \frac{1}{2}c$ perpendicular to AO, and it will be AM $= \sqrt{\frac{1}{4}cc + ab}$. Then with centre M, and radius MO, let a femicircle QOP be defcribed, and it will be AQ $= \sqrt{\frac{1}{4}c^2 + ab} - \frac{1}{2}c$, the pofitive value of x; and AP $= \sqrt{\frac{1}{4}cc + ab} + \frac{1}{2}c$. Wherefore AP, taken negatively, will be the negative value. Then, if with centre A, and radius AQ, an arch were defcribed, it would cut the femicircle BED in G the point required. And if, on the other fide, the femicircle RGH be defcribed on the diameter RH = BD, an arch on the fame centre, defcribed with radius AP, will cut it in the point required g, which belongs to the negative value. For it being EA × AG = DA × AB, that is EA $\times \sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c = ab$, it will be EA $= \frac{ab}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c}$. And therefore EG $= \frac{ab}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c} - \sqrt{\frac{1}{4}cc + ab} + \frac{1}{2}c$; that is, reducing to a common denominator, EG $= \frac{-\frac{1}{2}cc + c\sqrt{\frac{1}{4}cc + ab}}{\sqrt{\frac{1}{4}cc + ab} - \frac{1}{2}c}$. And actually making the divifion, it will be at laft EG $= c$, as it ought to be.

The fame calculus will ferve for the conftruction of the negative value, only making ufe of the rectangle HAR inftead of DAB.

Alfo,

Alſo, the ſolution of the problem may thus be demonſtrated ſynthetically.

Becauſe it is $OAq = RAD$, and $EAG = DAB$, and, by conſtruction, $AR = AB$, $AQ = AG$, $QP = BC$, $MO = MQ$, it will be $AOq + OMq = AMq = EAG + QMq$; that is, by *Euclid*, ii. 4, $AQq + 2AQM + QMq = EAG + QMq$. And, taking away the common QMq, it will be $AQq + 2AQM = EAG$; and, by the third of the ſame book, $AQq + 2AQM = EGA + GAq$. But $AG = AQ$; therefore it will be $2AQM = EGA$, that is, $AQ . AG :: EG . 2QM$. And therefore $EG = 2QM = BC$.

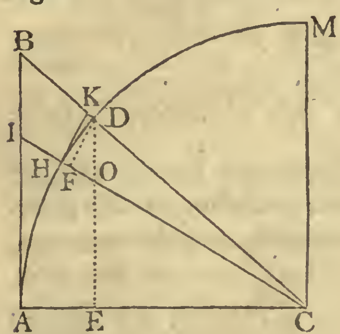
Q. E. D.

PROBLEM XIII.

108. Two arches of a circle being given, and their tangents, to find the tangent of the ſum of thoſe arches.

A trigonometrical problem, with a general solution.

Fig. 8.



Let the two given arches be AH, HD, and the tangents $AI = a$, $HK = b$, the radius $CA = r$, the tangent of the ſum of the given arches $AB = x$. It will be $CB = \sqrt{rr + xx}$, $CI = \sqrt{rr + aa}$, $CK = \sqrt{rr + bb}$. And, letting fall DE perpendicular to CA, and DF perpendicular to CH; becauſe of ſimilar triangles CBA, CDE, it will be $CE = \frac{rr}{\sqrt{rr+xx}}$, $DE = \frac{rx}{\sqrt{rr+xx}}$; and alſo, becauſe the triangles CAI, CEO, DFO, are ſimilar, we ſhall have $EO = \frac{ar}{\sqrt{rr+xx}}$, $CO = \frac{r\sqrt{rr+aa}}{\sqrt{rr+xx}}$, and $DO = \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}}$. Wherefore we ſhall have the equation $ED = EO + OD$, that is, $\frac{ar}{\sqrt{rr+xx}} + \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}} = \frac{rx}{\sqrt{rr+xx}}$, or $\frac{rx - ar}{\sqrt{rr+xx}} = \frac{b\sqrt{rr+aa}}{\sqrt{rr+bb}}$; and, ſquaring this to free it from the radicals, it will be $\frac{rrxx - 2arrx + aarr}{rr + xx} = \frac{bbrr + aabb}{rr + bb}$. Then, reducing to a common denominator, and taking away ſuch terms as deſtroy one another, it will be $r^4xx - 2ar^4x - 2abbrrx + aar^4 = aabbxx + bbr^4$; that is, $xx - \frac{2ar^4 + 2abbrr}{r^4 - aabb}x = \frac{bbr^4 - aar^4}{r^4 - aabb}$, which is an affected quadratick. Therefore, adding to each member the ſquare of half the co-efficient of the

 ſecond

ſecond term, that is the ſquare of $\frac{ar^4 + ab^2r^2}{r^4 - a^2b^2}$, it will become $xx - \frac{2ar^4 + 2ab^2r^2}{r^4 - aabb}x + \frac{aar^8 + 2aabbr^6 + aab^4r^4}{\overline{r^4 - aabb}|^2} = \frac{b^2r^4 - a^2r^4}{r^4 - a^2b^2} + \frac{a^2r^8 + 2a^2b^2r^6 + a^2b^4r^4}{\overline{r^4 - a^2b^2}|^2}$; then extracting the root, and reducing the *homogeneum comparationis* to a common denominator, it will be $x - \frac{ar^4 + ab^2r^2}{r^4 - a^2b^2} = \pm\sqrt{\frac{b^2r^8 + 2a^2b^2r^6 + a^4b^2r^4}{\overline{r^4 - a^2b^2}|^2}}$. But the quantity under the *vinculum* is a ſquare, and it's root is $\frac{br^4 + aabrr}{r^4 - a^2b^2}$, or otherwiſe $-\frac{br^4 + aabrr}{r^4 - a^2b^2}$. Therefore, in the firſt place, taking the poſitive root, it will be $x = \frac{ar^4 + abbrr + aabrr + br^4}{r^4 - aabb}$; and, taking the negative root, it will be $x = \frac{ar^4 + ab^2r^2 - aabr^2 - br^4}{r^4 - aabb}$. Now, in the firſt caſe, both the numerator and the denominator are diviſible by $rr + ab$, and the quotient is $\frac{arr + brr}{rr - ab}$; and, in the ſecond caſe, the numerator and the denominator are diviſible by $rr - ab$, and the quotient is $\frac{arr - brr}{rr + ab}$. Therefore the two values of the unknown quantity are $x = \frac{rr \times \overline{a + b}}{rr - ab}$, and $x = \frac{rr \times \overline{a - b}}{rr + ab}$. The firſt of theſe will ſerve for the tangent of the ſum of the given arches, and the ſecond for the tangent of their difference, as will eaſily be ſeen by ſolving the problem in this caſe. This value will be poſitive or negative, according as the arch, or it's tangent a, will be greater or leſs than the tangent b.

This foundation being laid, it will not be difficult to go on to the general ſolution of the problem; that is, as many ſucceſſive arches as you pleaſe, with their tangents being given, to find the tangent of the ſum of all thoſe arches; which may be done in the following manner.

Firſt, let there be three arches given, and let their tangents be a, b, c. By the foregoing ſolution, $\frac{rr \times \overline{a + b}}{rr - ab}$ will be the tangent of the ſum of two of thoſe arches, the tangents of which are a, b. Let this tangent be called z, and therefore it will be $z = \frac{rr \times \overline{a + b}}{rr - ab}$. But, by the ſame ſolution, it will be $\frac{rr \times \overline{z + c}}{rr - zc}$, the tangent of the ſum of the two arches, whoſe tangents are z, c; and z is the tangent of the ſum of the two arches, whoſe tangents are a, b.

 Therefore

Therefore $\frac{rr \times \overline{z + c}}{rr - zc}$ will be the tangent of the ſum of the three arches, whoſe tangents are a, b, c. And in this expreſſion, inſtead of z ſubſtituting it's value $\frac{rr \times \overline{a + b}}{rr - ab}$, we ſhall have the tangent of the ſum of the three arches expreſſed by the given tangents only a, b, c, which will be $\frac{rr \times \overline{a + b + c} - abc}{rr - ab - ac - bc}$. By the ſame way of arguing, we ſhall have the tangent of the ſum of four arches, their given tangents being a, b, c, f, which will be

$$\frac{rr \text{ into } arr + brr + crr + frr - abc - abf - acf - bcf}{rr \times rr - ab - ac - af - bc + bf - cf + abcf}.$$

Alſo, the tangent of the ſum of five, their given tangents being a, b, c, f, g, will be found to be

$$\frac{r^4 \times \overline{a+b+c+f+g} - r^2 \times \overline{abe+abf+acf+abg+bcf+acg+bcg+bff+afg+cfg} + abcfg}{rr \times \overline{rr-ab-ac-af-ag-bc-bf-bg-cf-cg-fg} + abcf+abcg+abfg+acfg+bcfg}.$$

And thus for as many more arches as you pleaſe. From hence may be derived a general rule, to form the fraction which ſhall expreſs the tangent of the ſum of as many given arches as you pleaſe ; which will be this.

To form the numerator of the fraction there muſt be taken the ſum of all the poſſible products of an odd number of factors, which can be made with all the given tangents. For example, if the number of tangents be ſeven, take the ſum of all theſe tangents; then the ſum of all the threes that can be made, then the ſum of all the fives, and laſtly, the product of all the ſeven. Theſe ſums are to be multiplied by ſuch a power of the radius, as each has occaſion for, that they may be of a dimenſion greater, by unity, than the number of the given tangents. And to theſe ſums muſt be prefixed the ſigns + and — alternately ; that is, to the ſum of all, the ſign + ; to the ſum of all the threes, the ſign —, and ſo on ; and thus the numerator will be completed.

To form the denominator muſt be taken the ſquare of the radius, then the ſum of all the products of an even number of factors, which can be made by the given tangents, that is of all the twos, of all the fours, &c. This ſquare of the radius, and the ſum of all the twos, of all the fours, of all the ſixes, &c. muſt be multiplied into ſuch a power of the radius, as each has occaſion for, that they may be of a dimenſion equal to the number of the given tangents. To the ſquare of the radius is to be prefixed the ſign +, to all the twos the ſign —, to the fours the ſign +, and ſo on alternately. And thus the denominator will be completed.

Now the rule for knowing what muſt be the number of all the twos poſſible, of all the threes, &c. in a given number of quantities, will be this following.

Write

Write down the number of quantities given, and thence continue the decreafing feries of natural numbers. Under thefe numbers write down in order an increafing feries of natural numbers, beginning from unity. Afterwards find the product of fo many terms of the upper feries, as is the index of the combination that is to be made. Alfo, there muft be made the product of as many terms of the feries below; and one product being divided by the other, the quotient will be the number required. So to know how many twos, threes, &c. can be made of 5 quantities, for example, write down the numbers thus:

5, 4, 3, 2, 1,
1, 2, 3, 4, 5.

The product of the two firft numbers of the upper feries is 20, which, divided by the product of the two firft numbers of the lower feries, will give 10 for the quotient. And therefore the twos will be 10. The product of the three firft is 60, which, divided by 6, the product of the three firft of the lower feries, will give the quotient 10; and therefore the threes will be 10, &c.

From the folution of this problem we obtain, by way of corollary, the folution of another which is more fimple; and that is, the tangent of an arch being given, to find the tangent of any multiple of that arch. For, in this cafe, it will be fufficient to make all the given tangents equal to one another, and equal to the tangent of the given arch. For example, make the tangent of the given arch $= a$, and let it be required to find the tangent of the double arch, the triple, &c. In the formula which we have already found for the tangent of the fum of two given arches, inftead of the letter b we muft every where put a, and we fhall have a formula or expreffion for the double arch $\frac{2arr}{rr - aa}$. In the formula for the tangent of the fum of three given arches, inftead of b and c we muft put a, and we fhall have the expreffion of the triple arch $\frac{3arr - a^3}{rr - 3aa}$. In like manner, that for the quadruple arch will be $\frac{4ar^4 - 4a^3rr}{r^4 - 6a^2r^2 + a^4}$. That for the quintuple arch will be $\frac{5ar^4 - 10a^3r^2 + a^5}{r^4 - 10a^2r^2 + 5a^4}$. And fo of all others fucceffively.

Whence we may form the following progreffion, or general canon, for the tangent of any multiple arch, according to any whole number whatever denoted by n.

$$\frac{nr^{n-1}a - \frac{n.\overline{n-1}.\overline{n-2}}{1.\ 2.\ 3} r^{n-3}a^3 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5.} r^{n-5}a^5 - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}.\overline{n-6}}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7} r^{n-7}a^7}{r^{n-1} - \frac{n.\overline{n-1}}{1.\ 2.} r^{n-3}a^2 + \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.\ 2.\ 3.\ 4.} r^{n-5}a^4 - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}.\overline{n-5}}{1.\ 2.\ 3.\ 4.\ 5.\ 6} r^{n-7}a^6} \ \&c.$$

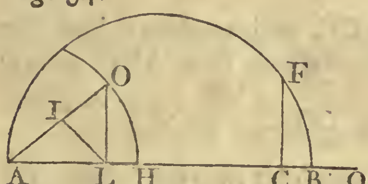
The

The tangent being found of any multiple arch, the inverſe problem will be eaſily reſolved. That is, the tangent of an arch being given, to find the tangent of any ſubmultiple arch, according to any whole number whatever. That is to ſay, to divide an arch or angle into as many equal parts as we pleaſe. Wherefore let the tangent of the given arch be b, and n the number according to which we would have the ſubmultiple arch; we muſt take the tangent found for the multiple arch by the number n, inſtead of a we muſt put x, and thus x will repreſent the tangent of the ſubmultiple arch. This tangent of the multiple arch is therefore equal to the given tangent b, whence we ſhall have an equation to determine the unknown quantity x.

Therefore the tangent b being given, and the radius r, the equation for the tangent of the ſubtriple arch will be $x^3 - 3bxx - 3rrx + brr = 0$. That for the ſubquintuple arch will be $x^5 - 5bx^4 - 10rrx^3 + 10brrxx + 5r^4x - br^4 = 0$. And ſo of the reſt.

PROBLEM XIV.

Fig. 34.



109. To find a triangle ALO, the ſides of which AO, LO, AL, and the perpendicular LI, are in continued geometrical proportion. A geometrical problem.

Take one ſide at pleaſure, or AL $= a$, and make OL $= x$. It will be, by the conditions of of the problem, AO $= \frac{xx}{a}$, and LI $= \frac{aa}{x}$. Therefore AI $= \sqrt{aa - \frac{a^4}{xx}}$, and IO $= \sqrt{xx - \frac{a^4}{xx}}$. Therefore AI + IO = AO, that is, $\sqrt{aa - \frac{a^4}{xx}} + \sqrt{xx - \frac{a^4}{xx}} = \frac{xx}{a}$. Or $\frac{xx}{a} - \sqrt{xx - \frac{a^4}{xx}} = \sqrt{a^2 - \frac{a^4}{xx}}$; and, by ſquaring, $\frac{x^4}{aa} - \frac{2xx}{a}\sqrt{xx - \frac{a^4}{xx}} + xx - \frac{a^4}{xx} = aa - \frac{a^4}{xx}$, that is $\frac{x^4}{aa} + xx - aa = \frac{2xx}{a}\sqrt{xx - \frac{a^4}{xx}}$. Now, by ſquaring again, it will be $\frac{x^8}{a^4} + \frac{2x^6}{aa} + x^4 - 2x^4 - 2aaxx + a^4 = \frac{4x^6}{aa} - 4aaxx$. And laſtly, by reducing to a common denominator, and ordering the equation, it will be $x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8 = 0$. This equation has the appearance of one of the eighth degree, but it may be obſerved to be a ſquare, and therefore, extracting it's root, it will be found to be $x^4 - aaxx - a^4 = 0$. This is an affected

affected quadratick; therefore, transposing $-a^4$, and adding $\frac{1}{4}a^4$ to both sides, and extracting the root by the common rule for affected quadraticks, it will be $xx - \frac{1}{2}aa = \pm\frac{1}{2}\sqrt{5a^4}$, that is, $xx = \frac{1}{2}aa \pm \frac{1}{2}\sqrt{5a^4}$, and finally, $x = \pm\sqrt{\frac{aa \pm \sqrt{5a^4}}{2}}$.

Therefore the unknown quantity will have four values; but it may be observed, that the quantity $\sqrt{5a^4}$ is greater than aa, and therefore, if we take the radical $\sqrt{5a^4}$ negative, that is $-\sqrt{5a^4}$, then the quantity under the common radical *vinculum* will be negative; whence the value of x will be imaginary, and therefore two values will be imaginary, that is $x = \pm\sqrt{\frac{aa - \sqrt{5a^4}}{2}}$. And two will be real, that is $x = \pm\sqrt{\frac{aa + \sqrt{5a^4}}{2}}$, both equal, but one positive and the other negative.

On the indefinite line AQ take $AL = a$, $LC = a\sqrt{5}$, and $CB = \frac{1}{2}a$. Then on the diameter AB describe the semicircle AFB, and erect the perpendicular CF. By the property of the circle, it will be $CF = \sqrt{\frac{aa + aa\sqrt{5}}{2}} = x$. Bisect AC in H, and with centre A, radius $AH = \frac{xx}{a} = \frac{1 + \sqrt{5}}{2}a$, describe the arch HO. From the point L draw $LO = CF$, and terminated at the arch HO. And if AO be drawn, and the perpendicular LI, then ALO will be the triangle required. For, because it is $AL = a$, $LO = x = \sqrt{\frac{aa + aa\sqrt{5}}{2}}$, $AO = AH = \frac{xx}{a} = \frac{1 + \sqrt{5}}{2}a$; it will be AO . LO :: LO . LA. But the two squares of AL and LO taken together, that is $aa + \frac{aa + aa\sqrt{5}}{2}$, are equal to the square of AO, that is $\frac{6aa + 2a\sqrt{5aa}}{4}$. Wherefore the angle ALO is a right angle, and thence it will be AO . LO :: AL . LI. But, because it is also AO . LO :: LO . LA, it will be likewise LO . LA :: LA . LI. The other negative value, which is equal to the positive, would serve for the construction that may be made under the line AB.

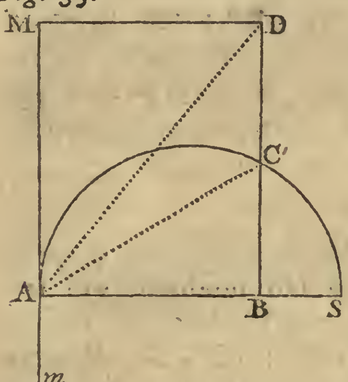
PRO-

PROBLEM XV.

110. To divide a given angle into three equal parts.

The Problem propofed contains three cafes; one is when the given angle is a right angle; another when it is obtufe; and the third when it is acute. The trifection of an angle.

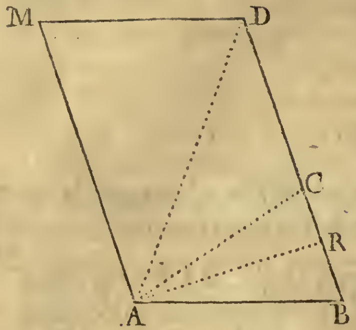
Fig. 35.



In the firft, let the given angle MAB be a right angle, which is fuppofed to be divided into three equal parts by the right lines AC, AD. Make AB $= a$, and at B raife the perpendicular BC, which produced fhall meet the line AD in D; and from the point D let DM be drawn parallel to AB. Then making BC $= x$, it will be AC $= \sqrt{aa + xx}$. But, becaufe the angle CAD muft be equal to the angle DAM, and becaufe of the parallels AM, BD, the angle DAM is equal to the angle ADC; the angles CDA, CAD, will be equal. Wherefore CD $=$ CA $= \sqrt{aa + xx}$, whence BD $= x + \sqrt{aa + xx}$. But befides, the two angles BAC, CAD, or CDA, ought alfo to be equal, and therefore in the two triangles BDA, CAB, the angle CAB will be equal to the angle BDA, and the right angle at B is common. Therefore alfo the third BCA $=$ BAD, and confequently the triangles are fimilar. Whence we fhall have AB . BC :: BD . AB; that is, $a \,.\, x :: x + \sqrt{aa + xx} \,.\, a$; and thence the equation $aa = xx + x\sqrt{aa + xx}$; and tranfpofing the term xx, and fquaring, it will be $aaxx + x^4 = a^4 - 2aaxx + x^4$, and finally, $3aaxx = a^4$, or $x = \pm\sqrt{\frac{1}{3}aa}$.

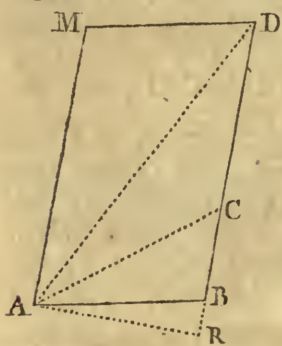
Produce AB to S, fo that it may be BS $= \frac{1}{3}$AB $= \frac{1}{3}a$. On the diameter AS let the femicircle ACS be defcribed; the ordinate BC will be $= \sqrt{\frac{1}{3}aa} = x$. Then draw AC to the point C, and take CD $=$ AC, drawing AD. The given angle will be then divided into three equal parts. For, whereas it is BC $= \sqrt{\frac{1}{3}aa}$, it will be AC $= \sqrt{\frac{4}{3}aa} =$ CD, and AD $= \sqrt{ABq + BDq} = \sqrt{aa + \frac{1}{3}aa + 2a\sqrt{\frac{4aa}{9}}} = 2a$. Therefore AD . AB :: $2a \,.\, a$ :: 2 . 1, and DC . CB :: $\sqrt{\frac{4}{3}aa} \,.\, \sqrt{\frac{1}{3}aa}$ :: 2 . 1; that is, in the very fame ratio as AD to AB. Wherefore, by *Euclid*, vi. 3, the angle BAC $=$ CAD; and, becaufe of CD $=$ CA, it will be alfo the angle CAD $=$ CDA $=$ DAM. The negative value, which is equal to the pofitive, would ferve for the divifion of the angle *m*AB.

Fig. 36.



Let the angle BAM be obtuſe, and draw BD parallel to AM, and making the reſt as above, draw AR perpendicular to BD. Since the angle ABD is known, as being the ſupplement of the given angle MAB, and the angle R is right, and the line AB is given; the line BR will alſo be known, which make $= b$. Whence AR $= \sqrt{aa - bb}$, CR $= x - b$, AC $=$ CD $= \sqrt{aa - 2bx + xx}$, and BD $= x + \sqrt{aa - 2bx + xx}$. Then, becauſe of ſimilar triangles ABC, ABD, it will be AB . BC :: BD . BA; that is, $a . x :: x + \sqrt{aa - 2bx + xx} . a$; or $aa = xx + x\sqrt{aa - 2bx + xx}$. Then taking away the aſymmetry, it is $2bx^3 - 3aaxx + a^4 = 0$, which is a ſolid equation, or of the third degree, which at preſent I ſhall leave unreſolved.

Fig. 37.



Laſtly, let the angle BAM be acute; the perpendicular from the point A to DB produced will fall under the point B in R, and therefore it will be RC $= b + x$, and AC $= \sqrt{aa + 2bx + xx}$. Wherefore, repeating the ſame argumentation as in the foregoing caſe, we ſhall have the equation $2bx^3 + 3aaxx - a^4 = 0$, which differs from the foregoing only in the ſigns.

SECT. III.

The Conſtruction of Loci, *or Geometrical Places, not exceeding the ſecond Degree.*

What are variable quantities; and what is the law by which they vary.

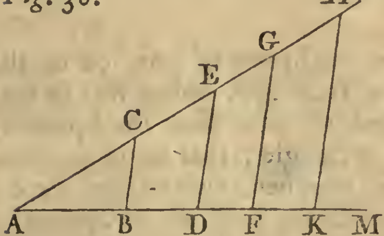
111. What are Indeterminate Problems, and how they require two unknown quantities, has been already explained at § 84. Now, becauſe the value of one of the unknown quantities may be varied an infinite number of ways, ſo, in like manner, the value of the other may be as often varied; whence they are called the *Variable Quantities* of the equation or problem, and their relation, or law which they obſerve in their variations, is expreſſed by an equation. Thus the equation

equation $bx = ay$ informs us, that, varying x as you pleafe, y muft alfo be varied, but with this condition, that x muft always have to y the conftant ratio of a to b. Thus the equation $ab = xy$ expreffes fuch a law, that the product of the two unknown quantities muft always be conftant, and equal to the product of a into b. The equation $ax = yy$ implies, that the fquare of y muft always be equal to the rectangle of x into a conftant line a; and fo of all other equations.

112. One of the two unknown quantities, fuppofe x for example, muft have it's origin from a fixed point, and muft be taken upon an indefinite right line. Then, if a determinate value be affigned to this, from the extremity is to be raifed another right line in the given angle of the problem, which line is to be taken of fuch a length as the other unknown line y ought to have, by the nature of the equation, relatively to the affigned or affumed value of x. And this ought to be repeated for every different value that x can affume. The line which fhall pafs through the extremities of all the y's is called the *Locus* of the equation. The unknown line, which is taken from the fixed point on the indefinite right line, is called the *Abfcifs*; and the other, at the given angle to it, is called the *Ordinate:* and both indifferently are called the *Co-ordinates* of the equation.

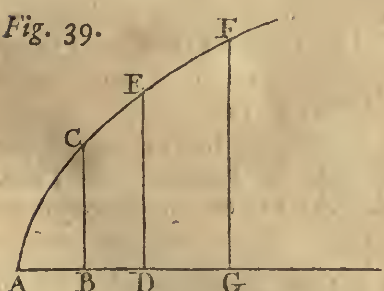
General precepts for the conftruction of *Loci*, with fome examples.

Fig. 38.



Now, for example, as to the equation $bx = ay$; upon the indefinite line AM take $AB = a$, and in any angle draw $BC = b$. Here, if we take $x = AD$, the fourth proportional will be parallel to BC, that is $DE = y$. And taking $x = AF$, then it will be $FG = y$. Alfo, taking $x = AK$, it will be $KH = y$. And thus for infinite others. And the line in which all thefe infinite points are found, C, E, G, H, &c. which are determined in this manner, will be the *locus* of the equation $bx = ay$, and which will be a right line.

Fig. 39.



In the fame manner, as to the equation $ax = yy$, if we take $x = AB$, and $BC = \sqrt{ax}$, that is, a mean proportional between AB and the given line a, it will be $BC = y$. And taking $x = AD$, and DE a mean proportional between AD and a, it will be $DE = y$. Taking $x = AG$, and GF a mean proportional between AG and a, it will be $GF = y$. And fo of all others. Now the points C, E, F, and infinite others determined in the fame manner, will form the line ACEF, which is the *locus* of the equation $ax = yy$. And the fame is to be underftood of all other equations.

 113. From

Different equations require different *loci*, and *vice versâ*.

113. From the ſeveral different laws expreſſed by the given equations, or from the different relations that the two variables or unknown quantities may have to each other, other *loci* or lines will ariſe, which will differ both in kind and in degree. So it is eaſy to perceive, that the *locus* of the equation $bx = ay$ will be a right line, as obſerved before; for y to x having a conſtant ratio, becauſe it is $y = \frac{bx}{a}$; any line ED (Fig. 38.) will be to AD, as any other FG to AF; therefore the triangles AED, AGF, will be ſimilar. This may be verified alſo by any other point H, &c. So that it muſt neceſſarily follow, that theſe points will all be in the ſame right line. But the equation $ax = yy$ requires, not that the lines BC, DE, &c. (Fig. 39.) but that their ſquares, may have a conſtant ratio to the correſponding lines AB, AD, &c. Whence it is, that the points C, E, F, &c. will not be in one right line, but in a certain curve line, called a *Parabola*. Thus a curve of a different kind from this would be the *locus* of the equation $xy = ab$; and a curve of a different kind and degree would be the *locus* of this other equation $a^3 - x^3 = y^3$. And the like of infinite others.

When the *locus* will be a right line.

114. As often as the equation ſhall not contain, in any term, either the ſquare, or ſome higher power, of one of the unknown quantities, or the product of the ſame, the *locus* will always be a right line.

When the *locus* is a conic ſection.

115. And when, in the equation, there is found the ſquare of one, or of the other, or of both the variable quantities, or their rectangle, either this or that as it may happen; and no term ſhall include a greater power than the ſquare of thoſe variable quantities, or a product above the rectangle; that is, in no term the variable quantities, either alone or multiplied together, exceed the ſecond dimenſion; the *locus* will always be one of the Conic Sections of *Apollonius*. Theſe aſſertions cannot be better demonſtrated than by actually conſtructing all the ſeveral equations of this nature.

Loci or curves diſtinguiſhed into orders.

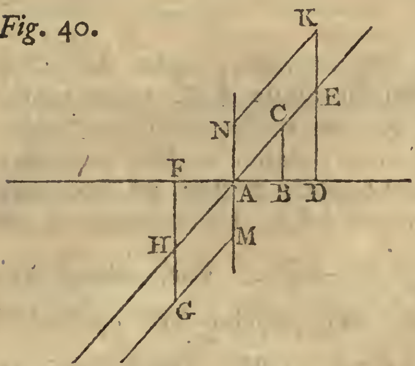
116. Equations which include the unknown quantities of one dimenſion only, that is, the *loci* to a right line, are called *Loci* or Lines of the Firſt Order. Thoſe which, either alone or multiplied together, include them of two dimenſions, that is, *loci* to the conic ſections, are called *Loci* or Lines of the Second Order, and therefore Curves of the Firſt Kind. Thoſe equations in which the variables aſcend to three dimenſions, are called *Loci* or Lines of the Third Order, and therefore Curves of the Second Kind. And ſo on ſucceſſively.

The *loci* to a right line conſtructed, in ſix caſes.

117. Now, as to the *loci* to a right line, they are all comprehended under theſe ſix equations following: $y = \frac{ax}{b}$, $y = -\frac{ax}{b}$, $y = \frac{ax}{b} + c$, $y =$

$-\frac{ax}{b}-c$, $y=\frac{ax}{b}-c$, and $y=-\frac{ax}{b}+c$. For, by multiplication and division, we may always reduce y to be free from fractions and co-efficients. By $\frac{a}{b}$ is to be understood the aggregate of all the known quantities which multiply x, and by c the aggregate of all the quantities which form the given or constant term.

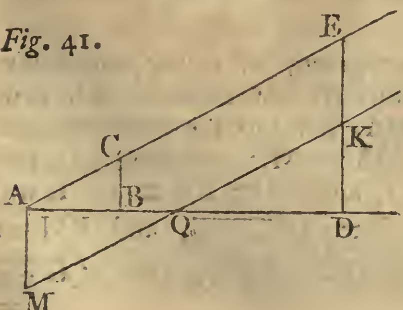
Fig. 40.



To construct the two first, upon AD produced both ways indefinitely, take AB $=$ AF $= b$ on each side, and draw BC $= a$, making the angle ABC such as the two variables of the problem ought to make. Through the points A, C, draw an indefinite right line HE; this will be the *locus* of the two equations $y=\frac{ax}{b}$, and $y=-\frac{ax}{b}$. For, taking any line AD $= x$, and drawing DE parallel to BC, it will be DE $=\frac{ax}{b}=y$. And taking AF $=-x$, and drawing FH parallel to BC, it will be FH $=-\frac{ax}{b}=y$.

The third and fourth are thus constructed. Take AN $=$ AM $= c$, and parallel to BC; and draw NK, MG, indefinitely, and parallel to HE. NK will be the *locus* of the equation $y=\frac{ax}{b}+c$; and MG the *locus* of the equation $y=-\frac{ax}{b}-c$. For, taking AD $= x$, it will be DE $=\frac{ax}{b}$. But it is EK $=$ AN $= c$, making DK parallel to BC. Then DK $=\frac{ax}{b}+c=y$. And taking AF $=-x$, and drawing FG parallel to BC, it will be FG $=-\frac{ax}{b}-c=y$.

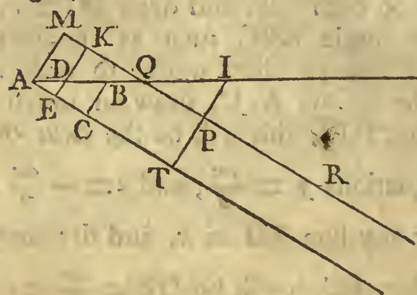
Fig. 41.



As to the fifth, construct the same triangle ABC, and produce the lines AE, AD, indefinitely; draw AM $= c$, and parallel to BC. Then from the point M draw the indefinite line MK parallel to AE, which will meet the right line AD in Q. Then will QK be the *locus* of the equation $y=\frac{ax}{b}-c$. For, taking any line AD $= x$, and drawing DE parallel

parallel to BC, it will be DE $= \frac{ax}{b}$. But KE $=$ AM $= c$; therefore DK $= \frac{ax}{b} - c = y$. The portion QM will ſerve when $\frac{ax}{b}$ is leſs than c, that is, when x is taken leſs than AQ, or leſs than $\frac{bc}{a}$; for, in this caſe, y will be negative, and therefore ought to be taken below AD, that is, the contrary way from DK.

Fig. 42.

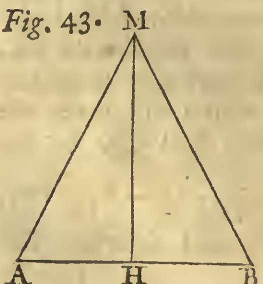


For the laſt formula, make AB $= b$, BC $= a$, and the angle ABC equal to the ſupplement of the angle of the variables. Make AM $= c$, parallel to BC, and draw MQK parallel to AC, cutting AB produced in Q. Then will MQK be the *locus* of the equation $y = c - \frac{ax}{b}$. For, taking any how AD $= x$, and drawing DE parallel to BC, it will be DE $= \frac{ax}{b}$. But, producing ED to K, it will be EK $=$ AM $= c$, and therefore DK $= c - \frac{ax}{b} = y$. Now, if x be taken greater than AQ, for inſtance $=$ AI, it will be IT $= \frac{ax}{b}$, and therefore $c - \frac{ax}{b}$ is a negative quantity $= y =$ IP; taken directly contrary to DK, and the indefinite line MR is the *locus* of the propoſed equation in both caſes.

The *locus* when one of the variables vaniſhes.

118. It may ſometimes happen, that, in the ſolution of a problem the *locus* of which is a right line, either one or the other of the two variables will diſappear, and will not enter into the equation. In ſuch caſes, the *locus* will be to the perpendicular, or to a parallel to the given right line upon which the abſciſſes are taken, according as either the ordinate or abſciſs vaniſhes. Here is an example or two of this.

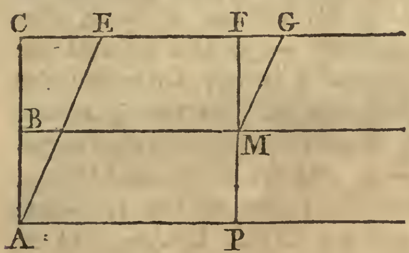
Fig. 43.



The right line AB being given, let it be propoſed to find the *locus* of the points M out of this, ſuch that, drawing the right lines MA, MB, to the extremities of AB, it may always be MA $=$ MB. Taking any line AH $= x$, draw HM $= y$, and make AB $= a$. It will be HB $= a - x$, AM $= \sqrt{xx + yy}$, and BM $= \sqrt{aa - 2ax + xx + yy}$; and thence the equation $\sqrt{xx + yy}$ $=$

$= \sqrt{aa - 2ax + xx + yy}$, and ſquaring, $xx + yy = aa - 2ax + xx + yy$, that is, $x = \frac{1}{2}a$; where y diſappears, and x remains determined. This ſhows us, that, taking $x =$ AH, which is half AB, and from the point H raiſing an indefinite perpendicular, every one of it's points will ſatisfy the queſtion, and therefore this will be the *locus* required.

Fig. 44.

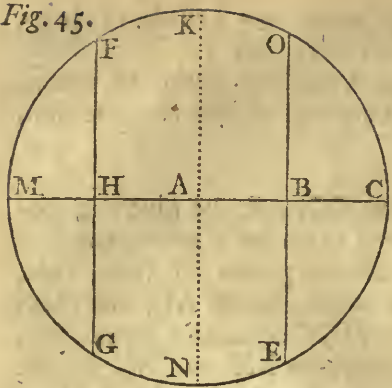


Let the parallels CG, AP, be given in poſition, and between them let it be required to find the *locus* of all the points M ſuch, that, drawing MP perpendicular to AP, and MG making the angle MGC equal to a given angle AEC; it may always be MP to MG in the conſtant ratio of a to b. Make the diſtance AC $= c$, AP $= x$, PM $= y$, and producing PM to F, it will be FM $= c - y$. Now, becauſe the angle AEC is given, and ACE is a right angle, and the ſide AC is given, the ſide AE will alſo be known, which may be called f. Now, becauſe of the ſimilar triangles ACE, FMG, it will be AC . AE :: MF . MG; that is, $c \,.\, f :: c - y \,.\, \text{MG} = \frac{cf - fy}{c}$. But beſides, it ought to be PM . MG :: $a \,.\, b$. Then it will be $y \,.\, \frac{cf - fy}{c} :: a \,.\, b$, and therefore $bcy = acf - afy$, or $y = \frac{acf}{bc + af}$. So that here is an equation, in which the unknown quantity x does not enter at all. Therefore, taking x as you pleaſe, y will always be conſtant, and equal to $\frac{acf}{bc + af}$; and therefore, drawing the indefinite line BM parallel to AP, and as far diſtant from it as the quantity $\frac{acf}{bc + af}$, this line will be the *locus* required.

119. Having thus explained the conſtruction of the *Loci* to a Right Line, I come now to the conſtruction of Equations of the Second Degree, or of the *Loci* to the Conic Sections. And here I muſt ſuppoſe the learner to be ſo well inſtructed in the chief geometrical properties of theſe ſections of the cone, as to form from thence the firſt and more ſimple equations of theſe curves; to which ſimple equations the more compounded ones may be reduced and referred, by the methods now to be explained. The *loci* to a circle conſtructed.

And, in the firſt place, it muſt be known, that in the circle any ordinate is a mean proportional between the ſegments of the diameter; that is, it's ſquare is equal to the rectangle of the ſaid ſegments. Therefore, in the circle MKCN, if

Fig. 45.



if you make the radius AC $=a$, and from the centre A any abſciſs whatever AB $=x$, and the perpendicular ordinate BD $=y$, it will be MB $=a+x$, BC $=a-x$, and therefore MB $\times$ BC $=aa-xx$; then it will be $yy=aa-xx$, an equation to the circle, in reſpect of the quadrant KC. But, becauſe the ſame property may be verified alſo, taking BE for the ordinate, that is the negative ordinate $-y$, and as well the ſquare of $-y$ as of y is yy; therefore the ſame equation belongs alſo to the quadrant CN. And now, if we take the abſciſſes negative, as AH $=-x$, and the ordinates HF $=y$, HG $=-y$, their ſquare yy will, in both caſes, be equal to the rectangle MH $\times$ HC. But when it is AH $=-x$, it will be CH $=$ CA $+$ AH $=a-x$; and MH $=$ AM $-$ AH $=a+x$ by the rules of Addition and Subtraction. And therefore the rectangle MH $\times$ HC will be ſtill $aa-xx$. So that $yy=aa-xx$ is the moſt ſimple equation that belongs to the whole circle with radius a, taking the abſciſſes from the centre.

If the abſciſſes ſhould be taken, not from the centre A, but from M the extremity of the diameter, making any one of them MH or MB equal to x, it will be HC or BC $=2a-x$, and the rectangle of the ſegments will be equal to $2ax-xx$. But the ſquare of the ordinate, as well poſitive as negative, is yy, ſo that it will be $yy=2ax-xx$; the moſt ſimple equation of the ſame circle, taking the abſciſſes not from the centre, but from the extremity of the diameter.

By the quantity or magnitude a, which denotes the radius, is meant any given quantity whatever, whether ſimple or compound, integer or fraction, rational or ſurd; ſo that $yy=aa-bb-xx$ will be a circle with radius $=\sqrt{aa-bb}$; $yy=\frac{aab}{m}-xx$ will be a circle with radius $=\sqrt{\frac{aab}{m}}$; $yy=a\sqrt{ab}-xx$ will be a circle with radius $=\sqrt{a\sqrt{ab}}$. Thus $yy=2ax-bx-xx$ will be a circle with diameter $=2a-b$, or with radius $=\frac{2a-b}{2}$; $yy=\frac{aax+abx}{b}-xx$ will be a circle with diameter $=\frac{aa+ab}{b}$; $yy=x\sqrt{ab}-xx$ will be a circle with diameter $=\sqrt{ab}$. And ſo of others.

Here it is plain, that, in the equation $yy=aa-bb-xx$, and in all others like it, if the quantity b ſhould be greater than a; then $aa-bb$ being a negative quantity, the circle would become imaginary. For then the ordinate

$y = \sqrt{aa - bb - xx}$ being equal to the ſquare-root of a negative quantity, it would be therefore imaginary.

For the ſame reaſon, in the equation $yy = 2ax - xx$, the abſciſs x cannot be taken negative; for, taking x negative, the term $2ax$ would be negative, and therefore the equation $yy = -2ax - xx$, that is $y = \sqrt{-2ax - xx}$, would be an imaginary quantity.

120. The primary property of the *Apollonian* Parabola is this, that the ſquare of any ordinate whatever is equal to the rectangle of the parameter into the abſciſs; taken on the axis if the angle of the co-ordinates be a right angle, or on a diameter if that angle be oblique. Then, making the parameter $= a$, any abſciſs AB $= x$, the correſponding poſitive ordinate BC $= y$, and the negative BD $= -y$; then yy will be the ſquare as well of BC as of BD, and ax will be the rectangle of the parameter into AB. Wherefore $yy = ax$ is the moſt ſimple equation which belongs to the parabola with the parameter a. And here it is plain, that the abſciſs x cannot be taken negative, becauſe of the avoiding imaginary quantities. And here alſo, by the quantity a, which expreſſes the parameter, is to be underſtood any given quantity, into which the abſciſs x is multiplied; ſo that $\frac{aax \pm bbx}{c} = yy$ will be a parabola, the parameter of which is $= \frac{aa \pm bb}{c}$. And $x\sqrt{ab} = yy$ will be a parabola, the parameter of which is $\sqrt{ab}$. And the like of all others. The ſimpleſt *loci* to the parabola conſtructed.

Fig. 46.

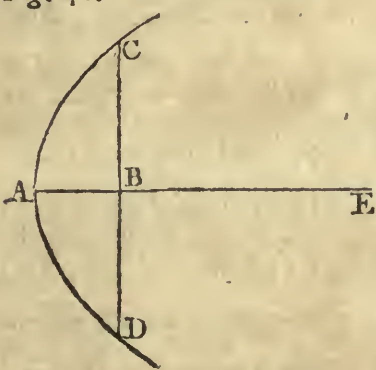
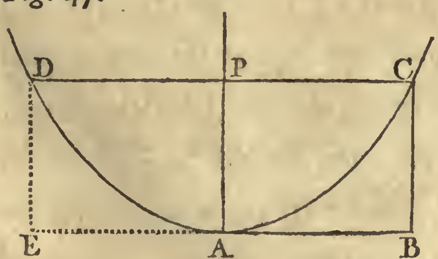


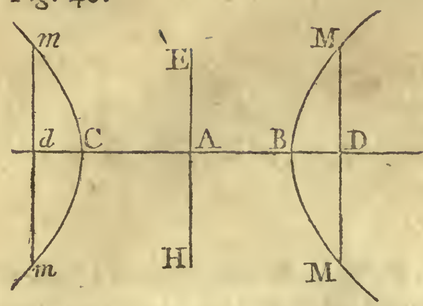
Fig. 47.



If the parabola ſhould be differently placed, as in Fig. 47, and on the ſame line AB, from the given point A, we ſhould take the abſciſſes, or x; the equation would be $xx = ay$, in which we may take the abſciſs either poſitive or negative, but the ordinates muſt always be poſitive.

The fimpleft *loci* to the hyperbola conftructed.

Fig. 48.



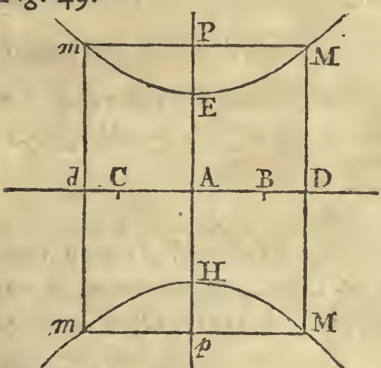
121. Let the oppofite hyperbolas be referred to their axis, or to a diameter, according as the angle of the co-ordinates is either right or oblique; and let CB be the axis, or the tranfverfe diameter, and HE the conjugate. By the known property of the hyperbola, taking D any point whatever, and drawing DM parallel to HE, the rectangle CD $\times$ DB muft be to the fquare of DM, as the fquare of CB is to the fquare of HE. Then, making CB $= 2a$, HE $= 2b$, and from the centre A taking any line AD $= x$, DM pofitive $= y$, DM negative $= -y$, it will be CD $= a + x$, BD $= x - a$, and therefore, by the faid property, $xx - aa \,.\, yy :: 4aa \,.\, 4bb$, that is, $xx - aa = \frac{aayy}{bb}$. And, taking A$d$ negative $= -x$, and the ordinates as before, it will be Bd $= -x + a$, Cd $= -x - a$, and the rectangle Bd $\times$ dC $= xx - aa$. Whence, in the fame manner, we fhall have $\frac{aayy}{bb} = xx - aa$; the moft fimple equation expreffing the two entire oppofite hyperbolas referred to their axes or diameters, taking the abfciffes from the centre. And, if we fhall take the abfciffes from the vertex C, we fhall have the analogy (by the faid property) $x \times \overline{x - 2a} \,.\, yy :: 4aa \,.\, 4bb$; that is, the equation $-2ax + xx = \frac{aayy}{bb}$. And laftly, taking the abfciffes from the vertex B, we fhall have $x \times \overline{2a + x} \,.\, yy :: 4aa \,.\, 4bb$; and therefore the equation $2ax + xx = \frac{aayy}{bb}$.

It is alfo a primary property of the oppofite hyperbolas, that the fame rectangle CD $\times$ DB, taking the abfciffes pofitive, and Bd $\times$ dC, taking the abfciffes negative, is to the fquare of the ordinate, whether pofitive or negative, as the axis or tranfverfe diameter is to the parameter. Making, therefore, the parameter $= p$, and other things as before, it will be $xx - aa \,.\, yy :: 2a \,.\, p$; that is, $\frac{2ayy}{p} = xx - aa$; the moft fimple equation expreffing the two oppofite hyperbolas as referred to a parameter, and taking the abfciffes from the centre. Now, taking the abfcifs from the vertex C, the equation will be $\frac{2ayy}{p} = xx - 2ax$; and laftly, taking the abfcifs from the vertex B, the equation will be $2ax + xx = \frac{2ayy}{p}$.

If

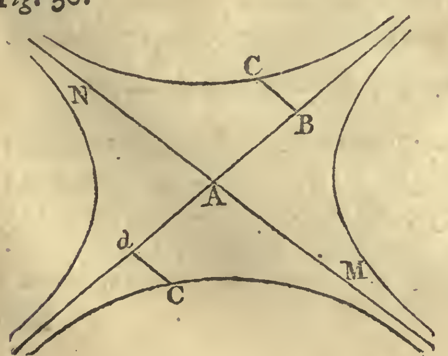
If the hyperbolas be equilateral, becauſe, in this caſe, the two axes or diameters are equal to each other, and equal to the parameter, each equation will become $yy = xx - aa$, taking the abſciſs from the centre; or $yy = 2ax + xx$, taking the abſciſs from the vertex B; or $yy = -2ax + xx$, taking the abſciſs from the vertex C. By the quantity aa is to be underſtood any plane however complicated, as alſo by the quantity bb. And by $2a$, as alſo by p, is underſtood any line whatever. So that, in the equation $\frac{\overline{aa+ff} \times yy}{b\sqrt{ab}} = xx - aa - ff$, we ſhall have $\sqrt{aa+ff}$ for the ſemiaxis, or tranſverſe ſemidiameter, and $2\sqrt{aa+ff}$ will be the whole axis or diameter. As alſo, $\sqrt{b\sqrt{ab}}$ is the ſemiaxis or ſemidiameter conjugate, and $2\sqrt{b\sqrt{ab}}$ is the whole. In the equation $\frac{a^3yy}{bbc} = xx - \frac{a^3}{c}$, it will be $\sqrt{\frac{a^3}{c}}$, the ſemiaxis or tranſverſe ſemidiameter, and b the conjugate. In the equation $xx - bx = \frac{byy}{c+m}$, it will be b the ſemiaxis or tranſverſe ſemidiameter, and $c+m$ the parameter. In the equation $\frac{2yy\sqrt{aa-bb}}{a-b} = xx - aa + bb$, it will be $2\sqrt{aa-bb}$ the axis or tranſverſe diameter, and $a - b$ the parameter. And ſo on.

Fig. 49.



If the oppoſite hyperbolas ſhall be differently ſituated, as in Fig. 49, and upon the ſame diameter CB equal to $2a$, produced, if you would have the x's poſitive, and negative from the centre A, (it being HE $= 2b$,) the equation would be $yy - bb = \frac{bbxx}{aa}$.

Fig. 50.



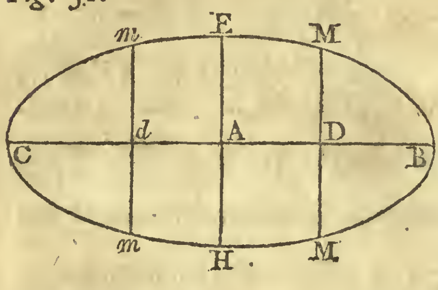
122. In the hyperbola between the aſymptotes, the rectangle of any line AB taken on the aſymptote dB, into the ordinate BC parallel to the aſymptote MN, or $Ad \times dC$, is always conſtant, that is, equal to a known rectangle. Therefore, making AB $= x$, BC $= y$, and the known rectangle $= ab$, it will be $xy = ab$; and, taking Ad negative $= -x$, and dC negative $= -y$, the rectangle $Ad \times dC$ ſhall be

The ſimpleſt *loci* of the hyperbola between it's aſymptotes conſtructed.

be alſo xy; and therefore $xy = ab$ is the moſt ſimple equation belonging to the oppoſite hyperbolas between the aſymptotes. It is plain, that the equation $-xy = ab$, or $xy = -ab$, will ſerve for the oppoſite hyperbolas in the angles BAM, bAN, one of the co-ordinates being always poſitive, and the other negative, and therefore the product is negative.

The ſimpleſt *loci* to the ellipſis conſtructed.

Fig. 51.



123. In the ellipſis CEBH, taking from the centre A any line AD upon the axis or tranſverſe diameter CB, and drawing DM parallel to the axis or conjugate diameter EH; by the known property of the ellipſis, the rectangle CD × DB muſt be to the ſquare of DM, as the ſquare of the axis or tranſverſe diameter CB is to the ſquare of the conjugate HE. Therefore, making CB $= 2a$, HE $= 2b$, and from the centre A taking any line AD $= x$, and making DM poſitive $= y$, DM negative $= -y$; it will be CD $= a + x$, DB $= a - x$, and therefore $aa - xx \,.\, yy :: 4aa \,.\, 4bb$; that is, $\frac{aayy}{bb} = aa - xx$. And taking A$d$ negative $= -x$, and the ordinates as before, it will be Bd = BA + Ad $= a - x$, dC = AC − Ad $= a + x$, and therefore the rectangle Bd × dC ſhall be alſo $= aa - xx$. Whence, in the ſame manner, we ſhall have $aa - xx = \frac{aayy}{bb}$, the moſt ſimple equation to the ellipſis, taking the abſciſſes from the centre. And if we ſhould take the abſciſſes from the vertex C, we ſhould have the analogy $2ax - xx \,.\, yy ::$ $4aa \,.\, 4bb$; and therefore the equation $\frac{aayy}{bb} = 2ax - xx$.

It is alſo a known property of the ellipſis, that the ſame rectangles are to the ſquares of the correſpondent ordinates, as the axis or tranſverſe diameter is to the parameter. Therefore, calling this parameter p, and every thing continuing as before, it will be $aa - xx \,.\, yy :: 2a \,.\, p$. Therefore it is $\frac{2ayy}{p} = aa - xx$, the moſt ſimple equation of the ellipſis referred to it's parameter, taking the abſciſſes from the centre. And, taking the abſciſſes from the vertex C, the equation of the ellipſis referred to it's parameter will be $\frac{2ayy}{p} = 2ax - xx$.

If the two axes ſhall be equal to each other, in which caſe they are alſo equal to the parameter, both of the equations will become $yy = aa - xx$, taking the abſciſſes from the centre; and $2ax - xx = yy$, taking the abſciſſes from the point C. But, if we confine it to an axis in which the angle of the co-ordinates is a right angle, the ellipſis will degenerate into a circle with radius $= a$.

The obſervation made in the hyperbola, concerning the given quantities aa, bb, $2a$, p, in reſpect to the diameters and parameter, is to be underſtood equally of the ellipſis, to ſave needleſs repetitions.

124. Now, in equations belonging to the hyperbola and the ellipſis, as referred to the axis or diameters, taking the abſciſs from the centre; as

In theſe *loci* the diameters may be found, if not given.

$$\frac{aayy}{bb} = xx - aa, \quad \frac{aayy}{bb} = aa - xx,$$

the ſquare-root of the conſtant term, or of aa, will always be the tranſverſe ſemiaxis or ſemidiameter. And if the co-efficient of the ſquare of the ordinate be the ſame conſtant term divided by any given quantity, the root of this diviſor is always the conjugate ſemiaxis or ſemidiameter, that is, the root of bb. But if this co-efficient be not ſuch, or do not contain the conſtant term after this manner, then the ſemiaxis or conjugate ſemidiameter will be different. Thus, for example, in the equation $\frac{ffyy}{bb} = xx - aa$, the ſemiaxis, or half the tranſverſe diameter, is indeed always a, but b is not the conjugate. To find this we muſt make an analogy: As the numerator of the co-efficient of the ſquare of the ordinate is to it's denominator, ſo is the conſtant term to a fourth, the root of which will be the ſemiaxis or ſemidiameter required. Then, in equations to the ellipſis or hyperbola referred to the axis or diameter, taking the abſciſs from the vertex, as in $\frac{aayy}{bb} = 2ax - xx$, $\frac{aayy}{bb} = xx - 2ax$, $\frac{aayy}{bb} = xx + 2ax$, the tranſverſe ſemiaxis or ſemidiameter ſhall be half of that quantity, which multiplies the unknown quantity in it's firſt dimenſion, and the conjugate as before. Obſerving, that when the co-efficient of the ſquare of the ordinate is not the ſquare of the axis or tranſverſe diameter thus found, the analogy for the ſemiaxis or conjugate ſemidiameter will be thus: As the numerator of the co-efficient of the ſquare of the ordinate is to the denominator, ſo the ſquare of half the quantity that multiplies the unknown quantity of the firſt dimenſion, is to a fourth; and the ſquare-root of this fourth proportional ſhall be the conjugate ſemiaxis or ſemidiameter.

Therefore, in the equation to the hyperbola $\frac{ffyy}{bb} = xx - aa$, the tranſverſe ſemiaxis or ſemidiameter will be $= a$, and the conjugate $= \frac{ab}{f}$. And ſince, by the property of the curve, it ought to be: As the rectangle of the ſum into the difference, (of the tranſverſe ſemiaxis or ſemidiameter and the abſciſs,) is to the ſquare of the ordinate, ſo is the ſquare of the axis or tranſverſe diameter to the ſquare of the conjugate; it will be $xx - aa \,.\, yy :: 4aa \,.\, \frac{4aabb}{ff}$, or $\frac{4aayy}{4aabb} \times ff = xx - aa$, that is, $\frac{ffyy}{bb} = xx - aa$, which is the propoſed equation.

Thus,

Thus, in the equation $\frac{abyy}{cc} = xx - aa$, the tranſverſe ſemiaxis or ſemidiameter $= a$, and the conjugate $= \sqrt{\frac{acc}{b}}$. In the equation $xx - 2ax = \frac{bbyy}{cm}$, the tranſverſe ſemiaxis or ſemidiameter $= a$, and the conjugate $= \frac{a}{b}\sqrt{cm}$. In the equation $\frac{aa - bb}{cc}yy = xx - bb$, the tranſverſe ſemiaxis or ſemidiameter will be $= b$, and the conjugate $= \sqrt{\frac{bbcc}{aa - bb}}$, &c.

To find the *loci* when referred to a parameter.

125. If the equations be referred to parameters, as $\frac{2ayy}{p} = aa - xx$, or $\frac{2ayy}{p} = xx - aa$, taking the abſciſſes from the centre; or $\frac{2ayy}{p} = 2ax - xx$, or $\frac{2ayy}{p} = 2ax + xx$, or $\frac{2ayy}{p} = xx - 2ax$, taking the abſciſſes from the vertex; in the firſt, the tranſverſe ſemiaxis or ſemidiameter will always be the root of the conſtant term; and in the ſecond, the half of the co-efficient of the unknown quantity of the firſt dimenſion; and the parameter will always be the quantity of the denominator of the co-efficient of the ſquare of the ordinate, when the numerator of the ſame co-efficient in the firſt is double to the root of the conſtant term; and in the ſecond, is equal to the quantity which multiplies the unknown quantity of the firſt dimenſion. But when the ſaid denominator has not the afore-mentioned conditions, the parameter ſhall be the fourth proportional to the numerator, the denominator, and the axis or tranſverſe diameter.

Therefore, in the equation to the ellipſis $aa - xx = \frac{byy}{c}$, the axis or tranſverſe diameter ſhall be $= 2a$, and the parameter $= \frac{2ac}{b}$. And, ſince it ought to be, by the property of the ellipſis, as the rectangle of the ſum into the difference of the ſemiaxis or tranſverſe ſemidiameter and the abſciſs, is to the ſquare of the ordinate, ſo the axis or tranſverſe diameter is to the parameter; it will be $aa - xx . yy :: 2a . \frac{2ac}{b}$, that is, $\frac{byy}{c} = aa - xx$, which is the equation propoſed. In the equation $xx - aa = \frac{3yy}{4}$, which is to the hyperbola, the axis or tranſverſe diameter $= 2a$, the parameter $= \frac{8a}{3}$. In the equation to the hyperbola $2ax + xx = \frac{b - c}{m}yy$, the axis or tranſverſe diameter will be $2a$, and the parameter $\frac{2am}{b - c}$. In the equation to the ellipſis $aa - bb$

$- xx$

$-xx = \frac{byy}{c}$, the axis or tranſverſe diameter will be $= 2\sqrt{aa - bb}$, and the parameter $= \frac{2c\sqrt{aa-bb}}{b}$; ſuppoſing a to be greater than b, for otherwiſe the curve would be imaginary.

126. Theſe things being premiſed, and well underſtood, the conſtruction of more complicate equations, or of all other *Loci* to the conic ſections, will be very eaſy; and that by reducing ſuch complicate equations to the ſimple primary equations here exhibited. So that, the deſcription of ſuch a conic ſection being ſuppoſed, we may proceed to the conſtruction of the propoſed equation. The *loci* to the conic ſections diſtributed into three ſpecies.

Now, to proceed with the greater perſpicuity, I ſhall diſtribute all equations to the conic ſections into three ſpecies or claſſes, I mean all complicate ones. Thoſe of the firſt claſs ſhall be all ſuch as contain the ſquare of only one of the unknown quantities, and the rectangle of the other unknown quantity into a conſtant quantity. As, for example, $ax \pm ab = yy$. And moreover, all thoſe ſhall be ſaid to be of the firſt ſpecies, which contain rectangles of the unknown quantities one among another, and with conſtant quantities, but have not the ſquare of either of the unknown quantities. As $xy + ax = aa - ay$; the ſigns being of any kind, which is alſo to be underſtood of the ſigns of the other two ſpecies.

Of the ſecond ſpecies I call thoſe, in which there are the ſquares of one or both the unknown quantities, and alſo their rectangles into conſtant quantities, but not their rectangle into each other; as $xx + 2ax = ay + by$, or $xx - 2bx = yy + ay - ax$.

Thoſe are of the third ſpecies, in which are contained rectangles of the two unknown quantities into each other, and other terms of what kind ſoever; ſuch as $xx + 2xy + 2yy = aa - xx + bx$.

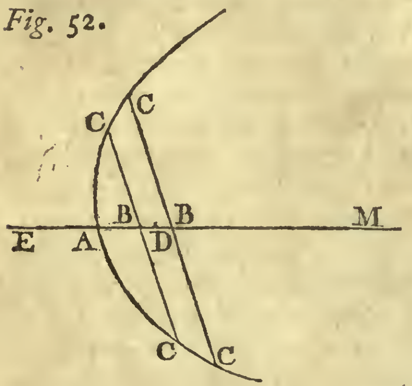
127. To diſtinguiſh and conſtruct equations of the firſt ſpecies, there is occaſion to make uſe of one ſubſtitution, which is, to put the unknown quantity which has no ſquare, *plus* or *minus* (according to the ſigns), a conſtant quantity, equal to ſome new unknown quantity; and thus to reduce the equation, (repeating this ſubſtitution if there be occaſion,) to a more ſimple expreſſion, ſo that the *locus* of the ſaid equation may be eaſily known and conſtructed; as may be ſeen in the following Examples. *Loci* of the firſt ſpecies conſtructed, with examples.

EX-

EXAMPLE I.

Let the equation be $ax + ab = yy$, and let the angle be given, which the co-ordinates make with each other. Becauſe $ax + ab$ is the ſame as $a \times \overline{x + b}$, make $x + b = z$; then, by ſubſtitution, it will be $az = yy$, which is the *Apollonian* parabola.

Fig. 52.



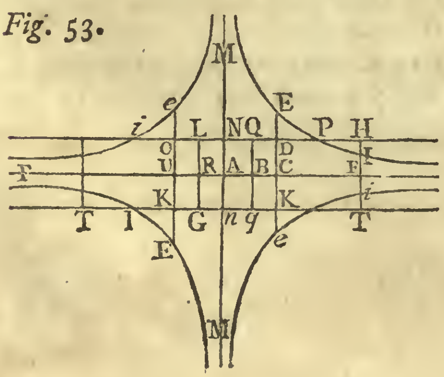
On the indefinite line AB as a diameter, with a parameter $= a$, let the parabola CAC be deſcribed, whoſe co-ordinates AB, BC, contain the given angle; then let $AD = b$. Taking any line $AB = z$, it will be $BC = y$. But, becauſe, by the ſubſtitution, we have $x = z - b$, DB will be x. Therefore the origin of the abſciſs x will be the point D, taking the poſitive towards M, the negative towards A, and the correſponding poſitive and negative ordinates will be y.

If the propoſed equation had been put $ax - ab = yy$, we ſhould have made the ſubſtitution $x - b = z$, and therefore $x = z + b$. In which caſe, taking $AE = b$ in the diameter produced, and doing the reſt as before, the point E would then have been the origin of the abſciſs x.

EXAMPLE II.

Let the equation be $xy + ax = aa - ay$. Make $y + a = z$, and, inſtead of y, ſubſtituting this value $z - a$, we ſhall have $zx + az = 2aa$; and making another ſubſtitution of $x + a = p$, it will be $pz = 2aa$, the *Apollonian* hyperbola between the aſymptotes.

Fig. 53.



Let the indefinite right lines MM, FF, comprehend the given angle of the co-ordinates, and between the aſymptotes MM, FF, let the two oppoſite hyperbolas be deſcribed, belonging to the conſtant rectangle $2aa$. Taking any line $AC = p$, and the ordinate CE parallel to AM, it will be $= z$. But, by the ſubſtitution, it is $x = p - a$; therefore, making $AB = a$, it will be $BC = x$. And,

And, becaufe we have alfo, by the other fubftitution, $y = z - a$, making $AN = a$, and drawing NH parallel to FF, it will be $DE = y$. Therefore, drawing BQ parallel to AN, Q will be the beginning of the abfcifs x. Thus, to any abfcifs $QD = x$ will correfpond the ordinate $DE = y$, pofitive between the points Q and P, and negative beyond the point P, as HI. But, when p is taken lefs than a, that is, AC lefs than AB, then, as it is $x = p - a$, x will be negative, that is, towards N; and to it will correfpond the pofitive ordinates y. Now, if we take p negative, and equal to AU for example, x will be negative, and equal to QO, and y negative $=$ OE. If the equation were $xy + ax = aa + ay$, or elfe, $xy + ax = -aa - ay$, or this, $xy - ax = aa - ay$, or this, $xy - ax = -aa + ay$; the two firft would be divifible by $y + a$, and we fhould have $x = \pm a$. The two others would be divifible by $y - a$, and we fhould have $x = \pm a$. Therefore they would not be *loci*, but equations of determinate problems. But if it were $xy - ax = aa + ay$, the firft fubftitution would be $y - a = z$, whence the equation $zx - az = 2aa$; and confequently the fecond fubftitution would be $x - a = p$; whence finally the equation $zp = 2aa$; and therefore, in this cafe, to the co-ordinates p, z, muft be added the quantity a, in order to have x and y. And therefore, taking from A towards U the line $AR = a$, and drawing RG parallel to MN and equal to a, then, through the point G drawing GT parallel to FF, G fhall be the origin of the abfciffes x, and the correfponding ordinates fhall be y.

If the equation were $xy + ax = -aa + ay$, the fubftitutions would be $y + a = z$, and $x - a = p$, which would give us the equation $pz = -2aa$.

Let the fame hyperbolas be defcribed, but in the other two angles, becaufe the conftant rectangle $2aa$ is negative; and let them be *ie*, *ie*. Producing GR to L, this will be the origin of x both affirmative and negative. And upon the right line LQ produced both ways, the ordinates y will infift, that is, negative from N towards H, and pofitive from N to the point i; and again negative beyond the point i.

If it were $xy - ax = -aa - ay$, the fubftitutions would be $y - a = z$, and $x + a = p$. Therefore, the fame hyperbolas *ie* being defcribed, and QB being produced to q, this will be the origin of the abfciffes x, and the ordinates y will infift upon TT.

If, in the equations, the term xy fhould be negative, it may be made pofitive by tranfpofing the terms.

The diverfity of fubftitutions, and of the pofition of the co-ordinates, which arifes from the different combinations of the figns in the propofed equations, and whatever elfe has been confidered here, is to be fupplied in what follows, where, for brevity-fake, I fhall omit it.

Hitherto I have fuppofed, that the conftant quantities of the equation are fuch, as may make room for the aforefaid fubftitutions. If they fhould not be ſuch, as, for example, if the equation were $aa - bx = yy$, we muſt make $aa = bc$, and then we ſhall have $bc - bx = yy$, and the ſubſtitution to be made would be that of $c - x$ equal to a new unknown quantity. Thus, if it were $\frac{abb}{m} + cx = yy$, we muſt make $bb = cf$, whence the equation $\frac{acf}{m} + cx = yy$. And then we muſt put $\frac{af}{m} + x$ equal to ſome new unknown quantity.

If it were $\frac{aax - bbx + m^3}{a + b} = yy$, we might make $aa - bb = cc$, and $m^3 = ccf$, and then it would be $\frac{ccx + ccf}{a + b} = yy$. And the like of others.

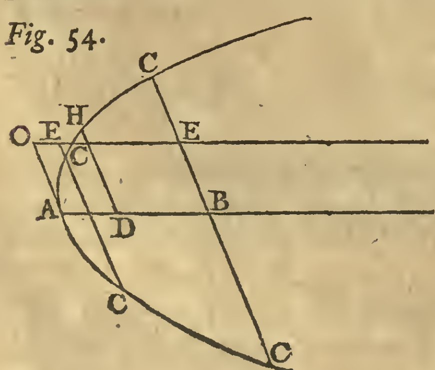
Loci of the ſecond ſpecies conſtructed.

128. To reduce and conſtruct equations of the ſecond ſpecies; let all the terms which contain the ſame unknown quantity be put in order on one ſide of the ſign of equality, and on the other ſide all the other terms in order likewiſe; and in the firſt member of the equation let the ſquare of the unknown quantity be poſitive, and free from co-efficients and fractions. To the ſame firſt member, (and to the ſecond alſo, to preſerve the equality,) muſt be added the ſquare of half the co-efficient of the ſecond term, if it be neceſſary, ſo as the firſt member may be a ſquare. Then put the root of that ſquare equal to a new unknown quantity; which operation muſt be performed in the ſecond member alſo, if it require it. This will give us an equation reduced to the ſimpleſt terms, or to an equation of the firſt ſpecies.

EXAMPLE III.

Let the equation be $xx + 2ax = ay + by$. Add the ſquare aa on each ſide, and it will be $xx + 2ax + aa = aa + ay + by$. And now, making $x + a = z$, we ſhall have $zz = aa + ay + by$, which is now reduced to the firſt ſpecies. Then, making $a + b = c$, and $aa = cf$, it will be $cf + cy = zz$; and putting $f + y = p$, it will be $zz = cp$, an equation to the *Apollonian* parabola.

Fig. 54.



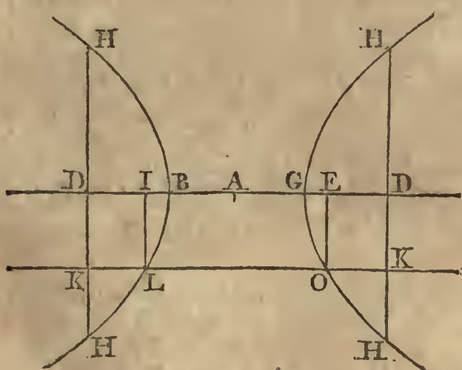
With parameter $c = a + b$, on the diameter AB, and with the co-ordinates in a given angle, let the parabola CAC be deſcribed. Then, taking any abſciſs $AB = p$, and BC ſhall be z, either poſitive or negative. And, becauſe $y = p - f = p - \frac{aa}{a+b}$, taking $AD = \frac{aa}{a+b}$, it will be $DB = y$. And, becauſe of the ſubſtitution $x + a = z$, from

from the point D draw DH $= a$ parallel to BC, which will be terminated by the parabola in H, (as will eafily be feen by fubftituting, inftead of p in the reduced equation $zz = cp$, the value of AD $= \frac{aa}{a+b} = \frac{aa}{c}$; for it will become $zz = aa$, and therefore DH $= z = a$,) and drawing through the point H the line OE parallel to the diameter, it will be HE $=$ DB $= p - \frac{aa}{a+b} = y$, and confequently EC $= z - a = x$ pofitive, and negative alfo when the abfciffes are pofitive. And to the negative abfciffes, that is, taking them from H towards O, both the negative ordinates will correfpond.

EXAMPLE IV.

Let the equation be $xx + 2bx = yy - ay$. Let there be added the fquare of half the co-efficient of the fecond term, that is bb; then it will be $xx + 2bx + bb = yy - ay + bb$. And making $x + b = z$, we fhall have $zz = yy - ay + bb$, that is, $zz - bb = yy - ay$. And adding the fquare of $\frac{1}{2}a$, it will be $zz - bb + \frac{1}{4}aa = yy - ay + \frac{1}{4}aa$. Then make $y - \frac{1}{2}a = p$, and it will be $zz - bb + \frac{1}{4}aa = pp$. And fuppofing bb greater than $\frac{1}{4}aa$, and making $bb - \frac{1}{4}aa = mm$, it will be $zz - mm = pp$, an equilateral hyperbola with the femidiameters $= m$, and taking the abfciffes from the centre.

Fig. 55.



In the indefinite line BD I take BG $= 2m = 2\sqrt{bb - \frac{1}{4}aa}$, and divide it equally in A. With centre A, the tranfverfe diameter $=$ 2AG, equal to the conjugate, and with the co-ordinates in a given angle, defcribe the two oppofite and equilateral hyperbolas. Taking any abfcifs pofitive and negative AD $= z$, the correfponding ordinates DH will be p, pofitive and negative. And becaufe, by the fubftitution, it is $x = z - b$, taking AE $= b$, it will be ED $= x$. But, by the other fubftitution, it being $y = p + \frac{1}{2}a$, from the point E drawing EO $= \frac{1}{2}a$, parallel to the ordinate, which will terminate at the curve in the point O; and through that point O draw the indefinite line KK parallel to the diameter BG, it will be KH $= p + \frac{1}{2}a = q$. Therefore the point O will be the origin of the abfcifs x on the right line KK, to which, taken pofitively, will correfpond the two ordinates y, one pofitive and the other negative. And taking it negative, but not greater than EG, two

 pofitive

pofitive ordinates will correfpond to it; but taking it negative and greater than EG, but lefs than EB, the ordinates y will be imaginary; and taking it negative greater than EB, and lefs than EI, making BI = GE, the two ordinates will be pofitive; and laftly, one of the ordinates will be pofitive, and the other negative, when the abfciffes, being negative, fhall be greater than EI.

Here it fhould be obferved, that the root of the fquare $yy - ay + \frac{1}{4}aa$ is not only $y - \frac{1}{2}a$, but alfo $\frac{1}{2}a - y$, and therefore the fubftitutions fhould be two, that is, both $y - \frac{1}{2}a = p$, and $\frac{1}{2}a - y = p$. Yet, notwithftanding, in the prefent example, and in others that follow, I only make ufe of the firft. For, confidering, in thefe conftructions, the new unknown quantity p is to be underftood both as pofitive and negative, herein will be comprehended thofe determinations alfo, which the other fubftitution would fupply, and which therefore would be fuperfluous here.

If the quantity bb, which I have fuppofed greater than $\frac{1}{4}aa$, fhould, on the contrary, be lefs, the *locus* would be to the fame hyperbolas, only by changing the places of the co-ordinates and of the conftant quantities. That is, the final equation would be $zz = pp - mm$, the conftruction of which is here omitted, becaufe it is not different from the foregoing, only that the femidiameters here are each equal to $\sqrt{\frac{1}{4}aa - bb} = m$. Now, if it were $bb = \frac{1}{4}aa$, the *locus* would degenerate into a right line, as is plain.

Loci of the third fpecies conftructed.

129. To diftinguifh and conftruct equations of the third fpecies, it is neceffary that, putting the fquare of one of the unknown quantities made pofitive, and free from fractions and co-efficients, together with the rectangle of the fame, on one fide of the mark of equality, and on the other fide all the remaining terms; adding to the firft member (and confequently to the fecond alfo) fuch a fraction of the other unknown quantity, that the firft member may be a fquare; then putting it's root equal to a new unknown quantity, and making the fubftitution; by means of which an equation may be had, reduced to a more fimple expreffion, or to one of the two fpecies before-mentioned.

Thus, in this equation, for example, $zz - \frac{2bzy}{a} = ay$, adding $\frac{bbyy}{aa}$ to both members, the firft member will be a fquare, the root of which is $z - \frac{by}{a}$, which is to be put equal to a new unknown quantity p; and, making the fubftitution, the equation will be $pp = \frac{bbyy}{aa} + ay$, which is now reduced to the fecond fpecies.

130. But

130. But it may be obferved, that fometimes the new unknown quantity to be introduced fhould be affected by fome conftant co-efficient, otherwife the conftructions would be much incumbered. For example, in the equation $xx \pm \frac{2bxy}{a} + \frac{bbyy}{aa} = \pm fy \pm hx$, the firft member of which, without any addition, is already a fquare, whofe root is $x \pm \frac{by}{a}$; if the term hx were not there, or being there, if we would eliminate x out of the equation, we might do it, by putting, inftead of x, it's value obtained by the fubftitution, fo that it may be expreffed by the new unknown quantity, and by y with conftant quantities; therefore the fubftitution of $x \pm \frac{by}{a} = z$ fhould be made.

Complicate *loci* of any fpecies reduced to fimple by fubftitution; with examples.

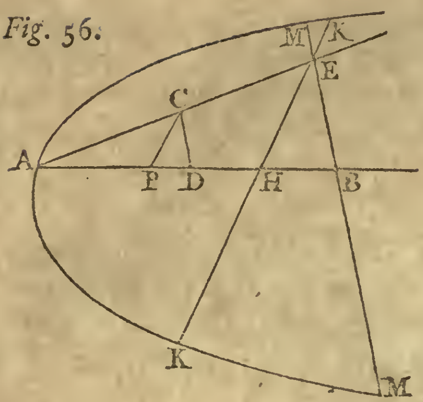
But if the term fy were not there, or being there, if we would eliminate y, we muft make a fubftitution of $x \pm \frac{by}{a} = \frac{bz}{a}$. And thus, refpectively, if the equation were $yy \pm \frac{2bxy}{a} + \frac{bbxx}{aa} = \pm fy \pm hx$, the term fy not being there, or elfe to be eliminated, a fubftitution muft be made of $y \pm \frac{bx}{a} = z$; or the term hx not being there, or being to be eliminated, a fubftitution of $y \pm \frac{bx}{a} = \frac{bz}{a}$ is to be made.

In general, the rectangle of conftant quantities into that unknown quantity by which the equation is ordered, not being in the equation; or being there, if we would eliminate that unknown quantity, we muft put the root of the firft member equal to a new unknown quantity. But if the rectangle of conftant quantities into the other unknown quantity, by which the equation is not ordered, be not in the equation, or if, being there, we would eliminate that unknown quantity, we muft put the root of the firft member equal to a new unknown quantity, multiplied into half the conftant co-efficient of the fecond term of the firft member.

EXAMPLE V.

Let the equation be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = cx$. Make $y + \frac{bx}{a} = z$, and the equation will be $zz = cx$, which is to the *Apollonian* parabola. If the angle of the co-ordinates x, y, of the propofed equation be not given, but left at pleafure, the

Fig. 56.



the conſtruction of the *locus* would be manifeſt. For, on the indefinite right line AB deſcribing the iſoſceles triangle ACD, with the baſe CD $= b$, and the ſides AC $=$ AD $= a$; and on the diameter AB, with a parameter $= c$, and with ordinates parallel to DC deſcribing the parabola of the reduced equation $zz = cx$; taking any abſciſs at pleaſure AB $= x$, it would be BM $= z$. But, by the ſimilar triangles ADC, ABE, we ſhall have EB $= \frac{bx}{a}$, and, by the ſubſtitution, it is $y = z - \frac{bx}{a} =$ EM, and alſo AE $=$ AB $= x$. Therefore, upon the indefinite line AE taking any abſciſs AE $= x$, the correſponding ordinate EM, poſitive or negative, will be the y of the propoſed equation. But, becauſe the angle of the co-ordinates x and y is ſuppoſed to be given, the conſtruction aforegoing will not obtain, but we may proceed thus. On the indefinite line AB let a triangle ACP be deſcribed, having the angle ACP equal to the ſupplement of the given angle, which the co-ordinates of the propoſed equation ought to make; and let AC$=a$, CP $= b$. Produce AC indefinitely, and, taking any line AE $= x$, make KK parallel to PC, and it will be EH $= \frac{bx}{a}$. Whence, if HK $= z$, it will be EK$=y$; and then AE, EK, are the co-ordinates of the propoſed equation, and in the angle given. But HK cannot be yet the z of the reduced equation $cx = zz$, ſince the abſciſſes AH are not yet equal to the x's, nor yet the lines AE. Obſerve, therefore, that AH will be $= \frac{AP \times x}{a}$, that is, $= \frac{fx}{a}$, (making AP $= f$, becauſe, in the triangle ACP, having given the ſides AC, CP, and the angle ACP, the line AP will alſo be given;) whence the curve thus deſcribed, calling AE $= x$, and HK $= z$, will give us the equation $\frac{cfx}{a} = zz$, which would be exactly our equation reduced, if, inſtead of the parameter c, we had deſcribed the curve with the parameter $\frac{ac}{f}$. Therefore, to conſtruct the propoſed *locus*, on the indefinite line AB deſcribe the triangle ACP, the ſides of which are AC$=a$, CP $= b$, and the angle ACP equal to the ſupplement of that angle which the co-ordinates of the propoſed equation ought to make. Then with diameter AB, parameter $= \frac{ac}{f}$, equal to the fourth proportional of AP, of AC, and of the parameter of the reduced equation, (which is general, whenever the *locus* is to

 the

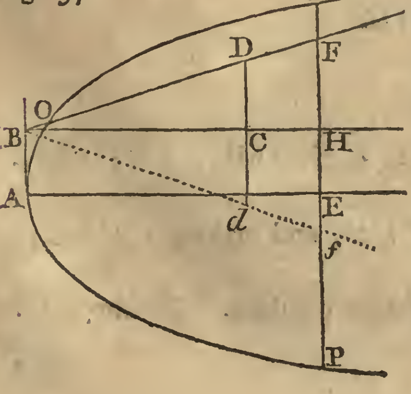
the parabola,) and with ordinates parallel to PC, the *Apollonian* parabola muſt be deſcribed. Then taking, on the indefinite line AE, any abſciſs AE $= x$, EK poſitive and negative will be $= y$, and the curve will be the *locus* of the equation propoſed. For it will be HKq equal to the rectangle of the parameter into AH, or $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = \frac{acfx}{af} = cx$.

The ſame artifice may be made uſe of in other equations, to the hyperbola and to the ellipſis, in regard to their diameters and parameters, with this difference only, that in theſe the tranſverſe diameter, or conjugate, according as this or that ought to be changed, (and it will always be that to which the triangle ACP belongs,) will be the fourth proportional of AC, AP, and the tranſverſe or conjugate diameter of the equation reduced. But as to the parameter, when the equation is given by that, the tranſverſe diameter being varied in the manner aforegoing, it will be the fourth proportional of AP, AC, and the parameter of the reduced equation. But if the triangle ACP do not belong to the tranſverſe diameter, but to the conjugate, (the equation being given by the parameter,) it will be the third proportional of the parameter of the reduced equation, and of AP; as will eaſily be known by the examples.

EXAMPLE VI.

Let the equation given be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = bx - cc - 2cy$. Making a ſubſtitution of $y + \frac{bx}{a} = z$, it will be $zz = bx - cc - 2cz + \frac{2bcx}{a}$, that is $zz + 2cz + cc = bx + \frac{2bcx}{a}$. And making again another ſubſtitution of $z + c = q$, it will be finally $qq = \frac{ab + 2bc}{a}x$, an equation to the *Apollonian* parabola. Now, to conſtruct it relatively to our co ordinates x, y; on the indefinite right line BH let the triangle BCD be conſtructed with it's ſides BD $= a$, DC $= b$, and with an angle BDC equal to the ſupplement of that angle, which ought to be made by the co-ordinates x, y, of the equation propoſed. Then let BD, BC, be produced indefinitely, and from the point B draw BA parallel to DC, and equal to c. Then from vertex A to the diameter AE parallel to BC, and with the ordinates EP parallel to CD, let the parabola

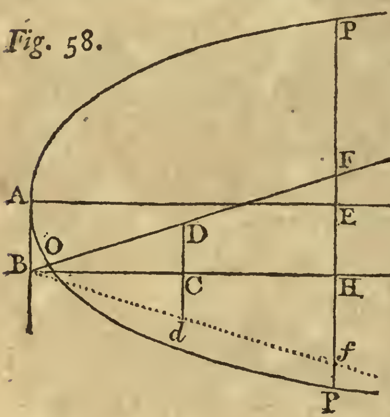
Fig. 57.



parabola PAP be defcribed, with the parameter $= \frac{ab + 2bc}{f}$, (meaning by f the known line BC,) and on the indefinite line BF taking any abfcifs BF $= x$, it will be BH = AE $= \frac{fx}{a}$, and EP $= q$, and therefore HP $= q - c = z$, and FH $= \frac{bx}{a}$. Then FP $= z - \frac{bx}{a} = y$, pofitive and negative when x is greater than BO; and both ordinates negative, when it is x lefs than BO.

In the equation propofed, if the rectangle $2cy$ fhall be affected by the affirmative fign, then the fecond fubftitution fhould be $z - c = q$, and the parameter of the parabola equal to $\frac{ab - 2bc}{a}$. Then doing the fame things as before, inftead of drawing BH above the diameter AE, it fhould be drawn below it, and the triangle BDC fhould be made above it, as is fhown by Fig. 58. Moreover, if the term $\frac{2bxy}{a}$ be negative, the firft fubftitution fhould be $y - \frac{bx}{a} = z$, and thence $y = z + \frac{bx}{a}$. Therefore, in this fuppofition, as well in regard to Fig. 57 as Fig. 58, the triangle BDC fhould be conftructed below BH, fuppofe as BdC. Wherefore, taking any line B$f = x$ above Bd produced, it will be fP $= y$; obferving that, in this cafe, the angle BdC fhould not be made equal to the fupplement, but to the angle itfelf, which is to be made by the co-ordinates of the equation.

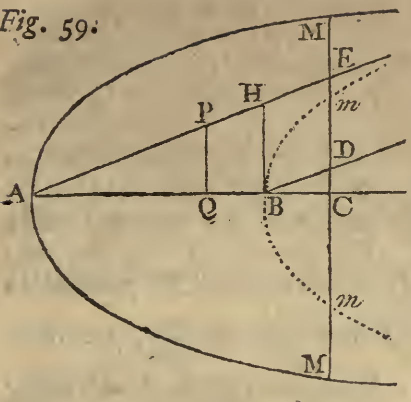
Fig. 58.



EXAMPLE VII.

Let the equation be $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = cx + cb$. Making the fubftitution of $x + \frac{by}{a} = \frac{bz}{a}$, it will be $\frac{bbzz}{aa} = cx + cb$; and making $x + b = p$, it will be $zz = \frac{aacp}{bb}$, an equation to the *Apollonian* parabola. On the indefinite

Fig. 59.



finite line AC describe the triangle APQ with the sides AP $= b$, PQ $= a$, and the angle APQ equal to the supplement of the angle which should be made by the co-ordinates of the proposed equation; and call the known line AQ $= f$, as usual. Let AP, AQ, be produced indefinitely, take AH $= b$, and draw the line HB parallel to PQ. From the point B let the indefinite line BD be drawn parallel to AP; and with vertex A, to the diameter AC, with the parameter $= \frac{aac}{bf}$, and with the ordinate CM parallel to PQ, let the parabola MAM be described. Taking any line AE $= p$, it will be CM $= z$; then HE or BD $= x$, and DC $= \frac{ax}{b}$, because of the similar triangles APQ, BDC. Then is DM $= z - \frac{ax}{b} = y$ positive and negative, and the lines BD, DM, are the co-ordinates of the proposed equation.

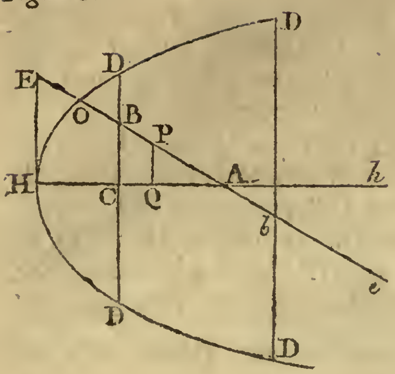
If the equation had been given $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = cx - cb$, making the same first substitution as in the foregoing equation, we should have $\frac{bbzz}{aa} = cx - cb$; and, putting $x - b = p$, it is $zz = \frac{aacp}{bb}$, which is the same as the first, nor is there any other difference, but only in the first case there is $x = p - b$, and here it is $x = p + b$. That is, in the present case the vertex of the parabola must be at B, and the origin of the absciss x must be in the point A, taken on the indefinite line AE.

EXAMPLE VIII.

Let the equation be $xx + \frac{2bxy}{a} + \frac{bbyy}{aa} = cb - cx$. Make the substitution of $x + \frac{by}{a} = \frac{bz}{a}$, and the equation will be $\frac{bbzz}{aa} = cb - cx$; and putting $b - x = p$, it will be $zz = \frac{aacp}{bb}$, an equation to the parabola.

On

Fig. 60.



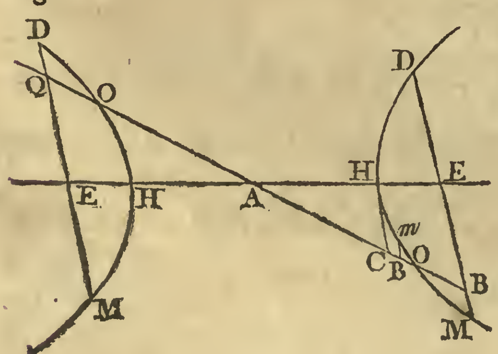
On the indefinite line AH let the triangle APQ be described towards H, with the sides AP $= b$, PQ $= a$, and the angle APQ equal to the supplement of the angle which the co-ordinates of the proposed equation ought to contain. Make the known line AQ $= f$. Produce AP, and take AE $= b$, and draw EH parallel to PQ. With vertex H, on the diameter HA, with the ordinates CD parallel to PQ, and with the parameter $= \frac{aac}{bf}$, let there be described the Apollonian parabola. Taking any line EB $= p$, it will be AB $= b - p = x$, BC $= \frac{ax}{b}$, CD $= z$. Then is BD $= z - \frac{ax}{b} = y$ positive and negative, taking x between the points A and O; and both the ordinates y negative, taking x beyond the point O. The right line AE being produced indefinitely on the opposite side to the point E, and taking any line Eb $= p$ positive and greater than AE, it will be Ab $= b - p = x$, a negative quantity; whence in this case the negative x's will be from A towards e, and the positive from A towards E; and to the same negative x will correspond two ordinates bD, bD, equal to y, one positive and the other negative.

If in these two last examples, as in the others which will follow, the rectangle of the two ordinates be affected by the sign $-$, it is done upon the same consideration as is mentioned at the end of the 6th Example; which it may suffice to have mentioned once for all.

EXAMPLE IX.

Let the equation be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = xx - aa$. Make the substitution of $y - \frac{bx}{a} = z$, and the equation will be $zz = xx - aa$, which is to the hyperbola. On the indefinite line EE describe the triangle ACH, and make AC $= a$, CH $= b$, and the angle ACH equal to the given angle of the co-ordinates of the equation proposed. Let AC be produced indefinitely both ways from the point A. With centre A, and transverse semidiameter AH $= f$, with the conjugate $= a$, let there be described the opposite hyperbolas with the ordinates

Fig. 61.

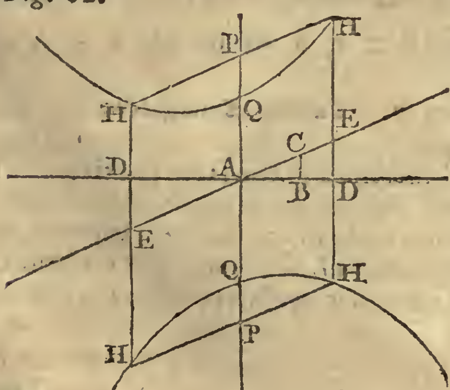


nates parallel to CH. Taking any line AB $= x$ positive, it will be BE $= \frac{bx}{a}$. But ED $= z$. Then is BD $= z + \frac{bx}{a} = y$ positive. And taking in the hyperbola the ordinate z negative, that is $=$ EM, then will y be equal to the difference between EB and EM, that is, equal to BM; and therefore negative when x is greater than AO. Then to any positive abscifs greater than AO will correfpond two ordinates, one positive and the other negative; and both the ordinates will be positive when x is lefs than AO. But when x is taken negative, that is on the fide of the point Q, then it muft be obferved that QE will be negative; for the analogy will be, AC (a) . CH (b) :: AQ $(-x)$. QE $= -\frac{bx}{a}$. Therefore, if QE $= -\frac{bx}{a}$, taking z positive $=$ ED, it will be $z + \frac{bx}{a} =$ QD $= y$ positive; and taking z negative, it will be $-z - \frac{bx}{a} =$ QM $= y$ negative.

EXAMPLE X.

Let the equation be $yy - \frac{2bxy}{a} + \frac{gxx}{a} = bb$. Adding $\frac{bbxx}{aa}$, it will be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = bb - \frac{gxx}{a} + \frac{bbxx}{aa}$; and making the fubftitution of $y - \frac{bx}{a} = z$, it will be $zz = \frac{bbxx}{aa} - \frac{gxx}{a} + bb$. And putting $bb - ag = mm$, it will be $zz = \frac{mmxx}{aa} + bb$, that is, $zz - bb = \frac{mmxx}{aa}$, an equation to the hyperbola.

Fig. 62.



On the indefinite right line DD let the triangle ABC be defcribed, with the fides AB $= a$, BC $= b$, and the angle ABC equal to that which is to be contained by the co-ordinates of the propofed equation; and make the known line $= f$. Through the point A draw the indefinite line PP parallel to BC, and with centre A, tranfverfe diameter QQ $= 2b$, conjugate $= \frac{2bf}{m}$ taken in the right line EE, at the vertices Q, Q, let there be defcribed the two oppofite hyperbolas HQH. Then taking

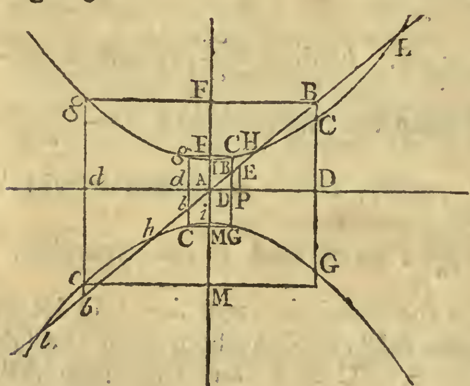
any

any line AD $= x$, and drawing DH parallel to BC, it will be EH $= z =$ AP, and DE $= \frac{bx}{a}$. Then DH $= z + \frac{bx}{a} = y$, and the lines AD, DH, ſhall be the co-ordinates of the propoſed equation.

EXAMPLE XI.

Let the equation be $yy + \frac{2bxy}{a} + \frac{bbxx}{aa} = \frac{2bxx}{a} + bb$. Making the ſubſtitution of $y + \frac{bx}{a} = z$, the equation will be $zz = \frac{2bxx}{a} + bb$, that is, $zz - bb = \frac{2bxx}{a}$, which is to the hyperbola.

Fig. 63.



On the indefinite line AD let the triangle AEP be deſcribed, and make AE $= a$, EP $= b$, and the angle AEP the ſupplement of the angle, which is to be contained by the co-ordinates of the propoſed equation. The right line AE being produced indefinitely both ways, and calling, as uſual, the known line AP $= f$; with centre A, tranſverſe ſemidiameter AI $= b$ parallel to PE, and with parameter $= \frac{ff}{a}$, deſcribe the oppoſite hyperbolas IC, ic; then taking any line AB $= x$, it will be BD $= \frac{bx}{a}$, and CD $=$ FA $= z$. Then BC $= z - \frac{bx}{a} = y$. Taking z negative $=$ DG, it will be BG $= -z + \frac{bx}{a} = -y$, and therefore to the ſame poſitive x will belong two ordinates y, one poſitive, the other negative, taking x between the points A, H. Then taking x between the points H, L, both the ordinates y will be negative; and again, one poſitive, the other negative, taking x greater than AL.

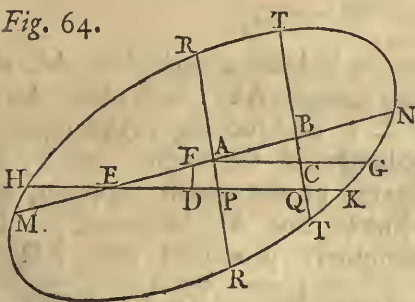
Then

Then taking A$b = -x$, it will be $(bd) = -\frac{bx}{a}$, and as it is $(dg) = z$, it will be $(bg) = z - \frac{bx}{a} = y$; and taking z negative $= (dc)$, it will be $(bc) = -z + \frac{bx}{a} = -y$. Therefore to the ſame A$b = x$ negative will correſpond two ordinates y, one of which is poſitive, the other negative, taking x leſs than Ab; both the ordinates will be poſitive between the points b and l; and again, one ordinate will be poſitive, and the other negative, taking x greater than Al. And therefore the hyperbolas thus deſcribed will be the *locus* of the propoſed equation.

EXAMPLE XII.

Let the equation be $yy - \frac{2bxy}{a} + \frac{bbxx}{aa} = cc - xx + 2bx - bb$. Making the ſubſtitution of $y - \frac{bx}{a} = z$, it will be $zz = cc - xx + 2bx - bb$. And making another ſubſtitution of $x - b = p$, it will be finally $zz = cc - pp$, which is an equation to an ellipſis, and not to a circle, though it may have the appearance of ſuch. The reaſon of which is, becauſe the co-ordinates p, z, do not form a right angle, yet however are in an angle to each other, one of them being AC, the other BT, as may be ſeen in the following conſtruction. On the indefinite line EB let a triangle EDF be deſcribed, with the ſides ED $= a$, DF $= b$, and the angle EDF equal to the angle which is made by the co-ordinates of the propoſed equation; and making the known line EF $= f$. Produce indefinitely the lines ED, EF, and taking EP $= b$, draw the indefinite line PA parallel to DF, and from the point A the line AG parallel to EP.

Fig. 64.



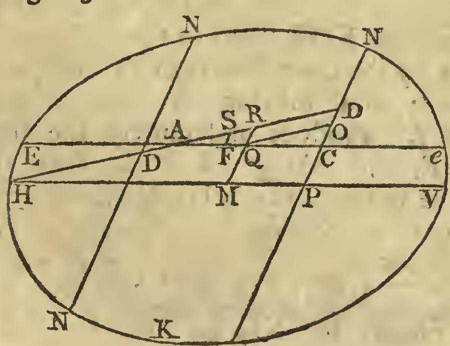
With centre A, tranſverſe diameter MN $= \frac{2cf}{a}$, with conjugate diameter RR equal to $2c$ and parallel to DF, let the ellipſis MRNR be deſcribed; then taking any line AC $= p$, it will be EQ $= x$, and therefore BQ $= \frac{bx}{a}$. But BT $= z$; then QT $= z + \frac{bx}{a} = y$; then will EQ, QT, be the co-ordinates of the *locus* required.

EXAMPLE XIII.

Let the equation be $yy + \frac{bxy}{a} + xx + cy + lx - ag = 0$. Adding on both ſides the ſquare $\frac{bbxx}{4aa}$, it will be $yy + \frac{bxy}{a} + \frac{bbxx}{4aa} = \frac{bbxx}{4aa} - xx - lx - cy + ag$. And making the ſubſtitution of $y + \frac{bx}{2a} = z$, it will be $zz = \frac{bbxx - 4aaxx}{4aa} + \frac{bcx - 2alx}{2a} - cz + ag$.

Let $4aa$ be greater than bb, and make $\frac{bb - 4aa}{4aa} = -\frac{m}{n}$, and $\frac{bc - 2al}{2a} = h$; then adding $\frac{1}{4}cc$ on each ſide, it will be $zz + cz + \frac{1}{4}cc = -\frac{mxx}{n} + hx + ag + \frac{1}{4}cc$. And making the ſubſtitution of $z + \frac{1}{2}c = p$, it will be $pp = -\frac{mxx}{n} + hx + ag + \frac{1}{4}cc$. That is, $-\frac{npp}{m} + \overline{\tfrac{1}{4}cc + ag} \times \frac{n}{m} = xx - \frac{nhx}{m}$; and laſtly, adding $\frac{nnhh}{4mm}$ to both ſides, and making the ſubſtitution of $x - \frac{nh}{2m} = q$, and of $\overline{\tfrac{1}{4}cc + ag} \times \frac{n}{m} + \frac{nnhh}{4mm} = ee$, we ſhall have $\frac{npp}{m} = ee - qq$, which is an equation to the ellipſis.

Fig. 65.



Upon the indefinite right line AC deſcribe the triangle ASF, and make AS $= 2a$, SF $= b$, and the angle ASF equal to the ſupplement of the angle made by the co-ordinates of the given equation, and let the known line AF be called f. On AS indefinitely produced take AR $= \frac{hn}{2m}$, and draw the indefinite line RQ parallel to FS, and from the point Q draw the indefinite line QO parallel to AS, and make QM $= \frac{1}{2}c$.

Then through the point M draw HV parallel to AQ, and with centre M, tranſverſe diameter HV $= \frac{ef}{a}$, and parameter $= \frac{4aem}{fn}$, let the ellipſis HNVK be be defcribed. And taking any line RD $= q$, it will be PN $= p$, and therefore AD $= x$, DC $= \frac{bx}{2a}$, CN $= z$; then DN $= z - \frac{bx}{2a} = y$.

Here it is to be obferved, that if the angle of the co-ordinates fhould be fuch, as that the angle AFS becomes a right angle, and confequently the angle MPN is fo too; then it would be $4aa - bb = ff$, whence $\frac{m}{n} = \frac{4aa - bb}{4aa} = \frac{ff}{4aa}$, and therefore the parameter would be $\frac{4aem}{fn} = \frac{ef}{a}$, that is, equal to the tranfverfe diameter. Then the angle MPN being alfo right, the ellipfis would degenerate into a circle with the diameter $= \frac{ef}{a}$.

131. As to equations of the hyperbola between the afymptotes, which may be required to be conftructed, they may all be underftood to be comprehended in the four examples following.

General conftruction of the *loci* to the hyperbola between it's afymptotes; with examples.

(1.) $\frac{gxx}{b} + xy = ab \pm mx \pm ny.$

(2.) $-\frac{gxx}{b} + xy = ab \pm mx \pm ny.$

(3.) $\frac{gxx}{b} - xy = ab \pm mx \pm ny.$

(4.) $-\frac{gxx}{b} - xy = ab \pm mx \pm ny.$

EXAMPLE XIV.

Firft, let the equation be $\frac{gxx}{b} + xy = ab + mx + ny$, in which I take all the terms pofitive of the *homogeneum comparationis*. Making a fubftitution of $\frac{gx}{b} + y = z$, we fhall have $zx = mx + nz - \frac{ngx}{b} + ab$; and, making another fubftitution of $z - m + \frac{ng}{b} = p$, it will be $px = np + mn + ab - \frac{nng}{b}$. Again, make a third fubftitution of $x - n = q$, and, finally, it will be $pq = ab + nm - \frac{nng}{b}$. Suppofing now that $ab + mn - \frac{nng}{b}$ is a pofitive quantity;

9 on

Fig. 66.

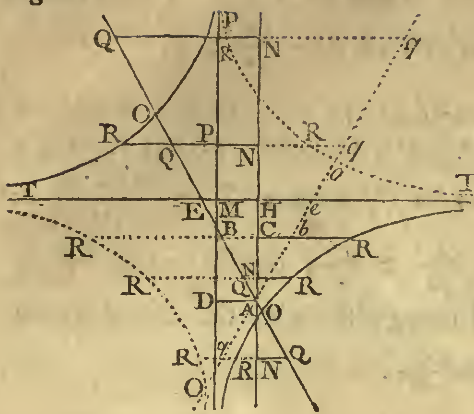
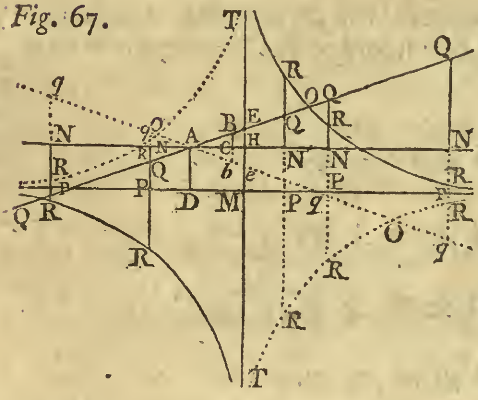


Fig. 67.



on the indefinite line NN, at the point A taken at pleaſure, deſcribe the triangle ABC, the ſides of which are AB $= b$, BC $= g$, and the angle ABC equal to the ſupplement of the angle which the co-ordinates of the equation propoſed ought to make, and make the known line AC $= f$. At the point A raiſe AD parallel to BC, and equal to $m - \frac{ng}{b}$, as in Fig. 66, when $m - \frac{ng}{b}$ is a poſitive quantity; and let fall AD, as in Fig. 67, when $m - \frac{ng}{b}$ is a negative quantity, becauſe of the ſubſtitution made of $z - m + \frac{ng}{b} = p$. Through D draw the indefinite line PP parallel to AC, and on AB produced take AE $= n$, and through E draw TT parallel to BC. Between the aſymptotes PP, TT, deſcribe the two oppoſite hyperbolas RR of the conſtant rectangle $= \overline{ab + mn - \frac{nng}{b}}$ $\times \frac{f}{b}$, that is, a fourth proportional to AB, AC, and the conſtant rectangle of the equation reduced. Taking any line EQ $= q$, it will be PM $= \frac{fq}{b}$, and PQ $= p$, and therefore AQ $= q + n = x$. But PN $=$ AD $= m - \frac{ng}{b}$, therefore NR $= p + m - \frac{ng}{b} = z$; and becauſe QN $= \frac{gx}{b}$, it will be, laſtly, QR $= z - \frac{gx}{b} = y$, and the two lines AQ, QR, will be the co-ordinates of the propoſed equation. Taking x poſitive, when it is leſs than AE, y will be negative: when it is greater than AE, and leſs than AO, y will be poſitive, and when it is greater than AO, y will be negative. Taking x negative, then it will be QN $= -\frac{gx}{b}$, a negative quantity; then $y = z - \frac{gx}{b}$ will be $=$ NR $+$ NQ; and therefore, when x negative is leſs than AO, y will be negative; and when it is greater than AO, y will be poſitive.

8 But

But if the second term of the *homogeneum comparationis* should be negative, that is, if the equation were $\frac{gxx}{h} + xy = ab - mx + ny$; then the second substitution would be $z = p - m - \frac{ng}{h}$, and the equation reduced $pq = ab - mn - \frac{nng}{h}$. Supposing then that $ab - mn - \frac{nng}{h}$ were a positive quantity, describe, as in Fig. 67, the hyperbolas RR, but with the constant rectangle $\overline{ab - mn - \frac{nng}{h}} \times \frac{f}{h}$, and taking $AD = m + \frac{ng}{h}$, this would be in the same manner the *locus* of the proposed equation.

Fig. 68.

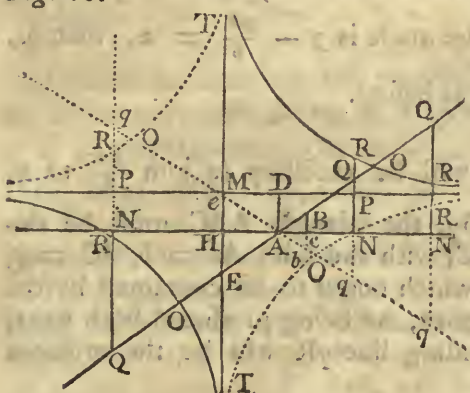
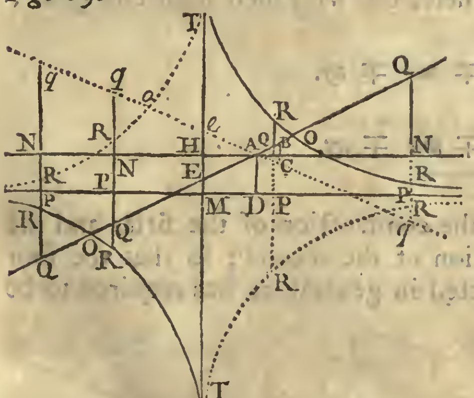


Fig. 69.



If the equation proposed had the last term affected by the negative sign, that is, if it were $\frac{gxx}{h} + xy = ab \pm mx - ny$, the third substitution to be made would be $x + n = q$, whereas before it was $x - n = q$, and therefore the position of the point A, the origin of x, would be changed. Then, in Fig. 68, if the value of AD be positive, and in Fig. 69, if it be negative, the side BA of the usual triangle being produced to E, so that $AE = n$; between the asymptotes TT, PP, let the hyperbolas be described of the constant rectangle belonging to them, that is, when in the equation the term mx is affected by the positive sign, then the constant rectangle $= \overline{ab - mn - \frac{nng}{h}} \times \frac{f}{h}$, and when, on the contrary, it is affected by the negative sign, the constant rectangle will be $= \overline{ab + mn - \frac{nng}{h}} \times \frac{f}{h}$; and taking, in the first case, $AD = m + \frac{ng}{h}$, and in the second, $AD = \frac{ng}{h} - m$, the *locus* of the proposed equation will be after the same manner.

Hitherto I have supposed, that the constant rectangle of the reduced equation is a positive quantity; but when it happens to be negative, the construction

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would not be different, only obſerve to deſcribe the hyperbolas in the other two angles, relatively to the conſtant rectangle, which the reduced equation will ſupply; taking the line AD poſitive or negative, according to it's value which the ſame equation will give, and the point A either to the right or left of the aſymptote TT, according as the laſt term of the *homogeneum* ſhall be poſitive or negative, as is clear by Fig. 66, 67, 68, 69.

The conſtant term ab has hitherto been taken for poſitive, but if it were negative it could make no other alteration, but to make negative the conſtant rectangle of the reduced equations, which caſe has already been conſtructed. Wherefore the firſt of the four equations propoſed has now been conſtructed in general.

As to the ſecond equation of thoſe exhibited above, which is $-\frac{gxx}{b}+xy = ab \pm mx \pm ny$; the firſt ſubſtitution to be made is $y-\frac{gx}{b}=z$, that is, $y=z+\frac{gx}{b}$, and let all the reſt be done as before.

Therefore, to obtain the ordinate y, it will be neceſſary to join $\frac{gx}{b}$ to z, whence in each caſe of Fig. 66, 67, 68, 69, the triangle ABC muſt be deſcribed under the line NN, as is ſeen at AbC, with the ſides A$b=b$, bC $=g$, and with the angle AbC equal to the angle which ought to be contained by the co-ordinates of the equation propoſed; whence, Ab being produced both ways, and taking any line A$q=x$, the correſponding line qR will be the ordinate y required.

The two laſt equations of the four were theſe, but with their ſigns changed.

$$-\frac{gxx}{b}+xy=-ab\mp mx\mp ny.$$

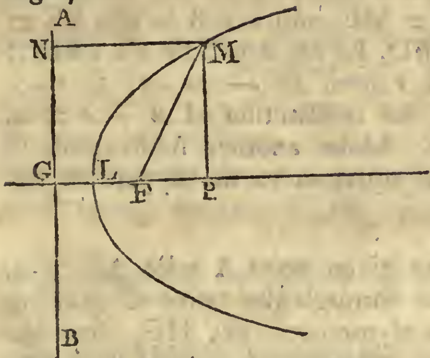
$$\frac{gxx}{b}+xy=-ab\mp mx\mp ny.$$

But this has been already conſtructed in the conſtruction of the firſt, and the other is already conſtructed in the conſtruction of the ſecond; ſo that the four equations at firſt propoſed are now conſtructed in general, as was required to be done.

PRO-

PROBLEM I.

Fig. 70.



132. The indefinite right line AB is given in poſition, and the point F is given out of it; it is required to find the *locus* of all the points M, ſuch that, drawing from each of them two right lines, one perpendicular to AB, the other to the point F, theſe two lines may always be equal to each other. A geometrical problem, conſtructed by the parabola.

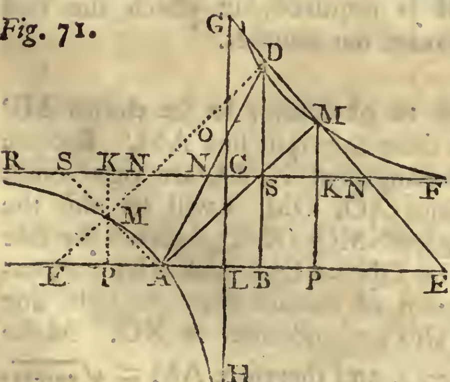
Let M be one of the points required, and let the right lines be drawn, MF to the given point F, and MN perpendicular to BA. Theſe therefore ought to be equal to each other by the condition of the Problem; and therefore, drawing FG perpendicular to AB, and calling it $= a$, let MP be drawn perpendicular to it, and make GP $= x$, PM $= y$, it will be PF $= x - a$, and therefore FM $= \sqrt{aa - 2ax + xx + yy}$. But FM $=$ MN $=$ GP; then $x = \sqrt{aa - 2ax + xx + yy}$, that is, $xx = xx - 2ax + aa + yy$, or $2ax - aa = yy$. And making the ſubſtitution of $x - \frac{1}{2}a = z$, it will be $2az = yy$, an equation to the common parabola.

Take GL equal to half GF, and with vertex L, and parameter $= 2a$, deſcribe the parabola LM. This ſhall be the *locus* required, in which taking any line LP $= z$, it will be PM $= y$. But GL $= \frac{1}{2}a$; therefore GP $= z + \frac{1}{2}a = x$, and therefore GP, PM, will be the co-ordinates of the equation propoſed.

It is known from the property of the parabola, that AB is the *directrix*, and F the *focus* of the curve.

PROBLEM II.

Fig. 71.



133. The indefinite right line PAP being given in poſition, and two fixed points A, D, one in the ſame line, and the other out of it; the *locus* is required of all the points M, ſuch that, drawing the lines MA to the given point A, and DME from the given point D through the point M, it may always be AM equal to the portion ME, comprehended between the point M, and the point E, in which the ſame line DME meets the given line PAP. Another, conſtructed by the hyperbola between the aſymptotes.

 From

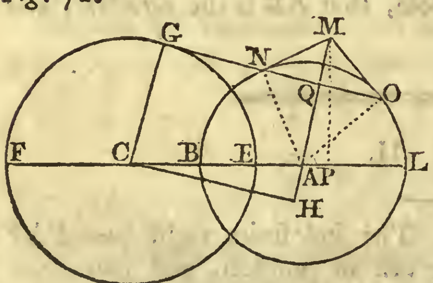
From the given point D, and from the point M, which is ſuppoſed to be one of thoſe required, draw the lines DB, MP, perpendicular to the given line PAP. Then the lines AB, BD, will be known, and therefore make AB $= 2a$, BD $= 2b$, AP $= x$, PM $= y$. Let the right lines AM, DME, be drawn. Now, by the condition of the Problem, AM = ME, and it will be alſo PE = AP $= x$. And becauſe of ſimilar triangles EBD, EPM, it will be EB . BD :: EP . PM. And, ſubſtituting the analytical values, $2x - 2a . 2b :: x . y$. Whence the equation $xy - ay = bx$. Make the ſubſtitution of $x - a = z$, it will be $zy = bz + ab$, or $zy - bz = ab$. Make another ſubſtitution of $y - b = p$, and it will be at laſt $pz = ab$, an equation to the hyperbola between the aſymptotes.

On the line PAP given in poſition, from the given point A take AL $= a$, and raiſe LC $= b$ perpendicular to it. Then through the point C drawing the right line RF parallel to PP, between the aſymptotes RF, HG, draw the two oppoſite hyperbolas DM, AM, with the rectangle ab, which ſhall paſs through the points D, A. Taking any line CK $= z$, it will be KM $= p$. But AL $= a$, LC $= b$; therefore AP $= a + z = x$, and PM $= p + b = y$, ſhall be the co-ordinates of the Problem, and the hyperbolas ſhall be the *locus* required.

PROBLEM III.

A problem with three caſes, conſtructed by the parabola, ellipſis, and hyperbola.

Fig. 72.



134. Two circles EGF, BNO, being given, and alſo their centres C, A; if, from any point G of the periphery of the circle EGF, be drawn a tangent GNO, which meets the other circle BNO in the points N, O; and from theſe two points, if we draw two tangents NM, OM, the *locus* of all the points M is required, in which the ſaid tangents meet one another.

From the point M, which is one of thoſe to be found, let be drawn MP perpendicular to CA, and from the centre A draw the right line AM. Becauſe the triangles ANM, AOM, are equal, for the angles at N, O, are right ones, and the ſides AN, NM, are equal to the ſides AO, OM, it will be alſo the angle NMA = OMA; whence in the triangles NMQ, OMQ, becauſe the ſide MQ is common, and MO = MN, it will be QN = QO, and AM perpendicular to NO. From the centre C to the point of contact draw the right line CG, which will be parallel to AM, it being alſo perpendicular to NO. Make AB $= a$, CE $= b$, CA $= c$, AP $= x$, PM $= y$, and therefore AM $= \sqrt{xx+yy}$.

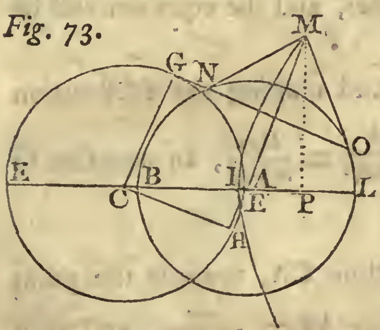
In

In the fimilar triangles AOM, AQO, it will be AM . OA :: OA . AQ; and fubftituting the analytical values, we fhall find $AQ = \frac{aa}{\sqrt{xx+yy}}$. Draw CH perpendicular to MA, produced if need be; it will be HQ = CG, and therefore $HA = b - \frac{aa}{\sqrt{xx+yy}}$. But the triangles CAH, AMP, will be fimilar; therefore PA . AM :: AH . AC; that is, $x \,.\, \sqrt{xx+yy} :: b - \frac{aa}{\sqrt{xx+yy}} \,.\, c$; and multiplying extremes and means, $cx = b\sqrt{xx+yy} - aa$, or $cx + aa = b\sqrt{xx+yy}$. Then fquaring, $ccxx + 2aaccx + a^4 = bbxx + bbyy$, that is, $yy + \frac{bb-cc}{bb}xx - \frac{2aacx}{bb} - \frac{a^4}{bb} = 0$.

In this equation there are three cafes that ought to be diftinguifhed; that is, when $b = c$, when b is greater than c, and when c is greater than b.

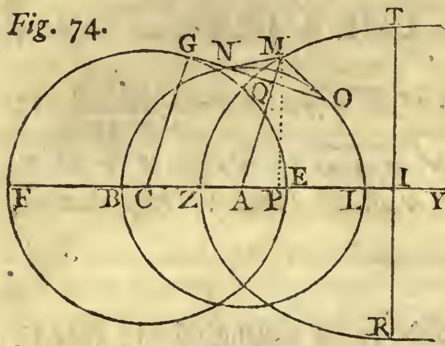
Firft, let $b = c$, then the equation will be $yy - \frac{2aax}{b} - \frac{a^4}{bb} = 0$, or $yy = \frac{2a^2x}{b} + \frac{a^4}{bb}$. And finding a rectangle $2bf = aa$, put it inftead of aa in the laft term of the fecond member, and it will be $yy = \frac{2aax + 2aaf}{b}$; and making the fubftitution of $x + f = z$, it will be at laft $yy = \frac{2aaz}{b}$, an equation to the *Apollonian* parabola. On the right line CA, towards C take $AI = \frac{aa}{2b} = f$, and with vertex I, axis IL, parameter $\frac{2aa}{b}$, let the parabola IM be defcribed. This will be the *locus* required; in which, taking any line $IP = z$, it will be $PM = y$; but $AI = f$, then $AP = z - f = x$, and the lines AP, PM, will be the co-ordinates of the Problem.

Fig. 73.



Secondly, let b be greater than c, which will make the term $\frac{bb-cc}{bb}xx$ to be pofitive. If we write the equation thus, $\frac{bb-cc}{bb}xx - \frac{2aacx}{bb} = \frac{a^4}{bb} - yy$; or thus, $xx - \frac{2aacx}{bb-cc} = \frac{a^4}{bb-cc} - \frac{bbyy}{bb-cc}$, and adding to both members the fquare $\frac{a^4cc}{\overline{bb-cc}|^2}$, it will be $xx - \frac{2aacx}{bb-cc} + \frac{a^4cc}{\overline{bb-cc}|^2} = \frac{a^4bb}{\overline{bb-cc}|^2} - \frac{bbyy}{bb-cc}$; and making the ſubſtitution of $x - \frac{aac}{bb - cc} = z$, it will be finally $\frac{bbyy}{bb - cc} = \frac{a^4bb}{\overline{bb - cc})^2} - zz$, which is an equation to the ellipſis.

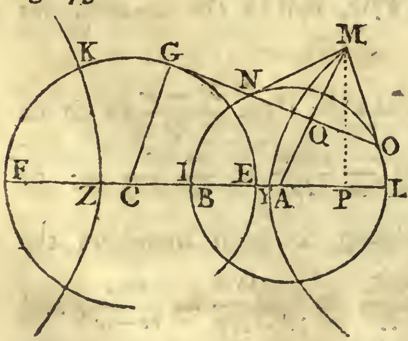
Fig. 74.



From the point A towards Y take the portion $AI = \frac{aac}{bb - cc}$, and with centre I, tranſverſe axis $ZY = \frac{2aab}{bb - cc}$, and conjugate $RT = \frac{2aa}{\sqrt{bb - cc}}$, let the ellipſis RZTY be deſcribed, which will be the *locus* required. In this, taking any line $IP = - z$, (that is, on the negative ſide,) and it will be $PM = y$. But $AI = \frac{aac}{bb - cc}$; therefore $AP = z + \frac{aac}{bb - cc} = x$, and therefore the lines AP, PM, will be the co-ordinates of the Problem.

Laſtly, let c be greater than b, then the quantity $\frac{bb - cc}{bb} xx$ will be negative, and therefore the equation is $\frac{cc - bb}{bb} xx + \frac{2aacx}{bb} = yy - \frac{a^4}{bb}$, or $xx + \frac{2aacx}{cc - bb} = \frac{bbyy - a^4}{cc - bb}$. Add the ſquare $\frac{a^4cc}{\overline{cc - bb})^2}$ on both ſides, and the equation will be $xx + \frac{2aacx}{cc - bb} + \frac{a^4cc}{\overline{cc - bb})^2} = \frac{bbyy}{cc - bb} + \frac{a^4bb}{\overline{cc - bb})^2}$. And making the ſubſtitution of $z = x + \frac{aac}{cc - bb}$, it will be at laſt $zz - \frac{a^4bb}{\overline{cc - bb})^2} = \frac{bbyy}{cc - bb}$, an equation to an hyperbola, when referred to it's axis.

Fig. 75.



On the right line CA, towards the point C take the portion $AI = \frac{aac}{cc - bb}$, and with centre I, tranſverſe axis $ZY = \frac{2aab}{cc - bb}$, and conjugate $= \frac{2aa}{\sqrt{cc - bb}}$, deſcribe the oppoſite hyperbolas YM, ZK; theſe ſhall be the *locus* required. In which, taking any line $IP = z$, it will be $PM = y$. But $AI = \frac{aac}{cc - bb}$; then AP

$AP = z - \frac{aac}{cc - bb} = x$. And therefore the lines AP, PM, will be the co-ordinates of the Problem.

In this Problem it is always ſuppoſed, that the circle EFG is greater than the circle BNO, or that b is greater than a; but if it ſhould be either $b = a$, or $b \angle a$, the *locus* of the points required in the firſt caſe would always be a parabola, in the ſecond an ellipſis, and in the third two oppoſite hyperbolas; ſo that it would be needleſs to diſtinguiſh theſe caſes, which make no variation in the *loci*.

PROBLEM IV.

Fig. 76.

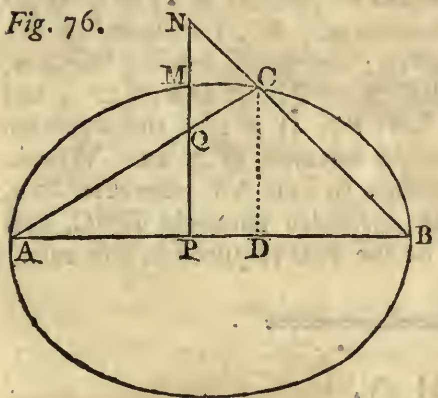
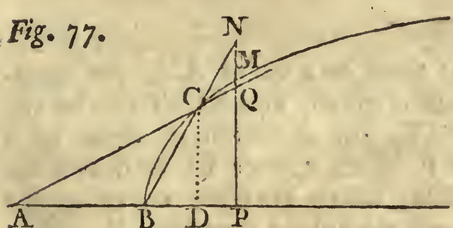


Fig. 77.



135. Two right lines AC, CB, (Fig. 76, 77.) are given in poſition on the right line AB, which cut one another in C; the *locus* is required of all the points M, ſuch that, drawing through them a perpendicular PMN to AB, which cuts the line AC in the point Q, and the line BC in the point N, the ſquare of PM may be equal to the rectangle PQ × PN. A *locus* to the Conic Sections conſtructed.

Let the right line CD be drawn parallel to PM; this will fall either between the points A, B, as in Fig. 76, or on one ſide of them, as in Fig. 77.

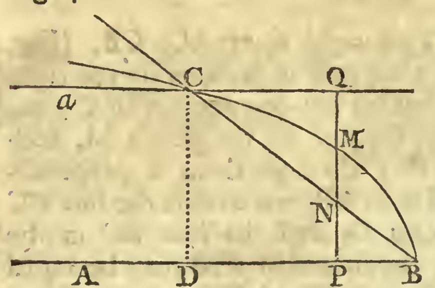
Firſt, let it fall between the points A, B, and make $AB = a$, $AP = u$, $PQ = x$, $PM = y$, $PN = z$. By the condition of the Problem, it will be $zx = yy$. But the ratio of AP to PQ is given, which therefore may be put as m to n. Alſo, the ratio of BP to PN is given, which may be as b to c. Then it will be $PQ = x = \frac{un}{m}$, and $PN = z = \frac{ac - uc}{b}$. Theſe values therefore being ſubſtituted in the equation $zx = yy$, it will be $yy = \frac{ac - uc}{b} \times \frac{un}{m}$, or $\frac{bmyy}{cn} = au - uu$, an equation to an ellipſis with tranſverſe axis $AB = a$, conjugate $a\sqrt{\frac{cn}{bm}}$. Such an ellipſis AMB being deſcribed, the upper half AMCB will be the *locus* required.

Now

Now let the point D (Fig. 77.) fall on one side of the points A, B, and make, as above, AB $= a$, AP $= u$, PM $= y$, PQ $= x$, PN $= z$; it will be BP $= u - a$, and therefore PN $= \frac{uc - ac}{b}$. But, by the condition of the Problem, $zx = yy$, and $x = \frac{un}{m}$, as before. Therefore, making a substitution of the values of z and x, it will be $yy = \frac{uc - ac}{b} \times \frac{un}{m}$, or $\frac{bmyy}{cn} = uu - au$, an equation to the hyperbola.

At the vertex B, with the transverse axis $= a$, and the conjugate axis $= a\sqrt{\frac{cn}{bm}}$, describe the hyperbola BCM; this will be the *locus* required.

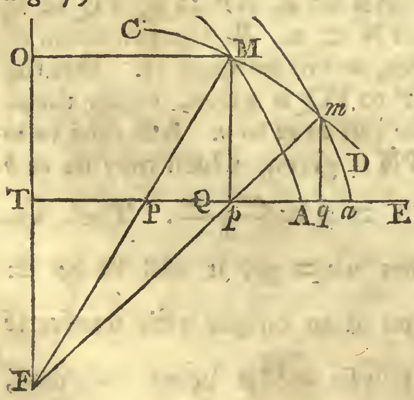
Fig. 78.



If the right line AC should not fall upon AB, but should be parallel to it, as it would be in the position aC, AB, the right line PQ would be given; therefore, making PQ $= m$, AB $= a$, BP $= u$, PN $= z$, PM $= y$, and supposing BP . PN :: m . n, the equation $xz = yy$ would become $yy = un$. Wherefore, with vertex B, axis AB, parameter $= n$, describe the *Apollonian* parabola BMC, and this would be the *locus* required in this case.

PROBLEM V.

Another. *Fig.* 79.



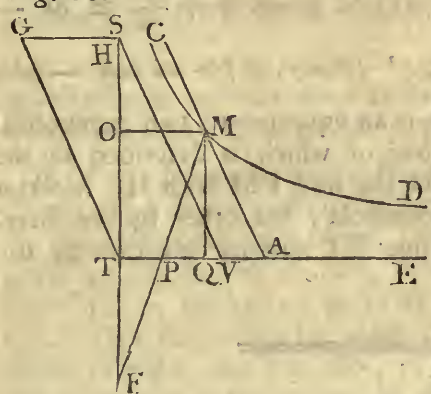
136. Let there be a curve AM, the equation of which is given, and let it's axis be the right line AT, out of which let there be a fixed point F, from whence let be drawn the right line FM, which cuts the curve in the point M, and the axis in the point P. Now the right line FM, moving about the point F, causes the whole plane AMP to move parallel to itself upon the line ET, the point P being fixed in respect of the point A, but moveable upon the axis TA, that is, AP being a given line. In the mean while, the point M will describe a curve CMD. It is required to know what kind of curve this is.

Let

Let the curve be now arrived at the point a of the right line ET; it will be, by the construction of the Problem, $Pp = Aa$, and therefore $AP = ap$. Make $AP = a$, $FT = b$; and from the point M letting fall the perpendicular MQ to ET, make $TQ = x$, $QM = y$, $AQ = t$. Because of the similar triangles FOM, PMQ, it will be FO . OM :: QM . PQ, that is, $b + y . x :: y . PQ = \frac{xy}{b+y}$. But $PQ = a - t$; therefore $\frac{xy}{b+y} = a - t$, or $xy = ab - bt + ay - ty$.

Now, in this canonical equation, if we substitute the value of t given by y, and by the known quantities of the equation of the curve AM, we shall have the required equation of the curve CMD.

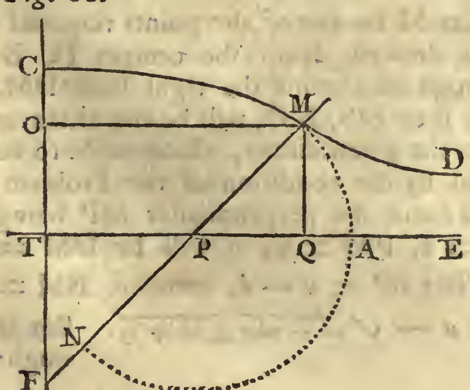
Fig. 80.



First, let AM be a right line. The ratio of t to y will be given, which let be that of m to n; then $t = \frac{my}{n}$. And, substituting this value of t in the canonical equation, it will be $\frac{myy}{n} = ab - xy - \frac{bmy}{n} + ay$; a *locus* to the hyperbola between the asymptotes.

To construct it in the given figure, on FO take any portion TH, and in a right angle draw HG such, that it may be FH . HG :: $n . m$; draw TG, and upon TA taking the portion $TV = \frac{an - bm}{n}$, from the point V draw VS parallel to TG; and between the asymptotes VS, VE, describe the hyperbola CMD with the constant rectangle $= \frac{abg}{n}$; (making the known line $TG = g$.) Then taking any absciss $TQ = x$, the corresponding ordinate will be $QM = y$, and the hyperbola will be the *locus* of the equation $\frac{myy}{n} = ab - xy - \frac{bmy}{n} + ay$.

Fig. 81.

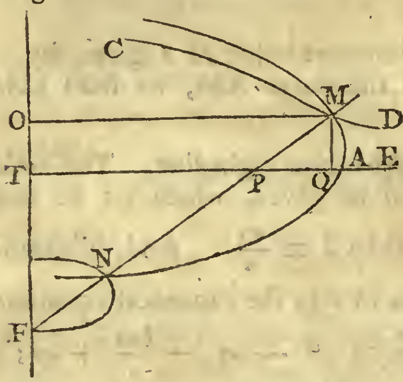


In the second place, let AM be a circle described with centre P, radius $AP = a$. By the property of the circle, it will be $AQ = t = a - \sqrt{aa - yy}$; and instead of t substituting this value in the general equation, it will be $xy = \overline{b + y} \times \sqrt{aa - yy}$, an equation to the conchoid of *Nicomedes*. And the curve CMD, which is described by the intersection M of the right line FM with the superior arch of the circle AM, will be the upper conchoid, ET

 will

will be the asymptote, F the pole. And the curve which is generated by the intersection N of the right line FM with the circle under ET, will be the lower conchoid. This appears evidently from the nature of the conchoid, and from the condition of the Problem. For the two lines PM, PN, intercepted between the asymptote and the curve, will always be equal to the radius of the circle AP.

Fig. 82.

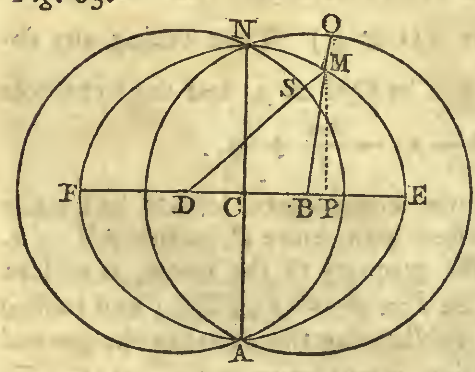


In the third place, let the curve AM be an *Apollonian* parabola, with a parameter AP $= a$. On this hypothesis, it will be $t = \frac{yy}{m}$; and this value of t being substituted in the canonical equation, it will be $xy - ay + \frac{y^3}{m} = ab - \frac{byy}{m}$, that is, $y^3 + mxy + byy - amy - abm = 0$. This is an equation to two parabolical conchoids, one of which is described by the intersection of the line FM with the superior part of the parabola; the other by the intersection with the inferior part. And the right line ET will in this case be the asymptote of the curve.

PROBLEM VI.

Another.

Fig. 83.



137. Two equal circles being given, cutting each other in two points A, N, and their centres D, B, being given; it is required to find the *locus* of all the points M such, that their distances from the said circles may always be equal to one another.

Let M be one of the points required; then drawing from the centres D, B, through this point the right lines DM, BO, then MS, MO, will be the distances from the given circles, which ought to be equal by the condition of the Problem. Therefore make DS $=$ BO $= a$, DB $= b$, and the perpendicular MP being let fall upon DB produced, make DP $= x$, PM $= y$; it will be DM $= \sqrt{xx + yy}$, and SM $= \sqrt{xx + yy} - a$. But BP $= x - b$, therefore BM $= \sqrt{xx - 2bx + bb + yy}$, and thence OM $= a - \sqrt{xx - 2bx + bb + yy}$. But it ought

ought to be SM = MO; whence we ſhall have the equation $\sqrt{xx + yy} - a = a - \sqrt{xx - 2bx + bb + yy}$. By the methods already taught this will be reduced to $xx - bx + \frac{1}{4}bb = aa - \frac{4aayy}{4aa - bb}$; and making the ſubſtitution of $x - \frac{1}{2}b = z$, it will be $zz = aa - \frac{4aayy}{4aa - bb}$, or $\frac{4aayy}{4aa - bb} = aa - zz$, which is an equation to an ellipſis.

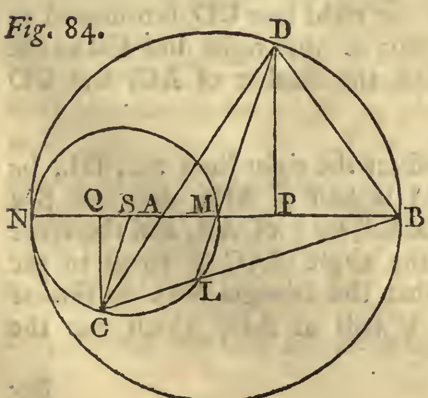
Let the right line DB be biſected in the point C, and with centre C, tranſverſe axis FE = $2a$, and conjugate AN = $\sqrt{4aa - bb}$, let the ellipſis FAEN be deſcribed, which will be the *locus* required. For, taking any line CP = z, it will be PM = y; but CD = $\frac{1}{2}b$, therefore DP = $z + \frac{1}{2}b = x$, and therefore the lines DP, PM, are the co-ordinates of the Problem propoſed.

It would be needleſs to diſtinguiſh the caſes, in which a is greater, equal to, or leſs than b, becauſe the Problem will ſtill be of the ſame nature, b being always leſs than $2a$, as plainly appears.

It follows from this conſtruction, that the points D, B, will be the *foci* of the ellipſis, and that it's conjugate axis will be terminated at the points, in which the two circles cut each other. And firſt, becauſe DS = BO, and SM = MO, it will be DS + SM + MB, that is, DM + MB = 2DS; but 2DS = FE, therefore, by the known property of the ellipſis, the points D, B, will be it's *foci*. This ſuppoſed, by another property of the ellipſis relating to the *foci*, conceiving the lines BA, BN, to be drawn, it will be BN = BA = CE. But this is verified in the points, in which the two given circles will cut each other; for D, B, are their centres, and CE, by conſtruction, is equal to the ſemidiameter of the ſame circles. Therefore the ellipſis will paſs through the ſaid points of interſection of the given circles. Q. E. D.

PROBLEM VII.

Fig. 84.



138. The right line AB being given, to find the *locus* of ſuch points D, that, in the produced line DA, taking AC half of AD, and drawing to the point B the right line CB, this may be equal to CD. Another.

Let D be one of the points required, from whence let fall DP perpendicular to AB. Make AB = a, AP = x, PD = y; it will be AD = $\sqrt{xx + yy}$, and, by the condition of the Problem, AC = $\frac{1}{2}\sqrt{xx + yy}$: wherefore CD $=$ CB $= \frac{3}{2}\sqrt{xx+yy}$. From the point C draw CQ perpendicular to BA produced. Now, becaufe of the fimilar triangles AQC, APD, and AD $=$ 2AC, it will be AP $=$ 2AQ, and PD $=$ 2QC; whence CQ $= \frac{1}{2}y$, and AQ $= \frac{1}{2}x$. Therefore BQ $= a + \frac{1}{2}x$. Now CBq $=$ CQq $+$ BQq $= aa + ax + \frac{1}{4}xx + \frac{1}{4}yy$. But CB$q$ $=$ CDq $= \frac{9}{4} \times \overline{xx+yy}$; whence we fhall have the equation $\frac{9}{4}xx + \frac{9}{4}yy = aa + ax + \frac{1}{4}xx + \frac{1}{4}yy$, which is reduced to $xx - \frac{1}{2}ax = \frac{1}{2}aa - yy$. Now, adding to both members the fquare $\frac{1}{16}aa$, and making the fubftitution of $x - \frac{1}{4}a = z$, it will be finally $zz = \frac{9}{16}aa - yy$, an equation to the circle.

Therefore, taking BM $= \frac{3}{4}a$, and with centre M, and radius BM, defcribe the circle NDB, this will be the *locus* required; in which, taking any line MP $= z$, it will be PD $= y$; but AM $= \frac{1}{4}a$, therefore AP $= z + \frac{1}{4}a = x$, and the lines AP, PD, will be the co-ordinates of the propofed Problem.

If we would have alfo the *locus* of the points C, this would be another Problem of a like nature, which might be refolved in the following manner.

Make AQ $= p$, QC $= q$, which is perpendicular to BN; it will be AP $= 2p$, PD $= 2q$; but AM $= \frac{1}{4}a$, and MB $= \frac{3}{4}a$. Then NA $= \frac{1}{2}a$, and therefore NP $\times$ PB $= \frac{1}{2}aa + ap - 4pp$. But, by the property of the circle, NP $\times$ PB $=$ PDq and $= 4qq$. Then it will be $4qq = \frac{1}{2}aa + ap - 4pp$. Whence $\frac{1}{8}aa - qq = pp - \frac{1}{4}ap$. Add to both fides the fquare $\frac{1}{64}aa$, and making the fubftitution of $p - \frac{1}{8}a = t$, it will be $qq = \frac{9}{64}aa - tt$. Whence, with diameter MN $= \frac{3}{4}a$ defcribing the femicircle NCM, this will be the *locus* of all the points C; in which, taking from the centre S any line SQ $= t$, it will be QC $= q$. But AS $= \frac{1}{8}a$ by the conftruction. Then AQ $= t + \frac{1}{8}a = p$, and the lines AQ, QC, will be the co-ordinates of the Problem.

Thefe two Problems may be demonftrated conjunctly in form of a theorem, after the following manner.

In the given line AB is taken MB equal to $\frac{3}{4}$ of AB, and with centre M, radius MB, a circle NDB is defcribed; and alfo with diameter MN the circle NCM; through the point A drawing any how the right line CD terminated at the periphery of each circle, and from the point C the right line CB to the extremity of the diameter, it will always be DA the double of AC, and CD equal to CB.

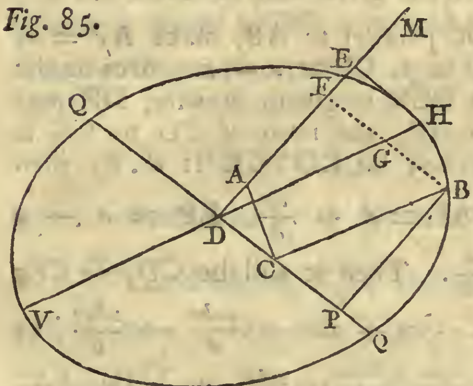
Let S be the centre of the circle NCM, and let the right lines SC, DL, be drawn through the centres S, M. Becaufe SM is half of MB, then will SM be $\frac{3}{8}$ of AB. But AM is $\frac{1}{4}$ of it; therefore SA will be $\frac{1}{8}$ of AB, and therefore $\frac{1}{2}$ of AM. But SC is alfo half of DM, and the angle SAC is equal to the angle DAM; therefore it is eafy to perceive, that the triangle SAC is fimilar to the triangle DAM, and that therefore AC is half of AD, which was the firft thing.

But

But if the triangles SAC, ADM, be ſimilar, then the angle SCA will be equal to the angle ADM; whence the right lines SC, DL, will be parallel, and conſequently the triangles BLM, BCS, are ſimilar, and therefore ML will be the fourth proportional to BS, SC, and MB. But BS $= \frac{9}{8}$AB, SC $= \frac{3}{8}$AB, MB $= \frac{6}{8}$AB. Therefore ML $= \frac{2}{8}$AB $=$ AM. But MD $=$ MB, and the angle AMD $=$ LMB. Therefore the triangles AMD, BML, are equal, and the angle ADM $=$ MBL. But alſo the angle MDB $=$ MBD, ſo that the angle CDB $=$ CBD, and therefore the ſide CB $=$ CD; which was the ſecond thing.

PROBLEM VIII.

Fig. 85.



139. The two ſides AC, CB, of the *norma* ACB being given, the *locus* is required of all the points, through which the extremity B of the ſide CB will paſs, whilſt the *norma* moves in ſuch manner, that it's point A ſhall always be upon the line DM, and the point C upon the line DP, which is ſuppoſed perpendicular to DM. Another.

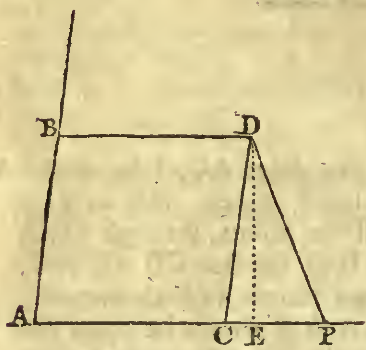
From the point B let fall BP perpendicular to DP, and make DP $= x$, PB $= y$, AC $= a$, CB $= b$; it will be CP $= \sqrt{bb - yy}$, DC $= x - \sqrt{bb - yy}$. But the angles DCA, BCP, taken together, are equal to a right angle, as alſo the angles BCP, CBP; and therefore the angles DCA, CBP, will be equal to each other. Then the triangles ADC, BCP, will be ſimilar, and it will be AC . CD :: BC . BP, that is, $a \,.\, x - \sqrt{bb - yy} :: b \,.\, y$, and thence $ay = bx - b\sqrt{bb - yy}$; and, by ſquaring and ordering, the equation will be $xx - \frac{2axy}{b} + \frac{aayy}{bb} = bb - yy$. Make the ſubſtitution of $x - \frac{ay}{b} = z$, and we ſhall have the equation $zz = bb - yy$, which is to the ellipſis.

On the indefinite line DM deſcribe the triangle DEH with it's ſides DE $= b$, EH $= a$, and with the right angle DEH, becauſe the co-ordinates of the Problem make a right angle; and let the known line DH $= f$. With tranſverſe ſemidiameter DH $= f$, and with the conjugate ſemidiameter DQ $= b$ and parallel to EH, deſcribe the ellipſis HBQ; it ſhall be the *locus* required. For, taking any line DF $=$ PB $= y$, it will be GB $= z$, FG $= \frac{ay}{b}$; therefore

fore FB $= z + \frac{ay}{b} = x =$ DP. And therefore the lines DP, PB, are the co-ordinates of the Problem.

PROBLEM IX.

Another. *Fig.* 86.

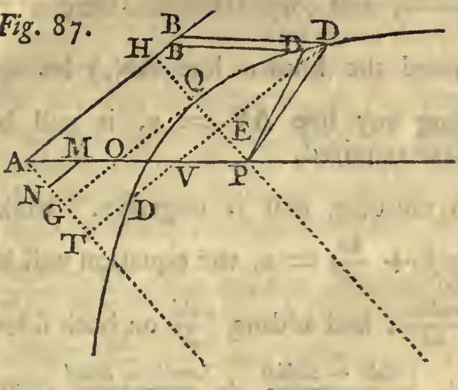


140. The angle BAP being given, and the point P being alſo given; it is required to find the *locus* of all ſuch points D, that, drawing the two right lines, BD parallel to AP, and DP to the given point P, the lines BD, DP, may always be to each other in the given ratio of d to e.

Drawing DC parallel to AB, make AP $= a$, AC $= x$, CD $= y$, CP $= a - x$. Becauſe the angle BAP or DCE is given, drawing DE perpendicular to AP, the ratio of CD to CE is given, which may be CD . CE :: $d . b$; then CE $= \frac{by}{d}$, AE $= x + \frac{by}{d}$, EP $= a - x - \frac{by}{d}$; or elſe $= x + \frac{by}{d} - a$, PD $= \frac{ex}{d}$. Then it will be CDq — CEq = DPq — PEq, that is, $yy = \frac{eexx}{dd} - aa - xx + 2ax + \frac{2aby}{d} - \frac{2bxy}{d}$, or $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{ee + bb - dd}{dd}xx + 2ax - aa + \frac{2aby}{d}$, by adding the ſquare $\frac{bbxx}{dd}$ on both ſides. But here it may be obſerved, that the quantity $ee + bb - dd$ may either be equal to, greater, or leſs than, nothing; and, firſt, let it be equal to nothing, in which caſe the equation will become $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{2aby}{d} + 2ax - aa$. And making the ſubſtitution of $y + \frac{bx}{d} = z$, it will be $zz - \frac{2abz}{d} = 2ax - \frac{2abbx}{dd} - aa$. Then adding $\frac{aabb}{dd}$ on both ſides, it will be $zz - \frac{2abz}{d} + \frac{aabb}{dd} = 2ax - \frac{2abbx}{dd} + \frac{aabb - aadd}{dd}$. Now, making the ſubſtitution of $z - \frac{ab}{d} = p$, it will be $pp = \frac{2addx - 2abbx + aabb - aadd}{dd}$, or $pp = \overline{x - \frac{1}{2}a} \times \frac{2add - 2abb}{dd}$; and making $x - \frac{1}{2}a = q$, it will become at laſt $pp = \frac{2add - 2abb}{dd}q$, an equation to the *Apollonian* parabola.

Let

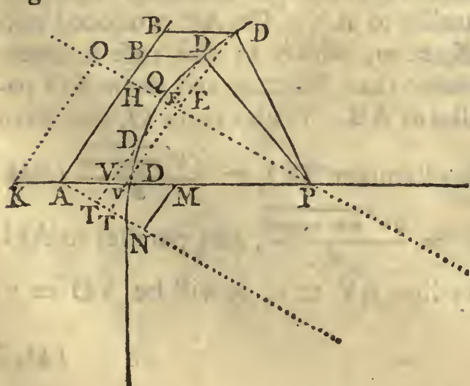
Fig. 87.



Let BAP be the given angle; the given line, AP $= a$. On AP, produced indefinitely, let there be defcribed the triangle AMN with the angle AMN $=$ BAP; and let AM . MN :: $d . b$. Produce AN indefinitely, and in AB take AH $= \frac{ab}{d}$, and draw HE indefinitely, and parallel to AN. Bifect AP in O, and draw OQ parallel to AB. With vertex Q, on the diameter QE, with parameter $= \frac{2add - 2abb}{df}$, (making $f =$ AN,) and with the ordinates parallel to AB, defcribe the parabola QD. Take any line QE $= x$, it will be ED $= y$, and this parabola will be the *locus* required.

In the fecond place, let $ee + bb - dd$ be greater than nothing, or a pofitive quantity. Affuming therefore the equation, and making $ee + bb - dd = hh$, it will be $yy + \frac{2bxy}{d} + \frac{bbxx}{dd} = \frac{hhxx}{dd} - aa + 2ax + \frac{2aby}{d}$. And making the fame fubftitution of $y + \frac{bx}{d} = z$, it will be $zz - \frac{2abz}{d} = \frac{hhxx}{dd} - aa + 2ax - \frac{2abbx}{dd}$; and adding $\frac{aabb}{dd}$, and making the fubftitution of $z - \frac{ab}{d} = p$, it will be $ddpp = hhxx + 2addx - 2abbx - aadd + aabb$; that is, $xx + \frac{2ad^2x - 2ab^2x}{hh} = \frac{ddpp}{hh} + \frac{aadd - aabb}{hh}$; make $\frac{add - abb}{hh} = m$, then $xx + 2mx = \frac{ddpp}{hh} + am$; and adding mm to each fide, it will be $xx + 2mx + mm = \frac{ddpp}{hh} + am + mm$, and making $x + m = q$, it will be finally $qq = \frac{ddpp}{hh} + am + mm$, that is, $qq - am - mm = \frac{ddpp}{hh}$, an equation to an hyperbola.

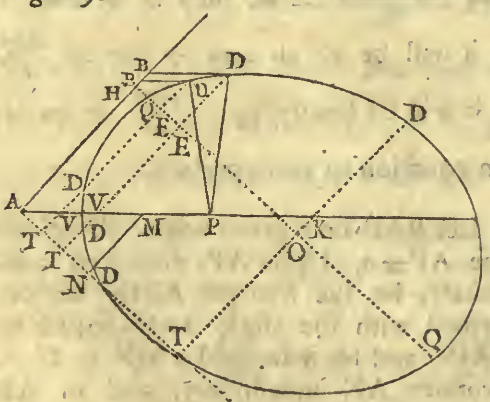
Fig. 88.



Let BAP be a given angle; the given line, AP$= a$. Upon AP, produced indefinitely, let the triangle AMN be defcribed with the angle AMN equal to BAP; and let it be AM . MN :: $d . b$. Produce AN indefinitely, and in AB take AH $= \frac{ab}{d}$, and through the point H draw the indefinite line OE parallel to AN. Then make AK $= m$, and draw KO parallel to AH. With centre O, O, tranſverſe ſemidiameter $OQ = \frac{f\sqrt{am + mm}}{d}$, and conjugate ſemidiameter $= \frac{b\sqrt{am + mm}}{d}$, parallel to AH, (by f is denoted the known line AN,) let the hyperbola QD be deſcribed. Then taking any line $AV = x$, it will be $VD = q$, and this hyperbola will be the *locus* required.

Laſtly, let $ee + bb - dd$ be leſs than nothing, that is negative. Make then $ee + bb - dd = -hh$, and making $y + \frac{bx}{d} = z$, the equation will be $zz - \frac{2abz}{d} = -\frac{hhxx}{dd} - aa + 2ax - \frac{2abbx}{dd}$; and adding $\frac{aabb}{dd}$ on both ſides, it will be $zz - \frac{2abz}{d} + \frac{aabb}{dd} = -\frac{hhxx}{dd} + \frac{2addx - 2abbx}{dd} + \frac{aabb - aadd}{dd}$; and making the ſubſtitution of $z - \frac{ab}{d} = p$, it will be $ddpp = -hhxx + 2addx - 2abbx + aabb - aadd$, that is, $xx + \frac{2abbx - 2addx}{hh} = \frac{aabb - aadd}{hh} - \frac{ddpp}{hh}$. Make $\frac{add - abb}{hh} = m$, and we ſhall have $xx - 2mx = -am - \frac{ddpp}{hh}$, and adding mm on both ſides, $xx - 2mx + mm = mm - am - \frac{ddpp}{hh}$; laſtly, making the ſubſtitution of $x - m = q$, it is $\frac{ddpp}{hh} = mm - qq - am$, an equation to an ellipſis.

Fig. 89.



Let BAP be the given angle, and the given line $AP = a$. On AP, indefinitely produced, deſcribe the triangle AMN with the angle AMN equal to BAP. Make $AM . MN :: d . b$, and produce AN indefinitely, and in AB take $AH = \frac{ab}{d}$, and through the point H draw the indefinite line HE parallel to AN. On AP produced take $AK = m$, which in this caſe is always greater than $AP = a$, and draw KO parallel to AB. With centre O, tranſverſe ſemidiameter $OQ = \frac{f\sqrt{mm - am}}{d}$ (making $AN = f$,) with conjugate ſemidiameter $= \frac{b\sqrt{mm - am}}{d}$, and parallel to AH, deſcribe the ellipſis QD. Then taking any line $AV = x$, it will be $VD = y$; and this ſhall be the *locus* required.

5 141. I

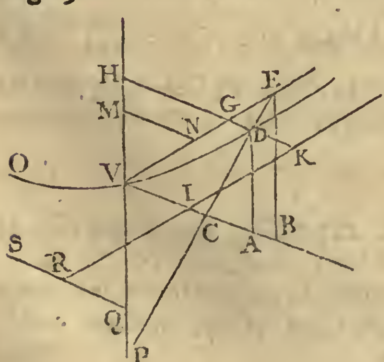
141. I ſaid above that AK $= m$ was greater than AP $= a$; in relation to which I think it neceſſary to explain how we may know which of two complicate quantities is the greater. Let there be made between them a compariſon of majority or minority, as you pleaſe, and then proceed as in an equation, by tranſpoſing, dividing, &c. and making other operations, till you arrive at a known conſequence; which, if it be true either abſolutely or hypothetically, the compariſon that was made will be abſolutely or hypothetically true; but if falſe, this will likewiſe be falſe. So, if we deſire to know whether m, that is, $\frac{add - abb}{dd - bb - ee}$, be greater than a, or not, make the compariſon or ſuppoſition $\frac{add - abb}{dd - bb - ee} > a$, and reducing to a common denominator, it will be $add - abb > add - abb - aee$, and expunging the terms that deſtroy each other, it will be $0 > -aee$; which is very true, for nothing is greater than a negative quantity. Therefore it was true that $\frac{add - abb}{dd - bb - ee}$ was greater than a.

A method to determine majority and minority in complicate quantities.

Thus, to know if $aa + 2ab$ be greater than bb, ſuppoſe $aa + 2ab > bb$, and add to each ſide the ſquare bb. It will be $aa + 2ab + bb > 2bb$, and extracting the root, it is $a + b > \sqrt{2bb}$, or $a > \sqrt{2bb} - b$. But, becauſe the quantities a, b, are given, it may always be known whether a be greater than $\sqrt{2bb} - b$, or not. And if it ſhould be ſo, then alſo $aa + 2ab$ would be greater than bb. The manner is the ſame in caſes more compounded, and therefore I ſhall inſiſt on it no longer.

PROBLEM X.

Fig. 90.



142. Two right lines VB, VE, being given in poſition, and alſo the point P, about which as a pole the right line PE revolves; to find the *locus* of all the points D, ſuch that it may always be CD to DE in a given ratio.

A geometrical problem.

Draw VP, and parallel to it the right lines AD, BE, and let the ratio of CD to DE, or rather of CD to EC, be as d to e; and the angles EVB, EBV, being given, let it be EB to BV as e to h.

T Make

Make VP $= a$, VA $= x$, AD $= y$; it will be EB $= \frac{ey}{d}$, and therefore VB $= \frac{hy}{d}$. Becaufe of the fimilar triangles CVP, CDA, it will be DA . PV :: CA . CV; and, by compounding, DA + PV . PV :: CA + CV . CV; that is, $a + y . a :: x$. CV, and therefore CV $= \frac{ax}{a+y}$. Again, becaufe of fimilar triangles PVC, EBC, it will be PV . VC :: EB . BC, that is, $a . \frac{ax}{a+y} :: \frac{ey}{d}$. BC; whence BC $= \frac{exy}{ad+dy}$, and therefore the equation BC + CV = BV, that is, $\frac{exy + adx}{ad+dy} = \frac{hy}{d}$, or $yy - \frac{exy}{h} = \frac{adx}{h} - ay$.

To conftruct this, make $y - \frac{ex}{h} = \frac{ez}{h}$, and, by fubftitution, it will be $\frac{ezy}{h} = -ay - \frac{adz}{h} + \frac{ady}{e}$, that is, $zy + \frac{ahy}{e} - \frac{adhy}{ee} = -\frac{adz}{e}$.

Again, make $z + \frac{ah}{e} - \frac{adh}{ee} = p$; then it will be $py = \frac{aadh}{ee} - \frac{aaddh}{e^3} - \frac{adp}{e}$. And making a third fubftitution of $y + \frac{ad}{e} = q$, it will be $pq = \frac{aaehd - aahdd}{e^3}$, an hyperbola between the afymptotes, the conftant rectangle of which is pofitive, becaufe e will always be greater than d.

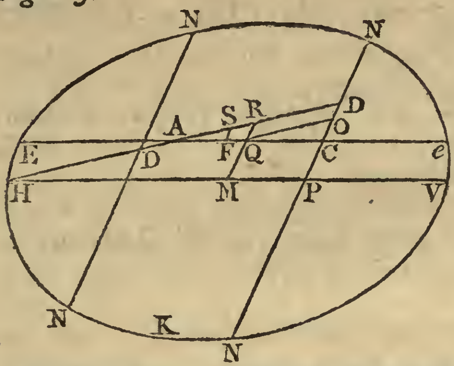
Let PV be produced indefinitely, and take VQ $= \frac{ad}{e}$. From the point Q draw the indefinite line QS parallel to VB, and, taking any point M in the right line PH, draw MN parallel to VB. Then, becaufe of fimilar triangles VMN, EBV, it will be VM . MN :: $e . h$. Make VI $= \frac{aeh - adh}{ee}$, and through the point I drawing the indefinite right line RIK parallel to VE, between the afymptotes RS, RK, defcribe the hyperbola OVD with the conftant rectangle $= \frac{aaehh - aaddh}{e^3} \times \frac{f}{e}$, (making the known line VN $= f$,) which will neceffarily pafs through the point V. Taking any line VH $= y$, it will be HD $= x$, that is, VA $= x$, AD $= y$, and the curve thus conftructed is the *locus* of the points D.

A fpecimen of the demonftration of thefe examples.

143. We may obferve here, that the equations expreffing the properties of the curves defcribed in thefe Examples, or Problems, ought to be the fame with the equations propofed to be conftructed, when the operations are truly performed; and therefore may ferve as a demonftration of the method itfelf. This

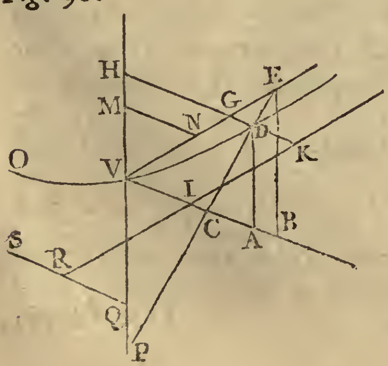
This I have purpofely omitted to do, to avoid being too prolix. However, to give a fhort fpecimen of it, I fhall take the conftructions of Example XIII. and of Problem VIII.

Fig. 65.



And, firft, for the example. Having made AD $= x$, and it being AS $= 2a$, AF $= f$, it will be AC $= \frac{fx}{2a}$, and therefore AR $= \frac{bn}{2m}$; it will be AQ $= \frac{bfn}{4am}$, and thence QC $= \frac{fx}{2a} - \frac{bfn}{4am} =$ MP. Therefore, the femidiameter being HM $= \frac{ef}{2a}$, we fhall have HP $= \frac{ef+fx}{2a} - \frac{bfn}{4am}$, and PV $= \frac{ef}{2a} - \frac{fx}{2a} + \frac{bfn}{4am}$. Thus, becaufe DN $= y$, CD $= \frac{bx}{2a}$, CP $=$ QM $= \frac{1}{2}c$, it will be PN $= y + \frac{bx}{2a} + \frac{1}{2}c$. But, by the property of the ellipfis, it muft be HP $\times$ PV . PNq :: HV . parameter $= \frac{4aem}{fn}$. Thence we fhall have the equation $\frac{eeff-ffxx}{4aa} + \frac{ffbnx}{4aam} - \frac{ffbbnm}{16aamm}$ into $\frac{4aam}{ffn} = \frac{1}{4}cc + \frac{bcx}{2a} + \frac{bbxx}{4aa} + cy + \frac{bxy}{a} + yy$. And, inftead of ee, reftoring it's value $\frac{ccmm + 4agmn + nnbb}{4mm}$, it will be $\frac{1}{4}cc + ag - \frac{mxx}{n} + bx = \frac{1}{4}cc + \frac{bcx}{2a} + \frac{bbxx}{4aa} + cy + \frac{bxy}{a} + yy$. And laftly, reftoring the values of $-\frac{m}{n} = \frac{bb - 4aa}{4aa}$, and $b = \frac{bc - 2al}{2a}$, we fhall have $ag - xx - lx = cy + \frac{bxy}{a} + yy$, which is the very equation propofed to be conftructed.

Fig. 90.



In the conftruction of the laft Problem it was $\frac{aacdh - aaddh}{e^3} \times \frac{f}{e}$ the conftant rectangle of the hyperbola, and VI $= \frac{aeh - adh}{ee}$, and parallel to the afymptote RS. Alfo, it will be RI $= \frac{adf}{ee}$. But, becaufe of fimilar triangles VMN, VHG, it is VM . VN :: VH . GV, and therefore

therefore GV $= \frac{fy}{e} =$ IK. Then RK $= \frac{adf}{ee} + \frac{fy}{e}$. But HG $= \frac{by}{e}$, GK $=$ VI $= \frac{aeh - adh}{ee}$. Whence HK $= \frac{hey + aeh - adh}{ee}$. But HD $=$ VA $= x$; then it will be KD $= \frac{hey + aeh - adh - eex}{ee}$, and therefore, by the property of the curve, the rectangle RK $\times$ KD ought to be equal to the constant rectangle, or $\frac{adf + efy}{ee} \times \frac{hey + aeh - adh - eex}{ee} = \frac{aaedh - aaddh}{e^3} \times \frac{f}{e}$. That is, $yy - \frac{exy}{h} + ay - \frac{adx}{h} = 0$, as it ought to be.

If the same care and industry were used in every Example and Problem, it would sufficiently prove the method of solution to be just.

SECT. IV.

Of Solid Problems and their Equations.

What are the roots of equations.

144. Any one of those quantities is called the Root of an Equation, which, being substituted in the equation instead of that root or letter, according to which the equation is ordered, (or instead of that letter which represents the unknown quantity,) shall make all the terms of the equation to vanish or become nothing. Or, which is the same thing, the root of an equation is each of the several values of the unknown quantity, or of that letter which performs the office of an unknown quantity in the equation.

Therefore the roots of the equation $xx - ax + bx - ab = 0$ will be two, one of which is a, the other $-b$; for each of these, being substituted instead of x, will make the terms of the equation to vanish; or, because either a or $-b$ are the values of the letter x in the proposed equation. The roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ will be 1, 2, 3, or -5; because any one of these numbers, being substituted instead of x, will make all the terms to vanish, and therefore any one of them is the root, or value of the unknown quantity x. The roots of the equation $x^4 - bbxx - aabb - a^4 = 0$ will be $+\sqrt{-aa}$, $-\sqrt{-aa}$, $+\sqrt{aa+bb}$, $-\sqrt{aa+bb}$; and so of all others.

145. Again,

145. Again, in another fenfe, thofe equations are ufed to be taken for the roots of an equation, which are formed by fubtracting, one by one, the pofitive values from the unknown quantity, or by adding the negative value, and making them equal to nothing. Therefore, in this fenfe, the roots of the equation $xx - ax + bx - ab = 0$ will be $x - a = 0$, and $x + b = 0$. Thofe of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ will be $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, and $x + 5 = 0$. And fo of others. And, in this fenfe, it is faid, that every equation is the product of all it's roots, becaufe, being continually multiplied into one another, they will exactly produce the given equation, or that of which they are the roots. Hence it is, that the roots of an equation will be fo many, including alfo the imaginary roots, as is the degree to which the equation arifes. And therefore a quadratick equation will have two roots, a cubick equation three roots, a biquadratick four roots; and fo on.

Or otherwife, the feveral divifors of an equation.

If $x + a = 0$ be multiplied into $x + b = 0$, there will arife the quadratick equation (I.) $\begin{array}{l} xx + ax + ab = 0. \\ \quad\;\; + bx \end{array}$

And if this again be multiplied into $x - c = 0$, there will arife the cubick equation (II.) $\begin{array}{llll} x^3 & + ax^2 & + abx & - abc = 0. \\ & + bx^2 & - acx & \\ & - cx^2 & - bcx & \end{array}$

And if this again be multiplied into $x + d = 0$, it will produce the biquadratick equation (III.) $\begin{array}{lllll} x^4 & + ax^3 & + abx^2 & - abcx & - abcd = 0. \\ & + bx^3 & - acx^2 & + abdx & \\ & - cx^3 & + adx^2 & - acdx & \\ & + dx^3 & - bcx^2 & - bcdx & \\ & & + bdx^2 & & \\ & & - cdx^2 & & \end{array}$

Thus, if $x + \sqrt{ab} = 0$ be multiplied into $x - \sqrt{ab} = 0$, the product will be $xx - ab = 0$; and if this be multiplied into $x + c = 0$, it will make $x^3 + cx^2 - abx - abc = 0$; and again, if this be multiplied into $x + c = 0$, it will make $\begin{array}{lllll} x^4 & + 2cx^3 & - abx^2 & - 2abcx & - abcc = 0. \\ & & + ccx^2 & & \end{array}$

If $x + \sqrt{-ab} = 0$ be multiplied into $x - \sqrt{-ab} = 0$, and then into $x + a = 0$, it will produce the cubick equation $x^3 + ax^2 + abx + aab = 0$.

146. Therefore, if we had the means of knowing what were the values of all, or of any of the unknown quantities of an equation, we might always divide it by fo many fimple equations as are thofe known values, by adding the negative values to the unknown quantity, and fubtracting the pofitive. Whence the firft equation before will be divifible by $x + a$, and by $x + b$. The fecond,

Equations might be refolved by divifion, if their roots were known.

ſecond, by $x + a$, $x + b$, and $x - c$. The third, by $x + a$, $x + b$, $x - c$, $x + d$. By this, compound equations will be reduced to ſo many ſimple equations as is the number of the roots, if all be known; or may be depreſſed by ſo many degrees as is the number of the known roots, if they be not all known. So that, for inſtance, an equation of the fifth degree may be reduced to one of the fourth, if one of it's roots be known; or to the third, if two roots be known; and ſo on.

Hence is known the nature or formation of the ſeveral co-efficients.

147. From the method by which equations are produced, (which equations are always underſtood to be reduced to nothing, and in which the greateſt term in reſpect of the unknown quantity, or in reſpect of that letter by which the terms are ordered, muſt be poſitive and free from a co-efficient,) it is eaſy to perceive that the co-efficient of the unknown letter, or that by which the equation is ordered, in the ſecond term is the ſum of all the roots of the equation affected with contrary ſigns; the co-efficient of the third term is the ſum of all the products of all the pairs of roots which can be formed; the co-efficient of the fourth term is the ſum of all the products of all the ternaries or threes; and ſo on to the laſt or conſtant term, which is the product of all the roots multiplied continually into one another.

When the ſecond term will be wanting.

148. Hence it may be inferred, that the ſum of the poſitive roots muſt neceſſarily be equal to the ſum of all the negative roots, in all ſuch equations in which the ſecond term is wanting: and that the ſum of the poſitive roots is greater than the ſum of the negative, when the ſecond term is affected with a negative ſign; and contrarily, when it is affected with a poſitive ſign.

How the abſence of a term is to be denoted.

149. When any term is wanting in an equation, it is uſual to ſupply it's place by an aſteriſm *; as in $x^4 * + aaxx - b^3x + a^4 = 0$, the ſecond term is wanting. In $x^4 - ax^3 * - b^3x + a^4$, the third term is wanting; and ſo in others.

Surd roots and imaginary roots always proceed by pairs.

150. If an equation have no term affected by an imaginary quantity, either it's roots ſhall be all real, or, if it have any imaginary roots, they ſhall always be even in number, and equal two by two; only with this difference, that one muſt be affirmative and the other negative. For, becauſe the ſecond term is the ſum of all the roots, if this be preſent in the equation, when the imaginary roots do not deſtroy one another, two by two, with contrary ſigns, ſome imaginary root muſt neceſſarily be in the co-efficient, which is contrary to the ſuppoſition. Now, if the ſecond term be wanting, it muſt needs follow, that the ſum of the poſitive roots is equal to the ſum of the negative, and conſequently the ſum of the poſitive imaginary roots muſt be equal to the ſum of the negative imaginary roots, otherwiſe they cannot deſtroy one another in the manner aforeſaid. Wherefore equations, whoſe degree is an odd number, will neceſſarily have one real root at leaſt; and thoſe of an even degree may have all

all their roots imaginary or impoffible. For the fame reafon, we may make like conclufions about furd roots. That is to fay, if the equation have no furd or irrational terms in it, it's roots will either be all rational, or the irrational roots will be in even numbers, and will be equal two by two, but with contrary figns.

151. There are equations which have all their roots pofitive, others have all their roots negative, others have both pofitive and negative. So fome have all their roots imaginary, others have all real, and laftly, others have both real and imaginary. Various rules are given by writers of Algebra, to determine in any given equation the number of pofitive and negative roots, alfo of real and imaginary roots. But, becaufe thefe rules and their demonftrations are very perplexed and prolix, and of but little ufe, I fhall here omit them, thinking it fufficient to take notice, firft, that if all the roots be negative, all the terms of the equation will be pofitive. For, in this cafe, fince all the terms of the fimple equations are pofitive, that is, of all the roots taken in the fecond fenfe, § 145, from whence the propofed equation is fuppofed to be produced, all the products will alfo be pofitive. Secondly, that if all the roots be pofitive, the terms of the equation will be pofitive and negative alternately. For the firft term will always be pofitive by fuppofition. The fecond term will be negative, becaufe it contains the fum of all the roots, which being pofitive, will be negative in the fimple equations. The third term, containing the ternaries or products of all the pairs, will be pofitive. And fo on. And therefore an equation compofed of pofitive and negative figns alternately, will have all it's roots pofitive.

Affections of the roots how diftinguifhed.

Whence, if the terms of an equation fhall not have all their figns pofitive, or fhall not have them pofitive and negative alternately, there will be both pofitive and negative roots. It fhall alfo be another fure proof, that the equation contains both pofitive and negative roots, if there be any term wanting; for no term can be abfent, but that the products of which it is formed muft deftroy one another by contrary figns; that is, there muft be both affirmative and negative roots. This obfervation will affift us in it's proper place, among the many divifors of the laft term of an equation, to felect thofe only by which the divifion is to be attempted. Becaufe, if the equation fhall have only pofitive roots, it would be of no ufe to try the divifion by pofitive divifors; and if it fhall have only negative roots, it would be needlefs to try by negative divifors. And the trials muft be made by each of them, when there are both pofitive and negative roots.

But all this muft be underftood of fuch equations in which all the roots are real; for where there are imaginary roots the rule does not obtain. For example, let the equation be $x^3 + bx^2 + aax + aab = 0$, in which all the terms are pofitive, and yet the roots are one pofitive and two negative, that is, $x = -b$, a real root, and $x = \pm\sqrt{-aa}$, two imaginary or impoffible roots, one pofitive, the other negative.

152. Equations

Affections of the roots of equations of the third or fourth degree.

152. Equations of the third and fourth degree, in which the fecond term is wanting, if the third term be affected with the pofitive fign, will certainly have imaginary roots; for, if all the roots were real, the third term could not but be affected with the negative fign; the reafon of which is, that in cubick equations, when the fecond term is wanting, the fum of the pofitive roots is equal to the fum of the negative, and therefore either one pofitive is equal to two negative, or two pofitive roots are equal to the one negative. Let the three roots, for inftance, be reprefented by a, b, and $-c$, or elfe by $-a$, $-b$, and $+c$; then the co-efficient of the third term will be $ab - ac - bc$. But, on fuppofition that the fecond term is wanting, it will be $a + b = c$. Therefore ac will be greater than ab, and confequently $ab - ac - bc$ will be a negative quantity.

Now, in equations of the fourth degree, there may be either three pofitive roots and one negative, as $+a$, $+b$, $+c$, and $-d$; or there may be three negatives and one affirmative, as $-a$, $-b$, $-c$, and $+d$; or there may be two negatives and two affirmative, as $-a$, $-b$, $+c$, and $+d$. In the firft and fecond cafe, the co-efficient of the third term will be $ab + ac + bc - ad - bd - cd$. But, by fuppofition, it ought to be $a + b + c = d$, fo that ad will be greater than ab, cd than ac, bd than bc; and therefore $ad + bd + cd$ will be greater than $ab + ac + bc$, and confequently the third term will be negative. In the third cafe, the co-efficient of the third term will be $ab - ac - bc - ad - bd + cd$, and it ought to be $a + b = c + d$. Here, if we make $m = a + b = c + d$, it will be $mm = \overline{a+b} \times \overline{c+d} = ac + ad + bc + bd$, and $mm = \overline{a+b}^2 = aa + 2ab + bb$, and alfo $mm = \overline{c+d}^2 = cc + 2cd + dd$. Therefore it is $ab = \frac{mm - aa - bb}{2}$, and $cd = \frac{mm - cc - dd}{2}$, and $ab + cd = mm - \frac{aa + bb + cc + dd}{2}$. Therefore mm is greater than $ab + cd$, and $ac + ad + bc + bd$ will be greater than $ab + cd$. Whence the co-efficient of the third term will be negative.

The pofitive roots may be made to become negative, and *vice verfâ.*

153. It is always in our power, in any equation, to make all the pofitive roots to become negative, and the negative to become pofitive. Nothing more is required to perform this, than to change all the figns which are in even places, that is, in the fecond, the fourth, the fixth, &c.; the reafon of which is, that the fecond term being the fum of all the roots, in this therefore are the negative with a pofitive fign, and the pofitive with a negative fign, as has been plainly feen at § 145. In forming equations, compounded of the products of fimple equations, by changing the figns they alfo will be changed. The other even terms in order are formed from the products of an odd number of roots, that is, the fourth from three, the fixth from five, &c. Wherefore, if they have the pofitive fign, they will be formed from the product of all the negative roots, or from an even number of pofitive roots, and an odd number of

of negative roots. And if they have a negative ſign, they will be formed from the product of all the poſitive roots, or an even number of negative roots, and an odd number of poſitive roots. Therefore, by changing the ſigns of all the even terms, the poſitive roots will become negative, and on the contrary.

As to the odd terms in order, they being formed of even products of roots, if they have the poſitive ſign, they will be formed either of an even number of negative roots alone, or of an even number of poſitive roots alone, or of an even number of poſitive, or an even number of negative together. Wherefore, by changing theſe reciprocally, the ſigns of the terms themſelves will not be changed. Now, if they have a negative ſign, they will be formed of the product of an odd number of poſitive roots, into an odd number of negative. Wherefore, by theſe alſo reciprocally, the ſign of the terms themſelves will not be changed, and therefore they muſt be left as they are.

The equation
$$\begin{array}{llll} x^3 & +ax^2 & +abx & -abc = 0 \\ & +bx^2 & -acx & \\ & -cx^2 & -bcx & \end{array}$$
has three roots. Two are negative, *viz.* $-a$, $-b$, or otherwiſe, $x + a = 0$, $x + b = 0$, and one is poſitive, *viz.* $+c$, or otherwiſe, $x - c = 0$. By changing the ſigns of thoſe terms which in the order of the equation are even, it will become
$$\begin{array}{llll} x^3 & -cx^2 & +abx & +abc = 0, \\ & -bx^2 & -acx & \\ & +cx^2 & -bcx & \end{array}$$
and the poſitive roots will be $x - a = 0$, $x - b = 0$, and the negative root will be $x + c = 0$. It is of no moment whether or no any term be wanting, becauſe in this caſe the aſteriſm ſupplies the vacancy, and then the ſame rule obtains. Thus, in the equation $x^3 * - 28x + 48 = 0$, the affirmative roots of which are $x - 2 = 0$, $x - 4 = 0$, and the negative root is $x + 6 = 0$. By changing the ſigns of the even terms in order, it will be $x^3 * - 28x - 48 = 0$, the negative roots of which are $x + 2 = 0$, $x + 4 = 0$, and the affirmative root is $x - 6 = 0$.

The roots of an equation may be increaſed or diminiſhed at pleaſure.

154. Any equation being given, by means of congruous ſubſtitutions it is eaſy to increaſe or diminiſh all it's roots, though yet unknown, by any given quantity; that is, it may be transformed into another equation, the roots of which ſhall be the ſame as thoſe of the propoſed equation, but increaſed or diminiſhed by ſome given quantity. Let the unknown quantity of the equation be put equal to a new unknown quantity, adding or ſubtracting the given quantity; adding, if we would have it increaſed, or ſubtracting, if we would have it diminiſhed. Then, in the propoſed equation, inſtead of the unknown quantity and it's powers, their values muſt be ſubſtituted, expreſſed by the other unknown quantity and the given conſtant quantity; from whence another equation will ariſe, the roots of which will be ſuch as are required. Let the equation be $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, the roots of which we

would have increafed by the number 3. Make $x + 3 = y$, whence $x = y - 3$, $x^2 = y^2 - 6y + 9$, $x^3 = y^3 - 9y^2 + 27y - 27$, and $x^4 = y^4 - 12y^3 + 54y^2 - 108y + 81$; therefore, in the propofed equation, fubftituting thefe values inftead of x and it's powers, it will be transformed into this other equation,

$$\left.\begin{array}{r} y^4 - 12y^3 + 54y^2 - 108y + 81 \\ + 4y^3 - 36y^2 + 108y - 108 \\ - 19y^2 + 114y - 171 \\ - 106y + 318 \\ - 120 \end{array}\right\} = 0; \text{ that is, } y^4 - 8y^3 - y^2 + 8y = 0;$$

and dividing by y, it is $y^3 - 8y^2 - y + 8 = 0$, in which it is plain, that the roots will be greater than the roots of the propofed equation by the number 3; becaufe it was made $y = x + 3$, and therefore the root y will be equal to every value of x increafed by 3. And here it may be obferved, that, in thus increafing the roots, the pofitive are increafed by fuch a quantity, but the negative are diminifhed by the fame quantity; for, by adding a pofitive to a negative, if the negative be greater than the pofitive, it will become lefs in it's kind than at firft; if they be equal, it becomes nothing, if it be lefs, it makes it pofitive. Whence, in the propofed equation $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, the roots of which (though they cannot be found by the methods hitherto taught,) are $+5, -2, -4, -3$, that is, $x - 5 = 0$, $x + 2 = 0$, $x + 4 = 0$, $x + 3 = 0$; one of which is affirmative, the other negative; as I defired to increafe them by the number 3, in the transformed equation $y^3 - 8y^2 - y + 8 = 0$, they ought to be $+8, +1, -1$, that is, $y - 8 = 0$, $y - 1 = 0$, $y + 1 = 0$, and are really fuch. And that which fhould correfpond to the fourth is $= 0$, becaufe $-3 + 3 = 0$. And, for this reafon, the reduced equation is only of three dimenfions, though the propofed equation is of four.

