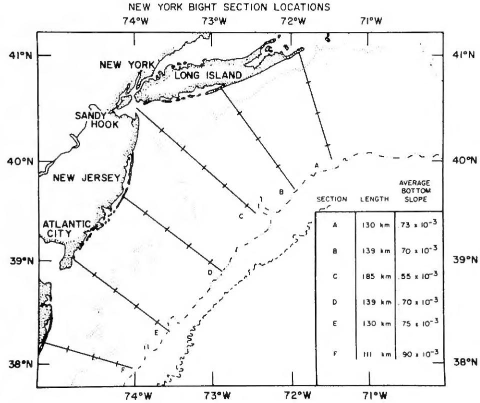
DEPARTMENT OF TRANSPORTATION

COAST GUARD

AN ANALYTICAL SEA CURRENT MODEL for COASTAL REGIONS with APPLICATION to the NEW YORK BIGHT

NEW YORK BIGHT SECTION LOCATIONS



SECTION	LENGTH	AVERAGE BOTTOM SLOPE
A	130 km	$.73 \times 10^{-3}$
B	139 km	70×10^{-3}
C	185 km	$.55 \times 10^{-3}$
D	139 km	$.70 \times 10^{-3}$
E	130 km	75×10^{-3}
F	111 km	90×10^{-3}

Oceanographic Unit Technical Report 75–2

W. M. Dunkle

AN ANALYTICAL SEA CURRENT MODEL FOR COASTAL REGIONS WITH APPLICATION TO THE NEW YORK BIGHT

By

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U.S. COAST GUARD TECHNICAL REPORT

ABSTRACT

Seasonal coastal currents on a continental shelf are modeled for use in Search and Rescue planning. The model considers a balance of Coriolis, pressure gradient, and frictional forces. Input parameters are the climatological wind and density fields. Comparison of results to currents depicted on climatological atlases for the New York Bight indicates the validity of the approach. In this light, one might extend this approach to other geographical regions where analogous oceanographic conditions prevail.

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INTRODUCTION

An analytical model was developed to determine the coastal "sea current" as given in the National Search and Rescue (SAR) Manual (CG-308). In the SAR Manual, "sea current" is characterized as the "permanent" large scale flow that one might determine from available current atlases. The model specifically describes the steady state non-tidal coastal current in the New York Bight where SAR cases are numerous and atlas data inadequate.

The momentum balance considered is steady, non-accelerated, and hydrostatic. Longshore pressure gradients are neglected. The Coriolis parameter, f, and vertical momentum exchange coefficient, A, are assumed constant. The governing equations resemble the formulation of Welander (1957), with his limitation of constant density removed by including a constant cross-shelf density gradient.

An analytical solution to the governing equation (essentially a second order differential equation) is found by applying the appropriate boundary conditions. These conditions include assuming that the surface wind stress is proportional to the surface velocity shear, and that the velocity at the bottom boundary vanishes. The solution is a relationship between wind stress, cross-shelf density gradient, sea surface slope, and velocity. A form of the continuity equation is then used to obtain the sea surface slope in terms of the wind stress and cross-shelf density gradient. The solution is now a single relationship between wind stress, cross-stream mean density gradient, and velocity.

FORMATION OF THE MODEL

Consider coastal waters in which a balance between Coriolis, pressure gradient, and frictional forces are of primary importance such that the steady linear equations of motion reduce to

$$A \frac{\partial^2 u}{\partial z^2} + \bar{\rho} f v = \frac{\partial p}{\partial n}$$

$$A \frac{\partial^2 v}{\partial z^2} - \bar{\rho} f u = 0 \qquad (1)$$

with the boundary conditions

$$A\left(\frac{\partial u}{\partial z}\right)_{z=0} = \tau_n^w \; ; \; A\left(\frac{\partial v}{\partial z}\right)_{z=0} = \tau_s^w \qquad (2)$$

$$u_{z=-h} = 0 \; ; \; v_{z=-h} = 0.$$

In the above equations, (n,s) are horizontal Cartesian coordinates and (u, v) are the corresponding velocity magnitudes. n is positive toward the shelf-break (with n=o at the 10 meters isobath) and s is positive 90° to the left of n. Also, z is the vertical coordinate positive upwards. The free surface is at $z=\zeta$ (n, s) and the bottom at $z=-h$ (n, s) (fig. 1). τ_n^w and τ_s^w are the components of the wind stress acting on the sea surface. The mean density, $\bar{\rho}$, is considered constant in a vertical column, but allowed to change at a constant rate in the cross-shelf direction (i.e., $\frac{\partial \bar{\rho}}{\partial n}$ = constant.) The pressure, p, is hydrostatic. Lateral friction and non-linear acceleration terms are neglected.

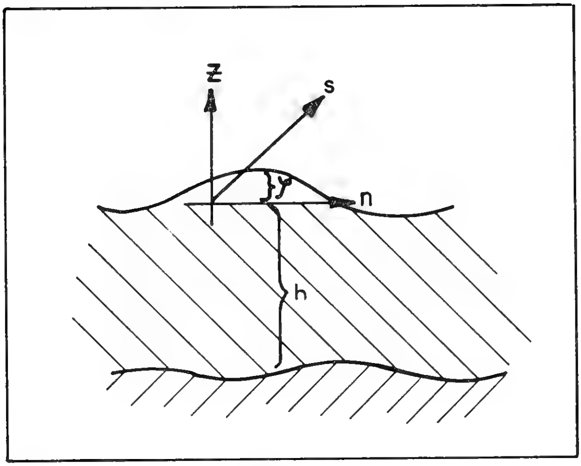

Fig. 1. Coordinate system

It is now convenient to introduce a complex notation as follows:

