ANHARMONIC COORDINATES

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BY

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PREFACE

ALTHOUGH fifty years have passed since the invention of Anharmonic Coordinates, no book, I believe, has hitherto been written on the subject. The explanation of them given by their inventor, Sir W. R. Hamilton, in his *Elements of Quaternions*, is short; the space devoted to them by Professor P. G. Tait and Mr. C. J. Joly in their works on Quaternions is still shorter; and they are not referred to at all in ordinary books on Coordinate Geometry. Whatever value be assigned to them, we ought not to allow a method devised by a great British mathematician to be altogether forgotten. These considerations may justify the publication of the present attempt to fill in the details of Hamilton's outline.

The book lays no claim to originality, and confines itself to the application of the method to well-known geometrical theorems. Mistakes will no doubt be detected, but I trust they will be few and unimportant.

19*th June*, 1910.

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CONVENTIONAL SIGNS

1. Λ = any straight line.
2. Λ_∞ = the line at infinity in the plane (saves 27 letters).
3. $\overline{AB}$, etc., is occasionally used to distinguish the vector AB from the Euclidean line AB.
4. $AB \cdot CD$ = the cross of the line AB and CD.
5. $l^2+m^2+n^2=\Sigma l^2$; $(l+m+n)^2=\Sigma^2 l$.
6. $s_1=s-a$, $s_2=s-b$, $s_3=s-c$. Area of triangle $=\sqrt{ss_1s_2s_3}$. $s_2+s_3=a$, etc., etc.
7. $\phi(xyz)=\phi(x, y, z)=ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy$.
8. $F(pqr) \qquad = Up^2+Vq^2+Wr^2+2U'qr+2V'rp+2W'pq$.
9. Δ is the discriminant of $\phi(xyz)$.
10. D is the bordered discriminant of $\phi(xyz)$.
11. A, B, C are the coordinates of the centre of $\phi(xyz)=0$. In the places in which they occur they cannot be confounded with the corners of the given triangle, ABC.
12. Z^2 is a certain function of the coordinates of a straight line.
13. Ω^2 is the tangential equation of the cyclic points.
14. The nine-points circle is occasionally referred to shortly as the IX circle.
15. II, 5° means Chapter II, section 5. II, (5) means Chapter II, equation 5. (5) alone means equation (5) of the chapter in which the reference occurs.

CHAPTER I

PLANE GEOMETRIC NETS

1°. In framing his method of Anharmonic Coordinates, Sir William Hamilton made use of a plane geometric net constructed somewhat on the plan of Prof. Möbius.* Regarding every point of the net as the term of a vector drawn from the origin, he deduced a general vector expression which, by a suitable choice of certain coefficients, would represent the vector of any one of these points, which he called "the rational points" of the net. He then proceeded to show how this general expression could be very simply modified so as to represent any point in the plane, not included in the net. These points he called "the irrational points" of the net.

Let any four points, A, B, C and O (fig. 1), no three of which are collinear, be taken in the plane, and let the six lines, OA, OB, OC, CA, AB, BC be drawn. Then if the vectors OA, OB, OC be called α, β, γ, three scalars l, m, n can always be found such that

$$l\alpha+m\beta+n\gamma=0; \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and if α, β, γ produced meet the sides of the triangle ABC in A', B', C',

$$\frac{BC'}{C'A}=\frac{l}{m}; \quad \frac{CA'}{A'B}=\frac{m}{n}; \quad \frac{AB'}{B'C}=\frac{n}{l}.\dagger \quad \ldots\ldots\ldots\ldots(2)$$

Conversely, if three coinitial vectors α, β, γ, when pro-

* "It was by combining some parts of (Möbius' Barycentric) Calculus with Quaternions that I happened to form the conception." MS. C, 1860, 64, Trinity College, Dublin, p. 51, kindly lent to the British Museum for my use by Dr. Abbot, Librarian, T.C.D. A large part of this MS., which consists of letters from Sir W. R. Hamilton to Dr. (Sir) Andrew Hart, is devoted to the anharmonic treatment of cubic curves.

† *Outlines of Quaternions*, by the present writer, p. 14.

duced, cut the sides of the triangle formed by their terms in points A', B', C' such that

$$\frac{BC'}{C'A}=\frac{l}{m}; \quad \frac{CA'}{A'B}=\frac{m}{n}; \quad \frac{AB'}{B'C}=\frac{n}{l};$$

then $$l\alpha+m\beta+n\gamma=0. \qquad (3)$$

O, A, B, C are the cardinal points of the net, and ABC is the given triangle.

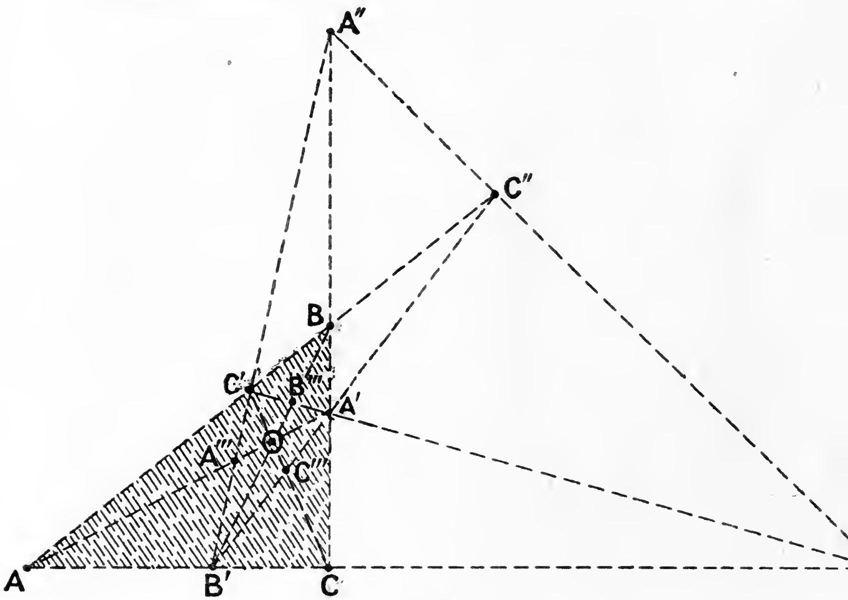

FIG. 1.

If O lies without the triangle, two of the ratios of (2) are negative. In this case we may take one of the three scalars as negative and the other two as positive.

The values of l, m, n are subject to certain limitations.

First, all three of them must have an actual value. For suppose that one of them, say n, is zero. Then, $l\alpha+m\beta=0$, and since α and β are not parallel vectors,

$$l=0, \quad m=0,$$

and the net shrinks to the point O.

Secondly, we must have $l+m+n \neq 0$. For let

$$l+m+n=0.$$

Then (fig. 2)

$$O = l\alpha + m\beta - (l+m)\gamma = l(\overline{OA} - \overline{OC}) + m(\overline{OB} - \overline{OC})$$
$$= l\overline{CA} + m\overline{CB},$$

and
$$\frac{CB}{CA} = \frac{-l}{m}.$$

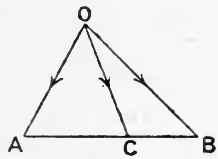

FIG. 2.

Therefore CB is parallel to CA, or B lies somewhere upon the indefinite line CA, and the net shrinks to the line CA.

Consequently, l, m, n must be actual scalars such that

$$l + m + n \neq 0.$$

2°. The first construction is to draw the intersections $OA \cdot BC$, $OB \cdot CA$, $OC \cdot AB$. To find the vector of the point $OA \cdot BC$, or A',

$$\overline{CA'} = \overline{OA'} - \overline{OC} = \overline{OA'} - \gamma; \quad \overline{A'B} = \beta - \overline{OA'};$$

and $\dfrac{CA'}{A'B} = \dfrac{m}{n}$. Hence

$$\left.\begin{aligned} &(m+n)\overline{OA'} = m\beta + n\gamma \\ \text{and} \quad &\overline{OA'} = \frac{m\beta + n\gamma}{m+n}. \\ \text{Similarly,} \quad &\overline{OB'} = \frac{n\gamma + l\alpha}{n+l}; \quad \overline{OC'} = \frac{l\alpha + m\beta}{l+m} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4)$$

3°. The second construction is to draw the intersections $BC \cdot B'C'$, $CA \cdot C'A'$, $AB \cdot A'B'$, $OA \cdot B'C'$, $OB \cdot C'A'$, $OC \cdot A'B'$. By pursuing the plan indicated in 2°, we get

$$\left.\begin{aligned} &OA'' = \frac{m\beta - n\gamma}{m-n}; \quad OB'' = \frac{n\gamma - l\alpha}{n-l}; \quad OC'' = \frac{l\alpha - m\beta}{l-m}; \\ &OA''' = \frac{2l\alpha + m\beta + n\gamma}{2l+m+n}; \quad OB''' = \frac{l\alpha + 2m\beta + n\gamma}{l+2m+n}; \\ &\qquad\qquad OC''' = \frac{l\alpha + m\beta + 2n\gamma}{l+m+2n}. \end{aligned}\right\} \ldots\ldots(5)$$

4°. A third construction would give 84 new points, and the process might be carried on indefinitely—Hamilton investigated some thousands of points; but however far it be continued the vectors of the rational points of the net are all of the form:

$$\rho=\frac{xl\alpha+ym\beta+zn\gamma}{xl+ym+zn}=\frac{\Sigma xl\alpha}{\Sigma xl}, \quad\text{.................(6)}$$

where x, y, z are whole numbers (or proportional to whole numbers) and the denominator is the algebraic sum of the coefficients.

5°. Let R (fig. 3) be a rational point, the lines through it from the corners of the triangle cutting the opposite sides in R_1, R_2, R_3. Thus, (6),

$$OR=\rho=\frac{xl\alpha+ym\beta+zn\gamma}{xl+ym+zn},$$

and

$$\begin{aligned}0&=xl(\alpha-\rho)+ym(\beta-\rho)+zn(\gamma-\rho)\\&=xlRA+ymRB+znRC.\end{aligned}$$

Therefore, (2),

$$\frac{BR_3}{R_3A}=\frac{xl}{ym};\quad \frac{CR_1}{R_1B}=\frac{ym}{zn};\quad \frac{AR_2}{R_2C}=\frac{zn}{xl}. \quad\text{...........(7)}$$

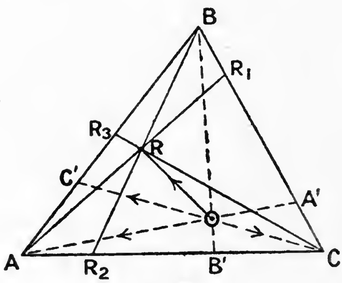

FIG. 3.