On the contrary, when the roots of an equation are to be diminifhed by a given quantity, for the fame reafon the negative roots are increafed in their kind by the fame quantity, but the pofitive may become nothing, if the given quantity be equal to them, and negative if greater. In the fame equation $x^4 + 4x^3 - 19x^2 - 106x - 120 = 0$, if I fhould defire to diminifh the roots by the number 3, I muft make $x - 3 = y$, and therefore $x = y + 3$, $x^2 = y^2 + 6y + 9$, $x^3 = y^3 + 9y^2 + 27y + 27$, $x^4 = y^4 + 12y^3 + 54y^2 + 108y + 81$. And therefore, making the fubftitutions, the equation will be

$$\left.\begin{array}{r} y^4 + 12y^3 + 54y^2 + 108y + 81 \\ + 4y^3 + 36y^2 + 108y + 108 \\ - 19y^2 - 114y - 171 \\ - 106y - 318 \\ - 120 \end{array}\right\} = 0. \text{ That is, } y^4 + 16y^3 + 71y^2$$

$- 4y - 420 = 0$. And, becaufe the roots of the propofed equation are $+5, -2, -3, -4$, that is, $x - 5 = 0$, $x + 2 = 0$, $x + 3 = 0$, $x + 4 = 0$; thofe of the transformed equation ought to be $+2, -5, -6, -7$,

-7, that is, $y-2=0$, $y+5=0$, $y+6=0$, $y+7=0$, as they really are.

Let the equation be $x^3+cx^2-bbx-bbc=0$, and we deſire to increaſe the roots by a given quantity a. Make $x+a=y$, and therefore $x=y-a$, $x^2=y^2-2ay+aa$, $x^3=y^3-3ay^2+3aay-a^3$. Wherefore, making the ſubſtitutions, the equation will be

$$\left.\begin{array}{r} y^3-3ay^2+3a^2y-a^3 \\ +cy^2-2acy+a^2c \\ -bby+abb \\ -bbc \end{array}\right\}=0.$$

The roots of this are greater than thoſe of the propoſed equation by the quantity a. And, in fact, the roots of the propoſed equation are $x-b=0$, $x+b=0$, $x+c=0$; but the roots of this are $y-b+a=0$, $y+b-a=0$, and $y+c-a=0$.

155. In like manner, if an equation be given, we may transform it into another, the roots of which are the ſame as thoſe of the propoſed equation, but multiplied or divided by a given quantity, ſuppoſe f; making a ſubſtitution of $fx=y$, (x being the unknown quantity of the given equation,) if we would have it multiplied; or of $\frac{x}{f}=y$, if we would have it divided. Thus, alſo, we may make $x=\frac{gy}{f}$, if we deſire that the roots of the tranſformed equation ſhould have to thoſe of the propoſed equation the ratio of f to g. And we may make $\sqrt{fx}=y$, if we would have them to be mean proportionals between the quantity f, and the roots of the propoſed equation. In like manner, we may make $x=\frac{1}{y}$, if we deſire they may be reciprocals, &c. Or the roots may be multiplied or divided at pleaſure.

156. The reaſon of theſe rules is evident. For, aſſuming the firſt caſe, or that of increaſing the roots, if we make the ſubſtitution of $x+a=y$, the values of y extracted from the transformed equation will be equal to $x+a$, or equal to the values of x in the propoſed equation increaſed by the quantity a. And by a like analogy in the other caſes. The reaſon of theſe operations.

157. Many are the uſes that may be made of theſe ſubſtitutions; one of which may be, that not having as yet a method of knowing what are the roots of the propoſed equation, by transforming it after ſome one of the aforementioned manners, we may diſcover the roots of the transformed equation; which being increaſed, diminiſhed, multiplied, divided, &c. by the conſtant quantity, according as the ſubſtitution is made, we ſhall alſo know the roots of the propoſed equation. And their uſes.

Equations may be freed from fractions or surds.

158. Another use may be, to free equations, whenever we please, from fractions, and very often from surds. As to fractions, we must make the unknown quantity of the equation equal to some new unknown quantity, divided by the least quantity that is divisible by every one of the denominators of the terms of the equation; which shall be the product of the same, in case that those denominators are prime to each other. Then making the substitutions, and reducing the terms to a common denominator, we shall have another equation which will be free from fractions, the roots of which will be those of the proposed equation, multiplied into the quantity by which the new unknown quantity was at first divided. Let the equation be $y^3 + \frac{1}{6}ay^2 - \frac{1}{3}aby + aab = 0$; if we make $y = \frac{1}{6}z$, $y^2 = \frac{1}{36}z^2$, $y^3 = \frac{1}{216}z^3$, then, by substitution, the equation will become $\frac{z^3}{216} + \frac{az^2}{6\times 36} - \frac{abz}{3\times 6} + aab = 0$. And, reducing to a common denominator, it will be $z^3 + az^2 - 12abz + 216aab = 0$. The roots of this equation divided by 6 will be the roots of the equation proposed.

Let the equation be $x^3 - \frac{ax^2}{b} + \frac{aax}{c} + \frac{a^3}{d} = 0$. Make $x = \frac{z}{bcd}$, and, substituting in the equation, it will be transformed into this, $\frac{z^3}{b^3c^3d^3} - \frac{az^2}{b^3c^2d^2} + \frac{aaz}{bc^2d} + \frac{a^3}{d} = 0$. Then reducing to a common denominator, it will be $z^3 - acdz^2 + a^2b^2cd^2z + a^3b^3c^3d^2 = 0$. Wherefore, if the value of z were known, the value of x would be known also. In like manner, to free equations from surds, we may often proceed thus. Make the unknown quantity equal to a new unknown quantity divided by the radical, and substitute this in the equation. Let the equation be $x^3 - \sqrt{3}\times x^2 + \frac{26}{27}x - \frac{8}{27\sqrt{3}} = 0$. Make $x = \frac{z}{\sqrt{3}}$, and therefore $x^2 = \frac{z^2}{3}$, $x^3 = \frac{z^3}{3\sqrt{3}}$; and, making the substitutions, it will be $\frac{z^3}{3\sqrt{3}} - \frac{z^2\sqrt{3}}{3} + \frac{26z}{27\sqrt{3}} - \frac{8}{27\sqrt{3}} = 0$. Now, multiplying by $3\sqrt{3}$, it will be $z^3 - 3z^2 + \frac{26}{9}z - \frac{8}{9} = 0$. Lastly, by freeing this from fractions after the foregoing manner, that is, making $z = \frac{1}{9}y$, or rather, $z = \frac{1}{3}y$, which in this case will be more compendious, the equation will be $y^3 - 9y^2 + 26y - 24 = 0$. And because, by the first substitution, it is $x = \frac{z}{\sqrt{3}}$, and, by the second, $z = \frac{1}{3}y$, it will be $x = \frac{y}{3\sqrt{3}}$; or the value of x will be equal to the value of y divided by $3\sqrt{3}$.

Let the equation be $x^4 - x^3\sqrt[3]{nn} + px^2\sqrt[3]{n} - qx + \frac{r}{\sqrt[3]{n}} = 0$. Make $x = \frac{y}{\sqrt[3]{n}}$, and therefore $xx = \frac{yy}{\sqrt[3]{nn}}$, $x^3 = \frac{y^3}{n}$, $x^4 = \frac{y^4}{n\sqrt[3]{n}}$; and making the substitutions,

ſtitutions, it will be $\frac{y^4}{n\sqrt[3]{n}} - \frac{y^3\sqrt[3]{nn}}{n} + \frac{pyy\sqrt[3]{n}}{\sqrt[3]{nn}} - \frac{qy}{\sqrt[3]{n}} + \frac{r}{\sqrt[3]{n}} = 0$. And multiplying by $n\sqrt[3]{n}$, it will be $y^4 - ny^3 + npy^2 - nqy + rn = 0$. If we would obſerve the law of homogeneity, equations may be delivered from radicals: but then fractions would thence ariſe, which muſt be reduced as above.

159. Becauſe, by taking away radicals by means of the foregoing ſubſtitutions, nothing elſe is done than multiplying the roots of the equation by that radical, it is eaſy to perceive, that if the radical be quadratick, for example $\sqrt{n}$, it is neceſſary, in order to expunge it out of the equation, that the ſecond term of the equation propoſed ſhall contain $\sqrt{n}$. For, as that term is the aggregate of all the roots of the equation, it muſt be multiplied by $\sqrt{n}$. It will be neceſſary that the third term ſhould not contain $\sqrt{n}$, becauſe, as it is the aggregate of the pairs of the roots of the equation, it muſt be multiplied by the ſquare of $\sqrt{n}$. Thus it will be neceſſary that the fourth ſhould contain $\sqrt{n}$, becauſe, as it is the aggregate of all the ternaries, or products of three roots, it muſt conſequently be multiplied by $n\sqrt{n}$. It will alſo be neceſſary that the fifth term ſhould not contain the radical; and ſo on alternately. For the ſame reaſon, if the radical to be taken away were $\sqrt[3]{n}$, it will be neceſſary, that in the ſecond term of the propoſed equation there ſhould be found $\sqrt[3]{nn}$, in the third $\sqrt[3]{n}$, in the fourth none at all, in the fifth $\sqrt[3]{nn}$, in the ſixth $\sqrt[3]{n}$, in the ſeventh none at all, &c. And the like is to be concluded of other radicals. Conditions for expunging radicals.

160. By means of theſe ſubſtitutions we may alſo take away the ſecond term from any equation. And that will be done by putting the unknown quantity equal to a new unknown quantity, adding or ſubtracting the co-efficient of the ſecond term divided by the index of the degree of the equation given: that is, adding, if the ſecond term have the negative ſign, and ſubtracting, if that ſign be poſitive. Let the equation be $x^2 + ax - bb = 0$; put $x = z - \frac{1}{2}a$, and, by ſubſtitution, it will become Thus the ſecond term of an equation may be taken away.

$$\left.\begin{array}{r} z^2 - az + \frac{1}{4}aa \\ + az - \frac{1}{2}aa \\ - bb \end{array}\right\} = 0. \text{ That is,}$$

$zz * - \frac{1}{4}aa - bb = 0$, or $zz = \frac{1}{4}aa + bb$. Hence it may be ſeen, how all affected quadratick equations may be reſolved more expeditiouſly in this manner, than by that before taught at § 74. Then, only ſubtracting $\frac{1}{2}a$ from the value of z ſo found, we ſhall have the value of x.

Let the equation be $x^3 + bx^2 - abx - a^3 = 0$. Make $x = z - \frac{1}{3}b$, and, by ſubſtitution, it will be

$$\left.\begin{array}{r} z^3 * - \frac{1}{3}bbz + \frac{2}{27}b^3 \\ - abz + \frac{1}{3}abb \\ - a^3 \end{array}\right\} = 0.$$

Whence, taking $\frac{1}{3}b$ from the value of z, we ſhall have the value of x.

Let

Let the equation be $\begin{matrix} x^4 - 2ax^3 + 2aaxx - 2a^3x + a^4 = 0. \\ \quad\quad\quad\quad\quad - ccxx \end{matrix}$ Make

$x = z + \frac{2a}{4}$, or $x = z + \frac{1}{2}a$. Then, by ſubſtitution, it will be

$$\left.\begin{matrix} z^4 \; * \; + \frac{1}{2}aaz^2 - a^3z + \frac{5}{16}a^4 \\ - ccz^2 - accz - \frac{1}{4}aacc \end{matrix}\right\} = 0.$$ Then add $\frac{1}{2}a$ to the value of z, and we ſhall have the value of x.

Or the third term may be taken away.

161. And thus we may take away the third term from any equation, proceeding after the following manner.

Let the equation be $x^4 - 3ax^3 + 3aax^2 - 5a^3x - 2a^4 = 0$. Make $x = y - h$, where h is a general quantity, to be determined as occaſion may require. Now, making the ſubſtitutions, it will be

$$\left.\begin{matrix} y^4 - 4hy^3 + 6hhy^2 - 4h^3y + h^4 \\ - 3ay^3 + 9ahy^2 - 9ah^2y + 3ah^3 \\ + 3aay^2 - 6a^2hy + 3a^2h^2 \\ - 5a^3y + 5a^3h \\ - 2a^4 \end{matrix}\right\} = 0.$$ Now, in this equation, that

the third term may be nothing, it is neceſſary that $6hhy^2 + 9ahy^2 + 3aay^2 = 0$, that is, $h^2 + \frac{3}{2}ah + \frac{1}{2}aa = 0$; and therefore $h = -\frac{3}{4}a \pm \frac{1}{4}a$. Hence we are informed, that the ſubſtitution to be made inſtead of $y - h$, is either $y + \frac{1}{2}a$, or $y + a$; for, indeed, either the one or the other takes away the third term, making the equation $y^4 - ay^3 * - \frac{15}{4}a^3y - \frac{65}{16}a^4 = 0$, or, ſecondly, $y^4 + ay^3 * - 4a^3y - 6a^4 = 0$.

By this artifice it may be known, that, to take away the ſecond term, we muſt make ſuch ſubſtitutions as have been ſhown at § 160.

Or the laſt but one, if the ſecond be wanting.

162. Now if an equation, in which the ſecond term is wanting, is to be transformed into another, in which the laſt term but one ſhall be abſent, it will be ſufficient to ſubſtitute any given quantity, divided by a new unknown quantity, inſtead of the unknown quantity of the equation. Let the equation be $x^4 * + aax^2 - a^3x + a^4 = 0$, and make $x = \frac{aa}{y}$. By ſubſtitution, it will be $\frac{a^8}{y^4} * + \frac{a^6}{y^2} - \frac{a^5}{y} + a^4 = 0$. And reducing this to a common denominator, and dividing by a^4, it will be $y^4 - ay^3 + aay^2 * + a^4 = 0$. In the ſubſtitution of $x = \frac{aa}{y}$, inſtead of the given quantity a, if we had taken any other, we ſhould have arrived at the ſame concluſion, but then the transformed equation would have involved fractions.

163. If,

163. If, in the propofed equation, not the fecond term, but the third, or fourth, &c. fhould be wanting, by the fame method we might make that term to vanifh, which is equally diftant from the laft term, as the abfent term is diftant from the firft.

Or any other on a certain condition.

164. And on the contrary, if one or more terms be wanting in an equation, we may always make it compleat, by taking a new unknown quantity, *plus* or *minus* fome known quantity, and making it equal to the unknown quantity of the equation, and then the transformed equation will have all it's terms compleat. Moreover, if we would have the transformed equation to be of a fuperior degree, let every term of the propofed equation be multiplied by fuch a power of the unknown quantity, by which we would have the degree to be increafed, and then the fubftitution may be made. Thus, the equation $x^4 - a^4 = 0$ being given, if we would have it to be changed into another which is compleat, and of the fixth degree, let it be made $x^6 - a^4x^2 = 0$, and then making the fubftitution of $x = z \pm a$, (where by a is underftood any known quantity,) and we fhall have the equation required. The calculation, for brevity, is omitted.

Or an equation may be completed or raifed higher.

165. When equations are reduced to fuch a form, as that they have their greateft term pofitive, and without a co-efficient except unity; that they may be free from fractions and furds, and compared to nothing, in order to judge whether the problem propofed be of that degree as is fhown by the equation, we muft examine whether it have a divifor of one, of two, or more dimenfions, by which, being divided, it may be reduced to a lower degree. For the problem is properly of that degree to which the equation may be reduced, and not of the degree of the firft equation. If a cubick equation have a divifor of one dimenfion, by being divided by that, it may be reduced to two dimenfions; and the two roots of this, (which will be had by the rules delivered at § 73, 74,) and the divifor, will be the three roots of the propofed equation. Whence the problem, which has brought us to fuch an equation, is not really cubical but plane, and may be conftructed by ruler and compaffes only, that is, by right lines and circles. If an equation of the fourth degree have two divifors of one dimenfion, and if it be divided by them, it will be reduced to two dimenfions; the roots of which, together with the two divifors, will be the four roots of the propofed equation, and therefore the problem will be plane. After the fame manner, if it have one divifor of two dimenfions, another of two dimenfions will be the quotient, the roots of which, together with the roots of the divifor, will be the four roots of the propofed equation, and therefore the problem is plane. Further, if it have one divifor only of one dimenfion, the reduced equation will be of three, and the problem will be folid indeed, but of the third degree only, and not of the fourth as it feemed to be. If an equation of the fifth degree fhall have three divifors of one dimenfion, or one of one and one of two, (which is the fame cafe as if it had two of two dimenfions, becaufe

Problems are often reduced to a lower degree by divifion.

 then

then it will necessarily have one of one dimension also,) it will be reduced to two dimensions, and therefore the five roots may be had, and the problem will be plane. If it have only one of one dimension, it will be reduced to the fourth degree, and the problem will be of the same degree. If it shall have two of one dimension, or one of two, it will be reduced to the third degree, and the problem will be of the same. And the like of others. The manner of finding divisors of one dimension has been taught before, at § 56.

And sometimes by compound divisors.

166. But besides, as equations may have divisors of two or more dimensions, whether rational or irrational, we may operate with them in like manner, and, by a like way of reasoning, we must attempt the division of the proposed equation; but, first, having tried the division, by divisors of one dimension, which ought always to precede, whatever the equation may be.

How equations of the fourth degree may often be reduced by two quadratick divisors.

167. The manner of finding these divisors for equations of the fourth degree may be this following. For those of the third degree are either irreducible, or may be reduced by a rational and linear divisor, being free from radicals, as is here supposed.

Admitting, then, that the equation of the fourth degree is not reducible by a divisor of one dimension only; let the second term be taken away (§ 160.), and, for example's sake, let there be produced this equation, $x^4 * - 17aax^2 - 20a^3x - 6a^4 = 0$. Let this be supposed equal to the product of these two equations of the second degree, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, in which y, z, u, are general quantities, which are to be determined afterwards as occasion may require; and z and u may have any sign. The product of these two equations will be $\begin{matrix} x^4 * & + zx^2 & - yzx & + uz = 0. \\ & - yyx^2 & + yux & \\ & + ux^2 & & \end{matrix}$ Now let this equation be compared, term by term, with the equation proposed, and, from the comparison of the third terms in each, we shall have $z = -17a^2 + y^2 - u$. From the comparison of the fourth terms, it will be $u = \frac{-20a^3}{y} + z$; and, instead of z, putting it's value already found, that we may have u expressed by y only, and known quantities, it will be $u = -\frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$. And, putting this value of u in the equation $z = -17aa + yy - u$, we shall have $z = -\frac{17}{2}aa + \frac{1}{2}yy + \frac{20a^3}{2y}$. From the comparison of the last terms, we shall have $uz = -6a^4$, and, instead of z and u, putting their values expressed by y only, and known quantities, it will be $\frac{289}{4}a^4 - \frac{34}{4}aayy - \frac{400a^6}{4yy} + \frac{1}{4}y^4 = -6a^4$; or, reducing to a common denominator,

$y^6 -$

$\begin{array}{l} y^6 - 34a^2y^4 + 289a^4y^2 - 400a^6 = 0. \\ \qquad\qquad\quad + \ \ 24a^4y^2 \end{array}$ This transformed equation may be confidered as of the third degree, becaufe it involves neither y^5, nor y^3, nor y. In this equation, let the divifors of the laft term be found, and, becaufe it may be confidered as of the third degree, though it is really of the fixth, try if it be divifible by $yy \pm$ thefe divifors, among which we are to choofe thofe only of two dimenfions, as is plain. And it will be found divifible by $yy - 16aa = 0$, whence it will be $yy = 16aa$, and $y = \pm 4a$. This value of y being fubftituted in the equations $u = -\frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$, and $z = \frac{20a^3}{2y} - \frac{17}{2}aa + \frac{1}{2}yy$, we fhall have $u = -3aa$, $z = 2aa$. Therefore the two fubfidiary equations $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, muft be $x^2 + 4ax + 2aa = 0$, and $x^2 + 4ax - 3aa = 0$, into which the equation $x^4 * - 17a^2x^2 - 20a^3x - 6a^4 = 0$ may be refolved, by dividing by either of them.

But the roots of thefe are (§ 74.) $x = -2a \pm \sqrt{2aa}$ for the firft, and $x = 2a \pm \sqrt{7aa}$ for the fecond; which are therefore the roots of the given equation, being all four real, one pofitive and three negative.

If the transformed equation fhould not have any divifor, it would be to no purpofe to feek another in this cafe; for neither would the propofed equation admit of any.

Although in the value of y we have $y = \pm 4a$, yet I have made ufe of the pofitive fign only, becaufe it is indifferent whether we take the pofitive or the negative root, the refult being the fame in both cafes. For, if we put $y = -4a$, it will be $u = 2aa$, $z = -3aa$, and the two equations will be the fame as before, that is, $x^2 - 4ax - 3aa = 0$, and $x^2 + 4ax + 2aa = 0$.

Let the equation be $\begin{array}{l} x^4 - 2ax^3 + 2a^2x^2 - 2a^3x + a^4 = 0. \\ \qquad\qquad\qquad\ \ - ccx^2 \end{array}$ Taking away the fecond term, by the fubftitution of $x = z + \frac{1}{2}a$, it will be changed into

$$\begin{array}{l} z^4 * + \frac{1}{2}a^2z^2 - a^3z + \frac{5}{16}a^4 = 0. \\ \qquad\ \ - ccz^2 - ac^2z - \frac{1}{4}a^2c^2 \end{array}$$

Wherefore, making a comparifon of this with the equation

$$\begin{array}{l} z^4 * + uz^2 - pyz + pu = 0, \\ \qquad\ \ - y^2z^2 + uyz \\ \qquad\ \ + pz^2 \end{array}$$

which is the product of the two equations $z^2 + yz + p = 0$, and $z^2 - yz + u = 0$; from the comparifon of the third terms, as ufual, we fhall have $p = yy - u + \frac{1}{2}aa - cc$. From the comparifon of the fourth terms, we fhall have $u = p \frac{-a^3 + acc}{y}$; or, inftead of p, putting it's value, $u = \frac{1}{2}yy + \frac{1}{4}aa - \frac{1}{2}cc - \frac{a^3 + acc}{2y}$; and therefore $p = \frac{1}{2}yy + \frac{1}{4}aa - \frac{1}{2}cc + \frac{a^3 + acc}{y}$. Laftly, from the comparifon of the

 laft

laſt terms, we ſhall have $pu = \frac{5}{16}a^4 - \frac{1}{4}aacc$; or, ſubſtituting the values of p and u, it will be

$$\left.\begin{array}{llll} y^6 + aay^4 - a^4y^2 - a^6 \\ \quad - 2ccy^4 + c^4y^2 - 2a^4c^2 \\ \quad\quad - a^2c^4 \end{array}\right\} = 0.$$

Now the diviſors of the laſt term, meaning thoſe of two dimenſions, are aa and $aa + cc$, and the diviſion will ſucceed by $yy - aa - cc = 0$. Therefore it will be $yy = aa + cc$, and $y = \pm\sqrt{aa + cc}$. Whence $u = \frac{3}{4}aa - \frac{a^3 + ac^2}{2\sqrt{aa + cc}}$, $p = \frac{3}{4}aa + \frac{a^3 + acc}{2\sqrt{aa + cc}}$; and the two equations $z^2 + yz + p = 0$, and $z^2 - yz + u = 0$, will be $zz + z\sqrt{aa + cc} + \frac{3}{4}aa + \frac{a^3 + ac^2}{2\sqrt{aa + cc}} = 0$, and $zz - z\sqrt{aa + cc} + \frac{3}{4}aa - \frac{a^3 + ac^2}{2\sqrt{aa + cc}} = 0$, or $zz + z\sqrt{aa + cc} + \frac{3}{4}aa + \frac{1}{2}a\sqrt{aa + cc} = 0$, and $zz - z\sqrt{aa + cc} + \frac{3}{4}aa - \frac{1}{2}a\sqrt{aa + cc} = 0$.

Theſe two equations, being reſolved, will give us four values of z; $z = -\frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa + cc}}$ from the firſt equation, and $z = \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa + cc}}$ from the ſecond equation. And, becauſe theſe are the diviſors of the equation

$$\left.\begin{array}{l} z^4 * + \frac{1}{2}a^2z^2 - a^3z + \frac{5}{16}a^4 \\ \quad\quad - ccz^2 - ac^2z - \frac{1}{4}a^2c^2 \end{array}\right\} = 0,$$

the ſame roots ſhall alſo belong to this equation. And now, making the ſubſtitution of $x = \frac{1}{2}a + z$, we ſhall have $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa + cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa + cc}}$, which are the four roots or values of the propoſed equation.

This reduction may be performed by a general canon.

168. But a general formula or canon may be formed, as well for the tranſformed equation as for the two ſubſidiary equations, which are aſſumed in order to obtain the diviſors; to which formulas any equation whatever of the fourth degree, in which the ſecond term is wanting, or taken away, may be univerſally applied. Therefore let there be this general equation $x^4 * \pm px^2 \pm qx \pm r = 0$; and taking the two ſubſidiary equations, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$, and finding their product,

$$\begin{array}{llll} x^4 * + zx^2 & - yzx & + uz = 0, \\ \quad\;\; - yyx^2 & + uyx & \\ \quad\;\; + ux^2 & & \end{array}$$

let it be compared, term by term, with the equation propoſed. Now, from the compariſon of the third terms, we ſhall have $z = \pm p + yy - u$. From the compariſon of the fourth, $u = z \pm \frac{q}{y}$; and, inſtead of z, it's value being

5 ſubſtituted,

ſubſtituted, it will be $u = \pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{2y}$; where it is $+ p$, if in the propoſed equation the third term be poſitive, and $-p$, if negative. And thus alſo for q, if the fourth term be poſitive, and $-q$, if negative. And this being put inſtead of u in the firſt compariſon, we ſhall have $z = \pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y}$; that is, $+ p$, if the third term of the propoſed equation be poſitive, and $-p$, if negative. And, on the contrary, $-q$, if the fourth term be poſitive, and $+q$, if negative. From the compariſon of the laſt terms, we ſhall find $zu = \pm r$, that is, $\pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{y}$ into $\pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y} = \pm r$; and, by actual multiplication, and reducing to a common denominator, it will be $y^6 \pm 2py^4 \begin{smallmatrix} + p^2y^2 \\ \mp 4ry^2 \end{smallmatrix} - qq = 0$, the transformed equation, which may be called cubick; in which it will be $+ 2p$, if the third term of the propoſed equation be poſitive, and $-p$, if negative. And it will be $-4r$, if the laſt term of the propoſed equation be poſitive, but $+ 4r$, if negative. In the two ſubſidiary equations, inſtead of z and u, if we put their values found before, they will be $xx + yx \pm \frac{1}{2}p + \frac{1}{2}yy \mp \frac{q}{2y} = 0$, and $xx - yx \pm \frac{1}{2}p + \frac{1}{2}yy \pm \frac{q}{2y} = 0$. Wherefore, if the transformed equation ſhall be diviſible by $yy \pm$ a diviſor of two dimenſions of the laſt term, we ſhould have the value of y, which, being ſubſtituted in theſe two laſt equations, will ſupply us with diviſors of the propoſed equation. And if the transformed equation be not diviſible, neither will the propoſed be ſo.

Let the given equation be $x^4 * - 4a^2x^2 - 8a^3x + 35a^4 = 0$. Comparing this with the canonical equation, it will be $p = 4aa$, $q = 8a^3$, $r = 35a^4$; and therefore the transformed equation will be $y^6 - 8a^2y^4 \begin{smallmatrix} + 16a^4y^2 \\ - 140a^4y^2 \end{smallmatrix} - 64a^6 = 0$, that is, $y^6 - 8a^2y^4 - 124a^4y^2 - 64a^6 = 0$. And the two ſubſidiary equations will be $x^2 + yx - 2aa + \frac{1}{2}yy + \frac{4a^3}{y} = 0$, and $x^2 - yx - 2aa + \frac{1}{2}yy - \frac{4a^3}{y} = 0$. Now, finding the diviſors of the laſt term, becauſe the tranſformed equation is diviſible by $yy - 16aa = 0$, we ſhall have $yy = 16aa$, and thence $y = 4a$; which values, being ſubſtituted in the two ſubſidiary equations, will give $x^2 + 4ax + 7aa = 0$, and $x^2 - 4ax + 5aa = 0$, which are the diviſors of the given equation; the four roots of which are $x = -2a \pm \sqrt{-3aa}$, and $x = 2a \pm \sqrt{-aa}$, all imaginary.

 169. Some-

Sometimes a biquadratick may be reduced to a quadratick.

169. Sometimes it will be ſufficient only to take away the ſecond term of the equation, in order to reduce it to a plane, and ſo to ſpare any further operation. Thus, for example, it will be in the equation

$$\begin{matrix} x^4 + 2cx^3 - 2acx^2 - 2aacx - aacc = 0 \\ \quad\quad\quad\quad + ccx^2 \end{matrix}$$

which, becauſe it is not reducible by any diviſor of the laſt term, if we take away the ſecond term by making $x = y - \frac{1}{2}c$, will be changed into this,

$$\left.\begin{matrix} y^4 * - 2a^2y^2 * + \frac{1}{16}c^4 \\ \quad - \frac{1}{2}c^2y^2 \quad - \frac{1}{2}a^2c^2 \end{matrix}\right\} = 0;$$

an affected quadratick equation, the roots of which, being diminiſhed by the quantity $\frac{1}{2}c$, by the ſubſtitution of $x = y - \frac{1}{2}c$, will be the ſame as of the propoſed equation.

Sometimes higher equations may be reſolved by this method.

170. This method requires, that the ſecond term ſhould be taken away from the equation, nor can it be extended beyond equations of the fourth degree. But here is another method, which does not oblige us to take away any term, and which may be applied, not only to equations of the fourth degree, but to thoſe of the fifth or ſixth, and ſometimes to thoſe of ſtill higher degrees. Let the given equation be

$$\begin{matrix} x^4 + ax^3 + a^2x^2 - a^2bx - a^3b = 0; \\ \quad\quad\quad - abx^2 \end{matrix}$$

and let there be taken two ſubſidiary equations of the ſecond degree, $x^2 + yx + u = 0$, and $x^2 + sx + z = 0$, in which the indeterminates, y, u, s, z, are to be determined afterwards as occaſion may require. The product of theſe will be

$$\begin{matrix} x^4 + yx^3 + ux^2 + usx + zu = 0, \\ \quad + sx^3 + syx^2 + zyx \\ \quad\quad + zx^2 \end{matrix}$$

which is to be compared, term by term, with the propoſed equation. From the compariſon of the ſecond terms, we ſhall have $s = a - y$; from the compariſon of the laſt terms, $z = -\frac{a^3b}{u}$; and from the compariſon of the fourth, $yz + su = -a^2b$: and, inſtead of s and z, ſubſtituting their values, that we may have an equation expreſſed by y and u only, and known quantities, it will be $y = \frac{auu + aabu}{uu + a^3b}$. And, becauſe we have found $zu = -a^3b$, from the compariſon of the laſt terms, therefore u ought to be a diviſor of $-a^3b$. Whence let the diviſors of $-a^3b$ of two dimenſions be found, (for thoſe of one or three dimenſions will not ſerve to be ſubſidiary equations of the ſecond degree,) which are $\pm ab$, $\pm aa$. Let us begin by taking, inſtead of u, one of theſe diviſors, for example ab, which, being ſubſtituted in the equation $y = \frac{auu + aabu}{uu + a^3b}$, gives $y = \frac{2ab}{a+b}$. Therefore, putting theſe values of y and u in the ſubſidiary equation

$x^2 +$

$x^2 + yx + u = 0$, it will be $x^2 + \frac{2abx}{a+b} + ab = 0$. And by this, if we try the divifion of the propofed equation, and if it fhould fucceed, then $x^2 + \frac{2abx}{a+b} + ab = 0$ would be one divifor, and the quotient would be the other. But, becaufe the divifion does not fucceed, we muft make another trial, by taking, inftead of u, the other divifor $-ab$ of the laft term, and it will be $y = 0$; and therefore the fubfidiary equation $x^2 + yx + u = 0$ will become $x^2 - ab = 0$, by which the propofed equation being divided, it will fucceed by giving the quotient $x^2 + ax + aa = 0$. So that the divifors of the propofed equation are $xx - ab = 0$, and $xx + ax + aa = 0$.

Alfo, inftead of u, taking the divifor aa of the laft term, by which we fhall find $y = a$, and the fubfidiary equation will be $xx + ax + aa = 0$. The divifion by this will fucceed, giving the quotient $xx - ab = 0$; that is, the very fame divifors as before.

When all the divifors of the laft term are put in the place of u, and if the operation will not fucceed by any, it may then be concluded, that the equation propofed cannot be depreffed, at leaft by this method, and that the Problem remains of fuch a degree as the equation indicates.

But, without trying the divifion, taking, inftead of u, every one of the divifors of two dimenfions of the laft term, and the correfpondent values of y, s, z, we may fubftitute them in their ftead in the fubfidiary formulas, $xx + yx + u = 0$, and $xx + sx + z = 0$. And if the product of thefe will give the propofed equation, they will be the divifors required. Thus, taking, inftead of u, the divifor $-ab$, we fhall have $y = 0$, and therefore $s = a$, $z = aa$, and the two fubfidiary equations will be $xx - ab = 0$, and $xx + ax + aa = 0$, the product of which will give us the propofed equation.

Let the equation $\begin{matrix} x^4 - 2ax^3 + 2aax^2 - 2a^3x + a^4 \\ \qquad\qquad\qquad - ccx^2 \end{matrix}$ be given, and let it be compared with the product of the two fubfidiary equations

$$\begin{matrix} x^4 + yx^3 + ux^2 + sux + zu = 0. \\ + sx^3 + syx^2 + zyx \\ + zx^2 \end{matrix}$$

From the comparifon of the fecond terms, we fhall have $s = -2a - y$. From the comparifon of the laft terms, $z = \frac{a^4}{u}$. We muft take the comparifon of the third, and not of the fourth, in order to have the value of y expreffed by c, (which letter muft neceffarily be in the divifor, which could not be had from the comparifon of the fourth,) it will be then $u + sy + z = 2aa - cc$. And fubftituting the values of s and z, it will be $yy + 2ay = \frac{a^4}{u} - 2aa + cc + u$; in which fubftituting, inftead of u, one of the divifors $\pm aa$ of the laft term, fuppofe $+ aa$, and refolving the

the equation, it will be $y = -a \pm \sqrt{aa+cc}$. And putting, in the equation $xx + yx + u = 0$, the values of u and y, (taking for the sign of the radical quantity either *plus* or *minus* as we please, because it will be all the same at last,) we shall have $xx - ax + x\sqrt{aa+cc} + aa = 0$, by which the division of the proposed equation will succeed, making the quotient $xx - ax - x\sqrt{aa+cc} + aa = 0$; and consequently the four roots of the proposed equation will be $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa+cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc - \frac{1}{2}a\sqrt{aa+cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa+cc} \pm \sqrt{-\frac{1}{2}aa + \frac{1}{4}cc + \frac{1}{2}a\sqrt{aa+cc}}$.

Let the equation be $x^4 + 2bx^3 + bbx^2 - a^3b = 0$, and let it be compared with the product of the two subsidiary formulas as before. From the comparison of the second terms, we shall have $s = 2b - y$. From the comparison of the last, $z = -\frac{a^3b}{u}$. From the comparison of the fourth, $zy + su = 0$; and substituting the values of s and z, it will be $-\frac{a^3by}{u} + 2bu - uy = 0$, that is, $y = \frac{2buu}{a^3b + uu}$. But, taking every one of the rational divisors, $\pm aa$, $\pm ab$, of the last term, and substituting in the place of u, and doing the rest as usual, the operation does not succeed. Therefore we must try by means of the irrational divisors $\pm a\sqrt{ab}$ of the last term; and therefore putting, instead of u, the irrational divisor $a\sqrt{ab}$, it will be $y = b$. Wherefore the subsidiary equation $xx + yx + u = 0$ will become $xx + bx + a\sqrt{ab} = 0$, by which the proposed equation being divided, there will arise the quotient $xx + bx - a\sqrt{ab} = 0$.

Exemplified in equations of the fifth degree.

171. As to equations of the fifth degree, it is manifest, that if they be not divisible by a linear divisor, as already supposed, they cannot be divided but by one of the second degree, and one of the third. Therefore for such equations must be taken two subsidiary equations, one of the third degree, and another of the second, and the product of these must be compared, term with term, with the proposed equation, in like manner as before.

Let this therefore be the given equation, $x^5 - 4ax^4 + 6aax^3 - 8a^3x^2 + 5a^4x - a^5 = 0$. And let us take the two subsidiary equations $xx + yx + u = 0$, and $x^3 + tx^2 + sx + z = 0$. Of these the product is

$$\begin{array}{llllll} x^5 & + yx^4 & + ux^3 & + tux^2 & + sux & + zu = 0 \\ & + tx^4 & + tyx^3 & + syx^2 & + zyx & \\ & & + sx^3 & + zx^2 & & \end{array}$$

which is to be compared with the proposed equation. Now, from the comparison of the second terms, we shall have $t = -4a - y$. From the comparison of the last terms, $z = -\frac{a^5}{u}$. From the comparison of the fifth, $s = \frac{5a^4 - zy}{u}$; or the value of z being

being ſubſtituted, $s = \frac{5a^4}{u} + \frac{a^5y}{uu}$. From the compariſon of the third, we ſhall have finally $u + ty + s = 6aa$; and, inſtead of t and s, putting their values, in order to obtain an equation expreſſed by y, u, and known quantities only, it will be $yy + 4ay - \frac{a^5y}{uu} = -6aa + u + \frac{5a^4}{u}$. And becauſe, from the compariſon of the laſt terms, we have $z = -\frac{a^5}{u}$, therefore u will be a diviſor of $-a^5$. So that, finding all the diviſors of two dimenſions of $-a^5$, they are to be ſubſtituted, one by one, in the foregoing equation, in order to have the value of y, which is then to be put inſtead of y in the ſubſidiary equation $xx + yx + u = 0$, as alſo the value of u. And if the diviſion of the given equation ſhall ſucceed by this, we ſhall have our deſire. Now the diviſors of two dimenſions of the laſt term are $\pm aa$. Let us take $+ aa$, which being ſubſtituted inſtead of u in the equation aforegoing, we ſhall have $yy + 3ay = 0$, that is, $y = 0$, and $y = -3a$. If, in the ſubſidiary equation $xx + yx + u = 0$, we put the diviſor $+ aa$ inſtead of u, and beſides, if we put 0, which is one of the values found, inſtead of y, it will become $xx + aa = 0$, by which the diviſion of the propoſed equation does not ſucceed. Therefore, inſtead of y, we may put it's other value $-3a$, and we ſhall have $xx - 3ax + aa = 0$, by which the diviſion ſucceeds, and gives $x^3 - ax^2 + 2aax - a^3 = 0$ in the quotient. If the operation had not ſucceeded by means of the diviſor $+ aa$, we muſt have tried the diviſor $- aa$; and if neither by this we had obtained our deſire, we muſt have concluded the equation to be irreducible, at leaſt by this method.

Let the equation be $x^5 + ax^4 * \begin{matrix} + a^3x^2 \\ - a^2bx^2 \end{matrix} - aabbx - a^4b = 0$, which is to be compared, term by term, with the product of the two uſual ſubſidiary equations; and from the compariſon of the ſecond terms, we ſhall find $t = a - y$. From the compariſon of the laſt terms, $z = -\frac{a^4b}{u}$. From the compariſon of the fifth, $su + zy = -aabb$. Now, inſtead of z, ſubſtituting it's value, it will be $s = -\frac{aabb}{u} + \frac{a^4by}{uu}$. From the compariſon of the third, we ſhall have $u + ty + s = 0$, in which, inſtead of t and s, putting their values, it will be $yy - ay - \frac{a^4by}{uu} = \frac{uu - aabb}{u}$. The diviſors of two dimenſions of a^4b are $\pm aa$, and $\pm ab$. We muſt try the operation by means of the diviſor $- ab$. And therefore, inſtead of u, putting it's value $- ab$ in the laſt equation, it will be $yy - ay - \frac{aay}{b} = 0$. Thence $y = 0$, and $y = \frac{ab + aa}{b}$. In the ſubſidiary equation $xx + yx + u = 0$, inſtead of y let it's value $\frac{aa + ab}{b}$ be be fubftituted, and $-ab$ inftead of u, and it will be $xx + ax + \frac{aa}{b}x - ab = 0$, by which the divifion does not fucceed. Therefore take the other value of y, which is 0, and the fubfidiary equation will be $xx - ab = 0$, by which the divifion of the propofed equation will fucceed, and the quotient will be $x^3 + ax^2 + abx + a^3 = 0$.

We were at liberty to make a comparifon between the fourth terms; but, for greater fimplicity, I made choice of the third terms.

Equations of the fixth degree refolved.

172. Equations of the fixth degree, fuppofed not to be reducible by any linear divifor, cannot be otherwife reducible but either by three divifors of two dimenfions, or by one of two dimenfions and one of four, or by two of three dimenfions. But it will be fufficient to examine the two cafes, in which they are reducible by two of three dimenfions, or by one of two and one of four. For as much as reducing them by one of two, the reduced equation will be of four dimenfions, which may afterwards be reduced by two divifors of two dimenfions, if the propofed equation be reducible by three of two dimenfions.

Let the equation given be this: $x^6 - 13ax^5 + 45aax^4 - 71a^3x^3 + 57a^4x^2 - 16a^5x + 2a^6 = 0$, which is required to be reduced by one of two dimenfions, and one of four. Let therefore be taken the two fubfidiary equations $xx + yx + u = 0$, and $x^4 + px^3 + tx^2 + sx + z = 0$, of which the product is

$$\begin{array}{llllll} x^6 + px^5 & + tx^4 & + sx^3 & + zx^2 & + zyx & + zu = 0. \\ \quad\; + yx^5 & + pyx^4 & + tyx^3 & + syx^2 & + sux & \\ & + ux^4 & + pux^3 & + tux^2 & & \end{array}$$

Now, from the comparifon of the fecond terms, we fhall have $p = -13a - y$. From the comparifon of the laft terms, $z = \frac{2a^6}{u}$. From the comparifon of the third, $t + py + u = 45aa$; and by fubftituting the value of p, it will be $t = 45aa + 13ay + yy - u$. From the comparifon of the fixth, $zy + su = -16a^5$; and putting here the value of z, it will be $s = -\frac{2a^6y}{uu} - \frac{16a^5}{u}$. From the comparifon of the fifth, $z + sy + tu = 57a^4$; and fubftituting the values of z, s, and t, that we may have an equation expreffed by u and y alone, and by the known quantities of the propofed equation, it will be at laft $\frac{2a^6}{u} - \frac{2a^6y^2}{uu} - \frac{16a^5y}{u} + 45a^2u + 13ayu + uy^2 - u^2 = 57a^4$. That is, $yy + \frac{13au^3y - 16a^5uy + 2a^6u - 57a^4u^2 + 45a^2u^3 - u^4}{u^3 - 2a^6} = 0$. And, becaufe the divifors of two dimenfions of the laft term $2a^6$ are $\pm aa$, and $\pm 2aa$, we muft make a trial, by putting in this laft equation, inftead of u, the divifor $+aa$, and it will be $yy + 3ay + 11aa = 0$, which, being refolved, will give

$y =$

$y = \frac{-3a \pm \sqrt{-35aa}}{2}$. Whence the fubfidiary formula $xx + yx + u = 0$ will be $xx - \frac{3a + \sqrt{-35aa}}{2}x + aa = 0$. But by this, even though we fhould take the alternative of the figns of the radical, the propofed equation is not divifible; nor will it fucceed if we fhould take the divifor $-aa$; therefore we muft take $+2aa$, and we fhall have $yy + 12ay + 20aa = 0$, that is, $y = -6a \pm 4a$, or $y = -10a$, and $y = -2a$. Take $y = -10a$, and fubftitute it in the fubfidiary formula $xx + yx + u = 0$, and $-10a$ inftead of y, and $+2aa$ inftead of u, and it will be $xx - 10ax + 2aa = 0$. But by this the divifion of the propofed equation does not fucceed. Therefore take the other value of y, or $-2a$, and the formula will be $xx - 2ax + 2aa = 0$, by which the divifion fucceeds, making in the quotient $x^4 - 11ax^3 + 21aax^2 - 7a^3x + a^4 = 0$.

Here it may not be amifs to obferve, that, inftead of the comparifon of the fifth terms, if I had made a comparifon of the fourth, I fhould have fallen upon the cubick equation $2y^3 + 26ay^2 + 81aay + 74a^3 = 0$. But the comparifon of the fifth terms has brought me to a quadratick equation only. Hence it may be feen, that the choice of the comparifon of fome terms rather than of others may be of good advantage. Yet, however, this cubick equation might have been of ufe; for, finding it's roots, which are $y + 2a = 0$, and $y + \frac{11a}{2} \pm \sqrt{47aa} = 0$, one of thefe, $y = -2a$, would have given me the fame equation $xx - 2ax + 2aa = 0$, by which the propofed equation may be divided.

Let $x^6 + 3ax^5 + 4aax^4 + 6a^3x^3 + 6a^4x^2 + 3a^5x + 2a^6 = 0$, be the given equation of the fixth degree, not reducible by a divifor of two dimenfions. Let us therefore attempt the reduction by two equations of three dimenfions, and let us take thefe two fubfidiary equations, $x^3 + yx^2 + px + u = 0$, and $x^3 + tx^2 + sx + z = 0$, of which this is the product;

$$\begin{array}{llllllll} x^6 & + yx^5 & + px^4 & + ux^3 & + tux^2 & + sux & + zu & = 0. \\ & + tx^5 & + tyx^4 & + ptx^3 & + psx^2 & + pzx & & \\ & & + sx^4 & + syx^3 & + zyx^2 & & & \\ & & & + zx^3 & & & & \end{array}$$

Now, from the comparifon of the fecond terms, we fhall have $t = 3a - y$. From the comparifon of the laft terms, $z = \frac{2a^6}{u}$. From the comparifon of the fixth, $su + pz = 3a^5$; and fubftituting the value of z, it will be $s = \frac{3a^5}{u} - \frac{2a^6p}{uu}$. From the comparifon of the third, $p + ty + s = 4aa$; and fubftituting the values of t and s, it will be $p = \frac{4aauu - 3a^5u + uuyy - 3auuy}{uu - 2a^6}$. From the

comparifon

comparifon of the fourth, $u + pt + sy + z = 6a^3$; and, inftead of t, s, z, fubftituting their values, that we may have another value of p, expreffed by u, y, and the known quantities of the equation, it will be $p = \frac{6a^3uu - u^3 - 3a^5uy - 2a^6u}{3auu - uuy - 2a^6y}$. Now, between thefe two values of p let an equation be made, to obtain the value of y expreffed by u only, and the given quantities of the equation. This will be $\frac{4aauu - 3a^5u - 3auuy + uuyy}{uu - 2a^6} = \frac{6a^3uu - u^3 - 3a^5uy - 2a^6u}{3auu - uuy - 2a^6y}$. Then, reducing to a common denominator, and ordering the equation by y, it will be

$$\frac{\begin{array}{l} y^3 - 6a^7uy^2 + 8a^3uy - 6a^3u^3 \\ \quad - 6au^3y^2 - 6a^5uuy + 9a^6u^2 \\ \quad\quad + 13a^2u^3y - 12a^9u \\ \quad\quad\quad + 4a^{12} \\ \quad\quad\quad - u^4 \end{array}}{u^3 + 2a^6u} = 0.$$

And, becaufe it is $uz = 2a^6$, we fhall have u a divifor of $2a^6$. But the divifors of three dimenfions of $2a^6$ are $\pm a^3$, and $\pm 2a^3$. Whence, taking one of thefe inftead of u, fuppofe $+ a^3$, and fubftituting it in the laft equation, we fhall have $y^3 - 4ay^2 + 5aay - 2a^3 = 0$. From hence muft be extracted the values of y, one of which is $y = 2a$, which, being fubftituted in one of the values of p inftead of y, and putting inftead of u the divifor a^3, it will be $p = aa$. Wherefore, fubftituting thefe values of y, p, and u, in the fubfidiary formula $x^3 + yx^2 + px + u = 0$, it will become $x^3 + 2ax^2 + aax + a^3 = 0$, by which the propofed equation being divided, will give the quotient $x^3 + ax^2 + aax + 2a^3 = 0$. If the divifion had not fucceeded by taking $y = 2a$, I muft have taken $y = a$. And if I had not attained my purpofe by this, I muft have made trials with every one of the other divifors, repeating the fame operations. And if it had fucceeded by none of thefe, the propofed equation could not have been depreffed, at leaft not by this method, but would have remained of the fixth degree.

Let $x^6 + ax^5 + aax^4 + 3a^3x^3 + a^4x^2 + a^5x + 2a^6 = 0$ be the equation, which is to be compared with the product of the two fubfidiary equations, as in the foregoing example. From the comparifon of the fecond terms, we fhall have $t = a - y$. From the comparifon of the laft terms, $z = \frac{2a^6}{u}$. From the comparifon of the fixth, $su + pz = a^5$; and, inftead of z, putting it's value, it will be $s = \frac{a^5}{u} - \frac{2a^6p}{uu}$. From the comparifon of the third, $p + ty + s = aa$; and putting the values of t and s, it is $p = \frac{aauu - auuy + uuyy - a^5u}{uu - 2a^6}$. From the comparifon of the fourth, $u + pt + sy + z = 3a^3$; and fubftituting the

the values of z, s, t, in order to have another value of p, expreſſed only by u, y, and known quantities, it will be $p = \frac{3a^3uu - a^3uy - 2a^6u - u^3}{auu - uuy - 2a^6y}$. Make an equation between theſe two values of p, that we may have the value of y given by u only and known quantities; and when all the neceſſary operations are performed, it will be

$$\frac{\begin{array}{r} y^3 - 2au^3y^2 + 2aau^3y + 2a^3u^3 \\ - 2a^7uy^2 - 2a^5uuy + a^6u^2 \\ + 2a^8uy - 6a^9u \\ - u^4 \\ + 4a^{12} \end{array}}{u^3 + 2a^6u} = 0.$$

The diviſors of three dimenſions of $2a^6$ are $\pm a^3$, and $\pm 2a^3$. Inſtead of u, take the diviſor $+ a^3$, to be ſubſtituted in this laſt equation, which then will be reduced to $y^3 - \frac{4}{3}ay^2 + \frac{2}{3}aay = 0$. And dividing by y, it will be $y = 0$, and $y^2 - \frac{4}{3}ay + \frac{2}{3}aa = 0$; that is, $y = \frac{2a \pm \sqrt{-2aa}}{3}$. Of theſe three values of y take the firſt, or $y = 0$, and ſubſtitute this inſtead of y in one of the two values of p, and a^3 inſtead of u, and it will be $p = 0$. Then the ſubſidiary equation $x^3 + yx^2 + px + u = 0$ will become $x^3 + a^3 = 0$; by which the propoſed equation being divided, will give $x^3 + ax^2 + a^2x + 2a^3 = 0$ for the quotient.

In ſuch equations as theſe, if it were known at firſt that they are diviſible by a diviſor, in which ſome term is wanting, much labour might be ſpared, by taking one of the two ſubſidiary equations without that term. But, becauſe this is not known, we may firſt try the operation with one of thoſe ſubſidiary equations, which wants either one or more terms. Nevertheleſs, becauſe the labour would be loſt, if the propoſed equation be not reducible by this means, and there will be need at laſt, notwithſtanding this compendium, to have recourſe to compleat ſubſidiary equations, it will be better at once to uſe this general method, becauſe it gives the diviſors in both caſes.

Without repeating the operations at every example, I might have formed a general canon, to which every particular equation might be referred, after the ſame manner as that at § 168. But beſides, as this may create ſome confuſion, it ſeems to me that actual operations made on purpoſe afford more light, and have a better effect; therefore I have rather choſe to confine myſelf to them.

173. After the ſame analogy, we may apply this method to equations of a ſuperior order, but the calculation increaſes beyond meaſure. For, if we are to Applied to higher equations.

to reduce an equation of the eighth degree, for example, by means of two equations of the fourth, in which no term is wanting, each of the two subsidiary equations must have four indeterminates, or general co-efficients. Whence, if we consider one of these equations, such as this, $x^4 + yx^3 + px^2 + qx + u = 0$, and take for u one of the divisors of the last term of the proposed equation, there will remain three indeterminates, y, p, q, to be determined by the usual comparisons, in which there will occur solid equations, whose roots are to be extracted, in order that the operation may proceed.

PROBLEM I.

Applied to the solution of an arithmetical problem.

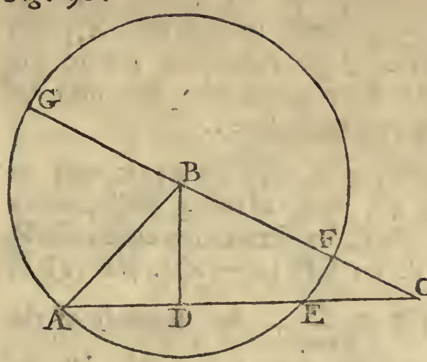
174. To find four numbers, which exceed one another by unity, and their product is 100.

Make the first number equal to x, the second will be $x + 1$, the third $x + 2$, and the fourth $x + 3$. Therefore their product will be $x^4 + 6x^3 + 11x^2 + 6x = 100$, or $x^4 + 6x^3 + 11x^2 + 6x - 100 = 0$. Now, because this equation is not divisible by any divisor of the last term, we must make the second term to vanish by the substitution of $x = z - \frac{3}{2}$, and there will arise the equation $z^4 * - \frac{5}{2}z^2 * - \frac{1591}{16} = 0$, which is an affected quadratick, the roots of which are $zz = \frac{5}{4} \pm \sqrt{101}$, and therefore $z = \pm\sqrt{\frac{5}{4} \pm \sqrt{101}}$. Whence we shall have $x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} \pm \sqrt{101}}$. Therefore, of the four values of x, two are real, that is, $x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{101}}$, and the other two are imaginary. If we take one of the real roots, $-\frac{3}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$, for the first number of the four that are required, then $-\frac{1}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the second, $\frac{1}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the third, and $\frac{3}{2} + \sqrt{\frac{5}{4} + \sqrt{101}}$ will be the fourth: the product of which numbers will be found to be 100. If we should take the other real value of x, that is, $-\frac{3}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$, for the first number, then $-\frac{1}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the second, $\frac{1}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the third, and $\frac{3}{2} - \sqrt{\frac{5}{4} + \sqrt{101}}$ would be the fourth; the product of which numbers would also be found to be 100.

PRO-

PROBLEM II.

Fig. 91:



175. In the right-angled triangle ABC the leſſer ſide AB is given, and, letting fall the perpendicular BD to the baſe AC, the difference of the ſegments AD, DC, of the ſame baſe AC is given alſo; it is required to find FC, the difference of the ſides AB, BC. A geometrical problem.

With centre B, diſtance BA, let the circle AEFG be deſcribed, and make AB $= a$, CE $= b$, the given difference of the ſegments AD, DC; and make FC, the difference required, $= x$. It will be GC $= 2a + x$, and, by the property of the circle, GC × CF = AC × CE, that is, $2ax + xx = \text{AC} \times b$, and therefore $\text{AC} = \frac{2ax + xx}{b}$. But, becauſe the angle ABC is a right angle, we ſhall have the equation $\frac{4aaxx + 4ax^3 + x^4}{bb} = 2aa + 2ax + xx$, or, by reduction, $x^4 + 4ax^3 + \begin{matrix} 4aaxx \\ -bbxx \end{matrix} - 2abbx - 2aabb = 0$. Now this is not diviſible by any diviſor of the laſt term, and therefore we muſt take away the ſecond term by the ſubſtitution of $x = z - a$; whence we ſhall have the affected quadratick

$$\left.\begin{matrix} z^4 - 2aazz + a^4 \\ \quad - bbzz - aabb \end{matrix}\right\} = 0,$$

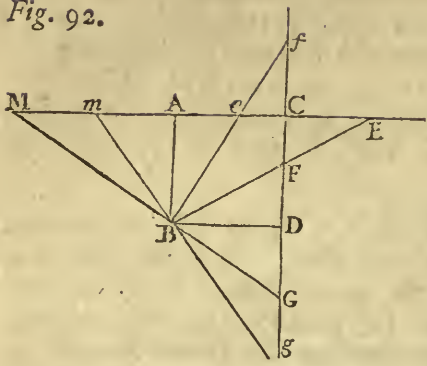
the roots of which are $zz = \frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}$, and thence $z = \pm\sqrt{\frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}}$. So that $x = -a \pm \sqrt{\frac{2aa + bb \pm \sqrt{8aabb + b^4}}{2}}$, which are the four roots, and all real, when a is greater than b. The root $x = -a + \sqrt{aa + \frac{1}{2}bb + b\sqrt{2aa + \frac{1}{4}bb}}$, which is poſitive, is adapted to the propoſed Problem. The negative root $x = -a + \sqrt{aa + \frac{1}{2}bb - b\sqrt{2aa + \frac{1}{4}bb}}$ is adapted to the caſe, when the ſide BC is leſs than the ſide AB; the other two roots ſerve for the angle ABG.

PRO-

PROBLEM III.

Another geometrical problem.

Fig. 92.



176. Having given the ſquare AD, in the ſide AC produced, to find ſuch a point E, that, drawing the right line EB to the angle B, the intercepted line EF may be equal to a given right line c.

Make BD $= a$, DF $= x$; it will be CF $= a - x$. And, drawing BFE, make FE $= c$. Now, by ſimilar triangles, ECF, BDF, it will be CF $(a - x)$. FE (c) :: FD (x) . FB $= \frac{cx}{a-x}$. But, becauſe of the right angle at D, it will be alſo FB $= \sqrt{aa + xx}$; whence we ſhall have the equation $\sqrt{aa + xx} = \frac{cx}{a-x}$; and, by ſquaring, $\frac{ccxx}{aa - 2ax + xx} = aa + xx$; and, reducing to a common denominator, and then ordering the equation, it is $x^4 - 2ax^3 + 2aaxx - 2a^3x + a^4 = 0$, $ - ccxx$

the roots of which may be ſeen, at § 167, 170, to be $x = \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and $x = \frac{1}{2}a + \frac{1}{2}\sqrt{aa + cc} \pm \sqrt{\frac{1}{4}cc - \frac{1}{2}aa + \frac{1}{2}a\sqrt{aa + cc}}$.

The two laſt roots are always real and poſitive; the latter of which, being leſs than a, determines the point F, through which the line BE being drawn, EF will be equal to the given line c, and reſolves the Problem propoſed. The other of the two, which is greater than a, determines the point f, to which drawing the right line Bf, gives us alſo ef equal to the given line, and ſerves as if the Problem had been propoſed by the angle ACf.

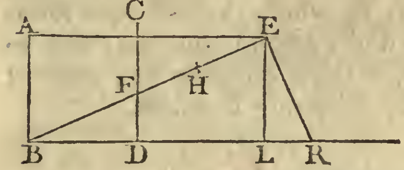
The two firſt roots are imaginary whenever cc is leſs than $8aa$, and the Problem will be impoſſible. But, ſuppoſing cc not leſs than $8aa$, the two roots are real and negative. Taking, therefore, DG $= \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} + \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and D$g$ $= \frac{1}{2}a - \frac{1}{2}\sqrt{aa + cc} - \sqrt{\frac{1}{4}cc - \frac{1}{2}aa - \frac{1}{2}a\sqrt{aa + cc}}$, and through the point B drawing the right lines GM, gm, they will both be equal to the given line c, and would ſerve were the Problem propoſed for the angle ACD.

177. Very

177. Very often, when the Problem is not really solid, but plane, it may appear as an equation of three dimensions, by making use of some certain line for the unknown quantity; but, by using some other line for the unknown quantity, it may put on the form of an equation of two dimensions only. I shall take an example of this in the foregoing Problem, in which, making $DF = x$, there has been found an equation of the fourth degree, by which means we have been obliged to take the trouble of reducing it. But, supposing E to be the point required, draw ER perpendicular to BE, which may meet BD produced in R, and EL perpendicular to BR. Then make $DR = x$, and, as before, $BD = a$, $FE = c$, and $BF = y$, another unknown quantity to be eliminated afterwards; it will be $BR = a + x$, $BE = c + y$. Now, because of similar triangles, BDF, ELR, it will be $ER = y$, because of $EL = CD = BD$. And, because of similar triangles, BRE, ERL, it will be BR . BE :: ER . EL. Therefore it will be $a + x . c + y :: y . a$; whence $cy + yy = aa + ax$. But, because of the right angle BER, the square of BR is equal to the sum of the squares of BE and ER; that is, $aa + 2ax + xx = 2yy + 2cy + cc$. Therefore, instead of $cy + yy$, putting it's value $aa + ax$, the equation will be $aa + 2ax + xx = 2aa + 2ax + cc$, that is, $x = \pm\sqrt{aa + cc}$.

How higher equations may sometimes be avoided.

Fig. 93.

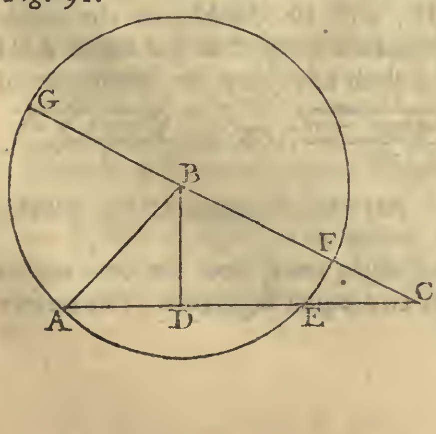


Again, after another manner. Bisect FE in H, and making $CD = a$, let the given line be $2c$, to which FE ought to be equal. And making $BH = x$, it will be $BF = x - c$, and $BE = x + c$. But $BEq - ABq = AEq$; therefore it will be $AE = \sqrt{xx + 2cx + cc - aa}$. Now, because of the similar triangles, BDF, BEA, it will be $BF (x - c) . BD (a) :: BE (x + c) . AE = \sqrt{xx + 2cx + cc - aa}$. Whence $ax + ac = \overline{x - c} \times \sqrt{xx + 2cx + cc - aa}$; and, by squaring and ordering the equation, it will be finally

$$\left.\begin{matrix} x^4 - 2aaxx - 2aacc \\ - 2ccxx + c^4 \end{matrix}\right\} = 0,$$

an affected quadratick equation, of which the four roots are $x = \pm\sqrt{aa + cc \pm a\sqrt{aa + 4cc}}$.

Fig. 91.



After the same manner in Prob. II. § 175, if, instead of making $FC = x$, I had denominated $BC = x$; by pursuing the same argumentation, I should have found the equation

$$\left.\begin{matrix} x^4 - 2aaxx + a^4 \\ - bbxx - aabb \end{matrix}\right\} = 0,$$

an affected quadratick, of which the roots are $x = \pm\sqrt{aa + \frac{1}{2}bb \pm b\sqrt{2aa + \frac{1}{4}bb}}$, which agree with those before found.

Again,

Again, in a ſimpler manner. Make AE $=x$, and, arguing as before, we ſhould have the equation $xx+bx=2aa$, and therefore $x=-\frac{1}{2}b\pm\sqrt{2aa+\frac{1}{4}bb}$. And, becauſe we ſhould find the expreſſion $-a+\sqrt{bb+2bx+xx-aa}$ for FC, inſtead of x putting the value now found, we ſhould have what is required, or the ſame value for FC as before.

Or otherwiſe, by finding two values of the ſame quantity.

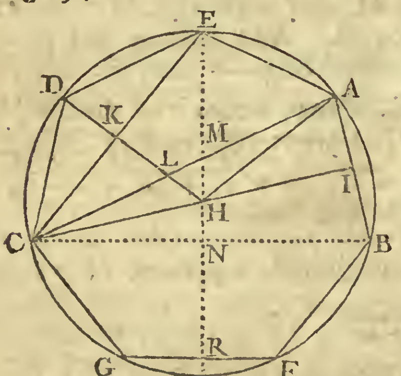
178. Another artifice may be tried for ſuch like Problems, when they bring us to a ſolid equation, and yet are not ſuch in their own nature. This is, retaining the ſame line for the unknown quantity, by which the firſt equation is found; then, by means of another property, to find a ſecond equation, and to equal one to the other. From their compariſon, a third equation will ariſe of an inferior degree. See an example of this in the following Problem.

PROBLEM.

This exemplified in a geometrical problem.

179. In a given circle, to inſcribe a regular heptagon.

Fig. 94.



Let the given circle be ABFGCDE, with centre H, radius HA $=r$, and let the ſide of the heptagon be AB $=$ BF $=$ FG, &c. $=x$. Let AB be biſected in I; it will be AI $=\frac{1}{2}x=$ IB. And drawing IC, which will neceſſarily paſs through the centre H, it will be HI $=\sqrt{rr-\frac{1}{4}xx}$, CI $=r+\sqrt{rr-\frac{1}{4}xx}$, CB $=\sqrt{2rr+2r\sqrt{rr-\frac{1}{4}xx}}$. Let there be drawn CE and HD; the triangles CDK, HIA, will be ſimilar, becauſe of the two right angles CKD, HIA, and of the angles DCK, AHI, the firſt of which, becauſe it inſiſts on the arch DE, will be double to the angle ACI, which inſiſts on the half of DE, and therefore is equal to the angle AHI the double of the ſame angle ACI. Hence we ſhall have, by the ſimilitude of theſe triangles, CK $=\frac{x}{r}\sqrt{rr-\frac{1}{4}xx}=\frac{\sqrt{4rrxx-x^4}}{2r}$, CE $=\frac{\sqrt{4rrxx-x^4}}{r}$, and HK $=\sqrt{rr-\frac{4rrxx-x^4}{4rr}}=\frac{2rr-xx}{2r}$. But the triangles CEN, CHK, are alſo ſimilar, the two angles at K, N, being right ones, and the two angles KCH, CEN, are equal, becauſe they inſiſt on two equal ſegments. Therefore it it will be $CN = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{2r^3}$, and $CB = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{r^3}$, and thence the equation $\sqrt{rr + r\sqrt{4rr - xx}} = \frac{\overline{2rr - xx} \times \sqrt{4rrxx - x^4}}{r^3}$. Therefore, ſquaring, it will be $2rr + r\sqrt{4rr - xx} = \frac{4r^4 - 4r^2x^2 + x^4}{r^6} \times \overline{4rrxx - x^4}$; and ſquaring again, and ordering, we ſhall have $x^{14} - 16r^2x^{12} + 104r^4x^{10} - 352r^6x^8 + 660r^8x^6 - 672r^{10}x^4 + 336r^{12}x^2 - 63r^{14} = 0$. But this equation is diviſible by $x^2 - 3r^2 = 0$. When the diviſion is performed, we ſhall have $x^{12} - 13r^2x^{10} + 65r^4x^8 - 157r^6x^6 + 189r^8x^4 - 105r^{10}x^2 + 21r^{12} = 0$, which is not diviſible by any diviſor of two dimenſions; wherefore the Problem ſeems to be of twelve dimenſions. Therefore I reſolve this Problem in another manner, retaining the ſame unknown quantity $x = AB = BF =$ &c. Becauſe, in the triangles HCD, CDL, the angle CDH is common, and the angle at the circumference DCL, which inſiſts upon the arch CD, the half of DA, theſe triangles will be ſimilar, and therefore we ſhall have $DL = \frac{xx}{r}$, and $LH = r - \frac{xx}{r}$. But the angle DLC = DCH = EDH; wherefore the angle HLM, which is equal to the angle at the vertex DLC, will be equal to the angle EDH; whence the two right lines LM, DE, will be parallel, and the triangles HLM, HDE, will be ſimilar, and therefore it will be $LM = \frac{rrx - x^3}{rr}$. But $CL = CD = x$, (the triangle LDC being ſimilar to the iſoſceles triangle HDC,) and CL = MA, becauſe the angles HLC, HMA, are equal, and therefore the triangles HLC, HMA, are equal and ſimilar. Therefore $CA = 2x + \frac{rrx - x^3}{rr}$. And, becauſe CA = CB, the equation will be $\frac{3rrx - x^3}{rr} = \sqrt{2rr + r\sqrt{4rr - xx}}$. And, by ſquaring, $9r^4x^2 - 6r^2x^4 + x^6 = 2r^6 + r^5\sqrt{4rr - xx}$. And, by ſquaring again, and ordering the terms, the equation will be $x^{10} - 12rrx^8 + 54r^4x^6 - 112r^6x^4 + 105r^8x^2 - 35r^{10} = 0$.

And thus I am arrived at another equation, which, becauſe it is of an inferior degree to the firſt, muſt be multiplied by ſuch a power of the unknown quantity, as is neceſſary to bring it to the ſame degree, ſo that it may be compared with that. Therefore, multiplying it by xx, it will be $x^{12} - 12r^2x^{10} + 54r^4x^8 - 112r^6x^6 + 105r^8x^4 - 35r^{10}x^2 = x^{12} - 13r^2x^{10} + 65r^4x^8 - 175r^6x^6 + 189r^8x^4 - 105r^{10}x^2 + 21r^{12}$. Now, ſubtracting the firſt from the ſecond, it will be $x^{10} - 11r^2x^8 + 45r^4x^6 - 84r^6x^4 + 70r^8x^2 - 21r^{10} = 0$. Which, becauſe it is of the tenth degree, being compared with the ſecond equation found above, and ſubtracted from the ſame, will be $x^8 - 9r^2x^6 + 28r^4x^4 - 35r^6x^2 + 14r^8 = 0$, which may be divided by $xx - 2rr$; and making this diviſion, we ſhall have at laſt this equation of the ſixth degree, $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$.

Z I have

I have proceeded in this way, to ſhow the uſe of the method. For otherwiſe, I might have gone more directly to the ſame equation, by comparing together the two values of the ſquares of CA, found in the two different ſolutions of the Problem; that is, $\frac{16r^6x^2 - 20r^4x^4 + 8r^2x^6 - x^8}{r^6}$ of the firſt, and $\frac{9r^4x^2 - 6r^2x^4 + x^6}{r^4}$ of the ſecond. For, making an equation between theſe two values, and taking away the terms that deſtroy one another, it will be $x^8 - 7r^2x^6 + 14r^4x^4 - 5r^6x^2 = 0$. And, dividing by x^2, it will be $x^6 - 7r^2x^4 + 14r^4x^2 - 5r^6 = 0$, as before. We might alſo, after a more compendious manner, have divided the equation firſt found by $x^6 - 6r^2x^4 + 9r^4x^2 - 5r^6 = 0$, and the ſecond by $x^4 - 5r^2x^2 + 5r^4 = 0$; and in each caſe we ſhould find the equation $x^6 - 7r^2x^4 + 14r^4x^2 - 7r^6 = 0$.

Yet the propoſed Problem is not of the ſixth degree, though it may ſeem to be ſuch, notwithſtanding all this care we take to depreſs it. To make this appear, we will retain the ſame compoſition of the figure, and make HI $= x$. Then it will be AI $= \sqrt{rr - xx} =$ IB, CI $= r + x$, CB $= \sqrt{rr + 2rx + xx + rr - xx} = \sqrt{2rr + 2rx}$. Then, by purſuing the ſame way of arguing as before, we ſhall have CK $= \frac{2x\sqrt{rr - xx}}{r}$, HK $= \sqrt{\frac{r^4 - 4r^2x^2 + 4x^4}{rr}} = \frac{rr - 2xx}{r}$, CE $=$ 2CK $= \frac{4x}{r}\sqrt{rr - xx}$, CN $= \frac{4rrx - 8x^3}{r^3} \times \sqrt{rr - xx}$, CB $=$ 2CN $= \frac{8rrx - 16x^3}{r^3}\sqrt{rr - xx}$. But we have before found CB $= \sqrt{2rr + 2rx}$. Therefore the equation will be $\sqrt{2rr + 2rx} = \frac{8rrx - 16x^3}{r^3} \times \sqrt{rr - xx}$.

Now I ſhall ſeek another equation after a different manner, but ſhall retain the ſame unknown quantity HI $= x$. By the ſame reaſoning as above, it will be DL $= \frac{4rr - 4xx}{r}$, LH $= r - \frac{4rr - 4xx}{r} = \frac{4xx - 3rr}{r}$, LM $= 2\sqrt{\frac{rr - xx}{rr}} \times \overline{4xx - 3rr}$, CA $= 4\sqrt{rr - xx} + 2\sqrt{\frac{rr - xx}{rr}} \times \overline{4xx - 3rr}$; that is, by reduction, CA $= \frac{8xx - 2rr}{rr}\sqrt{rr - xx} =$ CB. Whence the equation $\sqrt{2rr + 2rx} = \frac{8xx - 2rr}{rr}\sqrt{rr - xx}$; and laſtly, by equalling the *homogeneum comparationis* of each equation, it will be $\frac{8rrx - 16x^3}{r^3}\sqrt{rr - xx} = \frac{8xx - 2rr}{rr}\sqrt{rr - xx}$, which, being reduced, will be $8x^3 + 4rxx - 4rrx - r^3 = 0$, an equation only of the third degree.

180. When

180. When the methods above-defcribed have been put in practice, if the equations cannot be depreffed, but ftill remain above the fecond degree, we may proceed two ways in the folution of Problems, which arife to three or more dimenfions. The way of leaft general ufe belongs only to equations of the third or fourth degree, and confifts in refolving them by unravelling the analytical values of the unknown quantity, which therefore will prefent themfelves under the form of cubick roots; which method is called *Cardan's* Rule. The The fecond way is more general, and of much more extenfive ufe, and confifts in finding the geometrical values of the unknown quantity, by means of the interfections of certain curve-lines, which are purpofely introduced into the equation; that fo the propofed Problem may be conftructed. Solid Problems are folved by Cardan's rules, or by conftruction.

181. But, to begin with the analytical folution. I fuppofe the equations to be without the fecond terms, becaufe they may always be reduced to fuch, if they are not fuch already. And all equations of the third degree, wanting the fecond terms, are comprehended under thefe four canonical formulæ. How by the four cafes of Cardan's rule.

I. $x^3 - px - q = 0.$ II. $x^3 + px - q = 0.$
III. $x^3 - px + q = 0.$ IV. $x^3 + px + q = 0.$

Make $x = y + z$, then $px = py + pz$, and $x^3 = y^3 + 3y^2z + 3yz^2 + z^3$. And, fubftituting thefe values in the firft equation, it will be $y^3 + 3y^2z + 3yz^2 + z^3 - py - pz - q = 0$. Of this we may form two equations, which are $3y^2z + 3yz^2 = py + pz$, and $y^3 + z^3 = q$. Dividing the firft by $y + z$, we fhall have $3yz = p$, or $y = \frac{p}{3z}$. This, fubftituted in the fecond, will give $\frac{p^3}{27z^3} + z^3 = q$, or $z^6 - qz^3 = -\frac{1}{27}p^3$. Whence, by the rule for affected quadraticks, $z^6 - qz^3 + \frac{1}{4}qq = \frac{1}{4}qq - \frac{1}{27}p^3$, and $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$. Laftly, it will be $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. In the extraction of the fquare-root, I have taken only the pofitive fign, becaufe the negative would bring no variation, and gives at laft for the value of x the fame quantity as the pofitive, as may be feen from the calculation. And it is to be underftood in like manner in the other canonical equations. Now, becaufe $y^3 + z^3 = q$, it will be therefore $y^3 = q - \frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and thence $y = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. But it was at firft $x = y + z$; therefore $x = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. Hence it is feen, that the alternative of the figns, which was omitted, makes no variation.

182. The fecond equation $x^3 + px - q = 0$, making the fame fubftitutions, will be $y^3 + 3y^2z + 3z^2y + z^3 + py + pz - q = 0$. From hence let the two equations be formed, $3y^2z + 3yz^2 = -py - pz$, and $y^3 + z^3 = q$. By the fecond cafe of the fame rule.

 From

From the firſt, we have $3yz = -p$, or $y = -\frac{p}{3z}$, which, ſubſtituted in the ſecond, gives $-\frac{p^3}{27z^3} + z^3 = q$, or $z^6 - qz^3 = \frac{1}{27}p^3$. And therefore $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$, and $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$. But $y^3 + z^3 = q$, therefore $y = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, and $x = \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

The third caſe.

183. The third equation $x^3 - px + q = 0$, making the ſubſtitutions, will be $y^3 + 3y^2z + 3yz^2 + z^3 - py - pz + q = 0$. Let the two equations be formed, $3y^2z + 3yz^2 = py + pz$, and $y^3 + z^3 = -q$. From the firſt, we have $3yz = p$, or $y = \frac{p}{3z}$, which, ſubſtituted in the ſecond, gives $\frac{p^3}{27z^3} + z^3 = -q$, or $z^6 + qz^3 = -\frac{1}{27}p^3$; and therefore $z^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and thence $z = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. But $y^3 + z^3 = -q$; whence $y = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and laſtly, $x = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$.

The fourth and laſt caſe.

184. The fourth equation $x^3 + px + q = 0$, making the ſubſtitutions, will be $y^3 + 3y^2z + 3yz^2 + z^3 + py + pz + q = 0$. Forming the two equations, $3y^2z + 3yz^2 = -py - pz$, and $y^3 + z^3 = -q$, from the firſt we ſhall have $3yz = -p$, or $y = -\frac{p}{3z}$. This, ſubſtituted in the ſecond, gives $-\frac{p^3}{27z^3} + z^3 = -q$, or $z^6 + qz^3 = \frac{1}{27}p^3$, and therefore $z^3 = -\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$, and thence $z = \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$. But $y^3 + z^3 = -q$; whence $y = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, and laſtly, $x = \sqrt[3]{-\frac{1}{2}q - \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}} + \sqrt[3]{-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

Other expreſſions of the ſame roots.

185. The ſame roots or formulæ may be had, by putting $x = z \pm \frac{p}{3z}$, that is, $+\frac{p}{3z}$, if in the equation it be $-px$, and $-\frac{p}{3z}$, if it be $+px$ in the equation. Whence $x^3 = z^3 \pm pz + \frac{pp}{3z} \pm \frac{p^3}{27z^3}$. Make therefore the ſubſtitutions in the firſt canonical equation, and it will be $z^3 + \frac{p^3}{27z^3} - q = 0$, or $z^6 - qz^3 = -\frac{1}{27}p^3$, and $z^3 = \frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}$, and then $z = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$. Therefore, becauſe it was made $x = z + \frac{p}{3z}$, it will be $x = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \frac{p}{3\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}}$.

To reduce this to the ſame expreſſion found in the firſt manner, it will be ſufficient to multiply the numerator and denominator of the ſecond term of the *homogeneum comparationis* by $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and it will be $\frac{p\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}}{3\sqrt[3]{\frac{1}{27}p^3}}$, that is, $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and therefore x will be the ſame as before. And the like may be obſerved in the other caſes.

186. It is evident that the values of the unknown quantity x, found by the firſt ſubſtitution of $x = y + z$, require the extraction of two different cubick roots, whereas the ſecond, by the ſubſtitution of $x = z \pm \frac{p}{3z}$, require the extraction of one only; and that the value by the ſecond and fourth canonical equation will always appear under a real form, becauſe the quantities under the quadratick radical are wholly poſitive. But that of the firſt and third will be under a real form, if $\frac{1}{4}qq$ be greater than $\frac{1}{27}p^3$; and under an imaginary form, when $\frac{1}{4}qq$ is leſs than $\frac{1}{27}p^3$. And this is called the Irreducible Caſe; but, notwithſtanding this, it does not follow, but that all it's roots are real. For all the three values in the firſt and third equation are real, when $\frac{1}{4}qq$ is leſs than $\frac{1}{27}p^3$. But when $\frac{1}{4}qq$ is greater than $\frac{1}{27}p^3$, in the firſt and third equation, and, in general, in the ſecond and fourth, the roots or values alone thus found are real, and the other two are imaginary.

To diſtinguiſh when theſe roots are real, and when imaginary.

As to the ſecond and fourth equation, this has been already demonſtrated at § 152, when they have the third term poſitive. Then, as to the firſt and third, when the third term is negative, each of theſe will have three real roots, which are a, $-b$, $-c$, or $-a$, $+b$, $+c$; and, becauſe the ſecond term is wanting, as is here ſuppoſed, it will be $a = b + c$, and the equation therefore, which ariſes from ſuch roots, will be of this form,

$$\begin{aligned} x^3 &- bbx \pm bc \times \overline{b+c} = 0. \\ &- bcx \\ &- ccx \end{aligned}$$

When b, c, are real quantities, then $\overline{b-c}^2$ will be a poſitive quantity; and therefore, if we put $bb - 2bc + cc = D$, it will be alſo $bb + bc + cc = D + 3bc$, and $\frac{\overline{bb+bc+cc}|^3}{27} = \frac{1}{27}D^3 + \frac{1}{3}D^2bc + Dbbcc + b^3c^3$. But beſides, it will be $bb + 2bc + cc = \overline{b+c}|^2 = D + 4bc$, and therefore $\frac{1}{4}bbcc \times \overline{b+c}|^2 = \frac{1}{4}Dbbcc + b^3c^3$. And $\frac{1}{27}D^3 + \frac{1}{3}D^2bc + Dbbcc + b^3c^3$ is greater than $\frac{1}{4}Dbbcc$, and therefore it will alſo be greater than $\frac{1}{4}bbcc \times \overline{b+c}^2$, and therefore $\frac{1}{27} \times \overline{bb+bc+cc}|^3$ will be greater than $\frac{1}{4}bbcc \times \overline{b+c}|^2$. That is, the cube of the third part of the co-efficient of the third term, taken poſitively, is greater than the ſquare of half the laſt term; that is, $\frac{1}{27}p^3$ is greater than $\frac{1}{4}qq$. Therefore,

fore, if all the roots be real, the third term will always be negative, and besides, $\frac{1}{27}p^3$ will be greater than $\frac{1}{4}qq$. When it happens to be otherwise, two of the roots will be imaginary.

After the foregoing manner, having found one value for each equation, we shall have the other two roots by dividing the proposed equation by this value; for the quotient will be an equation of the second degree, which may always be easily resolved.

A compendium by the three cubick roots of unity.

187. But, if it shall be thought convenient, the trouble of this division may also be spared by considering, that as unity itself has three cubick roots, which are 1, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$; so it may be understood of any other quantity; of $\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}$ for example, which, being multiplied into unity, it's three cubick roots will be $1 \times \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, $\overline{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$ into $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$, and $\overline{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$ into $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}p^3}}$.

Whence the three cubick roots of the first equation $x^3 - px - q = 0$, by ordering them in a due manner, will be as follows: $x = \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, $x = \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$, and $x = \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} + \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}}$.

And, in fact, if we find the product of these three roots into each other, making, for brevity-sake, $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = m$ and $\sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{4}qq - \frac{1}{27}p^3}} = n$, the product of the last, $x + \frac{1 + \sqrt{-3}}{2}m + \frac{1 - \sqrt{-3}}{2}n$ into the second, $x + \frac{1 - \sqrt{-3}}{2}m + \frac{1 + \sqrt{-3}}{2}n$ will be $xx + mx + nx + mm - mn + nn$, which, multiplied into the first, $x - m - n$, will give $x^3 - 3mnx - m^3 - n^3$; and, restoring the values of m and n, it will be finally $x^3 - px - q = 0$, which is the equation proposed. Nor will it be otherwise in the other equations.

Example of this reduction.

188. The foregoing general formulæ being thus found, to apply them to the particular use of any given equations, it will be sufficient to compare the proposed equation to that of the four canonical equations which corresponds to it, thence to obtain the values of q and p; which, being substituted in the formula, will give the roots required.

Let

Let the equation be $x^3 + 2aax - 9a^3 = 0$. The correſponding one of the four canonical equations will be the ſecond, $x^3 + px - q = 0$; ſo that it will be $p = 2aa$, $q = 9a^3$. Then, making the ſubſtitution of theſe values inſtead of p and q, in the general expreſſion of the root of this ſecond equation, we ſhall have $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$, or, laſtly, $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^6}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$. The other two roots will be $x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^6}} + \frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$, and $x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{2219}{108}a^6}} + \frac{-1+\sqrt{-3}}{2}\sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{2219}{108}a^6}}$; the product of which roots will reſtore the propoſed equation.

189. But, without having recourſe to the general formulæ, particular equations may be ſolved independently of them, by making uſe of the given rule. Thus, for the equation $x^3 + 2aax - 9a^3 = 0$, making $x = y + z$, it will be $2aax = 2aay + 2aaz$, and $x^3 = y^3 + 3y^2z + 3yz^2 + z^3$; and, ſubſtituting theſe values in the propoſed equation, it will be changed into this other, $y^3 + 3zy^2 + 3z^2y + z^3 + 2aay + 2aaz - 9a^3 = 0$. Of this equation may be made theſe two, $3zy^2 + 3zzy = -2aay - 2aaz$, and $y^3 + z^3 = 9a^3$. From the firſt, by dividing by $y + z$, we have $3zy = -2aa$, or $y = -\frac{2aa}{3z}$; which, ſubſtituted in the ſecond, gives $-\frac{8a^6}{27z^3} + z^3 = 9a^3$, or $z^6 - 9a^3z^3 = \frac{8}{27}a^6$. And therefore $z^3 = \frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}$, and $z = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$. But it is $y^3 + z^3 = 9a^3$, therefore $y^3 = \frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}$, and $y = \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$. But it is $y + z = x$, therefore $x = \sqrt[3]{\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}} + \sqrt[3]{\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 + \frac{8}{27}a^6}}$, the ſame as above. Examples without the formula.

Let the equation be $z^3 + 3az^2 - 5aaz + 2a^3 = 0$. Let the ſecond term be taken away, by making $z = x - a$, and there ariſes $x^3 - 8a^2x + 9a^3 = 0$. By comparing this with the third canonical equation, we ſhall have $p = 8aa$, $q = 9a^3$; whence, ſubſtituting theſe values in the general formula for the root, it will be $x = \sqrt[3]{-\frac{9}{2}a^3 + \sqrt{\frac{81}{4}a^6 - \frac{512}{27}a^6}} + \sqrt[3]{-\frac{9}{2}a^3 - \sqrt{\frac{81}{4}a^6 - \frac{512}{27}a^6}}$, that is, $x = \sqrt[3]{-\frac{9}{2}a^3 + \sqrt{\frac{139}{108}a^6}} + \sqrt[3]{-\frac{9}{2}a^3 - \sqrt{\frac{139}{108}a^6}}$. The like for the other two roots. And, becauſe it was made $z = x - a$, by ſubtracting the quantity a from each of the three roots, we ſhall have the roots of the propoſed equation.

Let the equation be $x^3 - 9a^2x + 2a^3 = 0$. This will correſpond to the third of the four canonical equations, and therefore it will be $p = 9a^2$, $q = 2a^3$; therefore, making a ſubſtitution of theſe values, inſtead of p and q in the general expreſſion.

expreſſion of the root of that third equation, it will be $x = \sqrt[3]{-a^3 + \sqrt{-\frac{700}{27}a^6}} + \sqrt[3]{-a^3 - \sqrt{-\frac{700}{27}a^6}}$; which expreſſion is imaginary, notwithſtanding all the three roots are real; as the irreducible caſe requires.

Reduction of equations of the fourth degree.

190. In equations of the fourth degree, we may proceed after this manner. Let the canonical equation be $x^4 * + px^2 + qx - r = 0$, in which the ſecond term is wanting; and if it had not been abſent, it might have been taken away. Let this be transformed into a cubick equation, after the manner explained at § 167, by means of the two ſubſidiary formulæ, $x^2 + yx + z = 0$, and $x^2 - yx + u = 0$; and it will be transformed into $y^6 + 2py^4 + ppy^2 - qq = 0$.
$$\ + 4ry^2$$
And the two ſubſidiary equations, by putting, inſtead of u and z, their values found from the compariſon of the terms, will become $x^2 + yx + \frac{1}{2}p + \frac{1}{2}yy - \frac{q}{2y} = 0$, and $x^2 - yx + \frac{1}{2}p + \frac{1}{2}yy + \frac{q}{2y} = 0$. Now, as it is ſuppoſed that this equation has no diviſor of two dimenſions, the ſecond term muſt be taken from it by the ſubſtitution of $yy = t - \frac{2}{3}p$, and then we ſhall have this new equation,
$$\begin{aligned} t^3 &- \tfrac{1}{3}ppt - \tfrac{2}{27}p^3 = 0. \\ &+ 4rt \quad - \tfrac{8}{3}pr \\ &\qquad\quad\ - qq \end{aligned}$$

Let this be compared with the firſt or ſecond of the four canonical equations of § 181, according as $4r$ is leſſer or greater than $\frac{1}{3}pp$, that we may have it's cube-root, which, for brevity-ſake, we may call b. Whence it will be $t = b$; and, becauſe it was made $yy = t - \frac{2}{3}p$, it will be $yy = b - \frac{2}{3}p$, and therefore $y = \sqrt{b - \frac{2}{3}p}$, which, for brevity, may be called g. In the two ſubſidiary formulæ put g inſtead of y, and gg inſtead of yy, and they will be $xx + gx + \frac{1}{2}gg + \frac{1}{2}p - \frac{q}{2g} = 0$, and $xx - gx + \frac{1}{2}gg + \frac{1}{2}p + \frac{q}{2g} = 0$; the roots of which are $x = -\frac{1}{2}g \pm \sqrt{\frac{q}{2g} - \frac{1}{2}p - \frac{1}{4}gg}$ of the firſt, and $x = \frac{1}{2}g \pm \sqrt{-\frac{q}{2g} - \frac{1}{2}p - \frac{1}{4}gg}$ of the ſecond. And, reſtoring the value of $g = \sqrt{b - \frac{2}{3}p}$, they will be $x = -\frac{1}{2}\sqrt{b - \frac{2}{3}p} \pm \sqrt{\frac{q}{2\sqrt{b - \frac{2}{3}p}} - \frac{1}{3}p - \frac{1}{4}b}$, and $x = \frac{1}{2}\sqrt{b - \frac{2}{3}p} \pm \sqrt{\frac{-q}{2\sqrt{b - \frac{2}{3}p}} - \frac{1}{3}p - \frac{1}{4}b}$, the four roots of the propoſed equation $x^4 * + px^2 + qx - r = 0$.

Let the equation be $x^4 * - 86aax^2 + 600a^3x - 851a^4 = 0$. This being compared with the foregoing canonical equation, we ſhall have $p = -86aa$, $q = 600a^3$, $r = 851a^4$. Therefore the transformed cubical equation will be $y^6 - 172aay^4 + 10800a^4y^2 - 360000a^6 = 0$. Now, becauſe this is diviſible by by $y^2 - 100a^3 = 0$, without resolving it by the rules of cubick equations, as we know already the root to be $yy = 100aa$, and $y = 10a$; substitute these values instead of y and yy, as also the values of p, q, in the two subsidiary equations, they will be $x^2 + 10ax - 23aa = 0$, and $x^2 - 10ax + 37aa = 0$, and their roots are $x = -5a \pm \sqrt{48aa}$, and $x = 5a \pm \sqrt{-12aa}$, which are therefore the four roots of the proposed equation. This example is inserted only to show the use of the method; for the given equation may be reduced to two of two dimensions, after the way already explained in it's place.

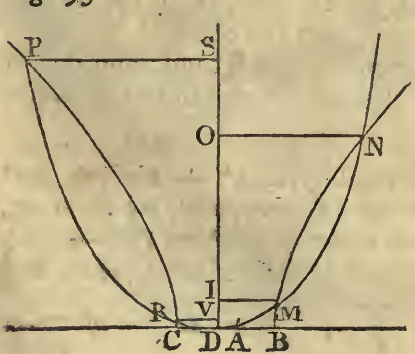
How equations may be resolved geometrically by a combination of loci.

191. This method of resolving equations can be of use only in arithmetical questions, and not in geometrical: because, in this way, we have the value of the unknown quantity expressed by a cube-root, which it is supposed cannot be actually extracted; for, otherwise, the equation would have a divisor, and would not be of the degree it seems to be. Now, to find this cube-root geometrically cannot be done otherwise than by the intersection of curve-lines; which is the second manner, and the general one which I have mentioned before, at § 180.

This method consists in introducing a new unknown quantity into the equation, by which we shall have two equations, each of which contains both the unknown quantities, and both of them together all the known quantities of the proposed equation. These two equations are two *loci geometrici*, which are therefore to be constructed; the intersections of which determine the geometrical values, or the roots of the equation proposed. And the reason of this is manifest. For, as from the combination of two places, or from two indeterminate equations, by putting in one of these, instead of one of the two unknown quantities, it's value given by the other equation, there arises a determinate equation, which determinate equation may be resolved into two indeterminates.

Let there be given the two equations $ax = zz$, and $xx - 5zz + 2az + 3aa = 0$. If from the first, for example, we derive the value of $x = \frac{zz}{a}$, and substitute it in the second, there will arise the determinate equation $z^4 - 5aazz + 2a^3z + 3a^4 = 0$, of the fourth degree. Then, taking the *locus* to the parabola $ax = zz$, if we make the substitution of the value of zz in the equation $z^4 - 5aazz + 2a^3z + 3a^4 = 0$, there will arise the second *locus* $aaxx - 5aaz^2 + 2a^3z + 3a^4 = 0$, or $x^2 - 5z^2 + 2az + 3aa = 0$. To construct this second *locus*, with centre A (Fig. 95.) and transverse axis CB $= \frac{8}{5}a$, and with the parameter $= 8a$, let there be described the two opposite hyperbolas BN, CP, which shall be the *locus* of the equation $x^2 - 5z^2 + 2az + 3a^2 = 0$, taking the abscis z from the point D, which is distant from the centre A by the quantity $\frac{1}{5}a$ towards the vertex C.

Fig. 95.



Rightly to combine this with the firſt *locus* $ax = zz$, it is neceſſary that the origin and the axis, of the unknown quantity x, may be in common to both the *loci*. And therefore at the vertex D, with the parameter $= a$, upon the axis DO, parallel to the conjugate axis of the oppoſite hyperbolas, the parabola of the firſt equation $ax = zz$ ſhould be deſcribed. This will meet the two hyperbolas in the four points M, N, R, P, from which drawing the perpendiculars MI, NO, RV, PS, to the axis DO, they will be the four values of z, that is, the four roots of the equation $z^4 * - 5aaz^2 + 2a^3z + 3a^4 = 0$. The two IM, ON, will be poſitive, and the other two VR, SP, will be negative. For, as z of the determinate equation, (that is to ſay, every one of the roots of the ſame,) ought to be common to both the *loci*, this can happen only in the points M, N, R, P, in which theſe two *loci* interſect each other. Therefore the right lines MI, NO, RV, SP, which expreſs z, will be the four roots of the determinate equation propoſed.

When two of the roots will be equal, when nothing, when imaginary.

192. Hence it is plain, that the nearer the points M, N, approach to each other, ſo much the leſs will be the difference of the ordinates IM, ON. So that when one point falls on another, (in which caſe the two curves will no longer cut but touch each other,) the two ordinates become equal, or the equation will have two equal roots. Alſo, if the curves cut each other at the vertex, in which place the ordinate is nothing, the equation will have one of it's roots equal to nothing. And laſtly, if the two curves neither cut nor touch in any point, the roots of the propoſed equation will be imaginary or impoſſible.

The *loci* ſhould be ſuch, as will ſupply the ſimpleſt conſtruction.

193. Now, in the introduction of the new unknown quantity, it ſhould be endeavoured, that it may be done in ſuch a manner, as that the two *loci* may be the ſimpleſt poſſible, in reſpect of the degree of the propoſed equation. That is to ſay, if the equation be of the third or fourth degree, the two *loci* ſhould be of the ſecond, that is, conic ſections. And it might be convenient, as any one would think, that one of them ſhould always be a circle, as being the ſimpleſt curve. But it ought to be conſidered, that, by determining one of the *loci* to be a circle, the equation to the other *locus* in many caſes may become perplexed; and therefore in ſuch caſes I ſhould prefer any other *locus* before the circle, if it would afford a greater ſimplicity. If the equation be of the fifth or ſixth degree, the two *loci* may be one of the ſecond, and the other of the third. If it be of the ſeventh or eighth, they ſhould be one of the ſecond, and one of the fourth; or two of the third, firſt reducing that of the eighth to the ninth. And ſo on, obſerving the ſame analogy.

Taking, therefore, this equation of the fourth degree, $x^4 + 2bx^3 + acx^2 - a^2dx - a^3f = 0$, aſſume the equation (I.) $xx + bx = ay$, and, by ſquaring, it will be $x^4 + 2bx^3 + b^2x^2 = a^2y^2$, and therefore $x^4 + 2bx^3 = a^2y^2 - b^2x^2$. In the propoſed equation let this value be ſubſtituted inſtead of $x^4 + 2bx^3$, and there will ariſe this other equation, (II.) $yy - \frac{b^2x^2}{a^2} + \frac{cx^2}{a} - dx - af = 0$.

Now,

Now, putting the value of xx obtained from the firft equation, that is, $ay - bx$, in the fecond term of this, and letting the third term alone, there will arife (III.) $yy - \frac{bb}{a}y + \frac{b^3}{a^2}x + \frac{c}{a}x^2 - dx - af = 0$. Or, fubftituting the value of xx in the third term of the fame equation, letting the fecond term alone, there will arife (IV.) $yy - \frac{bb}{aa}xx + cy - \frac{bc}{a}x - dx - af = 0$. And in this, putting the value of xx, it will be (V.) $yy + cy - \frac{bb}{a}y - \frac{bc}{a}x - dx - af = 0$. Laftly, if from this be fubtracted the firft made equal to nothing, or $xx + bx - ay = 0$, and then adding it to the fame, there will arife from the firft operation (VI.) $yy + cy - \frac{bb}{a}y + ay - xx - bx - \frac{bc}{a}x + \frac{b^3}{a^2}x - dx - af = 0$; and from the fecond, (VII.) $yy + cy - \frac{bb}{a}y - ay + xx + bx - \frac{bc}{a}x + \frac{b^3}{a^2}x - dx - af = 0$.

194. It is plain, that the firft equation is a *locus* to the *Apollonian* parabola. To diftinguifh the reft, we muft make ufe of the reductions explained at § 127, 128, by which we fhall find, that the fecond will be a *locus* to the parabola, when it is $ac = bb$; to the ellipfis, when ac is greater than bb; and, finally, to the hyperbola, when ac is lefs than bb. The third will be to an ellipfis, which will degenerate into a circle, when it is $c = a$, and the co-ordinates are at right angles. The fourth will be to an hyperbola, which befides will be equilateral, if it is $b = a$. The fifth will be to a parabola. The fixth will be to the equilateral hyperbola. The feventh will be to the circle, when the angle of the co-ordinates is a right angle. To diftinguifh thefe *loci*.

From hence we may make choice of fuch a combination of the two *loci*, for the conftruction of the propofed Problem, as fhall be thought moft convenient.

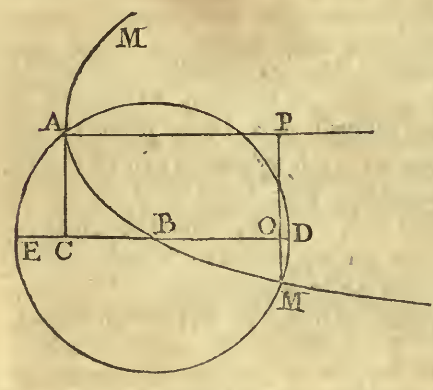
195. If the fecond term of the propofed equation had been negative, we fhould have made $xx - bx = ay$; and the equations thence arifing would have been the fame as before, only changing the fign of thofe terms, in which the letter b is of odd dimenfions. And if the propofed equation had at firft been without the fecond term, I fhould have taken $xx = ay$. Therefore, expunging the terms in which b is found in the other equations, they would have been fuch as this cafe requires. Cautions to be obferved.

196. In the propofed equations, the fecond term being $\pm 2bx^3$, we fhould take the *locus* to the parabola $xx \pm bx = ay$, rather than $xx = ay$; becaufe thus the other *loci* which arife have not the rectangle xy, and therefore are conftructed with the more eafe. Conftruction of a cubic equation, for example, by a parabola and a circle.

EXAMPLE I.

Let the equation be of the third degree, $x^3 - aax + 2a^3 = 0$. Let it be multiplied by $x = 0$, to reduce it to the fourth degree; whence it will be $x^4 - aax^2 + 2a^3x = 0$; which is required to be conſtructed by means of a parabola and a circle. As the ſecond term is wanting, make $xx = ay$, a *locus* to the parabola. Then ſubſtituting, inſtead of x^4 and x^2, their values $aayy$ and ay, it will be $yy - ay + 2ax = 0$; to which adding the firſt equation $xx - ay = 0$, we ſhall have the equation $yy - 2ay + 2ax + xx = 0$, which is a *locus* to the circle.

Fig. 96.



With radius BD $= \sqrt{2aa}$ let the circle ADME be deſcribed, and make BC $= a$, and alſo the ordinate CA $=$ CB $= a$. From the point A drawing the indefinite line AP parallel to ED, and on it taking the abſciſſes AP $= y$, and making the ordinate PM $= x$, this will be the *locus* of the equation $yy - 2ay + 2ax + xx = 0$. Upon the axis AP, on which are taken the y's, with vertex A let the *Apollonian* parabola MAM of the equation $xx = ay$ be deſcribed, which ſhall cut the circle in two points A, M; from whence the ordinates being drawn, they ſhall be the the real roots of the equation $x^4 * - aax^2 + 2a^3x = 0$, and two will be imaginary.

But at the point A the ordinate is nothing, and therefore one of the roots will be $x = 0$, as it ought to be; it being now introduced by multiplying the propoſed equation by $x = 0$. Therefore PM will be the real negative root of the equation $x^3 - aax + 2a^3 = 0$, and the other two will be imaginary. If I had multiplied the propoſed equation by x equal to ſome quantity, the circle would have cut the parabola in two points out of the vertex, one of which would have given me the introduced root, and the other that of the propoſed equation.

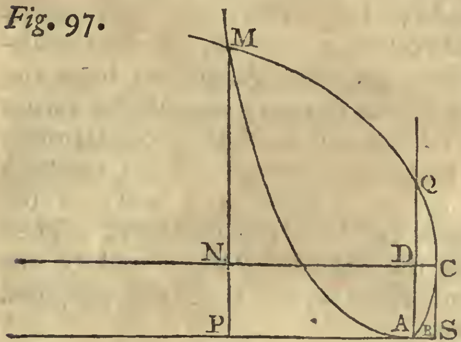
Now, to ſhow that PM is one of the roots of the equation $x^4 - aax^2 + 2a^3x = 0$, it may be conſidered, that, from the nature of the circle, it is EO $\times$ OD $=$ OMq. But OM $= -x - a$, EO $= y + \sqrt{2aa} - a$, and OD $= a - y + \sqrt{2aa}$. Therefore $xx + 2ax + aa = aa + 2ay - yy$. But, by the equation of the parabola AM, it is $xx = ay$, and therefore $\frac{x^4}{aa} = yy$.

Then

Then subſtituting theſe values of y and yy, and reducing the equation to nothing, it will be $x^4 - aax^2 + 2a^3x = 0$, which is the very equation of the fourth degree, whoſe roots we were to extract.

197. If we would conſtruct the equation $x^4 - aax^2 + 2a^3x = 0$ by means of two parabolas, it would be convenient to make uſe of the equation found above, $yy - ay + 2ax = 0$; and the *locus* of this, together with the parabola of the equation $xx = ay$, might determine the roots required. —By two parabolas.

Fig. 97.



Therefore, with parameter $= 2a$, let there be deſcribed the parabola MCA, in which make CD $= \frac{1}{8}a$. And letting fall DA $= \frac{1}{2}a$, which will meet the parabola in the point A, and through that point drawing the indefinite line AP parallel to the axis CD; and taking the abſciſs x from the point A, poſitive towards B and negative towards P, and the ordinates PM $= y$, this will be the *locus* of the equation $yy - ay + 2ax = 0$. Then with vertex A, to the axis AQ, let the other parabola MAS of the equation $xx = ay$ be deſcribed; this will cut the firſt in the points A, M. And letting fall the perpendicular MP, it will give the negative root AP of the propoſed equation. And becauſe at the point A the perpendicular is nothing, therefore there is no other root; juſt as it ought to be, the propoſed equation being multiplied by $x = 0$.

For, in the parabola MCA, it being CN $= -x + \frac{1}{8}a$, and NM $= y - \frac{1}{2}a$, it will be, by the property of this parabola, $\frac{1}{4}aa - 2ax = yy - ay + \frac{1}{4}aa$; and ſubſtituting the values of y and yy, which are given by the firſt equation to the parabola MAS, that is, $xx = ay$, and ordering the equation, we ſhall have at laſt $x^4 * - aax^2 + 2a^3x = 0$, which is the equation of the fourth degree, of which the roots were required.

198. Now, if I had intended to have made uſe of the parabola, and of the equilateral hyperbola, it would have ſufficed, from the ſame equation $yy - ay + 2ax = 0$, to have ſubtracted the firſt equation $xx - ay = 0$, and the equation $yy + 2ax - xx = 0$ would have ariſen from thence, which is a *locus* to the equilateral hyperbola; which, being conſtructed, would have given me the roots required, by means of it's interſections with the parabola of the equation $xx = ay$. —By a parabola and an equilateral hyperbola.

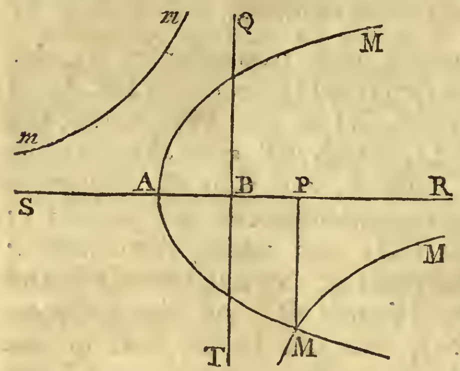
199. Finally, if I had deſired to ſolve the Problem by the circle and the hyperbola, I ſhould have conſtructed the third equation $yy - 2ay + 2ax + xx = 0$, a *locus* to the circle, and the fourth equation $yy + 2ax - xx = 0$, a *locus* to the hyperbola, as is ſeen before; the interſections of which *loci* would have given me the roots required. —By a circle and hyperbola.

 200. But,

These equations constructed by various *loci*, with examples.

200. But, without multiplying by x the equation proposed, $x^3 - aax + 2a^3 = 0$, we might have constructed it after the following manner, when we do not choose to introduce one *locus* rather than another. Make therefore $xx = ay$, and, instead of xx, put it's value ay in the equation, and there will arise the equation $xy - ax + 2aa = 0$, a *locus* to the hyperbola between it's asymptotes.

Fig. 98.



Therefore let the two indefinite right lines SR, QT, cut each other at right angles, and let these be the asymptotes of the two hyperbolas MM, *mm*, having the constant rectangle $-2aa$; taking the abscisses from the point A, distant from the point B by the quantity a. At the vertex A, to the axis AR, with the parameter $= a$, let the parabola of the first equation $xx = ay$ be described; it will cut the hyperbola MM in the point M. Then drawing the ordinate PM, it will be the real and negative root of the proposed equation.

For, by the property of the hyperbola MM, it will be $BP \times PM = -2aa$, that is, $xy - ax = -2aa$. And, by the property of the parabola AM, we shall have $y = \frac{xx}{a}$. Therefore, instead of y, substituting it's value, and ordering the equation, it will be $x^3 - aax + 2a^3 = 0$, the equation proposed.

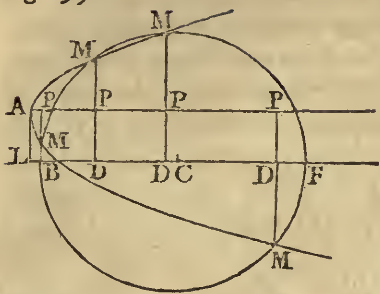
In general, all equations of the third degree may always be constructed after this manner, without being reduced to the fourth: by a parabola, and an hyperbola between the asymptotes.

EXAMPLE II.

Let there be given the equation of the fourth degree, $z^4 * - 5a^2z^2 + 2a^3z + 3a^4 = 0$, which is to be constructed by means of a parabola and a circle. Take the equation $ax = zz$, square it, and in the equation proposed, instead of z^4 and z^2, substitute their values, and there will arise a second equation, $xx - 5ax + 2az + 3aa = 0$, from whence subtracting and then adding the first equation $zz - ax = 0$, we shall have, in the first case, a third equation, $xx - 4ax + 2az + 3aa + zz = 0$; and in the second case, a fourth equation, $xx - 6ax + 2az + 3aa + zz = 0$; which is a *locus* to the circle, and therefore I shall make use of it to construct the proposed equation of the fourth degree.

With

Fig. 99.



With radius $= \sqrt{7aa}$ let there be defcribed a circle BMF, and from the centre C towards B taking the line CL $= 3a$, and from the point L make LA $= a$, perpendicular to the diameter, from the point A draw the indefinite right line AP parallel to the diameter BF; it will be AP $= x$, and the correfponding ordinates in the circle PM $= z$. And therefore A will be the vertex, and AP the axis of the parabola of the equation $ax = zz$. Whence, with the vertex A, axis AP, and parameter $= a$, defcribing the parabola AM, it will meet the circle in four points M, from whence drawing the perpendiculars PM to the axis AP, they will be the roots of the propofed equation, two being pofitive and two negative.

For, producing PM to D, if there be occafion, it will be, by the nature of the circle, BD × DF = DMq. But DM $= z + a$, BD $= x - 3a + \sqrt{7aa}$, and DF $= -x + 3a + \sqrt{7aa}$. Therefore $zz + 2az + aa = -xx + 6ax - 2aa$; but, by the nature of the parabola AM, it is $ax = zz$, and $xx = \frac{z^4}{aa}$. Therefore, making a fubftitution of thefe values, and ordering the equation, and bringing the terms all to one fide, it will be $z^4 - 5aaz^2 + 2a^3z + 3a^4 = 0$, which is the equation propofed.

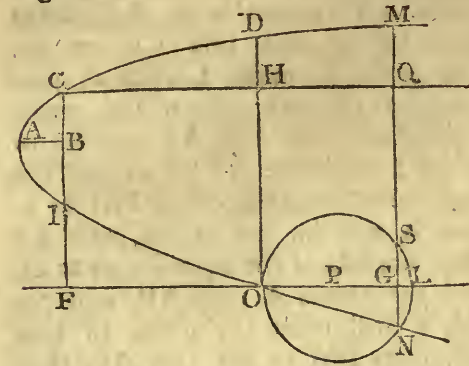
EXAMPLE III.

Let there be given an equation of the third degree, $x^3 - 3aax + 5a^3 = 0$, and let it be multiplied by $x + 2a$, that it may be reduced to one of the fourth degree; it will be $x^4 + 2ax^3 - 3aax^2 - a^3x + 10a^4 = 0$.

Take the equation to a parabola $xx + ax = ay$, which, by fquaring, will become $x^4 + 2ax^3 + aax^2 = aayy$. Let the value of it's two firft terms, $x^4 + 2ax^3$, that is, $aayy - aaxx$, be fubftituted in the equation, and there will arife (II.) $yy - 4xx - ax + 10aa = 0$. And in this, inftead of xx, fubftituting it's value $ay - ax$, there arifes (III.) $yy - 4ay + 3ax + 10aa = 0$; from thence fubtracting the firft, $xx + ax - ay = 0$, and alfo adding it, there will arife thefe two equations, (IV.) $yy - 3ay + 2ax + 10aa - xx = 0$ in the firft cafe, and (V.) $yy - 5ay + 4ax + 10aa + xx = 0$ in the fecond cafe. I fhall make ufe of the firft *locus*, and alfo of the laft.

For

Fig. 100.



For the conftruction of the laft, let the circle OSN be defcribed, with radius OP $= \frac{1}{2}a$; and, producing it to F, that it may be OF $= 2a$, and at the point F erecting the perpendicular FC $=$ FO $= 2a$, draw the indefinite right line CQ parallel to FP. Taking any line whatever, CQ $= y$, the correfponding negative ordinates, QS, QN, will reprefent x, and the circle will be the *locus* of the fifth equation. Now take in FC the line CB $= \frac{1}{2}a$, and from the point B draw the perpendicular BA $= \frac{1}{4}a$. Then with vertex A, and with parameter $= a$, let the parabola NAM be defcribed, which fhall be the *locus* of the firft equation, taking the abfciffes y on the right line CQ. From the points O, N, in which the parabola cuts the circle, raifing the perpendiculars OH, NQ, thefe will be the two real negative roots of the equation, $x^4 + 2ax^3 - 3a^2x^2 - a^3x + 10a^4 = 0$, of the fourth degree which was propofed.

And becaufe OH, taken negative, is equal to $2a$, which is the root introduced by the multiplication of the given equation into $x + 2a$, NL will be the real negative root of the propofed equation $x^3 - 3aax + 5a^3 = 0$, the other two roots being imaginary.

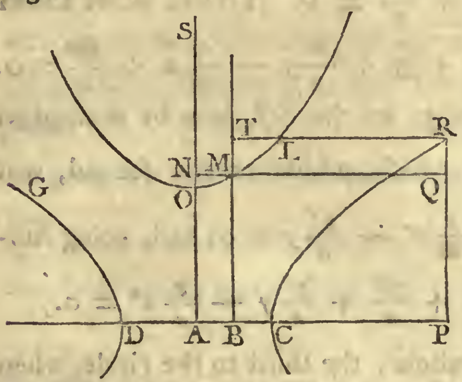
For, by the property of the circle OSL, it will be OG $\times$ GL $=$ GNq. But OG $= y - 2a$, GL $= 3a - y$, and GN $= -2a - x$. Therefore, making the fubftitutions, it will be $xx + 4ax + 10aa + yy - 5ay = 0$. But, from the equation to the parabola NAM, it will be $y = \frac{xx + ax}{a}$, and $yy = \frac{x^4 + 2ax^3 + aaxx}{aa}$; then fubftituting thefe values of y and yy in the equation to the circle, it will be at laft $x^4 + 2ax^3 - 3aaxx - a^3x + 10a^4 = 0$, as it ought to be.

EXAMPLE IV.

Let the equation be $x^6 - 4aax^4 - 8a^3x^3 + 8a^4x^2 + 32a^6 = 0$; and becaufe it is divifible by $x^2 - 4ax + 4aa$, and the quotient $x^4 + 4ax^3 + 8a^2x^2 + 8a^3x + 8a^4 = 0$ is an equation of the fourth degree, which we thus conftruct; take the equation $xx + 2ax = ay$, of which finding the fquare $x^4 + 4ax^3 + 4a^2x^2 = a^2y^2$, and, inftead of $x^4 + 4ax^3$, fubftitute it's value $aayy - 4aaxx$ in in the equation, and there will arife (II.) $yy + 4xx + 8ax + 8aa = 0$; in which, if we put the value of xx, or $ay - 2ax$, there will arife (III.) $yy + 4ay + 8aa = 0$, from which, if we fubtract the firft, there arifes (IV.) $yy + 5ay + 8aa - xx - 2ax = 0$; and laftly, if we add the firft to the third, it will be (V.) $yy + 3ay + 8aa + xx + 2ax = 0$.

The fecond *locus* is imaginary. The third is a determinate equation, but it's roots are imaginary. The fifth *locus* is alfo imaginary. But the fourth *locus* is real, and is to an equilateral hyperbola.

Fig. 101.



To the axis $DC = \sqrt{11aa}$, with centre A, let there be defcribed the hyperbolas CR, DG. Take $AB = a$, and let the indefinite perpendicular BM be raifed, in which take $BM = \frac{1}{4}a$; and from the point M let there be drawn MQ parallel to the axis DC. Taking the x's from the point M upon MQ, the correfponding QR or MT will be the y's, and the curve is the *locus* of the fourth equation. Producing QM to N, and making $MN = a$, and drawing NA to the centre of the hyperbola, take $NO = a$, and with vertex O, parameter $= a$, to the axis OS let the parabola OM be defcribed, which will pafs through the point M. Then taking the y's on MT, and the correfponding ordinates $TL = x$, this will be the *locus* of the firft equation $xx + 2ax = ay$. But now, as thefe two *loci* can never interfect each other, as is evident, all the four roots of the equation $x^4 + 4ax^3 + 8a^2x^2 + 8a^3x + 8a^4 = 0$ will be imaginary. Whence the propofed equation $x^6 - 4a^2x^4 - 8a^3x^3 + 8a^4x^2 + 32a^6 = 0$ is found to have only two real roots, which are equal to each other, being each equal to $2a$.

201. But if, befides, we fhould be willing to conftruct equations of the third and fourth degree, not only by the help of conical *loci*, which are to be thus found, but of fuch of them as may be given, or fimilar to given, *loci*; which may be of ufe when a conic fection is given in the ftate of a Problem: It may be done after the following manner, fuppofing, however, that the equations of the third degree are reduced to the fourth, and that thefe are freed from their fecond term, if they have any. —By given *loci*, or fuch as are fimilar to given.

Yet I muft here obferve, that though, for the moft part, it may be better to be determined to this conical *locus* which already enters into the Problem; yet we fhould always have it in view, that the ufe of this given *locus* ought not to fuperfede a greater fimplicity of conftruction. For, in this cafe, without any regard to the given *locus*, it may be better to introduce two new *loci*.

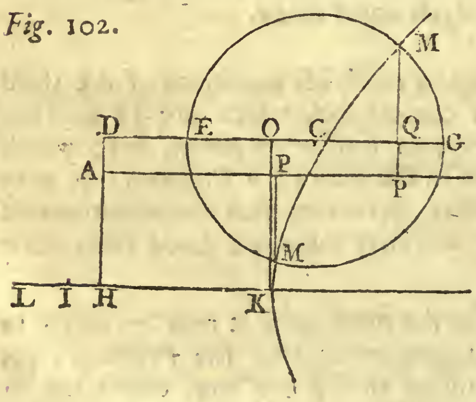
 Being

Being willing then to make uſe of given *loci*, or ſuch as are ſimilar to thoſe that are given, the artifice conſiſts in introducing two indeterminate or general quantities into the equation, and to determine them afterwards as occaſion may require. Therefore let the equation be $z^4 + abz^2 - aacz + a^3d = 0$. Make $z = \frac{ax}{f}$, in order to introduce the firſt indeterminate f. Making the ſubſtitutions, it will be $x^4 + \frac{bffx^2}{a} - \frac{f^3cx}{a} + \frac{f^4d}{a} = 0$. Let us take the firſt *locus* (I.) $x^2 - fy = 0$; and, ſubſtituting the values of x^2 and x^4, there will ariſe the ſecond *locus* (II.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$. To this let be added the firſt, and we ſhall have (III.) $x^2 - fy + yy + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$. Now, to introduce a ſecond indeterminate g, let the firſt *locus* be multiplied by $\frac{g}{a}$, and we ſhall have $\frac{gx^2 - gfy}{a} = 0$; which, added to the ſecond, will give (IV.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} + \frac{g}{a}x^2 - \frac{gf}{a}y = 0$; and, being ſubtracted, will give (V.) $y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} + \frac{gf}{a}y - \frac{g}{a}x^2 = 0$.

The firſt *locus* and the ſecond are to a parabola; the third to the circle, when the co-ordinates are at right angles; the fourth to the ellipſis; and the fifth to the hyperbola.

Now, let it be required, for example, to conſtruct the equation by means of a given circle and a given hyperbola. Let us therefore aſſume the third and fifth *loci*; and as to the third, with radius $CG = \frac{f}{2a}\sqrt{cc - 4ad + bb - 2ab + aa}$, let the circle EMG be deſcribed, and, taking $CD = \frac{fc}{2a}$, from the point D let fall the perpendicular $DA = \frac{af - bf}{2a}$, (ſuppoſing a to be greater than b; for it muſt be raiſed the contrary way, when b is greater than a,) then from the point A, on the right line AP parallel to DG, taking the abſciſſes $AP = x$, the correſponding PM will be the y, and the circle EMG will be the *locus* of the equation $x^2 - fy + y^2 + \frac{bf}{a}y - \frac{fc}{a}x + \frac{ffd}{a} = 0$.

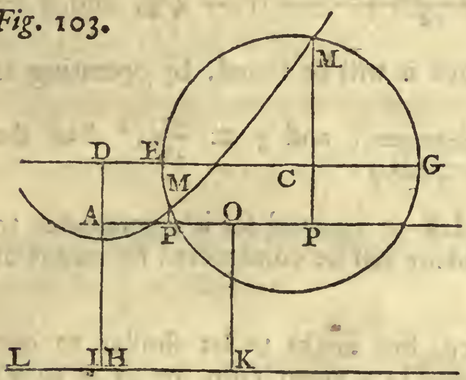
Fig. 102.



As

As to the fifth *locus*; to construct it and combine it with the circle, through the point A, the origin of x, produce the right line DA to H, so that it may be $AH = \frac{gf + bf}{2a}$; and through the points A, H, draw AP, HK, parallel to DG. On HK, towards the point L, set off the portion $HI = \frac{fc}{2g}$, and with centre I, transverse axis $LK = \frac{f}{ga}\sqrt{aacc + 4a^2gd - ab^2g - ag^3 - 2abg^2}$, (supposing $cc + 4dg$ to be greater than $\frac{bbg + g^3 + 2bgg}{a}$,) let the hyperbola KM be described, with parameter $KO = \frac{f}{aa}\sqrt{aacc + 4aagd - abbg - ag^3 - 2abgg}$; in which, if it be $AP = x$, $PM = y$, it will be the *locus* of the fifth equation. From the points M, in which it cuts the circle, drawing to AP the perpendiculars MP, the lines AP, AP, will be the roots of the equation $x^4 + \frac{bff}{a}x^2 - \frac{cf^3}{a}x + \frac{df^4}{a} = 0$. And, because it was made $z = \frac{ax}{f}$, and x is given, and also z, they will be the roots of the first proposed equation.

Fig. 103.



But, if we had supposed $cc + 4gd$ to be less than $\frac{bbg + 2bgg + g^3}{a}$, the *locus* of the fifth equation would be the hyperbola MM, half the transverse axis of which $= \frac{f}{2a}\sqrt{\frac{bbg + 2bgg + g^3 - acc - 4agd}{g}}$, the conjugate semiaxis $IK = \frac{f}{2g}\sqrt{\frac{b^2g + 2bg^2 + g^3 - ac^2 - 4agd}{a}}$, and the parameter KO of the conjugate axis $= \frac{f}{a}\sqrt{\frac{bbg + 2bgg + g^3 - acc - 4agd}{a}}$.

This supposed, to satisfy the first condition, that it shall be a given circle, let it's radius be $= r$, and then it would be $r = \frac{f}{2a}\sqrt{cc - 4ad + bb - 2ab + aa}$, from which equation the value of the assumed indeterminate may be derived, or $f = \frac{2ar}{\sqrt{cc - 4ad + bb - 2ab + aa}}$. And then the circle described, EGM, will be that, the radius of which is $= r$.

To

To ſatisfy the ſecond condition, that the hyperbola may be given alſo, let $2t$ be the given tranſverſe axis, and p the given parameter. Then it will be $2t = \frac{f}{g}\sqrt{cc + 4gd - \frac{bbg + g^3 + 2bgg}{a}}$, and $f = \frac{2gt}{\sqrt{cc + 4gd - \frac{bbg + g^3 + 2bgg}{a}}}$.

But it is alſo $p = \frac{f}{a}\sqrt{cc + 4dg - \frac{bbg + g^3 + 2bgg}{a}}$; therefore, inſtead of f, putting it's value now found, it will be $p = \frac{2gt}{a}$; from whence we have the value of $g = \frac{ap}{2t}$. And putting this inſtead of g in the value of f, it will be $f = \frac{2apt}{\sqrt{4ttcc + 8aptd - 2bbpt - \frac{aap^3}{2t} - 2abpp}}$. Wherefore the tranſverſe diameter and the parameter of the hyperbola deſcribed (Fig. 102.) ſhall be truly the given lines $2t$ and p; and thus as to the firſt caſe.

Then, as to the ſecond, which is when $cc + 4dg$ is leſs than $\frac{bbg + g^3 + 2bgg}{a}$, let the conjugate axis of the given hyperbola be LK $= 2u$, and it's parameter $= q$; then it will be $2u = \frac{f}{g}\sqrt{\frac{bbg + 2bgg + g^3}{a} - cc - 4dg}$, and $q = \frac{f}{a}\sqrt{\frac{bbg + 2bgg + g^3}{a} - cc - 4dg}$. Whence it will be found, by operating as before, $f = \frac{2aqu}{\sqrt{2bbuq + 2baqq + \frac{aaq^3}{2u} - 4ccuu - 8aduq}}$, and $g = \frac{aq}{2u}$. And the hyperbola will have for it's conjugate axis LK $= 2u$, and for it's parameter to the ſaid axis KO $= q$. And thus the Problem will be conſtructed by means of a given circle and a given hyperbola.

Now, if the hyperbola ſhall not be given, but ought to be ſimilar to one given; that is, if the axis be to it's parameter in a given ratio, or as m to n; becauſe it has been ſeen above, that the ratio of the axis to the parameter is that of a to g, it will be ſufficient to make the analogy, $a . g :: m . n$, and thence to have the value of $g = \frac{an}{m}$.

By making uſe of the ſame method, we may conſtruct equations by means of any other given *loci*, or which are ſimilar to thoſe given. As, for example, by means of the aforeſaid given circle, and of a given ellipſis, or like to a given one, by taking the fourth equation before, inſtead of the fifth.

EX-

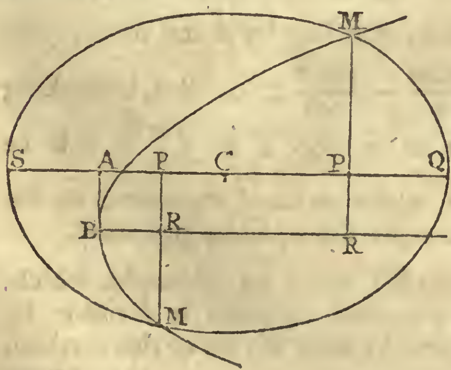
EXAMPLE V.

Let the equation be $x^4 - ax^3 - aax^2 - a^3x - 2a^4 = 0$, which it is required to conftruct by means of a parabola whofe parameter $= a$, and of an ellipfis fimilar to one given, whofe tranfverfe axis is to the parameter in the given ratio of b to d.

Let the fecond term be taken away by the fubftitution of $x = z + \frac{1}{4}a$, and the transformed equation will be $z^4 * - \frac{11}{8}aaz^2 - \frac{13}{8}a^3z - \frac{595}{256}a^4 = 0$. I put $z = \frac{ay}{f}$, to introduce the firft indeterminate f, and it will be $y^4 - \frac{11}{8}ffy^2 - \frac{13}{8}f^3y - \frac{595}{256}f^4 = 0$. Now, taking for the firft *locus* $yy = fq$ to the parabola, and making a fubftitution of the values of y^4 and y^2, we fhall have the fecond *locus* alfo to the parabola, $qq - \frac{11}{8}fq - \frac{13}{8}fy - \frac{595}{256}ff = 0$. Now, becaufe the given parabola has it's parameter $= a$, we may here make ufe of the firft *locus*, by taking $f = a$, and therefore it will be $yy = aq$. And fubftituting the value of f in the fecond, (for, the ellipfis not being given, the firft indeterminate f, in refpect of this, is ftill arbitrary,) it will be $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}aa = 0$.

Now let the firft *locus* be multiplied by $\frac{g}{a}$, in order to introduce the fecond indeterminate g, and it will be $\frac{gyy - agq}{a} = 0$, which, being added to the fecond, will give the third *locus*, $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}a^2 + \frac{gyy - agq}{a} = 0$, which is to an ellipfis.

Fig. 104.



For the conftruction of this third *locus*, we fhould have the ellipfis MSQ to defcribe, with the tranfverfe axis SQ $= 2\sqrt{\frac{716a^2g + 176ag^2 + 64g^3 + 169a^3}{256g}}$, and with parameter $= \frac{2a}{g}\sqrt{\frac{716a^2g + 176ag^2 + 64g^3 + 169a^3}{256g}}$. But, becaufe in this the ratio of the axis to the parameter is that of g to a, which, by the given condition, ought to be that of b to d, it will be $g = \frac{ab}{d}$. And therefore, inftead of g, fubftituting it's value,

value, the ellipsis MSQ must be described with the transverse axis $= \frac{1}{8d}\sqrt{\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{b}}$, and with parameter $= \frac{1}{8b}\sqrt{\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{b}}$.

Now, from the centre C taking CA $= \frac{11ad + 8ab}{16d}$, and from the point A letting fall the perpendicular AB $= \frac{13d}{16b}$, if from the point B be drawn BR parallel to the axis SQ, taking any line BR $= q$, it will be RM $= y$, and the ellipsis will be the *locus* of the third equation $qq - \frac{11}{8}aq - \frac{13}{8}ay - \frac{595}{256}aa + \frac{gyy - agq}{a} = 0$.

With vertex B, axis BR, and parameter $= a$, let the parabola MBM of the equation $yy = aq$ be described; it will cut the ellipsis in two points M, M. From which points drawing RM, RM, perpendicular to the right line BR, they will be the two real roots of the proposed equation.

For, by the property of the ellipsis, it will be SP $\times$ PQ to PMq, so is the transverse axis to the parameter. But CP $= q - \frac{11ad + 8ab}{16d}$, and therefore SP $= \frac{1}{16d}\sqrt{\frac{716a^2bd^2 + 176a^2db^2 + 64a^2b^3 + 169a^2d^3}{b}} + q - \frac{11ad + 8ab}{16d}$, and PQ $= \frac{1}{16d}\sqrt{\frac{716a^2bd^2 + 176a^2db^2 + 64a^2b^3 + 169a^2d^3}{b}} - q + \frac{11ad + 8ab}{16d}$. And besides, PM $= y - \frac{13ad}{16b}$. Therefore we shall have the analogy, $\frac{716a^2bd^2 + 176a^2b^2d + 64a^2b^3 + 169a^2d^3}{256bd^2} - q^2 + \frac{11daq + 8baq}{8d} - \frac{121a^2d^2 + 176a^2bd + 64a^2b^2}{256d^2}$. $yy - \frac{13ady}{8b} + \frac{169a^2d^2}{256b^2} :: \frac{1}{d} . \frac{1}{b} :: b . d.$ And therefore the equation $\frac{595a^2bd^2}{256bd} - dqq + \frac{11adq + 8abq}{8} = byy - \frac{13ad}{8}y$. But, by the equation to the parabola, it is $yy = aq$. Therefore, substituting, instead of q and qq, their values $\frac{yy}{a}$ and $\frac{y^4}{a^2}$, and ordering the equation, dividing by d and multiplying the terms by aa, it will be $y^4 - \frac{11aayy}{8} - \frac{13a^3y}{8} - \frac{595a^4}{256} = 0$. But, by making the substitution of $z = \frac{ay}{f}$, (or making $y = z$, for $a = f$,) it will be $z^4 - \frac{11}{8}a^2z^2 - \frac{13}{8}a^3z - \frac{595}{256}a^4 = 0$, which is the reduced equation; to the roots of which adding $\frac{1}{4}a$, they will be the roots of the equation proposed.

It was indeed unnecessary to take all this trouble about an Example, which, by nature, is not solid but plane; for the proposed equation is divisible by $x + a$, and by $x - 2a$. But, however, it will serve to show the use of this method.

202. Equa-

202. Equations of the fifth and sixth degree are constructed by means of two *loci*, one of the third degree, and the other a conic section. Equations constructed of the fifth and sixth degree.

EXAMPLE VI.

Let the equation be $x^5 + aax^3 - a^5 = 0$. I take the *Apollonian* parabola $xx = ay$, and making the substitutions, there arises the second *locus* $xyy + axy - a^3 = 0$.

Hitherto I have not mentioned the construction of *loci* above the Conic Sections, having reserved the treating on these for the following Section; for thus order necessarily required. At present, therefore, let there be supposed, and also let there be described, a curve with three branches MCH, FE, PNO, whose equation is $xyy + axy - a^3 = 0$, in which AB represents the x's, and BC the y's. With vertex A, axis AL, and parameter $= a$, let the *Apollonian* parabola RAC be described. It will meet the branch MCH in the point C; and therefore, letting fall the perpendicular CB, it will be $AB = x$, the real and positive root of the proposed equation, and the other four will be imaginary. If we desire to construct the same equation by means of an hyperbola between it's asymptotes, and also by a *locus* of the third degree, make $xy = aa$, and, by substituting, it will be $x^3 + aax - ayy = 0$.

Fig. 105.

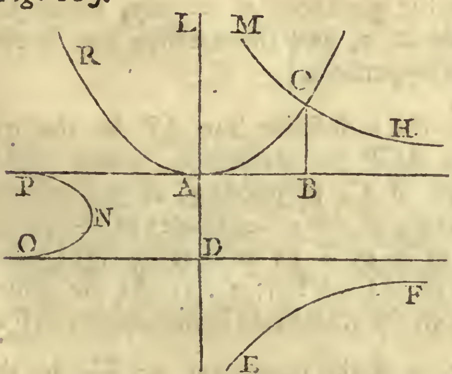
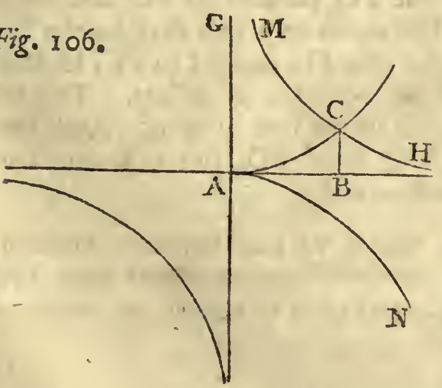


Fig. 106.

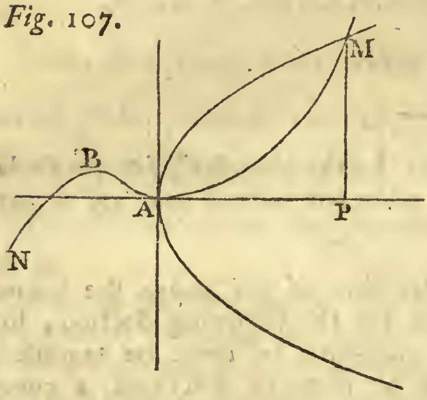


To axis AB, with abscis $AB = x$, and ordinate $BC = y$, let the curve CAN be described, which is the *locus* of the equation $x^3 + aax - ayy = 0$. And between the asymptotes AB, AG, let the hyperbola MCH of the equation $xy = aa$ be described, taking the x's on the same axis AB; this will cut the first curve in the point C, from whence letting fall the perpendicular CB, it will be $AB = x$, the root of the equation proposed.

Now

Now I multiply the ſame equation by $x = 0$, in order to reduce it to the ſixth degree, and I ſhall have $x^6 + aax^4 - a^5x = 0$. I take the ſame *locus* to the parabola $xx = ay$, and, making the ſubſtitution, there ariſes the ſecond *locus* $y^3 + ay^2 - aax = 0$, which is the curve NBAM, taking the abſciſſæ AP $= y$, and the ordinates PM $= x$.

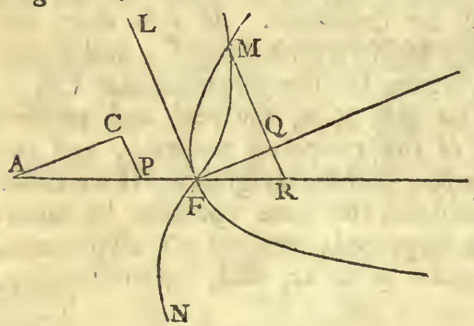
Fig. 107.



With vertex A, to the axis AP, with parameter $= a$, the *Apollonian* parabola AM of the equation $xx = ay$ being deſcribed, it will cut the ſaid curve in the vertex A, which gives us one of the roots $x = 0$, the ſame that was introduced into the equation. Beſides, it will cut it in the point M, and letting fall the perpendicular MP, it will be another root of the equation.

If we deſire to make uſe of the firſt cubic parabola $x^3 = aay$, make the ſubſtitution in the equation $x^6 + a^2x^4 - a^5x = 0$, and there ariſes the ſecond *locus*, $yy + xy - ax = 0$, to the *Apollonian* hyperbola.

Fig. 108.



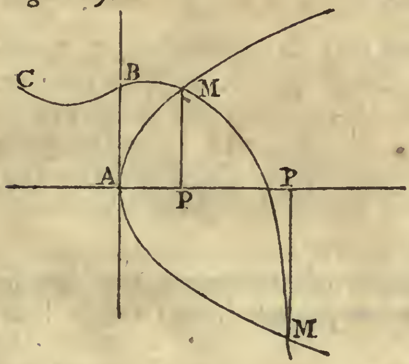
On the indefinite line AP let the triangle ACP be deſcribed, being right-angled at C, (ſuppoſing, if you pleaſe, the angle of the co-ordinates of the equation $yy + xy - ax = 0$ to be right,) and let it be AC . CP :: 2 . 1. At the centre A, with the tranſverſe ſemidiameter AF $= a\sqrt{5}$, with the parameter $= \frac{2a}{\sqrt{5}}$, let the *Apollonian* hyperbola FM be deſcribed; then from the point F drawing the indefinite line FQ parallel to AC, and taking any line FQ $= x$, and QM parallel to CP and equal to y, this ſhall be the *locus* of the equation $yy + xy - ax = 0$. To the axis FL parallel to PC, let there be deſcribed the cubical parabola NFM of the equation $x^3 = aay$. This will cut the hyperbola in the vertex F, which gives us the root $x = 0$. And from the point M letting fall the perpendicular MQ upon FQ, this will determine the other root FQ of the equation $x^6 + aax^4 - a^5x$.

If our equation had had the ſecond term, and if we had deſired to make uſe of the cubic parabola, a ſecond *locus* of the third degree would have been derived. Therefore we ought to make the ſecond term to vaniſh, or make uſe of another *locus*.

EXAMPLE VII.

Let the equation of the ſixth degree be this, $x^6 + ax^5 + a^5x - a^6 = 0$. I take the *locus* to the *Apollonian* parabola $xx = ay$. Making the ſubſtitutions, the ſecond *locus* will be $y^3 + xy^2 + aax - a^3 = 0$, which is the curve CBM, taking the abſciſſes AP $= y$, and the ordinates PM $= x$.

Fig. 109.

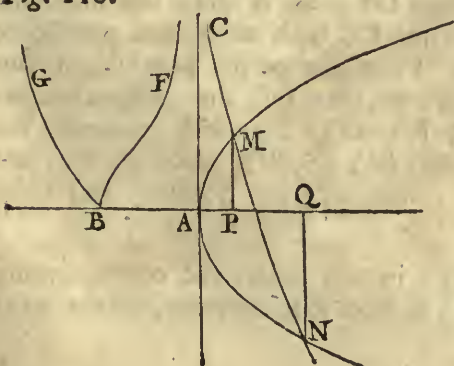


At the vertex A, with parameter $= a$, to the axis AP, let the parabola MAM of the equation $xx = ay$ be deſcribed. This will cut the ſaid curve in two points M, M, from whence drawing to the axis the perpendiculars MP, MP, they will be the two roots of the propoſed equation, of which one will be poſitive, the other negative, and the four others will be imaginary.

203. Equations of the ſeventh degree are conſtructed by means of two *loci* of the third, or elſe by one of the ſecond and one of the fourth. But, becauſe, by multiplying them by the unknown quantity, they are reduced to the eighth degree, and thoſe of the eighth are conſtructed in like manner by a *locus* of the ſecond, and another of the fourth, I ſhall content myſelf with giving an Example of thoſe of the eighth degree. —of ſevent[h and] eighth [de]gree.

EXAMPLE VIII.

Fig. 110.



Let the equation of the eighth degree be $x^8 + ax^7 + a^3x^5 - a^8 = 0$. Taking the equation to the *Apollonian* parabola $xx = ay$, and making the ſubſtitutions, there ariſes the ſecond *locus* $x^4 + xy^3 + axy^2 - a^4 = 0$, which is the curve GBFCMN, taking the abſciſſes AP $= y$, and the ordinates PM $= x$. With vertex A, parameter $= a$, and axis AP, let the parabola of *Apollonius*, MAN, be deſcribed, belonging to the equation $xx = ay$.

 This

This will meet the aforesaid curve in the points M, N, from which drawing the perpendiculars MP, NQ, to the axis, they will be the two real roots, one positive, the other negative, of the proposed equation, and the others are imaginary.

—or of higher degrees.

204. Here it may be observed, that equations of the ninth degree, (as well as those of the eighth, reduced to the ninth by multiplying them by the unknown quantity,) may always be constructed by means of two *loci* of the third degree, making the second term to vanish, if it have one.

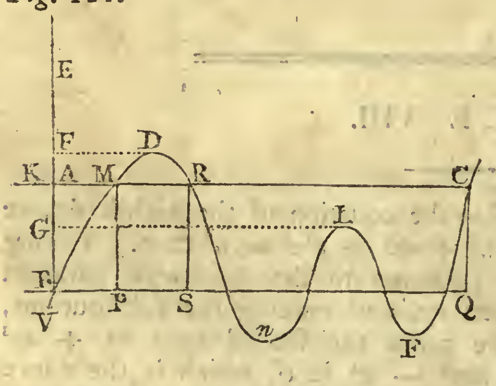
Thus, in general, equations of the tenth degree may be constructed by means of a *locus* of the third degree, and one of the fourth. And, in like manner, those of eleven and twelve degrees, observing to reduce those of eleven to twelve, by multiplying them by the unknown quantity, and by making the second term of an equation of the twelfth degree to vanish, if it have any. And the like is to be understood of equations of higher degrees.

All equations may be constructed by a *locus* of the same degree.

205. Another manner of constructing equations of any degree may be, by means of a *locus* of the same degree as the equation proposed, and a right line; after the following manner.

Let it be an equation of the fifth degree, $x^5 - bx^4 + acx^3 - aadx^2 + a^3cx - a^4f = 0$. Let the last term a^4f be transposed, and taking one of the linear divisors, f, of the last term, make it equal to z, for example, and divide the equation by a^4; then we shall have $z = \frac{x^5 - bx^4 + acx^3 - a^2dx^2 + a^3cx}{a^4}$.

Fig. 111.



On the indefinite line BQ describe the curve BMDRNLFC of this last equation, taking the x's from the fixed point B. The ordinates PM, SR, &c. will be equal to z; and therefore, from the point B draw the right line BA $= f$, parallel to the ordinates PM, SR, and through the point A draw the indefinite right line KC both ways, and parallel to BQ. From the points in which it cuts the curve, let fall the perpendiculars MP, RS, CQ; they will determine the abscisses BP, BS, BQ, which are the roots of the equation proposed. Those from A towards Q are positive, and those the contrary way are negative.

If the right line AC shall touch the curve in any point, the corresponding abscis x shall denote two equal roots; and if it meet it in no point, all the roots will be imaginary.

If the last term had had it's sign positive, we must have made $z = -f$, and therefore must have taken $BA = -f$, that is, below the point B, or on the negative side.

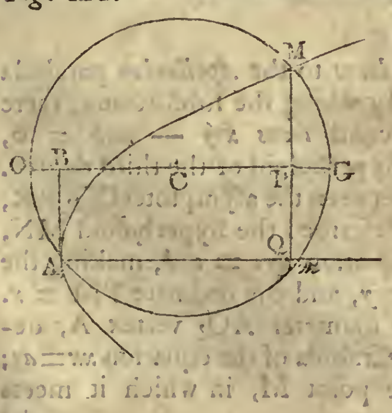
206. This method may be of use to verify constructions, which have been made by the combination of two curves, by confronting with each other the number of the roots, whether real or imaginary, positive or negative, which are found by each method. Use of this method.

PROBLEM I.

207. Between two given quantities, to find as many mean geometrical proportionals as shall be required. A Problem to exemplify this method.

Let the two given quantities be a and b, and let x be the first of the mean proportionals; they will form this geometrical progression following: $a,\ x,\ \frac{x^2}{a},\ \frac{x^3}{a^2},\ \frac{x^4}{a^3},\ \frac{x^5}{a^4}$, &c. Now, if we would have two mean proportionals, the fourth term of the progression must be b, and therefore we should have this equation $\frac{x^3}{a^2} = b$, or $x^3 = a^2b$. To construct this by the help of a parabola and a circle, I reduce it to the fourth degree, by multiplying it by $x = 0$, and then it will be $x^4 - a^2bx = 0$. Taking the *locus* to the parabola $xx = ay$, and making the substitutions, there arises the second *locus* $yy - bx = 0$, which is also to the parabola; from which subtracting the first, there arises a third, $yy - bx - xx + ay = 0$, which is to the hyperbola; or, adding the first and second together, there arises, lastly, $yy - bx + xx - ay = 0$, a *locus* to the circle, supposing the co-ordinates to contain a right angle.

Fig. 112.



With radius $CG = \frac{1}{2}\sqrt{aa + bb}$ let the circle OMA be described; and taking $CB = \frac{1}{2}a$, let fall the perpendicular $BA = \frac{1}{2}b$, which will meet the circle in the point A; from whence drawing AQ parallel to the diameter OG, and taking any portion $AQ = y$, it will be $QM = x$, and this circle will be the *locus* of the equation $yy - bx + xx - ay = 0$. With vertex A, axis AQ, and parameter $= a$, let the parabola $xx = ay$ be described, which will meet the circle in the point M; from whence letting fall the perpendicular MQ, it will be the root of the proposed equation. For the vertex of the parabola,

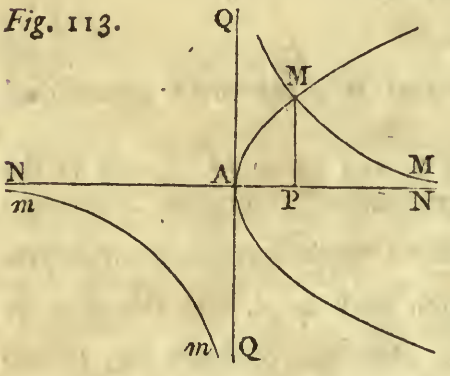
rabola, being in the periphery of the circle, will give the other root $x = 0$, which was introduced, and the other two are imaginary.

Taking the firſt and ſecond equation, the Problem will be conſtructed by means of two *Apollonian* parabolas. Taking the firſt and third, it will be conſtructed by means of the parabola, and the hyperbola referred to it's diameters.

ſame rwiſe .ructed.

208. Without multiplying the equation $x^3 - aab = 0$, it may be conſtructed by a parabola and an hyperbola between it's aſymptotes; for, taking the *locus* $xx = ay$, and making the ſubſtitutions, there ariſes $xy = ab$.

Fig. 113.



Between the aſymptotes NN, QQ, let there be deſcribed the hyperbola MM with the conſtant rectangle ab, and let AP be the y's, and PM the x's. To the axis AP, with the vertex A, the parameter $= a$, let the parabola AM be deſcribed; from the point M, in which it cuts the hyperbola, drawing the ordinate MP, it ſhall be the root of the propoſed equation.

The firſt of the two mean proportionals being thus found, we have alſo the ſecond, being equal to the abſciſs AP $= y = \frac{xx}{a}$.

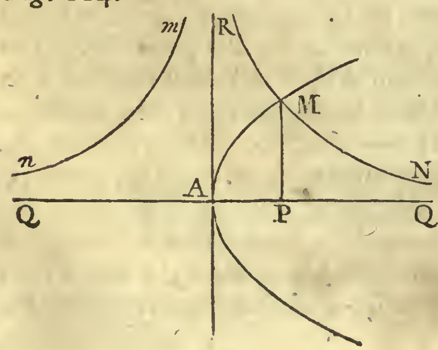
A ſimpler caſe of the ſame Problem.

209. To find three mean proportionals, the Problem becomes plane; for, having found, geometrically, that in the middle, which let be m for example, the mean between a and m will be the firſt of the three, and the mean between m and b will be the third.

Carried higher.

210. Let it be required to find four mean proportionals; then b ought to be the fixth term of the progreſſion, and therefore we ſhall have the equation $x^5 = a^4b$, or $x^5 - a^4b = 0$.

Fig. 114.



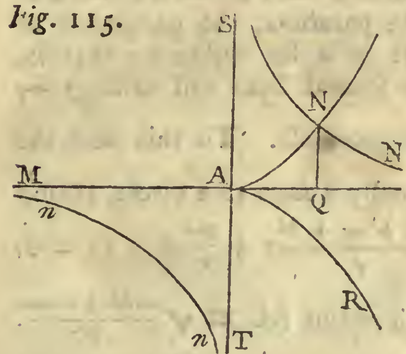
I take the *locus* to the *Apollonian* parabola $xx = ay$, and making the ſubſtitution, there ariſes the ſecond *locus* $xyy - aab = 0$, which is an hyperboloid of the third degree. Therefore, between the aſymptotes QQ, RR, let there be deſcribed the hyperboloid MN, mn, of the equation $xyy = aab$, making the abſciſs AP $= y$, and the ordinate PM $= x$. Now, to the diameter AQ, vertex A, deſcribing the parabola of the equation $xx = ay$; and from the point M, in which it meets the

the hyperboloid, drawing the ordinate MP, it ſhall be the root of the equation $x^5 - a^4b = 0$, and the firſt of the mean proportionals required; by means of which the others may be found alſo.

211. Alſo, the Problem may be conſtructed by means of the *Apollonian* hyperbola between it's aſymptotes, and the ſecond cubical parabola. Conſtructed otherwiſe.

Make therefore $aa = xy$, the *locus* to the aforeſaid hyperbola; and, inſtead of a^4, ſubſtituting it's value $xxyy$, there ariſes the *locus* $x^3 = byy$, which is the ſecond cubical parabola.

Fig. 115.



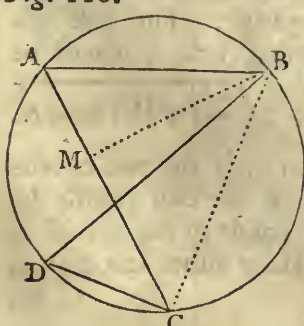
With the axis AQ let there be deſcribed the ſecond cubic parabola RAN, in which AQ gives the x's, and QN the y's. And between the aſymptotes ST, MQ, let there be deſcribed the hyperbola NN. And from the point N, in which it meets the parabola, let the ordinate NQ be drawn. Then will AQ be the root of the propoſed equation, that is, the firſt of the four mean proportionals.

212. To find five mean proportionals the Problem is only cubical. For, having found the middle term geometrically, which, for example, let be m; to have the two means between a and m, is a cubical or ſolid Problem, as has been ſeen juſt now. Extended to higher caſes.

It may be eaſily perceived with a little attention, that the Problem for finding ſix mean proportionals may be conſtructed, either with a *locus* of the ſecond, and one of the fourth degree, or with two of the third degree. But to find ſeven ſuch, having found the middle one, the Problem will be reduced to the finding of three. And in the ſame way of reaſoning, we may go on to greater numbers.

PROBLEM II.

Fig. 116.



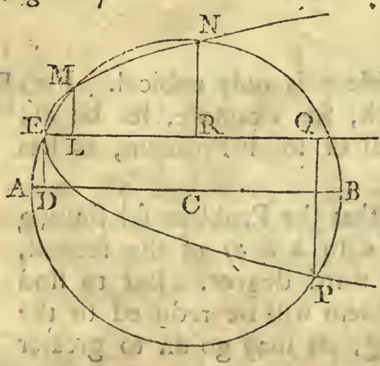
213. In the circle ABCD, having two chords given, BA, DC, which proceed from the extremities of the diameter BD, and the third chord AC being given alſo; to find the diameter BD. The *loci* exemplified by another Problem.

Draw the chord BC, and make $AB = a$, $AC = b$, $DC = c$, and the diameter $BD = x$; and let fall the perpendicular BM upon the chord AC. Becauſe the angle in the ſemicircle BCD is a right one, it will be $BC = \sqrt{xx - cc}$; and becauſe the angles BAC, BDC, inſiſt

insist on the same arch BC, and also the angles M, BCD, are right angles, the two triangles BCD, BAM, will be similar. Wherefore it will be AM $= \frac{ac}{x}$. But, by *Euclid*, ii. 13, it is BCq = ABq + ACq — 2CA × AM; therefore the equation will be $xx - cc = aa + bb - \frac{2abc}{x}$, that is, $x^3 - ccx - aax - bbx + 2abc = 0$.

I multiply it by x, to reduce it to the fourth degree, and thus construct it, by means of the parabola and the circle. It is then $x^4 - c^2x^2 - a^2x^2 - b^2x^2 + 2abcx = 0$. Taking therefore the *locus* to the parabola, the parameter of which is the least of the three chords, which let be c for instance; that is, taking $xx = cy$, make the substitutions, and the second *locus* will arise $yy - \frac{ccy + aay + bby}{c} + \frac{2abx}{c} = 0$, which is also to the parabola. To this add the first equation $xx - cy = 0$, and we shall have finally a *locus* to a circle, taking the co-ordinates at right angles, that is, $yy - \frac{2cc + aa + bb}{c}y + \frac{2ab}{c}x + xx = 0$.

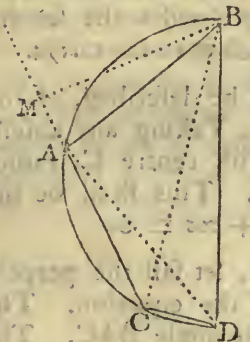
Fig. 117.



Therefore, with radius AC $= \sqrt{\frac{aabb + ccmm}{cc}}$, (for brevity-sake writing m for $\frac{2cc + aa + bb}{2c}$,) draw the circle AMBP, and taking CD $= m$, from the point D raise the perpendicular DE $= \frac{ab}{c}$, which will terminate in the periphery of the circle at the point E; and drawing the indefinite line EQ parallel to the diameter AB, upon this line take any how EL $= y$, the corresponding ordinate will be LM $= x$, and this circle is the *locus* of the equation. With vertex E, axis EQ and parameter $= c$, let the parabola of the equation $xx = cy$ be described. This will cut the circle at the vertex in the point E, which will give the introduced root $x = 0$. It will cut it besides in the three points M, N, P, from whence, to the right line EQ letting fall the perpendiculars ML, NR, PQ, they shall be the three roots of the equation proposed, two positive and one negative. The first positive root ML cannot serve for this Problem; for, supposing $y = c$, it will be in the parabola, $x = c$, and in the circle, $x = -\frac{ab}{c} + \sqrt{\frac{aabb}{cc} + bb + aa + cc}$. But this value of x, relatively to the circle, is greater than c, if the two chords a, b, be not equal to each other; and it is equal to c, if the two chords be equal. Wherefore the point in the parabola which corresponds to the abscis c, either falls in M, or falls within the circle. Therefore ML is either less than c, or, or, at moſt, is equal to it, and therefore muſt needs be leſs than either of the chords a, b, and conſequently cannot be the diameter of the circle.

The ſecond poſitive root RN will ſupply us with the diameter required. The negative root QP ſupplies us with a diameter for another caſe; that is, when the two chords which terminate at the diameter are drawn from the ſame ſide, as in Fig. 118. For, doing the ſame things as above, draw likewiſe the chord AD. The angle DAB being right, the two angles DAC, MAB, will be equal to a right angle. But alſo, the two angles MAB, MBA, are equal to a right angle; therefore MBA = DAC = CBD, as inſiſting on the ſame arch DC. Hence the two triangles CBD, MBA, are ſimilar, and therefore MA = $\frac{ac}{x}$; but, by *Euclid*, ii. 12, it will be $CBq. = CAq. + BAq + 2CA \times AM$; whence the equation $xx - cc = bb + aa + \frac{2abc}{x}$, that is, $x^3 - ccx - bbx - aax - 2abc = 0$; the conſtruction of which is the ſame as the preceding, except that now, the laſt term being negative, we muſt draw DE (Fig. 117.) the negative way, becauſe the axis of the parabola will be below the diameter of the circle; and the two poſitive roots in the firſt caſe are negative in this, and the negative becomes poſitive.

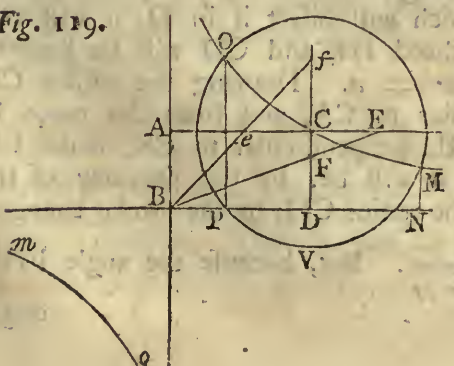
Fig. 118.



And becauſe the ſecond term is wanting in both the equations, it proceeds from thence, that the two poſitive roots in the firſt caſe are equal to the negative, and the poſitive in the ſecond is equal to the two negative. Hence we learn that the firſt of the three roots, which gave us no ſolution of the Problem, yet however belonged to it, as being the difference of the two diameters.

PROBLEM III.

Fig. 119.



214. The rectangle ACDB being given, in the ſide AC produced to find the point E, ſo that, drawing the right line BE from the angle B, the intercepted line EF may be equal to a given right line c. Another geometrical Problem.

When a ſquare is given inſtead of the rectangle ABDC, the Problem is plane, and has been already ſolved in Sect. IV. § 176. But, ſuppoſing ABDC to be a rectangle,

rectangle, it changes the nature of the Problem, and makes it solid. Therefore, making AB $= a$, BD $= b$, DF $= x$, and repeating the argumentation in the place above cited, we shall have an equation of the fourth degree, which is this;

$$x^4 - 2ax^3 + aax^2 - 2abbx + aabb = 0.$$
$$+ bbx^2$$
$$- ccx^2$$

To construct this by an hyperbola between the asymptotes, combined with a circle, I put $ab = zx$, and making the substitutions, there arises the second *locus* $xx - 2ax + aa + bb - cc - 2bz + zz = 0$, which is to the circle.

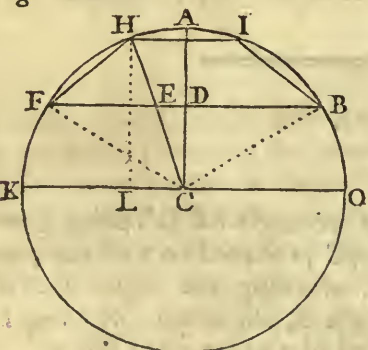
Between the asymptotes BA, BD, let the hyperbola OM be described, of the equation $zx = ab$, which shall pass through the point C. Taking any abscissa BP, BN $= z$, the ordinate will be PO, NM $= x$. With centre C, radius equal to the given line c, let the circle OMV be described. This shall be the *locus* of the equation $xx - 2ax + aa + bb - cc - 2bz + zz = 0$.

From the points O, M, in which this cuts the hyperbola, let fall the perpendiculars OP, MN; they shall be the two positive roots of the equation. The lesser will serve for the Problem in the case proposed, of the angle BAC. The greater for the angle ACf. And if the given line c be such, that the circle cannot reach to cut the opposite hyperbola *mo*, the other two roots will be imaginary. But if it shall cut it, they will be real and negative, and will serve for the angle ACD.

PROBLEM IV.

A Problem for angular sections.

Fig. 120.

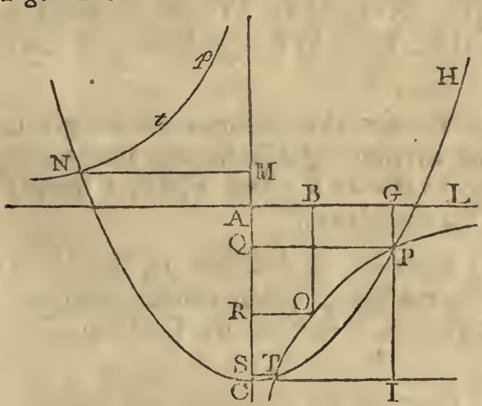


215. To divide a given angle FCB, or arch FAB, into three equal parts.

Let H, I, be the points of division required; then the chords FH, HI, IB, ought to be equal: and the arch FAB being given, it's chord FB will also be given, which let be equal to $2f$. Then, drawing the radius CA $= r$ perpendicular to FB, which will bisect it in D, it will also bisect the chord HI, and CD will be known, which make $= a$. Drawing the radius CK perpendicular to CA, and from the point H drawing HL perpendicular to CK, make CL $= y$, and it will be, by the property of the circle, HL $= \sqrt{rr - yy}$. And drawing the radius CH, by the similar triangles HLC, CDE, we shall have DE $= \frac{ay}{\sqrt{rr - yy}}$. But, because the angle FHC ought

ought to be equal to the angle CHI, by the conditions of the Problem, and CHI = CED by the parallels FB, HI, and CED = FEH; then FHC = FEH, and therefore FE = FH. But FH = HI = $2y$, therefore FE = $2y$. And the whole line FD = $2y + \frac{ay}{\sqrt{rr - yy}}$. But FD = f; therefore $2y + \frac{ay}{\sqrt{rr - yy}} = f$; and taking away the afymmetry, it will be $y^4 - fy^3 + \frac{1}{4}ffyy + \frac{1}{4}aayy - rryy + frry - \frac{1}{4}ffrr = 0$; or, becaufe $rr = ff + aa$, it is $y^4 - fy^3 - \frac{3}{4}rry^2 + frry - \frac{1}{4}ffrr = 0$, an equation of the fourth degree, which may be conftructed after the manner already explained, making ufe of fuch conical *loci* as fhall be moft agreeable. But this equation is divifible by $y - f$, and the quotient is the equation $y^3 - \frac{3}{4}rry + \frac{1}{4}frr = 0$, which I fhall conftruct by a parabola, and an hyperbola between the afymptotes. Make therefore $yy = rz$, and making the fubftitutions, it will be $zy - \frac{3}{4}ry + \frac{1}{4}fr = 0$, an equation to the hyperbola.

Fig. 121.



Make AR = $\frac{1}{2}r$, and AB = $\frac{1}{2}f$. Producing AR, AB, each way indefinitely, between them, as afymptotes, let the hyperbola TP*tp* be defcribed, which fhall pafs through the point O. Then taking RC = $\frac{1}{4}r$, and from the point C drawing the indefinite line CI parallel to AL, take any line whatever, CI = y, and it will be IP = z, and the hyperbola will be the *locus* of the equation $zy - \frac{3}{4}ry + \frac{1}{4}fr = 0$. With vertex C, diameter CM, and parameter = r, let the parabola NCH be defcribed; it will cut the hyperbola in three points T, P, N, from whence drawing the lines TS, PQ, NM, parallel to AL, thefe fhall be the three roots of the equation.

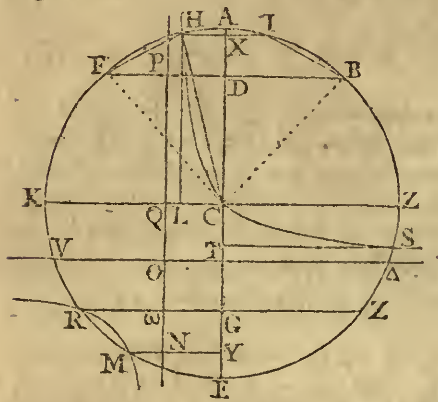
It is plain that the parabola will cut the hyperbola TP in the points T, P, becaufe, it being CR = $\frac{1}{4}r$, putting this value inftead of z in the equation to the parabola, $yy = rz$, it will give us $y = \frac{1}{2}r$. But $\frac{1}{2}r$ is always greater than $\frac{1}{2}f$, and therefore the ordinate in the parabola, which correfponds to the point R, will always be greater than RO; and therefore the parabola will pafs within the hyperbola.

Now, becaufe the circle is given in the Problem, it will be much more convenient to make ufe of this for the conftruction, by introducing it, firft, to be added to the final equation, and that by putting the line HL (Fig. 120.) or $\sqrt{rr - yy} = z$. Then it will be DE = $\frac{ay}{z}$, and DF = $2y + \frac{ay}{z}$, and

 therefore

therefore the equation is $2y + \frac{ay}{z} = f$, that is, $2yz + ay = fz$, a *locus* to the hyperbola between the asymptotes.

Fig. 122.



Bisecting DF in P, through the point P draw the indefinite line PN parallel to AC, and taking $QO = \frac{1}{2}a$, through the point O draw the indefinite line VΔ parallel to KC. Between the asymptotes PN, VΔ, describe the hyperbola whose rectangle is $\frac{1}{4}af$, which shall pass through the point C; and taking the y's on the line CQ, positive towards the point K, the corresponding ordinates shall be z, and the hyperbola be the *locus* of the equation $2zy + ay - fz = 0$.

This will cut the circle in four points H, R, M, S, from which drawing perpendiculars HX, RG, MY, ST, to AC, these shall be the roots of the equation, three, HX, RG, MY, positive, and one, ST, negative.

It is plain that the root HX, or CL, serves for the division of the given arch FAB; and the root YM serves for the division of FMB, the remainder to the whole circle. For, if I had proposed to divide the arch FMB, I should have had the same equation, and therefore the same *locus*.

The root RG serves to no purpose, but, however, it informs us, that it is equal to f, or that by which the equation is divisible, which results from the two *loci* $rr - yy = zz$, and $2zy + ay - fz = 0$; that is, the solid equation found before.

Now, to demonstrate it, taking $O\omega = \frac{1}{2}a = OQ$, the corresponding ordinate of the circle will be $GR = f$. But $\omega G = PD = \frac{1}{2}f$; therefore $\omega R = \frac{1}{2}f$. But the constant rectangle of the hyperbola is $\frac{1}{4}af$; therefore the hyperbola will cut the circle in the point R, and therefore it will be the root which corresponds to this point.

The other root TS serves for the division of the whole circle into three equal parts, which may be demonstrated in this manner.

Because FD = RG, the arches FK, KR, will be equal; and therefore, RG being produced to Z, the arches FAB, RMZ, will be equal. Therefore FR, or BZ, will be half the difference of the two arches FAB, FMB. But if we should solve the Problem relatively to the arch BZ, we should find the same hyperbola HCS, and ZS would be a third part of the arch BZ, that is, a third part of half the difference of the arches FAB, FMB; and therefore BS is a third part of the said difference. But HB is two-thirds of FAB, and therefore one-third one-third of the ſum of the two arches FAB, RMZ. Therefore the ſum of HB and BS, that is, the arch HS, will be a third part of the whole circle.

Q. E. D.

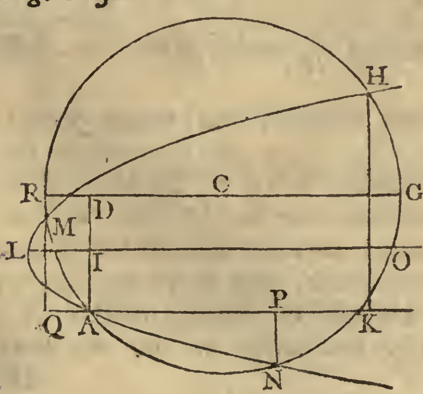
216. This Problem has been reſolved before, at § 110, but after another manner. There it is ſeen, that, in the caſe wherein the given angle is a right angle, the Problem will be plane. In the other two caſes, of an obtuſe or acute angle, we arrived at theſe two cubic equations, $2bx^3 - 3aax^2 + a^4 = 0$, and $2bx^3 + 3aax^2 - a^4 = 0$. Other caſes of this Problem conſtructed.

But if it be conſidered, that in the firſt equation, which ſerves for the obtuſe angle, taking x negative, it will be changed into the ſecond, which ſerves for the acute angle; it will be ſufficient to conſtruct the equation for the firſt caſe, becauſe the negative root of this will give the ſolution for the other caſe.

Therefore I multiply the firſt equation by $x = 0$, in order to reduce it to the fourth degree, and I divide it by $2b$; then it will become $x^4 - \frac{3aax^3}{2b} + \frac{a^4x}{2b} = 0$.

I take the equation to the parabola $xx - \frac{3aax}{4b} = ay$, and ſquaring it, it will be $x^4 - \frac{3aax^3}{2b} + \frac{9a^4x^2}{16bb} = aayy$. Then, inſtead of the two firſt terms, ſubſtituting their value, it will be $yy - \frac{9aaxx}{16bb} + \frac{aax}{2b} = 0$. Here, inſtead of xx, I ſubſtitute it's value $ay + \frac{3aax}{4b}$, and I ſhall have the equation $yy - \frac{9a^3y}{16bb} - \frac{27a^4x}{64b^3} + \frac{aax}{2b} = 0$; to which adding the firſt, $xx - \frac{3aax}{4b} - ay = 0$, it will be finally $yy - \frac{9a^3y}{16bb} - \frac{27a^4x}{64b^3} + \frac{a^2x}{2b} + x^2 - \frac{3a^2x}{4b} - ay = 0$, an equation to the circle, taking the co-ordinates at right angles.

Fig. 125.

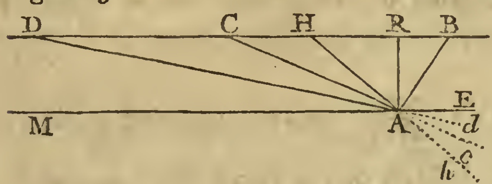


With radius $CG = \sqrt{mm + nn}$, (making, for brevity, $\frac{9a^3 + 16abb}{16bb} = 2m$, and $\frac{27a^4 + 16a^2b^2}{64b^3} = 2n$,) let the circle MNH be deſcribed, and taking $CD = m$, from the point D draw DA perpendicular to CD, and equal to n. This will meet the periphery of the circle in the point A. Through this point A draw AK parallel to RG; and, taking any line at pleaſure, $AK = y$, the correſponding ordinate will be $KH = x$, and the circle will be the *locus* of the equation.

On

On the right line AD take $AI = \frac{3aa}{8b}$, and through the point I drawing LO parallel to AK, let there be taken a portion of it, $IL = \frac{9a^3}{64bb}$, and with vertex L, axis LO, and parameter $= a$, let there be deſcribed the *Apollonian* parabola ALH. From the point A taking the abſciſs y on the axis AK, the correſponding ordinates will be $KH = x$, and the parabola will be the *locus* of the equation $xx - \frac{3aax}{4b} = ay$; this will meet the circle in four points, A, M, H, N. The point A will give the introduced root $= 0$. The three perpendiculars, QM, PN, KH, to AK, will give the three roots of the equation. The firſt poſitive root, QM, will ſerve for the obtuſe angle. The ſecond, PN, which is negative, will ſerve for the acute angle. The third, KH, will ſerve for the diviſion, into three equal parts, of that angle which is the difference between the given angle and a right angle.

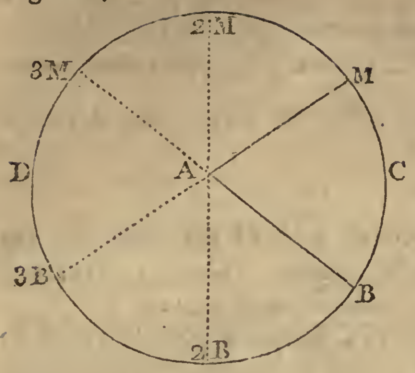
Fig. 123.



Now, to ſhow that this is true, let the given angle be MAB. Let AH be perpendicular to AB; and let us divide the angle MAH into three equal parts, which is the difference between the given angle MAB, and the right angle HAB. Suppoſe it ſo divided by the right lines AC, AD, and repeating the reaſoning of § 110, it will be $AC = CD$, and the triangle ACH will be ſimilar to the triangle DAH, and therefore we ſhall have the analogy, CH . HA :: HA . DH. Naming the quantities, therefore, as in § 110, $AB = a$, $BR = b$, and $BC = x$, it will be $RC = x - b$, $BH = \frac{aa}{b}$, $CH = x - \frac{aa}{b}$, $AR = \sqrt{aa - bb}$, $HA = \frac{a}{b}\sqrt{aa - bb}$, $AC = \sqrt{aa + xx - 2bx}$, $DH = x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}$. Therefore, ſubſtituting theſe analytical values in the foregoing proportion, it will be $x - \frac{aa}{b} \,.\, \frac{a}{b}\sqrt{aa - bb} :: \frac{a}{b}\sqrt{aa - bb} \,.\, x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}$. Whence the equation $\frac{aa}{bb} \times \overline{aa - bb} = \overline{x - \frac{aa}{b}} \times \overline{x - \frac{aa}{b} + \sqrt{aa + xx - 2bx}}$; which, being reduced, and finally divided by $aa - bb$, will be found to be $2bx^3 - 3aaxx + a^4 = 0$, the very equation which was to be conſtructed.

Beſides the angles leſs than two right ones, which inſiſt on arches leſs than a ſemicircle, and which the architects call *Entrant Angles*, there are alſo angles which are greater than two right ones, and which inſiſt on arches greater than a ſemicircle, and are called *Salient Angles*. The inclination of the two lines AB,

Fig. 124.



AB, AM, which point towards C, may be confidered as pofitive, and that negative which points towards D. As long as the inclination of the two lines AB, AM, fhall be pofitive, and fhall point towards C, fo long the angle MAB fhall be entrant, or lefs than two right angles, and fhall infift upon an arch, BCM, lefs than a femicircle. If the two lines A2B, A2M, fhall make a right line 2B2M, the inclination will be none at all. But if the inclination fhall become negative, the two lines A3B, A3M, winding towards D, then the angle 3MA3B will be changed into a falient angle, greater than two right ones, and will infift upon an arch, 3MC3B, greater than a femicircle. Therefore the trifection of any given angle may alfo include that of a falient angle.

Now it is to be confidered, that, as the line AB (Fig. 123.) infifts upon the line MAE, whilft it forms the angle MAB, three other angles will confequently arife, that is, the entrant BAE, which, united to the given and alfo entrant angle MAB, makes up the two right angles; and the falient angles MAB, BAE, which, united to the correfponding entrant angles, complete the four right angles.

Wherefore the three roots of our equation, $2bx^3 - 3aax^2 + a^4 = 0$, ferve for the trifection of all the fore-mentioned angles. By means of the leaft pofitive root, the obtufe angle MAB is divided into three equal parts; and, by means of the negative, the acute angle BAE, as has been feen. Befides, it has been fhown, that the greater pofitive root ferves for the angle MAH; and this ferves alfo to trifect both the falient angles MAB, BAE. For, indeed, the falient angle BAE is equal to three right angles, together with the angle MAH. The third part, therefore, of the falient angle BAE muft be equal to one right angle, together with the third part of the angle MAH; and fuch is the angle CAB. In like manner, the falient angle MAB is equivalent to three right angles, taking away the angle MAH, or hAE; and confequently cAB will be it's third part, as being equal to the right angle hAB, taking away the angle hAc, a third part of the angle hAE.

217. Now, to divide the given angle into three equal parts, if I had made ufe of Prob. XIII. § 108, I fhould have come to the equation $x^3 - 3bx^2 - 3rrx + brr = 0$; and, multiplying by $x = 0$, it is $x^4 - 3bx^3 - 3rrx^2 + brrx = 0$. Wherefore, affuming the *locus* to the parabola $xx - \frac{3}{2}bx = by$, and doing the reft as ufual, we fhall have another *locus* to the circle, taking the co-ordinates at right angles. This will be $yy - \frac{26b^3y + 24brry}{8bb} - \frac{39b^3x + 28brrx}{8bb} + xx = 0$. The fame conftructed another way.

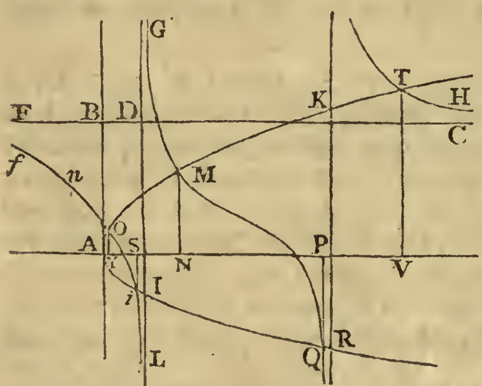
Thefe

Thefe two *loci* being defcribed and combined, will give the fame conftruction as in Fig. 125, differing only in the known quantities. For, in this cafe, the radius of the circle will be CG $= \sqrt{mm + nn}$; (making, for brevity-fake, $\frac{26b^3 + 24brr}{8bb} = 2m$, and $\frac{39b^3 + 28brr}{8bb} = 2n$,) and it will be CD $= m$, DA $= n$, AI $= \frac{3}{4}b$, and IL $= \frac{9}{16}b$.

This Problem raifed higher. 218. From the fame Problem we have a general method for dividing any given arch or angle into as many equal parts as we pleafe. Thus, to divide it into five equal parts, we fhall have this equation, $\frac{5r^4x - 10rrx^3 + x^5}{r^4 - 10rrxx + 5x^4} = b$, that is, $x^5 - 5bx^4 - 10rrx^3 + 10br^2x^2 + 5r^4x - br^4 = 0$.

To conftruct this, I take a *locus* to the *Apollonian* parabola $xx = ry$, and, making the fubftitutions, there arifes a fecond of the third degree, $xyy - 5byy - 10rxy + 10bry + 5rrx - brr = 0$, that is, $x = \frac{5byy - 10bry + brr}{yy - 10ry + 5rr}$.

Fig. 126.



Therefore, having defcribed the *locus* of the equation, which fhall be the curve with three branches, Fig. 126, that is, HT between the afymptotes RK, BC; GMQ between the afymptotes DI, KR; and *fni*L between the afymptotes DF, DI; in which, on the axis AV, are the *y*'s, and the correfponding ordinates are the *x*'s. With vertex A, parameter $= r$, and axis AV, if the parabola of the equation $xx = ry$ be defcribed, it will meet the curve in five points, O, M, T, *i*, Q, which will determine the five roots, *or*, *mn*, TV, S*i*, and PQ; three pofitive, and two negative, of the equation propofed.

—raifed ftill higher. 219. So, to divide an arch or angle given into any greater odd number of equal parts, other curves may be found, relative to the degree of the equation.

SECT.

SECT. V.

Of the Construction of Loci *which exceed the Second Degree.*

220. The Geometrical *Loci* may be constructed after two different manners; that is to say, by describing curves expressing equations which exceed the second degree; if we may call that describing, in each manner, which is rather tracing them out, so as to give some notion of such curves. Higher *loci* constructed two ways.

The first manner of tracing them is, by finding an infinite number of points. The second is, by means of other curves of an inferior degree, which are already described. Thus, a *locus* or equation of the third degree may be constructed by means of a right line and a conic section; a *locus* or equation of the fourth, by means of two conic sections; a *locus* or equation of the fifth, by means of a conic section and a *locus* of the third degree. And so on, as far as you please.

221. Now, as to the first manner, by an infinite number of points; first, the equation must be reduced in such manner, that one of the two unknown quantities, which shall seem fittest for the purpose, must be freed from fractions or co-efficients, must be of one dimension only, and placed alone on one side of the sign of equality; which may always be done by the methods explained in Sect. II. Then, in respect of such unknown quantity, (the other being considered as constant,) the equation must be of it's own nature plane, that is, must not exceed the second degree. As, for example, the equation $xyy + 2aay = x^3$, that is, $yy + \frac{2aay}{x} = xx$, which, managed by the rules for affected quadraticks, will give $y = \frac{-aa \pm \sqrt{x^4 + a^4}}{x}$. —first, by finding an indefinite number of points.