$$W = u + iv$$

$$\tau = \tau_n^w + i\tau_s^w$$

$$i = \sqrt{-1}.$$

Equation (1) can now be expressed as

$$A\frac{\partial^2 w}{\partial z^2} - if\bar{\rho}w = \frac{\partial p}{\partial n} \qquad (3)$$

with the complex boundary conditions

$$A\left(\frac{\partial W}{\partial z}\right)_{z=0} = \tau$$

$$(4)$$

$$W_{z=-h} = 0.$$

Consider the case where the wind stress and the horizontal pressure gradient are prescribed in equation (3). The general solution to equation (3) is thus

$$W = C_1 e^{\sqrt{\frac{if\bar{\rho}}{A}}z} + C_2 e^{-\sqrt{\frac{if\bar{\rho}}{A}}z} + \frac{i}{\bar{\rho}f}\frac{\partial p}{\partial n}$$

where the pressure is linear in z

$$p(z) = \bar{\rho}g(\zeta - z)$$

and g is the acceleration of gravity.

The horizontal pressure gradient is given as

$$\frac{\partial p(z)}{\partial n} = g\left(\bar{\rho}\frac{\partial \varsigma}{\partial n} - z\frac{\partial \bar{\rho}}{\partial n}\right).$$

The boundary conditions, (4), determine C_1 and C_2 uniquely giving (See Appendix I)

$$W = \left(\frac{\tau}{\sqrt{if\bar{\rho}A}} + \sqrt{\frac{iA}{\bar{\rho}^3 f^3}}\, g\frac{\partial \bar{\rho}}{\partial n}\right)\frac{\sinh\sqrt{\frac{if\bar{\rho}}{A}}(h+z)}{\cosh\sqrt{\frac{if\bar{\rho}}{A}}\,h}$$

$$+\frac{ig}{f}\frac{\partial \varsigma}{\partial n}\left(1 - \frac{\cosh\sqrt{\frac{if\bar{\rho}}{A}}\,z}{\cosh\sqrt{\frac{if\bar{\rho}}{A}}\,h}\right) \qquad (5)$$

$$-\frac{ig}{f\bar{\rho}}\frac{\partial \bar{\rho}}{\partial n}\left(z + h\frac{\cosh\sqrt{\frac{if\bar{\rho}}{A}}\,z}{\cosh\sqrt{\frac{if\bar{\rho}}{A}}\,h}\right)$$

Surface currents are now given from (5) as

$$W_o = \left(\frac{\tau}{\sqrt{if\bar{\rho}A}} + \sqrt{\frac{iA}{\bar{\rho}^3 f^3}}\, g\frac{\partial \bar{\rho}}{\partial n}\right)\tanh\sqrt{\frac{if\bar{\rho}}{A}}\,h$$

$$+\frac{ig}{f}\frac{\partial \varsigma}{\partial n}\left(1 - \operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\,h\right) \qquad (6)$$

$$-\frac{ig}{f\bar{\rho}}\frac{\partial \bar{\rho}}{\partial n}\,h\operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\,h$$

Neglecting vertical velocities and mass input at n=o, a condition for steady non-divergent flow is given by Gauss's divergence theorem in the form

$$\iiint_{Vol} \left[\frac{\partial}{\partial s}(\bar{\rho} v) + \frac{\partial}{\partial n}(\bar{\rho} u) \right] dz\, ds\, dn = \iint_{Area} \bar{\rho}\, u\, dz\, ds = 0 .$$

Applying the above condition to the u-component of (5) (See Appendix II) gives

$$\int_{-h}^{0} u\, dz = \frac{\tau_n^{\omega}}{\bar{\rho} f} \sinh\sqrt{\frac{f\bar{\rho}}{2A}}\,h \sin\sqrt{\frac{f\bar{\rho}}{2A}}\,h$$

$$+ \left(\frac{\tau_s^{\omega}}{\bar{\rho} f} + \frac{Ag}{\bar{\rho}^2 f^2}\frac{\partial\bar{\rho}}{\partial n}\right)\left(\sinh^2\sqrt{\frac{f\bar{\rho}}{2A}}\,h + \cos^2\sqrt{\frac{f\bar{\rho}}{2A}}\,h - \cosh\sqrt{\frac{f\bar{\rho}}{2A}}\,h \cos\sqrt{\frac{f\bar{\rho}}{2A}}\,h\right)$$

$$- \frac{1}{2}\left(\sqrt{\frac{A}{2\bar{\rho} f^3}}\, g \frac{\partial \zeta}{\partial n} + \sqrt{\frac{A}{2\bar{\rho}^3 f^3}}\, gh \frac{\partial\bar{\rho}}{\partial n}\right) \sinh 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h$$

$$+ \frac{1}{2}\left(\sqrt{\frac{A}{2\bar{\rho} f^3}}\, g \frac{\partial \zeta}{\partial n} + \sqrt{\frac{A}{2\bar{\rho}^3 f^3}}\, gh \frac{\partial\bar{\rho}}{\partial n}\right) \sin 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h$$

$$= 0 .$$

The previous relationship, when solved for the sea surface slope, $\frac{\partial\zeta}{\partial n}$, leads to an estimate of cross-shelf slope given as