Now suppose R to be an irrational point whose position in respect to the given triangle is given by the ratios:

$$\frac{BR_3}{R_3A}=\frac{p}{q};\quad \frac{CR_1}{R_1B}=\frac{q}{r};\quad \frac{AR_2}{R_2C}=\frac{r}{p}.$$

Then, (3),

$$\begin{aligned}0&=pRA+qRB+rRC\\&=p(\alpha-\rho)+q(\beta-\rho)+r(\gamma-\rho),\end{aligned}$$

and

$$OR=\rho=\frac{p\alpha+q\beta+r\gamma}{p+q+r}. \quad\text{.......................(8)}$$

Comparing this expression with the standard form (6),

$$\left.\begin{array}{lll} p=xl\,; & q=ym\,; & r=zn\,; \\ x=pl^{-1}; & y=qm^{-1}; & z=rn^{-1}. \end{array}\right\} \quad\text{............(9)}$$

Substituting these values of x, y, z in (6), we get for the vector of the irrational point R,

$$\rho=\frac{(pl^{-1})l\alpha+(qm^{-1})m\beta+(rn^{-1})n\gamma}{(pl^{-1})l+(qm^{-1})m+(rn^{-1})n}. \quad\text{.........(10)}$$

The vector of any point in the plane may be thus reduced to the standard form.

CHAPTER II

THE POINT

1°. The anharmonic function of any four collinear points, A, B, C, D, is defined to be

$$(ABCD)=\frac{AB\,.\,CD}{BC\,.\,DA}. \quad\text{.....................(1)}$$

Let $OR=\dfrac{xl\alpha+ym\beta+zn\gamma}{xl+ym+zn}$, (fig. 3). Then AB is cut in C' in the ratio $l:m$, and by R_3 in the ratio $xl:ym$; CA and BC being divided in corresponding ratios. Hence

$$\left.\begin{aligned} C\cdot AOBR&=(AC'BR_3)=\frac{m}{l}\,\frac{xl}{ym}=\frac{x}{y};\\ A\cdot BOCR&=(BA'CR_1)=\frac{n}{m}\,\frac{ym}{zn}=\frac{y}{z};\\ B\cdot COAR&=(CB'AR_2)=\frac{l}{n}\,\frac{zn}{xl}=\frac{z}{x}.\end{aligned}\right\}\text{...........(2)}$$

The product of these three anharmonic functions is unity, and any two of them suffice to determine the position of R when the triangle ABC and the origin O are given. Hence the name Anharmonic Coordinates.

Definition. The three coefficients x, y, z, or any scalars proportional to them, are the anharmonic coordinates of the point R.

The point R is denoted by the symbol

$$R=(xyz).$$

2°. The 13 rational points shown in fig. 1 are symbolised as follows.

The vector of the origin O (from itself to itself) is zero. Now the standard expression, I, (6), becomes zero when $x=y=z$, since $l\alpha+m\beta+n\gamma=0$. Consequently, $O=(1, 1, 1)$, or any three equal numbers.

For the point A, $\rho=\alpha$; and to reduce the standard expression to this value we have merely to equate x to unity (or any multiple of 1), y to 0 and z to 0. Consequently,

$A=(1, 0, 0)$. Similarly, $B=(010)$ and $C=(001)$—omitting the commas.

For A', we have, I, (4), $\rho=\dfrac{m\beta+n\gamma}{m+n}$. Consequently $A'=(011)$. Similarly, $B'=(101)$, $C'=(110)$.

For A'', I, (5), $\rho=\dfrac{m\beta-n\gamma}{m-n}$; and $A''=(01\bar{1})$—the minus sign being put above the line to save space. Similarly, $B''=(\bar{1}01)$, $C''=(1\bar{1}0)$.

For A''', I, (5), $\rho=\dfrac{2la+m\beta+n\gamma}{2l+m+n}$; and $A'''=(211)$. Similarly, $B'''=(121)$, $C'''=(112)$. And so on.

To recapitulate:

$$\left.\begin{array}{rrr} & O=(111) & \\ A=(100) & B=(010) & C=(001) \\ A'=(011) & B'=(101) & C'=(110) \\ A''=(01\bar{1}) & B''=(\bar{1}01) & C''=(1\bar{1}0) \\ A'''=(211) & B'''=(121) & C'''=(112) \end{array}\right\}\ldots\ldots\ldots(3)$$

3°. Irrational points are symbolised in a similar way.

For instance, let M_1 be the middle point of BC. Then, I, (10),

$$\rho=OM_1=\frac{\beta+\gamma}{2}=\frac{(m^{-1})m\beta+(n^{-1})n\gamma}{(m^{-1})m+(n^{-1})n}.$$

Hence $M_1=(om^{-1}n^{-1})=(onm)$. Similarly for the middle point of CA, $M_2=(l^{-1}on^{-1})=(nol)$; $M_3=(l^{-1}m^{-1}o)=(mlo)$.

Again, lines through the incentre, I, cut BC in the ratio $a:b$, etc., etc.

Therefore, I, (9), $x=al^{-1}$; $y=bm^{-1}$; $z=cn^{-1}$, and

$$I=(al^{-1},\ bm^{-1},\ cn^{-1}).$$

The following are the coordinates of some irrational points:

$$\left.\begin{array}{ll} \text{Mean Point, } M \quad . & (l^{-1}m^{-1}n^{-1}). \\ \text{Incentre, } I \quad . \quad . & (al^{-1},\ bm^{-1},\ cn^{-1}). \\ b\text{-excentre, } I_b \quad . \quad . & (al^{-1},\ -bm^{-1},\ cn^{-1}). \\ \text{Symmedian Point, } S & (a^2l^{-1},\ b^2m^{-1},\ c^2n^{-1}). \\ \text{Brocard Points}\begin{cases}\Omega_1 \quad . \\ \Omega_2 \quad .\end{cases} & \begin{array}{l}(c^2a^2l^{-1},\ a^2b^2m^{-1},\ b^2c^2n^{-1}). \\ (a^2b^2l^{-1},\ b^2c^2m^{-1},\ c^2a^2n^{-1}).\end{array} \\ \text{Orthocentre, } P \quad . & (l^{-1}\tan A,\ m^{-1}\tan B,\ n^{-1}\tan C). \\ \text{Circumcentre, } Q \quad . & (l^{-1}\sin 2A,\ m^{-1}\sin 2B,\ n^{-1}\sin 2C). \\ \text{Midcentre (IX circle)} & \{l^{-1}(\tan A+\Sigma\tan s), \\ & \quad m^{-1}(\tan B+\Sigma\tan s),\ (n^{-1}(\tan C+\Sigma\tan s)\}. \end{array}\right\}(4)$$

CHAPTER III

THE STRAIGHT LINE

1°. Let $OA=\rho_1=\frac{\Sigma x_1 l\alpha}{\Sigma x_1 l}$ and $OB=\rho_2=\frac{\Sigma x_2 l\alpha}{\Sigma x_2 l}$ (fig. 4) be two given constant vectors, and let a third constant vector, $OR=\rho=\frac{\Sigma xl\alpha}{\Sigma xl}$, cut BA so that $BR:RA=f:g$. What are the coordinates of the point R in terms of A and B?

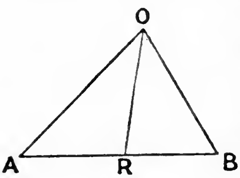

FIG. 4.

By an elementary principle of vectors,*

$$(f+g)\rho=f\rho_1+g\rho_2$$
$$=\frac{f(x_1 l\alpha+y_1 m\beta+z_1 n\gamma)}{\Sigma x_1 l}+\frac{g(x_2 l\alpha+y_2 m\beta+z_2 n\gamma)}{\Sigma x_2 l},$$

$$\rho=\frac{(fx_1\Sigma x_2 l+gx_2\Sigma x_1 l)l\alpha+(fy_1\Sigma x_2 l+gy_2\Sigma x_1 l)m\beta+\dots}{(fx_1\Sigma x_2 l+gx_2\Sigma x_1 l)l+(fy_1\Sigma x_2 l+gy_2\Sigma x_1 l)m+\dots}.$$

But $\rho=\frac{xl\alpha+ym\beta+zn\gamma}{xl+ym+zx}$.

Therefore
$$\left.\begin{aligned}x&=fx_1\Sigma x_2 l+gx_2\Sigma x_1 l,\\ y&=fy_1\Sigma x_2 l+gy_2\Sigma x_1 l,\\ z&=fz_1\Sigma x_2 l+gz_2\Sigma x_1 l,\end{aligned}\right\}\dots\dots\dots\dots(1)$$

the sought coordinates.

Ex. 1. The coordinates of A', which cuts BC in the ratio $m:n$.

$x_1=0,\ y_1=1,\ z_1=0;\ \Sigma lx_1=m.$	$x=0,\ y=mn,\ z=mn.$
$x_2=0,\ y_2=0,\ z_2=1;\ \Sigma lx_2=n.$	$A'=(0,\ mn,\ mn)=(011).$

* *Outlines of Quaternions*, p. 12.

Ex. 2. The coordinates of M_2, the middle point of CA.

$x_1=0,\ y_1=0,\ z_1=1;\ \Sigma lx_1=n.$	$x=n,\ y=0,\ z=l.$
$x_2=1,\ y_2=0,\ z_2=0;\ \Sigma lx_2=l.$	$M=(nol)=(l^{-1},\ o,\ n^{-1}).$

Ex. 3. The coordinates of R, the term of $\rho=\frac{2}{3}\gamma$.

$x_1=1,\ y_1=1,\ z_1=1;\ \Sigma lx_1=\Sigma l.$	$x=n,\ y=n,\ z=n+2\Sigma l.$
$x_2=0,\ y_2=0,\ z_2=1;\ \Sigma lx_2=n.$	$R=(n,\ n,\ 2l+2m+3n).$
$f=1;\ g=2.$	

The following is a method of determining the coordinates of a multiple or submultiple of a given vector, $\frac{t}{v}\frac{fl\alpha+gm\beta+hn\gamma}{\Sigma fl}$; $\frac{t}{v}$ being a proper or improper fraction, or a whole number.

Let $\frac{xl\alpha+ym\beta+zn\gamma}{\Sigma xl}=\frac{t}{v}\frac{fl\alpha+gm\beta+hn\gamma}{\Sigma fl}$.

Dividing across by z and eliminating γ by means of the equation $l\alpha+m\beta+n\gamma=0$, we get an equation of the form

$$M\alpha+N\beta=P\alpha+Q\beta;$$

whence $$(M-P)\alpha=(Q-N)\beta.$$

Therefore, since α and β are not parallel,

$$M-P=0;\quad Q-N=0,$$

two equations to determine the value of $\frac{x}{z}$ and $\frac{y}{z}$.

It will be found ultimately that

$$x:y:z$$
$$=\left(\frac{v}{t}-1\right)\Sigma fl+f\Sigma l:\left(\frac{v}{t}-1\right)\Sigma fl+g\Sigma l:\left(\frac{v}{t}-1\right)\Sigma fl+h\Sigma l.\ \ldots(2)$$

Ex. 1. Let $l:m:n=3:1:2$. To find the coordinates of $\frac{1}{3}\alpha$.

Here $f=1,\ g=0,\ h=0;\ \Sigma fl=3;\ \Sigma l=6;\ \frac{v}{t}-1=2.$

Therefore $x=2\times3+6;\ y=2\times3;\ z=2\times3,$

and $$x:y:z=2:1:1.$$

Consequently, $$\tfrac{1}{3}\alpha=\frac{2l\alpha+m\beta+n\gamma}{2l+m+n}=\frac{6\alpha+\beta+2\gamma}{9}.$$

Verification.

$$\frac{2l\alpha+m\beta+n\gamma}{2l+m+n}=\frac{(l\alpha+m\beta+n\gamma)+l\alpha}{9}=\frac{3\alpha}{9}=\frac{1}{3}\alpha.$$

Ex. 2. The coordinates of $-\alpha$.
Let $l:m:n=1$.