Equations being given or reduced in this manner, the way of constructing the *locus*, or curve expressed by it, consists in giving an arbitrary value to that unknown quantity which is included in the *homogeneum comparationis*; taking it from a fixed point on a right line, which serves as an axis or diameter, according as the angle of the co-ordinates is to be a right or an oblique angle. As in the equation $y = \frac{-aa \pm \sqrt{x^4 + a^4}}{x}$, if we should give to x a value at pleasure, by that means we ſhould have a congruous value of y alſo. Then, from the extremity of the aſſumed value of x having drawn the value of y, in the given angle of the co-ordinates, this would ſupply us with a point in the curve to be deſcribed. Another value that we may give to the ſame unknown quantity x will ſupply us with another y, and that with another point in the curve; and thus, one after another, by aſſigning different values to x, we ſhall have ſo many y's, or ſo many points of the curve. Now, the greater the number be of theſe points, ſo much the more exact will be the deſcription of the curve, and then only we can have it perfectly exact, when we take an infinite number of ſuch points, at due diſtances.

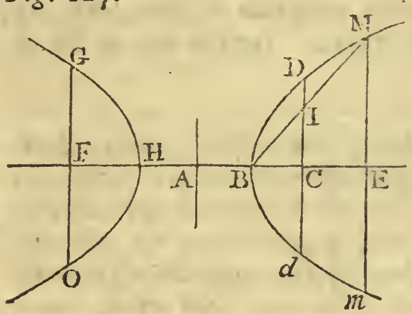
The ordinates to be at right angles to the abſciſs.

222. For the ſake of greater ſimplicity, I ſhall at preſent ſuppoſe, that theſe curves are referred to their axis, or that the angle of the co-ordinates is a right angle; for, in caſe the angle be oblique, no alteration will thence follow.

An Example of deſcribing the curve by points.

223. For the more eaſy underſtanding the application of this method, I ſhall take a ſimple example of a curve already known, that is, of the equilateral hyperbola $yy = xx - aa$, or $y = \pm \sqrt{xx - aa}$.

Fig. 127.



Let A be a fixed point, or the beginning of the x's, to be taken on the indefinite line AE. Firſt, then, I examine what ordinate correſponds to the point A, that is, what will y be when $x = 0$. Therefore, ſubſtituting 0 inſtead of x in the given equation, it will be found $y = \pm \sqrt{0 - aa}$, or y is imaginary and impoſſible. Therefore, to the point A there belongs no point of the curve. By making $x = 0$, if y had not come out imaginary, but only 0, the curve would have begun at the point A. It may be obſerved, that as often as x is leſs than a, the radical $\sqrt{xx - aa}$ will always be negative, and therefore y an imaginary quantity. Therefore, making AB $= a$, to every x leſs than AB an imaginary y will always correſpond, ſo that there will be no point in the curve. I take $x = a =$ AB, then $y = \pm \sqrt{aa - aa} = 0$; and therefore B will be a point in the curve, or rather, the curve will have it's origin in the point B. I take $x = 2a =$ AC, and it will be $y = \pm \sqrt{4aa - aa} = \pm \sqrt{3a^2}$, poſitive and negative. Therefore make CD poſitive, and Cd negative, each equal to $\sqrt{3aa}$, and D and d will be two points in the curve. I take $x = 3a =$ AE, and it will be $y = \pm \sqrt{8aa}$. Making therefore EM poſitive, and Em negative, $= \sqrt{8aa}$, and M, m, will be two points in the curve. And thus going on continually, by giving other values to x, we ſhall have the congruous values of y. And it is eaſy to perceive, that, as the x's increaſe, ſo the

 quantities

quantities $\sqrt{xx - aa}$ will perpetually increafe, that is, the values of y, both affirmative and negative. Thus, the curve will always proceed, enlarging and lengthening itfelf both above and below the axis; and, laftly, taking x infinite, becaufe, to fubtract a finite quantity from one that is infinite, is the fame thing as to fubtract nothing; therefore $\sqrt{xx - aa}$ will become $\sqrt{xx}$, or x, and we fhall have $y = \pm x$, and y pofitive and negative will be infinite, and therefore the curve will go on *ad infinitum*.

In even powers, the fign of the axis is ambiguous.

224. And becaufe, in the equation $y = \pm \sqrt{xx - aa}$, the unknown quantity x is raifed to an even power, that is, to the fquare; if we take x negative, the equation itfelf receives no alteration. Hence it is, that, if we affign negative values to x, or if we take it on the fide of A towards F, the fame curve would be defcribed as before, but in a contrary pofition with it's vertex H, it being $AH = AB$. And to no abfcifs x, pofitive or negative, taken between B and H, any real ordinate, pofitive or negative, will correfpond; that is, there will be no point of the curve.

To find where the curve cuts the axis.

225. Now it is plainly feen, that the given curve cuts the axis in no point out of the vertices B, H; for, as x increafes, y always increafes. Neverthelefs, it very often happens, that, befides the vertex, they cut it in other points, in which cafe y muft neceffarily become nothing. Therefore, to have thefe points, in the given equation we muft fuppofe $y = 0$, and find the values of x on this fuppofition, which will give us the points required. Wherefore, in the equation $yy = xx - aa$, fuppofing $y = 0$, it will be $xx = aa$, that is, $x = \pm a$. Therefore, in the points B, H, only, the curve will cut the axis, and not in any other.

The more points we take, the better.

226. If, between the points B, C, other values of x be taken, we fhall alfo have the correfponding values of y, that is, other points of the curve between B and D, as alfo, between B and d; fo that the more points we have, the more exact will be the defcription of the part BD, or Bd; but we can never have it perfect, unlefs the number of thofe points were infinite. And the fame may be faid of any other portion.

To find when a curve can have an afymptote.

227. Now it is plain, that if either of the two indefinite quantities be made infinite, and the other be neither infinite nor imaginary, but be either finite or equal to nothing, the firft indeterminate will be an afymptote to the curve, which will correfpond to fome determinate point of the value of the fecond. Therefore, to inquire if a curve have afymptotes, and where they are, it will fuffice to make y infinite, and to fee what value for x will then refult from the equation. Then, to make x infinite, and fee what value for y will thence refult. In the equation $y = \pm \sqrt{xx - aa}$, making y infinite, it will be $\sqrt{xx - aa} = \infty$, and therefore $xx - aa = \infty$, or $xx = \infty$, and therefore x is infinite; for the root of an infinite fquare muft alfo be infinite. So that y cannot be infinite

 except

except when x is infinite alſo; and therefore the axis of the y's cannot be an aſymptote. Making x infinite, $\sqrt{xx - aa}$ will be the ſame; for a finite quantity, added to or taken from an infinite quantity, can make no alteration; it will be now $y = \pm x$, or, if x be infinite, y will be ſo alſo, and it's axis cannot be an aſymptote.

—found by changing the equation.

228. But it is not ſo in the equation $ay + xy = bb$, which we otherwiſe know to belong to the hyperbola between it's aſymptotes. For, taking y infinite, the two terms $ay + xy$ will be infinite, and, in reſpect of them, the term bb will be nothing, and the equation will become $ay + xy = 0$, and, dividing by y, it is $x = -a$; ſo that, taking $x = -a$, the ordinate, which in this point is infinite, will be an aſymptote to the curve. Then, taking x infinite, becauſe the two rectangles ay, xy, having the ſame altitude y, are to each other as their baſes a, x, the ſecond muſt be infinitely greater than the firſt, or ay will be nothing in reſpect of xy. Therefore, expunging ay out of the equation, there will remain $xy = bb$, or $y = \frac{bb}{x}$. But x is infinite by ſuppoſition, therefore $y = \frac{bb}{\infty} = 0$. So that when $y = 0$, then x will be infinite, and therefore is an aſymptote to the curve.

Cautions to be obſerved in finding aſymptotes.

229. But here it muſt be obſerved, that this way of arguing takes place only in the caſe of aſymptotes parallel to the co-ordinates, and not otherwiſe. For the truth is, the hyperbola $yy = xx - aa$ has indeed it's aſymptotes, but which are not parallel to either of the co-ordinates; therefore, in this caſe, the preſent way of arguing cannot be applied, but there is need of a further artifice; which, as it depends on the Method of Infiniteſimals, I ſhall reſerve for another place.

To find if the curve be concave or convex towards it's axis.

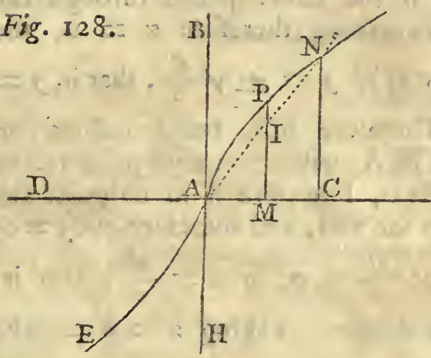
230. It remains to inquire, whether the ſaid curve $y = \pm \sqrt{xx - aa}$ be concave or convex towards it's axis; for which purpoſe, we muſt take from it's origin any abſciſs AE of a determinate value, and, by means of the given equation, we muſt find the value of the correſponding ordinate EM. Then, taking another abſciſs AC of a determinate value leſs than the former, we muſt find the value of the correſponding ordinate CD; and drawing the right line BM, which ſhall cut CD (produced, if occaſion) in I; and the lines AE, AC, being known, or BE, BC, and the ordinate EM, by the ſimilar triangles BEM, BCI, we ſhall find the value of CI; and if this be leſs than CD, the curve will be concave towards the axis AE, as is plain; but if it be greater, the curve will be convex. In the given Example, I take $x = AE = 3a$; then $y = \sqrt{8a^2}$. Again, I take $x = AC = 2a$; then $y = CD = \sqrt{3aa}$. Now, becauſe $BE = 2a$, $BC = a$, it will be $CI = \frac{\sqrt{8aa}}{2} = \sqrt{2aa}$, that is, CI leſs than CD, and therefore the curve is concave towards the axis AE.

231. But

231. But these conclusions are valid only in such curves, which have no point of contrary flexure, or of regression. But, because these have their particular methods, of which, at present, this is not a place to treat, we cannot as yet form a just and complete idea of such curves. Further to determine the forms of the curves, with examples.

EXAMPLE II.

Fig. 128.



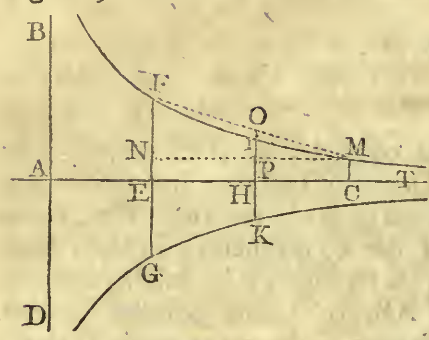
Let the equation be $y^3 = aax$, or $y = \sqrt[3]{aax}$. Drawing the two indefinite lines BH, DC, making a given angle BAC equal to that of the co-ordinates; in AC, from the point A let the x's be taken, and the y's upon AB, or a line parallel to AB. First, I inquire if the curve passes through the point A or not, that is, what will y be when $x = 0$. But, making $x = 0$, we find $y = \sqrt[3]{aa \times 0}$, that is, $y = 0$. Therefore the curve passes through the point A. I inquire further, if the curve cuts the axis AC in another point, that is to say, what is x when $y = 0$, and I find $\sqrt[3]{aax} = 0$, that is, $x = 0$. Therefore the curve cuts the axis in no other point but A. Make $x = AM = \frac{1}{2}a$, and the given equation will be $y = \sqrt[3]{\frac{1}{2}a^3}$. Therefore, drawing $MP = \sqrt[3]{\frac{1}{2}a^3}$, and parallel to AB, then P will be a point in the curve. I make $x = AC = a$, and it will be $y = \sqrt[3]{a^3} = a$; then drawing $CN = a$, and parallel to AB, N will be another point in the curve. And doing this successively, we may find as many points as we please, through which the curve of this equation will pass. Lastly, make x infinite, or $x = \infty$, and it will be $y = \sqrt[3]{aa \times \infty}$, that is, y is infinite, and therefore our curve passes on to infinity. And because, taking $x = 0$, it is also $y = 0$, and taking $x = \infty$, it is also $y = \infty$, the curve will have no asymptotes that are parallel to the co-ordinates.

Let the line AN be drawn beneath, which cuts in I the line MP, produced if necessary. Now, since $AM = \frac{1}{2}a$, $AC = a = CN$, it will be $MI = \frac{1}{2}a$. But $MP = \sqrt[3]{\frac{1}{2}a^3}$, therefore MI will be less than MP, and therefore the curve is concave to the axis AC.

Now, if we take the abscifs negative, because in the given equation $y^3 = aax$, the exponent of x is odd, when x is taken negative it's sign should be changed, and the equation will then be $y = \sqrt[3]{-aax}$; here it is evident, that, taking the values of x the negative way, that is, from A towards D, but equal to those already taken the positive way, it will give as many negative values of y, equal to the positive. Whence the branch AE will be just the same as the branch AN, but contrarily posited.

EXAMPLE III.

Fig. 129.



Let the equation be $a^3 - zyy = 0$, that is, $y = \pm\sqrt{\frac{a^3}{z}}$, and let us take the z's from the point A on the axis AC. First, I inquire if the curve passes through the point A; making therefore $z = 0$, the equation will be $y = \pm\sqrt{\frac{a^3}{0}}$, that is, $y = \pm\infty$. Therefore BD, being infinite on both sides of A, will be an asymptote to the curve. Next, I inquire if in no point the curve cuts the axis; and therefore put $y = 0$, and the equation will be $\pm\sqrt{\frac{a^3}{z}} = 0$, or $\frac{a^3}{z} = 0$, or $z = \frac{a^3}{0}$, that is, $z = \infty$. Therefore AC will be another asymptote. Taking $z = a = \text{AE}$, it will be $y = \pm\sqrt{\frac{a^3}{a}} = \pm a$. Making therefore EF positive and EG negative, and each $= a$, the points F, G, will be in the curve. Taking $z = 2a = \text{AH}$, it will be $y = \pm\sqrt{\frac{a^3}{2a}} = \pm\sqrt{\tfrac{1}{2}aa}$. Therefore, making HI positive, and HK negative, each equal to $\sqrt{\tfrac{1}{2}aa}$, the points I, K, will be in the curve. Taking new values of z always greater and greater continually, there will result new values of y always less and less, so that the two branches, FI, GK, of the curve being in every thing equal and similar, will stretch out on each side, approaching to the asymptotes BD, AC, yet without ever touching them, but at an infinite distance from the point A.

As to the negative abscis z; because the exponent of z is an odd number, if it be taken negative it will be convenient to change the sign of the term $-zyy$, and then the equation will be $a^3 + zyy = 0$; that is, $y = \pm\sqrt{-\frac{a^3}{z}}$. That is to say, the ordinate y is imaginary, and therefore on the negative part of the abscis there will be no curve.

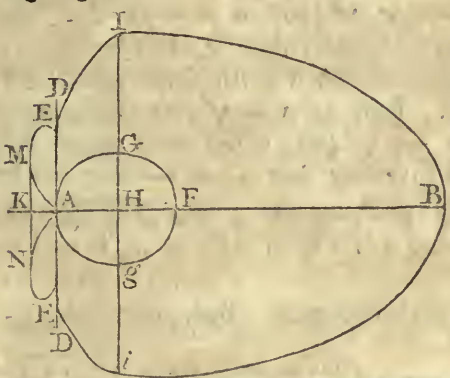
To examine whether the curve be concave or convex towards it's axis AC, I take $\text{AC} = 3a$; then it will be $\text{CM} = \sqrt{\tfrac{1}{3}aa}$; and drawing FM, which cuts HI (produced, if occasion) in O, and MN parallel to AC, it will be $\text{NF} = a - \sqrt{\tfrac{1}{3}aa}$, $\text{PI} = \sqrt{\tfrac{1}{2}aa} - \sqrt{\tfrac{1}{3}aa}$. Then making the analogy, MN . NF :: MP . PO, that is, $2a \,.\, a - \sqrt{\tfrac{1}{3}aa} :: a \,.\, \text{PO}$; it will be $\text{PO} = \frac{a - \sqrt{\frac{1}{3}aa}}{2}$;

and

and therefore, if PO be greater than PI, the curve will be convex towards the axis AC. This is to be examined thus. If it be $\frac{a - \sqrt{\frac{1}{3}aa}}{2} > \sqrt{\frac{1}{2}aa} - \sqrt{\frac{1}{3}aa}$, then multiplying by 2, it will be $a - \sqrt{\frac{1}{3}aa} > 2\sqrt{\frac{1}{2}aa} - 2\sqrt{\frac{1}{3}aa}$, and $a + \sqrt{\frac{1}{3}aa} > 2\sqrt{\frac{1}{2}aa}$, and ſquaring, $aa + 2a\sqrt{\frac{1}{3}aa} + \frac{1}{3}aa > 2aa$, and multiplying by 3, $3aa + 6a\sqrt{\frac{1}{3}aa} + aa > 6aa$, and reducing the terms, $6a\sqrt{\frac{1}{3}aa} > 2aa$, and dividing by $2a$, $3\sqrt{\frac{1}{3}aa} > a$, and, laſtly, ſquaring, $\frac{9}{3}aa > aa$, or $3 > 1$. Now, as this is a true reſult, ſo it is alſo true PO is greater than PI, and conſequently the curve is convex towards the axis AT.

EXAMPLE IV.

Fig. 130.



Let the equation of the curve be $y = \pm\sqrt{\frac{4ax + a^2 - 2x^2 \pm a\sqrt{a^2 + 8ax}}{2}}$. On the indefinite right line AB, taking the x's from the fixed point A, and the y's on AD, which makes the angle DAB of the co-ordinates; if it be put $x = 0$, it will be $y = \pm\sqrt{\frac{aa \pm a\sqrt{aa}}{2}}$, that is, $y = \pm\sqrt{\frac{2aa}{2}}$, and $y = \pm\sqrt{\frac{0}{2}}$; or $y = \pm a$, and $y = 0$. Therefore, making AE poſitive and negative $= a$, the points E, A, E, will be in the curve. To find where the curve cuts the axis AB, I put $y = 0$, and therefore $\pm\sqrt{\frac{4ax + a^2 - 2x^2 \pm a\sqrt{a^2 + 8ax}}{2}} = 0$. Then, ſquaring and tranſpoſing, $4ax + aa - 2xx = \pm a\sqrt{aa + 8ax}$, and ſquaring again, $16aaxx + 8a^3x + a^4 + 4x^4 - 16ax^3 - 4aaxx = a^4 + 8a^3x$; then, reducing and dividing by $4xx$, it is $3aa - 4ax + xx = 0$, and reſolving the equation, $x = \pm a + 2a$, that is, $x = a$, and $x = 3a$. Therefore, taking $x = AF = a$, and $x = AB = 3a$, the curve will cut the axis in the points F, B. Make $x = \frac{1}{2}a = AH$, it will be $y = \pm\sqrt{\frac{5aa \pm 2a\sqrt{5aa}}{4}}$; therefore the four values of y are real, becauſe $2a\sqrt{5aa}$ is leſs than $5aa$; which roots are, $\sqrt{\frac{5aa + 2a\sqrt{5aa}}{4}}$, $\sqrt{\frac{5aa - 2a\sqrt{5aa}}{4}}$, $-\sqrt{\frac{5aa - 2a\sqrt{5aa}}{4}}$, and $-\sqrt{\frac{5aa + 2a\sqrt{5aa}}{4}}$. The two poſitive roots are relatively

relatively equal to the two negative; therefore, taking $HI = Hi = \sqrt{\frac{5aa + 2a\sqrt{5aa}}{4}}$, and $HG = Hg = \sqrt{\frac{5aa - 2a\sqrt{5aa}}{4}}$, the four points, I, G, g, i, will be in the curve.

Examples to determine when the ordinates are real.

232. As often as the quantity under the common radical vinculum is a negative quantity, (for that under the ſecond vinculum, or $\sqrt{aa + 8ax}$, cannot be negative, the abſciſs being poſitive, as I now ſuppoſe it,) the ordinate y will be imaginary. Now, therefore, that there may be an ordinate, it will be neceſſary that it be $\sqrt{\frac{4ax + aa - 2xx \pm a\sqrt{aa + 8ax}}{2}} > 0$.

In the firſt place, I take the ſign poſitive of the ſecond radical, in which caſe the whole quantity will be certainly poſitive, if it be $4ax + aa - 2xx > 0$, that is, $2xx - 4ax < aa$, and therefore $xx - 2ax < \frac{1}{2}aa$, and $xx - 2ax + aa < \frac{3}{2}aa$, and extracting the root, $x - a < \sqrt{\frac{3}{2}aa}$, or $a - x < \sqrt{\frac{3}{2}aa}$. From the firſt root, in which x is ſuppoſed to be greater than a, I infer that it muſt be $x < a + \sqrt{\frac{3}{2}aa}$. From the ſecond, in which it is ſuppoſed that $x < a$, I conclude that it muſt be $x > a - \sqrt{\frac{3}{2}aa}$. But, as $a - \sqrt{\frac{3}{2}aa}$ is always a negative quantity, it will be always $x > a - \sqrt{\frac{3}{2}aa}$, when x is taken leſs than a. Therefore, taking x leſs than a, the quantity $4ax + aa - 2xx$ will be poſitive, ſo that much more the quantity $4ax + a^2 - 2x^2 + a\sqrt{a^2 + 8ax}$ will be poſitive. And therefore, in general, taking x leſs than AF, or a, it will be $y = \pm\sqrt{\frac{4ax + a^2 - 2x^2 + a\sqrt{a^2 + 8ax}}{2}}$, a real ordinate. But, even though $4ax + aa - 2xx$ were a negative quantity, yet $\sqrt{\frac{4ax + aa - 2xx + a\sqrt{aa + 8ax}}{2}}$ may be a poſitive quantity; that is, whenever it is $\sqrt{\frac{4ax + aa - 2xx + a\sqrt{aa + 8ax}}{2}} > 0$, it will be, by ſquaring and tranſpoſing, $a\sqrt{aa + 8ax} > 2xx - aa - 4ax$, and by ſquaring again, $a^4 + 8a^3x > 4x^4 - 16ax^3 + 16a^2x^2 - 4a^2x^2 + 8a^3x + a^4$, that is, $4x^4 - 16ax^3 + 12aaxx < 0$, and dividing by $4xx$, it is $xx - 4ax + 3aa < 0$, and therefore $xx - 4ax + 4aa < aa$, and extracting the root, $x - 2a < a$, as alſo $2a - x < a$. From the firſt root, which ſuppoſes x to be greater than $2a$, ariſes $x < 3a$. Therefore, taking x greater than $2a$, but leſs than AB, or $3a$, the radical will be poſitive, and therefore the ordinate y will be real. From the ſecond root, which ſuppoſes x leſs than $2a$, I obtain $x > a$; and therefore, whenever x is greater than a, and leſs than $2a$, the radical will be poſitive, and therefore y real. But we have ſeen by the firſt, that, taking x leſs than a, the ordinate y is real; therefore, in general, the ordinate y will be real, if we take x leſs than AB, or $3a$.

Taking

Taking the ſign negative of the ſecond radical, it would be $\sqrt{\frac{4ax + aa - 2xx - a\sqrt{aa + 8ax}}{2}} > 0$, and ſquaring, $4ax + aa - 2xx > a\sqrt{aa + 8ax}$, and ſquaring again and reducing, and dividing by $4xx$, it will be $xx - 4ax > -3aa$, and thence alſo $xx - 4ax + 4aa > aa$, and extracting the root, $x - 2a > a$, as alſo, $2a - x > a$. From the firſt root we obtain $x > 3a$. But we have ſeen, that $x > 3a$ gives the value of y imaginary, when the ſecond radical has a poſitive ſign, and therefore much more when it has a negative ſign. Wherefore, omitting this root, I make uſe of the other, $2a - x > a$, which gives me $x < a$. Therefore, taking x leſs than AF, or a, the quantity under the common radical vinculum will be poſitive, as well if we take the ſign of the ſecond radical poſitive as negative, and therefore between A and F there will correſpond four real ordinates, that is, two poſitive and two negative, which are relatively equal to the poſitive. But when x is greater than AF, or a, the negative ſign of the ſecond radical gives an imaginary ordinate, and the poſitive ſign gives it real; becauſe it is x leſs than AB, or $3a$, and therefore between F and B will correſpond to the ſame abſciſs only two real ordinates, one poſitive, the other negative and equal to the poſitive; and beyond the point B they will be only imaginary.

Now let the abſciſſes be taken negative, that is, from A towards K. In this caſe, changing in the equation the ſigns of all the terms in which the exponent of x is odd, then $y = \pm\sqrt{\frac{aa - 2xx - 4ax \pm a\sqrt{aa - 8ax}}{2}}$. I put $x = 0$, and it will be $y = \pm\sqrt{\frac{aa \pm a\sqrt{aa}}{2}}$, that is, $y = \pm a$, and $y = 0$. Therefore the points E, A, E, will be in the curve, as in the firſt caſe. To ſee if the curve cuts the axis, put $y = 0$; then $\sqrt{\frac{aa - 2xx - 4ax \pm a\sqrt{aa - 8ax}}{2}} = 0$, and ſquaring, and tranſpoſing, $aa - 2xx - 4ax = a\sqrt{aa - 8ax}$, and ſquaring again, and reducing, and dividing by $4xx$, it will be $xx + 4ax + 3aa = 0$; and reſolving, $x = -2a \pm a$.

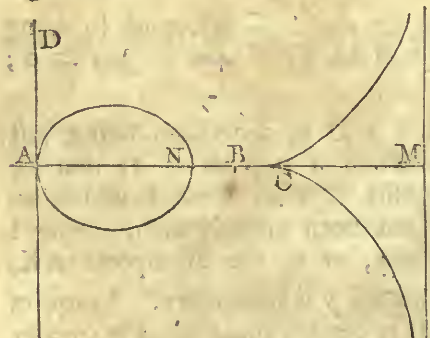
Therefore the curve will cut the axis when it is $x = 0$, a diviſion being juſt now made by $4xx$; when it is $x = -3a$, and when it is $x = -a$; that is, by being a negative quantity, on the ſide oppoſite to this, towards which we now take x; and therefore only in A, F, B, as has been already ſeen. Now I put $x = \infty$, to ſee if the curve goes on to infinity, or to the aſymptote AK, and it is $y = \pm\sqrt{-2 \times \infty^2 \pm \sqrt{-8a \times \infty}}$, that is, y is imaginary. I inquire then what are the limits of the real ordinates. It is certain that then x is greater than $\frac{1}{8}a$; the ſecond radical will be a negative quantity, and therefore the ordinate y imaginary; ſo that x muſt not be taken greater than $\frac{1}{8}a$; but in this hypotheſis, becauſe the whole quantity under the common radical is poſitive, taking the poſitive ſign of the ſecond radical, it will be enough that $aa - 2xx$

 $-4ax$

— $4ax$ be positive, that is, $aa - 2xx - 4ax > 0$, and therefore $xx + 2ax < \frac{1}{2}aa$, or $x < \sqrt{\frac{3}{2}aa} - a$. But when x is not greater than $\frac{1}{8}a$, and also $< \sqrt{\frac{3}{2}aa} - a$, making then x not greater than $\frac{1}{8}a$, the ordinate will be real. Taking the negative sign of the second radical, it will be $\sqrt{\frac{aa - 2xx - 4ax - a\sqrt{aa - 8ax}}{2}} > 0$, that is, squaring and transposing, $aa - 2xx - 4ax > a\sqrt{aa - 8ax}$, and squaring again and reducing, $x + 2a > a$. But $x + 2a$ is always greater than a, and therefore, supposing x to be taken not greater than $\frac{1}{8}a$, the ordinates will always be real. I take $x = \frac{1}{8}a$, and it will be $y = \pm\frac{\sqrt{15aa}}{8}$; and therefore, making KM positive, and KN negative and equal to $\frac{\sqrt{15aa}}{8}$, the points M, N, will be in the curve. I take $x = \frac{a}{16}$, it will be $y = \pm\frac{\sqrt{95aa \pm 128a\sqrt{\frac{1}{2}aa}}}{16}$, that is, the four values are real, two positive, which are relatively equal to the two negative. And, because the fourth proportional of $\frac{1}{8}a$, $\frac{\sqrt{15aa}}{8}$, and $\frac{1}{16}a$, or $\frac{\sqrt{15aa}}{16}$, is less than $\frac{\sqrt{95aa + 128a\sqrt{\frac{1}{2}aa}}}{16}$, but greater than $\frac{\sqrt{95aa - 128a\sqrt{\frac{1}{2}aa}}}{16}$, the curve will have two branches above AK, one concave, and the other convex, and also two below, like and equal to those above, as in Fig. 130.

EXAMPLE V.

Fig. 131.



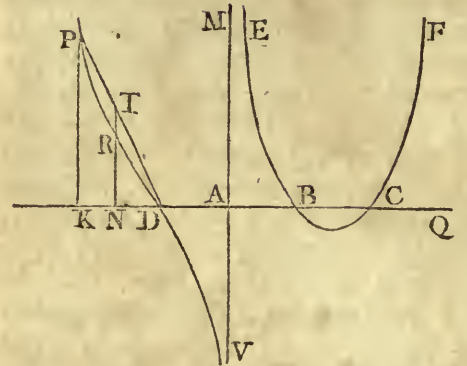
Let it be the curve of this equation $y = \pm\sqrt{\frac{bbxx - x^3 + 2ax^2 - aax}{x - 2a}}$; here, for one case, let a be greater than b, and let the x's be taken from the point A, upon the indefinite line AM, and the y's upon AD in a given angle, or parallel to a given line. Making $x = 0$, it will be $y = 0$, and therefore the point A is in the curve. Making $y = 0$, it will be $\sqrt{\frac{bbx - x^3 + 2axx - aax}{x - 2a}} = 0$, that is, $bbx - x^3 + 2axx - aax = 0$, and dividing by x, it is $bb - xx + 2ax - aa = 0$, and therefore $xx - 2ax + aa = bb$, and extracting the root, $x - a = \pm b$; therefore the values of x will be $x = a + b$, $x = a - b$, and $x = 0$, because the equation was divided by x. Whence, making AB = BM = a, BN

$BN = BC = b$, the curve will cut the axis in the point A, as has been already feen, and alfo in the points N, C. Making $x = AM = 2a$, y will be pofitive and negative infinite, and therefore there will be an afymptote at M. Put $x = \infty$, it will be $y = \pm\sqrt{-xx}$, that is, imaginary. Therefore the curve is not continued to infinity. Now, that the ordinate y may be real, it follows that the quantity under the vinculum muft be pofitive; it is therefore neceffary that, the numerator of the fraction being pofitive, the denominator muft be fo alfo; and the one being negative, the other muft be the fame. But, that the numerator may be pofitive, it muft be $bbx - x^3 + 2axx - aax > 0$, or, dividing by x and tranfpofing, $xx - 2ax < bb - aa$. Therefore $xx - 2ax + aa < bb$, and extracting the root, $x - a < b$, taking x greater than a; and $a - x < b$, taking x lefs than a. From the firft root, $x - a < b$, we have $x < a + b$. From the fecond, $a - x < b$, we have $x > a - b$. Therefore, taking x greater than a, it muft be $x < a + b$; and taking x lefs than a, it muft be $x > a - b$, fo that the numerator may be pofitive. Now, that the denominator may be pofitive, it muft be $x > 2a$; and, as it cannot be greater than $2a$, and at the fame time lefs than $a + b$, and than a, the numerator and denominator cannot be both pofitive; and therefore between the points N and C there will be no real ordinates. If we take $x > a + b$, the numerator will be negative; as alfo, if we take $x < a - b$. And if we take $x < 2a$, the denominator will alfo be negative. Therefore, between A and N, and between C and M, there will be real ordinates, and the curve will be nearly as in Fig. 131.

Take x negative; changing therefore the figns of thofe terms, in which the exponent of x is an odd number, the equation will be $y = \pm\sqrt{\frac{x^3 - bbx + 2axx + aax}{-2a - x}}$, that is, $y = \pm\sqrt{\frac{bbx - x^3 - 2axx - aax}{2a + x}}$. The denominator will always be pofitive; but, that the numerator may be pofitive, it will be neceffary that $b^2x - x^3 - 2ax^2 - a^2x > 0$; and, dividing by x and tranfpofing, $xx + 2ax + aa < bb$, that is, $x + a < b$, and therefore $x < b - a$. But, if we fuppofe $b < a$, then $b - a$ will be a negative quantity, and therefore it can never be $x < b - a$, that is, the numerator can never be pofitive. So that the ordinates y will always be imaginary, and there can be no part of the curve on the fide of the negative abfciffes.

EXAMPLE VI.

Fig. 132.



Let the equation be $y^3 - 2ay^2 - aay + 2a^3 = axy$, that is, $x = \frac{y^3 - 2ay^2 - aay + 2a^3}{ay}$. From the fixed point A, upon the indefinite line AQ, I take the y's, and on the indefinite line AM, or it's parallel, in the given angle of the co-ordinates, I take the x's. Putting $y = 0$, it will be $x = \frac{2aa}{0}$, that is, $x = \infty$; so that the curve will approach to the asymptote AM. To see if the curve cuts the axis, and where, I put $x = 0$, and therefore $y^3 - 2ay^2 - aay + 2a^3 = 0$; and, resolving this cubic equation, we shall have three values of y, that is, $y = a$, $y = 2a$, and $y = -a$. Therefore, making AB = AD = BC = a, the curve will cut the axis in the points B, C, on the side of the positives, and in the point D on the negative side.

To determine the same when the equations are irreducible.

233. If the equation $y^3 - 2ay^2 - aay + 2a^3 = 0$ had been irreducible, so that we could not have had the analytical values of y, we must have constructed this equation, and by that means have found the values of y geometrically, that is, expressed by lines, which would have given us the points required. And this is to be understood of any other such case. Thus, I put $y = \frac{3}{2}a$, and it will be $x = -\frac{5}{12}a$, that is, the ordinate is negative, and therefore the curve passes below the axis AQ at B, and returns above at C. I put $y = \infty$, it will be $x = \frac{yy}{a} = \infty$, and therefore the curve goes on to infinity. It is plain that the infinite branch BE will be convex towards the axis AM, the branch BC will be concave to the axis AQ, and CF convex, when the curve shall have no contrary flexures. Let us now take the abscisses y negative from A towards D. Then the equation will be $x = \frac{-y^3 - 2ay^2 + aay + 2a^3}{-ay}$, or $x = \frac{y^3 + 2ay^2 - aay - 2a^3}{ay}$. I take $y = 0$, then it will be $x = -\frac{2aa}{0} = -\infty$; therefore MA, produced infinitely on the side of the negatives, will be also an asymptote to the curve. I take $y = \frac{1}{2}a$, it will be $x = -\frac{15}{4}a$; I take $y = a$, then it will be $x = 0$, and the curve will pass through D. I take $y = \infty$, it will be $x = \frac{yy}{a} = \infty$, and the curve above AD will go on *ad infinitum*.

nitum. I take $y = 3a = AK$, then $x = \frac{40}{3}a = KP$. I take $y = 2a = AN$, then it will be $x = 6a = NR$. Now, becaufe, drawing the right line DP, it will be $NT = \frac{40}{6}a$, and $\frac{40}{6}a > 6a$; therefore $NT > NR$, and the curve in R is convex to the axis AK, that is, concave to the axis AM. But, if it go on towards the afymptote AV, below AK, it muft therefore neceffarily be convex to it, and therefore will have a contrary flexure; but, to determine this does not belong to this place.

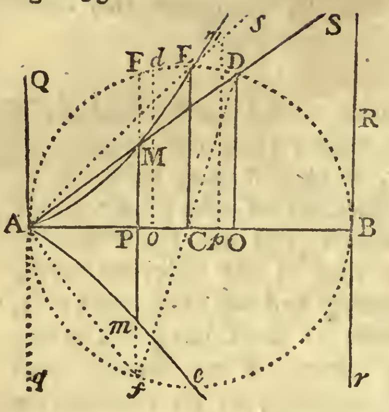
234. But, if the propofed equation of the curve to be conftructed fhall involve both the indeterminates raifed to a power higher than the fecond, fo that it cannot generally be reduced in fuch manner, as that it may have one of the two indeterminates alone, on one fide of the equation, of one power only; then, indeed, the trouble of the operation may increafe, but not the difficulty of the method. For, fixing a known value upon one of the indeterminates, for example x, we fhall have a folid equation, given by y and conftant quantities, which is to be refolved or conftructed; from whence we fhall have the values of y, which will determine fo many points of the curve. Then, fixing upon another value for x, we fhall have another folid equation to be refolved or conftructed, which will furnifh us with other points of the curve; and thus working from one to another fucceffively, we may find as many points as we pleafe of the curve to be defcribed. It may be done by taking points.

235. But, on this and fuch other occafions, as it is required to refolve and conftruct folid equations, as in the fixth Example, it may feem as if we fell into what logicians call *Circulus Vitiofus*, becaufe, in treating of Solid Problems, I have fuppofed the defcription of curves which are fuperior to conic fections. But, upon further reflection, the matter will be found to be much otherwife. For, if the curve to be defcribed be of three or four dimenfions, the folid equation to be conftructed will be of the third or fourth order at moft, and be performed by means of the conic fections. Therefore, without any *circulus vitiofus*, any curve of three or four dimenfions may be defcribed. If the equation of the curve to be defcribed fhall be of five dimenfions, the folid equation to be conftructed will be, at moft, of five; and this is done by means of a curve of three, and one of two dimenfions. And fo, in like manner, of the higher orders; whence it plainly appears, that there can be no objection of our falling into fuch a fallacy. An objection obviated.

PROBLEM I.

Example, for determining the forms of the *loci* from the equation.

Fig. 133.



236. Having given the semicircle AEB, it is required to find the *locus* of the points M such, that, if through every one of them a right line be drawn from the extremity of the diameter A, which shall cut the periphery in D, and if the lines MP, DO, be let fall perpendicular to the diameter, the intercepted lines from the centre CP, CO, may be always equal.

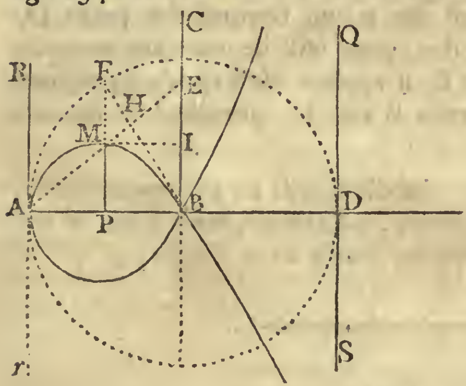
Let M be one of those points, and make AB $= a$, AP $= x$, PM $= y$; and, because it must be CP $=$ CO, it will be OB $=$ AP $= x$, and OD $= \sqrt{ax - xx}$. And, because of similar triangles APM, AOD, it will be $x \,.\, y :: a - x \,.\, \sqrt{ax - xx}$, and therefore $y = \frac{x\sqrt{ax - xx}}{a - x}$, that is, $y = \frac{x\sqrt{x}}{\sqrt{a - x}}$, or $y = \frac{xx}{\sqrt{ax - xx}}$, the equation of the curve to be described, which is the *Cissoid of Diocles*.

To describe it upon the given figure by various points, it may be observed that the right line AB is the axis of the x's, and A is the given point from whence they take their origin. And, because the y's are perpendicular to this axis, from the point A drawing the tangent AQ, this will be the axis to which the ordinates y are to be referred. These things being premised, if we make, first, $x = 0$, to see if the curve cuts the axis AQ; and, because we find also $y = 0$, therefore A will be a point in the curve to be described. Make $y = 0$, to see if the curve cuts the axis in any other point. But, because we find $x = 0$, the curve will not meet the two axes in any other point but A.

Make $x = \frac{1}{3}a$, it will be $y = \frac{a}{3\sqrt{2}}$; make $x = \frac{1}{2}a$, it will be $y = \frac{1}{2}a$, and therefore, from the centre drawing CE perpendicular to the diameter AB, the curve will pass through the point E. Make $x = \frac{2}{3}a$, then $y = \frac{4a}{3\sqrt{2}}$; and, lastly, making $x = a$, we shall find $y = \frac{aa}{0} = \infty$, and therefore the tangent BR to the circle will be the asymptote to the curve. Taking x greater than a, the quantity under the radical sign in the denominator will be negative, and the curve imaginary. Which being also imaginary, if we take x negative, it will be be wholly comprehended between the two tangents AQ, BR, produced *in infinitum*. And, becaufe it approaches to the afymptote BR, having no contrary flexure, it will neceffarily be wholly convex to the axis AB, and will appear as in Fig. 133.

PROBLEM II.

Fig. 134.



237. The angle ABC being a right angle, and the point A in the fide AB being given, the *locus* is required of all the points M, fuch that, drawing through every one of them the right lines AE, terminated at the fide BC in the point E, it may be always EM = EB. Another example for the fame purpofe.

Let any right line AE be drawn, and let M be one of the points required; from the point M let fall MP perpendicular to AB, and make AP $= x$, PM $= y$, and AB $= a$. It will be PB $= a - x$, and AM $= \sqrt{xx + yy}$. Now, becaufe of fimilar triangles APM, ABE, it will be $x \,.\, y :: a \,.\,$ BE, and therefore BE $=$ EM $= \frac{ay}{x}$. But it is alfo AP . PB :: AM . ME; that is, $x \,.\, a - x ::$ $\sqrt{xx + yy} \,.\, \frac{ay}{x}$. Therefore $ay = \overline{a - x} \times \sqrt{xx + yy}$, and fquaring, $aayy =$ $aaxx - 2ax^3 + x^4 + aayy - 2axyy + xxyy$, or $\frac{aaxx - 2ax^3 + x^4}{2ax - xx} = yy$. And, laftly, fince the root of $aaxx - 2ax^3 + x^4$ is as well $ax - xx$ as $xx - ax$, it will be $y = \frac{ax - xx}{\sqrt{2ax - xx}}$, and $y = \frac{xx - ax}{\sqrt{2ax - xx}}$: that is, $\pm y =$ $\frac{ax - xx}{\sqrt{2ax - xx}}$, the equation to the curve which is required.

The ordinates y will therefore be pofitive and negative, and equal to each other; and the pofitive and negative will correfpond to the fame abfcifs; and therefore the curve will be both above and below the axis AB, and will be altogether fimilar and equal.

From the point A drawing AR perpendicular to AB, which fhall be the axis to which the ordinates y are referred, as AB is the axis of the abfcifs x; firft, I make

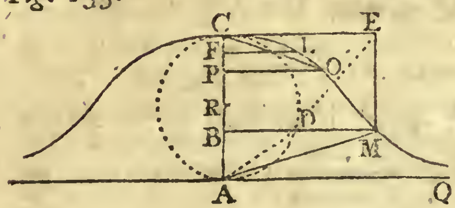
make $x = 0$, to see if the curve passes through A; and, because I find also $y = 0$, the point A will be the vertex of the curve. Now make $y = 0$, it will be $ax - xx = 0$, and therefore $x = 0$, and $x = a$. Hence I find that the curve will pass through the point B also. Make $x = \frac{1}{3}a$, and it will be $\pm y = \frac{2a}{3\sqrt{5}}$. Make $x = \frac{1}{2}a$, and it will be $\pm y = \frac{a}{2\sqrt{3}}$. Make $x = \frac{4}{3}a$, it will be $\pm y = \frac{4a}{3\sqrt{3}}$. Make $x = 2a$, and it will be $\pm y = \frac{2aa}{0} = \infty$; and therefore, taking AD = $2a$, and drawing the indefinite right line SQ parallel to PM, it will be an asymptote to the curve. If x be greater than $2a$, the quantity under the radical vinculum will be negative, and therefore the ordinate y will be imaginary, so that there is no part of the curve beyond the point D. It is plain that, between the points A and B, the curve will be concave towards the axis AB. And because, beyond the point B, it applies itself to it's asymptote SQ, it will be convex to the axis BD between B and D, provided it has no contrary flexure.

Taking x negative, the quantity under the vinculum will be always negative, and therefore the ordinate y is imaginary; so that, on the negative part of the abscis, there will be no curve; whence it will be nearly as in Fig. 134.

PROBLEM III.

Another example of the curve called the Witch.

Fig. 135.



238. The semicircle ADC, on the diameter AC, being given; out of it a point M is required, such that, drawing MB perpendicular to the diameter AC, which shall cut the circle in D, it may be AB . BD :: AC . BM. And, because there will be an infinite number of points that will satisfy the Problem, the *locus* of those points is required.

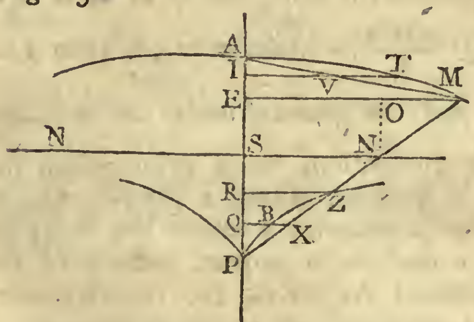
Let M be one such point, and making AC $= a$, AB $= x$, and BM $= y$, by the property of the circle, it will be BD $= \sqrt{ax - xx}$; and, by the condition of the Problem, it is AB . BD :: AC . BM; that is, $x . \sqrt{ax - xx} :: a . y$, and therefore $y = \frac{a\sqrt{ax - xx}}{x}$, or $y = \frac{a\sqrt{a - x}}{\sqrt{x}}$, will be the equation of the curve to be described, which is vulgarly called the *Witch*.

 Because

Becaufe AB $= x$, BM $= y$, the axis of the x's will be AC; and AQ, parallel to BM, will be the axis of the ordinates y. Firft, make $x = 0$, it will be $y = \infty$, and therefore AQ is the afymptote of the curve. Make $y = 0$, it will be $a\sqrt{a-x} = 0$, and therefore $x = a$. So that, when it is $x = a$, the curve will cut the axis AC, and confequently will pafs through the point C, which will be it's vertex. Make $x = AR = \frac{1}{2}a$, it will be $y = a$. Make $x = AP = \frac{3}{4}a$, it will be $y = a\sqrt{\frac{1}{3}}$. Make $x = AF = \frac{4}{5}a$, it will be $y = a\sqrt{\frac{1}{4}} = \frac{1}{2}a$. Putting x greater than a, the quantity under the vinculum will be negative, and the curve imaginary. To fee whether the curve be concave or convex towards the axis AC, make this proportion. As CP $= \frac{1}{4}a$ (which correfponds to $x = \frac{3}{4}a$,) is to $y = a\sqrt{\frac{1}{3}}$, fo is CF $= \frac{1}{5}a$, (which correfponds to $x = \frac{4}{5}a$,) to a fourth, which will be $a \times \frac{4}{5}\sqrt{\frac{1}{3}}$. But $x = \frac{4}{5}a$ gives $y = a\sqrt{\frac{1}{4}}$, and $a \times \frac{4}{5}\sqrt{\frac{1}{3}}$ is lefs than $a\sqrt{\frac{1}{4}}$. Therefore the curve will be concave towards the axis AC. But, becaufe of the afymptote AQ, it ought alfo to be convex; therefore it will be partly concave and partly convex, and therefore it will have a contrary flexure, which will be found by the method to be given in it's proper place. And taking x negative, becaufe the quantity under the vinculum will be negative in the denominator, y will be imaginary. Wherefore the curve will be as may be feen in Fig. 135, obferving that this curve has a branch fimilar and equal to the branch CLM, on the other fide of AC, correfponding to y negative.

PROBLEM IV.

Fig. 136.



239. The indefinite right line NN being given in pofition, and a point P out of the fame, the point M is required, fuch that, drawing from it to the point P the right line MP, the line NM, intercepted between the indefinite line NN and the point M, may be equal to a given right line. And, becaufe there are infinite points that fatisfy this demand, the *locus* of thefe points is required. Another example, being the *Conchoid* of *Nicomedes*.

From the point P draw the right line PA perpendicular to NN, and the right line PM to any point M, which is one of thofe required; and drawing the right line ME parallel to NN, make PS $= b$, SE $= x$, EM $= y$, and let SA $= a$ be the given line, to which the right line NM is to be equal by the condition of the Problem. From the point N draw the right line NO perpendicular to EM, and it will be MO $= \sqrt{aa - xx}$. And, becaufe of the fimilar ſimilar triangles PEM, NOM, it will be PE . EM :: NO . OM, that is $b + x . y :: x . \sqrt{aa - xx}$; and therefore $\overline{b + x} \times \sqrt{aa - xx} = xy$, and ſquaring, $xxyy = aaxx - x^4 + 2aabx - 2bx^3 + aabb - bbxx$; and laſtly, $y = \pm \frac{\sqrt{aaxx - x^4 + 2aabx - 2bx^3 + aabb - bbxx}}{x}$, the equation of the curve to be deſcribed, which is the *Conchoid* of *Nicomedes*.

Three different caſes may be diſtinguiſhed in this Problem. That is, it may be $b = a$; it may be b leſs than a; and laſtly, it may be b greater than a. Firſt, let it be $b = a$, and the equation will be changed into this following: $y = \pm \frac{\sqrt{-x^4 + 2a^3x - 2ax^3 + a^4}}{x}$.

Since it is SE $= x$, and EM $= y$, the axis will be NN, to which the y's are referred, and PA that of the x's, the origin of which is at S. Firſt, I make $x = 0$, to ſee if the curve paſſes through the point S; and becauſe there ariſes $y = \pm \frac{aa}{0}$, that is, y poſitive and negative is infinite, NN will be the aſymptote of the curve. I make $y = 0$, to ſee where the curve cuts the axis PA, and it will be $- x^4 + 2a^3x - 2ax^3 + a^4 = 0$. Now, this equation being reſolved by the rules before taught, it's roots will determine the points in which the curve meets the aforeſaid axis PA. Now the roots of this equation are four, that is, $x = a$ poſitive, and three negative roots equal to it, or $x = -a$. Therefore the curve will meet the axis in two points, diſtant from the point S by the quantity a. But, becauſe, at preſent, we are concerned only with the poſitive x's, it will be ſufficient to conſider the poſitive root; and therefore the curve will paſs through the point A, it being SA $= a$, as is ſuppoſed. Make $x = \frac{1}{2}a$, it will be $y = \pm \frac{\sqrt{27aa}}{2}$. Make $x = \frac{2}{3}a$, then $y = \pm \frac{\sqrt{125aa}}{6}$. Let x be greater than a, and the quantity under the vinculum will be negative, the firſt term, on this ſuppoſition, being greater than the fourth, and the third than the ſecond. Wherefore, taking x greater than a, the curve will be imaginary. It remains to examine whether the curve be always convex towards the axis PA; for it muſt be ſo in part, becauſe of the aſymptote NN. Make then this proportion: As AE $= \frac{1}{2}a$, (which correſponds to $x = \frac{1}{2}a$,) is to $y = \frac{\sqrt{27aa}}{2}$, ſo is AI $= \frac{1}{3}a$ to a fourth, which will be $\frac{\sqrt{27aa}}{3}$. But AI $= \frac{1}{3}a$ correſponds to $x = \frac{2}{3}a$, and therefore to $y = \frac{\sqrt{125aa}}{6}$. Now $\frac{\sqrt{125aa}}{6}$ is greater than $\frac{\sqrt{27aa}}{3}$, and therefore the curve will be partly concave towards the axis PA. Conſequently it will have a contrary flexure,

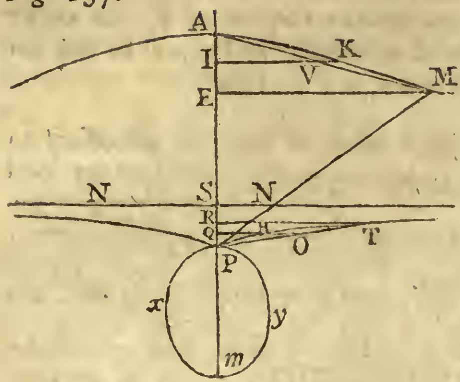
flexure, as ſhall be ſeen in it's due place. And, becauſe two equal values of y, one poſitive, the other negative, correſpond to the ſame value of x, the curve will have another branch on the negative ſide of y, ſimilar and equal to that on the poſitive ſide; and it will appear as is deſcribed in Fig. 136.

To deſcribe the curve on the negative part of x, it will be neceſſary to change the ſigns of thoſe terms in which the indeterminate is raiſed to an odd power; ſo that the equation will then be $y = \pm \frac{\sqrt{-x^4 - 2a^3x + 2ax^3 + a^4}}{-x}$. Now, firſt, let it be $x = 0$, then $y = \pm \frac{aa}{0}$, and therefore NN is ſtill the aſymptote to the curve on the negative part. Make $y = 0$, and it will be $-x^4 - 2a^3x + 2ax^3 + a^4 = 0$, whence we obtain four roots, as above: three are equal and poſitive, $x = a$, and one negative, $x = -a$. The negative root, which was poſitive in the foregoing caſe, is already fixed in the ſuperior *conchoid*. Then the three equal values ſignify, that, in the pole, which is diſtant from the beginning of the x's by the quantity a, the curve will have a regreſſion, of which we ſhall treat in the Method for Contrary Flexures. Make $x = \frac{1}{2}a$, then $y = \pm \frac{\sqrt{3aa}}{2}$. Make $x = \frac{2}{3}a$, then $y = \pm \frac{\sqrt{5aa}}{6}$. If we take x greater than a, the curve will be imaginary; becauſe, as the quantity under the vinculum is the product of $xx - 2ax + aa$ (a quantity always poſitive,) into $aa - xx$, which, in this ſuppoſition, is negative, the whole quantity under the radical will be negative, and therefore the ordinate y is imaginary. Now, make this proportion: As PR $= \frac{1}{2}a$ (making SR $= \frac{1}{2}a$,) is to $\frac{\sqrt{3aa}}{2}$, ſo is PQ $= \frac{1}{3}a$ (making SQ $= \frac{2}{3}a$,) to a fourth, which will be $\frac{\sqrt{3aa}}{3}$. But $y = \frac{\sqrt{5aa}}{6}$ correſponds to SQ $= \frac{2}{3}a$, or to PQ $= \frac{1}{3}a$, and $\frac{\sqrt{5aa}}{6}$ is leſs than $\frac{\sqrt{3aa}}{3}$; ſo that the curve will be always convex towards the axis NN, ſuppoſing it not to have a contrary flexure; and it will have two equal and ſimilar branches; for two equal values of y correſpond to the ſame x, one of which is poſitive, the other negative. So that the curve will appear as deſcribed in Fig. 136.

240. Now let b be leſs than a; the equation therefore will be $y = \pm \frac{\sqrt{aaxx - x^4 + 2aabx - 2bx^3 + aabb - bbxx}}{x}$. Make $x = 0$, it will be $y = \pm \frac{ab}{0} = \pm \infty$. Therefore, in this caſe alſo, NN (Fig. 137.) will be the Another caſe of the ſame.

aſymptote

Fig. 137.



of the curve. Make $y = 0$, then $a^2x^2 - x^4 + 2a^2bx - 2bx^3 + a^2b^2 - b^2x^2 = 0$; the four roots of which (that is, $x = \pm a$, and two equal to each other, $x = -b$,) will determine the points in which the curve cuts the axis PA. But, at present, it will be enough to consider the positive value $x = a$; and, because SA $= a$, A will be the vertex of the curve. Make $x = \frac{1}{2}a =$ SE, then it will be $y = \pm \frac{\sqrt{3aa + 12ab + 12bb}}{2} =$ EM. Make $x = \frac{2}{3}a =$ SI, then it will be $y = \pm \frac{\sqrt{20aa + 60ab + 45bb}}{6} =$ IK. Make the proportion, AE $= \frac{1}{2}a$ to EM $= \frac{\sqrt{3aa + 12ab + 12bb}}{2}$; so is AI $= \frac{1}{3}a$, to a fourth, which will be $\frac{\sqrt{3aa+12ab+12bb}}{3}$; in order to see if the curve be concave or convex to the axis SA. But, taking AI $= \frac{1}{3}a$, we have SI $= \frac{2}{3}a$, to which corresponds IK $= y = \frac{\sqrt{20aa+60ab+45bb}}{6}$; and it is found to be IV $= \frac{\sqrt{3aa+12ab+12bb}}{3}$ less than IK, or $\frac{\sqrt{20aa+60ab+45bb}}{6}$. Therefore the curve will be concave towards the axis SA. But, as it applies itself continually to the asymptote NN, it will be also convex, and therefore it will have a contrary flexure.

It is plain, that, taking the abscis beyond the point A, that is, x greater than a, there will be no curve; for the second term of the radical will be greater than the first, the fourth greater than the third, and the sixth greater than the fifth; and therefore the quantity under the vinculum will be negative, that is, y will be imaginary.

And, because to the same abscis x two equal ordinates y correspond, one of which is positive, the other negative, the curve on the side of the negative ordinates will also be the same, and nearly as in Fig. 137.

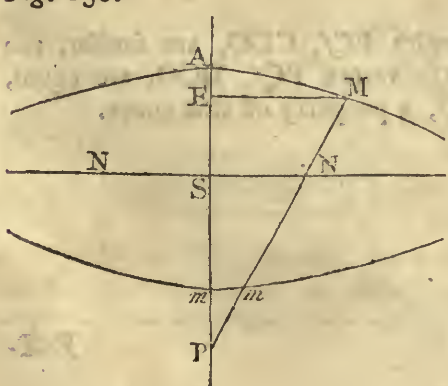
To describe the curve on the side of the abscis x negative, in the equation I change the sign in those terms wherein the power of x is odd, and it is $y = \pm \frac{\sqrt{aaxx - x^4 - 2aabx + 2bx^3 + aabb - bbxx}}{-x}$. Make $x = 0$, then $y = \pm \frac{ab}{0}$, that is, infinite, and therefore NN shall be an asymptote. I make $y = 0$, and it will be $aaxx - x^4 - 2aabx + 2bx^3 + aabb - bbxx = 0$; the four roots

roots of this equation, which are these two, $x = \pm a$, and two equal ones, $x = b$, will determine the points where the curve cuts the axis AP. The negative root $x = -a$ gives me the point A, the positive root $x = a$ the point m, and the two equal roots $x = b$ give the point P, so that there will be a node in the curve. Taking PR = SR $= \frac{1}{2}b = x$, it will be $y = \pm \frac{\sqrt{4aa-bb}}{2}$ = RT. Taking PQ $= \frac{1}{3}b$, that is, SQ $= x = \frac{2}{3}b$, it will be $y = \pm \frac{\sqrt{9aa-4bb}}{6}$ = QH. I make the analogy, PR $(\frac{1}{2}b)$. RT $(\frac{\sqrt{4aa-bb}}{2})$ :: PQ $(\frac{1}{3}b)$. QO $= (\frac{\sqrt{4aa-bb}}{3})$, in order to see whether the curve be concave or convex towards the axis PS. But QO $(\frac{\sqrt{4aa-bb}}{3})$ is greater than QH $(\frac{\sqrt{9aa-4bb}}{6})$; so that the curve is convex towards the axis PS. And this follows also from it's approaching to it as an asymptote.

Taking the abscis beyond the point m, that is, x greater than a, there will be no curve, because the radical aforegoing is the same as $\sqrt{aa-xx} \times \sqrt{x^2-2bx+b^2}$. But, supposing x greater than a, the quantity $aa - xx$ will be negative, and $xx - 2bx + bb$ is positive; therefore the product is negative, and the ordinate y is imaginary. Taking the abscis beyond the point P, that is, x greater than b, but less than a, it will be $aa - xx$, a positive quantity, as also, $xx - 2bx + bb$; therefore the product is positive, and the ordinate y is real; so that between P and m the curve will correspond, and will form a foliate Pxmy, having a node at P; and the curve will have the appearance nearly as in Fig. 137.

241. Lastly, let b be greater than a; the equation will be the same as in the former case, and, taking the abscis x positive, the curve will be also similar. A third case of the same.

Fig. 138.



Then taking x negative, and supposing $y = 0$, the four roots of the equation, that is, $x = \pm a$, and the two equal roots $x = b$, will give, indeed, the same points, A, m, P, in the axis PA: but the point m will be above the point P. And, assuming the abscis greater than Sm, that is, x greater than a, the quantity $aa - xx$ will be negative; and because $xx - 2bx + bb$ is positive, their product will be negative, and therefore the ordinate y will be imaginary. Therefore the curve will not have the

foliate

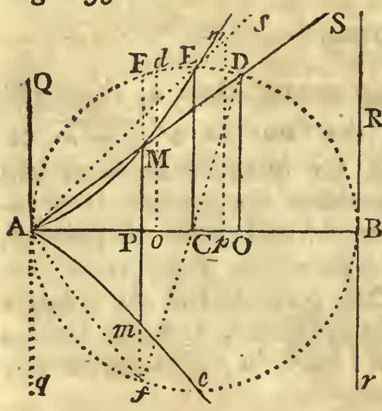
foliate of the former cafe, but will have it's vertex in *m*. And, becaufe the curve is firft concave, and then convex towards it's axis PS, as is eafily feen, and approaches to the afymptote NN, it will be nearly as in Fig. 138.

The method improved of defcribing curves by points.

242. This method of defcribing curves by an infinite number of points, may perhaps be reduced to a greater perfection, by making ufe alfo of geometrical conftructions. I fhall give fome Examples of it, which may ferve to put the matter in a proper light.

EXAMPLE I.

Fig. 133.



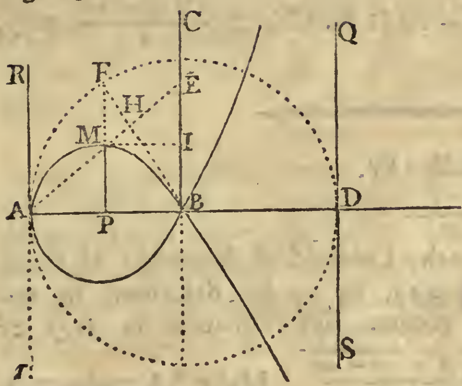
Let us conftruct, by various points, the curve of Prob. I. § 236, which is the *Ciffoid* of *Diocles*, the equation of which was found to be $y = \frac{xx}{\sqrt{ax - xx}}$. With radius AC $= \frac{1}{2}a$ let the circle AEB*e* be defcribed; and, taking at pleafure AP $= x$, I obferve that the correfponding ordinate P*f* is $= \sqrt{ax - xx}$. Through the point *f* I draw the diameter *f*CD, and joining the points A, D, with the line AD, the point *m*, in which it cuts the upper ordinate PF, continued if need be, will be in the *ciffoid*. For, the angle in the femicircle *f*AD being a right angle, as alfo the angle APM of the co-ordinates, the triangles A*f*P, APM, will be fimilar, and therefore we fhall have the analogy *f*P . AP :: AP . PM; that is, $\sqrt{ax - xx}$. x :: x . y. Whence it is $y = \frac{xx}{\sqrt{ax - xx}}$. Q. E. I.

After another manner. Becaufe the triangles PC*f*, CDO, are fimilar, the angles P, O, being right, and the angles at the vertex PC*f*, DCO, are equal, and alfo C*f* $=$ CD, it will be alfo CP $=$ CO, a property of this curve.

E X-

EXAMPLE II.

Fig. 134.

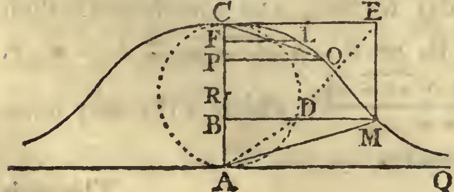


Let the curve be that of Prob. II. § 237, the equation of which is $\pm y = \frac{ax - xx}{\sqrt{2ax - xx}}$. With radius AB $= a$ let the circle AFD be drawn. Taking any line AP $= x$, from the point P draw the ordinate PF $= \sqrt{2ax - xx}$; and drawing the radius BF, let AHE be drawn perpendicular to it. This will cut the ordinate PF, continued if need be, in the point M, which will be in the curve AMB required. For, the triangles AMP, FMH, being ſimilar, and likewiſe the triangles FMH, FBP, the triangle AMP will be ſimilar to the triangle BFP, and therefore we ſhall have PF . PB :: AP . PM, that is, $\sqrt{2ax - xx} \,.\, a - x :: x \,.\, y$. Whence we have the propoſed equation $y = \frac{ax - xx}{\sqrt{2ax - xx}}$. Q. E. I.

After another manner. Becauſe the triangle AMP is ſimilar to the triangle AHB; and it has been ſeen above, that the triangle AMP is alſo ſimilar to the triangle FPB. But the ſide AB $=$ BF; therefore it will be alſo BH $=$ BP. Let the right line MI be drawn parallel to AB, and then the triangles BHE, MIE, will be ſimilar. But they will be alſo equilateral to each other, it being BH $=$ BP $=$ MI. Therefore it will be EB $=$ EM, which is the fundamental property of the curve propoſed.

EXAMPLE III.

Fig. 135.

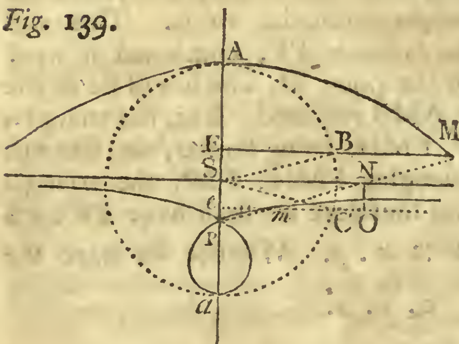


Let the curve to be deſcribed be that of Prob. III. § 238, called the *Witch*, the equation of which is $y = \frac{a\sqrt{ax - xx}}{x}$, the diameter of the circle, being AC $= a$. Take any line at pleaſure, AB $= x$, and draw

draw the indefinite lines BM, CE, perpendicular to AC. Then through the point D, in which BM cuts the circle, let AD be drawn, which, produced, ſhall cut CE in E. Through the point E draw a parallel to AC; it ſhall meet BM in the point M, which will belong to the curve. For, by the property of the circle, it is BD $= \sqrt{ax - xx}$, and, by ſimilar triangles ABD, ACE, it is AB . BD :: AC . CE. That is, $x . \sqrt{ax - xx} :: a . \mathrm{CE} = \frac{a\sqrt{ax - xx}}{x} = y$, the equation to the given curve.

EXAMPLE IV.

Fig. 139.



Let the *Conchoid* of *Nicomedes* of Prob. IV. § 239, be to be deſcribed by various points. It's equation is $\pm y = \frac{\overline{b \pm x} \times \sqrt{aa - xx}}{\pm x}$. Make SA $=$ S$a = a$, SP $= b$. With radius SA $= a$, let there be deſcribed the circle ABCa, and taking at pleaſure two abſciſſes SE, Se, equal to each other, which may be called x poſitive and negative, draw the ordinates EB, eC, each of which ſhall be $= \sqrt{aa - xx}$, and let them be produced indefinitely beyond the points B, C. Through the points S, B, let the right line SB be drawn, and through the point P a parallel to it, PM. The two points M, m, in which PM cuts the two right lines EB, eC, ſhall belong to the curve required; that is to ſay, the point M to the ſuperior branch, and m to the inferior branch of the *conchoid.*

And as to the point M; becauſe the two triangles SEB, PEM, are ſimilar, it will be SE . EB :: PE . EM; that is, $x . \sqrt{aa - xx} :: b + x . y$. And conſequently the equation will be $y = \frac{\overline{b + x} \times \sqrt{aa - xx}}{x}$, in reſpect of the upper branch of the *conchoid.*

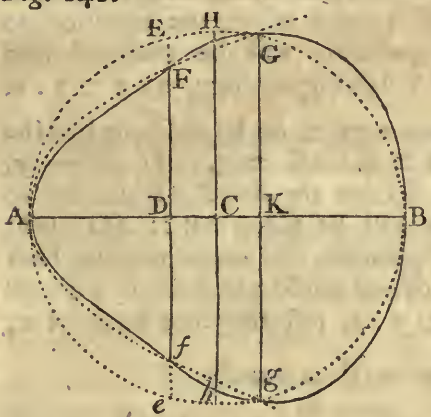
Then, as to the point m; drawing the line SC, the triangle SeC will be ſimilar and equal to the triangle SEB. For the triangle Pem is ſimilar to the triangle SEB; therefore alſo it will be ſimilar to SeC, and therefore we ſhall have the analogy Pe . em :: Se . eC; that is, $-x . \sqrt{aa - xx} :: b - x . y$. Whence we have the equation $y = \frac{\overline{b - x} \times \sqrt{aa - xx}}{-x}$; which is the very ſame as ſhould belong to the lower branch of the curve.

Through

Through the point S drawing the indefinite line SN parallel to the ordinates EM, *em*, from the conftruction above we fhall eafily obtain the principal property of the *conchoid*; which is, that, from the point or pole P, if we draw PM cutting the curve in the points M, *m*, and the line SN in the point N, the intercepted lines *m*N, NM, between the curve and the indefinite line SN, will always be of a conftant length, and equal to SA = SB = *a*. For, by the conftruction, SBMN will be a parallelogram, and therefore NM = SB. But, drawing NO parallel to S*e*, the triangles SBE, *m*NO, will be fimilar; and befides, NO = S*e* = SE. Therefore it will be *m*N = SB, and confequently *m*N = NM. Q. E. D.

243. The conftructions of the three firft Examples come out pretty fimple, there being nothing required to be done, but to draw a circle with a given diameter, and fome right lines. On other occafions the Conic Sections muft be admitted, which are fometimes to be defcribed with variable diameters, parameters, and rectangles. But thefe may be taken as conftant, in determining one or more points of the curve. —Improved by the conic fections.

Fig. 140.



To give an example of it. Let us conftruct, by points, the curve belonging to this equation $x\sqrt{2ax - xx} = yy$. Draw the circle AHB*b*, whofe diameter is AB = 2*a*. Take at pleafure AD = KB = *x*; it will be DE = KG = $\sqrt{2ax - xx}$. With parameter DE, to the axis AB, defcribe the *Apollonian* parabola GFA*fg*, and DF, D*f*, will give the pofitive and negative values of *y*, making *x* = AD. And KG, K*g*, the pofitive and negative values of *y*, making *x* = AK. Wherefore the four points F, *f*, G, *g*, will be in the curve required. By a like method, and by varying the value of *y*, we may determine other points of the curve.

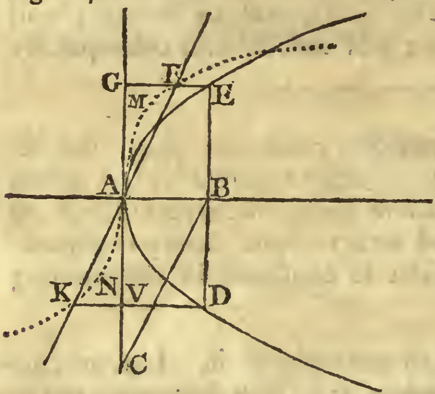
244. A fecond manner of conftructing curves beyond the fecond degree, will be that mentioned at § 220, by means of other lines of a lower degree. And, to begin with parabolas of any degree, it may be firft obferved, that the *Apollonian* parabola is the only one of it's kind, and is expreffed by the equation $ax = yy$. The cubic parabolas are two, that is, $aax = y^3$, and $axx = y^3$. Thofe of the fourth degree are three, $a^3x = y^4$, $aaxx = y^4$, and $ax^3 = y^4$. And, in general, thofe of the degree expreffed by *n* are in number $n - 1$, and are $ax^{n-1} = y^n$, $aax^{n-2} = y^n$, $a^3x^{n-3} = y^n$, $a^4x^{n-4} = y^n$, and fo on fucceffively, till the exponent of *x* is unity. —By parabolas of higher degrees.

The first cubical parabola constructed.

245. All those which have x, with unity, for it's exponent, are called first parabolas. Thus, $aax = y^3$, $a^3x = y^4$, $a^{n-1}x = y^n$, are all first parabolas.

To construct any parabola of any degree whatever, the beginning must be from the first cubic parabola $aax = y^3$.

Fig. 141.



It is plain that this must have two branches, one positive, the other negative; for, taking x positive, y will also be positive, that is, $y = \sqrt[3]{aax}$, and this will be it's positive branch. But, taking x negative, y will also be negative, or $y = \sqrt[3]{-aax}$, (which is no imaginary quantity,) and this will be the negative branch. It is evident that these two branches go on *ad infinitum*, and are concave to the axis AB.

To proceed to the construction. Make $yy = az$; and, substituting in the equation $aax = y^3$ this value of yy, the equation to the cubic parabola will be changed into this, $ax = zy$, which may be resolved into the following analogy, $a . z :: y . x$.

This supposed, let the parabola of the equation $yy = az$ be described to the axis AB, and let it be DAE. Make AB $= z$, BE $= y$, BD $= -y$, AC $= a$. Draw CB, and through the point A draw the line KAF parallel to CB; and making AG = BE, draw GE. It will be CA . AB :: AG . GF, that is, $a . z :: y . x$. Whence taking AB at pleasure, the corresponding lines BE, or AG, and GF, will be the co-ordinates of our cubic parabola, and F will be a point of it. For, in the analogy $a . z :: y . x$, restoring the value of z, or $\frac{yy}{a}$, it will be $a . \frac{yy}{a} :: y . x$, or the equation $y^3 = aax$.

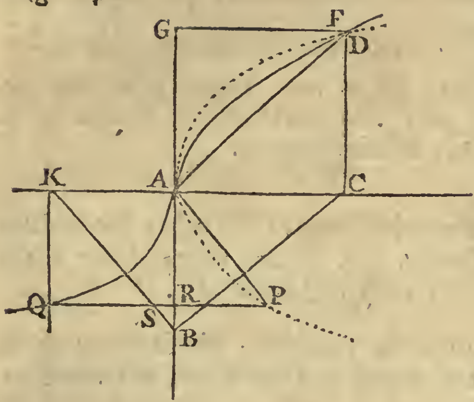
Now, because, when x is taken negative, y will be negative also, the analogy $a . z :: y . x$ will be changed into this following, $a . z :: -y . -x$; whence, taking AV = BD, it will be CA . AB :: AV . VK; that is, $a . z :: -y . -x$, and the point K will be in the cubical parabola. The branch AMF will be positive, and ANK the negative branch.

The first parabola of the fourth degree constructed.

246. Let it be proposed to construct the first parabola of the fourth degree $a^3x = y^4$. This will have also two branches, one above the axis, the other below it, because to x positive corresponds both y and $-y$, for the index of the power of y is an even number. These two branches will be concave towards the axis, and will proceed *in infinitum*. To go on to the construction. I make $y^3 = aaz$, and, instead of y^3, substituting this value in the equation proposed, we shall have $zy = ax$, or $a . z :: y . x$.

To

Fig. 142.



To axis KC let the parabola of the equation $y^3 = aaz$ be defcribed, which, becaufe it is the firft cubic, we know already how to conftruct; and let this be QAD. It will be $AC = GD = z$, $AK = -z$, and $CD = AG = y$, $KQ = -y$. Take $AB = a$, and draw the right lines BC, BK, and through the point A draw AF parallel to BC, and AP parallel to KB. This fuppofed, it will be BA . AC :: AG . GF, that is, $a . z :: y . x$; and the point F will be in the curve-line propofed to be conftructed. For, it being $a . z :: y . x$, and $z = \frac{y^3}{aa}$, it will be $a . \frac{y^3}{aa} :: y . x$; that is, $a^3x = y^4$.

But, becaufe when x is pofitive we may take y negative, which in this cafe will be KQ, and AK will be $-z$, we fhould have alfo BA . AK :: KQ (= AR) . RP; or $a . -z :: -y . x$. Therefore the point P will alfo be in the curve $a^3x = y^4$.

247. Let it be propofed to conftruct the firft parabola of the fifth degree, $a^4x = y^5$. This will alfo have two branches, one pofitive, the other negative. For, taking x pofitive, y will be pofitive, that is, $y = \sqrt[5]{a^4x}$. But, taking x negative, y will be negative, that is, $y = \sqrt[5]{-a^4x}$. Thefe two branches go on infinitely, and are concave to the axis AB. To proceed to the conftruction. Make $y^4 = a^3z$, and fubftituting this value in the propofed equation, it will be $ax = yz$, or $a . z :: y . x$. The firft parabola of the fifth degree conftructed.

To the axis AB (Fig. 141.) defcribe the parabola of the equation $y^4 = a^3z$, and let it be DAE. It being $AB = z$, it will be $BE = y$, and $BD = -y$. Make $AC = a$, and draw CB, and KAF parallel to it. Then draw the right line EFG, and the parallel DVK. This fuppofed, it will be CA . AB :: AG . GF, or $a . z :: y . x$; and the point F will be in the curve to be conftructed. For, it being $a . z :: y . x$, as alfo, $a^3z = y^4$, it will be $a . \frac{y^4}{a^3} :: y . x$, or $y^5 = a^4x$, the equation to the curve propofed.

Now, becaufe, x being negative, y will alfo be negative, the analogy $a . z :: y . x$ will be changed into this, $a . z :: -y . -x$. Wherefore, taking AV = DB, it will be CA . AB :: AV . VK, or $a . z :: -y . -x$. Whence the point K will be in the curve propofed to be conftructed. The branch AMF will be pofitive, and ANK will be the negative branch.

248. And,

The first parabola of any degree constructed.

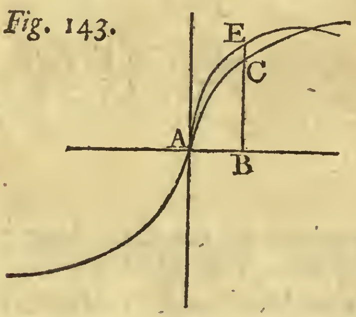
248. And, in general, let it be proposed to construct the parabola whose equation is $a^{n-1}x = y^n$. Make $y^{n-1} = a^{n-2}z$, and substituting this value in the proposed equation, we shall still have $zy = ax$. Whence it may be perceived, that we may always construct any first parabola by means of a triangle, and of the first parabola of the next inferior degree.

Construction of other succeeding parabolas.

249. Now it will be easy to go on to the construction of the other succeeding parabolas, or those of the second, third, fourth, &c. of any degree; for these also may be constructed by the construction of their first parabolas.

Let it be proposed to construct the second cubic parabola, whose equation is $axx = y^3$. I make $y^3 = aaz$, and, by substituting, instead of y^3, it's value in the proposed equation, it will be $xx = az$.

Fig. 143.



To the axis AB let there be described the *Apollonian* parabola AC, whose equation is $xx = az$; then to the same axis describe the first cubic parabola of the equation $y^3 = aaz$; and it being $AB = z$, it will be $BE = y$. But, in the *Apollonian* parabola AC, because $AB = z$, it will be $BC = x$. Therefore we shall always have the two co-ordinates x, y, of the second cubic parabola.

Let it be proposed to construct the third parabola of the fourth degree, whose equation is $ax^3 = y^4$. I make $a^3z = y^4$, and, by substitution, it will be $x^3 = aaz$. Let this first cubical parabola $x^3 = aaz$ be constructed, and to the same axis let there also be constructed the first of the fourth degree, $y^4 = a^3z$. The two ordinates of these curves, corresponding to the same abscis z, will give the co-ordinates x, y, of the proposed curve.

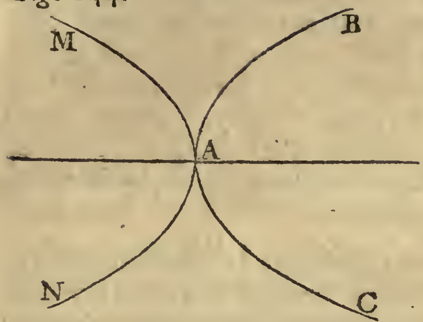
In the construction of all others, of any superior degree, we may proceed in the same method; these examples are sufficient, the thing itself being very plain.

Squaring the equation produces a reduplication of the curve.

250. It only remains to be observed, that the second parabola of the fourth degree, $aaxx = y^4$, is no other than the *Apollonian* parabola, but redoubled the contrary way. For, first, if it be $aaxx = y^4$, it will be also, by extracting the fourth root, $\sqrt[4]{aaxx} = \sqrt{ax} = \pm y$. But $\sqrt{ax} = \pm y$, or $ax = yy$, is no other than the equation to the *Apollonian* parabola. Our curve is therefore a common parabola, but redoubled; because the term $aaxx$ is alike generated, as well from $+ax \times +ax$, as from $-ax \times -ax$; which may be equally verified, because $\sqrt[4]{aaxx} = \sqrt[4]{+ax \times +ax} = \sqrt[4]{-ax \times -ax} = \sqrt{ax} = \pm y$.

Wherefore,

Fig. 144.



Wherefore, to negative x will correspond real y, and the branch MAN on the negative side will be perfectly like the branch BAC on the positive side, having respect to both the expressions $\sqrt[4]{aaxx} = \sqrt{ax} = \pm y$. But the *Apollonian* parabola has no branch on the negative side; for, putting x negative, it will be $\sqrt{-ax} = \pm y$; so that the curve will be imaginary.

If we raise the equation $ax = yy$ to the third power, the curve corresponding to the equation $a^3x^3 = y^6$ will be no other than the *Apollonian* parabola only. Raising the equation $ax = yy$ to the fourth power, the curve corresponding to the formula $a^4x^4 = y^8$ becomes the common parabola redoubled the contrary way. And, in general, if the power to which the formula $ax = yy$ is raised shall be even, the *Apollonian* parabola redoubled will exhibit the curve; if the power be odd, the common parabola will be sufficient.

The same doctrine may be applied to all first parabolas and hyperbolas, whose canonical equations are $a^{n-1}x = y^n$, taking for n any integer number, affirmative or negative. This being raised to an even power, the proper curve of the new equation will be the parabola or hyperbola $a^{n-1}x = y^n$ redoubled the contrary way. If the power be odd, the reduplication vanishes, and there will remain the simple genuine curve of the equation $a^{n-1}x = y^n$.

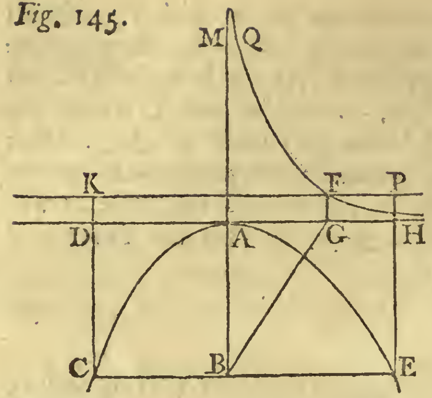
251. From the construction of parabolas of any degree, we may go on to the construction of hyperbolas also of any degree. Construction of hyperboloids.

The hyperboloids of the third degree are two; that is, $a^3 = xxy$, and $a^3 = xyy$. Let it be proposed to construct the hyperboloid of the equation $a^3 = xxy$. This curve will have two branches which approach to asymptotes; both of them will have their ordinates positive, but the abscisses in one will be positive, in the other negative.

To construct it, make $xx = az$, and, by substitution, it will be $aa = zy$. Between the asymptotes AM, AG, (Fig. 145.) describe the hyperbola FQ of the equation $aa = zy$. Then taking AG $= z$, it will be GF $= y$; then from the point G, at half a right angle, let be drawn GB, and it will be AB $=$ AG $= z$. To the axis AB let there be described the parabola CAE of the

 equation

Fig. 145.

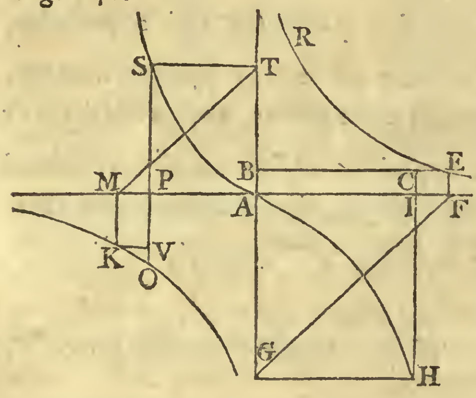


equation $az = xx$, and drawing the ordinates BC, BE, and the indefinite lines CK, EP, parallel to BA, it will be AH = BE $= x$. And drawing FK parallel to GD, it will be HP = GF $= y$. In the fame manner, it will be AD = BC $= -x$, DK $= y$; and the points P, K, will be in the curve propofed.

I forbear to give the conftruction of the equation $a^3 = xyy$, becaufe it is the very fame curve, only the co-ordinates have changed their places.

—of higher hyperboloids. 252. Let there be propofed an hyperboloid of the fourth degree, and let it's equation be $a^4 = x^3y$. This curve will have two branches, which apply to afymptotes, in one of which x will be pofitive, and y pofitive, and in the other x will be negative, and y negative.

Fig. 146.



Put $x^3 = aaz$, and, by fubftitution, we fhall have $zy = aa$. Between the afymptotes MF, TG, produced indefinitely, let the hyperbola of the equation $zy = aa$, or ER, KO, be defcribed. Then it will be AF $= z$, FE $= y$, AM $= -z$, MK $= -y$. From the point F, at half a right angle, draw FG, to which let MT be parallel, and it will be AG = AF $= z$, and AT = AM $= -z$. To the axis TG let be defcribed the cubic parabola SAH of the equation $x^3 = aaz$, and it will be AI = GH $= x$, and AP = TS $= -x$. Whence, drawing the right lines EC, KV, parallel to AI, it will be IC $= y$, and PV $= -y$, and the points C, V, will be in the propofed curve.

Here, alfo, I omit the conftruction of the equation $a^4 = xy^3$, becaufe, only changing the places of the co-ordinates, it is the fame as before. Alfo, I omit the conftruction of the equation $a^4 = xxyy$, becaufe it is reduced to the *Apollonian* hyperbola.

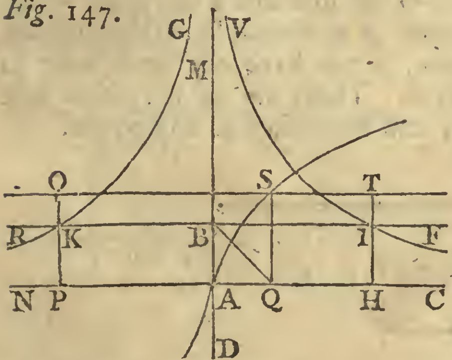
Other hyperboloids conftructed. 253. Let the hyperboloid of the fifth degree be propofed, and, firft, let the equation be $a^5 = x^4y$. This will have two branches, which approach to afymptotes; in one of which, taking x pofitive, y will alfo be pofitive. In the other, taking x negative, yet, however, y will be pofitive.

Make

Make $x^4 = a^3z$; then, fubftituting, it will be $aa = zy$. Between the afymptotes AG, AM, (Fig. 145.) defcribe the *Apollonian* hyperbola FQ of the equation $aa = zy$; then, taking AG $= z$, it will be GF $= y$. From the point G, at half a right angle, draw the right line GB, and it will be AB $=$ AG $= z$. To the axis AB defcribe the parabola CAE of the equation $x^4 = a^3z$, and it will be BE $=$ AH $= x$, BC $=$ AD $= -x$; and, drawing FK parallel to GD, and CK, EP, perpendicular to the fame, it will be HP $=$ DK $=$ GF $= y$, and the points P, K, will be in the curve propofed.

Let $a^5 = x^3y^2$ be another equation of the hyperboloid of the fame degree; this will have two branches, becaufe to the fame pofitive x will correfpond two ordinates y, one pofitive, the other negative.

Fig. 147.



Make $x^3 = aaz$; then, fubftituting, it will be $a^3 = zyy$. Between the afymptotes DM, CN, let there be defcribed the hyperboloid RG, FV, of the equation $a^3 = zy^2$, and making AH $= y$, AP $= -y$, it will be HI $= z =$ PK $=$ AB. To the axis PH let there be defcribed the cubic parabola AS of the equation $x^3 = aaz$, and from the point B draw BQ at half a right angle, and raife the perpendicular QS: then it will be AQ $= z$, QS $= x$. Through the point S draw the right line OT parallel to the afymptote NC, which may meet the produced lines HI, PK, in the points T, O. Then, it being AH $= y$, it will be HT $= x$, AP $= -y$, PO $= x$; and the points O, T, will be in the curve propofed.

The conftructions of the other two equations, $a^5 = x^2y^3$, and $a^5 = xy^4$, will be after the fame manner, only making the co-ordinates to change places. And by the fame artifice may all the hyperboloids of any degree be eafily conftructed.

254. It may be obferved, that all the firft parabolas, which are defcribed about one and the fame axis, will cut one another in the fame point. For, taking for every one of them the fame abfcifs $x = a$, they will all have the fame correfponding ordinate $y = a$; which could not be, except they all cut in the fame point. Obfervation on the forms of the firft paraboloids.

255. Alfo, the parabolas of higher dimenfions (meaning higher than the firft,) tend firft to arrive at the point of fection, above thofe of an inferior degree, approaching nearer to the tangent of the vertex, and after the fection they approach to the axis, thefe more than thofe. For, in the *Apollonian* parabola, it being $y = \sqrt{ax}$, in the firft cubic, $y = \sqrt[3]{aax}$, in the firft of the fourth degree, —of higher paraboloids and hyperboloids.

degree, $y = \sqrt[4]{a^3x}$, and ſo on; if we take x leſs than a, then $\sqrt{ax}$ will be leſs than $\sqrt[3]{aax}$, and this will be leſs than $\sqrt[4]{a^3x}$, and ſo on. But, on the contrary, taking x greater than a, it will be $\sqrt{ax}$ greater than $\sqrt[3]{aax}$, this greater than $\sqrt[4]{a^3x}$; and ſo on.

After the ſame manner, and for a like reaſon, the hyperboloids (meaning alſo the firſt,) all cut one another at the vertex, and thoſe of higher dimenſions tend after the point of ſection between thoſe of lower dimenſions and the aſymptote in which the x's are taken. And on the part of the aſymptote, parallel to y, the inferior tend within, between thoſe of higher dimenſions and the aſymptote.

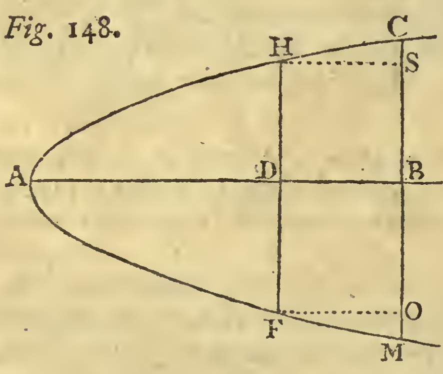
Curves of ſeveral terms conſtructed; divided into three caſes.

256. There remains now to conſtruct ſuch equations as have ſeveral terms, in which I ſhall diſtinguiſh three caſes. Thoſe of the firſt caſe I call ſuch, which have one term only, in which the indeterminate y is found, and that of one dimenſion alone. Of the ſecond caſe are thoſe, which have one term only in which y is found, but that raiſed to any power. Thoſe are of the third caſe which have many terms in which y is found; and that raiſed to any power.

CASE I. EXAMPLE I.

An example of the firſt caſe.

Fig. 148.



257. Let it be propoſed to conſtruct the curve of this equation $a^4 - x^4 = a^3y$. Make $y = t - q$, by which the given equation may be reſolved into theſe two, $a^4 = a^3t$, $x^4 = a^3q$. To the axis AB let the parabola MAC of the equation $x^4 = a^3q$ be deſcribed; and it being AD $= q$, it will be DH $= x$, DF $= -x$. But, by the equation $a^4 = a^3t$, it is $t = a$; and therefore, taking AB $= a = t$, it will be $t - q = y$. Whence, taking at pleaſure any abſciſs BS $=$ DH $= x$, and BO $=$ DF $= -x$, the lines SH, OF, parallel to BA, will be the correſponding ordinates of the curve propoſed, which is one portion of the ſame parabola of the fourth degree.

EX-

EXAMPLE II.

258. Let it be propofed to conftruct the curve of the equation $x^4 + ax^3 = a^3y$. By the rules already known, we may perceive this curve to have three branches, two infinite and pofitive, and one negative, together with a *maximum*, which at prefent we can take no notice of; and the axis will be cut in two points. Make $y = z + t$, whence we may have two equations, $x^4 = a^3z$, and $x^3 = aat$. To the axis AB let the parabola MAD of the equation $x^4 = a^3z$ be defcribed; and it being $AF = z$, it will be $FD = AE = x$. Through the fame point A let the cubic parabola CAP of the equation $x^3 = aat$ be defcribed, and $PE = t$ will correfpond to the fame x. Whence, it being $AE = x$, it will be $PE + ED = z + t = y$, making PD parallel to AF. Whence it may be feen, that, taking x pofitive, the ordinate y increafes *in infinitum*. Another example.

Fig. 149.

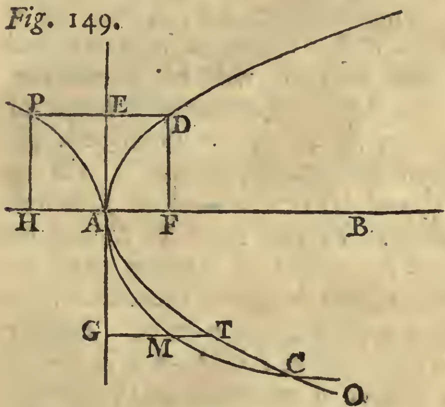
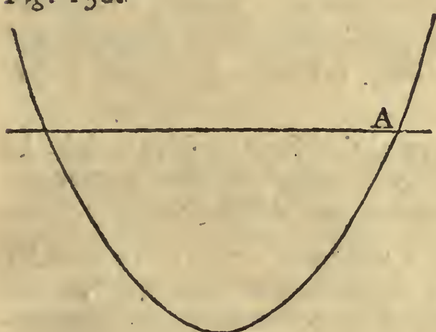


Fig. 150.



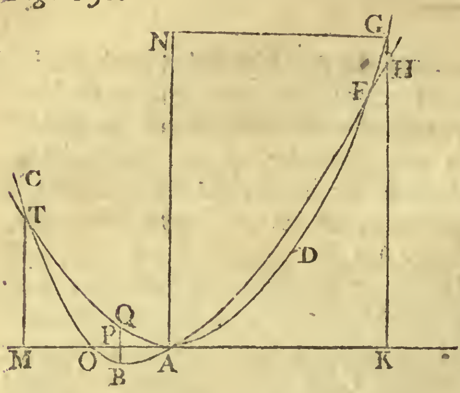
Then, taking x negative, t will be negative, and confequently $y = z - t$. Let $AG = x$ negative, it will be $GM = z$, $GT = t$, whence $y = MT$ negative; and among all the values of MT, there will be a greateft. Taking $x = -a$, it will be $GM = GT$, whence $y = 0$. Taking x negative, and greater than a, it will be $GM - GT$, a pofitive quantity; whence y will be pofitive, and will increafe *ad infinitum*. The curve will be nearly of the form of Fig. 150, taking x from the point A.

EXAMPLE III.

259. Let it be propofed to conftruct the curve of the equation $x^4 + ax^3 - aax^2 = a^3y$. This curve will have four branches, two pofitive and infinite, two negative and finite. It will cut the axis in two points, and will touch it in one. Another example to the firft cafe.

one. It will have two negative *maxima*, &c. as will be known by the rules to be delivered in their due place.

Fig. 151.



Put $y = z - q$, and make the two equations $x^4 + ax^3 = a^3z$, and $-xx = -aq$. The curve of the equation $x^4 + ax^3 = a^3z$ we know already how to construct by help of this method, and let it be CBADG (Fig. 151.); in which, taking AK $= x$ positive, it will be KG $= z$. Taking x negative $=$ AP, it will be z negative $=$ PB; taking x negative and greater than AO, it will be z positive. To the axis AN let the parabola TAH of the equation $xx = aq$ be described. It being then AK $= x$ positive, it will be KH $= q$, and GH $= z - q = y$, which will increase *in infinitum* as x increases *in infinitum*. In the point F it will be $z = q$, and $y = 0$. Between the points F and A, q will be greater than z; whence $z - q$ will be a negative quantity, and y negative, and there will be a negative *maximum*. In the point A, it will be $z = 0$, $q = 0$, $y = 0$. Taking x negative equal to AP, it will be $z =$ BP, and negative; whence y is always negative. Between the points A and O there will be a *maximum* BQ; whence there will be a greatest q negative. Taking x negative and greater than AO, z will be positive, but less than q; whence y is negative. Taking x negative and equal to AM, it will be $z = q$, and $y = 0$. Taking x negative and greater than AM, it will be always z greater than q; whence it will be always y positive *in infinitum*.

If the equation should more abound in terms, the same artifice might be used; and, though the construction in this case might become more compounded and perplexed, yet, however, the method would still obtain.

We might construct the last equation in a different manner, by making $y = z + t - q$, and thence deriving three equations, $x^4 = a^3z$, $x^3 = aat$, $-xx = -aq$, and, by means of these three auxiliary curves, we might proceed to the construction of the principal curve; but I omit this for brevity.

The co-ordinates may make any angle.

260. Perhaps, in these constructions, and in the few that follow, it may seem necessary that the angle of the co-ordinates should be a right angle, it being always supposed to be such. But it will appear, after a little reflection, that this angle may be as we please; especially if we give a little attention to the angle of the co-ordinates of the subsidiary curves introduced, relatively to the angle of the co-ordinates of the curve of the given equation.

CASE

CASE II. EXAMPLE IV.

261. Let it be propofed to conftruct this equation, $x^n \pm a^s x^{n-s} \pm a^m x^{n-m}$, &c. $= y^t$. Make $y^t = a^{t-1}z$, and fubftituting this value inftead of y^t, the equation will be $x^n \pm a^s x^{n-s} \pm a^m x^{n-m}$, &c. $= a^{t-1}z$. By the method of the firft cafe, this curve may be conftructed; then defcribe the parabola of the equation $y^t = a^{t-1}z$, and we fhall have the relation between x and y in the propofed equation.

The fecond cafe of curves conftructed.

CASE III. EXAMPLE V.

262. Let it be propofed to conftruct the equation $x^m \pm ax^n \pm bx^s$, &c. $= y^p \pm y^q$, &c. Make $y^p \pm y^q$, &c. $= z$; then, by fubftitution, the equation will be $x^m \pm ax^n \pm bx^s$, &c. $= z$. By the method of the firft cafe, each of thefe two auxiliary curves may be conftructed to the fame axis, in which z is to be taken; and we fhall have the relation of the two co-ordinates x and y of the curve propofed.

The third cafe conftructed, with a general example.

263. Hitherto I have confidered only thofe equations which have their indeterminates feparate; fo that, when the indeterminates are involved with each other, the rules hitherto given cannot take place.

To feparate the indeterminates when involved.

In thefe cafes there is need, either by the common divifion, or by the extraction of roots, or by a congruous fubftitution, or by other expedients, to contrive a feparation of the faid indeterminates. As, if we had the equation $a^3y + ax^2y = a^2x^2 + x^4$, dividing by $a^3 + ax^2$, it would be $y = \frac{aaxx + x^4}{a^3 + axx}$. And, if the equation were $aaxy + xxyy = x^4 + a^4$, making the fubftitution of $z = \frac{yx}{a}$, we fhould have the equation $a^3z + aazz = x^4 + a^4$, in which the indeterminates or unknown quantities are feparate.

The propofed equations being thus prepared, we may proceed to their conftruction in the following manner.

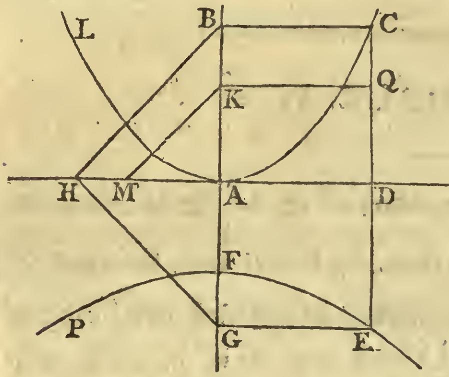
EXAMPLE VI.

Example of the construction of these *loci*.

264. Let the equation to be constructed be $y = \frac{a^2x^2 + x^4}{a^3 + ax^2}$. Make $a^2x^2 + x^4 = a^3p$. Make also $a^3 + ax^2 = a^2t$; and substituting these values in the equation proposed, it will be $y = \frac{ap}{t}$, that is, $t . p :: a . y$.

The proposed curve will have two branches, which stretch out *ad infinitum*. Positive y will correspond to x either positive or negative.

Fig. 152.



To axis HD let the curve LAC of the equation $aaxx + x^4 = a^3p$ be described; and, taking AD $= x$, it will be DC $= p$ $=$ AB. Take AF $= a =$ AM, then with vertex F, to the axis HD, let the curve PFE of the equation $a^3 + ax^2 = a^2t$ be described; and, taking AD $= x$, it will be DE $= t$. Whence, it being DC $= p$, and DE $= t$, draw EG parallel to AD, and from the point G draw GH at half a right angle, and it will be AH $= t$. From the point C draw CB parallel to DA, and draw the line BH, to which let MK be parallel. It being AD $= x$, it will be AK $= y$; for, because of similar triangles AMK, AHB, it will be AH . AB :: AM . AK; that is, $t . p :: a .$ AK $= \frac{ap}{t} = y$. Whence, drawing KQ parallel to the axis, the lines AD, DQ, will be the two co-ordinates of the curve proposed. To obtain the other branch of our curve, it will suffice to take x on the negative side, and to repeat the same construction on the contrary part.

EXAMPLE VII.

Another *locus* constructed.

265. Now let it be proposed to construct the other equation $a^2xy + x^2y^2 = x^4 + a^4$, which, being managed by the rules for affected quadratick equations, may have the indeterminates separated. Or, by the substitution of $z = \frac{xy}{a}$, it will be reduced to $a^3z + aazz = x^4 + a^4$. This equation may be constructed

ſtructed by the method of the third caſe, and we ſhall have the two co-ordinates x and z. Then make the analogy, $x \,.\, z :: a \,.\, y$, which will be the ordinate required. If one ſubſtitution be not enough, to free the indeterminates from being involved together, we muſt try more than one; and when none will ſucceed, the equations elude this method, and we muſt have recourſe to other artifices.

266. A convenient ſubſtitution may alſo be of uſe in other caſes, in which the indeterminates are already ſeparate; and may often ſuggeſt a conſtruction which is more eaſy and elegant. Wherefore it may not be amiſs to try ſeveral ways, that we may chooſe that which will prove to beſt advantage. An obſervation.

EXAMPLE VIII.

267. Let the equation be $y^4 - 4ay^3 + 4aayy = 2a^3x$. Make $2a^3x = z^4$, and therefore it will be $y^4 - 4ay^3 + 4aayy = z^4$, that is, $yy - 2ay = zz$, or $2ay - yy = zz$. Concluſion of the examples.

Therefore I conſtruct this *locus*, which in the firſt caſe will be, by two oppoſite equilateral hyperbolas, with tranſverſe axis equal to $2a$; and in the ſecond caſe, by a circle with diameter $= 2a$: and, in general, by this and that together.

Fig. 153.

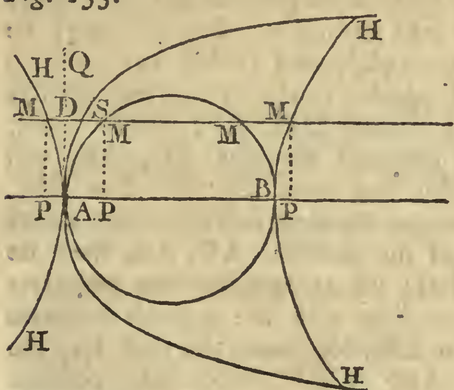
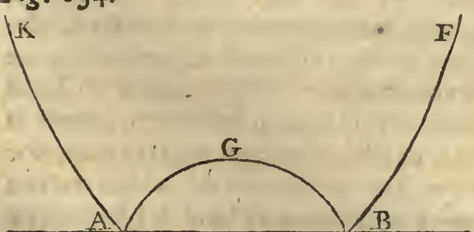


Fig. 154.



With tranſverſe diameter $AB = 2a$, (Fig. 153.) let there be deſcribed the two equilateral hyperbolas AMH, BMH, and the circle AMB. Then with vertex A, let the parabola of the equation $2a^3x = z^4$ be deſcribed, and raiſing the indefinite perpendicular AQ, and taking any line $AD = z$; then drawing MM parallel to AB, it will be $DS = x$, and $DM = y$, poſitive in the circle and in the hyperbola from A towards B, and negative in the hyperbola on the oppoſite part; and the curve will be nearly as KAGBF (Fig. 154.); in which the two branches, BF poſitive and AK negative, will go on *ad infinitum*; and there will be no branch under the axis AB, becauſe it can never be x negative.

SECT. VI.

Of the Method De Maximis et Minimis, *of the Tangents of Curves, of Contrary Flexure and Regreſſion; making uſe only of the Common* Algebra.

To find the *maxima* and *minima* of quantities, by compariſon with an equation of two equal roots.

268. Although the Calculus of Infiniteſimals be the ſimpleſt and the ſhorteſt method, and alſo the moſt univerſal, for managing ſuch ſpeculations; yet I was willing, before I finiſhed this Tract of Analyticks, or of what is called the *Carteſian* or Common *Algebra*, to ſhow very briefly, and by way of introduction, how the ſolution of ſuch queſtions may be performed, in geometrical curves, or ſuch as are expreſſed by finite algebräical equations, without the aſſiſtance of the *Differential Calculus*, or what is alſo called *The Method of Fluxions.*

Fig. 155.

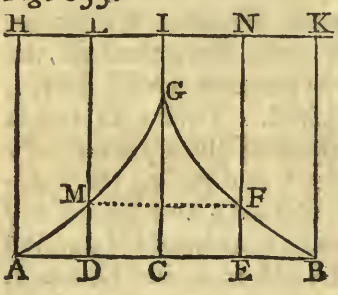
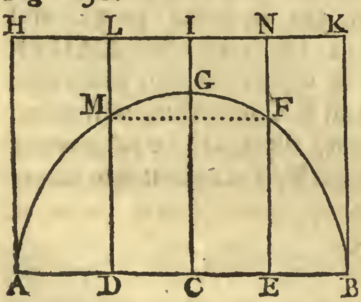


Fig. 156.



And to begin by the *Maxima* and *Minima*; that is to ſay, to find in geometrical curves the greateſt or the leaſt ordinates. Let the curve be AGB (Fig. 155, 156.), and taking any ordinate DM, draw MF parallel to the axis of the abſciſſes AB, the two ordinates DM, EF, will be equal, to which two different abſciſſes AD, AE, will correſpond. But the more the ordinates DM, EF, ſhall move approaching nearer to each other, the difference of the abſciſſes AD, AE, ſhall be ſo much the leſs; till at laſt the two ordinates DM, EF, coinciding with the greateſt ordinate CG, or the two LM, NF, with the leaſt IG, the abſciſſes AD, AE, or HL, HN, ſhall become equal in reſpect of the axis HK. Therefore, when the ordinate is the greateſt or the leaſt, the equation of the curve, diſpoſed according to the letter which expreſſes the abſciſs, ought to have two equal roots. To determine which, there is to be formed an equation of two equal roots, for example, $xx - 2ex + ee = 0$, which is the product of $x - e$ into $x - e$; and let the curve whoſe whofe greateft or leaft ordinates are required, be the ellipfis $xx - 2ax + \frac{2ayy}{p} = 0$, for example, the abfciffes being taken from the vertex. Let this equation be compared, term by term, with the equation formed from two equal roots, in the following manner: $xx - 2ax + \frac{2ayy}{p} = 0.$

$$xx - 2ex + ee = 0.$$

From the comparifon of the fecond terms, we find $a = e$; but e is the root of the equation $xx - 2ex + ee = 0$, and therefore $e = x$, and alfo $a = x$; and becaufe x is already determined, the comparifon of the laft terms will be fuperfluous. Wherefore, taking $x = a$, the correfponding ordinate in the ellipfis will be the greateft, as is already known, it being then half the conjugate axis.

But if the equation of the curve had been of the third, fourth, or higher degree, that we might make the comparifon, it would be neceffary that the equation of two equal roots, $xx - 2ex + ee = 0$, fhould be reduced to the fame degree as is the equation propofed, by multiplying it by fo many roots, whatever they may be, as there may be occafion for. Let the curve belong to this equation of the third degree, $x^3 * - axy + y^3 = 0$, (the afterifk * is put in the place of the fecond term which is wanting, and which fhould always be done, as often as any term is abfent,) of which we require the greateft ordinate. Therefore I multiply the equation $xx - 2ex + ee = 0$ by $x - f = 0$, and compare the product with the equation propofed, $x^3 \quad * \quad - axy + y^3 = 0.$

$$x^3 - 2ex^2 + eex - eef = 0.$$
$$- fx^2 + 2efx$$

From the comparifon of the fecond terms, I find $-2e - f = 0$, and therefore $f = -2e$. From the comparifon of the third, I find $2ef + ee = -ay$, and fubftituting the value of f, it is $-3ee = -ay$. But $e = x$, therefore $y = \frac{3xx}{a}$. Inftead of y, if we fubftitute this value in the equation of the curve, it will give us $x = \frac{\sqrt[3]{2a^3}}{3}$, to which correfponds the greateft ordinate y, which will be $\frac{a \times 2^{\frac{2}{3}}}{3}$, or $\frac{\sqrt[3]{4a^3}}{3}$.

269. But, without comparing the given equation with another, which contains two equal roots, to fatisfy the condition of the Problem, it will be fufficient to multiply it, term by term, by any arithmetical progreffion. For, if the equation has two equal roots, as it ought to have in the cafe of a *maximum* or *minimum*, one of thofe roots will alfo, of neceffity, be included in the product of that equation multiplied by the arithmetical progreffion. Whence, by thus multiplying the equation, the condition will be included, under which the value of

To find the fame by multiplying by an arithmetical progreffion.

of the abscifs will be found, to which the greatest or least ordinate corresponds. Now, to demonstrate this, let the equation of the two equal roots be in general this, $xx - 2hx + hh = 0$, which let be multiplied by the arithmetical progression a, $a + b$, $a + 2b$, and the product will be

$$\begin{aligned} axx - 2ahx &+ ahh = 0. \\ - 2bhx &+ 2bhh \end{aligned}$$

In this substitute the quantity h instead of x, and all the terms will destroy one another. Or else, dividing it by $x - h$, the division will succeed. Therefore $x - h$ will be one root of that product, as it is of $xx - 2hx + hh = 0$. The same will obtain if the arithmetical progression be decreasing, as a, $a - b$, $a - 2b$, $a - 3b$, &c.

Now, because the equation of the two equal roots is general, and the arithmetical progression a, $a + b$, $a + 2b$, &c. is general also, it will always be true, that when an equation of two equal roots is multiplied, term by term, by any arithmetical progression, the product will be divisible by one of those roots. For the same reason, if an equation shall have three equal roots, and be multiplied by an arithmetical progression, the product will have two of those equal roots. And if this product be multiplied again by an arithmetical progression, the new product will have one of those roots. And so we may go on to superior equations.

I resume the equation to the ellipsis $xx - 2ax + \frac{2ayy}{p} = 0$, which I multiply by the progression 2, 1, 0.

$$\begin{matrix} xx & - 2ax & + \frac{2ayy}{p} = 0. \\ 2, & 1, & 0. \end{matrix}$$

The product is $2xx - 2ax = 0$, which gives $x = a$, as is found above. I multiply the same equation by another arithmetical progression, 3, 2, 1,

$$\begin{matrix} xx & - 2ax & + \frac{2ayy}{p} = 0 \\ 3, & 2, & 1, \end{matrix}$$

The product is $3xx - 4ax + \frac{2ayy}{p} = 0$, in which, instead of yy, I substitute it's value, $\overline{2ax - xx} \times \frac{p}{2a}$, given from the equation of the curve, and find $x = a$, as before.

I take the second equation above, $x^3 * - axy + y^3 = 0$, and multiply it by the progression 3, 2, 1, 0,

$$\begin{matrix} x^3 & * & - axy & + y^3 = 0, \\ 3, & 2, & 1, & 0, \end{matrix}$$

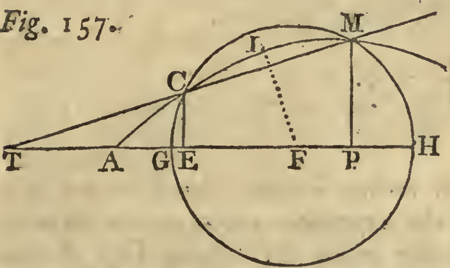
The product is $3x^3 - axy = 0$, or $3x^2 = ay$, as before.

270. By

270. By a like method may be found the tangents and perpendiculars to curves in any given points. *Tangents and perpendiculars, how found.*

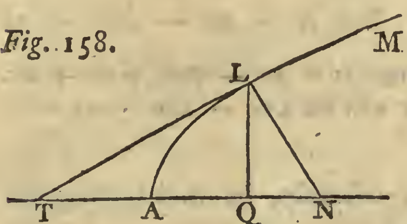
The queftion is reduced to this; to find a circle that fhall touch the curve in this point. For, in this cafe, the tangent of the circle in this point, as alfo the perpendicular or radius, will be in common to the curve alfo in the fame point.

Fig. 157.



Let the curve be ACM, of which we defire the tangent at the point L; and let the circle be GMH, which cuts it in the two points M, C. Drawing the two ordinates CE, MP, and the right line MCT, through the points M, C, it will cut the curve alfo in the points M, C. But the nearer thefe points fhall approach to each other, the lefs always will be the difference of the ordinates CE, MP, and alfo of the abfciffes AE, AP; fo that when the two points coincide, for example at L, they will make the values equal of thefe ordinates, or of thefe abfciffes; and then the circle will touch the curve in the point L. (Except when the curve and the circle are of equal curvature; for, in this cafe, the circle will both cut and touch the curve in the fame point, as will be feen in the Differential Calculus.) The right line MT fhall be a tangent both to the curve and the circle in the fame point L; as alfo, FL will be a common perpendicular.

Fig. 158.



Therefore, in the curve ALM, make $AQ = x$, $QL = y$, and from the given point L drawing the right line LN, which we fuppofe to be perpendicular to the curve, and confequently to the tangent at L; make $LN = s$, $AN = u$, and it will be $QN = u - x$. Then the right-angled triangle QLN will give the canonical equation $ss = uu - 2ux + xx + yy$, from which we are to have the value of y, or of x, and to fubftitute it in the equation of the given curve; by means of which we muft have the value of s, or of u, confidering x or y as given, becaufe we affume the point L as given.

Let the curve ALM, for example, be the *Apollonian* parabola of the equation $ax = yy$. Inftead of yy, make a fubftitution of it's value given by the canonical equation, and we fhall have $ax = ss - uu + 2ux - xx$; which being ordered according to the letter x, will be $\begin{matrix} xx - 2ux + uu = 0. \\ + ax - ss \end{matrix}$ This equation, therefore, ought to have two equal roots when the right line $LN = s$ is perpendicular to the parabola in the point L, that is, in the cafe of a tangent. Therefore, the value of the indeterminate AN $= u$ being found, on the hypothefis of two equal roots, we fhall have the point N, from whence drawing NL to the given point L, and LT perpendicular to NL, that fhall be the tangent required.

Now, to determine the unknown quantity u on the fuppofition of two equal roots; I compare the equation, term by term, with one of two equal roots, that is, with $xx - 2ex + ee = 0$, after the following manner:

$$\left.\begin{array}{l} xx - 2ux + uu \\ \quad\ + ax - ss \end{array}\right\} = 0.$$

$$xx - 2ex + ee = 0.$$

Now, from the comparifon of the fecond terms, we fhall have $-2u + a = -2e$, or $u = \frac{1}{2}a + e$. But $e = x$, by the equation $xx - 2ex + ee = 0$. Therefore $u = \frac{1}{2}a + x$. Wherefore, from the point Q, taking QN $= \frac{1}{2}a$, NL will be the perpendicular, and LT, perpendicular to it, will be the tangent to the curve in the point L.

Inftead of comparing the faid equation with one of two equal roots, it may be multiplied by this arithmetical progreffion 3, 2, 1, thus:

$$\left.\begin{array}{l} xx - 2ux + uu \\ \quad\ + ax - ss \end{array}\right\} = 0.$$

$$3, \quad 2, \quad 1,$$

The product is $\left.\begin{array}{l} 3xx - 4ux + uu \\ \quad\ + 2ax - ss \end{array}\right\} = 0.$ But $ss = uu - 2ux + xx + yy$; and, by the parabola, it is $yy = ax$; whence $ss = uu - 2ux + xx + ax$. Subftituting, therefore, this value inftead of ss, it will be $2xx - 2ux + ax = 0$. That is, $u = \frac{1}{2}a + x$, as before.

We might have had our defire more compendioufly, by multiplying the equation by this arithmetical progreffion, 2, 1, 0.

Example. 271. Let the curve be the fecond cubical parabola $x^3 = ayy$. Making the fubftitution of the value of yy, derived from the canonical equation, there arifes the equation $x^3 + ax^2 \begin{array}{l} - 2aux + auu \\ \qquad\quad - ass \end{array} = 0$, which, becaufe it is of the third degree, muft be compared with the product of the equation $xx - 2ex + ee = 0$ into $x - f = 0$; thus, $\left.\begin{array}{l} x^3 + ax^2 - 2aux + auu \\ \qquad\qquad\qquad\quad - ass \end{array}\right\} = 0.$

$$\begin{array}{l} x^3 - 2ex^2 + eex - eef = 0. \\ \quad\ - fx^2 + 2efx \end{array}$$

By

By comparing the ſecond terms, we have $-2e - f = a$, that is, $f = -a - 2e$. From the compariſon of the third, it is $ee + 2ef = -2au$; and putting the value of f now found, it is $u = \frac{3ee + 2ae}{2a}$, that is, $u = \frac{3xx + 2ax}{2a}$, becauſe $e = x$.

Now I ſhall multiply the equation by the arithmetical progreſſion 3, 2, 1, 0,

$$\left.\begin{array}{l} x^3 + ax^2 - 2aux + auu \\ \qquad\qquad\qquad\quad - ass \end{array}\right\} = 0,$$

$$3, \quad 2, \quad 1, \quad 0,$$

The product is $3x^3 + 2ax^2 - 2aux = 0$, and therefore, in like manner, $u = \frac{3xx + 2ax}{2a}$.

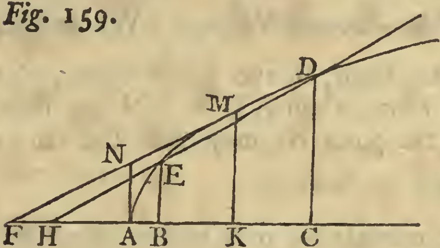
272. Concerning the choice of a proper arithmetical progreſſion, it may be obſerved, that, generally, that will be the moſt convenient, which forms the exponents, beginning with the greateſt index of that letter according to which the equation is ordered.

How to chooſe a progreſſion.

273. Another manner of ſolving this Problem may be this, which is ſomething different, but perhaps more ſimple, and which will be of uſe in contrary flexures and regreſſions.

This Problem ſolved another way.

Fig. 159.



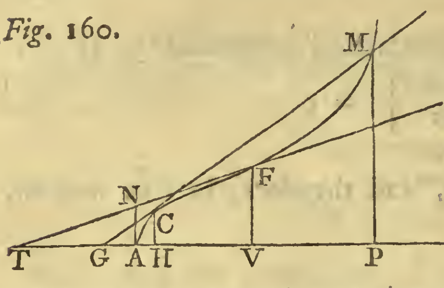
Let the curve AEMD be cut by the right line HED in the points E, D; and make the abſciſſes AB or AC $= x$, the ordinates BE or CD $= y$. It is plain that the right line HD going on to be the tangent FM of the curve in the point M, the two points E, D, will coincide in M, and conſequently will make the two lines AB, AC, equal to each other, as alſo the two lines BE, CD. Draw AN parallel to the ordinates, and make AF $= u$, AN $= s$. By the ſimilar triangles FAN, FKM, it will be $u \,.\, s :: u + x \,.\, y$; that is, $y = \frac{us + sx}{u}$, and $x = \frac{uy - us}{s}$. In the equation of the given curve, ſubſtitute theſe values inſtead of y or x, and another equation will ariſe from hence, which will have two equal roots, ſince AF, AN, are ſuch, as that the right line FNM touches the curve. Therefore, making a compariſon with another of two equal roots, or multiplying it by an arithmetical progreſſion, we ſhall have the value of AF or AN required; and one being given, the other will alſo be given. I forbear Examples, becauſe the manner of operation is the ſame as that uſed before.

274. As the nature of *maxima* and *minima*, and likewiſe of tangents, neceſſarily requires equations of two equal roots, ſo, in contrary flexures and re-

Points of contrary flexure and regreſſion what, and how found.

greſſions, three equal roots are required. By contrary flexure is meant that point, in which from concave the curve becomes convex, or the contrary; and by regreſſion is meant that point in which the curve turns directly back again, whether concave or convex.

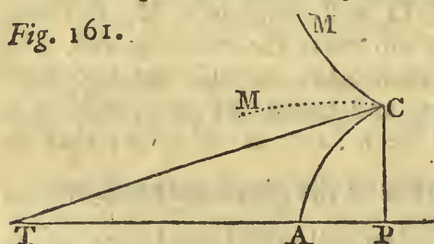
Fig. 160.



Let the curve be ACFM, which has a contrary flexure in the point F, and let be drawn the right line GCM, which touches it in the point C, and cuts it in the point M; from which draw the ordinates CH, MP. It is eaſy to perceive, that the more the point C of the tangent ſhall approach to the point F of contrary flexure, ſo much the more alſo the point M ſhall approach to the point F; ſo that when the point C falls in with F, the point M will alſo fall in with it; and conſequently AH, AP, will become equal, as alſo CH, MP, and the right line GCM will both touch and cut the curve in the point F. But the nature of the tangent already requires two equal roots, and now they are joined by a third; ſo that the property of contrary flexure is ſuch, that three equal roots are correſponding to it.

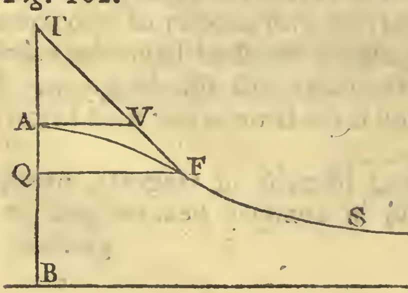
From the point A drawing AN parallel to the ordinates, and making AN $= s$, AT $= u$, and drawing TNF; becauſe of ſimilar triangles TAN, TVF, it will be $y = \frac{us + sx}{u}$, and $x = \frac{uy - us}{s}$, making VA $= x$, and VF $= y$. Wherefore, ſubſtituting theſe values of x or y in the equation of the given curve, the equation that ariſes ought to have three equal roots, when AT or AN are ſuch that TNF, drawn from the point T through the point N, may meet the curve in F, the point of contrary flexure required.

Fig. 161.



In like manner we may reaſon about the curve ACM, which has a regreſſion in the point C. For the tangent TC of the curve in the point C, will alſo cut it in the ſame point; and thence the three equal roots will ariſe after the ſame manner.

Fig. 162.



Let AFS be the curve of the equation $ayy - xyy - aax = 0$, in which are AQ $= x$, and QF $= y$; and let the point F of contrary flexure be required. Make AT $= u$, AV $= s$, and QF parallel to the ordinates. Now, inſtead of x, ſubſtituting it's value $\frac{uy - us}{s}$, in the equation of the curve, it will be

$$\left.\begin{array}{l} y^3 - \frac{asyy}{u} + aayy - aas \\ \quad - syy \end{array}\right\} = 0.$$

This equation ought to have three equal roots, and therefore we muſt compare it with an equation of three equal roots; or elſe multiply it by two arithmetical progreſſions.

Let us multiply it, therefore, by the progreſſion 1, 0, — 1, — 2, and the product will be $y^3 * - aay + 2a^2s = 0$. Multiply it again by the progreſſion 3, 2, 1, 0, which will give us $3y^3 - aay = 0$, and therefore $yy = \frac{1}{3}aa$. This value, being ſubſtituted in the equation of the given curve, will laſtly produce $x = \frac{1}{4}a$.

275. The manner is the ſame for finding the regreſſions of curves, and this method is applicable to both. So that, to diſtinguiſh them, there is no other way, but to find, by means of a conſtruction, the figure and proceeding of the curve.

To diſtinguiſh contrary flexures from regreſſions, and *maxima* from *minima*.

The ſame ambiguity ariſes in queſtions *de maximis et minimis*, which only can be removed by acquiring ſome knowledge of the diſpoſition of the curve. By the ſame condition of three equal roots we may find the *Radii* of Curvature; but as I ſhall further treat of ſuch things in the following Volume, not to be too tedious, I ſhall here put an end to this.

END OF THE FIRST VOLUME.

ANALYTICAL INSTITUTIONS.

ANALYTICAL INSTITUTIONS,

IN FOUR BOOKS:

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BY

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PROFESSOR OF THE MATHEMATICKS AND PHILOSOPHY IN THE UNIVERSITY OF BOLOGNA.

TRANSLATED INTO ENGLISH

BY THE LATE

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ANALYTICAL INSTITUTIONS.

BOOK II.

THE ANALYSIS OF QUANTITIES INFINITELY SMALL.

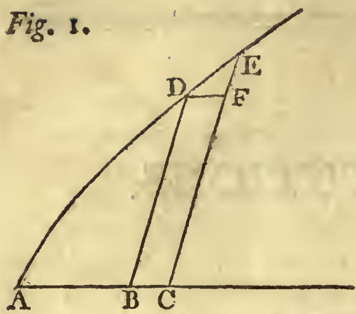
THE Analysis of infinitely ſmall Quantities, which is otherwiſe called the *Differential Calculus*, or the *Method of Fluxions*, is that which is converſant about the differences of variable quantities, of whatever order thoſe differences may be. This Calculus contains the methods of finding the Tangents of Curve-Lines, of the *Maxima* and *Minima* of Quantities, of Points of Contrary Flexure, and of the Regreſſion of Curves, of the *Radii* of Curvature, &c.; and therefore we ſhall divide it into ſeveral Sections, as the nature of the ſeveral ſubjects may require. Introduction.

SECT. I.

Of the Notion or Notation of Differentials *of ſeveral Orders, and the Method of calculating with the ſame.*

1. By the name of *Variable Quantities* we underſtand ſuch, as are capable of continual increaſe or decreaſe, while others continue the ſame. They are to be conceived as *Flowing Quantities*, or as generated (as it were) by a continual motion. Variable quantities, what.

Fig. 1.



For inſtance, in Fig. 1, let there be a right line ABC, which is conceived as generated by the motion of the point A, and is produced *in infinitum*. Upon this, at any inclination, let another right line BD inſiſt, and let it be conceived that, whilſt the point B moves from B to C, carrying with it the line BD from the place BD to CE, always remaining parallel to itſelf, the point D ſhall deſcribe the line FE in ſuch a manner, as to paſs through all the points of the curve ADE. It is plain that the abſciſſes AB, AC, as alſo the ordinates BD, CE, and likewiſe the arches AD, AE, will be quantities continually increaſing and decreaſing, and therefore are called *Variable Quantities*, or *Fluents*, or *Flowing Quantities*.

Conſtant quantities, what.

2. *Conſtant Quantities* are ſuch, which neither increaſe nor diminiſh, but are conceived as invariable and determinate, while others vary. Such are the parameters, diameters, axes, &c. of curve-lines.

Conſtant quantities are repreſented by the firſt letters of the alphabet, a, b, c, d, &c. and variable quantities by the laſt letters, z, y, x, v, &c. juſt as is uſually done in the common Algebra, in reſpect to known and unknown quantities.

A fluxion or difference, what.

3. Any infinitely little portion of a variable quantity is called it's *Difference* or *Fluxion*; when it is ſo ſmall, as that it has to the variable itſelf a leſs proportion than any that can be aſſigned; and by which the ſame variable being either increaſed or diminiſhed, it may ſtill be conceived the ſame as at firſt.

Fig. 2.

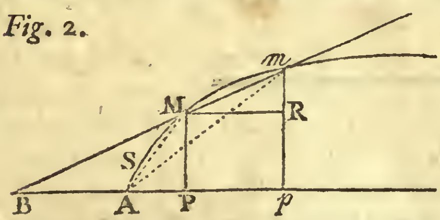
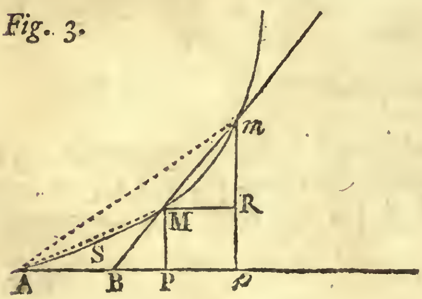


Fig. 3.



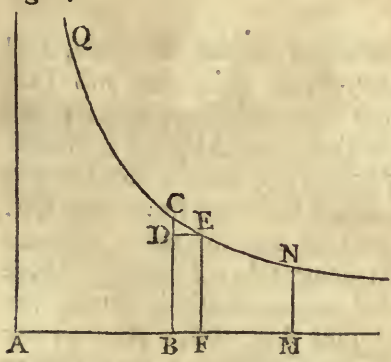
Let AM (Fig. 2, 3.) be a curve whoſe axis or diameter is AP; and if, in AP produced, we take an infinitely little portion Pp, it will be the difference or fluxion of the abſciſs AP, and therefore the two lines AP, Ap, may ſtill be conſidered as equal, there being no aſſignable proportion between the finite quantity AP, and the infinitely little portion Pp. From the points P, p, if we raiſe the two parallel ordinates PM, pm, in any angle, and draw the chord mM produced to B, and the right line MR parallel to AP; then, becauſe the two triangles BPM, MRm, are ſimilar, it will be BP . PM :: MR . Rm. But the two quantities BP, PM, are finite, and MR is infinitely little; then,

then alſo Rm will be infinitely little, and is therefore the fluxion of the ordinate PM. For the ſame reaſon, the chord Mm will be infinitely little; but (as will be ſhown afterwards,) the chord Mm does not differ from it's little arch, and they may be taken indifferently for each other; therefore the arch Mm will be an infinitely little quantity, and conſequently will be the fluxion or difference of the arch of the curve AM. Hence it may be plainly ſeen, that the ſpace PMmp likewiſe, contained by the two ordinates PM, pm, by the infiniteſimal Pp, and by the infinitely little arch Mm, will be the fluxion of the area AMP, comprehended between the two co-ordinates AP, PM, and the curve AM. And drawing the two chords AM, Am, the mixtilinear triangle MAm will be the fluxion of the ſegment AMS, comprehended by the chord AM, and by the curve ASM.

How fluxions are repreſented, and what are their ſeveral orders.

4. The mark or characteriſtic by which Fluxions are uſed to be expreſſed, is by putting a point over the quantity of which it is the fluxion. Thus, if the abſciſs AP $= x$, then will it be Pp or MR $= \dot{x}$. And, in like manner, if the ordinate PM $= y$, then it will be Rm $= \dot{y}$. And making the arch of the curve ASM $= s$, the ſpace APMS $= t$, the ſegment AMS $= u$, it will be M$m = \dot{s}$, PM$mp = \dot{t}$, AM$m = \dot{u}$. And all theſe are called *Firſt Fluxions*, or *Differences* of the *firſt Order.* And it may be obſerved, that the foregoing fluxions are written with the affirmative ſign $+$ if their flowing quantities increaſe, and with the negative ſign $-$ if they decreaſe. Thus, in the curve NEC, (Fig. 4.) becauſe AB $= x$, BF $= \dot{x}$, BC $= y$, it will be DC $= -\dot{y}$, the negative fluxion of y.

Fig. 4.

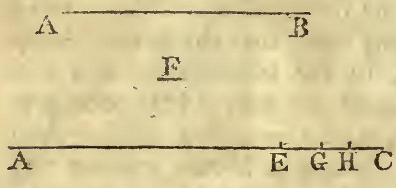


That theſe differential quantities are real things, and not merely creatures of the imagination, (beſides what is manifeſt concerning them, from the methods of the Ancients, of polygons inſcribed and circumſcribed,) may be clearly perceived from only conſidering that the ordinate MN (Fig. 4.) moves continually approaching towards BC, and finally coincides with it. But it is plain, that, before theſe two lines coincide, they will have a diſtance between them, or a difference, which is altogether inaſſignable, that is, leſs than any given quantity whatever. In ſuch a poſition let the lines BC, FE, be ſuppoſed to be, and then BF, CD, will be quantities leſs than any that can be given, and therefore will be *inaſſignable*, or *differentials*, or *infiniteſimals*, or, finally, *fluxions*.

Thus, by the common Geometry alone, we are aſſured that not only theſe infinitely little quantities, but infinite others of inferior orders, really enter the compoſition of geometrical extenſion. If incommenſurable quantities exiſt in Geometry, which are infinites in their kind, as is well known to Geometricians and

and Analyſts, then infiniteſimal magnitudes of various orders muſt neceſſarily be admitted.

Fig. 5.



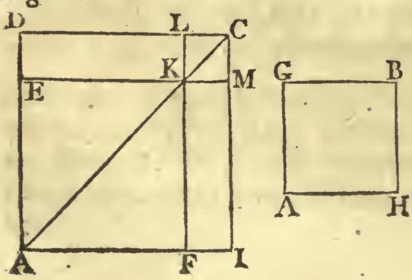
For the ſake of an example, let AB be the ſide of a ſquare, and AC it's diagonal or diameter; which two lines (by the laſt propoſition of the tenth Book of *Euclid*,) are incommenſurable to each other. Now it may be proved that this aſymmetry of their's does not proceed from any little finite line CE, how ſmall ſoever it may be taken, but from another which is infinitely leſs than it, and therefore of the infiniteſimal order.

Let it be ſuppoſed then, if poſſible, that it is the finite line CE which is the cauſe of the aſymmetry or incommenſurability between the two lines AB, AC; conſequently the remaining line AE will be commenſurable to the ſide AB. Let the right line F be their common meaſure, which can never be equal to EC, for then the diameter and ſide would be commenſurable. It muſt therefore be either greater or leſs than it.

In the firſt caſe, let F be ſubtracted from CE as often as can be done, and let the remainder be CG. Now, becauſe F meaſures AB, AE, and alſo EG, the two right lines AB, AG, will have to each other a rational proportion; and therefore it was not the magnitude CE that made the lines AB, AC, incommenſurable, but ſome quantity leſs than it, ſuppoſe GC, which therefore is finite, the finite line F being once or oftener ſubtracted from the finite line CE. Let F be biſected, and each part biſected again, and ſo on, till there ariſe an aliquot part of F which is leſs than CG, and which being taken from CG, there will remain CH. But this, by the ſame way of argumentation, is not the quantity that cauſes the incommenſurability of the lines AB, AC. And as the ſame way of reaſoning obtains in all other finite magnitudes, we may thence fairly conclude that the incommenſurability proceeds from an inaſſignable quantity, or which is leſs than any that can be given. The ſame may be alſo proved in the other caſe, or when the common meaſure F is greater than CE.

From hence I ſhall proceed, further, to take notice, that the ſquares upon the right lines AB, AC, which are to each other as one to two, notwithſtanding that their ſides are irrational, are nevertheleſs commenſurable to each other; and that this proceeds from an infinitely little quantity of the ſecond order. The two ſquares AB, AC, being propoſed, (Fig. 6.) let the two quantities ED, FI, equal and infiniteſimal, be thoſe which render the ſides AD, AG, AI, AH, incommenſurable; and the conſtruction being completed as in the figure, it is known that the two rectangles DK, IK, are incommenſurable

Fig. 6.



menfurable to the fquare AB. But the whole fquare AC is to the other AB in a rational proportion: therefore the fquare AC is made fo by the infinitefimal fquare KC, a quantity of the fecond order, by which it exceeds the faid incommenfurable gnomon.

It may be obferved, that cubes upon the lines AI, AH, are incommenfurable, although their bafes are rational; and it may be eafily proved, that they are made fuch by means of an inaffignable magnitude of the third order, and we may go on in like manner as far as we pleafe.

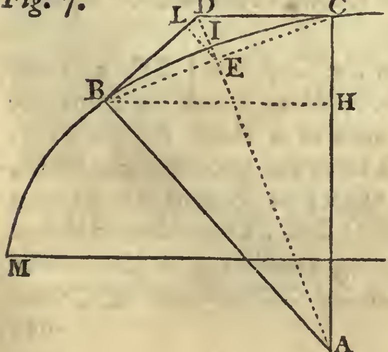
5. After the fame manner that firft differences or fluxions have no affignable proportion to finite quantities; fo differences or fluxions of the fecond order have no affignable proportion to firft differences, and are infinitely lefs than they: fo that two infinitely little quantities of the firft order, which differ from each other only by a quantity of the fecond order, may be affumed as equal to each other. The fame is to be underftood of third differences or fluxions in refpect of the fecond; and fo on to higher orders. How higher orders of fluxions are reprefented.

Second fluxions are ufed to be reprefented by two points over the letter, third fluxions by three points, and fo on. So that the fluxion of $\dot{x}$, or the fecond fluxion of x, is written thus, $\ddot{x}$; where it may be obferved, that $\ddot{x}$ and $\dot{x}^2$ are not the fame, the firft fignifying (as faid before,) the fecond fluxion of x, and the other fignifying the fquare of $\dot{x}$. The third fluxion of x will be $\dddot{x}$, and fo on. Thus, $\ddot{y}$ will be the fluxion of $\dot{y}$, or the fecond fluxion of y; and fo of others.

But, to give a juft idea of fecond, third, &c. fluxions, the following Theorems will be convenient.

THEOREM I.

Fig. 7.



6. Let there be any curve MBC, and BC an infinitely little portion of it of the firft order. From the points B, C, let the right lines BA, CA, be drawn perpendicular to the curve, and meeting in A. I fay, the lines BA, CA, may be affumed as equal to each other. Infinitefimals proved to exift.

Let the tangents BD, CD, be drawn, and the chord BC. If the two lines BA, CA, be unequal, let one of them, as CA, be the greater, and to this let the perpendicular BH be be drawn. The difference between the lines BA, CA, will be lefs than the intercepted line CH, which is lefs than the chord CB, becaufe of the right angle at H. But the chord BC is an infinitefimal of the firft order, the arch being fuppofed an infinitefimal; therefore the difference between BA and CA, at leaft, will not be greater than an infinitefimal of the firft order, and therefore thofe lines BA and CA may be affumed as equal.

Coroll. I. Therefore the triangle BAC will be equicrural, and thence the angles at the bafe ABC, ACB, will be equal; and being fubtracted from the right angles ABD, ACD, will leave the two angles BCD, DBC, equal to each other, and confequently the two tangents BD, CD, will be equal.

Coroll. II. The right line DA being drawn, the two triangles ADB, ADC, will be equal and fimilar; and that line will bifect the angles BAC, BDC. And, becaufe the two triangles AEB, AEC, are fimilar and equal, the fame line AD will be perpendicular to BC, and will divide it into equal parts in E.

Coroll. III. And the two triangles DAC, EDC, being fimilar, the angle DCE will be equal to the angle DAC; and the two angles DCE, DBE, being taken together, will be equal to the angle BAC.

Coroll. IV. From hence it follows, that any infinitefimal arch BC, of any curve whatever, will have the fame affections and properties as the arch of a circle, defcribed on the centre A, with the radius AB or AC.

Coroll. V. The two triangles AEB, BED, being fimilar, we fhall have AE . EB :: EB . ED. But AE is a finite line, and EB an infinitefimal of the firft order; therefore ED will be an infinitefimal of the fecond order, and it's value will be $= \frac{EBq}{AE}$. But the rectangle of twice AE into EI is equal to the fquare of EB, from the property of the circle. Therefore $EBq = 2AE \times EI = AE \times ED$, and confequently 2AE . AE :: ED . EI. But the firft term of the analogy is double to the fecond, therefore the third is double to the fourth. Confequently the two lines EI, DI, of the fecond order will be equal.

Coroll. VI. And therefore the difference between the femichord BE, and the tangent BD, is an infinitefimal of the third degree; for as much as from the centre B, and with the diftance BE, drawing the arch of a circle EL, a magnitude of the fecond clafs, which coincides with it's fine; the two triangles BDE, EDL, will be fimilar, which, befides the right angles at E and L, have a common angle in D. Thence it will be BD . DE :: DE . DL. But BD is a firft fluxion, DE is a fecond fluxion by the foregoing corollary, and therefore DL will be a third fluxion. Wherefore the arch of the curve BI being greater

 than

than the ſemichord BE, and leſs than the tangent BD, it cannot differ from either of them but by a magnitude of the third order.

THEOREM II.

Fig. 8.

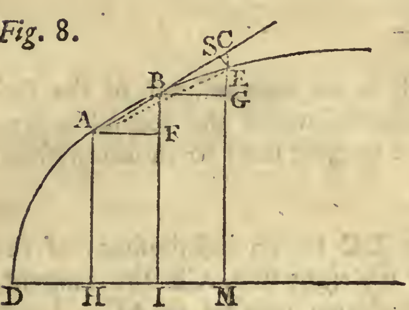
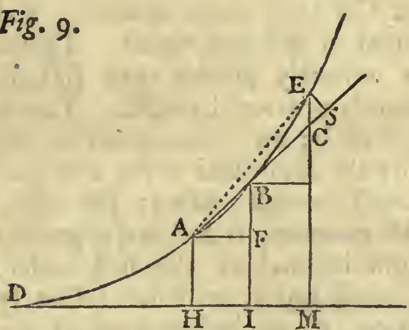


Fig. 9.



7. Let there be any curve whatever, DAE (Fig. 8, 9.), in whoſe axis are taken two equal infiniteſimal portions of the firſt order HI, IM; let parallel ordinates HA, IB, ME, be drawn, which in the given curve ſhall cut off the little arches AB, BE, which are likewiſe infiniteſimals of the firſt order. Let there be drawn the chord ABC, which ſhall meet the ordinate produced, ME, in the point C. I ſay, that the intercepted line CE, between the curve and the chord AB produced, ſhall be an infiniteſimal of the ſecond order.

Let the chord AE be drawn. If the right line IM were a finite and aſſignable quantity, then the triangle ACE would alſo be finite. But ME continually approaching, [from a finite diſtance,] to the ordinate HA, [while IB remains fixed,] ſo that IM may alſo become a fluxion, or may be an infiniteſimal of the firſt order; the angle ACE always continuing the ſame, the angle AEC increaſes, making the angle CAE always leſs and leſs, till at laſt it becomes leſs than any given angle, that is, an infiniteſimal. In this caſe, as the ſine of an infinitely little angle of the firſt order, having a finite and aſſignable radius, is an infiniteſimal quantity of the firſt order; ſo the ſine of an infiniteſimal angle, CAE, of the firſt order, with a radius AE or AC, which is an infiniteſimal quantity of the firſt order, ſhall be an infiniteſimal quantity of the ſecond order. But in triangles the ſides are proportional to the ſines of the oppoſite angles, and therefore the right line CE ſhall be an infiniteſimal of the ſecond order.

Wherefore, calling $DH = x$, $HA = y$, $HI = IM = \dot{x}$; then $FB = GC = \dot{y}$, and $EC = -\ddot{y}$; the negative ſign being prefixed, becauſe $\dot{y}$ does not increaſe but diminiſh (Fig. 8.). And thus, on the contrary, it will have the poſitive ſign if $\dot{y}$ increaſe, that is, if the curve be convex in this point to the axis DM (Fig. 9.).

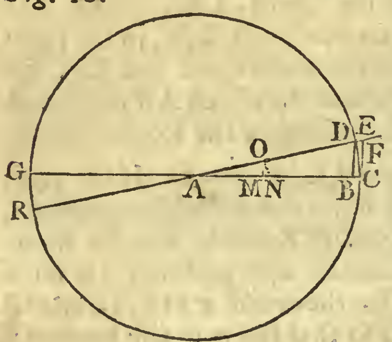
Coroll.

Coroll. If from the point E the normal ES be drawn to BC, then alſo ES, CS, will be the fluxions of the ſecond order; for each of them is leſs than EC.

THEOREM III.

8. If in the circle be taken an arch which is an infiniteſimal of the firſt order, I ſay, that it's verſed ſine ſhall be an infiniteſimal of the ſecond order; and the difference between the right ſine and the tangent ſhall be an infiniteſimal of the third order.

Fig. 10.



Let the arch DC be an infiniteſimal of the firſt order, DB it's right ſine, CE the tangent, and let DF be drawn parallel to AC. From the nature of the circle, it is GB . BD :: BD . BC. But GB is a finite quantity, and BD an infiniteſimal of the firſt order. Therefore, as GB is infinitely greater than BD, ſo BD will be infinitely greater than BC. Therefore BC or DF will be an infiniteſimal of the ſecond order. By the ſimilitude of the triangles ABD, DEF, it will be AB . BD :: DF . FE. But AB, a finite quantity, is infinitely greater than BD, an infiniteſimal of the firſt order, and therefore DF, an infiniteſimal of the ſecond order, will be infinitely greater than FE, which is therefore a third fluxion, or an infiniteſimal of the third order.

9. *Coroll.* I. And whereas the tangent is always greater than it's arch, the arch greater than it's chord, and the chord greater than the right ſine, the tangent and the right ſine may be aſſumed as equal, they not differing but by an infiniteſimal of the third order. Alſo, theſe following may be aſſumed as equal, the tangent, the arch, the chord, and the right ſine.

10. *Coroll.* II. If we conceive the radius of the circle AN to be an infiniteſimal of the firſt order, the arch NO and it's right ſine OM will be infiniteſimals of the ſecond; and therefore the verſed ſine MN will be an infiniteſimal of the third order.

11. *Coroll.*

Fig. 11.

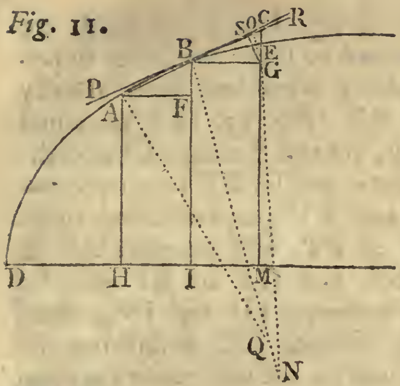
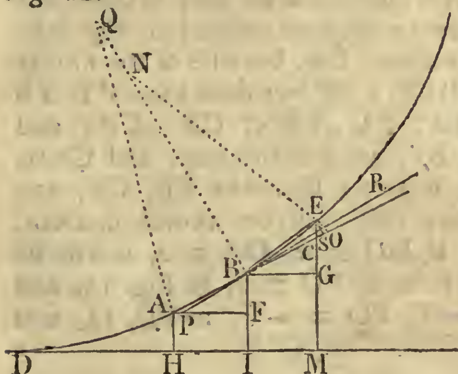


Fig. 12.



11. *Coroll.* III. In the axis DM (Fig. 11, 12.) let there be two first differences HI, IM, equal to each other, to which correspond the two infinitesimal arches AB, BE, of the curve DABE; and let be drawn the two chords BE, AB, of which this is produced till it meets in C the ordinate ME, produced also if necessary. Let ES be drawn perpendicular to BC, and from centre B, with radius BE, let the arch EO be drawn. By the corollary of Theor. II. CS is an infinitesimal of the second degree, and, by the foregoing, OS is an infinitesimal of the third degree. Then CO is also an infinitesimal of the second degree, because an infinitesimal of the third degree being added to, or subtracted from, an infinitesimal of the second degree, makes no alteration in it. Now, because HI $=$ IM, or AF $=$ BG, and, because of equal and similar triangles AFB, BGC, it will be also AB $=$ BC. But the arches may be assumed equal to their chords; then CO will be the difference of the two arches AB, BE; and therefore, if the arch DA $= s$, it will be AB $=$ BC $= \dot{s}$, and CO $= - \ddot{s}$ with a negative sign, because AB decreases when BE is less than AB, as in Fig. 11. And, on the contrary, with a positive sign, as in Fig. 12.

SCHOLIUM.

12. In determining the second differences (or fluxions) of the ordinate, and of the arch of the curve, I have supposed, both in Theor. II. and in this last corollary, that the first differences HI, IM, are equal; that is to say, that the first difference of the abscis does not alter, but remains constant, in which case the second difference of the abscis is none at all. So that, calling the abscis x, it's first difference will be $\dot{x}$, and it's second $\ddot{x} = 0$.

Wherefore we may further make these two other conclusions, one of which is, that if the first difference of the ordinate be constant, those of the abscis and of the curve will be variable. The other is, that if the first difference of the curve be constant, those of the abscis and ordinate will be variable.

 Now,

Fig. 13.

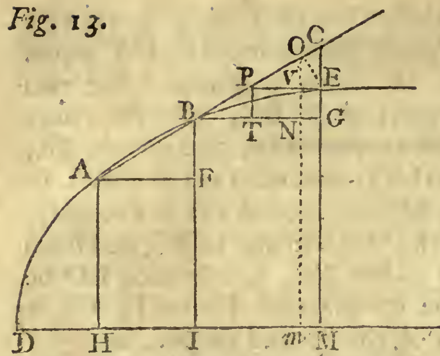
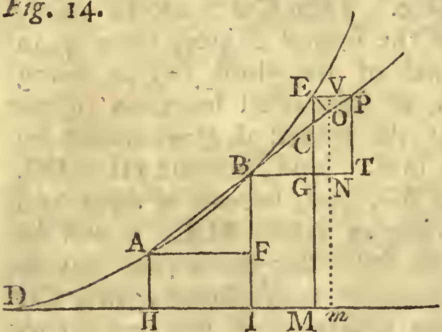


Fig. 14.



Now, these things being premised, we may easily proceed to these two other hypotheses. Supposing what has been already advanced, let BF (Fig. 13, 14.) be equal to EG; that is, let the fluxion of the ordinate be constant; and let EP be drawn parallel to BG, and PT perpendicular to it. Then will BF = PT, and therefore AF = BT, AB = BP, and GT or EP will be the difference between HI and IM. And with centre B, distance BE, describing the arch EO, PO will be the difference between the arch AB and the arch BE, because the chords may be assumed instead of the infinitesimal arches. But, because of the similar triangles BTP, CEP, we shall have PT.TB :: CE . EP, PT . PB :: CE . CP; and PT, TB, BP, are first fluxions, and CE is a second fluxion; therefore EP, CP, and much more OP, will be second fluxions. Whence, if DH $= x$, DA $= s$, it will be TG = PE $= \ddot{x}$, PO $= \ddot{s}$, in Fig. 13, and PE $= -\ddot{x}$, PO $= -\ddot{s}$, in Fig. 14, and $\ddot{y} = 0$.

Let the first differential of the curve be constant, that is, AB = BE. From the point O let fall ON parallel to TP. Because, by supposition, it is AB = BE = BO, it will be also AF = BN. Then VE or NG will be the difference between HI and IM. But it will be also FB = NO; then VO will be the difference between BF and EG. But it is plain that, EC being a fluxion of the second order, EV and VO will be so too. Then, if it be DH $= x$, HA $= y$, it will be NG $= \ddot{x}$, OV $= -\ddot{y}$, in Fig. 13, and NG $= -\ddot{x}$, OV $= \ddot{y}$, in Fig. 14, and $\ddot{s} = 0$.

The supposition of a constant first fluxion makes calculations more short and easy, as will be seen in applying it to use. However, on many occasions, for the sake of greater universality, we shall proceed from first to second differences, without making the supposition of any constant first fluxion, which it will be always easy to determine.

Let HI, IM, (Fig. 15, 16.) be first fluxions of the absciss DH, though not precisely equal to each other, and let their difference be ML, a second fluxion. Let the rest be as above, and draw the ordinate LN, and E*i* parallel to BG. Therefore, LM being the difference of HI and IM, it will be HI = IL; that is, AF = BR; and therefore the triangles ABF, BRN, will be similar and equal.

Fig. 15.

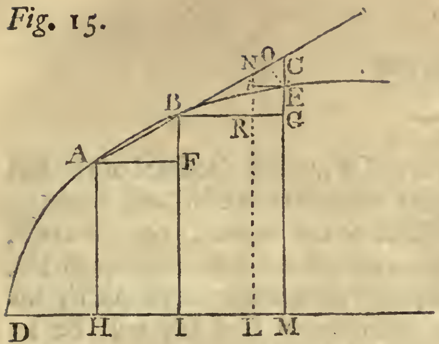
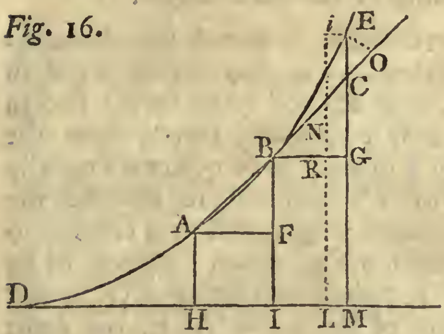


Fig. 16.



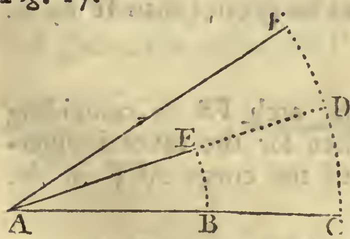
equal. Consequently BF = NR, and N*i* will be the difference between BF and EG; that is, the difference of BF, or the second difference of AH. In like manner, it will be AB = BN, and therefore NO will be the difference between the arch AB and the arch BE; and therefore the difference of the arch AB, or the second difference of the arch DA. Wherefore it is plain that N*i*, NO, are fluxions of the second degree. The same things will obtain, if, instead of supposing IM greater than HI by a second differential, we should suppose it less.

13. It is to be observed here, that the foregoing determinations do not require any restrictions concerning the angles of the coordinates, though the figures may seem to insinuate that they are at right angles; for the conclusions will be all the same, whatever the angles may be.

LEMMA.

14. Right-lined angles are to one another in a ratio compounded of the direct ratio of their arches, and the inverse ratio of their *radii*.

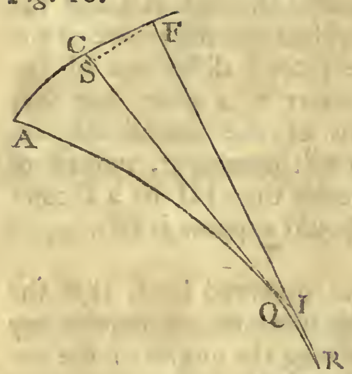
Fig. 17.



Let there be two angles EAB, FAC (Fig. 17.). Producing AE to D, from the similitude of the sectors ABE, ACD, it will be AB . BE :: AC . CD; therefore $CD = \frac{BE \times AC}{AB}$. But the angle EAB, or DAC, is to the angle FAC, as CD to CF; therefore the angle EAB will be to the angle FAC, as $\frac{BE \times AC}{AB}$ to CF; that is, as $\frac{BE}{AB}$ to $\frac{CF}{AC}$.

THEOREM IV.

Fig. 18.



15. Taking the arch CF, an infinitesimal of the first degree, in any curve whatever ACF, and drawing CI, FI, perpendicular to the curve; with centre I, and radius IF, if we describe the circular arch FS, I say that it will fall all within the curve ACF, towards C, and the intercepted line CS will be an infinitesimal quantity of the third degree.

Upon the curve AQR a thread may be conceived to be stretched, so as that, being fixed in any point below, as in R, and taken by it's end in the point A, it may continually recede from the curve, but in such a manner as to be always equally stretched, and with it's point A to describe the curve ACF. Now, the thread being in the position CQ, it will be a tangent to the curve AQR in the point Q; and in the position FR, which I suppose to be infinitely near to CQ, it will be a tangent in R; then producing CQ, it will meet FR in I. Now, since, by the generation of the curve ACF, the right line QC is equal to the curve QA, and the right line RF to the curve RQA, and the two infinitely little tangents QI, RI, are together greater than the element QR; therefore CI, IR, taken together, will be greater than the curve RQA, or than the right line FR. Then, taking away the common IR, IC will be greater than IF, and therefore the circular arch FS, described with centre I and radius IF, will fall within the curve. But, by Theor. I. and III., the two tangents QI, RI, do not exceed the arch QR but by a third fluxion. Therefore the curve AQ, together with the right lines QI, IR, exceed the curve AQR, or the right line FR, by the same quantity. Then taking away the common IR, AQ, together with QI, that is, IC, will be greater than IF by an infinitesimal of the third order.

16. *Coroll.* Therefore we may conceive the circular arch FS as coinciding with the arch of the curve FC; and one may be taken for the other indifferently. And the tangent RF will be perpendicular to the curve ACF in the point F, and QC in the point C.

The curve AQR is called the *Evolute*, the curve ACF is the *Involute*, or curve generated by the evolute; that is, produced by the unwinding of the string or thread AQR; and the circle FS, described with centre I and radius IF, is the *Osculating* or equicurved circle; also, IF is called the *Radius of Curvature* of the curve ACF in the point F.

THE-

THEOREM V.

17. If in the curve DABE (Fig. 11, 12.), at the points A, B, E, infinitely near, (that is, the arches AB, BE, being infinitesimals of the first order,) be drawn the perpendiculars QA, QB, and NE, which meet BQ in the point N; I say, that the angles AQB, BNE, may be assumed as equal.

For, by the foregoing Lemma, the angle AQB is to the angle BNE, as $\frac{AB}{AQ}$ is to $\frac{EB}{BN}$, that is, as $AB \times BN$ is to $EB \times AQ$. But the rectangle $EB \times AQ$ is not less than the rectangle $AB \times BN$, but only by the rectangle $BE \times QN$, and by the rectangle of BN into the difference of the arches AB, BE. And, as QN, BE, are infinitesimal quantities of the first degree, their rectangle will be an infinitesimal of the second degree; as also, the difference of the arches AB, BE, being an infinitesimal of the second degree, the rectangle of these into BN will be an infinitesimal of the second degree. Therefore the two rectangles $AB \times BN$ and EB into AQ do not differ from each other, but by two infinitesimal rectangles of the second degree, and therefore may be assumed as equal, and consequently the angles AQB, BNE.

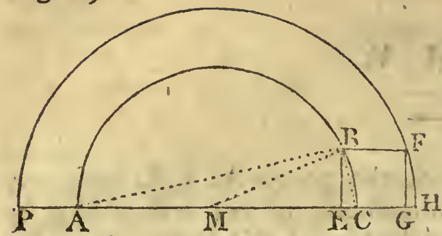
18. *Coroll.* I. If PBR be drawn a tangent at the point B, it will bisect the angle CBE, made by the two chords ABC and BE. For, by Theor. I. Coroll. III. the angle BQA being double to the angle PBA, to which the angle CBR is equal; thence the angle BNE shall be double to the angle CBR. But, by the same Corollary, the angle BNE is double to the angle RBE. Therefore the angles CBR, RBE, are equal.

19. *Coroll.* II. Therefore the angle CBE will be equal to the angle BNE, and thence the sector BNE will be similar to the sector EBO.

THEOREM VI.

20. If in two circles, the diameters of which exceed each other by a first infinitesimal, be taken two right sines equal to each other, and infinitesimals of the first degree, the difference of their versed signs shall be an infinitesimal of the third degree.

Fig. 19.

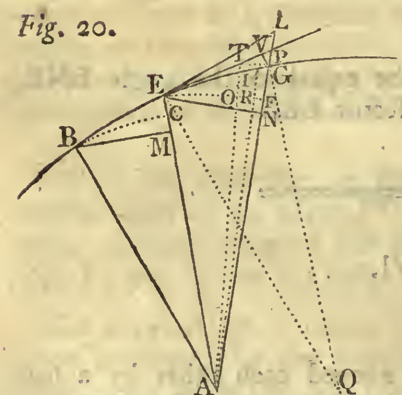


Let the two circles be ABC, PFH, and let the equal right ſines be BE, FG, infiniteſimals of the firſt degree, and the verſed ſines EC, GH. Let the chords AB, BC, be drawn. The ſine BE, and therefore the arch BC, being a firſt fluxion, the angle BME will be an infiniteſimal of the firſt order, and therefore alſo the angle BAC, which is the half of it, and the angle EBC, which is equal to this. Therefore, ſince the angle EBC, and the ſides EB, BC, are firſt infiniteſimals, the verſed ſine EC will be a ſecond infiniteſimal.

The ſame obtains of the verſed ſine GH. But the verſed ſine EC (by the property of the circle,) is found to be $\frac{EBq}{AE}$, and the verſed ſine GH $= \frac{GFq}{PG} = \frac{EBq}{PG}$. Therefore we ſhall have this analogy, EC . GH :: PG . AE. But PG, a finite quantity, exceeds AE, a finite quantity, by an infiniteſimal quantity in reſpect of itſelf, that is, of the firſt order, by hypotheſis. Therefore EC, an infiniteſimal quantity of the ſecond order, will exceed GH, an infiniteſimal of the ſecond order, by an infiniteſimal quantity in reſpect of itſelf, that is, of the third order.

THEOREM VII.

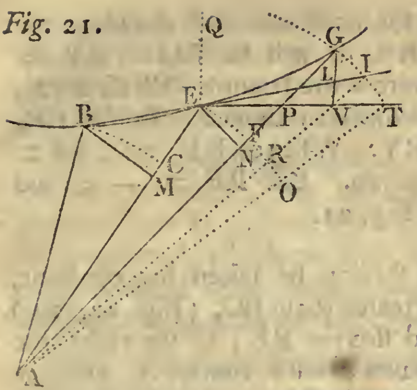
Fig. 20.



21. Let the curve BEG (Fig. 20, 21.) be referred to a *focus*, that is, ſuch, that all the ordinates proceed from a given point, which is called the *Focus*, and let this point be A. From hence let be drawn three ordinates, which are infinitely near, AB, AE, AG, which contain the two infinitely little arches of the firſt degree, BE, EG; and draw the chord BE, which, produced, meets the ordinate AG (produced if need be,) in the point L. With centre A let the arches BC, EF, be deſcribed, and let BM, EN, be their right ſines. Laſtly, make the angle NEP equal to the angle MBE. I ſay, that the intercepted line GP ſhall be the infinitely little difference of the ſecond order of the ordinate AB.

Let

Fig. 21.



Let the chord EG be drawn. Since the angles MBE, NEP, are equal by conſtruction, and the angles at M and N are right ones, the triangles EBM, PEN, will be ſimilar; then taking the ſine BM for conſtant, that is, ſuppoſing it equal to EN, the foreſaid triangles will alſo be equal. Therefore it will be ME = NP. But, ſuppoſing BM = EN, by the foregoing Theorem the difference of the verſed ſines MC, NF, is infiniteſimal in reſpect of them. Therefore, alſo, CE, FP, will be equal, and thence GP will be the difference between CE and FG. But the right lines EQ, QG, being drawn perpendicular to the curve in the points E, G, the angle LEG will be equal to the angle EQG, by Theor. V. Coroll. II. [which is true whether the curve be referred to an axis, or to a *focus*.] And the angle EQG is infinitely little. Therefore, alſo, the angle LEG will be infinitely little. And, becauſe the right lines EG, EL, are infiniteſimals of the firſt order, GL will be an infiniteſimal of the ſecond order; and much more GP, reſpect being had to Fig. 20.

By Theor. III. Coroll. I. the line BM is equal to the arch BC. Then, inſtead of the ſine, taking the arch for conſtant, and making it $= \dot{x}$, AB $= y$, CE $= \dot{y}$, it will be GP $= -\ddot{y}$. And with centre E, and diſtance EG, deſcribing the arch GV, it will be VP $= -\ddot{s}$, if BE $= \dot{s}$.

22. *Coroll.* The angle LEP will be equal to the angle EAG. For the angle EPA, by conſtruction, is equal to the angle BEA; but the external angle EPA is equal to the two internal angles L and LEP; and the other, BEA, is equal to the two, L and EAG. Then, taking away the common L, there will remain the two equal angles LEP, EAG. Wherefore this will be true, whether the curve be concave towards the point A, (Fig. 20.) or whether it be convex, (Fig. 21.) as it is eaſy to perceive. In the ſame Fig. 21, the angle LEP will be an infiniteſimal, and therefore LP is an infiniteſimal of the ſecond order. But it has been ſeen, that GL is alſo an infiniteſimal of the ſecond order. Therefore the whole, GP, will be ſo alſo, which will be $= \ddot{y}$; and with centre E, diſtance EG, the arch GV being deſcribed, it will be PV $= \ddot{s}$.

If we ſuppoſe $\dot{y}$ to be conſtant, with centre A, and diſtance AG, let the arch GT be deſcribed, and from the point T let the right line TOA be drawn. Becauſe FG = EC, by hypotheſis, the triangle TEO will be ſimilar and equal to the triangle EBC; and therefore BC = EO, and BE = ET. Then OF $= \ddot{x}$, and TV $= \ddot{s}$, in Fig. 20. But OF $= -\ddot{x}$, and TV $= -\ddot{s}$, in Fig. 21.

Taking

Fig. 22.

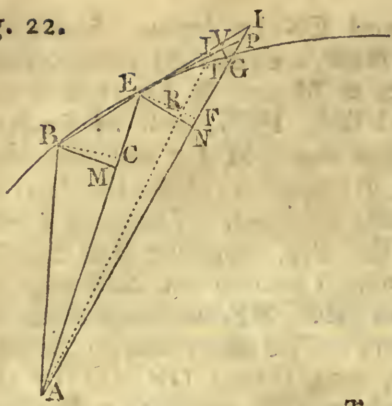
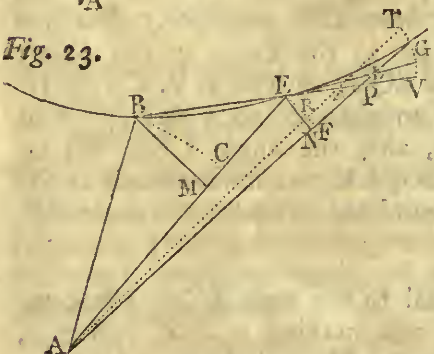


Fig. 23.



Taking $\dot{s}$ for conſtant, and drawing the right line VRA, it will be EG = EV = BE; and therefore the triangles EBC, EVR, are equal and ſimilar; thence is BC = ER, and CE = RV. Whence RF = $\ddot{x}$, VI = $-\ddot{y}$, in Fig. 20. But RF = $-\ddot{x}$, and VI = $\ddot{y}$, in Fig. 21.

If no firſt fluxion be taken for conſtant, let EG be greater than BC, (Fig. 22, 23.) by the ſecond fluxion RF; let the right line ART be drawn; with centre A, diſtance AG, draw the arch GT; and with centre E, diſtance EG, draw the arch GV. Therefore, ſince BC = ER, it will be alſo CE = RI, and BE = EI. Therefore TI will be the difference between CE and FG, and VI the difference between BE and EG.

SCHOLIUM I.

23. It may not be beſide our purpoſe to obviate a difficulty, which ſeems likely to ariſe. And this is, that in the foregoing Theorem the lines CE, FP, are aſſumed as equal, in virtue of Theor. VI.; which Theorem ſuppoſes as equal the ſines BM, EN. Whence it may ſeem, that the determinations concerning ſecond differentials can only take place in the caſe when we make a ſuppoſition of a conſtant fluxion BC, and in no other. But, to remove this difficulty, it will be ſufficient to conſider, that, though BC be ſuppoſed variable, the difference will be an infiniteſimal of the ſecond degree, which does not hinder the equality of the firſt fluxions BC, EF, nor of the ſines BM, EN.

SCHOLIUM II.

24. In the foregoing Theorems are contained the principles, by which infiniteſimals of any order may be managed, and which prepare the way to make a right

right uſe of the Method of Fluxions, whether direct or inverſe; and beſides, to apply the ſyntheſis of the ancients to infinitely little magnitudes of all degrees; and to make uſe of the ſtricteſt Geometry, which proceeds with a particular ſimplicity and elegance.

Now, to avoid paralogiſms, into which it is but too eaſy to fall, it will be needful to reflect, that infinitely little lines of any order, (agreeably to what obtains likewiſe in thoſe that are finite,) have two important circumſtances to be conſidered, which are their magnitude and their poſition. And as to their magnitude, I think they cannot be rejected except by thoſe, who fancy ſuch infiniteſimal quantities to be mere nullities.

Now, although quantities, by diminiſhing *ad infinitum*, may paſs from one order to another, the proportions in every order continue the ſame. And, becauſe of three lines of any the ſame order a triangle may be formed, it may be conſidered, that if, by leſſening proportionally the ſides, ſo as to paſs from one degree to another, the angles are not thereby changed, the ſides muſt always preſerve the ſame ratio to one another; that is, infiniteſimals with the finite, and infiniteſimals of the ſecond order with thoſe of the firſt, and with finite; and ſo on.

But if two magnitudes, of any order whatever, ſhall differ by a magnitude which in reſpect of them ſhall be inaſſignable, then with the utmoſt ſecurity, and without any danger of error, one of them may be taken for the other; nor need it be apprehended that ſuch a compariſon will introduce the leaſt error.

Therefore it is neceſſary to be much upon our guard, when the poſition of lines and angles is concerned; for, to confound them when they ought to be nicely diſtinguiſhed, muſt needs lead us into unavoidable paralogiſms.

25. The principal foundations of this calculus being thus laid, I ſhall paſs on to the methods or rules of finding the fluxions or differences of analytical formulas or expreſſions. And, firſt, let us take the differences of various quantities added together, or ſubtracted from one another; for example, of $a + x + z + y - u$. As the fluxion of x is $\dot{x}$, of z is $\dot{z}$, &c; and as the conſtant quantity a has no fluxion; then, conceiving every variable to be increaſed by it's fluxion, according to it's ſign, the formula propoſed will be changed into this other, $a + x + \dot{x} + z + \dot{z} + y + \dot{y} - u - \dot{u}$; from which ſubtracting the firſt, the remainder will be $\dot{x} + \dot{z} + \dot{y} - \dot{u}$, which is exactly that quantity by which the propoſed quantity is increaſed, that is to ſay, it's difference or fluxion.

Hence we derive this general rule, that, to find the fluxion of any aggregate of analytical quantities of one dimenſion, it will be ſufficient to take the fluxion of every one of the variable quantities with it's ſign, and the aggregate of theſe fluxions ſhall be the fluxion of the quantity propoſed. So, the fluxion of

$b - s - z$ will be $-\dot{s} - \dot{z}$. The fluxion of $aa - 4bz + by$ will be $-4b\dot{z} + b\dot{y}$. And ſo of others.

26. But if the quantity propoſed to be differenced ſhall be the product of ſeveral variables, as xy; becauſe x becomes $x + \dot{x}$, and y becomes $y + \dot{y}$, and xy becomes $xy + y\dot{x} + x\dot{y} + \dot{x}\dot{y}$, which is the product of $x + \dot{x}$ into $y + \dot{y}$; from this product ſubtracting, therefore, the propoſed quantity xy, there will remain $y\dot{x} + x\dot{y} + \dot{x}\dot{y}$. But $\dot{x}\dot{y}$ is a quantity infinitely leſs than either of the other two, which are the rectangle of a finite quantity into an infiniteſimal. But $\dot{x}\dot{y}$ is the rectangle of two infiniteſimals, and therefore is infinitely leſs, and muſt be ſuppoſed entirely to vaniſh. The fluxion, therefore, of xy will be $x\dot{y} + y\dot{x}$.

Let us difference xyz by this rule. The product of $x + \dot{x}$ into $y + \dot{y}$ into $z + \dot{z}$ is $xyz + yz\dot{x} + xz\dot{y} + xy\dot{z} + z\dot{x}\dot{y} + y\dot{x}\dot{z} + x\dot{y}\dot{z} + \dot{x}\dot{y}\dot{z}$; which, ſubtracting the quantity propoſed, will give the remainder $yz\dot{x} + xz\dot{y} + xy\dot{z} + z\dot{x}\dot{y} + y\dot{x}\dot{z} + x\dot{y}\dot{z} + \dot{x}\dot{y}\dot{z}$. But the firſt, ſecond, and third terms are each the product of two finite quantities and one infiniteſimal; the fourth, fifth, and ſixth are the products of one finite quantity and two infiniteſimals, and therefore every one of theſe is infinitely leſs than any one of thoſe, and therefore will vaniſh: and much more the laſt, which is the product of three infiniteſimals. Therefore let all theſe terms vaniſh, beginning at the fourth, and then $yz\dot{x} + xz\dot{y} + xy\dot{z}$ will be the fluxion of xyz.

Hence ariſes this rule, that, to take the fluxions of the product of ſeveral quantities multiplied together, we muſt take the ſum of the products of the fluxion of every one of thoſe quantities into the products of the others. Thus, the fluxion of $bxzt$ will be $bxz\dot{t} + bxt\dot{z} + btz\dot{x} + xzt \times 0$; becauſe the fluxion of the conſtant quantity b is nothing. That is, the fluxion of $bxzt$ will be $bxz\dot{t} + bxt\dot{z} + btz\dot{x}$. The fluxion of $\overline{a + x} \times \overline{b - y}$ will be $\dot{x} \times \overline{b - y} - \dot{y} \times \overline{a + x}$, that is, $b\dot{x} - y\dot{x} - a\dot{y} - x\dot{y}$.

27. Let the formula to be differenced be a fraction, ſuppoſe $\frac{x}{y}$. If we put $\frac{x}{y} = z$, it will be then $x = zy$. And therefore their differences will alſo be equal, that is, $\dot{x} = \dot{z}y + z\dot{y}$. Wherefore $\dot{z} = \frac{\dot{x} - z\dot{y}}{y}$. But $z = \frac{x}{y}$; therefore, ſubſtituting this value inſtead of z, it will be $\dot{z} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{yy} = \frac{\dot{x}y - x\dot{y}}{yy}$. But if $z = \frac{x}{y}$, then $\dot{z}$ will be the differential of $\frac{x}{y}$, and therefore the differential of $\frac{x}{y}$ will be $\frac{y\dot{x} - x\dot{y}}{yy}$.

Now

Now the rule will be, that the differential of a fraction will be another fraction, the numerator of which will be the product of the difference of the numerator into the denominator, subtracting the product of the difference of the denominator into the numerator of the proposed fraction; and the denominator must be the square of the denominator of the same proposed fraction.

Therefore the difference or fluxion of $\frac{a}{x}$ will be $-\frac{a\dot{x}}{xx}$. The fluxion of $\frac{a+x}{x}$ will be $\frac{x\dot{x} - a\dot{x} - x\dot{x}}{xx}$, that is, $-\frac{a\dot{x}}{xx}$. The fluxion of $\frac{y}{b-y}$ will be $\frac{b\dot{y} - \dot{y}y + y\dot{y}}{\overline{b-y}|^2}$, that is, $\frac{b\dot{y}}{\overline{b-y}|^2}$. The fluxion of $\frac{3xy}{a-x}$ will be $\frac{\overline{3x\dot{y} + 3y\dot{x}} \times \overline{a-x} + \dot{x} \times 3xy}{\overline{a-x}|^2}$, that is, $\frac{3ax\dot{y} + 3ay\dot{x} - 3xx\dot{y}}{\overline{a-x}|^2}$.

28. Now let us find the fluxions of powers, and, first, of perfect and positive powers, that is, whose exponents are positive integer numbers; for example, of x^2. But xx is the product of x into x, and therefore, by the rule of products, it's fluxion will be $\dot{x}x + x\dot{x}$, that is, $2x\dot{x}$. To find the fluxion of x^3. Now this is the product of x into x into x, and therefore the fluxion will be $xx\dot{x} + xx\dot{x} + xx\dot{x}$, that is, $3xx\dot{x}$. And, as we may proceed in the same manner *in infinitum*, the fluxion of x^m, m being any positive integer, will be $mx^{m-1}\dot{x}$.

If the exponent be negative, suppose ax^{-2}, or $\frac{a}{x^2}$, the fluxion, by the rule of fractions, will be the product of the fluxion of the numerator into the denominator, subtracting the product of the fluxion of the denominator into the numerator, the whole being divided by the square of the denominator. But the fluxion of the denominator is $2x\dot{x}$; so that the fluxion of ax^{-2} or $\frac{a}{xx}$ will be $-\frac{2ax\dot{x}}{x^4}$, that is, $-\frac{2a\dot{x}}{x^3}$. The fluxion of x^{-3}, or $\frac{1}{x^3}$, will be $-\frac{3xx\dot{x}}{x^6}$, or $\frac{3\dot{x}}{x^4}$. And, in general, the fluxion of $\frac{ax^{-m}}{b}$, or $\frac{a}{bx^m}$, will be $-\frac{mab\dot{x}x^{m-1}}{bbx^{2m}}$, that is, $-\frac{ma\dot{x}x^{-m-1}}{b}$.

Let it be an imperfect power, and, first, let it be positive; that is, let the exponent be an affirmative fraction, as $\sqrt[n]{x^m}$, or $x^{\frac{m}{n}}$, where $\frac{m}{n}$ stands for any positive fraction. Make $x^{\frac{m}{n}} = z$, and, raising each part to the power n, it will

will be $x^m = z^n$, of which taking the fluxions, we ſhall have $m\dot{x}x^{m-1} = n\dot{z}z^{n-1}$, whence $\dot{z} = \frac{m\dot{x}x^{m-1}}{nz^{n-1}}$. But, becauſe $x^m = z^n$, and thence $z^{n-1} = x^{m-\frac{m}{n}}$, which being ſubſtituted, it will be $\dot{z} = \frac{m\dot{x}x^{m-1}}{nx^{m-\frac{m}{n}}}$, that is, $\dot{z} = \frac{m}{n}\dot{x}x^{\frac{m}{n}-1}$.

If the exponent were negative, as $\frac{1}{\sqrt[n]{x^m}}$, that is, $x^{-\frac{m}{n}}$, or elſe $\frac{1}{x^{\frac{m}{n}}}$, the fluxion, by the rule of fractions, would be $-\frac{\frac{m}{n}\dot{x}x^{\frac{m}{n}-1}}{x^{\frac{2m}{n}}}$, or $-\frac{m}{n}\dot{x}x^{-\frac{m}{n}-1}$.

Therefore the general rule is, that the fluxion of any power whatever, whether perfect or imperfect, poſitive or negative, will be the product of the exponent of the power into the quantity raiſed to a power leſs by an unit than the given power, and this multiplied into the fluxion of the quantity.

Let it be required to find the fluxion of $x^{\frac{3}{2}}$; it will be $\frac{3}{2}x^{\frac{3}{2}-1}\dot{x}$, that is, $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, or elſe $\frac{3}{2}\dot{x}\sqrt{x}$.

Let be given $x^{\frac{5}{4}}$; it's fluxion will be $\frac{5}{4}x^{\frac{5}{4}-1}\dot{x}$, that is, $\frac{5}{4}x^{\frac{1}{4}}\dot{x}$, or $\frac{5}{4}\dot{x}\sqrt[4]{x}$.