$$\frac{\partial \zeta}{\partial n} = \frac{\frac{\tau_n^{\omega}}{\bar{\rho} f} \sinh\sqrt{\frac{f\bar{\rho}}{2A}}\,h \sin\sqrt{\frac{f\bar{\rho}}{2A}}\,h}{\frac{1}{2}\sqrt{\frac{A}{2\bar{\rho} f^3}}\, g\left(\sinh 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h - \sin 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h\right)} - \frac{h}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial n} \qquad (7)$$

$$+ \frac{\left(\frac{\tau_s^{\omega}}{\bar{\rho} f} + \frac{Ag}{\bar{\rho}^2 f^2}\frac{\partial\bar{\rho}}{\partial n}\right)\left(\sinh^2\sqrt{\frac{f\bar{\rho}}{2A}}\,h + \cos^2\sqrt{\frac{f\bar{\rho}}{2A}}\,h - \cosh\sqrt{\frac{f\bar{\rho}}{2A}}\,h \cos\sqrt{\frac{f\bar{\rho}}{2A}}\,h\right)}{\frac{1}{2}\sqrt{\frac{A}{2\bar{\rho} f^3}}\, g\left(\sinh 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h - \sin 2\sqrt{\frac{f\bar{\rho}}{2A}}\,h\right)} .$$

The modeled longshore (v) and cross-shelf (u) current components can now be computed using equations (6) and (7) with the mean seasonal wind stress and density field as input parameters.

APPLICATION OF THE MODEL TO THE NEW YORK BIGHT

The region under consideration was divided into six sections normal to the coast (fig. 2). Along each section the mean wind stress data given by Hidaka (1958) was used to compute τ. Climatological density data obtained from the National Oceanographic Data Center (NODC), in conjunction with specific oceanographic cruises, were used to approximate the mean density gradient $\frac{\partial\bar{\rho}}{\partial n}$, along the various sections.

The N.O.D.C. data were obtained by 1/4 degree squares for the region in question at various standard depths on a monthly basis. Seasonal density fields were then prepared by combining three months data. A weighted average density was then computed for each 1/4 degree square. These weighted average density data were combined with vertical average density fields from specific cruises to produce preliminary seasonal average density fields for each season (fig. 3).

Density gradients, $\frac{\partial\bar{\rho}}{\partial n}$, were then estimated from these diagrams along each section. Depths, h, were estimated from charts of submarine topography. These τ, $\frac{\partial\bar{\rho}}{\partial n}$, and h values were then substituted into the component form of equation (6) (Appendix III) with A=100 [gm cm^{-1} sec^{-1}]. The coastal "sea current" for the four seasons were then computed (fig. 4).

The results indicate that the terms which contain the τ dependence dominate the flow pattern in the winter months (December, January, February). In the summer months (June, July, August) the flow is dominated by terms that contain $\frac{\partial\bar{\rho}}{\partial n}$ dependence. This indicates that winter flow is basically wind driven and summer flow density driven. In this light, modeled summer flow more closely paralleled isopycnals then did winter. Additionally, in accord with the observations of Bumpus (1965), the flow along the Long Island coast during the winter season shows a slight off-shore tendency.

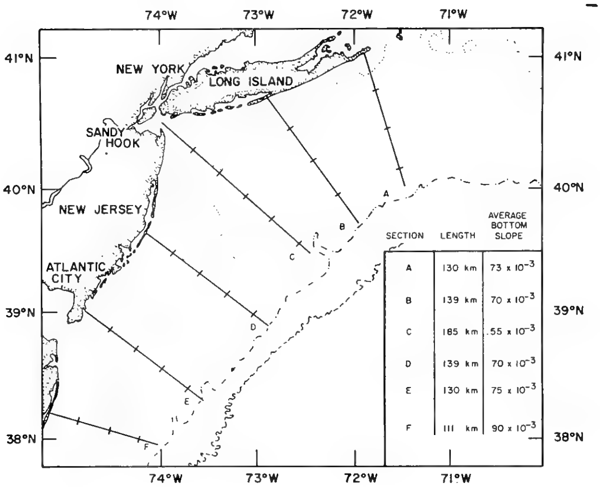

Figure 2. New York Bight Section Locations.