Here $\dfrac{v}{t}-1=-2$ and $x:y:z=1:-2:-2$.

Verification. $\alpha+\dfrac{\alpha-2\beta-2\beta}{-3}=\dfrac{2(\alpha+\beta+\gamma)}{3}=0.$

Ex. 3. The coordinates of the unit-vector of α, $U\alpha$ or $\dfrac{\alpha}{a}$, a being the tensor of α. Here $\dfrac{v}{t}-1=a-1$ and

$$x:y:z=al+m+n:(a-1)l:(a-1)l.$$

$$U\alpha=\frac{(al+m+n)l\alpha+l(a-1)m\beta+l(a-1)n\gamma}{al\Sigma l},$$

or if $l:m:n=1$,

$$U\alpha=\frac{(a+2)\alpha+(a-1)\beta+(a-1)\gamma}{3a}=\frac{2\alpha-\beta-\gamma}{3a}=\frac{\alpha}{a}.$$

Similarly, $U(-\alpha)=\dfrac{(a-2)\alpha+(a+1)\beta+(a+1)\gamma}{3a}.$

The coordinates of a point can only be obtained from the expression of its vector when this expression is in the standard form, I, (6).

2°. Instead of being a fixed point, let R be a variable point with the indefinite straight line AB for its locus. In this case f and g may be any two scalars whatever, and the coordinates of any and every point upon AB are of the form

$$\left.\begin{aligned} x&=tx_1+vx_2,\\ y&=ty_1+vy_2,\\ z&=tz_1+vz_2,\end{aligned}\right\} \quad \text{.........................(3)}$$

where t and v are arbitrary scalars.

Conversely, any point in the plane whose coordinates are of this form is collinear with $A=(x_1y_1z_1)$ and $B=(x_2y_2z_2)$. By hypothesis,

$$\rho=\frac{(tx_1+vx_2)l\alpha+(ty_1+vy_2)m\beta+(tz_1+vz_2)n\gamma}{(tx_1+vx_2)l+(ty_1+vy_2)m+(tz_1+vz_2)n}$$

$$=\frac{t\Sigma x_1l\alpha+v\Sigma x_2l\alpha}{t\Sigma x_1l+v\Sigma x_2l},$$

$$(t\Sigma x_1l+v\Sigma x_2l)\rho-t\Sigma x_1l\alpha-v\Sigma x_2l\alpha=0.$$

But $\rho_1 = \frac{\Sigma x_1 la}{\Sigma x_1 l}$, and $\rho_2 = \frac{\Sigma x_2 la}{\Sigma x_2 l}$ (by 1°).

Therefore $(t\Sigma x_1 l + v\Sigma x_2 l)\rho - t\rho_1 \Sigma x_1 l - v\rho_2 \Sigma x_2 l = 0.$

Now the sum of the coefficients of these three coinitial vectors is zero. Therefore R, A and B are collinear.*

3°. If t and v be eliminated from the three equations of (3), we get

$$(y_1 z_2 - y_2 z_1)x + (z_1 x_2 - z_2 x_1)y + (x_1 y_2 - x_2 y_1)z = 0, \quad \text{......(4)}$$

which may be written

$$px + qy + rz = 0, \quad \text{........................(5)}$$

or

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0. \quad \text{........................(6)}$$

Equations (4), (5) and (6) are the equations of a straight line, since they express the condition that the variable point (xyz) shall be always collinear with the two fixed points $A = (x_1 y_1 z_1)$ and $B = (x_2 y_2 z_2)$. The coefficients of (5) are the anharmonic coefficients of the line, and the line is denoted by the symbol

$$\Lambda = (pqr).$$

4°. The equations and symbols of the lines of the net (fig. 1) are as follows:

BC passes through $B = (010)$ and $C = (001)$, II, (3).

Consequently, (6),

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0;$$

or, $x = 0$, the equation of BC. The equations of the other lines are similarly obtained.

Lines.	*Equations.*	*Symbols.*
BC	$x = 0$	(100)
CA	$y = 0$	(010)
AB	$z = 0$	(001)
OA	$y - z = 0$	$(01\bar{1})$

* *Outlines of Quaternions*, p. 12.

Lines.	*Equations.*	*Symbols.*
OB	$z-x=0$	$(\bar{1}01)$
OC	$x-y=0$	$(1\bar{1}0)$
AA''	$y+z=0$	(011)
BB''	$z+x=0$	(101)
CC''	$x+y=0$	(110)
$A''B''C''$	$x+y+z=0$	(111)
$B'C'$	$y+z-x=0$	$(\bar{1}11)$
$C'A'$	$z+x-y=0$	$(1\bar{1}1)$
$A'B'$	$x+y-z=0$	$(11\bar{1})$
$B'''C'''$	$y+z-3x=0$	$(\bar{3}11)$
$C'''A'''$	$z+x-3y=0$	$(1\bar{3}1)$
$A'''B'''$	$x+y-3z=0$	$(11\bar{3})$
Λ_∞	$lx+my+nz=0$	(lmn)

5°. If we have three vectors $OA=\alpha$, $OB=\beta$, $OC=\gamma$, as in fig. 2, and if

$$\gamma=\frac{l\alpha+m\beta}{l+m},$$

l and m being constant; then the point C lies on the line AB, which it cuts in the ratio $\frac{y}{x}$. If l and m are variables,

$$\gamma=\frac{x\alpha+y\beta}{x+y}$$

expresses that the locus of C is the indefinite line AB.* In a similar way, when x, y, z are constants and the denominator of I, (6) happens to be zero, the expression is the vector of a point R which is infinitely distant; and when x, y, z vary, it implies that the locus of R is the line at infinity, Λ_∞. Hence the linear equation

$$lx+my+nz=0 \quad\text{.......................}(7)$$

is the equation of Δ_∞, being a constant relation between the coordinates of every infinitely distant point.

To illustrate this geometrically: let the point $P=(xyz)$ recede to infinity (fig. 5). At the limit, AP_2 and P_1C become parallel, and

$$\frac{BP_2}{P_2C}=\frac{BA}{AP_1}=\frac{BP_1-AP_1}{AP_1}=\frac{BP_1}{-P_1A}-1.$$

* See *Outlines of Quaternions*, p. 13.

Therefore, I, (7), $$\frac{zn}{ym} = \frac{-xl}{ym} - 1,$$

and $$lx + my + nz = 0.$$

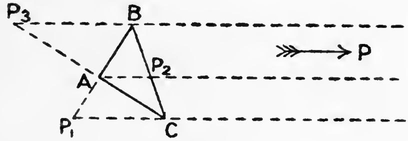

FIG. 5.

6°. The coordinates of the cross of two given straight lines $(p_1q_1r_1)$ and $(p_2q_2r_2)$.

The sought coordinates (tuv) must satisfy both the given equations. Therefore

$$p_1t + q_1u + r_1v = 0,$$
$$p_2t + q_2u + r_2v = 0.$$

Consequently,

$$\frac{t}{q_1r_2 - q_2r_1} = \frac{u}{r_1p_2 - r_2p_1} = \frac{v}{p_1q_2 - p_2q_1}.$$

Therefore the coordinates of the cross are the cofactors of x, y, z in the matrix

$$\begin{vmatrix} x & y & z \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}. \quad \text{.........................(8)}$$

Ex. The cross of (pqr) and Λ_∞.

$$\begin{vmatrix} x & y & z \\ p & q & r \\ l & m & n \end{vmatrix}$$

The cofactors of x, y and z and the coordinates of the cross are $(nq - mr,\ lr - nr,\ mp - lq)$.

7°. The coordinates of the cross of two given lines, $(p_1q_1r_1)$ and $(p_2q_2r_2)$, must satisfy the equation of any third line $(p_3q_3r_3)$ which passes through it. Therefore

$$p_3(q_1r_2 - q_2r_1) + q_3(r_1p_2 - r_2p_1) + r_3(p_1q_2 - p_2q_1) = 0;$$

or, the condition that the three lines shall be concurrent is,

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0. \quad \text{.....................(9)}$$

Ex. For $BC=(100)$, $CA=(010)$ and $OC=(1\bar{1}0)$, we have

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \bar{1} & 0 \end{vmatrix}=0.$$

8°. The coordinates of a straight line passing through the cross of two given straight lines.

Whatever loci be represented by the equations, $X=0$, $X'=0$, both these equations are satisfied by the coordinates of the points of intersection of X and X'. Therefore if k be an arbitrary scalar, the equation, $X+kX'=0$, represents a locus passing through all the points of intersection common to X and X'; for it is satisfied when $X=0$ and $X'=0$ are simultaneously satisfied. Now two straight lines intersect in one point only. Therefore the linear equation, $\Lambda+k\Lambda'=0$, represents a straight line passing through the cross of Λ and Λ'. Let $k=\frac{v}{t}$. Then

$$\begin{aligned} 0 &= \Lambda+k\Lambda'=t\Lambda+v\Lambda' \\ \text{,,} &= t(p_1x+q_1y+r_1z)+v(p_2x+q_2y+r_2z) \\ \text{,,} &= (tp_1+vp_2)x+(tq_1+vq_2)y+(tr_1+vr_2)z. \quad \ldots\ldots(10) \end{aligned}$$

Therefore the coordinates of any straight line passing through the cross of two given straight lines, $(p_1q_1r_1)$ and $(p_2q_2r_2)$, must be reducible to the form,

$$\{tp_1+vp_2,\ tq_1+vq_2,\ tr_1+vr_2\}.$$

And the converse.

Ex. If we take $t=1$ and $v=-2$, we find that one of the lines passing through the cross of $A''B''=(111)$ and $AB=(001)$ (fig. 1) is $x+y-z=0$, which is $A'B'$.

9°. If $\Lambda_1=(p_1q_1r_1)$ and $\Lambda=(pqr)$ are parallel, they concur in $\Lambda_\infty=(lmn)$. Therefore the coordinates of Λ_1 must be reducible in the form,

$$\{tp+vl,\ tq+vm,\ tr+vn\}, \text{ by } 8°. \quad \ldots\ldots\ldots(11)$$

Conversely, any two lines whose coordinates are of the form (pqr) and $\{tp+vl,\ tq+vm,\ tr+vn\}$ are parallel.

If the line $\{tp+vl,\ tq+vm,\ tr+vn\}$ passes through a known point, (fgh), we have

$$(tp+vl)f+(tq+vm)g+(tr+vn)h=0,$$

and consequently,

$$\frac{t}{v}=\frac{-(fl+gm+hn)}{fp+gq+hr},$$

by means of which relation we can calculate the coordinates of the parallel to (pqr) through (fgh).

Ex. 1. The equation of a line through B, parallel to CA (fig. 1).

Since the equation of CA is $y=0$, any line parallel to it must be of the form,

$$lx+(t+m)y+nz=0. \qquad \text{..................}(a)$$

In the present case this equation must be satisfied by the coordinates of $B=(010)$.