Let be given $\frac{1}{x^{\frac{3}{2}}}$, that is, $x^{-\frac{3}{2}}$; the fluxion will be $-\frac{3}{2}\dot{x}x^{-\frac{3}{2}-1}$, or $-\frac{3}{2}\dot{x}x^{-\frac{5}{2}}$, or, laſtly, $-\frac{3\dot{x}}{2x^{\frac{5}{2}}}$.

The fluxion of $\overline{ax+xx}|^2$ will be $2 \times \overline{ax+xx} \times \overline{a\dot{x}+2x\dot{x}}$, that is, $2aax\dot{x} + 6ax^2\dot{x} + 4x^3\dot{x}$.

The fluxion of $\overline{xy+ax}|^3$ will be $3 \times \overline{xy+ax}|^2 \times \overline{x\dot{y}+y\dot{x}+a\dot{x}}$, that is, $3x^3y^2\dot{y} + 6ax^3y\dot{y} + 3a^2x^3\dot{y} + 3y^3x^2\dot{x} + 9ay^2x^2\dot{x} + 9a^2yx^2\dot{x} + 3a^3x^2\dot{x}$.

The fluxion of $\frac{1}{\overline{ax-yy}|^2}$, or $\overline{ax-yy}|^{-2}$, will be $-2 \times \overline{ax-yy}|^{-3} \times \overline{a\dot{x}-2y\dot{y}}$, or $\frac{-2a\dot{x}+4y\dot{y}}{\overline{ax-yy}|^3}$.

The

The fluxion of $\sqrt{ax-xx}$, or $\overline{ax-xx}|^{\frac{1}{2}}$, will be $\frac{1}{2}\times\overline{ax-xx}|^{-\frac{1}{2}}\times\overline{a\dot{x}-2x\dot{x}}$, that is, $\frac{a\dot{x}-2x\dot{x}}{2\times\overline{ax-xx}|^{\frac{1}{2}}}$.

The fluxion of $\sqrt{xx+xy}$, or $\overline{xx+xy}|^{\frac{1}{2}}$, will be $\frac{1}{2}\times\overline{xx+xy}|^{-\frac{1}{2}}\times\overline{2x\dot{x}+\dot{x}y+\dot{y}x}$, that is, $\frac{2x\dot{x}+\dot{x}y+\dot{y}x}{2\times\overline{xx+xy}|^{\frac{1}{2}}}$.

The fluxion of $\sqrt[3]{ax-xx}$, or $\overline{ax-xx}|^{\frac{1}{3}}$, will be $\frac{1}{3}\times\overline{ax-xx}|^{-\frac{2}{3}}\times\overline{a\dot{x}-2x\dot{x}}$, that is, $\frac{a\dot{x}-2x\dot{x}}{3\times\overline{ax-xx}|^{\frac{2}{3}}}$.

The fluxion of $\frac{1}{\sqrt[3]{ay+xy}}$, or $\frac{1}{\overline{ay+xy}|^{\frac{1}{3}}}$, or $\overline{ay+xy}|^{-\frac{1}{3}}$, will be $-\frac{1}{3}\times\overline{ay+xy}|^{-\frac{4}{3}}\times\overline{a\dot{y}+x\dot{y}+y\dot{x}}$, or $-\frac{a\dot{y}+x\dot{y}+y\dot{x}}{3\times\overline{ay+xy}|^{\frac{4}{3}}}$.

The fluxion of $\overline{a-x}\sqrt[3]{a+x}$, or $\overline{a-x}\times\overline{a+x}|^{\frac{1}{3}}$, is $-\dot{x}\times\overline{a+x}|^{\frac{1}{3}}+\frac{1}{3}\times\overline{a-x}\times\overline{a+x}|^{-\frac{2}{3}}\times\dot{x}$, or $-\dot{x}\sqrt[3]{a+x}+\frac{a\dot{x}-x\dot{x}}{3\times\overline{a+x}|^{\frac{2}{3}}}$.

The fluxion $\sqrt{ax+xx+\sqrt[4]{a^4-x^4}}$, or $\overline{ax+xx+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{\frac{1}{2}}$ will be $\frac{1}{2}\times\overline{a\dot{x}+2x\dot{x}+\frac{1}{4}\times-4\dot{x}x^3\times\overline{a^4-x^4}|^{-\frac{3}{4}}}\times\overline{ax+x^2+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{-\frac{1}{2}}$, or $\frac{a\dot{x}+2x\dot{x}-\frac{\dot{x}x^3}{\overline{a^4-x^4}|^{\frac{3}{4}}}}{2\times\overline{ax+xx+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{\frac{1}{2}}}$, or $\frac{\overline{a+2x}\times\dot{x}\sqrt[4]{\overline{a^4-x^4}|^3}-\dot{x}x^3}{2\sqrt{ax+xx+\sqrt[4]{a^4-x^4}}\times\sqrt[4]{\overline{a^4-x^4}|^3}}$.

The fluxion of $\frac{aa+xx}{\sqrt{ax+xx}}$, or $\overline{aa+xx}\times\overline{ax+xx}|^{-\frac{1}{2}}$, will be $2x\dot{x}\times\overline{ax+xx}|^{-\frac{1}{2}}-\frac{1}{2}\times\overline{a\dot{x}+2x\dot{x}}\times\overline{ax+xx}|^{-\frac{3}{2}}\times\overline{aa+xx}$, that is, $\frac{2x\dot{x}\times\overline{ax+xx}-\frac{1}{2}\times\overline{a\dot{x}+2x\dot{x}}\times\overline{aa+xx}}{\overline{ax+xx}|^{\frac{3}{2}}}$, or $\frac{3a\dot{x}x^2+2\dot{x}x^3-a^3\dot{x}-2aa\dot{x}x}{2\sqrt{\overline{ax+xx}|^3}}$.

The fluxion of $\frac{x\sqrt{ax+xx}}{a\sqrt{ay-xy}}$ will be $\frac{3a^2yx\dot{x}+2ayx^2\dot{x}-3yx^3\dot{x}-a^2x^2\dot{y}+x^4\dot{y}}{2a\times\sqrt{ax+xx}\times\sqrt{\overline{ay-xy}|^3}}$.

29. After the ſame manner as the fluxions of finite quantities are found, ſo are found the fluxions of infiniteſimal quantities of the firſt order, and the fluxions of infiniteſimal quantities of the ſecond order, and ſo on ſucceſſively, making uſe of the ſame rules which have now been explained.

Here

Here it muſt be conſidered, whether any firſt fluxion be aſſumed as conſtant, and which it is; for then it's fluxion will be nothing, and ſo ought to be omitted in taking the fluxion.

Let the formula $y\dot{x} - x\dot{y}$ be propoſed, to find it's difference or fluxion. Let no fluxion at preſent be ſuppoſed to be conſtant, and it's fluxion will be $\dot{x}\dot{y} + y\ddot{x} - \dot{x}\dot{y} - x\ddot{y}$, that is, $y\ddot{x} - x\ddot{y}$. Now let the fluxion $\dot{x}$ be aſſumed as conſtant; then the difference will be $\dot{x}\dot{y} - \dot{x}\dot{y} - x\ddot{y}$, or $- x\ddot{y}$. Let the fluxion $\dot{y}$ be conſtant, then the difference will be $\dot{x}\dot{y} + y\ddot{x} - \dot{x}\dot{y}$, that is, $y\ddot{x}$.

Let the quantity be $\frac{y\dot{x}}{\dot{y}}$, in which no firſt fluxion is taken for conſtant. The fluxion will be $\frac{\dot{x}\dot{y}^2 + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, or $\dot{x} + \frac{y\ddot{x}}{\dot{y}} - \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Here, taking $\dot{x}$ for conſtant, it will be $\dot{x} - \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Taking $\dot{y}$ for conſtant, it will be $\dot{x} + \frac{y\ddot{x}}{\dot{y}}$.

Let the formula be $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}}$, and let $\dot{z}$ be conſtant. The fluxion will be $\frac{\dot{y}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}} + y \times \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, that is, $\frac{\dot{x}\dot{x}\dot{y} + \dot{y}^3 + y\dot{x}\ddot{x} + y\dot{y}\ddot{y}}{\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. Taking $\dot{y}$ for conſtant, it will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} + \frac{y\dot{z}\dot{x}\ddot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}\dot{z}},$$ that is,

$$\frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{x}\ddot{x} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}.$$ Taking $\dot{x}$ for conſtant, it will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} + \frac{y\dot{z}\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}\dot{z}},$$ that is, $$\frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{y}\ddot{y} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}.$$

And, laſtly, if no fluxion be conſtant, the differential will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} + y\dot{z} \times \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}\dot{z}},$$ that is,

$$\frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{x}\ddot{x} + y\dot{z}\dot{y}\ddot{y} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}.$$

Now in this, if we expunge all the terms in which $\ddot{z}$ is found, that is, if we aſſume the hypotheſis of $\dot{z}$ being conſtant, this expreſſion will be changed into the firſt. And if we cancel thoſe in which $\ddot{y}$ is found, it will be changed into the ſecond. And, by expunging thoſe in which $\ddot{x}$ is found, it will become the third, as is manifeſt.

Let

Let be given $\frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}}$, and let $\dot{x}$ be conſtant. Then the fluxion will be

$$\frac{\dot{x}\dot{x} + \dot{y}\dot{y} + y\ddot{y} \times \sqrt{xx + yy} - \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}} \times \overline{x\dot{x} + y\dot{y}}}{xx + yy}, \text{ or}$$

$\frac{x^2\dot{y}^2 + x^2y\ddot{y} + y^2\dot{x}^2 + y^3\ddot{y} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$. Taking $\dot{y}$ for conſtant, it will be

$$\frac{\overline{\dot{x}^2 + x\ddot{x} + \dot{y}^2} \times \sqrt{xx + yy} - \frac{\overline{x\dot{x} + y\dot{y}} \times \overline{x\dot{x} + y\dot{y}}}{\sqrt{xx + yy}}}{xx + yy}, \text{ that is,}$$

$\frac{x^3\ddot{x} + x^2\dot{y}^2 + y^2\dot{x}^2 + y^2x\ddot{x} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$. And laſtly, taking neither of the fluxions for conſtant, it will be $\frac{\overline{\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}} \times \sqrt{xx + yy} - \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}} \times \overline{x\dot{x} + y\dot{y}}}{xx + yy}$,

that is, $\frac{x^3\ddot{x} + x^2\dot{y}^2 + x^2y\ddot{y} + y^2\dot{x}^2 + y^2x\ddot{x} + y^3\ddot{y} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$.

Let it be required to find the fluxion of this differential formula of the ſecond degree, $\frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$, or of this, $\frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, taking $\dot{x}$ for conſtant. The fluxion will be $\frac{3\dot{y}\ddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} \times -\dot{x}\ddot{y} + \dot{x}\dddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}$. The hypotheſis of $\dot{y}$ being conſtant, cannot take place in this formula, becauſe here is already found $\ddot{y}$. Taking neither of the fluxions as conſtant, the differential will be

$$\frac{3 \times \overline{\dot{x}\ddot{x} + \dot{y}\ddot{y}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} \times -\dot{x}\ddot{y} + \overline{\dot{x}\dddot{y} + \ddot{x}\ddot{y}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}.$$

In a like method we muſt proceed in all other caſes, ſtill more compounded.

SECT. II.

The Method of Tangents.

Fig. 24.

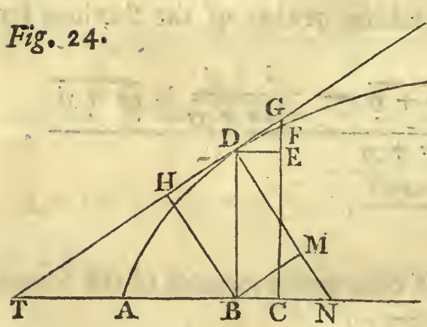
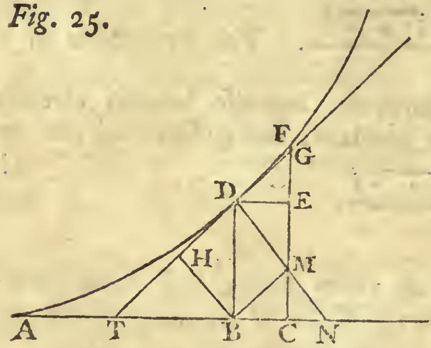


Fig. 25.



30. Let the right line TDG (Fig. 24, 25.) be a tangent to the curve ADF in any point D, and the ordinate BD be perpendicular to the axis AB in the point B, to which let CF be infinitely near, which produced (if need be,) ſhall meet the tangent in the point G, and let DE be drawn parallel to the axis AB. By what has been already demonſtrated in the foregoing Theorems, and their Corollaries, GF will be an infiniteſimal in reſpect of EF, and alſo the difference between DF and DG will be an infiniteſimal in reſpect of the little arch DF. Therefore we may aſſume as equal the two lines EF, EG, as alſo the two, DF, DG; and therefore, if AB $= x$, BD $= y$, it will be EF $=$ EG $= \dot{y}$, DF $=$ DG $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. But the ſimilar triangles GED, DBT, give us this analogy, GE . ED :: DB . BT; that is, in analytical terms, $\dot{y} . \dot{x} :: y$. BT, and therefore BT $= \frac{y\dot{x}}{\dot{y}}$; and this will be a general formula for the ſubtangent of any curve.

Wherefore, in the caſe of any given curve, in order to have the ſubtangent, nothing elſe is required to be done, but to find the fluxion of the equation, and to ſubſtitute the value of $\dot{x}$ or $\dot{y}$ in the general formula $\frac{y\dot{x}}{\dot{y}}$, by which the differentials will vaniſh, and we ſhall have the value of the ſubtangent expreſſed in finite terms. This will belong to the curve in any point whatever; and if we would have it at a determinate point, inſtead of the unknown quantities we are to ſubſtitute ſuch as ſhall belong to the given points.

31. Becauſe

31. Becauſe we may aſſume EF = EG, and DF = DG, it will follow, that we may conſider the point G as coinciding with F, that is, that the tangent DG, the arch DF, and it's chord, are all confounded together, or that curves may be conſidered as polygons of an infinite number of infinitely little ſides. This concluſion obtains only when we confine ourſelves to firſt fluxions; but when we are to proceed to ſecond fluxions, the point G muſt not then be confounded with the point F, for GF will then be a ſecond fluxion. Now, whereas, in the Method of Tangents, there is no occaſion for ſecond fluxions, it may be ſafely ſuppoſed that the tangent coincides with the little arch and it's chord.

32. The ſame triangle GDE will ſupply formulas for the other lines, which are analogous to the ſubtangent.

Becauſe the triangles GED, DBT, are ſimilar, it will be GE . GD :: DB . DT; that is, $\dot{y} \,.\, \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} :: y \,.\,$ DT, and therefore DT $= \frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}}$; which is a general formula for the tangent.

Let DN be perpendicular to the curve in the point D. The triangles GDE, DBN, will be ſimilar, whence it will be DE . EG :: DB . BN; that is, $\dot{x} \,.\, \dot{y} :: y \,.\,$ BN, and therefore BN $= \frac{y\dot{y}}{\dot{x}}$, a general formula for the ſubnormal.

It will be alſo DE . DG :: DB . DN, or $\dot{x} \,.\, \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} :: y \,.\,$ DN; therefore DN $= \frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$, a general formula for the normal.

From the point B draw BM perpendicular to DN, and BH perpendicular to DT. The triangle GDE will be ſimilar to DBM, whence GD . GE :: DB . BM, or $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} \,.\, \dot{y} :: y \,.\,$ BM $= \frac{y\dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, a general formula for the line BM.

The ſame triangle GDE will alſo be ſimilar to DBH; whence it will be GD . DE :: DB . BH, or $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} \,.\, \dot{x} :: y \,.\,$ BH $= \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, a general formula for the line BH.

33. The ſimilitude of the two triangles GED, DBT, will alſo be a means of diſcovering the angle, which the tangent makes with the axis at any point of the curve at pleaſure. For, becauſe the angle DTB is known, therefore the ratio of the right ſine DB to the ſine of the complement BT will be known alſo; that is, the ratio of GE to ED, or that of $\dot{y}$ to $\dot{x}$.

 Therefore,

Therefore, the equation of the curve being given, if it's fluxions be found and reſolved into an analogy, of which two terms are $\dot{y}$ and $\dot{x}$, we may have the ratio of the ſines of the angle DTB, and conſequently the angle will be known.

Fig. 26.

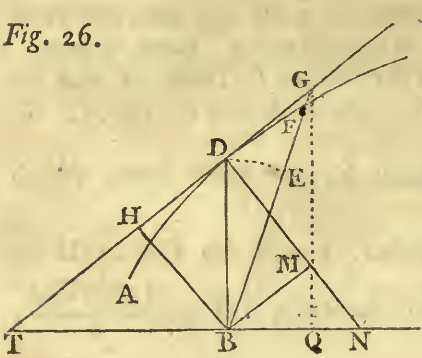
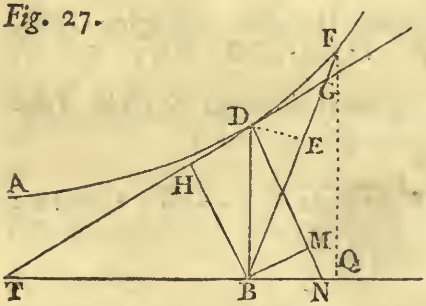


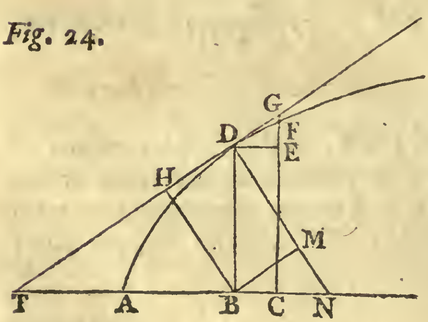
Fig. 27.



34. By the ſame way of argumentation, the ſame formulas may be derived for ſuch curves as are referred to a *focus*, (Fig. 26, 27.) if we only conſider, that, drawing from the *focus* B the right line BT perpendicular to the ordinate BD, meeting the tangent in T; the triangles DTB, DGE, will be ſimilar, becauſe the angles TBD, DEG, are right angles, and the angle TDB is not greater than the angle DGE, except by an infinitely little angle DBG, which is plainly ſeen by drawing GQ perpendicular to TB. Therefore the two angles TDB, DGE, may be aſſumed as equal, and conſequently the two, BTD, GDE; therefore the two triangles DTB, GDE, are ſimilar. But GF is an infiniteſimal in reſpect of EF; therefore, &c.

EXAMPLE I.

Fig. 24.



35. Let the curve ADF be the *Apollonian* parabola, whoſe equation is $ax = yy$. Taking the fluxions, it will be $a\dot{x} = 2y\dot{y}$, or $\dot{x} = \frac{2y\dot{y}}{a}$. Wherefore, ſubſtituting this value inſtead of $\dot{x}$, in the general formula for the ſubtangent $\frac{y\dot{x}}{\dot{y}}$, we ſhall have $\frac{2yy}{a}$, or $2x$, putting, inſtead of yy, it's value ax, given

given by the equation of the curve. Therefore the ſubtangent in the parabola is double to the abſciſs; ſo that, taking AT = AB, and from the point T drawing the right line TD to the point D, it ſhall be a tangent to the curve at the point D. Inſtead of the value of $\dot{x}$, given from the equation of the curve, if we ſubſtitute the value of $\dot{y}$, or $\frac{a\dot{x}}{2y}$, in the general formula $\frac{y\dot{x}}{\dot{y}}$, it will be alſo $\frac{2yy}{a}$, as before; which may ſuffice to obſerve in this Example.

In the ſame parabola, if we require the ſubnormal BN; the general formula of the ſubnormal is $\frac{y\dot{y}}{\dot{x}}$. But, by the equation of the curve, it is $\dot{x} = \frac{2y\dot{y}}{a}$; ſo that, making the ſubſtitution, the ſubnormal in the parabola will be $= \frac{1}{2}a$, that is, half of the parameter; and therefore, making BN $= \frac{1}{2}a$, and from the point N drawing the right line ND to the point D, this ſhall be perpendicular to the curve in D.

If we ſeek the tangent DT, the general formula of which is $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}}$, by the equation of the curve we have $\dot{x} = \frac{2y\dot{y}}{a}$. Then, ſubſtituting this value inſtead of $\dot{x}$ in the formula, we ſhall have $\frac{y\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{a\dot{y}} = \frac{y}{a}\sqrt{4yy + aa} = \sqrt{4xx + ax}$, (putting, inſtead of yy, it's value ax from the given equation,) which will be the tangent required.

If we would have the normal DN, ſubſtituting the value of $\dot{x} = \frac{2y\dot{y}}{a}$ in the general formula $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$, it will be $\frac{y\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{2y\dot{y}} = \frac{\sqrt{4yy + aa}}{2} = \frac{\sqrt{4ax + aa}}{2}$, putting, inſtead of yy, it's value from the given equation.

If we would have the right line BM; ſubſtituting the value of $\dot{x} = \frac{2y\dot{y}}{a}$ in the general formula $\frac{y\dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, it will be $\frac{ay\dot{y}}{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}} = \frac{ay}{\sqrt{4yy + aa}} = \frac{a\sqrt{ax}}{\sqrt{4ax + aa}}$.

If we would have the right line BH; ſubſtituting the value of $\dot{x}$ in the general formula $\frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, it will be $\frac{2yy\dot{y}}{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}} = \frac{2yy}{\sqrt{4yy + aa}} = \frac{2ax}{\sqrt{4ax + aa}}$.

Having found the ſubtangent, there is no need of any formulas for finding the other lines, though here, by way of exerciſe, I have made uſe of them. For, when BT is known, the triangle TDB, right-angled at B, will furniſh us with the tangent TD, and the ſimilar triangles TBD, DBN, DMB, DHB, with all the other lines. So that, in the following examples, I ſhall apply the method to finding the ſubtangents only.

If we would have the angle which is made by the tangent of the parabola with it's axis; taking the fluxional equation $a\dot{x} = 2y\dot{y}$, and reſolving it into an analogy, it will be $\dot{y} . \dot{x} :: a . 2y$. That is, that the right ſine BD is to the ſine of the complement BT, as the parameter is to the double of the ordinate; whence is determined the point D. And if we would determine the tangent to any certain point, for example, to the point D, to which correſponds the abſciſs $AB = x = \frac{1}{4}a$; from the equation of the curve finding the ordinate y, correſponding to $x = \frac{1}{4}a$, which, in this caſe, is $y = \frac{1}{2}a$, we ſhall have the analogy, $\dot{y} . \dot{x} :: a . a$; that is, the angle DTB will be half a right angle, when it is $y = \frac{1}{2}a$, or $x = \frac{1}{4}a$.

At the vertex A it is $y = 0$, and therefore the analogy for the angle of the tangent at the vertex will be $\dot{y} . \dot{x} :: a . 0$; that is, the ratio of $\dot{y}$ to $\dot{x}$ is infinite, which is as much as to ſay, that the ſine of the complement will be nothing at all, or that, at the vertex, the tangent is perpendicular to the axis.

EXAMPLE II.

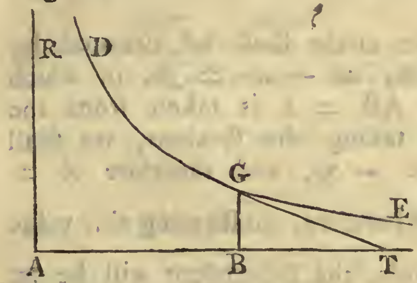
36. Let the equation be $x = y^m$, which is a general equation to all parabolas of any degree whatever; where m ſtands for any poſitive number, integer, or fraction, and unity ſupplies any dimenſions that are wanting. By taking the fluxions, it will be $\dot{x} = m\dot{y}y^{m-1}$; and, ſubſtituting this value inſtead of $\dot{x}$ in the general formula $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $my^m = mx$. Let $m = 3$, that is, let it be the firſt cubic parabola $x = y^3$; it's ſubtangent will be $3x$. Let $m = \frac{3}{2}$, that is, let it be the ſecond cubic parabola $xx = y^3$; the ſubtangent will be $\frac{3}{2}x$, &c.

The fluxional equation of the curve $\dot{x} = m\dot{y}y^{m-1}$ gives this analogy, $\dot{y} . \dot{x} :: 1 . my^{m-1}$. But, putting $y = 0$, if m be greater than unity, the analogy will be $\dot{y} . \dot{x} :: 1 . 0$; or the ratio of $\dot{y}$ to $\dot{x}$ will be infinite, and therefore the tangent at the vertex is perpendicular to the axis. And if m be leſs than unity, the

the analogy will be $\dot{y} . \dot{x} :: 1 . \frac{m}{y^{1-m}}$; that is, making $y = 0$, $\dot{y} . \dot{x} :: 1 . \frac{m}{0}$, which is as much as to ſay, that the ratio of $\dot{y}$ to $\dot{x}$ is infinitely little, and therefore, at the vertex, the tangent is parallel to the axis.

EXAMPLE III.

Fig. 28.



37. Let the curve be DCE, of which we deſire the ſubtangent, the equation of which is $xy = aa$, being the hyperbola between it's aſymptotes. By taking the fluxions, we ſhall have $x\dot{y} + y\dot{x} = 0$, or $\dot{x} = -\frac{x\dot{y}}{y}$. Wherefore, ſubſtituting this value of $\dot{x}$ in the formula of the ſubtangent $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $-x$ with a negative value, which is as much as to ſay, that the ſubtangent BT muſt be taken on the contrary part of the abſciſs.

Therefore, taking BT = BA, and drawing the right line TC to the point C, it ſhall be a tangent to the curve at the point C.

Now, becauſe in the curve DCE, as the axis increaſes, the ordinate y will decreaſe, in taking the fluxion we might have put $\dot{y}$ negative; but becauſe, for the ſame reaſon, we ought to have taken the ſame $\dot{y}$ negative alſo in the general formula, I have omitted to do it in both places, becauſe it comes to the ſame thing, without incumbering ourſelves with changing ſigns; and what is now mentioned may be underſtood on other like occaſions.

Let $x = \frac{1}{y^m}$ be a general equation to all hyperbolas *ad infinitum*, between their aſymptotes, where m ſtands for any poſitive number, integer, or fraction. By taking the fluxions, we ſhall have $\dot{x} = -\frac{my\dot{y}^{m-1}}{y^{2m}} = -\frac{m\dot{y}}{y^{m+1}}$. And, ſubſtituting this value in the general formula $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $-\frac{m}{y^m}$, or $-mx$, by the equation of the curve.

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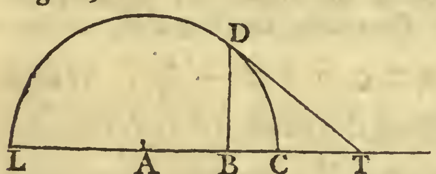
EXAMPLE IV.

38. Let the curve ADF (Fig. 24.) be a circle whoſe diameter is $2a$, AB $= x$, BD $= y$; the equation will be $2ax - xx = yy$, whoſe fluxion is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$, and therefore $\dot{x} = \frac{y\dot{y}}{a - x}$. Then, ſubſtituting this value in the formula $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $\frac{yy}{a - x}$, that is, $\frac{2ax - xx}{a - x}$, by putting, inſtead of yy, it's value from the given equation. Therefore the ſubtangent in the circle will be a fourth proportional to $a - x$, $2a - x$, and x.

Fig. 29.



But if the circle ſhall be denoted by this equation, $aa - xx = yy$, in which the abſciſs AB $= x$ is taken from the centre; by taking the fluxions, we ſhall have $x\dot{x} = -y\dot{y}$, and therefore $\dot{x} = -\frac{y\dot{y}}{x}$. Wherefore, ſubſtituting this value in the formula, the ſubtangent will be $= -\frac{yy}{x}$, that is, a third proportional to AB and BD, but negative; that is to ſay, it muſt be taken from B towards T.

EXAMPLE V.

39. Let the curve ADF (Fig. 24.) be an ellipſis, with this equation $ax - xx = \frac{ayy}{b}$; taking the abſciſs from the vertex A. The fluxional equation will be $a\dot{x} - 2x\dot{x} = \frac{2ay\dot{y}}{b}$, and therefore $\dot{x} = \frac{2ay\dot{y}}{b \times \overline{a - 2x}}$. Now, ſubſtituting this value in the general formula $\frac{y\dot{x}}{\dot{y}}$, then $\frac{2ayy}{b \times \overline{a - 2x}}$ will be the ſubtangent; or elſe, $\frac{2ax - 2xx}{a - 2x}$, inſtead of $\frac{ayy}{b}$, putting it's value $ax - xx$ from the given equation.

Making $x = \frac{1}{2}a$, half the tranſverſe axis, in the value of the ſubtangent, it will be $\frac{2aa}{0}$, that is, infinite. Therefore the tangent will be parallel to the tranſverſe

transverse axis in that point, in which the conjugate axis meets the curve. And this we shall find to be true also, if we inquire what is that angle, which the tangent itself makes with the same axis.

Let the equation, in general, to ellipses of any degree be this; $\frac{ay^{m+n}}{b} = x^m \times \overline{a-x}\rangle^n$, where m and n represent any positive numbers, whether integers or fractions. The fluxion of this will be $\frac{m+n}{b} \times a\dot{y}y^{m+n-1} = m\dot{x}x^{m-1} \times \overline{a-x}\rangle^n - n\dot{x}x^m \times \overline{a-x}\rangle^{n-1}$; and therefore $\dot{x} = \frac{\overline{m+n} \times a\dot{y}y^{m+n-1}}{bmx^{m-1} \times \overline{a-x}\rangle^n - bnx^m \times \overline{a-x}\rangle^{n-1}}$. And, substituting this value in the general formula, it will be $\frac{\overline{m+n} \times ay^{m+n}}{bmx^{m-1} \times \overline{a-x}\rangle^n - bnx^m \times \overline{a-x}\rangle^{n-1}}$. Then, instead of $\frac{ay^{m+n}}{b}$, putting it's value from the given equation, the subtangent will be $\frac{\overline{m+n} \times x^m \times \overline{a-x}\rangle^n}{mx^{m-1} \times \overline{a-x}\rangle^n - nx^m \times \overline{a-x}\rangle^{n-1}}$. And, dividing the numerator and denominator by $x^{m-1} \times \overline{a-x}\rangle^{n-1}$, it will be, finally, $\frac{\overline{m+n} \times \overline{ax-xx}}{ma-mx-nx}$.

Make $m=1$, $n=1$, that is, let it be the ellipsis of *Apollonius*; then the subtangent will be $\frac{2ax-2xx}{a-2x}$, as before. Make $m=3$, $n=2$; then the equation is $\frac{ay^5}{b} = x^3 \times \overline{a-x}\rangle^2$, and the subtangent will be $\frac{5ax-5xx}{3a-5x}$. And so of others.

If the equation were $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}\rangle^n$, it would express all hyperbolas of any degree, when referred to their axis; taking, in the same manner, the beginning of the axis from the vertex A. Then, by a like operation, we should find the subtangent to be $\frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx}$, which differs from the foregoing only in it's signs; as also, the equation, from whence it is derived, differs only in it's signs.

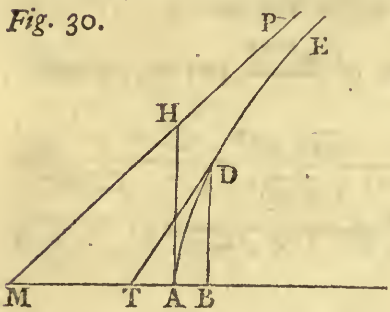
Make $m=1$, $n=1$, which is the *Apollonian* hyperbola. The subtangent will be $\frac{2ax+2xx}{a+2x}$. Make $m=3$, $n=2$, then the equation will be $\frac{ay^5}{b} = x^3 \times \overline{a+x}\rangle^2$; and the subtangent will be $\frac{5ax+5xx}{3a+5x}$, &c.

40. From

Aſymptotes. 40. From this method of tangents may be further derived a way of diſcovering whether curves propoſed have aſymptotes, and the manner of drawing them, when they are inclined to the axis. For, as to the more ſimple caſes, in which they are either perpendicular or parallel to the axes, ſufficient has been ſaid in the firſt Part, Sect. V.

EXAMPLE I.

Fig. 30.



41. Let the curve be ADE, with the equation $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}|^n$, as above, the ſubtangent of which is TB $= \frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx}$. Then the intercepted line AT $= \frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx} - x$, that is, $\frac{nax}{ma+mx+nx}$.

It is plain that the tangent TD will become an aſymptote, when touching the curve at an infinite diſtance; that is, when the abſciſs AB $= x$ becomes infinite, the intercepted line AT ſhall remain finite. Now, putting x infinite in the expreſſion of AT, the firſt term ma of the denominator is infinitely leſs than the others, and therefore vaniſhes. Whence, in this caſe, it will be $\frac{nax}{mx+nx}$, or $\frac{na}{m+n}$, which is a finite quantity. Wherefore the curve has an aſymptote, which will begin from the point M, making AM $= \frac{na}{m+n}$. Now, to draw it, let AH be raiſed perpendicular to AB, and let it be, for example, MHP. This being ſuppoſed, if we take x infinite, it will be $\dot{x} . \dot{y}$ :: MA . AH, and, in the ſuppoſition of x being infinite, the equation of the curve $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}|^n$, ($a$ being nothing in reſpect of x, will be changed into this other, $\frac{ay^{m+n}}{b} = x^{m+n}$. Or, extracting the root, and, for convenience, making $m+n=t$, it will be $y\sqrt[t]{a} = x\sqrt[t]{b}$; and, taking the fluxions, $\dot{y}\sqrt[t]{a} = \dot{x}\sqrt[t]{b}$; ſo that $\dot{x} . \dot{y} :: \sqrt[t]{a} . \sqrt[t]{b}$. Whence MA . AH :: $\sqrt[t]{a} . \sqrt[t]{b}$. And, becauſe MA

$MA = \frac{na}{t}$, it will be $\frac{na}{t} . AH :: \sqrt[t]{a} . \sqrt[t]{b}$, or $AH = \frac{na}{t} \times \sqrt[t]{\frac{b}{a}}$. If, therefore, we take $AM = \frac{na}{t}$, and raising the perpendicular $AH = \frac{na}{t} \times \sqrt[t]{\frac{b}{a}}$, and drawing the indefinite right line MHP; this will be the asymptote of the curve ADE.

Make $m = 1$, $n = 1$, that is, let the curve be the *Apollonian* hyperbola, whose equation is $\frac{ayy}{b} = ax + xx$; it will be $t = 2$, and therefore $AM = \frac{1}{2}a$, $AH = \frac{a}{2} \times \sqrt{\frac{b}{a}} = \frac{1}{2}\sqrt{ab}$. That is, AM is half the transverse axis, and AH half the conjugate, just as it should be from the Conic Sections.

EXAMPLE II.

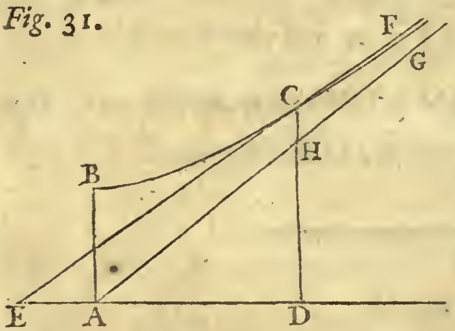
42. Let ADE (Fig. 30.) be a curve whose equation is $y^3 - x^3 = axy$; making $AB = x$, $BD = y$. By taking the fluxions, we shall have $3y^2\dot{y} - 3x^2\dot{x} = ax\dot{y} + ay\dot{x}$, and therefore $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - axy}{3xx + ay}$; and $AT = \frac{y\dot{x}}{\dot{y}} - x = \frac{3y^3 - 3x^3 - 2axy}{3xx + ay}$. Or, instead of $3y^3 - 3x^3$, putting it's value $3axy$ from the equation of the curve, it will be $AT = \frac{axy}{3xx + ay}$. And, making x infinite, that is, in case of an asymptote, in which AT becomes AM, the term ay is nothing in respect of $3xx$, so that it will be $AM = \frac{axy}{3xx} = \frac{ay}{3x}$.

But, because, in the proposed equation, the indeterminates cannot be separated, nor, consequently, can the value of AM be determined; if we put $AM = \frac{ay}{3x} = t$, (which expedient may also be used in other like cases,) it will be $y = \frac{3tx}{a}$; which value being substituted in the proposed equation, it will be $\frac{27t^3x^3}{a^3} - x^3 = 3tx^2$, or $\frac{27t^3x}{a^3} - x = 3t$. But, as x is infinite, the last term will be nothing in comparison of the others, so that it will be $\frac{27t^3x}{a^3} - x = 0$, or $t = \frac{1}{3}a$. Taking, therefore, $AM = \frac{1}{3}a$, the asymptote must be drawn

drawn from the point M. Moreover, it muſt be MA . AH :: $\dot{x}$. $\dot{y}$, and the propoſed equation $y^3 - x^3 = axy$, or $y^3 = x^3 + axy$, will be reduced to $x^3 = y^3$, or $x = y$, when x is infinite, and therefore $\dot{x} = \dot{y}$. Therefore, making MA = AH, if from the point M, through the point H, a right line be drawn, it will be an aſymptote to the curve.

I add further, that the line AT muſt neceſſarily approach to a certain limit, beyond which it cannot paſs, and that the aforeſaid limit is then an infiniteſimal, or nothing. Here follows a plain Example of this.

Fig. 31.



Let BCF be an equilateral hyperbola; and making AB = a, AD = x, DC = y, we ſhall have the equation $aa + xx = yy$, the fluxion of which is $x\dot{x} = y\dot{y}$. Thence the ſubtangent will be ED $= \frac{y\dot{x}}{\dot{y}} = \frac{yy}{x} = \frac{aa + xx}{x}$, and conſequently ED — AD $= \frac{aa + xx}{x} - x = \frac{aa}{x} =$ AE.

Putting $x = 0$, AE will become infinite, and the tangent at the point B will be parallel to the axis AD. And, making $x = \infty$, it will be AE = o. Wherefore the point E deſcribes the whole line AE infinitely produced, and finiſhes it's courſe at it's origin A, beyond which it paſſes not, though the curve turns it's convexity towards the axis. Therefore the aſymptote AG proceeds from the point A, and makes half a right angle with the line of the abſciſſes; foraſmuch as, in the equation of the *locus* $aa + xx = yy$, making $x = \infty$, the conſtant quantity aa will vaniſh, and it becomes $xx = yy$, or $x = y$.

Fig. 32.

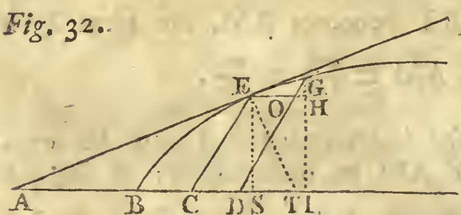
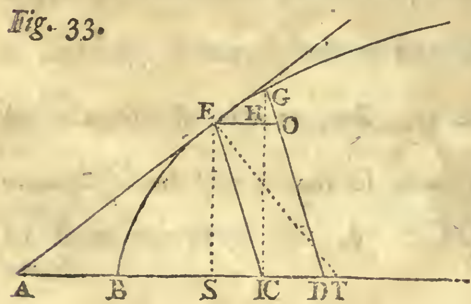


Fig. 33.



43. Hitherto I have ſuppoſed that the angle of the co-ordinates is a right angle; but, if it were obtuſe or acute, making, as before, BC = x, CE = y, CD = $\dot{x}$, OG = $\dot{y}$, (Fig. 32, 33.) the ſubtangent will be neither more nor leſs than $\frac{y\dot{x}}{\dot{y}}$, for the two triangles GEO, EAC, will be ſtill ſimilar; but the other formulas will have need of ſome reformation.

In the triangle EOG, the angle at O, equal to the angle ACE, is ſuppoſed to be known; therefore, from the point G letting fall GI perpendicular to AD, and producing

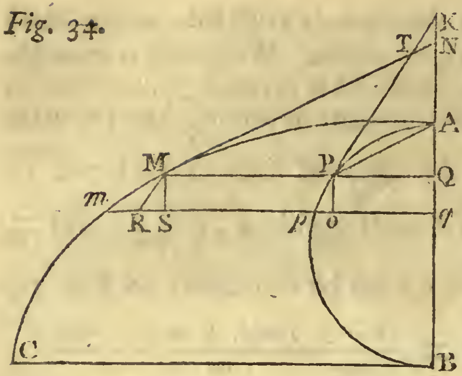
producing EO to H, if there be occaſion, in the triangle GOH the angle GOH will be known, and the angle at H be a right angle. Wherefore the angle OGH is known, and conſequently the triangle OGH is given *in ſpecie*, that is, the ratio of GO to GH is given. Let this be the ſame as a to m, and therefore it will be $a \,.\, m :: \dot{y} \,.\, GH = \frac{m\dot{y}}{a}$. Alſo, the ratio of GO to OH will be given, which may therefore be as a to n; and conſequently $a \,.\, n :: \dot{y} \,.\, OH = \frac{n\dot{y}}{a}$. Then $EH = \dot{x} \pm \frac{n\dot{y}}{a}$, (where the ſign muſt be affirmative in Fig. 32, and negative in Fig. 33.) Wherefore $EGq = \frac{aa\dot{x}\dot{x} \pm 2an\dot{x}\dot{y} + nn\dot{y}\dot{y} + mm\dot{y}\dot{y}}{aa}$. But if OG be expreſſed by a, GH by m, OH by n, then it will be $aa = mm + nn$, and $aa\dot{y}\dot{y} = mm\dot{y}\dot{y} + nn\dot{y}\dot{y}$, which, being ſubſtituted in this value of EGq, will make $EGq = \frac{aa\dot{x}\dot{x} \pm 2an\dot{x}\dot{y} + aa\dot{y}\dot{y}}{aa}$, and $EG = \dot{s} = \sqrt{\frac{a\dot{x}^2 \pm 2n\dot{x}\dot{y} + a\dot{y}^2}{a}}$, the expreſſion of the element or fluxion of the curve. This being ſuppoſed, by the ſimilitude of the triangles EGO, AEC, it will be $GO \,.\, GE :: EC \,.\, EA$, that is, $\dot{y} \,.\, \sqrt{\frac{a\dot{x}\dot{x} \pm 2n\dot{x}\dot{y} + a\dot{y}\dot{y}}{a}} :: y \,.\, EA$; or $EA = \frac{y}{\dot{y}}\sqrt{\frac{a\dot{x}\dot{x} \pm 2n\dot{x}\dot{y} + a\dot{y}\dot{y}}{a}}$, which will be the formula of the tangent.

Let TE be perpendicular to the curve, and ES to the diameter AI. Then, by ſimilar triangles GOH, ECS, we ſhall have $ES = \frac{my}{a}$, and $CS = \frac{ny}{a}$. And, by the ſimilar triangles GEH, EST, we ſhall have EH . HG :: ES . ST. That is, $\frac{a\dot{x} \pm n\dot{y}}{a} \,.\, \frac{m\dot{y}}{a} :: \frac{my}{a} \,.\, ST = \frac{mmy\dot{y}}{a \times \overline{a\dot{x} \pm n\dot{y}}}$. And therefore $CT = \frac{mmy\dot{y}}{a \times \overline{a\dot{x} \pm n\dot{y}}} \pm \frac{ny}{a} = \frac{mmy\dot{y} + nny\dot{y} \pm any\dot{x}}{a \times \overline{a\dot{x} \pm n\dot{y}}} = \frac{ay\dot{y} \pm ny\dot{x}}{a\dot{x} \pm n\dot{y}}$, which is the formula of the ſubnormal.

In a like manner, the other formulas may be reduced, which it is ſufficient only to take notice of here.

44. But the curves, whoſe tangents we deſire, may be *Tranſcendent* or *Mechanical*, that is, are not expreſſible by any Algebräical equation, but may depend on the rectification of other curves, which are not rectifiable. Let the curve Tangents to tranſcendent curves.

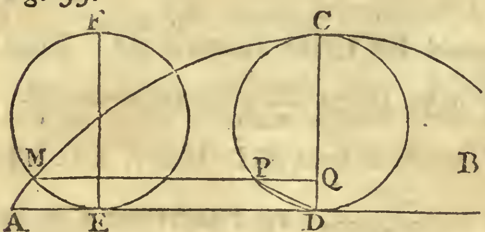
Fig. 34.



curve be APB, whose tangent PTK we know how to draw, at any given point P. Then, producing to M the line QP perpendicular to AQ, let the relation of MP to the arch PA be expressed by any equation, to find the tangent MT of the curve CMA, described from the point M. Draw qm infinitely near to QM, and MR parallel to PT; and supposing the rectification of the arch AP; make $AP = x$, $PM = y$, and it will be $Pp = \dot{x}$, $Rm = \dot{y}$, and the two triangles mRM, MPT, will be similar, and therefore $mR \,.\, RM ::$ $MP \,.\, PT$, that is, $\dot{y} \,.\, \dot{x} :: y \,.\, PT = \frac{y\dot{x}}{\dot{y}}$, the formula for the subtangent of the curve CMA, by taking it on the tangent of the curve APB. From the given equation of the curve AMC is found the value of $\dot{x}$ or $\dot{y}$, to be substituted in the formula. All the rest is to be done as usual.

EXAMPLE.

Fig. 35.



45. While the circle DPC revolves uniformly upon the right line AB, beginning at the point A; the point C of it's periphery, which at the beginning of the motion fell upon A, leaves an impression in the plane of it's motion, which it continues till the point C arrives again at the right line AB. It will describe a curve ACB, which, from it's generation, is called a *Cycloid*. It will be the ordinary cycloid, when the circle so moves upon the right line AB, as that it shall measure out the whole exactly by it's periphery, after that the point C shall have passed from A to B, so that AB may be equal to the periphery of the same circle. It will be a prolonged cycloid when the motion is such, that the right line AB is longer than the periphery of the circle; and a contracted cycloid when the same AB is shorter than the periphery.

From the description of this curve it plainly follows, that, drawing from any point the right line MQ parallel to AB, the intercepted line MP, between the curve and the circle CPD, will have to the arch CP the same ratio as the line AB has to the whole circle.

N. B. The chord ME is omitted in Fig. 35.

Suppose

Suppofe the generating circle to be in the two pofitions EMF, DPC; draw the chords ME, PD. Now, becaufe the arches EM, DP, are equal, the chords EM, DP, will be equal and parallel, and therefore MP = ED. But, by the nature of the curve, it is AE . EM :: AD . EMF :: AB . EMFE. And in the fame ratio is alfo ED . MF. And MF = PC, FD = MP: therefore it will be MP . PC :: AD . EMF :: AB . EMFE. Therefore, if we call the right line AB $= a$, the periphery of the generating circle EMFE $= b$, and any arch or abfciffa CP $= x$, the ordinate PM $= y$; the equation of the curve of the cycloid will be $x = \frac{by}{a}$.

Having therefore the equation of the curve, in order to find the fubtangent, it's fluxion will be $\dot{x} = \frac{b\dot{y}}{a}$; and, inftead of $\dot{x}$, fubftituting this value in the formula $\frac{y\dot{x}}{\dot{y}}$, it will be PT $= \frac{by}{a} = x$. Therefore, taking, on the tangent of the circle, PK, (Fig. 34.) which is fuppofed to be drawn, a portion PT equal to the arch of the circle AP, and drawing the right line TM to the point M, it fhall be a tangent to the cycloid in the point M.

Now, befides, if the cycloid be the ordinary one; becaufe, in this cafe, we fhall have $b = a$, and therefore $y = x$, it will be PM = PT, and the angle PTM = PMT. But the external angle TPQ is double to the angle TMP, and the angles TPA, APQ, are equal, by *Euclid*, iii. 29 and 32, therefore the angle APQ will be equal to the angle TMP, and therefore the tangent MT is parallel to the chord PA.

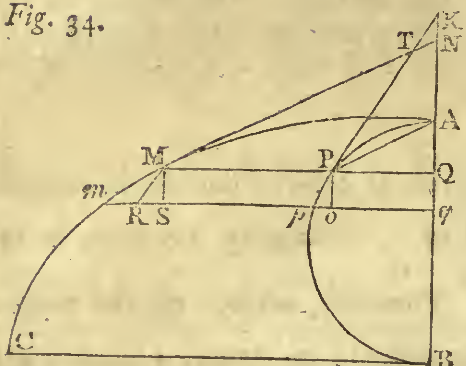
46. Without the affiftance of the tangent of the curve APB, (Fig. 34.) we may have the fubtangent of the curve AM, taking it in the axis KAB. Make AQ $= x$, QP $= y$, the arch AP $= s$, QM $= z$, and let the relation of the arch AP to the ordinate QM be expreffed by any equation whatever. Let qm be infinitely near to QM, and MS parallel to AB. It will be MS $= \dot{x}$, S$m = \dot{z}$, and the fimilar triangles mSM, MQN, will give us $\dot{z} . \dot{x} :: z$. QN $= \frac{z\dot{x}}{\dot{z}}$, a formula for the fubtangent.

Inftead of taking for the ordinate QM $= z$, if we take PM $= u$; drawing MR parallel to the little arch Pp, it will be mR $= \dot{u}$, RS $= po = \dot{y}$, and therefore mS $= \dot{u} + \dot{y}$. And the fimilar triangles mSM, MQN, will give us $\dot{u} + \dot{y} . \dot{x} :: u + y$. QN $= \frac{\overline{u+y} \times \dot{x}}{\dot{u} + \dot{y}}$, another formula for the fubtangent.

EX-

EXAMPLE I.

Fig. 34.



47. Let the curve APB be a circle whofe diameter is $2r$, and let the ratio of PM to the arch PA be that of a to b; that is, let the curve AMC be a cycloid. Make AQ $= x$, QP $= y$, QM $= z$, the arch AP $= s$; then drawing mq infinitely near to MQ, MR, parallel to Pp; MS, Po, parallel to AB; it will be mS $= \dot{z}$, RS $= po = \dot{y}$, P$p = \dot{s}$; and mR, the difference or fluxion of MP, will be $\dot{z} - \dot{y}$. But, becaufe, by the property of the curve, we have MP, to the arch PA, as a to b; in the fame ratio, alfo, will be their differentials mR, pP; and therefore it will be $\dot{z} - \dot{y} . \dot{s} :: a . b$; that is, $\dot{z} - \dot{y} = \frac{a\dot{s}}{b}$. But $\dot{s} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and, by the property of the circle, $y = \sqrt{2rx - xx}$. Therefore $\dot{y} = \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$, and $\dot{y}\dot{y} = \frac{r^2\dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{2rx - xx}$; whence $\dot{s} = \frac{r\dot{x}}{\sqrt{2rx - xx}}$.

Wherefore, thefe values being fubftituted inftead of $\dot{s}$ and $\dot{y}$ in the equation $\frac{a\dot{s}}{b} = \dot{z} - \dot{y}$, we fhall have $\dot{z} = \frac{ar\dot{x} + br\dot{x} - bx\dot{x}}{b\sqrt{2rx - xx}}$, the differential equation of the cycloid.

Therefore, the value of $\dot{z}$, given from the equation, being fubftituted in the formula for the fubtangent $\frac{z\dot{x}}{\dot{z}}$, we fhall have QN $= \frac{bz\sqrt{2rx - xx}}{ar + br - bx}$.

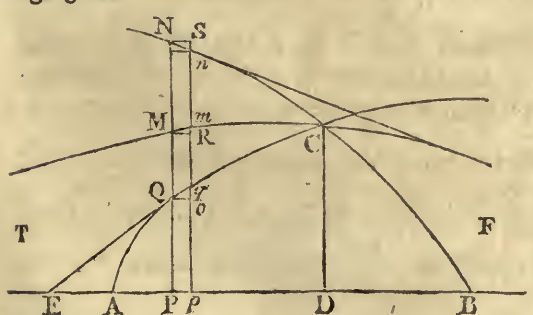
Now, if the cycloid be the ordinary one, it will be $a = b$, and therefore QN $= \frac{z\sqrt{2rx - xx}}{2r - x}$; that is, $2r - x . \sqrt{2rx - xx} :: z .$ QN; or $2r - x . y :: z .$ QN. But, by the property of the circle, it is $2r - x . y :: y . x$. Therefore it will be $y . x :: z .$ QN; that is, QP . QA :: QM . QN. Therefore MN will be parallel to PA.

EXAMPLE II.

48. Let the curve APB be a parabola, the equation of which is $px = yy$. Make $AQ = x$, $QP = y$, and let the arch $AP = s$, $PM = u$, and the ratio of MP to the arch PA be that of a to b. Therefore it will be $m\text{R} . \text{P}p :: a . b$. That is, $\dot{u} . \dot{s} :: a . b$, and therefore $\frac{a\dot{s}}{b} = \dot{u}$. But, in the parabola, it is $y = \sqrt{px}$, and $\dot{y} = \frac{p\dot{x}}{2\sqrt{px}}$. Therefore $\dot{s} = \frac{\dot{x}\sqrt{4px+pp}}{2\sqrt{px}}$. And this value being ſubſtituted inſtead of $\dot{s}$ in the equation $\frac{a\dot{s}}{b} = \dot{u}$, the equation to the curve AMC will be $\frac{a\dot{x}\sqrt{4px+pp}}{2b\sqrt{px}} = \dot{u}$. Wherefore, taking the formula of the ſubtangent $\frac{\overline{u+y} \times \dot{x}}{\dot{u}+\dot{y}}$, which is proper to this caſe, and making the ſubſtitutions inſtead of $\dot{u}$ and $\dot{y}$, it will be $\text{QN} = \frac{\overline{u+y} \times 2b\sqrt{px}}{a\sqrt{4px+pp}+bp}$. But $y = \sqrt{px}$, by the property of the curve APB, and $\frac{as}{b} = u$, by the property of the curve AMC; wherefore $\text{QN} = \frac{2as\sqrt{px} + 2bpx}{a\sqrt{pp+4px}+bp}$.

49. From the different manners by which many curves may be generated, ariſe different formulas of their ſubtangents, though the method of finding them is alike. It will be enough to ſhow it in one, to give an idea of the manner, and of the artifice, which is to be uſed on all other occaſions. Wherefore, two curves AQC, BCN, being given, having a common diameter TF, whoſe tangents can be drawn; let there be another curve MC ſuch, as that the relation of the ordinates PQ, PM, PN, in reſpect of any point at pleaſure, M, may be expreſſed by any equation whatever; and let the tangent MT be required, at any point M. Let pS be drawn infinitely near to PN, and the lines NS, MR, QO, parallel to AB, and make

Fig. 36.



make PE $= s$, PF $= t$, known by ſuppoſition, PQ $= x$, PM $= y$, PN $= z$. Becauſe of ſimilar triangles QPE, qOQ, it will be QO $= \frac{s\dot{x}}{x} =$ MR $=$ NS; and, becauſe of the ſimilar triangles mRM, MPT, it will be PT $= \frac{sy\dot{x}}{x\dot{y}}$, a formula for the ſubtangent. Now, by differencing the equation of the curve MC, in order to have the value of $\dot{x}$, to be ſubſtituted in this formula, it will be given by $\dot{y}$ and $\dot{z}$; but the ſubtangent itſelf is not to be had in finite terms. It is to be conſidered, then, that the ſimilar triangles NSn, NPF, will give us NP . PF :: nS . SN, that is, $z . t :: \pm \dot{z}$. SN $= \pm \frac{t\dot{z}}{z}$. (That is, $\dot{z}$ muſt have a poſitive ſign, if, when x and y increaſe, z will increaſe alſo; and a negative ſign, if, when x and y increaſe, z will decreaſe.) But it is alſo SN $= \frac{s\dot{x}}{x}$; then $\pm \frac{t\dot{z}}{z} = \frac{s\dot{x}}{x}$, and therefore $\dot{z} = \pm \frac{sz\dot{x}}{tx}$. Therefore, inſtead of $\dot{z}$, putting this value in the fluxional equation of the curve MC, we ſhall have the value of $\dot{x}$ expreſſed by $\dot{y}$, which, being ſubſtituted in the formula for the ſubtangent $\frac{sy\dot{x}}{x\dot{y}}$, will make the fluxions to vaniſh, and the ſubtangent will be expreſſed in finite terms.

EXAMPLE I.

50. Let $xz = yy$ be the equation of the curve MC, the fluxion of which will be $z\dot{x} + x\dot{z} = 2y\dot{y}$; and, inſtead of $\dot{z}$, ſubſtituting it's value $\pm \frac{sz\dot{x}}{tx}$, it will become $z\dot{x} \pm \frac{sz\dot{x}}{t} = 2y\dot{y}$, and therefore $\dot{x} = \frac{2ty\dot{y}}{tz \pm sz}$. Wherefore, inſtead of $\dot{x}$, ſubſtituting this value in the formula for the ſubtangent, it will be PT $= \frac{2styy}{tzx \pm szx} = \frac{2st}{t \pm s}$, when, inſtead of yy, we put it's value xz. Now let the curve AQC be a parabola whoſe parameter is b; the curve BCN a circle whoſe diameter is AB $= 2a$. If, therefore, the point N falls in the periphery of the firſt quadrant beginning at A, in which $\dot{z}$ is poſitive; the formula of the ſubtangent PT will be $\frac{2st}{t + s}$, and the ſubtangent of the circle will be $\frac{2aq - qq}{a - q} = t$, (making AP $= q$,) and that of the parabola will be $2q = s$. Therefore, theſe values of t and s being put in the expreſſion $\frac{2st}{t + s}$, we ſhall have PT $= \frac{8aq - 4qq}{4a - 3q}$.

 51. But

51. But if the point N falls in the periphery of the other quadrant, $\dot{z}$ will be negative, and the formula of the ſubtangent will be $PT = \frac{2st}{t-s}$. In this caſe, the ſubtangent of the circle is $\frac{2aq - qq}{q - a} = t$, and that of the parabola continues to be $2q = s$. Therefore, making the ſubſtitution of the values of t and s in the expreſſion $\frac{2st}{t-s}$, we ſhall have $PT = \frac{8aq - 4qq}{4a - 3q}$; the ſame as before.

52. Let AP be denominated as before, AQ being a parabola; it will be $PQ = x = \sqrt{bq}$. And BCN being a circle, it will be $PN = z = \sqrt{2aq - qq}$. Then the equation $yy = zx$ of the curve MC will be $yy = q\sqrt{2ab - bq}$. And thus, the equation being given by the two co-ordinates AP, PM, the ſubtangent PT may be found by the uſual and ordinary formulas $\frac{y\dot{q}}{\dot{y}}$. Therefore, differencing the equation $yy = q\sqrt{2ab - bq}$, it will be $y\dot{y} = \frac{4ab\dot{q} - 3bq\dot{q}}{4\sqrt{2ab - bq}}$. Now, multiplying the numerator and denominator of the formula $\frac{y\dot{q}}{\dot{y}}$ by y, it will be $\frac{yy\dot{q}}{y\dot{y}}$, and ſubſtituting the reſpective values inſtead of yy and $y\dot{y}$, it will be $\frac{yy\dot{q}}{y\dot{y}} = \frac{8aq - 4qq}{4a - 3q} = PT$, as before.

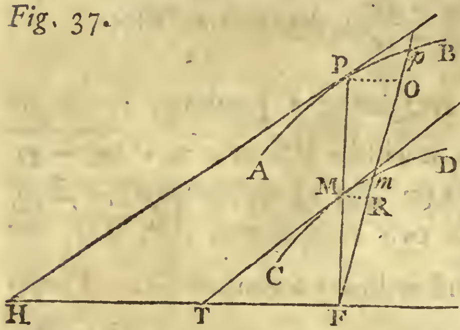
53. Let the equation of the curve MC be more general, thus, $x^m z^n = y^{m+n}$, the fluxion of which is $mz^n \dot{x} x^{m-1} + nx^m \dot{z} z^{n-1} = \overline{m+n} \times \dot{y} y^{m+n-1}$. And, inſtead of $\dot{z}$, putting it's value $\pm \frac{sz\dot{x}}{tx}$, it will be $\frac{tmz^n \dot{x} x^{m-1} \pm snz^n \dot{x} x^{m-1}}{t} = \overline{m+n} \times \dot{y} y^{m+n-1}$; and therefore $\dot{x} = \frac{\overline{mt + nt} \times \dot{y} y^{m+n-1}}{\overline{mt \pm ns} \times z^n x^{m-1}}$. Whence $PT = \frac{sy\dot{x}}{x\dot{y}} = \frac{\overline{m+n} \times sty^{m+n}}{\overline{mt \pm ns} \times z^n x^m} = \frac{m+n}{mt \pm ns} st$, if we put it's value $x^m z^n$ inſtead of y^{m+n}.

54. If the two curves AC, BCN, become right lines, in the caſe of the ſimple equation $xz = yy$ of the curve MC, it will be one of the Conic Sections of *Apollonius*, as is to be ſeen in Sect. III. of Vol. I. § 135. It will be an ellipſis, when the ordinate CD falls between the points A and B: an hyperbola, when it falls either on one ſide or the other: and laſtly, a parabola, when the points A, B, are infinitely diſtant one from the other, that is, when one of the

 right

right lines AC, BC, is parallel to the diameter. Hence it is manifeſt, that, in the ſame circumſtances, the ſame curves will be conic ſections, but of a ſuperior degree *in infinitum*, when the equation to the curve MC ſhall be this general one, $x^m z^n = y^{m+n}$.

Fig. 37.



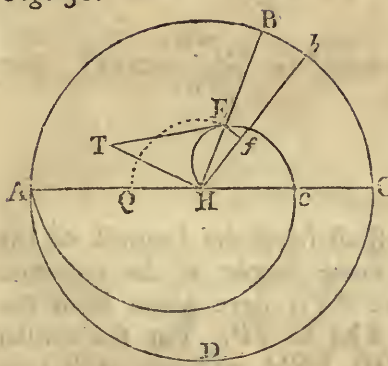
55. If the curve AP be given, having it's origin in A, of which we know how to draw the tangent; let there be another curve CMD ſuch, that, from a given point F drawing the right line FMP any how, the relation of FM to the portion AP may be expreſſed by any equation: we are to find the tangent of the curve CMD.

Let PH be a tangent to the curve APB in the point P, and let FH be drawn perpendicular to FP, and Fp infinitely near; and with centre F let the infinitely little arches MR, PO, be deſcribed; and let MT be the tangent required of the curve CMD. Make PH $= t$, FH $= s$, FM $= y$, FP $= z$, and the arch AP $= x$. Becauſe, inſtead of the infiniteſimal arches, their right ſines may be aſſumed, the triangle MRm will be right-angled at R; and, becauſe the angle MmR is not different from the angle TMF, but only by the infiniteſimal angle MFm, the two triangles MRm, TFM, may be conſidered as ſimilar; and, for the ſame reaſon, the two triangles POp, HFP, are ſimilar. Therefore it will be mR . RM :: MF . FT; that is, $\dot{y}$. MR :: y . FT, and FT $= \frac{MR \times y}{\dot{y}}$. So that, to have the value of FT, it is neceſſary to have that of MR firſt, which we might have if PO were known. Now, by the ſimilar triangles PFH, POp, it will be PH . FH :: Pp . PO; that is, $t \,.\, s :: \dot{x} \,.\, OP = \frac{s\dot{x}}{t}$. And, by the ſimilar ſectors FPO, FMR, it will be FP . PO :: FM . MR; that is, $z \,.\, \frac{s\dot{x}}{t} :: y \,.\, MR = \frac{sy\dot{x}}{zt}$. Whence FT $= \frac{syy\dot{x}}{tz\dot{y}}$, the formula for the ſubtangent. Now if, inſtead of $\dot{x}$, we ſubſtitute it's value, which may be obtained from the fluxional equation of the curve CMD, we ſhall have the ſubtangent expreſſed in finite terms.

EX-

EXAMPLE I.

Fig. 38.

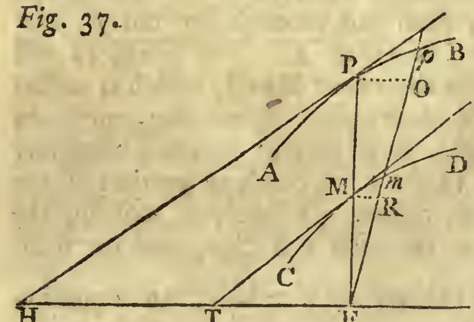


56. Let there be a circle ABCD defcribed with centre H, and radius HA; and whilft the radius HA, with one end fixed in the centre, moves uniformly round, and with the other extremity A defcribes the periphery ABCD; let the point H move uniformly upon the radius HA, fo that when the radius returns to it's firft fituation HA, the point H, in the mean time, fhall pafs through the radius, and fhall then be found at A. The point H will then defcribe the curve HEcA, which is called the *Spiral* of *Archimedes*. From the generation of this curve, it is eafy to perceive that any arch of the circle whatever, as AB, fhall be to the correfponding portion of the radius HE, as the whole circle is to the whole radius. Therefore, making the radius $= r$, the periphery of the circle $= c$, the arch AB $= x$, and the ordinate HE $= y$; it will be $x \,.\, y :: c \,.\, r$; and therefore $y = \frac{rx}{c}$, an equation to the fpiral, in which the ordinates proceed from the fixed point H. This being premifed, if we would find ET, the tangent of the fpiral; becaufe, in this cafe, FP (Fig. 37.) is the radius HB of the circle, it will be $z = r$, and the two lines, PH the tangent, and FH the fubtangent, (in the fame Fig. 37,) are in this both perpendicular to the radius HB, (by the nature of the circle,) and confequently parallel to each other, and alfo equal; whence it will be $s = t$, and therefore the general formula, in this cafe, will be $\frac{yy\dot{x}}{r\dot{y}}$. Then, differencing the equation $y = \frac{rx}{c}$, it will be $\dot{y} = \frac{r\dot{x}}{c}$; and the value of $\dot{x}$ being fubftituted in the formula, it will be $\frac{cyy}{rr} =$ HT. Or elfe, putting, inftead of y, it's value $\frac{rx}{c}$, it will be $\frac{xy}{r} =$ HT. Therefore, with centre H, and radius HE $= y$, defcribing the arch EQ, and taking HT equal to the arch EQ, it fhall be the fubtangent. For, by fimilar fectors HEQ, HBA, it will be HA . AB :: HQ . QE. That is, $r \,.\, x :: y \,.\, \text{QE} = \frac{xy}{r}$.

 If,

If, instead of making the equation $y = \frac{rx}{c}$, it were, in general, $y^m = \frac{r^m x}{c}$; that is, the periphery to the arch AB, as any power integral or fractional of the radius, to a like power of the ordinate: Then taking the fluxion of the equation, it would give us $\dot{x} = \frac{mc\dot{y}y^{m-1}}{r^m}$, and $y\dot{x} = \frac{mc\dot{y}y^m}{r^m}$. Then substituting this in the formula of the subtangent $\frac{yy\dot{x}}{r\dot{y}}$, it would be $\frac{mcy^{m+1}}{r^{m+1}} = \text{HT}$. But $y^m = \frac{r^m x}{c}$; therefore $\frac{mxy}{r} = \text{HT} = m \times \text{EQ}$.

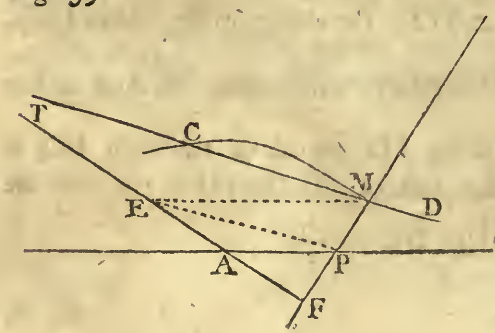
Fig. 37.



57. We shall have the formula of the subtangent more simple, if the equation of the curve APB were given from the relation of TM to FP. For the similar triangles pOP, PFH, will give us $\text{PO} = \frac{s\dot{z}}{z}$, and the similar sectors FPO, FMR, will give us $\text{MR} = \frac{sy\dot{z}}{zz}$; and lastly, the similar triangles MRm, TFM, will give us $\text{FT} = \frac{syy\dot{z}}{zz\dot{y}}$.

EXAMPLE II.

Fig. 39.

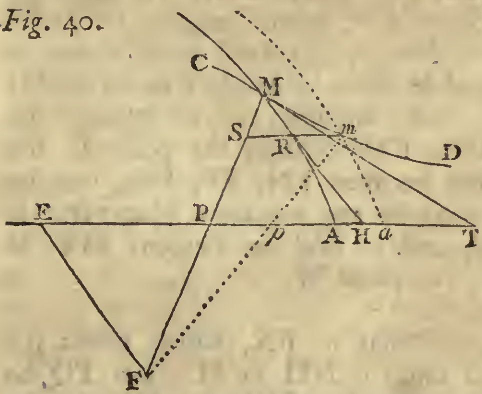


58. Let the curve CMD be above the line APB, which makes no alteration, and let APB be a right line, and let FM, FP, always differ from each other by the same quantity, that is, make the constant line PM $= a$. Then will $y - z = a$ be the equation of the curve, which is the Conchoid of *Nicomedes*, whose pole is the point F, and asymptote AB. Taking the fluxions

fluxions of the equation, it will be $\dot{y} = \dot{z}$, and thence the subtangent FT $= \frac{syy}{zz}$.

Drawing, then, ME parallel to PA, and MT parallel to PE, MT will be a tangent to the curve in M. For it will be FA $= s$, FE $= \frac{sy}{z}$, and FT $= \frac{syy}{zz}$.

Fig. 40.



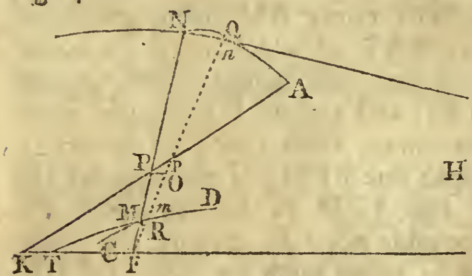
59. Any curve AM being given, to the axis EAT of which curve we know how to draw the tangent MH, at any point M; and a point F being given out of the curve, from which let be drawn the right-line FPM; if we conceive the right line FPM to revolve about the immoveable point F, making the plane PAM to move upon the right line ET, always parallel to itself, the intercepted line PA always continuing the same: Then the point M, which is the common intersection of the two lines FM, AM, by this motion will describe a curve CMD, the tangent of which is required. Let the plane PAM move, and, in the first instant, let it arrive at an infinitely near position *pam*, and let SR*m* be drawn parallel to ET. The similar triangles MR*m*, MHT, would give the right line HT, which determines the tangent required, if the sides MR, R*m*, were known. Therefore, to obtain them, let us make FP, or F*p* $= x$, FM, or F*m* $= y$, P*p* $= \dot{z}$, and the known lines PA $= a$, HM $= t$, PH $= s$. It is plain, by the construction, that it will be P*p* = A*a* = R*m* $= \dot{z}$; and, by the similar triangles FP*p*, FS*m*, it will be F*p* . P*p* :: F*m* . S*m*. That is, $x . \dot{z} :: y . Sm = \frac{y\dot{z}}{x}$. Then SR $= \frac{y\dot{z} - x\dot{z}}{x}$. And, by similar triangles MPH, MSR, it will be HP . HM :: RS . RM. That is, $s . t :: \frac{y\dot{z} - x\dot{z}}{x} . MR = \frac{ty\dot{z} - tx\dot{z}}{sx}$. Lastly, by the similar triangles MR*m*, MHT, it will be MR . R*m* :: MH . HT. That is, $\frac{ty\dot{z} - tx\dot{z}}{sx} . \dot{z} :: t . HT = \frac{sx}{y - x}$.

From the point F draw FE parallel to the tangent MH, and taking HT = PE, draw TM, which shall be a tangent to the curve at the point M. For, because of similar triangles PMH, PFE, it will be PM . PH :: PF . PE; that is, $y - x . s :: x . \frac{sx}{y - x}$ = PE = HT.

60. It

60. It has been already demonſtrated, Vol. I. Sect. III. § 136, that, if the line AM were a right line, the curve CMD would be an hyperbola, which would have ET for one of it's two aſymptotes. If AM were a circle with centre P, the curve CMD would be the conchoid of *Nicomedes*, the pole of which is F, and it's aſymptote ET. And laſtly, if AM were a parabola, the curve CMD would be the companion of the paraboloid of *Carteſius*, that is, one of the two parabolical conchoids.

Fig. 41.



61. To the diameter AP let there be any curve AN, whoſe tangent we know how to draw, and a fixed point F out of it; and let there be another curve CMD ſuch, that, drawing, as we pleaſe, the right line FMPN from the point F, the relation between FN, FP, FM, may be expreſſed by any equation whatever. It is required to find the tangent MT, at any given point M.

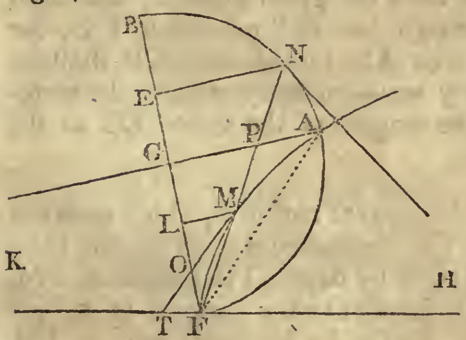
Through the point F draw HK perpendicular to FN, which meets the diameter AP produced in K, and the given tangent NH in H. Let FQ be infinitely near FN, and with centre F let the arches MR, P*o*, NQ, be deſcribed. Make FK $= s$, FH $= t$, FP $= x$, FM $= y$, FN $= z$; then it will be *m*R $= \dot{y}$, *po* $= \dot{x}$, Q*n* $= -\dot{z}$. And, becauſe of like triangles NQ*n*, NFH, it will be NQ $= -\frac{t\dot{z}}{z}$. Alſo, becauſe of like ſectors FNQ, FMR, it will be MR $= -\frac{ty\dot{z}}{zz}$. Laſtly, becauſe of like triangles MR*m*, MFT, it will be FT $= -\frac{yyt\dot{z}}{zz\dot{y}}$, the formula required for the ſubtangent. But here it might be ſuſpected, that, taking the fluxion of the equation of the curve, the value of $\dot{y}$ to be ſubſtituted in the formula will be given by $\dot{x}$ and $\dot{z}$, by which means the fluxions would not vaniſh. Yet, however, the ſimilar ſectors FNQ, FP*o*, will give us P*o* $= -\frac{tx\dot{z}}{zz}$; and the ſimilar triangles P*op*, PFK, will give us the analogy, $\dot{x} . -\frac{tx\dot{z}}{zz} :: x . s$. Whence the equation $szz\dot{x} = -txx\dot{z}$, and therefore $-\dot{z} = \frac{szz\dot{x}}{txx}$. Therefore, ſubſtitute the value of $\dot{y}$ in the formula for the ſubtangent, which value is to be obtained from the fluxional equation of the curve, and then this value inſtead of $\dot{z}$; by which the fluxions will vaniſh, and we ſhall have the ſubtangent in finite terms.

If

If the line AP were a curve inftead of a right line, drawing the tangent PK, by the fame way of argumentation we fhould find the fame value of the fubtangent FT.

EXAMPLE.

Fig. 42.



62. Let the curve AN be a circle which paffes through the point F, and is fo pofited, that, from the point F drawing the perpendicular FB (produced) to AP, it may pafs through the centre G of the fame circle; and let PN be always equal to PM: the curve CMD of the foregoing figure, that is, FMA in this, will be the ciffoid of *Diocles*, the equation of which will be $z + y = 2x$. Then we fhall have, by taking the fluxion, $\dot z + \dot y = 2\dot x$, or $\dot y = 2\dot x - \dot z$; and fubftituting this value of $\dot y$ in the formula $-\frac{yyt\dot z}{zz\dot y}$ of the fubtangent, it will be $-\frac{yyt\dot z}{2zz\dot x - zz\dot z}$; and laftly, putting, inftead of $-\dot z$, it's value $\frac{szz\dot x}{tx x}$, we fhall have $\frac{styy}{2txx + szz}$ = FT, the fubtangent required.

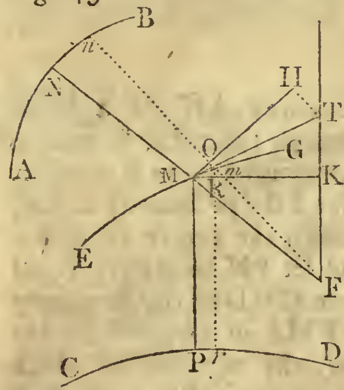
Here it is plain, that if the point M, at which the tangent is required, fhould fall upon the point A; in this cafe, KH being perpendicular to FA, it would be FN = FP = FM = FA = FK = FH; and therefore FT $= \frac{1}{3}x$ $= \frac{1}{3}$AF.

63. Perhaps we might find the fubtangent of the ciffoid more fpeedily, by means of the ufual formula, at § 30. For, drawing NE, ML, perpendicular to FB, and making FB $= 2a$, FL $= x$, LM $= y$; by the property of the curve FMA, it will be BE = FL $= x$; and, by the property of the circle, it will be EN $= \sqrt{2ax-xx}$; and the fimilar triangles FLM, FEN, will give FL . LM :: FE . FN, and therefore FL . LM :: EN . EB; that is, $x \,.\, y$ :: $\sqrt{2ax - xx} \,.\, x$, whence $y = \frac{xx}{\sqrt{2ax - xx}}$, or $yy = \frac{x^3}{2a - x}$, the equation of the curve FMA. Therefore, by taking the fluxions, we fhall have $2y\dot y = \frac{6axx\dot x - 2x^3\dot x}{\overline{2a - x}|^2}$; and taking the ufual formula $\frac{y\dot x}{\dot y}$, and making all the neceffary

8 fubftitutions,

ſubſtitutions, it will be $\frac{y\dot{x}}{\dot{y}} = yy \times \frac{\overline{2a - x}|^2}{3ax^2 - x^3} = \text{LO} = \frac{2ax - xx}{3a - x}$, by putting, inſtead of yy, it's value $\frac{x^3}{2a - x}$.

Fig. 43.



64. Let there be two curves ANB, CPD, and a right line FK, in which are three fixed points A, C, F. Further, let the curve EMG be ſuch, that, drawing through any of it's points, M, the right line FMN from the given point F, and from the point M the right line MP parallel to FK; the relation of the arch AN to the arch CP ſhall be expreſſed by any equation at pleaſure. It is required to find the tangent of the curve EG at the point M.

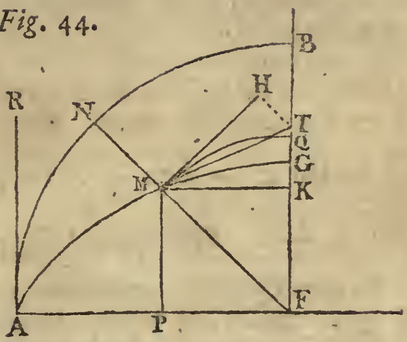
Let MT be the tangent required, which meets in T the right line FK, produced if need be, and from the point T let there be drawn TH parallel to FM, and through the point M let be drawn MRK parallel to the tangent in P, and MOH parallel to the tangent in N, and let FmOn be infinitely near to FN. Make FM $= s$, FN $= t$, MK $= u$, and the arches AN $= y$, CP $= x$; and therefore Nn $= \dot{y}$, Pp $= \dot{x}$. By the ſimilar triangles FNn, FMO, it will be FN . Nn :: FM . MO; that is, $t . \dot{y} :: s . \text{MO} = \frac{s\dot{y}}{t}$. And, by the ſimilar triangles MmR, MTK, and MOm, MHT, it will be MR . mM :: MK . MT, and Mm . MO :: MT . MH; and it will be alſo MR . MO :: MK . MH. That is, $\dot{x} . \frac{s\dot{y}}{t} :: u . \text{MH} = \frac{us\dot{y}}{t\dot{x}}$. Wherefore, by taking the fluxion of the given equation, we ſhall have the value of $\dot{y}$ given by $\dot{x}$; and, by making the neceſſary ſubſtitutions, we ſhall have MH expreſſed in finite terms. Taking, therefore, MH equal to the value now found, and parallel to the tangent in N of the curve ANB, and drawing HT parallel to MF; if from the point M be drawn the right line TM to the point T, it will be a tangent to the curve EMG in the point M.

N. B. The letter r has been put, by miſtake, for the letter p, in Fig. 43.

E X-

EXAMPLE.

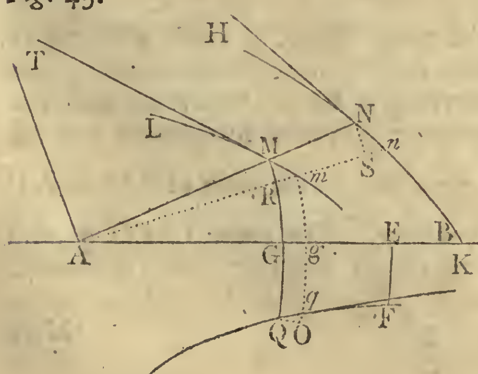
Fig. 44.



65. Let the curve ANB be a quadrant of a circle, whose centre is F; and let CPD of Fig. 43 be the radius APF of Fig. 44, which is perpendicular to the right line FKTB, and let the tangent AR be drawn. Let the radius FA be conceived to revolve equably about the centre F, and, at the same time, the tangent AR to move equably upon AF towards FB, always parallel to itself; so that, when the radius FA falls upon FB, the tangent AR may coincide with FB. By this motion, the point M, which is the intersection of the radius and the tangent, will describe the curve AMG, called the *Quadratrix* of *Dinostratus*.

It is plain, from the generation of this curve, that the arch AN will be to the intercepted line AP, as the quadrantal arch AB is to the radius AF. Therefore, making AN $= y$, AP $= x$, AB $= a$, AF $= r$, it will be $ry = ax$, and $\dot{y} = \frac{a\dot{x}}{r}$; then, substituting this value of $\dot{y}$ in the formula $\frac{us\dot{y}}{t\dot{x}}$, it will be MH $= \frac{asu}{rt}$; but, in this case, FN is the radius of the circle, and MK $=$ AF $-$ AP; then $t = r$, $u = r - x$; whence MH $= \frac{asr - asx}{rr} = \frac{as - sy}{r}$, putting, instead of ax, it's value ry from the given equation. From the point M raise MH perpendicular to FM, and equal to the arch MQ described with centre F, radius FM, and let HT be drawn parallel to FM; then MT will be a tangent to the quadratrix in the point M. For, because of similar sectors FNB, FMQ, it will be FN . NB :: FM . MQ. That is, $r \,.\, a - y :: s \,.\,$ MQ $= \frac{as - sy}{r} =$ MH.

Fig. 45.



66. Let there be two curves BN, FQ, of which it is known how to draw the tangents, and which have the right line BA for a common axis, in which are two fixed points A, E. And let there be another curve LM, such, that, drawing the right line AMN through any of it's points M, and with centre A and radius AM describing the arch MG; and from the point G letting fall GQ perpendicular to AG; the relation of the spaces

 ANB,

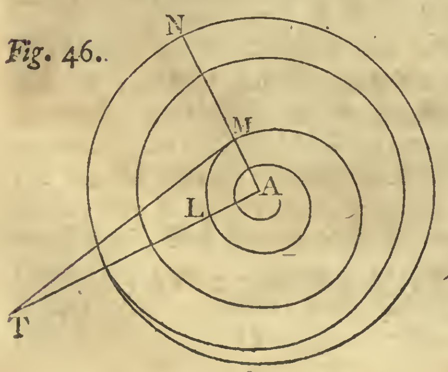
ANB, EFQG, and of the lines AM, AN, QG, may be given by the means of any equation. The tangent of the curve LM is required at the point M.

Drawing the right line ATH perpendicular to AMN, let there be another Amn infinitely near to AMN, and the arch mq, and the perpendicular gq: Then, with centre A describing the little arch NS, making the given subtangents HA $= a$, GK $= b$, and make AM $= y$, AN $= z$, QG $= u$, and the spaces EGQF $= s$, ANB $= t$, it will be Rm = Gg $= \dot{y}$, Sn $= \dot{z}$. And, because of the similar triangles KGQ, QOq, it will be Oq $= -\dot{u} = \frac{u\dot{y}}{b}$. And, by the similar triangles HAN, NSn, it will be SN $= \frac{a\dot{z}}{z}$. The space GQqg may be taken for the space GQOg, because their difference QOg is an infinitesimal of the second order. Whence it will be GQqg $= u\dot{y} = -\dot{s}$. Thus, therefore, it will be ANn $= \frac{1}{2}$AN × NS $= \frac{1}{2}a\dot{z} = -\dot{t}$. Wherefore, these values being substituted, instead of $\dot{u}$, $\dot{s}$, $\dot{t}$, in the fluxion of the proposed equation, we shall have an equation from whence may be deduced the value of $\dot{z}$ given by $\dot{y}$. Now, because of similar sectors ARM, ANS, it will be MR $= \frac{ay\dot{z}}{zz}$; and, by the similar triangles mRM, MAT, it will be AT $= \frac{ayy\dot{z}}{zz\dot{y}}$, the formula for the subtangent; in which, instead of $\dot{z}$, if we substitute it's value given by $\dot{y}$ from the equation of the curve, the fluxions will disappear, and the subtangent will be given in finite terms.

EXAMPLE.

67. Let the space EGQF be double to ABN, that is, $s = 2t$; then $\dot{s} = 2\dot{t}$. But $\dot{s} = -u\dot{y}$, and $\dot{t} = -\frac{1}{2}a\dot{z}$; therefore it will be $u\dot{y} = a\dot{z}$, and $\dot{z} = \frac{u\dot{y}}{a}$. Then the subtangent is AT $= \frac{uyy}{zz}$.

Fig. 46.



Let the curve BN be a circle with centre A, radius AN $= c$; whence $z = c$; and let the curve FQ be an hyperbola with the equation $uy = ff$; the subtangent will be AT $= \frac{ffy}{cc}$; that is, the ratio of AM to AT will be constant. The curve LM (Fig. 46.) will be called, in this case, the *Logarithmic Spiral*.

 Here

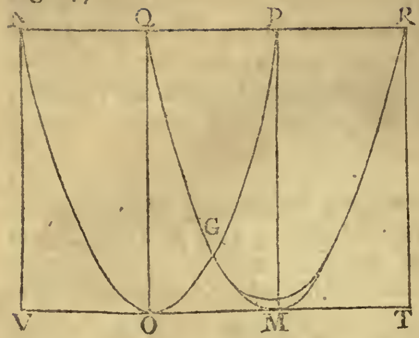
Here it is manifeſt, that the curve LM will make an infinite number of circumvolutions before it arrives at the point A; foraſmuch as, when the point G (Fig. 45.) coincides with A, the ſpace s will be infinite, as may be ſeen from the Inverſe Method of Fluxions. For then, alſo, the ſpace t muſt be infinite, which cannot be but after infinite revolutions of the radius AM.

68. It remains, laſtly, to conſider a particular caſe belonging to Tangents. It has been ſeen that, the co-ordinates of any curve being x and y, the general formula of the ſubtangent will be $\frac{y\dot{x}}{\dot{y}}$, or $\frac{x\dot{y}}{\dot{x}}$, according as y or x ſupplies the place of the ordinate. Wherefore, the fluxion of the equation of the curve being taken, if from thence we deduce the value of $\dot{x}$ or $\dot{y}$, this value, being ſubſtituted in the general formula, will give us a fraction in finite terms, which is the expreſſion or value of the ſubtangent for any point of the propoſed curve. Now, if we deſire the ſubtangent for any determinate point of the curve, nothing elſe is required to be done, but to ſubſtitute in this fraction, inſtead of x and y, their values which they have at the point given. But it may ſometimes happen, that, by ſubſtituting, inſtead of x or y, a determinate value in the fraction which expreſſes the ſubtangent, or otherwiſe, in the ratio of $\dot{x}$ to $\dot{y}$ deduced from the fluxional equation of the curve, all the terms in the numerator and denominator may vaniſh of themſelves, and that there will only ariſe $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$, and thence, alſo, the ſubtangent will be $\frac{0}{0}$, from whence, however, we are not to infer that the ſubtangent is nothing in this point.

For an example, let us take the curve belonging to this equation $y^4 - 8ay^3 - 12axyy + 16aayy + 48aaxy + 4aaxx - 64a^3x = 0$, and let y be the abſciſs, and x the ordinate. Therefore $\frac{x\dot{y}}{\dot{x}}$ will be the formula for the ſubtangent. Therefore, by taking the fluxion of this equation, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{3ayy - 12aay - 2aax + 16a^3}{y^3 - 6ayy - 6axy + 8aay + 12aax}$, and the ſubtangent will be $\frac{x\dot{y}}{\dot{x}} = \frac{3axyy - 12aaxy - 2aaxx + 16a^3x}{y^3 - 6ayy - 6axy + 8aay + 12aax}$. Now, if we would have the ſubtangent to that point of the curve, which correſponds to the abſciſs $y = 2a$, it being alſo in this caſe $x = 2a$, by the given equation; make the ſubſtitutions in the fraction which expreſſes the ratio of $\dot{x}$ to $\dot{y}$, and we ſhall find it to be $\frac{12a^3 - 24a^3 - 4a^3 + 16a^3}{8a^3 - 24a^3 - 24a^3 + 16a^3 + 24a^3}$, that is, $\frac{0}{0}$, becauſe all the terms deſtroy one another; and therefore the ſubtangent alſo, at this point, is $\frac{0}{0}$, which informs us of nothing, although one or more ſubtangents may belong to that point.

 69. This

Fig. 47.



69. This case will always happen, whenever the curve has several branches which intersect one another, and when we would have a tangent at the point of concourse. And, indeed, the curve NOPQMR (Fig. 47.) of the proposed equation has two such branches, which cut one another in the point G, to which exactly corresponds $y = 2a$, OT being the axis of the y's, and it's beginning at O. Also, $x = 2a$, taking the x's in the axis OQ.

To give a reason for this case, it is enough to take notice of two things. The first is, that, at the point of concourse of the different branches of the curve, several roots of the equation become equal to one another. Thus, as to the proposed equation, in the point G the two values of x are equal, and also, two are equal of the four values of y. The second is, (what is demonstrated in *Des Cartes's* Algebra,) that if an equation which contains equal roots be multiplied, term by term, into any arithmetical progression, the product will be equal to nothing, and will contain in it fewer by one of the equal roots. And if this product be again multiplied by an arithmetical progression, the product will, in like manner, be equal to nothing, and will contain still fewer by one of the equal roots, than were contained by the first product; that is, fewer by two of the equal roots, than were contained by the first equation. And thus on successively to that product, which shall contain only one of the equal roots.

If, therefore, any equation of a curve, treating x as variable and y as constant, shall be multiplied by an arithmetical progression which terminates in nothing; in the case of equal roots the product shall be equal to nothing; and it will also be so, if the product be divided by x, which division will succeed when the last term is multiplied by nothing. The same thing will obtain also by treating y as variable and x as constant, and multiplying the equation by such an arithmetical progression as has nothing, or o, to put under the last term.

This being supposed, it is easy to perceive that such an operation as this performs the very same thing as taking the fluxion; that is, if it treats x as variable, and multiplies the equation by an arithmetical progression, the first term of which is the greatest exponent of x, and the last term is nothing, and produces a product multiplied into $\dot{x}$. Then, if it treats y as variable, and multiplies the equation by an arithmetical progression, the first term of which is the greatest exponent of y, and the last is nothing, or o, and produces a product multiplied into $\dot{y}$. But, in the case of equal roots of x, and in that of equal roots of y, as well the product multiplied by $\dot{x}$, as that by $\dot{y}$, are equal to nothing. So that the ratio $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$ ought to arise, in that point wherein two branches of the curve intersect each other.

That this may be feen more fully, I here fet in order the equation of the propofed curve according to the letter y, and multiply it by an arithmetical progreffion, the laft term of which is o.

$$\left.\begin{matrix} y^4 - 8ay^3 - 12axy^2 + 48aaxy + 4aaxx \\ + 16aay^2 \qquad\qquad - 64a^3x \end{matrix}\right\} = 0.$$

$$4, \quad 3, \quad 2, \quad 1, \quad 0,$$

The product will be

$$4y^4 - 24ay^3 - 24axy^2 + 32aay^2 + 48aaxy = 0.$$

That is, dividing by $4y$,

$$y^3 - 6ay^2 - 6axy + 8aay + 12aax = 0.$$

Then I fet the fame equation in order according to the letter x, and multiply it by the arithmetical progreffion, the laft term of which is o.

$$\left.\begin{matrix} 4aax^2 + 48aayx + y^4 \\ - 64aaax - 8ay^3 \\ - 12ayyx + 16a^2y^2 \end{matrix}\right\} = 0.$$

$$2, \quad 1, \quad 0,$$

The product will be

$$8aax^2 + 48aayx - 64a^3x - 12ayyx = 0.$$

That is, dividing by $4x$,

$$2aax + 12aay - 16a^3 - 3ayy = 0.$$

This being done, I take the fluxion of the propofed equation, which is $4y^3\dot{y} - 24ay^2\dot{y} - 24axy\dot{y} - 12ay^2\dot{x} + 32aay\dot{y} + 48aax\dot{y} + 48a^2y\dot{x} + 8a^2x\dot{x} - 64a^3\dot{x} = 0$; that is, dividing it by 4, and tranfpofing the terms belonging to $\dot{x}$,

$$y^3 - 6ay^2 - 6axy + 8a^2y + 12a^2x \text{ into } \dot{y}$$
$$= 3ay^2 - 12aay + 2aax + 16a^3 \text{ into } \dot{x}.$$

Now here the multiplier of $\dot{y}$ is the firft product into the arithmetical progreffion, and confequently $= 0$ in relation to the point G, in which y has two equal values. And the multiplier of $\dot{x}$ is the fecond product into it's arithmetical progreffion with it's figns changed, which does not hinder it being $= 0$, in relation to the fame point G, in which x has two equal values. Therefore it will be $\dot{y} \times 0 = \dot{x} \times 0$, that is, $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$ in the point G.

But, if to multiply any equation by an arithmetical progreffion, or to find it's fluxion, (which is the fame thing,) bring it to pafs, that, on the fuppofition of of equal roots, that case will arise of which we are treating, that is, $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$; it also brings it to pass, that, in the equation derived from thence, there will be one less of those equal roots. Wherefore, if the equation proposed have two equal roots, when differenced it will have but one of those equal roots. And, if the proposed equation have three, by differencing again that which was differenced before, (assuming as constant the differences or fluxions $\dot{x}$, $\dot{y}$,) the equation thence arising will have only one; and so on. Therefore, if we assume as constant the fluxions $\dot{x}$, $\dot{y}$, as well the terms multiplied into $\dot{x}$ as those multiplied into $\dot{y}$, will mutually destroy each other, in the supposition of such a determinate value of x and y; also, the terms multiplied into $\ddot{x}$ and $\ddot{y}$ will destroy one another. By proceeding in this way of operation, equations will be reduced to contain only one of the number of equal roots which they had at first; and therefore, finally differencing the last, to obtain the ratio of $\dot{y}$ to $\dot{x}$, there can no longer arise the case of $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$.

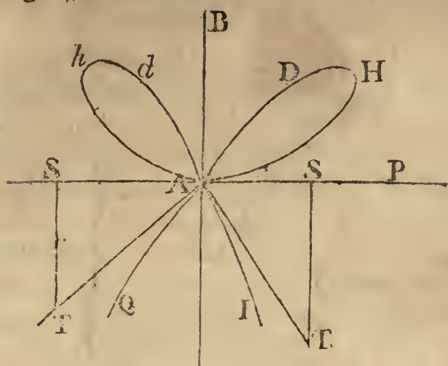
Therefore I resume the foregoing equation whose fluxion was found to be $y^3\dot{y} - 6ay^2\dot{y} - 6axy\dot{y} - 3ay^2\dot{x} + 8aay\dot{y} + 12aax\dot{y} + 12aay\dot{x} + 2aax\dot{x} - 16a^3\dot{x} = 0$. But, because, by substituting, instead of y, it's value $2a$, and, instead of x, it's correspondent value $2a$, in order to have the tangent at the point G; I find only $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$: I go on to difference that already differenced, taking always for constant the fluxions $\dot{x}$, $\dot{y}$, and I shall obtain $3y^2\dot{y}^2 - 12ay\dot{y}^2 - 6ax\dot{y}^2 + 8aa\dot{y}^2 - 12ay\dot{y}\dot{x} + 24aa\dot{y}\dot{x} + 2aa\dot{x}^2 = 0$.

Instead of y and x, I substitute their values $2a$, in relation to the point G; and I find $\dot{x} = \pm \dot{y}\sqrt{8}$. Then, in the general formula for the subtangent $\frac{x\dot{y}}{\dot{x}}$, putting the values of $x = 2a$, and $\dot{x} = \pm \dot{y}\sqrt{8}$, I shall finally have the subtangent $= \pm \frac{a}{\sqrt{2}}$; or, to speak more properly, the two subtangents corresponding to the point G, one positive, the other equal to it, but negative.

If the curve shall have three equal roots at the point in which the tangent is required, that is, if the curve shall have three branches which meet one another in that point; because, after the equation has been differenced once, it will still have two equal roots; it must be differenced again, that we may have the ratio of $\dot{y}$ to $\dot{x}$: It will give us, notwithstanding, by what has been already said, the ratio $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$; and therefore it will be necessary to take the difference or fluxion a third time. And, in general, the equation must be so often differenced as is the number of equal roots, or the branches of the curve; and from the last difference must be obtained the ratio of $\dot{y}$ to $\dot{x}$. And so many will be the tangents as are the branches of the curve, which cut one another in that point.

Let

Fig. 48.



Let the curve be QADHA*bd*AI, whose equation is $x^4 - ayxx + by^3 = 0$, and which has three branches QAD, IA*d*, *b*AH, which cut one another in A. And let AP be the axis belonging to x, and AB perpendicular to AP, the axis belonging to y, and the point A their common origin. By differencing the equation, it will be $4x^3\dot{x} - 2ayx\dot{x} - axx\dot{y} + 3byy\dot{y} = 0$; that is, $\frac{\dot{x}}{\dot{y}} = \frac{axx - 3byy}{4x^3 - 2ayx}$. But, if we would have the tangent at the point A, because there it is $x = 0$, $y = 0$; it will be $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$. We must therefore go on to second fluxions, and the equation will be $12xx\dot{x}\dot{x} - 2ay\dot{x}\dot{x} - 4ax\dot{x}\dot{y} + 6by\dot{y}\dot{y} = 0$. But from this we shall only obtain $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$, every term being multiplied by $x = 0$, by supposition, or by $y = 0$. Therefore, differencing for the third time, it will be $24x\dot{x}^3 - 6a\dot{y}\dot{x}^2 + 6b\dot{y}^3 = 0$. Here, making $x = 0$, the first term vanishes, and therefore it is $a\dot{y}\dot{x}^2 = b\dot{y}^3$, from whence we have three values of $\dot{y}$; that is, $\dot{y} = 0$, and $\dot{y} = \pm \frac{\dot{x}\sqrt{a}}{\sqrt{b}}$, which give us three ratios of $\dot{x}$ to $\dot{y}$; that is to say, three tangents at the point A. One of them will be infinite, which coincides with the axis AP, and serves for the branch *b*AH. The other, taking any line AS, and drawing ST perpendicularly in such a manner, as that it may be ST . SA :: $\sqrt{a} \,.\, \sqrt{b}$; the lines TA will be tangents in the point A, one of the branch QAD, the other of the branch IA*d*.

70. The truth of these conclusions may also be demonstrated after another manner, and, as they say, *à posteriori*. The differentials of finite equations, which are found by the foregoing rules of differencing, are not really the complete differentials, the rules giving us only those terms which contain the first differences, or of one dimension only; and omitting, for brevity-sake, and for greater convenience, the differences of other degrees, or of greater dimensions: which, by the principles of the calculus, would make those terms in which they are found to be relatively nothing.

Resuming the equation
$$\left.\begin{array}{l} y^4 - 8ay^3 - 12axy^2 + 48a^2yx + 4a^2x^2 \\ \qquad\qquad + 16a^2y^2 \qquad\qquad - 64a^3x \end{array}\right\} = 0,$$
it's fluxion or difference will be $4y^3\dot{y} - 24ay^2\dot{y} - 12ayy\dot{x} - 24axy\dot{y} + 32a^2y\dot{y} + 48aax\dot{y} + 48aay\dot{x} + 8aax\dot{x} - 64a^3\dot{x} = 0$. But here, if y be considered as increased by it's fluxion or difference, and likewise x; and that in the proposed equation, instead of y and it's powers, we should write $y + \dot{y}$ and it's corresponding powers; and should do the same by writing $x + \dot{x}$ and it's powers instead of those

thoſe of x; we ſhould then have the terms as they are ſet in order in the following Table.

I.	II.	III.	IV.	V.	
$+ y^4$	$+ 4y^3\dot{y}$	$+ 6yy\dot{y}\dot{y}$	$+ 4y\dot{y}^3$	$+ \dot{y}^4$	
$- 8ay^3$	$- 24ay^2\dot{y}$	$- 24ay\dot{y}\dot{y}$	$- 8a\dot{y}^3$		
$- 12axy^2$	$- 24axy\dot{y}$	$- 12ax\dot{y}\dot{y}$	$- 12a\dot{x}\dot{y}^2$		
$+ 16aay^2$	$- 12ayy\dot{x}$	$- 24ay\dot{x}\dot{y}$			$= 0.$
$+ 48aaxy$	$+ 32a^2y\dot{y}$	$+ 16aa\dot{y}\dot{y}$			
$+ 4aaxx$	$+ 48a^2y\dot{x}$	$+ 48aa\dot{x}\dot{y}$			
$- 64a^3x$	$+ 48a^2x\dot{y}$	$+ 4aa\dot{x}\dot{x}$			
	$+ 8a^2x\dot{x}$				
	$- 64a^3\dot{x}$				

Now the ſum of all theſe columns, excepting the firſt, which is the propoſed equation itſelf, will be their complete and entire fluxion. But, becauſe the laſt or fifth column is infinitely little in reſpect of the fourth, and the fourth in reſpect of the third, and the third in reſpect of the ſecond; we aſſume the ſecond column alone for the fluxion of the propoſed equation, which compendium proceeds from the common rule of differencing. But it can be ſo only when the columns after the ſecond are abſolutely nothing. If, therefore, a caſe ſhall ariſe, in which the ſecond column is abſolutely nothing, the third may not be nothing in reſpect of it, and therefore ought not to be omitted, but will itſelf be the differential of the firſt. And the ſame may be ſaid of the fourth, when the ſecond and third are nothing; and ſo of the reſt. But this caſe preciſely happens, when we ſeek the relation of $\dot{x}$ to $\dot{y}$ in the propoſed equation, in that point in which it is $y = 2a$, and $x = 2a$; becauſe, making the neceſſary ſubſtitutions, we find the ſecond column itſelf to be nothing; and therefore we go on to make uſe of the third. And this is exactly the ſame thing as to difference the equation twice, as appears from hence.

71. By the ſame principles, and after the ſame manner, a like caſe may be reſolved, which ariſes in the conſtruction of curves, when the ordinate is expreſſed by a fraction, the denominator and numerator of which become each equal to nothing, when a determinate value is aſſigned to the abſciſs.

Now, to remove this difficulty, it is enough to conſider the fraction as if it expreſſed the ordinates of two curves, which meet in ſome point of their common axis. And becauſe, in this point, their ratio cannot be expreſſed otherwiſe than by $\frac{0}{0}$, it is neceſſary to find what may be their ratio in a point infinitely near it, that is, when they are increaſed by an infiniteſimal. That is to ſay, we muſt proceed to differencing the numerator, and then the denominator of the ſaid fraction, and that once, twice, or oftener, till at laſt, putting the determinate value of the abſciſs in the fraction, it may no longer be $\frac{0}{0}$, for the ſame reaſon mentioned before, concerning the columns of differentials.

Let

Let the equation be $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$. Taking $x = a$, and making the ſubſtitution, it will be $y = \frac{0}{0}$, from whence we cannot therefore infer, that when the abſciſs $x = a$, the correſponding ordinate will be $y = 0$. For, by differencing the numerator, and then the denominator of the fraction, it will be $y = \frac{\overline{a^3\dot{x} - 2x^3\dot{x}} \times \overline{2a^3x - x^4}|^{-\frac{1}{2}} - \frac{1}{3}a^3\dot{x} \times a^{-\frac{4}{3}}x^{-\frac{2}{3}}}{-\frac{3}{4}axx\dot{x} \times a^{-\frac{3}{4}}x^{-\frac{9}{4}}}$. Then, dividing both above and below by $\dot{x}$, and making $x = a$, it will be $y = \frac{16}{9}a$.

Let the equation be $y = \frac{a\sqrt[3]{4a^3 + 4x^3} - ax - aa}{\sqrt{2aa + 2xx} - x - a}$, in which, if we put $x = a$, it will become $y = \frac{0}{0}$. Wherefore, differencing, firſt, the numerator, and then the denominator of the fraction, it will be $y = \frac{4axx \times \overline{4a^3 + 4x^3}|^{-\frac{2}{3}} - a}{2x \times \overline{2aa + 2xx}|^{-\frac{1}{2}} - 1}$, omitting $\dot{x}$, which ſhould be in both the numerator and the denominator. But now, in this fraction, if we put $x = a$, it will be ſtill $y = \frac{0}{0}$. Therefore, proceeding to difference this ſecond fraction alſo, we ſhall have $y = \frac{32a^4x \times \overline{4a^3 + 4x^3}|^{-\frac{5}{3}}}{4aa \times \overline{2aa + 2xx}|^{-\frac{3}{2}}}$, omitting the $\dot{x}$. And now, making $x = a$, it will be $y = 2a$.

SECT. III.

The Method of the Maxima *and* Minima *of Quantities.*

Fig. 49.

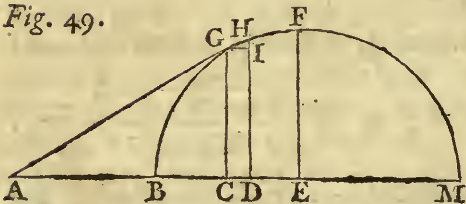


Fig. 50.

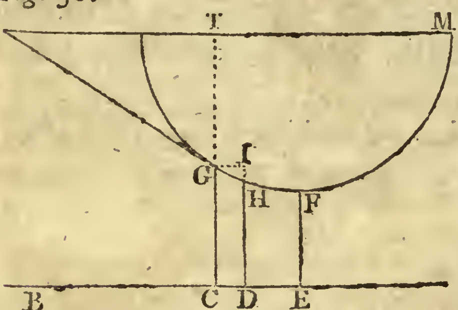
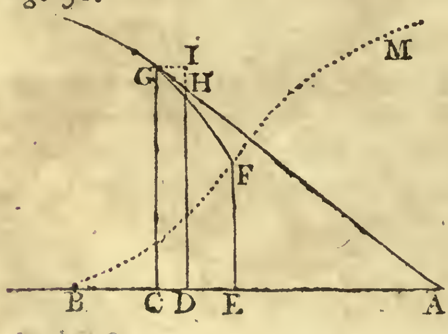


Fig. 51.

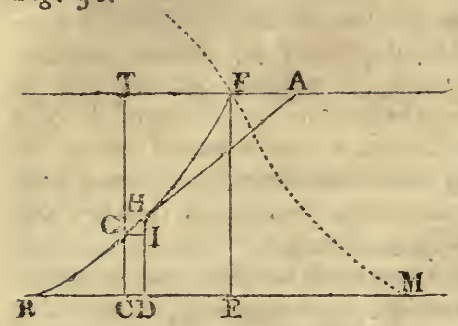


72. In any curve whatever, whose ordinates are parallel, if, the abscis BC (Fig. 49, 50, 51, 52,) continually increasing, the ordinate CG increases also to a certain point E, after which it decreases, or is no longer an ordinate of any kind; or, on the contrary, the abscis increasing, the ordinate CG goes on continually decreasing to a certain point E, after which it either increases, or else is no more: In this case, the ordinate EF is called a *Maximum* or a *Minimum*.

In the curve GHF, let EF be the greatest of the ordinates, (Fig. 49.) or the least, (Fig. 50.) taking any abscis BC, and drawing the ordinate CG; let GA be supposed to be a tangent at the point G, and DH to be infinitely near to CG. Make $BC = x$, $CG = y$, and drawing GI parallel to BC, it will be $GI = CD = \dot{x}$, and $IH = \dot{y}$. Now, because the triangles ACG, GHI, are similar, in Fig. 49, it will be AC . CG :: GI . IH. And, because the triangles ATG, GHI, are similar, in Fig. 50, it will be AT . TG :: GI . IH. This being supposed, let the ordinate GC, being always parallel to itself,

N. B. The letter A is omitted in Fig. 50.

Fig. 52.



itself, be conceived to approach to the greatest or least ordinate EF. It is plain, that, as CG approaches to EF, the subtangent AC, or AT, will always become greater and greater; so that, when CG falls upon EF, the tangent will become parallel to BC, and consequently the subtangent will be infinite. Therefore, in this case, we shall have AC to CG, or AT to TG, an infinite ratio, CG still remaining a finite quantity. But, since it is always AC to CG, or AT . TG :: GI . IH, GI to IH will also have an infinite ratio. Therefore it will be as nothing in respect of $\dot{x}$, that is, $\dot{y} = 0$ in the point of the greatest or least ordinate.

Let the curve be GHF, (Fig. 51, 52.) EF the least of the ordinates, (Fig. 51.) or the greatest (Fig. 52.); therefore, taking any abscifs BC, and drawing the ordinate CG, the tangent GA, DH infinitely near to CG, and GI parallel to BC; and making BC $= x$, CG $= y$, it will be GI $=$ CD $= \dot{x}$, IH $= \dot{y}$. Now, because of the similar triangles ACG, GIH, it will be (Fig. 51.) AC . CG :: GI . IH; and, because of the similar triangles ATG, GIH, it will be (Fig. 52.) AT . TG :: GI . IH. Now, the ordinate CG always remaining parallel to itself, and continually approaching towards the greatest or least ordinate, the subtangent AC or AT will always become less and less; so that, when CG falls upon EF, the tangent will become perpendicular to BC, and consequently the subtangent will be nothing. Therefore, in this case, we shall have AC to CG, or AT to TG, in the ratio of nothing to a finite quantity; and therefore, GI to IH being in the same ratio, $\dot{x}$ will be nothing in respect of $\dot{y}$, that is, $\dot{y} = \infty$, in the point of the greatest or least ordinate. Wherefore the general formula for the greatest and least ordinate will be $\dot{y} = 0$, or else $\dot{y} = \infty$.

73. Therefore, the equation of the curve being given, of which we would find the greatest or least ordinate, we must difference it to find the value of the fraction or ratio $\frac{\dot{y}}{\dot{x}}$; then making the supposition of $\dot{y} = 0$, or else of $\dot{x} = 0$, that is, $\dot{y} = \infty$, we shall have the value of the abscifs x, to which belongs the greatest or least, y; and this value, being substituted in the proposed equation, will give us the greatest or least ordinate, as required. Only here we must observe, that, in the case of the supposition of $\dot{y} = \infty$, that is, of $\dot{x} = 0$, x will supply the place of the ordinate; if in the other supposition, it is y that does the same. That, if neither the first supposition of $\dot{y} = 0$, nor the second of $\dot{y} = \infty$, will supply us with any real value of y, it is then to be concluded, that the proposed curve has no greatest or least ordinate.

74. This

74. This method will help us to acquire a complete and exact idea of curve-lines; to find in what points the tangents are parallel to their conjugate axes, &c. Besides which, it may be applied to an infinite number of questions, which we may want to have resolved, whether geometrical or physical. Such it would be to inquire, among the infinite parallelopipeds of a given solidity, which is that which has the least surface: as it would be to inquire, among the infinite different ways along which a moving body may pass, to go from one point to another not in the same vertical line, which is that which may be described in the shortest time, according to some given law of motion: and many others of a like kind. In such questions must be found an analytical expression of what we would have to be a *maximum*, or a *minimum*, which may be put equal to y. Then taking the fluxion, we must proceed according to the rules here given.

EXAMPLE I.

75. Let there be a curve with this equation $2ax - xx = yy$, and let it be required to know, to what point of the axis, or of the abscis x, the greatest ordinate y corresponds, and what that ordinate is.

The fluxional equation of this will be $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$, that is, $\frac{\dot{y}}{\dot{x}} = \frac{a-x}{y}$. Making the supposition of $\dot{y} = 0$, the numerator of the fraction ought to be nothing, or $a - x = 0$, whence $x = a$. Therefore the greatest ordinate belongs to that abscis which is equal to a. This value being substituted instead of x in the proposed equation, it will be $2aa - aa = yy$, that is, $y = \pm a$. Therefore the greatest ordinate, positive and negative, will be equal to a. Making the supposition of $\dot{y} = \infty$, the denominator of the fraction ought to be nothing, and therefore it will be $y = 0$. Wherefore, substituting this value instead of y in the proposed equation, we shall have $x = 0$, and $x = 2a$; which is as much as to say, that $x = 0$ will be the least, and $x = 2a$ the greatest: Or, more properly, that, when $x = 0$, and $x = 2a$, then $\dot{y}$ being infinite in respect of $\dot{x}$, the subtangent will be nothing, or the tangent will be parallel to the ordinate y.

EXAMPLE II.

76. Let it be the curve of this equation $xx - ax = yy$. By taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2x - a}{2y}$. The ſuppoſition of $\dot{y} = 0$ gives here $x = \frac{1}{2}a$. But this value being ſubſtituted inſtead of x in the propoſed equation, y will be found imaginary; ſo that the curve has no ordinate correſponding to ſuch an abſciſs, and therefore much leſs will it have a greateſt or a leaſt. The ſuppoſition of $\dot{y} = \infty$, that is, of $\dot{x} = 0$, will here give $y = 0$: which declares that the tangent will be perpendicular to the axis of the abſciſs x in the point in which $y = 0$; which correſponds to the two abſciſſes $x = 0$, and $x = a$. For, inſtead of y, ſubſtituting 0 in the propoſed equation, it will be $xx - ax = 0$, and therefore $x = 0$, and $x = a$.

EXAMPLE III.

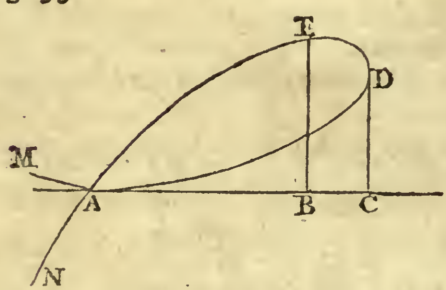
77. Let the curve belong to this equation $2axy = a^3 + axx - bxx$, in which x is the abſciſs, and y the ordinate. By taking the fluxions, it will be $2ax\dot{y} + 2ay\dot{x} = 2ax\dot{x} - 2bx\dot{x}$, and therefore $\frac{\dot{y}}{\dot{x}} = \frac{ax - bx - ay}{ax}$. The ſuppoſition of $\dot{y} = 0$ gives $x = \frac{ay}{a - b}$; and this value being ſubſtituted in the propoſed equation, it will be $\frac{2aayy}{a - b} = a^3 + \frac{a^3y^2 - a^2by^2}{\overline{a - b}|^2}$, that is, $yy = a \times \overline{a - b}$, and $y = \pm\sqrt{aa - ab}$, the greateſt or leaſt ordinate. And, ſince we have $x = \frac{ay}{a - b}$, ſubſtituting this in the value of y, it will be $x = \pm\frac{a\sqrt{a}}{\sqrt{a - b}}$, the abſciſs, to which belongs the greateſt or leaſt ordinate now found. The ſuppoſition of $\dot{y} = \infty$, or $\dot{x} = 0$, gives us $ax = 0$, that is, $x = 0$. And making the ſubſtitution in the propoſed equation, it will be $a^3 = 0$; which implies that a given finite quantity is as nothing: ſo that the curve will have no other *maxima* or *minima* but thoſe found in the firſt ſuppoſition, which, becauſe of the ambiguity of the ſigns, are two, and thoſe equal; one of which is poſitive, and correſponds to the poſitive abſciſſes, the other negative, and belongs to the negative abſciſs.

 78. This

78. This method, indeed, gives us the *maxima* and *minima*, but ambiguoufly and indifcriminately; nor by this can we diftinguifh one from the other. But they become known when the progrefs of the curve is known. But, without fuch knowledge, we may proceed after this manner. Let there be affigned a value to the abfcifs in the given equation, which is either a little greater or a little lefs than that which anfwers to the greateft or leaft ordinate with which we are concerned, and the value of the ordinate which arifes from thence will determine the queftion. For, if it fhall be greater than that which the method difcovers, the queftion is about a *minimum*; but, being lefs than that, the queftion is about a *maximum*. Therefore the curve of this Example will have two leaft ordinates.

EXAMPLE IV.

Fig. 53.



79. Let the curve MADEAN belong to this equation $x^3 + y^3 = axy$; make AB $= x$, and BE $= y$. By differencing, we fhall have $\frac{\dot{y}}{\dot{x}} = \frac{ay - 3xx}{3yy - ax}$; and therefore, making the fuppofition of $\dot{y} = 0$, it will be $y = \frac{3xx}{a}$. Then fubftituting this value in the equation, we fhall find $x = \frac{1}{3}a\sqrt[3]{2}$. Wherefore, fince $y = \frac{3xx}{a}$, it will be $y = \frac{1}{3}a\sqrt[3]{4} =$ BE, the greateft ordinate in the curve, which correfponds to the abfcifs $x = \frac{1}{3}a\sqrt[3]{2} =$ AB. The fuppofition of $\dot{x} = 0$ will give us $x = \frac{3yy}{a}$, and making the fubftitution in the given equation, it will be $y = \frac{1}{3}a\sqrt[3]{2}$, whence $x = \frac{1}{3}a\sqrt[3]{4}$, the greateft AC, to which correfponds $y =$ CD $= \frac{1}{3}a\sqrt[3]{2}$, which is the tangent in the point D.

80. But, before we proceed to more Examples, it will be convenient to provide for a cafe, which fometimes is wont to happen; and that is, that as well the fuppofition of $\dot{y} = 0$, as that of $\dot{y} = \infty$, will furnifh the fame value of the ordinate, or of the abfcifs; in which cafe, no *maximum* or *minimum* will be determined, but only a point of interfection or the meeting of two branches of

of the curve. And the reaſon of this is plain; foraſmuch as, $\frac{\dot{y}}{\dot{x}}$ being equal to a fraction, if from the numerator we derive the ſame value of x, for example, as from the denominator, this value or root being ſubſtituted, will make each of them equal to nothing, and therefore in ſuch a point of the curve it will be $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$. But it has been already ſhown before, at § 69, that when $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$, it always indicates the meeting of two branches of the curve. Therefore, &c.

EXAMPLE V.

81. Let the curve GFM (Fig. 51.) be the cubic parabola with the equation $y - a = \sqrt[3]{a^3 - 2aax + axx}$, BE $=$ EF $= a$, BC $= x$, CG $= y$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2ax - 2aa}{3 \times \overline{a^3 - 2aax + axx})^{\frac{2}{3}}}$. The ſuppoſition of $\dot{y} = 0$ will give us $x = a$, and the ſuppoſition of $\dot{y} = \infty$ will give, in like manner, $x = a$. Therefore the curve has a point of interſection F, which correſponds to the abſciſs $x = a$, and to the leaſt ordinate $y = a$; which is derived from the propoſed equation, by ſubſtituting it's value in the place of x.

Let us take the ſame equation, but freed from radicals, that is, $y^3 - 3ay^2 + 3aay - a^3 = a^3 - 2aax + axx$. By taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2ax - 2aa}{3yy - 6ay + 3aa}$. The ſuppoſition of $\dot{y} = 0$ will give $x = a$, and putting this value in the propoſed equation, we have $y = a$. The ſuppoſition of $\dot{y} = \infty$ will alſo give $y = a$, and therefore $x = a$; and $y = a$ gives us the point F, which is a point of meeting or contact of the two branches GF, FM, and, at the ſame time, the leaſt ordinate y.

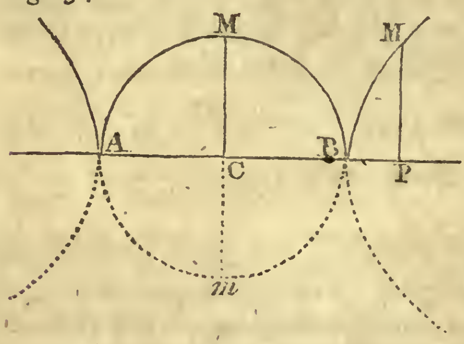
But, if we ſhould operate upon the equation $y - a = a^{\frac{1}{3}} \times \overline{a - x})^{\frac{2}{3}}$, which expreſſes the branch GF alone, (the other branch FM would be expreſſed by $y - a = a^{\frac{1}{3}} \times \overline{x - a})^{\frac{2}{3}}$,) we ſhould have $\frac{\dot{y}}{\dot{x}} = \frac{-2a^{\frac{1}{3}}}{3 \times \overline{a - x})^{\frac{1}{3}}}$. The ſuppoſition of $\dot{y} = 0$, informs us of nothing. The ſuppoſition of $\dot{y} = \infty$ gives us $x = a$, and therefore $y = a$. And the point F, in this caſe, ſupplies us with a *maximum* in reſpect of x, and a *minimum* in reſpect of y.

82. I said that the supposition of $\dot{y} = 0$, which here gives $2a^{\frac{1}{3}} = 0$, informs us of nothing, meaning in respect of finite *maxima*; for, taking in the infinite also, it supplies us with two of them. If $2a^{\frac{1}{3}} = 0$, it will be then $x = 0$; and substituting this value in the proposed equation, it will be $\frac{y}{0} = \sqrt[3]{xx}$, that is, $x = \pm \sqrt{\frac{y^3}{0}}$; and therefore x and y are infinite. The *maxima* are two, one belonging to the branch FG, the other to the branch FM; for, putting $a = 0$, the equations express them both.

This case will generally arise, as often as the supposition of $\dot{y} = 0$, or of $\dot{y} = \infty$, exhibits a constant finite expression, or a constant divisor, to be equal to nothing; which value, being substituted in the proposed equation, does not bring us to an imaginary quantity, or to a contradiction. And the reason of it is this, because a finite quantity cannot be taken for nothing, but only in respect of an infinite quantity.

EXAMPLE VI.

Fig. 54.



83. Let the curve belong to the equation $x^4 - 2ax^3 + aaxx = y^4$. Make AB $= a$, AC or AP $= x$, CM or PM $= y$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{4x^3 - 6ax^2 + 2aax}{4y^3}$. The supposition of $\dot{y} = 0$ will give us three values of x, that is, $x = 0$, $x = a$, $x = \frac{1}{2}a$. The value $x = 0$, being substituted in the proposed equation, makes $y = 0$. The value $x = a$, makes $y = 0$. The value $x = \frac{1}{2}a$, makes $y = \pm \frac{1}{2}a$. The supposition of $\dot{y} = \infty$ gives us $y = 0$; so that y has the same value in both the suppositions, when $x = 0$ and $x = y$. Whence the points A, B, will be points of meeting of the branches of the curve, and $x = \frac{1}{2}a = $ AC will give the greatest ordinate $y = \pm \frac{1}{2}a = $ CM, or Cm. The *locus* of the foregoing Example may be called a double *locus*, which arises from one or other of the two simple formulas, ($ax - xx = yy$, to the circle, and $xx - ax = yy$ to the hyperbola,) being raised to it's square. Whence it would not be sufficient to reduce the equation to a simple circle, or to a simple hyperbola; but it will be necessary to have a view to the complication of the two *loci* or curves with each other.

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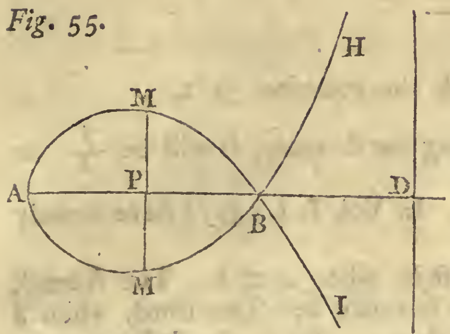
EXAMPLE VII.

Fig. 55.



84. Let it be the curve of Fig. 55, the equation of which is $yy = \frac{aax - 2axx + x^3}{2a - x}$. Make AP $= x$, PM $= y$, AD $= 2a$. The fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{a^3 - 4a^2x + 4ax^2 - x^3}{y \times \overline{2a - x}|^2}$; that is, $\frac{\dot{y}}{\dot{x}} = \frac{a^3 - 4aax + 4axx - x^3}{\overline{a-x} \times \sqrt{x} \times \overline{2a-x}|^{\frac{3}{2}}}$. Before I proceed, I shall here observe that both the numerator and the denominator of the fraction are divisible by $a - x$; therefore, in the supposition of $\dot{y} = 0$, and in that of $\dot{y} = \infty$, we shall have $a - x = 0$, or $x = a$. And this, being substituted, will give $y = 0$, and therefore the curve will have a node in the axis at the point B, making AB $= a$. Therefore, making the division, it will be $\frac{\dot{y}}{\dot{x}} = \frac{aa - 3ax + xx}{\overline{2a-x} \times \sqrt{2ax-xx}}$. The supposition of $\dot{y} = 0$ will give $x = \frac{3a \pm a\sqrt{5}}{2}$. But the value $x = \frac{3a + a\sqrt{5}}{2}$ cannot be of use, because, being substituted in the proposed equation, it makes the ordinate imaginary; and this, in general, is imaginary, when x is assumed greater than $2a$, as may be plainly seen. Wherefore, substituting the other value, $x = \frac{3a - a\sqrt{5}}{2}$, it gives $y = \pm a\sqrt{\frac{7a - 3a\sqrt{5}}{a + a\sqrt{5}}}$. Making, then, AP $= \frac{3a - a\sqrt{5}}{2}$, PM, P*m*, will be the greatest ordinates, one positive, the other negative; as above.

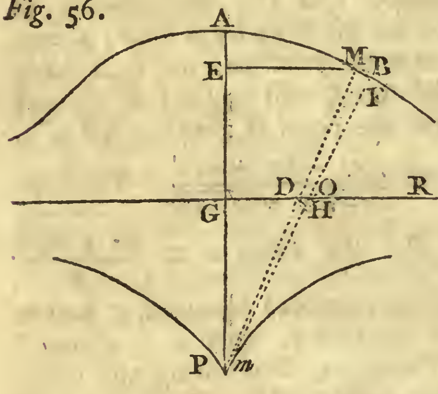
The supposition of $\dot{y} = \infty$ will give $x = 0$, and $x = 2a$. These values being substituted in the proposed equation, we shall have $y = 0$, and $y = \infty$; that is, taking $x = 0$, or in the point A, the tangent will be parallel to the ordinate PM. And taking $x = 2a = $ AD, the ordinate will be infinite, that is, will become an asymptote to the curve, in respect of the branches BH, BI.

N. B. By mistake of the Wood-cutter, a Roman M has been put in the lower part of Fig. 55, instead of an Italic *m*.

EXAMPLE VIII.

85. Let the curve be the conchoid with the equation $yy = \frac{aaxx - x^4 + 2aabx - 2bx^3 - bbxx + aabb}{xx}$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{-x^4 - bx^3 - aabx - aabb}{\pm xx\sqrt{aaxx - x^4 + 2aabx - 2bx^3 - b^2x^2 + a^2b^2}}$. In Vol. I. § 239, I have already considered three cases of this curve. The first is, when $a = b$. The second, when b is less than a. The third, when b is greater than a. As to the first case, the curve will be that of Fig. 56, and the equation $yy = \frac{a^4 + 2a^3x - 2ax^3 - x^4}{xx}$. Making GA $=$ GP $= a$, GE $= x$, EM $= y$; and, taking the fluxions, it is $\frac{\dot{y}}{\dot{x}} = \frac{-x^4 - ax^3 - a^3x - a^4}{\pm xx\sqrt{a^4 + 2a^3x - 2ax^3 - x^4}}$. The supposition of $\dot{y} = 0$ will give the numerator equal to nothing, that is, $\overline{x + a} \times \overline{x^3 + a^3} = 0$; and therefore $x = -a$, which value, substituted in the equation of the curve, gives $y = 0$. The supposition of $\dot{y} = \infty$ gives the denominator equal to nothing, that is, $xx\sqrt{\overline{x + a} \times \overline{aa - xx}} = 0$, and therefore $x = 0$, $x = -a$, and $x = a$. But the value $x = -a$ was also found in the supposition of $\dot{y} = 0$. Therefore, when it is $x = -a$, that is, taking GP $= a$, the curve will have a point P, where two branches meet each other.

Fig. 56.



The value $x = a$, being substituted in the equation, will give us $y = 0$; and therefore the same x will be $= a =$ GA, to which corresponds $y = 0$. The value $x = 0$, being substituted, will give $y = \infty$. Therefore, through the point G, where $x = 0$, if a line be drawn parallel to the ordinates, it will touch the curve at an infinite distance, that is, it will be an asymptote.

As

Fig. 57.

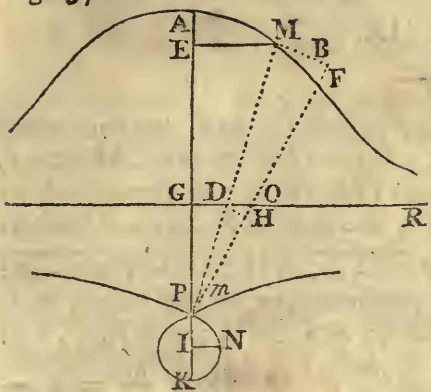
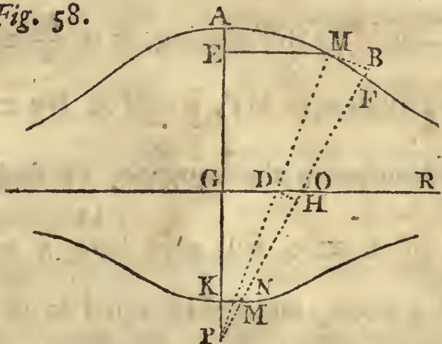


Fig. 58.



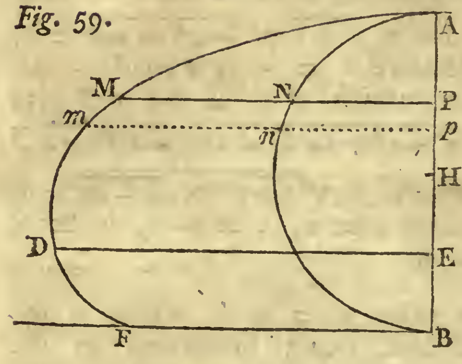
As to the other two cafes, Fig. 57, 58. Let GA $=$ GK $= a$, GP $= b$, and the reft as above. The fuppofition of $\dot{y} = 0$ will give $-x^4 - bx^3 - aabx - aabb = 0$; that is, $\overline{x+b} \times \overline{-x^3 - aab} = 0$, and therefore $x = -b$, $x = \sqrt[3]{-aab}$. The fuppofition of $\dot{y} = \infty$, will give $xx\sqrt{a^2x^2 - x^4 + 2a^2bx - 2bx^3 - b^2x^2 + a^2b^2} = 0$, that is, $xx\sqrt{\overline{x+b}^2 \times \overline{aa-xx}} = 0$, and thence $x = 0$, $x = -b$, $x = a$, $x = -a$.

The value $x = -b$, which is the fecond cafe, being fubftituted in the equation, makes $y = 0$, and is exhibited by both the fuppofitions. Therefore (Fig. 57.) taking GP on the negative fide, and equal to $-b$, the point P fhall be a meeting or an interfection of two branches of the curve. The fame value $x = -b$, being fubftituted in the equation of the curve $\pm y = \frac{b+x}{x}\sqrt{aa-xx}$, in the third cafe, gives the radical negative, becaufe of b greater than a, and therefore the curve is imaginary, and of no ufe.

The value $x = \sqrt[3]{-aab}$, fubftituted in the equation of the curve, gives us $y = \pm\sqrt{\frac{\overline{aa-bb} \times \sqrt[3]{abb} + 3ab\sqrt[3]{-aab} + 3abb}{\sqrt[3]{abb}}}$, which is therefore imaginary when b is greater than a, (Fig. 58.) and therefore, in like manner, ferves to no purpofe in this third cafe. But it gives y real when b is lefs than a; and therefore, (Fig. 57.) making GI $= \sqrt[3]{-aab}$, IN will be the greateft ordinate, or y, as above. The value $x = 0$ here gives $y = \infty$, that is, an afymptote. The value $x = \pm a$ gives $\dot{y} = 0$; that is, the tangent in the points A, K, is parallel to the ordinate.

EXAMPLE IX.

Fig. 59.

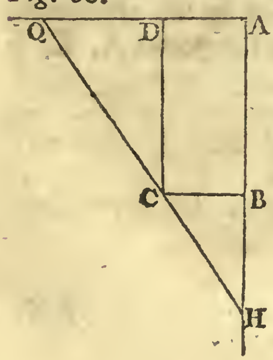


86. Let AMF be half the contracted cycloid. Make AB $= 2a$, BF $= b$, AP $= x$, PM $= z$, the semiperiphery ANB $= c$, the arch AN $= q$; it will be PN $= \sqrt{2ax-xx}$, NM $= z - \sqrt{2ax-xx}$; and, by the property of the curve, it is ANB . BF :: AN . NM; that is, $c \,.\, b :: q \,.\,$ NM $= \frac{bq}{c}$. Therefore $\frac{bq}{c} = z - \sqrt{2ax-xx}$. By differencing, it is $\frac{b\dot{q}}{c} = \dot{z} - \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$. Now, drawing mp infinitely near to MP, it will be Nn $= \dot{q} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$. Whence, making the substitution in the equation, we shall have $\frac{\dot{z}}{\dot{x}} = \frac{ab + ac - cx}{c\sqrt{2ax - xx}}$. The supposition of $\dot{z} = 0$ will give here $x = \frac{ab}{c} + a$. Therefore, if H be the centre of a circle, taking HE equal to the fourth proportional of the semiperiphery ANB, of the right line BF, and of the radius; the corresponding ordinate will be the greatest, as was required.

The supposition of $\dot{z} = \infty$ gives us $x = 0$, and $x = 2a$; which is as much as to say, that, in the points A, F, the tangent will be parallel to the ordinates.

PROBLEM I.

Fig. 60.



87. A rectangle ADCB being given, the least right line QH is required, which can be drawn through the point C in the angle QAH.

Make AB $= a$, BC $= b$, BH $= x$; it will be CH $= \sqrt{bb + xx}$; and, because of the similar triangles HBC, HAQ, we shall have HB . HC :: HA . HQ; that is, $x \,.\, \sqrt{bb + xx} :: x + a \,.\,$ HQ $= \frac{x+a}{x}\sqrt{bb + xx}$.

Wherefore,

Wherefore, ſuppoſing HQ $= y$, as if it were the ordinate of a curve, we ſhall have $y = \frac{x+a}{x}\sqrt{bb+xx}$, and, by differencing, it will be $\frac{\dot{y}}{\dot{x}} = \frac{x^3 - abb}{xx\sqrt{bb+xx}}$. The ſuppoſition of $\dot{y} = 0$ will give $x = \sqrt[3]{abb}$; and therefore, making BH $= \sqrt[3]{abb}$, and drawing HCQ, it will be the leaſt line, as required. The ſuppoſition of $\dot{y} = \infty$ will give $x = \sqrt{-bb}$, and $x = 0$, which anſwers no purpoſe; it not being meant that the right line drawn through the point C, which, in this caſe, would be BC infinitely produced, ſhould be a *maximum*, for that reaſon becauſe infinite. Wherefore, in ſuch caſes as theſe, it will be ſufficient to difference that expreſſion, which we would have to be a *maximum* or *minimum*, and afterwards to ſuppoſe the numerator equal to nothing, and then the denominator.

PROBLEM II.

Fig. 61.

88. The right line AB being divided into three given parts, AC, CF, FB, the point E is required, in which the middle portion CF is to be divided, ſo that the rectangle AE × EB to the rectangle CE × EF, may have the leaſt poſſible ratio.

Make AC $= a$, CF $= b$, CB $= c$, and CE $= x$; then AE $= a + x$, EB $= c - x$, EF $= b - x$; and therefore the ratio will be $\frac{AE \times EB}{CE \times EF} = \frac{ac + cx - ax - xx}{bx - xx}$, which muſt be a *minimum*. The fluxion, therefore, will be $\frac{cxx - axx - bxx + 2acx - abc}{(bx - xx)^2} \times \dot{x}$; and making the numerator equal to nothing, we ſhall have $x = \frac{-ac \pm \sqrt{abcc - abbc - aabc + aacc}}{c - b - a}$. One of the values is poſitive, which gives the point required, E, from C towards B. The other is negative, which would give us the point E, from C towards A. Making the denominator equal to nothing, we ſhall have $x = 0$, and $x = b$, in which two caſes the ratio of the rectangles will be a *maximum*; for, taking $x = 0$, the point E falls in C; and taking $x = b$, the point E falls in F; and therefore, in each caſe, the rectangle CE × EF is nothing.

PRO-

PROBLEM III.

89. The given right line AB is to be fo cut in the point C, as that the product $ACq \times CB$ fhall be the greateft of all fuch products.

Make $AB = a$, $AC = x$, then $CB = a - x$. Therefore $ACq \times CB = axx - x^3$. The differential will be $2ax\dot{x} - 3xx\dot{x}$, which, compared to nothing, will give $x = \frac{2}{3}a$, and $x = 0$. Wherefore, taking $AC = x = \frac{2}{3}a$, the product will be the greateft poffible; and taking $x = 0$, the product will be a kind of *minimum*, becaufe it will be nothing, the point C falling in A. The differential not being a fraction, the other ufual fuppofition cannot take place, of the denominator being made equal to nothing. But if we will confider the expreffion of the product $axx - x^3$ as an ordinate of a curve, by the laws of homogeneity that product may be divided by a conftant plane, and thus the differential will be a fraction with a conftant denominator. But that conftant quantity can never be nothing, but only relatively in refpect of x being affumed infinite; and furely then the product muft be a *maximum*, when it is $AC = x = \infty$.

I faid that the product $ACq \times CB$ is a *maximum*, when it is $AC = \frac{2}{3}a$; which will be plainly feen by defcribing the curve of the equation $\frac{axx - x^3}{aa} = y$. For all the ordinates between A and B are lefs than that which correfponds to the abfcifs $x = \frac{2}{3}a$. The other value, $x = 0$, being fubftituted, it will be $y = 0$, from whence it may be concluded, that this value will be of no ufe.

90. In the foregoing Problem, and in all others of a like nature, this method may be made ufe of to difcover, whether the queftions propofed are concerning a *maximum* or a *minimum*.

PROBLEM IV.

91. Among all the parallelopipeds that are equal to a given cube, and of which one fide is given; it is required to find that which has the leaft furface.

Let the given cube be a^3, and the known fide of the parallelopiped $= b$. Let one of the fides fought be x, and then the third will be $\frac{a^3}{bx}$, becaufe the

product

product of the three makes the given cube a^3. The products of the sides, taken two and two, that is, bx, $\frac{a^3}{x}$, and $\frac{a^3}{b}$, form the three planes which are half the superficies of the parallelopiped, and therefore the sum of these, that is, $bx + \frac{a^3}{x} + \frac{a^3}{b}$, must be the *minimum* required. Therefore, taking the fluxions, we shall have $b\dot{x} - \frac{a^3\dot{x}}{xx}$, or $\frac{bxx - a^3}{xx}\dot{x}$. The supposition of the numerator equal to nothing gives $x = \sqrt{\frac{a^3}{b}}$. Therefore the three sides of the required parallelopiped will be b, $\sqrt{\frac{a^3}{b}}$, and $\frac{a^3}{b\sqrt{\frac{a^3}{b}}}$, or $\sqrt{\frac{a^3}{b}}$. Therefore the two sides required will be equal. The supposition of the denominator, being equal to nothing, serves to no purpose; for then $x = 0$, which contradicts the Problem.

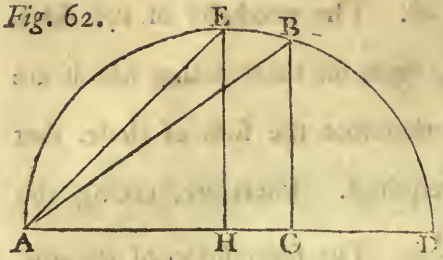
If we would have a parallelopiped with the conditions assigned, but without assuming any side as given; making one side $= x$, the two others will be equal, and each $= \sqrt{\frac{a^3}{x}}$. The sum of the three sides or planes, which is to be a *minimum*, will be $2x\sqrt{\frac{a^3}{x}} + \frac{a^3}{x}$, which, by differencing, is $\frac{a^3\dot{x}}{x\sqrt{\frac{a^3}{x}}} - \frac{a^3\dot{x}}{xx}$; or thus, $\frac{a^3x\dot{x} - a^3\dot{x}\sqrt{\frac{a^3}{x}}}{xx\sqrt{\frac{a^3}{x}}}$. Here, making the numerator equal to nothing, we shall have $x = a$, and, in like manner, the other two sides will be $= a$; so that the cube itself will be the parallelopiped required.

PROBLEM V.

92. Among the infinite cones that may be inscribed in a sphere, to determine that whose convex superficies is the greatest; the base being excluded.

In

Fig. 62.

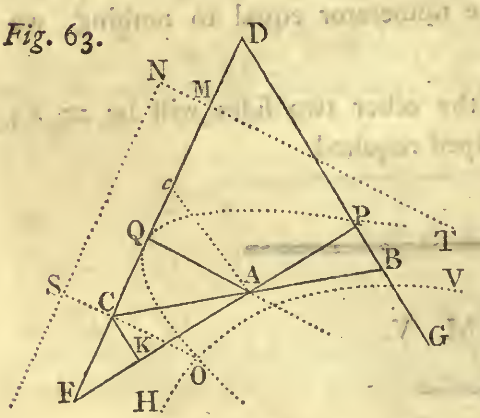


In the ſemicircle ABD let there be the triangles ABC, AEH, and let a ſemicircle revolve about it's diameter AD. At the ſame time that it deſcribes a ſphere, the triangles will deſcribe ſo many cones. But, as it is demonſtrated by *Archimedes*, that the ſuperficies of the inſcribed cones will be to each other as the rectangles AE × EH, AB × BC; the queſtion is reduced to this, to determine ſuch a point C in the diameter AD, that the product AB × BC may be a *maximum*.

Therefore make AC $= x$, AD $= a$; by the property of the circle, it will be CB $= \sqrt{ax - xx}$, AB $= \sqrt{ax}$, and AB × BC $= \sqrt{ax} \times \sqrt{ax - xx}$ $= \sqrt{aaxx - ax^3}$. Therefore, taking the fluxions, we ſhall have $\frac{2aax\dot{x} - 3axx\dot{x}}{2\sqrt{aaxx - ax^3}}$. And making the numerator equal to nothing, it will be $x = \frac{2}{3}a$, and $x = 0$. Making the denominator $= 0$, it will be $x = a$, and $x = 0$. Taking, therefore, AC $= \frac{2}{3}$AD, the ſuperficies of the cone deſcribed by the triangle ABC will be the greateſt, as required. The other two values $x = 0$, and $x = a$, can be of no uſe in this Problem, as is evident.

PROBLEM VI.

Fig. 63.



93. The angle FDG being given, and the point A being given in poſition, to find the leaſt right line, which, in the given angle, can paſs through the point A.

Let CB be the line required, and let AQ be drawn perpendicular to FD, FAP perpendicular to DG, and CK perpendicular to FP. Becauſe the angle FDG is given, and the angle FPD is a right one, the angle AFQ will be known. But the point A is alſo given in poſition; then the lines QA, QF, FA, QD, will alſo be known. Therefore make QF $= a$, QA $= c$, QD $= b$, and QC $= x$. Therefore it will be FA $= \sqrt{aa + cc}$, CA $= \sqrt{cc + xx}$, FD $= b + a$, and FC $= a - x$. But, becauſe of ſimilar triangles FAQ, FDP, it will be FA

FA . FQ :: FD . FP. Wherefore $FP = \frac{aa + ab}{\sqrt{aa + cc}}$, and $AP = \frac{ab - cc}{\sqrt{aa + cc}}$. Now, becaufe of fimilar triangles ACK, ABP, it will be AK . CA :: AP . AB. Therefore $AB = \frac{\overline{ab - cc} \times \sqrt{cc + xx}}{cc + ax}$, and thence $CB = \sqrt{cc + xx} + \frac{ab - cc}{cc + ax}\sqrt{cc + xx}$, which is to be a *minimum*. Therefore, taking the fluxions, it will be $\frac{x\dot{x}}{\sqrt{cc + xx}} + \frac{x\dot{x} \times \overline{ab - cc} \times \overline{cc + ax} - a\dot{x} \times \overline{ab - cc} \times \overline{cc + xx}}{\overline{cc + ax}|^2 \times \overline{cc + xx}|^{\frac{1}{2}}}$. And, putting the numerator = o, (firft reducing to a common denominator,) it will be $x^3 + \frac{2c^2x^2}{a} + \frac{bc^2x}{a} + \frac{c^4}{a} - bc^2 = 0$, which is a folid equation.

To conftruct it, I take the equation to the parabola $xx = ay$; making the fubftitution, it will be $xy + \frac{2ccy}{a} + \frac{bccx}{aa} + \frac{c^4}{aa} - \frac{bcc}{a} = 0$, a *locus* to the hyperbola between it's afymptotes.

This fuppofed, on the right line QD is taken $QM = \frac{2cc}{a}$, and drawing the right line $MN = \frac{bcc}{aa}$ from the point M, and parallel to AQ, NS is drawn parallel to QD, and between the afymptotes NS, NT, the hyperbola HOV is defcribed with the conftant rectangle $\frac{2bc^4 + aabcc - ac^4}{a^3}$. And, on the right line QF, from the point Q let the x's be taken, and the y's perpendicular to them. Then, with the axis AQ, vertex Q, and parameter = a, let the parabola QO of the equation $xx = ay$ be defcribed. From the point O, in which the parabola cuts the hyperbola, let OC be drawn parallel to AQ; and from the point C let the right line CAB be drawn through the point A. This fhall be the *minimum* required.

And, indeed, by the conftruction, it is $NS = x + \frac{2cc}{a}$, $SO = y + \frac{bcc}{aa}$. And, by the property of the hyperbola, it ought to be NS × SO, equal to the conftant rectangle. Therefore $xy + \frac{2ccy}{a} + \frac{bccx}{aa} + \frac{2bc^4}{a^3} = \frac{2bc^4 + aabcc - ac^4}{a^3}$. But $CO = y = \frac{xx}{a}$, by the property of the parabola. Therefore, inftead of y, fubftituting this value, we fhall have $\frac{x^3}{a} + \frac{2ccxx}{aa} + \frac{bccx}{aa} = \frac{bcc}{a} - \frac{c^4}{aa}$; that is, $x^3 + \frac{2ccxx}{a} + \frac{bccx}{a} + \frac{c^4}{a} - bcc = 0$, which is the very equation from whence the value of x was to be derived. Therefore, &c.

I have

I have here made the ſuppoſition, that the numerator of the fraction, which expreſſes the *minimum*, is to be nothing. The other ſuppoſition, that the denominator muſt be nothing, will give $\overline{cc + ax}\vert^2 \times \sqrt{cc + xx} = 0$, that is, $\sqrt{cc + xx} = 0$, $cc + ax = 0$. But $\sqrt{cc + xx} = 0$ gives us $x = \sqrt{-cc}$, which is imaginary, and therefore of no uſe. $cc + ax = 0$ gives us $x = -\frac{cc}{a}$. But, taking $Qc = x = -\frac{cc}{a}$, and drawing Ac, the triangle QAc will be ſimilar to the triangle QFA, or PFD, and therefore the angle QcA will be equal to the angle FDP. Whence cA will be parallel to DP; which is as much as to ſay, that a line drawn from the point c, and through the point A in the given angle FDG, will be infinite, which is a kind of *maximum*.

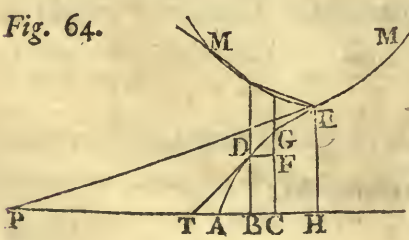
It may be ſhown ſtill in a ſhorter manner, that the right line here ſought will be infinite. For, in the expreſſion $\sqrt{cc + xx} + \frac{ab - cc}{cc + ax}\sqrt{cc + xx} = CB$, inſtead of x, if we ſubſtitute it's value $-\frac{cc}{a}$, the denominator becomes nothing, and therefore the line is infinite.

SECT. IV.

Of Points of Contrary Flexure, and of Regreſſion.

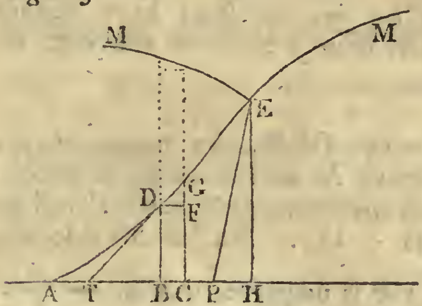
94. In Sect. VI. Vol. I. it has been ſaid already, what are Contrary Flexures and Regreſſions of Curves. Suppoſing, therefore, that to be already known, let ADEM be a curve whoſe ordinates are parallel, and which in E has a contrary flexure or regreſſion. Taking any abſciſs, AB $= x$, and it's ordinate BD $= y$, and drawing CF parallel and indefinitely near to BD; it is plain, that, aſſuming $\dot{x} =$ BC as conſtant, that, as the abſciſs AB $= x$ continually increaſes, the fluxion GF of the ordinate

Fig. 64.



nate BD, that is, $\dot y$, will always become lefs and lefs, till the ordinate becomes HE, which correfponds to the point of contrary flexure or of regreffion: after which point, in both cafes, the fluxion $\dot y$ will go on continually increafing. Therefore, in the point of contrary flexure or regreffion, $\dot y$ will be a *minimum.* Whence, by the Method of *Maxima* and *Minima*, $\ddot y = 0$, or elfe $\ddot y = \infty$, will be the formula of contrary flexure or regreffion.

Fig. 65.



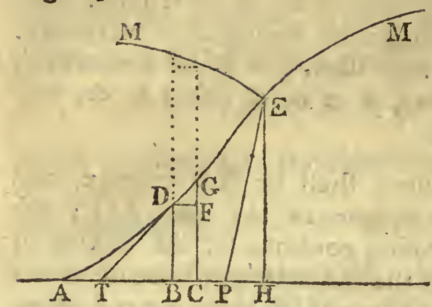
If the curve fhall be firft convex, and afterwards concave to the axis AH; the abfcifs increafing continually, the fluxion or difference of the ordinate will increafe to the point E of contrary flexure or regreffion, after which it will go on decreafing. Therefore, in this point, $\dot y$ is a *maximum*, and, for that reafon, we may put $\ddot y = 0$, or elfe $\ddot y = \infty$.

The fame thing may alfo be inferred from this confideration, that, in a curve firft concave towards it's axis, the fecond fluxion of the ordinate y, that is, $\ddot y$, is negative to the point E of regreffion or contrary flexure, after which it becomes pofitive. And, in curves that are firft convex, that fecond fluxion is pofitive as far as the point E, after which it becomes negative. But no quantity from pofitive can become negative, or from negative can become pofitive, but it muft pafs through either nothing or infinite. Therefore, in the point E of regreffion or contrary flexure, it ought to be $\ddot y = 0$, or elfe $\ddot y = \infty$.

Let the right line DT (Fig. 64.) be a tangent in the point D to the curve AEM, which is firft concave towards the axis; and alfo, the right line EP at the point E. As the abfcifs AB increafes, the line AT, intercepted between the tangent and the origin of the abfcifs will always increafe fo far till the point B falls in H, after which, in the cafe of contrary flexure, the abfcifs ftill increafing, that intercepted line will decreafe. Therefore, in the point E of contrary flexure, that intercepted line AP $= \frac{y\dot x}{\dot y} - x$ ought to be a *maximum.* Wherefore, by differencing, taking $\dot x$ for conftant, it will be $\frac{\dot y\dot y\dot x - y\dot x\ddot y - \dot y\dot y\dot x}{\dot y\dot y}$, equal to nothing, or to infinite; that is, by reducing, and dividing by $-y\dot x$, and multiplying by $\dot y\dot y$, it will be, finally, $\ddot y = 0$, or $\ddot y = \infty$. In cafe that the point E be a point of regreffion, if the intercepted line AT increafe, the abfcifs AB will alfo increafe, till the point T falls in P, and the abfcifs fhall be AH; beyond which point T the abfcifs will go on decreafing. Therefore AH will be a *maximum*, and it's difference will be equal to nothing, or infinite. Therefore, relatively to fuch a difference, the difference of AP will be infinite, or nothing. Therefore $\ddot y = \infty$, or $\ddot y = 0$, as before.

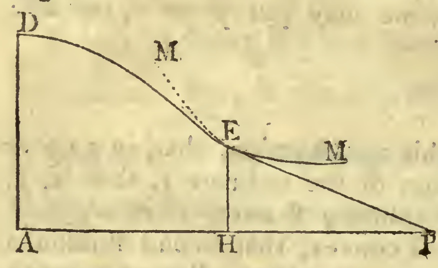
 If

Fig. 65.



If the curve be firſt convex to the axis, the intercepted line AT will be $= x - \frac{y\dot{x}}{\dot{y}}$, and the difference $\frac{\dot{x}\dot{y}\dot{y} - \dot{x}\dot{y}\dot{y} + y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, that is, $\frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$; and therefore, dividing by $y\dot{x}$, and multiplying by $\dot{y}\dot{y}$, we ſhall have neither more nor leſs than $\ddot{y} = 0$, or elſe $\ddot{y} = \infty$.

Fig. 66.



In the curve DEM, the origin of the abſciſſes x being A, and E the point of contrary flexure, the intercepted line AP will be equal to AH + HP. But, in this caſe, the ſubtangent HP is negative, that is, $-\frac{y\dot{x}}{\dot{y}}$. Therefore it will be AP $= x - \frac{y\dot{x}}{\dot{y}}$. Hence we ſee, that in no caſe the intercepted line AP can be $x + \frac{y\dot{x}}{\dot{y}}$.

95. The formula here found will ſerve for curves which have parallel ordinates, or ſuch as are referred to an axis or diameter. But it is different in curves that are referred to a focus.

Fig. 67.

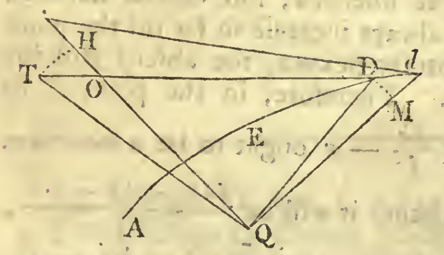
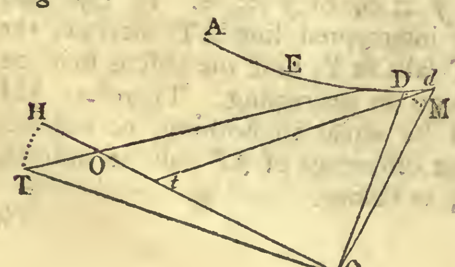


Fig. 68.



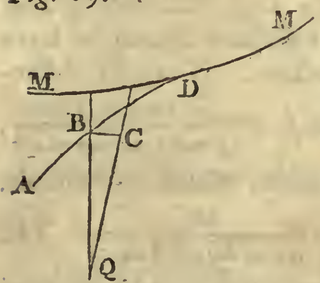
Let the curve be ADE, (Fig. 67, 68.) it's focus Q, from whence the ordinates QD proceed; and let Q*d* be infinitely near to QD. Draw QT perpendicular to QD, and and Q*t* perpendicular to Q*d*. Draw DT a tangent to the curve in the point D, and *dt* a tangent in the point *d*. Let Q*t* (produced if need be,) meet DT in the point *o*. Now it is plain, that, as the ordinates increaſe, if the curve be concave towards the focus Q, (Fig. 67.) Q*t* will be greater than QT. But, if the curve be convex towards the focus Q, (Fig. 68.) Q*t* will be leſs than QT. Therefore, as the curve changes from being concave to convex, or *vice verſâ*, that is, in the point of contrary flexure or regreſſion, the line or quantity *ot*, from

from being poſitive, ought to become negative, or the contrary, and therefore muſt paſs through nothing or infinite.

Wherefore, make QD $= y$, DM $= \dot{x}$, and with centre Q let the infiniteſimal arches DM, TH, be deſcribed. The two triangles dMD, dQT, will be ſimilar, as alſo, dQo, THo, and therefore it will be dM . MD :: dQ (or DQ) . QT. That is, $\dot{y} . \dot{x} :: y$. QT $= \frac{y\dot{x}}{\dot{y}}$. But the two ſectors DQM, TQH, are alſo ſimilar; whence QD . DM :: QT . TH. That is, $y . \dot{x} :: \frac{y\dot{x}}{\dot{y}}$. TH $= \frac{\dot{x}\dot{x}}{\dot{y}}$. And, becauſe of the ſimilar triangles dQo, THo, it will be dQ (or DQ) . Qo (or QT) :: TH . Ho. That is, $y . \frac{y\dot{x}}{\dot{y}} :: \frac{\dot{x}\dot{x}}{\dot{y}}$. HO $= \frac{\dot{x}^3}{\dot{y}^2}$. But H$t$ (Fig. 67.) is the difference of QT, that is, Ht $= \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, taking $\dot{x}$ for conſtant. Therefore $to = t$H + Ho $= \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y} + \dot{x}^3}{\dot{y}\dot{y}}$, which muſt be equal to o, or to ∞. And therefore, alſo, multiplying by $\dot{y}\dot{y}$, and dividing by $\dot{x}$, it will be $\dot{y}\dot{y} - y\ddot{y} + \dot{x}\dot{x}$, equal to nothing, or infinite.

In Fig. 68, the line ot becomes negative, and therefore $= \frac{-\dot{x}\dot{y}\dot{y} + y\dot{x}\ddot{y} - \dot{x}^3}{\dot{y}\dot{y}}$. Therefore, dividing by $-\dot{x}$, and multiplying by $\dot{y}\dot{y}$, it will be $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}$ equal to o, or to ∞.

Fig. 69.



Wherefore, if any curve be referred to a focus Q, whoſe ordinates are QB $= y$, and the little arches BC $= \dot{x}$, and ſhall have a contrary flexure or regreſſion; the general formula to determine it will be $\dot{y}\dot{y} + \dot{x}\dot{x} - y\ddot{y} = o$, or $= \infty$.

Here, if we ſuppoſe y infinite, the two firſt terms of the formula will be nothing in reſpect of the third, and therefore it will be $-y\ddot{y}$, equal to nothing, or infinity; and dividing by $-y$, we ſhall have $\ddot{y} = o$, or $\ddot{y} = \infty$; which is the formula of the firſt caſe of curves referred to a diameter, as it ought to be. For, ſuppoſing y infinite, the ordinates become parallel to one another.

96. The nature of a curve being given by means of an equation, and $\dot{x}$ being ſuppoſed conſtant; by differencing twice, if the curve be algebräical, or once, if it be a differential of the firſt degree, that we may have the value of $\ddot{y}$ expreſſed by $\dot{x}$; this, compared to o or ∞, will give thoſe values of the abſciſs x, to which will correſpond that ordinate y, which meets the curve in the points of contrary flexure or regreſſion. Wherefore, if thoſe values be ſubſtituted in the

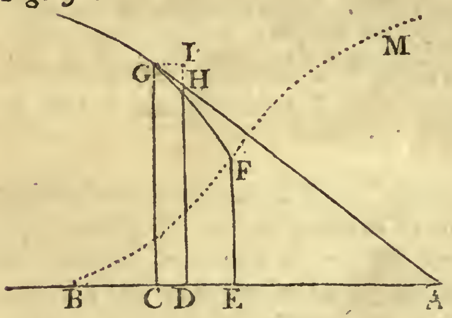
the equation of the curve inftead of x, we fhall have y either real or imaginary. If y be imaginary, or fhall involve a contradiction, then the curve will have no fuch points.

97. To diftinguifh the points of contrary flexure from thofe of regreffion, becaufe this method gives us each of them indifcriminately, it will be fufficient to fee the progrefs of the curve, by taking an ordinate very near. And this will afford light enough to remove any doubt about it.

98. Curves may have another kind of regreffion, different from this which has been confidered. And that is, when the curve returns backwards towards it's origin, turning it's cavity the fame way as it did before it's regreffion. After I have firft treated on the Radii of Curvature, I fhall give a general formula, alfo, for regreffions of this fecond fort, at the end of the following Section.

EXAMPLE I.

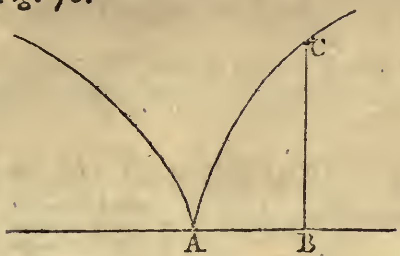
Fig. 51.



99. Let there be a cubic parabola with the equation $y = a + \sqrt[3]{a^3 - 2aax + axx}$, which, in § 81, has been found to have a point of interfection. Now, by differencing, it will be $\dot{y} = \frac{-2aa\dot{x} + 2ax\dot{x}}{3 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}}}$, and differencing again, taking $\dot{x}$ conftant, it will be $\ddot{y} = -\frac{2a\dot{x}\dot{x}}{9 \times \overline{a^3 - 2aax + axx}|^{\frac{5}{3}}}$. The fuppofition of $\ddot{y} = 0$ will give us $-2a\dot{x}\dot{x} = 0$, which is of no ufe; making, therefore, the fuppofition of $\ddot{y} = \infty$, it will be $9 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}} = 0$, that is, $aa - 2ax + xx = 0$, and therefore $x = a$. This value being fubftituted inftead of x in the propofed equation, it will be $y = a$, and therefore the curve has a contrary flexure, or regreffion, which correfponds to the abfcifs $x = a$, to which belongs the ordinate $y = a$. And, becaufe we know otherwife, that this is alfo a point of interfection; it cannot therefore be a point of contrary flexure, but muft be a regreffion.

In

Fig. 70.

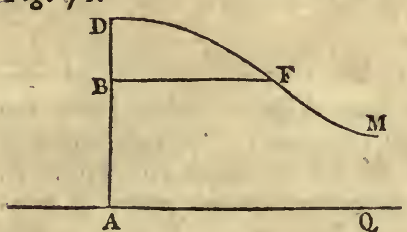


In the same cubic parabola, taking the abscifs AB $= x$ from the vertex A, and the ordinate BC $= y$; the equation is $axx = y^3$, the fluxion of which is $2ax\dot{x} = 3yy\dot{y}$. And taking the fluxions again, making $\dot{x}$ constant, it will be $\ddot{y} = \frac{-6y\dot{y}\dot{y} + 2a\dot{x}\dot{x}}{3yy}$. But, by the equation, it is $3yy = 3x\sqrt[3]{aax}$, and, by the first differencing, $\dot{y} = \frac{2ax\dot{x}}{3x\sqrt[3]{aax}}$. Therefore, making the substitutions, it will be $\ddot{y} = \frac{-2a\dot{x}\dot{x}}{9x\sqrt[3]{aax}}$.

The supposition of $\ddot{y} = 0$ has no use. The supposition of $\ddot{y} = \infty$ will give $9x\sqrt[3]{aax} = 0$, that is, $x = 0$; which value, being substituted in the equation, gives $y = 0$. Therefore the curve has a regression at the vertex A.

EXAMPLE II.

Fig. 71.



100. Let the curve be DFM, commonly called the Witch, the equation of which is $y = a\sqrt{\frac{a-x}{x}}$, AB $= x$, BF $= y$, AD $= a$; by differencing, $\dot{y} = -\frac{aa\dot{x}}{2x\sqrt{ax - xx}}$; and taking $\dot{x}$ constant, and differencing again, it will be $\ddot{y} = \frac{3a^3\dot{x}\dot{x} - 4aax\dot{x}\dot{x}}{4x \times \overline{ax - xx}|^{\frac{3}{2}}}$.

The supposition of $\ddot{y} = 0$ will give $3a^3 - 4aax = 0$, that is, $x = \frac{3}{4}a$; which value, being substituted in the equation of the curve, gives $y = a\sqrt{\frac{1}{3}}$. Whence, taking AB $= \frac{3}{4}a$, the ordinate BF $= a\sqrt{\frac{1}{3}}$ will meet the curve in the point F, which will be a contrary flexure. The supposition of $\ddot{y} = \infty$ gives us $4x \times \overline{ax - xx}|^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = a$. The first value substituted in the equation makes $y = \infty$, the second, $y = 0$. But neither the one nor the other case infer a contrary flexure, but only that the asymptote AQ, as also the tangent in the point D, is parallel to the ordinates.

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EXAMPLE III.

Fig. 72.

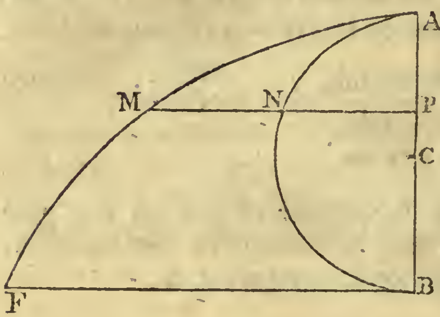


Fig. 73.

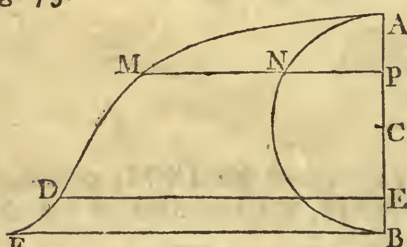
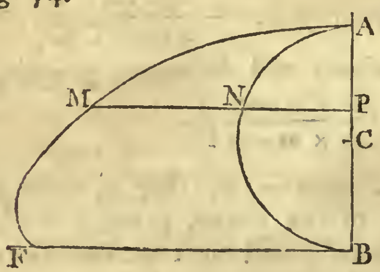


Fig. 74.



101. Let AMF (Fig. 72, 73, 74.) be a cycloid with the equation $\dot{z} = \frac{ar\dot{x} + br\dot{x} - bx\dot{x}}{b\sqrt{2rx - xx}}$, § 47. By differencing, it will be $\ddot{z} = \frac{arx - arr - brr}{b \times \overline{2rx - xx}|^{\frac{3}{2}}} \times \dot{x}\dot{x}$.

The ſuppoſition of $\ddot{z} = 0$ will give $arx - brr - arr = 0$, that is, $x = r + \frac{br}{a}$. If a be greater than b, it will be the protracted cycloid. Whence, taking CE from the centre, and equal to the fourth proportional of BF, the ſemicircle, and the radius, and drawing the ordinate ED, (Fig. 73.) it will meet the curve in the point of contrary flexure D. If a be leſs than b, (Fig. 74.) the cycloid will be contracted. But when $a < b$, the line $x = r + \frac{br}{a}$ will be greater than $2r$, that is, greater than AB, in which caſe the ordinates are imaginary; becauſe there is no part of the curve under the point F. Therefore the curve has no point of contrary flexure or regreſſion. If it be $a = b$, it will be the common cycloid, (Fig. 72.) and therefore $x = r + \frac{br}{a} = 2r =$ AB, and $y =$ BF; which gives no contrary flexure or regreſſion, but only informs us that the tangent in F will be parallel to the abſciſs or diameter AB.

The ſuppoſition of $\ddot{z} = \infty$ gives us $b \times \overline{2rx - xx}|^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = 2r$. The value $x = 0$, in all the three caſes, gives the tangent in the point A parallel to the ordinates. The value $x = 2r$, in the firſt and ſecond caſe, gives the tangent in the point F, in the ſame manner, parallel to the ordinates. But, in the third caſe, it gives us a contradiction. For, the equation

equation being $\dot z = \frac{\dot x\sqrt{2r - x}}{\sqrt{x}}$, inftead of x fubftituting it's value $2r$, it will be $\dot z = 0$. But it cannot be $\dot z = 0$, and at the fame time $\ddot z = \infty$; therefore fuch a value ferves to no purpofe in this cafe.

EXAMPLE IV.

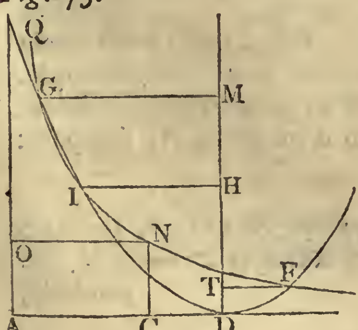
102. Let the curve be the conchoid of *Nichomedes*, confidered above at § 85, the equation of which is $yy = \frac{aaxx - x^4 + 2aabx - 2bx^3 - bbxx + aabb}{xx}$, or $y = \frac{\overline{b + x} \times \sqrt{aa - xx}}{x}$. Taking the fluxions, it will be $\dot y = \frac{-x^3\dot x - aab\dot x}{xx\sqrt{aa - xx}}$; and taking them again, making $\dot x$ conftant, $\ddot y = \frac{2a^4b - a^2x^3 - 3a^2bx^2}{x^3 \times \overline{aa - xx}|^{\frac{3}{2}}} \times \dot x\dot x$.

As to the three ufual cafes, which this curve may have, I begin with the firft, when $a = b$, (Fig. 56.) This fuppofed, it will be $\ddot y = \frac{2a^5 - aax^3 - 3a^3xx}{x^3 \times \overline{aa - xx}|^{\frac{3}{2}}}\dot x\dot x$.

The fuppofition of $\ddot y = 0$ will give $2a^5 - aax^3 - 3a^3xx = 0$, that is, $x^3 + 3ax^2 - 2a^3 = 0$; and, refolving the equation, it is $x = \sqrt{3aa} - a$, $x = -\sqrt{3aa} - a$, and $x = -a$. The firft value gives us the abfcifs GE $= x = \sqrt{3aa} - a$, to which belongs the ordinate EM $= y = \frac{\sqrt{3aa} \times \sqrt{2a\sqrt{3aa} - 3aa}}{\sqrt{3aa} - a}$, which meets the curve in M, the point of contrary flexure; the fecond value is of no fervice, becaufe it makes the equation of the curve imaginary; the third gives us a regreffion in the point P.

As to the other two cafes, the fuppofition of $\ddot y = 0$ gives $2aab - x^3 - 3bxx = 0$, or $x^3 + 3bx^2 - 2aab = 0$. Now, to have the roots of this equation, I make $xx = bz$, a *locus* to the *Apollonian* parabola; and, making the fubftitution, there arifes the fecond *locus* $xz + 3bz - 2aa = 0$, which is to the hyperbola.

Fig. 75.



Between the afymptotes AQ, AD, take AC $= 2a$, the perpendicular CN $= a$, AD $= 3b$, and taking the abfcifs x from the point D on the afymptote AD, let the hyperbola GNF be defcribed, with the conftant rectangle $= 2aa$; it will pafs through the point N. Then raifing DM perpendicular to DA, on the axis DM, with the vertex D, and parameter $= b$, let the parabola of the equation $xx = bz$ be defcribed.

 If,

If, therefore, we aſſume b greater than a, becauſe AD $= 3b$, AC $= 2a$, CD will be greater than b. Now, taking in the parabola the abſciſs $z = a = $ CN, the ordinate will be $x = \sqrt{ab}$. But if a be leſs than b, alſo $\sqrt{ab}$ will be leſs than b, and thence alſo leſs than CD. Therefore the parabola will cut the hyperbola between N and D, ſuppoſe in the point I.

Now, if we aſſume $x = -a$, it will be in the parabola $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{-a+3b}$; but $\frac{aa}{b}$ is greater than $\frac{2aa}{-a+3b}$; therefore the parabola will cut the hyperbola in ſuch a point I, as that it will be HI $= -x$ leſs than a. Therefore this abſciſs will have in the conchoid a real ordinate, which here determines the contrary flexure in the point N, for example, of the lower branch KN. The line GM, drawn from the point G, another interſection of the parabola and hyperbola, will neceſſarily be greater than a, and therefore to ſuch an abſciſs there can be no correſponding real ordinate in the conchoid; ſo that this value is of no uſe. Laſtly, the third value TF will give us an abſciſs, to which an ordinate belongs in the upper branch, which meets the curve in the point of contrary flexure M.

Fig. 58.

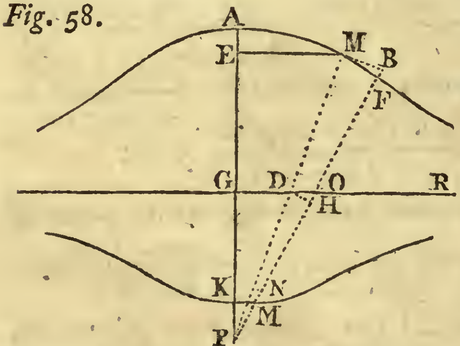
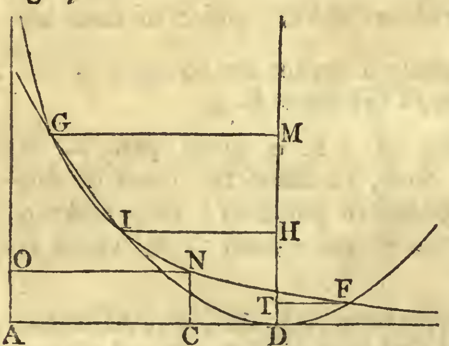


Fig. 76.



Let b be leſs than a; then CD will be leſs than b; and in the parabola, taking $z = a = $ CN, the ordinate will be $x = \sqrt{ab}$, that is, greater than b, and therefore greater than CD. Whence the parabola will paſs between N and C: ſo that it will either not cut the hyperbola, and the two negative values of x in the equation $x^3 + 3bx^2 - 2aab = 0$ will be imaginary; or, if it cut it, they will always be greater than a, to which, in the conchoid, (Fig. 57.) imaginary ordinates correſpond, and therefore are of no ſervice. Wherefore the parabola will certainly cut the hyperbola, on the poſitive ſide, in the point F for example. Whence TF, which is leſs than a, will be the value of x, to which the ordinate correſponds in the branch AM of the conchoid, which it meets in M, the point of contrary flexure.

I ſaid that if the parabola cut the hyperbola between N and O, the two negative values of x would be greater than a. For, taking $x = -a$ in the parabola, it will be $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{3b - a}$. But $\frac{aa}{b}$ is lefs than $\frac{2aa}{3b - a}$, for b is lefs than a. Now, if fo be that x negative be not greater than a, the parabola would not cut the hyperbola; fo that it will cut it in a point in which x fhall be greater than a. Taking x pofitive equal to a, it will be in the parabola $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{3b + a}$. But $\frac{aa}{b}$ is greater than $\frac{2aa}{3b + a}$; fo that the parabola will cut the hyperbola in fuch a point F, that TF will be lefs than a.

The fuppofition of $\ddot{y} = \infty$ gives us $x^3 \times \overline{aa - xx}|^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = \pm a$; which is as much as to fay that the afymptote and tangent in A are parallel to the ordinates in all the three cafes, as likewife the tangent in K, in the fecond and third cafe: and in the firft, that in P there is a point of interfection, (as the regreffions alfo intimate,) becaufe the fame value $x = -a$ has alfo been already fupplied from the fuppofition of $\ddot{y} = 0$; which point of interfection has alfo been found before, at § 85.

103. The fame after another manner. I take the fame conchoidal curve, but with all it's ordinates proceeding from a fixed point, or from the pole P. Therefore make PM $= y$, (Fig. 56, 57, 58.) and draw PF infinitely near to PM. Then with centre P defcribe the little arches MB, DH; make MB $= \dot{x}$, AG $= a$, GP $= b$, and make PD $= z$, HO $= \dot{z}$. By the property of the curve, the equation will be $y = z \pm a$; that is, $y = z + a$ in refpect of the curve above the afymptote GR, and $y = z - a$ in refpect to the curve below it.

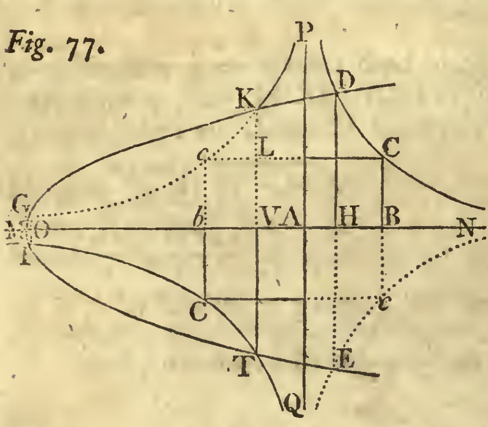
Therefore, finding the fluxions, it will be in both cafes $\dot{y} = \dot{z}$. Becaufe of fimilar triangles PGD, DHO, (for the angles GDP, DOH, do not differ but by the infinitely little angle DPH, and the angles at H and G are right angles,) we fhall have PG . GD :: DH . HO; that is, $b \,.\, \sqrt{zz - bb} :: \frac{z\dot{x}}{y} \,.\, \dot{z}$; and therefore $\dot{z} = \frac{z\dot{x}\sqrt{zz - bb}}{by}$. But $\dot{z} = \dot{y}$, therefore $\dot{y} = \frac{z\dot{x}\sqrt{zz - bb}}{by}$; and taking the fluxions again, making $\dot{x}$ conftant and putting $\dot{z}$ inftead of $\dot{y}$, $\ddot{y} = \frac{2byzz - b^3y - bz^3 + b^3z}{bbyy\sqrt{zz - bb}} \times \dot{x}\dot{z}$; and then putting the value of $\dot{z}$, we fhall have $\ddot{y} = \frac{2yz^3 - bbyz - z^4 + bbzz}{bby^3} \times \dot{x}\dot{x}$; and laftly, fubftituting the value of $y = z \pm a$, it will be $\ddot{y} = \frac{z^4 \pm 2az^3 \mp abbz}{bb \times \overline{z \pm a}|^3} \times \dot{x}\dot{x}$.

 The

The formula of curves referred to a *focus* has been found to be $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y} = 0$, or else $= \infty$. Therefore, putting the values of y, of $\dot{y}$, and of $\ddot{y}$, it will be $\frac{aabb \pm 3abbz \mp 2az^3}{bb \times \overline{z \pm a}|^2} \times \dot{x}\dot{x} = 0$, or else $= \infty$. The supposition of the formula being equal to o, will give $abb \pm 3bbz \mp 2z^3 = 0$. In the first place, let it be $a = b$, and let us consider the upper branch; it will be $z^3 - \frac{3}{2}aaz - \frac{1}{2}a^3 = 0$, and the three values of z are $z = -a$, $z = \frac{a - \sqrt{3aa}}{2}$, and $z = \frac{a + \sqrt{3aa}}{2}$. But it is $y = z + a$; therefore it will be $y = 0$, $y = \frac{3a + \sqrt{3aa}}{2}$, and $y = \frac{3a - \sqrt{3aa}}{2}$. The third value is of no use, because it gives the ordinate less than $2a$, where there is no curve. The second gives the ordinate y, which meets the curve in the point of contrary flexure, for example, at M. The first is also supplied by considering the lower branch, and determines the point of regression P; and, in respect of the inferior branch, will be $z^3 - \frac{3}{2}aaz + \frac{1}{2}a^3 = 0$. Hence the three values, $z = a$, $z = \frac{-a \pm \sqrt{3aa}}{2}$. But, in this case, $y = z - a$, so that we shall have $y = 0$, $y = \frac{-3a \pm \sqrt{3aa}}{2}$. The two last values serve to no purpose, because they give y negative, where there is no curve.