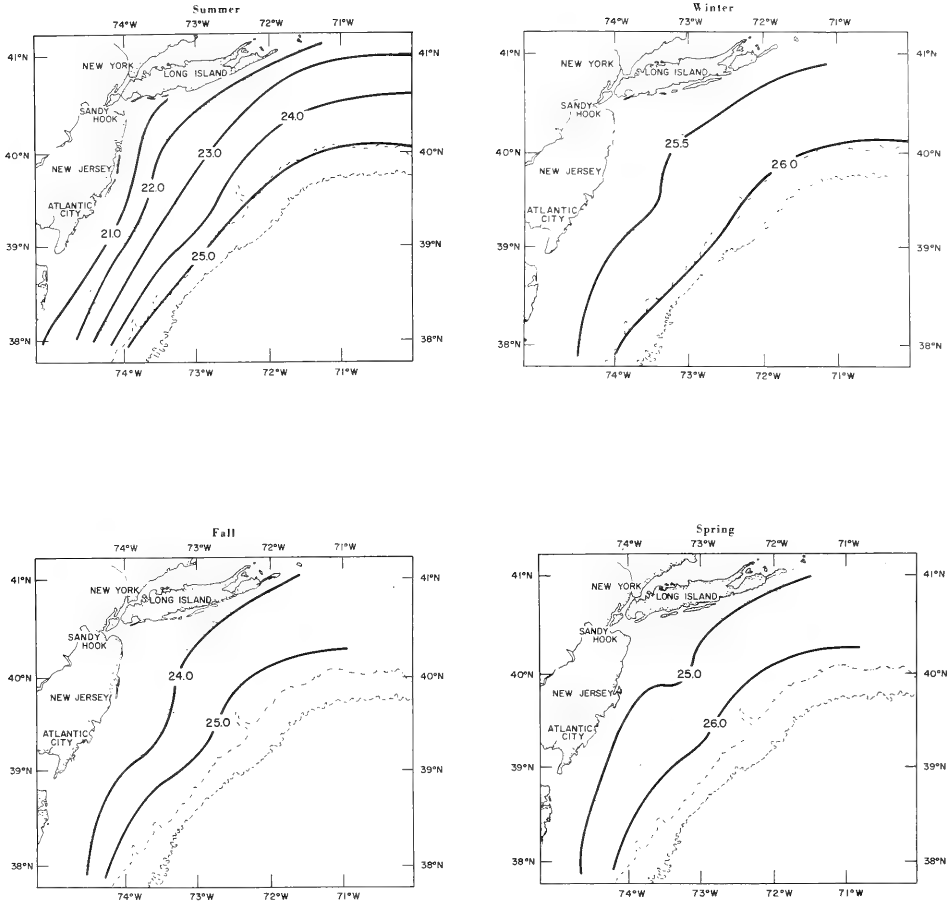

Figure 3. Preliminary Mean Seasonal Average Vertical Density Field in terms of Sigma-t as calculated from data bank files of National Oceanographic Data Center (N O D C) and specific oceanographic cruises.

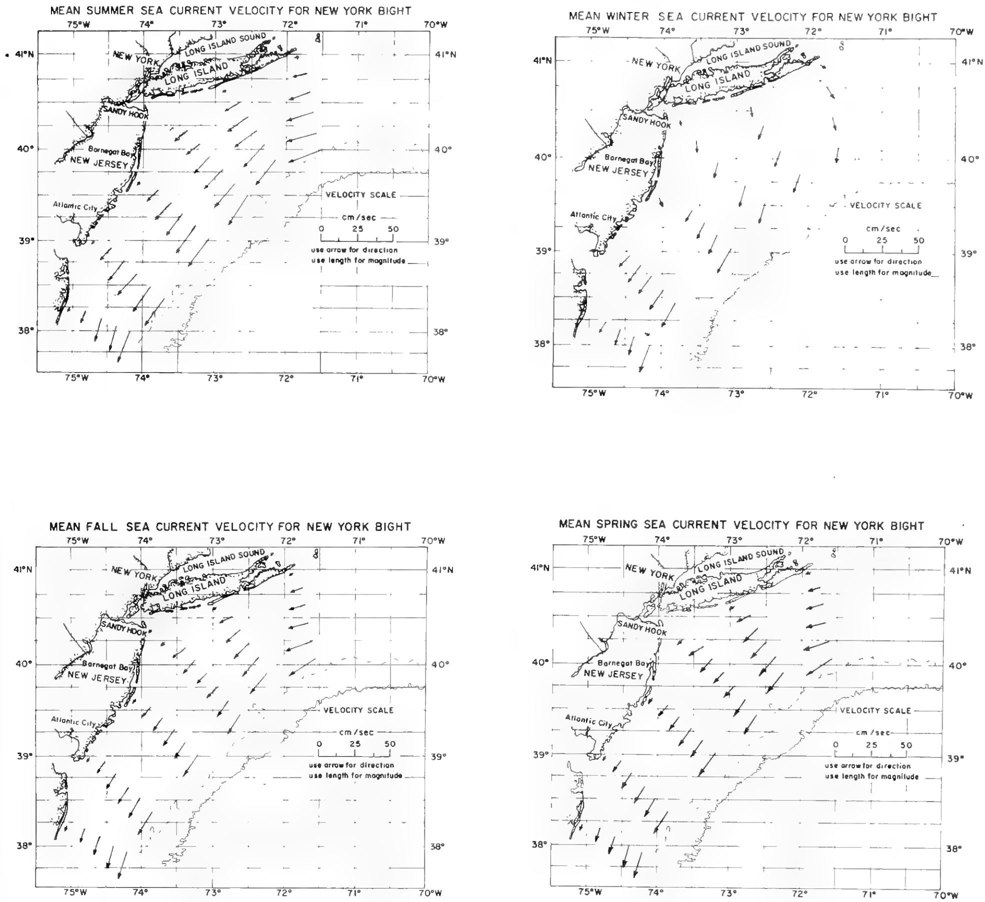

Figure 4. Modeled Seasonal Sea Current Velocity for the New York Bight Region.

During the fall (September, October, November) and spring (March, April May) seasons both the τ and $\frac{\partial \bar{\rho}}{\partial n}$ dependent terms contribute equally to the flow pattern. Computed longshore velocity components, at the shelf-break during all seasons, approximate 20-30 cm/sec while cross-shelf values are in the 2-5 cm/sec range.

A comparsion between modeled and "observed" currents for the New York Bight was made for the summer season. The "observed" data was taken from available current atlases (such as shown in Fig. 5), drift bottle observations (Fig. 6) and lightship data. A subjective combining of these data give a rough general picture of seasonal coastal currents in the New York Bight. These data were then compared against the model with $\tau=0$ and $\frac{\partial \bar{\rho}}{\partial n} = 3 \times 10^{-10}$ [gm cm^{-4}]. These input parameters roughly represent summer conditions. The result of this comparsion were favorable in that the model approximately splits the "observations" for all but the shallowest of depths (Fig. 7).