Therefore $$t+m=0,$$

and (a) becomes $$lx+nz=0.$$

Verification. This parallel, CA and Λ_∞ are concurrent. Therefore

$$\begin{vmatrix} l & m & n \\ 0 & 1 & 0 \\ l & o & n \end{vmatrix}=nl-nl=0.$$

Ex. 2. The equation of a parallel through C to OA, $y-z=0$.

$$lx+(t+m)y+(-t+n)z=0.$$

This equation must be satisfied by the coordinates of $C=(001)$. Therefore

$$-t+n=0 \text{ and } t=n.$$

Consequently, $$lx+(m+n)y=0,$$

the required equation.

Cor. Λ_∞ is parallel to every straight line in the plane.

10°. The angle contained by two given straight lines (fig. 6).

(a) Let the two lines be (por) and $(p'or')$, which pass through the corner B of the given triangle. These lines cut CA in $$P=(\bar{r}op), \text{ and } P'=(\bar{r}'op').$$

Therefore $$\frac{AP}{PC}=\frac{-np}{lr}; \quad \frac{AP'}{P'C}=\frac{-np'}{lr'}.$$

Let the angles which BP and BP' respectively make with AB be θ and θ'. Then

$$\frac{AP}{PC}=\frac{c\sin\theta}{a\sin(B-\theta)};\quad \frac{AP'}{P'C}=\frac{c\sin\theta'}{a\sin(B-\theta')}.$$

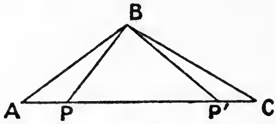

FIG. 6.

Consequently,

$$\tan\theta=\frac{anp\sin B}{anp\cos B-clr};\quad \tan\theta'=\frac{anp'\sin B}{anp'\cos B-clr'}.$$

If ϕ be the angle between the given lines, $\phi=\theta\sim\theta'$, and

$$\tan\phi=\pm\tan(\theta-\theta')=\pm\frac{\tan\theta-\tan\theta'}{1+\tan\theta\tan\theta'}.$$

Substituting in this equation the values of $\tan\theta$ and $\tan\theta'$ given above, we get

$$\tan\phi=\pm\frac{nacl\sin B(rp'-r'p)}{l^2c^2rr'+n^2a^2pp'-nlca\cos B(rp'+r'p)}.\quad(12)$$

If the two lines are at right angles, $\tan\phi=\infty$ and

$$l^2c^2rr'+n^2a^2pp'-nlca\cos B(rp'+r'p)=0,\ \ldots\ldots(13)$$

the relation between the coordinates of two straight lines which intersect at right angles in B.

(*b*) Let $(p_1q_1r_1)$ and $(p_2q_2r_2)$ intersect in any point in the plane.

The equations of parallels to them through B are

$$(lq_1-mp_1)x+(nq_1-mr_1)z=0,$$
$$(lq_2-mp_2)x+(nq_2-mr_2)z=0.$$

Substituting the coefficients of x and z in this equation for r and p, r' and p', in (12),

$$\tan\phi=\pm\frac{lmnca\sin B\begin{vmatrix} l & m & n\\ p_1 & q_1 & r_1\\ p_2 & q_2 & r_2\end{vmatrix}}{\begin{aligned}&m^2n^2a^2p_1p_2+n^2l^2b^2q_1q_2+l^2m^2c^2r_1r_2\\ &\quad-lmn\{lbc\cos A(q_1r_2+q_2r_1)\\ &\qquad+mca\cos B(r_1p_2+r_2p_1)\\ &\qquad+nab\cos C(p_1q_2+p_2q_1)\}\end{aligned}}.\ (14)$$

If the two lines are $y=0$ and $px+qy+rz=0$,

$$\tan\phi=\pm\frac{mc\sin A(np-Cr)}{mna\cos Cp-nlbq+lmc\cos Ar}. \quad \ldots\ldots(15)$$

(*c*) If the two lines are rectangular, $\tan\phi=\infty$ and

$$\begin{aligned}0=m^2n^2a^2p_1p_2+n^2l^2b^2q_1q_2+l^2m^2c^2r_1r_2\\ -lmn\{lbc\cos A(q_1r_2+q_2r_1)+mca\cos B(r_1p_2+r_2p_1)\\ +nab\cos C(p_1q_1+p_2q_1)\},\quad (16)\end{aligned}$$

which may be written

$$\begin{aligned}&mna^2\{(mp-lq)(np'-lr')+(mp'-lq')(np-lr)\}\\ &+nlb^2\{(nq-mr)(lq'-mp')+(nq'-mr')(lq-mp)\}\\ &+lmc^2\{(lr-np)(mr'-nq')+(lr'-np')(mr-nq)\}=0;\end{aligned}$$

or if the given triangle be equilateral and its mean point the origin,

$$2pp'+2qq'+2rr'=qr'+q'r+rp'+r'p+pq'+p'q.$$

It appears from these expressions that Λ_∞ is perpendicular to every straight line in the plane.

Ex. Let lines be drawn from the corners A and C of the given triangle to some point $(x'y'z')$, with the condition that these lines shall be at right angles. What is the relation between the coordinates of AX and CX under this condition?

The equations of the lines are

$$z'y-y'z=0, \text{ and } y'x-x'y=0,$$

and by (16)

$$\begin{aligned}-n^2l^2b^2z'x'-lmn(lbc\cos Ax'y'-mca\cos By^2\\ +nab\cos Cy'z')=0.\end{aligned}$$

Suppose X to be a variable point, and omitting the dashes, we have

$$m^2ca\cos By^2-mnab\cos Cyz-nlb^2zx-lmbc\cos Axy=0,$$

the equation of a circle with the line CA for a diameter.

11°. If a line $(p'q'r')$, perpendicular to a given line (pqr), passes through a given point (fgh), we have the equation, $\frac{p'}{r'}f+\frac{q'}{r'}g+h=0$. The condition (16) gives another equation to determine the ratios $\frac{p'}{r'}, \frac{q'}{r'}$. Solving these equations we get

$$\left.\begin{aligned}p' &= l^2(nq-mr)(mg+nh)bc\cos A\\ &\quad -lm^2g(lr-np)ca\cos B - n^2lh(mp-lq)ab\cos C,\\ q' &= -l^2mf(nq-mr)bc\cos A\\ &\quad +m^2(lr-np)(nh+lf)ca\cos B\\ &\quad -mn^2h(mp-lq)ab\cos C,\\ r' &= -nl^2f(nq-mr)bc\cos A - m^2ng(lr-np)ca\cos B\\ &\quad +n^2(mp-lq)(lf+mg)ab\cos C,\end{aligned}\right\} \quad (17)$$

the coordinates of a line which passes through the point (fgh) and is at right angles to the line (pqr).

This equation holds good whether the point (fgh) lies on or off the line (pqr).

Owing to the complexity of these expressions, which are often wanted, it is frequently simpler to let fall a perpendicular on (pqr) from one of the corners of the given triangle and find the equation of a parallel to it through (fgh).

12°. The connexion between Anharmonic and Trilinear Coordinates (fig. 7).

Let ABC be the given triangle and O the given origin.

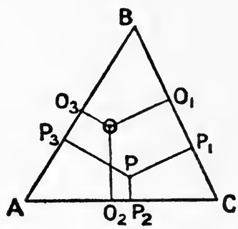

Fig. 7.

Let P be any point in the plane; (xyz) its anharmonic coordinates; $PP_1=\alpha$, $PP_2=\beta$, $PP_3=\gamma$, its trilinear coordinates; $OO_1=\delta$, $OO_2=\epsilon$, $OO_3=\zeta$, the trilinear coordinates of the origin O. Then

$$\left.\begin{aligned} l:m:n &= OBC:OCA:OAB = a\delta:b\epsilon:c\zeta \\ \text{and} \qquad \delta:\epsilon:\zeta &= bcl:cam:abn.\end{aligned}\right\} \quad \text{.........(18)}$$

Again, $lx:my:nz=PBC:PCA:PAB=a\alpha:b\beta:c\gamma$. ...(19)

Hence
$$\left.\begin{aligned}\alpha:\beta:\gamma &= bclx:camy:abnz,\\ x:y:z &= \epsilon\zeta\alpha:\zeta\delta\beta:\delta\epsilon\gamma.\end{aligned}\right\} \quad \text{...............(20)}$$

Ex. 1. The trilinear coordinates of the circumcentre are $\alpha=\frac{1}{2}a\cot A$, $\beta=\frac{1}{2}b\cot B$, $\gamma=\frac{1}{2}c\cot C$. By (20), we get

$$Q=(\tfrac{1}{2}mna^2\cot A,\ \tfrac{1}{2}nlb^2\cot B,\ \tfrac{1}{2}lmc^2\cot C)$$
$$,,=(l^{-1}a\cos A,\ m^{-1}b\cos B,\ n^{-1}c\cos C)$$
$$,,=(l^{-1}\sin 2A,\ m^{-1}\sin 2B,\ n^{-1}\sin 2C),\ \text{II, } 3^\circ.$$

Ex. 2. The anharmonic equation of Λ_∞ is

$$lx+my+nz=0;$$

and by (19) this equation becomes, $a\alpha+b\beta+c\gamma=0$, the trilinear equation of the line.

Ex 3. The trilinear equation of the Brocard circle is

$$abc(\alpha^2+\beta^2+\gamma^2)=a^3\beta\gamma+b^3\gamma\alpha+c^3\alpha\beta.$$

This is transformed by (20) into

$$b^2c^2l^2x^2+c^2a^2m^2y^2+a^2b^2n^2z^2-a^4mnyz-b^4nlzx-c^4lmxy=0,$$

the anharmonic equation of this circle.

CHAPTER IV

LENGTHS, AREAS AND ANGULAR FUNCTIONS

1°. To find the distance between any two given points in the plane, $P_1=(x_1y_1z_1)$, $P_2=(x_2y_2z_2)$ (fig. 8).

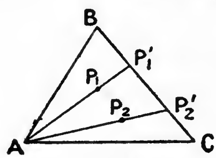

FIG. 8.