As to the other two cases, (Fig. 57, 58.) it will be $z^3 - \frac{3}{2}bbz \mp \frac{1}{2}abb = 0$. To obtain the roots of this equation, I put $zz = \frac{1}{2}bp$, a *locus* to the *Apollonian* parabola; and making the substitution, there arises a second *locus* which is to the hyperbola, $pz - 3bz = \pm ab$; that is, the *homogeneum comparationis* is positive in regard to the upper branch of the curve, and negative in regard to the lower. Between the asymptotes PQ, NM, perpendicular in A, are described the opposite hyperbolas (Fig. 77.) in the angles PAN, MAQ, if the *homogeneum* be positive, and in the angles PAM, NAQ, if it be negative. And, supposing b to be greater than a, make $AB = b$, $BC = a$; the hyperbolas will pass through the point C. And taking $AM = 3b$, from the point M in the asymptote MN let the p's proceed. Then at the vertex M, with axis MN, and parameter $\frac{1}{2}b$, let there be described the parabola EMD of the equation $zz = \frac{1}{2}bp$. Then taking $p = MB = 2b$, the ordinate in the parabola is $z = b$, greater than a, that is, than bc, the parabola will pass without the points C, and will cut the hyperbolas DC, CT,

Fig. 77.



CT, in the points D, T, I, from which the right lines DH, TV, IO, being drawn parallel to the aſymptote QP, will be the three roots or values of z in the equation $z^3 - \frac{3}{2}bbz - \frac{1}{2}abb = 0$, that is, in reſpect of the upper branch of the conchoid. But $y = z + a$, then DH + a ſhall be the ordinate y, which meets the curve in the point of contrary flexure, for example in M, (Fig. 58.) The other two roots VT, OI, ſerve to no purpoſe; for, being negative, and a adjoined to VT, the difference, or y, will be negative; and a, adjoined to OI, the difference will be poſitive, but leſs than a; and, in this caſe, the curve will not correſpond to y negative, or leſs than a. As to the inferior branch of the conchoid, that is, in the equation $z^3 - \frac{3}{2}bbz + \frac{1}{2}abb = 0$, the three roots will be OG, VK, HE; but if from the firſt, and from the third, a be ſubtracted to have y, the difference will be negative, that is, y negative, to which the curve does not correſpond, and therefore they will be of no uſe. If a be ſubtracted from the ſecond, VK, the difference LK will be the ordinate y, which meets the curve in the point of contrary flexure, that is, in N.

Fig. 58.

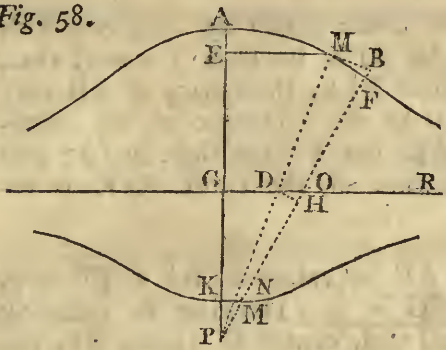
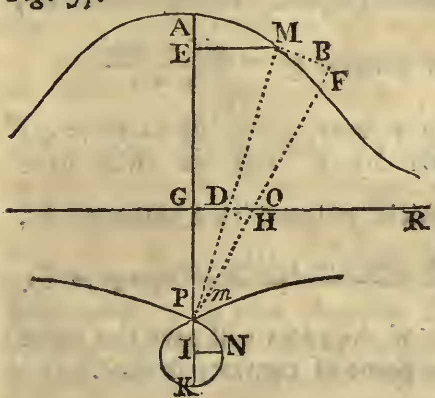


Fig. 57.



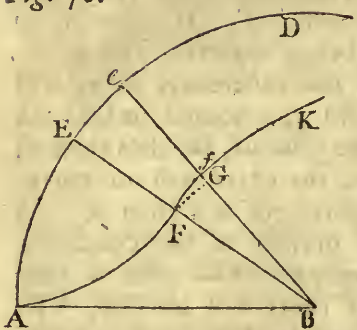
Suppoſing b leſs than a, the parabola will paſs between the points c, C, of the hyperbolas GcK, ICT; and therefore the two negative values of z in the equation $z^3 - \frac{3}{2}bbz - \frac{1}{2}abb = 0$, by adding a, will give y leſs than a, to which the curve does not correſpond. The third, by adding a, will give y, which will meet the curve in the contrary flexure, as at M, (Fig. 57.)

As to the inferior branch, that is, to the equation $z^3 - \frac{3}{2}bbz + \frac{1}{2}abb = 0$, from the two poſitive roots, which are leſs than b, ſubtract a; and alſo, being ſubtracted from the negative root, we ſhall always have negative y greater than PK, to which the curve does not correſpond. Therefore the inferior branch of the conchoid, when b is leſs than a, has neither contrary flexure nor regreſſion.

The ſuppoſition of the formula being $= \infty$, gives, in all the three caſes, $z = \mp a$, and therefore $y = 0$. In Fig. 58, the value $y = 0$ ſerves to no purpoſe, becauſe there is no curve. In Fig. 56, 57, it gives the tangent in P, which is alſo a point of regreſſion in Fig. 56, but not ſo in Fig. 57.

EXAMPLE V.

Fig. 78.



104. Let the circle AED be defcribed with centre B, and let AFK be fuch a curve, that, drawing any radius BFE, the fquare of FE may be always equal to the rectangle of the correfpondent arch AE, into a given right line b; and the contrary flexure of the curve AFK is required.

Let the arch AE be called z, BA = BE = a, BF = y, and FG = $\dot{x}$. Drawing Be infinitely near to BE, and with centre B, radius BF, defcribing the little arch FG; by the nature of the curve, it will be $bz = aa - 2ay + yy$. Then taking the fluxions, it is $b\dot{z} = -2a\dot{y} + 2y\dot{y}$, whence $\dot{z} = \frac{2y\dot{y} - 2a\dot{y}}{b} =$ Ee. But, becaufe of fimilar fectors BEe, BFG, it will be BE . BF :: Ee . FG; that is, $a \,.\, y :: \frac{2y\dot{y} - 2a\dot{y}}{b} \,.\, \dot{x}$. Whence $\dot{x} = \frac{2yy\dot{y} - 2ay\dot{y}}{ab}$. And taking the fluxions again, making $\dot{x}$ conftant, it will be $4y\dot{y}\dot{y} + 2yy\ddot{y} - 2a\dot{y}\dot{y} - 2ay\ddot{y} = 0$, whence $y\ddot{y} = \frac{a\dot{y}\dot{y} - 2y\dot{y}\dot{y}}{y - a}$.

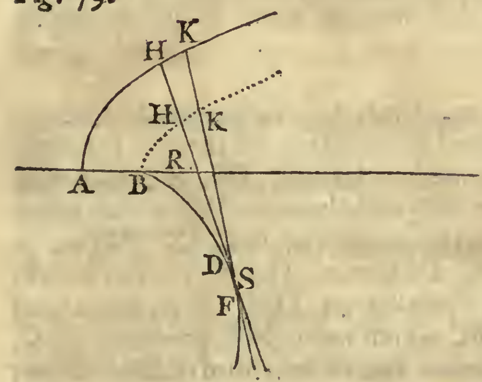
In the general formula of curves referred to a *focus*, $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y} = 0$; fubftitute the values of $\dot{x}\dot{x}$ and of $y\ddot{y}$ given by $\dot{y}$, and we fhall have $\frac{4y^4\dot{y}^2 - 8ay^3\dot{y}^2 + 4a^2y^2\dot{y}^2 + a^2b^2\dot{y}^2}{aabb} - \frac{a\dot{y}^2 - 2y\dot{y}^2}{y - a}$; which, reduced to a common denominator, will be $\frac{4y^5 - 12ay^4 + 12aay^3 - 4a^3yy + 3aabby - 2a^3bb}{aabb \times \overline{y - a}} = 0$, or $= \infty$. Wherefore, this equation being conftructed, one of the roots will give the value of the ordinate y, which meets the curve in the point of contrary flexure.

SECT.

SECT. V.

Of Evolutes, and of the Rays of Curvature.

Fig. 79.



105. Let the curve be BDF, and let it be involved or wound about by the thread ABDF; that is, the thread being faftened by one of it's ends in the fixed and immoveable point F, let it be conceived to be ftretched along the curve BDF, fo that the portion AB may fall upon the tangent of the curve AR in the point B. Let the thread move or unwind by it's extremity A, continually evolving the curve, but in fuch a manner that it may always have the fame degree of tenfion. By this motion, the point A will defcribe the curve AHK.

The curve BDF is called the *Evolute* of the curve AHK, as has been already faid before, at § 16. And the curve AHK is called the *Involute* of BDF, or the curve generated by the evolution of BDF; and the portions AB, HD, KF, of the thread are called the *Rays* of the Evolute, or *Rays of Ofculation.*

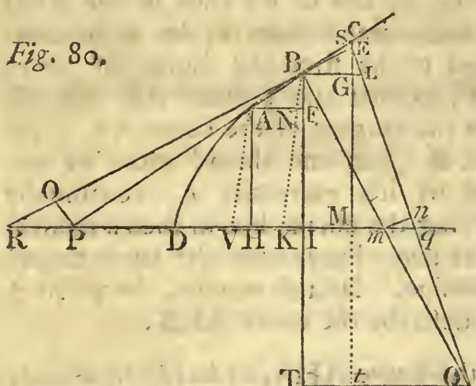
106. Now, becaufe the length of the thread ABDF always continues the fame, it follows from thence, that the difference of the rays of ofculation AB, HD, will be equal to BD, the correfponding portion of the curve. As alfo, the other portion DF is equal to the difference of the radii HD, KF, and the whole curve BDF is equal to the difference of the radii AB, KF. And if the radius AB fhould be none at all, that is, if the point A fhould fall in B, the radius HD would be equal to the portion BD, and the radius FK to the whole curve BDF.

107. From

107. From the generation of the curve AHK, by the unwinding of the thread, it may be clearly ſeen that every radius HD, KF, at it's extremities D, F, is a tangent to the evolute BDF.

108. Let the arch HK of the curve AHK be an infiniteſimal; therefore, alſo, the arch DF of the evolute will be an infiniteſimal; and, as it has been demonſtrated in Coroll. 4. Theor. I. § 6. that any infinitely little arch of a curve has the ſame properties as the arch of a circle: and in Theor. IV. § 15. that the radius HD being produced, ſo that it may meet the radius KF in S, the lines SH, SK, differ from each other only by an infinitely little quantity of the third degree; therefore thoſe lines SH, SK, may be aſſumed as equal: and therefore they are perpendicular to the curve AHK in the points H, K. But the two lines HD, HS, differ from each other by DS, an infiniteſimal of the firſt order, and HD is finite; therefore they may be aſſumed as equal. Wherefore, to determine any point D in the evolute, that is, to determine the length of any ray of oſculation or of curvature HD; it will ſuffice to have given in poſition the perpendicular HS of the given curve AHK, (which is done by the Method of Tangents;) the point S may be determined, in which it is cut by the infinitely near perpendicular KS. This may be done in the following manner.

Fig. 80.



109. Firſt, let the curve DABE be referred to it's axis; let the two infinitely little arches be AB, BE, the perpendicular BQ, and the other EQ, which meets it in the point required, Q. Make, as uſual, DH $= x$, HA $= y$; draw AF, BG, parallel to DM, and the chord PABC which meets ME produced in C, and draw the other chord EBR. Now, with centre B, and diſtances BE, BP, the little arches ES, PO, being deſcribed, it will be AF $= \dot{x}$, FB $= \dot{y}$, AB $= \dot{s} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. But, by Coroll. 2. Theor. V. § 19, the ſectors QBE, BES, are ſimilar. Therefore we ſhall have QB . BE :: BE . ES, that is, QB . $\dot{s}$ :: $\dot{s}$. ES, (calling the element of the curve $\dot{s}$,) and therefore QB $= \frac{\dot{s}\dot{s}}{ES}$. Now, becauſe the little arch PO may be expreſſed by it's right ſine, (Cor. 1. Theor. III. § 9.) the triangles RPO, BEG, will be ſimilar, and therefore BE . EG :: RP . PO; that is, $\dot{s} . \dot{y}$ :: RP . PO $= \frac{\dot{y} \times RP}{\dot{s}}$. But the ſectors BPO, BES, are alſo ſimilar; and therefore it will be BP . PO :: BE . ES; that is, $\frac{y\dot{s}}{\dot{y}} . \frac{\dot{y} \times RP}{\dot{s}}$ :: $\dot{s}$. ES $= \frac{\dot{y}\dot{y} \times RP}{y\dot{s}}$.

And

And laftly, $QB = \frac{y\dot{s}^3}{\dot{y}^2 \times RP}$, a general formula for the rays of ofculation, or the radii of curvature, in which nothing elfe remains to be done, but to fubftitute the value of RP, the fluxion of $DP = \frac{y\dot{x}}{\dot{y}} - x$, according to the different hypothefis of the firft fluxion which is to be taken for conftant.

If no firft fluxion be taken for conftant, it will be $RP = \frac{y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

If $\dot{x}$ be affumed as conftant, it will be $RP = -\frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

If $\dot{y}$ be affumed as conftant, it will be $RP = \frac{y\ddot{x}}{\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x}}$.

If $\dot{s}$ be affumed as conftant, that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, it will be $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$, and $-\ddot{y} = \frac{\dot{x}\ddot{x}}{\dot{y}}$; whence $RP = \frac{y\ddot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}^3}$, and therefore $QB = \frac{\dot{y}}{\ddot{x}}\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}}$; or elfe, fubftituting the value $\ddot{x}$, $QB = \frac{\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{-\ddot{y}}$. Therefore, in the expreffion of $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, in which, as no fluxion is taken for conftant, it will be fufficient to expunge the term $\dot{y}\ddot{x}$, in the fuppofition of $\dot{x}$ conftant; to expunge the term $\dot{x}\ddot{y}$, in the fuppofition of $\dot{y}$ conftant; and to put, inftead of $-\ddot{y}$, it's value $\frac{\dot{x}\ddot{x}}{\dot{y}}$, in the fuppofition of $\dot{s}$ conftant.

110. The curve may be referred to a diameter, or the co-ordinates may be inclined to each other in an oblique angle. Make the abfcifs $DV = x$, $VK = \dot{x}$, the ordinate $VA = y$, and the reft as above. Becaufe the angle DKB is known, the angle BNF will be known alfo. Wherefore, it being $NB = \dot{y}$, NF and FB will be given, and therefore AB, or $\dot{s}$. But the triangle RPO is fimilar to the triangle ABF, for the angles at O and F are right ones, and the angle ORP does not differ from the angle FAB but by an infinitely little angle RBP. Wherefore there will be given RP, PO, and thence ES, and finally, QB.

111. From

Subosculatrix, or Co-radius, what.

111. From the extremity of the radius of curvature BQ is drawn QT parallel to the axis DM, which meets in T the ordinate BI produced; the right line BT is called the *Subosculatrix*, or the *Co-radius*. The radius BQ being given, the co-radius BT will, in like manner, be given also; for, by the method of tangents, the normal of the curve B*m* is given, and therefore BT will be given by means of the similar triangles B*m*I, BQT.

But if we would have an expression for the co-radius independently of the radius, we may make BT $= z$. The triangle BTQ is similar to the triangle BCG, or BAF; for, the two angles TBG, QBC, being right ones, take away the common angle QBG, and there will remain the equal angles TBQ, CBG, and the angles at T and G are right ones. Therefore it will be $\dot{x} \,.\, \dot{s} :: z \,.\,$ BQ $= \frac{z\dot{s}}{\dot{x}} = \frac{z\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$. But, by Theor. IV. § 15, BQ is equal to EQ, because they differ from each other only by an infinitesimal of the third degree; therefore the difference of QB shall be nothing; and, by differencing, without assuming a constant fluxion, $\frac{\dot{x}\dot{z} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}} + z\dot{x}\dot{x}\ddot{x} + z\dot{x}\dot{y}\ddot{y} - z\ddot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = 0$. But $\dot{z} = \dot{y}$, because TB and IB have the same difference. Therefore $z = \frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} =$ BT, a formula for the co-radius, in which no fluxion is yet assumed as constant. If $\dot{x}$ be constant, the term $\dot{y}\ddot{x}$ shall be nothing, and therefore the formula, on this supposition, will be $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}} =$ BT. If $\dot{y}$ be constant, the term $-\dot{x}\ddot{y}$ will be nothing, and therefore the formula, on this supposition, will be $\frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}\ddot{x}} =$ BT. If the element of the curve be constant, it will be $-\ddot{y} = \frac{\dot{x}\ddot{x}}{\dot{y}}$, and therefore the formula, on this supposition, will be $\frac{\dot{x}\dot{y}}{\ddot{x}} =$ BT, the value of $\ddot{y}$ being substituted: or else $-\frac{\dot{x}\dot{x}}{\ddot{y}} =$ BT, the value of $\ddot{x}$ being substituted.

The co-radius being given, by the similitude of the triangles B*m*I, BQT, the radius QB will be given in a like manner.

112. If the co-ordinates shall be at an oblique angle to each other, in the analogy $\dot{x} \,.\, \dot{s} :: z \,.\,$ BQ, instead of $\dot{x}$ and $\dot{s}$, it will be enough to put the respective values, which in this case agree to AF, AB, and to do the rest as above; and then you will have the formula of the co-radius BT, in that case when the co-ordinates are at any oblique angle.

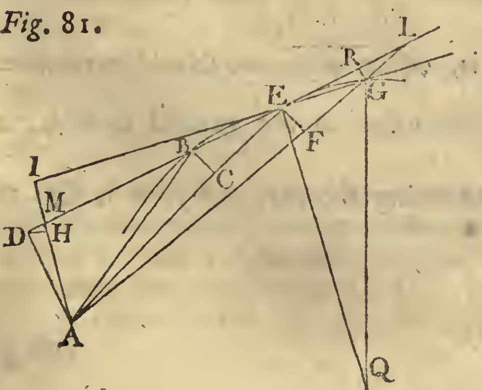
113. After

4

113. After feveral other manners the fame formula of the radius of curvature may be had. As, with centre Q, diftance Q*m*, defcribe the little arch *mn*. Affuming the infinitefimal arch *mn* by the tangent at *n*, the two triangles BCG, *mnq*, will be fimilar, and therefore BC . BG :: *mq* . *mn*; that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. $\dot{x}$ :: mq . $mn = \frac{mq \times \dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. But *mq* is the fluxion of D*m*, that is, of the fubnormal I*m*, with the abfcifs DI or DH; that is, of $x + \frac{y\dot{y}}{\dot{x}}$. Therefore, by differencing in the hypothefis, that no fluxion be conftant, it will be $mq = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\dot{x}}$. Therefore $mn = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. But, becaufe of fimilar fectors Q*mn*, QBE, it will be BE — *mn* . BE :: B*m* $\left(\frac{y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}\right)$. QB; that is, fubftituting their analytical values, $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}})^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$. Which formula, being modified according to the fuppofition of fome conftant fluxion, will give an expreffion for the radius QB, correfponding to that fuppofition.

114. In another manner, thus. Let EM be produced to *t*, and BG to L. Becaufe the triangle EGL is fimilar to the triangle BI*m*, the angles GEL, IB*m*, being different from each other only by the infinitefimal angle BQE, it will be $GL = \frac{y\ddot{y}}{\dot{x}}$. Therefore $BL = \frac{\dot{x}\dot{x} + y\ddot{y}}{\dot{x}}$. But it has been feen, that $mq = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\dot{x}}$. And the fimilar triangles QBL, Q*mq*, give BL — *mq* . BL :: B*m* . BQ. Therefore, fubftituting the analytical values, we fhall have $BQ = \frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}})^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

Fig. 81.



115. Now let us refume the curves which are referred to a *focus*. Therefore let the curve be BEG, the *focus* A. And taking the two infinitely little arches BE, EG, and drawing the ordinates AB, AE, AG, with centre A let the little arches BC, EF, be defcribed, and to the chords GE, EB produced, let AI, AD, be perpendicular. Laftly, let the chord DE, produced, meet the ordinate AG in L,

and

and with centre E let the little arch GR be defcribed. Make AB $= y$, CE $= \dot{y}$, BC $= \dot{x}$, AD $= p$. The little arch DH being defcribed with centre A, it will be HI $= \dot{p}$. But HM is an infinitefimal quantity of the fecond degree; Theor. III. § 8. Therefore we may take as equal HI, IM, and thence it will be MI $= \dot{p}$. The triangles EBC, EAD, are fimilar, which gives ED $= \frac{y\dot{y}}{\dot{s}} =$ EI, as being different only by an infinitefimal. And, affuming the little arch GR by it's tangent, the triangles EIM, EGR, will be fimilar. Hence GR $= \frac{\dot{p}\dot{s}\dot{s}}{y\dot{y}}$. Now, drawing EQ, QG, perpendicular to the curve in the points E, G, the fectors QEG, EGR, are fimilar; fo that QE $= \frac{y\dot{y}}{\dot{p}}$. The fimilar triangles EBC, EAD, will give us $p = \frac{y\dot{x}}{\dot{s}} = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$; and, by differencing, without affuming any conftant fluxion, $\dot{p} = \frac{\overline{y\ddot{x}+\dot{x}\dot{y}} \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}} - \overline{\dot{x}\ddot{x}+\dot{y}\ddot{y}} \times y\dot{x}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}$; or $\dot{p} = \frac{\dot{x}^3\dot{y} + y\dot{y}\dot{y}\ddot{x} + \dot{x}\dot{y}^3 - y\dot{x}\dot{y}\ddot{y}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}$. Whence, fubftituting this value inftead of $\dot{p}$ in the expreffion of QE, it will be QE $= \frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{x}^3 + y\dot{y}\ddot{x} + \dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}$, a general formula for the radius of curvature of curves referred to a *focus*, without taking any fluxion as conftant.

If we would have $\dot{x}$ conftant, taking the value of $\dot{p}$ in this hypothefis, and fubftituting; or, without any thing elfe but expunging the term $y\dot{y}\ddot{x}$ in the general formula, it will be QE $= \frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}$.

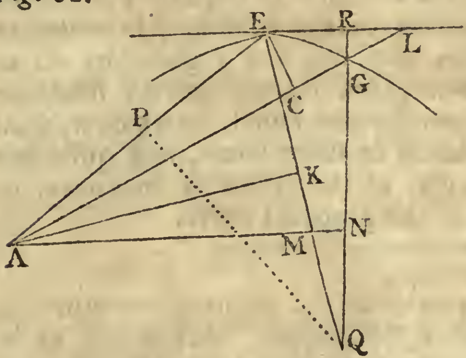
If we would have $\dot{y}$ conftant, expunging the term $- y\dot{x}\ddot{y}$ in the general formula, it will be QE $= \frac{y \times \overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} + y\dot{y}\ddot{x}}$.

And laftly, taking $\dot{s}$ for conftant, that is, $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, we fhould have $\ddot{x} = -\frac{\dot{y}\ddot{y}}{\dot{x}}$; and, inftead of $\ddot{x}$, fubftituting this value in the general formula, it will be QE $= \frac{y\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}\dot{x} - y\ddot{y}}$; or elfe, fubftituting the value of $\ddot{y}$, it is QE $= \frac{y\dot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}\dot{y} + y\ddot{x}}$.

116. If

116. If, in any of these formulæ, we should suppose y infinite, all those terms would vanish in which it is not found, and the formulæ will be the same as those found for curves referred to an axis; which ought to obtain, because, if y be infinite, the point A will be at an infinite distance, and therefore the ordinates will be parallel.

Fig. 82.



117. After another manner. In the point E let ER be a tangent to the infinitely little arch EG, and let QE, QG, be the two radii of curvature, and produce QG to R. From the *focus* A draw AN perpendicular to QG, and AK perpendicular to QE, and make EK $= t$; then is KM $= i$. Because the triangle AKM is similar to the triangle QNM, and this is similar to the triangle QER, it will be QE . ER :: AK . KM $= i$. But, because of the similar triangles ELC, or EGC, EAK, it is AK $= \frac{y\dot y}{\dot s}$, and ER may be assumed for EG. Then it will be QE . $\dot s$ :: $\frac{y\dot y}{\dot s}$. i, and therefore QE $= \frac{y\dot y}{i}$. But EK $= t = \frac{y\dot x}{\dot s}$. Then doing the rest as before, that is, differencing the value of i, and substituting in the expression of QE, we shall obtain the same formulæ as above.

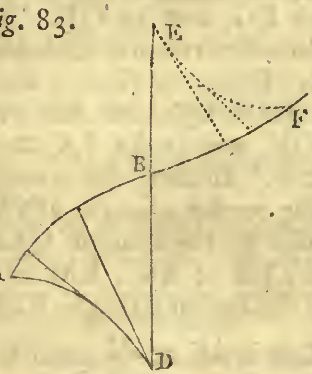
118. Making QP perpendicular to EA produced to P, the triangles EAK, EQP, will be similar, and therefore EA . EK :: EQ . EP. But it has been shown, that EQ $= \frac{y\dot y}{i}$. Then y . t :: $\frac{y\dot y}{i}$: EP $= \frac{t\dot y}{i}$. And, instead of t, substituting it's value $\frac{y\dot x}{\dot s}$, and, instead of i, the differential $\frac{\dot x^3\dot y + y\dot y\dot y\ddot x + \dot x\dot y^3 - y\dot x\dot y\ddot y}{(\dot x^2 + \dot y^2)^{\frac{3}{2}}}$, without assuming a constant fluxion, it will be EP $= \frac{y\dot x\dot s\dot s}{\dot x\dot s\dot s + y\dot y\ddot x - y\dot x\ddot y} = \frac{y\dot x^3 + y\dot x\dot y\dot y}{\dot x^3 + \dot x\dot y\dot y + y\dot y\ddot x - y\dot x\ddot y}$, a general formula for the co-radius, in which no fluxion is made constant; from which, being modified, we obtain the other formulæ, which correspond to the supposition of a constant differential. And if in these we should suppose y to be infinite, that is, if we should cancel the terms in which it is not found, we should have the same formulæ which have been found for curves referred to an axis or diameter.

119. Now,

119. Now, whatever the curve may be, as we find but one expreffion only for the radius of curvature, and for the co-radius; and that as well in curves referred to an axis, as in thofe referred to a *focus*; it follows from hence, that, whatever the curve be, it can have but one evolute.

120. Therefore, any curve being given, expreffed by any equation whatever, of which curve the radius of curvature, or the co-radius is required; it will be neceffary to difference the equation, in order to have the values of $\dot{y}$, $\dot{y}\dot{y}$, and $\ddot{y}$ given by $\dot{x}$; or thofe of $\dot{x}$, &c. given by $\dot{y}$; and to fubftitute them in the formulas now found, by which we fhall have the expreffion in finite terms, and quite free from differentials, of the radius of curvature, or the co-radius of the propofed curve.

Fig. 83.



121. If the value of the radius of curvature, or of the co-radius, be pofitive, they ought to be taken on that fide of the axis DM, (Fig. 80.) or of the *focus*, (Fig. 81.) as has been hitherto fuppofed, and the curve will be concave to this axis or *focus*. But if it fhall be negative, they ought to be taken on the contrary fide, and, in this cafe, the curve will be convex. Hence it follows, that, in the point of contrary flexure or regreffion, if the curve have any, the co-radius, from pofitive, will become negative; and two radii of curvature that are infinitely near, from being convergent will become divergent. But this cannot be, without they firft become parallel, that is, the radius of the evolute muft be infinite in this point; or elfe they muft coincide one with the other, and thus make the radius of the evolute nothing. It is evident, that when the evolute is fuch, as that the radii go on always increafing, as they approach to the point B (Fig. 83, 84.) of contrary flexure or regreffion, to pafs from being converging to become diverging, they muft firft become parallel, being then AD, FE, the evolute of the curve ABF. But if the evolute of the curve ABF, (Fig. 85, 86.) fhall be DBE, the thread, unwinding itfelf from the point B, and proceeding towards A in refpect of the portion BA of the curve, and going on towards F, in refpect of the portion

Fig. 84.

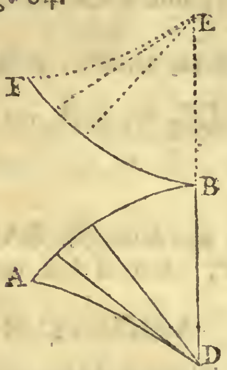


Fig. 85.

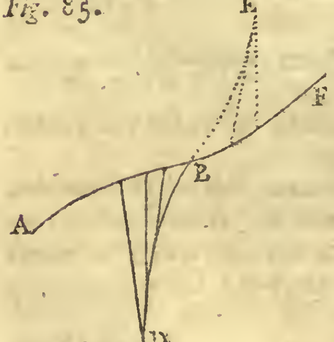
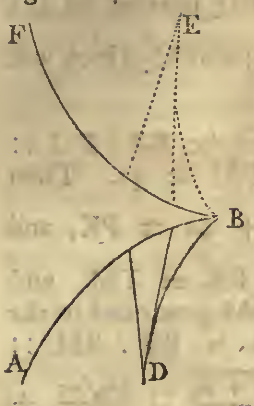


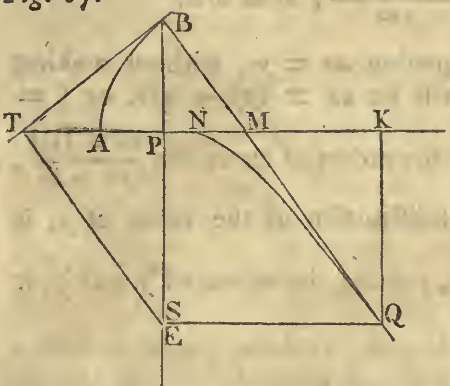
Fig. 86.



portion BF; becaufe, as the radius is always lefs, the nearer it is to the point B, it muft of neceffity become nothing before it paffes from being pofitive to become negative.

EXAMPLE I.

Fig. 87.



122. Let the curve AB be the *Apollonian* parabola of the equation $ax = yy$, of which we would find the radius of curvature at any point B. By taking the fluxions, it will be $a\dot{x} = 2y\dot{y}$; and taking the fluxions again, making, if you pleafe, $\dot{x}$ conftant, it will be $2\dot{y}\dot{y} + 2y\ddot{y} = 0$. But $\dot{y} = \frac{a\dot{x}}{2y}$, therefore $\ddot{y} = -\frac{aa\dot{x}\dot{x}}{4y^3}$. Wherefore, thefe values being fubftituted in the formula for the co-radius $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$, it will be $\frac{4y^3 + aay}{aa}$ = BE; orelfe, by putting, inftead of y, it's value given by the equation of the curve, it will be BE = $\frac{4x\sqrt{ax}}{a} + \sqrt{ax}$.

From the point B let the tangent BT be drawn, which meets the axis in T, and from the point T is drawn TE parallel to the perpendicular BM: this will meet BP produced in the point required, E. For, becaufe of the right angle BTE, it will be BP . PT :: PT . PE; that is, by the property of the parabola $\sqrt{ax} \cdot 2x :: 2x \cdot PE = \frac{4xx}{\sqrt{ax}} = \frac{4x\sqrt{ax}}{a}$. Therefore BP + PE = BE =

$= \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$. Now, BE being determined, draw EQ parallel to the axis AP; the normal BM, produced, will meet EQ in the point Q, which will be a point in the evolute.

Or elſe, becauſe of the ſimilar triangles BPM, BEQ, it will be BP . PM :: BE . EQ. But, by the property of the parabola, it is PM $= \frac{1}{2}a$. Then $\sqrt{ax} . \frac{1}{2}a :: \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$. EQ. Whence EQ $= 2x + \frac{1}{2}a =$ PK, and MK $= 2x$. Wherefore, taking MK double to AP, or PK = TM, and drawing KQ parallel to PB, it will meet the perpendicular BM produced in the point Q, which will be in the evolute. And, becauſe it is BP . BM :: BE . BQ, and BM $= \frac{\sqrt{4ax + aa}}{2}$, it will be $\sqrt{ax} . \frac{\sqrt{4ax + aa}}{2} :: \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$. BQ $= \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, the radius of curvature.

Taking the formula $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, and making the ſubſtitutions, it will be QB $= \frac{\overline{4yy + aa}|^{\frac{3}{2}}}{2aa} = \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, as at firſt.

Proceeding to the ſecond fluxions of the equation $ax = yy$, without making any conſtant fluxion; becauſe $a\dot{x} = 2y\dot{y}$, it will be $a\ddot{x} = 2y\ddot{y} + 2\dot{y}\dot{y}$, or $\ddot{y} = \frac{a\ddot{x} - 2\dot{y}\dot{y}}{2y}$. Wherefore, taking the formula for the radius of curvature $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, which belongs to this caſe, and making the ſubſtitution of the value of $\ddot{y}$, it will be QB $= \frac{2y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{2y\dot{y}\ddot{x} - a\dot{x}\ddot{x} + 2\dot{x}\dot{y}\dot{y}}$; and laſtly, putting the values of y and $\dot{y}$, it is QB $= \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, as above.

The ſame thing will be found in the other ſuppoſitions of $\dot{y}$ or $\dot{s}$ conſtant; which, conſulting brevity, I ſhall here omit.

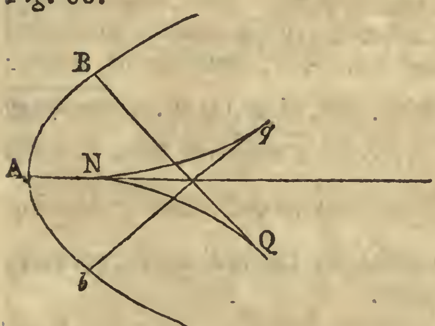
If we would have the radius of curvature at any determinate point of the curve, it will be ſufficient to ſubſtitute, in the finite expreſſion already found for the radius of curvature for any point, the value of x agreeing to that determinate point. Thus, if we would have the radius of curvature in the vertex A, or in the point N in which the axis AN of the parabola touches the evolute NQ; ſince, at the vertex A, it is $x = 0$, by expunging the term $4ax$ in the expreſſion $\frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$ of the radius of curvature, we ſhall have AN $= \frac{1}{2}a$; which

which cannot be otherwiſe, the radius AN in this caſe being the ſame as the ſubnormal, which, in the parabola, is known to be equal to half the parameter.

123. Now it will be eaſy to find the equation to the evolute NQ, after the manner of *Des Cartes*, or the relation of the ordinates NK, KQ, in the following manner.

Make NK $= u$, KQ $= t$. Since KQ $=$ PE $= \frac{4x\sqrt{ax}}{a}$, we ſhall have the equation $t = \frac{4x\sqrt{ax}}{a}$. But AK $=$ AP $+$ PK $= 3x + \frac{1}{2}a$, and AN $= \frac{1}{2}a$. Then NK $= 3x = u$, and $x = \frac{1}{3}u$; therefore, putting, inſtead of x, this value in the equation $t = \frac{4x\sqrt{ax}}{a}$, we ſhall have $t = \frac{4u\sqrt{\frac{1}{3}au}}{3a}$, and, by ſquaring, $27att = 16u^3$, which is an equation to the ſecond cubic parabola, with a parameter $= \frac{27a}{16}$; which expreſſes the relation of the co-ordinates NK, KQ, and is the evolute of the propoſed *Apollonian* parabola.

Fig. 88.

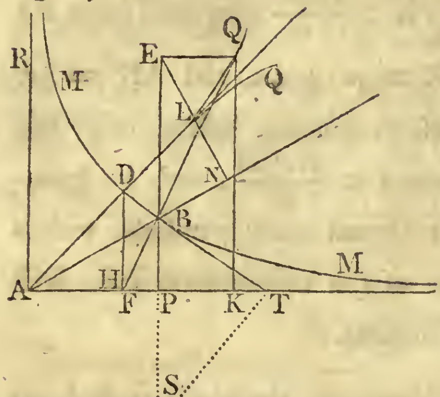


It is evident that the whole ſecond cubical parabola will be the evolute of the whole *Apollonian* parabola; that is, that the branch NQ will be the evolute of the upper part AB, and the branch Nq of the lower part Ab: and that the two branches Nq, NQ, change their convexity, and have a regreſſion at N.

124. It is alſo evident, that if the propoſed curves be algebräical, their evolutes alſo will be algebräical curves, and that we may always have an equation in finite terms, expreſſing the relation of the co-ordinates; and that, beſides, thoſe evolutes will be rectifiable, or we may find right lines equal to any portion of the ſame; for example, to QN. For, if the propoſed curve AB be algebräical, we may always have the radii of curvature BQ, AN, in finite terms; and, from BQ ſubtracting AN, the remainder will be the arch NQ.

EXAMPLE II.

Fig. 89.



125. Let the curve MBM be the hyperbola between the asymptotes, whose equation is $aa = xy$. By differencing, it is $x\dot y + y\dot x = 0$, and by differencing again, and taking $\dot x$ as constant, it is $\ddot y = -\frac{2\dot x\dot y}{x}$. Substituting these values of $\dot y$ and $\ddot y$ in the formula $\frac{\dot x\dot x + \dot y\dot y}{-\ddot y}$ for the co-radius, we shall have $BE = \frac{xx + yy}{-2y}$, a negative value. If, therefore, it is $AP = x$, $PB = y$, in AB, produced, taking $BN = \frac{1}{2}BA = \frac{1}{2}\sqrt{xx + yy}$, and raising the perpendicular NE, which may meet the ordinate BP, produced in E, the co-radius will be BE, as was required. For, because of similar triangles BPA, BNE, it will be BP . BA :: BN . BE, that is, $y . \sqrt{xx + yy} :: \frac{1}{2}\sqrt{xx + yy} . BE = \frac{xx + yy}{2y}$; and therefore, on the negative side, it must be $BE = \frac{xx + yy}{-2y}$. Wherefore, drawing EQ parallel to AP, and producing to Q the perpendicular to the curve FB in the point B, the radius of curvature will be BQ, and the point Q will be in the evolute.

To determine the radius of curvature at the vertex of the hyperbola D, make $x = AH = a$, and therefore $y = HD = a$. Then the co-radius $\frac{xx + yy}{-2y}$ at the vertex D will be equal to $-a$, and the radius equal to $-\sqrt{2aa}$.

If we do but consider a little the figure of the curve MBM, we shall find that the evolute will have two branches, with a point of regression at L, in which the radius DL will revert, and will be the least of all the radii BQ. Wherefore, by differencing the formula of the radius of curvature $\frac{\overline{\dot x\dot x + \dot y\dot y}|^{\frac{3}{2}}}{-\dot x\ddot y}$, the difference or fluxion will be nothing, or infinite; that is, supposing $\dot x$ to be constant, it will be $\frac{-3\dot x\dot y\ddot y^2\sqrt{\dot x^2 + \dot y^2} + \dot x\dddot y \times \overline{\dot x\dot x + \dot y\dot y}|^{\frac{3}{2}}}{\dot x\dot x\ddot y\ddot y} = 0$, or ∞. And, dividing by $\sqrt{\dot x\dot x + \dot y\dot y}$, and multiplying by $\dot x\ddot y\ddot y$, it will be $\dot x\dot x\dddot y + \dot y\dot y\dddot y - 3\dot y\ddot y\ddot y = 0$,

$= 0$, or ∞. But, by the equation of the curve, it is $\dot{y} = -\frac{aa\dot{x}}{xx}$, $\ddot{y} = \frac{2aa\dot{x}\dot{x}}{x^3}$, $\dot{\ddot{y}} = -\frac{6aa\dot{x}^3}{x^4}$. Therefore, making the fubftitutions, and fuppofing the faid quantity to be equal to nothing, we fhall have $x = a =$ AH. That is to fay, the regreffion will be in the radius of curvature at the vertex D of the curve. But it has been feen, that that radius is equal to $-\sqrt{2aa}$; therefore it will be DL $= -\sqrt{2aa} =$ DA.

In the formula of the radius of curvature, fubftituting the values of $\dot{y}$ and $\ddot{y}$, we fhall have BQ $= \frac{\overline{xx + yy}|^{\frac{3}{2}}}{-2xy} = \frac{\overline{xx + yy}|^{\frac{3}{2}}}{-2aa}$, and therefore, differencing, that we may have the leaft radius, that is, the point of regreffion L, it will be $\overline{3x\dot{x} + 3y\dot{y}} \times \sqrt{xx + yy} = 0$; and, inftead of $\dot{y}$, putting it's value, it will be $\overline{3xx\dot{x} - 3yy\dot{x}} \times \sqrt{xx + yy} = 0$, that is, $x = y = a$. And fubftituting this value in the expreffion for the radius of curvature, it will be $= -\sqrt{2aa} =$ DL, as found above.

The radius BQ may alfo be conftructed in another manner. For, becaufe $\ddot{y} = -\frac{2\dot{x}\dot{y}}{x}$, inftead of $\dot{x}$ and x, fubftituting their values by y, it will be $\ddot{y} = \frac{2\dot{y}\dot{y}}{y}$, and therefore the co-radius BE $= \frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{-2\dot{y}\dot{y}}$. And, becaufe of fimilar triangles BPF, BEQ, we fhall have EQ $= -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$. Now draw the tangent BT to the point B, and from the point T the line TS perpendicular to BT, or parallel to BQ, and make BE $= \frac{1}{2}$BS, or PK $= \frac{1}{2}$FT. Now, if EQ be drawn parallel to AT, or KQ perpendicular to it, they will meet the line BQ in the point of the evolute Q. For it will be BS $= \frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{\dot{y}\dot{y}}$, then BE $= \frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{-2\dot{y}\dot{y}}$; it will be alfo FP + PT = FT $= -\frac{y\dot{y}}{\dot{x}} - \frac{y\dot{x}}{\dot{y}}$, and therefore EQ $= -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$.

If the equation be $y^m = x$, which expreffes all parabolas *ad infinitum*, when m denotes an affirmative number, and confequently the parabola of the firft example: (and it expreffes all hyperbolas between the afymptotes, when m ftands for a negative number, and therefore that of the prefent example.) By taking the fluxions, we fhall have $m\dot{y}y^{m-1} = \dot{x}$; and taking the fluxions again, fuppofing $\dot{x}$ conftant, it will be $\overline{mm - m} \times \dot{y}\dot{y}y^{m-2} + m\ddot{y}y^{m-1} = 0$.

Now,

Now, dividing by my^{m-1}, it will be $-\ddot{y} = \overline{m-1} \times \frac{\dot{y}\dot{y}}{y}$. Wherefore, taking the formula for the co-radius $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}}$, and making the fubftitution of the value of $\ddot{y}$, we fhall have BE $= \frac{y\dot{x}\dot{x}+y\dot{y}\dot{y}}{\overline{m-1}\,\dot{y}\dot{y}}$, and therefore EQ, or PK $=$ $\frac{y\dot{x}}{\overline{m-1}\,\dot{y}} + \frac{y\dot{y}}{\overline{m-1}\,\dot{x}}$.

Fig. 87.

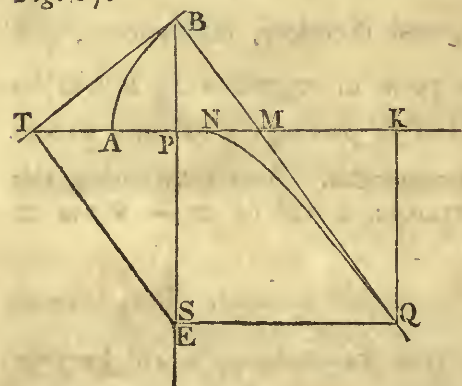
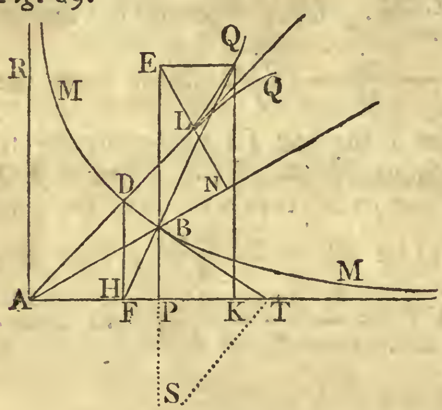


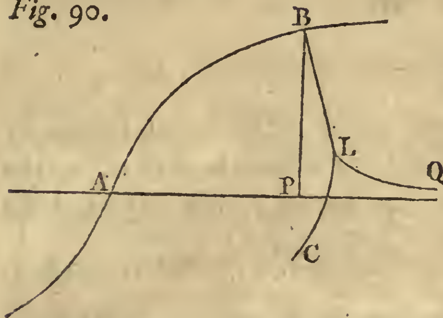
Fig. 89.



From the point T (Fig. 87, 89.) in which the tangent BT meets the axis AP, is drawn, in like manner, TS parallel to BQ, a perpendicular to the curve, which meets in S the ordinate BP produced. Then take BE $= \frac{BS}{m-1}$, on the negative fide, if m be a negative number, as in the hyperbolas which are convex towards the the axis AP, (Fig. 89.) that is, to the afymptote. But BE muft be taken on the pofitive fide, if m be a pofitive number, and greater than unity, as in the parabolas (Fig. 87.) that are concave to the axis AP; and on the negative part, if m, being pofitive, be lefs than unity, in which cafe the parabolas are convex to the axis AP.

To determine the point in which the axis of the parabola touches the evolute, I take the formula of the radius of curvature, which is $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, from whence, by fubftituting the values of $\dot{x} = m\dot{y}y^{m-1}$, and of $-\ddot{y} = \frac{\overline{m-1}\times\dot{y}\dot{y}}{y}$, we fhall have BQ $= \frac{\overline{mmy^{2m-2}+1}|^{\frac{3}{2}}}{m\times\overline{m-1}\times y^{m-2}}$. It is here underftood, that unity may fupply any powers required by the law of homogeneity. Whence, fuppofing m to be greater than unity, for that reafon the parabolas will be concave to the axis AP; if m be lefs than 2, the y in the denominator will become a multiplier in the numerator, and therefore, making $y = 0$, as the prefent cafe requires, it will be BQ $= 0$, that is, the axis will be a tangent to the evolute in A, the vertex of of the parabola, as it would be (for inſtance) in the ſecond cubic parabola $axx = y^3$, Fig. 70.

Fig. 90.



Now, if m be greater than 2, the y of the denominator would be raiſed to a poſitive power, and therefore, making $y = 0$, BQ would be infinite, that is, the axis of the parabola will be an aſymptote to the evolute; as in the firſt cubical parabola AB, (Fig. 90.) whoſe axis AP is an aſymptote to the evolute LQ.

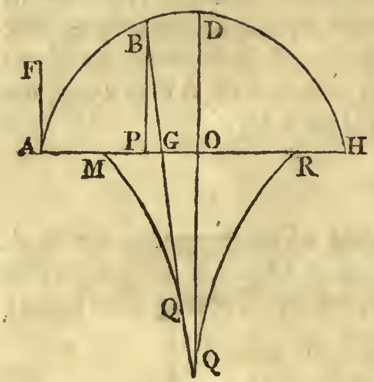
The evolute CLQ of the cubical ſemiparabola ABM of the equation $aax = y^3$, has a point of regreſſion L, and therefore two branches LQ, LC; by evolving the branch LQ, the portion BA will be generated, and by evolving the branch LC, the infinite portion BM will be produced.

To determine the contrary flexure L, take the value of the radius of curvature, which in this curve is $\frac{\overline{9y^4 + a^4}|^{\frac{3}{2}}}{6a^4y}$, which ought to be a *minimum*; and therefore, by taking the fluxion, it will be $\frac{3 \times 18a^4y^4\dot{y} \times \overline{9y^4 + a^4}|^{\frac{1}{2}} - a^4\dot{y} \times \overline{9y^4 + a^4}|^{\frac{3}{2}}}{6a^8yy}$ $= 0$, that is, $45y^4 - a^4 = 0$; whence $y = \sqrt[4]{\frac{a^4}{45}}$. And this value, being ſubſtituted inſtead of y in the equation $aax = y^3$, we ſhall have $x = \sqrt[4]{\frac{a^4}{91125}}$. Taking, therefore, $AP = \sqrt[4]{\frac{a^4}{91125}}$, and drawing the ordinate PB, the point of regreſſion L will be in the perpendicular to the curve at the point B. And, in the expreſſion of the radius of curvature, putting $\sqrt[4]{\frac{a^4}{45}}$ inſtead of y, we ſhall have the value of BL.

After another manner. By differencing the equation $aax = y^3$, or $y = a^{\frac{2}{3}}x^{\frac{1}{3}}$, it will be $\dot{y} = \frac{1}{3}a^{\frac{2}{3}}\dot{x}x^{-\frac{2}{3}}$, $\ddot{y} = -\frac{2}{9}a^{\frac{2}{3}}\dot{x}\dot{x}x^{-\frac{5}{3}}$, $\dddot{y} = \frac{10}{27}a^{\frac{2}{3}}\dot{x}^3x^{-\frac{8}{3}}$, ſuppoſing $\dot{x}$ to be conſtant. Whence, taking the formula $\dot{x}\dot{x}\dddot{y} + \dot{y}\dot{y}\dddot{y} - 3\dot{y}\ddot{y}\ddot{y} = 0$, and ſubſtituting theſe values, we ſhall have $AP = \sqrt[4]{\frac{a^4}{91125}}$, as before.

EXAMPLE III.

Fig. 91.



126. Let the curve ABD be an ellipsis or hyperbola, the axis of which is AH $= a$, the parameter AF $= b$, AP $= x$, PB $= y$, and the equation $y = \sqrt{\frac{abx \mp bxx}{a}}$. By differencing, it will be $\dot{y} = \frac{ab\dot{x} \mp 2bx\dot{x}}{2\sqrt{aabx \mp baxx}}$, and $\ddot{y} = \frac{-a^3bb\dot{x}\dot{x}}{4 \times \overline{aabx \mp abxx}|^{\frac{3}{2}}}$, taking $\dot{x}$ for constant. Making the substitutions in the formula $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, it will be BGQ $= \frac{\overline{4aabx \mp 4abxx + aabb \mp 4abbx + 4bbxx}|^{\frac{3}{2}}}{2a^3bb}$. But the normal will be found to be BG $= \frac{\overline{4aabx \mp 4abxx + aabb \mp 4abbx + 4bbxx}|^{\frac{1}{2}}}{2a}$. Therefore the radius will be BQG $= \frac{4\text{BG cub.}}{bb}$; so that, taking the parameter b for the first term, the normal BG for the second, and continuing the geometrical proportion, the quadruple of the fourth term will be the radius of curvature BQ.

Making $x = 0$ in the expression for the radius of curvature, it will be BGQ $=$ AM $= \frac{1}{2}b$. And making $x =$ AO $= \frac{1}{2}a$, we shall have in the ellipsis BGQ $=$ DOQ $= \frac{a\sqrt{ab}}{2b}$, that is, equal to half the parameter of the conjugate axis; and in Q will be a regression; and the evolute of the portion AD $=$ DH will be MQ,—of the portion DH, will be RQ. But, in the hyperbola, the radius is extended *in infinitum*.

In the ellipsis, if we make $a = b$, the radius of curvature BGQ will be $= \frac{1}{2}a$, wherever the point B be situate. Therefore the radii will all be equal to one another, and the evolute will become a point; that is to say, that the ellipsis, in this case, degenerates into a circle, having the centre for it's evolute.

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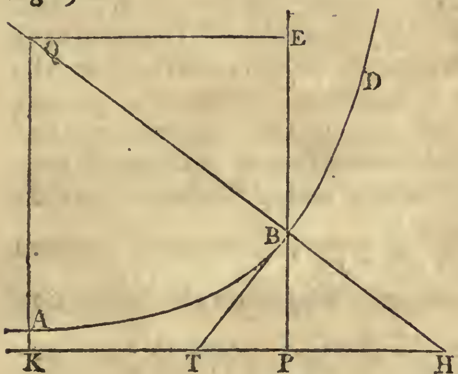
EXAMPLE IV.

Fig. 92.



127. Let the curve ABD be the common logarithmic curve, the equation of which is $\frac{a\dot{y}}{y} = \dot{x}$.

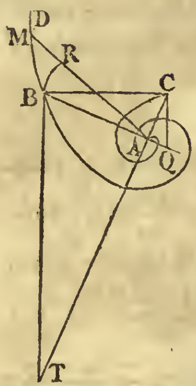
By taking the fluxions, making $\dot{x}$ conſtant, it will be $\ddot{y} = \frac{\dot{x}\dot{y}}{a} = \frac{y\dot{x}\dot{x}}{aa}$, by ſubſtituting the value of $\dot{y}$. Making the uſual ſubſtitutions in the formula $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$ of the co-radius, we ſhall have BE $= \frac{-aa - yy}{y}$; and becauſe, in the logarithmic, it is found that the ſubnormal PH $= \frac{yy}{a}$, it will be EQ $= -a - \frac{yy}{a}$. Therefore, taking PK $=$ TH, and raiſing KQ at right angles, it will meet the normal HBQ in Q, the point of the evolute required.

If we would determine the point of greateſt curvature in the logarithmic, that is, the point where there is the leaſt radius of curvature; making the ſubſtitutions in the formula $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, it will be $\frac{\overline{aa + yy}|^{\frac{3}{2}}}{-ay}$; and taking the fluxions, it will be $\frac{-3ayy\dot{y} \times \overline{aa+yy}|^{\frac{1}{2}} + a\dot{y} \times \overline{aa+yy}|^{\frac{3}{2}}}{aayy} = 0$, and therefore PB $= y = \sqrt{\tfrac{1}{2}aa}$.

Or, taking the formula of § 125, $\dot{x}\dot{x}\dot{\ddot{y}} + \dot{y}\dot{y}\dot{\ddot{y}} - 3\dot{y}\ddot{y}\ddot{y} = 0$, and making the ſubſtitutions of $\dot{y} = \frac{y\dot{x}}{a}$, $\ddot{y} = \frac{y\dot{x}\dot{x}}{aa}$, and $\dot{\ddot{y}} = \frac{y\dot{x}^3}{a^3}$, we ſhall come to the ſame concluſion of PB $= y = \sqrt{\tfrac{1}{2}aa}$.

EXAMPLE V.

Fig. 93.



128. Let ABD be the logarithmic ſpiral, the property of which is, that, at any point B, drawing the tangent BT, and from the pole A the ordinate AB, the angle ABT may always be the ſame : therefore, making AM to be infinitely near AB, the ratio of MR to RB will be conſtant. Wherefore, putting AB $= y$, the little arch BR $= \dot{x}$, the equation will be $a\dot{x} = b\dot{y}$; and, by taking the fluxions, and making $\dot{x}$ conſtant, it will be $\ddot{y} = 0$. Therefore, taking the formula of the co-radius, § 118, $\frac{y\dot{x}^3 + y\dot{x}\dot{y}\dot{y}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}$, for curves that are referred to a *focus*, which, being managed on the ſuppoſition of $\dot{x}$ being conſtant, will be $\frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}$. And in this, expunging the term $y\ddot{y}$, becauſe the curve gives us here $\ddot{y} = 0$, and making the ſubſtitution of the value of $\dot{x}$ or $\dot{y}$, or, dividing the numerator and denominator by $\dot{x}\dot{x} + \dot{y}\dot{y}$, the co-radius will be BA $= y$.

Therefore, drawing AC perpendicular to AB, it will meet the perpendicular BC in C, the point of the evolute required ; and, becauſe the ſubnormal AC $= \frac{ay}{b}$, it will be BC $= \frac{y\sqrt{aa + bb}}{b}$.

Drawing BT, a tangent to the curve in the point B, the triangles TCB, CBA, will be ſimilar, and therefore the angles TBA, ACB, will be equal. But the angle TBA is a conſtant angle, ſo that the angle ACB will be ſo too. Therefore the evolute AC will be the ſame logarithmic ſpiral ABD, but in an inverted ſituation.

EXAMPLE VI.

129. Let ABD (Fig. 93.) be the hyperbolical ſpiral, the property of which is, that the ſubtangent is a conſtant line.

Do the ſame things as in the foregoing example, and the equation of the curve will be $\frac{y\dot{x}}{\dot{y}} = a$, or $y\dot{x} = a\dot{y}$. Then, by differencing, making $\dot{x}$ conſtant,

ſtant, $\ddot{y} = \frac{\dot{x}\dot{y}}{a}$. Wherefore, taking the formula of the co-radius, correſponding to the hypotheſis of $\dot{x}$ conſtant, that is, $\frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}$, and, inſtead of $\ddot{y}$, ſubſtituting it's value $\frac{\dot{x}\dot{y}}{a}$, and, inſtead of $\dot{y}$, it's value $\frac{y\dot{x}}{a}$ given by the equation, the co-radius will be $= \frac{y \times \overline{aa + yy}}{aa}$.

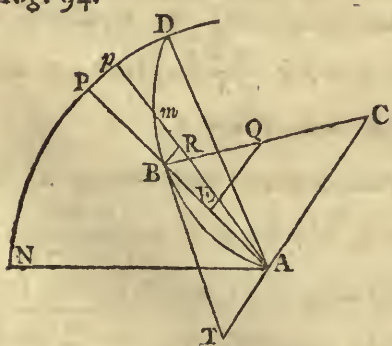
But, becauſe the ſubtangent AT $= a$, and the ſubnormal AC $= \frac{yy}{a}$, it will be TC $= \frac{aa + yy}{a}$. Therefore the fourth proportional to the ſubtangent TA, and TC, and the ordinate AB, here determines the co-radius. Whence, from the point C drawing CQ parallel to the tangent BT, which cuts in Q the ordinate BA produced, BQ will be the co-radius required.

For the triangles BAT, CAQ, are ſimilar; ſo that we ſhall have CA . AQ :: TA . AB; and, by permutation, CA . TA :: AQ . AB. And, by compounding, TC . AT :: QB . AB; and, by inverſion, TA . TC :: BA . BQ.

Q. E. I.

EXAMPLE VII.

Fig. 94.



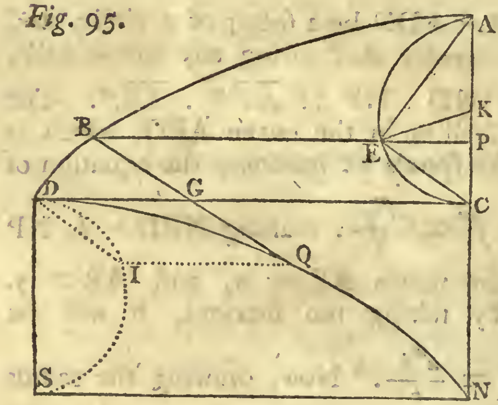
130. Let ADN be a ſector of a circle, and from the centre A drawing any radius ABP, let it be ND . NP :: $\overline{AP}^m$. $\overline{AB}^m$. The point B ſhall be in the curve ABD, which is one of the ſpirals *ad infinitum*, the equation of which is $y^m = \frac{a^m z}{b}$, making NPD $= b$, NP $= z$, the radius AP $= a$, and AB $= y$. Then, by taking the fluxions, it will be $m\dot{y}y^{m-1} = \frac{a^m \dot{z}}{b}$. Now, drawing the radius Ap infinitely near to AP, and making BR $= \dot{x}$; becauſe of ſimilar ſectors APp, ABR, it will be $\dot{z} = \frac{a\dot{x}}{y}$. Wherefore, putting the value, inſtead of $\dot{z}$,

 in

in the fluxional equation, it will be $m\dot{y}y^m = \frac{a^{m+1}\dot{x}}{b}$; and therefore, taking the fluxions again, making $\dot{x}$ conftant, we fhall have $mm\dot{y}\dot{y}y^{m-1} + my^m\ddot{y} = 0$, that is, $y\ddot{y} = -m\dot{y}\dot{y}$. Wherefore, making a fubftitution of this value, and of the value of $\dot{x}$, in the formula of the co-radius, it will be BE $= \frac{y \times \overline{mmbby^{2m} + a^{2m+2}}}{mmbby^{2m} + \overline{m+1} \times a^{2m+2}}$. Make TAC perpendicular to AB, and draw BT a tangent to the curve in B, and BC perpendicular to it; it will be AT $= \frac{mby^{m+1}}{a^{m+1}}$, AC $= \frac{a^{m+1}}{mby^{m-1}}$, and therefore TC $= \frac{mmbby^{2m} + a^{2m+2}}{mba^{m+1}y^{m-1}}$. Whence the fourth proportional to TA $+ \overline{m+1} \times$ AC, to TC, and to AB, will be $\frac{y \times \overline{mmbby^{2m} + a^{2m+2}}}{mmbby^{2m} + \overline{m+1} \times a^{2m+2}} =$ BE. And therefore, drawing EQ parallel to TC, it will meet the perpendicular BC in the point Q, which will be a point in the evolute.

EXAMPLE VIII.

Fig. 95.



131. Let the curve ABD be half of the common cycloid, the equation of which is $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$; making AC $= 2a$, AP $= x$, PB $= y$.

By differencing, and taking $\dot{x}$ for conftant, it will be $\ddot{y} = \frac{-a\dot{x}\dot{x}}{x\sqrt{2ax - xx}}$; and fubftituting thefe values in the formula for the radius of curvature $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, it will be BQ $= 2\sqrt{4aa - 2ax}$. But the normal BG $= \sqrt{4aa - 2aa}$, which is equal to the chord EC. Therefore the radius of curvature BQ $=$ 2BG $=$ 2EC.

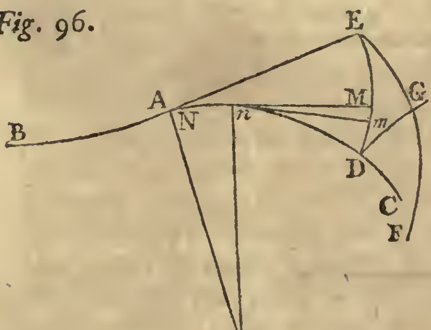
Making $x = 0$, to have the radius of curvature in the point A, it will be BQ = AN = $4a$, and therefore CN = CA = $2a$.

Making $x = 2a$, the radius of curvature in the point D will be = o, and therefore the evolute begins in D, and terminates in N.

Becaufe the tangent of the cycloid in B is parallel to the chord EA, (§ 47.) the normal BQ will be parallel to the chord EC. This fuppofed, complete the rectangle DCNS, and with the diameter DS = CN = AC defcribe the femicircle DIS, and draw the chord DI parallel to BQ, or to EC. The angles CDI, DCE, will be equal, and confequently the arches DI, CE, and their chords. Therefore DI, GQ, are equal and parallel; and drawing IQ, it will be equal and parallel to DG. But, by the property of the cycloid, DG is equal to the arch EC, and therefore to the arch DI. Then the arch DI = IQ, and the femicircle DIS = SN. Whence the evolute DQN is the fame cycloid, DBA, in an inverted fituation.

132. The radius of curvature and it's formula being now fufficiently explained, it will not be difficult to find the formula for the regreffions of the fecond fpecies, mentioned before at § 98.

Fig. 96.



Let the curve be BAC, with a contrary flexure at A, and let it be evolved by the thread beginning at any point D, different from the point of contrary flexure A. The evolution of the portion DC generates the curve DG, and that of the portion AB generates the curve EF; in fuch manner, that the evolution of the whole curve BAC will form the entire curve FEDG, which has two regreffions; one at D of the ufual form, becaufe the two branches DE, DG, turn their convexity; the other at E of the fecond fort, becaufe the two branches ED, EF, are concave towards the fame parts. Let NM, N*nm*, be any two rays infinitely near, of the evolute DA, and NH, *n*H, two perpendiculars to the fame; the two infinitefimal fectors N*m*M, HN*n*, will be fimilar, and therefore HN . NM :: N*n* . M*m*. But, in the point of contrary flexure A, the radius HN (§ 121.) ought to be either infinite, or equal to nothing, and the radius NM, which becomes AE, continues finite. Therefore, in the cafe of contrary flexure A, that is, in the point of regreffion E, of the fecond fort, the ratio of N*n*, M*m*, that is, the ratio of the differential of the radius MN to the element of the curve, ought to be either infinitely great or infinitely little. But the formula of the radius

 MN

MN is $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, taking $\dot{x}$ for conſtant; the differential of which is $\frac{-3\dot{x}\ddot{y}\dot{y}\ddot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}+\dot{x}\dddot{y}\times\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{x}\dot{x}\ddot{y}\ddot{y}}$, and $Mm = \sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$. Therefore $\frac{Nn}{Mm} = \frac{\dot{x}\dot{x}\dddot{y}+\dot{y}\dot{y}\dddot{y}-3\dot{y}\ddot{y}\ddot{y}}{\dot{x}\ddot{y}\ddot{y}} = 0$, or ∞, the formula for the points of regreſſion of the ſecond ſort.

This formula is the ſame as that already found, § 125; but in that place it ſerved for the regreſſions of the firſt ſort of evolutes, and here for the regreſſion of the ſecond ſort of curves, derived from evolutes; x and y, in both caſes, being the co-ordinates of the curves ſo produced.

END OF THE SECOND BOOK.

ANA-

ANALYTICAL INSTITUTIONS.

BOOK III.

OF THE INTEGRAL CALCULUS.

THE *Integral Calculus*, which is alſo uſed to be called the *Summatory Calculus*, is the method of reducing a differential or fluxional quantity, to that quantity of which it is the difference or fluxion. Whence the operations of the Integral Calculus are juſt the contrary to thoſe of the Differential; and therefore it is alſo called *The Inverſe Method of Fluxions*, or of *Differences*. Thus, for example, the fluxion or differential of y is $\dot{y}$, and conſequently the *fluent* or *integral* of $\dot{y}$ is y. Hence it will be a ſure proof that any integral is juſt and true, if, being differenced again, it ſhall reſtore the given fluxion, or the quantity whoſe integral was to be found. Differential formulæ have two different manners, by which their integrals are inveſtigated. One is, by the help of finite Algebräical expreſſions, or by being reduced to quadratures which are granted or ſuppoſed. In the other, we are allowed the uſe of infinite ſeries. In this firſt Section, I ſhall deliver the rules required in the firſt manner. In the ſecond Section, I ſhall treat of the ſecond manner; to which I ſhall add a third Section, to ſhow the uſe of theſe Rules in the Rectification of Curve-lines, the Quadrature of Curve-ſpaces, &c. And laſtly, I ſhall add a fourth, which ſhall teach the Rules of the *Exponential Calculus*. Introduction.

SECT.

SECT. I.

The Rules of Integrations expressed by finite Algebräical Formulæ, or which are reduced to supposed Quadratures.

1. As in simple quantities raised to any power, their differential or fluxion is the product of the exponent of the variable into the variable itself, raised to the same power lessened by unity, and multiplied by it's fluxion or difference; so the fluent or integral of the product of a variable raised to any power, into the difference of the same variable, is the variable raised to a power the exponent of which is increased by unity, divided by the same exponent so increased. And this obtains, whatever the exponent shall be of the power of the variable, whether positive or negative, integer or fraction. Thus, for example, the fluent of $mx^{m-1}\dot{x}$ will be $\frac{mx^{m-1+1}}{m-1+1}$, or x^m. The integral of $x^{\pm\frac{m}{n}}\dot{x}$ will be $\frac{x^{\pm\frac{m}{n}+1}}{\pm\frac{m}{n}+1}$, that is, $\frac{nx^{\pm\frac{m+n}{n}}}{\pm m+n}$; and so of others.

2. Any constant quantities, simple or complicate, by which the fluxions may be multiplied or divided, will make no alteration in the rule; for they remain in the fluents just as they were in the fluxions. Therefore the fluent of $\frac{aax^n\dot{x}}{mb-cc}$ will be $\frac{aax^{n+1}}{\overline{n+1}\times\overline{mb-cc}}$.

3. Thus, if the differential formula were a fraction, the denominator of which were also any power of the variable, multiplied (if you please) by any constant quantity; as the formula $\frac{x^m\dot{x}}{aax^n-bbx^n}$, or $\frac{x^m\dot{x}}{\overline{aa-bb}\times x^n}$, which will be the same as $\frac{\dot{x}x^{m-n}}{aa-bb}$, and therefore subject to the general rule.

4. But

4. But here we are to obferve, that, in order to have the integrals complete, we ought always to add to them, or to fubtract from them, fome conftant quantity at pleafure, which, in particular cafes, is afterwards to be determined as occafion may require. Of this we fhall take further notice in it's due place.

Thus, the complete integral of $\dot{x}$, for example, will be $x \pm a$, where a fignifies fome conftant quantity. That of $x^2\dot{x}$ will be $\frac{1}{3}x^3 \pm a^3$; and fo of others. The reafon of which is, that, as conftant quantities have no differentials, but $\dot{x}$ may as well be the differential of $x + a$, or of $x - b$, &c. as of x; fo x, or $x + a$, or $x - b$, &c. may be the integral of $\dot{x}$. The fame obtains in any other formula.

5. The fame rule of integration ferves for complicate differential formulæ, or thofe compounded of many terms; whether they have a denominator, whether that be wholly conftant, or contains the variable in it, whether it be fimple and of one term, or whether it be reducible to fuch.

For, in thefe cafes, the complicate differential formula may be refolved into as many fimple ones, as are the terms of the complicate, and then each of thefe comes under the given rule. Let the formula be $\frac{bx^m\dot{x} + aax^{m-1}\dot{x}}{aa - bb}$; this will be equivalent to thefe two, $\frac{bx^m\dot{x}}{aa - bb}$ and $\frac{aax^{m-1}\dot{x}}{aa - bb}$, and therefore the integral of thefe two formulæ will be the integral of the firft; that is, $\frac{bx^{m+1}}{\overline{m+1} \times \overline{aa - bb}}$ $+ \frac{aax^m}{m \times \overline{aa - bb}} \pm f$.

Let it be $\frac{bx^3\dot{x} - a^4\dot{x}}{axx - cxx}$; this is the fame as thefe two, $\frac{bx^3\dot{x}}{\overline{a - c} \times x^2} - \frac{a^4\dot{x}}{\overline{a - c} \times x^2}$, or as thefe, $\frac{bx\dot{x}}{a - c} - \frac{a^4x^{-2}\dot{x}}{a - c}$, and therefore the integral will be $\frac{bx^2}{2 \times \overline{a - c}} - \frac{a^4x^{-1}}{-1 \times \overline{a - c}} \pm f$, that is, $\frac{bxx}{2a - 2c} + \frac{a^4}{\overline{a - c} \times x} \pm f$.

Let it be $\frac{bx^m\dot{x} - aax^{m-1}\dot{x}}{xx}$; this is equivalent to thefe two, $bx^{m-2}\dot{x} - aax^{m-3}\dot{x}$, and therefore the integral will be $\frac{bx^{m-1}}{m - 1} - \frac{aax^{m-2}}{m - 2} \pm f$.

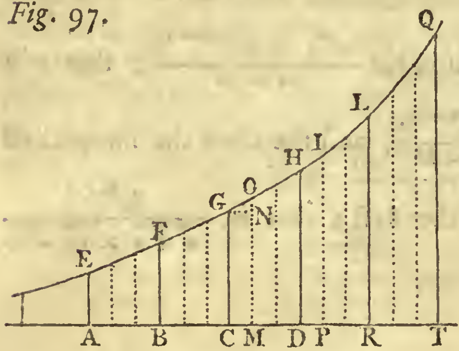
6. Befides,

6. Beſides, if the complicate differential formula be raiſed to any power, the exponent of which is a poſitive integer, it being actually reduced to the given power, every term may be integrated by the ſame rule.

7. All that I have hitherto ſaid will obtain, when in the differential formula there is no term in which the exponent of the variable is negative unity, ſuch as $\frac{a\dot{x}}{x}$, or $ax^{-1}\dot{x}$; for, according to the rule, the integral would be $\frac{ax^{-1+1}}{-1+1}$, or $\frac{ax^{0}}{0}$, that is, infinite; and which therefore teaches us nothing.

8. In theſe caſes, therefore, we are obliged to have recourſe to other methods. There are two of theſe which will aſſiſt us. One is, by means of a curve which is called the *Logarithmic Curve*, or the *Logiſtic*. The other is, by means of infinite ſeries. As to infinite ſeries, of which we ſhall make very great uſe in many other caſes alſo, I ſhall treat of them hereafter, as may be ſeen in the next Section.

Fig. 97.



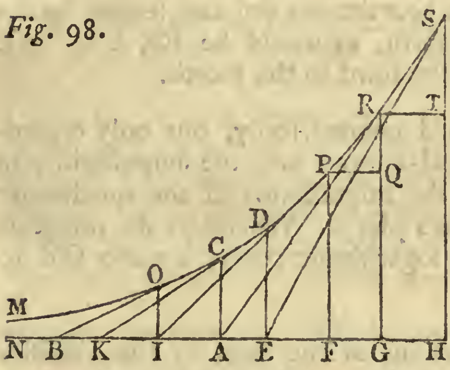
9. Now, as to the logarithmic curve, it is a curve of ſuch a property, that, in the axis, taking the abſciſſes in arithmetical progreſſion, the correſponding ordinates will be in geometrical progreſſion. Therefore let the axis AD be divided into equal parts, AB, BC, CD, DE, &c. At the points A, B, C, D, &c. erect the perpendiculars AE, BF, CG, DH, &c. ſuch, that they may be to each other in geometrical proportion. The points E, F, G, H, &c. will be in the curve. And again dividing the lines AB, BC, &c. into equal parts, and at the diviſions raiſing perpendiculars in the ſame geometrical proportion, we ſhall have other intermediate points. And laſtly, multiplying the diviſions *in infinitum*, we ſhall have infinite points, or the very curve itſelf.

Therefore, the axis being divided into infiniteſimal equal parts, let one of theſe be CM $= \dot{x}$, the ordinate CG $= y$, and MO infinitely near it; therefore it will be NO $= \dot{y}$. Let there be another ordinate DH $= z$, and others as many as you pleaſe, correſponding to the abſciſſes that are arithmetically proportionals. Therefore theſe ordinates will have the ſame proportion to each other, and, by conſequence, their differentials alſo will be in the ſame proportion. So that it will be $\dot{y} . \dot{z} :: y . z$; or $\dot{y} . y :: \dot{z} . z$; whence the ratio of $\dot{y}$ to y will be a conſtant ratio. And therefore, aſſuming $\dot{x}$ conſtant, it will be $\dot{y} . y :: \dot{x} . a$, or $\frac{a\dot{y}}{y} = \dot{x}$; which is the equation to the curve.

Here it will be eaſy to perceive that the ſubtangent of this curve will always be conſtant; for, in the general formula of the ſubtangent $\frac{y\dot{x}}{\dot{y}}$, inſtead of y, ſubſtituting it's value given from the equation of the curve, we ſhall have $\frac{y\dot{x}}{\dot{y}} = \frac{ay\dot{x}}{\dot{x}y} = a$. Now, as the increaſing geometrical progreſſion of the ordinates may be continued *in infinitum*, the abſciſſes alſo increaſing arithmetically *in infinitum*; therefore the curve will go on infinitely, always receding further from the axis. And as the ſame progreſſion, decreaſing, may be alſo continued *in infinitum*, the axis ſtill increaſing the contrary way, the other part of the curve will go on infinitely, but always approaching towards the axis without ever touching it, and therefore that axis will be an aſymptote to the curve.

9. Among many other ways, the logarithmic curve may be conceived to be deſcribed in this manner alſo.

Fig. 98.



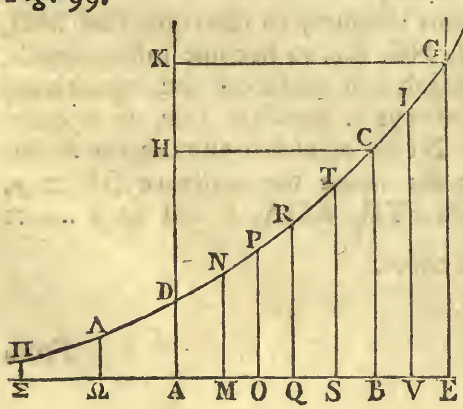
Let the indefinite right line MH be divided into equal parts MN, NB, BK, &c.; and taking NI at pleaſure, at the point I let the perpendicular IO be erected of any magnitude; then draw NO, and at the point A let the perpendicular AC be erected till it meets NO produced to C. From the point B draw BC, and at the point E let the perpendicular ED be erected, which meets BC produced in D. From the point K draw KD, and at the point F let the perpendicular FP be raiſed, which meets KD produced in the point P. After the ſame manner, let the operation be continued *in infinitum*, and the points O, C, D, P, &c. will be in the logarithmic curve. To have the intermediate points between O, C, D, P, &c. let the portions MN, NB, &c. be biſected, and the ſame operation being repeated, we ſhall have other points. And finally, by multiplying the equal diviſions infinitely in the right line MH, that is, by ſuppoſing the equal portions MN, NB, &c. to become infiniteſimals, we ſhall have an infinite number of points which will mark out the logarithmic curve, the ſubtangent of which ſhall be always a conſtant line, as appears from the conſtruction. Making, therefore, $NI = a$, and ſuppoſing the infiniteſimal conſtant portion of the axis to be $\dot{x}$; make the ordinate $GR = y$, $GH = \dot{x}$, $TS = \dot{y}$; by the ſimilar triangles STR, RGA, it will be $\dot{y} . \dot{x} :: y . a$; that is, $\frac{a\dot{y}}{y} = \dot{x}$, the equation of the curve.

From this conftruction we deduce alfo this, which the firft fuppofes; that is, the primary property of the logarithmic curve, that the ordinates are in geometrical proportion, which correfpond to the abfciffes in arithmetical proportion. For, fuppofing the equal portions of the axis to be infinitefimals, the little arch OC, produced, will be the tangent NO, the little arch CD, produced, will be the tangent BC, the little arch BD, produced, the tangent KD; and fo of all the others. Therefore the triangles OIN, CAN, will be fimilar, and therefore it will be OI . CA :: NI . NA. Thus, alfo, by the fimilitude of the triangles CAB, DEB, it will be CA . DE :: BA . BE. But NI = BA, NA = BE; therefore it will be OI . CA :: CA . DE; and fo fucceffively. Therefore the ordinates will be in continual geometrical proportion. Hence, alfo, if we conceive the axis to be divided, not into infinitely little parts, but into finite and equal parts; becaufe the intermediate proportional ordinates, for example, between IO and CA, are neither more nor fewer in number than the intermediate between CA and DE, and thus of others; therefore IO, CA, DE, will be in geometrical proportion, correfponding to the abfciffes in arithmetical proportion. Therefore, taking any two ordinates at pleafure, and other two alfo where you pleafe, provided the diftance between the firft and fecond be the fame as the diftance between the third and fourth, as would be IO, CA, RG, SH; then the firft will be to the fecond, as the third to the fourth.

The logarithmic curve cannot be defcribed geometrically, but only organically, and therefore it is called a mechanical curve; and the impoffibility of being geometrically defcribed is the fame as the impoffibility of the quadrature of the hyperbolical fpace, as will be feen in it's place. Wherefore the integrals of fuch differential formulæ as belong to the logarithmic curve, are alfo faid to depend on the quadrature of the hyperbola.

Hence, in the logarithmic curve, the portions of the axis, or the abfciffes taken from fome fixed point, correfpond to the ordinates juft in the fame manner as, in the trigonometrical tables, the logarithms correfpond to the natural feries or progreffion of numbers.

Fig. 99.



10. This fuppofed, let DC be the logarithmic curve, the fubtangent of which is equal to unity, or, if you pleafe, is equal to the conftant line a; and let the ordinate AD be equal to the fubtangent, that is, equal to unity, or to the conftant line a, which is in the place of unity. Taking any abfcifs AB $= x$, make BC $= y$. But the equation of the curve is $\frac{a\dot{y}}{y} = \dot{x}$, and therefore the integral

integral or fluent of $\frac{a\dot{y}}{y}$ will be x. But $x = AB$, and AB is the logarithm of BC, or of y. Now, to make ufe of the mark $\int$ to fignify the integral, fum, or fluent, all which mean the fame thing; and of the mark l, which means the logarithm, it will be $\int \frac{a\dot{y}}{y} = ly$, in the logarithmic curve, the fubtangent of which is a. After the fame manner, it will be $\int \frac{\dot{y}}{y} = ly$, in the logarithmic whofe fubtangent $= 1$; $\int \frac{b\dot{y}}{y} = ly$, in the logarithmic whofe fubtangent is b; $\int \frac{a\dot{y}}{b+y} = l\overline{b+y}$, in the logarithmic whofe fubtangent is equal to a. That is, taking, in the logarithmic, the ordinate $BC = AH = y$, if to it we fhall add $HK = b$, and if we draw KG parallel to the afymptote, and draw GE parallel to AD, it will be $GE = y + b$, and then $AE = l\overline{b+y}$.

11. From the nature of the logarithmic it is plainly feen, that whenever the quantity is infinite, of which we would have the logarithm; which quantity will be reprefented by an infinite ordinate in the logarithmic; then the line intercepted in the axis, between that ordinate and the point A, will alfo be infinite, that is, the logarithm will be infinite. And if it fhall be equal to the firft ordinate AD, that is, to the fubtangent, the logarithm will then be equal to nothing. And if it fhall be lefs than AD, as if it were ΩA, the logarithm will be ΩA, and therefore negative. And laftly, if the ordinate were $= 0$, the logarithm would be negative and infinite. If the differential formula were $-\frac{\dot{y}}{y}$, the integral would be $-ly$. And if it were $-\frac{\dot{y}}{a+y}$, the integral would be $-l\overline{a+y}$. If it were $-\frac{\dot{y}}{a-y}$, the integral would be $l\overline{a-y}$; and if it were $\frac{\dot{y}}{a-y}$, the integral would be $-l\overline{a-y}$. Thefe logarithms are to be underftood in the logarithmic of which the fubtangent is unity. The reafon of this is, that as the integral of $\frac{\dot{y}}{y}$ is ly, fo the differential of ly is $\frac{\dot{y}}{y}$. And, to fpeak in general, the differential of a logarithmic quantity is that fraction, the numerator of which is the product of the fubtangent into the differential of the quantity, and the denominator is the fame quantity. Thus, the differential of $-l\overline{a+y}$ will be $-\frac{\dot{y}}{a+y}$. The differential of $l\overline{a-y}$ will be $-\frac{\dot{y}}{a-y}$. The differential of $-l\overline{a-y}$ will be $\frac{\dot{y}}{a-y}$, fuppofing the

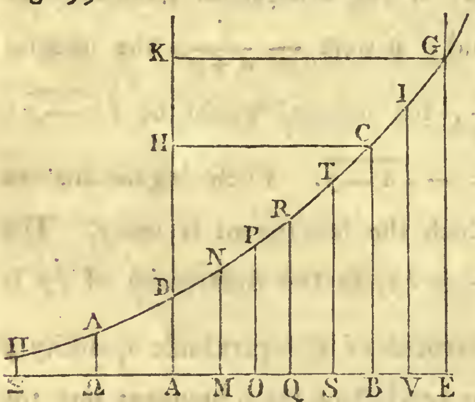
 fubtangent

ſubtangent of the logarithmic $= 1$: and whenever it is not ſo, the numerators of the differentials muſt be multiplied by the given ſubtangent.

12. But, becauſe the logarithmic has no negative ordinates, it would ſeem that we cannot find the quantity which correſponds to the expreſſion $l\,\overline{a-y}$, that is, what is the logarithm of $a-y$, when $a-y$ is a negative quantity, or when y is greater than a. But, in this caſe, it may be obſerved, that $l\,\overline{a-y}$ and $l\,\overline{y-a}$ are the ſame thing; and that in ſuch a ſuppoſition, when $y-a$ is poſitive, it may be the ordinate in the logarithmic; and, indeed, if we difference the firſt logarithm, we ſhall have $-\frac{\dot{y}}{a-y}$, and if we difference the ſecond, we ſhall have $\frac{\dot{y}}{y-a}$; and changing the ſigns of the numerator and denominator, it will be $-\frac{\dot{y}}{a-y}$, the ſame as the firſt.

13. Other properties concerning logarithmic quantities may be deduced from theſe of the logarithmic curve; and firſt, that the multiple or ſubmultiple of a logarithm ſhall be the logarithm of the quantity raiſed to the power of the given number. Thus, $2lx = lx^2$; $3lx = lx^3$; $\frac{1}{2}lx = lx^{\frac{1}{2}}$; $\frac{1}{3}lx = lx^{\frac{1}{3}}$; $nlx = lx^n$; $\frac{1}{n}lx = lx^{\frac{1}{n}}$; and the reaſon of this is, becauſe, in the logarithmic curve, if we take any ordinate whatever, $OP = y$, (Fig. 3.) whoſe logarithm is AO; if AO, OS, SV, &c. be equal to each other, then AO, AS, AV, &c. will be arithmetical proportionals, and the ordinates AD, OP, ST, VI, &c. will be geometrical proportionals. Wherefore, putting AD equal to unity, $OP = y$, it will be $ST = y^2$, $VI = y^3$, &c. But AS, the double of AO, is the logarithm of y^2, or ly^2; and AV, the triple of AO, is ly^3. So that $2ly = ly^2$, $3ly = ly^3$, &c. Thus, alſo, making $AO = ly$, and biſecting it at M, it will be $MN = y^{\frac{1}{2}}$, and therefore $AM = \frac{1}{2}AO$, that is, $\frac{1}{2}ly = ly^{\frac{1}{2}}$. In the ſame manner, making $QR = y$, and dividing AQ into three equal parts in M and O, it will be $MN = \sqrt[3]{y} = y^{\frac{1}{3}}$. But $AM = \frac{1}{3}ly$, and therefore $\frac{1}{3}ly = ly^{\frac{1}{3}}$; and, in like manner, of all others.

Fig. 99.



We

We muſt here obſerve, that the integral of $-\frac{\dot{y}}{y}$ is not only $-ly$, as was ſeen before, but may be thus expreſſed alſo, $l\frac{1}{y}$, or ly^{-1}; for, taking in the logarithmic any ordinate OP $= y$, and making $A\Omega = AO$, it will be, by the nature of the curve, OP . AD :: AD . ΩΑ; that is, $y . 1 :: 1 . \Omega A = \frac{1}{y}$. But ΩA is the negative logarithm of OP, that is, of y, and is alſo the logarithm of ΩΑ. Therefore it will be $-ly = l\frac{1}{y} = ly^{-1}$; that is to ſay, the negative logarithm of any quantity whatever will be the ſame with the poſitive logarithm of the fraction, of which the ſame quantity is the denominator, or of the ſame quantity with a negative exponent. Thus it will be $-mly = l\frac{1}{y^m} = ly^{-m}$.

14. Moreover, the ſum of two, three, &c. logarithms will be equal to the logarithm of the product of the quantities, of which they are the poſitive logarithms; and the difference of two, three, &c. logarithms ſhall be equal to the logarithm of the fraction, the numerator of which is the product of the quantities, of which they are the poſitive logarithms, and the denominator is the product of the quantities, of which they are the negative logarithms. For, becauſe it is OP $= y$, QR $= z$, it will be AO $= ly$, AQ $= lz$. Take QB $=$ AO, it will be AB $= ly + lz$. But AB is alſo the logarithm of BC, and, by the property of the logarithmic, BC is the fourth proportional to AD, OP, QR, that is, $= yz$; therefore it will be AB $= ly + lz = lzy$. Let there be another ordinate MN $= p$, and take BV $=$ AM; it will be AV $=$ AM $+$ AB $= lp + lyz$; but AV is the logarithm of VI, and VI $= pyz$. Therefore $lp + ly + lz = lpyz$.

Now make QR $= z$, OP $= y$, and take QM $=$ AO; it will be AM $=$ AQ $-$ AO $= lz - ly$. But AM is the logarithm of MN, and, by the ſame property of the logarithmic, it is MN $= \frac{z}{y}$. Therefore AM $= lz - ly = l\frac{z}{y}$. Let there be another ordinate BC $= p$, and take ΣA $=$ BM. It will be ΣA $= -$ AB $+$ AM $= -lp + l\frac{z}{y}$. But ΣA is the logarithm of ΣΠ, and ΣΠ $= \frac{z}{py}$, (becauſe it is the fourth proportional to BC, MN, AD,) therefore $lz - ly - lp = l\frac{z}{py}$.

15. As

15. As in other cafes, fo alfo in thefe integrations by means of the logarithms, fome conftant quantity fhould always be added, that is, the logarithm of an arbitrary conftant quantity, which is to be determined afterwards as particular cafes may require.

16. But when the differential formulæ propofed to be integrated are fractions with a complicated denominator, fome cafes may be given in which it is eafy to have their integrals by means of the logarithmic, and this will be as often as the numerator of the fraction fhall be the exact differential of the denominator, or as often as it is proportional to it. And, in this cafe, the integral of the formula will be the logarithm of the denominator, or it's multiple, or fubmultiple, or proportional to that logarithm.

Thus, the integral of $\frac{2x\dot{x}}{aa+xx}$ will be $l\,\overline{aa+xx}$; the integral of $-\frac{2x\dot{x}}{aa-xx}$ will be $l\,\overline{aa-xx}$; the integral of $\frac{3x^2\dot{x}}{a^3+x^3}$ will be $l\,\overline{a^3+x^3}$; the integral of $\frac{4x\dot{x}}{aa+xx}$ will be $2l\,\overline{aa+xx}$, that is, $l\,\overline{aa+xx}\rceil^2$; the integral of $\frac{x\dot{x}}{aa+xx}$ will be $\frac{1}{2}l\,\overline{aa+xx}$, or $l\,\overline{aa+xx}\rceil^{\frac{1}{2}}$; the integral of $\frac{x^2\dot{x}}{a^3+x^3}$ will be $\frac{1}{3}l\,\overline{a^3+x^3}$, or $l\sqrt[3]{a^3+x^3}$; and, in general, the integral of $\frac{mx^{n-1}\dot{x}}{a^n\pm x^n}$ will be $\pm\frac{m}{n}l\,\overline{a^n\pm x^n}$; that is, $\pm m\,l\,\overline{a^n\pm x^n}\rceil^{\frac{1}{n}}$, or $\pm l\,\overline{a^n\pm x^n}\rceil^{\frac{m}{n}}$. Thus the integral of $\frac{a\dot{x}-2x\dot{x}}{ax-xx}$ will be $l\,\overline{ax-xx}$; the integral of $\frac{\frac{1}{2}a\dot{x}-x\dot{x}}{ax-xx}$ will be $l\sqrt{ax-xx}$; and thus of all others whatever, taking thefe logarithms from the logarithmic, the fubtangent of which is $=1$.

17. But if the numerator of the fraction be not of the form we have now confidered, though the denominator may be fuch; and that no one of its linear components is imaginary; that is, when all the roots of the product from whence it arifes are real ones; then we may proceed in the following manner.

18. And, firft, the roots of the denominator are all equal to each other, or they are not. If they be all equal, as in the formula $\frac{x^m\dot{x}}{\overline{x\pm a}\rceil^n}$, make $x\pm a$ $=z$, and therefore $\dot{x}=\dot{z}$, $x^m=\overline{z\mp a}\rceil^m$, $\overline{x\pm a}\rceil^n=z^n$; and fubftituting

thefe

these values in the formula, it will be $\frac{\overline{z \mp a}|^m \times \dot{z}}{z^n}$. Wherefore, actually raising $z \mp a$ to the power m, each term can be integrated, either algebräically, or, at least, transcendentally, by means of the logarithmic. Whence, instead of z, restoring it's value given by x, we shall have the integral of the formula proposed $\frac{x^m \dot{x}}{\overline{x \pm a}|^n}$.

Let it be, for example, $\frac{x^3 \dot{x}}{\overline{x - a}|^3}$. Put $x - a = z$, and therefore $\dot{x} = \dot{z}$, $x^3 = z^3 + 3az^2 + 3aaz + a^3$, $\overline{x - a}|^3 = z^3$; and, making the substitutions, it will be $\frac{z^3\dot{z} + 3az^2\dot{z} + 3aaz\dot{z} + a^3\dot{z}}{z^3}$; and, by integration, $z + 3lz - \frac{3aa}{z} - \frac{a^3}{2zz}$; and, instead of z, restoring it's value given by x, we shall have at last $\int \frac{x^3 \dot{x}}{\overline{x - a}|^3} = x - a + l\overline{x - a}|^3 - \frac{3aa}{x - a} - \frac{a^3}{2 \times \overline{x - a}|^2}$; which integral, being differenced again, will restore the formula proposed to be integrated.

19. Now, if the roots of the denominator shall not be all equal, but either all unequal, or mixed of equal and unequal; then it will be necessary, first, to prepare the formula, by making the term of the highest power of the variable in the denominator to be positive, if it should happen to be negative, and then to free it from co-efficients, if it have any. Then, if the variable in the numerator, when there is any, be raised to a greater or equal power to the highest in the denominator, the numerator must be divided by the denominator so long, as that the exponent of the variable in that may be less than in this. Lastly, the roots of the denominator are to be found algebräically. Take this formula $-\frac{aa\dot{x}}{aa - 4xx}$ for an example. Changing the signs, and dividing by 4, it will become $\frac{\frac{1}{4}aa\dot{x}}{xx - \frac{1}{4}aa}$, that is, $\frac{\frac{1}{4}aa\dot{x}}{\overline{x - \frac{1}{2}a} \times \overline{x + \frac{1}{2}a}}$. Again, let the formula proposed be $\frac{aa\dot{x}}{2x^2 + 4ax + 2bx + 4ab}$; dividing by 2, it will be $\frac{\frac{1}{2}aa\dot{x}}{xx + 2ax + bx + 2ab}$, that is, $\frac{\frac{1}{2}aa\dot{x}}{\overline{x + 2a} \times \overline{x + b}}$. If the variable should be in the numerator, and raised to a higher power than in the denominator, we must make an actual division, by which we shall have both integers and fractions. The integers must be treated in the manner before explained; the fractions in the manner following.

20. Let the fraction be $\frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}}$; I say, this will be equal to two fractions, the numerators of which will be the same as of the first, and the denominators will be these: Of the first, it will be the product of one of the roots into the difference of the constant quantity of the other root, and of the constant quantity of the same root: Of the second, it will be the product of the other root into the difference of the constant quantity of the first root, and of the constant quantity of this second root. Thus, $\frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}} = \frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{b-2a}} + \frac{\frac{1}{2}aa\dot{x}}{\overline{x+b}\times\overline{2a-b}}$. And if the roots shall be three, four, &c. proceed always in the same method. And if the fractions found after this manner shall be reduced to a common denominator, they will restore the first fraction from which they were derived.

Now the integrals of such fractions so split, which will always be in our power to find, supposing the logarithmic curve to be given, will be the integrals of the formula proposed. Thus, it will be $\int \frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}} = \frac{\frac{1}{2}a}{2a-b}\times l\,\overline{x+b} - \frac{\frac{1}{2}a}{2a-b}\times l\,\overline{x+2a}$; that is, $\frac{\frac{1}{2}a}{2a-b}\times l\frac{x+b}{x+2a}$, or $\frac{a}{2a-b}l\sqrt{\frac{x+b}{x+2a}}$, in the logarithmic whose subtangent $= a$.

Let it be $\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{x-\frac{1}{2}a}}$; this may be split into these two, $\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{-\frac{1}{2}a-\frac{1}{2}a}} + \frac{\frac{1}{4}aa\dot{x}}{\overline{x-\frac{1}{2}a}\times\overline{\frac{1}{2}a+\frac{1}{2}a}}$, or $\frac{\frac{1}{4}a\dot{x}}{x-\frac{1}{2}a} - \frac{\frac{1}{4}a\dot{x}}{x+\frac{1}{2}a}$, and therefore it will be $\int\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{x-\frac{1}{2}a}} = \frac{1}{4}l\frac{x-\frac{1}{2}a}{x+\frac{1}{2}a}$, or $= l\sqrt[4]{\frac{x-\frac{1}{2}a}{x+\frac{1}{2}a}}$, in the logarithmic of which the subtangent $= a$.

Let it be $\frac{a^3\dot{x}}{\overline{x+a}\times\overline{x-b}\times\overline{x+c}}$; this may be split into three,

$$\frac{a^3\dot{x}}{\overline{x+a}\times\overline{-b-a}\times\overline{c-a}} + \frac{a^3\dot{x}}{\overline{x-b}\times\overline{a+b}\times\overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c}\times\overline{a-c}\times\overline{-b-c}},$$

and therefore $\int\frac{a^3\dot{x}}{\overline{x+a}\times\overline{x-b}\times\overline{x+c}} = \frac{aa}{\overline{a+b}\times\overline{a-c}}\times l\,\overline{x+a} + \frac{aa}{\overline{a+b}\times\overline{c+b}}\times l\,\overline{x-b} - \frac{aa}{\overline{a-c}\times\overline{b+c}}\times l\,\overline{x+c}$, in the logarithmic whose subtangent $= a$.

Let

Let it be $\frac{-a^3\dot{x}}{x^3 - aax}$, that is, $\frac{-a^3\dot{x}}{\overline{x+a} \times \overline{x-a} \times \overline{x+0}}$; this may be ſplit into theſe three, $\frac{-a^3\dot{x}}{\overline{x+a} \times \overline{-2a} \times \overline{0-a}} + \frac{-a^3\dot{x}}{\overline{x-a} \times \overline{2a} \times \overline{0+a}} + \frac{-a^3\dot{x}}{\overline{x+0} \times \overline{a-0} \times \overline{-a-0}}$; that is, $\frac{-a\dot{x}}{2 \times \overline{x+a}} - \frac{a\dot{x}}{2 \times \overline{x-a}} + \frac{a\dot{x}}{x}$; and therefore it will be $\int \frac{-a^3\dot{x}}{x^3 - aax} = lx - \frac{1}{2}l\,\overline{xx-aa}$, that is, $l\frac{x}{\sqrt{xx-aa}}$, in the logarithmic of ſubtangent $= a$.

22. If the denominator of the formula ſhall be mixed of equal and unequal roots, as, for example, $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}}$, then the formula muſt be conſidered as if it were $\frac{a^3\dot{x}}{\overline{x-b} \times \overline{x+c}}$, and being ſplit as uſual, it will be $\frac{a^3\dot{x}}{\overline{x-b} \times \overline{x+c}} = \frac{a^3\dot{x}}{\overline{x-b} \times \overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c} \times \overline{-b-c}}$; and then, multiplying the denominators by $\overline{x-b}$, the other root of the propoſed formula, it will be $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}} = \frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c} \times \overline{x-b} \times \overline{-b-c}}$; but the firſt term of the *homogeneum comparationis* has all the roots of it's denominator equal, and the ſecond term conſiſts of roots all unequal; ſo that, both of them being managed as before, we may have the integral of $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}}$, which will be partly algebräical, and partly logarithmical, that is, $\frac{aa}{\overline{b+c}|^2} \times l\frac{x+c}{x-b} - \frac{a^3}{\overline{x-b} \times \overline{b+c}}$; taking the logarithm from the logarithmic, whoſe ſubtangent $= a$.

If there ſhall be a greater number of equal roots, the operation muſt be repeated in the ſame manner, as often as ſhall be neceſſary.

23. That caſe remains to be conſidered, in which the fractions have alſo in the numerator the variable raiſed to any power; always meaning, as has been already obſerved, that the power of this variable in the numerator be leſs than the greateſt which is in the denominator; and not being ſo, it muſt be made ſuch by actually dividing.

In theſe caſes the formula muſt be treated in the ſame manner, as if in the numerator there were no power of the variable, ſplitting it, in the manner before explained, into ſo many parts, as are the roots of the denominator. Then, if the exponent of the variable in the numerator of the given formula be an odd

 number,

number, let the ſigns be changed in the numerators of the fractions found; and if it be an even number, their own ſigns muſt remain to the numerators. After which, every numerator muſt be multiplied by ſuch a power of the conſtant quantity of that root, which is in the denominator, as is the power of the variable in the numerator of the propoſed formula, prefixing ſuch a ſign to that conſtant, raiſed to that power, as it's natural ſign requires, which it has in the denominator.

Let the example be $\frac{bbx\dot{x}}{\overline{x+a}\times\overline{x-a}}$. This being conſidered as if there were no variable in the numerator, it will be ſplit into theſe two, $\frac{bb\dot{x}}{\overline{x+a}\times -2a}$ + $\frac{bb\dot{x}}{\overline{x-a}\times 2a}$; but, becauſe in the numerator there is the variable raiſed to the power denominated by unity, or the firſt power, the ſigns are changed in the numerators, and are multiplied relatively by the conſtant of that root which is in it's denominator, that is, the firſt by a, and the ſecond by $-a$, and we ſhall have $\frac{bbx\dot{x}}{\overline{x+a}\times\overline{x-a}} = \frac{-bb\dot{x}\times a}{\overline{x+a}\times -2a} - \frac{bb\dot{x}\times -a}{\overline{x-a}\times 2a}$, that is, $\frac{bb\dot{x}}{2\times\overline{x+a}}$ + $\frac{bb\dot{x}}{2\times\overline{x-a}}$; and therefore it will be $\int\frac{bbx\dot{x}}{\overline{x+a}\times\overline{x-a}} = bl\sqrt{x+a} + bl\sqrt{x-a}$, or $bl\sqrt{xx-aa}$, in the logarithmic of the ſubtangent $= b$. Or otherwiſe, it will be $bbl\sqrt{xx-aa}$, in the logarithmic of the ſubtangent $= 1$.

But it was needleſs to reduce this formula to two fractions; for, as it was $\frac{bbx\dot{x}}{xx-aa}$, the numerator is exactly half the differential of the denominator, and therefore, without any other operation, the integral will be $bbl\sqrt{xx-aa}$, (as is ſaid at § 17.) in the logarithmic whoſe ſubtangent is unity.

Let it be $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}}$, that is, $\frac{x^4\dot{x}}{x^3+bx^2-aax-aab}$; and dividing the numerator by the denominator, we ſhall have $x\dot{x} + \frac{-bx^3\dot{x}+aax^2\dot{x}+aabx\dot{x}}{x^3+bx^2-aax-aab}$; and dividing again the term $-bx^3\dot{x}$ by the denominator, we ſhall have $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = x\dot{x} - b\dot{x} + \frac{aax^2\dot{x}+bbx^2\dot{x}-aabb\dot{x}}{\overline{xx-aa}\times\overline{x+b}}$. Now the two firſt terms are integers, and the laſt has not the variable in the laſt term of the numerator, and therefore may be managed; ſo that there only remains the term $\frac{\overline{aa+bb}\times x^2\dot{x}}{\overline{xx-aa}\times\overline{x+b}}$ ſtill to be reduced. This being conſidered as not having the variable

variable in the numerator, will be $\frac{\overline{aa+bb}\times\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = \frac{\overline{aa+bb}\times\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$; and therefore it will be $\frac{\overline{aa+bb}\times x^2\dot{x}}{\overline{x+b}\times\overline{xx-aa}} = \frac{\overline{aa+bb}\times bb\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times a^2\dot{x}}{\overline{x-a}\times\overline{2ab+2a^2}}$. Whence, lastly, $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = x\dot{x} - b\dot{x} - \frac{aabb\dot{x}}{\overline{x+b}\times\overline{xx-aa}} + \frac{\overline{aa+bb}\times bb\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$; and if we would ſtill ſplit the term $-\frac{aabb\dot{x}}{\overline{x+b}\times\overline{xx-aa}}$, in order to have, finally, the integral of the propoſed formula, it will be $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = x\dot{x} - b\dot{x} + \frac{b^4\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{a^4\dot{x}}{\overline{x+a}\times\overline{2aa-2ab}} + \frac{a^4\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$. Then, by integration, we ſhall have $\int\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = \frac{1}{2}xx - bx - \frac{b^4}{aa-bb}\times l\overline{x+b} + \frac{a^4}{2aa-2ab}\times l\overline{x+a} + \frac{a^4}{2aa+2ab}\times l\overline{x-a}$; taking ſuch logarithms in the logarithmic of the ſubtangent $= 1$.

Now in this, as well as in all other integrations that can be made, we are to conceive a conſtant quantity is to be added, though, for the ſake of brevity, I here omit it; but it will be enough to mention it here.

24. But differential formulæ may have, and often have, ſuch denominators, of which we cannot find the roots algebräically; yet, notwithſtanding this, we may make good uſe of the Rule of Fractions in theſe caſes alſo. For we may treat the denominator as if it were an equation, and, by means of the interſections of curves, may be found geometrically, in lines, the values of the variable, juſt after the ſame manner as ſolid problems are conſtructed. And ſuch values or lines may be called A, B, C, &c. with poſitive or negative ſigns, according as they come out poſitive or negative. Every one of theſe, being ſubtracted from the variable, will form a root of the denominator in ſuch manner, that the propoſed differential formula will be converted into one of this form, $\frac{x^n\dot{x}}{\overline{x-A}\times\overline{x+B}\times\overline{x-C}, \&c}$, and with this we may proceed in the ſame manner, as the operation has been performed in the caſe of algebräical roots.

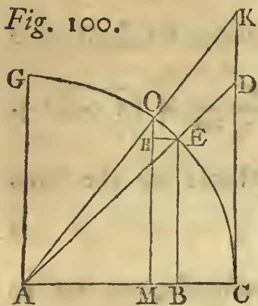
25. It

25. It may be eaſily obſerved, that the rule here produced ſerves only in ſuch caſes, when the roots of the denominator are real; for when it is otherwiſe, the formula being ſplit into other fractions, ſo many of theſe will be imaginary, (and conſequently the integrals will be imaginary,) as are the imaginary roots in the denominator of the differential formula propoſed.

26. Therefore, when the denominator of the propoſed differential formula is compoſed of imaginary roots, either wholly or in part, there is a neceſſity of having recourſe to other means. And, in the firſt place, let the given formulæ have their denominators of two dimenſions only, that is, of two imaginary roots; and let it be, for example, $\frac{bb\dot{x}}{xx + aa}$.

The integral of this formula, and of all others like it, depends on the rectification or quadrature of the circle; I ſay rectification or quadrature, becauſe, one of them being given, the other is reciprocally given alſo.

Fig. 100.



Wherefore let ACG be a quadrant of a circle, the radius AC $= a$, the tangent CD $= x$; it will be AB $= \frac{aa}{\sqrt{aa + xx}}$, CB $= a - \frac{aa}{\sqrt{aa + xx}}$, EB $= \frac{ax}{\sqrt{aa + xx}}$.

Drawing AK infinitely near to AD, then EO will be the fluxion or difference of the arch CE. And from the point O drawing the right line OM parallel to EB, and EH parallel to AC, then will HE be the differential of CB, and HO the differential of EB, and therefore EH $= \frac{aax\dot{x}}{\overline{aa + xx}|^{\frac{3}{2}}}$, and HO $= \frac{a^3\dot{x}}{\overline{aa + xx}|^{\frac{3}{2}}}$. Thence the little arch EO $= \sqrt{HEq + OHq}$, will be $= \sqrt{\frac{a^6\dot{x}\dot{x} + a^4xx\dot{x}\dot{x}}{\overline{aa + xx}|^3}} = \frac{aa\dot{x}}{aa + xx}$. Whence the integral of the formula $\frac{aa\dot{x}}{aa + xx}$ will be the arch CE of the tangent CD $= x$, and of radius CA $= a$.

Now I reſume the formula $\frac{bb\dot{x}}{aa + xx}$; multiplying the numerator and denominator by aa, it will be $\frac{bb}{aa} \times \frac{aa\dot{x}}{aa + xx}$; but the integral of $\frac{aa\dot{x}}{aa + xx}$ is the circular arch, which has for it's tangent x, and it's radius $= a$; therefore $\int \frac{bb\dot{x}}{aa + xx}$ $=$ to the fourth proportional of aa, of bb, and of the arch of the circle with radius $= a$, and tangent $= x$.

 Let

Let the formula be $\frac{aam\dot{x}}{nxx + nab}$; as, by multiplying the numerator and denominator by b, it will be equivalent to this other, $\frac{am}{nb} \times \frac{ab\dot{x}}{xx + ab}$; it will be $\int\frac{aam\dot{x}}{nxx + nab}$ = to a fourth proportional to nb, to am, and to the arch of a circle, with radius $= \sqrt{ab}$, and tangent $= x$. And ſo of all others of a like kind.

27. On the contrary, therefore, the differential of any arch of a circle is the product of the ſquare of the radius into the fluxion of the tangent, divided by the ſum of the ſquares of the ſaid radius, and the ſquare of the tangent.

And, as a conſtant quantity is always to be joined to other integrals or fluents, ſo alſo to this of the rectification of the circle; to have the integral complete, we muſt add a conſtant arch of the ſame circle; for the difference by which the arch, thus compoſed of a variable and a conſtant, can increaſe or diminiſh, can never be any other than what belongs to the differential of the variable arch; ſo that to the ſame differential may belong, by way of integral, the ſum of the variable arch, together with any conſtant arch of the ſame circle. Let us ſuppoſe that x is the tangent of an arch of a circle whoſe radius is a, and that b is the tangent of another conſtant arch of the ſame circle; we know that the tangent of the ſum of theſe two arches (Vol. I. § 108.) will be $= \frac{aab + aax}{aa - bx}$. But the differential of this, multiplied by the ſquare of the radius, and the product divided by the ſquare of the radius, adding the ſquare of the ſame tangent, is found to be $\frac{aa\dot{x}}{aa + xx}$, which is the differential of the variable arch.

Let the formula be $\frac{aa\dot{x}}{aa + xx - 2bx + bb}$, in which $xx - 2bx + bb$ is a ſquare. Make $x - b = z$, and, by ſubſtitution, we ſhall have $\frac{aa\dot{z}}{aa + zz}$. Therefore $\int\frac{aa\dot{z}}{aa + zz}$ = arch of a circle with radius $= a$, and tangent $= z$. But $z = x - b$; therefore $\int\frac{aa\dot{x}}{aa + xx - 2bx + bb}$ = arch of a circle with radius $= a$, with tangent $= x - b$, when x is greater than b. But, taking x leſs than b, the integral will be *minus* the arch of the circle, with the ſame radius and tangent. And, indeed, by differencing, we ſhould have $\frac{aa\dot{x}}{aa + bb - 2bx + xx}$, the ſame formula as at firſt.

Let

Let this formula be propofed, $\frac{4ab\dot{x} + 3bx\dot{x}}{xx - 4ax + 6aa}$. Make the fecond term of the denominator to vanifh, by putting $x = y + 2a$. Making the fubftitutions, it will be $\frac{4ab\dot{y} + 3by\dot{y} + 6ab\dot{y}}{yy + 2aa}$, that is, $\frac{10ab\dot{y}}{yy + 2aa} + \frac{3by\dot{y}}{yy + 2aa}$.

Therefore the integral of the firft term will be a third proportional to a, to $5b$, and to the arch of a circle with radius $= \sqrt{2aa}$, and with tangent $= y$: Of the fecond, it will be $l\,\overline{yy + 2aa}|^{\frac{3}{2}}$, in the logarithmic of fubtangent $= b$. Then, inftead of y, fubftituting it's value $x - 2a$, the integral of the formula $\frac{4ab\dot{x} + 3bx\dot{x}}{xx - 4ax + 6aa}$ will be the third proportional of a, $5b$, and the arch of the circle with radius $= \sqrt{2aa}$, with tangent $= x - 2a$; with $l\,\overline{xx - 4ax + 6aa}|^{\frac{3}{2}}$ alfo, in the logarithmic of fubtangent $= b$.

28. We will proceed now to fuch differential formulæ, as contain radical figns, that is, quantities raifed to a power with a fraction for it's exponent. If the formula either is, or may be reduced to fuch, that the variable quantity under the radical does not exceed the firft dimenfion; and out of the radical is a pofitive power; then fuch formulæ will always be integrable algebräically, and will obtain their integrations by making ufe of a very fimple fubftitution; and that is, by putting the quantity under the vinculum equal to a new variable.

Wherefore let the formula be $a\dot{x}\sqrt{ax - aa}$. Put $\sqrt{ax - aa} = z$, and therefore $x = \frac{zz + aa}{a}$, $\dot{x} = \frac{2z\dot{z}}{a}$; and, making the fubftitutions, we fhall have $2zz\dot{z}$, and, by integration, $\frac{2}{3}z^3$; and, inftead of z, reftoring it's value given by x, it will be $\frac{2}{3} \times \overline{ax - aa}|^{\frac{3}{2}}$, the integral of the propofed formula.

If the given formula were $\frac{a\dot{x}}{\sqrt{ax - aa}}$, by proceeding after the fame manner we fhould have $2 \times \overline{ax - aa}|^{\frac{1}{2}}$ for the integral.

Let it be $x\dot{x}\sqrt[4]{a - x}$; putting $\sqrt[4]{a - x} = z$, and therefore $x = a - z^4$, and $\dot{x} = -4z^3\dot{z}$; and making the fubftitutions, we fhould have $4z^8\dot{z} - 4az^4\dot{z}$; and by integrating, $\frac{4}{9}z^9 - \frac{4}{5}az^5$; and, inftead of z, reftoring it's value given by x, it will be $\frac{4}{9} \times \overline{a - x}|^{\frac{9}{4}} - \frac{4a}{5} \times \overline{a - x}|^{\frac{5}{4}}$.

If

If the formula were $\frac{x\dot{x}}{\sqrt[4]{a-x}}$, proceeding after the ſame manner, we ſhould have the integral $\frac{4}{7} \times \overline{a-x}|^{\frac{7}{4}} - \frac{4a}{3} \times \overline{a-x}|^{\frac{3}{4}}$.

Let it be $x^2\dot{x}\sqrt{a+x}$; make $\sqrt{a+x} = z$, and therefore $x = z^2 - a$, and $\dot{x} = 2z\dot{z}$, and $xx = \overline{zz-a}|^2$; and making the ſubſtitutions, we ſhall have $\overline{zz-a}|^2 \times 2zz\dot{z}$, that is, $2z^6\dot{z} - 4az^4\dot{z} + 2aaz^2\dot{z}$; and, by integration, $\frac{2}{7}z^7 - \frac{4}{5}az^5 + \frac{2}{3}aaz^3$; and, inſtead of z, reſtoring it's value given by x, it will be, laſtly, $\frac{2}{7} \times \overline{a+x}|^{\frac{7}{2}} - \frac{4}{5}a \times \overline{a+x}|^{\frac{5}{2}} + \frac{2}{3}aa \times \overline{a+x}|^{\frac{3}{2}}$, the integral required.

If the formula were $\frac{xx\dot{x}}{\sqrt{a+x}}$, the integral would be $\frac{2}{5} \times \overline{a+x}|^{\frac{5}{2}} - \frac{4a}{3} \times \overline{a+x}|^{\frac{3}{2}} + 2a^2 \times \overline{a+x}|^{\frac{1}{2}}$.

Let it be $x\dot{x}\sqrt{\overline{a+x}|^3}$, that is, $x\dot{x} \times \overline{a+x}|^{\frac{3}{2}}$. Make, as uſual, $\overline{a+x}|^{\frac{3}{2}} = z$, and therefore $x = z^{\frac{2}{3}} - a$, $\dot{x} = \frac{2}{3}z^{\frac{2}{3}-1}\dot{z}$; and making the ſubſtitutions, it will be $\overline{z^{\frac{2}{3}} - a} \times \frac{2}{3}z^{\frac{2}{3}}\dot{z}$, that is, $\frac{2}{3}z^{\frac{4}{3}}\dot{z} - \frac{2}{3}az^{\frac{2}{3}}\dot{z}$; and integrating, $\frac{2}{7}z^{\frac{7}{3}} - \frac{2}{5}az^{\frac{5}{3}}$; and, inſtead of z, ſubſtituting it's value, it will be $\frac{2}{7} \times \overline{a+x}|^{\frac{7}{2}} - \frac{2}{5}a \times \overline{a+x}|^{\frac{5}{2}}$.

If the formula were $\frac{x\dot{x}}{\overline{a+x}|^{\frac{3}{2}}}$, we ſhould have for it's integral $2\sqrt{a+x} + \frac{2a}{\sqrt{a+x}}$.

29. In general, let it be $ax^t\dot{x} \times \overline{a+x}|^{\frac{m}{n}}$, and let the exponents t, m, n, be poſitive integers; make, as uſual, $\overline{a+x}|^{\frac{m}{n}} = z$, and therefore $a + x = z^{\frac{n}{m}}$, $\dot{x} = \frac{n}{m}z^{\frac{n}{m}-1}\dot{z}$, $x^t = \overline{z^{\frac{n}{m}} - a}|^t$; and making the ſubſtitutions, the formula will be $\overline{z^{\frac{n}{m}} - a}|^t \times \frac{n}{m}az^{\frac{n}{m}}\dot{z}$; and actually raiſing $z^{\frac{n}{m}} - a$ to the power t, it is plain that every term will be algebräically integrable; in which terms, being integrated, inſtead of z, reſtore it's value given by x, and we ſhall have the algebräical integral of the propoſed formula.

30. If

30. If the exponent m were negative, ſo that the quantity under the vinculum would paſs into the denominator, in which caſe the exponent m would then become poſitive; that is, if the formula were $\frac{ax^t\dot{x}}{\overline{a+x}|^{\frac{m}{n}}}$; making the ſame ſubſtitutions, we ſhould have $\overline{z^{\frac{n}{m}} - a}|^t \times \frac{n}{m} az^{\frac{n}{m}-2}\dot{z}$; and actually raiſing $z^{\frac{n}{m}} - a$ to the power t, every term would then be algebräically integrable, excepting ſuch caſes in which the power $z^{-1}\dot{z}$ ſhould inſinuate itſelf, and then we ſhould be obliged to have recourſe to the logarithms.

But if the exponent t were negative, the two foregoing formulæ would not then be algebräically integrable, but might be freed from their radicals, and reduced to the quadrature of the circle and hyperbola, as will be ſeen in it's place.

31. But when the variable under the vinculum is raiſed to any power greater than unity, provided the quantity out of the vinculum is the exact differential, or any proportional to the differential, of the quantity under the vinculum; then, by means of the ſaid very ſimple ſubſtitution, we might have the integral of the differential formula, which ſaid integrals will always be algebräical.

Wherefore let the formula be $2x\dot{x}\sqrt{xx + aa}$; make $\sqrt{xx + aa} = z$, whence $xx + aa = zz$, $2x\dot{x} = 2z\dot{z}$; and making the ſubſtitutions, we ſhall have $2zz\dot{z}$, and integrating, $\frac{2}{3}z^3$; and reſtoring the value of z, it will be $\frac{2}{3} \times \overline{xx + aa}|^{\frac{3}{2}}$.

If the formula were $\frac{2x\dot{x}}{\sqrt{xx + aa}}$, we ſhould have for the integral $2\sqrt{xx + aa}$.

Let it be $\overline{2a\dot{x} - 4x\dot{x}} \times \sqrt{ax - xx + bb}$, that is, $2 \times \overline{a\dot{x} - 2x\dot{x}} \times \sqrt{ax - xx + bb}$; put $\sqrt{ax - xx + bb} = z$, and therefore $ax - xx + bb = zz$, and $a\dot{x} - 2x\dot{x} = 2z\dot{z}$; and making the ſubſtitutions, we ſhall have $4zz\dot{z}$, and integrating, it will be $\frac{4}{3}z^3$; and, inſtead of z, reſtoring it's value, it is $\frac{4}{3} \times \overline{ax - xx + bb}|^{\frac{3}{2}}$.

Let the formula be $\frac{2a\dot{x} - 4x\dot{x}}{\sqrt{ax - xx + bb}}$; it's integral will be $4 \times \overline{ax - xx + bb}|^{\frac{1}{2}}$.

Let it be $\overline{xx\dot{x} - \frac{2}{3}ax\dot{x}} \times \sqrt[4]{x^3 - ax^2}$, that is, $\frac{3xx\dot{x} - 2ax\dot{x}}{3} \times \sqrt[4]{x^3 - ax^2}$; make $\sqrt[4]{x^3 - ax^2} = z$, and therefore $z^4 = x^3 - ax^2$, and $3xx\dot{x} - 2ax\dot{x}$ =

$= 4z^3\dot{z}$; and making the fubftitutions, we fhall have $\frac{4}{3}z^4\dot{z}$, and by integrating, $\frac{4}{15}z^5$; and, inftead of z, reftoring it's value, it will be $\frac{4}{15} \times \overline{x^3 - axx}|^{\frac{5}{4}}$.

If the formula were $\frac{3xx\dot{x} - 2ax\dot{x}}{3\sqrt[4]{x^3 - axx}}$, the integral would be $\frac{4}{9} \times \overline{x^3 - axx}|^{\frac{3}{4}}$.

Let it be $2x\dot{x}\sqrt[3]{\overline{xx + aa}|^2}$, that is, $2x\dot{x} \times \overline{xx + aa}|^{\frac{2}{3}}$; put $\overline{xx + aa}|^{\frac{2}{3}} = z$, and therefore $xx + aa = z^{\frac{3}{2}}$, and $2x\dot{x} = \frac{3}{2}z^{\frac{3}{2}-1}\dot{z}$; and making the fubftitutions, we fhall have $\frac{3}{2}z^{\frac{3}{2}}\dot{z}$, and by integration, $\frac{3}{5}z^{\frac{5}{2}}$; and, inftead of z, reftoring it's value, $\frac{3}{5} \times \overline{xx + aa} \times \sqrt[3]{\overline{xx + aa}|^2}$.

If the formula were $\frac{2x\dot{x}}{\sqrt[3]{\overline{xx + aa}|^2}}$, the integral would be $3\sqrt[3]{xx + aa}$.

And, in general, let the formula be $px^{m-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$, in which p and m may alfo be fractions; put $\overline{x^m + a^m}|^{\frac{n}{u}} = z$, and therefore $z^{\frac{u}{n}} = x^m + a^m$, and $mx^{m-1}\dot{x} = \frac{u}{n}z^{\frac{u}{n}-1}\dot{z}$; and making the fubftitutions, we fhall have $\frac{pu}{mn}z^{\frac{u}{n}}\dot{z}$, and by integration, $\frac{pu}{mu + mn} \times z^{\frac{u+n}{n}}$; and, inftead of z, reftoring it's value, the integral will be $\frac{pu}{mu + mn} \times \overline{x^m + a^m} \times \overline{x^m + a^m}|^{\frac{n}{u}}$.

If n were negative, or if the formula were $\frac{px^{m-1}\dot{x}}{\overline{x^m + a^m}|^{\frac{n}{u}}}$, in which n is now pofitive, we fhould have the integral $\frac{pu}{mu - mn} \times \overline{x^m + a^m}|^{\frac{u-n}{u}}$.

Hence we may form this general rule, that the integral of fuch a formula will be the quantity under the vinculum, the exponent being increafed by unity, and dividing it by the exponent fo increafed; or the integral will be a proportional to this, according to the proportion which the differential quantity out of the vinculum will have to the precife differential.

32. But ſtill in a more general manner: Let the formula be $px^{rm-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$, ſuppoſing r to be a poſitive integer. It will be equivalent to this other, $px^{rm-m} \times x^{m-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$; make, as uſual, $z = \overline{x^m + a^m}|^{\frac{n}{u}}$, and therefore $x^m + a^m = z^{\frac{u}{n}}$, and $mx^{m-1}\dot{x} = \frac{u}{n} z^{\frac{u}{n}-1}\dot{z}$; and, becauſe $x^m = z^{\frac{u}{n}} - a^m$, it will be $x^{rm-m} = \overline{z^{\frac{u}{n}} - a^m}|^{r-1}$. Therefore, making the ſubſtitutions, we ſhall have $p \times \overline{z^{\frac{u}{n}} - a^m}|^{r-1} \times \frac{u}{mn} z^{\frac{u}{n}}\dot{z}$. Now, ſuppoſing r to be a poſitive integer number, then alſo $r - 1$ will be a poſitive integer number; and actually raiſing $z^{\frac{u}{n}} - a^m$ to the power $r - 1$, each term will be algebräically integrable, in which integral reſtoring, inſtead of z, it's value given by x, we ſhall have the integral required.

If n were negative, that is, if the formula were $\frac{px^{rm-1}\dot{x}}{\overline{x^m + a^m}|^{\frac{n}{u}}}$, in which n is now poſitive, making the ſubſtitutions, it will be $p \times \overline{z^{\frac{u}{n}} - a^m}|^{r-1} \times \frac{u}{mn} z^{\frac{u}{n}-2}\dot{z}$, which is likewiſe integrable.

In all theſe caſes, if the quantity under the vinculum, inſtead of being $x^m + a^m$, had been $x^m - a^m$, or $a^m - x^m$, we might proceed after the ſame manner, without hindering the operation.

By this method we may find likewiſe, that it will be

$$\int ax^{m-1}\dot{x} \times \sqrt{e + fx^m} = \frac{2a}{3mf} \times \overline{e + fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{m-1}\dot{x}}{\sqrt{e + fx^m}} = \frac{2a}{mf} \times \overline{e + fx^m}|^{\frac{1}{2}}.$$

$\int ax$

$$\int ax^{2m-1}\dot{x}\sqrt{e+fx^m} = -\frac{4e-6fx^m}{15mff} \times a \times \overline{e+fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{2m-1}\dot{x}}{\sqrt{e+fx^m}} = -\frac{4e-2fx^m}{3mff} \times a \times \overline{e+fx^m}|^{\frac{1}{2}}.$$

$$\int ax^{3m-1}\dot{x}\sqrt{e+fx^m} = a \times \frac{16ee-24efx^m+30ffx^{2m}}{105f^3m} \times \overline{e+fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{3m-1}\dot{x}}{\sqrt{e+fx^m}} = \frac{16ee-8efx^m+6ffx^{2m}}{15mf^3} \times a \times \overline{e+fx^m}|^{\frac{1}{2}}.$$

And ſo we might go on as far as we pleaſe.

33. Likewiſe in the caſe, in which the variable out of the vinculum ſhall be in the denominator, the formula will be algebräically integrable by the help of two ſubſtitutions, provided the exponent of that variable out of the vinculum ſhall have a certain condition; thus, let the formula be $\dfrac{\dot{x} \times \overline{x^m+a^m}|^{\frac{n}{u}}}{x^{rm+\frac{mn}{u}+1}}$.

Then make $x = \dfrac{aa}{y}$, $\dot{x} = -\dfrac{aa\dot{y}}{yy}$, $x^m = \dfrac{a^{2m}}{y^m}$, $\overline{x^m+a^m}|^{\frac{n}{u}} = \dfrac{\overline{a^{2m}+a^m y^m}|^{\frac{n}{u}}}{y^{\frac{mn}{u}}}$.

Then making the ſubſtitutions, the formula will be

$$\frac{-\dfrac{aa\dot{y}}{yy} \times \dfrac{\overline{a^{2m}+a^m y^m}|^{\frac{n}{u}}}{y^{\frac{mn}{u}}} \times y^{rm+\frac{mn}{u}+1}}{a^{2rm+\frac{2mn}{u}+2}}, \text{ that is, } -y^{rm-1}\dot{y} \times \frac{\overline{a^{2m}+a^m y^m}|^{\frac{n}{u}}}{a^{2rm+\frac{2mn}{u}}};$$

a formula which has the conditions here required, and which may be integrated algebräically, by means of the ſubſtitution mentioned at § 32.

If the formula propoſed were $\dfrac{a^5\dot{x}}{x^4\sqrt{ax+xx}}$, that is, $\dfrac{a^5\dot{x}}{x^{\frac{9}{2}}\sqrt{a+x}}$; this having the conditions required, will be algebräically integrable; which is alſo to be obſerved of others.

34. But

34. But here it may be obſerved, that, in the general formula, it may alſo be $u = 1$, in which caſe the power of $x^m + a^m$ will be rational, that is, integrable. [qu. integral.]

Alſo, in this caſe, ſuppoſing n to be a negative number, (for when it is affirmative there will be no difficulty,) we may make uſe of the ſame ſubſtitution, and of the ſame method, by which the integrals may be found of ſuch formulæ, the integrals of which will not always be algebräical. For very often they will depend in part upon the quadrature of the hyperbola, that is, on the logarithmic curve.

Therefore, by a known method, we ſhall find that

$$\int \frac{x^{m-1}\dot{x}}{\overline{x^m + a^m}^{\,2}} = -\frac{1}{m} \times \overline{x^m + a^m}^{\,-1},$$

$$\int \frac{x^{2m-1}\dot{x}}{\overline{x^m + a^m}^{\,2}} = \frac{1}{m}\, l\,\overline{a^m + x^m} + \frac{a^m}{m \times \overline{a^m + x^m}}.$$

$$\int \frac{x^{3m-1}\dot{x}}{\overline{x^m + a^m}^{\,2}} = \frac{a^m + x^m}{m} - \frac{2a^m\, l\,\overline{a^m + x^m}}{m} - \frac{a^{2m}}{m \times \overline{a^m + x^m}}.$$

$$\int \frac{x^{m-1}\dot{x}}{\overline{a^m + x^m}^{\,3}} = -\frac{1}{2m \times \overline{a^m + x^m}^{\,2}}.$$

$$\int \frac{x^{2m-1}\dot{x}}{\overline{a^m + x^m}^{\,3}} = -\frac{1}{m \times \overline{a^m + x^m}} + \frac{a^m}{2m \times \overline{a^m + x^m}^{\,2}}.$$

$$\int \frac{x^{3m-1}\dot{x}}{\overline{a^m + x^m}^{\,3}} = \frac{1}{m}\, l\,\overline{a^m + x^m} + \frac{2a^m}{m \times \overline{a^m + x^m}} - \frac{a^{2m}}{2m \times \overline{a^m + x^m}^{\,2}}. \quad \text{\&c.}$$

35. But the manner of proceeding will be very different when the propoſed differential formulæ containing the radical, are not ſuch as that the quantity out of the vinculum ſhall have thoſe conditions before mentioned. Theſe formulæ may always be delivered from their radical, provided they contain but one, which is that of the ſquare-root, and that the variable under the ſame does not exceed two dimenſions. Now, for theſe there will be occaſion for ſome caution in the choice of ſuch ſubſtitutions as are to be made, that they may be freed from radical ſigns. When this is done, we may go on to integrations, either algebräical, or ſuch as depend on the quadrature of the circle or hyperbola, after the manner already explained, if they come under the given rules. If not, we muſt have recourſe to other methods, which are to be given hereafter.

If the radical of the propofed formula were $\sqrt{ax \pm xx}$, or $\sqrt{xx \pm ax}$; this radical may be made equal to $\frac{xz}{b}$, meaning by z a new variable, and by b any conftant quantity whatever.

If the radical were $\sqrt{aa \pm xx}$, make it $= x + z$, or $x - z$.

If the radical were $\sqrt{aa - xx}$, or $\sqrt{fp - xx}$, put the radical $= \sqrt{fp} + \frac{xz}{b}$, or $= \sqrt{fp} - \frac{xz}{b}$. From fuch equations the values of x and $\dot{x}$ may be derived, expreffed by z and conftant quantities; which values are to be fubftituted in the given formulæ, and we fhall have other formulæ free from radicals, and given by z. In the integrations of which, if they can be had, the value of z by x being reftored, we fhall have the integrations of the propofed formulæ.

36. If the quantity fhould have three terms, that is, the fquare of the variable with the rectangle of the fame into a conftant, and befides, a term which is wholly conftant; then either the fecond term muft be taken away, after the ufual manner, as in the common Algebra; or, if the conftant term be pofitive, as in $\sqrt{xx + ax + aa}$ for inftance, however the others may be pofitive or negative, provided the quantity be not imaginary; make $\sqrt{xx + ax + aa} = a + \frac{xz}{b}$; and if the conftant term be negative, as, fuppofe $\sqrt{xx + ax - aa}$, it may be made $\sqrt{xx + ax - aa} = x + z$.

From hence it may be feen, that the whole artifice confifts in comparing the radical quantity to fuch other quantity compofed of the given variable, and of a new one with conftant quantities, as that an equation may refult from thence, from whence we may have the value of x and of $\dot{x}$, free from radical figns.

Let there be propofed to be integrated the differential formula $x^3\dot{x}\sqrt{ax - xx}$. Put $\sqrt{ax - xx} = \frac{xz}{b}$, and therefore $a - x = \frac{xzz}{bb}$, that is, $x = \frac{abb}{zz + bb}$, and $\dot{x} = -\frac{2abbz\dot{z}}{\overline{zz + bb})^2}$, $x^3 = \frac{a^3b^6}{\overline{zz + bb})^3}$, and $\sqrt{ax - xx} = \frac{xz}{b} = \frac{abz}{zz + bb}$. Make the fubftitutions in the propofed formula, and it will be $-\frac{2a^5b^9zz\dot{z}}{\overline{zz + bb})^6}$, a formula which, though free from radical figns, yet, as to it's integration, will not fubmit to the ufual methods.

Let

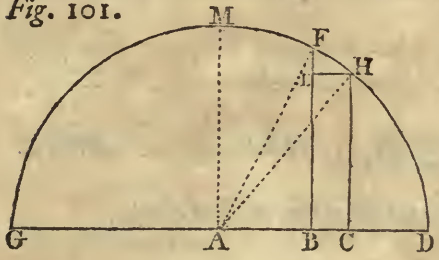
Let it be $\frac{aa\dot{x}}{x\sqrt{ax+xx}}$. Make $\sqrt{ax+xx} = \frac{xz}{b}$, and therefore it will be $x = \frac{abb}{zz-bb}$, $\dot{x} = -\frac{2abbz\dot{z}}{\overline{zz-bb}|^2}$, $\sqrt{ax+xx} = \frac{xz}{b} = \frac{abz}{zz-bb}$. Making the ſubſtitutions in the propoſed formula, it will be $-\frac{2a\dot{z}}{b}$, and by integration, $-\frac{2az}{b}$; and, inſtead of z, reſtoring it's value by x, it is $\int\frac{aa\dot{x}}{x\sqrt{ax+xx}} = -\frac{2a\sqrt{ax+xx}}{x}$.

Let it be $\frac{x\dot{x}}{\sqrt{ax+xx}}$; put $\sqrt{ax+xx} = \frac{xz}{b}$, and making the neceſſary ſubſtitutions as before, the formula will be $-\frac{2ab^3\dot{z}}{\overline{zz-bb}|^2}$, that is, $-\frac{2ab^3\dot{z}}{\overline{z+b}|^2\times\overline{z-b}|^2}$. But we have already ſeen how to manage this by the Rule of Fractions, and it will have for it's fluent $\frac{abz}{zz-bb} + \frac{1}{2}a\,l\frac{z-b}{z+b}$, in the logarithmic the ſubtangent of which is unity. And, inſtead of z, reſtoring it's value by x, it will be $\int\frac{x\dot{x}}{\sqrt{ax+xx}} = \sqrt{ax+xx} + \frac{1}{2}al\frac{\sqrt{ax+xx}-x}{\sqrt{ax+xx}+x}$, in the logarithmic of the ſame ſubtangent $= 1$.

Let it be $\frac{x\dot{x}}{\sqrt{xx+ax-aa}}$. Make $\sqrt{xx+ax-aa} = x+z$, and therefore it will be $x = \frac{zz+aa}{a-2z}$, $\dot{x} = \frac{2az\dot{z}-2zz\dot{z}+2aa\dot{z}}{\overline{a-2z}|^2}$, and $\sqrt{xx+ax-aa} = x+z = \frac{aa+az-zz}{a-2z}$. Make the ſubſtitutions, and the propoſed formula will be $\frac{\overline{zz+aa}\times 2\dot{z}}{\overline{a-2z})^2}$, that is, $\frac{2zz\dot{z}+2aa\dot{z}}{\overline{a-2z}|^2}$; and by integration, (which may be performed by the foregoing rules, it is $\frac{5aa}{4\times\overline{a-2z}} - \frac{1}{4}a + \frac{1}{2}z + \frac{1}{2}al\,\overline{a-2z}$, in the logarithmic with ſubtangent $= 1$. And, inſtead of z, reſtoring it's value by x, it will be, laſtly, $\int\frac{x\dot{x}}{\sqrt{xx+ax-aa}} = \frac{5aa}{4a+8x-8\sqrt{xx+ax-aa}} - \frac{1}{4}a - \frac{1}{2}x + \frac{1}{2}\sqrt{xx+ax-aa} + \frac{1}{2}al\,\overline{a+2x-2\sqrt{xx+ax-aa}}$, in the logarithmic whoſe ſubtangent is unity.

37. As to ſome radical differential formulæ, the trouble, indeed, would be ſuperfluous to tranſmute them, by means of theſe ſubſtitutions, into others that are free from radical ſigns, in order to prepare them for integration; and ſuch are all thoſe which of their own nature require the quadrature or rectification of the circle. Wherefore let there be a ſemicircle GMD, (Fig. 101.) it's radius AD $= a$, AB $= x$, whence BF $= \sqrt{aa - xx}$; and drawing CH infinitely near to BF, it will be BC $= \dot{x}$, EF $= \frac{x\dot{x}}{\sqrt{aa - xx}}$. Therefore the expreſſion of the infiniteſimal rectangle BCHE will be $\dot{x}\sqrt{aa - xx}$, and therefore $\int \dot{x}\sqrt{aa - xx}$ is equal to the ſpace ABFM.

Fig. 101.



Alſo, $\frac{a\dot{x}}{\sqrt{aa - xx}}$ will be the expreſſion of the infinitely little arch FH, and therefore $\int \frac{a\dot{x}}{\sqrt{aa - xx}} =$ arch MF. And if the little arch FH be drawn into half the radius, then $\frac{aa\dot{x}}{2\sqrt{aa - xx}}$ will be the expreſſion of the infinitely little ſector AFH, and therefore $\int \frac{aa\dot{x}}{2\sqrt{aa - xx}} =$ to the ſector AFM.

In the ſame circle let it be now DC $= x$, and CB $= \dot{x}$. It will be CH $= \sqrt{2ax - xx}$, EF $= \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$. Wherefore $\int \dot{x}\sqrt{2ax - xx}$ will be equal to the ſpace HCD. And thus $\int \frac{a\dot{x}}{\sqrt{2ax - xx}} =$ arch HD, and $\int \frac{aa\dot{x}}{2\sqrt{2ax - xx}} =$ ſector AHD. In ſuch as theſe, therefore, the trouble [of transformation] would be needleſs; for, in the firſt caſe, we ſhould make $\sqrt{aa - xx} = a - \frac{xz}{b}$, and therefore $x = \frac{2abz}{zz + bb}$, $\dot{x} = \frac{2ab^3\dot{z} - 2abzz\dot{z}}{\overline{zz + bb}|^2}$, $\sqrt{aa - xx} = a - \frac{xz}{b} = \frac{abb - azz}{zz + bb}$. Now, making theſe ſubſtitutions, it will be $\frac{a\dot{x}}{\sqrt{aa - xx}} = \frac{2ab\dot{z}}{zz + bb}$; a formula for the rectification of the circle, the tangent of which is equal to z, as has been ſeen already at § 26.

Alſo,

Alfo, let it be $\frac{aa\dot{x}}{2\sqrt{ax - xx}} = \frac{aab\dot{z}}{zz + bb}$, a formula which requires the fame rectification. In like manner, it will be $\dot{x}\sqrt{aa - xx} = \frac{2aab\dot{z} \times (bb - zz)^2}{(zz + bb)^3}$, a formula which, though at prefent we cannot manage, yet afterwards we fhall find to depend on the fame circle.

In the fecond cafe, I put $\sqrt{2ax - xx} = \frac{xz}{b}$, and therefore $x = \frac{2abb}{zz + bb}$, $\dot{x} = -\frac{4abbz\dot{z}}{(zz + bb)^2}$, and $\sqrt{2ax - xx} = \frac{xz}{b} = \frac{2abz}{zz + bb}$. Making the fubftitutions, it will be $\frac{a\dot{x}}{\sqrt{2ax - xx}} = -\frac{2ab\dot{z}}{zz + bb}$, the rectification of the circle.

Let it be alfo $\frac{aa\dot{x}}{2\sqrt{2ax - xx}} = -\frac{aab\dot{z}}{zz + bb}$, the rectification of the circle, as before.

In like manner, it will be $\dot{x}\sqrt{2ax - xx} = -\frac{8a^2b^3z^2\dot{z}}{(zz + bb)^3}$, which includes the fame circle.

38. If our differential formulæ fhall be compofed of two radical quantities, in this cafe the operation will be double, but ftill it will fucceed as well. For, in the radical quantities, the fecond term may be wanting, or it may be taken away, and the formula may be multiplied by an odd power of the variable; and that by putting one of the radical quantities equal to a new variable. And thus the propofed formula will be reduced to another, which will contain one radical only, and which confequently may be managed in the ufual manner.

Let it be, for example, $\frac{x^3\dot{x}\sqrt{aa + xx}}{\sqrt{bb + xx}}$. I put $\sqrt{aa + xx} = y$, and therefore $xx = yy - aa$, $x\dot{x} = y\dot{y}$. Making the fubftitutions, it will be $\frac{yy\dot{y} \times \overline{yy - aa}}{\sqrt{yy - aa + bb}}$, that is, $\frac{y^4\dot{y}}{\sqrt{yy - aa + bb}} - \frac{aayy\dot{y}}{\sqrt{yy - aa + bb}}$, each of which we know how to manage.

39. If we confider a little this manner of operation, we may eafily perceive, that, in thefe radical formulæ, it will not fucceed in general, that we fhall be able to free them from their radical vinculum, except when it is a fquare-root, and the invariable under the vinculum does not exceed the fecond dimenfion. I fay in general; becaufe, in feveral cafes, it may fucceed, whatever the radical may

may be, and whatever the power of the variable may be, which is under the vinculum. And certainly it will, in all cafes, be comprehended in the two following formulæ, the firft of which is this, $\frac{\dot{y} \times \overline{y^m + b^m}|^{\pm \frac{1}{n}}}{y^{tm+1}}$, in which m, n, t, are pofitive integers, and may alfo be nothing; and this obtains, by making $\overline{y^m + b^m}|^{\frac{1}{n}} = z$, whence $y^m = z^n - b^m$, $\dot{y} = \frac{nz^{n-1}\dot{z}}{my^{m-1}}$; and making the fubftitutions, it will be $\frac{nz^{n-1}\dot{z} \times z^{\pm 1}}{my^{tm+m}}$, that is, $\frac{nz^{n-1}\dot{z} \times z^{\pm 1}}{my^{\overline{t+1} \times m}}$. But $y^{\overline{t+1} \times m} = \overline{z^n - b^m}|^{t+1}$; and when t is an integer, the power $t + 1$ will be an integer, fo that the propofed formula will be free from radicals.

If t were negative, the formula would be the cafe confidered above at § 32, which has an algebräical integration.

In other cafes, the integral will depend on the quadrature of the circle, and of the hyperbola, as will be feen in it's place.

The fecond formula is $y^n\dot{y} \times \overline{y^m + b^m}|^{\pm \frac{t}{p}}$, which, when $\frac{n+1}{m}$ is a whole number, may always be freed from it's radical figns, either in the whole, or, at leaft, from radicals of the complicate quantity, which will be fufficient. Wherefore, make $\overline{y^m + b^m}|^{\frac{t}{p}} = z$, and then it will be $y^m = z^{\frac{p}{t}} - b^m$, $y = \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m}}$, $\dot{y} = \frac{\frac{p}{t} z^{\frac{p}{t} - 1} \dot{z} \times \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m} - 1}}{m}$, and $y^n = \overline{z^{\frac{p}{t}} - b^m}|^{\frac{n}{m}}$; and making the fubftitutions, we fhall have the formula $\frac{pz^{\frac{p}{t} - 1}\dot{z}}{tm} \times z^{\pm 1} \times \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m} + \frac{n}{m} - 1}$. But when $\frac{n+1}{m}$ is an integer, the power $\frac{1+n}{m} - 1$ will always be an integer, [or o,] fo that the formula will have only

only radical ſigns of the complicate quantities. And therefore, when $\frac{1+n}{m} - 1$ is a poſitive integer number, the integration, at moſt, will depend on the quadrature of the hyperbola, or on the logarithmic, and may be had by the given rules. And when $\frac{1+n}{m} - 1$ is a negative integer, the integration will depend on the quadrature of the circle, and of the hyperbola, and may be had by the rules which will be given in due place.

40. Now let us go on to ſuch formulæ, which being fractions free from radicals, the variable is raiſed to any power in the denominator, which I will ſuppoſe to be compoſed of imaginary roots, becauſe in theſe only there is any difficulty. I ſay, that as often as the denominator is reducible to real components, in which the variable does not exceed the ſecond dimenſion, the formula may always be ſplit into ſo many fractions, as are the forementioned real components, each of which will be integrable, ſuppoſing the quadrature of the circle and hyperbola; and conſequently the propoſed formula will always be reducible to the ſaid quadratures. To do this, let there be propoſed this formula, $\frac{aa\dot{x}}{\overline{xx + ax + bb} \times \overline{xx + cx + bc}}$. Take a fictitious equation,

$$\frac{aa\dot{x}}{\overline{xx + ax + bb} \times \overline{xx + cx + bc}} = \frac{Ax\dot{x} + B\dot{x}}{xx + ax + bb} + \frac{Cx\dot{x} + D\dot{x}}{xx + cx + bc},$$

in which formulæ the capitals A, B, C, D, are conſtant arbitrary quantities, which are to be determined by the proceſs.

Thus, if the formula were $\frac{ab\dot{x}}{\overline{xx + ax + bb} \times \overline{xx \pm aa} \times \overline{x \pm c}}$, we ſhould make it equal to $\frac{Ax\dot{x} + B\dot{x}}{xx + ax + bb} + \frac{Cx\dot{x} + D\dot{x}}{xx \pm aa} + \frac{H\dot{x}}{x \pm c}$. And thus we may proceed in the ſame order, if the components in the denominator were more in number. When this is done, the terms of this equation are to be reduced to a common denominator, and laſtly, by tranſpoſition, the equation muſt be made equal to nothing. Then, by comparing the firſt terms to nothing, the value of the aſſumed quantity A may be found. And ſo, by comparing the ſecond, third, fourth, &c. terms in the ſame manner, the values of the other capitals B, C, D, &c. may be found, expreſſed by the given quantities of the propoſed formula; which values, being ſubſtituted in the places of the aſſumed capitals A, B, C, D, &c. in the equation, will ſupply us with ſo many fractions as are equivalent to the propoſed formula; and which, being reduced to a common denominator, will exactly reſtore the formula at firſt propoſed.

Of this we will take an example. Let it be propofed to find the integral of this formula $\frac{aa\dot{x}}{\overline{xx + 2ax - aa} \times \overline{xx + aa}}$. Therefore I affume this fictitious equation $\frac{aa\dot{x}}{\overline{xx + 2ax - aa} \times \overline{xx + aa}} = \frac{Ax\dot{x} + B\dot{x}}{xx + 2ax - aa} + \frac{Cx\dot{x} + D\dot{x}}{xx + aa}$. Then I reduce the equation to a common denominator, and, by tranfpofing the term $aa\dot{x}$, I reduce it to o, and find it to be

$$\left.\begin{array}{l} Ax^3\dot{x} + Bx^2\dot{x} + Aaax\dot{x} + Baa\dot{x} \\ + Cx^3\dot{x} + Dx^2\dot{x} + 2Dax\dot{x} - Daa\dot{x} \\ \qquad + 2Cax^2\dot{x} - Caax\dot{x} - aa\dot{x} \end{array}\right\} = 0.$$

Wherefore, from the comparifon of the firft terms with o, we fhall have $A + C = 0$, or $A = -C$. From the fecond, $B + D + 2Ca = 0$, that is, putting $-A$ inftead of C, $B = 2Aa - D$. From the third, $Aa^2 + 2Da - Ca^2 = 0$, that is, $C = a + \frac{2D}{a}$. From the laft, $Baa - Daa - aa = 0$, that is, putting, inftead of B, it's value given by D and A, it will be $D = Aa - \frac{1}{2}$, and therefore it will be $C = \frac{3Aa - 1}{a}$; but $C = -A$, and therefore $A = \frac{1}{4a}$, $D = -\frac{1}{4}$, $B = \frac{3}{4}$, $C = -\frac{1}{4a}$; whence we fhall have at laft $\frac{aa\dot{x}}{\overline{xx + 2ax - aa} \times \overline{xx + aa}} = \frac{x\dot{x} + 3a\dot{x}}{4a \times \overline{xx + 2ax - aa}} - \frac{x\dot{x} + a\dot{x}}{4a \times \overline{xx + aa}}$.

But, by making the fecond term of the denominator to vanifh, where there is occafion, the *homogeneum comparationis* is integrable by the quadrature of the circle and hyperbola; the integral of which, by the given rules, will be found to be $\frac{1}{4a} l\sqrt{xx + 2ax - aa} + \frac{1}{2\sqrt{2aa}} l\sqrt{x + a - \sqrt{2aa}} - \frac{1}{2\sqrt{2aa}} l\sqrt{x + a + \sqrt{2aa}} - \frac{1}{4a} l\sqrt{xx + aa}$, fubtracting, befides, from thefe logarithms the fourth proportional of $4aa$, of unity, and of the arch of the circle, the radius of which is a, and the tangent $= x$. Therefore the integration of this formula depends on no higher quadratures than thofe of the circle and hyperbola.

41. If, befides, the fraction fhall be multiplied into any power of the variable, which power is pofitive; as if the formula were $\frac{aax^n\dot{x}}{\overline{xx + 2ax - aa} \times \overline{xx + aa}}$; make it equal to $\frac{Ax^{n+1}\dot{x} + Bx^n\dot{x}}{xx + 2ax - aa} + \frac{Cx^{n+1}\dot{x} + Dx^n\dot{x}}{xx + aa}$, and let the values of the capitals A, B, C, &c. be found in the fame manner as above, or you may work as if the faid power were not there; and the refulting fractions may be multiplied by the

ſaid power, and we ſhall have, in like manner, ſo many fractions, which will not require any higher quadratures than thoſe of the circle and hyperbola, and which may be managed by the rules already given.

42. And if the power of the variable ſhall be negative, that is, if it ſhall be poſitive in the denominator, all the denominators of the reſulting fractions may be multiplied by this power, and they will acquire the form following.

As, for example, $\frac{x^{-n}\dot{x}}{xx + ax + bb \times xx \pm aa \times x \pm c}$. This being reſolved as if x^{-n} were abſent, and then multiplying every term by x^{-n}, it will be

$$\frac{x^{-n}\dot{x}}{xx + ax + bb \times xx \pm aa \times x \pm c} = \frac{Ax\dot{x} + B\dot{x}}{xx + ax + bb \times x^{n}} + \frac{Cx\dot{x} + D\dot{x}}{xx \pm aa \times x^{n}} + \frac{H\dot{x}}{x \pm c \times x^{n}},$$

underſtanding now by the capitals ſuch values, as, being found by the foregoing method, ſhall make the ſum of theſe fractions equal to the propoſed formula.

The laſt fraction will have no occaſion for any particular artifice, becauſe it's integration is known by the common rules.

As to the firſt, to clear up the example, let it be A = aa, and B = abb, whence it will be thus expreſſed, $\frac{aax\dot{x} + abb\dot{x}}{xx + ax + bb \times x^{n}}$, which is to be made equal to

$$\frac{Mx\dot{x} + N\dot{x}}{xx + ax + bb} + \frac{Px^{n-1}\dot{x} + Hx^{n-2}\dot{x} + Ex^{n-3}\dot{x}, \&c.}{x^{n}}.$$

And thus we muſt go on till the laſt term becomes conſtant, that is, the laſt power of the variable x muſt have it's index = o. When theſe fractions are reduced to a common denominator, and all made = o, we ſhall have the values of the capitals, as was done before. The ſame thing muſt be done in regard to the other fraction $\frac{Cx\dot{x} + D\dot{x}}{xx \pm aa \times x^{n}}$, and thus, finally, the integral will be found of the propoſed formula.

Wherefore generally, ſuppoſing only the quadratures of the circle and hyperbola, we may always have the integral of the foregoing formula, if the components of the denominator be real, provided in them the unknown quantity do not exceed the ſecond dimenſion.

43. But if the denominator of the propoſed formula, or fraction, may not be reſolvable into it's real components, in which the variable does not exceed two dimenſions, nor can be reduced to ſuch by the common rules of Algebra; yet it may always be reduced to ſuch by a little further artifice, as often as it is a convertible

convertible formula, or the product of feveral convertible terms. I fhall call that a convertible formula, in which the variable has the greateft exponent of it's dimenfions an even affirmative number; as, fuppofe n were fuch, then the laft term would be a^n, and the terms equidiftant from that in the middle muft have the fame co-efficient, and be affected by the fame fign, fupplying the dimenfions by that conftant quantity, of which the laft term is formed. Such would be the formula $x^6 + a^6$, or this, $x^4 + bx^3 + ccxx + aabx + a^4$, or this other, $x^6 - bx^5 + b^3x^3 - a^4bx + a^6$. Now, if it were $x^5 + bx^4 + a^4x + a^4b$, it would be written in this equivalent form, $\overline{x^4 + a^4} \times \overline{x + b}$, in which $x^4 + a^4$ is a convertible formula, and $x + b$ is linear, which does not increafe the difficulty. The fame thing is to be underftood of infinite others. A convertible formula, what.

44. Therefore now let us have $x^m - a^m$ to be refolved into it's real components, in which x may not exceed two dimenfions, and which fhall not have fractions for their exponents; and, in the firft place, let m be an even affirmative whole number. In this cafe, it will be divifible into $x^{\frac{1}{2}m} + a^{\frac{1}{2}m}$ and $x^{\frac{1}{2}m} - a^{\frac{1}{2}m}$, without any fractions in the exponents, becaufe of m being an even whole number. The firft divifor may be refolved by the rules which will be foon given for the binomial $x^m + a^m$. The fecond, $x^{\frac{1}{2}m} - a^{\frac{1}{2}m}$, if $\frac{1}{2}m$ fhall be an even number, may be again refolved into $x^{\frac{1}{4}m} + a^{\frac{1}{4}m}$ and $x^{\frac{1}{4}m} - a^{\frac{1}{4}m}$, without a fraction in the exponents. But, if $\frac{1}{2}m$ fhall be an odd number, it will be refolved by the rules that will be prefcribed for the binomial $x^m - a^m$, when m is an odd number.

In the fecond place, let it be $x^m + a^m$, and let m be an even affirmative whole number, in which cafe, the formula is convertible. Let us fuppofe $x^m + a^m = 0$, and then let there be formed a convertible formula, in which the greateft exponent of x may be $m - 2$, and which may have all it's terms, and the laft term may be a^{m-2}, and the co-efficient of the fecond term may be b, for example, that of the third cc, that of the fourth d^3, and fo on; and let this be compared to o, whence refults an equation. Let this equation be multiplied by $xx + fx + aa$; the product will be another convertible equation, in which the greateft exponent of x will be $= m$. Let this equation be compared, term by term, with the fictitious equation $x^m + a^m = 0$, in which the co-efficients of the intermediate terms are $= 0$; and, by the comparifon of the fecond terms having the value of the affumed quantity b, from the comparifon of the third terms the value of cc, from that of the fourth terms the value of d^3, and fo on to the middle term, taking this in alfo; now, from that of the middle the other

other equations will become the ſame, becauſe of their being convertible equations which are compared. From this laſt term will be found the value of f expreſſed by an equation, which will have $\frac{1}{2}m$ for the number of it's dimenſions, of which all the roots will be real, and will give us the values of f; which being ſubſtituted in the trinomial $xx + fx + aa$, will give us ſo many trinomials, the products of which will reſtore the propoſed binomial $x^m + a^m$.

Let the example be $x^4 + a^4$. I take a convertible equation of the ſecond degree, $xx + bx + aa = 0$, which I multiply by $xx + fx + aa = 0$, from whence I have another convertible equation,

$$\left.\begin{matrix} x^4 + bx^3 + 2aax^2 + aafx + a^4 \\ + fx^3 + bfx^2 + aabx \end{matrix}\right\} = 0.$$

I compare this with the fictitious equation $x^4 + a^4 = 0$, and from the compariſon of the ſecond terms I find $b + f = 0$, or $b = -f$. From the compariſon of the middle terms I find $2aa + bf = 0$, and, inſtead of b, ſubſtituting it's value $-f$, it will be $ff - 2aa = 0$, or $f = \pm\sqrt{2aa}$.

Let it be $x^6 + a^6$. I take the convertible equation $x^4 + bx^3 + c^2x^2 + a^2bx + a^4 = 0$, which I multiply by $x^2 + fx + aa = 0$, and the reſulting equation is

$$\left.\begin{matrix} x^6 + bx^5 + ccx^4 + 2aabx^3 + a^4x^2 + a^4fx + a^6 \\ + fx^5 + bfx^4 + fccx^3 + a^2bfx^2 + a^4bx \\ + a^2x^4 \qquad\qquad + a^2c^2x^2 \end{matrix}\right\} = 0.$$

I compare this with the fictitious equation $x^6 + a^6 = 0$, and from the compariſon of the ſecond terms I find $b + f = 0$; from the compariſon of the third terms I find $cc + bf + aa = 0$, that is, ſubſtituting the value of b, $cc - ff + aa = 0$; from the compariſon of the middle terms I find $2aab + fcc = 0$, that is, inſtead of b and cc, ſubſtituting their values, $f^3 - 3aaf = 0$.

Now, by actually performing theſe operations, we ſhall find that

If $m = 4$, it will be $ff - 2aa = 0$.

If $m = 6$, then $f^3 - 3aaf = 0$.

If $m = 8$, then $f^4 - 4aaf^2 + 2a^4 = 0$.

If $m = 10$, then $f^5 - 5aaf^3 + 5a^4f = 0$.

If $m = 12$, then $f^6 - 6aaf^4 + 9a^4f^2 - 2a^6 = 0$.

If $m = 14$, then $f^7 - 7aaf^5 + 14a^4f^3 - 7a^6f = 0$.

And ſo we might proceed to the other even values of m.

Inſtead

Inſtead of $x^4 + a^4$, let it be $x^4 + 2bx^3 + 2aabx + a^4$, which is alſo a convertible formula. I multiply the convertible equation $xx + bx + aa = 0$ by $xx + fx + aa = 0$, and I ſhall have, as above,

$$\left.\begin{array}{l} x^4 + bx^3 + 2aax^2 + aafx + a^4 \\ \quad + fx^3 + bfx^2 + aabx \end{array}\right\} = 0.$$

I compare this with the fictitious equation $x^4 + 2bx^3 + 2aabx + a^4 = 0$, and from the compariſon of the ſecond terms I find $b + f = 2b$, that is, $b = 2b - f$; from the compariſon of the middle terms I find $2aa + bf = 0$, and, inſtead of b, ſubſtituting it's value, we ſhall have $2aa + 2bf - ff = 0$, that is, $ff - 2bf - 2aa = 0$.

Let it be $x^6 + a^3x^3 + a^6$. I take the convertible equation $x^4 + bx^3 + ccx^2 + aabx + a^4 = 0$, which I multiply by $xx + fx + aa$, and I ſhall have this product;

$$\left.\begin{array}{l} x^6 + bx^5 + ccx^4 + 2aabx^3 + a^4x^2 + a^4fx + a^6 \\ \quad + fx^5 + bfx^4 + ccfx^3 + a^2bfx^2 + a^4bx \\ \qquad + aax^4 \qquad\qquad + a^2c^2x^2 \end{array}\right\} = 0.$$

This being compared with the equation $x^6 + a^3x^3 + a^6 = 0$, I find, from the compariſon of the ſecond terms, $b + f = 0$; from the compariſon of the third terms, $cc + bf + aa = 0$; and, inſtead of b, putting it's value, it will be $cc - ff + aa = 0$; from the compariſon of the middle terms, $2aab + ccf = a^3$; and, inſtead of b and cc, putting their values, it will be $f^3 - 3aaf - a^3 = 0$. And ſo for as many others as you pleaſe.

Now let us have $x^4 + 2bx^3 + 2aabx + a^4$ to reſolve into it's real components, in which x has no fraction for it's exponent, and does not exceed the ſecond dimenſion. The equation which ſhould give us the values of f is therefore $ff - 2bf = 2aa$, from which we obtain both the real values of f, that is, $f = b + \sqrt{2aa + bb}$, and $f = b - \sqrt{2aa + bb}$. Wherefore, ſubſtituting each of theſe values inſtead of f, in the trinomial $xx + fx + aa$, we ſhall find that $x^4 + 2bx^3 + 2aabx + a^4$ is the product of the two real components $xx + bx + x\sqrt{2aa + bb} + aa$, and $xx + bx - x\sqrt{2aa + bb} + aa$.

Thus, if it were $x^6 + aax^4 + a^4x^2 + a^6 = 0$. The equation which gives the values of f being $f^3 - 2aaf = 0$, from thence we ſhall have the values of f all real, that is, $f = 0$, $f = \sqrt{2aa}$, and $f = -\sqrt{2aa}$; ſo that $x^6 + aax^4 + a^4x^2 + a^6$ is the product of the three real components $xx + aa$, $xx + x\sqrt{2aa} + aa$, and $xx - x\sqrt{2aa} + aa$.

Let us have $x^{10} + a^{10}$. The equation which ought to give the values of f is $f^5 - 5aaf^3 + 5a^4f = 0$. From whence we derive the values of f all real, that

that is, $f = 0$, $f = a\sqrt{\frac{5+\sqrt{5}}{2}}$, $f = -a\sqrt{\frac{5+\sqrt{5}}{2}}$, $f = a\sqrt{\frac{5-\sqrt{5}}{2}}$, and $f = -a\sqrt{\frac{5-\sqrt{5}}{2}}$. Wherefore, ſubſtituting every one of theſe values inſtead of f in the trinomial $xx + fx + aa$, we ſhall find that $x^{10} + a^{10}$ is the product of theſe five real components, $xx + aa$, $xx + ax\sqrt{\frac{5+\sqrt{5}}{2}} + aa$, $xx - ax\sqrt{\frac{5+\sqrt{5}}{2}} + aa$, $xx + ax\sqrt{\frac{5-\sqrt{5}}{2}} + aa$, and $xx - ax\sqrt{\frac{5-\sqrt{5}}{2}} + aa$. Whence it is to be concluded, that the integral of any differential formula, whoſe numerator is $\dot{x}$ multiplied into any conſtant quantity, and the denominator is of a like nature with theſe here conſidered, will not depend on quadratures higher than thoſe of the circle and hyperbola, and may be had from the rules here given.

45. Now let $x^m \pm a^m$ be given to reſolve as above, and let m be any affirmative integer, but odd.

The formula may be divided by $x \pm a$, and the quotient (which in the firſt caſe will be $x^{m-1} - ax^{m-2} + a^2x^{m-3} - a^3x^{m-4}$, &c. to the laſt term, which will be $+ a^{m-1}$; and, in the ſecond caſe, it will be $x^{m-1} + ax^{m-2} + a^2x^{m-3} + a^3x^{m-4}$, &c. to the laſt term, which will be $+ a^{m-1}$,) may be ſuppoſed $= 0$; and let this fictitious equation, which is a convertible one, be compared, term by term, with the product of a convertible equation, in which the number of dimenſions of the variable x is $m - 3$, into the trinomial $xx + fx + aa$; and, from the compariſon of the ſecond terms, we ſhall have the value of the aſſumed quantity, for example b; from the third the value of cc, from the fourth the value of d^3, &c.; and laſtly, from the compariſon of the middle terms, we may derive the values of f, expreſſed by an equation of which the number of dimenſions will be $\frac{m-1}{2}$. All the roots of which will be real, and will determine the values of f all real; which, being ſubſtituted in the trinomial $xx + fx + aa$, will ſupply us with ſo many trinomials, which, multiplied together, and alſo by $x \pm a$, will reſtore the propoſed formula $x^m \pm a^m$.

By this method we may find the following equations, which will ſerve for the reſolution of the binomial $x^m + a^m$, when m is an odd, integer, and poſitive number.

If

If $m = 3$, it will be $f + a = 0$.

If $m = 5$, then $ff + af - aa = 0$.

If $m = 7$, then $f^3 + aff - 2aaf - a^3 = 0$.

If $m = 9$, then $f^4 + af^3 - 3aaff - 2a^3f + a^4 = 0$.

If $m = 11$, then $f^5 + af^4 - 4aaf^3 - 3a^3f^2 + 3a^4f + a^5 = 0$.

If $m = 13$, then $f^6 + af^5 - 5a^2f^4 - 4a^3f^3 + 6a^4f^2 + 3a^5f - a^6 = 0$.

And thus we might proceed to find the other values of f, if m be an odd number.

If the propofed formula were $x^m - a^m$, and m were an odd integer affirmative number, dividing by $x - a$ as before, the fame equations would be had, only changing the figns in the fecond, fourth, and fixth term, and in all others in even places.

46. If, inftead of $x^m \pm a^m$, fuppofing m to be any odd affirmative integer, the formula were any other, but fuch, as that, dividing by $x \pm$ fome conftant quantity, that which refults fhould be a convertible formula; as $x^5 + bx^4 - aax^3 - aabx^2 + a^4x + a^4b$, which, being divided by $x + b$, will give $x^4 - aax^2 + a^4$; this laft being managed as ufual, and the values of f found and fubftituted in the trinomial $xx + fx + aa$, we fhould have fo many trinomials, which being multiplied together, and alfo by $x + b$, would reftore the propofed formula.

Let it be required, for example, to refolve $x^5 + a^5$ into it's real components, in which x may have no fractional exponents, and may not exceed the fecond dimenfion. The equation which is to give the values of f (according to what goes before) will be $ff + af - aa = 0$, from whence we derive thefe values of f, $f = \frac{-a \pm a\sqrt{5}}{2}$. Thefe being fubftituted, inftead of f, in the trinomial $xx + fx + aa$, we fhall have the two real trinomials $xx - \frac{1}{2}ax + \frac{1}{2}ax\sqrt{5} + aa$, and $xx - \frac{1}{2}ax - \frac{1}{2}ax\sqrt{5} + aa$, the product of which, together with $x + a$, will reftore the formula propofed.

Let it be required to refolve into real components the formula $x^5 + bx^4 - aax^3 - aabx^2 + a^4x + a^4b$, which, being divided by $x + b$, will give $x^4 - aax^2 + a^4$. The equation that gives us f will be $ff = 3aa$, and the values of f will be $f = \pm\sqrt{3aa}$. Thefe being fubftituted inftead of f in the trinomial $xx + fx + aa$, we fhall have thefe two real trinomials $xx + x\sqrt{3aa} + aa$, and $xx - x\sqrt{3aa} + aa$; the product of which, together with $x + b$, will reftore the formula propofed.

 47. From

47. From hence I conclude, that the integral of any differential formula whatever, the numerator of which is $\dot{x}$ into any conftant quantity, and the denominator of a nature like to thefe here confidered, will not depend on quadratures higher than thofe of the circle and hyperbola, and which may be obtained by the rules here given.

48. But, becaufe in higher dimenfions the value of f cannot be obtained by actual feparation, from the equations before cited; in fuch cafes it will be enough to have recourfe to the geometrical conftruction of the fame equations. Thus, to find the components of $x^7 + a^7$, and thence the integral of the formula $\frac{\dot{x}}{x^7 + a^7}$, the denominator being divided by $x + a$, the quotient will be $x^6 - ax^5 + aax^4 - a^3x^3 + a^4x^2 - a^5x + a^6$. The values of f for the refolution of this formula muft be furnifhed by the equation $f^3 + af^2 - 2aaf - a^3 = 0$. Wherefore, by the ufual methods of Algebra, by means of the interfections of two curves, or by any other way, having found the values of f affirmative and negative, which are to be all real; for example, let one be A, another $-$ B, the other $-$ C; the quantity $x^7 + a^7$ will be the product of $x + a$ into $xx + Ax + aa$ into $xx - Bx + aa$ into $xx - Cx + aa$; and the quantities A, B, C, will be real and given. Then we may proceed to the integration of the formula $\frac{\dot{x}}{x^7 + a^7}$, by the quadrature only of the circle and hyperbola.

49. By the fame artifice by which we find the equations for the refolution of the binomial $x^m \pm a^m$, we may find them for the refolution of the trinomial $x^{2m} \pm 2aax^m + aa$, fuppofing $2m$ to be an even affirmative integral number. And thus, in general, as often as it is propofed to refolve a formula which is convertible, or is the product of a convertible into a linear quantity, and which has not a fraction in the exponents; they may always be reduced by the method here explained.

The cafe of the product of a convertible formula into a linear, we fhall have when m is an odd number, and otherwife. Let this be an example, $x^8 + b^4x^4 - a^4x^4 - a^4b^4$, that is, $\overline{x^4 + b^4} \times \overline{x^4 - a^4}$, or $\overline{x^4 + b^4} \times \overline{xx + aa} \times \overline{xx - aa}$. Wherefore, the divifor $x^4 + b^4$ being refolved into it's real components of two dimenfions, which may be, for example, $xx + Ax + bb$, and $xx + Bx + bb$, it will be $\overline{x^4 + b^4} \times \overline{x^4 - a^4} = \overline{xx + Ax + bb} \times \overline{xx + Bx + bb} \times \overline{xx + aa} \times \overline{xx - aa}$. And if it had been $\overline{x^4 + b^4} \times \overline{x^4 + a^4}$, then, by the refolution of $x^4 + a^4$ into $xx + Cx + aa$, and $xx + Dx + aa$, it would be $\overline{x^4 + b^4} \times \overline{x^4 + a^4} = \overline{xx + Ax + bb} \times \overline{xx + Bx + bb} \times \overline{xx + Cx + aa} \times \overline{xx + Dx + aa}$.

50. To

50. To have the integral of the formula $\frac{ma^m \dot{x}}{x^m \pm a^m}$, in which m denotes any affirmative integer number, let A, B, C, &c. reprefent the feveral values of f with their figns, which ferve for the refolution of the denominator $x^m \pm a^m$. And it muft be obferved, that of thefe values one may fometimes be $= 0$, which will obtain as often as m is a term in this feries 4, 8, 12, 16, &c. it being $x^m - a^m$ in the given formula. And as often as m is a term in this feries 2, 6, 10, 14, 18, &c. when it is $x^m + a^m$. This being fuppofed, the integral required will be $\pm \frac{A}{a} l \sqrt{xx + Ax + aa} \pm \frac{B}{a} l \sqrt{xx + Bx + aa} \pm \frac{C}{a} l \sqrt{xx + Cx + aa}$, &c. taking thefe logarithms from the logarithmic curve, the fubtangent of which is $= a$; adding to, or fubtracting from this aggregate of logarithmic terms, (according as the fign of the term a^m in the denominator fhall be + or −,) twice the fum of fo many arches of a circle, as are the values A, B, C, &c. of which arches thefe are the radii in order, $\sqrt{aa - \frac{1}{4}AA}$, $\sqrt{aa - \frac{1}{4}BB}$, $\sqrt{aa - \frac{1}{4}CC}$, &c. and the tangents are in the fame order, $x + \frac{1}{2}A$, $x + \frac{1}{2}B$, $x + \frac{1}{2}C$, &c. Such will be the integral of the formula $\frac{ma^m \dot{x}}{x^m + a^m}$, if m fhall be an even affirmative number. But in the fame formula, if m fhall be an odd affirmative number, it will be neceffary to add to the whole the logarithm of $x + a$, becaufe the denominator has alfo the real root $x + a$. And if the formula fhould be $\frac{ma^m \dot{x}}{x^m - a^m}$, m being an odd affirmative number; inftead of the logarithm of $x + a$, that of $x - a$ muft be added. And laftly, the formula being $\frac{ma^m \dot{x}}{x^m - a^m}$, and m being an even affirmative number, it will be neceffary to add the logarithm of $x - a$, and to fubtract that of $x + a$; ftill taking thefe logarithms from the logarithmic with fubtangent $= a$.

51. But if in the propofed formula $\frac{\dot{x}}{x^m \pm a^m}$ the number m fhould be a negative number, that is, if it were $\frac{\dot{x}}{x^{-m} \pm a^{-m}}$, it would be expreffed thus,

$\frac{\dot{x}}{\frac{1}{x^m} \pm \frac{1}{a^m}}$, which, reduced to a common denominator, is equivalent to this, $\frac{a^m x^m \dot{x}}{a^m \pm x^m}$; and dividing the numerator by the denominator till the greateſt power of the variable is leſs in this than in that, we ſhall have at laſt $\pm a^m \dot{x} \pm \frac{a^{2m}\dot{x}}{x^m \pm a^m}$, in which m will be a poſitive number. And what has been ſaid before will alſo take place, in the formula $\frac{\dot{x}}{x^m \pm a^m}$, when m is an integer negative number.

52. Moreover, if the fraction $\frac{\dot{x}}{x^m \pm a^m}$ be ſuppoſed to be multiplied by x^n, n being an integer number either affirmative or negative, the denominator being reſolved into it's real components, in which x does not exceed the ſecond dimenſion; this will be the caſe already conſidered by me at § 41, 42, and is therefore reducible to the quadrature of the circle and hyperbola.

53. But when n is negative, it may be reduced more expeditiouſly thus. Firſt, let n be leſs than m. The formula $\frac{\dot{x}}{\overline{x^m + a^m} \times x^n}$ may be thus expreſſed by equivalents, $\frac{\dot{x}}{a^m x^n} - \frac{x^{m-n}\dot{x}}{a^m \times \overline{x^m + a^m}}$. And likewiſe, the formula $\frac{\dot{x}}{\overline{x^m - a^m} \times x^n}$ by $-\frac{\dot{x}}{a^m x^n} + \frac{x^{m-n}\dot{x}}{a^m \times \overline{x^m - a^m}}$. Secondly, let n be greater than m. The formula $\frac{\dot{x}}{\overline{x^m + a^m} \times x^n}$ may be expreſſed by the equivalent ſeries $\frac{\dot{x}}{a^m x^n} - \frac{\dot{x}}{a^{2m} x^{n-m}} + \frac{\dot{x}}{a^{3m} x^{n-2m}} - \frac{\dot{x}}{a^{4m} x^{n-3m}}$, &c. till we come to that term, in which the exponent of x is but juſt greater than m; $\pm \frac{\dot{x}}{\overline{x^m + a^m} \times a^r x^t}$. Here the ſign muſt be $+$ or $-$, according as the alternate change of the ſigns ſhall require; and r is the ſame exponent of the quantity a, as in the antecedent term, and t is the remainder of the diviſion made of the number n by the number m, taken as often as it can be done.

Now

Now if it were $\frac{\dot{x}}{\overline{x^m - a^m} \times x^n}$, suppofing n to be greater than m; all the terms of the feries ought to be affected by the negative fign, and the term out of the feries, that is, $\frac{\dot{x}}{\overline{x^m - a^m} \times a^r x^t}$, ought always to have the affirmative fign prefixed. Thus, if the formula were $\frac{\dot{x}}{\overline{x^3 + a^3} \times x^5}$, it would be equivalent to $\frac{\dot{x}}{a^3x^5} - \frac{\dot{x}}{\overline{x^3 + a^3} \times a^3x^2}$. But we know that $-\frac{\dot{x}}{\overline{x^3 + a^3} \times a^3x^2}$ is equal to $-\frac{\dot{x}}{a^6x^2}$ $+ \frac{x\dot{x}}{a^6 \times \overline{x^3 + a^3}}$. Therefore it will be $\frac{\dot{x}}{\overline{x^3 + a^3} \times x^5} = \frac{\dot{x}}{a^3x^5} - \frac{\dot{x}}{a^6x^2} + \frac{x\dot{x}}{a^6 \times \overline{x^3 + a^3}}$; all which are quantities that may be managed by the given rules.

54. But if m fhall be a fraction either affirmative or negative, let t be the numerator of the fraction which is equal to m, and reduced to the fimpleft terms, and let p be the denominator of the fame: fo that the given formula may be thus expreffed, $\frac{\dot{x}}{x^{\frac{t}{p}} \pm a^{\frac{t}{p}}}$. Put $x = y^p$, and $a = b^p$, and the formula will be converted into this, $\frac{py^{p-1}\dot{y}}{y^t \pm b^t}$, which has no fractions for it's exponents, and may therefore be refolved by the given rules.

Let the formula be, for example, $\frac{\dot{x}}{x^{\frac{3}{2}} \pm a^{\frac{3}{2}}}$; make $x = yy$, $a = bb$, and it will be $\dot{x} = 2y\dot{y}$; and making the fubftitutions, the formula will be changed into $\frac{2y\dot{y}}{y^3 \pm b^3}$, which has no fractions for it's exponents.

55. Now if the given formula be $\frac{x^n\dot{x}}{x^m \pm a^m}$, in which m and n are broken numbers; making r the numerator of the fraction n, and p the denominator of the fame; and thus making t the numerator of the fraction m, and q it's denominator, (fuppofing thefe fractions to be reduced to their fmalleft terms,) the formula will be $\frac{x^{\frac{r}{p}}\dot{x}}{x^{\frac{t}{q}} \pm a^{\frac{t}{q}}}$, in which r, p, q, t, will be integer numbers, pofitive or negative.

Now

Now let it be made $x = y^{pq}$, and $a = b^{pq}$; the formula will be converted into this, $\frac{pqy^{qr+pq-1}\dot{y}}{y^{pt} \pm b^{pt}}$, which has no fractions in it's exponents. Let it be, for example, the formula $\frac{x^{\frac{3}{2}}\dot{x}}{x^{\frac{4}{5}} \pm a^{\frac{4}{5}}}$; make $x = y^{10}$, $a = b^{10}$; it will be $\dot{x} = 10y^9\dot{y}$, $x^{\frac{3}{2}} = y^{15}$, $x^{\frac{4}{5}} = y^8$; and making the substitutions, the formula will be changed into $\frac{10y^{24}\dot{y}}{y^8 \pm b^8}$, which has no fractional exponents.

56. Lastly, if the formula shall be $\frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u}$, the exponents n, m, u, being positive integers, we may always have it's integral, supposing only the quadratures of the circle and hyperbola. And the integral will be composed of algebraical quantities, and of one fluential quantity; which will be done in the following manner.

Suppose the formula $\int \frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u} =$

$$\frac{Bx^{n+um-2m+1} + Cx^{n+um-2m} + Dx^{n+um-2m-1} \&c}{\overline{x^m \pm a^m}|^{u-1}}$$ as far as to a constant term, or to that term in which the exponent of x is o, and let this be K; then must be added $A\int \frac{x^n\dot{x}}{x^m \pm a^m}$; that is, it must be made $\int \frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u} =$

$$\frac{Bx^{n+um-2m+1} + Cx^{n+um-2m} + Dx^{n+um-2m-1} \&c + K}{\overline{x^m \pm a^m}|^{u-1}} + A\int \frac{x^n\dot{x}}{x^m \pm a^m}.$$

Difference the equation, make it = o, and set the terms in order. From making the first terms = o we shall find the value of the assumed quantity B. Making the second terms = o, we shall have the value of C. And so, one by one, the values of the others; which values being substituted instead of the capitals, as the fluent of $\frac{x^n\dot{x}}{x^m \pm a^m}$ will depend only on the quadratures of the circle and hyperbola, and the other terms in the *homogeneum comparationis* are purely algebraical, so the proposed formula will require no higher quadratures.