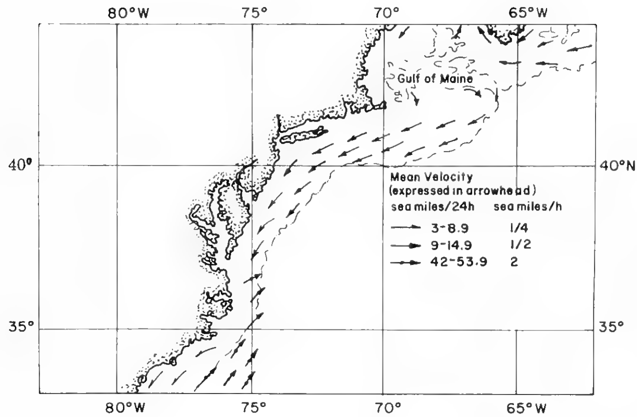

Figure 5. Mean surface currents off the North American east coast during the spring season (after Neumann and Schumacher, 1944).

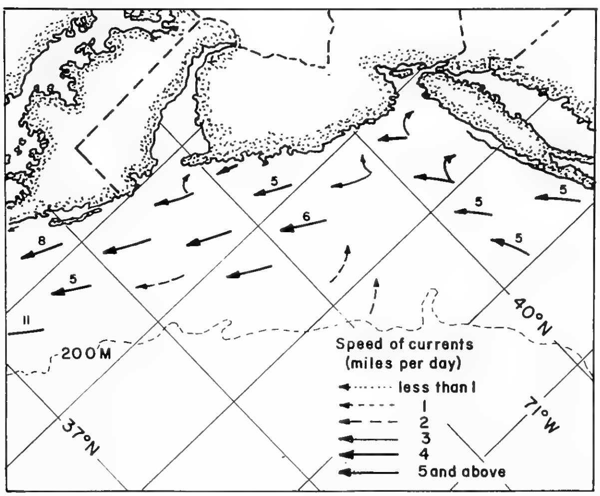

Figure 6. Mean surface currents off the North American east coast during May (After Bumpus and Lauzier, 1965).

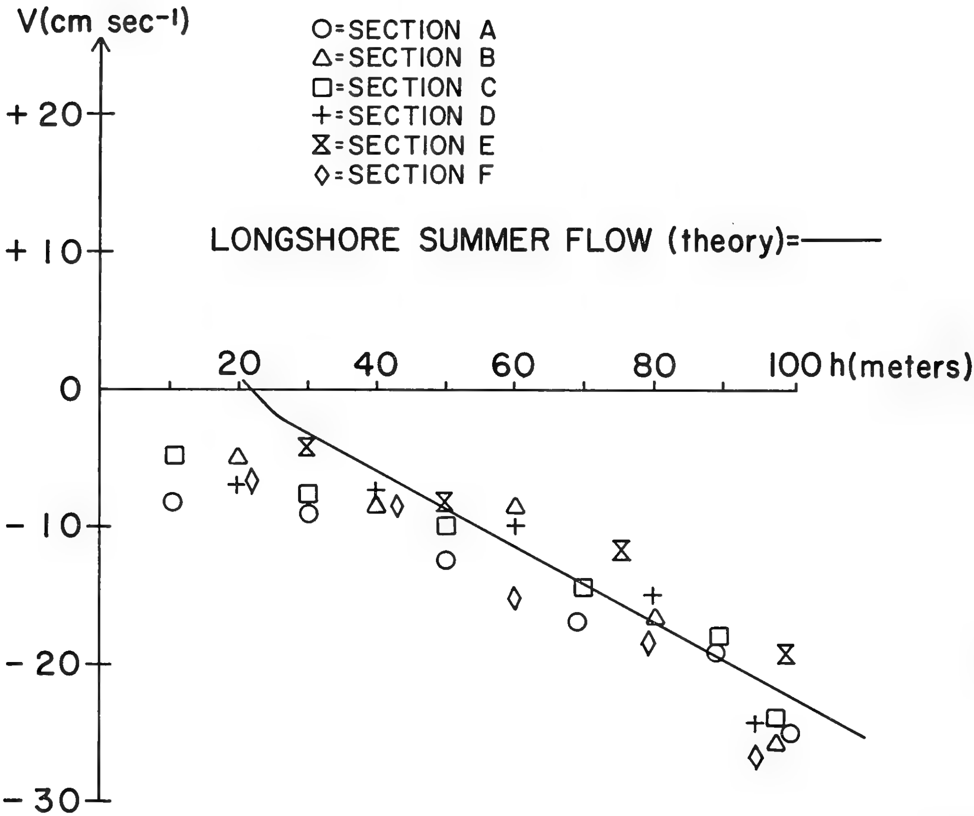

Figure 7. The relationship between observed surface currents and modeled surface currents for the New York Bight in the summer season.

CONCLUSIONS

This simple model may be expected to give results limited in applicability by the validity of the assumptions used. In a small area the variation of the Coriolis parameter is insignificant as is assumed. Also, for shallow seas the lateral friction will probably become negligible in comparsion with vertical friction (Welander, 1957). The assumption of a constant coefficient of vertical friction, A, is not actually realistic. A constant frictional coefficient simplifies the mathematical treatment and may be considered to represent the "effective friction". Since equation (6) indicates that the solution increases with depth continously, the model must be limited to a range of depth over which the approach is valid. This depth limitation is connected with the validity of the assumptions that $\frac{\partial \bar{\rho}}{\partial n}$ = constant and that the water column density may represented by $\bar{\rho}$. Consequently the model is generally applied to depths between 10 - 100 meters.

This steady state formulation is possibly applicable to time periods of less than seasonal length. Thus, the approach has the possibility of being extended to model currents in the New York Bight that are in equilibrium with the mean local wind and density fields such that the steady linear equations of motion are valid.