By I, (8), the lines drawn from A through the given points cut BC in the ratios

$$\frac{CP_1}{P_1B}=\frac{my_1}{nz_1};\quad \frac{CP_2}{P_2B}=\frac{my_2}{nz_2}.$$

Let the vectors $BC=\alpha'$, $CA=\beta'$, $AB=\gamma'$;

so that, $$\alpha'+\beta'+\gamma'=0.$$

Then, III, 1°, $$AP_1'=\frac{my_1\gamma'-nz_1\beta'}{my_1+nz_1}.$$

But $$\frac{AP_1}{P_1P'_1}=\frac{my_1+nz_1}{lx_1}.$$

Therefore $$AP_1=\frac{my_1\gamma'-nz_1\beta'}{\Sigma lx_1}.$$

Similarly, $$AP_2=\frac{my_2\gamma'-nz_2\beta'}{\Sigma lx_2}.$$

Let $P_2P_1=\delta$. Then

$$\delta=AP_1-AP_2$$
$$=\frac{(my_1\Sigma lx_2-my_2\Sigma lx_1)\gamma'-(nz_1\Sigma lx_2-nz_2\Sigma lx_1)\beta'}{\Sigma lx_1\Sigma lx_2}.$$

Let the minors of the matrix $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ be as usual, III, 3°, (5),

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = p; \quad \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} = q; \quad \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = r,$$

p, q, r being thus the coordinates of the straight line passing through the given points $(x_1y_1z_1)$ and $(x_2y_2z_2)$. Then

$$y_1\Sigma lx_2 - y_2\Sigma lx_1 = np - lr; \quad z_1\Sigma lx_2 - z_2\Sigma lx_1 = lq - mp;$$

and
$$\delta = \frac{(mnp - lmr)\gamma' - (nlq - mnp)\beta'}{\Sigma lx_1 \Sigma lx_2}$$

$$,, = \frac{mnp(\beta' + \gamma') - nlq\beta' - lmr\gamma'}{\Sigma lx_1 \Sigma lx_2}$$

$$,, = \frac{-(mnp\alpha' + nlq\beta' + lmr\gamma')}{\Sigma lx_1 \Sigma lx_2}. \qquad (1)$$

$$\delta^2\Sigma^2 lx_1 \Sigma lx_2 = m^2n^2p^2\alpha'^2 + n^2l^2q^2\beta'^2 + l^2m^2r^2\gamma'^2$$
$$+ 2lmn(lqr \,.\, S\beta'\gamma' + mrp \,.\, S\gamma'\alpha' + npq \,.\, S\alpha'\beta').$$

Now

$$\delta^2 = -d^2 = -P_2P_1{}^2; \quad \alpha'^2 = -a^2, \text{ etc.}; \quad S\beta'\gamma' = bc \cos A, \text{ etc.*}$$

Therefore

$$\left.\begin{aligned} d^2\Sigma^2 lx_1 \Sigma^2 lx_2 &= m^2n^2a^2p^2 + n^2l^2b^2q^2 + l^2m^2c^2r^2 \\ &\quad - 2lmn(lqrbc \cos A + mrpca \cos B + npqab \cos C) \\ ,, \quad &= mna^2(mp - lq)(np - lr) + nlb^2(nq - mr)(lq - mp) \\ &\quad + lmc^2(lr - np)(mr - nq). \end{aligned}\right\} (2)$$

Let the right-hand member, which occurs frequently, be Z^2, and

$$d = \frac{\pm Z}{\Sigma lx_1 \Sigma lx_2}. \qquad (3)$$

If $\quad l : m : n = 1$ and $a = b = c$,

$$Z^2 = p^2 + q^2 + r^2 - qr - rp - pq = (p + \omega q + \omega^2 r)(p + \omega^2 q + \omega r),$$

where ω and ω^2 are cube roots of unity.

Ex. 1. The distance from B to C (fig. 8).

Here $p = 1$, $q = 0$, $r = 0$; $\Sigma lx_1 = m$, $\Sigma lx_2 = n$; $Z = mna$; and consequently, $dmn = mna$, and, $d = a$.

Ex. 2. The distance from O to A.

$$p = 0, \ q = 1, \ r = -1; \ \Sigma lx_1 = \Sigma l, \ \Sigma lx_2 = l;$$
$$Z^2 = l^2(n^2b^2 + m^2c^2 + 2mnbc \cos A);$$

and
$$OA^2 = \frac{n^2b^2 + m^2c^2 + 2mnbc \cos A}{\Sigma^2 l}.$$

* $S\beta'\gamma'$ is the scalar of the quaternion $\beta'\gamma'$; *Outlines of Quaternions*, p. 47.

If I (incentre) be taken as origin, $l:m:n=a:b:c$, and

$$IA=\frac{2bc\cos\frac{1}{2}A}{a+b+c}.$$

If Q (circumcentre) be taken as origin,

$$l:m:n=\sin 2A:\sin 2B:\sin 2C, \text{ and}$$

$$QA^2=\frac{b^2(\cos^2 B+\cos^2 C+2\cos A\cos B\cos C)}{4\sin^2 A\sin^2 B}=\frac{b^2}{4\sin^2 B}=R^2.$$

2°. The perpendicular distance of the corners of the given triangle from a given line $\Lambda=(pqr)=0$ (fig. 9).

Produce AB, AC to meet Λ in $B'=(q\bar{p}o)$ and $C'=(\bar{r}op)$. Let the perpendiculars from A, B, C be d_1, d_2, d_3, and let the function of the coordinates of $B'C'$ be Z. By (3) we get the following lengths:

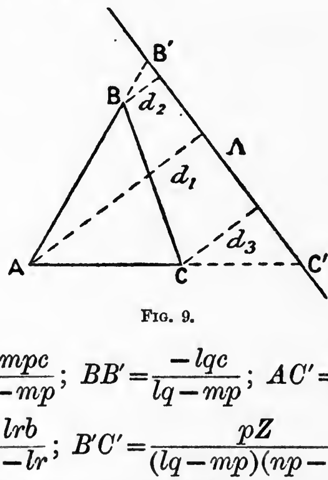

FIG. 9.

$$AB'=\frac{-mpc}{lq-mp};\quad BB'=\frac{-lqc}{lq-mp};\quad AC'=\frac{-npb}{np-lr};$$

$$CC'=\frac{-lrb}{np-lr};\quad B'C'=\frac{pZ}{(lq-mp)(np-lr)}.$$

$$d_1\,.\,B'C'=2\text{ area }AB'C'=AB'\,.\,AC'\,.\sin A, \text{ and}$$

$$d_1=\frac{mnpbc\sin A}{Z}. \qquad \text{..........................(4)}$$

Since $\dfrac{d_2}{d_1}=\dfrac{BB'}{AB'}$ and $\dfrac{d_3}{d_1}=\dfrac{CC'}{AC'}$, we get

$$d_2=\frac{nlqca\sin B}{Z};\quad d_3=\frac{lmrab\sin C}{Z}.$$

3°. The distance from the origin, O, to any given line, $(pqr)=0$.

Since $l\alpha+m\beta+n\gamma=0$, O is the complex mean point of the system of points A, B, C, weighted with the given scalars, l, m, n. Consequently, the perpendicular distance from O to any line is the complex mean of the distances of A, B, C from it; that is,

$$d=\frac{ld_1+md_2+nd_3}{\Sigma l}=\frac{lmnbc\sin A\Sigma p}{Z\Sigma l}. \qquad \text{............(5)}$$

Ex. The distance from O to CA.

The equation of CA being $y=0$, $\Sigma p=1$ and $Z=nlb$. Therefore

$$d=\frac{mc\sin A}{\Sigma l}.$$

If the incentre be origin, $d=\dfrac{bc\sin A}{a+b+c}=r$.

If the mean point be origin, $d=\dfrac{c\sin A}{3}$.

The distance from O to $A''B''$ is $\dfrac{3lmnbc\sin A}{Z\Sigma l}$. If the triangle be equilateral and its mean point the origin, this expression becomes $d=\dfrac{\sqrt{3}}{2Z}$.

But in this case $Z=0$; therefore $d=\infty$.

4°. The perpendicular distance between two parallel lines.

$$\Lambda_1=(p_1q_1r_1) \quad \text{and} \quad \Lambda_2=(p_2q_2r_2).$$

Let e_1 and e_2 be the distances of the two lines from O. Then whatever be the position of O,

$$d=e_1\sim e_2.$$

Let Z_1 be the function of Λ_1 and Z_2 the function of Λ_2.

Then, (5), $\qquad e_1=\dfrac{lmnbc\sin A\Sigma p_1}{Z_1\Sigma l}$.

Now, since Λ_2 is parallel to Λ_1, its coordinates are of the form

$$(tp_1+l,\ tq_1+m,\ tr_1+n).$$

Consequently, $\Sigma p_2=t\Sigma p_1+\Sigma l$, and it will be found that $Z_2=tZ_1$.

Therefore $\qquad d=e_1\sim e_2=\dfrac{-1}{t}\,\dfrac{lmnbc\sin A}{Z_1}. \qquad$ ....(6)

Ex. Let the parallels be CA and $lx-my+nz=0$, a line which passes through (onm), the midpoint of BC. Then $t=-2m$ and $Z_1=nlb$. Therefore

$$d=\frac{1}{2m}\,\frac{lmnbc\sin A}{nlb}=\frac{c\sin A}{2}.$$

5°. The distance from any point to a given straight line. Find the value of the factor t for a parallel to the given line through the given point and apply (6).

Let the given point be (fgh) and the given line (pqr).

Then $$t=\frac{-\Sigma fl}{\Sigma fp}.$$

Therefore $$d=\frac{\Sigma fp}{\Sigma fl}\cdot\frac{lmnbc\sin A}{Z}. \qquad\text{.................(7)}$$

Ex. The distance of the symmedian point from BC.

Here $f=\dfrac{a^2}{l}$, $g=\dfrac{b^2}{m}$, $h=\dfrac{c^2}{n}$; $p=1$, $q=0$, $r=0$; $\Sigma fp=\dfrac{a^2}{l}$; $\Sigma fl=\Sigma a^2$; $Z=mna$. Therefore

$$d=\frac{a^2}{l\Sigma a^2}\,\frac{lmnbc\sin A}{mna}=\frac{abc\sin A}{\Sigma a^2}.$$

6°. The area of a triangle in terms of the coordinates of its corners.

Let the corners of the triangle EFG be

$$E=(x_1y_1z_1),\quad F=(x_2y_2z_2),\quad G=(x_3y_3z_3),$$

and let the function of FG be Z_1. The equation of FG is

$$(y_2z_3-y_3z_2)x+(z_2x_3-z_3x_2)y+(x_2y_3-x_3y_2)z=0.$$

Therefore

$$p_1=y_2z_3-y_3z_2;\quad q_1=z_2x_3-z_3x_2;\quad r_1=x_2y_3-x_3y_2.$$

The length of FG is $\dfrac{Z_1}{\Sigma lx_2\Sigma lx_3}$. For a parallel to FG through E,

$$t=\frac{-\Sigma lx_1}{\Sigma p_1x_1}=\frac{-\Sigma lx_1}{|x_1y_2z_3|};$$

and the perpendicular from E on FG is

$$d=\frac{lmnbc\sin A\,|x_1y_2z_3|}{Z_1\Sigma lx_1}.$$

$$\text{Area } EFG=\tfrac{1}{2}d\,.\,FG=\frac{lmnbc\sin A}{2\Sigma lx_1\Sigma lx_2\Sigma lx_3}\begin{vmatrix}x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\\ x_3 & y_3 & z_3\end{vmatrix}. \qquad\text{...(8)}$$

As verification, (8) becomes $\dfrac{bc\sin A}{2}$ when EFG is the given triangle.

7°. The sine of an angle—the angle E of the triangle EFG, 6°.

Let the functions of the coordinates of EG and EF be Z_2 and Z_3.

Then the length of the perpendicular from F on EG is

$$p=\frac{lmnbc\sin A\,|x_1y_2z_3|}{Z_2\Sigma lx_2}.$$

The length of EF is $\dfrac{Z_3}{\Sigma lx_1\Sigma lx_2}$.