57. Sometimes it may happen, that some one of the co-efficients B, C, D, &c. may come out arbitrary, or to be determined at pleasure; but it will be only

only when n is greater than $m - 1$. And it may alfo be obferved, that as often as it is $m = n + 1$, the co-efficient A will be found $= 0$, and confequently the integral of the propofed formula will be algebräical.

58. But if, in the propofed differential formula, the exponent n fhould be a negative integer, fo that it might be reduced to $\frac{\dot{x}}{x^n \times \overline{x^m \pm a^m}|^u}$; in which it is now pofitive; the integral would be

$$\frac{Bx^{um-2m} + Cx^{um-2m-1} + Dx^{um-2m-2}, \text{ \&c. } + K}{x^{n-1} \times \overline{x^m \pm a^m}|^{u-1}} + A\int \frac{\dot{x}}{x^n \times \overline{x^m \pm a^m}}.$$ Which co-efficients B, C, D, &c. will be determined in the fame manner as before.

As, for example, $\frac{x\dot{x}}{\overline{x^3 + a^3}|^2}$; in which cafe we have $n = 1$, $m = 3$, $u = 2$. Wherefore it will be $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{Bx^2 + Cx + K}{x^3 + a^3} + A\int \frac{x\dot{x}}{x^3 + a^3}$. And taking the fluxions, $\frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{\overline{2Bx\dot{x} + C\dot{x}} \times \overline{x^3 + a^3} - 3x^2\dot{x} \times \overline{Bx^2 + Cx + K}}{\overline{x^3+a^3}|^2} + \frac{Ax\dot{x}}{x^3 + a^3}$.
Then reducing to a common denominator, fetting the equation in order, and making it equal to 0, it will be

$$\left.\begin{array}{llll} 2Bx^4\dot{x} & + Cx^3\dot{x} & - 3Kx^2\dot{x} & + 2Ba^3x\dot{x} + Ca^3\dot{x} \\ - 3Bx^4\dot{x} & - 3Cx^3\dot{x} & & + Aa^3x\dot{x} \\ + Ax^4\dot{x} & & & - x\dot{x} \end{array}\right\} = 0.$$

Now making the firft, fecond, third, &c. terms $= 0$ fucceffively, we fhall find $A - B = 0$, or $B = A$; $C = 0$, $K = 0$; $2Ba^3 + Aa^3 - 1 = 0$, or $Aa^3 = 1 - 2Ba^3$; and putting A inftead of B, it will be $A = \frac{1}{3a^3} = B$. Whence, laftly, it is $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{xx}{3a^3 \times \overline{x^3+a^3}} + \frac{1}{3a^3} \times \int \frac{x\dot{x}}{x^3 + a^3}$. But $\int \frac{x\dot{x}}{x^3 + a^3}$ $= \frac{1}{3aa} l\sqrt{xx - ax + aa} - l\overline{x + a}$; together with $\frac{2}{3aa}$ multiplied into the arch of a circle with radius $= \sqrt{\frac{3}{4}aa}$, and tangent $= x - \frac{1}{2}a$. So that it will be $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{xx}{3a^3 \times \overline{x^3+a^3}} + \frac{1}{9a^5} \times l\sqrt{xx - ax + aa} - \frac{1}{9a^5} \times l\overline{x + a}$ $+ \frac{2}{9a^5} \times$ arch of a circle with radius $\sqrt{\frac{3}{4}aa}$, and tangent $= x - \frac{1}{2}a$: taking the logarithms from the logarithmic with fubtangent $= a$.

59. But

59. But if the exponent m be negative, the formula muſt be changed into another that is equivalent to it, in which the exponent is poſitive; according to the manner ſhown at § 51 of this Book.

60. And if both m and n ſhould be fractions, the ſubſtitutions muſt be made according to § 55 of this Book.

61. Again, if the exponent u were not an integer, but a fraction either affirmative or negative, it will ſuffice that the formula be one of thoſe caſes conſidered at § 39. Foraſmuch as it may be tranſmuted into another form, which is capable of being managed by the given rules.

Thus the formula $\frac{x^n \dot{x}}{\overline{x^m \pm a^m}|^u}$, the exponents n, m, u, being poſitive or negative integers, or elſe rational fractions of any kind, with the ſigns + and — at pleaſure; it will be integrable, or, at leaſt, may be reduced to known quadratures, as often as the ſaid exponents ſhall have ſuch a relation to one another, that one of theſe two quantities compoſed of them, that is, $u - \frac{1}{m} - 1 - \frac{n}{m}$, or $\frac{1}{m} - 1 + \frac{n}{m}$, ſhall be equal to any integer number. If this integer number ſhall be poſitive, the formula will admit of an algebräical integration, except the caſes in which the power $x^{-1}\dot{x}$ ſhall intrude, which obliges us to recur to the logarithms. If this integer number ſhall be negative, the formula will be reduced to the quadrature of the circle, or of the hyperbola.

To obtain our purpoſe as to the firſt caſe, in which $u - \frac{1}{m} - \frac{n}{m} - 1$ is equal to an integer, make $x^m + a^m = zx^m$; then $x^m = \frac{a^m}{z-1}$, $x = \frac{a}{\overline{z-1}|^{\frac{1}{m}}}$, $x^n = \frac{a^n}{\overline{z-1}|^{\frac{n}{m}}}$, $x^{n+1} = \frac{a^{n+1}}{\overline{z-1}|^{\frac{1+n}{m}}}$; and therefore $x^n\dot{x} = -\frac{a^{n+1}\dot{z}}{m} \times \overline{z-1}|^{\frac{-n-1-m}{m}}$. But $x^m + a^m = zx^m = \frac{a^m z}{z-1}$, and $\overline{x^m + a^m}|^u = \frac{a^{mu}z^u}{\overline{z-1}|^u}$. Therefore, making the neceſſary ſubſtitutions in the propoſed formula, it will be $-\frac{a^{n+1-mu}\dot{z}z^{-u}}{m} \times \overline{z-1}|^{\frac{-n-1}{m}-1+u}$, which is plainly ſeen to be algebräically integrable, (except the excepted caſe,) when $\frac{-n-1}{m} - 1 + u$ is equal

 to

to a positive integer number. And that if $\frac{-n-1}{m} - 1 + u$ is an integer number, but negative, by what is advanced in the foregoing articles, the integration of this formula will depend on no higher quadratures than those of the circle and hyperbola.

I come now to the second case, when $\frac{1}{m} - 1 + \frac{n}{m}$ is equal to an integer number. Make $x^m + a^m = z$, and then it will be $x^m = z - a^m$, $x = \overline{z - a^m}|^{\frac{1}{m}}$, $x^n = \overline{z - a^m}|^{\frac{n}{m}}$, $x^{n+1} = \overline{z - a^m}|^{\frac{1+n}{m}}$, $x^n \dot{x} = \frac{\dot{z}}{m} \times \overline{z - a^m}|^{\frac{n+1}{m} - 1}$. But $x^m + a^m = z$, and $\overline{x^m + a^m}|^u = z^u$; therefore, making the substitutions in the proposed formula, it will become $\frac{\dot{z}}{m} \times \frac{\overline{z - a^m}|^{\frac{n+1}{m} - 1}}{z^u}$, or else $\frac{z^{-u}\dot{z}}{m} \times \overline{z - a^m}|^{\frac{n+1}{m} - 1}$, which is algebräically integrable, (excepting in the case excepted,) when $\frac{n+1}{m} - 1$ is equal to a positive integer, or a negative; for then the integration will depend on the known quadratures of the circle and hyperbola, as appears by the foregoing articles.

62. Now if the denominator of the proposed fraction, raised to any integral power, should not be a binomial, as has been considered hitherto, but should be any multinomial whatever; provided it be reducible into it's real components, in which the variable does not exceed the second dimension; either by means of convertible equations, or some other manner; the formula may always be reduced to known quadratures.

Let it be, for example, $\frac{\dot{x}}{\overline{xx + bx + aa}|^2 \times \overline{x + c}|^3}$; raising actually the powers of the denominator, make a fictitious equation thus:

$$\frac{\dot{x}}{\overline{xx+bx+aa}|^2 \times \overline{x+c}|^3} = \frac{Ax^3\dot{x} + Bx^2\dot{x} + Cx\dot{x} + D\dot{x}}{x^4 + 2bx^3 + 2aax^2 + bbx^2 + 2aabx + a^4} + \frac{Fx^2\dot{x} + Gx\dot{x} + H\dot{x}}{x^3 + 3cx^2 + 3ccx + c^3}.$$

Here are so many terms taken in general, as are the components of the denominator; and in these terms so many capitals, as is the highest power of the variable in it's respective denominator, multiplying also the first capital in each term by the highest power, lessened by unity, of the variable in it's denominator, the second capital by the same power diminished by 2, and so on to the

 last

laſt conſtant quantity. Theſe aſſumed conſtant quantities are to be determined in the uſual manner, and the firſt term will furniſh ſo many fractions divided by $\overline{xx+bx+aa}|^2$; in which denominator making the middle term to vaniſh, the fractions will be a particular caſe of the general canon $\frac{x^m\dot{x}}{\overline{x^n \pm a^n}|^u}$. And the ſecond term will give us ſo many fractions divided by $\overline{x+c}|^3$, which may be reduced to the uſual rule of denominators compounded of equal roots.

63. Moreover, if the numerator of the propoſed formula be multiplied by a poſitive or negative power of the variable; having found the values of the capitals, and operating as if the fraction had not been multiplied by any ſuch power; the reſulting terms may be multiplied by the ſaid power, and the reſt may be done as uſual.

64. I ſhall finiſh this Section by fulfilling my promiſe made to the reader, concerning the Method of Multinomials, of Sig. Count *James Riccati*, which is as follows.

By the name of Differential Multinomials I call ſuch fractions, as have for their numerators the fluxion $\dot{x}$, and for denominators an aggregate of powers, the exponents of which conſtitute an arithmetical progreſſion, which proceeds till it terminates in nothing. And till this condition is fulfilled, the abſent terms muſt be ſupplied, and their co-efficients made equal to nothing. Suppoſe we had this expreſſion $\frac{\dot{x}}{x^{\frac{1}{2}} + x^{\frac{1}{3}} + a}$. At firſt view it might ſeem to be a trinomial, but is really a quadrinomial, and is thus to be compleated: $\frac{\dot{x}}{x^{\frac{3}{6}} + x^{\frac{2}{6}} + 0x^{\frac{1}{6}} + a}$.

In any multinomial expreſſed by a fraction, the denominator of which is raiſed to the power p, being a poſitive integral number, there is a method which would be general, if it were not frequently made uſeleſs by the intervention of imaginary quantities. But there are ſome particular artifices, which often come opportunely to our aſſiſtance.

I begin with the trinomial $\frac{\dot{x}}{\overline{x^{2m} + ax^m + b}|^p} = \dot{y}$, becauſe to ſuch an expreſſion as this every trinomial may eaſily be reduced. Make $x^m = z + A$, where z is a new variable aſſumed, and A is a conſtant to be afterwards determined. The neceſſary computations being made, to arrive at the ſubſtitutions we ſhall have as follows,

x^{2m}

$$x^{2m} = zz + 2Az + AA, \text{ and confequently}$$
$$ax^m = az + aA$$
$$b = b$$
$$\overline{x^{2m} + ax^m + b}|^p = \overline{zz + \overline{2A + a} \times z + AA + aA + b}|^p.$$

It ought to be contrived in fuch manner, that the quantities $AA + aA + b$ may difappear, by putting them $= 0$, and in cafes in which A is no imaginary quantity, this reduction fucceeds very well. It is therefore $x^m = z + A$; and taking the fluxions, $mx^{m-1}\dot{x} = \dot{z}$, and $x = \overline{z + A}|^{\frac{1}{m}}$. Then $\dot{x} = \frac{\dot{z}}{mx^{m-1}} = \frac{\dot{z}}{m \times \overline{z + A}|^{\frac{m-1}{m}}}$.

In proceeding to the neceffary fubftitutions, in our principal formula, inftead of x and it's powers, are to be fubftituted the affumed variable z, with it's functions; and we fhall find $\frac{\dot{x}}{\overline{x^{2m} + ax^m + b}|^p} = \frac{\dot{z}}{m \times \overline{z + A}|^{\frac{m-1}{m}} \times \overline{zz + \overline{2A + a} \times z}|^p}$;

and freeing it from the quantity z, which multiplies the binomial $z + \overline{2A + a}$ under the vinculum, it will be $\frac{\dot{x}}{\overline{x^{2m} + ax^m + b}|^p} = \frac{z^{-p}\dot{z}}{m \times \overline{z + A}|^{\frac{m-1}{m}} \times \overline{z + 2A + a}|^p}$.

The moft fimple cafe is, when the exponent p is equal to unity, the other being when m is any number, integer or fraction, affirmative or negative; and, for brevity, making $2A + a = g$, the general expreffion, [when $p = 1$,] will become this particular one, $\frac{z^{-1}\dot{z}}{g \times \overline{z + A}|^{\frac{m-1}{m}} + z \times \overline{z + A}|^{\frac{m-1}{m}}} = m\dot{y}$.

I make a firft divifion by dividing the numerator of the fraction by it's denominator, and the firft quotient will be $\frac{z^{-1}\dot{z}}{g \times \overline{z + A}|^{\frac{m-1}{m}}}$; and making the multiplication and the fubtraction, according to the ufual method, the remainder will be $-\frac{\dot{z}}{g}$, to be divided by the denominator; and therefore

$$\frac{z^{-1}\dot{z}}{g \times \overline{z + A}|^{\frac{m-1}{m}} + z \times \overline{z + A}|^{\frac{m-1}{m}}} = \frac{z^{-1}\dot{z}}{g \times \overline{z + A}|^{\frac{m-1}{m}}} - \frac{\dot{z}}{gg \times \overline{z + A}|^{\frac{m-1}{m}} + gz \times \overline{z + A}|^{\frac{m-1}{m}}}.$$

The

The firſt term of the ſecond member is already reduced to known quadratures, and the other term may eaſily be reduced, by making $z + A = u$, and performing the neceſſary ſubſtitutions. For then we ſhall have

$$\frac{-\dot{z}}{gg \times \overline{z+A}^{\frac{m-1}{m}} + gz \times \overline{z+A}^{\frac{m-1}{m}}} = \frac{u^{\frac{-m+1}{m}}\dot{u}}{gg - gA + gu}.$$

To purſue our inquiry, let the exponent p be equal to any poſitive and integer number; to obtain our deſire it will be ſufficient ſomething to produce the operation. Reſuming, then, the general formula $\frac{\dot{x}}{\overline{x^{2m} + ax^m + b}^p} = \frac{z^{-p}\dot{z}}{m \times \overline{z+A}^{\frac{m-1}{m}} \times \overline{z+g}^p} = \dot{y}$. And, for example-ſake, making $p = 2$, this will be reduced to the following,

$$\frac{z^{-2}\dot{z}}{gg \times \overline{z+A}^{\frac{m-1}{m}} + 2gz \times \overline{z+A}^{\frac{m-1}{m}} + zz \times \overline{z+A}^{\frac{m-1}{m}}} = m\dot{y}.$$

Then, as before, I divide the numerator of this fraction by it's denominator, and the firſt quotient will be $\frac{z^{-2}\dot{z}}{gg \times \overline{z+A}^{\frac{m-1}{m}}}$; and, after the neceſſary operations, we ſhall have the remainder $-\frac{2z^{-1}\dot{z}}{g} - \frac{\dot{z}}{gg}$, to be again divided by the whole denominator. Then I make a ſecond diviſion with the fraction $\frac{-2z^{-1}\dot{z}}{g^3 \times \overline{z+A}^{\frac{m-1}{m}} + 2gg \times \overline{z+A}^{\frac{m-1}{m}} + gzz \times \overline{z+A}^{\frac{m-1}{m}}}$. Here, after the neceſſary operations, we ſhall have the remainder $\frac{4\dot{z}}{g} + \frac{2z\dot{z}}{gg}$, to be divided by the whole denominator. Whence there will ariſe the following equation,

$$\frac{z^{-2}\dot{z}}{\overline{z+A}^{\frac{m-1}{m}} \times \overline{z+g}^2} = \frac{z^{-2}\dot{z}}{gg \times \overline{z+A}^{\frac{m-1}{m}}} - \frac{2z^{-1}\dot{z}}{g^3 \times \overline{z+A}^{\frac{m-1}{m}}} + \frac{3\dot{z}}{g^2 \times \overline{z+A}^{\frac{m-1}{m}} \times \overline{z+g}^2} + \frac{2z\dot{z}}{g^3 \times \overline{z+A}^{\frac{m-1}{m}} \times \overline{z+g}^2}.$$

The

The two firſt terms of the *homogeneum comparationis* are two binomials, and the other two may eaſily be reduced to the form of binomials, by making $z + A = u$, or $z + g = u$. In caſes more compounded, in which are made $p = 3$, or 4, or 5, &c. the tedioufneſs of calculation will indeed increaſe, but the method will ſtill be the ſame.

This method may be extended to all multinomials *in infinitum*, ſuppoſing p to be a poſitive integer; for, if it were a negative integer, the matter becomes ſo eaſy that there is no need to mention it. To apply the method, nothing elſe is required but to repeat the ſubſtitutions $x = z + A$, $z = u + B$, &c. always making thoſe terms to vaniſh, in which only conſtant quantities are found; by which means quadrinomials (for inſtance) may be reduced to trinomials, and theſe to binomials. It will alſo be needful, from time to time, to make uſe of a partial diviſion, that we may not be interrupted by negative exponents, which will often intrude in the numerator of the fraction. After all, the manner of operation will be better perceived by examples than by precepts.

Let us take the quadrinomial $\frac{\dot{x}}{\overline{x^{3m} + ax^{2m} + bx^{m} + c}|^{p}} = \dot{y}$. The conſtant quantities a, b, may be $= 0$. I ſuppoſe $x^{m} = z + A$; then we ſhall have

$$\begin{aligned} x^{3m} + ax^{2m} + bx^{m} + c = z^3 &+ 3Az^2 + 3AAz + A^3 \\ &+ az^2 + 2aAz + aA^2 \\ &+ bz + Ab + c. \end{aligned}$$

I make $A^3 + aA^2 + Ab + c = 0$, and thus I determine the value of the aſſumed conſtant quantity A. Then repeating the operations as in the trinomial, I find $\frac{z^{-p}\dot{z}}{\overline{z+A}|^{\frac{m-1}{m}} \times \overline{zz + gz + h}|^{p}}$. The letters g, h, denote conſtant quantities, which are ſubſtituted in the place of others more compounded. And, ſuppoſing p to be a poſitive integer, I raiſe the trinomial $zz + gz + h$ to the power p.

After this, I make uſe of as many diviſions as are neceſſary, to make the exponent of the variable in the numerator to be negative; and in the denominator, that no other quantity ſhall enter but the binomial $\overline{z + a}|^{\frac{m-1}{m}}$. And I ſet aſide ſuch fractions, as, neglecting the co-efficients, ſhall be analogous to this, $\frac{z^{-n}\dot{z}}{\overline{z + A}|^{\frac{m-1}{m}}}$; ſuppoſing n to be any poſitive integer. The other terms are repreſented

represented by the general formula $\frac{z^n \dot{z}}{\overline{z + A}|^{\frac{m-1}{m}} \times \overline{zz + gz + h}|^p}$. Then I repeat the operation, making $z = u + B$, making the laſt term to vaniſh as uſual, and raiſing the binomial $u + B$ to any power $n + 1$, and ſubſtituting, inſtead of z and it's powers, their values expreſſed by the new variable u; all the parts will appear under the aſpect expreſſed by the following formula,

$$\frac{u^{n-p}\dot{u}}{\overline{u + A + B}|^{\frac{m-1}{m}} \times \overline{u + k}|^p}.$$

When p is greater than n, ſo that the exponent $n - p$ is negative, then the diviſions muſt be put in practice, and the formula thence ariſing will be $\frac{u^{-n}\dot{u}}{\overline{u + A + B}|^{\frac{m-1}{m}}}$; then $n - p$, being poſitive, we ſhall have $\frac{u^n \dot{u}}{\overline{u + A + B}|^{\frac{m-1}{m}} \times \overline{u + k}|^p}$. And laſtly, making $u + k = \omega$, and, as well n as p being integer numbers, the binomials that will ariſe from the forementioned operations will always be reducible to more ſimple quadratures.

It is true, that, upon the account of imaginary quantities, this method remains limited; but very often the roots, either in the whole or in part, are real; and beſides that, in many particular caſes, theſe imaginary quantities may be eliminated. Nor ought we to deſpiſe the much we may have, becauſe we cannot obtain all.

Let us take, for example, the trinomial $\frac{\dot{x}}{\overline{x + 2\sqrt{x} + 2}|^p}$. Make $x^{\frac{1}{2}} = z + A$, then $x + 2\sqrt{x} + 2 = zz + 2Az + 2z + AA + 2A + 2$. By making $AA + 2A + 2 = 0$, we find $A = \sqrt{-1} - 1$. Now here we have a magnitude made up of real and imaginary quantities; therefore, proceeding according to the method, we ſhall have $\frac{z^{-p}\dot{z}}{\overline{z + A}|^{-1} \times \overline{z + 2A + 2}|^p} = \frac{z^{1-p}\dot{z}}{\overline{z + 2\sqrt{-1}}|^p} + \frac{Az^{-p}\dot{z}}{\overline{z + 2\sqrt{-1}}|^p}$. Now, that the imaginary quantities may be avoided, let us change our manner, and in the magnitude $zz + \overline{2A + 2} \times z + AA + 2A + 2$, let us bring it about, that the middle term $2Az + 2z$ may be deſtroyed, by putting it $= 0$; whence it is $A = -1$, and $AA + 2A + 2 = 1$. So that the formula will be as follows, $\frac{\dot{z}}{\overline{z - 1}|^{-1} \times \overline{zz + 1}|^p} = \frac{z\dot{z}}{\overline{zz + 1}|^p} - \frac{\dot{z}}{\overline{zz + 1}|^p}$. And now, in the two binomials of the *homogeneum comparationis*, which are equivalent to the two others already conſidered, we ſhall meet with no difficulty.

SECT. II.

Of the Rules of Integration, having recourſe to Infinite Series.

65. Now, to proceed to the other manner of Integration, or of finding fluents, which was mentioned at the beginning, that is, by means of infinite ſeries; it is neceſſary to premiſe theſe Rules following.

RULE I. To reduce a fraction to an infinite ſeries.

Divide the numerator by the denominator, according to the ordinary method of diviſion, and let the remainder be again divided, and thus from term to term *in infinitum*; and you will have a ſeries conſiſting of an infinite number of terms, which is equal to the propoſed fraction. Therefore it muſt be obſerved, to make that term the firſt which is the greateſt, and that as well in the numerator as in the denominator of the fraction propoſed. Wherefore, by operating after this manner, we ſhall have as follows:

$$\frac{f}{m+n} = \frac{f}{m} - \frac{fn}{m^2} + \frac{fn^2}{m^3} - \frac{fn^3}{m^4} + \frac{fn^4}{m^5}, \text{ \&c.}$$

$$\frac{f}{m-n} = \frac{f}{m} + \frac{fn}{m^2} + \frac{fn^2}{m^3} + \frac{fn^3}{m^4} + \frac{fn^4}{m^5}, \text{ \&c.}$$

$$\frac{af}{m^2 \pm n^2} = \frac{af}{m^2} \mp \frac{afn^2}{m^4} + \frac{afn^4}{m^6} \mp \frac{afn^6}{m^8} + \frac{afn^8}{m^{10}}, \text{ \&c.}$$

Here the ſigns of the ſeries muſt be alternately + and −, when the ſecond term of the denominator is poſitive; and all the ſigns muſt be poſitive when it has a negative ſign.

In like manner, it will be

$$\frac{f}{m^2 \pm mn} = \frac{f}{m^2} \mp \frac{fn}{m^3} + \frac{fn^2}{m^4} \mp \frac{fn^3}{m^5} + \frac{fn^4}{m^6}, \text{ \&c.}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8, \text{ \&c.}$$

$$\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}, \text{ \&c.}$$

$$\frac{f}{\overline{m \mp n}|^3} = \frac{f}{m^3} \pm \frac{3fn}{m^4} + \frac{6fn^2}{m^5} \pm \frac{10fn^3}{m^6} + \frac{15fn^4}{m^7}, \text{ \&c.}$$

Let

Let there be a fraction, of which the numerator and denominator are each an infinite series; for example, this following:

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}aax^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8, \text{ \&c.}}{1 - \frac{1}{2}bx^2 - \frac{1}{8}bbx^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8, \text{ \&c.}}$$

The quotient will be

$$\left.\begin{array}{l} 1 + \frac{1}{2}bx^2 + \frac{3}{8}b^2x^4 + \frac{5}{16}b^3x^6 + \frac{35}{128}b^4x^8 \\ \quad + \frac{1}{2}ax^2 + \frac{1}{4}abx^4 + \frac{3}{16}ab^2x^6 + \frac{5}{32}ab^3x^8 \\ \qquad - \frac{1}{8}a^2x^4 - \frac{1}{16}a^2bx^6 - \frac{3}{64}a^2b^2x^8 \\ \qquad\quad + \frac{1}{16}a^3x^6 + \frac{1}{32}a^3bx^8 \\ \qquad\qquad - \frac{5}{128}a^4x^8 \end{array}\right\} \text{\&c.}$$

66. Rule II. To reduce a complicate radical quantity into an infinite series.

Take, for example, $\sqrt{aa \pm xx}$; let the square-root of the first term be extracted, and then let the operation be prosecuted *in infinitum*, in the usual manner of the extraction of the square-root, and we shall have

$$\sqrt{aa \pm xx} = a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}$$

$$\sqrt{ax \pm xx} = a^{\frac{1}{2}}x^{\frac{1}{2}} \pm \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} \pm \frac{x^{\frac{7}{2}}}{16^{\frac{5}{2}}} - \frac{5x^{\frac{9}{2}}}{128a^{\frac{7}{2}}}, \text{ \&c.}$$

It may here be observed, that in each of these two series, if the numerator and denominator of each term be multiplied by 3, beginning at the fourth, the numerical co-efficients of the numerators will be in order, 3, 3×5, $3 \times 5 \times 7$, &c. arising from the continual multiplication of the odd numbers. Then in the denominators, beginning at the second, they will be 2, 2×4, $2 \times 4 \times 6$, $2 \times 4 \times 6 \times 8$, &c. arising from the continual multiplication of the even numbers.

67. Rule III. All this may be done more generally by the help of the following canon:

$$\overline{P + PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ, \text{ \&c.}$$

In which $P + PQ$ is the given quantity, $\frac{m}{n}$ is the numeral exponent, P represents the first term, Q is the quotient of all the other terms divided by the first, and every one of the capitals A, B, C, D, &c. signify the preceding terms respectively

respectively; so that by A is understood $P^{\frac{m}{n}}$, by B is meant $\frac{m}{n}AQ$, by C, $\frac{m-n}{2n}BQ$, and so on.

Let the formula $\sqrt{aa + xx}$ be proposed to be reduced into a series; then it will be $P = aa$, $Q = \frac{xx}{aa}$, $m = 1$, $n = 2$; therefore

$$\sqrt{aa + xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}$$

Let it be $\sqrt[5]{a^5 + a^4x - x^5}$, that is, $\overline{a^5 + a^4x - x^5}|^{\frac{1}{5}}$; it will be $P = a^5$, $Q = \frac{a^4x - x^5}{a^5}$, $m = 1$, $n = 5$; therefore $\overline{a^5 + a^4x - x^5}|^{\frac{1}{5}} = a + \frac{a^4x - x^5}{5a^4} - \frac{2a^8x^2 - 4a^4x^6 + 2x^{10}}{25a^9}$, &c.

Let it be $\frac{b}{\sqrt[3]{y^3 - aay}} = b \times \overline{y^3 - aay}|^{-\frac{1}{3}}$; it will be $P = y^3$, $Q = -\frac{aa}{yy}$, $m = -1$, $n = 3$; therefore

$$b \times \overline{y^3 - aay}|^{-\frac{1}{3}} = \frac{b}{y} + \frac{aab}{3y^3} + \frac{2a^4b}{9y^5} + \frac{14a^6b}{81y^7}, \text{ \&c.}$$

Let it be $\frac{b}{\sqrt[5]{\overline{a+x}|^3}}$, which would be expressed thus, $b \times \overline{a + x}|^{-\frac{3}{5}}$, and the rest would be done as before.

Let it be $b \times \overline{a + x}|^{-3}$; then $P = a$, $Q = \frac{x}{a}$, $m = -3$, $n = 1$; therefore $b \times \overline{a + x}|^{-3} = \frac{b}{a^3} - \frac{3bx}{a^4} + \frac{6bx^2}{a^5} - \frac{10bx^3}{a^6}$, &c.

68. Let us have a complicate quantity to raise to a given power, or let $a + x$ (for example) be raised to the power m. Then $P = a$, $Q = \frac{x}{a}$, $m = m$, $n = 1$; therefore

$$\overline{a + x}|^m = a^m + \frac{ma^{m-1}x}{1} + \frac{m \times \overline{m-1}\, a^{m-2}x^2}{1 \times 2} + \frac{m \times \overline{m-1} \times \overline{m-2}\, a^{m-3}x^3}{1 \times 2 \times 3}, \text{\&c.}$$

Let us have an infinite series to raise to a given power. For example, let $y + ay^2 + by^3 + cy^4 + dy^5$, &c. be raised to the power m. Then will $P = y$, $Q = ay + by^2 + cy^3 + dy^4$, &c. $m = m$, $n = 1$; wherefore

$$\overline{y + ay^2 + by^3 + cy^4 + dy^5, \&c.}\,\Big|^m = y^m + \frac{may^{m+1}}{1}$$

$$+ \frac{m \times \overline{m-1}\, a^2y^{m+2}}{1 \times 2} + \frac{m \times \overline{m-1} \times \overline{m-2}\, a^3y^{m+3}}{1 \times 2 \times 3}$$

$$+ \frac{mby^{m+2}}{1} + \frac{m \times \overline{m-1}\, aby^{m+3}}{1 \times 1}$$

$$+ \frac{mcy^{m+3}}{1}$$

$$+ \frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}\, a^4y^{m+4}}{1 \times 2 \times 3 \times 4} \ \&c.$$

$$+ \frac{m \times \overline{m-1} \times \overline{m-2}\, a^2by^{m+4}}{1 \times 2 \times 1}$$

$$+ \frac{m \times \overline{m-1}\, acy^{m+4}}{1 \times 1}$$

$$+ \frac{m \times \overline{m-1}\, b^2y^{m+4}}{1 \times 2}$$

$$+ \frac{mdy^{m+4}}{1}$$

69. This being now ſuppoſed, let the differential formula $\frac{b\dot{x}}{a + x}$ be propoſed to be integrated. The fraction $\frac{b}{a + x}$ being reduced to a ſeries, and every numerator being multiplied by $\dot{x}$, we ſhall have $\frac{b\dot{x}}{a + x} = \frac{b\dot{x}}{a} - \frac{bx\dot{x}}{aa} + \frac{bx^2\dot{x}}{a^3} - \frac{bx^3\dot{x}}{a^4} + \frac{bx^4\dot{x}}{a^5}$, &c. And by integration,

$$\int \frac{b\dot{x}}{a + x} = \frac{bx}{a} - \frac{bx^2}{2aa} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4} + \frac{bx^5}{5a^5}, \ \&c.$$

70. Let the formula be $\frac{a\dot{x}}{x}$. Making $x = b + z$, where b denotes any conſtant quantity at pleaſure, and z a new variable; it will be $\frac{a\dot{x}}{x} = \frac{a\dot{z}}{b + z}$.

The fraction $\frac{a}{b + z}$ being reduced to a ſeries, and multiplied by $\dot{z}$, it will be

$\frac{a\dot{z}}{b+z} = \frac{a\dot{z}}{b} - \frac{az\dot{z}}{b^2} + \frac{az^2\dot{z}}{b^3} - \frac{az^3\dot{z}}{b^4} + \frac{az^4\dot{z}}{b^5}$, &c. And by integration,

$\int\frac{a\dot{z}}{b+z} = \frac{az}{b} - \frac{az^2}{2b^2} + \frac{az^3}{3b^3} - \frac{az^4}{4b^4} + \frac{az^5}{5b^5}$, &c.; that is,

$$\int\frac{a\dot{x}}{x} = \frac{a \times \overline{x-b}}{b} - \frac{a \times \overline{x-b}^2}{2b^2} + \frac{a \times \overline{x-b}^3}{3b^3} - \frac{a \times \overline{x-b}^4}{4b^4}, \text{ \&c.}$$

71. Let the formula be $\frac{b\dot{x}}{\sqrt[5]{\overline{x+a}^3}}$; this, reduced to a series, is $\frac{b\dot{x}}{\sqrt[5]{\overline{a+x}}^3} = \frac{b\dot{x}}{a^{\frac{3}{5}}} - \frac{3bx\dot{x}}{5a^{\frac{8}{5}}} + \frac{12bx^2\dot{x}}{25a^{\frac{13}{5}}} - \frac{52bx^3\dot{x}}{125a^{\frac{18}{5}}}$, &c. And by integration, $\int\frac{b\dot{x}}{\sqrt[5]{\overline{a+x}}^3} = \frac{bx}{a^{\frac{3}{5}}} - \frac{3bx^2}{10a^{\frac{8}{5}}} + \frac{12bx^3}{75a^{\frac{13}{5}}} - \frac{52bx^4}{500a^{\frac{18}{5}}}$, &c. And the same may be done by any other proposed formula.

72. If the series thus found, which express the fluents of proposed differential formulæ, and which are composed of an infinite number of terms, shall be infinite in value; the fluents or integrals of the proposed fluxions will be infinite. And if these series shall be finite in value, and also summable, that is to say, if we know how to find the values of these series, though composed of terms infinite in number, and which very often may be done; we shall have them in a finite quantity, and therefore the algebräical integral of the proposed differential formulæ. But, if the series shall be finite in value, and yet not summable, the more terms shall be taken of the series, so much the nearer we shall approach to the true value of the formula; but we cannot arrive at the exact value, except we could take in the whole series.

73. In order to know what series are infinite in value, what are of a finite value, and which are summable; the treatise of Mr. *James Bernoulli de Seriebus infinitis*, may be consulted, and other authors who have written expressly on this subject.

74. But whenever the differential formula shall be composed of two terms only, we may, in general, and with expedition, make use of the following canon; in which the exponents *m*, *n*, *t*, may be integers or fractions, affirmative or negative; and which may be continued to as many terms as we please; for from these four terms set down, the law of continuation is sufficiently manifest.

$\int ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{b+cy^n}|^{m+1}$ into $\frac{ay^t}{tb} - \frac{t+mn+n}{t+n} \times \frac{ac}{tbb}y^{t+n} + \frac{t+mn+n}{t+n} \times \frac{t+mn+2n}{t+2n} \times \frac{ac^2}{tb^3}y^{t+2n} - \frac{t+mn+n}{t+n} \times \frac{t+mn+2n}{t+2n} \times \frac{t+mn+3n}{t+3n} \times \frac{ac^3}{tb^4}y^{t+3n}$, &c.

The manner of finding this canon is this. Take the fictitious equation $\int ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{b+cy^n}|^{m+1}$ into $Ay^t + By^{t+n} + Cy^{t+2n} + Dy^{t+3n} + Ey^{t+4n}$, &c.; in which the assumed quantities A, B, C, D, E, &c. are arbitrary and constant, to be determined afterwards as occasion may require. Then, by taking the fluxions of this fictitious equation, we shall have $ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{m+1} \times ncy^{n-1}\dot{y} \times \overline{b+cy^n}|^m$ into $Ay^t + By^{t+n} + Cy^{t+2n}$, &c. $+ \overline{b+cy^n}|^{m+1}$ into $tA\dot{y}y^{t-1} + \overline{t+n} \times B\dot{y}y^{t+n-1} + \overline{t+2n} \times C\dot{y}y^{t+2n-1}$, &c. Then dividing all by $\overline{b+cy^n}|^m$, and setting the terms in order, it will be

$$a\dot{y}y^{t-1} = tbA\dot{y}y^{t-1} + \overline{t+n} \times bB\dot{y}y^{t+n-1} + \overline{t+2n} \times bC\dot{y}y^{t+2n-1}, \text{ \&c.}$$
$$+ tcA\dot{y}y^{t+n-1} + \overline{t+n} \times cB\dot{y}y^{t+2n-1}, \text{ \&c.}$$
$$+ \overline{m+1} \times ncA\dot{y}y^{t+n-1} + \overline{m+1} \times ncB\dot{y}y^{t+2n-1}, \text{ \&c.}$$

Here the term $a\dot{y}y^{t-1}$ might be transposed to the other side of the equation by which the whole will be equal to nothing, and therefore the co-efficients of each term will be equal to nothing, by which we should have as many equations as there are arbitrary quantities A, B, C, D, &c. by which they will be determined. Or, making the first terms on each side equal, it will be $tbA = a$, or $A = \frac{a}{tb}$. Then $\overline{t+n} \times bB + tcA + \overline{m+1} \times ncA = 0$, and substituting the value of A, it is $tbB + nbB + \frac{ac}{b} + \frac{mnac}{tb} + \frac{nac}{tb} = 0$, or $B = \frac{t+mn+n}{t+n} \times -\frac{ac}{tb^2}$. Again, $\overline{t+2n} \times bC + \overline{t+n} \times cB + \overline{m+1} \times ncB = 0$, or $C = \frac{\overline{t+n} \times -cB + \overline{m+1} \times -ncB}{b \times \overline{t+2n}}$, and substituting the value of B, it will be

C =

$C = \frac{\overline{t+mn+n} \times \overline{t+mn+2n} \times acc}{\overline{t+n} \times \overline{t+2n} \times tb^3}$. And thus from one to another, till we have the values of as many as we pleafe of the feveral affumed conftants; and thefe values, fubftituted in the fictitious equation, will fupply us with the aforefaid canon.

If the exponents m, n, t, of the propofed formula fhall be fuch, that the canon or infinite feries will break off, or that any term fhall become $= 0$, (in which cafe all the others that follow will alfo be $= 0$,) the feries becomes finite and terminated, or we fhall have the algebräical integral of the propofed differential formula. But it is neceffary that the feries fhould firft break off in the numerator, or that the numerator fhould become equal to nothing before the denominator. For, if the denominator be equal to nothing firft, that term and all that follow after will be equal to infinite. Now, that the feries fhould break off in the numerator, it is neceffary that $-\frac{t}{n} - m$ fhould be equal to fome integer affirmative number.

But if the exponents t, m, n, of the propofed formula fhould be fuch, that the feries never breaks off; then the expreffion of the formula fhould be changed into another equivalent to it. Thus, for example, the formula $a\dot{y}y^{t-1} \times \overline{b+cy^n}|^m$ fhould be changed into this other, $a\dot{y}y^{t-1+mn} \times \overline{by^{-n}+c}|^m$, which is equivalent to the firft, and it fhould be tried whether or not this will anfwer our expectation. If not, the formula will not be algebräically integrable, at leaft not by this canon. If the formula were $a\dot{y}y^{t-1} \times \overline{b-cy^n}|^m$, then all the terms of the canon would be pofitive.

Let it be $\frac{a^5\dot{x}\sqrt{bx+xx}}{x^5}$, that is, $a^5\dot{x}x^{-\frac{9}{2}} \times \overline{b+x}|^{\frac{1}{2}}$; it will be $t-1 = -\frac{9}{2}$, $n = 1$, $m = \frac{1}{2}$, $c = 1$; whence the quantity $t + mn + 3n$ will be equal to nothing, and confequently the fourth term $= 0$, and the others of the feries that follow. Therefore we fhall have $\int \frac{a^5\dot{x} \times \sqrt{bx+xx}}{x^5} = \int a^5\dot{x}x^{-\frac{9}{2}} \times \overline{b+x}|^{\frac{1}{2}}$

$$= \overline{-\frac{2a^5x^{-\frac{7}{2}}}{7b} + \frac{2a^5}{7bb} \times \frac{4}{5}x^{-\frac{5}{2}} - \frac{2a^5}{7b^3} \times \frac{8}{15}x^{-\frac{3}{2}}} \times \overline{b+x}|^{\frac{3}{2}} =$$

$$\frac{-30a^5b^2 + 24a^5bx - 16a^5x^2}{105b^3x^{\frac{7}{2}}} \times \overline{b+x}|^{\frac{3}{2}}.$$

Let

Let it be $\frac{a\dot{y}}{yy\sqrt{aa+yy}}$; then $t = -1$, $n = 2$, $m = -\frac{1}{2}$, $c = 1$, $b = aa$; and therefore the ſecond term of the ſeries will be $= 0$. Hence

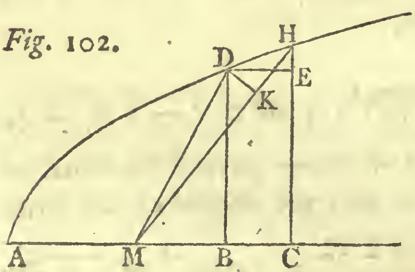
$$\int \frac{a\dot{y}}{yy\sqrt{aa+yy}} = \frac{ay^{-1}}{-aa} \times \overline{aa+yy}|^{\frac{1}{2}} = -\frac{\sqrt{aa+yy}}{ay}.$$

SECT. III.

The Rules of the foregoing Sections applied to the Rectification of Curve-lines, the Quadrature of Curvilinear Spaces, the Complanation of Curve Superficies, and the Cubature of their Solids.

75. To ſhow the uſe of the foregoing Rules of the Integral Calculus, by applying it to the quadrature of ſpaces, to the rectification of curves, to the complanation or quadrature of ſuperficies, and to the cubature of ſolids; let there be any curve ADH referred to an axis AB, with the ordinates parallel to each other, and at right angles to the axis. Draw CH parallel to the ordinate BD, and infinitely near to it, and alſo DE parallel to BC; the mixtilinear figure BDHC will be the fluxion, the differential, or the element of the ſpace ABD; and becauſe the ſpace DEH is nothing in reſpect of the rectangle BDEC, we may take that rectangle for the element of the ſaid ſpace ABD. Therefore the ſum of all theſe infiniteſimal rectangles BDEC will be the ſpace comprehended by the curve AD, and by the co-ordinates AB and BD. Wherefore, making $AB = x$, $BD = y$, it will be $BC = \dot{x}$, $EH = \dot{y}$, and the rectangle $BDEC = y\dot{x}$ will be the formula for ſuch ſpaces. Therefore, in this formula, inſtead of y, if we ſubſtitute it's value given by x, and by the conſtant quantities of the equation of the curve; or, inſtead of $\dot{x}$, it's value given

Fig. 102.



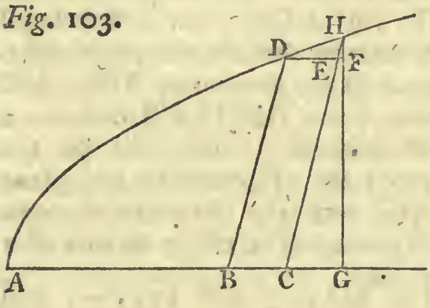
given by y and $\dot{y}$, and the conftants, and then integrate the formula, this integral will be the required fpace ABD.

Other expreffions or formulæ may be had for the elements of fpaces, by means of fectors, or of trapezia, which, on certain occafions, are fometimes more convenient than rectangles; we fhall hereafter fee the ufe and manner of them in fome examples.

76. For, if the curve be referred to a *focus*, that is, to a fixed point, fuppofe to M, from whence all the ordinates proceed; drawing MH infinitely near to the ordinate MD, the infinitefimal fpace MHD will be the element of the fpace AMD. Then with centre M and radius MD, drawing the infinitely little arch DK, the little fpace DKH will be nothing in refpect of the fpace MDK; and alfo, becaufe the little arch DK may be affumed for the tangent in D, or in K, it thence follows that the fpace MDK fhall be the element of the fpace AMD.

Wherefore, calling MD $= y$, KD $= \dot{z}$, it will be $\frac{1}{2}y\dot{z}$ for the general formula of the fpaces, in curves referred to a *focus*. And in this formula, inftead of $\dot{y}$, or of $\dot{z}$, if the refpective values be fubftituted from the equation of the curve, the integral will be the fpace required AMD.

Fig. 103.



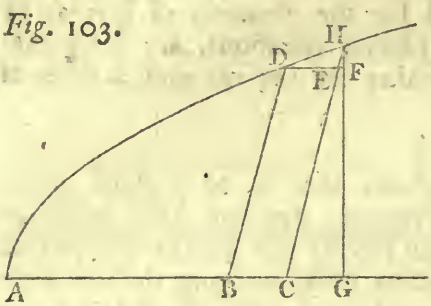
77. But if the curve fhall be referred to a diameter, fo that the ordinates fhall not be at right angles to their abfciffes; drawing HG perpendicular to AG, the product of HG, or of FG into BC, will be the little parallelogram BCED, and confequently the element of the area ABD. Therefore the angle DBG being given, and confequently the ratio of the whole fine to the right fine, which, for example, may be that of m to n; making, as ufual, AB $= x$, BD $= y$, then will HG or FG be $= \frac{ny}{m}$, and the parallelogram BCED will be $\frac{ny\dot{x}}{m}$, a general formula for this fpace.

78. It is plain, that the fum of all the infinitefimal portions DH of the curve will form the curve itfelf, and therefore that DH will be it's element. Making, therefore, AB $= x$, (Fig. 102.) BD $= y$, and thence BC $= \dot{x}$, EH $= \dot{y}$; in fuch curves as are referred to an axis, that is, with the co-ordinates at right angles, it will be DH $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, a general formula for the rectification of thefe curves.

79. As

79. As to fuch curves as are referred to a *focus*, making alfo MD $= y$, KD $= \dot{z}$, we fhall have, in like manner, $\sqrt{\dot{y}\dot{y} + \dot{z}\dot{z}}$ for a general formula.

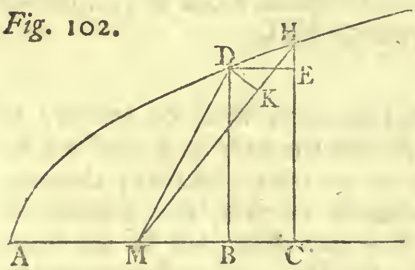
Fig. 103.



80. But as to the curves with the co-ordinates at oblique angles, the given angle being HCG, the ratio of the whole fine to the fine of the complement is given, which fuppofe is that of m to e; whence it will be CG $= \frac{ey}{m}$, and EF $= \frac{e\dot{y}}{m}$, and therefore DH $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$.

81. Now in each of thefe formulæ, inftead of $\dot{y}$, or $\dot{x}$, or $\dot{z}$, fubftituting their refpective values given by the other variable, and their differentials from the equation of the curve, and then making the integrations, we fhall have the length of the curve required.

Fig. 102.



82. Let the plane AHC be conceived to move about the right line AC, the curve AH will defcribe a fuperficies, while the plane AHC defcribes a folid. But the infinitefimal portion DH will defcribe an infinitefimal zone, which will be the element of the fuperficies defcribed by the curve AH. And the infinitefimal plane DBCH will defcribe a folid alfo infinitefimal, which will be the element of the folid defcribed by the plane AHC. Now, as to curves referred to an axis with the ordinates at right angles; let the ratio of the radius to the circumference of a circle be that of r to c; the circumference defcribed with radius BD $= y$ will be $\frac{cy}{r}$, and therefore $\frac{cy}{r}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ will be the expreffion of the infinitefimal zone, and confequently the general formula for the fuperficies.

83. Alfo, $\frac{cyy}{2r}$ will be the area of the circle defcribed with radius BD $= y$, and therefore $\frac{cyy\dot{x}}{2r}$ will be the expreffion of the infinitely little cylinder defcribed by the rectangle BCED. Now this does not differ from the folid generated by the plane BCHD, but by an infinitefimal quantity of the fecond order; therefore the general formula for thefe folids will be $\frac{cyy\dot{x}}{2r}$.

84. But

84. But as to the cafe of Fig. 103; that is, when the co-ordinates make a given oblique angle to each other; the radius of the circle, on which the little zone and the little cylinder infift, it is not CH $= y$, but indeed GH $= \frac{ny}{m}$; as likewife the element DH, which forms the zone, is not $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, but $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$; and the height of the little cylinder is not BC $= \dot{x}$, but FD $= \dot{x} + \frac{e\dot{y}}{m}$. Therefore the formula for the fuperficies, in this cafe, will be $\frac{cny}{rm}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$.

85. The product of the circle with radius GH into the height FD, that is, $\frac{cnnyy}{2rmm} \times \overline{\dot{x} + \frac{e\dot{y}}{m}}$, is the element of the folid generated by the plane AGH. Therefore, from this fubtracting the element of the folid generated by the triangle HCG, that is, $\frac{cnnyy}{2rmm} \times \frac{e\dot{y}}{m}$, what remains will be the element of the folid generated by the plane ABD, and therefore will be $\frac{cnnyy\dot{x}}{2rmm}$, the general formula for thefe folids.

86. As to the curves referred to a focus, becaufe of the variable angle DMB, (Fig. 102.) and confequently becaufe we cannot have the value of BD or CH, the radius of the circle, which muft neceffarily enter the formula of the quadrature of the fuperficies, and the cubature of the folid; it will be neceffary, from the equation referred to the focus, to derive the equation of the fame curve referred to an axis, and then we are to proceed in the manner before fpecified; obferving that, in the cubature, it will be neceffary to fubtract from the integral the cone generated by the triangle MHC, to have the folid generated by the plane AMD.

87. From the differential equation of a curve to the focus, to obtain the equation of the fame curve to an axis, the manner is this following.

Let the curve ADH (Fig. 102.) be confidered, at the fame time, both as related to the focus M, and alfo to the axis AMB. It is certain that the fquare of HD, the element of the curve, is equal as well to the two fquares DK, KH, as to the two others DE, EH; and moreover, that the fquare of MD is equal to the two fquares MB, BD. Making MB $= x$, BD $= y$, MD $= z$, and the little arch DK $= \dot{u}$, we fhall have $\dot{z}\dot{z} + \dot{u}\dot{u} = \dot{x}\dot{x} + \dot{y}\dot{y}$, and $xx + yy = zz$.

 Now

Now the equation of the curve to the focus is expreſſed, in general, by the formula $p\dot{z} = \dot{u}$, in which p is a known function or power of z; and it will be $\dot{z}\dot{z} + pp\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$. And putting, inſtead of $\dot{y}$, it's value ariſing from the equation $xx + yy = zz$, that is, $\dot{y} = \frac{z\dot{z} - x\dot{x}}{\sqrt{zz - xx}}$, we ſhall find $\dot{z}\dot{z} + pp\dot{z}\dot{z} = \dot{x}\dot{x} + \frac{\overline{z\dot{z} - x\dot{x}}^2}{zz - xx}$, which may be reduced to this following, $pp\dot{z}\dot{z} \times \overline{zz - xx} = zz\dot{x}\dot{x} - 2xz\dot{x}\dot{z} + xx\dot{z}\dot{z}$; and extracting the ſquare-root, it will be $p\dot{z} = \frac{z\dot{x} - x\dot{z}}{\sqrt{zz - xx}}$.

It is neceſſary to clear again the foregoing equation, by freeing it from a mixture of unknown quantities, by making $x = \frac{zq}{b}$, and therefore $\dot{x} = \frac{z\dot{q} + q\dot{z}}{b}$. By the help of this aſſumed ſubſidiary equation, make x and it's functions to vaniſh, and we ſhall have $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$. In this equation, if the value of p given by z ſhall be ſuch, that the quantity $\frac{p\dot{z}}{z}$ may be reduced to the differential of a circular arch by due ſubſtitutions; and that, making the neceſſary integrations, the two circular arches ſhall be to each other as number to number; then the curve ſhall be algebraical, and we ſhall find it's equation to the axis by a formula, after the manner of *Carteſius*. In every other caſe the curve will be tranſcendental.

EXAMPLE.

Let the equation of a curve referred to a focus be $\frac{z\dot{z}}{\sqrt{cc - 2bz - zz}} = \dot{u}$. We ſhall have, in this caſe, $p = \frac{z}{\sqrt{cc - 2bz - zz}}$; and in the equation $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$ ſubſtituting the value of p, it will be $\frac{\dot{z}}{\sqrt{cc - 2bz - zz}} = \frac{\dot{q}}{\sqrt{bb - qq}}$. Make $b + z = t$, then $bb + 2bz + zz = tt$, and $bb - tt = -2bz - zz$; wherefore, making the ſubſtitution, it will be $\frac{\dot{t}}{\sqrt{cc + bb - tt}} = \frac{\dot{q}}{\sqrt{bb - qq}}$.

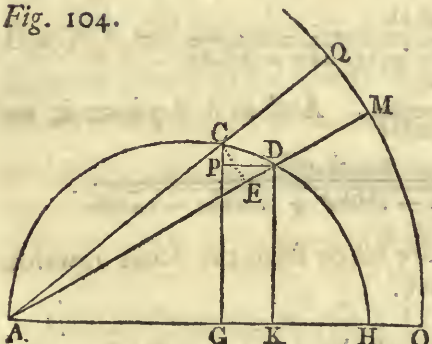
For

For a particular cafe, let it be $cc + bb = hh$, on which fuppofition it will be $t = q$, that is, $b + z = q = \frac{hx}{z}$. Therefore $bz + zz = hx$, and, inftead of z, fubftituting it's value, the equation of the curve will be $b\sqrt{xx + yy} + xx + yy = hx$.

88. The affigned canon alfo teaches us the manner of paffing from the differential equation of a curve to the axis to that of the focus, in the way following.

EXAMPLE I.

Fig. 104.



Let it be propofed to find the equation to a focus in a circle, taking the focus in a point of the circumference A.

Make AH $= b$, AG $= x$, AC $= z = \sqrt{bx}$. Refume the formula $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$, where is taken $q = \frac{bx}{z}$. Becaufe, by the local equation of the circle, it is $bx = zz$, it will be $q = z$. Then making q to vanifh, by fubftituting it's value z, it will be $\frac{p\dot{z}}{z} = \frac{\dot{z}}{\sqrt{bb - zz}}$, or $p = \frac{z}{\sqrt{bb - zz}}$. Therefore, in the formula $p\dot{z} = \dot{u}$, if, inftead of p, we fhould fubftitute it's value now found, it will be $\frac{z\dot{z}}{\sqrt{bb - zz}} = \dot{u}$, an equation of the circle to the focus, which is taken in A, a point of the circumference.

EXAMPLE II.

89. Let it be propofed to find the equation of a conic fection, referred to it's umbilicus M, that is, to it's focus (Fig. 102.)

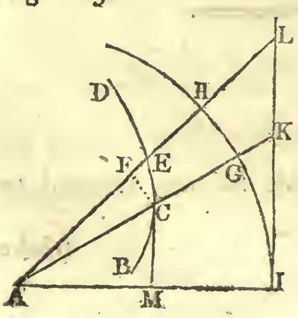
Make

Make MB $= x$, BD $= y$; the general equation, which comprehends all the ſections of a cone, will be this; $a \pm \frac{cx}{b} = \sqrt{xx + yy}$; to the parabola with the parameter $2a$, when $c = b$; to the ellipſis with tranſverſe axis $= \frac{2abb}{bb - cc}$, if b be greater than c; to the hyperbola with tranſverſe axis $= \frac{2abb}{cc - bb}$, with conjugate axis $= \frac{2ab}{\sqrt{cc - bb}}$, diſtance of the vertex from the focus $= \frac{ab}{b + c}$, if b be leſs than c. If $c = 0$, it will be to the circle with diameter $= 2a$. Put $\sqrt{xx + yy} = z$; therefore $a \pm \frac{cx}{b} = z$. And beſides, $bx = zq$; then $a \pm \frac{czq}{bb} = z$, or $\pm \frac{bb}{c} \mp \frac{abb}{cz} = q$. And taking the fluxions, $\pm \frac{abb\dot{z}}{ccz z} = \dot{q}$, and $qq = \frac{bbbb}{cc} - \frac{2abbbb}{ccz} + \frac{aabbbb}{cczz}$, and $bb - qq = bb - \frac{bbbb}{cc} + \frac{2abbbb}{ccz} - \frac{aabbbb}{cczz}$. Hence $\frac{\dot{q}}{\sqrt{bb - qq}} = \frac{\pm abcb\dot{z}}{cz\sqrt{bbcczz - bbbbzz + 2abbbbz - aabbbb}} = \frac{p\dot{z}}{z}$, and therefore $p = \frac{\pm abb}{\sqrt{bbcczz - bbbbzz + 2abbbbz - aabbbb}}$. And as it is $p\dot{z} = \dot{u}$, we ſhall have the equation required, $\dot{u} = \frac{\pm abb\dot{z}}{\sqrt{bbcczz - bbbbz^2 + 2abbbbz - aabbbb}}$.

The negative ſign ſerves when the abſciſſes are taken from the focus towards the vertex, and the poſitive are the contrary way.

90. I ſaid we ought to reduce the equation of the curve to the focus, to another referred to the axis; not becauſe this is abſolutely neceſſary for the complanation of ſuperficies, or for the cubature of ſolids; for the whole may be obtained by means of this known theorem: The periphery of the curve, drawn into the line deſcribed by the centre of gravity of that periphery, is equal to the ſuperficies of the ſolid which is generated by it's rotation. And the area of the curve, drawn into the line deſcribed by the centre of gravity of the ſaid area, is equal to the ſaid ſolid. But here we muſt not ſuppoſe our readers ſo ſkillful as to be acquainted with the theory of Centres of Gravity.

Fig. 105.



Now, to have a competent notion of curves referred to a focus, I ſhall make an attempt at finding out their conſtruction. Let BCD be one of theſe; the co-ordinates infinitely near are AC, AE, which proceed from the point A, and may be called z, their difference FE $= \dot{z}$, and the little arch CF, deſcribed with centre A, may be $= \dot{u}$. The nature of the curve is commonly expreſſed by the differential equation

tion $p\dot{z} = \dot{u}$, in which p is any how given by z. Wherefore it muft be obferved, that the firft member $p\dot{z}$, having the variable z, all which take their origin from the pole A, is integrable either algebraically or tranfcendentally. But the other member $\dot{u}$ cannot be integrated without falling into a parallogifm, as not being yet the complete fluxion of the arch u. For that element $\dot{u}$ increafes or decreafes in a double refpect, that is, in itfelf, and alfo by the increafing or diminifhing of the ordinates AC, AE. To proceed, therefore, with accuracy, with any radius at pleafure, AI $= r$, let a circle IGH be defcribed, and in the periphery let any determinate point I be taken, from which, as from a fixed point, the increafing arches IG, IH, have their origin. And producing, if neceffary, the variables AC, AE, to G and H, the fectors ACF, AGH, will be fimilar, and therefore it is $z . \dot{u} :: r$. GH, which may be called $\dot{q}$. Then $\frac{z\dot{q}}{r} = \dot{u}$. But, by the general equation of the curve, it is $p\dot{z} = \dot{u}$; then $\frac{z\dot{q}}{r} = p\dot{z}$, and therefore $\frac{rp\dot{z}}{z} = \dot{q}$. Now, by finding the fluent, it will be $\int \frac{rp\dot{}}{z} = q =$ IG. The adding or taking away of the conftants in the integration, will have no other effect, but to diverfify the fituation of the point I.

EXAMPLE I.

Let the logarithmic fpiral be to be conftructed, the equation of which is $\frac{a\dot{z}}{b} = \dot{u}$. But $\dot{u} = \frac{z\dot{q}}{r}$, therefore $\frac{a\dot{z}}{b} = \frac{z\dot{q}}{r}$. Or, becaufe the radius AI is affumed at pleafure, making $b = r$, and taking a as unity, it will be $\frac{\dot{z}}{z} = \dot{q}$. And by integration, $lz = q$, the geometrical conftruction of which is tranfcendental, but yet is very fimple.

EXAMPLE II.

Let it be the hyperbolical fpiral, with the conftant fubtangent $= a$, and therefore the equation is $\frac{a\dot{z}}{z} = \dot{u}$. But $\dot{u} = \frac{z\dot{q}}{r}$, therefore $\frac{ar\dot{z}}{zz} = \dot{q}$; and by integrating, it will be $b - \frac{ar}{z} = q$.

In

In fuch conftructions we have always the circular arch IG, which forms the *homogeneum comparationis*; the other member $\int \frac{rp\dot{z}}{z}$ may be analytically integrable, as in the fecond example, or tranfcendentally, by means of the quadrature of the hyperbola, as in the firft, or by any other method more compounded. Whence, in one cafe only, our curves may be algebräical, and that is, when the quantity $\int \frac{rp\dot{z}}{z}$ may be reduced to the rectification of an arch of a circle, which to it's correfpondent IG is as number to number. If the proportion happen to be furd, then the curve will indeed be mechanic, as BCED, but not dependent on the quadrature of the circle, being reduced to a different problem, confifting in the dividing circular arches in any given ratio; which may be obtained by means of the helix or fpiral of *Archimedes*, or of the quadratrix of *Dinoftratus*.

The things afore-mentioned furnifh us with another manner of paffing from expreffions of curves to a focus, to thofe which are referred to an axis, or on the contrary. For, becaufe $\frac{rp\dot{z}}{z} = \dot{q} = \frac{rr\dot{t}}{rr + tt}$, making the tangent IK $= t$, (§ 26.) this tangent t will be given analytically or tranfcendentally by z. But AI $= r$, AK $= \sqrt{rr + tt}$, AM $= x$, MC $= y$. Therefore $\frac{rz}{x} = \sqrt{rr + tt}$, and, after due reductions, $\frac{r\sqrt{zz - xx}}{x} = t = \frac{ry}{x}$. But t is given by z, and $z = \sqrt{xx + yy}$; fo that we are arrived at the curve in refpect to the axis, which may foon be reduced to the ufual co-ordinates x and y. By going the fame fteps backwards, we may pafs from the equation to the axis, to that in refpect of the focus.

I refume the example of § 87; that is, the curve $\frac{z\dot{z}}{\sqrt{cc - 2bz - zz}} = \dot{u}$ referred to a focus, to reduce it to the axis. Now, if $p\dot{z} = \dot{u}$ be taken for a general equation of curves referred to a focus, it will be, in this particular cafe, $p = \frac{z}{\sqrt{cc - 2bz - zz}}$. So that, fubftituting this value, inftead of p, in the equation $\frac{rp\dot{z}}{z} = \dot{q} = \frac{rr\dot{t}}{rr + tt}$, it will be $\frac{r\dot{z}}{\sqrt{cc - 2bz - zz}} = \frac{rr\dot{t}}{rr + tt}$. Make $b + z = s$, $\dot{z} = \dot{s}$, $bb + 2bz + zz = ss$; whence $- 2bz - zz = bb - ss$. And fubftituting thefe values, it will be $\frac{r\dot{s}}{\sqrt{cc + bb - ss}} = \frac{rr\dot{t}}{rr + tt}$. Making $cc + bb = hh$, it will be $\frac{r\dot{s}}{\sqrt{hh - ss}} = \frac{rh\dot{s}}{h\sqrt{hh - ss}} = \frac{rr\dot{t}}{rr + tt}$. But the integral of the

the firſt member will be the arch of a circle, the radius of which is b, and s is the ſine of the complement (§ 37.) multiplied by the conſtant fraction $\frac{r}{b}$; and the integral of the ſecond is an arch of a circle with radius $= r$, and tangent equal to t. Wherefore the firſt arch will be to the ſecond as b to r, or they will be to each other as their radii reſpectively; then they will be ſimilar, and therefore their tangents alſo will be in the ſame ratio as their radii. Therefore the tangent of the firſt arch is $\frac{b}{s}\sqrt{bb - ss}$; and it will be $\frac{b}{s}\sqrt{bb - ss} \,.\, t :: b \,.\, r$, or $t = \frac{r}{s}\sqrt{bb - ss}$. So that, reſtoring the value of s, and putting $\frac{ry}{x}$ inſtead of t, we ſhall have $\frac{ry}{x} = \frac{r\sqrt{bb - bb - 2bz - zz}}{b + z}$, which is an equation reduced to the axis, and which may be expreſſed by the co-ordinates x and y only, by putting, inſtead of zz, it's value $xx + yy$. Then the equation will be $by + y\sqrt{xx + yy} = x\sqrt{bb - bb - 2b\sqrt{xx + yy} - xx - yy}$, which is the ſame as that found at § 87, as before cited.

To paſs from equations to an axis, to thoſe belonging to a focus, I take Example I. at § 88, the equation of which to the circle is $z = \sqrt{bx}$ (Fig. 104.) The tangent given by z of the arch OQ, deſcribed with centre A, and radius r, was found to be $\frac{r\sqrt{bb - zz}}{z} = t$. Then, in the canonical equation $\dot{q} = \frac{rr\dot{t}}{rr + tt}$, inſtead of t and $\dot{t}$, ſubſtituting their reſpective values, we ſhall have $-\dot{q} = -\frac{r\dot{z}}{\sqrt{bb - zz}}$. I put it $-\dot{q}$, becauſe, as AC $= z$ increaſes, the arch OQ $= q$ will diminiſh. But $\dot{q} = \frac{r\dot{u}}{z}$; wherefore $\frac{r\dot{u}}{z} = \frac{r\dot{z}}{\sqrt{bb - zz}}$; that is, $\frac{z\dot{z}}{\sqrt{bb - zz}} = \dot{u}$, which is the ſame equation as that found at § 88.

91. The particular formulæ, which are found in the caſe of curves having their co-ordinates at oblique angles, are not leſs uſeful, becauſe ſuch equations may always be changed into others, which have their co-ordinates at right angles; and after that we may make uſe of the ordinary formulæ.

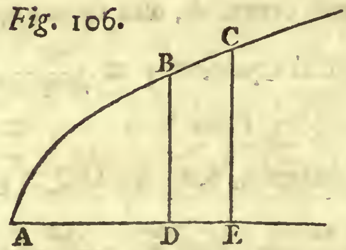
To ſhow this, make HG $= p$, (Fig. 103.) AG $= q$; then it is $p = \frac{ny}{m}$, $q = x + \frac{ey}{m}$, naming, as before, AB $= x$, BD $= y$, and the ratio of the whole ſine to the right ſine that of m to e. Therefore it will be $y = \frac{mp}{n}$, and $x = q - \frac{ey}{m} = q - \frac{ep}{n}$. Wherefore, inſtead of x and y, ſubſtituting, in the propofed equation, thefe values given by p and q, we fhall have the equation of the curve with the ordinates at right angles to each other. But it will often happen that the primitive equation will be fimple; and yet, by transforming it, it may become fufficiently compound. Alfo, though the variables are feparate in the propofed equation, they may not be fo in the transformed equation; and what may increafe the difficulty, they cannot be feparated by the ordinary rules of Divifion, Extraction of Roots, &c. However, in many particular cafes, perhaps it may not be amifs to try each method, that we may make choice of that which, in the given cafe, fhall be moft convenient.

But now it will be time to proceed to Examples, in which it is always underftood, except when warning is given to the contrary, that the co-ordinates are at right angles to each other.

EXAMPLE I.

The quadrature of curvilinear fpaces.

Fig. 106.



92. Let ABC be an *Apollonian* parabola, with the equation $ax = yy$, any abfcifs AD $= x$, it's ordinate DB $= y$, and the fpace ADB is to be fquared. Therefore it will be $y = \sqrt{ax}$; and this value, being fubftituted, inftead of y, in the general formula for fpaces $y\dot{x}$, it will be $\dot{x}\sqrt{ax}$; and by integration, it will be $\frac{2}{3}x\sqrt{ax} + b$. The quantity b is the ufual conftant, which, in the integration, ought to be added, and which now ought to be determined. In the point A, that is, when $x = 0$, the fpace is nothing, and therefore the integral $\frac{2}{3}x\sqrt{ax} + b$, which expreffes this fpace, ought alfo to be nothing. Therefore, making $x = 0$, it will be $\frac{2}{3}0 \times \sqrt{a} \times 0 + b = 0$, that is, $b = 0$; which is as much as to fay that, in this cafe, no conftant quantity is to be joined to the integral. Therefore the fpace ABD $= \frac{2}{3}x\sqrt{ax}$. But $\sqrt{ax} = y$. Whence ABD $= \frac{2}{3}xy$, that is, is equal to two third parts of the rectangle of the abfcifs into the ordinate.

Now, if we fhould require the fpace comprehended by an affigned and determinate abfcifs and ordinate, for example, when it is $x = 2a$; as, by the equation of the curve, it is in this cafe $y = \sqrt{2aa}$, this fpace will be $= \frac{4}{3}aa\sqrt{2}$. If the abfciffes of the parabola fhould not begin at the vertex A, but at fome given point D; making, for example, AD $= a$, any line DE $= x$, the parameter $= f$, the equation will be $af + fx = yy$, and $y = \sqrt{af + fx}$. Subftituting

tuting this value in the formula $y\dot{x}$, it will be $\dot{x}\sqrt{af+fx}$, and by integrating, $\frac{2}{3} \times \overline{a+x} \times \sqrt{af+fx} + b$ will be equal to the ſpace DECB. But, to determine the conſtant quantity b, it muſt be conſidered, that at the point D, where $x = 0$, the ſpace will alſo be $= 0$; ſo that, in the integral, making $x = 0$, it will be $\frac{2}{3}a\sqrt{af} + b = 0$, and therefore the conſtant $b = -\frac{2}{3}a\sqrt{af}$. So that, to have the integral complete, inſtead of adding b, we muſt ſubtract $\frac{2}{3}a\sqrt{af}$, and therefore the ſpace required will be DECB $= \frac{2}{3} \times \overline{a+x} \times \sqrt{af+fx} - \frac{2}{3}a\sqrt{af}$.

Let AE $= a$, and let x begin at E towards A, and take any line ED $= x$; the equation will be $af - fx = yy$, and $y = \sqrt{af-fx}$. Whence $y\dot{x} = \dot{x}\sqrt{af-fx}$, and by integration, it will be $-\frac{2}{3} \times \overline{a-x} \times \sqrt{af-fx} + b$. But when $x = 0$, the ſpace alſo $= 0$. Therefore, in the integral, making $x = 0$, it will become $-\frac{2}{3}a\sqrt{af} + b = 0$, or $b = \frac{2}{3}a\sqrt{af}$. Therefore the ſpace EDBC $= \frac{2}{3}a\sqrt{af} - \frac{2}{3}\overline{a-x}\sqrt{af-fx}$.

It may be obſerved, that, in general, the parabolical ſpace AEC $= \frac{2}{3}$AE $\times$ EC; wherefore the ſpace ADB $= \frac{2}{3}$AD $\times$ DB; ſo that the ſpace DECB will be $= \frac{2}{3}$AE $\times$ EC $- \frac{2}{3}$AD $\times$ DB; which agrees with the calculus in both caſes, when the origin of x is in the point D towards E, and in the point E towards D.

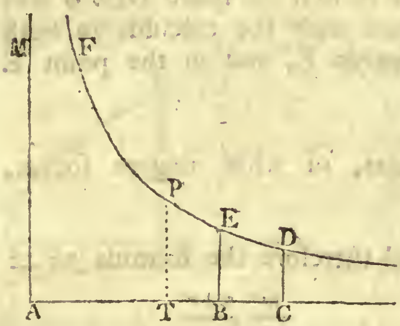
I take the general equation to all parabolas, of what degree ſoever, $a^m x^n = y^r$; whence it will be $y = a^{\frac{m}{r}} x^{\frac{n}{r}}$, and therefore the formula $y\dot{x} = a^{\frac{m}{r}} x^{\frac{n}{r}} \dot{x}$; and, by integration, the ſpace will be $= \frac{ra^{\frac{m}{r}} x^{\frac{n+r}{r}}}{n+r} + b$. But, taking $x = 0$, it is found that $b = 0$; ſo that there is no conſtant quantity to be annexed to it, but the integral before found is complete. Now, putting y inſtead of $a^{\frac{m}{r}} x^{\frac{n}{r}}$, it will be $\frac{rxy}{n+r} =$ to the ſpace required.

EXAMPLE II.

93. Let the curve be $y = \sqrt[m]{x + a}$; therefore it will be $y\dot{x} = \dot{x}\sqrt[m]{x + a}$; and, by integration, the ſpace will be $\frac{m}{m+1} \times \overline{x + a} \times \overline{x + a}\big|^{\frac{1}{m}} + b$. But, making $x = 0$, it will be $b = -\frac{m}{m+1} \times a\sqrt[m]{a}$. Therefore the complete integral or ſpace required $= \frac{m}{m+1} \times \overline{x + a} \times \sqrt[m]{x + a} - \frac{m}{m+1} \times a\sqrt[m]{a}$.

EXAMPLE III.

Fig. 107.



94. Let FED be the hyperbola between the aſymptotes, and make AB $= x$, BE $= y$, and the equation is $xy = aa$. Then $y = \frac{aa}{x}$, and therefore $y\dot{x} = \frac{aa\dot{x}}{x}$; and, by integration, the ſpace will be $= alx + b$, taking the logarithm from the logarithmic curve with ſubtangent $= a$. But, putting $x = 0$, the logarithm of 0 is an infinite negative quantity, and therefore the ſpace is infinite which is contained by the curve EF continued *in infinitum*, by the aſymptote, and by the co-ordinates AB, BE.

Let there be a hyperboloid of this equation $a^3 = xyy$; then $y = \sqrt{\frac{a^3}{x}}$, and therefore $y\dot{x} = \dot{x}\sqrt{\frac{a^3}{x}}$; and, by integration, the ſpace will be $= 2\sqrt{a^3x} + b$. Now, putting $x = 0$, it is $b = 0$; therefore no conſtant quantity need be added to complete the integral. So that the ſpace ABEF, infinitely produced upwards, will be the finite quantity $2\sqrt{a^3x}$, or from the equation of the curve $= 2xy$.

Let there be a hyperboloid of this equation, $a^3 = xxy$; then $y = \frac{a^3}{x^2}$, and $y\dot{x} = \frac{a^3\dot{x}}{xx}$; and, by integration, the ſpace will be $= -\frac{a^3}{x} + b$. But, putting

$x = 0$,

$x = 0$, it will be $\frac{a^3}{0}$, an infinite quantity, and therefore b is infinite. Wherefore, to have the integral complete, an infinite quantity ought to be added to it, and therefore the ſpace itſelf is infinite.

Let the equation be $a^{m+n} = x^n y^m$, which is to all hyperboloids in general; then $y = a^{\frac{m+n}{m}} x^{-\frac{n}{m}}$, and therefore $\int y\dot{x} = \frac{ma^{\frac{m+n}{m}} x^{\frac{m-n}{m}}}{m-n} + b$. Now, if $m = 1$, $n = 1$, that is, $xy = aa$, we ſhould have $\int y\dot{x} = \frac{aa}{0} + b$, an infinite quantity; whence the ſpace will be infinite, as was ſeen before.

If $n = 1$, $m = 2$, that is, $a^3 = xyy$, then $\int y\dot{x} = 2\sqrt{a^3x} + b$. But, putting $x = 0$, it will be alſo $b = 0$; therefore the complete integral, or the ſpace required, will be $= 2\sqrt{a^3x} = 2xy$, by the equation of the curve; which is therefore finite, though infinitely produced upwards towards F.

If $n = 2$, $m = 1$, that is, $a^3 = xxy$, it will be $\int y\dot{x} = -\frac{a^3}{x} + b$. But, making $x = 0$, b will be infinite; ſo that an infinite quantity is to be added to the integral, and the ſpace itſelf will be infinite.

If $n = 1$, $m = 3$, that is, $a^4 = xy^3$; it will be $\int y\dot{x} = \frac{3}{2}a^{\frac{4}{3}}x^{\frac{2}{3}} + b$. But, making $x = 0$, it will be $b = 0$, and therefore the integral is complete. That is, the ſpace will be $= \sqrt[3]{a^4xx} = \frac{3}{2}xy$, a finite quantity, however infinitely produced upwards.

If $n = 3$, $m = 1$, that is, $a^4 = x^3y$; it will be $\int y\dot{x} = -\frac{a^4}{2xx} + b$. But, making $x = 0$, b will be infinite, and therefore the ſpace is infinite.

If $n = 1$, $m = 4$, that is, $a^5 = xy^4$; it will be $\int y\dot{x} = \frac{4}{3}\sqrt[4]{a^5x^3} + b$. But, making $x = 0$, it will be $b = 0$; ſo that the integral is complete, and the whole ſpace $= \frac{4}{3}\sqrt[4]{a^5x^3} = \frac{4}{3}xy$, a finite quantity.

If $n = 4$, $m = 1$, that is, $a^5 = x^4y$; it will be $\int y\dot{x} = -\frac{a^5}{3x^3} + b$. Now making $x = 0$, b will be infinite, and therefore the ſpace is infinite. In the ſame manner we might proceed to other caſes, as far as we pleaſe.

Now let us take the abſciſſes from the point B, to find the ſpace BCDE. Make $AB = b$, $BC = x$, $CD = y$, and let it be the ſame *Apollonian* hyperbola, whoſe equation is $by + xy = aa$. Then it will be $y = \frac{aa}{b+x}$, and

therefore

therefore $y\dot{x} = \frac{aa\dot{x}}{b + x}$. Then, by integration, $\int y\dot{x} = al\,\overline{b + x} + f$, taking the logarithm from the logarithmic with ſubtangent $= a$. But, to determine the conſtant quantity f, making $x = 0$, it ought to be $f = -alb$; ſo that the complete integral or ſpace BCDE will be $al\,\overline{b + x} - alb$.

If we take x negative $=$ BA $= -b$, then $al\,\overline{b + x}$ is equal to a multiplied into the logarithm of o. But the logarithm of o is an infinite negative quantity; ſo that, in this caſe, the ſpace is negative, that is, towards M, and alſo infinite, as has been ſeen above; and therefore the ſpace between the *Apollonian* hyperbola and it's aſymptotes is infinite, being infinitely produced both ways.

Let it be the cubical hyperboloid whoſe equation is $byy + xyy = a^3$. It will be $y = \sqrt{\frac{a^3}{b + x}}$, whence $y\dot{x} = \dot{x}\sqrt{\frac{a^3}{b + x}}$, and by integration, $\int y\dot{x} = 2\sqrt{a^3b + a^3x} + f$. But, making $x = 0$, it will be $f = -2\sqrt{a^3b}$; ſo that the complete integral or ſpace EBCD will be $= 2\sqrt{a^3b + a^3x} - 2\sqrt{a^3b}$; and taking x infinite, the ſpace EBCD, infinitely produced towards C, will be infinite alſo.

Taking x negative $=$ BA $= -b$, the integral will be $-2\sqrt{a^3b}$, ſo that the ſpace will be negative; that is, it will be FEBAM, and will be finite, however infinitely produced towards M; as is alſo ſeen before.

Let it be the hyperboloid of this equation $\overline{b + x}|^2 \times y = a^3$. It will be $y = \frac{a^3}{\overline{b + x}|^2}$, whence $y\dot{x} = \frac{a^3\dot{x}}{\overline{b + x}|^2}$. And, by integrating, $\int y\dot{x} = -\frac{a^3}{b + x} + f$. Now, putting $x = 0$, it will be $f = \frac{a^3}{b}$, and therefore the complete integral, or the ſpace EBCD, will be $\frac{a^3}{b} - \frac{a^3}{b + x}$. Taking x infinite, the term $-\frac{a^3}{b + x}$ will be $= 0$; ſo that the ſpace will be finite, though infinitely produced towards C. Let x be negative $=$ BA $= -b$; the integral will be $\frac{a^3}{b} - \frac{a^3}{0}$. But $-\frac{a^3}{0}$ is infinite and negative, and therefore the ſpace towards M will be infinite. By proceeding in this manner, we may find that the ſpace between the *Apollonian* hyperbola and it's aſymptotes, produced both ways infinitely, will be infinite; between the firſt cubical hyperboloid and it's aſymptotes, it will be finite towards M, and infinite towards C; between the ſecond cubical hyperboloid and it's aſymptotes, it will be infinite towards M, and finite towards C; between the firſt hyperboloid of the fourth kind and it's aſymptotes,

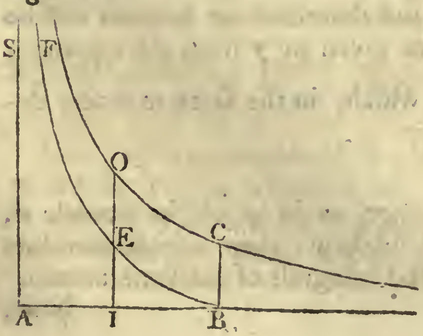
aſymptotes, it will be finite towards M, and infinite towards C; between the ſecond hyperboloid and it's aſymptotes, it will be finite towards C, and infinite towards M. And ſo on.

Now, to have recourſe to infinite ſeries: I take the expreſſion of the ſpace BCDE, of the aforeſaid *Apollonian* hyperbola, that is, $\frac{aa\dot{x}}{b+x}$. This, reduced into a ſeries, will be $= \frac{a^2\dot{x}}{b} - \frac{a^2x\dot{x}}{bb} + \frac{a^2x^2\dot{x}}{b^3} - \frac{ax^3\dot{x}}{b^4}$, &c. And, by integration, $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{ax^4}{4b^3}$, &c.; which ſeries, infinitely continued, will be accurately equal to the ſpace BCDE. And if it were ſummable, it would give us the ſpace required in finite terms, that is, algebraically: and this would be the true quadrature of the hyperbola. But as this is not ſummable, the more terms we take of it, beginning with the firſt, the nearer approach we ſhall make to the juſt value of this ſpace.

Now I take the abſciſs BT on the negative ſide, and the equation of the curve will be $by - xy = aa$, and therefore $y\dot{x} = \frac{aa\dot{x}}{b-x}$; and, reducing to a ſeries, it will be $y\dot{x} = \frac{a^2\dot{x}}{b} + \frac{a^2x\dot{x}}{b^2} + \frac{a^2x^2\dot{x}}{b^3} + \frac{a^2x^3\dot{x}}{b^4} + \frac{a^2x^4\dot{x}}{b^5}$, &c. And by integration, $\int y\dot{x} = \frac{a^2x}{b} + \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} + \frac{a^2x^4}{4b^4} + \frac{a^2x^5}{5b^5}$, &c. which is equal to the ſpace BTPE. Taking BT = BA, the ſpace FEBAM, infinitely produced towards M, will be $= aa + \frac{1}{2}aa + \frac{1}{3}aa + \frac{1}{4}aa + \frac{1}{5}aa$, &c.; the value of which ſeries being infinite, the ſpace it denotes will be infinite alſo.

EXAMPLE IV.

Fig. 108.

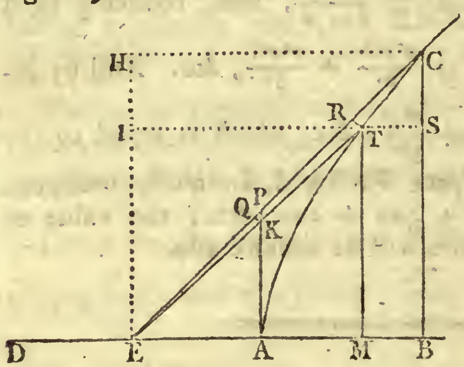


95. Let OC be an equilateral hyperbola between the aſymptotes AS, AB, and make AB = BC = a, BI = $-x$. Let the mechanical curve BEF be conceived to be deſcribed, ſuch, that the rectangle of AB into any ordinate IE may be equal to the correſponding hyperbolical ſpace BCOI. The indeterminate ſpace SABEF is required. Make the ordinate IE = z. It has been found already, that the ſpace BCOI is equal to

to the ſeries $ax + \frac{1}{2}x^2 + \frac{x^3}{3a} + \frac{x^4}{4a^2} + \frac{x^5}{5a^3}$, &c. making a and b equal. Then, by the property of the curve, it will be $z = x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{4a^3}$, &c. and therefore $z\dot{x} = x\dot{x} + \frac{x^2\dot{x}}{2a} + \frac{x^3\dot{x}}{3a^2} + \frac{x^4\dot{x}}{4a^3}$, &c. And finally, by integration, the ſpace BIE will be $= \frac{xx}{2} + \frac{x^3}{6a} + \frac{x^4}{12a^2} + \frac{x^5}{20a^3} + \frac{x^6}{30a^4}$, &c. Now, taking $x = a = $ BA, as to the whole ſpace SABEF infinitely produced, it will be $= \frac{1}{2}aa + \frac{1}{6}aa + \frac{1}{12}aa + \frac{1}{20}aa + \frac{1}{30}aa$, &c. which ſeries is ſummable, and is $= aa$; ſo that it is algebräically quadrable, and the ſpace SABEF, infinitely produced, is equal to the ſquare of BA.

EXAMPLE V.

Fig. 109.



96. Let ATC be a hyperbola, it's tranſverſe axis AD $= 2a$, the parameter $= p$, EB $= x$, BC $= y$, and therefore the equation $xx - aa = \frac{2ayy}{p}$, and let the ſpace ABC be required. It will be therefore $y = \sqrt{\frac{px^2 - pa^2}{2a}}$, and the formula will be $y\dot{x} = \dot{x}\sqrt{\frac{pxx - paa}{2a}}$. Now, if we proceed to integration, we ſhould find, after the uſual manner, that the integral is partly algebräical, and partly logarithmical; ſo that the ſpace ABC of the hyperbola depends on the deſcription of the logarithmic curve.

If we would have the ſpace ACHE; making MT infinitely near to BC, it's element will be the infiniteſimal ſpace ITCH; and therefore the formula will be $x\dot{y}$, in which, inſtead of x, ſubſtituting it's value given by y from the equation, it will be $x\dot{y} = \dot{y}\sqrt{\frac{2ayy + aap}{p}}$, the integral of which, in the ſame manner, depends upon the logarithmic curve.

And, as well in the formula $y\dot{x}$ of the firſt ſpace, as in $x\dot{y}$ of the ſecond, if, inſtead of $\dot{x}$ in that, or of $\dot{y}$ in this, we ſhould ſubſtitute their reſpective values given from the equation; we ſhould likewiſe find integrals of the ſame nature.

Now,

Now, to return to infinite ſeries. I take the formula of the ſpace ACHEA, that is, $x\dot{y}$. Then $x\dot{y} = y\sqrt{\frac{2ayy + aap}{p}}$; and, for greater facility, making $2a = p$, (for the conſtants make no alteration in the method,) that is, ſuppoſing the hyperbola to be equilateral, it will be $x\dot{y} = \dot{y}\sqrt{yy + aa}$; and, reducing the radical to an infinite ſeries, it will be $x\dot{y} = a\dot{y} + \frac{y^2\dot{y}}{2a} - \frac{y^4\dot{y}}{8a^3} + \frac{y^6\dot{y}}{16a^5} - \frac{5y^8\dot{y}}{128a^7}$, &c. And by integration, $\int x\dot{y}$, or the ſpace ACHEA, $= ay + \frac{y^3}{6a} - \frac{y^5}{40a^3} + \frac{y^7}{7 \times 16a^5} - \frac{5y^9}{9 \times 128a^7}$, &c. a ſeries, the ſummation of which is unknown. And ſubtracting this ſeries from the rectangle xy, we ſhould have the ſpace ABC.

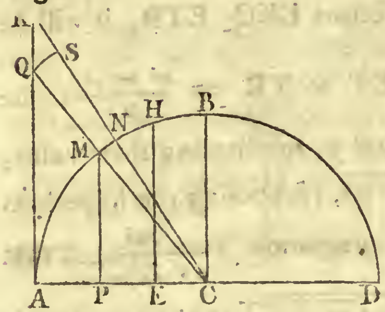
From the centre E let the lines ET, EC, be drawn infinitely near, and let AKP be a tangent at the vertex. With centre E let the little circular arches KQ, TR, be drawn. It will be $AK = \frac{ay}{x}$, $KP = \frac{ax\dot{y} - ay\dot{x}}{xx}$, $ET = \sqrt{xx + yy}$, $EK = \frac{a\sqrt{xx + yy}}{x}$. And, becauſe of ſimilar triangles PKQ, KEA, or TEM, it will be $KQ = \frac{ax\dot{y} - ay\dot{x}}{x\sqrt{xx + yy}}$. And, becauſe of ſimilar ſectors EKQ, ETR, it will be $TR = \frac{x\dot{y} - y\dot{x}}{\sqrt{xx + yy}}$; and therefore it will be $\frac{1}{2}ET \times TR = \frac{x\dot{y} - y\dot{x}}{2}$, the element of the ſector ETA. And, inſtead of y and $\dot{y}$, ſubſtituting their values given from the equation of the curve $y = \sqrt{xx - aa}$, (ſuppoſing the hyperbola to be equilateral,) it will be $\frac{aa\dot{x}}{2\sqrt{xx - aa}}$; and by integration, $\int \frac{aa\dot{x}}{2\sqrt{xx - aa}}$, that is, the ſector ETA, will be equal to $- \frac{1}{2}al\,\overline{x - \sqrt{xx - aa}}$ in the logarithmic with ſubtangent $= a$; which ſpace is therefore expreſſed by a negative quantity, becauſe it is aſſumed on the negative ſide.

By reducing the formula into a ſeries, we ſhall find $\frac{aa\dot{x}}{2\sqrt{xx - aa}} = \frac{aa\dot{x}}{2x} + \frac{a^4\dot{x}}{4x^3} + \frac{3a^6\dot{x}}{16x^5} + \frac{5a^8\dot{x}}{32x^7} + \frac{35a^{10}\dot{x}}{256x^9}$, &c.

Now, to integrate the firſt term of the ſeries, there would be occaſion, firſt, to reduce it to an infinite ſeries. Therefore it would be better to do it more expeditiouſly after the following manner. Make $EM = x$, $MT = y$, $AK = z$, then $KP = \dot{z}$. Make $KE = p$, $AE = a$, the tranſverſe ſemiaxis, and the femi-conjugate $= b$. Therefore it will be KQ $= \frac{a\dot{z}}{p}$, ET $= \frac{px}{a}$, TR $= \frac{x\dot{z}}{p}$, and therefore $\frac{1}{2}$ET $\times$ TR $= \frac{xx\dot{z}}{2a}$. But, by the equation of the curve, it is $y = \frac{b}{a}\sqrt{xx - aa}$; and by fimilar triangles EAK, EMT, it will be $y = \frac{xz}{a}$. Therefore $zx = b\sqrt{xx - aa}$, and $xx = \frac{aabb}{bb - zz}$, and confequently the formula will be $\frac{\frac{1}{2}abb\dot{z}}{bb - zz}$, which, reduced to a feries, will be $\frac{a\dot{z}}{2} + \frac{az^2\dot{z}}{2b^2} + \frac{az^4\dot{z}}{2b^4} + \frac{az^6\dot{z}}{2b^6} + \frac{az^8\dot{z}}{2b^8}$, &c.; and by integration, $\int\frac{\frac{1}{2}abb\dot{z}}{bb - zz}$, that is, the fpace ETA, will be $= \frac{az}{2} + \frac{az^3}{6b^2} + \frac{az^5}{10b^4} + \frac{az^7}{14b^6} + \frac{az^9}{18b^8}$, &c.

EXAMPLE VI.

Fig. 110.



97. Let ABD be a circle defcribed with diameter AD $= a$, and let the area of any half fegment AHE be required. Make AE $= x$, EH $= y$; the equation will be $y = \sqrt{ax - xx}$, and therefore $y\dot{x} = \dot{x}\sqrt{ax - xx}$. Here it would be to no purpofe to free the formula from it's radical, or to try any other methods, in order to change it into fome other formula, which may admit of an algebräical integration, or by means of the logarithms. For this would be an ufelefs trouble, becaufe we fhould ftill be brought to a formula of quadrature or rectification of the circle; as has been obferved at § 37. And therefore we fhall thus proceed by way of infinite feries.

Refolving the formula into a feries, it will be $\dot{x}\sqrt{ax - xx} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^{\frac{5}{2}}}$, &c. And by integration, $\int y\dot{x}$, or the fpace AEH $= \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{28a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{72a^{\frac{5}{2}}}$, &c.

Now

Now make the radius CA $= a$, and let CE $= x$, EH $= y$, and the equation will be $y = \sqrt{aa - xx}$. Therefore $y\dot{x} = \dot{x}\sqrt{aa - xx}$; and reducing this to a ſeries, $y\dot{x} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$, &c. And by integration, $\int y\dot{x}$, that is, the ſpace CEHB $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c.

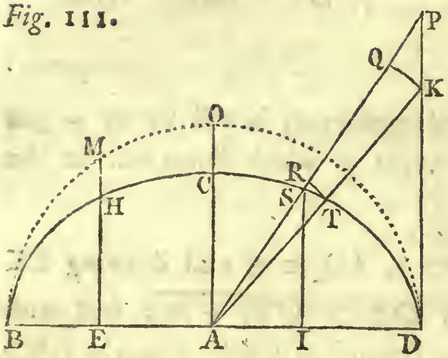
And making $x = a$, in reſpect of the whole quadrant, it will be $aa - \frac{1}{6}aa - \frac{1}{40}aa - \frac{1}{112}aa - \frac{5}{1152}aa$, &c. the quadruple of which ſeries will be the area of the whole circle.

Now, by means of a ſector. Make CA $= a$, AQ $= x$, and drawing CK infinitely near to CQ, it will be QK $= \dot{x}$, CQ $= \sqrt{aa + xx}$; and with centre C deſcribing the infiniteſimal arch QS; becauſe of ſimilar triangles KSQ, QAC, it will be QS $= \frac{a\dot{x}}{\sqrt{aa + xx}}$, and therefore MN $= \frac{aa\dot{x}}{aa + xx}$. Whence the little ſector CMN, the element of the ſector CAM, will be $= \frac{a^3\dot{x}}{2 \times \overline{aa + xx}}$, which, reduced into a ſeries, will be $= \frac{a^3\dot{x}}{2a^2} - \frac{a^3x^2\dot{x}}{2a^4} + \frac{a^3x^4\dot{x}}{2a^6} - \frac{a^3x^6\dot{x}}{2a^8}$, &c. And by integration, it will be $\int \frac{\frac{1}{2}a^3\dot{x}}{aa + xx}$, or the ſector CMA $= \frac{ax}{2} - \frac{x^3}{6a} + \frac{x^5}{10a^3} - \frac{x^7}{14a^5} + \frac{x^9}{18a^7}$, &c.; and ſuppoſing the arch AM to be half the quadrant, that is, taking $x = a$, the ſeries is $\frac{aa}{2} - \frac{aa}{6} + \frac{aa}{10} - \frac{aa}{14}$, &c.; and the double of this, or $aa - \frac{1}{3}aa + \frac{1}{5}aa - \frac{1}{7}aa$, &c. will be the quadrant ABC.

Inſtead of taking the radius CA $= a$, if I had taken it $= \sqrt{\frac{1}{8}aa}$, the quadrant would have been ABC $= \frac{aa}{8} - \frac{aa}{3 \times 8} + \frac{aa}{5 \times 8} - \frac{aa}{7 \times 8}$, &c.; and actually ſubtracting every negative term from the poſitive term before it, [and multiplying the reſult by 4,] it would be $\frac{aa}{3} + \frac{aa}{35} + \frac{aa}{99} + \frac{aa}{195}$, &c. [= the area of the whole circle;] which is the ſame ſeries as is inſerted by Mr. *Leibnitz* in the *Leipſic* Acts, for the year 1682.

EXAMPLE VII.

Fig. 111.



98. Let BCD be an ellipſis, the tranſverſe ſemiaxis AB $= a$, the ſemi-conjugate AC $= b$, AE $= x$, EH $= y$; whence the equation will be $\frac{bb}{aa} \times \overline{aa - xx} = yy$, and therefore $y\dot{x} = \frac{b\dot{x}}{a}\sqrt{aa - xx}$, the element of the area AEHC. But $\dot{x}\sqrt{aa - xx}$ is the formula for ſquaring the circle BOD, the diameter of which is the tranſverſe axis of the ellipſis; ſo that the quadrature of the ellipſis will depend on that of the circle. And becauſe $\int\frac{b\dot{x}}{a}\sqrt{aa - xx} =$ EHCA, and $\int\dot{x}\sqrt{aa - xx} =$ EMOA, any ſpace of the ellipſis to the correſpondent ſpace of the circle on the diameter DB, will be as b to a, that is, as the conjugate ſemiaxis to the tranſverſe ſemiaxis; and conſequently the whole ellipſis to the whole circle will be in the ſame ratio. But, as circles are to each other as the ſquares of their diameters or radii, if we take a circle the radius of which is $= \sqrt{ab}$, that is, a mean proportional between the two ſemiaxes of the ellipſis BCD, this circle will be to the circle BOD as $ab \,.\, aa :: b \,.\, a$. But the area of the ellipſis BCD is to the ſame circle BOD, in this very ratio. Therefore the area of the ellipſis will be equal to the area of the circle, the radius of which is a mean proportional between the two ſemiaxes of the ellipſis.

Now, by the help of ſeries. The formula $\frac{b\dot{x}}{a}\sqrt{aa - xx}$, being reduced to a ſeries, will be $= \frac{b\dot{x}}{a}$ into $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c. And by integration, $\int\frac{b\dot{x}}{a}\sqrt{aa - xx}$, or area ACHE, $= bx - \frac{bx^3}{6a^2} - \frac{bx^5}{40a^4} - \frac{bx^7}{112a^6} - \frac{5bx^9}{1152a^8}$, &c. And making $x = a$, the area ACB, or a fourth part of the ellipſis, will be $= ab$ into $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152}$, &c.

In the ſame ellipſis, taking any arch DS, let DP be a tangent in D, AI $= x$, IS $= y$; and through the point S drawing AP infinitely near AK, which cuts the ellipſis in T. With centre A let the little arches of a circle KQ, TR, be deſcribed.

defcribed. Then it will be AS $= \sqrt{xx + yy} =$ AT, DP $= \frac{ay}{x}$, AK=AP $= \frac{a\sqrt{xx + yy}}{x}$, KP $= \frac{-ax\dot{y} + ay\dot{x}}{xx}$, PK being a negative difference. And by the fimilitude of the triangles PQK, PAD, it will be KQ $= \frac{-ax\dot{y} + ay\dot{x}}{x\sqrt{xx + yy}}$. And by the fimilitude of the fectors ATR, AKQ, it will be TR $= \frac{-x\dot{y} + y\dot{x}}{\sqrt{xx + yy}}$, and therefore $\frac{1}{2}$TR $\times$ AT, that is, $\frac{-x\dot{y} + y\dot{x}}{2}$, will be the formula for the fpace ACT. This will be finally $\frac{ab\dot{x}}{2\sqrt{aa - xx}}$, by fubftituting, inftead of y and $\dot{y}$, their values given from the equation of the curve.

But $\int \frac{a\dot{x}}{\sqrt{aa - xx}}$ is the rectification of the circle, as was feen at § 37, and as will be here feen alfo. Therefore the quadrature of elliptical fectors will depend on the rectification or quadrature of the circle. It would fignify nothing to take pains to free the formula from it's radical, becaufe, notwithftanding this, we fhould ftill fall upon a formula, which would depend on the fame circle.

Now, by the means of feries, we fhould find it to be $\frac{ab\dot{x}}{2\sqrt{aa - xx}} = \frac{b\dot{x}}{2} + \frac{bx^2\dot{x}}{4aa} + \frac{3bx^4\dot{x}}{16a^4} + \frac{5bx^6\dot{x}}{32a^6} + \frac{35bx^8\dot{x}}{256a^8}$, &c. And by integration, the fpace ATC $= \frac{bx}{2} + \frac{bx^3}{12a^2} + \frac{3bx^5}{80a^4} + \frac{5bx^7}{224a^6} + \frac{35bx^9}{2304a^8}$, &c. And making $x = a$, in refpect to the whole fpace ADC, a fourth part of the entire elliptical fpace, it will be $= \frac{ab}{2} + \frac{ab}{12} + \frac{3ab}{80} + \frac{5ab}{224} + \frac{35ab}{2304}$, &c.

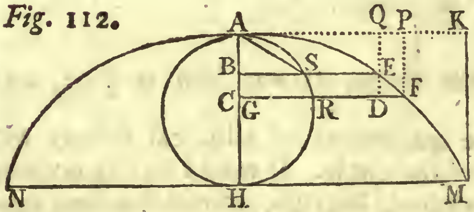
If we would free the formula from the radical vinculum, making the fubftitution of $\sqrt{aa - xx} = a - \frac{xz}{a}$, it would be changed into this other, $\frac{aab\dot{z}}{aa + zz}$, which, being reduced into a feries, would be found to be $b\dot{z} - \frac{bz^2\dot{z}}{a^2} + \frac{bz^4\dot{z}}{a^4} - \frac{bz^6\dot{z}}{a^6} + \frac{bz^8\dot{z}}{a^8}$, &c. And by integration, $bz - \frac{bz^3}{3a^2} + \frac{bz^5}{5a^4} - \frac{bz^7}{7a^6} + \frac{bz^9}{9a^8}$ &c.; and making $x = a$, in which cafe it is alfo $z = a$, it will be $ab - \frac{1}{3}ab + \frac{1}{5}ab - \frac{1}{7}ab + \frac{1}{9}ab$, &c. in refpect of a quadrant of the ellipfis.

 And

And if we ſuppoſe $a = b$, the ellipſis becomes a circle with radius $= a$, and the ſeries will be as at § 97, which will expreſs the quadrant. And therefore, from hence it may alſo be ſeen, that the area of the ellipſis is to the area of the circle, the diameter of which is equal to the tranſverſe axis of the ellipſis, as the conjugate axis is to the tranſverſe axis of the ſame ellipſis.

EXAMPLE VIII.

Fig. 112.



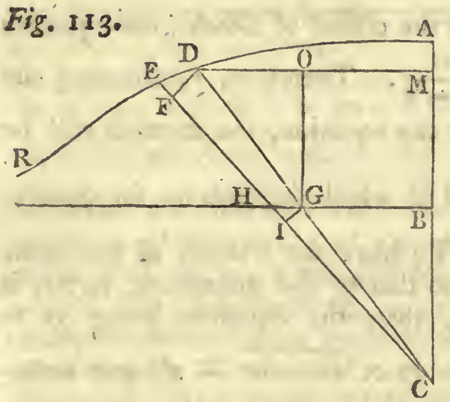
99. Let NAM be a cycloid, it's generating circle ARH, and make $AH = a$, $AB = x$, $BC = \dot{x}$, $BE = y$, $DF = \dot{y}$. The equation will be $\dot{y} = \frac{a\dot{x} - x\dot{x}}{\sqrt{ax - xx}} = \frac{\dot{x}\sqrt{a - x}}{\sqrt{x}}$. But the little ſpace QEFP is the element of the ſpace AEQ, and therefore $FP \times PQ$, that is, $\frac{x\dot{x}\sqrt{a - x}}{\sqrt{x}} = \dot{x}\sqrt{ax - xx}$ will be it's formula. But $\int \dot{x}\sqrt{ax - xx}$ is the circular ſegment ASB; therefore the cycloidal ſpace AEQ will be equal to the correſpondent circular ſpace ASB, and the whole ſpace AMK will be equal to the ſemicircle. But the rectangle AHMK is quadruple of the ſemicircle, becauſe it is the product of the ſemiperiphery into the diameter. Therefore the ſpace AMH will be triple of the ſemicircle, and therefore the whole cycloidal ſpace will be triple of the generating circle.

If we would have the ſpace AFC immediately; as the little ſpace FCBE, that is, $y\dot{x}$, is it's element, and from the equation of the curve we have $\dot{y} = \frac{\dot{x}\sqrt{a - x}}{\sqrt{x}}$; let the *homogeneum comparationis* be reduced into a ſeries, firſt multiplying the numerator and denominator by $\sqrt{x}$; whence it would be $\frac{\dot{x}\sqrt{av - xx}}{x}$ $= \frac{a^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}} - \frac{x^{\frac{1}{2}}\dot{x}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}\dot{x}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{5}{2}}\dot{x}}{16a^{\frac{5}{2}}}$, &c.; and therefore, by integration, $\int\frac{\dot{x}\sqrt{ax - xx}}{x} = y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{20a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{56a^{\frac{5}{2}}}$, &c. Whence $y\dot{x} =$

$2a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{3a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}\dot{x}}{20a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}\dot{x}}{56a^{\frac{5}{2}}}$, &c. And laſtly, by integration, $\int y\dot{x} =$ ABE $= \frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{15a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{70a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{252a^{\frac{5}{2}}}$, &c.

EXAMPLE IX.

Fig. 113.



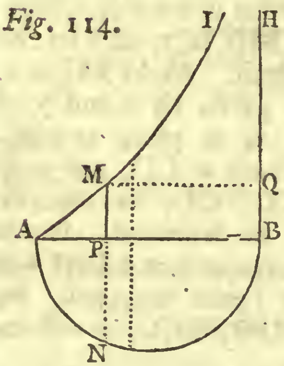
100. Let ADR be the conchoid, CB $=$ BA $= a$, CM $= x$, MD $= y$, and let the ſpace ADGB be required. Make CG $= z$, which will always be given by x and y of the propoſed curve, as is plain enough. Let CE be infinitely near to CD, and with centre C, intervals CG, CD, let the two little arches GI, DF, be deſcribed. It will be HI $= \dot{z}$, and the trapezium FDGI will be the element of the ſpace required. By the ſimilar triangles HIG, BGC, it will be GI $= \frac{a\dot{z}}{\sqrt{zz - aa}}$; and by the ſimilar ſectors CGI, CDF, it will be DF $= \frac{az\dot{z} + aa\dot{z}}{z\sqrt{zz - aa}}$. But the trapezium FDGI $=$ $\overline{\text{DF} + \text{GI}} \times \frac{1}{2}\text{GD} = \frac{2a^2z\dot{z} + a^3\dot{z}}{2z\sqrt{zz - aa}}$. Therefore $\int \frac{2a^2z\dot{z} + a^3\dot{z}}{2z\sqrt{zz - aa}}$, that is, $[\overline{alz + \sqrt{zz - aa}} - ala + \frac{1}{2}a \times$ arch of a circle of which the radius $=$ and the tangent $= \sqrt{zz - aa}$,] (taking the logarithm in the logarithmic with ſubtangent $= a$,) will be equal to the ſpace required.

Alſo, the whole ſpace may be had of the ſame conchoid, and likewiſe the parts, by conſidering the curve in relation to it's axis. In the ſame Figure make AB $=$ DG $=$ BC $= a$, BM $= x$, MD $= y$; and from the point G let there be drawn GO perpendicular to the ordinate MD; it will be DO $= \sqrt{aa - xx}$, becauſe of the right angle GOD; and by the ſimilitude of the triangles CBG, GOD, it will be BG $= \frac{a\sqrt{aa - xx}}{x} =$ MO. Therefore MD $=$

$= \sqrt{aa - xx} + \frac{a\sqrt{aa - xx}}{x} = y$. Whence $y\dot{x}$, that is, the element of the ſpace, will be $\dot{x}\sqrt{aa - xx} + \frac{a\dot{x}\sqrt{aa - xx}}{x}$. The fluent of the firſt term depends on the quadrature of the circle, and of the ſecond on that of the hyperbola.

EXAMPLE X.

Fig. 114.



101. Let AMI be the ciſſoid of *Diocles*, the equation of which is $yy = \frac{x^3}{a - x}$. Therefore, ſubſtituting the value of y given by the equation, the formula will be $\frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a - x}}$, the integral of which depends on the quadrature of the circle. To have the relation of the whole ſpace of the ciſſoid to that of the generating circle, it muſt be conſidered, that, the equation being $yy = \frac{x^3}{a - x}$, it will be alſo $yy \times \overline{ax - xx} = x^4$, and therefore $y\sqrt{ax - xx} = xx$. This ſuppoſed, by differencing the propoſed equation $ayy - xyy = x^3$, there ariſes $2ay\dot{y} - 2xy\dot{y} - yy\dot{x} = 3xx\dot{x}$, that is, $2\dot{y} \times \overline{a - x} - y\dot{x} = \frac{3xx\dot{x}}{y}$. And, becauſe $xx = y\sqrt{ax - xx}$, therefore $2\dot{y} \times \overline{a - x} - y\dot{x} = 3\dot{x}\sqrt{ax - xx}$. But $\dot{y} \times \overline{a - x}$ is the element of the ſpace AMQB, and $y\dot{x}$ is the element of the ſpace AMP; then, by integrating, as to the whole ſpace, it is $\int\dot{y} \times \overline{a - x} = \int y\dot{x}$. Then, in this circumſtance, it will be $2\int\dot{y} \times \overline{a - x} - \int y\dot{x} = \int\dot{y} \times \overline{a - x}$, and therefore $\int\dot{y} \times \overline{a - x} = 3\int\dot{x}\sqrt{ax - xx}$; and becauſe, in the caſe of the total ſpace of the ciſſoid, $\int\dot{x}\sqrt{ax - xx}$ is the area of the ſemicircle ABN; thence the ſpace of the ciſſoid, infinitely produced, will be triple of the generating circle.

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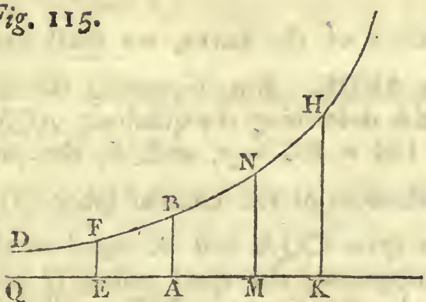
EXAMPLE XI.

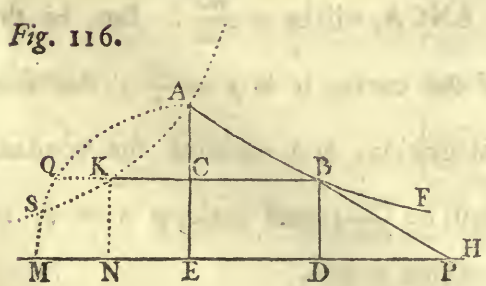
Fig. 115.



102. Let HBD be the logarithmic to the afymptote MQ, and let AB $=a=$ fubtangent, KH $=y$, AK $=x$, and the equation $\frac{a\dot{y}}{y}=\dot{x}$. Then the formula will be $y\dot{x}=a\dot{y}$, and by integration, $\int y\dot{x}=ay+bb$. But, fuppofing $y=a$, it will be $bb=-aa$; fo that the integral complete, or fpace AKHB $=ay-aa$. Taking any other ordinate MN $=z$, it will be alfo AMNB $=az-aa$, fo that MKHN $=ay-az$. Let there be an ordinate EF lefs than AB, and equal to y, AE $=-x$; in the fame manner, the equation will be $\frac{a\dot{y}}{y}=\dot{x}$, becaufe, x being negative, it's difference will be negative alfo. But the abfcifs x increafing, the ordinate y decreafes, and therefore $\dot{y}$ muft be negative. Now, becaufe the element of the fpace will alfo be negative, this element will be $-y\dot{x}$, that is, $-a\dot{y}$; and by integrating, $-ay+bb$. But when $y=a$, it will be $bb=aa$; therefore the complete integral, that is, the fpace AEFB, will be $=aa-ay$. And making $y=0$, that is, when it is infinitely produced towards Q, it will be $=aa$. And confequently the fame fpace, infinitely produced towards Q, but which begins from any ordinate EF $=y$, will be $=ay$.

EXAMPLE XII.

Fig. 116.

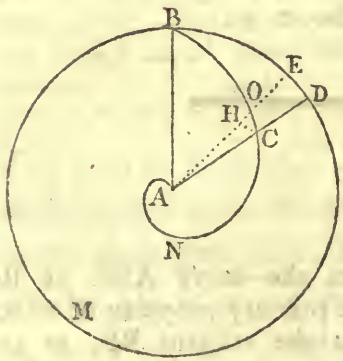


103. Let the curve ABF be the *tractrix*, the primary property of which is this, that the tangent BP, at any point B, is always equal to a conftant right line given. Make any abfcifs ED $=x$, the ordinate DB $=y$, the arch of the curve AB $=u$, and the given right line $=a$. Becaufe, as the abfcifs ED increafes, the ordinate DB diminifhes, it's element will be negative,

tive, that is, $-\dot{y}$. Whence, from the property of the curve, we ſhall have the equation $-\frac{y\dot{u}}{\dot{y}} = a$; and, inſtead of $\dot{u}$, putting it's value $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, it is $\dot{x} = \frac{-\dot{y}\sqrt{aa - yy}}{y}$. This being done, in the formula for areas $y\dot{x}$, inſtead of $\dot{x}$, putting it's value given by the equation of the curve, we ſhall have $-\dot{y}\sqrt{aa - yy}$ for the element of any ſpace ABDE. But, ſuppoſing the firſt of the ordinates AE $= a$, and with radius EA deſcribing the quadrant AQM, and drawing BQ parallel to MH; becauſe DB = EC $= y$, and, by the property of the circle, CQ $= \sqrt{aa - yy}$, the element of the circular ſpace CQA will alſo be $-\dot{y}\sqrt{aa - yy}$. Whence the ſpace CQA will be equal to the ſpace ABDE; and ſo of others. And conſequently the ſpace infinitely produced, comprehended by the *tractrix* ABF, by the aſymptote EH, and by the right line AE, will be equal to the quadrant AME.

EXAMPLE XIII.

Fig. 117.



104. Let ACB be a ſpiral, and AB $= a$ the radius of the circle BMD, the periphery of which $= b$, any arch BD $= x$, AC $= y$; the equation will be $by = ax$. Drawing AE infinitely near to AD, it will be ED $= \dot{x}$; and with centre A deſcribe the infiniteſimal arch CH. Becauſe of ſimilar ſectors ACH, ADE, it will be CH $= \frac{y\dot{x}}{a}$, and therefore the ſector ACH, the element of the ſpace ANCA, will be $= \frac{yy\dot{x}}{2a}$. But, by the equation of the curve, it is $y = \frac{ax}{b}$; therefore that element will be $= \frac{axx\dot{x}}{2bb}$, and by integration, and omitting the conſtant quantity as ſuperfluous, the ſpace ACN will be $\frac{ax^3}{6bb}$; and making $x = b$, in reſpect of the whole ſpace ANB, which will be $= \frac{1}{6}ab$.

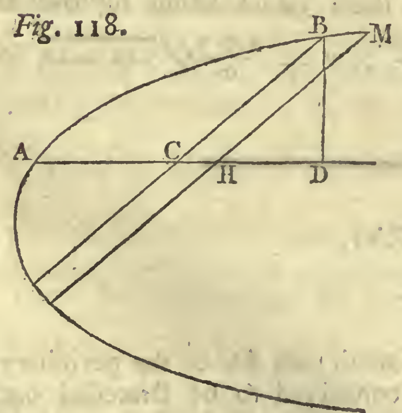
Let

Let the equation be general to infinite ſpirals $a^m x^n = b^n y^m$; then it will be $yy = \frac{aax^{\frac{2n}{m}}}{b^{\frac{2n}{m}}}$, and the formula of the ſpace will be $\frac{ax^{\frac{2n}{m}}\dot{x}}{2b^{\frac{2n}{m}}}$, and by integration, $\frac{max^{\frac{2n+m}{m}}}{\overline{4n+2m} \times b^{\frac{2n}{m}}}$; and making $x = b$, the whole ſpace will be $= \frac{mab}{4n+2m}$.

It is eaſy to perceive, that the ſpace ABMDCNA, terminated by the radius AB, the circular arch BMD, and the portion of the ſpiral ANC, will be $\frac{ax}{2} - \frac{ax^3}{6bb}$; becauſe it is equal to the ſector ABMDA, diminiſhed by the ſpace ACN. But if we would have it by means of the differential formula, it is enough to obſerve, that it's element will be the infiniteſimal trapezium ECHD, which is known to be $= \overline{DE + CH} \times \frac{1}{2}CD$, that is, $\overline{\dot{x} + \frac{y\dot{x}}{a}} \times \frac{a-y}{2} = \frac{aa\dot{x} - yy\dot{x}}{2a}$. And, inſtead of yy, putting it's value $\frac{aaxx}{bb}$ given by the equation, it will be $\frac{a\dot{x}}{2} - \frac{axx\dot{x}}{2bb}$; and by integration, $\frac{ax}{2} - \frac{ax^3}{6bb}$, omitting the ſuperfluous conſtant quantity.

EXAMPLE XIV.

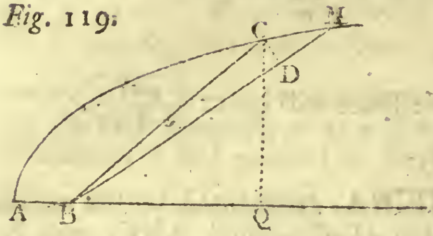
Fig. 118.



105. Let ABM be the parabola, whoſe equation is $ax = yy$, and make $AC = x$, $CB = y$, and let the ratio of the whole ſine to the right ſine of the angle BCD be that of a to b; to the ſine of the complement be that of a to f; then it will be $BD = \frac{by}{a}$, and $CD = \frac{fy}{a}$. Let $CH = \dot{x}$, then $CH \times DB = CHMB$, the element of the ſpace ACB, and therefore the formula will be $\frac{by\dot{x}}{a}$. And, inſtead of y, putting it's value given from the equation, that is, $\sqrt{ax}$, it will be $\frac{b\dot{x}\sqrt{ax}}{a}$; and by integration, $\frac{2bx\sqrt{ax}}{3a}$, or $\frac{2bxy}{3a} = \frac{2}{3}AC \times BD$.

EXAMPLE XV.

Fig. 119.

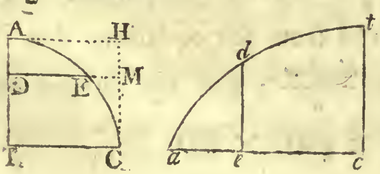


106. Let ACM be a parabola referred to the focus B, the equation of which will be $\frac{a\dot{z}}{\sqrt{2az - aa}} = \dot{u}$, making BC $= z$, CD $= \dot{u}$, an infinitely little arch of a circle, and the parameter $= 2a$. Then the infinitesimal sector BMC, or BDC, will be the element of the space ABC, and therefore $\frac{1}{2}z\dot{u}$, or $\frac{az\dot{z}}{2\sqrt{2az - aa}}$, will be the formula; the integral of which will be found to be $\frac{z + a}{6}\sqrt{2az - aa} + mm$. Now, taking $z =$ BA $= \frac{1}{2}a$, in which case the space ought to be nothing, it will be $mm = 0$, and therefore the complete integral, that is, the space ABC, is $\frac{z + a}{6}\sqrt{2az - aa}$.

And in fact, from the point C letting fall CQ perpendicular to AQ, the space BCA is equal to the space QCA lessened by the triangle BQC. But, making BQ $= x$, QC $= y$, it will be QCA $-$ QCB $= \frac{2}{3} \times \overline{\frac{1}{2}a + x} \times y - \frac{1}{2}xy = \frac{2a + x}{6} \times y$. Therefore BCA $= \frac{2a + x}{6} \times y$. But, by the property of the parabola, BC $=$ AQ $+$ AB $= x + a$, that is, $z = x + a$, and $y = \sqrt{aa + 2ax} = \sqrt{2az - aa}$. Therefore, these values being substituted instead of x and y, we shall find BCA $= \frac{2a + x}{6} \times y = \frac{a + z}{6}\sqrt{2az - aa}$, as above.

EXAMPLE XVI.

Fig. 120.



107. If the fourth part AC of the periphery of a circle be conceived to be stretched out into a right line (ac), and taking any portion (ae) equal to the arch AE, let there be raised the perpendicular (ed) equal to the right sine DE;

DE; the curve (at) which paffes through all the points (d) fo determined, is called the line of right fines. Producing (ac) till it be equal to the femicircumference of the circle, the curve will have another branch beyond (ct), fimilar and equal to the firft.

Let the radius be $= r$, any arch AE $= x = (ae)$, the correfponding fine DE $= y = (ed)$; becaufe the fluxion or differential of the arch, expreffed by means of the fine, is found to be $\frac{r\dot{y}}{\sqrt{rr - yy}}$, we fhall have $\dot{x} = \frac{r\dot{y}}{\sqrt{rr-yy}}$, which is the equation of our curve. Therefore the formula $y\dot{x}$, by fubftituting the value of $\dot{x}$, will be $\frac{ry\dot{y}}{\sqrt{rr - yy}}$; and by integration, $-r\sqrt{rr - yy} + n$. But, putting $y = 0$, it is $n = rr$. Therefore the complete integral is $rr - r\sqrt{rr - yy}$ $=$ fpace (ade); and making $y = r$, it will be $rr =$ to the whole fpace (atc). Whence, making TH the fquare of the radius, and producing the fine DE to M, the fpace (ade) will be equal to the rectangle DH, and the whole fpace (atc) equal to the fquare TH.

108. The Examples now produced may fuffice to fhow the ufe of the method. It only remains to obferve, that often the equations of the curves, the areas of which are to be fquared, (and this is alfo to be underftood in refpect to rectifications, quadratures of fuperficies, and cubatures,) may be fuch, that they have not the variable quantities feparate, nor can they be feparated by divifion only, and confequently are not reducible to the formulas required. Such would be the curve, whofe equation is $x^3 + y^3 = axy$, for example.

In thefe cafes there is occafion to take the advantage of fome proper fubftitution, by means of which the equation may be transformed into another, in which the variable quantities are feparate, or at leaft are feparable. But it cannot be determined, in general, what thofe fubftitutions ought to be. There is need of practice, and perhaps many trials, to know when this may be fuccefsfully performed.

As to the propofed equation $x^3 + y^3 = axy$, make $y = \frac{axx}{zz}$; and making the fubftitution, the equation will be $x^3 + \frac{a^3x^6}{z^6} = \frac{a^2x^3}{zz}$, that is, $x^3 = \frac{aaz^4 - z^6}{a^3}$. Then, by differencing, $x^2\dot{x} = \frac{4a^2z^3\dot{z} - 6z^5\dot{z}}{3a^3}$. Wherefore, taking the formula for fpaces, which is $y\dot{x}$, becaufe, by fubftitution, it is $y = \frac{axx}{zz}$, this formula will be $\frac{axx\dot{x}}{zz}$; and fubftituting, inftead of $xx\dot{x}$, it's value now found, $\frac{4aaz^3\dot{z} - 6z^5\dot{z}}{3a^3}$, it will be $y\dot{x} = \frac{4aaz\dot{z} - 6z^3\dot{z}}{3a^2}$; and by integration, $\int y\dot{x} = \frac{2}{3}zz$ —

$-\frac{z^4}{2aa}$; and, inftead of zz, reftoring it's value $\frac{axx}{y}$, it will be finally $\int y\dot{x} = \frac{2axx}{3y} - \frac{x^4}{2yy}$.

EXAMPLE XVII.

109. Let the curve be $a^5x^2y^2 - x^6 = a^6y^3$, whofe area is required. Put $y = \frac{xx}{z}$, and the equation will be transformed into this other, $a^5z - x^3z^3 = a^6$, from whence we have $x = \frac{a\sqrt[3]{aaz - a^3}}{z}$; and therefore $\dot{x} = \frac{a^3\dot{z}}{3z \times \overline{aaz - a^3}|^{\frac{2}{3}}} - \frac{a\dot{z} \times \overline{aaz - a^3}|^{\frac{1}{3}}}{zz}$, and $y = \frac{aa \times \overline{aaz - a^3}|^{\frac{2}{3}}}{z^3}$. Hence we fhall have the element of the area $y\dot{x} = \frac{a^5\dot{z}}{3z^4} - \frac{a^3\dot{z}}{z^5} \times \overline{aaz - a^3} = \frac{a^6\dot{z}}{z^5} - \frac{2a^5\dot{z}}{3z^4}$; and therefore, by integration, $\int y\dot{x} = \frac{-a^6}{4z^4} + \frac{2a^5}{9z^3}$. And, inftead of z, reftoring it's value $\frac{xx}{y}$, the area will be $-\frac{a^6y^4}{4x^8} + \frac{2a^5y^3}{9x^6}$.

To this purpofe may be feen the Method of Mr. *Craig*, in his Book *De Calculo Fluentium.*

EXAMPLE XVIII.

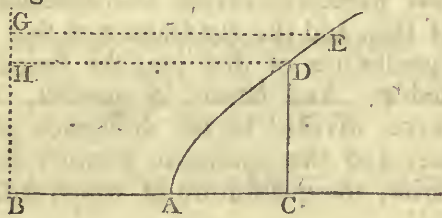
The rectification of curves.

110. Let the *Apollonian* parabola be given to be rectified; that is, to find a right line equal to any arch of the fame parabola, the equation of which is $ax = yy$. It's fluxion will be $a\dot{x} = 2y\dot{y}$, and $\dot{x}\dot{x} = \frac{4yy\dot{y}\dot{y}}{aa}$. Now the formula for rectification is $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$; fo that, fubftituting here, inftead of $\dot{x}\dot{x}$, it's value given from the fluxional equation, it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{a} = \frac{\dot{y}}{a}\sqrt{4yy + aa}$, the element of the *Apollonian* parabola

$ax =$

$ax = yy$. Proceeding to the integration; by making the subſtitution of $\sqrt{4yy + aa} = 2y + z$, in order to take away the radical, we ſhall find it to be $\frac{\dot{y}}{a}\sqrt{4yy + aa} = -\frac{a^3\dot{z}}{8z^3} - \frac{a\dot{z}}{4z} - \frac{z\dot{z}}{8a}$, the integral of which we may ſee is partly algebräical, and partly logarithmical; and therefore the rectification of the parabola depends on the quadrature of the hyperbola; which truth may be diſcovered after this other manner. Let ADE be an equilateral hyperbola, with ſemiaxis $= a$, BC $= x$ from the centre, CD $= 2y$, the equation of which will be $xx - aa = 4yy$. Drawing GE infinitely near to HD, then HGED will be the element of the ſpace ADHB. But we know HGED to be $2\dot{y}\sqrt{4yy + aa}$, which is the ſame formula as that for the rectification of the parabola, excepting the conſtant denominator $2a$. Therefore, &c.

Fig. 121.



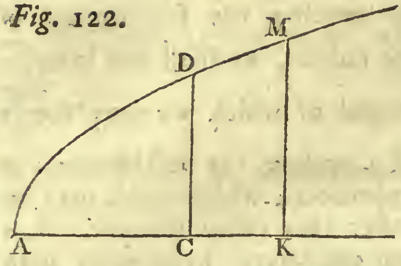
By the help of infinite ſeries. I take the above-written formula for the rectification of the parabola, that is, $\frac{\dot{y}}{a}\sqrt{4yy + aa}$, which, being reduced to a ſeries, will be $\dot{y} + \frac{2y^2\dot{y}}{aa} - \frac{2y^4\dot{y}}{a^4} + \frac{4y^6\dot{y}}{a^6} - \frac{10y^8\dot{y}}{a^8}$, &c. And, by integration, $\int \frac{\dot{y}}{a}\sqrt{4yy + aa} = y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} - \frac{10y^9}{9a^8}$, &c. will be any arch whatever.

In the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, inſtead of ſubſtituting, in the place of $\dot{x}$, it's value given by $\dot{y}$ from the equation of the curve; if we ſhould ſubſtitute, in the place of $\dot{y}$, it's value given by $\dot{x}$, it would be $\frac{\dot{x}\sqrt{4ax + aa}}{\sqrt{4ax}}$, or $\frac{\dot{x}\sqrt{4xx + ax}}{2x}$, which is not indeed more manageable than the other.

If the parabola was not that of *Apollonius*, but the ſecond cubic, the equation of which is $axx = y^3$; by taking the difference, it would be $\dot{x}\dot{x} = \frac{9yy\dot{y}}{4a}$, and therefore the formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y}\sqrt{\frac{9y + 4a}{4a}}$, the integral of which is $\frac{9ay + 4aa}{27aa}\sqrt{9ay + 4aa} + m$. But, putting $y = 0$, it will be $m = -\frac{8}{27}a$; therefore the complete integral, or the length of the arch, will be $\frac{9ay + 4aa}{27aa}\sqrt{9ay + 4aa} - \frac{8}{27}a$.

In

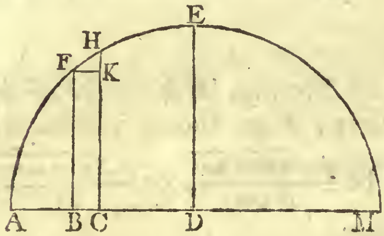
Fig. 122.



In the *Apollonian* parabola ADM, if it ſhall be AC $= \frac{4}{9}a$, and taking any line CK $= y$, the parameter $= \frac{9}{4}a$; it will be AK $= \frac{4}{9}a + y$, KM $= \sqrt{\frac{4aa + 9ay}{4}}$. Whence the element of the area MKCD will be $\dot{y}\sqrt{\frac{4aa + 9ay}{4}}$, which is the ſame with the element of the length of the ſecond cubical parabola, except the conſtant quantity a. And therefore the rectification of this, and the quadrature of that, is the ſame thing. Whence, becauſe the quadrature of that may be found algebräically, this is alſo algebräically rectifiable. And hence, in general, if the expreſſion of the element of any given curve, divided by the difference of the unknown quantity, be put for the ordinate, and the unknown quantity be put for the abſciſs; a new curve will thence ariſe, the quadrature of which will give the rectification of the given curve.

EXAMPLE XIX.

Fig. 123.



111. Let AEM be a circle, it's diameter AM $= a$, AB $= x$; it will be BF $= y = \sqrt{ax - xx}$. Then $\dot{y} = \frac{\frac{1}{2}a\dot{x} - x\dot{x}}{\sqrt{ax - xx}}$, $\dot{y}\dot{y} = \frac{\frac{1}{4}aa\dot{x}\dot{x} - ax\dot{x}\dot{x} + xx\dot{x}\dot{x}}{ax - xx}$. And therefore the element of the curve FH $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{x}}{2\sqrt{ax - xx}}$, and reducing it to a ſeries, it will be $\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x} + \frac{x^{\frac{1}{2}}\dot{x}}{2 \times 2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}\dot{x}}{2 \times 2 \times 4a^{\frac{3}{2}}} + \frac{15x^{\frac{5}{2}}\dot{x}}{2 \times 2 \times 4 \times 6a^{\frac{5}{2}}} + \frac{105x^{\frac{7}{2}}\dot{x}}{2 \times 2 \times 4 \times 6 \times 8a^{\frac{7}{2}}}$, &c. And by integration, it will be $a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{2 \times 3a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{2 \times 4 \times 5a^{\frac{3}{2}}} + \frac{15x^{\frac{7}{2}}}{2 \times 4 \times 6 \times 7a^{\frac{5}{2}}} + \frac{105x^{\frac{9}{2}}}{2 \times 4 \times 6 \times 8 \times 9a^{\frac{7}{2}}}$, &c. Or, becauſe it is $\dot{x}\dot{x} = \frac{\dot{y}\dot{y} \times \overline{ax - xx}}{\frac{1}{4}aa - ax + xx}$, that is, by ſubſtituting yy inſtead of $ax - xx$, $\dot{x}\dot{x} = \frac{yy\dot{y}\dot{y}}{\frac{1}{4}aa - yy}$; then putting this value, inſtead

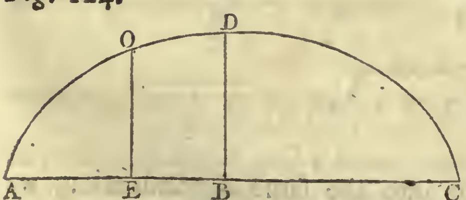
stead of $\dot{x}\dot{x}$ in the general formula, it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{y}}{2\sqrt{\frac{1}{4}aa - yy}}$; which, being reduced to a series, will be found to be $= \dot{y} + \frac{2yy\dot{y}}{aa} + \frac{6y^4\dot{y}}{a^4} + \frac{20y^6\dot{y}}{a^6} + \frac{70y^8\dot{y}}{a^8}$, &c. And by integration, it will be finally the arch FA $= y + \frac{2y^3}{3a^2} + \frac{6y^5}{5a^4} + \frac{20y^7}{7a^6} + \frac{70y^9}{9a^8}$, &c.

But if the radius were made $= a$, the series would be $y + \frac{y^3}{2 \times 3a^2} + \frac{3y^5}{2 \times 4 \times 5a^4} + \frac{15y^7}{2 \times 4 \times 6 \times 7a^6} + \frac{105y^9}{2 \times 4 \times 6 \times 8 \times 9a^8}$, &c.

Lastly, if it were DB $= x$, the radius DA $= a$, it would be $y = \sqrt{aa - xx}$, and $\dot{y} = \frac{-x\dot{x}}{\sqrt{aa - xx}}$; therefore $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{x}}{\sqrt{aa - xx}}$; and, reducing to a series, it will be $\frac{a\dot{x}}{\sqrt{aa - xx}} = \dot{x} + \frac{x^2\dot{x}}{2aa} + \frac{3x^4\dot{x}}{2 \times 4a^4} + \frac{15x^6\dot{x}}{2 \times 4 \times 6a^6} + \frac{105x^8\dot{x}}{2 \times 4 \times 6 \times 8a^8}$, &c. Whence the arch EF $= x + \frac{x^3}{2 \times 3a^2} + \frac{3x^5}{2 \times 4 \times 5a^4} + \frac{15x^7}{2 \times 4 \times 6 \times 7a^6} + \frac{105x^9}{2 \times 4 \times 6 \times 8 \times 9a^8}$, &c.

EXAMPLE XX.

Fig. 124.



112. Let ADC be an ellipsis with transverse semiaxis BA $= a$, and conjugate semiaxis BD $= b$, BE $= x$, EO $= y$; the equation will be $\frac{aayy}{bb} = aa - xx$, and therefore $y\dot{y} = \frac{-bbx\dot{x}}{aa}$, and $\dot{y}\dot{y} = \frac{bbxx\dot{x}\dot{x}}{aa \times \overline{aa - xx}}$, and the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \sqrt{\dot{x}\dot{x} + \frac{bbxx\dot{x}\dot{x}}{a^4 - aaxx}} = \frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$.

Instead

Inftead of fubftituting the value of $\dot{y}$ given by x from the equation, if we fhould fubftitute the value of $\dot{x}$, it would be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\dot{y}\sqrt{aayy - bbyy + b^4}}{b\sqrt{bb - yy}}$. But both of the expreffions fo found would want one of the conditions of § 38, without which it may be feen, that thefe formulas cannot be freed from radical figns, and fo prepared for integration. Then to proceed to feries, I take one of the two formulas, for inftance $\frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$, which alfo may be thus expreffed, $\dot{x}\sqrt{1 + \frac{bbxx}{a^4 - aaxx}}$; and this being reduced to a feries, will be $=$

$$\dot{x} + \frac{\frac{1}{2}bbxx\dot{x}}{aa \times \overline{aa - xx}} - \frac{\frac{1}{8}b^4x^4\dot{x}}{a^4 \times \overline{aa - xx}|^2} + \frac{\frac{1}{16}b^6x^6\dot{x}}{a^6 \times \overline{aa - xx}|^3} - \frac{\frac{5}{128}b^8x^8\dot{x}}{a^8 \times \overline{aa - xx}|^4}, \text{ \&c.}$$

And again, reducing every term of this into a feries, beginning at the fecond, it will be

$$\dot{x}\sqrt{1 + \frac{bbxx}{a^4 \times \overline{aa - xx}}} = \dot{x} + \frac{\frac{1}{2}bbxx\dot{x}}{aa} \text{ into } \frac{1}{aa} + \frac{x^2}{a^4} + \frac{x^4}{a^6} + \frac{x^6}{a^8}, \text{ \&c.}$$

$$- \frac{\frac{1}{8}b^4x^4\dot{x}}{a^4} \text{ into } \frac{1}{a^4} + \frac{2x^2}{a^6} + \frac{3x^4}{a^8} + \frac{4x^6}{a^{10}}, \text{ \&c.}$$

$$+ \frac{\frac{1}{16}b^6x^6\dot{x}}{a^6} \text{ into } \frac{1}{a^6} + \frac{3x^2}{a^8} + \frac{6x^4}{a^{10}} + \frac{10x^6}{a^{12}}, \text{ \&c.}$$

$$- \frac{\frac{5}{128}b^8x^8\dot{x}}{a^8} \text{ into } \frac{1}{a^8} + \frac{4x^2}{a^{10}} + \frac{10x^4}{a^{12}} + \frac{20x^6}{a^{14}}, \text{ \&c.}$$

And by integration, the arch DO will be $\int \dot{x}\sqrt{1 + \frac{bbxx}{a^4 - a^2x^2}} =$

$$x + \frac{bb}{2aa} \text{ into } \frac{x^3}{3a^2} + \frac{x^5}{5a^4} + \frac{x^7}{7a^6} + \frac{x^9}{9a^8}, \text{ \&c.}$$

$$- \frac{b^4}{8a^4} \text{ into } \frac{x^5}{5a^4} + \frac{2x^7}{7a^6} + \frac{3x^9}{9a^8} + \frac{4x^{11}}{11a^{10}}, \text{ \&c.}$$

$$+ \frac{b^6}{16a^6} \text{ into } \frac{x^7}{7a^6} + \frac{3x^9}{9a^8} + \frac{6x^{11}}{11a^{10}}, \text{ \&c.}$$

$$- \frac{5b^8}{128a^8} \text{ into } \frac{x^9}{9a^8} + \frac{4x^{11}}{11a^{10}}, \text{ \&c.}$$

And laftly, reducing the homogeneous terms into the fame denomination, we fhall find DO $=$

$$x + \frac{b^2x^3}{6a^4} + \frac{4a^2b^2 - b^4}{40a^8}x^5 + \frac{8a^4b^2 - 4a^2b^4 + b^6}{112a^{12}}x^7$$

$$+ \frac{64a^6b^2 - 48a^4b^4 + 24a^2b^6 - 5b^8}{9 \times 128a^{16}}x^9, \text{ \&c.}$$

 Now,

Now, if we should suppose $a = b$, in which case the ellipsis would become a circle, we shall have the arch DO $= x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6} + \frac{35x^9}{9 \times 128a^8}$, &c. just as was found before, at § 111.

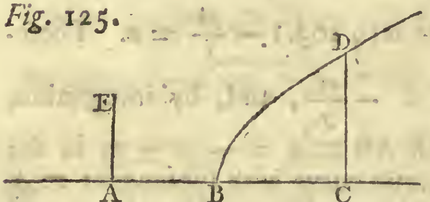
After another manner, thus. In the formula $\frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$, if we make $bb - aa = - cc$, so that it may be $\frac{\dot{x}\sqrt{a^4 - ccxx}}{a\sqrt{aa - xx}}$, the two radicals being resolved into series, it will be $\frac{\dot{x}\sqrt{a^4 - c^2x^2}}{a\sqrt{aa - xx}} =$

$$\frac{\frac{\dot{x}}{a} \text{ into } a^2 - \frac{c^2x^2}{2a^2} - \frac{c^4x^4}{8a^6} - \frac{c^6x^6}{16a^{10}} - \frac{5c^8x^8}{128a^{14}}, \text{ \&c.}}{a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}}$$

; and actually making the division of the numerator by the denominator, after a very long calculation we shall find another series, which, being integrated, and the value of cc restored in it's place, will give us the same series as above, which expresses the value of the arch DO.

EXAMPLE XXI.

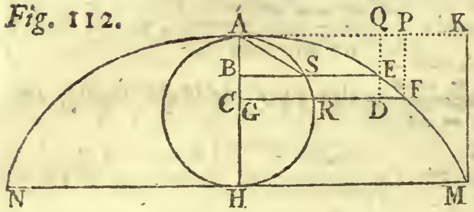
Fig. 125.



113. Let BD be an hyperbola with transverse semiaxis AB $= a$, conjugate semiaxis AE $= b$, CD $= y$, AC $= x$; the equation will be $xx - aa = \frac{aayy}{bb}$. Then, by taking the fluxions, it will be $\dot{x} = \frac{a\dot{y}}{b\sqrt{bb + yy}}$, whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y}\sqrt{1 + \frac{aayy}{b^4 + b^2y^2}} = \frac{\dot{y}}{b} \times \frac{\sqrt{bbyy + aayy + b^4}}{\sqrt{bb + yy}}$. Therefore, this being reduced into a series, after either of the ways before made use of for the ellipsis, we shall find it's integral, or the arch BD $= y + \frac{a^2y^3}{6b^4} - \frac{4a^2b^2 + a^4}{40b^8}y^5 + \frac{8a^2b^4 + 4a^4b^2 + a^6}{112b^{12}}y^7 - \frac{64a^2b^6 + 48a^4b^4 + 24a^6b^2 + 5a^8}{9 \times 128b^{16}}y^9$, &c. which is the same series as that for the ellipsis, excepting the signs, and the change of the constants a, b.

EXAMPLE XXII.

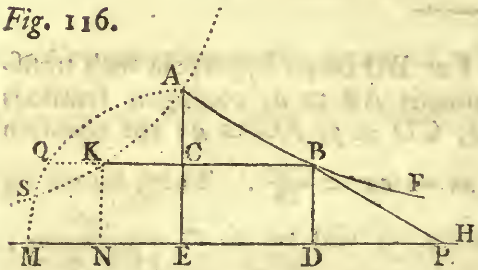
Fig. 112.



114. Let it be the cycloid of Example VIII. of Quadratures, the equation of which we know to be $\dot{y} = \dot{x}\sqrt{\frac{a-x}{x}}$; therefore the formula will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{x}\sqrt{\frac{a}{x}}$, and therefore, by integration, it will be the arch $EA = 2\sqrt{ax}$, or the double of the chord AS of the correſponding circular arch AS. And putting $x = a$, AM will be the double of the diameter of the generating circle, and therefore the whole cycloid will be quadruple.

EXAMPLE XXIII.

Fig. 116.



115. Let ABF be the *tractrix*, whoſe equation is (§ 103.) $-\frac{y\dot{u}}{\dot{y}} = a$. Therefore $\dot{u} = -\frac{a\dot{y}}{y}$, and, by integration, any arch $AB = u = -ly \pm n$, in the logarithmic curve with ſubtangent $= a$. But, making $u = 0$, it is $y = a$, and $ly = 0$; therefore $n = 0$, and the complete integral will be $u = -ly$. Therefore, if the logarithmic AKS be deſcribed through the point A, with the ſubtangent AE, to the aſymptote MH; taking any point B in the *tractrix*, and drawing to the logarithmic BK parallel to the aſymptote, and letting fall the perpendicular KN, the intercepted line NE will be equal to the arch AB.

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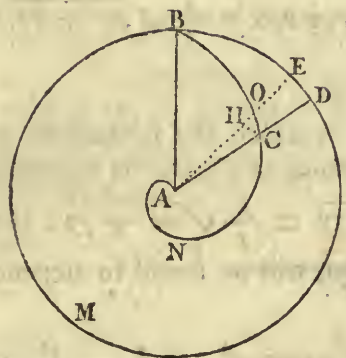
EXAMPLE XXIV.

Fig. 117.

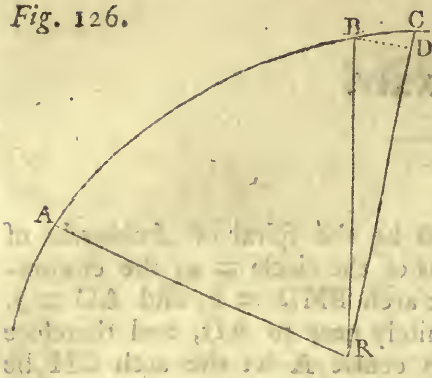


116. Let ACB be the ſpiral of *Archimedes* of § 104. the radius of the circle $= a$, the circumference $= b$, the arch BMD $= x$, and AC $= y$. Let AE be infinitely near to AD, and therefore DE $= \dot{x}$. With centre A let the arch CH be deſcribed; then it will be CH $= \frac{y\dot{x}}{a}$, and OH $= \dot{y}$. Therefore CO, the element of the curve, is equal to $\frac{\sqrt{yy\dot{x}\dot{x} + aa\dot{y}\dot{y}}}{a}$. But the equation of the curve is $ax = by$, and therefore $\dot{x}\dot{x} = \frac{bb\dot{y}\dot{y}}{aa}$; whence, making the ſubſtitution, it will be CO $= \frac{\dot{y}}{aa}\sqrt{a^4 + bbyy}$. The integral of this, after a long calculation, which, to avoid being tedious, I ſhall omit, will be found to depend on the logarithms, or, which is the ſame, on the quadrature of the hyperbola.

Now, by infinite ſeries. Firſt, I make $a^4 = bbmm$; whence the formula will be this, $\frac{b\dot{y}}{aa}\sqrt{mm + yy}$, which, being reduced to a ſeries, will be $\frac{b\dot{y}}{aa}$ into $m + \frac{yy}{2m} - \frac{y^4}{8m^3} + \frac{y^6}{16m^5} - \frac{5y^8}{128m^7}$, &c.; and therefore, by integration, the arch AC $= \frac{bmy}{aa} + \frac{by^3}{6a^2m} - \frac{by^5}{40a^2m^3} + \frac{by^7}{112a^2m^5} - \frac{5by^9}{9 \times 128a^2m^7}$, &c. And making $y = a$, the whole curve ACB $= \frac{bm}{a} + \frac{ab}{6m} - \frac{a^3b}{40m^3} + \frac{a^5b}{112m^5} - \frac{5a^7b}{9 \times 128m^7}$, &c. Then, inſtead of m, reſtoring it's value $\frac{aa}{b}$, it will be ACB $= a + \frac{bb}{6a} - \frac{b^4}{40a^3} + \frac{b^6}{112a^5} - \frac{5b^8}{9 \times 128a^7}$, &c.

 If

Fig. 126.



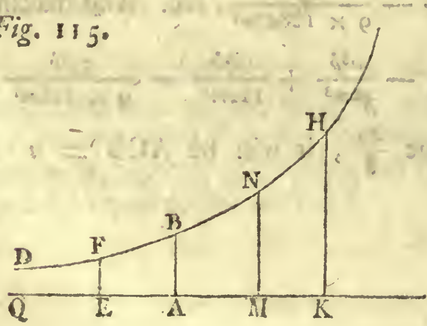
If the curve ABC were the logarithmic ſpiral, whoſe equation is $a\dot{y} = b\dot{x}$; making RB $= y$, and the infinitely little arch BD $= \dot{x}$; putting, in the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, the value of $\dot{x}$ given from the equation, it will be $\frac{\dot{y}\sqrt{aa + bb}}{b}$, and by integration, the curve AB $= \frac{y}{b}\sqrt{aa + bb}$.

Let the curve ABC be the hyperbolical ſpiral, in which the ſubtangent is always conſtant; and therefore, retaining the ſame names as above, the equation will be $y\dot{x} = a\dot{y}$. Therefore it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\dot{y}}{y}\sqrt{aa + yy}$; the integral of which formula, freed from the radical ſign, will be found to depend on the logarithmic.

By means of ſeries we ſhall find $\frac{\dot{y}}{y}\sqrt{aa + yy} = \dot{y}$ into $\frac{a}{y} + \frac{y}{2a} - \frac{y^3}{8a^3} + \frac{y^5}{16a^5} - \frac{5y^7}{128^7}$, &c. But if we would proceed to the integration, the firſt term cannot be integrated, but by the help of another infinite ſeries. Wherefore, the ſum of the ſaid ſeries being integrated, all but the firſt term, together with the integral of the ſeries expreſſing that firſt term, will form a ſeries which will be the value of the curve propoſed.

EXAMPLE XXV.

Fig. 115.



117. Let HBD be the logarithmic, AB the ſubtangent $= a$, AK $= x$, KH $= y$, and the equation $\frac{a\dot{y}}{y} = \dot{x}$. The value of $\dot{x}$ being ſubſtituted in the general formula, it will be $\frac{\dot{y}}{y}\sqrt{aa + yy}$, of which the integral depends on the ſame logarithmic. I ſhall forbear to apply infinite ſeries, becauſe their uſe may be ſufficiently ſeen in the former Examples.

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EXAMPLE XXVI.

118. Let the curve be the *Apollonian* parabola, with it's co-ordinates at any oblique angle, and whose equation is $ax = yy$. This being differenced, and substituted in the general formula for rectifications, when the ordinates are at oblique angles; that is, in the formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$, instead of $\dot{x}$, it's value given by y being substituted, we shall have $\frac{2\dot{y}}{a}\sqrt{yy + \frac{aey}{m} + \frac{1}{4}aa}$; the integral of which will be partly algebraical, and will depend partly on the quadrature of the hyperbola.

EXAMPLE XXVII.

119. Let the equation be $x^t = y$, which is to infinite parabolas, and to infinite hyperbolas between the asymptotes. By differencing, it will be $x^{t-1}\dot{x} = \dot{y}$, and $x^{2t-2}\dot{x}\dot{x} = \dot{y}\dot{y}$; whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, or the element of the curve, will be $\dot{x}\sqrt{x^{2t-2} + 1}$. Proceeding to the integration, I shall have recourse to the method of § 61, and shall exhibit the formula in the following manner, $\frac{\dot{x}}{\overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$. The canonical formula of the said article, or $\frac{x^n\dot{x}}{\overline{x^m + a^m}|^u}$, is algebräically integrable when $\frac{1-m+n}{m}$ is an integer affirmative number; and if it be an integer negative number, it will be reduced to known simple quadratures. Now, by comparing this formula $\frac{\dot{x}}{\overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$ with the canonical, we have $n = 0$, $2t - 2 = m$, and $a = 1$. By which it will be necessary that $\frac{1-2t+2}{2t-2}$ shall be an integer, which I call h. Then $\frac{1-2t+2}{2t-2}$, that is, $\frac{3-2t}{2t-2} = h$, and consequently $\frac{3+2h}{2+2h} = t$, the determining exponent of the infinite curves.

Let

Let b be a positive integer, beginning from 0. Now, if $b = 0$, it will be $t = \frac{3}{2}$; if $b = 1$, it will be $t = \frac{5}{4}$; if $b = 2$, it will be $t = \frac{7}{6}$, &c. Let b be any one of the series of natural numbers, 0, 1, 2, 3, 4, 5, &c. the innumerable values of the exponent t will be expressed by the following progression, $t = \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10}, \frac{13}{12}$, &c. the law of which series is manifest; and in all these cases the parabolical curves will be algebräically rectifiable; the first of which is the second cubical parabola.

Let b be equal to an integer negative number; and, first, make $b = -0$, in which case the same cubical parabola arises, because -0 and $+0$ are the same thing. Make $b = -1$, and the exponent t becoming $= \frac{1}{0}$, it is consequently infinite. Make $b = -2$, then $t = \frac{1}{2}$. Make $b = -3$, then $t = \frac{3}{4}$. And so on. Therefore the infinite values of the exponent t will be expressed by this progression, $t = \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$, &c. and the parabolical curves thence arising will be rectifiable by means of known quadratures.

The first curve which presents itself is the conic parabola, the rectification of which requires the quadrature of the hyperbola, § 110.

The other case, in which the general formula of § 61 is either rectifiable algebräically, or by means of known quadratures, is when $u - \frac{1}{m} - 1 - \frac{n}{m}$ is an integer. That is, by substituting the particular species of this example, $\frac{-3t+2}{2t-2} = b$, and therefore $\frac{2+2b}{3+2b} = t$, the determining exponent of the infinite curves.

Let b be a positive integer number, beginning at 0; we shall have the following progression, $t = \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}$, &c.

Let b be a negative integer, and, first, let $b = -0$. Then the same exponent $t = \frac{2}{3}$ returns upon us, because -0 is equivalent to $+0$. Let $b = -1$, the exponent t becomes equal to the fraction $\frac{0}{1}$, and consequently is nothing. Let $b = -2$, $b = -3$, &c. and we shall have this following progression, $t = \frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \frac{10}{9}$, &c.

The fraction which gives the value of the determining exponent t, is the same in both cases, only that in the second it is the reciprocal of the first; so that the progressions ought to come out reciprocal, as in effect they do. Therefore the curves determined by means of each formula are the same, but with reciprocal exponents, that is, they are referred to two different axes. As for example, the two exponents $\frac{1}{2}$ and $\frac{2}{1}$ belong to the *Apollonian* parabola, which offers itself in two manners, $x^{\frac{1}{2}} = y$, that is, $x = yy$, and likewise $x^{\frac{2}{1}} = y$, or $xx = y$; both local equations to the parabolical *trilineum*.

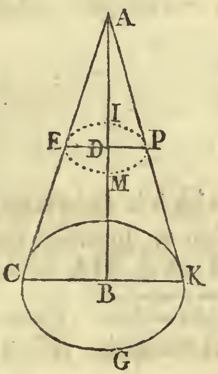
Wherefore

Wherefore these curves, which are included in the foregoing progressions, are either algebräically integrable, or do not require quadratures beyond the circle or hyperbola. But the other curves, infinite in number, require quadratures of a higher order.

It appears from our progressions, that the value of the exponent t is never negative. Hence no hyperbola admits of a rectification, either algebräical, or depending on the forementioned simple quadratures.

EXAMPLE XXVIII.

Fig. 127.



120. Let ACGKA be an erect cone, AB $= a$, BC $= b$; Of cubatures.
let AD $= x$ be any portion of the axis AB; it will be DE $= y = \frac{bx}{a}$, and therefore, substituting this value instead of y in the general formula, $\frac{cyy\dot{x}}{2r}$, it will be $\frac{cbbxx\dot{x}}{2aar}$, and by integration, $\frac{cbbx^3}{6aar}$, in respect to any portion taken from the vertex; omitting the constant quantity, which here is needless. And making $x = a$, the whole cone ACGKA will be $= \frac{cbba}{6r} = \frac{cbb}{2r} \times \frac{a}{3}$, that is, equal to the product of the base into a third part of the altitude.

And, because the solid content of a cylinder is the product of the base into it's height, the cylinder will be to the inscribed cone as 3 to 1.

The cone ACGKA is therefore $\frac{cbba}{6r}$, and the cone AIEMP $= \frac{cbbx^3}{6aar}$; therefore the frustum of the cone IMCK will be $\frac{cbb}{6r} \times \overline{a - \frac{x^3}{aa}}$, and therefore will be to the whole cone in the ratio of $a^3 - x^3$ to a^3. Whence, for example, if we should make AD $= \frac{1}{2}$AB $= \frac{1}{2}a$, the frustum will be to the whole cone as $a^3 - \frac{1}{8}a^3$, or $\frac{7}{8}a^3$, to a^3, or as 7 to 8; and to the cone AEMPI, as 7 to 1.

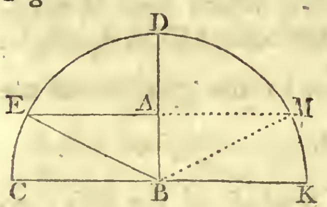
Therefore, as often as we are to measure any solid, it is necessary to consider, of what elements we design to have it composed, according to the different sections that may be adapted to it; varying it sometimes one way, sometimes another, as circumstances and conveniency may require. Then, among the aforesaid elements, to choose those which may be managed with the greatest facility,

facility, and to which the calculation may be moſt naturally adapted. In the erect cone for example, of which we are treating, we have as many circles as we pleaſe parallel to the baſe; and alſo as many triangles, which have their vertex the ſame as the cone, and for a baſe the parallel ordinates of the circle CGK. We may alſo cut the cone according to ſo many parabolas, which are equidiſtant from each other, and with axes parallel to the ſide AK; and many other ſections may be made.

Nevertheleſs it is true, that, to find the ſolidity of the cone, ſuch means as theſe are to be conſidered as not to the purpoſe, as being too compounded for the caſe propoſed. But it may be propoſed to cut the cone, or other ſolid, according to any plane whatever, and then to meaſure the two ſegments into which it is divided; and, in this caſe, it is convenient to make uſe of ſuch elements as ſhall correſpond to that ſection; as may be ſeen in Examples XXXVII. and XXXVIII. following.

EXAMPLE XXIX.

Fig. 128.



121. Let CDK be a ſemicircle, which is converted about a fixed radius DB, by which a hemiſphere will be produced; and make DB $= a$, DA $= x$, and it will be AE $= y = \sqrt{2ax - xx}$. Then, ſubſtituting this value in the general formula, it will be $\frac{cx}{2r} \times \overline{2ax - xx}$; and, by integration, the ſolidity of the indefinite ſegment AEM will be $= \frac{3caxx - cx^3}{6r}$. And making $x = a$, the ſolidity of the hemiſphere will be $= \frac{ca^3}{3r}$, and it's double, $\frac{2ca^3}{3r}$, will be the whole ſphere.

And becauſe the cylinder, the height of which is equal to the diameter of the baſe, or $2a$, is $\frac{ca^3}{r}$; the cylinder circumſcribed will be to the ſphere inſcribed, as $\frac{ca^3}{r}$ is to $\frac{2ca^3}{3r}$, or as 3 to 2. And conſequently the half cylinder will be to the hemiſphere in the ſame ratio. But the cone alſo, whoſe height is equal to the radius of the baſe, (or equal to a, the radius of the ſphere,) is $= \frac{ca^3}{6r}$; therefore the hemiſphere will be to the cone inſcribed as 2 to 1.

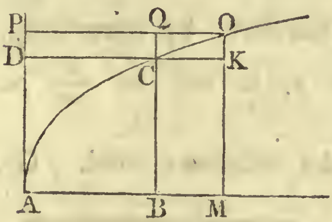
8 Furthermore,

Furthermore, as it is known that $\frac{\sqrt{3aa}}{2}$ is the radius of the baſe of an equilateral cone inſcribed in a ſphere, the radius of which is $= a$; and the height of the ſame being $= \frac{3a}{2}$, the cone will be $= \frac{9ca^3}{48r}$, and the ſphere will be $\frac{2ca^3}{3r}$, and therefore the ſphere to the cone as $\frac{2}{3}$ to $\frac{9}{48}$, or as 32 to 9. In like manner may be demonſtrated as many Theorems of *Archimedes* as we pleaſe, which are of a like nature.

Hence the manner is plain, of obtaining any ſector of the ſphere, which is generated (for example) by the ſector of the circle BEDM. For to the ſegment of the ſphere generated by the figure AED, which we know to be $= \frac{3caxx - cx^3}{6r}$, muſt be added the cone generated by the triangle EBA, and which is found to be $= \frac{c}{6r} \times \overline{2ax - xx} \times \overline{a - x}$, and the ſum, $\frac{caax}{3r}$, will be the ſector required.

EXAMPLE XXX.

Fig. 129.



122. Let there be a parabola of any order, whoſe equation is $y^m = a^{m-1}x$; which, being converted about the axis AM, generates a parabolical conoid. Then it will be $y = a^{\frac{m-1}{m}} x^{\frac{1}{m}}$, and $yy = a^{\frac{2m-2}{m}} x^{\frac{2}{m}}$; and therefore, ſubſtituting this value, the general formula will be $\frac{ca^{\frac{2m-2}{m}} x^{\frac{2}{m}} \dot{x}}{2r}$; and, by integrating, $\frac{mca^{\frac{2m-2}{m}} x^{\frac{m+2}{m}}}{2r \times \overline{m+2}}$ will be the ſolid content of the indefinite conoid. Or elſe, becauſe $x^{\frac{2}{m}} = \frac{yy}{a^{\frac{2m-2}{m}}}$, and therefore $x^{\frac{2+m}{m}} = \frac{xyy}{a^{\frac{2m-2}{m}}}$, by ſubſtituting this value in the integral now found, it will be $\frac{mcxyy}{2r \times \overline{m+2}}$.

Make

Make $m = 2$, that is, let it be the *Apollonian* parabola; the conoid will be $= \frac{cxyy}{4r}$, that is, the product of the base into half the height; and, by consequence, the said conoid will be half a cylinder of the same height, and of the same base.

If we would have the solid content of the dish, or of the solid generated by the figure ACD, converted about the axis AB; from the cylinder described by the rectangle ABCD, which we know to be $= \frac{cxyy}{2r}$, we must subtract the parabolical conoid $\frac{mcxyy}{2r \times \overline{m+2}}$, the remainder, $\frac{cxyy}{r \times \overline{m+2}}$, will be the content of the dish. And making $m = 2$, in respect of the *Apollonian* parabola, the dish will be $\frac{cxyy}{4r}$, which is half the cylinder, just as it ought to be, the conoid being also half of the same cylinder.

Let the figure move about the ordinate MO, and make AM $= b$, MO $= f$, AB $= x$, BC $= y$, CK $= b - x$, KO $= f - y$. The circle, with radius CK, will be $= \frac{c}{2r} \times \overline{b - x}|^2$, and therefore the product of this circle into $\dot{y}$ will be the differential of KM; that is, $\frac{c}{2r} \times \overline{bb\dot{y} - 2bx\dot{y} + xx\dot{y}}$ will be the element of the solid generated by the figure MACK. Therefore, by integrating, and, instead of x, putting it's value given by y, it will be $\frac{c}{2r} \times \overline{bby - \frac{2by^{m+1}}{\overline{m+1} \times a^{m-1}} + \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m-2}}}$, equal to the indefinite solid. Or, putting x in the place of $\frac{y^m}{a^{m-1}}$, it will be $\frac{c}{2r} \times \overline{bby - \frac{2bxy}{m+1} + \frac{xxy}{2m+1}}$. Now, putting $x = b$, $y = f$, in respect to the whole solid generated by the figure ACOM, it will be $\frac{c}{2r} \times \overline{bbf - \frac{2bbf}{m+1} + \frac{bbf}{2m+1}}$, that is, $\frac{2mmbbf}{\overline{2m+1} \times \overline{m+1}} \times \frac{c}{2r}$. And if we would have the parabola to be that of *Apollonius*, that is, if $m = 2$, then the solid will be $= \frac{4cbbf}{15r}$.

It is easy to perceive, that, in the *Apollonian* parabola, a cylinder on the same base, and of the height of the said solid, shall be to the solid as 15 to 8; and that the solid generated by the figure OAP shall be $= \frac{7cbbf}{30r}$.

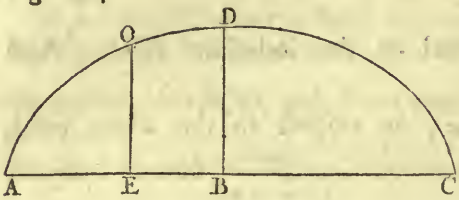
Let the figure move about the right line AP, and let it be, as before, AB $= x$, BC $= y$; then $\frac{cxx}{2r}$ will be a circle with radius DC, and $\frac{cxx\dot{y}}{2r}$ will be the element of the solid generated by the figure ACD. And, inftead of x, putting it's value given by y, and then integrating, it will be $\frac{c}{2r} \times \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m+2}}$, that is, $\frac{c}{2r} \times \frac{xxy}{2m+1}$, equal to the indefinite folid. And making $x = b$, $y = f$, it will be $\frac{cbbf}{2r \times \overline{2m+1}}$, in refpect to the whole folid, generated by the figure AOP.

But the cylinder on the fame bafe and altitude is $= \frac{cbbf}{2r}$; therefore the folid generated by the figure AMO is $= \frac{c}{2r} \times \frac{2mbbf}{2m+1}$.

But ftill, in another manner, we may obtain the folid generated by the figure AOM, revolving about the axis AP. Make AM $= b$, MO $= f$. A circle with radius DC will be $= \frac{cxx}{2r}$, and the circle with radius DK will be equal to $\frac{cbb}{2r}$. Therefore $\frac{c}{2r} \times \overline{bb - xx}$ will be the annulus defcribed by the line CK, and $\frac{c\dot{y}}{2r} \times \overline{bb - xx}$ will be the element of the folid generated by the figure CKMA; and, inftead of x, putting it's value given by y, it will be $\frac{c}{2r} \times \overline{bb\dot{y} - \frac{y^{2m}\dot{y}}{a^{2m-2}}}$, and by integration, $\frac{c}{2r} \times \overline{bby - \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m-2}}}$. Laftly, making $y = f$, in refpect of the whole folid, generated by the figure AMOA, it will be $\frac{c}{2r} \times bbf - \frac{f^{2m+1}}{\overline{2m+1} \times a^{2m-2}}$. But, when $y = f$, becaufe of the parabola, it will be $x = b = \frac{f^m}{a^{m-1}}$, and $bb = \frac{f^{2m}}{a^{2m-2}}$. Therefore, in the integral, fubftituting the value given by b, the folid will be $\frac{c}{2r} \times bbf - \frac{bbf}{2m+1} = \frac{c}{2r} \times \frac{2mbbf}{2m+1}$, as above.

EXAMPLE XXXI.

Fig. 124.

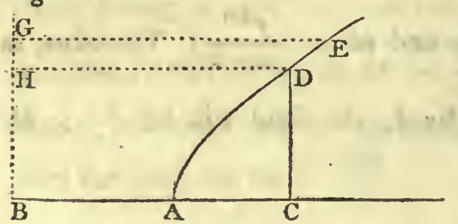


123. Let ADC be an ellipſis, $AB = a$, $BD = b$, $AE = x$, $EO = y$; and therefore the equation is $\frac{bb}{aa} \times \overline{2ax - xx} = yy$. Therefore, in the general formula, ſubſtituting the value of y given from the equation, it will be $\frac{c\dot{b}b}{2aar} \times \overline{2ax\dot{x} - xx\dot{x}}$; and by integration, it will be $\frac{cbb}{2aar} \times \overline{axx - \frac{1}{3}x^3}$, equal to the indefinite ſolid generated by the figure AEO, turning about the axis AC. Making $x = a$, it will be $\frac{cbba}{3r}$, half of the ſpheroid; and putting $x = 2a$, it will be $\frac{2cbba}{3r}$, the whole ſpheroid.

And, becauſe the cone of the ſame altitude AC, and of a baſe the radius of which is the conjugate ſemiaxis BD, is $= \frac{cbba}{3r}$, and the cylinder is $= \frac{cbba}{r}$, the ſpheroid will be two third parts of the cylinder, and double to the cone.

EXAMPLE XXXII.

Fig. 121.



124. Let AD be an hyperbola, which is converted about BC, and let it's tranſverſe ſemiaxis be $BA = \frac{1}{2}a$, the centre B, and it's parameter $= b$, $AC = x$, $CD = y$, and the equation is $\overline{ax + xx} \times \frac{b}{a} = yy$. Subſtituting the value of y in the general formula, it will be $\frac{cb\dot{x}}{2ar} \times \overline{ax + xx}$; and by integration, it will be $\frac{cb}{2ar} \times \overline{\frac{1}{2}axx + \frac{1}{3}x^3}$, equal to the indefinite hyperbolical conoid, generated by the figure ADC.

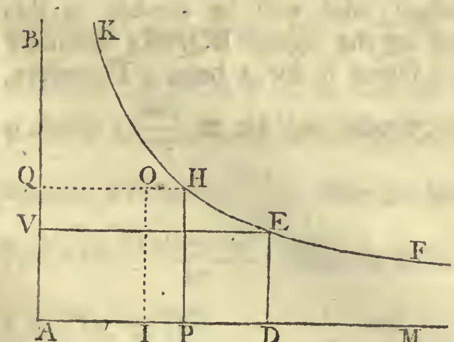
Make

Make BC $= x$, and the rest as above. The equation will be $\frac{b}{a} \times \overline{xx - \frac{1}{4}aa} = yy$, and therefore the formula will be $\frac{cbx}{2ar} \times \overline{xx - \frac{1}{4}aa}$, and by integration, $\frac{cb}{2ar} \times \overline{\frac{1}{3}x^3 - \frac{1}{4}aax} + f$. I add the constant quantity f, which, in this case, will be something. To determine what, it must be observed that in the point A, when $x = \frac{1}{2}a$, the solid ought to be nothing. Wherefore, instead of x, putting $\frac{1}{2}a$ in the integral, it ought to be $f + \frac{cb}{2ar} \times \overline{\frac{1}{24}a^3 - \frac{1}{8}a^3} = 0$, and therefore $f = \frac{caab}{24r}$. Therefore the complete integral will be $\frac{cb}{2ar} \times \overline{\frac{1}{3}x^3 - \frac{1}{4}a^2x + \frac{1}{12}a^3}$.

Let the hyperbola be converted about the conjugate semiaxis HB, and make the transverse semiaxis AB $= a$, the conjugate semiaxis $= b$, BC $= x$, CD $= y$. The circle with radius HD will be $= \frac{cxx}{2r}$, and therefore $\frac{cxx\dot{y}}{2r}$ will be the element of the solid generated by the plane or figure BHDA. And, instead of xx, substituting it's value given from the equation of the curve, we shall have $\frac{c\dot{y}}{2r} \times \frac{aayy + aabb}{bb}$; and by integration, $\frac{c}{2r} \times \overline{\frac{aay^3}{3bb} + aay}$; and making $y = b$, the solid will be $= \frac{2caab}{3r}$.

EXAMPLE XXXIII.

Fig. 130.



125. Let KHF be an hyperbola between the asymptotes; AD $= a$, DE $= b$, AP $= x$, PH $= y$, and the equation $xy = ab$. Let the curve revolve about the asymptote AB. Then the circle with radius QH will be $= \frac{cxx}{2r}$, and therefore $\frac{cxx\dot{y}}{2r}$ will be the element of the solid generated by the figure AQHFMA, infinitely produced towards M. And, instead of x, putting it's value given from the equation, it

it will be $\frac{caabb\dot{y}}{2ryy}$, and by integration, $f - \frac{caabb}{2ry}$. Now, to determine f, it may be obferved, that, when it is $y = 0$, the folid ought to be nothing, and therefore $f = \frac{caabb}{2r \times 0}$, an infinite quantity, and therefore the complete integral will be $-\frac{caabb}{2ry} + \infty$; fo that the folid is of an infinite value.

Inftead of fubftituting in the formula the value given by y from the equation, in the place of xx, if we fhould fubftitute the value of y; it would be $-\frac{abc\dot{x}}{2r}$, and by integration, $-\frac{abcx}{2r} + f$. But the folid cannot be nothing except when x is infinite, and then the conftant quantity f to be added ought to be infinite, and therefore the folid will be infinite.

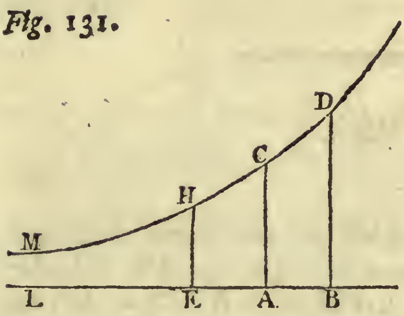
To have the folid generated by the plane or figure BAPHK infinitely produced towards B, it will be enough to confider, that as $\frac{cx}{r}$ is the periphery of the circle whofe radius is QH $= x$, then $\frac{cxy}{r}$ will be the fuperficies of the cylinder, generated by the plane AQHP, and confequently $\frac{cxy\dot{x}}{r}$ will be the folid content of the hollow cylinder, generated by the infinitely little rectangle IPHO. Therefore the fum of all thefe, or $\int \frac{cxy\dot{x}}{r}$, will be the folid required. Therefore, inftead of y, putting it's value $\frac{ab}{x}$, the integral will be $\frac{cabx}{r}$, a finite quantity, although the folid be of an infinite altitude.

In the expreffion $\frac{cabx}{r}$ of the folid, inftead of ab putting it's value xy, given from the equation; it will be $\frac{cxxy}{r}$. But $\frac{cxxy}{2r}$ is the cylinder generated by the rectangle APHQ. Therefore the hyperbolical folid will be double to this cylinder. And therefore the folid generated by the figure BQHK, infinitely produced, will be equal to the cylinder which ferves it for a bafe. Therefore, taking $x = a$, and confequently $y = b$, this cylinder will be $= \frac{caab}{2r}$, which is equal to the folid erected upon it.

EX-

EXAMPLE XXXIV.

Fig. 131.



126. Let HCD be the logarithmic curve, it's ſubtangent CA $= a$, AB $= x$, BD $= y$, and it's equation $\dot{x} = \frac{a\dot{y}}{y}$. Let it be converted about the aſymptote EB. In the general formula, inſtead of $\dot{x}$, putting it's value given from the equation, it will be $\frac{cay\dot{y}}{2r}$; and by integration, it will be $\frac{cayy}{4r} + f$. But when it is $y = AC = a$, the ſolid will be $= 0$. Therefore it muſt be $f = -\frac{ca^3}{4r}$; and the complete integral, that is, the ſolid generated by the indefinite plane ABDC, will be $= \frac{cayy - ca^3}{4r}$.

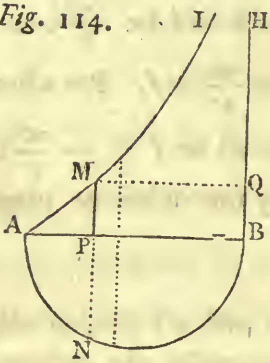
Let the abſciſs AE be negative, and therefore $= -x$; and it's fluxion alſo will be negative, or $-\dot{x}$. And becauſe, as the abſciſs increaſes, the ordinate will diminiſh, therefore the fluxion of EH will alſo be negative, or $-\dot{y}$; ſo that the equation of the curve will be ſtill the ſame, $\dot{x} = \frac{a\dot{y}}{y}$. But, becauſe $\dot{x}$ is negative, the general formula will be negative alſo, or $-\frac{cyy\dot{x}}{2r}$. Subſtituting therefore, the value of $\dot{x}$, it will be $-\frac{cay\dot{y}}{2r}$, and by integration, $-\frac{cayy}{4r} + f$. But when the ſolid is nothing, it will be $y = a$; therefore $f = \frac{ca^3}{4r}$, and the complete integral will be $\frac{ca^3 - cayy}{4r}$, equal to the ſolid generated by the plane ACHE. Putting $y = 0$, that is, ſuppoſing the ſolid to be infinitely produced towards M, the integral will be $= \frac{ca^3}{4r}$, and then the ſolid itſelf, infinitely produced, will be $= \frac{ca^3}{4r}$. But the ſolid generated by the plane ACHE we have ſeen to be $\frac{ca^3 - cayy}{4r}$; then the ſolid infinitely produced, generated by the plane LEMH, is $\frac{cayy}{4r}$.

Now,

Now, becaufe the cylinder, the radius of whofe bafe is AC $= a$, and it's height alfo $= a$, is $\frac{ca^3}{2r}$; the folid of the logarithmic curve, infinitely produced towards M, on the bafe with radius AC $= a$, will be to the faid cylinder, in the ratio of $\frac{1}{4}$ to $\frac{1}{2}$, or as 1 to 2.

EXAMPLE XXXV.

Fig. 114.



127. Let the curve AMI be the ciffoid of *Diocles*, which, by revolving about the right line AB, defcribes a folid. Make AP $= x$, PM $= y$, AB $= a$, and the equation will be $yy = \frac{x^3}{a-x}$. Therefore, the value of yy being fubftituted, the general formula of folids will be $\frac{cx^3\dot{x}}{2r \times \overline{a-x}}$, and by integration, $-\frac{cx^3}{6r} - \frac{cax^2}{4r} - \frac{caax}{2r} - \frac{caaa}{2r} \times l\,\overline{a-x} + f$. But, making $x = 0$, the folid ought to be nothing, and therefore $f = \frac{caaa}{2r} la$. And the complete integral $\frac{caaa}{2r} la - \frac{caaa}{2r} l\,\overline{a-x} - \frac{caax}{2r} - \frac{caxx}{4r} - \frac{cx^3}{6r}$ is equal to the folid generated by the figure APM. And making $x = a$, the whole folid will be $= \frac{caaa}{2r} la - \frac{caaa}{2r} l0 - \frac{11ca^3}{12r}$. But the logarithm of o is an infinite quantity and negative, which, multiplied into $-\frac{caaa}{r}$, makes an affirmative quantity; fo that the intire folid will be infinite. It is to be obferved, that the aforefaid logarithms are to be taken from the logarithmic curve, the fubtangent of which $= a$.

By the help of infinite feries, it will be $\frac{cx^3\dot{x}}{2r \times \overline{a-x}} = \frac{cx^3\dot{x}}{2ar} + \frac{cx^4\dot{x}}{2raa} + \frac{cx^5\dot{x}}{2ra^3} + \frac{cx^6\dot{x}}{2ra^4}$, &c.; and by integration, the folid generated by the plane APM will be $= \frac{cx^4}{8ar} + \frac{cx^5}{10ra^2} + \frac{cx^6}{12ra^3} + \frac{cx^7}{14ra^4}$, &c. And making $x = a$, in refpect of the intire folid, it will be $\frac{ca^3}{2r}$ into $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, &c. the total value of which feries is infinite.

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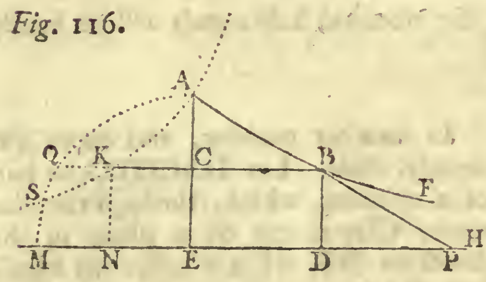
EXAMPLE XXXVI.

Fig. 116.



128. Let the *tractrix* ABF be converted about the asymptote EH. In the general formula $\frac{cyy\dot{x}}{2r}$, substituting the value of $\dot{x}$ given from the equation $\dot{x} = -\frac{\dot{y}\sqrt{aa - yy}}{y}$, § 103, we shall have $-\frac{cy\dot{y}\sqrt{aa - yy}}{2r}$. And by integration, it will be $\frac{c}{6r} \times \overline{aa - yy}|^{\frac{3}{2}}$, equal to the solid generated by the figure AEDB, omitting the addition of a constant, which is here unnecessary. Wherefore, making $y = 0$, the solid infinitely produced will be $= \frac{ca^3}{6r}$. But the solid content of the sphere whose radius is AE $= a$, (§ 121.) will be $= \frac{2ca^3}{3r}$; and therefore the solid infinitely produced will be a fourth part of that sphere.

EXAMPLE XXXVII.

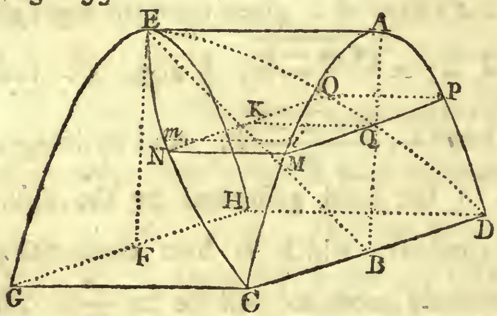
Fig. 132.

129. Let QBMCPT be a cylinder, from which, by a plane through the diameter BC, and in the direction AP, a portion or *ungula*, BMCPB, is cut off; the solid content of this is required.