The proposed model may be considered an initial imporvement on the present state of the art for near-shore SAR planning. It augments existing velocity data and has the advantage of allowing increased detailed representation of the velocity field which cannot be done with present atlas presentations.

Appendix I: The solution to the equation of motion. The equation of motion takes the form

$$A\frac{\partial^2 W}{\partial z^2} - if\bar{\rho}\,W = \frac{\partial p}{\partial n}$$

The general solution is

$$W = C_1 e^{\sqrt{i}az} + C_2 e^{-\sqrt{i}az} + \frac{i}{\bar{\rho}f}\frac{\partial p}{\partial n}$$

where

$$a = \sqrt{\frac{f\bar{\rho}}{A}}\ .$$

The surface boundary condition

$$A\left(\frac{\partial W}{\partial z}\right)_{z=0} = \tau$$

when applied to the general solution gives

$$C_1 = C_2 + \frac{\tau}{\sqrt{i}\,aA} + \frac{ig}{a\bar{\rho}f\sqrt{i}}\frac{\partial\bar{\rho}}{\partial n}$$

The bottom boundary condition that

$$W_{z=-h} = 0$$

gives

$$C_1 = -C_2 e^{2\sqrt{i}ah} - \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}ah}$$

The boundary conditions thus yield

$$C_1 = \left\{ \left[\frac{\frac{\tau}{\sqrt{i}\,aA} + \frac{ig}{\sqrt{i}\,a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n} + \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah}}{1 + e^{2\sqrt{i}\,ah}} \right] e^{2\sqrt{i}\,ah} \right.$$

$$\left. - \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah} \right\}$$

$$C_2 = -\left[\frac{\frac{\tau}{\sqrt{i}\,aA} + \frac{ig}{\sqrt{i}\,a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n} + \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah}}{1 + e^{2\sqrt{i}\,ah}} \right] .$$

We may now write

$$W = \left\{ \left[\frac{\frac{\tau}{\sqrt{i}\,aA} + \frac{ig}{\sqrt{i}\,a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n} + \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah}}{1 + e^{2\sqrt{i}\,ah}} \right] e^{2\sqrt{i}\,ah} \right.$$

$$\left. - \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah} \right\} e^{\sqrt{i}\,az}$$

$$- \left[\frac{\frac{\tau}{\sqrt{i}\,aA} + \frac{ig}{\sqrt{i}\,a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n} + \frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h} e^{\sqrt{i}\,ah}}{1 + e^{2\sqrt{i}\,ah}} \right] e^{-\sqrt{i}\,az}$$

$$+ \frac{i}{\bar{\rho}f}\frac{\partial p}{\partial n} .$$

Thus it follows that

$$W = \frac{\tau}{\sqrt{i}\,aA}\left[\frac{e^{2\sqrt{i}ah+\sqrt{i}az} - e^{-\sqrt{i}az}}{1+e^{2\sqrt{i}ah}}\right]$$

$$+\frac{ig}{\sqrt{i}a}\frac{1}{\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n}\left[\frac{e^{2\sqrt{i}ah+\sqrt{i}az} - e^{-\sqrt{i}az}}{1+e^{2\sqrt{i}ah}}\right]$$

$$+\frac{i}{\bar{\rho}f}\left(\frac{\partial p}{\partial n}\right)_{z=-h}\left[\frac{e^{3\sqrt{i}ah+\sqrt{i}az} - e^{\sqrt{i}ah-\sqrt{i}az}}{1+e^{2\sqrt{i}ah}}\right.$$

$$\left. - \frac{e^{\sqrt{i}ah+\sqrt{i}az}}{1}\right]$$

$$+\frac{i}{\bar{\rho}f}\frac{\partial p}{\partial n},$$

but from

$$\frac{e^{3\sqrt{i}ah+\sqrt{i}az} - e^{\sqrt{i}ah-\sqrt{i}az}}{1+e^{2\sqrt{i}ah}} - \frac{e^{\sqrt{i}ah+\sqrt{i}az}}{1} =$$

$$= -\left[\frac{e^{-\sqrt{i}az+\sqrt{i}ah} + e^{+\sqrt{i}az+\sqrt{i}ah}}{1+e^{2\sqrt{i}ah}}\right]$$

$$= -\frac{\cosh\sqrt{i}az}{\cosh\sqrt{i}ah},$$

and

$$\frac{\partial p(z)}{\partial n} = \bar{\rho} g \frac{\partial \varsigma}{\partial n} - z g \frac{\partial \bar{\rho}}{\partial n}$$

we have

$$W = \frac{1}{\sqrt{i}\,a}\left[\frac{\tau}{A} + \frac{ig}{\bar{\rho} f}\frac{\partial \bar{\rho}}{\partial n}\right]\frac{\sinh \sqrt{i}\,a(z+h)}{\cosh \sqrt{i}\,ah}$$

$$+ \frac{ig}{f}\frac{\partial \varsigma}{\partial n}\left[1 - \frac{\cosh \sqrt{i}\,az}{\cosh \sqrt{i}\,ah}\right]$$

$$- \frac{ig}{f\bar{\rho}}\frac{\partial \bar{\rho}}{\partial n}\left[z + \frac{h \cosh \sqrt{i}\,az}{\cosh \sqrt{i}\,ah}\right] .$$

Appendix II: Calculation of the sea-surface slope from the solution for W.