Therefore

$$\sin E=\frac{p}{EF}=\frac{lmnbc\sin A\,\Sigma lx_1\,|x_1y_2z_3|}{Z_2Z_3}. \qquad \text{.........(9)}$$

CHAPTER V

THE GENERAL EQUATION OF THE SECOND DEGREE

1°. The general equation of the second degree,

$$ux^2+vy^2+wz^2+2u'yz+2v'zx+2w'xy=0, \quad\text{.........(1)}$$

represents in general a conic section, because it is cut in two, and only two, points by every straight line in the plane.

2°. Differentiating successively with respect to x, y, z,

$$\left.\begin{aligned}\frac{1}{2}\frac{d\phi}{dx}&=ux+w'y+v'z=\phi_x,\\ \frac{1}{2}\frac{d\phi}{dy}&=w'x+vy+u'z=\phi_y,\\ \frac{1}{2}\frac{d\phi}{dz}&=v'x+u'y+wz=\phi_z.\end{aligned}\right\} \quad\text{...............(2)}$$

Obviously, $$x\phi_x+y\phi_y+z\phi_z=\phi(xyz). \quad\text{............(3)}$$

Multiplying the 3 equations of (2) respectively by x', y', z',

$$\begin{aligned}x'\phi_x+y'\phi_y+z'\phi_z&\\ &=(ux+w'y+v'z)x'+(w'x+vy+u'z)y'+(v'x+u'y+wz)z'\\ \text{,,}\quad &=(ux'+w'y'+v'z')x+(w'x'+vy'+u'z')y\\ &\qquad\qquad+(v'x'+u'y'+wz')z\\ \text{,,}\quad &=x\phi_{x'}+y\phi_{y'}+z\phi_{z'}. \quad\text{..(4)}\end{aligned}$$

3°. Suppose that

$$\phi(xyz)=(px+qy+rz)(p'x+q'y+r'z)=0,$$

the product of two straight lines. Then

$$\begin{aligned}\phi_x&=p(p'x+q'y+r'z)+p'(px+qy+rz),\\ \phi_y&=q(p'x+q'y+r'z)+q'(px+qy+rz),\\ \phi_z&=r(p'x+q'y+r'z)+r'(px+qy+rz).\end{aligned}$$

Hence the three equations, $\phi_x=0$, $\phi_y=0$, $\phi_z=0$ represent three straight lines passing through the cross of (pqr) and $(p'q'r')$, III, 8°. Therefore the three straight lines $(uw'v')$, $(w'vu')$, $(v'u'w)$ are concurrent, and consequently, III, (9),

$$\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} = 0, \quad \text{.....................(5)}$$

or, $$0=uvw+2u'v'w'-uu'^2-vv'^2-ww'^2.$$

This determinant is the discriminant of $\phi(xyz)$, and expresses the relation between the coefficients of the function when it is the product of two linear factors. In future it will be designated by Δ, and its minors will be designated as follows:

$$\left.\begin{array}{l} vw-u'^2=U; \quad wu-v'^2=V; \quad uv-w'^2=W. \\ v'w'-uu'=U'; \; w'u'-vv'=V'; \; u'v'-ww'=W'. \\ VW-U'^2=u\Delta; \; V'W'-UU'=u'\Delta, \text{ etc., etc.} \\ \qquad \begin{vmatrix} U & W' & V' \\ W' & V & U' \\ V' & U' & W \end{vmatrix} = \Delta^2. \end{array}\right\} \text{......(6)}$$

4°. Given the coordinates of $X'=(x'y'z')$, one point of section of a conic by a straight line FX (fig. 10); to find the coordinates of the second point of section, X.

Let $X=(xyz)$ be the second point of section, and let $F=(fgh)$ be any point on the given line. Then, III, (3),

$$x=x'+tf, \quad y=y'+tg, \quad z=z'+th. \; \ldots(a)$$

Since x is on the curve,

$$0=u(x'+tf)^2+\ldots 2w'(x'+tf)(y'+tg)$$

$$,, =\phi(fgh)t^2+2(f\phi_{x'}+g\phi_{y'}+h\phi_{z'})+\phi(x'y'z').$$

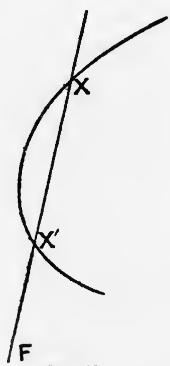

FIG. 10.

Now since $(x'y'z')$ is on the curve $\phi(x'y'z')=0$, and, consequently, one root of the quadratic (corresponding to X') is zero. The other root is

$$t=\frac{-2(f\phi_{x'}+g\phi_{y'}+h\phi_{z'})}{\phi(fgh)}.$$

Let $f\phi_{x'}+g\phi_{y'}+h\phi_{z'}=\sigma$, and we have from ($a$),

$$x=x'-\frac{2f\sigma}{\phi(fgh)};\quad y=y'-\frac{2g\sigma}{\phi(fgh)};\quad z=z'-\frac{2h\sigma}{\phi(fgh)}, \quad \ldots(7)$$

the sought coordinates.

5°. Were the line FX (fig. 10) to revolve to the left-hand in the plane round X', at a certain moment X would coincide with X'. At this moment

$$x=y';\quad y=y';\quad z=z',$$

and FX', which then passes through two coincident points of the curve at X', becomes the tangent at this point. Now we obtain these three equalities from (7) when

$$\sigma=f\phi_{x'}+g\phi_{y'}+h\phi_{z'}=0.$$

When, therefore, this is the relation between the coordinates of F and X', FX' touches the curve at X'.

By 4°, (a), $f=\dfrac{x-x'}{t},\quad g=\dfrac{y-y'}{t},\quad h=\dfrac{z-z'}{t}.$

Consequently, FX' touches the curve when

$$x\phi_{x'}+y\phi_{y'}+z\phi_{z'}-\phi(x'y'z')=0,$$

or since $$\phi(x'y'z')=0$$

when $$x'\phi_x+y'\phi_y+z'\phi_z=0. \quad \ldots\ldots\ldots\ldots(8)$$

This is the equation of the tangent to the conic at $(x'y'z')$.

6°. The condition that a straight line shall touch a given conic. Let (pqr) be the line and (fgh) its point of contact. The equation of the tangent at this point is, (7),

$$x\phi_f+y\phi_g+z\phi_h=0.$$

But $$px+qy+rz=0.$$

Therefore $$\frac{\phi_f}{p}=\frac{\phi_g}{q}=\frac{\phi_h}{r}=(\text{say})-k.$$

Therefore
$$uf+w'g+v'h+pk=0,$$
$$w'f+vg+u'h+qk=0,$$
$$v'f+u'g+wh+rk=0.$$

Also $$pf+qg+rh=0,$$

since (fgh) lies upon the line (pqr).

Therefore

$$0=\begin{vmatrix} p & q & r & o \\ u & w' & v' & p \\ w' & v & u' & q \\ v' & u' & w & r \end{vmatrix} = Up^2+Vq^2+Wr^2+2U'qr+2V'rp+2W'pq = F(pqr) \quad \ldots(9)$$

is the condition for the tangency of the given line (pqr).

7°. Let $F=(fgh)$ be a fixed point, and let a straight line passing through it cut a conic in $X_1=(x_1y_1z_1)$ and $X_2=(x_2y_2z_2)$. The tangents at these points are, (8),

$$x_1\phi_x+y_1\phi_y+z_1\phi_z=0; \quad x_2\phi_x+y_2\phi_y+z_2\phi_z=0;$$

and for their cross,

$$\frac{\phi_x}{y_1z_2-y_2z_1}=\frac{\phi_y}{z_1x_2-z_2x_1}=\frac{\phi_z}{x_1y_2-x_2y_1}. \quad \ldots\ldots\ldots\ldots(a)$$

But since F, X_1, X_2 are collinear,

$$f(y_1z_2-y_2z_1)+g(z_1x_2-z_2x_1)+h(x_1y_2-x_2y_1)=0. \quad \ldots\ldots(b)$$

From (a) and (b),

$$f\phi_x+g\phi_y+h\phi_z=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

the equation of the polar of (fgh) in respect to the conic $\phi(x, y, z)$. For this equation, being independent of X_1 and X_2, represents the locus of the cross of the tangents drawn at the two points in which any straight line whatever, passing through F, cuts the conic. Secondly, being of the first degree, it shows that the locus of the cross of all these tangents is a straight line. Thirdly, being identical in form with the equation of the tangent, (9), it shows that when the pole, F, is on the conic, *i.e.* when it moves towards the curve along FX_2 and ultimately coalesces with X_1, its polar is the tangent at this point.

8°. Let the pole of $px+qy+rz=0$ be (fgh). Then, (10), its polar is

$$\phi_f x+\phi_g y+\phi_h z=0$$

and

$$px+qy+rz=0.$$

Therefore

$$\frac{\phi_f}{p}=\frac{\phi_g}{q}=\frac{\phi_h}{r}=-k.$$

Hence

$$uf+w'g+v'h+pk=0,$$
$$w'f+vg+u'h+qk=0,$$
$$v'f+u'g+wh+rk=0.$$

Consequently,

$$\frac{f}{\begin{vmatrix} w' & v' & p \\ v & u' & q \\ u' & w & r \end{vmatrix}} = \frac{-g}{\begin{vmatrix} u & v' & p \\ w' & u' & q \\ v' & w & r \end{vmatrix}} = \frac{h}{\begin{vmatrix} u & w' & p \\ w' & v & q \\ v' & u' & r \end{vmatrix}} = \frac{-k}{\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}}, \ldots(11)$$

or, treating the constants p, q, r as variables,

$$f:g:h = Up + W'q + V'r : W'p + Vq + U'r : V'p + U'q + Wr$$
$$\text{,,} \quad = F_p : F_q : F_r.$$

9°. (*a*) If the point (fgh) lies on the polar of the point $(f'g'h')$, then $(f'g'h')$ lies on the polar of (fgh).

The polar of (fgh) is $\phi_f x + \phi_g y + \phi_h z = 0$, ...$(a)$

,, ,, $(f'g'h')$ is $\phi_{f'} x + \phi_{g'} y + \phi_{h'} z = 0$. ....$(b)$

If (fgh) lies on (b),

$$0 = \phi_{f'} f + \phi_{g'} g + \phi_{h'} h = \phi_f f' + \phi_g g' + \phi_h h',$$

which is the condition that $(f'g'h')$ should lie on (a).

(fgh) and $(f'g'h')$ are conjugate points.

(*b*) If a straight line (pqr) passes through the pole of the line $(p'q'r')$, then $(p'q'r')$ passes through the pole of (pqr).

Let (fgh) be the pole of $(p'q'r')$. Then the polar of (fgh) is

$$\phi_f x + \phi_g y + \phi_h z = 0$$

and

$$p'x + q'y + r'z = 0.$$

Consequently,

$$\frac{\phi_f}{p} = \frac{\phi_g}{q} = \frac{\phi_h}{r} = -k.$$

Therefore

$$\begin{aligned} uf + w'g + v'h + p'k &= 0, \\ w'f + vg + u'h + q'k &= 0, \\ v'f + u'g + wk + r'k &= 0. \end{aligned}$$

Also

$$pf + qg + rh \qquad = 0,$$

because the line (pqr) passes through (fgh), the pole of $(p'q'r')$.