Make BC $=$ QM $= 2a$, MP $=$ QT $= b$, AD $= x$, and DH being drawn, shall be an ordinate in the circle $= \sqrt{aa - xx}$. From the point H let the right line HO be drawn parallel to MP or QT, which shall be in the superficies of the cylinder. Then from D to the point O let the right line DO be drawn, which shall be in the plane BOPC. Then we shall have formed in the solidity of the *ungula* the triangle DHO, which

which is similar to the triangle AMP, and therefore it will be HO $= \frac{b\sqrt{aa-xx}}{a}$. But the aggregate of all these triangles, DHO, is just the solidity required of half the *ungula*, and therefore it will be $= \int \frac{b\dot{x}}{2a} \times \overline{aa - xx}$; and by integration, $\frac{abx}{2} - \frac{bx^3}{6a}$. And making $x = a$, the aforesaid half *ungula* will be finally $= \frac{1}{3}aab$, and the whole $= \frac{2}{3}aab$.

Fig. 133.



In another manner, and more generally, thus. Let DACHEG be half of a cylinder, which, through the diameter CD, is cut by a plane in the direction DE, whence arises the *ungula* DBCEAD, the solidity of which is required. Make BA $= a$, AE $= b$, BQ $= x$, QM $= y$; it will be QK $= \frac{bx}{a}$, and therefore the rectangle PONM $= \frac{2bxy}{a}$. And this being drawn into $\dot{x}$, or $\frac{2byx\dot{x}}{a}$, will be the element of the solidity of the *ungula*.

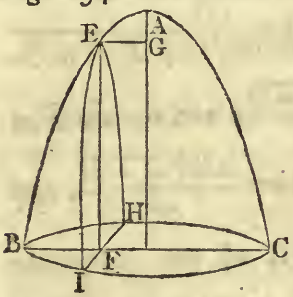
Let the curve DAC be a semicircle; then $y = \sqrt{aa - xx}$, and the formula will be $\frac{2bx\dot{x}}{a}\sqrt{aa - xx}$; and by integration, $-\frac{2b}{3a} \times \overline{aa-xx}|^{\frac{3}{2}} + m$. Now, by putting $x = 0$, the constant, m, will be found to be $= \frac{2}{3}baa$, and therefore the integral of the solid complete will be $\frac{2}{3}baa - \frac{2b}{3a} \times \overline{aa-xx}|^{\frac{3}{2}}$; and making $x = a$, in respect of the whole *ungula*, it will be $\frac{2}{3}baa$, as before.

Let the curve DAC be one of the parabolas *ad infinitum*, and it's equation $y^m = a - x$. Substituting the value of y, the formula will be $\frac{2bx\dot{x}}{a} \times \overline{a-x}|^{\frac{1}{m}}$, which being integrated according to § 29, and a constant being joined, and making $x = a$; it will give $\frac{2bm^2a^{\frac{m+1}{m}}}{\overline{2m+1} \times \overline{m+1}}$ for the solidity of the whole *ungula*. And taking $m = 2$, or the *Apollonian* parabola, it will be $\frac{8ba^{\frac{3}{2}}}{15}$. Now, supposing that, when $x = 0$, it is BC $= y = c$; it will be $a^{\frac{1}{2}} = c$, and therefore

therefore the *ungula* $= \frac{8}{15}abc$. After the fame manner we may find the *ungula* of the elliptical cylinder to be $\frac{2}{3}abc$, fuppofing the tranfverfe femiaxis $= a$, and the conjugate femiaxis $= c$.

EXAMPLE XXXVIII.

Fig. 134.



130. Let the parabolical conoid BAC be cut by any plane IEH, perpendicular to the circular bafe BICH; it is required to find the meafure of the fegment, comprehended by the fection IEH, and by the plane parallel to it, through the axis AD.

Make the parameter $= a$ of the generating parabola BAC, the given abfcifs AD $= b$, then the ordinate DB $= \sqrt{ab}$. Let the co-ordinates be DF $= x$, FE $= y$, and therefore the equation of the aforefaid curve BAC will be $ab - xx = ay$. By the nature of the circle BICH, the rectangle CFB $= ab - xx$, equal to the fquare FH $= zz$. But $ab - xx = ay$; therefore $ay = zz$, and confequently the fection IEH will be a parabola, with the fame parameter as the principal. Wherefore the rectangle EFH remains fixed, $= yz = y\sqrt{ay}$; and becaufe this is to the area IEH, as 3 to 4, this area will be $= \frac{4}{3}y\sqrt{ay}$, and the product of this area IEH into the infinitely little height $\dot{x}$, the fluxion of DF $= x$, will be the element of the folid in queftion, that is, $\frac{4}{3}y\dot{x}\sqrt{ay}$. But $y = \frac{ab - xx}{a}$; therefore the element will be $\frac{4}{3}\dot{x} \times \frac{ab - xx}{a}\sqrt{ab - xx}$, or $\frac{4}{3}b\dot{x}\sqrt{ab - xx} - \frac{4}{3a}x^2\dot{x}\sqrt{ab - xx}$.

The fluent of the firft term depends on the quadrature of the circle BHC; the fecond is reduced to known quadratures, by means of the firft formula of § 61.

131. I forbear from giving examples of folids generated by curves with the co-ordinates at oblique angles to each other; becaufe, the formula for thefe cafes being different from the ufual and ordinary ones, only by conftant quantities, no difficulties can be met with of a different nature from thefe already produced.

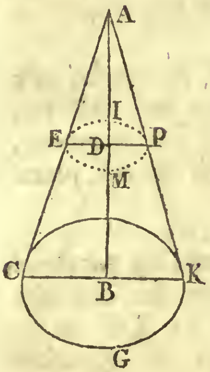
Thus, alfo, I omit examples of folids generated by curves which are referred to a *focus*, becaufe I am not willing to introduce the Theory of the Centers of Gravity, as I have faid before. The given curves may be reduced to others referred to an axis, about which I have already treated.

N. B. The letter D is omitted in the center of the bafe of Fig. 134.

EXAMPLE XXXIX.

The complanation of curved surfaces.

Fig. 127.

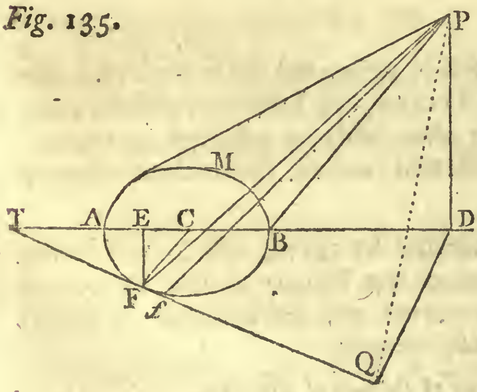


132. Let ACGK be an erect cone, AB $= a$, BC $= b$, any portion of the axis, as AD $= x$; it will be DE $= y$ $= \frac{bx}{a}$, and $\dot{y} = \frac{b\dot{x}}{a}$, $\dot{y}\dot{y} = \frac{bb\dot{x}\dot{x}}{aa}$. Therefore this value of $\dot{y}\dot{y}$, being substituted in the general formula $\frac{cy}{r}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$, it will be $\frac{cy}{r}\sqrt{\frac{aa\dot{x}\dot{x}+bb\dot{x}\dot{x}}{aa}} = \frac{cy\dot{x}\sqrt{aa+bb}}{ar}$; and the value of y being substituted, that is, $\frac{bx}{a}$, it will be $\frac{cbx\dot{x}\sqrt{aa+bb}}{aar}$, and by integration, $\frac{cbxx\sqrt{aa+bb}}{2aar}$, in respect of the superficies of the cone AEMPI. And making $x = a$, it will be $\frac{cb\sqrt{aa+bb}}{2r}$, in respect of the superficies of the whole cone, and therefore it is equal to the rectangle of half the circumference of the base into the side AC.

The same conclusion would have been had, if, instead of substituting in the general formula the value of $\dot{y}\dot{y}$, we had substituted the value of $\dot{x}\dot{x}$.

Wherefore the surface of the frustum of the cone IMKCG will be $=$ $\frac{cb}{2r}\sqrt{aa+bb} - \frac{cbxx}{2aar}\sqrt{aa+bb}$, that is, $\frac{cb\sqrt{aa+bb}\times\overline{aa-xx}}{2raa}$; and therefore it will be to the surface of the whole cone, as $aa - xx$ to aa.

Fig. 135.



133. But if the cone be scalene, it is necessary to proceed after another manner. Let PAFBM be a scalene cone, the base of which is the circle AFBM; and on the diameter produced (if need be) let fall PD perpendicular to the plane of the circle, or the base. Let two points F, f, be taken in the periphery of the circle, infinitely near to each other, and let the two sides of the cone FP, fP, be drawn. It is plain that the infinitesimal triangle PFf will be the difference of

or element of the ſuperficies of the cone. Then to the point F let the tangent TFQ be drawn, to which let DQ be perpendicular, and let the points P, Q be joined by the right line PQ.

Now, becauſe the plane of the triangle PDQ paſſes through the right line PD, which is perpendicular to the plane of the baſe of the cone, the plane PQD will alſo be perpendicular to the ſame plane of the baſe. But the tangent TQ, which is alſo in the plane of the baſe, makes a right angle with QD, the common ſection of the two planes, and therefore will be perpendicular to the plane PQD, and conſequently to the right line QP; and therefore the triangle $PFf = \frac{PQ \times Ff}{2}$.

Make the radius $CA = r$, $CD = b$, $CE = x$; it will be $FE = \sqrt{rr - xx}$; and becauſe the angle CFT is a right one, TF being a tangent to the circle, the triangles CFE, TCF, will be ſimilar. Whence it will be $CT = \frac{rr}{x}$. But CT . CF :: CF . CE :: TD . DQ. Therefore $DQ = \frac{rr + bx}{r}$. Make the given line $PD = p$. Therefore it will be $PQ = \sqrt{pp + \frac{\overline{rr + bx}|^2}{rr}}$. But the element of the circle Ff we know to be $-\frac{r\dot{x}}{\sqrt{rr - xx}}$; therefore $\frac{1}{2}Ff \times PQ$, the element of the ſuperficies, will be $-\frac{r\dot{x}}{2}\sqrt{pp + \frac{\overline{rr + bx}|^2}{rr}} \div \sqrt{rr - xx}$; a formula which hitherto has not been reduced to the known quadratures of the circle or hyperbola, becauſe it cannot be freed from radical ſigns, as has been ſeen at § 38, and as we have alſo ſeen, in our attempt to rectify the ellipſis.

If we have recourſe to infinite ſeries, the numerator muſt be reduced to a ſeries, and alſo the denominator; then we muſt proceed in the ſame manner as was done in the ſecond method concerning the ellipſis, in Example XX, § 112.

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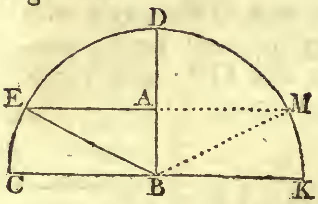
EXAMPLE XL.

Fig. 128.



134. Let there be a hemisphere, the generating semicircle of which is CDK, which is converted about the radius DB $= a$, and make any line DA $= x$; it will be AE $= y = \sqrt{2ax - xx}$, and therefore $\dot{y}\dot{y} = \frac{\overline{a-x}^2 \times \dot{x}\dot{x}}{2ax - xx}$. And making the substitutions in the general formula, it will be $= \frac{ca\dot{x}}{r}$, and by integration, $\frac{cax}{r} =$ to the superficies of the segment of the sphere, generated by the arch EDM. And making $x = a$, the superficies of the hemisphere will be $= \frac{caa}{r}$, and therefore $\frac{2caa}{r}$ will be the superficies of the whole sphere. Therefore the superficies of any segment will be equal to the product of the periphery of the generating circle of the sphere, into the altitude of that segment; of the hemisphere, equal to the rectangle of the same periphery into the radius; and of the sphere, equal to the rectangle of the periphery into the diameter; and therefore these superficies will be to each other in the ratio of their respective altitudes, the radius, and the diameter.

And because the area of the generating circle of the sphere is $= \frac{caa}{2r}$, the superficies of the sphere will be to the same area as 4 to 1, that is, quadruple of the greatest circle.

And because, also, the superficies of the cylinder, (excluding it's bases,) which is circumscribed to the hemisphere, is equal to the product of the periphery of the base into the height; it will therefore be $= \frac{caa}{r}$, and consequently the superficies will be equal to that of the hemisphere. Now the cone inscribed in the hemisphere has also it's superficies $= \frac{ca\sqrt{2aa}}{2r}$; therefore the superficies of the cylinder, or of the hemisphere, to the superficies of the inscribed cone, will be as $2a$ to $\sqrt{2aa}$, that is, as the diameter to the side of the cone.

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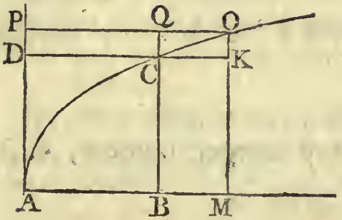
EXAMPLE XLI.

Fig. 129.



135. If the parabola ACO of the equation $ax = yy$, turns about the axis AM; it will be $a\dot{x} = 2y\dot{y}$, and $\dot{x}\dot{x} = \frac{4yy\dot{y}\dot{y}}{aa}$, and therefore, making the ſubſtitution, the formula will be $\frac{cy\dot{y}}{ar}\sqrt{4yy+aa}$, and by integration, $\frac{c}{12ra} \times \overline{4yy+aa}|^{\frac{3}{2}}$, equal to the [ſuperficies of the] indefinite parabolical conoid, equal to the fourth proportional of $6a$, $\sqrt{4yy+aa}$, and the area of the circle whoſe radius is $= \sqrt{4yy+aa}$.

136. More generally, let $\frac{x^t}{t} = y$ be the equation of the parabola ACO, (Fig. 129.) with it's abſciſs AB $= x$, and with it's ordinate BC $= y$; which equation for the *trilineum* ACD will be $\overline{xt}|^{\frac{1}{t}} = y$, if we take AD $= x$ as abſciſs, and DC $= y$ as ordinate. At § 119, Example XXVII, it has been ſeen, that the element of the curve, which I call $\dot{u}$, was $= \frac{\dot{x}}{\overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$; and the differential formula for the ſuperficies is $\frac{cy\dot{u}}{r}$. Then it will be $\frac{cy\dot{u}}{r} = \frac{cy\dot{x}}{r \times \overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$. But, by the local equation, it is $\frac{x^t}{t} = y$. Then it will be $\frac{cy\dot{u}}{r} = \frac{cx^t\dot{x}}{rt \times \overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$.

To proceed to the integrations or quadratures, I ſhall make uſe of the method explained at § 61, and applied to the aforeſaid Example XXVII. But, firſt, it is to be obſerved, that c, being the periphery of the circle whoſe radius is r, the integral $\int\frac{cy\dot{u}}{r}$ will give us the ſurface of the conoid. But if c repreſents any right line whatever, we ſhall have the meaſure of the ſurface of the *ungula*, when a cylindroid is erected upon the baſe CAB, which is cut by a plane

plane paſſing through the axis AB, and with the ſubject baſe CAB forms an angle, of which the right ſine is to that of the complement, as c is to r. Then the ungular ſuperficies is to that of the round ſolid, as a given right line is to the circumference c.

Operating, therefore, as explained above, at § 61, that our formula may be algebräically integrable, or reducible to known quadratures, we ſhall find that it muſt be $t = \frac{3 + 2b}{1 + 2b}$, or elſe $t = \frac{b + 1}{b + 2}$, where b denotes any integer number, poſitive or negative.

The firſt condition, or $t = \frac{3 + 2b}{1 + 2b}$, making b any integer number, firſt poſitive and then negative, will give us theſe two progreſſions:

I. $t = \frac{3}{1}, \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$, &c. II. $t = \frac{-1}{1}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}$, &c.

The ſecond condition, or $t = \frac{b + 1}{b + 2}$, making b any integer number, firſt poſitive and then negative, will give us theſe other two progreſſions:

III. $t = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$, &c. IV. $t = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$, &c.

To this I ſhall ſubjoin a few ſhort obſervations.

I. As the two progreſſions, the firſt and the third, contain the exponents of all thoſe parabolas, which, by circulating about the axis, generate conoids, the ſuperficies of which are analytically quadrable, ſuppoſing only the rectification of the circular periphery; and conſequently all the *ungulæ* above deſcribed, of a given altitude, admit an algebräical quadrature: So, in the caſes of the ſecond and fourth progreſſions, ſomething more is intended, as they require the quadrature of the hyperbola.

II. It is obſervable that, the firſt ſeries being compared with the ſecond, and the third with the fourth, the exponents are reciprocal, and belong to the ſame curve. This ſhows that, as the parabolical area may be converted, either about the axis AB, or about the axis AD, and in each caſe may produce very different ſuperficies; if, in the firſt caſe, it generates a ſuperficies that is abſolutely quadrable, at leaſt conſidered in the *ungula*; in the ſecond caſe, on the contrary, the values being reciprocal, the above-ſaid ſuperficies will ariſe, which are only hypothetically quadrable. For example, the conoid formed from the firſt cubical parabola being turned about AD, furniſhes us with the ſurface of an *ungula* which is algebräically quadrable, and alſo that of the round ſolid, provided we have a right line equal to the circumference. But if it be converted about the axis AB, then quadratures are required. The ſame thing obtains in the ſecond cubical parabola, and quite the contrary in that of *Apollonius*.

 III. Com-

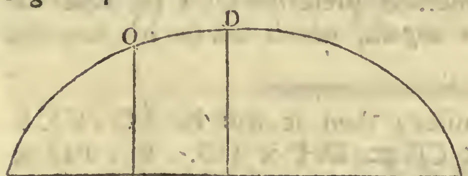
III. Comparing these series with those of § 119, we may discover, that among these there is no parabola of the first or second series, that is rectifiable either analytically, or by the means of known quadratures; on the contrary, those of the third and fourth are all rectifiable, and comprehend all that are contained in the progressions of § 119.

IV. Among the hyperbolas, the common one only between the asymptotes admits a superficies reducible to the quadrature of the said hyperbola; because no other negative exponent appears in the progressions, except — 1.

V. The exponents which are not found in the said series are these, $t = 4$, 5, 6, &c. $\frac{2}{5}$, $\frac{5}{3}$, &c. for which higher quadratures are required, to measure the conoidal surfaces thence arising.

EXAMPLE XLII.

Fig. 124.

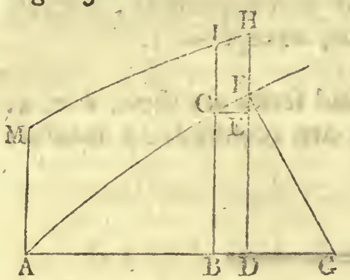


137. Let ADC be an ellipsis, which is converted about the axis AC, and make $AB = a$, $BD = b$, $AE = x$, $EO = y$; and the equation is $\frac{aayy}{bb} = 2ax - xx$. Therefore, by differencing, it will be $\dot{x} = \frac{aay\dot{y}}{bb \times \overline{a-x}}$, and therefore $\dot{x}\dot{x} = \frac{a^4yy\dot{y}\dot{y}}{b^4 \times \overline{a-x}^2}$; and, instead of $-2ax + xx$, putting it's value $-\frac{aayy}{bb}$ given by the equation, it will be $\dot{x}\dot{x} = \frac{aayy\dot{y}\dot{y}}{bb \times \overline{bb-yy}}$. Then substituting this value in the general formula, we shall have $\frac{cy\dot{y}\sqrt{b^4 + aayy - bbyy}}{rb\sqrt{bb - yy}}$; and, for brevity-sake, making $aa - bb = ff$, supposing a to be greater than b, or that the axis about which the ellipsis circulates to be the greater axis (for, if a were less than b, we ought to make $aa - bb = -ff$), the formula will be $\frac{cy\dot{y}\sqrt{b^4 + ffyy}}{rb\sqrt{bb - yy}}$, which, for reasons already mentioned in their place, may be freed from radicals; and the integral of which, by means of the canon of § 56, we shall find to depend on the quadrature of the circle. But if a shall be less than b, or the axis about which the ellipsis turns be the lesser axis, the superficies of the spheroid will

depend

depend on both the quadratures, that of the circle and that of the hyperbola. Wherefore the ſuperficies of the *ungula*, in the firſt caſe, is equal to a portion of the elliptic ſpace, which is eaſily determined by means of the perpendicular to the curve. But, in the ſecond caſe, theſe perpendiculars will give us an hyperbolical ſpace equal to the ſame ſuperficies of the *ungula*. That this may be plainly ſeen, let ACF be the curve on which a cylindroid is ſuppoſed to be erected, which is to be cut by a plane which paſſes through the axis AB, and forms with the ſubjacent plane CAB half a right angle. It is evident that, making $\dot{u}$ the element of the curve, $\int y\dot{u}$ will be the ſuperficies of the lower *ungula*, and $\int\frac{cy\dot{u}}{r}$ will be the ſuperficies of the conoid, generated by the converſion of the figure CAB about the axis AB; and therefore the ſuperficies of the *ungula* will be to that of the conoid, as radius to the circumference of the circle.

Fig. 136.



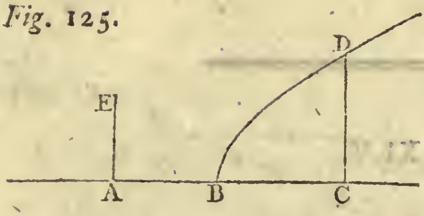
Now let the two ordinates BC, DF, be infinitely near, and drawing the perpendicular FG at the point F, let it be put in DH, and repreſent the ordinate of a new curve MIH drawn by the method preſcribed. I ſay that the area MABI is equal to the ſuperficies of the *ungula*, which has for it's baſe the arch AC.

The two triangles FCE, GFD, are ſimilar; then it will be FC . CE :: GF . FD. Therefore FD × FC = GF × CE = DH × DB. But FD × FC ($y\dot{u}$) is the element of the ſuperficies of the *ungula*, and HD × DB is the element of the area IMAB. Then, theſe elements being equal, their integrals will be equal alſo; that is, the aforeſaid areas. This being premiſed, let the figure ACB be a fourth part of the ellipſis, the equation of which is $\frac{aayy}{bb} = 2ax - xx$. Then the perpendicular will be FG = $\frac{b}{aa}\sqrt{2a^3x - aaxx + bbxx - 2abbx + aabb}$. Then, making the ordinate BI = z, it will be $z = \frac{b}{aa}\sqrt{\overline{xx - 2ax} \times \overline{b^2 - a^2} + a^2b^2}$, an equation to the curve MIH, which will be another ellipſis when a is greater than b, or if AB be the greater axis of the ellipſis ACB; and on the contrary, an hyperbola, when a is leſs than b, that is, when AB is the leſſer axis.

Laſtly, in the middle caſe, or when the ellipſis degenerates into a circle, we know already, that the ſaid ſurface of the *ungula* is quadrable, as being equal to a rectangle.

EXAMPLE XLIII.

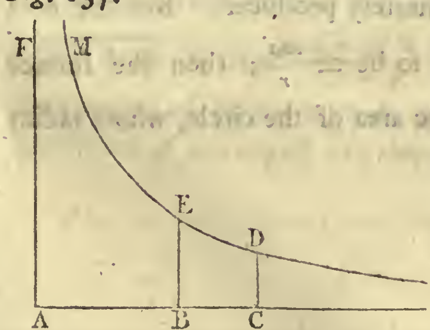
Fig. 125.



138. Let BD be an hyperbola, which circulates about the tranfverfe axis BA. Let A be it's centre, BA $= a$, the conjugate femi-axis AE $= b$, AC $= x$, CD $= y$. The equation will be $xx - aa = \frac{aayy}{bb}$, and therefore $y = \frac{b}{a}\sqrt{xx - aa}$, and $\dot{y} = \frac{b x\dot{x}}{a\sqrt{xx - aa}}$. Therefore the general formula, when the fubftitutions are made, will be $\frac{cb}{ar}\sqrt{xx - aa} \times \sqrt{\frac{a^2x^2\dot{x}^2 + b^2x^2\dot{x}^2 - a^4\dot{x}^2}{a^2 \times \overline{xx - aa}}}$, that is, $\frac{cb\dot{x}}{aar}\sqrt{aaxx + bbxx - a^4}$; or, making $aa + bb = ff$, it will be $\frac{cbf\dot{x}}{aar}\sqrt{xx - \frac{a^4}{ff}}$, the integral of which, when it is freed from it's radical fign, we fhall find, in like manner, to depend on the quadrature of the hyperbola.

EXAMPLE XLIV.

Fig. 137.



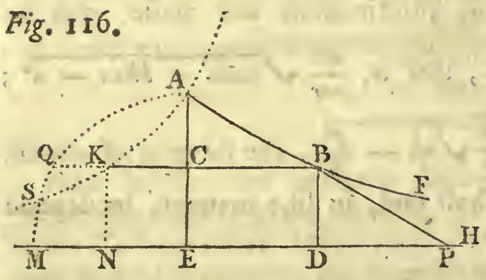
139. Let MD be an equilateral hyperbola, between it's afymptotes, and let it turn about the afymptote AC, of which the equation is $ay + xy = aa$; making AB $= a$, BC $= x$, and CD $= y$. Then it will be $x = \frac{aa}{y} - a$, and $\dot{x} = -\frac{aa\dot{y}}{yy}$, $\dot{x}\dot{x} = \frac{a^4\dot{y}\dot{y}}{y^4}$. Therefore, making the fubftitution, the general formula will be $\frac{c\dot{y}}{ry}\sqrt{y^4 + a^4}$. Put $\sqrt{y^4 + a^4} = z$, and therefore $y^4 = zz - a^4$, $\dot{y} = \frac{z\dot{z}}{2y^3}$. Make thefe fubftitutions, and we fhall have the formula

 transformed

transformed into this other, $\frac{czz\dot{z}}{2r \times \overline{zz - a^4}}$, which is free from radical ſigns; the integral of which depends partly on the logarithms, as is eaſy to perceive. Therefore the ſuperficies required, deſcribed by our hyperbola, will alſo depend on the quadrature of the hyperbola.

EXAMPLE XLV.

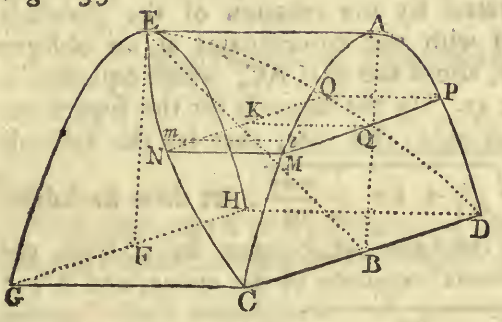
Fig. 116.



140. Let ABF be the ſolid generated by the *tractrix*, as in Example XXXVI, § 128, of which the ſuperficies is required. In the general formula $\frac{cy\dot{u}}{r}$ (where $\dot{u}$ repreſents the element of the curve,) inſtead of $\dot{u}$, ſubſtituting it's value $-\frac{a\dot{y}}{y}$ obtained from the equation of the curve, we ſhall have $-\frac{ac\dot{y}}{r}$, and by integration, $-\frac{acy}{r} + n$. But when the ſuperficies is nothing, we have $y = a$; therefore the conſtant $n = \frac{aac}{r}$, and therefore the complete integral is $\frac{aac}{r} - \frac{acy}{r}$, equal to the ſurface of the ſolid generated by the figure AEDB. And making $y = 0$, then $\frac{aac}{r}$ will be equal to the ſurface of the ſolid infinitely produced. But the area of the circle, whoſe radius is $\sqrt{2aa}$, was found to be $= \frac{caa}{r}$; then the ſurface of the ſolid, infinitely produced, is equal to the area of the circle, whoſe radius is equal to the diagonal of the ſquare of AE.

EXAMPLE XLVI.

Fig. 133.



141. Let CNEODAC be the *ungula* whose superficies is required. Imposing the same names as at § 129, it will be $QK = \frac{bx}{a} = MN$. But M*i*, the element of the curve, is $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and therefore it will be $\frac{bx}{a}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, equal to the infinitesimal *quadrilineum* M*im*N, the element of the superficies of half the *ungula*.

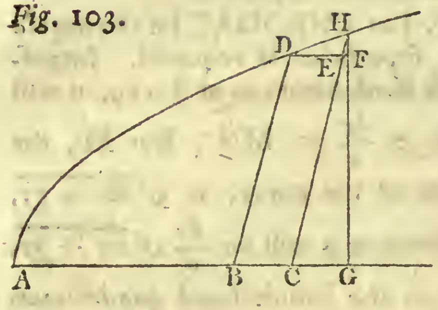
Let the curve DAC be a semicircle; in this case it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{x}}{\sqrt{aa - xx}}$, and therefore the formula is $\frac{bx\dot{x}}{\sqrt{aa - xx}}$. And by integration (according to § 31), it will be $-b\sqrt{aa - xx} + f$. But, making $x = 0$, it will be $f = ab$; therefore the complete integral will be found to be $ab - b\sqrt{aa - xx}$. And making $x = a$, in respect of the whole superficies of the half *ungula*, that superficies will be $= ab$.

Let the curve DAC be the parabola of the equation $yy = a - x$; it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{1}{2}\dot{x}\sqrt{\frac{4a - 4x + 1}{a - x}}$, and therefore the formula is $\frac{bx\dot{x}}{2a}\sqrt{\frac{4a - 4x + 1}{a - x}}$, the integral of which depends on the quadrature of the hyperbola; so that the superficies of the *ungula* will depend on the same quadrature.

EX-

EXAMPLE XLVII.

Fig. 103.



142. Let there be a parabolical conoid, generated by the rotation of the parabola ADH with the co-ordinates at an oblique angle, about the axis AC, whofe equation is $ax = yy$. In the formula for the fuperficies belonging to this cafe, that is, the formula $\frac{cny}{rm}\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x} + \frac{2e\dot{x}\dot{y}}{m}}$, let there be fubftituted the value of $\dot{x}$ given by $\dot{y}$, from the differential equation of the curve, and it will be transformed into this other, $\frac{2cny\dot{y}}{arm}\sqrt{yy + \frac{aey}{m} + \frac{1}{4}aa}$; the integral of which will be found to be partly algebräical, and partly logarithmical.

143. In purfuance of the method already explained, for quadratures, rectifications, &c. this would be the proper place to give alfo formulas for centres of gravity, of ofcillation and percuffion; but I rather choofe to omit them, as they neceffarily require fome knowledge of the principles of Staticks and Mechanicks, which I fhall not fuppofe my young readers to underftand at prefent.

SECT.

SECT. IV.

The Calculus of Logarithmic and Exponential Quantities.

144. EXPONENTIAL Quantities, (of which, as alſo of logarithmic quantities, we have treated elſewhere,) are thoſe which are raiſed to any indeterminate power. Such would be a^x, y^z, &c. the exponents of which, x, z, are indeterminate or variable quantities. And therefore the method of computation, which is converſant about ſuch quantities, is called *The Exponential Calculus.*

145. But exponential quantities are of ſeveral degrees. Thoſe are ſaid to be of the firſt degree, the exponents of which are the common indeterminates, as are the quantities a^x, y^z. Thoſe are of the ſecond degree, the exponents of which are the ſaid exponential quantities; ſuch would be a^{x^t}, y^{z^p}, where x is raiſed to the power t, and z to the power p. Thoſe are of the third degree, which have an exponential of the ſecond degree for their exponent. And ſo on.

146. Now here we ſhould call to mind what is already ſaid at § 11, that $\int \frac{a\dot{y}}{y} = ly$, in the logarithmic curve, the ſubtangent of which $= a$. Therefore the differential of ly will be $\frac{\dot{y}}{y}$ multiplied into the ſubtangent of the logarithmic, from which the logarithm is taken. Thus, the differential of $l\sqrt{aa - xx}$ will be $- \frac{x\dot{x}}{aa - xx}$ in the logarithmic, in which the ſubtangent $= 1$. And, in general, the differential of any logarithmic quantity whatever will be a formula, compounded of the differential of the quantity itſelf, multiplied into the ſubtangent, and divided by the ſame quantity.

147. This

147. This fuppofed, let it be required to difference the logarithmic quantity $l^m x$, where m is the exponent of the power of the logarithm. Make $l^m x = y^m$, then it will be $lx = y$, and $\frac{\dot{x}}{x} = \dot{y}$. But the differential of $l^m x$ will be $my^{m-1}\dot{y}$; and it is $y^{m-1} = l^{m-1}x$. So that, inftead of y and $\dot{y}$, fubftituting their values given by x, the differential of $l^m x = ml^{m-1}x \times \frac{a\dot{x}}{x}$, fuppofing the fubtangent of the logarithmic $= a$. Or otherwife, $= ml^{m-1}x \times \frac{\dot{x}}{x}$, fuppofing that fubtangent $= 1$.

148. If we were to difference $l^m x^n$, making $x^n = z$, it will be $l^m z$; and the differential of this will be $ml^{m-1}z \times \frac{\dot{z}}{z}$. But $\dot{z} = nx^{n-1}\dot{x}$, and by fubftitution, the differential of the propofed formula $l^m x^n$ will be $nml^{m-1}x^n \times \frac{\dot{x}}{x}$.

149. Let it be propofed to difference the formula llx. Make $lx = y$, and therefore $llx = ly$. But it will be $\frac{\dot{x}}{x} = \dot{y}$, in the logarithmic whofe fubtangent $= 1$ (which is always to be underftood, whenever thefe fubtangents are not particularly expreffed). But, becaufe $llx = ly$, the differential of llx will be $\frac{\dot{y}}{y}$. Therefore, inftead of y and $\dot{y}$, putting their values given by x, it will be $\frac{\dot{x}}{xlx}$ for the differential of the formula propofed.

But, more generally, let it be required to difference $l^m lx$. Put $lx = y$, and therefore $l^m lx = l^m y$, and $\frac{\dot{x}}{x} = \dot{y}$. But the differential of $l^m y$ is $ml^{m-1}y \times \frac{\dot{y}}{y}$; therefore, fubftituting the values of y and $\dot{y}$ given by x, the differential required will be $ml^{m-1}lx \times \frac{\dot{x}}{xlx}$.

Still more generally. Let it be required to difference $l^n l^m x$. Make $l^m x = y^m$, and therefore $lx = y$, and $\frac{\dot{x}}{x} = \dot{y}$. Then it will be $l^n l^m x = l^n y^m$.

 But

But the differential of $l^n y^m$ is $mnl^{n-1} y^m \times \frac{\dot{y}}{y}$. So that, making the substitutions, $mnl^{n-1} l^m x \times \frac{\dot{x}}{xlx}$ will be the differential required.

150. Now for the method of differencing exponential quantities. Let the quantity to be differenced be z^x. Make $z^x = t$, and consequently it will be $lz^x = lt$. But, by § 14, it is $lz^x = xlz$, and therefore it will be $xlz = lt$, and therefore, by differencing, $\dot{x}lz + \frac{x\dot{z}}{z} = \frac{\dot{t}}{t}$. But $t = z^x$, whence $\dot{x}lz + \frac{x\dot{z}}{z} = \frac{\dot{t}}{z^x}$, and finally, $\dot{t} = z^x \dot{x}lz + xz^{x-1}\dot{z}$, which is the differential required.

151. Let it be required to difference the exponential quantity of the second degree, z^{x^p}. Make $z^{x^p} = t$, and therefore it will be $x^p lz = lt$. And, by differencing, the differential of $x^p \times lz + x^p \times \frac{\dot{z}}{z}$ will be $= \frac{\dot{t}}{t}$. But, by the foregoing article, we know the differential of x^p to be $x^p \dot{p}lx + px^{p-1}\dot{x}$; and therefore it will be $\overline{x^p \dot{p}lx + px^{p-1}\dot{x}} \times lz + \frac{x^p \dot{z}}{z} = \frac{\dot{t}}{t}$. But $t = z^{x^p}$. Therefore it will be $\dot{t} = z^{x^p} x^p \dot{p}lxlz + z^{x^p} px^{p-1}\dot{x}lz + z^{x^p} z^{-1} x^p \dot{z}$ for the differential required.

In the same manner, we may proceed to exponential quantities of any other degrees.

152. Likewise, in the same manner, we may have the differentials of quantities, which are the products of exponential quantities; as, for example, of $x^p y^u$. For the differential of this will be the product of x^p into the differential y^u, together with the product of y^u into the differential of x^p. But it has been shown how to find the differentials of x^p and y^u. Therefore, &c.

153. From the order in which logarithmic differentials proceed, we may derive rules for the integration of logarithmic differential formulas. And, first, those canons which serve for the integration of common differential quantities, will also serve for logarithmical differentials which are like to them; because

these are divided also by the variable, and the integrals of these will be the same as the integrals of those, putting only in these, instead of the variable or it's power, the logarithm or power of the logarithm of the same variable; dividing the whole by the subtangent of the logarithmic.

Thus, because the integral of $mx^{m-1}\dot{x}$ is x^m, also the integral of $ml^{m-1}x \times \frac{\dot{x}}{x}$ will be $\frac{l^m x}{a}$.

In the same manner, because $\int x^{-1}\dot{x} = lx$; so likewise $\int l^{-1}x \times \frac{\dot{x}}{x}$, or $\int \frac{\dot{x}}{xlx}$ will be llx; supposing the subtangent $= 1$.

And, because $\int y\dot{y}\sqrt{aa + yy} = \frac{1}{3} \times \overline{aa + yy}|^{\frac{3}{2}}$; it will be also $\int ly\sqrt{aa + l^2y} \times \frac{\dot{y}}{y} = \frac{1}{3} \times \overline{aa + l^2y}|^{\frac{3}{2}}$.

Let $ml^{m-1}lx \times \frac{\dot{x}}{xlx}$ be given to be integrated. Make $lx = y$; then $\frac{\dot{x}}{x} = \dot{y}$. And making the substitution, it will be $ml^{m-1}y \times \frac{\dot{y}}{y}$. But we know the integral of $my^{m-1}\dot{y}$ to be y^m, and therefore the integral of $ml^{m-1}y \times \frac{\dot{y}}{y}$ will be $l^m y$. But $y = lx$, and therefore $ly = llx$, and $l^m y = l^m lx$. Therefore $\int ml^{m-1}lx \times \frac{\dot{x}}{xlx} = l^m lx$.

Let it be $nml^{n-1}x^m \times \frac{\dot{x}}{x}$. Make $x^m = y$, and therefore $\dot{x} = \frac{\dot{y}}{mx^{m-1}}$. And making the substitutions, it will be $nml^{n-1}y \times \frac{\dot{y}}{mx^{m-1} \times x}$, that is, $nl^{n-1}y \times \frac{\dot{y}}{y}$, the integral of which is $l^n y$. Then restoring the value of y, it will be $\int nml^{n-1}x^m \times \frac{\dot{x}}{x} = l^n x^m$.

Let it be $nml^{n-1}l^m x \times \frac{\dot{x}}{xlx}$. Make $lx = y$; then $\frac{\dot{x}}{x} = \dot{y}$, and $l^m x = y^m$. Making the substitution, it will be $nml^{n-1}y^m \times \frac{\dot{y}}{y}$. But the integral of this is $l^n y^m$. Therefore, restoring the value, it will be $\int nml^{n-1}l^m x \times \frac{\dot{x}}{x} = l^n l^m x$.

154. To

154. To this I ſhall add a general rule for the integration of the formula $y^m l^n y \times \dot{y}$, and ſay, in general, it will be $\int y^m l^n y \times \dot{y} = \frac{y^{m+1} l^n y}{m+1} - \frac{n y^{m+1} a l^{n-1} y}{\overline{m+1}|^2} + \frac{n \times \overline{n-1} \times y^{m+1} a^2 l^{n-2} y}{\overline{m+1}|^3} - \frac{n \times \overline{n-1} \times \overline{n-2} \times y^{m+1} a^3 l^{n-3} y}{\overline{m+1}|^4}$, &c. And thus the ſeries may be continued *in infinitum*, by obſerving the law of it's progreſſion, which is manifeſt of itſelf.

Hence, if the exponent n ſhall be a poſitive integer number, it is eaſy to obſerve, that the ſeries will break off of itſelf, and conſequently the integral of the propoſed formula will be given in a finite number of terms.

For example, make $n = 2$; then it will be $n - 2 = 0$, and therefore the co-efficient of the fourth term will be nothing, and of all that follow, becauſe every one is multiplied by $n - 2$. So, if $n = 3$, the ſeries will break off at the fifth term; and ſo of others.

Make $n = 2$, $m = 1$; then the formula to be integrated will be $yl^2 y \times \dot{y}$. Therefore the fourth term, and all the ſubſequent terms, will be nothing. Therefore the integral will be $\frac{yyl^2y}{2} - \frac{2yyaly}{4} + \frac{2yyaa}{8}$.

Now, if it were $m = -1$, the ſeries would be of no uſe, becauſe it would be $m + 1 = 0$, which makes every term infinite. But, in this caſe, there would be no need of a ſeries, becauſe we know already how to integrate ſuch formulas, by what has been ſaid before.

It remains to give the demonſtration of this rule. To do which, make $ly = z$, and therefore $\frac{a\dot{y}}{y} = \dot{z}$. Then making the ſubſtitution, it will be $y^m l^n y\dot{y} = y^m z^n \dot{y}$. But $y^m z^n \dot{y} = y^m z^n \dot{y} + \frac{n}{m+1} y^{m+1} z^{n-1} \dot{z} - \frac{n}{m+1} y^m z^{n-1} a\dot{y} - \frac{n \times \overline{n-1}}{\overline{m+1}|^2} y^{m+1} z^{n-2} a\dot{z} + \frac{n \times \overline{n-1}}{\overline{m+1}|^2} y^m z^{n-2} a^2 \dot{y}$, &c. And ſo on *in infinitum*; becauſe, in this manner, every term, except the firſt, will be deſtroyed by that immediately following, becauſe it is $\dot{z} = \frac{a\dot{y}}{y}$. Now, becauſe ſuch an infinite ſeries is integrable, by taking the terms two by two; for the integral of the firſt and ſecond term is $\frac{y^{m+1} z^n}{m+1}$, of the third and fourth is $- \frac{any^{m+1} z^{n-1}}{\overline{m+1}|^2}$, of the fifth and ſixth is $\frac{aan \times \overline{n-1} \times y^{m+1} z^{n-2}}{\overline{m+1}|^3}$; and ſo of the reſt: in this

integral,

integral, instead of z, restoring it's value ly, we shall find it to be at last $\int y^m l^n y\dot{y} = \frac{y^{m+1} l^n y}{m+1} - \frac{any^{m+1} l^{n-1} y}{\overline{m+1}|^2}$, &c. as before.

155. The artifice of finding the aforesaid series is this. We may conceive the integral of $y^m l^n y\dot{y}$ to be $\frac{y^{m+1} l^n y}{m+1}$, as it really would be, if $l^n y$ were not a variable quantity; but, supposing the subtangent $= a$, the differential of this integral is $y^m l^n y\dot{y} + \frac{ny^m al^{n-1} y\dot{y}}{m+1}$. This is found greater than the proposed formula by $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$, so that the integral assumed is greater than it ought to be, by the integral of $\frac{nyal^{n-1} y\dot{y}}{m+1}$, and therefore the integral of this ought to be subtracted from the supposed integral.

And here again I conceive that the integral of $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$ is $\frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$; whence the integral of the proposed formula will be $\frac{y^{m+1} l^n y}{m+1} - \frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$. But, by differencing $\frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$, we shall have $\frac{ny^m al^{n-1} y\dot{y}}{m+1} + \frac{n \times \overline{n-1} \times y^m a^2 l^{n-2} y\dot{y}}{\overline{m+1}|^2}$. Therefore the integral of $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$ is not $\frac{ny^m al^{n-1} y}{\overline{m+1}|^2}$, but is greater than it ought to be by the integral of $\frac{n \times \overline{n-1}}{\overline{m+1}|^2} \times y^m a^2 l^{n-2} y\dot{y}$. Therefore too much is subtracted, and this integral is to be added, which again I imagine to be $\frac{n \times \overline{n-1} \times y^{m+1} a^2 l^{n-2} y}{\overline{m+1}|^3}$. So that the integral of the proposed formula will be $\frac{1}{m+1} y^{m+1} l^n y - \frac{n}{\overline{m+1}|^2} y^{m+1} al^{n-1} y + \frac{n \times \overline{n-1}}{\overline{m+1}|^3} y^{m+1} a^2 l^{n-2} y$, &c. And thus proceeding in the same manner, the series may be continued *in infinitum*.

156. We

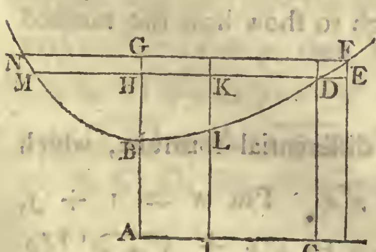
156. We may also have the integrals of logarithmic differential formulæ by the help of series, which shall not contain logarithmic quantities, but only common quantities; which series, therefore, will never break off, but are always infinite.

Let $xlx \times \dot{x}$ be proposed to be integrated. Make $x = z + a$; then, by substitution, it will be $\overline{z+a} \times l\overline{z+a} \times \dot{z}$. But, by § 70, it is $l\overline{z+a} = \frac{z}{a} - \frac{z^2}{2a^2} + \frac{z^3}{3a^3} - \frac{z^4}{4a^4}$, &c. Supposing the subtangent $= 1$. Then, by actually multiplying, we shall have $\overline{z+a} \times l\overline{z+a} \times \dot{z} = z\dot{z} + \frac{z^2\dot{z}}{a} - \frac{z^3\dot{z}}{2a^2} + \frac{z^4\dot{z}}{3a^3} - \frac{z^5\dot{z}}{4a^4}$, &c. $- \frac{z^2\dot{z}}{2a} + \frac{z^3\dot{z}}{3a^2} - \frac{z^4\dot{z}}{4a^3} + \frac{z^5\dot{z}}{5a^4}$, &c.; that is, $z\dot{z} + \frac{z^2\dot{z}}{2a} - \frac{z^3\dot{z}}{6a^2} + \frac{z^4\dot{z}}{12a^3} - \frac{z^5\dot{z}}{20a^4}$, &c.; and, by integration, it will be $\frac{z^2}{2} + \frac{z^3}{6a} - \frac{z^4}{24a^2} + \frac{z^5}{60a^3} - \frac{z^6}{120a^4}$, &c. $= \int \overline{z+a} \times l\overline{z+a} \times \dot{z}$.

So, if the formula were $x^m lx \times \dot{x}$, that is, $\overline{z+a}|^m \times l\overline{z+a} \times \dot{z}$, we must multiply the series expressing the logarithm into the power $\overline{z+a}|^m$. And moreover, if the logarithm also were raised to a power, as $x^m l^n x \times \dot{x}$, that is, $\overline{z+a}|^m \times l^n \overline{z+a} \times \dot{z}$, there would be occasion, besides, to raise the infinite series, expressing the logarithm, to the power n, and to do the rest, as above.

157. Differential formulas, or equations affected by logarithmic quantities, very often admit of integrations which are geometrical, and which depend on quadratures of curvilinear spaces, which may easily be described, supposing the logarithmic curve to be given. Here are some examples selected out of the more simple ones.

Fig. 138.



Let the equation be $\dot{y}ly = \dot{x}$, and in the logarithmic described let CD $= y$; and taking the subtangent for unity, we shall have AC $=$ HD $= ly$. Whence the infinitesimal rectangle DG, of which the base is GH $=$ FE $= \dot{y}$, will be $= \dot{y}ly$. But this rectangle is the element of the increasing area BDH, and therefore the sum or integral $\int \dot{y}ly$ is equal to the said area. In fact, the area itself is equal to the rectangle AD, subtracting the logarithmic space ABDC. But this

this ſpace, as is known, is meaſured by the rectangle AB × CD $= y$. Therefore the area BDH $= \int \dot{y}ly = yly - y$; as may be found by the way of analyſis.

I ſhall conſider another formula, $\dot{y}l^2y = \dot{x}$. The firſt member is no other than the ſolid generated by the fluxion HG, multiplied into the ſquare of the ordinate GF; which ſolid is analogous to the element of the conoid, generated by the area BDH, revolving about the axis BG. Therefore the integral $\int \dot{y}l^2y = yl^2y - 2yly + 2y$ is to the ſaid conoid in a given ratio.

More generally, let us have $\dot{y}l^my$. Raiſing the ordinate HD to the power m, (the index m being either an affirmative or negative number, either whole or broken, it will ſuffice that the ordinate HM may be made equal to the dignity HD^m, and that through the point M, and infinite others to be determined in the ſame manner, the curve BMN may paſs; in order that the area BMH $= \int$MH $\times \dot{y}$ may be equal to, or analogous to, the integral $\int \dot{y}l^my$.

The difficulty will not be greater, even though the logarithms of logarithms ſhould alſo be found in our expreſſions. Let there be propoſed $\dot{y}lly = \dot{x}$. Whereas AC is the logarithm of CD; if, in the logiſtic, the new ordinate IL, equal to the abſciſs AC, ſhould be adapted; AI will be the logarithm of IL, and conſequently the logarithm of the logarithm of CD. Let the right line IL be prolonged, ſo as to cut HD, parallel and equal to AC, in the point K; through which and infinite others, determined in the ſame manner, let a new curve paſs, drawn relatively to the logiſtic. I ſay, that the quadrature of the ſpace belonging to this curve will give us the integral of the formula $\dot{y}lly = \dot{x}$.

After another manner. I take the fluxion of the quantity $ylly$, that is, $\dot{y}lly + \frac{\dot{y}}{ly}$, and adding the term $\frac{\dot{y}}{ly}$ to both ſides of our expreſſion, we ſhall have $\dot{y}lly + \frac{\dot{y}}{ly} = \dot{x} + \frac{\dot{y}}{ly}$; and by integration, $ylly = x + \int \frac{\dot{y}}{ly}$. Therefore, to the abſciſs AH annexing the correſponding ordinate in the reciprocal ratio of HD $= ly$, a curve will be produced, the quadrature of which will expreſs the integral $\int \frac{\dot{y}}{ly}$. And this will be enough to ſhow how the method proceeds.

158. I ſhall now go on to the integration of differential formulæ, which contain exponential quantities; and let us integrate $x^x\dot{x}$. Put $x = 1 + y$, (taking unity for any conſtant quantity,) then it will be $x^x\dot{x} = \overline{1 + y}^{1+y}\dot{y}$.

 This

This ſuppoſed, make alſo $\overline{1+y}|^{1+y} = 1 + u$, and then it will be $\overline{1+y} \times l\overline{1+y} = l\overline{1+u}$. Now let the two logarithms be converted into ſeries, by § 70; and making an actual multiplication of the firſt ſeries by $1 + y$, we ſhall have $y + \frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{1}{12}y^4 - \frac{1}{20}y^5$, &c. $= u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5$, &c. Then make a fictitious equation, ſuppoſing it to be $u = y + Ay^2 + By^3 + Cy^4 + Dy^5$, &c. (where A, B, C, D, &c. are quantities to be determined by the proceſs.)

Therefore $uu = y^2 + 2Ay^3 + A^2y^4 + 2ABy^5$, &c.
$\qquad\qquad\qquad\qquad + 2By^4 + 2Cy^5$

$u^3 = y^3 + 3Ay^4 + 3A^2y^5$, &c.
$\qquad\quad + 3By^5 \qquad u^4 = y^4 + 4Ay^5$, &c. $\quad u^5 = y^5$, &c.

Whence $u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5$, &c. $=$

$$\left.\begin{array}{lllll} y & + Ay^2 & + By^3 & + Cy^4 & + Dy^5, \&c. \\ & - \frac{1}{2}y^2 & - Ay^3 & - \frac{1}{2}A^2y^4 & - ABy^5 \\ & & & - By^4 & - Cy^5 \\ & & + \frac{1}{3}y^3 & + Ay^4 & + A^2y^5 \\ & & & & + By^5 \\ & & & - \frac{1}{4}y^4 & - Ay^5 \\ & & & & + \frac{1}{5}y^5 \end{array}\right\} = y + \frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{1}{12}y^4 - \frac{1}{20}y^5, \&c.$$

Now, by comparing homologous terms, we ſhall find the values of the aſſumed quantities to be $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{3}$, $D = \frac{1}{12}$, &c.; ſo that, putting theſe values in the places of the capitals, we ſhall have $1 + u = \overline{1+y}|^{1+y} = 1 + y + y^2 + \frac{1}{2}y^3 + \frac{1}{3}y^4 + \frac{1}{12}y^5$, &c. Whence $\overline{1+y}|^{1+y}\dot{y} = \dot{y} + y\dot{y} + y^2\dot{y} + \frac{1}{2}y^3\dot{y} + \frac{1}{3}y^4\dot{y}$, &c.; and laſtly, by integration, $\int \overline{1+y}|^{1+y} \times \dot{y} = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{8}y^4 + \frac{1}{15}y^5 + \frac{1}{72}y^6$, &c.

159. We may find the integral of the formula $x^x\dot{x}$ thus, in another manner. Make $x^x = 1 + y$, then $xlx = l\overline{1+y}$. Reduce $l\overline{1+y}$ to a ſeries, and it will be $l\overline{1+y} = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. This ſuppoſed, make $y = l\overline{1+y} + Al^2\overline{1+y} + Bl^3\overline{1+y} + Cl^4\overline{1+y} + Dl^5\overline{1+y}$, &c. (where A, B, C, D, &c. are quantities to be determined,) and it will be

$y^2 =$

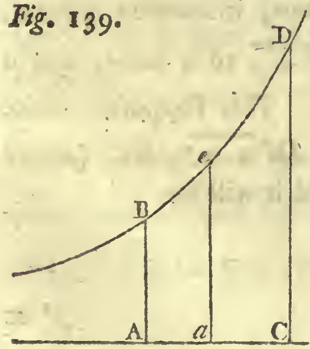
$$y^2 = l^2\overline{1+y} + 2Al^3\overline{1+y} + A^2l^4\overline{1+y} + 2ABl^5\overline{1+y}, \&c.$$
$$+ 2Bl^4\overline{1+y} + 2Cl^5\overline{1+y}$$
$$y^3 = l^3\overline{1+y} + 3Al^4\overline{1+y} + 3A^2l^5\overline{1+y}$$
$$+ 3Bl^5\overline{1+y}$$
$$y^4 = l^4\overline{1+y} + 4Al^5\overline{1+y}$$
$$y^5 = l^5\overline{1+y}$$

Therefore $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. $= l\overline{1+y} =$

$$l\overline{1+y} + Al^2\overline{1+y} + Bl^3\overline{1+y} + Cl^4\overline{1+y} + Dl^5\overline{1+y}, \&c.$$
$$- \tfrac{1}{2}l^2\overline{1+y} - Al^3\overline{1+y} - \tfrac{1}{2}A^2l^4\overline{1+y} - ABl^5\overline{1+y}$$
$$- Bl^4\overline{1+y} - Cl^5\overline{1+y}$$
$$+ \tfrac{1}{3}l^3\overline{1+y} + Al^4\overline{1+y} + A^2l^5\overline{1+y}$$
$$+ Bl^5\overline{1+y}$$
$$- \tfrac{1}{4}l^4\overline{1+y} - Al^5\overline{1+y}$$
$$+ \tfrac{1}{5}l^5\overline{1+y}$$

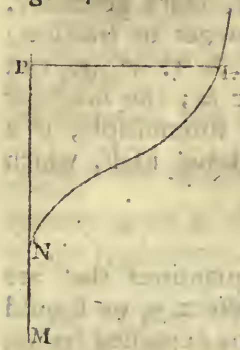
Now, by the compariſon of homologous terms, we ſhall find $A = \frac{1}{2}$, $B = \frac{1}{6}$, $C = \frac{1}{24}$, $D = \frac{1}{120}$, &c.; whence $1 + y = 1 + l\overline{1+y} + \frac{1}{2}l^2\overline{1+y} + \frac{1}{6}l^3\overline{1+y} + \frac{1}{24}l^4\overline{1+y} + \frac{1}{120}l^5\overline{1+y}$, &c. But $l\overline{1+y} = xlx$, and $1 + y = x^x$; therefore, making the ſubſtitutions, and multiplying by $\dot{x}$, it will be $x^x\dot{x} = \dot{x} + x\dot{x}lx + \frac{1}{2}x^2\dot{x}l^2x + \frac{1}{6}x^3\dot{x}l^3x + \frac{1}{24}x^4\dot{x}l^4x + \frac{1}{120}x^5\dot{x}l^5x$, &c.; and integrating, by the known rules above delivered, it will be $\int x^x\dot{x} = x + \frac{1}{2}x^2lx - \frac{1}{4}x^2 + \frac{1}{6}x^3l^2x - \frac{1}{9}x^3lx + \frac{1}{27}x^3 + \frac{1}{24}x^4l^3x - \frac{1}{32}x^4l^2x + \frac{1}{64}x^4lx - \frac{1}{256}x^4$, &c.

Fig. 139.



160. Now, to add ſomething concerning the conſtruction of curves expreſſed by logarithmic and exponential equations. Firſt, let it be required to deſcribe the curve of the equation $x = \frac{l^{\frac{3}{2}}y}{a^{\frac{1}{2}}}$. Let BD (Fig. 139.) be the logarithmic, in which we are to take the logarithms of the propoſed equation, whoſe ſubtangent (for example) is $= a =$ AB. This ſuppoſed,

Fig. 140.



ſuppoſed, taking $y = a =$ AB, the logarithm of y will be $= 0$, and therefore $x = 0$. Making, then, MN $= y = a$ (Fig. 140), N will be a point in the curve. Taking y leſs than AB, ly will be a negative quantity, and therefore $l^{\frac{3}{2}}y$ will be an imaginary quantity, becauſe the even number 2 is the index of the root of a negative quantity; whence x will be imaginary whenever y is leſs than a. Taking y greater than AB, ſuppoſe $=$ CD, it will be AC $= ly$. But, by the given equation, it is $a^{\frac{1}{2}} . l^{\frac{1}{2}}y :: ly . x$, or $a . \sqrt{aly} :: ly . x$; and therefore, making MP $=$ CD, we muſt take PH equal to the fourth proportional of AB, a mean proportional between AB and AC, and the ſaid AC; which fourth proportional will be $= x$, and H will be a point in the curve. After this manner we may find as many points as we pleaſe, and ſo deſcribe the curve, which will go on *ad infinitum*, as is eaſy to perceive.

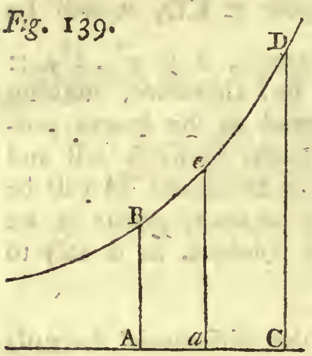
To have the ſubtangent of the given curve, I take the differential formula $\frac{y\dot{x}}{\dot{y}}$ of the ſubtangent, find the difference of the equation of the curve, which is $\dot{x} = \frac{3}{2}l^{\frac{1}{2}}y \times \frac{a^{\frac{1}{2}}\dot{y}}{y}$. Making the ſubſtitution in the place of $\dot{x}$, we ſhall have the ſubtangent $= \frac{3}{2}l^{\frac{1}{2}}y \times a^{\frac{1}{2}} = \frac{3ax}{2ly} = \frac{3}{2}a^{\frac{2}{3}}x^{\frac{1}{3}}$.

Alſo, our curve will have a contrary flexure; to find which I take the ſecond fluxion of the given equation, ſuppoſing $\dot{x}$ conſtant, and I find $\frac{\frac{3}{2}a^{\frac{1}{2}}\dot{y}\,l^{\frac{1}{2}}y \times \ddot{y} + \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}\,l^{-\frac{1}{2}}y - \frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}\,l^{\frac{1}{2}}y}{yy} = 0$; and therefore $\ddot{y} = \frac{\frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}\,l^{\frac{1}{2}}y - \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}\,l^{-\frac{1}{2}}y}{\frac{3}{2}a^{\frac{1}{2}}y\,l^{\frac{1}{2}}y}$. But, by the method of contrary flexures, it ought to be $\ddot{y} = 0$. Therefore it will be $\frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}\,l^{\frac{1}{2}}y - \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}\,l^{-\frac{1}{2}}y = 0$; that is, $l^{\frac{1}{2}}y - \frac{1}{2}al^{-\frac{1}{2}}y = 0$, or $ly = \frac{1}{2}a$. Therefore the point of contrary flexure will be there, where it is $ly = \frac{1}{2}a$.

If the curve propoſed to be deſcribed were $xlx = y$, reſolving the equation into an analogy, it will be $1 : lx :: x : y$, which may be conſtructed in a like manner.

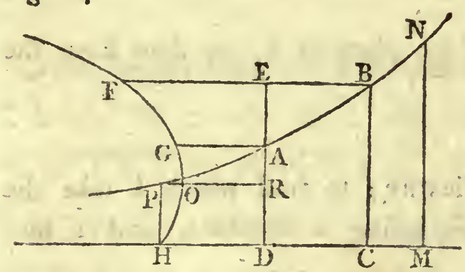
If the curve were $x^2lx = y$, or $x^3lx = y$, or $x^{\frac{1}{2}}lx = y$, or, more generally, $x^n lx = y$, ſuppoſing n to denote any power of x, whether integer or fraction; this equation being likewiſe reſolved into an analogy, $1 . lx :: x^n . y$, and taking in the logarithmic any line CD $= x$, whence AC $= lx$; the multiple of AC, according to the number n, if it be an integer, the ſubmultiple, if a fraction, will give the correſponding ordinate in the logarithmic itſelf, which ſhall be x^n, by the property of the logarithmic.

Fig. 139.



If the curve ſhould contain quantities that are logarithms of logarithms, ſuch as $x^n llx = y$, we ſhould eaſily have in the logarithmic the line expreſſed by llx, by taking any line CD $= x$ (Fig. 139.), whence it is AC $= lx$; and then putting AC for an ordinate in (ae). For Aa would be the logarithm of (ae), that is, llx; as has before been taken notice of at § 157.

Fig. 141.



161. Let it be required to conſtruct the exponential curve of the equation $x^x = y$. Now, taking the logarithms, it will be $xlx = ly$; and deſcribing the logarithmic curve PAB, (Fig. 141.) with the ſubtangent AD $= 1$, and taking any line CB $=$ DE $= x$, it will be DC $= lx$. Then, becauſe the equation may be reſolved into the analogy, $1 . x :: lx . ly$; the fourth proportional to AD, BC, and DC, which ſuppoſe is DM, will be ly; ſo that MN $= y$. Therefore, if it be made EF $=$ MN, it will be DE $= x$, EF $= y$, and F will be a point in the curve to be deſcribed.

The curve will cut the aſymptote HM in H, making DH $=$ DA. For, putting $x = 0$, it will be $ly = 0$, that is, $y =$ DA. Making, therefore, AG $=$ DH, G will be a point in the curve.

From the point H drawing HP, an ordinate to the logarithmic, and drawing POR parallel to HD, then OR will be the leaſt ordinate, y, to the curve. For, taking the difference of the equation, it will be $\dot{x} + \dot{x}lx = \frac{\dot{y}}{y}$, that is, $y\dot{x} + y\dot{x}lx = \dot{y}$. But, by the method *de maximis et minimis*, it muſt be $\dot{y} = 0$; therefore $y\dot{x} + y\dot{x}lx = 0$, and therefore $- lx = 1 =$ HD $=$ DA.

 Becauſe

Becaufe $\frac{y\dot{x}}{\dot{y}}$ is the general formula for the fubtangent, and having $\dot{x} = \frac{\dot{y}}{y \times 1 + lx}$ from the given equation of the curve, by fubftituting this value in the formula, the fubtangent belonging to any point of the curve will be $= \frac{1}{1 + lx}$; and for the point G, in refpect of which it is $x = $ AD, and confequently $lx = 0$, the fubtangent will be $= 1 =$ AD, which is the fubtangent of the logarithmic.

As to the area, take the general formula $y\dot{x}$; but $y = x^x$, in the equation of the curve. Therefore, fubftituting the value of y in the formula, it will become $x^x\dot{x}$, and therefore $\int x^x\dot{x}$ is the indefinite area HOFEADH; which, being integrated according to § 159, will be $= x + \frac{x^2lx}{2} - \frac{1}{4}x^2 + \frac{x^3l^2x}{6} - \frac{x^3lx}{9} + \frac{x^3}{27} + \frac{x^4l^3x}{24} - \frac{x^4l^2x}{32} + \frac{x^4lx}{64}$, &c. And taking $x =$ AD $= 1$, it will be $lx = 0$, and therefore the area HOGAD $= 1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256}$, &c.; that is, $= 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5}$, &c.

162. Let $x^y = a$ be the equation of the curve. Then $ylx = la$, and therefore it may be conftructed by means of the logarithmic. By taking the fluxion of the equation, we fhall have $\frac{y\dot{x}}{x} + \dot{y}lx = 0$, making the fubtangent of the logarithmic $= 1$. And therefore it will be $\dot{x} = -\frac{x\dot{y}lx}{y}$; and therefore the fubtangent $= -xlx$.

163. Let it be $x^x = a^y$; therefore $xlx = yla$, which may be conftructed as ufual. Taking the fluxion, it will be $\dot{x} + \dot{x}lx = \dot{y}la$; and the fubtangent $= \frac{xlx}{1 + lx}$.

Here, becaufe $y = \frac{xlx}{la}$, it will be $y\dot{x}$, or the element of the area, $= \frac{x\dot{x}lx}{la}$; and integrating, by § 154, it is $\frac{2xxlx - xx}{4la} =$ area.

164. Other queftions may be ftill propofed, relating to exponential equations; as, for example, in exponential equations compofed of only known quantities, but with variable exponents, to find thofe exponents. So, let it be $c^x = ab^{x-1}$; the value of the unknown exponent, x, is required, a, b, c, being given.

 Becaufe

Becaufe $\frac{c^x}{a} = b^{x-1}$, it will be $xlc - la = \overline{x-1}\,lb$, and therefore $xlc - xlb = la - lb$. Whence $x = \frac{la - lb}{lc - lb}$.

165. Another queftion fhall be this. To find fuch a number x, as that it may be $x^x = a$, and alfo $x^{x+p} = b$. Now, by the firft condition, we fhall have $xlx = la$, and therefore $x = \frac{la}{lx}$, or $lx = \frac{la}{x}$. By the fecond condition, we fhall have $\overline{x+p}\,lx = lb$. Therefore it will be $x = \frac{lb - plx}{lx}$, or $lx = \frac{lb}{x+p}$. Then it will be $\frac{la}{x} = \frac{lb}{x+p}$, that is, $xla + pla = xlb$, or $x = \frac{pla}{lb - la}$; or elfe $\frac{la}{lx} = \frac{lb - plx}{lx}$, that is, $lx = \frac{lb - la}{p}$. This fuppofed, I fhall propofe to myfelf to refolve the following Problem.

166. A veffel being given of a known capacity, full of any liquor, fuppofe wine, out of which is drawn a draught of a given quantity, and then the veffel is filled up with water. Of this mixture of wine and water another draught is drawn equal to the former, and the veffel is again filled up with water. Again, of this mixed liquor another fuch draught is drawn out; and the fame operation is continually repeated in the fame manner. It is demanded how many fuch draughts may be drawn out, or how many times the operation muft be repeated, that a given quantity of wine may be left in the veffel.

Let the capacity of the veffel be $= a$, and the quantity of each draught $= b$. Therefore, at the firft draught, will be drawn fuch a quantity of wine as will be expreffed by b; and as much water will be poured in again; whence, after the firft draught, will be left in the veffel the quantity of wine $= a - b$.

At the fecond draught will be drawn out the quantity b of the mixture; fo that, to have the quantity of pure wine contained in it, we muft make this analogy; as the capacity of the veffel (a) is to the quantity of the draught (b), fo is the wine which is in the veffel ($a - b$) to a fourth proportional $\frac{ab - bb}{a}$, which will be the quantity of pure wine which is drawn out at the fecond draught. Then there remains in the veffel the quantity of pure wine, $\frac{aa - 2ab + bb}{a}$, that is, $\frac{\overline{a-b}|^2}{a}$.

Therefore, for the third draught, making alfo this analogy; as the capacity of the veffel (a) is to the quantity of a draught (b), fo is the wine in the veffel, veſſel, $\frac{\overline{a-b}|^2}{a}$, to a fourth, $\frac{\overline{a-b}|^2}{a} \times \frac{b}{a}$. This will be the quantity of pure wine, which was drawn out at the third draught; ſo that there will remain in the veſſel the quantity of pure wine, $\frac{\overline{a-b}|^2}{a} - \frac{b}{a} \times \frac{\overline{a-b}|^2}{a}$, or $\frac{\overline{a-b}|^3}{aa}$. And thus, after the fourth draught, there will be left in the veſſel the quantity of pure wine, $\frac{\overline{a-b}|^4}{a^3}$; and, in general, after a number of draughts denoted by n, there will be left in the veſſel the quantity of pure wine $= \frac{\overline{a-b}|^n}{a^{n-1}}$. Therefore, if we would know how many draughts muſt be taken, ſo that there ſhould remain in the veſſel a given quantity of pure wine, ſuppoſe, for example, $\frac{a}{m}$ part of the whole; we muſt make the equation $\frac{\overline{a-b}|^n}{a^{n-1}} = \frac{a}{m}$; which, becauſe n is an unknown number, will be an exponential quantity. Wherefore, the equation being reduced to the logarithms, it will be $l\frac{\overline{a-b}|^n}{a^{n-1}} = l\frac{a}{m}$, that is, $nl\overline{a-b} = la - lm + \overline{n-1}\,la$, or $nl\overline{a-b} = -lm + nla$, and therefore $n = \frac{lm}{la - l\overline{a-b}}$; ſo that it will be eaſy from hence to find the number n, by the help of a Table of Logarithms.

END OF THE THIRD BOOK.

which, $\frac{\overline{a-b}^2}{a}$, to a fourth, $\frac{\overline{a-b}^2}{a} \times \frac{b}{a}$. This will be the quantity of pure wine, which was drawn out at the third draught; so that there will remain in the vessel the quantity of pure wine, $\frac{\overline{a-b}^3}{a^2} - \frac{b}{a} \times \frac{\overline{a-b}^3}{a^2}$, or $\frac{\overline{a-b}^4}{a^3}$. And thus, after the fourth draught, there will be left in the vessel the quantity of pure wine, $\frac{\overline{a-b}^4}{a^3}$; and, in general, after a number of draughts denoted by n, there will be left in the vessel the quantity of pure wine $= \frac{\overline{a-b}^n}{a^{n-1}}$. Therefore, if we would know how many draughts must be taken, so that there should remain in the vessel a given quantity of pure wine, suppose, for example, $\frac{a}{m}$ part of the whole; we must make the equation $\frac{\overline{a-b}^n}{a^{n-1}} = \frac{a}{m}$; which, because n is an unknown number, will be an exponential equation. Wherefore, the equation being reduced to the logarithms, it will be $l\frac{\overline{a-b}^n}{a^{n-1}} = l\frac{a}{m}$, that is, $nl\overline{a-b} = la - lm + nla - 1la$, or $nl\overline{a-b} = -lm + nla$, and therefore $n = \frac{lm}{la - l\overline{a-b}}$; so that it will be easy from hence to find the number n, by the help of a Table of Logarithms.

END OF THE THIRD BOOK.

ANALYTICAL INSTITUTIONS.

BOOK IV.

THE INVERSE METHOD OF TANGENTS.

1. AS, when any curve is given, the manner of finding it's tangent, ſubtangent, perpendicular, or any line of that kind, is called the Direct Method of Tangents; ſo, when the tangent, ſubtangent, perpendicular, or any ſuch line is given,—or when the rectification or area is given, to find the curve to which ſuch properties belong, is called the Inverſe Method of Tangents.

In the ſecond and third Books are found the general differential expreſſions of the tangent, or other lines analogous to it; as alſo, of rectifications and areas. Therefore, by comparing the given property of the tangent, rectification, &c. with the reſpective expreſſion or general differential formula, there will ariſe a differential equation of the firſt degree, or of a ſuperior degree, which, being integrated, either algebraically, or reduced to known quadratures, will give the curve required, to which belongs the given property. For example, let the curve be required of which the ſubtangent is double to the abſciſs. Calling the abſciſs x, and the ordinate y, the formula of the ſubtangent is $\frac{y\dot{x}}{\dot{y}}$, and therefore the equation will be $\frac{y\dot{x}}{\dot{y}} = 2x$. Again, let us ſeek the curve, the area of

which

which muſt be equal to two third parts of the rectangle of the co-ordinates; the element of the area is $y\dot{x}$, and therefore it ought to be $\int y\dot{x} = \frac{2}{3}xy$, or $y\dot{x} = \frac{2x\dot{y} + 2y\dot{x}}{3}$. If we would find the curve whoſe property it is, that any arch taken from the vertex ſhall be equal to the reſpective ſubnormal; the expreſſion of the arch is $\int\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and that of the ſubnormal is $\frac{y\dot{y}}{\dot{x}}$; ſo that we ſhall have $\int\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{y\dot{y}}{\dot{x}}$, and therefore $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y}}{\dot{x}\dot{x}}$, (taking $\dot{x}$ for conſtant,) which is a differential equation of the ſecond degree.

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2. The equations which ariſe by proceeding after this manner, will always have (as is eaſy to perceive,) the indeterminates and differentials intermixed and blended with each other, ſo that at preſent they cannot be managed, in order to proceed to their integration, ſo as to obtain the curves required; and much more if they contain differentials of the ſecond, third, and higher degrees. For, in the third Section aforegoing, the differential formulæ have always been ſuppoſed to be compounded of one indeterminate only, with it's difference or fluxion. Therefore other expedients are neceſſary, to try to reduce ſuch equations to integration, or quadratures, which is called the Conſtruction of Differential Equations, of the firſt, ſecond, &c. Degrees: And, as to the conſtruction of thoſe of the firſt degree, we may proceed two ways; one is, to paſs immediately to integrations or quadratures, without any previous ſeparation of the indeterminates and their differentials; the other is, firſt to ſeparate the indeterminates, and ſo to make the equations fit for integration or quadrature.

I ſhall proceed to ſhow ſeveral particular methods for both the ways, by which we may attain our purpoſe in moſt equations. But very often we ſhall meet with others, which will be found ſo ſtubborn, as not to ſubmit to any methods hitherto diſcovered, or which have not the univerſality that is neceſſary.

SECT.

SECT. I.

Of the Construction of Differential Equations of the First Degree, without any previous Separation of the Indeterminates.

3. The most simple formulæ which have the two variables mixed together, are these two, $x\dot{y} + y\dot{x}$, and $\frac{y\dot{x} - x\dot{y}}{yy}$. The integral of the first is xy, and of the second $\frac{x}{y}$, as is manifest. To these, therefore, we should endeavour to reduce the more compounded, and that by the usual helps of the common Analyticks, by adding, subtracting, multiplying, dividing, &c. by any quantities that will make for the purpose, which will be different according to different cases. We shall here see something of the practice.

Let it be $y\dot{x} = x\dot{x} - x\dot{y}$. By transposing the last term, it will be $y\dot{x} + x\dot{y} = x\dot{x}$, and therefore, by integration, $xy = \frac{1}{2}xx \pm bb$. Let the equation be $x^4\dot{y}\dot{y} + 2x^3y\dot{x}\dot{y} = a^4\dot{x}\dot{x} - xxyy\dot{x}\dot{x}$; then transposing the last term, and dividing by xx, it is $x^2\dot{y}^2 + 2xy\dot{x}\dot{y} + y^2\dot{x}^2 = \frac{a^4\dot{x}^2}{x^2}$, and extracting the square-root, $x\dot{y} + y\dot{x} = \frac{a^2\dot{x}}{x}$; and by integration, $xy = alx \pm b$, in the logarithmic with subtangent $= a$. Let the equation be $y\dot{x} = y^3\dot{y} + y^2\dot{y} + x\dot{y}$, that is, $y\dot{x} - x\dot{y} = y^3\dot{y} + y^2\dot{y}$. The first member would be integrable if it were divided by yy; therefore I divide the equation, and it will be $\frac{y\dot{x} - x\dot{y}}{yy} = y\dot{y} + \dot{y}$, and, by integration, it is $\frac{x}{y} = \frac{1}{2}yy + y \pm b$.

4. Let the equation be $y^r\dot{y} = my\dot{x} + x\dot{y}$. If there was not here the coefficient m, the matter would be easy, because the integral of the second member would

would be xy. The operation would not succeed any better, by transposing the member $x\dot{y}$ to the other side, or by writing $y^r\dot{y} - x\dot{y} = my\dot{x}$; yet I observe, that the differential of $mxy^{\frac{1}{m}}$ is $m\dot{x}y^{\frac{1}{m}} + xy^{\frac{1}{m}-1}\dot{y}$, different from that proposed, $my\dot{x} + x\dot{y}$, only in this, that it is multiplied by $y^{\frac{1}{m}-1}$. Therefore, to make the quantity $my\dot{x} + x\dot{y}$ become integrable, it will be sufficient to multiply it by $y^{\frac{1}{m}-1}$, and, to preserve the equality, to multiply also the corresponding member of the equation $y^r\dot{y}$; therefore it will be $y^{r+\frac{1}{m}-1}\dot{y} = my^{\frac{1}{m}}\dot{x} + xy^{\frac{1}{m}-1}\dot{y}$, and, by integration, $\int y^{r+\frac{1}{m}-1}\dot{y} = mxy^{\frac{1}{m}} \pm b$.

Let the equation be the same, but with a different co-efficient in each of the two last terms; that is, let it be $y^r\dot{y} = my\dot{x} + nx\dot{y}$. The second member is not integrable; yet I observe, that the differential of $mxy^{\frac{n}{m}}$ is $my^{\frac{n}{m}}\dot{x} + nxy^{\frac{n}{m}-1}\dot{y}$. Therefore the *homogeneum comparationis* would be integrable, if it were multiplied by $y^{\frac{n}{m}-1}$. Therefore I multiply the whole equation, and it will become $y^{r+\frac{n}{m}-1}\dot{y} = my^{\frac{n}{m}}\dot{x} + nxy^{\frac{n}{m}-1}\dot{y}$, and the integral will be $\int y^{r+\frac{n}{m}-1}\dot{y} = mxy^{\frac{n}{m}} \pm b$.

5. The differential of $x^n y$ is $x^n\dot{y} + nyx^{n-1}\dot{x}$. This supposed, let the equation be $y^r\dot{y} = x^n\dot{y} + yx^{n-1}\dot{x}$. If the last term had n for it's co-efficient, the integral of the second member of the equation would be x^ny. I observe, therefore, that the differential of x^ny^n is $nx^ny^{n-1}\dot{y} + ny^nx^{n-1}\dot{x}$; therefore, multiplying the equation by ny^{n-1}, there will arise $ny^{r+n-1}\dot{y} = nx^ny^{n-1}\dot{y} + ny^nx^{n-1}\dot{x}$, which is found to be integrable, it's integral being $\int ny^{r+n-1}\dot{y} = x^ny^n \pm b$.

But if the last term, instead of the co-efficient n, had any other, or, in general, if both the last terms were affected by different co-efficients; or if the equation

equation were $y^r\dot{y} = cx^n\dot{y} + eyx^{n-1}\dot{x}$; I obferve, that the differential of $\frac{e}{n}x^n y^{\frac{cn}{e}}$ is $cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}} x^{n-1}\dot{x}$. Therefore multiply the equation by $y^{\frac{cn}{e}-1}$, that it may be $y^{r+\frac{cn}{e}-1}\dot{y} = cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}} x^{n-1}\dot{x}$, which is integrable, and it's integral is $\int y^{r+\frac{cn}{e}-1}\dot{y} = \frac{e}{n}x^n y^{\frac{cn}{e}} \pm b$.

Here make $r = 1$, $c = 3$, $n = 1$, $e = 1$, that is, the equation $y\dot{y} = 3x\dot{y} + y\dot{x}$; the integral will be $\frac{1}{4}y^4 = xy^3$. Make $c = 2$, $e = 3$, $n = 1$, $r = 1$, that is, the equation will be $y\dot{y} = 2x\dot{y} + 3y\dot{x}$, and the integral will be $\frac{y^{1+\frac{2}{3}}}{1+\frac{2}{3}} = 3xy^{\frac{2}{3}}$, or $\frac{3}{5}y^{\frac{5}{3}} = 3xy^{\frac{2}{3}}$. Make $c = 2$, $e = 2$, $n = 3$, $r = 3$, or the equation $y^3\dot{y} = 2x^3\dot{y} + 2yx^2\dot{x}$; and the integral will be $\frac{1}{6}y^6 = \frac{2}{3}x^3y^3$.

If the equation were expreffed thus, $y^{1-\frac{cn}{e}}x^r\dot{x} = cx^n\dot{y} + eyx^{n-1}\dot{x}$, it is eafy to fee, that it would be integrable. For, multiplying by $y^{\frac{cn}{e}-1}$, it would be $x^r\dot{x} = cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}} x^{n-1}\dot{y}$. But the integral of the fecond member is known to be $\frac{e}{n}x^n y^{\frac{cn}{e}}$; &c.

6. Now let the equation be $y^r\dot{y} = \frac{2x\dot{y} - y\dot{x}}{xx}$. If it were not for the co-efficient 2, the integral of the fecond member would be $\frac{y}{x}$. But it will be to no purpofe to tranfpofe to the other fide the term $\dot{y}x$, and to write it $y^r\dot{y} - \frac{x\dot{y}}{xx} = \frac{x\dot{y} - y\dot{x}}{xx}$. But I obferve that the differential of $\frac{yy}{x}$ is $\frac{2xy\dot{y} - yy\dot{x}}{xx}$; fo that if the propofed equation be multiplied by y, that it may be $y^{r+1}\dot{y} = \frac{2xy\dot{y} - yy\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+1}\dot{y} = \frac{yy}{x} \pm b$. But, more generally, let there be any co-efficient n, and therefore the equation is $y^r\dot{y} = \frac{nx\dot{y} - y\dot{x}}{xx}$. I obferve that the differential of $\frac{y^n}{x}$ is $\frac{nxy^{n-1}\dot{y} - y^n\dot{x}}{xx}$; therefore, if it be

be multiplied by y^{n-1}, fo that the equation may be $y^{r+n-1}\dot{y} = \frac{nxy^{n-1}\dot{y} - y^n\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+n-1}\dot{y} = \frac{y^n}{x} \pm b$.

Thus, let both the laft terms have different co-efficients, and let the equation be $y^r\dot{y} = \frac{nx\dot{y} - my\dot{x}}{xx}$. I obferve, that the differential of $\frac{my^{\frac{n}{m}}}{x}$ is $\frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{xx}$; therefore, if the equation be multiplied by $y^{\frac{n}{m}-1}$, fo that it may be $y^{r+\frac{n}{m}-1}\dot{y} = \frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+\frac{n}{m}-1}\dot{y} = \frac{my^{\frac{n}{m}}}{x} \pm b$.

If the equation were $y^{1-\frac{n}{m}}x^r\dot{x} = \frac{nx\dot{y} - my\dot{x}}{xx}$, it would alfo be integrable. For, multiplying it by $y^{\frac{n}{m}-1}$, it will be $x^r\dot{x} = \frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{xx}$. But the integral of the fecond member is known to be $\frac{my^{\frac{n}{m}}}{x}$; therefore, &c.

Let the denominator xx be wanting in the aforefaid equations, and let the equation be $y^r\dot{y} = nx\dot{y} - y\dot{x}$. To integrate the fecond part of the equation, there would be occafion to multiply it by y^{n-1}, and to divide it by xx. But as this muft be done alfo in refpect to the firft part, it would be $\frac{y^{r+n-1}\dot{y}}{xx}$, which cannot by any means be integrated. Therefore let the figns of the equation be changed, and it will be $-y^r\dot{y} = y\dot{x} - nx\dot{y}$. I obferve that the differential of $\frac{x}{y^n}$ is $\frac{y^n\dot{x} - nxy^{n-1}\dot{y}}{y^{2n}}$. Therefore, if the equation be multiplied by y^{n-1}, and then divided by y^{2n}, fo that it may be $\frac{-y^{r+n-1}\dot{y}}{y^{2n}} =$

$\frac{y^n\dot{x} - nxy^{n-1}\dot{y}}{y^{2n}}$, it will be integrable, and the integral is $\int \frac{-y^{r+n-1}\dot{y}}{y^{2n}} = \frac{x}{y^n} \pm b$.

Let the equation have both the laſt terms with a co-efficient, and let it be $y^r\dot{y} = nx\dot{y} - myx\dot{}$. Let the ſigns be changed, and it will be $-y^r\dot{y} = my\dot{x} - nx\dot{y}$. I obſerve that the differential of $\frac{x}{my^{\frac{n}{m}}}$ is $\frac{my^{\frac{n}{m}}\dot{x} - nxy^{\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}}$. Therefore, if the equation be multiplied by $y^{\frac{n}{m}-1}$, and divided by $mmy^{\frac{2n}{m}}$, ſo that it may be $\frac{-y^{r+\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}} = \frac{my^{\frac{n}{m}}\dot{x} - nxy^{\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}}$, it will be integrable, and the integral is $\int \frac{-y^{r+\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}} = \frac{x}{my^{\frac{n}{m}}} \pm b$.

7. Let the equation be $y^r\dot{y} = x^n\dot{y} - nyx^{n-1}\dot{x}$. Change the ſigns, and it will be $-y^r\dot{y} = nyx^{n-1}\dot{x} - x^n\dot{y}$. I obſerve that the differential of $\frac{x^n}{y}$ is $\frac{nyx^{n-1}\dot{x} - x^n\dot{y}}{yy}$. Therefore, dividing the equation by yy, it will become $-y^{r-2}\dot{y} = \frac{nyx^{n-1}\dot{x} - x^n\dot{y}}{yy}$, which will be integrable, and it's integral is $\int -y^{r-2}\dot{y} = \frac{x^n}{y} \pm b$.

But if the co-efficient n had been wanting, and the equation were $y^r\dot{y} = x^n\dot{y} - yx^{n-1}\dot{x}$; change the ſigns, and it will be $-y^r\dot{y} = yx^{n-1}\dot{x} - x^n\dot{y}$. It may be obſerved, that the differential of $\frac{x^n}{y^n}$ is $\frac{ny^nx^{n-1}\dot{x} - nx^ny^{n-1}\dot{y}}{y^{2n}}$. Therefore, multiplying the equation by ny^{n-1}, and dividing it by y^{2n}, it will become

$\frac{-ny^{r+n-1}\dot{y}}{y^{2n}} = \frac{ny^n x^{n-1}\dot{x} - nx^n y^{n-1}\dot{y}}{y^{2n}}$, which will be integrable, and it's integral is $\int -\frac{ny^{r+n-1}\dot{y}}{y^{2n}} = \frac{x^n}{y^n} \pm b$.

But if, inftead of the co-efficient n, there fhould be another of a different nature; or if both the laft terms were affected by a different co-efficient, as if the equation were $y^r\dot{y} = cx^n\dot{y} - eyx^{n-1}\dot{x}$; change the figns, and it will be $-y^r\dot{y} = eyx^{n-1}\dot{x} - cx^n\dot{y}$. I obferve that the differential of $\frac{x^n}{ey^{\frac{nc}{e}}}$ is $\frac{ney^{\frac{nc}{e}}x^{n-1}\dot{x} - ncx^n y^{\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}}$. Therefore, multiplying the equation by $ny^{\frac{nc}{e}-1}$, and dividing it by $eey^{\frac{2nc}{e}}$, it will be $-\frac{ny^{r+\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}} = \frac{ney^{\frac{nc}{e}}x^{n-1}\dot{x} - ncx^n y^{\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}}$, which will be integrable, and it's integral will be

$$\int -\frac{ny^{r+\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}} = \frac{x^n}{ey^{\frac{nc}{e}}} \pm b.$$

But if the equation were thus expreffed, $y^{1-\frac{ne}{e}}x^r\dot{x} = cx^n\dot{y} - eyx^{n-1}\dot{x}$; without changing the figns, I obferve, that the differential of $\frac{ey^{\frac{nc}{e}}}{x^n}$ is $\frac{ncx^n y^{\frac{nc}{e}-1}\dot{y} - ney^{\frac{nc}{e}}x^{n-1}\dot{x}}{x^{2n}}$; therefore, multiplying the equation by $ny^{\frac{nc}{e}-1}$, and dividing it by x^{2n}, we fhall have $\frac{nx^r\dot{x}}{x^{2n}} = \frac{ncx^n y^{\frac{nc}{e}-1}\dot{y} - ney^{\frac{nc}{e}}x^{n-1}\dot{x}}{x^{2n}}$, which will be integrable; for it's integral is $\int\frac{nx^r\dot{x}}{x^{2n}} = \frac{ey^{\frac{nc}{e}}}{x^n} \pm b$.

8. I have

8. I have already ſaid, in the foregoing Book, § 17, that as often as the numerator of a fraction, compoſed of only one variable and conſtants, is the exact differential of the denominator, or proportional to that differential; the integral of ſuch a formula is the logarithm of the denominator, or in a given proportion to that logarithm. This alſo obtains when the formula contains two variables, intermixed with each other and with their differentials. Therefore the integral of $\frac{\dot{x} + \dot{y}}{x + y} = \dot{z}$, ($\dot{z}$, after any manner, being given by x or by y,) will be $l\,\overline{x + y} = z \pm b$. The integral of $\frac{\dot{x} + \dot{y}}{2x + 2y} = \dot{z}$ will be $l\sqrt{x + y} = z + b$. The integral of $\frac{4x\dot{x} - 4y\dot{y}}{xx - yy} = \dot{z}$ will be $2l\,\overline{xx - yy} = z \pm b$. The integral of $\frac{y\dot{x} + x\dot{y} - 2y\dot{y}}{2xy - 2yy} = \dot{z}$ will be $l\sqrt{xy - yy} = z \pm b$. And, in general, the integral of $\frac{my^n x^{m-1}\dot{x} + nx^m y^{n-1}\dot{y} - \overline{m+n}\,y^{m+n-1}\dot{y}}{r \times \overline{x^m y^n - y^{m+n}}} = \dot{z}$ will be $l\sqrt[r]{x^m y^n - y^{m+n}} = z \pm b$. And ſo of any other equation whatever, which ſhall have the condition aſſigned.

9. Wherefore many equations, though they have not the neceſſary condition, yet may eaſily be made to acquire it, with the aſſiſtance of ſome calculation. Thus, the equation $\frac{x\dot{y} + y\dot{x}}{x} = -\dot{y}$, has not the required condition in the firſt member; but it will have it if it be divided by y. Then it will be $\frac{x\dot{y} + y\dot{x}}{xy} = -\frac{\dot{y}}{y}$; and therefore, by integration, $l\,xy = ly^{-1} \pm lb$.

Let the equation be $ax\dot{y} + 2ay\dot{x} = xy\dot{y}$. I divide it by axy, and it will be $\frac{x\dot{y} + 2y\dot{x}}{xy} = \frac{\dot{y}}{a}$. This would be integrable if it were not for the co-efficient 2 in the ſecond term of the firſt member; therefore I ſubtract the quantity $\frac{y\dot{x}}{xy}$ from each member, and it will be $\frac{x\dot{y} + y\dot{x}}{xy} = \frac{\dot{y}}{a} - \frac{y\dot{x}}{xy}$, that is, $\frac{x\dot{y} + y\dot{x}}{xy} = \frac{\dot{y}}{a} - \frac{\dot{x}}{x}$; and therefore, by integration, $lxy = \frac{y}{a} - lx \pm lb$.

Let the equation be $yx\dot{x} = \overline{x^2y\dot{y} + y^3\dot{y}} \times \sqrt{y} - y^2\dot{y}$. I divide it by y, and it will be $x\dot{x} = \overline{x^2\dot{y} + y^2\dot{y}} \times \sqrt{y} - y\dot{y}$, that is, $x\dot{x} + y\dot{y} = \overline{x^2\dot{y} + y^2\dot{y}} \times \sqrt{y}$. And dividing again by $xx + yy$, it will be $\frac{x\dot{x} + y\dot{y}}{xx + yy} = \dot{y}\sqrt{y}$. And therefore, by integration, $l\sqrt{xx + yy} = \frac{2}{3}y^{\frac{3}{2}} \pm b$.

10. From § 31, 32, of the ſaid Book III, we may gather, that any formula compoſed of one variable only, if it be the product of any complicate quantities raiſed to a poſitive or negative power, integer or fraction, into the exact differential, or into a proportional of the differential of the terms of the quantity; it will always be integrable. And the integral will be the ſame quantity, the exponent of which will be that as at firſt, but increaſed by unity, and multiplied into the ſame exponent ſo increaſed, but taken inverſely: Or, which is the ſame thing, divided by it; or elſe this integral ſhall be proportional to it. Nevertheleſs the rule obtains when the differential formulæ are likewiſe compoſed of two variables and their differentials promiſcuouſly, provided they have the condition required.

Thus, the integral of $\overline{\dot{x} + \dot{y}} \times \sqrt{x + y} = \dot{z}$, (where $\dot{z}$ is any how given by x or by y,) will be $\frac{2}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$. The integral of $\overline{\frac{1}{2}\dot{x} + \frac{1}{2}\dot{y}} \times \sqrt{x + y} = \dot{z}$ will be $\frac{1}{2} \times \frac{2}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$, that is, $\frac{1}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$. The integral of $\frac{p^3\dot{q} + 3qp^2\dot{p} + 3pq^2\dot{q} + q^3\dot{p}}{2\sqrt{p^3q + q^3p}} = \dot{z}$, will be $\sqrt{p^3q + q^3p} = z \pm b$.

The integral of $\overline{x\dot{y} + y\dot{x} + 2y\dot{y}} \times b \times \overline{xy + yy}|^{\frac{n}{m}} = \dot{z}$ will be $\frac{mb}{m + n} \times \overline{xy + yy}|^{\frac{m+n}{m}} = z \pm b$. The integral of $\frac{x\dot{y} + y\dot{x} + 2y\dot{y}}{b \times \overline{xx + yy}|^{\frac{n}{m}}} = \dot{z}$, will be $\frac{m \times \overline{xy + yy}|^{\frac{m-n}{m}}}{\overline{m - n} \times b} = z \pm b$; And ſo of infinite others of the like kind.

But ſome equations of this kind will firſt have need of ſome preparation. Let the equation be $xx\dot{x} + xy\dot{y} + yy\dot{x} = \dot{z}$, (where $\dot{z}$ is any how given by x,) I multiply it by x, and it will be $x^3\dot{x} + x^2y\dot{y} + xy^2\dot{x} = x\dot{z}$, or $x\dot{x} \times \overline{xx + yy} + xx \times y\dot{y} = x\dot{z}$, which has not yet the neceſſary condition. But it would have it if $y\dot{y}$ were alſo multiplied into yy; therefore I add to each member the term $y^3\dot{y}$, and it will be $x\dot{x} \times \overline{xx + yy} + y\dot{y} \times xx + y^3\dot{y} = x\dot{z} + y^3\dot{y}$, that is, $\overline{x\dot{x} + y\dot{y}} \times \overline{xx + yy} = x\dot{z} + y^3\dot{y}$, which is capable of integration, and it's integral is $\frac{1}{4} \times \overline{x^2 + y^2}|^2 = \frac{1}{4}y^4 \pm b + \int x\dot{z}$.

But it is not always eaſy to perceive, what quantities are to be added or ſubtracted, or what other alterations muſt be made in the equations, that they may be brought under the foregoing method; eſpecially when the equations are ſomething compounded. In this way, to arrive at a ſolution is rather the work of of chance than of art. In ſuch caſes, therefore, we muſt have recourſe to the Methods of Separation of the Indeterminates, which ſhall now follow.

SECT. II.

Of the Conſtruction of Differential Equations, by a Separation of the Indeterminates.

11. The Separation of the Indeterminates in ſome equations, although but few, may be performed by the firſt operations only of the common Algebra. Such would be the equation $x^2\dot{x}^2 + xy\dot{x}\dot{y} = a^2\dot{y}^2$, in which I obſerve, that the firſt member is a formula of an affected quadratick, which would be made a complete ſquare if the term $\frac{y^2\dot{y}^2}{4}$ were added to it. Therefore I add this quantity on each ſide, and the equation will be $xx\dot{x}\dot{x} + xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y} = aa\dot{y}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y}$. And extracting the root, it will be $x\dot{x} + \frac{1}{2}y\dot{y} = \dot{y}\sqrt{\frac{1}{4}yy + aa}$, in which the variables are ſeparated, and therefore, by integration, $\frac{1}{2}xx + \frac{1}{4}yy = \int\dot{y}\sqrt{aa + \frac{1}{4}yy} \pm b$. The integral of the ſecond member depends on the quadrature of the hyperbola.

12. But moſt frequently it will be convenient to make uſe of ſubſtitutions. Let the equation be $aa\dot{x} = xx\dot{y} + 2xy\dot{y} + yy\dot{y}$. Make $x + y = z$, aſſuming z as a new indeterminate; and therefore $\dot{x} + \dot{y} = \dot{z}$, and $xx + 2xy + yy = zz$. Then making the ſubſtitutions, it will be $aa\dot{z} - aa\dot{y} = zz\dot{y}$, that is, $\frac{aa\dot{z}}{aa + zz} = \dot{y}$, an equation in which the variables are ſeparate. The integration of the firſt member depends on the rectification of the circle.

Let the equation be $\overline{x\dot{y} + y\dot{x}} \times \sqrt{a^4 - xxyy} = \frac{x\dot{x} + y\dot{y}}{\sqrt[4]{xx + yy} \times \sqrt{xx + yy}}$. Here I obſerve in the firſt member, that the integral of $x\dot{y} + y\dot{x}$ is xy, and that the ſquare of this integral is found exactly in the quantity $\sqrt{a^4 - xxyy}$; therefore, if I put $xy = z$, in the firſt member the variables will be ſeparated, and it will be $\dot{z}\sqrt{a^4 - zz}$. I obſerve further, that, in the ſecond member, the integral of $x\dot{x} + y\dot{y}$ is $\frac{xx + yy}{2}$, and that the quantities in the denominator are like to this integral. Therefore, by the ſubſtitution $xx + yy = 2p$, the indeterminates of the ſecond member will alſo be ſeparated, and the equation will be $\dot{z}\sqrt{a^4 - zz} = \frac{\dot{p}}{\sqrt[4]{2p} \times \sqrt{2p}}$.

Let the equation be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$, (where $\dot{z}$ is any how given by x or y;) the integral of $x\dot{y} - y\dot{x}$ will be had, if we divide by xx, and it will be $\frac{y}{x}$. Let us suppose, then, $\frac{y}{x} = \frac{p}{a}$, and therefore $\frac{x\dot{y} - y\dot{x}}{xx} = \frac{\dot{p}}{a}$, and $\frac{2x\dot{y} - 2y\dot{x}}{xx} = \frac{2\dot{p}}{a}$, and $2x\dot{y} - 2y\dot{x} = \frac{2xx\dot{p}}{a}$. Making, therefore, the substitutions, it will be $\frac{2xx\dot{p}}{a \times \overline{xx - 2xy + yy}} = \dot{z}$, and dividing the numerator and denominator of the first member by xx, it will be $\frac{2\dot{p}}{a \times \overline{1 - \frac{2y}{x} + \frac{yy}{xx}}} = \dot{z}$. But it was put $\frac{y}{x} = \frac{p}{a}$, and $\frac{yy}{xx} = \frac{pp}{aa}$; therefore it will be $\frac{2a\dot{p}}{aa - 2ap + pp} = \dot{z}$. And, because the integral of this equation is algebräical, I will go on to the integration. Make, therefore, $a - p = q$, and it will be $-\frac{2a\dot{q}}{qq} = \dot{z}$, and by integration, $\frac{2a}{q} \pm b = z$. But $q = a - p$, and $p = \frac{ay}{x}$; therefore it is $q = \frac{ax - ay}{x}$. Now, restoring this value, it will be $\frac{2x}{x-y} \pm b = z$, which is the curve belonging to the differential equation proposed. If, instead of making $a - p = q$, I had made $p - a = q$, another integral would have been found, but differing from this only in the signs.

13. The above equation gives me an occasion of making an useful observation; which is, that sometimes curves do not only change their nature by taking their integrals, either simply or with the addition of constants, which has been already observed from the first original of infinitesimal quantities; but sometimes also present us with such formulæ, as admit of integrations which are really different, and supply us with curves of various kinds, even without the addition of any constant quantity; which is a matter deserving consideration.

By means of the supposition $\frac{y}{x} = \frac{p}{a}$, the equation $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$ is presently integrated, and the integration is found to be $\frac{2x}{x-y} = z$, omitting the constant. Now I make the supposition of $\frac{x}{y} = \frac{p}{a}$, and attempt the integration. It will be, therefore, $\frac{y\dot{x} - x\dot{y}}{yy} = \frac{\dot{p}}{a}$, and thence $2x\dot{y} - 2y\dot{x} = -\frac{2yy\dot{p}}{a}$. And, by substitution, the equation will be $\frac{-2\dot{p}}{a \times \overline{\frac{xx}{yy} - \frac{2x}{y} + 1}} = \dot{z}$. But $\frac{x}{y} = \frac{p}{a}$; therefore $\frac{-2a\dot{p}}{pp - 2ap + aa} = \dot{z}$. And making $p - a = q$, it will

will be $-\frac{2a\dot{q}}{qq} = \dot{x}$; and, by integration, $\frac{2a}{q} = z$. Now, reſtoring the values, it is $\frac{2y}{x-y} = z$, the integral of the propoſed differential equation, which is different from the firſt.

Another integral of the propoſed formula, different from the two firſt, is $\frac{x+y}{x-y} = z$. For, by differencing, it is $\frac{x\dot{x} - y\dot{x} + x\dot{y} - y\dot{y} - x\dot{x} - y\dot{x} + x\dot{y} + y\dot{y}}{\overline{x-y}|^2} = \dot{z}$, and ſtriking out the terms that deſtroy one another, it is $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$, which is the equation at firſt propoſed.

Make $\dot{z} = \dot{y}$, and the propoſed equation is $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{y}$. If I make uſe of the ſecond integral found above, there ariſes the equation $\frac{2y}{x-y} = y$, and therefore $2 + y = x$, which is a *locus* to a triangle. Then, if I make uſe of the firſt, and of the third integral, by putting $\frac{2x}{x-y} = y$, or $\frac{x+y}{x-y} = y$, the curve will be of the ſecond degree.

In general, let it be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = y^m\dot{y}$. The firſt and the third integration being performed, the curve thence ariſing will aſcend to a degree denoted by $m + 2$, if m be a poſitive number. But, making uſe of the ſecond, the curve will ſtop one degree ſhort.

14. But, however, the method of ſubſtitutions is nevertheleſs univerſal, the greateſt difficulty of which is, that it is often very hard to know what ſubſtitutions ought to be made, that we may not work by chance, and beſtow much labour unſucceſsfully. However, we ſhall proceed with the greateſt ſecurity in all ſuch equations, in which the ſum of the exponents of the variable quantities is the ſame in every term, and the ſeparation of the indeterminates will always ſucceed. It matters not that theſe equations are affected by radicals, or by fractions, or by ſeries, and that the co-efficients and ſigns are of any kind. The ſubſtitution to be made in all theſe equations will be, by putting one of the variables equal to the product of the other into a new variable, ſo that, if the equation be given by x and y, we muſt make $x = \frac{yz}{a}$, or elſe $y = \frac{xz}{a}$, (where by the denominator a is underſtood any conſtant quantity at pleaſure,) and therefore $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; and, making the ſubſtitutions, we ſhall arrive at another equation, which will always be diviſible by as high a power of the indeterminate x, as was the ſum of the exponents of x and y in every term of the propoſed equation.

tion. Wherefore, making the divifion, the letter x will not exceed the firft power, and will always be multiplied by $\dot{z}$; whence the equation will be fo reduced, that on one fide there will be $\frac{\dot{x}}{x}$, and on the other fide $\dot{z}$, with only the functions of z; and thus the variables will be feparated. For, calling A all thofe terms which are multiplied into $\dot{y}$, and B thofe which are multiplied into $\dot{x}$, the equation will be $A\dot{y} = B\dot{x}$, and A and B will be given promifcuoufly by x and y. Now, becaufe the dimenfions of the letter y, together with the dimenfions of the letter x, in every term make the fame number; if, inftead of y, we put $\frac{xz}{a}$, it will follow from thence, that in every term of the quantities A, B, the letter x will have the fame dimenfion which, at firft, x and y had together. Whence, if this dimenfion be called n, the equation will be divifible by x^n, there only remaining z, a, $\dot{y}$, $\dot{x}$. Let it be fuppofed, that after the fubftitution of $\frac{xz}{a}$, and after the divifion by x^n, that which remains in the quantity A may be called C, and that which remains in the quantity B may be called D; the equation will be $C\dot{y} = D\dot{x}$, and C and D will be given by z and by conftants. But $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; therefore the equation will be $\frac{Cx\dot{z} + Cz\dot{x}}{a} = D\dot{x}$, that is, $Da\dot{x} - Cz\dot{x} = Cx\dot{z}$, and therefore $\frac{\dot{x}}{x} = \frac{C\dot{z}}{Da - Cz}$. And thus the indeterminates, with their differentials, will be feparated, and the equation will be conftructible, at leaft by quadratures.

It is indifferent whether you put $y = \frac{zx}{a}$, or $x = \frac{yz}{a}$; for, in either of the two ways, the indeterminates will always be feparated. But fometimes one fubftitution will give a more fimple equation, and of fewer terms, than the other, and the conftruction will be more eafy and elegant. Wherefore it will not be amifs to try them both, and, at laft, to make choice of that which fucceeds beft.

EXAMPLE I.

Let the equation be $xx\dot{y} = yy\dot{x} + xy\dot{x}$. Make $y = \frac{xz}{a}$, and therefore $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$. Making the fubftitutions, it will be $\frac{x^3\dot{z} + zx^2\dot{x}}{a} = \frac{xxzz\dot{x}}{aa} + \frac{zxx\dot{x}}{a}$. And reducing to a common denominator, and dividing by xx, it will be $ax\dot{z} + az\dot{x} = zz\dot{x} + az\dot{x}$; that is, $ax\dot{z} = zz\dot{x}$, or $\frac{\dot{x}}{ax} = \frac{\dot{z}}{zz}$.

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EXAMPLE II.

Let the equation be $xx\dot{y} = yy\dot{x} + xx\dot{x}$. Putting $y = \frac{xz}{a}$, it will be $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$. And, making the ſubſtitutions, it will be $\frac{x^3\dot{z} + zx^2\dot{x}}{a} = \frac{z^2x^2\dot{x}}{aa} + x^2\dot{x}$. And, reducing to a common denominator, and dividing by xx, it will be $ax\dot{z} + az\dot{x} = zz\dot{x} + aa\dot{x}$, that is, $zz\dot{x} - az\dot{x} + aa\dot{x} = ax\dot{z}$, and therefore $\frac{\dot{x}}{x} = \frac{a\dot{z}}{zz - az + aa}$. Now, making another ſubſtitution, $x = \frac{yp}{a}$, it will be $\dot{x} = \frac{y\dot{p} + p\dot{y}}{a}$, and therefore $\frac{ppyy\dot{y}}{aa} = \frac{y^3\dot{p} + pyy\dot{y}}{a} + \frac{y^3pp\dot{p} + p^3yy\dot{y}}{a^3}$; and, dividing by yy, it is $app\dot{y} = aay\dot{p} + aap\dot{y} + ypp\dot{p} + p^3\dot{y}$, that is, $app\dot{y} - aap\dot{y} - p^3\dot{y} = aay\dot{p} + ypp\dot{p}$; and therefore $\frac{\dot{y}}{y} = \frac{aa\dot{p} + pp\dot{p}}{app - aap - p^3}$.

EXAMPLE III.

Let the equation be $\dot{y}\sqrt{xx + yy} = y\dot{x}$. Make $y = \frac{xz}{a}$, and $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; and, making the ſubſtitutions, it will be $\frac{x\dot{z} + z\dot{x}}{a} \times \frac{\sqrt{xxzz + aaxx}}{a} = \frac{zx\dot{x}}{a}$, that is, $\overline{xx\dot{z} + zx\dot{x}} \times \sqrt{aa + zz} = azx\dot{x}$, and, dividing by x, it will be $x\dot{z}\sqrt{aa + zz} + z\dot{x}\sqrt{aa + zz} = az\dot{x}$, or $x\dot{z}\sqrt{aa + zz} = az\dot{x} - z\dot{x}\sqrt{aa + zz}$. Therefore $\frac{\dot{z}\sqrt{aa + zz}}{az - z\sqrt{aa + zz}} = \frac{\dot{x}}{x}$. If I had made $x = \frac{yp}{a}$, I ſhould have had this equation, $\frac{\dot{y}}{y} = \frac{\dot{p}}{\sqrt{aa + pp} - p}$.

15. But ſometimes the differentials themſelves, $\dot{x}$ and $\dot{y}$, aſcend to higher dimenſions, the condition mentioned before being, however, in the equations. In which caſes, the ſubſtitution of $\frac{xz}{a}$, inſtead of y, being made as before, not meddling with $\dot{y}$ at preſent, will make every term of the equation diviſible by the

the ſame power of x, and there will remain in the equation only z, $\dot{x}$, and $\dot{y}$, with the conſtants given or aſſumed, but not x. Now, becauſe, inſtead of $\dot{y}$, we muſt put $\frac{z\dot{x} + x\dot{z}}{a}$, by which the letter x will again be introduced; make $\frac{x\dot{z}}{a} = \dot{t}$, and, inſtead of $\dot{y}$, write $\frac{z\dot{x} + a\dot{t}}{a}$, and the equation will have only z, $\dot{t}$, $\dot{x}$, with conſtant quantities given or aſſumed, but no longer x. Now, if we make $a \,.\, u :: \dot{x} \,.\, \dot{t}$, and if, inſtead of $\dot{t}$, we put every where $\frac{u\dot{x}}{a}$, we ſhall have an equation free from differential quantities, in which will be only u, z, and conſtants, for an algebräical curve. By means of this curve, we may find the real values of u. Let there be, therefore, A, B, C, &c. ſo that it may be $u = A$, $u = B$, $u = C$, &c. and A, B, &c. will be given only by z, and by conſtants, and it will be $\dot{x} = \frac{a\dot{t}}{A}$, $\dot{x} = \frac{a\dot{t}}{B}$, &c.; and therefore $\dot{t} = \frac{x\dot{z}}{a}$ will be $\dot{x} = \frac{x\dot{z}}{A}$, $\dot{x} = \frac{x\dot{z}}{B}$, &c.; whence, laſtly, $\frac{\dot{x}}{x} = \frac{\dot{z}}{A}$, $\frac{\dot{x}}{x} = \frac{\dot{z}}{B}$, &c.; and the logarithms of x will be directly proportional to the ſpaces comprehended by the curves, of which, the abſciſſes being z, the ordinates will be reciprocally proportional to the values of the quantity u before found. And the curves ſatisfying the purpoſe will be ſo many, as are the real values (different from each other) of the letter u; ſtill obſerving, that the adding of a conſtant quantity in the integration of the equations $\frac{\dot{x}}{x} = \frac{\dot{z}}{A}$, $\frac{\dot{x}}{x} = \frac{\dot{z}}{B}$, &c. may again diverſify the curves that ſatisfy the demand, and will often double their number. Then lx will be equal to the area of that curve, which has z for it's abſciſs, and $\frac{1}{A}$, $\frac{1}{B}$, &c. for it's ordinate; that is, it will be equal to the integral of $\frac{\dot{z}}{A}$, $\frac{\dot{z}}{B}$, &c. Wherefore, taking z at pleaſure, the logarithm of x will be given, and conſequently the correſponding ordinate x in the logarithmic will be given alſo. Then, x being given, by means of the equation $y = \frac{xz}{a}$ will y be given alſo, that is, both the co-ordinates of the differential equation propoſed, or of the curve required. Then, in reference to the different values which will be given to z, ſo will be the different points alſo, which will be found in the ſame curve required.

I ſhall apply the rule to an example. Let the equation be $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} = xx\dot{x}\dot{x}$. Make, therefore, $y = \frac{xz}{a}$, and, putting this value in the equation, inſtead of y, we ſhall have $ax^2\dot{y}^2 + x^2z\dot{x}\dot{y} = ax^2\dot{x}^2$, and dividing by xx, it will be $a\dot{y}^2 + z\dot{x}\dot{y} = a\dot{x}^2$. Here we ſee, that x and it's functions entirely diſappear, there

there remaining only z, $\dot{x}$, $\dot{y}$, with their functions. But, becaufe, by fubftituting, inftead of $\dot{y}$, it's value $\frac{z\dot{x} + x\dot{z}}{a}$, we fhall again introduce x into the equation; make $\frac{x\dot{z}}{a} = \dot{t}$, and therefore $\dot{y} = \frac{z\dot{x} + a\dot{t}}{a}$, and the equation will be $\frac{zz\dot{x}\dot{x} + 2az\dot{x}\dot{t} + aa\dot{t}\dot{t}}{a} + \frac{zz\dot{x}\dot{x} + az\dot{x}\dot{t}}{a} = a\dot{x}\dot{x}$, that is, $2zz\dot{x}\dot{x} + 3az\dot{x}\dot{t} + aa\dot{t}\dot{t} = aa\dot{x}\dot{x}$; in which only enter z, $\dot{x}$, $\dot{t}$, with their functions. Again, fuppofing $\dot{t} = \frac{u\dot{x}}{a}$, and making the fubftitution, we fhall arrive at an expreffion which is purely algebräical, $2zz + 3zu + uu = aa$, fo that we fhall have the value of u given algebräically by z and conftant quantities. But $\dot{t} = \frac{u\dot{x}}{a} = \frac{x\dot{z}}{a}$, whence $\frac{\dot{x}}{x} = \frac{\dot{z}}{u}$, in which equation, u being given by z, the variables will be feparated. Therefore the curve being defcribed, of which the abfciffes are z, and the ordinates reciprocally proportional to the values of u; we fhall have x, and thence y, by making the fubftitution of $\frac{xz}{a} = y$.

16. Now, from this and other examples, it will fucceed alfo, without making ufe of this method, that they may eafily be reduced by the method of § 14. And, indeed, if to each of the members of the aforefaid equation, $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} = xx\dot{x}\dot{x}$, there be added the fquare $\frac{1}{4}yy\dot{x}\dot{x}$, it will be $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{x}\dot{x} = xx\dot{x}\dot{x} + \frac{1}{4}yy\dot{x}\dot{x}$, and extracting the root, $x\dot{y} + \frac{1}{2}y\dot{x} = \dot{x}\sqrt{xx + \frac{1}{4}yy}$; where now it is reduced to the aforefaid general method of § 14. Or elfe, tranfpofing the term $xy\dot{x}\dot{y}$, and adding the fquare $\frac{1}{4}yy\dot{y}\dot{y}$, it will be $xx\dot{y}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y} = xx\dot{x}\dot{x} - xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y}$; and, extracting the root, it is $\dot{y}\sqrt{xx + \frac{1}{4}yy} = x\dot{x} - \frac{1}{2}y\dot{y}$; now reduced to the fame method.

17. Equations which contain differentials mixed together, and raifed to any power, may not only be conftructed in the cafe confidered at § 15, which fuppofes the fum of the exponents of the variables to be equal in every term; but, in general, in what manner foever thofe equations are, provided one of the two indeterminates, x or y, be abfent. This is done by making $\dot{x} = \frac{z\dot{y}}{a}$, if x be wanting, or $\dot{y} = \frac{z\dot{x}}{a}$, if y be wanting; z being a new indeterminate, and a any conftant quantity. For, by fuch a fubftitution in the propofed equation, of $\frac{z\dot{y}}{a}$ inftead of $\dot{x}$, it is plain that another will arife, which will be divifible by the power of $\dot{y}$; fo that it will be compofed of finite quantities only, and

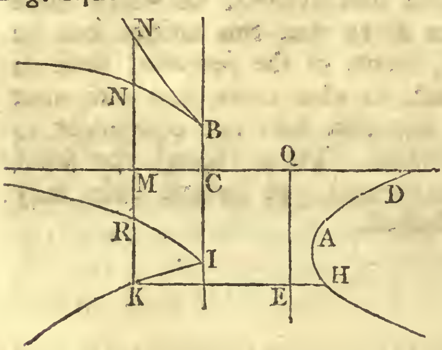
 therefore

therefore will have z given by y and conſtants only, and the relation of y to z will be expreſſed by an equation, or an algebräical curve. Therefore, in the equation $\dot{x} = \frac{z\dot{y}}{a}$, inſtead of $\dot{y}$, putting the value that will be derivèd from ſuch algebräical equation, we ſhall have the variables ſeparated.

EXAMPLE I.

Let the equation be $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$. Make $\dot{x} = \frac{z\dot{y}}{a}$; and, making the ſubſtitutions, inſtead of $\dot{x}$ and it's powers, we ſhall have the equation $\frac{zy\dot{y}^4}{a} = \frac{z^4\dot{y}^4}{a^3} + \frac{2zz\dot{y}^4}{a} + a\dot{y}^4$; and, dividing by $\dot{y}^4$, it will be $\frac{zy}{a} = \frac{z^4}{a^3} + \frac{2z^2}{a} + a$. Or $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$, and $\dot{y} = \frac{3zz\dot{z}}{aa} + 2\dot{z} - \frac{aa\dot{z}}{zz}$. Therefore $\frac{z\dot{y}}{a} = \dot{x} = \frac{3z^3\dot{z}}{a^3} + \frac{2z\dot{z}}{a} - \frac{a\dot{z}}{z}$. If we go on to the integration, it will be $x = \frac{3z^4}{4a^3} + \frac{zz}{a} - lz$, taking the logarithm from the logarithmic with the ſubtangent $= a$. Whence we have the values of the two co-ordinates x and y of the propoſed differential equation, by means of two curves, which have z for a common indeterminate. Now, as to the conſtruction, we may proceed thus.

Fig. 142.



Taking the abſciſſes in the axis QE, deſcribe the curve DAH of the equation $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$, and the curve RIK of the equation $x = \frac{3z^4}{4a^3} + \frac{zz}{a} - lz$. Then EH $= y$, and EK $= x$, will be the co-ordinates of the propoſed differential curve; by the conſtruction of which, making CM parallel to EK, then KM is produced to N, whence it will always be MN $=$ EH; and the curve NBN will be that required.

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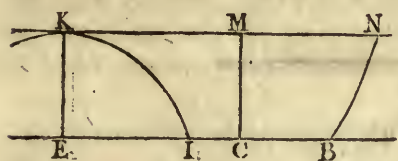
EXAMPLE II.

Let the equation be $y^3\dot{x}^5 + aay\dot{y}\dot{x}^4 = a^3\dot{y}^5$. Make $\dot{x} = \frac{z\dot{y}}{a}$; and, making the substitutions, we shall have $\frac{z^5y^3\dot{y}^5}{a^5} + \frac{aaz^4y\dot{y}^5}{a^4} = a^3\dot{y}^5$. And, dividing by $\dot{y}^5$, it will be $z^5y^3 + a^3z^4y = a^8$. Therefore z will be given only by y and constants, and therefore, in the equation $\dot{x} = \frac{z\dot{y}}{a}$, the variables are separated.

Fig. 143.



Now, to have the curve of the proposed differential equation; to the axis CE let there be described the curve IK of the equation $z^5y^3 + a^3z^4y = a^8$, it being CM $= y$, and MK $= z$. In KM, produced, take MN equal to the area CMKI, divided by a. Then will it be MN $= \int\frac{z\dot{y}}{a} = x$, and the point N will be in the curve.

18. The method of § 14 may be rendered still more general, by transforming the equations which have not the condition required, of the sum of the exponents being equal, into others which shall have those sums equal, and consequently shall come under the rule of that article. This may be done two ways. One will be, to make use of convenient substitutions, for which there can be no rule, and it must be by examples alone that this artifice can be acquired. The other is, by changing the exponents of the proposed formula or equation, that it may be determined, at least, in what cases, and with what substitutions it may succeed, to transform the equation into one equivalent to it, in which the condition required may be found. Thus, though the separation of the variables cannot be universally performed, yet infinite cases may be assigned, in which that separation will be effected.

EXAMPLE I.

Now, as to the firſt manner. Let the equation be $\dot{x}\sqrt{aaxx + az^3} = zz\dot{z}$, which has not the neceſſary condition. Make $z^3 = ayy$, and, taking the fluxions, $zz\dot{z} = \frac{2}{3}ay\dot{y}$. Therefore, making the ſubſtitutions, $\dot{x}\sqrt{aaxx + aayy} = \frac{2}{3}ay\dot{y}$; an expreſſion that may be managed by the method of § 14. We may alſo have our deſire, by putting $\sqrt{aaxx + az^3} = au$, and therefore $aaxx + az^3 = aauu$, and, by differencing, $2aax\dot{x} + 3azz\dot{z} = 2aau\dot{u}$, that is, $zz\dot{z} = \frac{2}{3}au\dot{u} - \frac{2}{3}ax\dot{x}$; and, making the ſubſtitutions, it is $u\dot{x} = \frac{2u\dot{u} - 2x\dot{x}}{3}$.

EXAMPLE II.

Let the equation be $x^3\dot{x} + \frac{xx\dot{y}}{\sqrt{a+y}} = \dot{y}$. Make $\sqrt{a+y} = z$, and therefore $a + y = zz$, and $\dot{y} = 2z\dot{z}$. And, by ſubſtitution, $x^3\dot{x} + 2xx\dot{z} = 2z\dot{z}$. But this ſtill requires a little further reduction. Therefore make $xx = u$, or $x^4 = uu$, and $4x^3\dot{x} = 2u\dot{u}$; whence, theſe values being ſubſtituted, it will be finally $\frac{1}{2}u\dot{u} + 2u\dot{z} = 2z\dot{z}$, &c.

19. I ſhall go on to the ſecond manner of altering the exponents, and therefore I ſhall take a general equation of three terms, $ay^nx^m\dot{x} + by^qx^p\dot{x} + cx^ry^s\dot{y} = 0$; in which the ſigns may be as we pleaſe, either poſitive or negative. If it were $n + m = q + p = r + s$, it would be the caſe of § 14. But, ſuppoſing ſuch an equality ſhould not be found between the ſums of the exponents; make $y = z^t$, whence $\dot{y} = tz^{t-1}\dot{z}$, $y^s = z^{st}$, $y^q = z^{tq}$, $y^n = z^{nt}$, and making the neceſſary ſubſtitutions in the propoſed equation, it will be $az^{nt}x^m\dot{x} + bz^{qt}x^p\dot{x} + tcx^rz^{st+t-1}\dot{z} = 0$. But, by the condition of the aforeſaid § 14, it is neceſſary that it ſhould be $nt + m = qt + p = r + st + t - 1$. From the firſt equation, therefore, $nt + m = qt + p$, we muſt derive the value of the aſſumed exponent $t = \frac{p-m}{n-q}$, which, being ſubſtituted in

in the ſecond, $qt + p = r + st + t - 1$, or $\overline{s - q + 1} \times t = p - r + 1$, will give $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$; which is the condition that the exponents of the propoſed equation ought to have. To verify which, it will always be reducible by the rule of § 14; and the ſubſtitution to be made will be $y = z^{\frac{p-m}{n-q}}$.

Inſtead of making $y = z^t$, if I had made $x = z^t$, I ſhould have found the ſame condition to be verified in the exponents, but it would have been $t = \frac{n-q}{p-m}$, and therefore the ſubſtitution to be made is $x = z^{\frac{n-q}{p-m}}$.

It may happen, that the ſubſtitution of $y = z^{\frac{p-m}{n-q}}$ may become impoſſible, that is, when $p = m$, or $n = q$. But it may be obſerved, that, in theſe caſes, the indeterminates are ſeparable without need of reduction.

In the canonical equation $ay^n x^m \dot{x} + by^q x^p \dot{x} + cx^r y^s \dot{y} = 0$, if, beſides the ſuppoſition of $y = z^t$, we ſhall alſo make $x = u^w$; making all the ſubſtitutions, we ſhall find $awz^{nt} u^{wm+w-1} \dot{u} + bwz^{qt} u^{wp+w-1} \dot{u} + ctu^{wr} z^{st+t-1} \dot{z} = 0$. By the compariſon of the exponents of the firſt and ſecond terms, we ſhould have $nt + wm + w - 1 = qt + wp + w - 1$, that is, $t = w \times \frac{p-m}{n-q}$. From the compariſon of thoſe of the ſecond and third, we ſhall have $wr + st + t - 1 = qt + wp + w - 1$, or $t \times \overline{s - q + 1} = w \times \overline{p - r + 1}$. And, inſtead of t, putting it's value, $w \times \overline{p - m} \times \overline{s - q + 1} = w \times \overline{n - q} \times \overline{p - r + 1}$, which is the condition the exponents of the propoſed equation ought to have. But the letter w vaniſhes out of the condition; therefore the ſecond ſubſtitution of $x = u^w$ is altogether ſuperfluous; whence it may be inferred, that all the formulæ, in general, cannot be reduced to the rule of § 14, but only ſuch, in which the condition $\overline{p - m} \times \overline{s - q + 1} = \overline{n - q} \times \overline{p - r + 1}$ may be verified. The ſame thing is to be concluded of others, when compounded of a greater number of terms, which I ſhall now proceed to treat of.

20. As the number of terms increaſes beyond three, ſo, in like manner, the number of conditions increaſes, which the exponents of the equation muſt have,

in order to be reducible by the method of § 14. I will take this canonical equation of four terms, $ax^m y^n \dot{x} + bx^p y^q \dot{x} + cx^r y^s \dot{y} + dx^e y^u \dot{y} = 0$. Putting $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$, and making the substitutions, it is $az^{nt}x^m \dot{y} + bz^{qt}x^p \dot{x} + tcx^r z^{st+t-1}\dot{z} + dtx^e z^{tu+t-1}\dot{z} = 0$. Therefore it ought to be $nt + m = qt + p$. Whence we may derive the value of the assumed exponent $t = \frac{p-m}{n-q}$. Also, it ought to be $r + st + t - 1 = qt + p$, or $st - qt + t = p - r + 1$; and, substituting the value of t, it will be $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$, the first condition. And, besides, it ought to be $e + tu + t - 1 = qt + p$, or $tu - qt + t = p - e + 1$, and, substituting the value of t, $\overline{u - q + 1} \times \overline{p - m} = \overline{p - e + 1} \times \overline{n - q}$, the second condition. If, therefore, the exponents of a proposed equation shall be such, as that both these conditions shall be found therein, it will be reducible to the case of § 14, and the substitution to be made will be $y = z^{\frac{p-m}{n-q}}$.

If the equations shall have five terms, the conditions to be verified will be three; and so on to more terms.

EXAMPLE.

Let the equation be $ay^3 x\dot{x} + byyx^{\frac{1}{2}}\dot{x} = cx\dot{y}$. This, being compared with the canonical equation, will give $n = 3$, $m = 1$, $q = 2$, $p = \frac{1}{2}$, $r = 1$, $s = 0$. And, because, in the present case, the condition is verified of $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$, giving $-1 \times -\frac{1}{2} = \frac{1}{2} \times 1$, which is true; the equation will be reducible to the method of § 14, and the substitution to be made will be $y = z^{\frac{p-m}{n-q}} = z^{-\frac{1}{2}}$. Therefore I make $y = z^{-\frac{1}{2}}$, $\dot{y} = -\frac{1}{2}z^{-\frac{3}{2}}\dot{z}$, $y^3 = z^{-\frac{3}{2}}$, $y^2 = z^{-1}$; and, making the substitutions, I find $az^{-\frac{3}{2}}x\dot{x} + bz^{-1}x^{\frac{1}{2}}\dot{x} = -\frac{1}{2}cxz^{-\frac{3}{2}}\dot{z}$; which is now reduced to the case of the said article.

21. But,

21. But, without applying particular equations to canonical ones, perhaps it may be more commodious to manage them by this method only.

EXAMPLE I.

Let the equation be $ay^{\frac{1}{3}}x^{\frac{11}{6}}\dot{x} - bx^3y^{-1}\dot{y} = cx^2y\dot{y}$. Make $x = z^t$, $\dot{x} = tz^{t-1}\dot{z}$; making the substitutions, it will be $tay^{\frac{1}{3}}z^{\frac{11}{6}t+t-1}\dot{z} - bz^{3t}y^{-1}\dot{y} = cz^{2t}y\dot{y}$. But it ought to be $\frac{1}{3} + \frac{11}{6}t + t - 1 = 3t - 1$, whence I obtain $t = 2$, which, being put instead of t, gives me this equation $2ay^{\frac{1}{3}}z^{\frac{14}{3}}\dot{z} - bz^6y^{-1}\dot{y} = cz^4y\dot{y}$, which is just the case of § 14. Therefore the substitution to be made, $x = z^2$.

EXAMPLE II.

Let the equation be $x^{\frac{1}{2}}\dot{x} + y^{\frac{4}{3}}\dot{x} + x^{\frac{3}{4}}y\dot{y} = y^3\dot{y}$. Put $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$, and, making the substitutions, it will be $x^{\frac{1}{2}}\dot{x} + z^{\frac{4t}{3}}\dot{x} + tx^{\frac{3}{4}}z^{2t-1}\dot{z} = tz^{4t-1}\dot{z}$. But it ought to be $\frac{1}{2} = \frac{4}{3}t$, whence I have $t = \frac{3}{8}$; which value, being put instead of t, gives me the equation $x^{\frac{1}{2}}\dot{x} + z^{\frac{1}{2}}\dot{x} + \frac{3}{8}x^{\frac{3}{4}}z^{-\frac{1}{4}}\dot{z} = \frac{3}{8}z^{\frac{1}{2}}\dot{z}$, which is just the case of § 14. Therefore the substitution to be made is $y = z^{\frac{3}{8}}$.

EXAMPLE III.

Let the equation be $ay^2x^2\dot{x} + b\dot{x} + cyx\dot{x} + dx^4y^2\dot{y} = 0$. Put $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$; making the substitutions, it will be $az^{2t}x^2\dot{x} + b\dot{x} + cz^tx\dot{x} + tdx^4z^{3t-1}\dot{z} = 0$. Now it ought to be $2t + 2 = t + 1$, whence $t = -1$; and,

and, putting this inſtead of t, gives me the equation $\frac{ax^2\dot{x}}{zz} + b\dot{x} + \frac{cx\dot{x}}{z} - \frac{dx^4\dot{z}}{z^4} = 0$, which is the caſe of § 14. The ſubſtitution to be made is $y = \frac{1}{z}$.

22. The method of § 14 being thus made more general, I ſhall proceed to another, which is alſo general in it's kind. This comprehends all thoſe equations, in which neither the indeterminates, nor their differentials, exceed the firſt dimenſion.

Wherefore let the general differential equation, which includes all poſſible caſes wherein the variables and their fluxions do not aſcend beyond one dimenſion, be $ax\dot{x} + by\dot{y} + cy\dot{x} + gx\dot{y} + f\dot{x} + h\dot{y} = 0$. The co-efficients a, b, c, &c. may be poſitive, or negative, or nothing, as the circumſtances of the particular equation may require, which is propoſed to be conſtructed. As to this equation, I obſerve, in the firſt place, that, if it ſhall be $c = g$, both of them being poſitive, or both negative, the equation may be integrated. For then it will be $\pm c \times \overline{y\dot{x} + x\dot{y}} = -ax\dot{x} - by\dot{y} - f\dot{x} - h\dot{y}$, and, by integration, $\pm cxy = -\frac{1}{2}axx - \frac{1}{2}byy - fx - hy$. But, it not being $c = g$, I make $x = p + A$, $y = q + B$, where p and q are two new indeterminates, and A and B are arbitrary conſtants, to be determined as the ſequel may require. It will be then $\dot{x} = \dot{p}$, $\dot{y} = \dot{q}$, $x\dot{x} = p\dot{p} + A\dot{p}$, $y\dot{y} = q\dot{q} + B\dot{q}$. Theſe values being ſubſtituted in the principal equation propoſed, there will ariſe this following.

$$\begin{array}{llllll} ap\dot{p} & + aA\dot{p} & + bq\dot{q} & + bB\dot{q} & + cq\dot{p} & + gp\dot{q} = 0. \\ & + cB\dot{p} & & + gA\dot{q} & & \\ & + f\dot{p} & & + h\dot{q} & & \end{array}$$

In this equation, if the ſecond and fourth terms be made to vaniſh, this will be the caſe of § 14; and we ſhall know how to ſeparate the indeterminates. But the ſecond term will vaniſh, if it be made $aA + cB + f = 0$, and the fourth, if it be $bB + gA + h = 0$. Whence, from theſe two equations, the values of the aſſumed quantities A and B will be determined, ſo as that the new equation will be a caſe of the aforeſaid § 14. Then it will be $A = \frac{-cB - f}{a}$, $B = \frac{-gA - h}{b}$, that is, $A = \frac{bf - ch}{cg - ab}$, $B = \frac{ah - fg}{cg - ab}$. If, therefore, we make the ſubſtitutions of $x = p + \frac{bf - ch}{cg - ab}$, and of $y = q + \frac{ah - fg}{cg - ab}$, an equation will ariſe, which may be managed by the method of § 14.

If it ſhould happen, in a particular equation, that it ſhould be $bf = ch$, or $ah = fg$, ſo that either of the aſſumed conſtants ſhould be nothing; it would be be a ſure token, that we might obtain our deſire by one ſubſtitution only. For example-ſake, let $\frac{bf - cb}{cg - ab} = A = o$. In this caſe, omitting the quantity x with it's fluxion, it will be enough to ſubſtitute $q + B$ inſtead of y, and to proceed in the manner above explained.

Now, if both the quantities A and B ſhould be nothing, in this hypotheſis we ſhould have $bf = cb$, and $ab = fg$; and conſequently $\frac{cb}{b} = \frac{ab}{g} = f$. Then $cg = ab$, by which we ſhould no longer have any need of theſe ſubſtitutions. Therefore, as often as it is $cg = ab$, make the ſubſtitution $ax + cy = z$, and take y and $\dot{y}$ out of the equation. It will be then $y = \frac{z - ax}{c}$, $\dot{y} = \frac{\dot{z} - a\dot{x}}{c}$. Make theſe ſubſtitutions in the principal equation, and we ſhall have $ax\dot{x} + \frac{bz\dot{z} - abx\dot{z} - abz\dot{x} + aabx\dot{x}}{cc} + z\dot{x} - ax\dot{x} + \frac{gx\dot{z} - agx\dot{x}}{c} + f\dot{x} + \frac{h\dot{z} - ah\dot{x}}{c} = o$. That is, ſtriking out the firſt and ſeventh terms, and, reducing all to a common denominator, $bz\dot{z} - abx\dot{z} - abz\dot{x} + aabx\dot{x} + ccz\dot{x} + cgx\dot{z} - acgx\dot{x} + ccf\dot{x} + ch\dot{z} - ach\dot{x} = o$. But, becauſe $gc = ab$, the ſecond term will deſtroy the ſixth, and the fourth the ſeventh, ſo that there will remain only $bz\dot{z} - abz\dot{x} + ccz\dot{x} + ccf\dot{x} + ch\dot{z} = ach\dot{x}$, or $\dot{x} = \frac{bz\dot{z} + ch\dot{z}}{abz - ccz - ccf + ach}$.

EXAMPLE I.

Let the equation be $ax\dot{x} + 2ay\dot{x} + bx\dot{y} - ab\dot{y} = o$. Make $x = p + A$, $y = q + B$, $\dot{x} = \dot{p}$, $\dot{y} = \dot{q}$; and, making the ſubſtitutions, the equation will be

$$\begin{aligned} &ap\dot{p} + aA\dot{p} + 2aq\dot{p} + bp\dot{q} + bA\dot{q} = o. \\ &\quad\;\; + 2aB\dot{p} \qquad\qquad\qquad\quad - ab\dot{q} \end{aligned}$$

The laſt term will vaniſh if it be $bA - aB = o$, or $A = a$. The ſecond will vaniſh if it be $2aB + aA = o$, or $B = -\frac{1}{2}a$. Therefore the ſubſtitutions are $x = p + a$, and $y = q - \frac{1}{2}a$; and the equation will be reduced to the caſe of § 14.

The aforeſaid terms vaniſhing out of the equation, it may be integrated by means of § 4, without having recourſe to § 14.

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EXAMPLE II.

Let the equation be $2ax\dot{x} - 2by\dot{y} - 4ay\dot{x} + bx\dot{y} - aa\dot{x} = 0$. In this the co-efficient $2a$ correſponds with a in the canonical equation, $-2b$ with b, $-4a$ with c, b with g; and gives us the caſe, that it is $cg = ab$, in reſpect to the conſtants of the canonical equation. Therefore I make the ſubſtitution $2ax - 4ay = z$, and therefore $y = \frac{2ax - z}{4a}$, $\dot{y} = \frac{2a\dot{x} - \dot{z}}{4a}$; wherefore, eliminating y and $\dot{y}$, we ſhall have $2ax\dot{x} - \frac{8aabx\dot{x} + 4abz\dot{x} + 4abx\dot{z} - 2bz\dot{z}}{16aa} - 2ax\dot{x} + z\dot{x} + \frac{2abx\dot{x} - bx\dot{z}}{4a} - aa\dot{x} = 0$. That is, $4abz\dot{x} - 2bz\dot{z} + 16aaz\dot{x} - 16a^4\dot{x} = 0$, or $\dot{x} = \frac{2bz\dot{z}}{4abz + 16a^2z - 16a^4}$.

23. Equations of this kind, as alſo thoſe of a higher degree, may be thus managed by the help of one, but a more compounded ſubſtitution. I reſume the canonical equation above, $ax\dot{x} + by\dot{y} + cy\dot{x} + gx\dot{y} + f\dot{x} + h\dot{y} = 0$, becauſe thoſe of higher degrees would involve us in too long calculations; and what I ſhall ſay concerning this, will be ſufficient to ſhow us how thoſe others are to be treated. Therefore I make $x = Ay + p + B$, in which ſubſidiary equation p is a new indeterminate, which has no conſtant prefixed to it, becauſe that would be unneceſſary, as the operation will ſhow. A and B are two conſtants, to be determined as occaſion may require. Making, then, $x = Ay + p + B$, it will be $\dot{x} = A\dot{y} + \dot{p}$, $x\dot{x} = AAy\dot{y} + Ap\dot{y} + AB\dot{y} + Ay\dot{p} + p\dot{p} + B\dot{p}$; ſo that, theſe values being ſubſtituted in the canonical equation, it will be transformed into this following.

$$\left.\begin{array}{llllll} aAAy\dot{y} & + aAp\dot{y} & + aAy\dot{p} & + ap\dot{p} & + aAB\dot{y} & + aB\dot{p} \\ + by\dot{y} & + gp\dot{y} & + cy\dot{p} & & + gB\dot{y} & + f\dot{p} \\ + cAy\dot{y} & & & & + fA\dot{y} & \\ + gAy\dot{y} & & & & + h\dot{y} & \end{array}\right\} = 0.$$

Now we muſt contrive to make ſome of the terms of this equation to vaniſh, by conveniently determining the aſſumed arbitrary quantities A and B, and to make it capable of the end propoſed; when ſome of the conditions are to be verified, which ariſe from the values of A and B. If, therefore, the ſecond and third terms could be deſtroyed, the variables would be ſeparated, and the equation would become integrable. But, that theſe two terms may become nothing, it is neceſſary that it be $aA + g = 0$ in reſpect of the ſecond, and

and $aA + c = 0$, in reſpect of the third; and conſequently $g = c$. But, ſuppoſing this, the principal equation will be already integrable, without the help of any operation.

If the two laſt terms were nothing, the equation would be reduced to the canon of § 14. But, that they may vaniſh, it will be neceſſary that $aB + f = 0$, or $B = -\frac{f}{a}$, in reſpect of the laſt, and $aAB + gB + fA + b = 0$, in reſpect of the fifth. But, ſubſtituting the value of B, it will be $-Af - \frac{gf}{a} + Af + b = 0$, that is, $ab = gf$. Therefore the laſt two terms cannot be made to vaniſh, ſo that by them the equation may be reduced, except in the particular caſe, in which is verified the condition of $ab = gf$.

If we endeavour, then, to take away the firſt and fifth terms, by which the equation will be reduced to the caſe of § 4 and § 6; then, in reſpect of the firſt term, it will be $aAA + b + cA + gA = 0$, or $A^2 + \frac{c+g}{a}A = -\frac{b}{a}$, from whence we may deduce the value of A. This being found, the value of B will be diſcovered from the fifth term, and will be $B = \frac{-fA - b}{aA + g}$. And the new equation will become $\overline{aA + g} \times p\dot{y} + \overline{aA + c} \times y\dot{p} = -ap\dot{p} - aB\dot{p} - f\dot{p}$, which may be conſtructed by means of § 4, if the co-efficients of the two firſt terms are both poſitive or negative; but, by means of § 6, if one be poſitive, and the other negative.

But, to obtain the ſeparation required, it will be ſufficient to make the firſt term of the ſubſidiary equation to vaniſh, by making it $aAA + cA + gA + b = 0$. Now, putting the aſſumed conſtant $B = 0$, which, in this caſe, will be unneceſſary, there will remain the equation $-ap\dot{p} - f\dot{p} = \overline{aA + g} \times p\dot{y} + \overline{fA + b} \times \dot{y} + \overline{aA + c} \times y\dot{p}$, in which the variables may be ſeparated by the method, which ſhall be explained in the following article. Or elſe, by the foregoing, with the help of an eaſy preparation, that is, making $\overline{Aa + g} \times p + fA + b = q$, and taking the fluxion $\overline{Aa + g} \times \dot{p} = \dot{q}$. Then, by ſubſtitution, $-ap\dot{p} - f\dot{p} = q\dot{y} + \frac{\overline{Aa + c} \times y\dot{q}}{Aa + g}$. But we ought to conſider, that, in making uſe of ſuch formulæ, very often imaginary quantities will inſinuate themſelves, ariſing from the extraction of the root A out of the affected quadratick equation $aAA + \overline{c + g} \times A + b = 0$. And theſe will not only obtrude themſelves into the co-efficients, but will often paſs from thence into the exponents. And, becauſe as yet we have not found out the ways

ways of managing them, it is neceſſary to avoid them as much as poſſible; and, among various methods, to adhere to that which ſhall be found moſt convenient.

For an example; let the equation be $abxx\dot{x} + bbyx\dot{x} + a^3y\dot{x} + aaby\dot{y} + a^3x\dot{y} = 0$. Make $y = Ax + p + B$, whence $\dot{y} = A\dot{x} + \dot{p}$. Here I chooſe to ſubſtitute inſtead of y rather than x, becauſe I foreſee the calculation will be ſhorter. Subſtituting, therefore, we ſhall have this equation following.

$$\begin{aligned} &abx^2\dot{x} + bbpx\dot{x} + bbBx\dot{x} + a^3p\dot{x} + a^3B\dot{x} + a^2bAx\dot{p} \\ &bbAx^2\dot{x} \qquad\quad + 2a^3Ax\dot{x} + a^2bAp\dot{x} + a^2bAB\dot{x} + a^3x\dot{p} \\ &\qquad\qquad\qquad + a^2bA^2x\dot{x} \\ &+ a^2bp\dot{p} + a^2bB\dot{p} = 0. \end{aligned}$$

Here I obſerve, that, in this equation, if I make the firſt, third, fifth, and ſixth terms to vaniſh, we ſhould have the indeterminates ſeparable; for it would be $bbpx\dot{x} + a^3p\dot{x} + a^2bAp\dot{x} + a^2bp\dot{p} + a^2bB\dot{p} = 0$. And, dividing by p, $bbx\dot{x} + a^3\dot{x} + a^2bA\dot{x} = -a^2b\dot{p} - \frac{a^2bB\dot{p}}{p}$. Now, that the firſt may vaniſh, it is neceſſary that $a + bA = 0$, or $A = -\frac{a}{b}$. And, together with this will alſo vaniſh the fifth and ſixth, without any condition ariſing from thence. That the third ſhould vaniſh, it is neceſſary that $bbB + 2a^3A + aabAA = 0$. And ſubſtituting the value of A, it is $bbB - \frac{2a^4}{b} + \frac{a^4b}{bb} = 0$, that is, $B = \frac{a^4}{b^3}$. Therefore the ſubſtitution will be $y = -\frac{ax}{b} + p + \frac{a^4}{b^3}$, and the equation thence ariſing will be $bbx\dot{x} = -aab\dot{p} - \frac{a^6\dot{p}}{b^6p}$.

24. The method of this article conſiſts, firſt, in diſpoſing the propoſed equation in ſuch manner, as that the fluxions may continue accompanied with their indeterminates reſpectively, and that a half-ſeparation (as I may ſo ſay) may be made, by throwing into the common multipliers, or diviſors, ſuch quantities as hinder the operation. Then taking the integrals of the differential thus prepared, compounded of two variables, it muſt be made equal to one aſſumed variable, and, by means of an auxiliary equation, it muſt give a new form to the principal equation. Laſtly, taking obſervation by that which ſucceeds, the operation muſt be repeated till the deſired ſeparation is completed, or till we ſee the formula eludes all our endeavours.

This method has this advantage above the others, that in trying theſe ſubſtitutions, at the ſame time it informs us, which will be ſucceſsful and which uſeleſs. But it muſt be obſerved, that there are ſome equations which will not admit of

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the artifice of the prefent method, unlefs they are firft prepared according to art. The whole will be better underftood by the following Examples.

EXAMPLE I.

Let this equation be propofed, $\frac{x^3\dot{y} + y^3\dot{x}}{\overline{xx+yy} \times \sqrt{xx+yy-xxyy}} = \dot{z}$, in which $\dot{z}$ ftands for any function of x or y whatever. I fet afide the denominator, which is an affection common to the two terms which compofe the firft part of the equation, and the bare differential $x^3\dot{y} + y^3\dot{x}$ will remain. I divide $\dot{x}$ by x^3, and $\dot{y}$ by y^3, and then it will be $x^3\dot{y} + y^3\dot{x} = x^3y^3 \times \overline{\frac{\dot{y}}{y^3} + \frac{\dot{x}}{x^3}}$. Hence the propofed equation will take this new form, $\frac{x^3y^3}{\overline{xx+yy} \times \sqrt{xx+yy-xxyy}} \times \overline{\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3}} = \dot{z}$. Having obtained this half-feparation, in which the fluxions $\dot{x}$, $\dot{y}$, appear combined fimply with the functions of their variables x^3, y^3, and the other terms conftitute, as it were, a foreign quantity, which has the appearance of a multiplier; I make $\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3} = -\frac{\dot{p}}{a^3}$, and then, by integration, $\frac{a^3}{2xx} + \frac{a^3}{2yy} = p$. Now, finding the value, fuppofe of x, which will be $x = \frac{ya\sqrt{a}}{\sqrt{2yyp-a^3}}$, and fubftituting this inftead of x, and $-\frac{\dot{p}}{a^3}$ inftead of $\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3}$ in the equation, it will be $-\frac{\dot{p} \times a\sqrt{a}}{2p\sqrt{2p-a^3}} = \dot{z}$. Wherefore, &c.

It may be recollected, that, taking a quantity at pleafure any how given by p, as $p = \frac{a^3}{2qq}$, it will be $\frac{a^3}{2qq} = \frac{a^3}{2xx} + \frac{a^3}{2yy}$, that is, $q = \frac{xy}{\sqrt{xx+yy}}$; by which, in an inftant, we may perceive infinite fubftitutions, which will promote the defired feparation of the variables. All the other poffible ones will be ufelefs, and will leave the variables as much blended and intermixed as before.

Moreover, let it be obferved, that it often happens with the fubftitutions here explained, that in one member of the equation there may remain fome function of one of the variables x or y; in which cafe, if $\dot{z}$ were given by the variable whofe function remains, one fimple divifion would anfwer the purpofe.

EXAMPLE II.

Let the equation be $\frac{2y\dot{y} + x\dot{y} + y\dot{x}}{a + x + y} = \dot{z}$, in which $\dot{z}$ is any how given by y. To reduce this equation to the method, I take the integral of the numerator of the fraction, that is, $yy + xy$, and make it equal to p. Now, making x and $\dot{x}$ to vanish out of the equation, by substituting their values, I shall have a new equation $\frac{\dot{p}}{a + \frac{p}{y}} = \dot{z}$, which is reduced to the following, $y\dot{p} - p\dot{z} = ay\dot{z}$. And this, being prepared according to the method, will be found to be $p \times \frac{\dot{p}}{p} - \frac{\dot{z}}{y} = a\dot{z}$. I make $\frac{\dot{p}}{p} - \frac{\dot{z}}{y} = \frac{\dot{q}}{q}$, and therefore $lp - \int\frac{\dot{z}}{y} = lq$. I make also $\int\frac{\dot{z}}{y} = ulm$, where lm is some constant logarithm. Then it will be $lp - lq = ulm$. And going on from logarithmic quantities to exponentials, it will be $\frac{p}{q} = m^u$. Therefore, in the reduced equation, making the substitutions of $\frac{\dot{q}}{q}$ instead of $\frac{\dot{p}}{p} - \frac{\dot{z}}{y}$, and of $m^u q$ instead of p, it will be $m^u\dot{q} = a\dot{z}$, that is, $\dot{q} = \frac{a\dot{z}}{m^u}$; in which the variables are separated, because both $\dot{z}$ and m^u are given by y.

EXAMPLE III.

Let the equation be $\frac{2xx\dot{x} + xy\dot{y} + yy\dot{x}}{x^4 + xxyy + a^4} = \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}}$. Before we attempt this formula, it will be best to reduce it. I observe that the second member is integrable, and it's integral is $\sqrt{xx + yy}$ (§ 10). Wherefore I make $\sqrt{xx + yy} = z$, and making y to vanish, finding that it's powers ascend to the square, and putting $zz - xx$ instead of yy, and $z\dot{z} - x\dot{x}$ instead of $y\dot{y}$, we shall have the equation $\frac{2x^2\dot{x} + xz\dot{z} - x^2\dot{x} + z^2\dot{x} - x^2\dot{x}}{xxzz + a^4} = \dot{z}$, that is, $\frac{xz\dot{z} + zz\dot{x}}{xxzz + a^4} = \dot{z}$; which, being

being prepared as ufual, will be $\frac{z}{x^2z^2 + a^4} \times \overline{x\dot{z} + z\dot{x}} = \dot{z}$. I make $x\dot{z} + z\dot{x} = \dot{p}$, and, by integration, $xz = p$; and, making x to vanifh, we fhall have $\frac{z\dot{p}}{pp + a^4} = \dot{z}$, and, finally, $\frac{\dot{p}}{pp + a^4} = \frac{\dot{z}}{z}$.

EXAMPLE IV.

Let it be the laft equation of the foregoing article, $-ap\dot{p} - f\dot{p} = \overline{aA + g} \times p\dot{y} + \overline{fA + b} \times \dot{y} + \overline{aA + c} \times y\dot{p}$, which I undertook to conftruct. This equation being prepared according to the method, and, for brevity, making $aA + g = e$, $fA + b = m$, $aA + c = n$, it will be reduced to this, $-\frac{ap\dot{p} + f\dot{p}}{ep + m} = y \times \overline{\frac{\dot{y}}{y} + \frac{n\dot{p}}{ep + m}}$. Therefore I put $\frac{\dot{y}}{y} + \frac{n\dot{p}}{ep + m} = \frac{\dot{q}}{q}$; and, by integration, $ly + \frac{n}{e} l\overline{p + \frac{m}{e}} = lq$. And therefore $y = \frac{q}{\overline{p + \frac{m}{e}}^{\frac{n}{e}}}$.

And eliminating y, we fhall have $-\frac{ap\dot{p} + f\dot{p}}{ep + m} = \frac{\dot{q}}{\overline{p + \frac{m}{e}}^{\frac{n}{e}}}$, that is, $-\frac{ap\dot{p} + f\dot{p}}{ep + m} \times \overline{p + \frac{m}{e}}^{\frac{n}{e}} = \dot{q}$.

EXAMPLE V.

Let the equation be this already prepared, $y^m \times \overline{x\dot{x} + y\dot{y}} = x^n \times \overline{y\dot{x} - x\dot{y}}$, which I write in this manner, $\frac{y^{m-2}}{x^n} \times \overline{x\dot{x} + y\dot{y}} = \frac{y\dot{x} - x\dot{y}}{yy}$, in order to make the fecond member integrable. In this I make ufe of a double fubftitution, and therefore I put $x\dot{x} + y\dot{y} = p\dot{p}$, and, by integration, $xx + yy = pp$. I put alfo $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{q}$, and by integration, $\frac{x}{y} = q$. Making the fubftitutions, we fhall have $\frac{y^{m-2}}{x^n} \times p\dot{p} = \dot{q}$. But $yy = pp - xx$, and $xx = qqyy$, fo that it will

will be $yy = pp - qqyy$, that is, $yy = \frac{pp}{1+qq}$, and $y^{m-2} = \frac{p^{m-2}}{\overline{1+qq}|^{\frac{m-2}{2}}}$, and $x^n = \frac{q^n p^n}{\overline{1+qq}|^{\frac{n}{2}}}$. Wherefore, substituting these values of y^{m-2} and x^n, we shall have $p^{m-2-1}\dot{p} = q^n\dot{q} \times \overline{1+qq}|^{\frac{m-n-2}{2}}$.

EXAMPLE VI.

Let the equation be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$; in which $\dot{z}$ is any how given by x or y. I observe that the numerator of the first member is integrable, if it were divided by xx, and that it's integral would be $\frac{2y}{x}$, and therefore I thus dispose the equation, $\frac{1}{\overline{x-y}|^2} \times \frac{2x\dot{y} - 2y\dot{x}}{xx} = \frac{\dot{z}}{xx}$. Put $\frac{2y}{x} = p$, whence it will be $\frac{2x\dot{y} - 2y\dot{x}}{xx} = \dot{p}$, and the equation will be changed into this following, $\frac{\dot{p}}{\overline{x-y}|^2} = \frac{\dot{z}}{xx}$. But $2y = px$, and $yy = \frac{1}{4}ppxx$; so that, making the substitutions, it will be $\frac{\dot{p}}{xx - pxx + \frac{1}{4}ppxx} = \frac{\dot{z}}{xx}$; and, multiplying by xx, it is $\frac{\dot{p}}{1 - p + \frac{1}{4}pp} = \dot{z}$, in which the variables are separated. I go on to the integration; and therefore it will be $\frac{2}{1-\frac{1}{2}p} + c = \int\dot{z}$; and, restoring the value of p, it is $\frac{2}{1-\frac{y}{x}} + c = \int\dot{z}$, and reducing to a common denominator, it is $\frac{2x + cx - cy}{x-y} = \int\dot{z}$. If we make the constant $c = 0$, we shall have $\frac{2x}{x-y} = \int\dot{z}$; and, making $c = -2$, it will be $\frac{2y}{x-y} = \int\dot{z}$, which is another integral of the proposed formula different from the first. Lastly, putting $c = -1$, a third integral will arise, $\frac{x+y}{x-y} = \int\dot{z}$.

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25. The method I now undertake to explain, although much limited and confined, is yet of great ufe in fome particular cafes. By this the variables may be feparated in the canonical equation $a\dot{y} = yp\dot{x} + by^n q\dot{x}$, in which the quantities p, q, are to be underftood as any how given by x. The quantities a, b, are conftant; the figns may be pofitive or negative at pleafure, and the exponent n may be integer, fraction, pofitive, negative, or even nothing. In this equation, then, make $y = zu$, where z and u are two new variables; and, by taking the fluxions, it will be $\dot{y} = z\dot{u} + u\dot{z}$; and, by fubftituting, inftead of $\dot{y}$, y, and y^n, their values $z\dot{u} + u\dot{z}$, zu, and $u^n z^n$, we fhall have the equation $az\dot{u} + au\dot{z} = uzp\dot{x} + bz^n u^n q\dot{x}$, in which, if two terms fhall vanifh, the indeterminates will be feparated. To do which, let us feign an equation between the two terms $au\dot{z} = uzp\dot{x}$, then $\frac{a\dot{z}}{z} = p\dot{x}$, and, by integration, $alz = \int p\dot{x}$; and, proceeding from logarithms to exponential quantities, it is $z^a = m^{\int p\dot{x}}$, or $z = m^{\frac{\int p\dot{x}}{a}}$, fuppofing $lm = 1$. This laft equation fhows us the value of z, and informs us, that, to reduce the equation propofed to two terms only, and to caufe the other two to deftroy each other, inftead of $y = zu$, we ought to put $y = um^{\frac{\int p\dot{x}}{a}}$, that is, $\frac{y}{u} = m^{\frac{\int p\dot{x}}{a}}$, or $ly - lu = \int \frac{p\dot{x}}{a}$. And, by differencing, $\frac{a\dot{y}}{y} - \frac{a\dot{u}}{u} = p\dot{x}$, and therefore $a\dot{y} = yp\dot{x} + \frac{ay\dot{u}}{u}$. Therefore, in the canonical equation $a\dot{y} = yp\dot{x} + by^n q\dot{x}$, inftead of $\dot{y}$, I fubftitute it's value now found, and it will be $yp\dot{x} + \frac{ay\dot{u}}{u} = yp\dot{x} + by^n q\dot{x}$, that is, $\frac{ay\dot{u}}{u} = by^n q\dot{x}$, and therefore $\frac{a\dot{u}}{u} = by^{n-1} q\dot{x}$. But $y = zu$, and $y^{n-1} = z^{n-1} u^{n-1}$; whence, finally, it will be $\frac{a\dot{u}}{u^n} = bz^{n-1} q\dot{x}$; in which equation the variables will be feparated, becaufe z is fuppofed given by x. When we came to the equation $alz = \int p\dot{x}$, it is plain, that if p given by x is fuch, that the integral $\int p\dot{x}$ depends on the quadrature of the hyperbola, that is, on the logarithms, and the quantity a is any number whatfoever, the relation of z to x will be algebräical, and in all other cafes tranfcendental.

And here it may be obferved, that, in order to have a given equation come under the cafe of the canonical formula, it is neceffary that the following conditions fhould take place. Firft, that the fluxion $\dot{y}$ may be alone, or, at leaft, multiplied by a conftant, on one fide of the equation. Then, that, on the other fide, the firft term may contain the fluxion $\dot{x}$, multiplied by any function of x expreffed by p, and by the indeterminate y. Then, that, in the other term, the quantity $q\dot{x}$ given by x may be multiplied by a power of y. In a word, making,

making the divifion by y, it is required, that, on one fide of the equation, there may remain the logarithmical fluxion $\frac{a\dot{y}}{y}$, and, on the other fide, the firft term may be free from the indeterminate y, and the fecond multiplied by the dignity y^{n-1}. If any one of thefe requifites be wanting, this method cannot take place; as we fhould not have them in the following equations, $a\dot{y} = yyp\dot{x} + by^n q\dot{x}$, and $a\dot{y} = yp\dot{x} + \overline{ayy + y^3} \times q\dot{x}$.

But fome formulæ are very eafily reduced to the canon, by a little preparation only. For example, take this equation $a\dot{y} = yp\dot{x} + byq\dot{x} + yyq\dot{x}$. Confider that the quantity $p\dot{x} + bq\dot{x}$, multiplied by y, and that the binomial $p + bq$ is given by x, fo that in it's place may be fubftituted the quantity r, alike given by x; the expreffion then will be changed into the following, $a\dot{y} = yr\dot{x} + yyq\dot{x}$, in which the method here explained will take place. And this will be fufficient to fhow the way of operation in all like cafes.

EXAMPLE I.

Let the equation be $a\dot{y} = \frac{fy\dot{x}}{x} + yy\dot{x}$. Make $y = zu$, and therefore $a\dot{y} = az\dot{u} + au\dot{z}$. And, making the neceffary fubftitutions, we fhall have $az\dot{u} + au\dot{z} = \frac{fuz\dot{x}}{x} + zzuu\dot{x}$. Let $au\dot{z} = \frac{fuz\dot{x}}{x}$, that is, $\frac{a\dot{z}}{z} = \frac{f\dot{x}}{x}$; and integrating, it will be $alz = flx$, and therefore $z^a = x^f$.

If the conftants a, f, fhall be rational numbers, whole or fracted, affirmative or negative, z will be given algebräically by x. For example, make $a = 1$, $f = 2$, fo that it may be $z = xx$. Then eliminating the terms $au\dot{z}$, $\frac{fuz\dot{x}}{x}$, there will remain the two, $az\dot{u} = zzuu\dot{x}$. But $z = xx$, therefore it will be $\frac{a\dot{u}}{uu} = xx\dot{x}$, an equation in which the variables are feparated.

In proceeding to the integration, it will be $-\frac{a}{u} + c = \frac{1}{3}x^3$. But $u = \frac{y}{z} = \frac{y}{xx}$, and therefore $-\frac{axx}{y} + c = \frac{1}{3}x^3$; that is, $3cy - 3axx = x^3y$; which is the algebräical equation concealed under the propofed differential.

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EXAMPLE II.

Let the equation be $\dot{y} = \frac{ay\dot{x}}{xx - aa} + \frac{y^3\dot{x}}{x^3}$. Make, as above, $y = zu$, and $\dot{y} = z\dot{u} + u\dot{z}$; then, making the ſubſtitutions, we ſhall have $z\dot{u} + u\dot{z} = \frac{azu\dot{x}}{xx - aa} + \frac{z^3u^3\dot{x}}{x^3}$. And, ſuppoſing $u\dot{z} = \frac{azu\dot{x}}{xx - aa}$, that is, $\frac{\dot{z}}{z} = \frac{a\dot{x}}{xx - aa}$, or $z = m^{\int \frac{a\dot{x}}{xx-aa}}$, we ſhall have the equation $z\dot{u} = \frac{z^3u^3\dot{x}}{x^3}$, or $\frac{\dot{u}}{u^3} = \frac{zz\dot{x}}{x^3}$, in which the variables are ſeparated, z being given by x. But it may be obſerved, that the quantity $\frac{a\dot{x}}{xx - aa}$ may be reduced to a logarithmic fluxion, by making $x = \frac{\overline{a+n} \times a}{a - n}$; wherefore, making the due ſubſtitutions, it will be $\frac{a\dot{x}}{xx - aa} = \frac{\dot{n}}{2n}$. Whence $\frac{\dot{z}}{z} = \frac{\dot{n}}{2n}$, and therefore $zz = n = \frac{a \times \overline{x - a}}{x + a}$. And, putting this value, inſtead of zz, in the final equation, we ſhall have $\frac{\dot{u}}{u^3} = \frac{ax\dot{x} - aa\dot{x}}{x^4 + ax^3}$.

Without making the ſubſtitution of $x = \frac{\overline{a + n} \times a}{a - n}$, the quantity $\frac{a\dot{x}}{xx - aa}$ may be reduced to a logarithmical fluxion, by means of § 21, Book III; and we ſhould have $\frac{a\dot{x}}{xx - aa} = -\frac{\dot{x}}{2 \times \overline{x+a}} + \frac{\dot{x}}{2 \times \overline{x-a}} = \frac{\dot{z}}{z}$, and conſequently $zz = \frac{x - a}{x + a}$.

EXAMPLE III.

Let the equation be $\dot{y} = -\frac{y\dot{x}}{x} + y^m\dot{x}$. Make $y = zu$, $\dot{y} = z\dot{u} + u\dot{z}$; therefore, ſubſtituting, it will be $z\dot{u} + u\dot{z} = -\frac{uz\dot{x}}{x} + u^m z^m \dot{x}$. Suppoſing $u\dot{z} = -\frac{uz\dot{x}}{x}$, or $\frac{\dot{z}}{z} = -\frac{\dot{x}}{x}$, and, by integration, $z = \frac{a}{x}$; we ſhall have the equation $z\dot{u} = z^m u^m \dot{x}$, that is, $\frac{\dot{u}}{u^m} = z^{m-1}\dot{x}$, or $\frac{\dot{u}}{u^m} = \frac{a^{m-1}\dot{x}}{x^{m-1}}$.

EXAMPLE IV.

Sometimes a two-fold operation is neceſſary; as in certain equations which have more than three terms. Wherefore, let the equation be $x\dot{y} + y\dot{x} = a\dot{u} + x\dot{u}$, and let u be any how given in the terms of y. I diſpoſe the equation in the following manner, $a\dot{u} + x\dot{u} - x\dot{y} = y\dot{x}$, or $\frac{a\dot{u}}{y} + \frac{x\dot{u}}{y} - \frac{x\dot{y}}{y} = \dot{x}$. Make $x = pq$, and $\dot{x} = p\dot{q} + q\dot{p}$; then, making the ſubſtitutions, it will be $\frac{a\dot{u}}{y} + \frac{pq\dot{u}}{y} - \frac{pq\dot{y}}{y} = p\dot{q} + q\dot{p}$. If any one would reduce the formula by one operation only, he muſt put $\frac{pq\dot{u}}{y} - \frac{pq\dot{y}}{y} = p\dot{q}$, that is, $\frac{\dot{u}}{y} - \frac{\dot{y}}{y} = \frac{\dot{q}}{q}$; by which we find q given by y. But the operation will be performed more neatly in the following manner. Make $-\frac{pq\dot{y}}{y} = p\dot{q}$; then $-\frac{\dot{y}}{y} = \frac{\dot{q}}{q}$, and, by integration, $\frac{a}{y} = q$. Taking, therefore, the other terms of the equation $\frac{a\dot{u}}{y} + \frac{pq\dot{u}}{y} = q\dot{p}$, and, inſtead of q, ſubſtituting it's value $\frac{a}{y}$, it will be $\frac{a\dot{u}}{y} + \frac{ap\dot{u}}{yy} = \frac{a\dot{p}}{y}$, that is, $\dot{u} + \frac{p\dot{u}}{y} = \dot{p}$. Make $p = mn$, then $\dot{p} = m\dot{n} + n\dot{m}$, and making the ſubſtitution, it will be $\dot{u} + \frac{mn\dot{u}}{y} = m\dot{n} + n\dot{m}$. Suppoſe $\frac{mn\dot{u}}{y} = m\dot{n}$, that is, $\frac{\dot{u}}{y} = \frac{\dot{n}}{n}$. Therefore n will be given by y, and in the remaining equation, after the terms $\frac{mn\dot{u}}{y}$, $m\dot{n}$, have been eliminated, that is, in the equation $\dot{u} = n\dot{m}$, the variables will be ſeparated, and it will be $\frac{\dot{u}}{n} = \dot{m}$.

26. Still, after another manner, the variables may be ſeparated in the canonical equation $\dot{y} = py\dot{x} + qy^n\dot{x}$. Make $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$, $\dot{x} = \frac{\dot{z}}{\overline{1-n} \times pz}$; Making the ſubſtitutions, it will be $\dot{y} = \frac{y\dot{z}}{\overline{1-n} \times z} + \frac{qy^n\dot{z}}{\overline{1-n} \times pz}$; that is, $\dot{y} = \frac{py\dot{z} + qy^n\dot{z}}{\overline{1-n} \times pz}$, or $\overline{1-n} \times pz\dot{y} = py\dot{z} + qy^n\dot{z}$; and therefore, dividing by py^n, it is $\frac{\overline{1-n} \times z\dot{y} - y\dot{z}}{y^n} = \frac{q\dot{z}}{p}$. Laſtly, dividing by zz, it will be

$\frac{\overline{1-n} \times zy^{-n}\dot{y} - y^{1-n}\dot{z}}{zz} = \frac{q\dot{z}}{pzz}$, and, by integration, $\frac{y^{1-n}}{z} = \int\frac{q\dot{z}}{pzz}$, that is, $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$. And, becaufe p and q are fuppofed to be given by x; and z alfo, by the fubftitution of $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$, is given by x; the variables will be feparated, at leaft tranfcendentally.

Refuming, therefore, the equation of the firft example, $a\dot{y} = \frac{fy\dot{x}}{x} + y^2\dot{x}$, that is, $\dot{y} = \frac{fy\dot{x}}{ax} + \frac{yy\dot{x}}{a}$, it will be $p = \frac{f}{ax}$, $q = \frac{1}{a}$, $n = 2$. So that, fubftituting thefe values in the final equation $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$, it will be $\frac{1}{y} = z\int\frac{x\dot{z}}{fzz}$, and the fubftitution $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$ will be $\frac{f\dot{x}}{ax} = -\frac{\dot{z}}{z}$. And, making $f = 2$, $a = 1$, we fhall have $\frac{2\dot{x}}{x} = -\frac{\dot{z}}{z}$, that is, $z = \frac{1}{xx}$. And therefore $\frac{1}{y} = \frac{1}{xx}\int -xx\dot{x}$. And, by integration, $\frac{1}{y} = \frac{1}{xx} \times \overline{-\frac{1}{3}x^3 + c}$, that is, $3cy - 3xx = x^3y$, as before. And fo we may proceed with the other Examples.

EXAMPLE V.

Let the equation be $ax^4y\dot{y} - bx^4y\dot{y} = ayyx^3\dot{x} - byyx^3\dot{x} + a^6\dot{x} - x^6\dot{x}$, which, divided by $ax^4y - bx^4y$, will be found to be $\dot{y} = \frac{y\dot{x}}{x} + \frac{a^6\dot{x} - x^6\dot{x}}{\overline{a-b} \times x^4y}$, which is a cafe of the canonical equation. Therefore it will be $p = -\frac{1}{x}$, $q = \frac{a^6 - x^6}{\overline{a-b} \times x^4}$, $n = -1$. And, by fubftitution, $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$ will be $-\frac{\dot{x}}{x} = \frac{\dot{z}}{2z}$, whence $z = xx$. Then, putting thefe values in the final canonical equation, $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$, we fhall have $yy = xx\int\frac{\overline{a^6 - x^6} \times 2x\dot{x}}{\overline{a-b} \times x^4 \times x^3}$, in which the variables are feparated.

27. If

27. If the canonical equation were $y^{n-1}\dot{y} = p\dot{x} + qy^n\dot{x}$, where p and q, in a like manner, are any how given by x; the indeterminates may be feparated by making $q\dot{x} = \frac{\dot{z}}{nz}$, and $\dot{x} = \frac{\dot{z}}{nqz}$. For, making the fubftitutions, it will be $y^{n-1}\dot{y} = \frac{p\dot{z}}{nqz} + \frac{y^n\dot{z}}{nz}$, that is, $\frac{nzy^{n-1}\dot{y} - y^n\dot{z}}{z} = \frac{p\dot{z}}{qz}$; and, dividing by z, $\frac{nzy^{n-1}\dot{y} - y^n\dot{z}}{zz} = \frac{p\dot{z}}{qzz}$; and, by integration, $\frac{y^n}{z} = \int\frac{p\dot{z}}{qzz}$, that is, $y^n = z\int\frac{p\dot{z}}{qzz}$, an equation in which the variables are feparated.

For an example, let the equation be $2a^2xy\dot{y} = aayy\dot{x} + 2bx^3\dot{x}$, that is, $y\dot{y} = \frac{bxx\dot{x}}{aa} + \frac{yy\dot{x}}{2x}$. It will be $n = 2$, $p = \frac{bxx}{aa}$, $q = \frac{1}{2x}$, and therefore we fhall have $\frac{yy}{z} = \int\frac{2bx^3\dot{z}}{aazz}$. But $q\dot{x} = \frac{\dot{x}}{2x} = \frac{\dot{z}}{2z}$, and $x = z$. Therefore it will be $\frac{yy}{x} = \int\frac{2bx\dot{x}}{aa}$, and, by integration, $\frac{yy}{x} = \frac{bxx}{aa} \pm c$; an algebräical curve.

Alfo, the general formula $y^{n-1}\dot{y} = p\dot{x} + qy^n\dot{x}$ might be conftructed, and confequently the particular example, by means of the method at § 24.

28. Before I finifh this Section, I fhall add one obfervation, that fometimes the indeterminates are involved and mingled with differential quantities, when it may be allowed to modify the co-efficients; and this fucceeds efpecially when the exponents are formed of the co-efficients; and thus making a kind of circuit in the reduction. This artifice chiefly takes place in Phyfico-mathematical Problems, in which magnitudes of very different kinds mingling together, we are more at liberty to make ufe of fuch conftant quantities, as beft ferve the prefent purpofe.

For an example, I fhall propofe to myfelf this equation, $x^m\dot{x} + \overline{by + yy} \times \frac{c\dot{x}}{x} = y\dot{y}$, which, being prepared according to the method of § 24, will be $x^m\dot{x} + \frac{bcy\dot{x}}{x} = yy \times \overline{\frac{\dot{y}}{y} - \frac{c\dot{x}}{x}}$. Make, then, $\frac{\dot{y}}{y} - \frac{c\dot{x}}{x} = \frac{\dot{p}}{p}$, and we fhall have the value of $y = px^c$, and $yy = ppx^{2c}$. Thefe values, conveniently fubftituted, will give the equation $x^m\dot{x} + bcpx^{c-1}\dot{x} = x^{2c}p\dot{p}$; and, dividing by x^{2c}, it will be $x^{m-2c}\dot{x} + bcpx^{-c-1}\dot{x} = p\dot{p}$. Here it is plain, that, an equality being given between the exponents of the indeterminate x, that is, between $m - 2c$ and $-c - 1$, the variables will be feparate, the *homogeneum comparationis* $p\dot{p}$ being only to be divided by the binomial $1 + bcp$. Now, putting

$m - 2c$

$m - 2c = -c - 1$, it follows $m + 1 = c$; fo that, expounding the conftant c by $m + 1$, we fhall have our defire. If c reprefents unity, which we are at liberty to fuppofe, it will be $m = 0$; and if $c = 2$, it will be $m = 1$. And fo we may go on.

The artifice here explained may be applied to all other equations of a like kind; for example, to this following, $x^m\dot{x} + \frac{cby^n\dot{x}}{x} + \frac{gy^r\dot{x}}{x} = y^t\dot{y}$. For, putting $t = r - 1$, or $= n - 1$, the formula will be thence abbreviated by making ufe of the logarithms.

SECT. III.

Of the Conftruction of more Limited Equations, by the Help of various Subftitutions.

29. In the equation $\overline{x^n\dot{x} \pm ay^n\dot{y}} \times p = \overline{x\dot{y} - y\dot{x}} \times q$, the indeterminates are always feparable; where p and q are promifcuoufly given by y and x after any manner; algebräically, when, in every term of the quantity p, the Sum of the exponents of x and y is the fame, and thus likewife in every term of the quantity q; but it is not required that the fum fhould be the fame in p and q. The fubftitutions to be made are $y = tz^{\frac{2}{n+1}}$, and $x = t \times \overline{a^3 \mp azz}|^{\frac{1}{n+1}}$. Thefe being fubftituted, refpectively, inftead of x, $\dot{x}$, y, $\dot{y}$, and making the neceffary operations, after a very long calculation we fhall come to this equation,

$$t^{n-2}\dot{t} = \frac{\frac{2}{n+1} z^{\frac{1-n}{n+1}}\dot{z} \times \frac{q}{p}}{\overline{a^3 \mp azz}|^{\frac{n}{n+1}}}.$$

Now, becaufe it is known, that, in every term of p, the fum of the exponents of x and y is equal, as alfo in every term of q; making in them the fubftitutions of the values given by t and z; in every term of p, t will have the fame power, as alfo in every term of q a fame power; that is to fay, that the *homogeneum comparationis* will be multiplied by a pofitive or negative power of t, or the firft member will be multiplied or divided by that power, and therefore the variables will be feparated.

As,

As, for example, let the equation be $\overline{x\dot{x} + ay\dot{y}} \times \sqrt{y} = \overline{x\dot{y} - y\dot{x}} \times \sqrt{a}$; it will be $n = 1$, $p = \sqrt{y}$, $q = \sqrt{a}$, and therefore $\frac{\dot{t}}{t} = \frac{\dot{z}\sqrt{a}}{\sqrt{a^3 - azz} \times \sqrt{y}}$. But $y = tz$; therefore it will be $\frac{\dot{t}}{\sqrt{t}} = \frac{\dot{z}\sqrt{a}}{\sqrt{a^3z - az^3}}$.

In the ſame equation the indeterminates may be ſeparated, when alſo the exponent n is negative; that is, when the equation is this, $\overline{x^{-n}\dot{x} \pm ay^{-n}\dot{y}} \times p = \overline{x\dot{y} - y\dot{x}} \times q$; and the ſubſtitutions are $y = tz^{\frac{2}{1-n}}$, and $x = t \times \overline{a^3 \mp azz}|^{\frac{1}{1-n}}$. Theſe will give the equation $t^{-n-2}\dot{t} = \frac{\frac{2}{1-n} z^{\frac{1+n}{1-n}} \dot{z} \times \frac{q}{p}}{\overline{a^3 \mp azz}|^{\frac{-n}{1-n}}}$, the ſame as that above, only with the ſigns of n changed. And though the equation were alſo thus expreſſed, $y^n\dot{x} \pm ax^n\dot{y} \times \frac{p}{x^ny^n} = \overline{x\dot{y} - y\dot{x}} \times q$; it follows that this alſo is conſtructible by the ſame ſubſtitutions.

30. Let the equation be more general, $\overline{x^n\dot{x} \pm ay^{\frac{-n-1-c}{c}}\dot{y}} \times p = \overline{x\dot{y} + cy\dot{x}} \times q$. The variables will always be ſeparated by making the ſubſtitutions of $y = t^s z^{\frac{r}{n+1}}$, and $x = t^{-\frac{s}{c}} \times \overline{a \pm acz^{-\frac{r}{c}}}|^{\frac{1}{n+1}}$, where s and r are numbers aſſumed at pleaſure; ſuppoſing, however, this condition, that the quantities p, q, are given algebräically, and in ſuch a manner, that, in every term of the quantity p, the exponent of y, taken as often as the number c denotes, may exceed, or be exceeded by, the exponent of x in the ſame exceſs; and ſo in every term of the quantity q; but it is no matter that the exceſs in p ſhall be the ſame as in q. Thus, for example, if $c = 3$, it may be $p = by^2x^4 + fy^9x^{25}$, &c.; and it may be $q = gy^{\frac{1}{2}}x^3 - hy^{10}x^{\frac{63}{2}}$, &c. It is eaſy to perceive, that it cannot be $c = 0$.

Making the due ſubſtitutions, inſtead of x and y, in the propoſed equation, we ſhall have this following, $-\frac{s}{c} t^{\frac{-sn-c-cs}{c}} \dot{t} = \frac{\frac{r}{n+1} \times z^{\frac{r-n-1}{n+1}} \dot{z} \times \frac{q}{p}}{\overline{a \pm acz^{-\frac{r}{c}}}|^{\frac{n}{n+1}}}$.

For

For example, let it be $\overline{x\dot{x} + ay^{-3}\dot{y}} \times \frac{1}{y} = \overline{x\dot{y} + y\dot{x}} \times x$. Make $s = 1$, $r = 2$; it will be $n = 1$, $c = 1$, $p = \frac{1}{y}$, $q = x$; and, making the substitutions in the last equation found above, we shall have $-t^{-3}\dot{t} = \frac{\dot{z} \times xy}{\overline{a + az^{-2}}|^{\frac{1}{2}}}$.