$$W = \left[\frac{\tau}{\sqrt{if\bar{\rho}A}} + \sqrt{\frac{iA}{\bar{\rho}^3 f^3}}\, g \frac{\partial \bar{\rho}}{\partial n} \right] \frac{\sinh \sqrt{\frac{if\bar{\rho}}{A}}\,(h+z)}{\cosh \sqrt{\frac{if\bar{\rho}}{A}}\, h}$$

$$+ \frac{ig}{f} \frac{\partial \zeta}{\partial n} \left[1 - \frac{\cosh \sqrt{\frac{if\bar{\rho}}{A}}\, z}{\cosh \sqrt{\frac{if\bar{\rho}}{A}}\, h} \right]$$

$$- \frac{ig}{f\bar{\rho}} \frac{\partial \bar{\rho}}{\partial n} \left[z + \frac{h \cosh \sqrt{\frac{if\bar{\rho}}{A}}\, z}{\cosh \sqrt{\frac{if\bar{\rho}}{A}}\, h} \right]$$

The integral of the solution gives

$$\int_{-h}^{0} W\,dz = \frac{\tau_n}{\sqrt{if\bar{\rho}A}} \sqrt{\frac{A}{if\bar{\rho}}} \left[1 - \operatorname{sech} \sqrt{\frac{if\bar{\rho}}{A}}\, h \right]$$

$$+ i \frac{\tau_s}{\sqrt{if\bar{\rho}A}} \sqrt{\frac{A}{if\bar{\rho}}} \left[1 - \operatorname{sech} \sqrt{\frac{if\bar{\rho}}{A}}\, h \right]$$

$$+ \sqrt{\frac{iA}{\bar{\rho}^3 f^3}}\, g \frac{\partial \bar{\rho}}{\partial n} \sqrt{\frac{A}{if\bar{\rho}}} \left[1 - \operatorname{sech} \sqrt{\frac{if\bar{\rho}}{A}}\, h \right]$$

$$- \frac{ig}{f} \frac{\partial \zeta}{\partial n} \sqrt{\frac{A}{if\bar{\rho}}} \tanh \sqrt{\frac{if\bar{\rho}}{A}}\, h + \frac{igh}{f} \frac{\partial \zeta}{\partial n}$$

$$- \frac{ig}{\bar{\rho}f} \frac{\partial \bar{\rho}}{\partial n}\, h \sqrt{\frac{A}{if\bar{\rho}}} \tanh \sqrt{\frac{if\bar{\rho}}{A}}\, h$$

$$+ \frac{1}{2} \frac{ig}{\bar{\rho}f} \frac{\partial \bar{\rho}}{\partial n}\, h^2 .$$

The non-divergence condition may now be applied to the real component of the previous expression such that

$$\int_{-h}^{0} u\,dz = \frac{\frac{\tau_n^w}{\bar\rho f}\sinh\theta\sin\theta + \left(\frac{\tau_s^w}{\bar\rho f} + \frac{Ag}{\bar\rho^2 f^2}\frac{\partial\bar\rho}{\partial n}\right)(\sinh^2\theta + \cos^2\theta - \cosh\theta\cos\theta)}{\sinh^2\theta + \cos^2\theta}$$

$$-\left[\sqrt{\frac{A}{2\bar\rho f^3}}\,g\frac{\partial\zeta}{\partial n} + \sqrt{\frac{A}{2\bar\rho^3 f^3}}\,gh\frac{\partial\bar\rho}{\partial n}\right]\frac{\sinh\theta\cosh\theta}{\sinh^2\theta + \cos^2\theta}$$

$$+\left[\sqrt{\frac{A}{2\bar\rho f^3}}\,g\frac{\partial\zeta}{\partial n} + \sqrt{\frac{A}{2\bar\rho^3 f^3}}\,gh\frac{\partial\bar\rho}{\partial n}\right]\frac{\sin\theta\cos\theta}{\sinh^2\theta + \cos^2\theta} = 0$$

where

$$\theta = \sqrt{\frac{f\bar\rho}{2A}}\,h .$$

This expression may now be solved for the sea-surface slope, $\frac{\partial\zeta}{\partial n}$, in terms of the surface stress components and density field as

$$\frac{\partial\zeta}{\partial n} = \frac{\frac{\tau_n^w}{\bar\rho f}\sinh\theta\sin\theta + \left(\frac{\tau_s^w}{\bar\rho f} + \frac{Ag}{\bar\rho^2 f^2}\frac{\partial\bar\rho}{\partial n}\right)(\sinh^2\theta + \cos^2\theta - \cosh\theta\cos\theta)}{\frac{1}{2}\sqrt{\frac{A}{2\bar\rho f^3}}\,g\,(\sinh 2\theta - \sin 2\theta)}$$

$$-\frac{h}{\bar\rho}\frac{\partial\bar\rho}{\partial n} .$$

Appendix III: Resolving of the cross-shelf (u) and longshore (v) components from the complex surface solution, Wo.

$$W_o = \left[\frac{\tau}{\sqrt{if\bar{\rho}A}} + \sqrt{\frac{iA}{\bar{\rho}^3 f^3}}\, g \frac{\partial\bar{\rho}}{\partial n}\right] \tanh\sqrt{\frac{if\bar{\rho}}{A}}\, h$$

$$+ \frac{ig}{f}\frac{\partial S}{\partial n}\left[1 - \operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\, h\right]$$

$$- \frac{ig}{f\bar{\rho}}\frac{\partial\bar{\rho}}{\partial n}\, h \operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\, h$$