Therefore

$$\begin{vmatrix} u & w' & v' & p' \\ w' & v & u' & q' \\ v' & u' & w & r' \\ p & q & r & o \end{vmatrix} = 0$$

is the condition that (pqr) should pass through the pole of $(p'q'r')$.

But this matrix is identically equal to

$$\begin{vmatrix} u & w' & v' & p \\ w' & v & u' & q \\ v' & u' & w & r \\ p' & q' & r' & o \end{vmatrix} = 0,$$

which for similar reasons is the condition that $(p'q'r')$ should pass through the pole of (pqr). (pqr) and $(p'q'r')$ are conjugate lines.

(*c*) The cross of two straight lines Λ_1, Λ_2 is the pole of the join of their poles, Λ.

Since the pole of Λ_1 lies on Λ, the pole of Λ lies on Λ_1, (*b*). Similarly, the pole of Λ lies on Λ_2. Therefore the only point which Λ_1 and Λ_2 have in common, their cross, is the pole of Λ.

(*d*) If a number of points are collinear, their polars are concurrent. Let the points $P_1 \ldots P_n$ lie on Λ. Then (*a*) since Λ passes through P_1, the polar of P_1 passes through the pole of Λ. Similarly, the polars of $P_2 \ldots P_n$ pass through the pole of Λ, which is the common cross of the polars of these points.

(*e*) Conversely, if a number of lines are concurrent, their poles are collinear. Let $\Lambda_1 \ldots \Lambda_n$ concur in P. Then, by (*c*), p, the polar of P, is the join of the poles of Λ_1 and Λ_2, Λ_1 and Λ_3, Λ_2 and Λ_3, etc. Or p is the locus of the poles of the given lines.

10°. To find the ratios of the segments into which a given finite straight line, FP, is cut by a conic.

Let $F=(fgh)$, $P=(pqr)$, and let the sought ratio be $t:1$. Then the vector of the point of section is

$$\rho=\frac{tOF+OP}{t+1}=\frac{tfl\alpha+tgm\beta+thn\gamma}{(t+1)(fl+gm+hn}+\frac{pl\alpha+qm\beta+rn\gamma}{(t+1)(pl+qm+rn)},$$

$$,,=\frac{(tf\Sigma pl+p\Sigma fl)l\alpha+(tg\Sigma pl+q\Sigma fl)m\beta+(th\Sigma fl+r\Sigma fl)n\gamma}{(t+1)\Sigma fl\Sigma pl}.$$

Consequently, the coordinates of the point of section are

$$\{(tf\Sigma pl+p\Sigma fl),\quad (tg\Sigma pl+q\Sigma fl),\quad (th\Sigma pl+r\Sigma fl)\}.$$

Now this point lies upon the conic. Substituting its coordinates in the general equation of the second degree, we get

$$\Sigma^2 pl\phi(f,g,h)t^2+2\Sigma fl\Sigma pl(p\phi_f+q\phi_g+r\phi_h)t \\ +\Sigma^2 fl\phi(p,q,r)=0. \quad \ldots\ldots(12)$$

The roots of this quadratic are the values of t for the two points X_1, X_2, in which the line FP is cut by the conic, *i.e.* $\frac{FX_1}{X_1P}$ and $\frac{FX_2}{X_2P}$.

11°. Let the roots of (12) be real and their sum zero. Then the coefficient of the second term vanishes and

$$p\phi_f+q\phi_g+r\phi_h=0,$$

which shows that the equation of the polar of the point F,

$$x\phi_f+y\phi_g+z\phi_h=0,$$

is satisfied by the coordinates of the point P. Therefore P lies on the polar of F when the sum of the roots of (12) is zero, and

$$t=\pm\frac{\Sigma fl}{\Sigma pl}\sqrt{\frac{\phi(p, q, r)}{\phi(f, g, h)}}.$$

The line FP, then, is cut positively and internally by the conic in X_1, and negatively and externally in X_2 in the same ratio, *i.e.*
$$\frac{FX_1}{X_1P}=-\frac{X_2F}{PX_2},$$

and
$$\frac{FX_1 . PX_2}{X_1P . X_2F}=(FX_1PX_2)=-1. \qquad (13)$$

FP is thus the harmonic mean between FX_1 and FX_2, P being a point upon the polar of F. Therefore a line which passes through a given point and cuts a given conic, is divided harmonically by the point, its polar and the conic, whether the point lies without or within the conic.

12°. Let fig. 11 represent a central conic. Let $F_1 \dots F_n$ be points on Λ_∞ and let $\Lambda_1 \dots \Lambda_n$ be their polars. Since the given points are collinear, their polars concur in K, the pole of Λ_∞, 9° (d). By (13) all chords drawn in the directions of the infinitely distant points $F_1 \dots F_n$ are bisected respectively by $\Lambda_1 \dots \Lambda_n$. Consequently, all chords passing through their common cross are bisected in K, which is the centre of the curve.

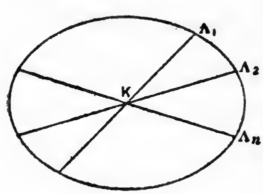

Fig. 11.

Since the centre of the conic, $K=(\bar{x}\,\bar{y}\,\bar{z})$, is the pole of Λ_∞, by (11) its coordinates are

$$\frac{\bar{x}}{\begin{vmatrix} w' & v' & l \\ v & u' & m \\ u' & w & n \end{vmatrix}}=\frac{-\bar{y}}{\begin{vmatrix} u & v' & l \\ w' & u' & m \\ v' & w & n \end{vmatrix}}=\frac{\bar{z}}{\begin{vmatrix} u & w' & l \\ w' & v & m \\ v' & u' & n \end{vmatrix}}=\frac{-k}{\Delta}. \quad (14)$$

13°. The three matrices of (14) are the cofactors of l, m and n in the matrix

$$\begin{vmatrix} l & m & n & o \\ u & w' & v' & l \\ w' & v & u' & m \\ v' & u' & w & n \end{vmatrix}=D,$$

which is the discriminant bordered by l, m, n. In future this bordered discriminant will be called D, and its minors (14) will be called A, B, C. The coordinates of the centre of a conic may consequently be written

$$(\bar{x}, \bar{y}, \bar{z})=(A, B, C). \quad \text{.................. (15)}$$

If D be expanded, we get the determinant

$$\begin{aligned} D&=(vw-u'^2)l^2+(wu-v'^2)m^2+(uv-w'^2)n^2 \\ &\quad +2(v'w'-uu')mn+2(w'u'-vv')nl+2(u'v'-ww')lm \\ \text{,,} &=Ul^2+Vm^2+Wn^2+2U'mn+2V'nl+2W'lm. \quad \text{......... (16)} \end{aligned}$$

The determinants of A, B, C are

$$A=Ul+W'm+V'n; \quad B=W'l+Vm+U'n;$$
$$C=V'l+U'm+Wn. \quad \text{.................. (17)}$$

Evidently $\qquad lA+mB+nC=D. \quad$ .... (18)

On expanding and arranging the function, it will be found that

$$\phi(A, B, C)=\Delta D. \quad \text{.................... (19)}$$

14°. The value of D enables us to determine the species of a conic. One of the three forms of the coordinates of the two points in which $\phi(xyz)$ is cut by Λ_∞ is

$$\left.\begin{aligned} x&=vnl-u'lm+v'm^2-w'mn\pm m\sqrt{-D}, \\ y&=umn+u'l^2-v'lm-w'nl\mp l\sqrt{-D}, \\ z&=-(um^2+vl^2-2w'lm). \end{aligned}\right\} \quad \text{...... (20)}$$

If $D<0$, the conic is cut in two real and distinct points by Λ_∞ and is a hyperbola. If $D>0$, the conic is cut in two imaginary points and is an ellipse or circle. If $D=0$, the conic is touched in two real and coincident points by Λ_∞ and is a parabola. Since the vector of the centre is

$$\frac{Al\alpha+Bm\beta+Cn\gamma}{lA+mB+nC},$$

and since, for the parabola,

$$0=D=lA+mB+nC,$$

the centre of this curve is at infinity.

15°. Chords which pass through the centre are diameters, the loci of the midpoints of parallel chords. If $(x'y'z')$ be any point upon a diameter, its equation is

$$(y'C-z'B)x+(z'A-x'C)y+(x'B-y'A)z=0. \quad \ldots(21)$$

Conjugate diameters are such that either is parallel to the tangents at the extremities of the other, and therefore passes through its pole. Only central conics possess such diameters, all diameters of the parabola being parallel because the centre is at infinity.

16°. The equation of a diameter conjugate to a given diameter,

$$px+qy+rz=0.$$

The sought diameter passes through the centre and the pole of the given diameter. Its coordinates are therefore given by the matrix

$$\begin{vmatrix} Up+W'q+V'r, & W'p+Vq+U'r, & V'p+U'q+Wr \\ Ul+W'm+V'n, & W'l+Vm+U'n, & V'l+U'm+Wn \end{vmatrix}.$$

On expanding and simplifying the determinants, it will be found that the coordinates of the sought diameter are

$$\left.\begin{aligned} x&=u(nq-mr)+v'(mp-lq)+w'(lr-np),\\ y&=v(lr-np)+w'(nq-mr)+u'(mp-nq),\\ z&=w(mp-lq)+u'(lr-np)+v'(nq-lr). \end{aligned}\right\} \quad \ldots(22)$$

Ex. Let the conic be the inscribed conic,

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

with $$l:m:n=2:3:2.$$

This conic touches AB in $C'=(110)$, and its centre K is (545). The diameter $C'K$ is consequently $(\bar{5}51)$, and its conjugate is, (22),

$$\Lambda=10x+15y-32z=0.$$

This equation is satisfied by the coordinates of the pole of $C'K$, $(3\bar{2}0)$, and Λ is parallel to AB, a tangent at the extremity of $C'K$. For (bearing in mind that the equation of Λ_∞ is $2x+3y+2z=0$)

$$\begin{vmatrix} 10 & 15 & -22 \\ 0 & 0 & 1 \\ 2 & 3 & 2 \end{vmatrix}=0.$$

Any two diameters, (pqr) and $(p'q'r')$, will be conjugate if

$$Upp'+Vqq'+Wrr'+U'(qr'+q'r)+V'(rp'+r'p) + W'(pq'+p'q)=0. \quad \ldots(23)$$

17°. The polar of any point (fgh) upon a diameter is parallel to the tangents at its extremities.

In this theorem we shall denote Λ_∞ by the equation

$$A\phi_x+B\phi_y+C\phi_z=0,$$

in its quality of polar to the centre of the conic.

Let one extremity of the diameter be $(x'y'z')$. The tangent at this point is $x'\phi_x+y'\phi_y+z'\phi_z=0$; the polar of (fgh) is $f\phi_x+g\phi_y+h\phi_z$; and Λ_∞ is $A\phi_x+B\phi_y+C\phi_z=0$. If the polar of (fgh) is parallel to the tangent at $(x'y'z')$, the eliminant of these three equations must be zero, since the three lines are concurrent. Expanding these functions, the eliminant is

$$\begin{vmatrix} uf+w'g+v'h, & ux'+w'y+v'z', & uA+w'B+v'C \\ w'f+vg+u'h, & w'x'+vy+u'z', & w'A+vB+u'C \\ v'f+v'g+wh, & v'x'+u'y'+wz', & v'A+u'B+wC \end{vmatrix}$$

$$=\begin{vmatrix} f & x' & A \\ g & y' & B \\ h & z' & C \end{vmatrix}\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}=0,$$

since the three points (fgh), $(x'y'z)$ and (ABC) are collinear.