But, by the substitutions made, $x = t^{-1} \times \overline{a + az^{-2}}|^{\frac{1}{2}}$, and $y = tz$. Therefore $xy = z \times \overline{a + az^{-2}}|^{\frac{1}{2}}$. Whence we shall have $-\frac{\dot{t}}{t^3} = z\dot{z}$.

31. But let the equation be still more general, $\overline{x^n\dot{x} \pm ay^{\frac{-nf-c-f}{c}}\dot{y}} \times p = \overline{fx\dot{y} + cy\dot{x}} \times q$, which comprehends, as particular cases, the two canonical equations of the foregoing articles; that is, that of § 30, when it is $f = 1$; and that of § 29, when it is $f = 1$, and $c = -1$.

The indeterminates are separated by means of the substitutions $y = t^{\frac{s}{f}} z^{\frac{r}{f \times \overline{n+1}}}$, and $x = t^{-\frac{s}{c}} \times \overline{a \pm \frac{acz^{-\frac{r}{c}}}{f}}|^{\frac{1}{n+1}}$; the condition concerning the quantities p and q being such, that, in these, the exponent of y being multiplied by c, may exceed, or be exceeded by, the exponent of x multiplied by f, by the same excess in each term. The same quantities p, q, may also be fractions, or mixed with fractions, and rational or irrational integers, whatever they may be. And the indeterminates will always be separable in the equations, provided that p and q are given by x and y in such a manner, that, the assigned substitutions being made, such quantities may arise in their place, that they may be the product of two, one of which shall contain z, and not t, the other t and not z.

The said substitutions being made, we shall have this formula,

$$-\frac{s}{c}t^{\frac{-fc-fsn-sc}{cf}}\dot{t} = \frac{\frac{r}{n+1} \times z^{\frac{r-fn-f}{fn+f}}\dot{z} \times \frac{q}{p}}{\overline{a \pm \frac{acz^{-\frac{r}{c}}}{f}}|^{\frac{n}{n+1}}}.$$

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EXAMPLE I.

Let the equation be $\overline{xx\dot{x} + ay^{8}\dot{y}} \times y = -\overline{3x\dot{y} + y\dot{x}} \times ax$. Let it be, as before, $s = 1$, $r = 2$, it will be $f = -3$, $c = 1$, $n = 2$, $q = ax$, $p = y$; and, making the fubftitutions in the laft formula found above, we fhall have

$$-t^{-\frac{8}{3}}\dot{t} = \frac{\frac{2}{3}z^{-\frac{11}{9}}\dot{z} \times \frac{ax}{y}}{\overline{a - \frac{1}{3}az^{-2}}^{\frac{2}{3}}}.$$ But $y = t^{-\frac{1}{3}}z^{-\frac{2}{9}}$, $x = t^{-1} \times \overline{a - \frac{1}{3}az^{-2}}^{\frac{1}{3}}$;

therefore it will be $-\frac{\dot{t}}{tt} = \frac{2a\dot{z}}{3z \times \overline{a - \frac{1}{3}az^{-2}}^{\frac{1}{3}}}$; as was to be found.

EXAMPLE II.

Let the equation be $\overline{x^{\frac{1}{2}}\dot{x} + ay^{-2}\dot{y}} \times \overline{ay^{\frac{1}{2}}x + yyx^{\frac{13}{4}}} = \overline{2x\dot{y} + 3y\dot{x}} \times \overline{y^{\frac{1}{3}}x - yxx}$. Let $s = 1$, $r = 1$; it will be $c = 3$, $f = 2$, $n = \frac{1}{2}$, $p = ay^{\frac{1}{2}}x + yyx^{\frac{13}{4}}$, $q = y^{\frac{1}{3}}x - yxx$. And, making the fubftitutions, it will be

$$\frac{-\frac{1}{3}\dot{t}}{t^{\frac{19}{12}}} = \frac{\frac{2}{3}z^{-\frac{5}{9}}\dot{z} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}^{\frac{1}{3}} - \frac{2}{3}z^{-\frac{1}{3}}\dot{z} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}}{az^{\frac{1}{6}} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}^{\frac{2}{3}} + z^{\frac{2}{3}} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}^{\frac{13}{6}}}\ *,$$

in which the variables are feparated, as was required.

32. In the equations

$$(1)\quad pxy^{n-1}\dot{y} = py^{n}\dot{x} + q\dot{x},$$
$$(2)\quad pxy^{n-1}\dot{y} = -py^{n}\dot{x} + q\dot{x},$$
$$(3)\quad apxy^{n-1}\dot{y} = bpy^{n}\dot{x} + q\dot{x},$$
$$(4)\quad apxy^{n-1}\dot{y} = -bpy^{n}\dot{x} + q\dot{x},$$

where p and q are any how given by x; the indeterminates may be feparated, by putting, as to the firft, $y = xz$; as to the fecond, $y = \frac{z}{x}$; as to the third, $y = x^{\frac{b}{a}}z$; as to the fourth, $y = x^{-\frac{b}{a}}z$.

* This equation evidently admits of a fimpler form. EDITOR.

As,

As, for example, let the equation be $2hhxyy\dot{y} - 2x^3yy\dot{y} = bx^4\dot{x} - 3hhy^3\dot{x} + 3xxy^3\dot{x}$, which I write thus, $\overline{hh - xx} \times 2xyy\dot{y} = hx^4\dot{x} + \overline{hh - xx} \times -3y^3\dot{x}$. This being referred to the laſt of the four canonical equations, it will be $p = hh - xx$, $a = 2$, $n = 3$, $b = 3$, $q = hx^4$. Therefore we muſt put $y = \frac{z}{x^{\frac{3}{2}}}$, $\dot{y} = \frac{x^{\frac{3}{2}}\dot{z} - \frac{3}{2}zx^{\frac{1}{2}}\dot{x}}{x^3}$, $yy = \frac{zz}{x^3}$, $y^3 = \frac{z^3}{x^{\frac{9}{2}}}$. And, making the ſubſtitutions, we ſhall have $\overline{2hhx - 2x^3} \times \frac{x^{\frac{3}{2}}z^2\dot{z} - \frac{3}{2}x^{\frac{1}{2}}z^3\dot{x}}{x^6} = hx^4\dot{x} + \overline{3hh - 3xx} \times - \frac{z^3\dot{x}}{x^{\frac{9}{2}}}$; that is, $\overline{2hh - 2xx} \times \overline{xzz\dot{z} - \frac{3}{2}z^3\dot{x}} = hx^{\frac{17}{2}}\dot{x} + \overline{3hh - 3xx} \times - z^3\dot{x}$; and, making the uſual multiplications, it will be $2hhxzz\dot{z} - 2x^3zz\dot{z} = hz^{\frac{17}{2}}\dot{x}$, that is, $zz\dot{z} = \frac{hx^{\frac{17}{2}}\dot{x}}{2hhx - 2x^3}$ *.

33. Let the equation be $ax\dot{y} + by\dot{x} + cy^n x^{m-1}\dot{x} + fx^m y^{n-1}\dot{y} = 0$. In this the indeterminates may be ſeparated, in general, by putting $x = u^{n-1}z^{n-1}$, and $y = z^{1-m}$; for, making the neceſſary operations, we ſhall come to the equation $\overline{1-m} \times \overline{a\dot{z} + fu^{mn-m-n+1}\dot{z}} + \overline{n-1} \times \overline{b\dot{z} + cu^{mn-m-n+1}\dot{z}} = \overline{n-1} \times \overline{-bzu^{-1}\dot{u} - czu^{mn-m-n}\dot{u}}$, that is, $\frac{\dot{z}}{z} =$

$$\frac{\overline{n-1} \times \overline{-bu^{-1}\dot{u} - cu^{mn-m-n}\dot{u}}}{\overline{1-m} \times \overline{a + fu^{mn-m-n+1}} + \overline{n-1} \times \overline{b + cu^{mn-m-n+1}}}.$$

As, for example, let the equation be $a^3x\dot{y} - b^3y\dot{x} = cyyx\dot{x} - fxxy\dot{y}$. Then it will be $n = 2$, $m = 2$. Therefore I put $x = \frac{uz}{a}$, and $y = \frac{aa}{z}$, that is, $x = \frac{au}{y}$, and therefore $\dot{x} = \frac{ay\dot{u} - au\dot{y}}{yy}$. Whence, making the due ſubſtitutions, we ſhall have $\frac{a^4u\dot{y}}{y} - b^3 \times \frac{ay\dot{u} - au\dot{y}}{y} = \frac{caayu\dot{u} - caauu\dot{y}}{y} - \frac{faauu\dot{y}}{y}$, that is, $a^4u\dot{y} + ab^3u\dot{y} + aacuu\dot{y} + faauu\dot{y} = ab^3y\dot{u} + aacyu\dot{u}$, and therefore $\frac{\dot{y}}{y} = \frac{ab^3\dot{u} + aacu\dot{u}}{a^4u + ab^3u + aacuu + aafuu}$ *.

34. Let the equation be $\frac{y\dot{x}}{\overline{bx^{\frac{1}{m}} + ay^n x^r}|^m} = \dot{y}$; or, more generally, $\frac{x^{mt-1}y\dot{x}}{\overline{bx^t + ay^n x^r}|^m} = \dot{y}$. The indeterminates will be ſeparated by putting $\overline{bx^t + ay^n x^r}|^m = zx^{mt}$.

* See the Note at the bottom of the preceding page.

Whence $y = \frac{\overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big|^{\frac{1}{n}}}{a^{\frac{1}{n}}}$, and therefore $\dot{y} = \frac{\frac{1}{n} \times \overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big|^{\frac{1-n}{n}}}{a^{\frac{1}{n}}} \times$

$\overline{\frac{1}{m}x^{t-r}z^{\frac{1}{m}-1}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \overline{t-r} \times -bx^{t-r-1}\dot{x}} =$

$\frac{x^{-1}\dot{x} \times \overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big|^{\frac{1}{n}}}{a^{\frac{1}{n}}z}$, putting these values of y and x^{mt} in the proposed general equation; and dividing by $\frac{\overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big|^{\frac{1}{n}}}{a^{\frac{1}{n}}}$, it will be

$$\frac{\frac{1}{n} \times \overline{\frac{1}{m}x^{t-r}z^{\frac{1}{m}-1}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \overline{t-r} \times -bx^{t-r-1}\dot{x}}}{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}} = \frac{x^{-1}\dot{x}}{z},$$ that is,

$$\frac{1}{m}z^{\frac{1}{m}}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}+1}x^{-1}\dot{x} + \overline{t-r} \times -bzx^{-1}\dot{x} = nz^{\frac{1}{m}}x^{-1}\dot{x} - nbx^{-1}\dot{x};$$

and therefore $\frac{z^{\frac{1}{m}}\dot{z}}{mnz^{\frac{1}{m}} + \overline{r-t} \times mz^{\frac{1}{m}+1} + \overline{mr-mt} \times bz - mnb} = \frac{\dot{x}}{x}$.

If you should have terms with negative signs, you must proceed after the same manner, and in the final equation there would be no other difference, but that of the signs themselves.

35. Also, taking a more universal equation, as $\frac{y^u\dot{x}}{\overline{bx^t + ay^nx^r}\big|^m} = cx^{\frac{ut-mnt-t+r+n-ur}{n}}\dot{y}$; the indeterminates would be separated by the same substitution.

EXAMPLE I.

Let the equation be $\frac{aay\dot{x}}{\sqrt{bbxx - a^3y}} = b\dot{y}$. Make $\sqrt{bbxx - a^3y} = xz$, and therefore $y = \frac{bbxx - zzxx}{a^3}$, and $\dot{y} = \frac{2bbx\dot{x} - 2zzx\dot{x} - 2xxz\dot{z}}{a^3}$. And, making the

the ſubſtitutions, $\frac{aa\dot{x}}{xz} \times \frac{bbxx - zzxx}{a^3} = \frac{2b^3x\dot{z} - 2bzzx\dot{z} - 2bxxz\dot{z}}{a^3}$; that is, $aabbx\dot{x} - aazzx\dot{x} = 2b^3zx\dot{x} - 2bz^3x\dot{x} - 2bxxzz\dot{z}$, or $2bxxzz\dot{z} = 2b^3zx\dot{x} - 2bz^3x\dot{x} + aazzx\dot{x} - aabbx\dot{x}$; and therefore $\frac{2bzz\dot{z}}{2b^3z - 2bz^3 + aazz - aabb} = \frac{\dot{x}}{x}$.

EXAMPLE II.

Let the equation be $\frac{xy\dot{x}}{\sqrt{-bbx^4 + a^3xyy}} = \frac{\dot{y}}{b}$. Make $\sqrt{-bbx^4 + a^3xyy} = zxx$, and therefore $y = \sqrt{\frac{zzx^3 + bbx^3}{a^3}}$, and $\dot{y} = \frac{x^3z\dot{z} + \frac{3}{2}zzxx\dot{x} + \frac{3}{2}bbxx\dot{x}}{a^3\sqrt{\frac{zzx^3 + bbx^3}{a^3}}}$. Wherefore, making the ſubſtitutions, we ſhall have $\frac{x\dot{x}}{zxx}\sqrt{\frac{zzx^3 + bbx^3}{a^3}} = \frac{x^3z\dot{z} + \frac{3}{2}zzxx\dot{x} + \frac{3}{2}bbxx\dot{x}}{a^3b\sqrt{\frac{zzx^3 + bbx^3}{a^3}}}$, that is, $bzzxx\dot{x} + b^3xx\dot{x} = x^3zz\dot{z} + \frac{3}{2}z^3xx\dot{x} + \frac{3}{2}bbzxx\dot{x}$, or $bzzxx\dot{x} + b^3xx\dot{x} - \frac{3}{2}z^3xx\dot{x} - \frac{3}{2}bbzxx\dot{x} = x^3zz\dot{z}$; and therefore $\frac{\dot{x}}{x} = \frac{zz\dot{z}}{bzz - \frac{3}{2}z^3 - \frac{3}{2}bbz + b^3}$.

36. By the ſame ſubſtitution as above, the indeterminates in this equation alſo may be ſeparated.

$\frac{y^u\dot{y}}{\overline{bx^t + ay^nx^r}|^m} = cx^{\frac{tu-n-tmn-rz+t-r}{n}}\dot{x}$. Make $\overline{bx^t + ay^nx^r}|^m = x^{mt}z$, it will be $y = \frac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{1}{n}}}{a^{\frac{1}{n}}}$; then $\dot{y} = \frac{x^{t-r}z^{\frac{1-m}{m}}\dot{z}}{mn} + \frac{t-r}{n}z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \frac{r-t}{n}bx^{t-r-1}\dot{x}$ into $\frac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{1-n}{n}}}{a^{\frac{1}{n}}}$; and, making the ſubſtitutions, we ſhall have $\frac{x^{t-r}z^{\frac{1-m}{m}}\dot{z}}{mn} + \frac{t-r}{n}z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \frac{r-t}{n}bx^{t-r-1}\dot{x}$ into

$\frac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{u-n+1}{n}}}{a^{\frac{u+1}{n}} x^{tm} z} = cx^{\frac{tu-n-tmn-ru+t-r}{n}} \dot{x}$. Wherefore, dividing the numerator and denominator of the firſt member of the equation by x^{tm}, and multiplying the whole by $a^{\frac{u+1}{n}} z$; and, inſtead of $\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{u+1-n}{n}}$, writing $x^{\frac{tu-tn+t-ru+nr-r}{n}} \times \overline{z^{\frac{1}{m}} - b}|^{\frac{u+1-n}{n}}$, which is the ſame; and, uniting the dimenſions of the letter x, we ſhall find the equation to be diviſible by $x^{\frac{ut-n-tmn-ru+t-r}{n}}$, and that being divided accordingly, it will be $\frac{xz^{\frac{1-m}{m}}\dot{z}}{mn} + \frac{t-r}{n} z^{\frac{1}{m}}\dot{x} + \frac{r-t}{n} b\dot{x}$ into $\overline{z^{\frac{1}{m}} - b}|^{\frac{u+1-n}{n}} = ca^{\frac{u+1}{n}} z\dot{x}$. And laſtly, dividing again by $\overline{z^{\frac{1}{m}} - b}|^{\frac{u+1-n}{n}}$, it will be $\frac{xz^{\frac{1-m}{m}}\dot{z}}{mn} = \frac{r-t}{n} \times z^{\frac{1}{m}}\dot{x} + \frac{t-r}{n} \times b\dot{x} + ca^{\frac{u+1}{n}} z\dot{x} \times \overline{z^{\frac{1}{m}} - b}|^{\frac{n-u-1}{n}}$, that is, $\frac{\dot{x}}{x} =$

$$\frac{z^{\frac{1-m}{m}}\dot{z}}{mnca^{\frac{u+1}{n}} z \times \overline{z^{\frac{1}{m}} - b}|^{\frac{n-u-1}{n}} + \overline{mr-mt} \times z^{\frac{1}{m}} + \overline{mt-mr} \times b}.$$

EXAMPLE.

Let the equation be $\frac{y^3\dot{y}}{\sqrt{bbxx - aaxy - abxy}} = \frac{xx\dot{x}}{c}$. Put $\sqrt{bbxx - aaxy - abxy} = xz$, and therefore $y = \frac{bbx - zzx}{aa + ab}$, and $\dot{y} = \frac{bb\dot{x} - zz\dot{x} - 2xz\dot{z}}{aa + ab}$. Making, therefore, the ſubſtitutions, it will be $\overline{\frac{bbx - zzx}{aa + bb}}|^3 \times \frac{bb\dot{x} - zz\dot{x} - 2xz\dot{z}}{aa + bb \times xz} = \frac{xx\dot{x}}{c}$. And, inſtead of $\overline{bbx - zzx}|^3$, writing $x^3 \times \overline{bb - zz}|^3$, and multiplying the whole equation by $\overline{aa + ab}|^4 \times zx$, we ſhall have $x^3 \times \overline{bb - zz}|^3 \times$

$\overline{bb\dot{x} - zz\dot{x} - 2xz\dot{z}} = \overline{aa + ab}|^4 \times \frac{zx^3\dot{x}}{c}$. And, dividing by $x^3 \times \overline{bb-zz}|^3$, it will be $bb\dot{x} - zz\dot{x} - 2xz\dot{z} = \overline{aa + ab}|^4 \times \overline{bb - zz}|^{-3} \times \frac{z\dot{x}}{c}$; that is, $bb\dot{x} - zz\dot{x} + \overline{aa+ab}|^4 \times \overline{bb - zz}|^{-3} \times - \frac{z\dot{x}}{c} = 2xz\dot{z}$. And therefore

$$\frac{\dot{x}}{x} = \frac{2z\dot{z}}{bb - zz - \frac{z}{c} \times \overline{bb - zz}|^{-3} \times \overline{aa + ab}|^4}.$$

37. The same substitution will serve, in like manner, for a more general equation, $\frac{\overline{bx^t + fy^n x^r}|^v \times y^u \dot{y}}{\overline{bx^t + ay^n x^r}|^m} = cx^{\frac{ut - n - tmn - ru + t - r + ntv}{n}} \dot{x}$. Also, it will serve for the equation $\frac{y^{n-1}\dot{y}}{\overline{bx^t + cx^r + ay^n x^r}|^m} = fx^{t - r - 1 - mt}\dot{x}$, by making $\overline{bx^t + cx^r + ay^n x^r}|^m = x^{mt}z$; which, if $m = 1$, will be a particular case of § 27; and if it be $c = 0$, will be a particular case of § 36. Moreover, we may also construct the equation $\frac{\overline{gx^t + hx^r + ky^n x^r}|^e \times y^{n-1}\dot{y}}{\overline{ax^t + bx^r + cy^n x^r}|^m} = fx^{t-r-1+et-mt}\dot{x}$, when it is $ch = bk$, making use of the same substitution, $\overline{ax^t + bx^r + cy^n x^r}|^m = x^{mt}z$.

Now, if it should be also $b = 0$, $h = 0$, the equation will be a particular case of the first equation of this article.

38. These equations may be constructed, $\frac{a\dot{y}}{\overline{b + cy^n + fx}|^u} = gy^{1-n}\dot{x}$, and $\frac{ay^{n-1}\dot{y}}{\overline{b + cy^n + fx^m}|^u} = gx^{m-1}\dot{x}$, by putting, for the first, $\overline{cy^n + fx}|^u = z$, and for the second, $\overline{cy^n + fx^m}|^u = z$. And, as for the first, it will be then $y = \frac{\overline{z^{\frac{1}{u}} - fx}|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and $\dot{y} = \frac{1}{n} \times \frac{\overline{z^{\frac{1}{u}} - fx}|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}} \dot{z} - f\dot{x}}$; and therefore, making the substitutions, we shall have $az^{\frac{1-u}{u}}\dot{z} = nubcg\dot{x} + nucgz\dot{x} + auf\dot{x}$, that

that is, $\frac{az^{\frac{1-u}{u}}\dot{z}}{nubcg + nucgz + auf} = \dot{x}$. As to the ſecond, we ſhall have $y = \frac{\overline{z^{\frac{1}{u}} - fx^m}|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and therefore $\dot{y} = \frac{1}{n} \times \frac{\overline{z^{\frac{1}{u}} - fx^m}|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}} \dot{z} - mfx^{m-1}\dot{x}}$; and, making the ſubſtitutions, $x^{m-1}\dot{x} = \frac{az^{\frac{1-u}{u}}\dot{z}}{bcgnu + cgnuz + mafu}$.

Likewiſe, if we take a more general equation, $\frac{ay^{n-1}\dot{y}}{\overline{b + cy^n + p}|^u} = gq\dot{x}$, where p and q are any how given by x and conſtants; if it be $q = \frac{\dot{p}}{\dot{x}}$, the indeterminates may be ſeparated, by putting, in like manner, $\overline{cy^n + p}|^u = z$. For it will be $y = \frac{\overline{z^{\frac{1}{u}} - p}|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and $\dot{y} = \frac{\frac{1}{n} \times \overline{z^{\frac{1}{u}} - p}|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}} \dot{z} - \dot{p}}$; and, making the ſubſtitutions, the equation will be $nbcguq\dot{x} + ncguzq\dot{x} + au\dot{p} = az^{\frac{1-u}{u}}\dot{z}$. But if we ſuppoſe $\dot{p} = q\dot{x}$, then it will be $\frac{az^{\frac{1-u}{u}}\dot{z}}{nbcgu + ncguz + au} = q\dot{x}$.

EXAMPLE I.

Let the equation be $a^3\dot{y} = 6b^3\dot{x} - 3bb\dot{x}\sqrt{cy + bx}$, or $\frac{a^3\dot{y}}{2b - \sqrt{cy + bx}} = 3bb\dot{x}$. Make $\sqrt{cy + bx} = z$, it will be $y = \frac{zz - bx}{c}$, $\dot{y} = \frac{2z\dot{z} - b\dot{x}}{c}$; and, making the ſubſtitutions, $\frac{2a^3z\dot{z} - a^3b\dot{x}}{2bc - cz} = 3bb\dot{x}$, or $2a^3z\dot{z} = 6b^3c\dot{x} - 3bbcz\dot{x} + a^3b\dot{x}$, and therefore $\frac{2a^3z\dot{z}}{6b^3c - 3bbcz + a^3b} = \dot{x}$.

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EXAMPLE II.

Let the equation be $\frac{ay y \dot{y}}{b + \sqrt[3]{y^3 + aax - bxx}} = aa\dot{x} - 2bx\dot{x}$. Make $\sqrt[3]{y^3 + a^2x - bx^2} = z$; it will be $y = \overline{z^3 - aax + bxx}|^{\frac{1}{3}}$, and $\dot{y} = \frac{zz\dot{z} - \frac{1}{3}aa\dot{x} + \frac{2}{3}bx\dot{x}}{\overline{z^3 - aax + bxx}|^{\frac{2}{3}}}$; whence, making the ſubſtitutions, the equation will be $\frac{azz\dot{z} - \frac{1}{3}a^3\dot{x} + \frac{2}{3}abx\dot{x}}{b + z} = aa\dot{x} - 2bx\dot{x}$; that is, $3azz\dot{z} = a^3\dot{x} - 2abx\dot{x} + 3aab\dot{x} - 6bbx\dot{x} + 3aaz\dot{x} - 6bzx\dot{x}$; and, dividing by $a + 3b + 3z$, it will be $\frac{3azz\dot{z}}{a + 3b + 3z} = aa\dot{x} - 2bx\dot{x}$.

39. The equation, or canonical formula, $ax^m\dot{x} + cyyx^n\dot{x} = \dot{y}$, has not it's indeterminates ſeparable in general, whatever the exponent m may be; yet they are ſeparable in an infinite number of caſes; that is, the exponent m may receive infinite values, in which the deſired ſeparation will ſucceed.

To determine which I make uſe of a method not unlike to that of § 23. Make $y = Ax^p + x^r t$; where the quantity A, and the exponents p, r, are arbitrary conſtants, to be determined as exigence may require, and t in a new indeterminate quantity. Therefore it will be $\dot{y} = pAx^{p-1}\dot{x} + rtx^{r-1}\dot{x} + x^r\dot{t}$, and $yy = AAx^{2p} + 2Ax^{p+r}t + ttx^{2r}$. Wherefore, ſubſtituting theſe values in the propoſed formula, they will give this following, $ax^m\dot{x} + cAAx^{2p+n}\dot{x} + 2cAtx^{p+r+n}\dot{x} + cttx^{2r+n}\dot{x} = pAx^{p-1}\dot{x} + rtx^{r-1}\dot{x} + x^r\dot{t}$. Let us ſuppoſe $cAA = pA$, $2p + n = p - 1$, $r = 2cA$; that is, $p = -n - 1$, $A = \frac{-n-1}{c}$, $r = -2n - 2$. By theſe, in the laſt formula, will vaniſh the ſecond, third, fifth, and ſixth terms, and it will be reduced to $ax^m\dot{x} + cttx^{-3n-4}\dot{x} = x^{-2n-2}\dot{t}$. That is, (dividing by x^{-2n-2},) $ax^{m+2n+2}\dot{x} + cttx^{-n-2}\dot{x} = \dot{t}$; or (D) $ax^K\dot{x} + cttx^X\dot{x} = \dot{t}$, making $m + 2n + 2 = K$, and $-n - 2 = X$.

I reſume

I reſume the propoſed equation $ax^m\dot{x} + cyyx^n\dot{x} = \dot{y}$, which, putting $y = \frac{1}{z}$, is transformed into this other, $azzx^m\dot{x} + cx^n\dot{x} = -\dot{z}$; in which is put, as above, $z = Bx^q + x^v u$, where B, q, v, are conſtants, to be determined as before, and u is a new indeterminate quantity. Therefore it will be $\dot{z} = qBx^{q-1}\dot{x} + vux^{v-1}\dot{x} + x^v\dot{u}$, $zz = BBx^{2q} + 2Bx^{q+v}u + uux^{2v}$. And theſe values being ſubſtituted, we ſhall have $aBBx^{2q+m}\dot{x} + 2aBux^{q+v+m}\dot{x} + auux^{2v+m}\dot{x} + cx^n\dot{x} = -qBx^{q-1}\dot{x} - vux^{v-1}\dot{x} - x^v\dot{u}$. Now, if we ſuppoſe $aBB = -Bq$, $2q + m = q - 1$, $-v = 2aB$; that is, $q + m = -1$, $B = \frac{m+1}{a}$, $v = -2m - 2$; with theſe in this laſt formula will vaniſh the firſt, ſecond, fifth, and ſixth terms, and it will be reduced to $auux^{-3m-4}\dot{x} + cx^n\dot{x} = -x^{-2m-2}\dot{u}$; that is, (dividing by x^{-2m-2},) $cx^{2m+n+2}\dot{x} + auux^{-m-2}\dot{x} = -\dot{u}$, or (G) $cx^\delta\dot{x} + auux^\omega\dot{x} = -\dot{u}$; making $2m + n + 2 = \delta$, and $-m - 2 = \omega$.

Now, in the propoſed equation, the indeterminates are ſeparable when $m = n$. Wherefore, alſo, in the formulæ marked D, G, the indeterminates will be ſeparable, when it is $m + 2n + 2 = -n - 2$, $2m + n + 2 = -m - 2$, becauſe m obtains two values, that is, $m = -3n - 4$, $m = \frac{-n-4}{3}$; which being ſubſtituted, the ſeparation of the indeterminates will ſucceed. For then, in the propoſed equation, the indeterminates will be ſeparated when it is $m = \frac{-n-4}{3}$; alſo, they will be ſeparated in the formulæ D, G, when it is $K = \frac{-X-4}{3}$, $\delta = \frac{-\omega-4}{3}$, becauſe there are other two values of m, that is, $m = \frac{-5n-8}{3}$, $m = \frac{-3n-8}{5}$.

By the ſame way of argumentation, we may have infinite other values of m; as $m = \frac{-7n-12}{5}$, $m = \frac{-5n-12}{7}$, $m = \frac{-9n-16}{7}$, $m = \frac{-7n-16}{9}$, &c.; and, in general, $m = \frac{\overline{2b \pm 1} \times -n-4b}{2b \mp 1}$, taking b any integer poſitive number, beginning from unity. Putting any of theſe values in the propoſed equation, we ſhall have the indeterminates ſeparable.

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It may be added, that the indeterminates will alfo be feparable in the propofed equation, when the exponent m is fuch, that, by the method of § 19, it may be reduced to a cafe of § 14.

This would be the place to make ufe of two Differtations of the very learned Mr. *Euler*, inferted in the Memoirs of the Academy of *Peterfburg*, Tom. VI. But, becaufe of the fubtile manner in which that author proceeds, I fhould be obliged to exceed thofe limits which I had fixed to myfelf, intending only a plain and fimple Inftitution. I fhall therefore leave the curious reader to feek them in the book itfelf.

PROBLEM I.

40. To find the curve, the fubtangent of which is equal to the fquare of the ordinate, divided by a conftant quantity.

Making the abfcifs equal to x, the ordinate equal to y, the fubtangent is always $\frac{y\dot{x}}{\dot{y}}$, which therefore ought to be equal to $\frac{yy}{a}$. Therefore we fhall have the equation $\frac{y\dot{x}}{\dot{y}} = \frac{yy}{a}$, or $a\dot{x} = y\dot{y}$, and, by integration, $ax = \frac{1}{2}yy$, or $2ax = yy$, which is the *Apollonian* parabola.

If the fubtangent ought to be equal to twice the abfcifs, we fhould have the equation $\frac{y\dot{x}}{\dot{y}} = 2x$, and therefore $\frac{\dot{x}}{2x} = \frac{\dot{y}}{y}$, and, by integration, $\frac{1}{2}lx + \frac{1}{2}la = ly$, (where the conftant $\frac{1}{2}la$ is added, to fulfil the law of homogeneity,) that is, $l\sqrt{ax} = ly$; and, returning from the logarithms, $\sqrt{ax} = y$, or $ax = yy$, which is alfo the fame parabola.

If the fubnormal is to be conftant, it will be $\frac{y\dot{y}}{\dot{x}} = a$, that is, $y\dot{y} = a\dot{x}$, and, by integration, $\frac{1}{2}yy = ax$, or $yy = 2ax$, which is again the fame parabola.

Let the fubtangent be triple of the abfcifs; it will be $\frac{y\dot{x}}{\dot{y}} = 3x$, or $\frac{\dot{x}}{3x} = \frac{\dot{y}}{y}$, and, by integration, $l\sqrt[3]{aax} = ly$, or $aax = y^3$, which is the firft cubical parabola.

Let the ſubtangent be a multiple of the abſciſs, according to any number m; it will be $\frac{y\dot{x}}{\dot{y}} = mx$, that is, $\frac{\dot{x}}{mx} = \frac{\dot{y}}{y}$, and, by integration, $l\sqrt[m]{a^{m-1}x} = ly$, or $a^{m-1}x = y^m$, a curve of the parabolic kind.

Let the ſubtangent be $\frac{2ax + xx}{a + x}$; then the equation is $\frac{y\dot{x}}{\dot{y}} = \frac{2ax + xx}{a + x}$, that is, $ay\dot{x} + yx\dot{x} = 2ax\dot{y} + xx\dot{y}$, or $\frac{a\dot{x} + x\dot{x}}{2ax + xx} = \frac{\dot{y}}{y}$. And, by integration, it will be $ly = \frac{1}{2}l\overline{2ax + xx}$, and therefore $xx + 2ax = yy$, an equation to the hyperbola.

Let the ſubtangent be $\frac{2axy - 3x^3}{ay + 3xx}$; then the equation will be $\frac{y\dot{x}}{\dot{y}} = \frac{2axy - 3x^3}{ay + 3xx}$, that is, $ayy\dot{x} + 3yxx\dot{x} = 2axy\dot{y} - 3x^3\dot{y}$. According to what has been already delivered at § 18, I endeavour to reduce this equation to a caſe of § 14. Therefore I make $y = \frac{zz}{a}$, $\dot{y} = \frac{2z\dot{z}}{a}$; and, making the ſubſtitutions, it will be $z^4\dot{x} + 3zzxx\dot{x} = 4xz^3\dot{z} - 6x^3z\dot{z}$, where now it is reduced to the ſaid caſe. Wherefore the indeterminates will be ſeparated, if we put $z = \frac{xp}{a}$, $\dot{z} = \frac{x\dot{p} + p\dot{x}}{a}$; and, making the ſubſtitutions, it will be $\frac{p^4x^4\dot{x}}{a^4} + \frac{3ppx^4\dot{x}}{aa} = \frac{4x^4p^3}{a^3} \times \frac{x\dot{p} + p\dot{x}}{a} - \frac{6x^4p}{a} \times \frac{x\dot{p} + p\dot{x}}{a}$, that is, $9aap\dot{x} - 3p^3\dot{x} = 4xpp\dot{p} - 6aax\dot{p}$, and therefore $\frac{\dot{x}}{x} = \frac{4pp\dot{p} - 6aa\dot{p}}{9aap - 3p^3}$; and, by integration, $lx = \frac{lm}{l\sqrt[3]{p^4 - 3aapp}}$. And, reſtoring the value of p, that is, $a\sqrt{\frac{ay}{xx}}$, it will be $x = \frac{m}{\sqrt[3]{\frac{a^6yy - 3a^5yxx}{x^4}}}$, that is, finally, $a^6y^2 - 3a^5yx^2 = mx$.

The two ſubſtitutions made of $y = \frac{zz}{a}$, and $z = \frac{xp}{a}$, in order to ſeparate the indeterminates, plainly ſhow us that it would have been ſufficient if, at firſt, we had made but one of them, or $y = \frac{xxpp}{a^3}$.

But we might have obtained our deſire ſomething more expeditiouſly, by writing the equation thus: $3yxx\dot{x} + 3x^3\dot{y} = 2axy\dot{y} - ayy\dot{x}$; which, divided by xx, will be $3y\dot{x} + 3x\dot{y} = \frac{2axy\dot{y} - ay^2\dot{x}}{xx}$; and, by integration, $3xy = \frac{ayy}{x}$, that is, $\frac{1}{3}ay = xx$, the *Apollonian* parabola, when we omit the conſtant m.

Let

Let the ſubtangent be $\frac{4x^3 - axy}{3xx - ay}$; the equation will be $\frac{4x^3 - axy}{3xx - ay} = \frac{y\dot{x}}{\dot{y}}$, that is, $4x^3\dot{y} - axy\dot{y} = 3xxy\dot{x} - ayy\dot{x}$, which I write in another manner, thus: $4x^3\dot{y} - 3yxx\dot{x} = axy\dot{y} - ayy\dot{x}$. I obſerve that the ſecond member would be integrable, if it were divided by xxy; I divide, therefore, the whole equation, whence it is $\frac{4x\dot{y} - 3y\dot{x}}{y} = \frac{ax\dot{y} - ay\dot{x}}{xx}$. I ſuppoſe the integral of this ſecond member $\frac{ay}{x} = z$; and, making y to vaniſh out of the equation, it will be $\frac{4x \times \overline{x\dot{z} + z\dot{x}} - 3zx\dot{x}}{zx} = \dot{z}$, that is, $\frac{4x\dot{z} + z\dot{x}}{z} = \dot{z}$, which may be conſtructed by the method of § 14, or elſe prepared according to the method of § 24, it will be $x \times \overline{\frac{4\dot{z}}{z} + \frac{\dot{x}}{x}} = \dot{z}$. Therefore I make $\frac{4\dot{z}}{z} + \frac{\dot{x}}{x} = \frac{\dot{p}}{p}$, and, by integration, $lz^4x = la^4p$, or $z^4x = a^4p$; and therefore, making x to vaniſh out of the final equation, we ſhall have, laſtly, $\frac{a^4p}{z^4} \times \frac{\dot{p}}{p} = \dot{z}$, that is, $a^4\dot{p} = z^4\dot{z}$, and, by integration, $a^4p = \frac{1}{5}z^5$; in which, reſtoring the value of p, then that of z, it will be $xx = \frac{1}{5}ay$, which is the *Apollonian* parabola.

Let the ſubtangent be $\frac{\overline{a+x} \times l\overline{a+x}}{a + l\overline{a+x}}$; the equation will be $\frac{\overline{a+x} \times l\overline{a+x}}{a + l\overline{a+x}} = \frac{y\dot{x}}{\dot{y}}$, that is, $\frac{\dot{y}}{y} = \frac{a\dot{x} + \dot{x}l\overline{a+x}}{\overline{a+x} \times l\overline{a+x}}$. In order to proceed to the integration, I make $\overline{a+x} \times l\overline{a+x} = z$, and therefore $\dot{z} = \dot{x} \times l\overline{a+x} + a\dot{x}$; (ſuppoſing the logarithmic with the ſubtangent $= a$.) Theſe values being ſubſtituted in the equation, it will be $\frac{\dot{y}}{y} = \frac{\dot{z}}{z}$, and integrating, it is $y = z$, that is, $y = \overline{a+x} \times l\overline{a+x}$, a tranſcendent curve, but which is eaſily deſcribed, ſuppoſing the logarithmic.

PROBLEM II.

41. To find the curve, the area of which is equal to two third parts of the rectangle of the co-ordinates.

The formula for the area is $y\dot{x}$, and therefore we ſhall have $\int y\dot{x} = \frac{2}{3}xy$; whence $y\dot{x} = \frac{2}{3}x\dot{y} + \frac{2}{3}y\dot{x}$, that is, $y\dot{x} = 2x\dot{y}$, or $\frac{\dot{x}}{2x} = \frac{\dot{y}}{y}$; and, by integra-

tion,

tion, as before, it is $l\sqrt{ax} = ly$, $ax = yy$. The curve is the ſame *Apollonian* parabola.

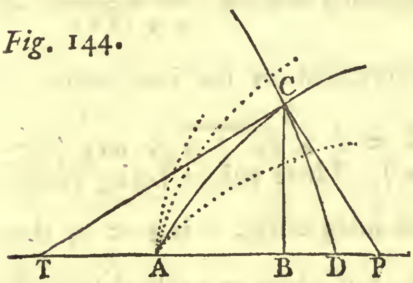
Let the area be equal to the fourth power of the ordinate, divided by a conſtant ſquare; then it will be $\int y\dot{x} = \frac{y^4}{aa}$, that is, $y\dot{x} = \frac{4y^3\dot{y}}{aa}$, or $aa\dot{x} = 4yy\dot{y}$; and, by integration, $\frac{3}{4}aax = y^3$, the firſt cubic parabola.

Let the area be equal to the power denoted by m of the ordinate, divided by a conſtant; it will be $\int y\dot{x} = \frac{y^m}{a^{m-2}}$, that is, $y\dot{x} = \frac{my^{m-1}\dot{y}}{a^{m-2}}$, a curve of the parabolic or hyperbolic kind, according as $m - 1$ ſhall be poſitive or negative.

PROBLEM III.

42. In infinite number of parabolas being given, of any the ſame kind; to find what that curve is, which cuts them all at right angles.

Fig. 144.

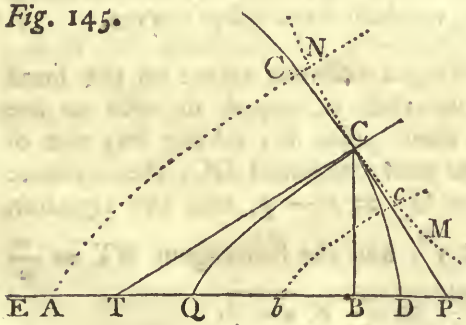


Let the equation of the curve required be $p^{m-n}x^n = y^m$, which, (p being conſidered as arbitrary, and ſuſceptible of infinite values,) expreſſes infinite parabolas; and (conſidering m and n in the ſame manner,) expreſſes any kind of parabolas. And, firſt, let them all belong to the ſame axis AB, (Fig. 144.) with vertex A, and different only in their parameters. Let AC be one of theſe infinite parabolas, in which $AB = x$, $BC = y$.

From any point C let the tangent CT be drawn, and the normal CP. It is known already, that it will be $BT = \frac{mx}{n}$. Let DC be the curve required; and, becauſe this ought to cut the parabola perpendicularly in the point C, in an infiniteſimal portion it muſt coincide with the normal CP in the point C. Therefore CT, the tangent of the parabola AC, will be likewiſe perpendicular to the curve DC in the point C, and conſequently, at the ſame time, BT will be both a ſubtangent to the parabola, and a ſubnormal of the curve required, DC. What is ſaid of the parabola AC agrees with any other of the ſame kind. Therefore the problem conſiſts in finding, of what kind is the curve DC, DC, whofe fubnormal is $= \frac{mx}{n}$. Now the general expreffion of the fubnormal is $\frac{y\dot{y}}{\dot{x}}$, which, in this cafe, ought to be taken negative, becaufe, in the curve DC, as AB, or x, increafes, at the fame time BC, or y, decreafes; and therefore the differential equation will be $\frac{mx}{n} = -\frac{y\dot{y}}{\dot{x}}$; and, feparating the variables, $\frac{mx\dot{x}}{n} = -y\dot{y}$; and, by integration, $\frac{mxx}{2n} = -\frac{1}{2}yy + aa$, or $\frac{nyy}{m} = \frac{2naa}{m} - xx$, which is an equation to the ellipfis. And, becaufe the parameter p does not at all enter here, the folution will be general for the infinite parabolas that may be thus defcribed.

If the exponent n of the equation $p^{m-n}x^n = y^m$ is fuppofed to be negative, fo that the equation may be $x^n y^m = p^{m+n}$, in which now it is pofitive; it will belong to infinite hyperbolas of the fame kind between the afymptotes, the fubtangents of which are $-\frac{mx}{n}$, and the fubnormal of the curve DC ought alfo to be equal to thefe. Then it will be $-\frac{mx}{n} = -\frac{y\dot{y}}{\dot{x}}$, or $\frac{mx\dot{x}}{n} = y\dot{y}$. And, by integration, $\frac{mxx}{2n} = \frac{1}{2}yy + aa$, or $xx - \frac{2naa}{m} = \frac{nyy}{m}$, an equation to the hyperbola.

Fig. 145.



If the infinite parabolas AC, QC, &c. of the equation $p^{m-n}z^n = y^m$, fhall have all the fame parameter, but each a different vertex in the fame axis; that is to fay, if one of them be conceived to move always upon the axis parallel to itfelf; from a fixed point A (Fig. 145.) making any abfcifs AB $= x$, and taking any curve QC, whofe abfcifs is QB $= z$, and ordinate BC $= y$; then will alfo $-\frac{y\dot{y}}{\dot{x}}$ be the fubnormal of the curve DC required, and therefore equal to the fubtangent BT of the parabola QC. Whence the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{mz}{n}$; but, by the equation of the parabola we have $z = \frac{y^{\frac{m}{n}}}{p^{\frac{m-n}{n}}}$, and therefore $-\frac{y\dot{y}}{\dot{x}} =$

$\frac{my^{\frac{m}{n}}}{np^{\frac{m-n}{n}}}$, that is, $\dot{x} = -\frac{n}{m}p^{\frac{m-n}{n}}y^{1-\frac{m}{n}}\dot{y}$; and, by integration, $x =$

$-\frac{nnp^{\frac{m-n}{n}}y^{\frac{2n-m}{n}}}{m \times 2n-m}$, the equation of the curve required, DC.

If the parabolas are the *Apollonian*, that is, $m = 2$, $n = 1$, the integrated equation would not be of uſe in this caſe; for, making the ſubſtitutions of the values of m and x, we ſhould have $x = -\frac{p}{0}$. But, taking the differential equation, it would be $\dot{x} = -\frac{1}{2}p \times \frac{\dot{y}}{y}$, an equation to the logarithmic. Therefore the curve which cuts the infinite *Apollonian* parabolas at right angles will be the logarithmic MCN, the ſubtangent of which is equal to half the parameter of the parabola.

Let the parabolas be the firſt cubics, that is, $m = 3$, $n = 1$; it will be $x = -\frac{ppy^{-1}}{-3}$, or $xy = \frac{1}{3}pp$, and the curve DC will be the hyperbola between it's aſymptotes.

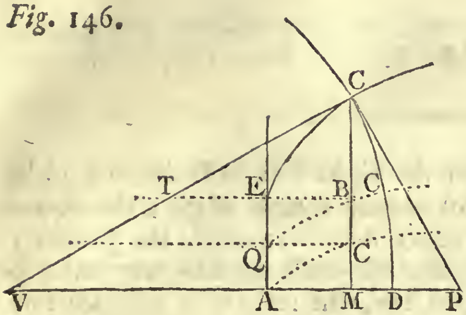
Let the parabolas be the ſecond cubics, that is, $m = 3$, $n = 2$; it will be $x = -\frac{4}{3}\sqrt{py}$, or $xx = \frac{16}{9}py$, and the curve DC will be the common parabola. Taking other values for m and n, we ſhall have other curves.

If the parabolas AC, QC, &c. beſides having a different vertex on the ſame axis, ſhould have their parameter variable, that is, equal in each to the reſpective diſtances of the vertex from the fixed point E; taking any one of them, QC, make $EB = x$ the abſciſs of the curve required DC, the ordinate $BC = y$, $EQ = p =$ parameter; it will be $QB = x - p$, and the equation of the infinite parabolas $p^{m-n} \times \overline{x-p}^{\,n} = y^m$, and the ſubtangent $BT = \frac{m}{n} \times \overline{x-p}$, and therefore the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{m}{n} \times \overline{x-p}$.

If the parabolas be *Apollonian*, that is, $m = 2$, $n = 1$, it will be $p = \frac{1}{2}x \pm \sqrt{\frac{1}{4}xx - yy}$; whence, making the ſubſtitutions in the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{m}{n} \times \overline{x-p}$, it will be $-\frac{y\dot{y}}{\dot{x}} = x \mp 2\sqrt{\frac{1}{4}xx - yy}$, which may be reduced to a ſeparation of the indeterminates by the method of § 14; and then we may go on to the integral, which will be algebräical.

If

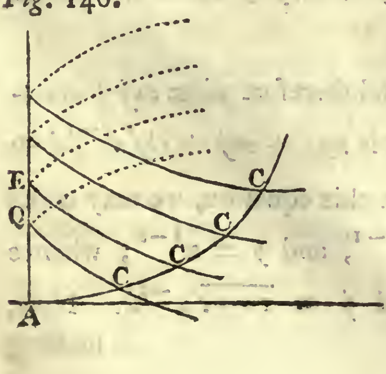
Fig. 146.



If the infinite parabolas AC, QC, &c. of the equation $p^{m-n}z^n = y^m$ ſhall have the ſame conſtant parameter, the axes parallel, and the vertices variable in the perpendicular to the axes; that is to ſay, if one of them be moved in ſuch a manner, as that every one of it's points may deſcribe perpendiculars to the axes: Taking any one of them, EC, (Fig. 146.) and calling AM = EB = z, BC = y, MC = x; and, drawing to the parabola EC the tangent CT, produced to V, then MV will be the ſubnormal of the curve DC required. Now, becauſe it is BT = $\frac{mz}{n}$, it will be MV = $\frac{mzx}{ny}$; whence we ſhould have the equation $\frac{mzx}{ny} = -\frac{x\dot{x}}{\dot{z}}$; and, inſtead of y, ſubſtituting it's value $p^{\frac{m-n}{m}} z^{\frac{n}{m}}$, given by the equation $p^{m-n}z^n = y^m$, it will be, finally, $\frac{mzx}{np^{\frac{m-n}{m}} z^{\frac{n}{m}}} = -\frac{x\dot{x}}{\dot{z}}$, that is, $-\frac{mz\dot{z}}{np^{\frac{m-n}{m}} z^{\frac{n}{m}}} = \dot{x}$, and, by integration, $x = -\frac{mmz^{\frac{2m-n}{m}}}{n \times \overline{2m-n} \times p^{\frac{m-n}{m}}}$, the equation of the curve required, DC.

Let the parabolas be the *Apollonian*, that is, $m = 2$, $n = 1$; it will be $x = -\frac{4z^{\frac{3}{2}}}{3p^{\frac{1}{2}}}$, or $\frac{9}{16}pxx = z^3$; and therefore the curve DC will be the ſecond cubic parabola, of which the *latus rectum* will be to that of the parabola AC as 9 to 16.

Fig. 146.



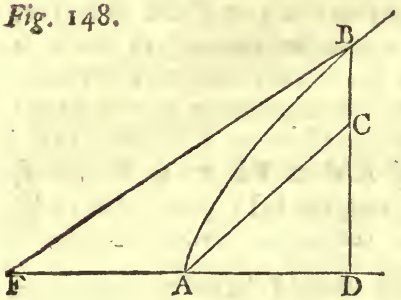
It is to be obſerved, that, in this caſe, the poſition of the curve DC will not be that marked in Fig. 146, but will have it's vertex in A, cutting the inferior part of the *Apollonian* parabola at right angles; that is, meeting the convexity, as in Fig. 147.

Another kind being pitched upon for the parabolas AC, alſo the curve DC will be a parabola of another kind.

PRO-

PROBLEM IV.

Fig. 148.



43. Upon the right line AD let the right line AC insist at half a right angle; the equation of the curve AB is required, the property of which is, that the ordinate BD may have to the subtangent DF, the ratio of a constant line a, to BC.

Make AD $= x$, DB $= y$; it will be CB $= y - x$. Whence, by the condition of the problem, we shall have $y \,.\, \frac{y\dot{x}}{\dot{y}} :: a \,.\, y - x$; and therefore the equation $a\dot{x} = y\dot{y} - x\dot{y}$. Now, to separate the indeterminates, I make use of the method of § 23. Wherefore, putting $x = Ay + p + B$, and $\dot{x} = A\dot{y} + \dot{p}$; and, making the substitutions, it will be $aA\dot{y} + a\dot{p} = y\dot{y} - Ay\dot{y} - p\dot{y} - B\dot{y}$. Now, in this equation, the indeterminates will be separated, if the first and second terms of the *homogeneum comparationis* be made to vanish; that is, if A $= 1$, and B remains arbitrary, which, for brevity-sake, I will make B $= 0$. Therefore the substitutions to be made will be $x = y + p$, $\dot{x} = \dot{y} + \dot{p}$, and the equation will be $a\dot{p} = -a\dot{y} - p\dot{y}$, that is, $\frac{a\dot{p}}{a + p} = -\dot{y}$, a transcendent curve, and which depends on the logarithmic.

PROBLEM V.

44. To find the curve, the area of which is $axy + bx^c y^e$; where the abscis is x, and the ordinate y, as usual.

Therefore it ought to be $\int y\dot{x} = axy + bx^c y^e$; and therefore $y\dot{x} = ax\dot{y} + ay\dot{x} + cby^e x^{c-1}\dot{x} + ebx^c y^{e-1}\dot{y}$; or, making $a - 1 = m$, it is $my\dot{x} + ax\dot{y} + cby^e x^{c-1}\dot{x} + ebx^c y^{e-1}\dot{y} = 0$. To separate the indeterminates in this equation, we may make use of the method of § 33, putting $x = u^{e-1} z^{e-1}$, and $y = z^{1-c}$; whence $\dot{x} = \overline{e-1}\, z^{e-1} u^{e-2}\dot{u} + \overline{e-1}\, u^{e-1} z^{e-2}\dot{z}$, and $\dot{y} = \overline{1-c}\, z^{-c}\dot{z}$. Now, making

making the ſubſtitutions, we ſhould obtain an equation much compounded, and which would require a very long calculation.

To come, then, to the point with brevity; reſuming the equation $\int y\dot{x} = bx^c y^e + axy$, put $x^c y^e = q$, whence the equation will be $\int y\dot{x} = bq + axy$, and therefore $y\dot{x} = b\dot{q} + ax\dot{y} + ay\dot{x}$. This ſuppoſed, I make uſe of the method of § 24, in the form of which I write the equation thus, $axy \times \overline{\frac{1-a}{a} \times \frac{\dot{x}}{x} - \frac{\dot{y}}{y}} = b\dot{q}$; then I put $\frac{1-a}{a} \times \frac{\dot{x}}{x} - \frac{\dot{y}}{y} = \frac{\dot{p}}{p}$; and then integrating, it will be $\frac{1-a}{a} l \frac{x}{y} = lp$, or $\frac{x^{\frac{1-a}{a}}}{y} = p$. Wherefore, making the neceſſary ſubſtitutions, we ſhall have the equation $\frac{ax^{\frac{1}{a}}\dot{p}}{pp} = b\dot{q}$. Now, to expreſs the quantity $x^{\frac{1}{a}}$ by the aſſumed quantities p, q, we muſt conſider, that $x^c y^e = q$, that is, $y^e = \frac{q}{x^c}$, or $y = \frac{q^{\frac{1}{e}}}{x^{\frac{c}{e}}}$. But we have alſo $\frac{x^{\frac{1-a}{e}}}{p} = y$; therefore $\frac{x^{\frac{1-a}{a}}}{p} = \frac{q^{\frac{1}{e}}}{x^{\frac{c}{e}}}$, or $x^{\frac{e-ae+ac}{ae}} = q^{\frac{1}{e}}p$; and, laſtly, $x^{\frac{1}{a}} = q^{\frac{1}{e-ae+ac}} \times p^{\frac{e}{e-ae+ac}}$. Then, making this ſubſtitution inſtead of $x^{\frac{1}{a}}$, we ſhall have the equation $ap^{\frac{e}{e-ae+ac}-2}\dot{p} = \frac{b\dot{q}}{q^{\frac{1}{e-ae+ac}}}$; that is, $ap^{\frac{2ae-2ac-e}{e-ae+ac}}\dot{p} = bq^{\frac{-1}{e-ae+ac}}\dot{q}$; and, by integration, $\frac{ae-aae+aac}{ae-ac} \times p^{\frac{ae-ac}{e-ae+ac}} = \frac{be-bae+bac}{e-ae+ac-1} \times q^{\frac{e-ae+ac-1}{e-ae+ac}} + g$; which is the equation of the curve required.

It is plain that this curve will be algebräical, at leaſt when the quantities a, c, e, ſhall be rational; and, on the contrary, it will be tranſcendental when one of theſe ſhall be irrational. I ſay at leaſt, becauſe, making a, c, e, rational, the curve, however, will be tranſcendental if $e = c$; or if $a = \frac{1-e}{c-e}$; or if $c = 1$, and at the ſame time $a = 1$; or $a = 0$, and alſo $e = 1$. And in ſeveral other caſes, which it is not neceſſary to enumerate.

SECT. IV.

Of the Reduction of Fluxional Equations, of the Second Degree, &c.

45. When the differential equations of the second degree are ſuch, that the rules here explained for integrations may be adapted to them, as well in caſes of ſeparate variables, as in thoſe that are mixed; nothing elſe remains to be done, but to apply the ſaid rules, and thus, by means of integration, to reduce them to firſt differentials; therefore there is no need to add any thing further about this matter. If, after the formulæ thus reduced to the firſt degree, the indeterminates will not then be ſeparable, as is often the caſe, nor ſhall be in any wiſe conſtructible; it is not the method that is in fault, by which the ſecond differences are reſolved, but rather that by which the firſt differences are managed.

Therefore we ought to employ our induſtry about the reduction of the differentio-differential equations, that, by the rules already taught, they may be made fit for integration, which may be attempted ſeveral ways.

46. One way will be, to make uſe of the common expedients of vulgar Algebra, by tranſpoſing the terms, by multiplying or dividing them by ſome quantity, and ſuch like. But, firſt, before any other thing, it is neceſſary to recollect, or to know, if, from paſſing from firſt to ſecond fluxions, there be any fluxion that was taken for conſtant, and what it was. And beſides, that as, in the integration of firſt differences to finite quantities, there is always added ſome conſtant quantity; ſo, likewiſe, in the integrations of ſecond to firſt differences, ſome conſtant quantities ſhould be added. This ſuppoſed, let us proceed to ſome Examples.

EXAMPLE I.

Let this equation be propofed, $\frac{by^m}{c^m} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{\dot{u}\dot{y}}$, in which $\dot{u} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ is the element of a curve, and is fuppofed conftant. I write it thus, $\frac{by^m\dot{y}\dot{u}}{c^m} = 2ay\ddot{x} + a\dot{x}\dot{y}$.

As $\dot{u}$ is conftant, the firft member will be integrable, even though it fhould be multiplied or divided by any function of y; and I obferve, that the fecond would be fo alfo, if it were divided by $2\sqrt{y}$. Therefore I divide the whole equation by $2\sqrt{y}$, and it will be $\frac{by^m\dot{y}\dot{u}}{2c^m\sqrt{y}} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{2\sqrt{y}}$; and, by integration, it will be $\frac{by^{m+\frac{1}{2}}\dot{u}}{\overline{m+\frac{1}{2}} \times 2c^m} = a\dot{x}\sqrt{y} + a\dot{u}\sqrt{a}$, which equation is now reduced to firft fluxions.

In the integration I have added $\dot{u}$ for this reafon, becaufe it is conftant; and I have multiplied it by $a\sqrt{a}$, to preferve the law of homogeneity.

EXAMPLE II.

Let the equation be $f = \frac{\dot{x}\dot{x} - y\ddot{y}}{y^3\dot{x}\dot{x}}$, in which $y\dot{x}$ is taken for a conftant. I multiply it by $2\dot{y}$, and it will be $2f\dot{y} = \frac{2\dot{x}\dot{x}\dot{y} - 2y\dot{y}\ddot{y}}{y^3\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2\dot{y}}{y^3} - \frac{2\dot{y}\ddot{y}}{yy\dot{x}\dot{x}}$; and, by integration, becaufe of $y\dot{x}$ being conftant, it will be $\int 2f\dot{y} = -\frac{1}{yy} - \frac{\dot{y}\dot{y}}{yy\dot{x}\dot{x}} + nyy\dot{x}\dot{x}$.

EXAMPLE III.

Let the equation be $f = \frac{\dot{u}\dot{u} - \dot{y}\ddot{y}}{y^3\dot{x}\dot{x}}$, in which let $\dot{x}$ be conſtant, and $\dot{u}$ the element of a curve, that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{u}$. Therefore, becauſe $\dot{x}$ is conſtant, it will be $\dot{y}\ddot{y} = \dot{u}\ddot{u}$; and therefore, ſubſtituting the value of $\ddot{y}$ in the equation, it will be $f = \frac{\dot{y}\dot{u}\dot{u} - \dot{y}\dot{u}\ddot{u}}{y^3\dot{y}\dot{x}\dot{x}}$; and, multiplying by $2\dot{y}$, it is $2f\dot{y} = \frac{2\dot{y}\dot{y}\dot{u}\dot{u} - 2\dot{y}\dot{y}\dot{u}\ddot{u}}{y^3\dot{y}\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2y\dot{y}\dot{u}\dot{u} - 2yy\dot{u}\ddot{u}}{y^4\dot{x}\dot{x}}$; and, by integration, $2\int f\dot{y} = -\frac{\dot{u}\dot{u}}{yy\dot{x}\dot{x}} + n\dot{x}\dot{x}$.

Again, after another manner. Inſtead of $\dot{u}$, putting it's value in the equation, it will be $f = \frac{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}{y^3\dot{x}\dot{x}}$; and, multiplying by $2y\dot{y}$, it is $2fy\dot{y} = \frac{2y\dot{y}\dot{x}\dot{x} + 2y\dot{y}^3 - 2yy\dot{y}\ddot{y}}{y^3\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2y\dot{y}\dot{x}\dot{x} + 2y\dot{y}^3 - 2yy\dot{y}\ddot{y}}{y^4\dot{x}\dot{x}}$; and, by integration, $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} \pm n\dot{x}\dot{x}$.

EXAMPLE IV.

Let the equation be $a\dot{x} = \frac{xy\ddot{y} + x\dot{y}\dot{y}}{\dot{x}}$, in which let $\dot{x}$ be conſtant. Multiplying by $\dot{x}$, and dividing by x, it will be $\frac{a\dot{x}\dot{x}}{x} = y\ddot{y} + \dot{y}\dot{y}$; and, by integration, becauſe $\dot{x}$ is conſtant, it is $a\dot{x}lx + A\dot{x} = y\dot{y}$. Now, if we ſhould make the aſſumed conſtant $A = a$, we ſhould have $a\dot{x}lx + a\dot{x} = y\dot{y}$; and, proceeding to integration, $axlx = \frac{1}{2}yy$.

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EXAMPLE V.

Let the equation be $f = \frac{\dot x\ddot y\dot u\dot u + \dot y\dot u\dot u\ddot x - \dot y\dot x\dot u\ddot u}{y\dot x\dot y\dot t\dot t}$, in which $\dot u$ is the little arch or element of a curve, $\dot t$ is given by x and y, and no firſt fluxion is yet taken for conſtant. I divide it by $y^3\dot x^3$, and multiply it by 2, and it will be $\frac{2f}{y^3\dot x^3} = \frac{2\dot x\ddot y\dot u\dot u + 2\dot y\dot u\dot u\ddot x - 2\dot y\dot x\dot u\ddot u}{y^4\dot x^4\dot y\dot t\dot t}$, or $\frac{2f\dot y\dot t\dot t}{yy\dot x\dot x} = \frac{2y\dot x\dot x\ddot y\dot u\dot u + 2yy\dot u\dot u\dot x\ddot x - 2yy\dot x\dot x\dot u\ddot u}{y^4\dot x^4}$; and, by integration, $2\int\frac{f\dot y\dot t\dot t}{yy\dot x\dot x} = -\frac{\dot u\dot u}{yy\dot x\dot x} \pm n$.

But it may truly be ſaid to be a thing impoſſible, to make uſe of this method in ſuch equations, in which the quantities are intricate and compounded, when we do not know the integrations pretty nearly before-hand, which we are to make. Wherefore I ſhall go on to other methods.

47. In the ſolution of problems, when we are to proceed from firſt to ſecond fluxions, it may be much more convenient not to aſſume any fluxion for conſtant, though we are at liberty to do it : that we may be able the better, when the formula is under our inſpection, to determine that to be ſuch conſtant, by which the expreſſion may be much abbreviated, and moſt readily integrable. The Examples will beſt make this method to be underſtood.

EXAMPLE I.

Let the equation be $f = \frac{\dot y^3 + \dot x^2\dot y - x\dot y\ddot x + x\dot x\ddot y}{2x^3\dot y^3}$, which may ariſe without having taken any fluxion for conſtant. To ſhorten this formula, I conſider, what may be that fluxion which, taken for conſtant, will deſtroy two terms of the *homogeneum comparationis*, and leave only two in the equation; and I find there may be two, that is, $x\dot y$ and $\frac{\dot x}{x}$. Therefore make $x\dot y = c$, and taking the difference, it is $x\ddot y + \dot x\dot y = 0$. Then multiplying by $\dot x$, it is $x\dot x\ddot y + \dot x\dot x\dot y = 0$, by which means the ſecond and fourth terms of the *homogeneum* diſappear out of the principal equation, ſo that we ſhall have $f = \frac{\dot y^3 - x\dot y\ddot x}{2x^3\dot y^3}$. But, as it is

$x\ddot{y} + \dot{x}\dot{y} = 0$, it will be $\dot{y} = -\frac{x\ddot{y}}{\dot{x}}$; whence, by fubftitution, $f = -\frac{x\dot{y}\dot{y}\ddot{y}}{2x^3\dot{x}\dot{y}^3}$ $- \frac{x\dot{y}\ddot{x}}{2x^3\dot{y}^3}$, that is, $f = -\frac{x\dot{y}\dot{y}\ddot{y} + x\dot{x}\dot{y}\ddot{x}}{2x^3\dot{x}\dot{y}^3}$, or $f = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2x^2\dot{x}\dot{y}^2}$. But $x\dot{y} = c$, and therefore $f = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2cc\dot{x}}$; and, laftly, $f\dot{x} = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2cc}$; and, by integration, $\int f\dot{x} = -\frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{4cc} \pm n$, or $\int f\dot{x} = -\frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$. When I came to the equation $f = \frac{\dot{y}^3 - x\dot{y}\ddot{x}}{2x^3\dot{y}^3}$, we might more briefly have gone on to the integration, by multiplying by $\dot{x}$, and difpofing it thus, $f\dot{x} = \frac{\dot{x}}{2x^3} - \frac{\dot{x}\ddot{x}}{2xx\dot{y}\dot{y}}$, where, becaufe $x\dot{y}$ is conftant, it will be $\int f\dot{x} = -\frac{1}{4xx} - \frac{\dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$, as before.

Now let us make conftant the quantity $\frac{\dot{x}}{x}$. Such a fuppofition giving $\frac{x\ddot{x} - \dot{x}\dot{x}}{xx} = 0$, and alfo $-x\dot{y}\ddot{x} + \dot{x}\dot{x}\dot{y} = 0$, takes away the fecond and third terms from the principal equation, and changes it into this, $f = \frac{\dot{y}^3 + x\dot{x}\ddot{y}}{2x^3\dot{y}^3}$; and, multiplying by $\dot{x}$, it is $f\dot{x} = \frac{\dot{x}\dot{y}^3 + x\dot{x}^2\ddot{y}}{2x^3\dot{y}^3}$, the integral of which, (becaufe of $\frac{\dot{x}}{x}$, or $\frac{\dot{x}\dot{x}}{xx}$ conftant,) will be found to be $\int f\dot{x} = -\frac{1}{4xx} - \frac{\dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$, as above.

48. But, to know nearly what fluxion may be taken for conftant, it may be obferved, if, in the propofed equation, there be two, three, or more terms, which, being multiplied or divided by a quantity which is common to them, they may be reduced to be integrable; then making the integration, their integral may be taken as conftant, and fo proceed in the manner fpecified. If not always, yet fometimes, at leaft, we fhall fucceed in our attempt.

I refume the equation $f = \frac{\dot{y}^3 + \dot{x}^2\dot{y} - x\dot{y}\ddot{x} + x\dot{x}\ddot{y}}{2x^3\dot{y}^3}$, and obferve, that the two terms $\dot{x}^2\dot{y} + x\dot{x}\ddot{y}$, being divided by $\dot{x}$, will become $\dot{x}\dot{y} + x\ddot{y}$, which is an integrable quantity, and that it's integral is $x\dot{y}$. See, then, upon what account we may take this quantity for conftant. In like manner, I obferve, that the two terms $\dot{x}^2\dot{y} - x\dot{y}\ddot{x}$, if they be divided by $-xx\dot{y}$, will give us $\frac{-\dot{x}\dot{x} + x\ddot{x}}{xx}$, an integrable quantity, the integral of which is $\frac{\dot{x}}{x}$; therefore the fluxion $\frac{\dot{x}}{x}$ might alfo be taken as conftant.

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For example, let the formula $xy \times \overline{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = y\dot{y}\dot{x}^2 - y^2\dot{z}\dot{y}^2 - x\dot{x}\dot{y}^2$ be propoſed, in which the variable z is any how given by y. I diſpoſe it thus, $xy\dot{x}\ddot{y} + yy\dot{z}\dot{y}^2 = yx\dot{y}\ddot{x} + y\dot{y}\dot{x}^2 - x\dot{x}\dot{y}^2$, and obſerve, that, if the *homogeneum comparationis* be divided by $yy\dot{y}$, it will be $\frac{yx\ddot{x} + y\dot{x}^2 - x\dot{x}\dot{y}}{yy}$, the integral of which is $\frac{x\dot{x}}{y}$. Therefore I take $\frac{x\dot{x}}{y}$ for conſtant, and make $\frac{x\dot{x}}{y} = c$, and thence $\frac{xy\ddot{x} + y\dot{x}^2 - x\dot{x}\dot{y}}{yy} = 0$. Whence the propoſed equation will become $xy\dot{x}\ddot{y} + yy\dot{z}\dot{y}^2 = 0$, that is, $\dot{z} = -\frac{x\dot{x}\ddot{y}}{yy\dot{y}^2}$; and, by integration, becauſe of $\frac{x\dot{x}}{y}$ conſtant, it will be $z = \frac{x\dot{x}}{y\dot{y}} \pm n$.

49. In an equation of the ſecond degree, when either of the two indeterminates are wanting with all it's functions, and only it's firſt or ſecond differences enter in the formula, any how compounded and raiſed to any dignity; the integration, or reduction to firſt fluxions, will always be in our power, by help of a ſubſtitution. This will be, to make the firſt fluxion, which is flowing or indeterminate, equal to a new variable multiplied into a conſtant aſſumed fluxion, or which may be aſſumed at pleaſure, in caſe that no other be appointed conſtant. For example, in a given equation, let $\dot{x}$, at firſt, be ſuppoſed variable, and $\dot{y}$ conſtant; make $\dot{x} = p\dot{y}$, and taking the fluxions, on the ſuppoſition of $\dot{y}$ being conſtant, it will be $\ddot{x} = \dot{p}\dot{y}$. Making this ſubſtitution inſtead of $\ddot{x}$, and the equation being managed by ſubſtituting the values taken from the equation $\dot{x} = p\dot{y}$, it will always be reduced to firſt fluxions.

Or, perhaps, it may be more convenient to make the firſt fluxion of the variable, which is wanting in the equation, equal to a new indeterminate, multiplied into the firſt fluxion of the other. Making the neceſſary ſubſtitutions, and having a due regard to the fluxion which, at firſt, was taken for conſtant, we ſhall have the propoſed equation reduced to firſt fluxions.

EXAMPLE I.

Let us take again the equation of the firſt example of § 46, $\frac{by^m}{c^m} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{\dot{u}\dot{y}}$, in which $\dot{u}$ is ſuppoſed conſtant. Make, therefore, $\dot{x} = p\dot{u}$, and by differencing, $\ddot{x} = \dot{p}\dot{u}$. Then, ſubſtituting this value, we ſhall have $\frac{by^m}{c^m} =$

$\frac{2ay\dot{p}\dot{u} + ap\dot{u}\dot{y}}{\dot{y}\dot{u}}$, that is, $\frac{by^m}{c^m} = \frac{2ay\dot{p} + ap\dot{y}}{\dot{y}}$, and therefore $\frac{by^m\dot{y}}{c^m} = 2ay\dot{p} + ap\dot{y}$, which equation, divided by $2\sqrt{y}$, is integrable, and the integral is $\frac{by^{m+\frac{1}{2}}}{m+\frac{1}{2} \times 2c^m} = ap\sqrt{y} \pm g$. But $p = \frac{\dot{x}}{\dot{u}}$, therefore $\frac{by^{m+\frac{1}{2}}\dot{u}}{m+\frac{1}{2} \times 2c^m} = a\dot{x}\sqrt{y} \pm g\dot{u}$.

EXAMPLE II.

Let the equation be $fyy\dot{y}\dot{x}\dot{x} = -\dot{u}\ddot{u}$, where f is given by y, $\dot{u}$ is the element of a curve, and $y\dot{x}$ is the fluxion taken for conſtant. Therefore I make $\dot{u} = py\dot{x}$, and, by differencing, it is $\ddot{u} = y\dot{p}\dot{x}$; and therefore, making the ſubſtitutions, it is $fy^2\dot{y}\dot{x}^2 = -y^2p\dot{p}\dot{x}^2$, that is, $f\dot{y} = -p\dot{p}$. Whence, by integration, $2\int f\dot{y} = -pp + 2m$. But $pp = \frac{\dot{u}\dot{u}}{yy\dot{x}\dot{x}} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}}$. Wherefore, making the ſubſtitutions and the reduction, we ſhall have $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

Now I reduce the ſame equation by means of the other ſubſtitution mentioned before. Make, therefore, $\dot{x} = p\dot{u}$, and $\ddot{x} = \dot{p}\dot{u} + p\ddot{u}$, whence $\ddot{u} = \frac{\ddot{x} - \dot{p}\dot{u}}{p}$. Making the ſubſtitutions, the equation will be $fyypp\dot{y}\dot{u}\dot{u} = \frac{-\dot{u}\ddot{x} + \dot{p}\dot{u}\dot{u}}{p}$. But the fluxion $y\dot{x}$ is aſſumed as conſtant, whence we ſhall have $y\ddot{x} + \dot{y}\dot{x} = 0$, that is, $\ddot{x} = -\frac{\dot{x}\dot{y}}{y}$, or $\ddot{x} = -\frac{p\dot{u}\dot{y}}{y}$. And, ſubſtituting this value again in the equation, it will be $fppyy\dot{y} = \frac{\dot{y}}{y} + \frac{\dot{p}}{p}$. This ſuppoſed, we may go on, and make $\frac{\dot{y}}{y} + \frac{\dot{p}}{p} = \frac{\dot{q}}{q}$, whence $py = q$, and therefore $fqq\dot{y} = \frac{\dot{q}}{q}$, or $f\dot{y} = \frac{\dot{q}}{q^3}$. And, by integration, $\int f\dot{y} = -\frac{1}{2qq} + m$. But $qq = ppyy = \frac{yy\dot{x}\dot{x}}{\dot{u}\dot{u}} = \frac{yy\dot{x}\dot{x}}{\dot{x}\dot{x} + \dot{y}\dot{y}}$. Therefore it will be $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} + 2m$; from whence we may derive, as above, $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

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EXAMPLE III.

I reſume the equation of Example III, § 46, $fy^3\dot{x}\dot{x} = \dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}$, in which $\dot{x}$ is conſtant; and make $\dot{y} = p\dot{x}$, and therefore $\ddot{y} = \dot{p}\dot{x}$. Making the ſubſtitutions, it will be $fy^3\dot{x}\dot{x} = \dot{x}\dot{x} + \dot{y}\dot{y} - y\dot{p}\dot{x}$; and, making $\dot{x}$ to vaniſh by it's value $\frac{\dot{y}}{p}$, we ſhall have $\frac{fy^3\dot{y}\dot{y}}{pp} = \frac{\dot{y}\dot{y}}{pp} + \dot{y}\dot{y} - \frac{y\dot{y}\dot{p}}{p}$; that is, $fy^3\dot{y}\dot{y} = \dot{y}\dot{y} + pp\dot{y}\dot{y} - yp\dot{y}\dot{p}$. And, dividing by $y^3\dot{y}$, it will be $f\dot{y} = \frac{\dot{y}}{y^3} + \frac{pp\dot{y} - yp\dot{p}}{y^3}$. And, by integration, $\int f\dot{y} = -\frac{1}{2yy} - \frac{pp}{2yy} + m$. And, inſtead of p, ſubſtituting it's value $\frac{\dot{y}}{\dot{x}}$, it is $\int f\dot{y} = -\frac{1}{2yy} - \frac{\dot{y}\dot{y}}{2yy\dot{x}\dot{x}} + m$, that is, $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} + 2m$; and therefore $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

50. If, in the propoſed equation, no fluxion has been taken for conſtant, one may be taken at pleaſure, and the operation may be performed, as is done at § 48.

As, for example, the equation of Example V, § 46, being given, in which no fluxion is aſſumed as conſtant, that is, $fy^3\dot{y}\dot{x}^3 = \dot{x}\dot{y}\dot{u}\ddot{u} + y\dot{u}\dot{u}\ddot{x} - y\dot{x}\dot{u}\ddot{u}$, (putting $y\dot{x}$ inſtead of $\dot{t}$,) if $\dot{x}$ be made conſtant, it will expunge the term $y\dot{u}\dot{u}\ddot{x}$, and the equation will become $fy^3\dot{y}\dot{x}^2 = \dot{y}\dot{u}^2 - y\dot{u}\ddot{u}$. Now, to reduce it, we muſt put $\dot{u} = p\dot{x}$, whence $\ddot{u} = \dot{p}\dot{x}$. Theſe values being ſubſtituted, we ſhall have $fy^3\dot{y}\dot{x}\dot{x} = pp\dot{y}\dot{x}\dot{x} - yp\dot{p}\dot{x}\dot{x}$, that is, $fy^3\dot{y} = pp\dot{y} - yp\dot{p}$; which equation, in order to proceed to integration, I write thus, $fy^3\dot{y} = ppy \times \overline{\frac{\dot{y}}{y} - \frac{\dot{p}}{p}}$. Therefore, integrating by the method of § 24 aforegoing, $\int f\dot{y} = -\frac{pp}{2yy} + m$; and, reſtoring the value of p, $\int f\dot{y} = -\frac{\dot{u}\dot{u}}{2yy\dot{x}\dot{x}} + m$.

If $\dot{u}$ be taken as conſtant, the term $y\dot{x}\dot{u}\ddot{u}$ will be expunged, and the equation will be $fy^3\dot{y}\dot{x}^3 = \dot{x}\dot{y}\dot{u}\ddot{u} + y\dot{u}\dot{u}\ddot{x}$, and therefore we muſt put $\dot{x} = p\dot{u}$, $\ddot{x} = \dot{p}\dot{u}$. Theſe values being ſubſtituted, we ſhall have $fy^3\dot{y} \times p^3\dot{u}^3 = p\dot{y}\dot{u}^3 + y\dot{p}\dot{u}^3$, that is, $fy^3\dot{y} = \frac{p\dot{y} + y\dot{p}}{p^3}$; then, by integration, it will be $\int f\dot{y} = -\frac{1}{2ppyy} + m$; and reſtoring the value of p, it will be $\int f\dot{y} = -\frac{\dot{u}\dot{u}}{2yy\dot{x}\dot{x}} + m$.

51. To aſſume at pleaſure any fluxion as conſtant, in equations wherein there is none already ſo taken, may make ſome equations ſubject to the method of § 49, which are not ſo already, becauſe of having both the indeterminates finite quantities. And this by aſſuming ſuch a fluxion for conſtant, as may make all the terms to vaniſh, in which is found one of the finite indeterminates, thoſe only remaining which include the other.

For example, let the equation be $\dot{x}^3 - \dot{x}\dot{y}\ddot{y} = y\dot{x}\ddot{x} + 2x\dot{y}\ddot{y}$, in which no fluxion is taken as conſtant. If we make $\dot{x}$ conſtant, the firſt term of the *homogeneum comparationis* will vaniſh; and if we make $\dot{y}$ conſtant, the laſt term will vaniſh; and, in either caſe, there remains only one of the indeterminates. Therefore, appointing $\dot{x}$ to be conſtant, the equation will be $\dot{x}^3 - \dot{x}\dot{y}\ddot{y} = 2x\dot{y}\ddot{y}$. Put $\dot{y} = \frac{p\dot{x}}{a}$, $\ddot{y} = \frac{\dot{p}\dot{x}}{a}$, and making the ſubſtitutions, it will be $\dot{x}^3 - \frac{pp\dot{x}^3}{aa} = \frac{2xp\dot{p}\dot{x}\dot{x}}{aa}$, that is, $aa\dot{x} - pp\dot{x} = 2xp\dot{p}$, or $\frac{\dot{x}}{x} = \frac{2p\dot{p}}{aa - pp}$; then, by integration, it will be $lx = -l\,\overline{aa - pp} + lm$, and therefore $x = \frac{m}{aa - pp}$. And, inſtead of p, reſtoring it's value $\frac{a\dot{y}}{\dot{x}}$, it will be $x = \frac{m}{aa - \frac{aa\dot{y}\dot{y}}{\dot{x}\dot{x}}}$, that is, $x = \frac{m\dot{x}\dot{x}}{aa\dot{x}^2 - aa\dot{y}^2}$, or $m\dot{x}^2 = aax\dot{x}^2 - a^2x\dot{y}^2$.

52. But when the taking at pleaſure a fluxion for conſtant, does not ſucceed in eliminating one of the two finite indeterminates, or if the conſtant fluxion be already fixed, ſo that both the indeterminates remain in the equation; there is no general method as yet diſcovered, how to proceed further.

The methods here explained may ſometimes have their uſe, as alſo the uſual expedients of common Algebra, ſuch as multiplication, diviſion, &c. As, for example, in the equation $xxy\dot{y}\ddot{y} = x\ddot{x} - \dot{x}\dot{x}$, which, being divided by xx, will be $y\dot{y}\ddot{y} = \frac{x\ddot{x} - \dot{x}\dot{x}}{xx}$, and therefore is integrable, (ſuppoſing $\dot{y}$ to be conſtant,) and the integral is $\frac{1}{2}yy\dot{y} = \frac{\dot{x}}{x} + m\dot{y}$.

Sometimes a ſubſtitution may make the propoſed equation within the reach of the method of § 49. And, indeed, the equation $x^m\dot{x} = y\ddot{y} + \dot{y}\dot{y} + yy\dot{y}\dot{y}$, which is not ſubject to the canon of the aforeſaid article, will however be ſo, if we make $y\dot{y} = \dot{z}$; whence it will be $x^m\dot{x} = \ddot{z} + z\dot{z}$.

53. Wherefore, in caſe that in the equation there ſhould be already a conſtant fluxion, it may be of good uſe to change the propoſed equation into another equivalent

equivalent to it, in which no fluxion is conſtant. To do which, let there be a general equation $\dot{y} = p\dot{x}$, where p is a quantity any how given by x and y, and let $\dot{x}$ be conſtant. By taking the difference, it will be $\ddot{y} = \dot{p}\dot{x}$. But it is $p = \frac{\dot{y}}{\dot{x}}$; then, by differencing, without making any conſtant fluxion, it will be $\dot{p} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}\dot{x}}$. Wherefore, the value of $\dot{p}$ being ſubſtituted in the equation $\ddot{y} = \dot{p}\dot{x}$, we ſhall have $\ddot{y} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$. So that, in any propoſed equation in which $\dot{x}$ is conſtant, inſtead of $\ddot{y}$, if we put it's value, $\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$, it will be changed into another that is equivalent to it, in which there is no conſtant fluxion.

But, becauſe often other more compound fluxions may be aſſumed as conſtant, or have been at firſt aſſumed, it may be of uſe to render this method more univerſal.

Let us take this general equation $\dot{y} = mp\dot{x}$, where p is likewiſe given, in any manner, by x and y, and m is any function whatever of x or of y, or of both together. Let $m\dot{x}$ be conſtant; then, by differencing, it will be $\ddot{y} = m\dot{x}\dot{p}$. But $p = \frac{\dot{y}}{m\dot{x}}$; and by differencing, without aſſuming any conſtant, it is $\dot{p} = \frac{m\dot{x}\ddot{y} - \dot{m}\dot{x}\dot{y} - m\dot{y}\ddot{x}}{mm\dot{x}\dot{x}}$. Wherefore, ſubſtituting this value in the equation $\ddot{y} = m\dot{x}\dot{p}$, inſtead of $\dot{p}$, we ſhall have $\ddot{y} = \frac{m\dot{x}\ddot{y} - \dot{m}\dot{x}\dot{y} - m\dot{y}\ddot{x}}{m\dot{x}}$. Wherefore in any propoſed equation, in which $m\dot{x}$ is conſtant, if, inſtead of $\ddot{y}$, we put it's value now found, it will be changed into another which is equivalent, in which no fluxion is conſtant.

After this manner equations being made complete, that is, ſuch as may have no conſtant fluxion, in proceeding to the reduction, we ſhall be at liberty to take that for conſtant, by the aſſiſtance of which we may beſt attain our purpoſe.

EXAMPLE I.

Let it be propoſed to reduce this equation, $x\dot{x}\dot{y} - \dot{y}^3 = a\dot{x}\ddot{y} + x\dot{x}\ddot{y}$, in which $\dot{x}$ is conſtant. Therefore, inſtead of $\ddot{y}$, putting it's value $\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$, (for in this case $m = 1$, and $\dot{m} = 0$,) it will be $\dot{x}\dot{x}\dot{y} - \dot{y}^3 = a\dot{x}\ddot{y} - a\dot{y}\ddot{x} + x\dot{x}\ddot{y} - x\dot{y}\ddot{x}$, in which no fluxion is constant. Whence, making $\dot{y}$ constant, it will be found to be $\dot{x}\dot{x} + x\ddot{x} + a\ddot{x} = \dot{y}\dot{y}$; and, by integration, $x\dot{x} + a\dot{x} = y\dot{y}$, which is an equation to the hyperbola.

EXAMPLE II.

Let the equation be $-\frac{x\dot{y}\dot{y} + xy\ddot{y} + y\dot{y}\dot{x}}{y\dot{y}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$, in which the fluxion $y\dot{x}$ is assumed as constant. To transform it into another, in which there is no constant fluxion, because in this case it is $m = y$, the value of $\ddot{y}$ to be substituted will be $\frac{y\dot{x}\ddot{y} - \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{y\dot{x}}$, and therefore the equation is $-\frac{x\dot{y}}{y} - \dot{x} - \frac{xy\dot{x}\ddot{y} - x\dot{x}\dot{y}\dot{y} - xy\dot{y}\ddot{x}}{y\dot{x}\dot{y}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$. To reduce this, making $x\dot{y}$ a constant fluxion, in consequence of which it will be $x\ddot{y} + \dot{x}\dot{y} = 0$, that is, $-\ddot{y} = \frac{\dot{x}\dot{y}}{x}$; then making the substitution, it is $-\frac{x\dot{y}}{y} - \dot{x} + \dot{x} + \frac{x\dot{y}}{y} + \frac{x\ddot{x}}{\dot{x}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$, that is, $-\frac{\ddot{x}}{\dot{x}} = \frac{xx\dot{x} - aa\dot{x}}{aax + x^3}$; and, by integration, $-l\dot{x} = l\frac{aa + xx}{x} - lx\dot{y}$. Here I subtract $lx\dot{y}$, because it is a constant quantity. And, taking away the logarithms, $\frac{1}{\dot{x}} = \frac{aa + xx}{xx\dot{y}}$, that is, $x^2\dot{y} = a^2\dot{x} + x^2\dot{x}$.

EXAMPLE III.

Let the equation be $-\frac{\dot{x}\ddot{y}}{\dot{y}} - \frac{\dot{y}\ddot{x}}{y} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{x}$, and $y\dot{x}$ a constant fluxion. Therefore, instead of $\ddot{y}$, I put it's corresponding value, $\frac{y\dot{x}\ddot{y} - \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{y\dot{x}}$, and it will be $-\frac{\dot{x}\ddot{y}}{\dot{y}} + \frac{\dot{y}\ddot{x}}{\dot{y}} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{x}$, in which there is no constant fluxion. Wherefore, taking $\dot{y}$ constant, it will be $x\ddot{x} = \dot{x}\dot{x} + \dot{y}\dot{y}$. Which equation is the case of § 49, and therefore it's reduction is known.

54. The

54. The method explained in the foregoing Section, at § 24, may be alſo of uſe in differentio-differential equations, by proceeding nearly in the manner there purſued. Here is the practice in ſome Examples.

EXAMPLE I.

I reſume the formula of the firſt Example of this Section, $\frac{by^m}{c^m} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{\dot{u}\dot{y}}$, in which $\dot{u} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ is aſſumed conſtant. It will be $\frac{by^m\dot{y}\dot{u}}{ac^m} = 2y\ddot{x} + \dot{x}\dot{y}$. I prepare it after the following manner, $\overline{\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y}} \times \dot{x} = \frac{by^m\dot{y}\dot{u}}{ac^m \times 2y}$, where I obſerve, that the two quantities under the vinculum are integrable, by means of the logarithms. Therefore I make $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y} = \frac{\dot{p}}{p}$, and therefore $l\dot{x} + l\sqrt{y} = lp + l\dot{u}$; (I add $l\dot{u}$, becauſe of $\dot{u}$ conſtant,) that is, $\dot{x}\sqrt{y} = p\dot{u}$. Wherefore, in the propoſed equation, inſtead of $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y}$, ſubſtituting it's value $\frac{\dot{p}}{p}$, and, inſtead of $\dot{x}$, it's value $\frac{p\dot{u}}{\sqrt{y}}$, it will be $\frac{\dot{p}\dot{u}}{\sqrt{y}} = \frac{by^{m-1}\dot{y}\dot{u}}{2ac^m}$, or $\dot{p} = \frac{by^{m-\frac{1}{2}}\dot{y}}{2ac^m}$; and, by integration, $h + p = \frac{by^{m+\frac{1}{2}}}{\overline{m+\frac{1}{2}} \times 2ac^m}$. But $p = \frac{\dot{x}\sqrt{y}}{\dot{u}}$, and therefore, laſtly, $h\dot{u} + \dot{x}\sqrt{y} = \frac{by^{m+\frac{1}{2}}\dot{u}}{\overline{m+\frac{1}{2}} \times 2ac^m}$, as in the Example quoted above.

EXAMPLE II.

Let the equation be $\frac{-\ddot{x}\sqrt{xx+yy}}{x} = \frac{\overline{y\dot{x} - x\dot{y}}|^2}{xx+yy}$, in which $y\dot{x} - x\dot{y}$ is conſtant. The ſecond fluxion $\ddot{x}$, divided by the conſtant $\dot{x}y - x\dot{y}$, will give us an integrable quantity, and therefore I write the equation thus, $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} =$

$\frac{x \times \overline{y\dot{x} - x\dot{y}}}{\overline{xx + yy} \times \sqrt{xx + yy}}$. But I obferve, that, in the fecond member, the quantity $y\dot{x} - x\dot{y}$ is fummable when it is divided by yy; therefore I prepare the equation according to this method, and it will be $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{xyy}{\overline{xx + yy} \times \sqrt{xx + yy}} \times \frac{y\dot{x} - x\dot{y}}{yy}$. Make $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{p}$, and, by integration, $\frac{x}{y} = p$. Whence, making the fubftitution, we fhall have $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{xyy\dot{p}}{\overline{xx+yy} \times \sqrt{xx+yy}}$, from whence we can expunge x or y, by means of the equation $\frac{x}{y} = p$. Expunge x from the fecond member, by putting it's value py in it's place, and we fhall have $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{p\dot{p}}{\overline{1 + pp} \times \sqrt{1 + pp}}$; and, proceeding to the integration, it will be $\frac{-\dot{x}}{y\dot{x} - x\dot{y}} = -\frac{1}{\sqrt{1 + pp}}$, that is, $\frac{-\dot{x}}{y\dot{x} - x\dot{y}} = \frac{-y}{\sqrt{1yy + xx}}$, inftead of p, by reftoring it's value $\frac{x}{y}$.

In this integration the conftant $y\dot{x} - x\dot{y}$ might have been added; but whether it be added or omitted, the reduction of firft differences to finite quantities, in each cafe, will always give the conic fections.

55. I faid before, at § 52, that when the differentio-differential equations contain both the variables, there is no general method to reduce them. One, however, may be affigned, which, though it does not ferve in all cafes, yet is very general in it's kind, and comprehends all the infinite number of equations, which may be referred to thefe three following canons. By the help of this method, the given equations are transformed into others, in which one of the two variables is wanting, and confequently they may be managed by the method of § 49.

The firft canon comprehends thofe which are of two terms only, and are expreffed by the general formula $ax^m\dot{x}^p = y^n\dot{y}^{p-2}\ddot{y}$, in which let $\dot{x}$ be taken as conftant. To reduce this equation, make $x = c^{bu}$, and $y = c^u t$, where c is a number, the logarithm of which is unity, and b is an arbitrary quantity to be determined afterwards, and u, t, are two new variables. Now, fince $x = c^{bu}$, and $y = c^u t$, by the rules of the exponential calculus it will be $\dot{x} = bc^{bu}\dot{u}$, $\ddot{x} = bc^{bu} \times \overline{\ddot{u} + b\dot{u}\dot{u}}$, $\dot{y} = c^u\dot{t} + c^u t\dot{u}$, $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{t}\dot{u} + t\dot{u}\dot{u} + t\ddot{u}}$.

But,

But, making $\dot{x}$ conſtant, it is $\ddot{x} = 0$, and therefore $hc^{hu} \times \overline{\ddot{u} + h\dot{u}\dot{u}} = 0$, or $\ddot{u} = -h\dot{u}\dot{u}$. This, being ſubſtituted, inſtead of $\ddot{u}$, in the value of $\ddot{y}$, will be $\ddot{y} = c^{u} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-h} \times t\dot{u}\dot{u}}$. In the propoſed equation, ſubſtituting the reſpective values inſtead of x, y, and their differentials, it will be changed into this other, $ac^{hmu} \times h^{p} \times c^{hpu}\dot{u}^{p} = c^{nu}t^{n} \times \overline{c^{u}\dot{t} + c^{u}t\dot{u}}^{p-2} \times c^{u} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-h} \times t\dot{u}\dot{u}}$, that is, $ac^{hu \times \overline{m+p}}h^{p}\dot{u}^{p} = c^{\overline{n+p-1} \times u}t^{n} \times \overline{\dot{t} + t\dot{u}}^{p-2} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-h} \times t\dot{u}\dot{u}}$.

Now, to free this equation from exponential quantities, that is, to take c out of it, it will be neceſſary that $n + p - 1 = hm + hp$, by which the value of the aſſumed quantity h will be determined, that is, $h = \frac{n+p-1}{m+p}$. Whence the equation will be $\frac{a \times \overline{n+p-1}^{p} \times \dot{u}^{p}}{\overline{m+p}^{p}} = t^{n} \times \overline{\dot{t} + t\dot{u}}^{p-2} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \frac{m-n+1}{m+p} \times t\dot{u}\dot{u}}$, which, becauſe it contains only one of the finite variables, that is, t, will now be ſubject to the above-cited rule.

Now, ſince we have found the value of $h = \frac{p+n-1}{p+m}$, it eaſily appears what ſubſtitutions might have been made at the beginning, that is, $x = c^{\frac{n+p-1}{m+p} \times u}$, and $y = c^{u}t$, in order to obtain our intention.

To go on with the operation according to the method of § 49, make $\dot{u} = z\dot{t}$, and therefore $\ddot{u} = z\ddot{t} + \dot{z}\dot{t}$. But the ſuppoſition of $\dot{x}$ conſtant has given us $\ddot{u} = -h\dot{u}\dot{u}$, that is, $\ddot{u} = \frac{1-n-p}{m+p} \times zz\dot{t}\dot{t}$. Therefore we ſhall have $\frac{1-n-p}{m+p} \times zz\dot{t}\dot{t} = z\ddot{t} + \dot{t}\dot{z}$, whence $\ddot{t} = \frac{1-n-p}{m+p} \times z\dot{t}\dot{t} - \frac{\dot{t}\dot{z}}{z}$. Wherefore, ſubſtituting in the equation their reſpective values, inſtead of $\dot{u}$ and $\ddot{t}$, it will be $a \times \frac{\overline{p+n-1}^{p}}{\overline{m+p}^{p}} \times z^{p}\dot{t}^{p} = t^{n} \times \overline{\dot{t} + zt\dot{t}}^{p-2} \times \overline{\frac{1-n-p}{p+m} \times z\dot{t}\dot{t} - \frac{\dot{t}\dot{z}}{z} + 2z\dot{t}\dot{t} + \frac{m-n+1}{m+p} \times zzt\dot{t}\dot{t}}$; or, dividing by $\dot{t}^{p-1}$, and multiplying by z, it will be $a \times \frac{\overline{n+p-1}^{p}}{\overline{m+p}^{p}} \times z^{p+1}\dot{t} = t^{n} \times \overline{1+tz}^{p-2}$

$\times$

$\times \overline{\frac{1 + 2m - n + p}{m + p} \times zz\dot{t} + \frac{m - n + 1}{m + p} \times tz^3\dot{t} - \dot{z}}$; which equation is now reduced to firſt fluxions. It is eaſy to perceive, that, to reduce the equation, it would be ſufficient to make $x = c^{\frac{n+p-1}{m+p} \times \int z\dot{t}}$, and $y = c^{\int z\dot{t}} \times t$.

In this general equation, which I have now reduced, I ſuppoſed the fluxion $\dot{x}$ to be conſtant; yet it would make no difficulty in the method, that, in any propoſed equation, ſome other fluxion different from $\dot{x}$ ſhould be made conſtant. For, by § 53, the propoſed equation may be changed into another equivalent to it, in which no fluxion is conſtant, and then the ſaid $\dot{x}$ may be made conſtant.

EXAMPLE I.

Let the equation be $x\dot{x}\dot{y} = y\ddot{y}$, in which $\dot{x}$ is conſtant. I write it thus, $\dot{x}x = y\dot{y}^{-1}\ddot{y}$. This being compared with the canonical equation, it will be $a = 1$, $m = 1$, $p = 1$, $n = 1$; whence, theſe values being ſubſtituted in the general differential equation of the firſt degree found above, we ſhall have $\frac{1}{2}zz\dot{t} = \frac{t}{1 + tz} \times \overline{\frac{3}{2}zz\dot{t} + \frac{1}{2}tz^3\dot{t} - \dot{z}}$.

EXAMPLE II.

Let $p = 1$, $n = -1$, $m = -1$, or the equation $ax^{-1}\dot{x} = y^{-1}\dot{y}^{-1}\ddot{y}$, or $\frac{a\dot{x}}{x} = \frac{\ddot{y}}{y\dot{y}}$, in which $\dot{x}$ is a conſtant fluxion. In reſpect of this, the method will be of no uſe, for we ſhall have $p + m = 0$, and conſequently every one of the terms of the general differential equation of the firſt degree, except the laſt, will be infinite.

But, in this caſe, the reduction is eaſy, without any further artifice. I write the equation thus, $x\ddot{y} = ay\dot{y}\dot{x}$. Now the integral of the firſt member is $x\dot{y} - y\dot{x}$, that of the ſecond is $\frac{1}{2}ayy\dot{x}$. Therefore the equation is $x\dot{y} - y\dot{x} = \frac{1}{2}ayy\dot{x} \pm b\dot{x}$.

56. The

56. The ſecond canon comprehends all thoſe equations, in which the ſum of the exponents of the indeterminates, and of their differentials, is the ſame in every term. Suppoſing x and y the two indeterminates, and $\dot{x}$ to be conſtant, theſe are reduced to the caſe of § 49, by putting $x = c^u$, and $y = c^u t$; c being ſtill the number, the logarithm of which is unity, and u, t, are new indeterminates. To ſhow the method, I ſhall take the equation $ax^m y^{-m-1}\dot{x}^p\dot{y}^{2-p} + bx^n y^{-n-1}\dot{x}^q\dot{y}^{2-q} = \ddot{y}$, which, though it be but of one dimenſion only, and of three terms only, yet the method is general notwithſtanding, and will ſerve for any number of terms and dimenſions, if the conditions be obſerved.

Therefore I make $x = c^u$, $y = c^u t$; it will be $\dot{x} = c^u \dot{u}$; and, becauſe $\dot{x}$ is conſtant, we ſhall have $c^u \ddot{u} + c^u \dot{u}\dot{u} = 0$, that is, $\ddot{u} = -\dot{u}\dot{u}$. It will be alſo $\dot{y} = c^u \dot{t} + c^u t\dot{u}$, and $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{u}\dot{t} + t\dot{u}\dot{u} + t\ddot{u}}$. But $\ddot{u} = -\dot{u}\dot{u}$; therefore $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{u}\dot{t}}$. Wherefore, theſe values being ſubſtituted in the propoſed equation, it will be $at^{-m-1}\dot{u}^p \times \overline{\dot{t} + t\dot{u}}|^{2-p} + bt^{-n-1}\dot{u}^q \times \overline{\dot{t} + t\dot{u}}|^{2-q} = \ddot{t} + 2\dot{u}\dot{t}$. And, becauſe in this the indeterminate u is wanting, we may proceed by the method of § 49.

Make $\dot{u} = z\dot{t}$; it will be $\ddot{u} = \dot{z}\dot{t} + z\ddot{t}$. But $\ddot{u} = -\dot{u}\dot{u} = -zz\dot{t}\dot{t}$; therefore $\ddot{t} = -\frac{\dot{z}\dot{t}}{z} - z\dot{t}\dot{t}$. Wherefore, ſubſtituting theſe values, we ſhall have $at^{-m-1}z^p\dot{t}^p \times \overline{\dot{t} + zt\dot{t}}|^{2-p} + bt^{-n-1}z^q\dot{t}^q \times \overline{\dot{t} + zt\dot{t}}|^{2-q} = -\frac{\dot{z}\dot{t}}{z} + z\dot{t}\dot{t}$, or $ct^{-m-1}z^p\dot{t} \times \overline{1 + zt}|^{2-p} + bt^{-n-1}z^q\dot{t} \times \overline{1 + zt}|^{2-q} = -\frac{\dot{z}}{z} + z\dot{t}$, a differential equation of the firſt degree. From hence it may be ſeen, that the propoſed equation might have been reduced at the beginning, by putting $x = c^{\int z\dot{t}}$, and $y = c^{\int z\dot{t}} t$.

For example, let the equation be $x\dot{x}\dot{y} - y\dot{x}\dot{x} = y\ddot{y}$. To bring this to the canonical equation, I write it thus, $xy^{-2}\dot{x}\dot{y} - y^{-1}\dot{x}\dot{x} = \ddot{y}$. Then it will be $a = 1$, $m = 1$, $p = 1$, $n = 0$, $b = -1$, $q = 2$. Wherefore, theſe values being ſubſtituted in the differential canonical equation, here before found, we ſhall have the equation reduced, $t^{-2}z\dot{t} \times \overline{1 + zt} - t^{-1}zz\dot{t} = -\frac{\dot{z}}{z} + z\dot{t}$; or $\frac{z\dot{t} + zzt\dot{t}}{tt} - \frac{zz\dot{t}}{t} = \frac{-\dot{z} + zz\dot{t}}{z}$, that is, $zz\dot{t} - zztt\dot{t} = -tt\dot{z}$.

If we proceed on to the integration, it will be $\frac{tt\dot{t} - \dot{t}}{tt} = \frac{\dot{z}}{zz}$, and therefore, by integrating, $t + \frac{1}{t} = -\frac{1}{z} + f$, (where f is a conſtant to complete the integral,) that is, $ttz + z = -t + ftz$. But, by the ſubſtitutions, $z = \frac{\dot{u}}{\dot{t}}$, $x = c^u$, $y = c^u t$, it will be $\dot{u} = \frac{\dot{x}}{x}$, $t = \frac{y}{x}$, $\dot{t} = \frac{x\dot{y} - y\dot{x}}{xx}$, and therefore $z = \frac{x\dot{x}}{x\dot{y} - y\dot{x}}$; wherefore, ſubſtituting the values of t and z, we ſhall have $\frac{x\dot{x} + y\dot{y}}{y\dot{x}} = f$.

57. The third canon comprehends all thoſe equations, in which one of the two variables, whatever it may be, together with it's differentials, always makes in every term the ſame number of dimenſions. But we muſt here diſtinguiſh two caſes. One is, when the differential of that variable is conſtant, which forms the ſame number of dimenſions. The other caſe is, when the differential of the other is conſtant.

As to the firſt caſe, let the canonical equation be $Px^m\dot{y}^{m+2} + Qx^{m-n}\dot{x}^n\dot{y}^{m+2-n} = \dot{x}^m\ddot{y}$, in which the ſum of the exponents of x and $\dot{x}$ is the ſame in every term. P and Q are any functions of y, and $\dot{x}$ is conſtant. To reduce this equation, make $x = c^u$, where alſo c is a number, the logarithm of which is unity, and u is a new variable. Therefore it will be $\dot{x} = c^u\dot{u}$; and differencing again, making $\dot{x}$ conſtant, it will be $c^u\ddot{u} + c^u\dot{u}\dot{u} = 0$, that is, $\ddot{u} = -\dot{u}\dot{u}$. Theſe values being ſubſtituted in the equation, we ſhall have $P\dot{y}^{m+2} + Q\dot{u}^n\dot{y}^{m+2-n} = \dot{u}^m\ddot{y}$, which, becauſe it does not contain u, will be under the canon of § 49.

Therefore I put $\dot{u} = z\dot{y}$, and it will be $\ddot{u} = \dot{z}\dot{y} + z\ddot{y}$; but $\ddot{u} = -\dot{u}\dot{u} = -z^2\dot{y}^2$; therefore we ſhall have $z\ddot{y} + \dot{z}\dot{y} = -zz\dot{y}\dot{y}$; and thence $\ddot{y} = \frac{-zz\dot{y}\dot{y} - \dot{z}\dot{y}}{z}$. Wherefore, theſe values of $\dot{u}$ and $\ddot{y}$ being ſubſtituted in the equation before found, it will be $P\dot{y}^{m+2} + Qz^n\dot{y}^{m+2} = -z^{m+1}\dot{y}^{m+2} - z^{m-1}\dot{y}^{m+1}\dot{z}$; and, dividing by $\dot{y}^{m+1}$, it will be $P\dot{y} + Qz^n\dot{y} = -z^{m+1}\dot{y} - z^{m-1}\dot{z}$, an equation of the firſt degree. Therefore we might at firſt have made $x = c^{\int z\dot{y}}$, and thus have reduced the equation at one ſtroke.

For example, let the equation be $2ax\dot{x}\dot{y} + ax\dot{x}\ddot{y} = 2x\dot{x}\dot{y}\dot{y} + 2xx\dot{y}\ddot{y}$, in which let $\dot{x}$ be constant. Put $x = c^{\int z\dot{y}}$, and therefore $\dot{x} = z\dot{y}c^{\int z\dot{y}}$, and $\ddot{x} = c^{\int z\dot{y}} \times \overline{z^2\dot{y}^2 + z\ddot{y} + \dot{y}\dot{z}}$. But $\dot{x}$ is constant, and therefore $zz\dot{y}\dot{y} + z\ddot{y} + \dot{y}\dot{z} = 0$, whence $\ddot{y} = \frac{-zz\dot{y}\dot{y} - \dot{z}\dot{y}}{z}$. Now, the values of x and $\dot{x}$ being substituted in the equation, we shall have $2az^2\dot{y}^3 + az\dot{y}\ddot{y} = 2z\dot{y}^3 + 2\dot{y}\ddot{y}$; and, substituting the value of $\ddot{y}$, it is $2az^2\dot{y}^3 + az\dot{y} \times \frac{-z^2\dot{y}^2 - \dot{z}\dot{y}}{z} = 2z\dot{y}^3 + 2\dot{y} \times \frac{-zz\dot{y}^2 - \dot{z}\dot{y}}{z}$, that is, dividing by $\dot{y}\dot{y}$, $az^3\dot{y} - az\dot{z} = -2\dot{z}$, or $a\dot{y} = \frac{az\dot{z} - 2\dot{z}}{z^3}$. And, by integration, $ay = -\frac{a}{z} + \frac{1}{zz}$. Lastly, restoring the value of z, which is given from the supposition made of $x = c^{\int z\dot{y}}$, that is, $z = \frac{\dot{x}}{x\dot{y}}$, we shall have the equation reduced, $ay\dot{x}\dot{x} = xx\dot{y}\dot{y} - ax\dot{x}\dot{y}$.

58. As to the second case, let the canonical equation be $\mathrm{P}x^m\dot{y}^{m+1} + \mathrm{Q}x^{m-n}\dot{x}^n\dot{y}^{m-n+1} = \dot{x}^{m-1}\ddot{x}$, in which let $\dot{y}$ be constant, and P, Q, any functions of y.

Put, as above, $x = c^u$, and therefore $\dot{x} = c^u\dot{u}$, $\ddot{x} = c^u\ddot{u} + c^u\dot{u}\dot{u}$. Make the substitutions in the canonical equation, and we shall have $\mathrm{P}\dot{y}^{m+1} + \mathrm{Q}\dot{u}^n\dot{y}^{m-n+1} = \dot{u}^{m+1} + \dot{u}^{m-1}\ddot{u}$, which, because it does not involve u, is subject to the canon of § 49. Therefore I put $\dot{u} = z\dot{y}$; and, as $\dot{y}$ is constant, it will be $\ddot{u} = \dot{z}\dot{y}$; and then making the substitutions, we shall have $\mathrm{P}\dot{y}^{m+1} + \mathrm{Q}z^n\dot{y}^{m+1} = z^{m+1}\dot{y}^{m+1} + z^{m-1}\dot{y}^m\dot{z}$; and, dividing by $\dot{y}^m$, it will be $\mathrm{P}\dot{y} + \mathrm{Q}z^n\dot{y} = z^{m+1}\dot{y} + z^{m-1}\dot{z}$, an equation of the first degree; which might have been reduced at once, by putting, as above, $x = c^{\int z\dot{y}}$.

For an example, let the equation be $2\dot{x}\dot{y} = a\ddot{x} - y\ddot{x}$, in which let $\dot{y}$ be constant. Therefore, putting $x = c^{\int z\dot{y}}$, thence $\dot{x} = z\dot{y} \times c^{\int z\dot{y}}$, and $\ddot{x} = c^{\int z\dot{y}} \times \overline{z^2\dot{y}^2 + z\ddot{y} + \dot{y}\dot{z}}$. But $\dot{y}$ is supposed constant, and therefore $\ddot{y} = 0$, and thence $\ddot{x} = c^{\int z\dot{y}} \times \overline{zz\dot{y}\dot{y} + \dot{z}\dot{y}}$. Wherefore, making the substitutions in the proposed equation, we shall have $2z\dot{y}\dot{y} = azz\dot{y}\dot{y} + a\dot{z}\dot{y} - zzy\dot{y}\dot{y} - y\dot{y}\dot{z}$; and, dividing by $\dot{y}$, it will be $2z\dot{y} = azz\dot{y} + a\dot{z} - zzy\dot{y} - y\dot{z}$, which is a differential equation of the first degree.

To go on to the integration, I divide the equation by $az - yz$, whence it is $\frac{2\dot{y}}{a-y} = z\dot{y} + \frac{\dot{z}}{z}$, or $\frac{2\dot{y}}{a-y} - \frac{\dot{z}}{z} = z\dot{y}$. And now, if you pleafe, making ufe of the method in § 24, by integrating, we fhall have $\frac{-1}{\overline{a-y}|^2 \times z} = \frac{-1}{a-y} + m$; and, laftly, by reftoring the value of $z = \frac{\dot{x}}{x\dot{y}}$, we fhall have the equation reduced, $y\dot{x} + x\dot{y} = a\dot{x}$, where the conftant m is neglected, which was introduced in the integration.

This example has ferved to fhow the application of the method; for otherwife fo many operations would have been unneceffary. Indeed, the equation itfelf, $2x\ddot{x} = a\ddot{x} - y\ddot{x}$, might have been reduced in an inftant, by only tranfpofing the term $y\ddot{x}$, and writing it thus: $2\dot{x}\dot{y} + y\ddot{x} = a\ddot{x}$; for, as $\dot{y}$ is conftant, the integral of the firft member is $y\dot{x} + x\dot{y}$, as plainly appears.

59. To what has been already faid, concerning differentio-differential equations, in which no firft fluxion was taken for conftant; another method may be added which is more univerfal, and which will ferve for all fuch as are comprehended under this canonical formula, $z^{m+1}\dot{x}^m\ddot{x} + \frac{\dot{z}}{z}\dot{y}^{m+1} = \dot{y}^m\ddot{y}$; in which z is any how given by the functions of x and y.

To reduce this, appoint the fluxion $\frac{\dot{x}}{q}$ for conftant, where q is any how given by the functions of x and y. Then make $\frac{\dot{x}}{q} = \dot{p}$. Now, becaufe $\frac{\dot{x}}{q}$ is conftant, it will be, by differencing, $q\ddot{x} - \dot{x}\dot{q} = 0$, that is, $\ddot{x} = \frac{\dot{x}\dot{q}}{q}$; or, inftead of $\frac{\dot{x}}{q}$, writing it's value $\dot{p}$, it will be $\ddot{x} = \dot{q}\dot{p}$. Befides, make $\dot{y} = u\dot{p}$, and taking the fecond fluxions, fuppofing $\dot{p}$ conftant, as being equal to $\frac{\dot{x}}{q}$, which is conftant, it will be $\ddot{y} = \dot{u}\dot{p}$. Therefore, in the canonical equation, fubftituting the values thus determined, inftead of $\dot{x}$, $\ddot{x}$, $\dot{y}$, and $\ddot{y}$, we fhall have the equation $z^{m+1}q^m\dot{q}\dot{p}^{m+1} + \frac{u^{m+1}\dot{z}\dot{p}^{m+1}}{z} = u^m\dot{u}\dot{p}^{m+1}$; and, dividing by $\dot{p}^{m+1}$, it will be $z^{m+1}q^m\dot{q} + \frac{u^{m+1}\dot{z}}{z} = u^m\dot{u}$, or $q^m\dot{q} = \frac{zu^m\dot{u} - u^{m+1}\dot{z}}{z^{m+2}}$. And, by integration, $\frac{q^{m+1}}{m+1} + g = \frac{u^{m+1}}{\overline{m+1} \times z^{m+1}}$, and therefore $u = z \times$

$\overline{q^{m+1} + \overline{m+1} \times g}^{\frac{1}{m+1}}$. But $u = \frac{\dot{y}}{p} = \frac{q\dot{y}}{\dot{x}}$. Then $\frac{q\dot{y}}{\dot{x}} = z \times \overline{q^{m+1} + \overline{m+1}\, g}^{\frac{1}{m+1}}$, an equation reduced to firſt fluxions.

60. Concerning this laſt equation we are to obſerve, that, if the quantity z be given by x and y in ſuch manner, that to the quantity q ſuch a value may be aſſigned, alſo given by x and y, that the indeterminates may be ſeparable in the equation, and therefore that it may be conſtructible, either algebräically, or, at leaſt, by quadratures, we may have the curve, on which the differentio-differential equation depends. And, becauſe the values are many which may be aſſigned to q, the curves may be many alſo, and every value of q will ſupply us with a different curve, either tranſcendent or algebräical, which will ſatisfy the queſtion. Let the equation be $\frac{x^4y^2\dot{x}\dot{x}}{a^2} + \frac{2aay\dot{x}\dot{y}\dot{y} + aax\dot{y}^3}{xy} = aa\ddot{y}$. Now, applying this to the canonical equation, it will be $m = 1$, $z = \frac{xxy}{aa}$; therefore the reduced equation is $\frac{q\dot{y}}{\dot{x}} = \frac{xxy}{aa} \times \overline{qq + 2g}^{\frac{1}{2}}$. I take $q = x$; it will be $\frac{x\dot{y}}{\dot{x}} = \frac{xxy}{aa}\sqrt{xx + 2g}$, that is, $\frac{aa\dot{y}}{y} = x\dot{x}\sqrt{xx + 2g}$; the integral of which plainly depends on the quadrature of the hyperbola, and the curve will be tranſcendent.

61. In paſſing from firſt to ſecond fluxions, either we aſſume no fluxion for conſtant, or we aſſume ſuch an one as is moſt eligible, as ſaid before. Wherefore, in finding the integrals of formulæ of the ſecond degree, becauſe we know what fluxion had been ſo taken, we know alſo how to proceed, and the rules for it have been explained.

But there are an infinite number of problems, which require ſecond fluxions, without our knowing what conſtants are involved in the formulæ thence ariſing. It often happens, that we cannot arrive at the analytical expreſſion without the aſſiſtance of the conſtants; and likewiſe, it ſucceeds ſometimes, that the equation may be reſolved without recurring to the conſtants. Theſe two caſes, therefore, ought to be examined, and we ſhould ſeek for ſome criterion, to diſtinguiſh one from the other. And, becauſe examples will perform this better than any thing elſe, I ſhall take this following.

It is required to find ſuch a curve, that it's abſciſs, raiſed to any dignity, may be directly as the ſecond difference of the ordinate, and reciprocally as the ſecond difference of the ſame abſciſs. Therefore we ſhall have this analogy,

x^m

$x^m . \frac{\ddot{y}}{\ddot{x}} :: a . b$. And consequently $bx^m\ddot{x} = a\ddot{y}$. In this equation I find the second differences both of the abscifs and of the ordinate; but I cannot know what constant was assumed, or whether any constant was assumed or no; so that I cannot know what course I am to pursue.

I say, in the case of this equation, that no possible curve will satisfy the Problem, since we pass from first to second fluxions, without the assistance of constants. On the contrary, the constants being determined, we may find curves that will fulfil the conditions of the Problem, but they are infinite in number, and different in their nature, as varying by the change of the arbitrary constant which is assumed.

To distinguish one species from another of these equations, we may make use of the method, or canon, which will arise from the following Examples, and which will serve in all such cases, wherein the Integral Calculus does not forsake us.

EXAMPLE I.

Let this equation, $z^{m+1}\dot{x}^m\ddot{x} + \frac{\dot{z}}{z} \times \dot{y}^{m+1} = \dot{y}^m\ddot{y}$, be proposed. I say, this is one of those formulæ to which we may attain, without taking any quantity by way of a constant. Let the variable z be any how given by x and y.

The demonstration will be made general, as far as that can be done, by taking the fluxion $\frac{\dot{x}}{q}$ as constant, in which q is a function of x and y, any how combined. Wherefore I put $\frac{\dot{x}}{q} = \dot{p}$; and, because the first member of this equation is constant, the second $\dot{p}$ will be so too. And, as it is $\dot{x} = q\dot{p}$, if we pass to second fluxions, it will be $\ddot{x} = \dot{q}\dot{p}$.

Now make $\dot{y} = u\dot{p}$; and, taking the second fluxions, on the supposition of $\dot{p}$ being constant, we shall have $\ddot{y} = \dot{u}\dot{p}$. Wherefore, substituting, in the principal equation, the values thus determined, there will arise the equation $z^{m+1}q^m\dot{q}\dot{p}^{m+1} + \frac{u^{m+1}\dot{z}\dot{p}^{m+1}}{z} = u^m\dot{u}\dot{p}^{m+1}$; and, dividing by $\dot{p}^{m+1}$, an equation will arise which is free from the unknown quantity p, and from it's functions, that is, $z^{m+1}q^m\dot{q} + \frac{u^{m+1}\dot{z}}{z} = u^m\dot{u}$. Taking the fluent, therefore, by by the rules before explained, not omitting to add the conſtant g, it will be $\frac{q^{m+1}}{m+1} + g = \frac{u^{m+1}}{m+1 \times z^{m+1}}$, which equation gives us $u = z \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$. And, becauſe $\dot{y} = u\dot{p} = \frac{u\dot{x}}{q}$, making the neceſſary ſubſtitutions, we ſhall have the equation reduced to it's ſimpleſt ſtate, that is, $\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$.

From the foregoing manner of operation, we may deduce the following Corollaries.

I. The quantity z being determined, if the laſt equation can be conſtructed, even by quadratures, ſo that it may but be executed, it is plain that infinite curves will agree to our formula, which will change their nature by changing the aſſumed conſtant fluxion $\frac{\dot{x}}{q}$. And every value of the quantity q will ſupply us with a new local equation, either algebräical or tranſcendental.

II. Although, if the value of the ſymbol q be altered, different curves will ariſe; yet it is certain, that, if we make the additional conſtant $g = 0$, we ſhall always have the equation $\dot{y} = z\dot{x}$. In which caſe, it matters not what fluxion $\frac{\dot{x}}{q}$ is taken for conſtant; becauſe, the given quantity g vaniſhing, the variable q alſo vaniſhes.

III. Here, then, is a token by which it may be known, that we ſhall arrive at our primary equation, without aſſuming any fluxion as conſtant, and that, in ſuch a ſuppoſition, it's integral is $z\dot{x} = \dot{y}$. For, recalling to our view the expreſſion $z^{m+1}\dot{x}^{m}\ddot{x} + \frac{\dot{z}}{z} \times \dot{y}^{m+1} - \dot{y}^{m}\ddot{y} = 0$, and again differencing the integral $z\dot{x} = \dot{y}$, without aſſuming any conſtant; thence we ſhall have $z\ddot{x} + \dot{z}\dot{x} = \ddot{y}$; if, by means of theſe two laſt equations, we ſhould make to vaniſh out of the principal formula, firſt y, then x, with their functions, we ſhall find $z^{m+1}\dot{x}^{m}\ddot{x} + z^{m}\dot{z}\dot{x}^{m+1} - z^{m+1}\dot{x}^{m}\ddot{x} - z^{m}\dot{z}\dot{x}^{m+1} = 0$, and $\dot{y}^{m}\ddot{y} - \frac{\dot{z}}{z}\dot{y}^{m+1} + \frac{\dot{z}}{z}\dot{y}^{m+1} - \dot{y}^{m}\ddot{y} = 0$.

IV. The

IV. The primary formula being managed as above, and the equation being found reduced to the firſt degree, that is, $\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$, we ſhould paſs on to the integrations, which ſometimes will be out of our power, according to the various values of the exponent m of the fraction z given by x and by y, and of the quantity $\frac{\dot{x}}{q}$, which is taken for conſtant. However the reſt may proceed, the aforeſaid values being determined in infinite particular caſes, the local equation of the curve is alſo diſcovered in finite terms; when we proceed to the firſt, and thence to ſecond differences, keeping ſtill the conſtant $\frac{\dot{x}}{q}$, which our principal formula will preſent us with. But, changing the conſtant, different formulæ will be found. I can aſſure nothing further, but this is very manifeſt, by turning back again the ſteps of the Analyſis.

V. The ſame thing happens by taking the firſt fluxion $\frac{\dot{y}}{q}$ for conſtant. For, making the operation according to the method, (which I ſhall omit for the ſake of brevity,) we ſhould arrive at the reduced equation $\dot{x} = \frac{\dot{y}}{z} - \frac{\dot{y}}{q} \times \overline{mg + g}|^{\frac{1}{m+1}}$; in which it may be obſerved, in like manner, that, making $g = 0$, it concludes by reſtoring the equation $\dot{x} = \frac{\dot{y}}{z}$, expreſſed by firſt differences.

VI. Aſſuming ſome limitations that are more ſimple, that is, $m = 1$, $z = xx$, and $q = x$; if we make uſe of the conſtant $\frac{\dot{x}}{q}$, as in Cor. IV, the formula $\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$ will be changed into this following, $\dot{y} = x\dot{x}\sqrt{xx + 2g}$, which admits of analytical integration. Now, making uſe of the expreſſion contained in Corol. V, that is, $\dot{x} = \frac{\dot{y}}{z} - \frac{\dot{y}}{q} \times \overline{mg + g}|^{\frac{1}{m+1}}$, ariſing from the aſſumed conſtant $\frac{\dot{y}}{q}$, and keeping ſtill the limitations of $m = 1$, $z = xx$, and $q = x$, there reſults the expreſſion $\frac{xx\dot{x}}{1 - x\sqrt{2g}} = \dot{y}$, which is not integrable without the help of the logarithms, and conſequently gives us none but tranſcendent curves.

Therefore

Therefore it is plain that we may arrive at the differential formula of the ſecond order, $z^{m+1}\dot{x}^{m}\ddot{x} + \frac{\dot{z}}{z} \times \dot{y}^{m+1} = \dot{y}^{m}\ddot{y}$, without taking any conſtant; in which caſe the integral $z\dot{x} = \dot{y}$ will take place; or, fixing for conſtant the fluxions $\frac{\dot{x}}{q}$, $\frac{\dot{y}}{q}$, for example-ſake, and then the ſame integrations will be made as before, that were found in theſe ſuppoſitions.

EXAMPLE II.

Let us take the equation $x^{m}\ddot{x} = \ddot{y} + \dot{y}\dot{y}$. I ſay, we cannot arrive at it, without taking ſome conſtant, except in one caſe, in which it is $m = -1$. To ſhow this plainly, I ſhall manage the formula in the manner following.

Firſt, I take $\dot{x}$ for conſtant, and thence $\ddot{x} = 0$. Then $-\frac{\ddot{y}}{\dot{y}} = \dot{y}$, and by integrating, $l\frac{\dot{x}}{\dot{y}} = y$*, or $\frac{\dot{x}}{\dot{y}} = c^{y}$. Make $c^{y} = z$, it will be $ylc = lz$, and therefore $\dot{y} = \frac{\dot{z}}{z}$; and, inſtead of $\dot{y}$, ſubſtituting this value, we ſhall have $\frac{z\dot{x}}{\dot{z}} = c^{y}$. But $c^{y} = z$, therefore $\dot{x} = \dot{z}$, and $x = z = c^{y}$; and therefore $\frac{\dot{x}}{x} = \dot{y}$, an equation to the logarithmic.

Secondly, I propoſe to inveſtigate how it may ſucceed on the ſuppoſition of another conſtant, $\dot{y}$ for example, whence $\ddot{y} = 0$. I make $\dot{x} = s\dot{y} + c\dot{y}$, where s is a new variable, and c a given quantity. I go on to ſecond differences, and it will be $\ddot{x} = \dot{s}\dot{y}$; and, making the ſubſtitution, it is $x^{m}\dot{s}\dot{y} = \dot{y}\dot{y}$, or $x^{m}\dot{s} = \dot{y}$. But $\dot{y} = \frac{\dot{x}}{s+c}$; then $s\dot{s} + c\dot{s} = x^{-m}\dot{x}$; and integrating (omitting to add a conſtant), $\frac{1}{2}ss + cs = \frac{x^{-m+1}}{-m+1}$, or $s + c = \sqrt{\frac{2x^{-m+1}}{-m+1} + cc}$. But $\dot{x} = \overline{s+c} \times \dot{y} = \dot{y}\sqrt{\frac{2x^{-m+1}}{-m+1} + cc}$; therefore $\frac{\dot{x}}{\sqrt{\frac{2x^{-m+1}}{-m+1} + cc}} = \dot{y}$.

* See § 46. EDITOR.

 I proceed

I proceed to inquire if poſſibly the logarithmic curve may be concealed under the laſt formula, which being found above, in the hypotheſis of $\dot{x}$ being conſtant, it may likewiſe have place in the other ſuppoſition of $\dot{y}$ being conſtant. Making $c = 0$, it is neceſſary that the equation $\sqrt{\frac{2x^{-m+1}}{-m+1}} = x$ ſhould be verified, or elſe $2x^{-m+1} = \overline{-m+1} \times xx$. And, that the equation may be found, the ſame quantity $-m+1$, both in the co-efficient and the exponent, ought to be $= 2$; for this to obtain, it follows, that it muſt be $m = -1$.

Therefore, in the formula $x^m\ddot{x} = \ddot{y} + \dot{y}\dot{y}$, by limiting the value of the exponent to $m = -1$, we come to a differential equation of the ſecond degree, without aſſuming a conſtant, the integral of which is the logarithmic expreſſion $\frac{\dot{x}}{x} = \dot{y}$. In any other caſe we could not obtain the foreſaid expreſſion, without fixing upon ſome infiniteſimal quantity of the firſt order as a conſtant.

EXAMPLE III.

It remains that we ſhould propoſe a differential equation of the other claſs, at which we cannot arrive without aſſuming a conſtant.

I reſume the problem: To conſtruct a curve, in which any dignity whatever of the abſciſs may be in a direct ratio of the ſecond fluxion of the ordinate, compounded with the inverſe ratio of the ſecond fluxion of the abſciſs.

The equation is $bx^m\ddot{x} = a\ddot{y}$. Make $\dot{x} = q\dot{p}$, $\dot{y} = u\dot{p}$; and perform the operations, as in the firſt Example. Taking the ſecond fluxions, we ſhall have $\ddot{x} = \dot{p}\dot{q}$, $\ddot{y} = \dot{u}\dot{p}$; and, ſubſtituting theſe values, it will be $bx^m\dot{q} = a\dot{u}$; and by integration, $\int bx^m\dot{q} = au \pm g$. But $\dot{y} = u\dot{p} = \frac{u\dot{x}}{q}$; then $a\dot{y} = \frac{\dot{x}}{q}\int bx^m\dot{q} \mp \frac{g\dot{x}}{q}$. Making $g = 0$, in this caſe, whatever be the value of the ſymbol q, it gives us a different curve, if alſo we do not put the exponent $m = 0$, by which the hypotheſis will be deſtroyed, and the problem changed. The ſame thing may be ſaid if we make conſtant the fraction $\frac{\dot{y}}{q}$; and from

8 hence

hence we may conclude, that it is not poſſible a differential equation of the firſt degree, without the benefit of a conſtant, ſhall reſtore our formulæ, when it is differenced again; for, if it were ſo, it would be manifeſted in any aſſumption of a conſtant; and alſo, the analyſis evidences the contrary.

PROBLEM I.

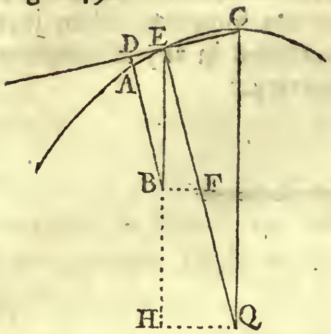
62. The radius of curvature being given, any how expreſſed by the ordinate of a curve, to find the curve itſelf.

As, when the curve is given, to find it's radius of curvature, it is called the Direct Method, or Problem of the Radii of Curvature, of which we have treated already; ſo, when the radius of curvature is given, to find what curve it is to which it belongs, is called the Inverſe Problem of the Radii of Curvature. Wherefore, let the radius of curvature be $= r$, and be any how given by y, the ordinate of the curve; and we may take any one of the formulæ for the radii of curvature, which we pleaſe; but, firſt, for the curves referred to a *focus*; as, for example, $\frac{y\dot{s}^3}{\dot{x}\dot{s}\dot{s} - y\dot{x}\ddot{y}}$, in which $\dot{x}$ is conſtant, and $\dot{s}$ is the element of the curve. Then we ſhall have the equation $r = \frac{y\dot{s}^3}{\dot{x}\dot{s}\dot{s} - y\dot{x}\ddot{y}}$; or elſe, it being $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y}$, it is $\dot{s}\ddot{s} = \dot{y}\ddot{y}$, becauſe of $\dot{x}$ conſtant, and $r = \frac{y\dot{y}\dot{s}^2}{\dot{x}\dot{y}\dot{s} - y\dot{x}\ddot{s}}$.

To reduce this equation, I make uſe of the method of § 49; and therefore I make $\dot{s} = p\dot{x}$, whence $\ddot{s} = \dot{p}\dot{x}$. Then, making the ſubſtitutions in the equation, it will be $r = \frac{pp y\dot{y}}{p\dot{y} - y\dot{p}}$, or elſe $\frac{p\dot{y} - y\dot{p}}{pp} = \frac{y\dot{y}}{r}$; and then, by integration, becauſe r is given by y, it will be $\frac{y}{p} = \int \frac{y\dot{y}}{r} \pm b$. But $p = \frac{\dot{s}}{\dot{x}} = \frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$; therefore the curve will be $\frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = \int \frac{y\dot{y}}{r} \pm b$, an equation reduced to firſt fluxions, becauſe, r being given by y, the integral $\int \frac{y\dot{y}}{r}$ may always be had, at leaſt tranſcendentally.

 Another

Fig. 149.



Another way. I write the equation, $r = \frac{y\dot{s}^3}{\dot{x}\dot{s}^2 - y\dot{x}\ddot{y}}$, in this manner, $\frac{y\dot{s}^3}{r} = \dot{x}\dot{s}^2 - y\dot{x}\ddot{y}$. Then, from the point B, (Fig. 149.) from whence proceed the ordinates BE of the curve required AEC, I draw BF perpendicular to EB, terminated at the radius of curvature EQ; and, making BF $= p$, EF $= q$, by the known formulæ of the normal and ſubnormal, it will be $q = \frac{y\dot{s}}{\dot{x}}$, $p = \frac{y\dot{y}}{\dot{x}}$, or $\dot{y} = \frac{p\dot{x}}{y}$. And, by taking the fluxions, on the ſuppoſition of $\dot{x}$ being conſtant, it will be $\ddot{y} = \frac{y\dot{p}\dot{x} - p\dot{x}\dot{y}}{yy}$. And, making the ſubſtitutions in the principal equation, it will be $\frac{y\dot{s}^3}{r} = \dot{x}\dot{s}^2 - \dot{p}\dot{x}^2 + \frac{p\dot{x}^2\dot{y}}{y}$. But $\dot{s} = \frac{q\dot{x}}{y}$; therefore $\frac{q^3\dot{x}}{r} = qq\dot{x} - yy\dot{p} + py\dot{y}$. And, becauſe it is $\dot{x} = \frac{y\dot{y}}{p}$, it will be $\frac{q^3\dot{y}}{r} = qq\dot{y} + pp\dot{y} - yp\dot{p}$. But, becauſe of the right angle EBF, it is $pp = qq - yy$, and $p\dot{p} = q\dot{q} - y\dot{y}$. Wherefore, making the ſubſtitution, we ſhall have $\frac{qq\dot{y}}{r} = 2q\dot{y} - y\dot{q}$; and, multiplying by y, and dividing by qq, it will be $\frac{y\dot{y}}{r} = \frac{2qy\dot{y} - yy\dot{q}}{qq}$; and, by integration, it is $\int\frac{y\dot{y}}{r} \pm b = \frac{yy}{q}$. But $q = \frac{y\dot{s}}{\dot{x}}$; therefore $\int\frac{y\dot{y}}{r} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

It may be done thus more ſimply, by avoiding ſecond fluxions.

Taking the infinitely little arch EC, let the chord CED be produced, to which let BD be perpendicular. Now, if we make BD $= p$, by what has been ſaid at § 115, Sect. V, B. II, QE $= r = \frac{y\dot{y}}{\dot{p}}$, and therefore $\frac{y\dot{y}}{r} = \dot{p}$; and by integration, becauſe r is given by y, it is $\int\frac{y\dot{y}}{r} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$; for $p = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, by the place now quoted.

Let it be $r = \frac{y}{b}\sqrt{aa + bb}$; then it will be $\int\frac{b\dot{y}}{\sqrt{aa + bb}} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, and by actual integration, (omitting the conſtant b for greater ſimplicity,) $\frac{b}{\sqrt{aa + vb}} = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, and therefore $b^2\dot{x}^2 + b^2\dot{y}^2 = a^2\dot{x}^2 + b^2\dot{x}^2$, that is, $b\dot{y} = a\dot{x}$, which is the logarithmic ſpiral of Example V, § 128, Book II.

Inſtead

Instead of the radius QE, let the co-radius HE $= z$ be any how given by the ordinate y. Becaufe of similar triangles, EBD, QEH, it will be EB . BD :: QE . EH; that is, $y \,.\, p :: \frac{y\dot{y}}{\dot{p}} \,.\, z$, and therefore $z = \frac{p\dot{y}}{\dot{p}}$, or $\frac{\dot{y}}{z} = \frac{\dot{p}}{p}$; and by integration, $\int\frac{\dot{y}}{z} \pm b = lp$. Make $z = y$, then $\int\frac{\dot{y}}{y} \pm b = \int\frac{\dot{p}}{p}$; and by integrating, $by = lp + l\frac{m}{b}$ *, that is, $y = \frac{pm}{b}$. But $p = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, then $b\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = m\dot{x}$, and therefore $b\dot{y} = \dot{x}\sqrt{mm - bb}$, which is the logarithmic spiral; and, when $b = b$, $m = \sqrt{aa + bb}$ is the fame as the above-cited.

63. For curves referred to an axis, the formula of the radius of curvature is $\frac{\dot{s}^3}{-\dot{x}\ddot{y}}$, putting $\dot{x}$ conftant; and therefore the equation will be $r = \frac{\dot{s}^3}{-\dot{x}\ddot{y}}$.

I put $\dot{y} = q\dot{x}$, whence $\ddot{y} = \dot{q}\dot{x}$; and, making the fubftitutions, it is $r = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\dot{x}\dot{q}}$; and, inftead of $\dot{x}$, putting it's value $\frac{\dot{y}}{q}$, it will be $r = \frac{\dot{y} \times \overline{1 + qq}|^{\frac{3}{2}}}{-q\dot{q}}$, that is, $\frac{\dot{y}}{r} = -\frac{q\dot{q}}{\overline{1 + qq}|^{\frac{3}{2}}}$. And, by integration, $\int\frac{\dot{y}}{r} \pm b = \frac{1}{\sqrt{1 + qq}}$. But $q = \frac{\dot{y}}{\dot{x}}$; therefore $\int\frac{\dot{y}}{r} \pm b = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

Let $r = \frac{\overline{4yy + aa}|^{\frac{3}{2}}}{2aa}$; then it will be $\int\frac{2aa\dot{y}}{\overline{4yy + aa}|^{\frac{3}{2}}} \pm b = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. And, by actual integration, omitting the conftant b, it is $\frac{2y}{\sqrt{4yy + aa}} = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, that is, $2y\dot{y} = a\dot{x}$; and by integration, $yy = ax$, which is the parabola of the firft Example, § 122, Sect. V, Book II.

Inftead of the radius, let the co-radius be given, which make $= z$, the formula of which (fuppofing $\dot{x}$ to be conftant,) is $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$. Then $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}} = z$; and making $\dot{y} = q\dot{x}$, $\ddot{y} = \dot{q}\dot{x}$, and making the fubftitutions of thefe values of $\ddot{y}$ and $\dot{x}$, it will be $\frac{\dot{y} \times \overline{1 + qq}}{-q\dot{q}} = z$, that is, $\frac{\dot{y}}{z} = \frac{-q\dot{q}}{1 + qq}$. And, by integration, $\int\frac{\dot{y}}{z} \pm b = -l\sqrt{1 + qq}$. Whence, if z, or the co-radius, be in fuch manner given by y, as that $\int\frac{\dot{y}}{z}$ be a logarithmic expreffion, we fhall have a

* This equation, as well as the fubfequent work, would have been clearer and fimpler, if m had been put for the conftant number of which the logarithm is b. EDITOR.

differential

differential equation of the firſt degree expreſſed after the uſual manner; in any other caſe, it will be expreſſed by logarithmic quantities.

Let it be $z = \frac{4y^3 + aay}{aa}$; we ſhall have the equation $\int \frac{aa\dot{y}}{4y^3 + aay} \pm b = - l\sqrt{1 + qq}$. And, by actual integration, (omitting the conſtant b,) it is $l\frac{y}{\sqrt{yy + \frac{1}{4}aa}} = l\frac{1}{\sqrt{1 + qq}}$, and therefore $\frac{yy}{yy + \frac{1}{4}aa} = \frac{1}{1 + qq}$. And, ſubſtituting the value of q, it is $2y\dot{y} = a\dot{x}$, and, by integration, it is $yy = ax$, the ſame parabola as before.

64. In the ſecond place, let the radius, or co-radius, of curvature be any how given by the abſciſs x; it is plain that, in this caſe, we cannot make uſe of the ſame reductions we did in the firſt, becauſe we cannot have the fluents $\int \frac{\dot{y}}{r}$, or $\int \frac{\dot{y}}{z}$, if r and z are given by x.

Taking, therefore, the formula of the radius of curvature, in which $\dot{x}$ is conſtant, that is, $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{- \dot{x}\ddot{y}}$ for curves referred to an axis, (for, in thoſe referred to a *focus*, the radius, or co-radius, cannot be given by the abſciſs,) it will be $r = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{- \dot{x}\ddot{y}}$, and therefore, in the ſame manner as before, I put $\dot{y} = q\dot{x}$, whence $\ddot{y} = \dot{q}\dot{x}$, $\dot{y}\dot{y} = qq\dot{x}\dot{x}$; and, making the ſubſtitutions, $r = \frac{\overline{\dot{x}\dot{x} + qq\dot{x}\dot{x}}|^{\frac{3}{2}}}{- \dot{x}\dot{x}\dot{q}}$, that is, $\frac{\dot{x}}{r} = \frac{- \dot{q}}{\overline{1 + qq}|^{\frac{3}{2}}}$, and, by integration, $\int \frac{\dot{x}}{r} \pm b = \frac{- q}{\overline{1 + qq}|^{\frac{1}{2}}}$, which is an equation reduced to firſt fluxions; becauſe r, being given by x, the fluent $\int \frac{\dot{x}}{r}$ may always be had, at leaſt tranſcendentally. And, ſubſtituting the value of q, it is $\int \frac{\dot{x}}{r} \pm b = \frac{- \dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

Let it be $r = 2\sqrt{4aa - 2ax}$; then it will be $\int \frac{\dot{x}}{2\sqrt{4aa - 2ax}} \pm b = \frac{- \dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. And, by actual integration, omitting the conſtant b, it will be $\frac{- \sqrt{4aa - 2ax}}{2a} = \frac{- \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. And, by ſquaring, and reducing to a common denominator, it is $4aa\dot{x}\dot{x} - 2ax\dot{x}\dot{x} - 2ax\dot{y}\dot{y} = 0$, that is, $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$, an equation to the cycloid of § 131, Sect. V, B. II.

Inſtead

Inſtead of the radius, let the co-radius be given; then $z = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$. And putting, in like manner, $\dot{y} = q\dot{x}$, it is $\ddot{y} = \dot{q}\dot{x}$, $\dot{y}\dot{y} = qq\dot{x}\dot{x}$; and making the ſubſtitutions, inſtead of $\ddot{y}$ and $\dot{y}\dot{y}$, it will be $z = \frac{\dot{x}\dot{x} + qq\dot{x}\dot{x}}{-\dot{q}\dot{x}}$, that is, $\frac{\dot{x}}{z} = \frac{-\dot{q}}{1 + qq}$; and, by integration, $\int\frac{\dot{x}}{z} \pm b = \int\frac{-\dot{q}}{1 + qq}$. But the integral of the *homogeneum comparationis* is the arch of a circle; therefore, if the co-radius ſhall be given in ſuch manner, as that $\int\frac{\dot{x}}{z}$ is alſo the arch of a circle, and theſe arches ſhall ſo correſpond, as to be to each other as number to number, we ſhall have the equation reduced to firſt fluxions, and expreſſed in common quantities.

Let $z = 2\sqrt{2ax - xx}$; then it will be $\int\frac{\dot{x}}{2\sqrt{2ax - xx}} = \int\frac{-\dot{q}}{1 + qq}$. But the integral of the firſt member is the arch of a circle, the tangent of which is $\frac{\sqrt{2ax - xx}}{x}$; and of the ſecond, is the arch of a circle, the tangent of which is q. Then it will be $\frac{\sqrt{2ax - xx}}{x} = q = \frac{\dot{y}}{\dot{x}}$; therefore $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$, an equation to the ſame cycloid.

PROBLEM II.

65. The radius of curvature being given in any manner, in a curve referred to an axis, to find the ſaid curve.

The formula for the radius of curvature is $\frac{\dot{x}\dot{s}}{\ddot{y}}$, making $\dot{s}$ the element of the curve conſtant; whence the equation will be $r = \frac{\dot{x}\dot{s}}{\ddot{y}}$. Call the tangent of the curve t, and the ſubtangent p. It will be $\frac{y\dot{s}}{\dot{y}} = t$, and, differencing in the hypotheſis of $\dot{s}$ conſtant, it will be $\dot{t} = \frac{\dot{y}\dot{y}\dot{s} - y\dot{s}\ddot{y}}{\dot{y}\dot{y}}$, that is, $\ddot{y} = \frac{\dot{y}\dot{y}\dot{s} - \dot{y}\dot{y}\dot{t}}{y\dot{s}}$. Wherefore, making the ſubſtitutions, it will be $r = \frac{y\dot{x}\dot{s}\dot{s}}{\dot{y}\dot{y}\dot{s} - \dot{y}\dot{y}\dot{t}}$. But, becauſe we have $p = \frac{y\dot{x}}{\dot{y}}$, and $t = \frac{y\dot{s}}{\dot{y}}$, it will be $\dot{x} = \frac{p\dot{y}}{y}$, $\dot{s} = \frac{t\dot{y}}{y}$. Then, ſubſtituting theſe values in the equation above, we ſhall have $r = \frac{pt\dot{s}}{t\dot{y} - y\dot{t}}$. But $p = \sqrt{tt - yy}$; therefore $r = \frac{t\dot{s}\sqrt{tt - yy}}{t\dot{y} - y\dot{t}}$, or $\frac{\dot{s}}{r} = \frac{t\dot{y} - y\dot{t}}{t\sqrt{tt - yy}}$.

The firſt member of this laſt equation is in our power, at leaſt tranſcendentally, becauſe r is a function of s. Then, in the ſecond, the indeterminates will be eaſily ſeparated, if we make $q = \frac{y}{t}$, by which we ſhall have a very ſimple equation, $\frac{\dot{s}}{r} = \frac{\dot{q}}{\sqrt{1 - qq}}$.

In the formula $r = \frac{pt\dot{s}}{t\dot{y} - y\dot{t}}$, if, inſtead of t, we had taken it's value $\sqrt{pp+yy}$, we ſhould have found $r = \frac{\overline{pp + yy} \times \dot{s}}{p\dot{y} - y\dot{p}}$; and, making $\frac{y}{p} = z$, we ſhould alſo have had a very ſimple equation, $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$.

The two differential quantities $\frac{\dot{q}}{\sqrt{1 - qq}}$ and $\frac{\dot{z}}{1 + zz}$ are the expreſſions of the element of the arch of a circle. Whence, if the integral $\int\frac{\dot{s}}{r}$ ſhall be algebräical, or ſhall depend on the logarithms, or on higher quadratures, the rectification of the curves required, and the value of the radius of curvature, will ſuppoſe the quadrature of the circle. But, on the contrary, each of them may be algebräical, if the integral $\int\frac{\dot{s}}{r}$ agrees with a formula of the circular arch.

Retaining one of the two equations, for example the ſecond, $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$; becauſe $\dot{s} = \frac{t\dot{y}}{y} = \frac{\dot{y}}{y}\sqrt{pp + yy}$, and $p = \frac{y}{z}$, it will be $\dot{s} = \frac{\dot{y}}{z}\sqrt{1 + zz}$. Then, ſubſtituting this value into the equation, we ſhall have $\dot{y} = \frac{rz\dot{z}}{\overline{1+zz} \times \sqrt{1+zz}}$. Now, it being $\dot{s} = \frac{\dot{y}}{z}\sqrt{1 + zz}$, we ſhall have alſo $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = \frac{\dot{y}\dot{y} + zz\dot{y}\dot{y}}{zz}$, and therefore $\dot{x} = \frac{\dot{y}}{z}$:

Make

Make the given radius of curvature $r = 1 + ss$. Then the equation $\frac{\dot{z}}{1 + zz} = \frac{\dot{s}}{r}$ will be changed into this, $\frac{\dot{z}}{1 + zz} = \frac{\dot{s}}{1 + ss}$; from whence we obtain $z = s$, and therefore $r = 1 + zz$. Subſtitute this value in the equation $\dot{y} = \frac{rz\dot{z}}{1+zz \times \sqrt{1+zz}}$, and it will be $\dot{y} = \frac{z\dot{z}}{\sqrt{1+zz}}$. And, by integration, omitting the conſtant, it is $y = \sqrt{1 + zz}$, whence $z = \sqrt{yy - 1}$. Then, becauſe I retained $\dot{x} = \frac{\dot{y}}{z}$, it will be finally $\dot{x} = \frac{\dot{y}}{\sqrt{yy - 1}}$, an equation of the curve required, on the aſſumed ſuppoſition of the radius of curvature. It's conſtruction depends on the quadrature of the hyperbola.

I take the formula of the radius of curvature, $\frac{\dot{s}}{r} = \frac{\dot{s}\ddot{y} - \dot{y}\ddot{s}}{\dot{x}\dot{s}}$, in which no firſt fluxion is conſtant. I diſpoſe the equation thus, $\frac{\dot{y}}{\dot{x}} \times \overline{\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}}} = \frac{\dot{s}}{r}$. The integral of $\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}}$ is $l\dot{y} - l\dot{s}$, which I make equal to lp. Then it will be $\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}} = \frac{\dot{p}}{p}$; and $\frac{\dot{y}}{\dot{s}} = p$, and then the equation will be $\frac{\dot{s}}{r} = \frac{\dot{y}}{\dot{x}} \times \frac{\dot{p}}{p}$. But $p = \frac{\dot{y}}{\dot{s}}$, and $\frac{\dot{y}\dot{y}}{pp} = \dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y}$; therefore $\dot{x} = \frac{\dot{y}\sqrt{1 - pp}}{p}$. And, ſubſtituting this value, it will be $\frac{\dot{s}}{r} = \frac{\dot{p}}{\sqrt{1 - pp}}$, an equation in which the variables are ſeparated, and conſequently may be treated in the manner made uſe of before.

Let the formula of the radius of curvature be $\frac{\dot{s}}{r} = -\frac{\dot{y}\ddot{s}}{\dot{s}\dot{x}}$, in which $\dot{y}$ is conſtant. Make $\dot{s} = q\dot{y}$, and therefore $\ddot{s} = \dot{q}\dot{y}$. Then $\frac{\dot{s}}{r} = -\frac{\dot{y}\dot{y}\dot{q}}{\dot{s}\dot{x}}$; but $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = qq\dot{y}\dot{y}$. Whence we have $\dot{x} = \dot{y}\sqrt{qq - 1}$, and $\dot{x}\dot{s} = q\dot{y}^2\sqrt{q^2 - 1}$. Wherefore, making this ſubſtitution, it will be $\frac{\dot{s}}{r} = -\frac{\dot{q}}{q\sqrt{qq - 1}}$.

Laſtly, let the formula of the radius of curvature be $\frac{\dot{s}}{r} = -\frac{\dot{x}\ddot{y}}{\dot{s}\dot{s}}$, in which $\dot{x}$ is conſtant. Make $z = \frac{\dot{x}}{\dot{y}}$, and therefore $\dot{z} = -\frac{\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Then $\frac{\dot{s}}{r} = \frac{\dot{y}\dot{y}\dot{z}}{\dot{s}\dot{s}}$. But $\dot{x} = z\dot{y}$, and $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = zz\dot{y}\dot{y} + \dot{y}\dot{y}$. Whence $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$.

Therefore, after whatever manner we operate, the integral $\int\frac{\dot{s}}{r}$ will always be brought, either to the rectification or quadrature of the circle.

Let the co-radius u be any how given, to find the curve. Take one of the three formulæ before, that, for example, in which $\dot{y}$ is taken for conſtant; that is, $\frac{\dot{s}}{r} = -\frac{\dot{q}}{q\sqrt{qq - 1}}$, in which it is put $\dot{s} = q\dot{y}$. The radius will be $r = \frac{u\dot{s}}{\dot{x}}$; and, putting this value in the formula, we ſhall have $\frac{\dot{s}}{u} = -\frac{\dot{s}\dot{q}}{q\dot{x}\sqrt{qq - 1}}$. But $\dot{s} = q\dot{y}$, and $\dot{x} = \dot{y}\sqrt{qq - 1}$. Whence, making the ſubſtitutions, it will be $\frac{\dot{s}}{u} = -\frac{\dot{q}}{qq - 1}$. But u is given by s; therefore, &c.

Here it may be obſerved, that, as the integral $\int\frac{\dot{s}}{r}$ is equal to an expreſſion of a circular arch; ſo the other integral $\int\frac{\dot{s}}{u}$ will be referred to the quadrature of the hyperbola, or to the logarithms.

66. By like artifices and expedients, or but little different from theſe, many equations, or formulæ, may be reduced to ſecond differentials, which are expreſſed by third, fourth, or higher degrees of fluxions. And, firſt, the method of § 49 may be extended, (yet within certain limitations,) to differential equations of the third, fourth, &c. order. That is to ſay, equations of the third order may always be reduced to the firſt order, provided that either one or the other of the finite variables, x or y, is wanting in them. Thoſe of the fourth order may be reduced, if, beſides one or other of the two finite variables, x or y, one or other of the firſt fluxions, $\dot{x}$ or $\dot{y}$, be wanting, together with their reſpective functions. Thoſe of the fifth may be reduced, if both the finite variables, and both their firſt fluxions, be wanting in them. Thoſe of the ſixth, if, beſides all this, one or other of their ſecond fluxions be wanting. And ſo on.

Let the equation be $\dot{x}\dddot{y} + \dot{x}\dot{x}\ddot{y} = \dot{x}^4 + \dot{y}^4$, in which $\dot{x}$ is taken for conſtant. I make, as uſual, $p\dot{x} = \dot{y}$, and therefore $\dot{p}\dot{x} = \ddot{y}$, and $\ddot{p}\dot{x} = \dddot{y}$. Wherefore, making

making the fubftitutions, we fhall have $\dot{x}\dot{x}\ddot{p} + \dot{x}^3\dot{p} = \dot{x}^4 + \dot{y}^4$. But $\dot{y}^4 = p^4\dot{x}^4$; therefore it will be $\ddot{p} + \dot{x}\dot{p} = \dot{x}\dot{x} + p^4\dot{x}\dot{x}$, an equation reduced to the fecond order. Make further $q\dot{x} = \dot{p}$, retaining $\dot{x}$ as conftant, and therefore $\dot{q}\dot{x} = \ddot{p}$. Then, by fubftitution, it will be $\dot{q}\dot{x} + \dot{p}\dot{x} = \dot{x}\dot{x} + p^4\dot{x}\dot{x}$, that is, $\dot{q} + \dot{p} = \dot{x} + p^4\dot{x}$. But $\dot{x} = \frac{\dot{p}}{q}$; therefore $\dot{q} + \dot{p} = \frac{\dot{p}}{q} + \frac{p^4\dot{p}}{q}$; which equation is now reduced to firft fluxions.

Let there be a fluxional equation of the fourth order, $\ddddot{y} + \dot{x}\dddot{y} - \dot{x}\dot{x}\ddot{y} = 0$; in which let $\dot{x}$ be conftant. Therefore I make $p\dot{x} = \dot{y}$, and thence $\dot{p}\dot{x} = \ddot{y}$; and $\ddot{p}\dot{x} = \dddot{y}$, and $\dddot{p}\dot{x} = \ddddot{y}$. Therefore, making the fubftitutions, we fhall have $\dddot{p} + \dot{x}\ddot{p} - \dot{x}\dot{x}\dot{p} = 0$; an equation which is a cafe of the foregoing Example; and which therefore we know how to manage; and which will eafily be reduced to firft fluxions.

The method of § 49, found fome time ago by S. Count *James Riccati*, was now firft known to me; but the foregoing application, as alfo the fecond inverfe Problem concerning Radii of Curvature, I have learned of him only fince the fecond Tome of the Commentaries of the Inftitute of *Bolonia* is fallen into my hands. And, indeed, fomething too late for me, becaufe I was now at the clofe of the impreffion of this my Work; nor could I take the advantage of the other learned Differtations, neither of P. *Vincent Riccati*, fon of the aforefaid gentleman, nor of S. *Gabriel Manfredi*, therein inferted. Therefore it muft fuffice that I have juft named them to the readers, that they may there find them, and be improved and inftructed by them.

67. Having fhown the aforefaid application, or improvement of the method of § 49, I fhall go on to other equations, and to other expedients. Therefore let the equation be $p\dot{y}\dddot{y} = p\dot{x}\dot{x}\ddot{y} - 2p\dot{x}\ddot{x}\ddot{y} - p\dot{x}\dot{x}\dddot{y}$, in which p is any how given by x and y, and now the element of the curve, $\dot{s}$, is taken for conftant. Becaufe $\dot{s}$ is conftant, it will be $\dot{x}\ddot{x} = -\dot{y}\ddot{y}$; then, fubftituting this value inftead of $\dot{x}\ddot{x}$, it will be $p\dot{y}\dddot{y} = \dot{p}\dot{x}\dot{x}\ddot{y} + 2p\dot{y}\ddot{y}\ddot{y} - p\dot{x}\dot{x}\dddot{y}$, that is, ftriking out the fuperfluous terms, $\dot{p}\dot{x}\dot{x}\ddot{y} = p\dot{y}\dddot{y} + p\dot{x}\dot{x}\dddot{y}$, or $\frac{\dot{p}}{p} = \frac{\dot{y}\dddot{y}}{\dot{x}\dot{x}} + \frac{\dddot{y}}{\ddot{y}}$. And, inftead of $\dot{y}\ddot{y}$, putting it's value $-\dot{x}\ddot{x}$, it will be $\frac{\dot{p}}{p} = -\frac{\ddot{x}}{\dot{x}} + \frac{\dddot{y}}{\ddot{y}}$. And laftly, integrating by the logarithms, $lp = l\ddot{y} - l\dot{x} - l\dot{s}$, $\dot{s}$ being conftant; and therefore $p = \frac{\ddot{y}}{\dot{x}\dot{s}}$: which equation is reduced to fecond fluxions.

Let the equation be $h\dot{z}\ddot{x} - 3\dot{h}\ddot{z}\dot{x} - \dot{h}\dot{z}\dot{x} = 0$, in which h is any how given by x and z. Let us aſſume the following fictitious equation, $h^m \dot{z}^n \dot{x}^r =$ conſtant; where m, n, r, are unknown exponents of powers, to be determined by the proceſs. Then, by taking the fluxions, we ſhall have $rh^m \dot{z}^n \dot{x}^{r-1}\ddot{x} + nh^m \dot{x}^r \dot{z}^{n-1}\ddot{z} + mh^{m-1}\dot{h}\dot{z}^n \dot{x}^r = 0$, which, being divided by $h^{m-1}\dot{z}^{n-1}\dot{x}^{r-1}$, will be reduced to $rh\dot{z}\ddot{x} + nh\dot{x}\ddot{z} + m\dot{h}\dot{z}\dot{x} = 0$. This equation being compared, term by term, with the principal equation propoſed, we ſhall have $r = 1$, $n = -3$, $m = -1$; wherefore, inſtead of the fictitious equation $h^m \dot{z}^n \dot{x}^r =$ conſtant, we ſhall have the true one, $\frac{\dot{x}}{h\dot{z}^3} =$ conſtant, which is the integral of the propoſed equation.

Alſo, by the way of the logarithms, we may obtain the ſame integration. I reſume the equation $h\dot{z}\ddot{x} - 3h\ddot{z}\dot{x} - \dot{h}\dot{z}\dot{x} = 0$. I divide it by $h\dot{z}\dot{x}$; it will be $\frac{\ddot{x}}{\dot{x}} - \frac{3\ddot{z}}{\dot{z}} - \frac{\dot{h}}{h} = 0$, and by integration, $l\dot{x} - l\dot{z}^3 - lh =$ to a conſtant logarithm. Therefore $\frac{\dot{x}}{h\dot{z}^3}$ is equal to a conſtant quantity.

ADVERTISEMENT.

68. I SHALL finiſh theſe Inſtitutions with an Advertiſement, which is this; that the ingenious Analyſt muſt endeavour, with all his ſkill, in the ſolution of Problems, to avoid ſecond fluxions, and much more thoſe of a higher order; and this by means of various expedients, which will offer themſelves commodiouſly on the ſpot. Such artifices may be ſeen, as they are made uſe of by famous Mathematicians, in the Problems of the Elaſtic Curves, the Catenaria, the Velaria, in that of Iſoperimetral Curves, and in others of this kind; the ſolutions of which may be ſeen in the *Leipſic* Acts, and other works of this nature: by which a learner may acquire ſuch ſkill and dexterity, as will be very beneficial to him.

END OF THE FOURTH BOOK.

AN

AN ADDITION

TO THE FOREGOING

ANALYTICAL INSTITUTIONS;

Being a Paper of Mr. *Colson's*, containing a Specimen of the Manner in which Two or more Persons may entertain themselves, by proposing and answering curious Questions in the Mathematicks.

THE Manuscript of this little piece appears to be a first draught, and only a part, of what Mr. *Colson* intended to draw up: yet, I persuade myself, it is sufficient to point out to the readers of it the way in which several persons may amuse themselves with proposing and answering Questions of this kind. Those readers, who wish to see more of this, may find it in the VIth Section of Mr. *Colson's* Comment on Sir ISAAC NEWTON'S *Fluxions*. They may also, with a little attention, propose and solve, in the same manner, any of the Questions in these Volumes.

" A *Problem* is supposed to be managed between two persons, the *Querist* and the *Respondent*: the *Data* are such numbers or quantities as are given or supplied by the Querist; the *Assumpta* or *Quæsita* are such as are assumed or found by the Respondent."

PROBLEM I.

" QUERIST. *I give you three numbers*, 4, 5, *and* 10; *I require a fourth*,

RESPONDENT. I assume x to denote that fourth.

Q. *So that, if from the product of this into the third, the first be subtracted*,

R. Then

R. Then the remainder will be denoted by $10x - 4$.

Q. *And if the remainder be divided by the firſt,*

R. The quotient will be denoted by $\frac{10x - 4}{4}$;

Q. *The Quotient will be equal to the ſecond number.*

R. Then the equation is $\frac{10x - 4}{4} = 5$; whence $10x - 4 = 20$, and $10x = 24$, and $x = \frac{24}{10} = 2.4$."

PROBLEM II.

" Q. *A certain number of ſhillings,*

R. That number ſhall be denoted by x;

Q. *Was to be diſtributed among a certain number of poor people;*

R. The number of poor ſhall be y.

Q. *Now if three ſhillings were given to each, there would be 8 wanting;*

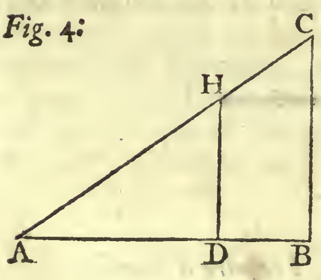
R. Then $x = 3y - 8$.

Q. *But if two were given to each, there would be 3 to ſpare.*

R. Then $x = 2y + 3 = 3y - 8$, or $y = 11$, the number of poor; and thence $x = 2y + 3 = 22 + 3 = 25$, the number of ſhillings."

PROBLEM III.

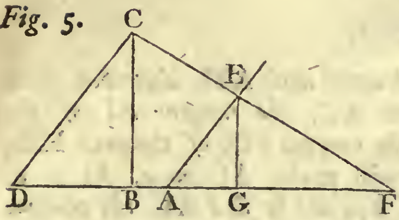
Fig. 4:



" Q. *In the triangle* ABC, *I give you the ſides* AC $= a$, BC $= b$, *and the baſe* AB $= c$; *you are to find in this ſuch a point* D, R. I will aſſume AD $= x$; then DB $= a - x$; Q. *That drawing* DH *parallel to* BC, R. Then it will be AB (c) . BC (b) :: AD (x) . DH $= \frac{bx}{c}$; Q. *The ſquare of* DH *may be equal to the rectangle of* AD *and* DB. R. Then $\frac{bbxx}{cc} = x \times \overline{a - x}$, and $\frac{bbx}{cc} = a - x$, and $bbx = acc - ccx$, and $bbx + ccx = acc$, and $x = \frac{acc}{bb + cc}$."

PROBLEM IV.

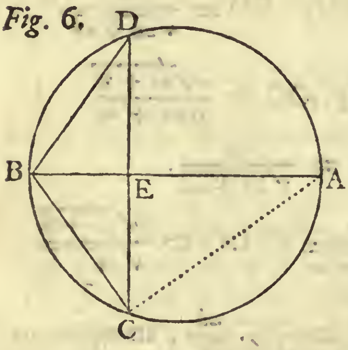
Fig. 5.



" Q. *I give you in position the two right lines* AF, AE, *and a point* C *in neither of those lines*; R. Then I can continue AF to D, and draw CD parallel to AE; and as AD will be given, I shall make AD $= a$. And I can let fall the perpendicular CB, which will be given also; and therefore I will make CB $= b$. Q. *You are to draw the line* CEF *in such a manner, as that it shall cut off the triangle* AEF *equal in area to the given plane cc.* I will let fall the perpendicular EG, and make the base AF $= x$. And then, by similar triangles, it will be DF $(a + x)$. AF (x) :: DC . AE :: CB (b) . EG $= \frac{bx}{a+x}$. But the area of the triangle AEF is $\frac{1}{2}$AF $\times$ EG $= \frac{1}{2}x \times \frac{bx}{a+x}$. Therefore $\frac{bxx}{2a+2x} = cc$." [From which quadratick equation the value of x is easily obtained by § 74, Sect. II, Book I.]

PROBLEM V.

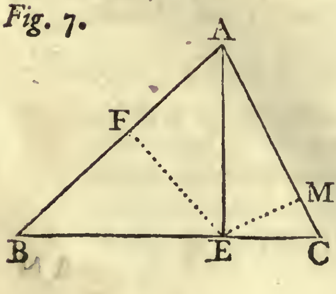
Fig. 6.



" Q. *I give you the isosceles triangle* CDB; R. Then I will make CD $= a$, BC $= b$; I will bisect CD in E, and draw the indefinite line BEA. Q. *The diameter of the circle is required in which it may be inscribed.* R. Let AB $= x$ be the diameter, and the circle ACBD. Now, because of similar triangles, it is AB (x) . BC (b) :: BC (b) . BE $= \frac{bb}{x}$. But BE $= \sqrt{BCq - CEq} = \sqrt{bb - \frac{1}{4}aa}$. Therefore $\frac{bb}{x} = \sqrt{bb - \frac{1}{4}aa}$, and $x = \frac{bb}{\sqrt{bb - \frac{1}{4}aa}}$."

PROBLEM VI.

Fig. 7.

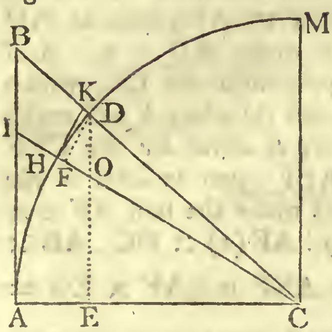


" Q. *In the triangle* ABC, *I give you the three sides*, AB $= a$, AC $= b$, *and* BC $= c$; *and letting fall the perpendicular* AE, *I require the segments of the base*, BE *and* EC. R. I make BE $= x$; then is EC $= c - x$. But ABq $-$ BEq $=$ AEq $=$ ACq $-$ ECq; that is, $aa - xx =$ (AEq) $= bb - cc + 2cx - xx$; from which $x = \frac{aa - bb + cc}{2c}$."

PRO-

PROBLEM VII.

Fig. 8.



" Q. *In the quadrantal arch* AM, *described with center* C, *and radius* AC, *the tangent* AI *of the arch* AH, *and also the tangent* HK *of the arch* HD, *are given*; R. I will make AC = a, AI = b, and HK = c. Q. *You are to find* AB, *the tangent of* AD, *which is the sum of those two arches.* R. I will make AB = x, and let fall the perpendiculars DF and DE; and then, from similar triangles, I shall have

$$CB\ (\sqrt{aa+xx})\ .\ AB\ (x)\ ::\ CD\ (a)\ .\ DE = \frac{ax}{\sqrt{aa+xx}}.$$

and $CB\ (\sqrt{aa+xx})\ .\ CA\ (a)\ ::\ CD\ (a)\ .\ CE = \dfrac{aa}{\sqrt{aa+xx}}.$

and $BC\ (\sqrt{aa+xx})\ .\ DC\ (a)\ ::\ AC\ .\ EC\ ::\ AI\ (b)\ .\ EO = \dfrac{ab}{\sqrt{aa+xx}}.$"

[and $BC\ (\sqrt{aa+xx})\ .\ DC\ (a)\ ::\ IC\ (\sqrt{aa+bb})\ .\ OC = \dfrac{a\sqrt{aa+bb}}{\sqrt{aa+xx}}.$

and $KC\ (\sqrt{aa+cc})\ .\ DC\ (a)\ ::\ KH\ (c)\ .\ DF = \dfrac{ac}{\sqrt{aa+cc}}.$

and $CE\ (\dfrac{aa}{\sqrt{aa+xx}}).\ CO\ (\dfrac{a\sqrt{aa+bb}}{\sqrt{aa+xx}})\ ::\ DF\ (\dfrac{ac}{\sqrt{aa+cc}})\ .\ DO = \dfrac{c\sqrt{aa+bb}}{\sqrt{aa+cc}}.$

But DO = DE — OE; therefore I have $\dfrac{c\sqrt{aa+bb}}{\sqrt{aa+cc}} = \dfrac{ax-ab}{\sqrt{aa+xx}}$, an equation which differs from that in § 108, Sect. II, Book I, only in notation, and which therefore may be solved in the same manner.]

AN

AN INDEX,

POINTING OUT

THE MATTER CONTAINED IN EACH ARTICLE, OR MINOR SECTION, OF THESE TWO VOLUMES.

VOLUME I. BOOK I.

THE ANALYSIS OF FINITE QUANTITIES.

SECT. I. Of the firſt Notions and Operations of the Analyſis of Finite Quantities.

Sect. II. Of Equations, and of Plane Determinate Problems.

60. *When*

SECT. III. The Construction of *Loci*, or Geometrical Places, not exceeding the Second Degree.

Sect. IV. Of Solid Problems and their Equations.

6 173. *The*

217. *The*

Sect. V. Of the Conſtruction of *Loci* which exceed the Second Degree.

254. *Ob-*

SECT. VI. Of the Method *De Maximis et Minimis,* of the Tangents of Curves, of Contrary Flexure and Regreſſion; making uſe only of Common Algebra.

VOLUME

VOLUME II. BOOK II.

THE ANALYSIS OF QUANTITIES INFINITELY SMALL.

SECT. I. Of the Notion or Notation of *Differentials*, of ſeveral Orders, and the Method of calculating with the ſame.

SECT. II. The Method of Tangents.

SECT. III. Of the *Maxima* and *Minima* of Quantities.

SECT. IV. Of Points of Contrary Flexure, and of Regreſſion.

SECT. V. Of Evolutes, and of the Rays of Curvature.

BOOK III.

OF THE INTEGRAL CALCULUS.

SECT. I. The Rules of Integrations expreſſed by Finite Algebräical *Formulæ*, or which are reduced to ſuppoſed Quadratures.

26. *Fluents*

63. —— *When*

Sect. II. Of the Rules of Integration, having Recourse to Infinite Series.

Sect. III. The Rules of the foregoing Sections applied to the Rectification of Curve-Lines, the Quadrature of Curvilinear Spaces, the Complanation of Curve Superficies, and the Cubature of their Solids.

87. *Reduction*

127.

SECT. IV. The Calculus of Logarithmic and Exponential Quantities.

BOOK IV.

THE INVERSE METHOD OF TANGENTS.

SECT. I. Of the Construction of Differential Equations of the First Degree, without any previous Separation of the Indeterminates.

SECT. II. Of the Construction of Differential Equations, by a Separation of the Indeterminates.

13. *Some*

SECT. III. Of the Conſtruction of more limited Equations, by the Help of various Subſtitutions.

SECT. IV. Of the Reduction of Fluxional Equations of the Second Degree, &c.

FINIS.

ERRATA.

E R R A T A.

NOTE. When the letter *b* is joined to the number of any line, it is counted from the bottom of the page.

VOLUME I.

In the Plan of the Lady's Syftem of Analyticks.

Page. Line.

xl. 11. *After the word* branch, *infert a comma.*

In the Body of the Work.

41. 3.*b*. *Dele* as.

125. 7. *Inftead of* 2*aaccx*, *read* 2*aacx*.

And in the head-lines, on the right-hand pages, from p. 209 to p. 223, *inftead of* SECT. IV., *read* SECT. V.

VOLUME II.

Page. Line.

9. *In fig.* 11, *the perpendicular to* AC *is drawn from the point* G, *inftead of* E.

11. *The fmall letter* i *is wanting in fig.* 15.

15. 4.*b*. *Inftead of* each, *read* one the.

16. 9. *Inftead of* EG, read EF.

24. *In the head-line, inftead of* BOOK I., *read* BOOK II.

64. 7.*b*. *After the letter* a, *inftead of* —, *read* =.

113. *Inftead of art.* 9, *read* 10. N. B. All the articles from 9 to 22 are numbered too little by 1.

125.

Page.	Line.
125.	20. *Towards the end of the line, after the word* radius, *dele the comma*; *and instead of* adding, *read* added to.
189.	9.*b*. *After* $=$, *insert the letter a*.
205.	8. *Instead of* $\overset{t}{x}$, *read* $\frac{\overset{t}{x}}{t}$.
216.	6.*b*. *After* $=$, *instead of a, read* 1.
295.	13. *Instead of* in, *read* is.
317.	3.*b*. *Instead of* $\frac{\overline{y\dot{x} - xy)}^2}{xx + yy}$, *read* $\frac{\overline{y\dot{x} - x\dot{y})}^2}{xx + yy}$.
339.	3. *Instead of* qx, *read* $q\dot{x}$.

N. B. The name of the city *Bologna* is in a few places printed *Bolonia*, as it was found in the Translator's Manuscript, but I take it to be erroneous.

EDITOR.

A LETTER

A LETTER

FROM

PHILALETHES CANTABRIGIENSIS.

Reprinted from the Gentleman's Magazine for November 1801.

IN the Gentleman's Magazine for November laſt, pages 997 and 998, is a Letter ſigned *Philalethes Cantabrigienſis*, the deſign of which is ſo laudable, that I gladly embrace this opportunity of contributing my mite to it by reprinting the Letter; conceiving that it cannot fail of the approbation of all the ſober and diſcerning part of mankind, and that, if the ſuggeſtions of it be duly attended to, it will prove very beneficial to thoſe who are of a different character, as well as to the public in general. EDITOR.

Dec. 10, 1801.

'Mr. URBAN, *Oct.* 7.

'THE following paſſage, taken from the preface to the fourth volume of the "Scriptores Logarithmici," lately publiſhed by Mr. Baron Maſeres, appears to be written with ſo benevolent a deſign, and points

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out to the Great objects so worthy of their attention, that I wish it were more generally known; and therefore shall be glad to see it in the Gentleman's Magazine.

'The passage begins in the ixth page of the preface, where, speaking of Dr. James Wilson's "Historical Dissertation of the Rise and Progress of the Modern Art of Navigation," the Baron says,'

"It is full of curious historical matter, and has suggested to my mind a wish that some person of affluence, fond of the subject of navigation, and who should have been indebted to it, perhaps, for his rank or fortune, would cause a collection of all the authors on that subject, whose works are mentioned in this Dissertation, to be made, and reprinted in a handsome manner in a set of quarto volumes, of the size of these volumes of the *Scriptores Logarithmici*, under the title of *Scriptores Nautici.* Such collections of learned tracts on particular subjects, under various titles suited to the several subjects of which they treated, would be very convenient in the present state of science; which is extended to such a variety of subjects, and dispersed in such a number of different books, that it is very difficult and very expensive for a person, fond of any particular branch of science, to procure himself all the books that relate to it. Besides the collection called *Scriptores Nautici*, relating to navigation, there might be a collection called *Scriptores Statici*, relating to the doctrine of *staticks*, or bodies at rest that form an equilibrium, or counterpoise to each other; under which head all the books of merit that treat of the *lever*, the *inclined plane*, and the other mechanical powers, would be comprized, and those that treat of the catenary curve, and of the partial immersion and the positions of bodies floating in liquids of greater specifick gravity than themselves, and of many other curious subjects of the like nature. And there might be another collection called *Scriptores Phoronomici*, relating to the doctrine of bodies in motion; under which head would be comprized Galileo's Mechanical Dialogues, of which the 3d and 4th contain the doctrine of the fall of heavy bodies to the earth with the law of their acceleration, and of their motion on inclined planes,

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and of the motion of pendulums in circular arches, and of the motion of projectiles, which (abſtracting from the reſiſtance of the air,) would deſcribe parabolas; and under the ſame head would be comprized Mr. Huygens's tract on the motions of perfectly elaſtic bodies ſtriking againſt each other, and his admirable treatiſe *De Horologio Oſcillatorio*, or on the motion of a pendulum-clock, and his tract on central forces; and all Sir Iſaac Newton's moſt profound, but very difficult work, called the *Principia*, or *Mathematical Principles of Natural Philoſophy*, with the ſeveral commentators on it, and Herman's *Phoronomia*, and Euler's work *De Motu*. Another collection might relate to the finding the centres of gravity of different bodies; which is, I believe, a more ſubtle and difficult ſubject than is generally ſuppoſed. This collection might be called *Scriptores Centrobarici*. And another collection might conſiſt of all the writers on opticks, under the title of *Scriptores Optici*. This collection ſhould comprize the work of Euclid, or that which has been aſcribed to him, on this ſubject, and thoſe of Alhazen, and Vitellio, and Roger Bacon (the learned Engliſh monk), and *Antonio De Dominis*, and Willebrord Snell, and Des Cartes, and Huygens's Dioptricks, and his treatiſe *De Lumine*, and other works of his on the ſubject of opticks, and James Gregory's *Optica Promota*, and Dr. Barrow's *Lectiones Opticæ*, and Sir Iſaac Newton's *Lectiones Opticæ*, and his Treatiſe of Opticks, or Experiments on Light and Colours, and Molineux's Dioptricks, and Dr. Smith's Compleat Syſtem of Opticks, and Harris's Opticks, and many papers in the Philoſophical Tranſactions relating to the ſame ſubject. If ſuch ſeparate collections of authors were publiſhed, every perſon who was devoted to any particular branch of theſe ſciences, (and no man can attend to all of them, or even to many of them, with any great proſpect of becoming maſter of them,) might buy the collection which related to his particular branch at a moderate expence."

'On this occaſion I beg leave to make another remark or two.

'The importance of the art of navigation to this iſland, in times of peace as well as of war, is generally acknowledged; yet it may be juſtly doubted whether it has been encouraged here in a degree ſuitable to its importance, or equal to what it has received, in the laſt fifty years, from other nations; certainly not ſo as to excite equal emulation amongſt men of ſcience *. In ſupport of this aſſertion, I might enumerate the prizes which, from time to time, have been given by foreign academies for improvements in navigation and aſtronomy, and recount the learned tracts which have been produced in conſequence of that encouragement; but I ſhall at preſent wave this ſubject.

'In all civilized nations, arts and ſciences have been conſidered as making a part of the education of the Great, and as being under their patronage. Amongſt the men of rank in this country, in former ages, are to be found the names of *Napier*, *Bacon*, *Boyle*, *Newton*, *Macclesfield,* and *Stanhope;* men who excelled in ſcience, and patronized it in others. May I then be allowed to ſuggeſt to the nobility and gentry who, of late, have made a conſpicuous figure in *Weſtminſter-Hall*, and to all others of rank and fortune, who, although their names have not yet *graced* the columns of the London *news-papers*, are waſting their time and money in the ſeduction of the *wives* and *daughters* of their *friends*, or in other idle and vicious amuſements, that, if they would exchange thoſe vicious amuſements for the innocent and rational ones purſued by the men whoſe names I have mentioned, and, inſtead of ſquandering away thouſands on *courteſans*, lay out a few hundreds in printing ſuch *ſcientific tracts* as the worthy baron has mentioned, and in the ſupport of *Genius ſtruggling with poverty*, it would undoubtedly be much more

* I am aware of the rewards which have been offered by acts of parliament for the diſcovery of the longitude at ſea, and not unacquainted with the manner in which 20,000l. has been beſtowed.

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for their prefent honour and future fatisfaction, as well as for the good of mankind.'

'PHILALETHES CANTABRIGIENSIS.'

Omne animi vitium tanto confpectius in fe
Crimen habet, quanto major, qui peccat, habetur.
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Tota licet veteres exornent undique ceræ
Atria, NOBILITAS *fola eft atque unica* VIRTUS.

JUV.

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