Identities to be used are

$$\tanh\sqrt{\frac{if\bar{\rho}}{A}}\, h = \frac{\sinh\theta\cosh\theta + i\sin\theta\cos\theta}{\sinh^2\theta + \cos^2\theta}$$

$$1 - \operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\, h = \frac{\sinh^2\theta + \cos^2\theta - \cosh\theta\cos\theta + i\sinh\theta\sin\theta}{\sinh^2\theta + \cos^2\theta}$$

$$\operatorname{sech}\sqrt{\frac{if\bar{\rho}}{A}}\, h = \frac{\cosh\theta\cos\theta - i\sinh\theta\sin\theta}{\sinh^2\theta + \cos^2\theta}$$

where

$$\theta = \frac{\sqrt{2}}{2} a h$$

$$a = \sqrt{\frac{\bar{\rho} f}{A}}$$

$$\tau = \tau_n^w + i\tau_s^w .$$

Using these definitions we find

$$W_0 = \frac{\sqrt{2}}{2}\frac{(\tau_s^{\omega} + \tau_n^{\omega})}{aA} \tanh\sqrt{i}\,ah + \frac{\sqrt{2}\,g}{2a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n}\tanh\sqrt{i}\,ah$$

$$+ \frac{\sqrt{2}}{2} i \frac{(\tau_s^{\omega} - \tau_n^{\omega})}{aA} \tanh\sqrt{i}\,ah + \frac{i\sqrt{2}\,g}{2a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n}\tanh\sqrt{i}\,ah$$

$$+ \frac{ig}{f}\frac{\partial \zeta}{\partial n}(1 - \operatorname{sech}\sqrt{i}\,ah)$$

$$- \frac{ig}{\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n} h \operatorname{sech}\sqrt{i}\,ah .$$

Putting in the identities we have:

$$W_0 = \left\{\left[\frac{\sqrt{2}}{4aA}(\tau_s^{\omega} + \tau_n^{\omega}) + \frac{\sqrt{2}\,g}{4a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n}\right]\sinh 2\theta\right\} \div (\sinh^2\theta + \cos^2\theta)$$

$$- \left\{\left[\frac{\sqrt{2}}{4aA}(\tau_s^{w} - \tau_n^{w}) + \frac{\sqrt{2}\,g}{4a\bar{\rho}f}\frac{\partial\bar{\rho}}{\partial n}\right]\sin 2\theta\right\} \div (\sinh^2\theta + \cos^2\theta)$$

$$-\left\{\left[\frac{g}{f}\frac{\partial \varsigma}{\partial n}+\frac{gh}{\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sinh\theta\sin\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$+i\left\{\left[\frac{\sqrt{2}}{4}\frac{(\tau_s^{\omega}+\tau_n^{\omega})}{aA}+\frac{\sqrt{2}g}{4a\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sin 2\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$+i\left\{\left[\frac{\sqrt{2}}{4}\frac{(\tau_s^{\omega}-\tau_n^{\omega})}{aA}+\frac{\sqrt{2}g}{4a\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sinh 2\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$-i\left\{\left[\frac{g}{f}\frac{\partial \varsigma}{\partial n}+\frac{gh}{\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\cosh\theta\cos\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$+i\frac{g}{f}\frac{\partial \varsigma}{\partial n}\,.$$

Since

$$W_0 = u + iv$$

we now have the cross-shelf (u) component and longshore (v) component of the surface current:

$$u=\left\{\left[\frac{\sqrt{2}}{4}\frac{(\tau_s^{\omega}+\tau_n^{\omega})}{aA}+\frac{\sqrt{2}g}{4a\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sinh 2\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$-\left\{\left[\frac{\sqrt{2}}{4}\frac{(\tau_s^{\omega}-\tau_n^{\omega})}{aA}+\frac{\sqrt{2}g}{4a\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sin 2\theta\right\}\div(\sinh^2\theta+\cos^2\theta)$$

$$-\left\{\left[\frac{g}{f}\frac{\partial \varsigma}{\partial n}+\frac{gh}{\bar{\rho}f}\frac{\partial \bar{\rho}}{\partial n}\right]\sinh\theta\sin\theta\right\}\div(\sinh^2\theta+\cos^2\theta)\,,$$

and

$$V = \left\{ \left[\frac{\sqrt{2}}{4} \frac{(\tau_s^w + \tau_n^w)}{aA} + \frac{\sqrt{2}\, g}{4a\bar{\rho} f} \frac{\partial \bar{\rho}}{\partial n} \right] \sin 2\theta \right\} \div (\sinh^2\theta + \cos^2\theta)$$

$$+ \left\{ \left[\frac{\sqrt{2}}{4} \frac{(\tau_s^w - \tau_n^w)}{aA} + \frac{\sqrt{2}\, g}{4a\bar{\rho} f} \frac{\partial \bar{\rho}}{\partial n} \right] \sinh 2\theta \right\} \div (\sinh^2\theta + \cos^2\theta)$$

$$- \left\{ \left[\frac{g}{f} \frac{\partial \varsigma}{\partial n} + \frac{gh}{\bar{\rho} f} \frac{\partial \bar{\rho}}{\partial n} \right] \cosh\theta \cos\theta \right\} \div (\sinh^2\theta + \cos^2\theta)$$

$$+ \frac{g}{f} \frac{\partial \varsigma}{\partial n} .$$

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ACKNOWLEDGEMENTS

LCDR C. W. Morgan is to be thanked for his helpful suggestions and constructive criticisms. Thanks also to Captain Eugene Delaney, Dr. Thomas Wolford, Dr. David Mountain, Mr. Scott Henderson, and my other associates at the U.S. Coast Guard Oceanographic Unit for their help and cooperation.