18°. (a) Let the roots of (12) be real and equal, and we have

$$\phi(f, g, h)\phi(p, q, r)-(p\phi_f+q\phi_g+r\phi_h)^2=0. \quad \ldots\ldots(24)$$

Since the roots are equal, the points of section of FP (fig. 12) are either both internal (as shown) or both external. In either case the two values of t, $\dfrac{FX_1}{X_1P}$ and $\dfrac{FX_2}{X_2P}$, can only become equal when the points X_1 and X_2 coalesce, which happens when FP revolves in the plane round F until its direction coincides with FQ or FR, the tangents at Q and R. When this occurs the two ratios are equal, whatever be the position of P on the line FX_1. Since the position of

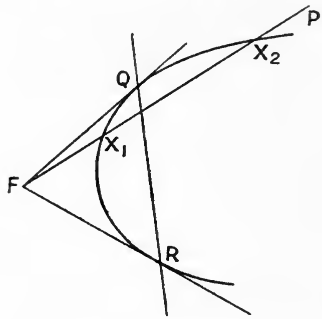

FIG. 12.

P is immaterial, we may eliminate its coordinates from (24) by means of the equations $p=\dfrac{x-f}{t}$, $q=\dfrac{y-g}{t}$, $r=\dfrac{z-h}{t}$.

Equation (24) then becomes

$$\phi(f, g, h)\phi\{(x-f), (y-g), (z-h)\}$$
$$-\{(x-f)\phi_f+(y-g)\phi_g+(z-h)\phi_h\}^2=0.$$

But $\phi\{(x-f), (y-g), (z-h)\}$

$$=\phi(x, y, z)-2(f\phi_x+g\phi_y+h\phi_z)+\phi(f, g, h).$$

Therefore

$$\phi(f, g, h)\phi(x, y, z)-(f\phi_x+g\phi_y+h\phi_z)^2=0, \ldots\ldots(25)$$

the equation of a pair of tangents from a point F to a conic. Being a quadratic, (25) shows that only two tangents can be drawn from any point to a conic. If the point lies within the conic, the tangents will be imaginary.

Ex. The equation of a pair of tangents from C (001) to the inscribed conic,

$$x^2+y^2+z^2-2yz-2zx-2xy=0.$$

Here, $f=g=0$, $h=1$; $u=v=w=1$; $u'=v'=w'=-1$; $\phi(f, g, h)=1$; $\phi_z=(-x-y+z)$.

Hence (25) becomes

$$0=x^2+y^2+z^2-2yz-2zx-2xy$$
$$-(x^2+y^2+z^2-2yz-2zx+2xy)=-4xy.$$

(*b*) The separate equations of the two tangents from (fgh) may be obtained as follows. Let $\Lambda=(pqr)$ be a tangent to the conic which passes through (fgh). Then the values of p, q, r in terms of f, g, h are obtained from the equations

$$pf+qg+rh=0,$$

$$Up^2+Vq^2+Wr^2+2U'qr+2V'rp+2W'pq=0.$$

In order that the coordinates of the result may be symmetric, we must add together the three equations obtained by solving successively for $\frac{p}{q}$, $\frac{q}{r}$ and $\frac{r}{p}$. The result is

$$p=Vh(h-f)-Wg(f-g)+U'(hf+fg-2gh)-V'g(g-h) \\ +W'h(g-h)\pm(g-h)\sqrt{-\Delta\phi(fgh)},$$

$$q=Wf(f-g)-Uh(g-h)+U'f(h-f)+V'(fg+gh-2hf) \\ -W'h(h-f)\pm(h-f)\sqrt{-\Delta\phi(fgh)},$$

$$r=Ug(g-h)-Vf(h-f)-U'f(f-g)+V'g(f-g) \\ +W'(gh+hf-2fg)\pm(f-g)\sqrt{-\Delta\phi(fgh)}.$$

Ex. Let the conic be $yz+zx+xy=0$, and the point $(\bar{1}11)$.

Then $\qquad f-g=-2, \quad g-h=0, \quad h-f=2;$

$$hf+fg-2gh=-4; \quad fg+gh-2hf=-2; \quad gh+hf-2fg=0.$$

$$U=V=W=-1; \quad U'=V'=W'=1.$$

$$\Delta=2; \quad \phi(\bar{1}11)=-2; \quad \sqrt{-\Delta\phi(\bar{1}11)}=\sqrt{4}=2.$$

Consequently, the coordinates of one tangent are

$$(-8, 0, -8)=(101),$$

and of the other, $\quad (-8, -8, 0)=(110).$

Therefore $xy=0$ is the equation of a pair of tangents from C to an inconic as it ought to be, since the equations of CA and CB are $y=0$ and $x=0$.

19°. Suppose F to be the centre of the conic. Then (25) becomes

$$\phi(A, B, C)\phi(x, y, z)-(A\phi_x+B\phi_y+C\phi_z)^2=0.$$

Now $\phi(A, B, C)=D\Delta$, and on expansion and rearrangement it will be found that

$$A\phi_x+B\phi_y+C\phi_z=(lx+my+nz)\Delta.$$

Therefore $D\phi(x, y, z)-\Delta(lx+my+nz)^2=0,$ ....(26)

the equation of the asymptotes.

The asymptotes of the circle and ellipse are imaginary. For the parabola, $D=0$ and (26) degrades to the equation of Λ_∞, which has double contact with the curve.

20°. The rectangular hyperbola.

Let (pqr) and $(p'q'r')$ be the asymptotes of any hyperbola. Multiplying the two equations together,

$$pp'x^2+qq'y^2+rr'z^2+(qr'+q'r)yz$$
$$+(rp'+r'p)zx+(pq'+p'q)xy=0. \quad \text{.........}(a)$$

Expanding (26),

$$(Du-\Delta l^2)x^2+(Dv-\Delta m^2)y^2+(Dw-\Delta n^2)z^2$$
$$+2(Du'-\Delta mn)yz+2(Dv'-\Delta nl)zx+2(Dw'-\Delta lm)xy=0. \quad (b)$$

The coefficients of like powers of the variables in (a) and (b) are proportional, since both equations represent the asymptotes, and we may put

$$Du-\Delta l^2=pp' \ldots 2(Du'-\Delta mn)=qr'+q'r \ldots \text{etc.}$$

To find, therefore, the condition that the asymptotes shall be at right angles, we have merely to substitute these values for pp', $qr'+q'r$, etc., in III, (16), and on doing so,

$$0=m^2n^2a^2(Du-\Delta l^2)+n^2l^2b^2(Dv-\Delta m^2)+l^2m^2c^2(\Delta w-\Delta n^2)$$
$$-2lmn\{lbc\cos A(Du'-\Delta mn)+mca\cos B(Dv'-\Delta nl)$$
$$+nab\cos C(Dw'-\Delta lm)\}.$$

The coefficient of Δ in this equation vanishes, and the condition for rectangular asymptotes is

$$0=ul^{-2}a^2+vm^{-2}b^2+wn^{-2}c^2-2u'm^{-1}n^{-1}bc\cos A$$
$$-2v'n^{-1}l^{-1}ca\cos B-2w'l^{-1}m^{-1}ab\cos C. \ldots(27)$$

Ex. 1. The equation of the conics circumscribing the given triangle is

$$yz+zx+xy=0.$$

Hence, $\quad u=v=w=0;\quad u'=v'=w'=1;$

and in this case (27) becomes

$$0=m^{-1}n^{-1}bc\cos A+n^{-1}l^{-1}ca\cos B+l^{-1}m^{-1}ab\cos C$$
$$,,\ =m^{-1}n^{-1}\tan B\tan C+n^{-1}l^{-1}\tan C\tan A$$
$$+l^{-1}m^{-1}\tan A\tan B. \quad \text{.........}(a)$$

Now (a), the condition for a rectangular hyperbola, is also the condition that the orthocentre,

$$(l^{-1}\tan A,\ m^{-1}\tan B,\ n^{-1}\tan C),$$

II, 3°, shall lie on the circumconic. Therefore the orthocentre, P, lies on a circumconic when it is a rectangular hyperbola.

Ex. 2. The equation of a polar conic (for which the given triangle is self-conjugate) may be written

$$-x^2+y^2+z^2=0.$$

Here $u=-1$; $v=w=1$; $u'=v'=w'-0$;

and (27) becomes

$$-l^{-2}a^2+m^{-2}b^2+n^{-2}c^2=0. \qquad (b)$$

Now (b), the condition for a rectangular hyperbola, is also the condition that the centres of the incircle and three escribed circles,

$$(l^{-1}a,\ m^{-1}b,\ n^{-1}c), \quad (-l^{-1}a,\ m^{-1}b,\ n^{-1}c),$$
$$(l^{-1}a,\ -m^{-1}b,\ n^{-1}c), \quad (l^{-1}a,\ m^{-1}b,\ -n^{-1}c),$$

II, 3°, shall lie on the polar conic. Therefore the centres of these four circles lie on a polar conic when it is a rectangular hyperbola.

Ex. 3. By (14), the centre of a polar conic,

$$-x^2+y^2+z^2=0,$$

is $(-l,\ m,\ n)$. Substituting these coordinates for the variables in the equation of the circumcircle,

$$mna^2yz+nlb^2zx+lmc^2xy=0,$$

we get (b), the condition for a rectangular hyperbola. Therefore the locus of the centres of polar rectangular hyperbolæ is the circumcircle.

21°. If the coefficient of one of the squares of the variables in the general equation of the second degree vanishes, the conic represented passes through one of the corners of the given triangle. What is the consequence of the vanishing of the coefficient of one of the products of the variables?

Let w' vanish. Then the coordinates of the two points in which the curve is cut by the line AB, $z=0$, are easily found to be

$$P=(\sqrt{-v},\ \sqrt{u},\ 0) \quad \text{and} \quad P'=(\sqrt{-v},\ -\sqrt{u},\ 0).$$

Now it will be shown in a future chapter that

$$\frac{l\alpha\sqrt{-v}+m\beta\sqrt{u}}{l\sqrt{-v}+m\sqrt{u}} \quad \text{and} \quad \frac{l\alpha\sqrt{-v}-m\beta\sqrt{u}}{l\sqrt{-v}-m\sqrt{u}}$$

are harmonic conjugates of α and β. When, therefore, w' vanishes in the general equation of the second degree, the side AB of the given triangle is cut harmonically by the two other sides and the conic; with corresponding results when u' and v' vanish.

When all three coefficients, u', v', w', vanish, each side of the triangle is cut harmonically by the two other sides and the conic, and the triangle is self-conjugate in respect to the conic.

22°. The equation of a conic in terms of a pair of tangents and the chord of contact.

Let $t=0$, $u=0$, $v=0$, $w=0$ be the equations of four straight lines, no three of which are concurrent, and let k be an arbitrary constant. Then the equation

$$tu+kvw=0 \quad\text{.........................}(a)$$

is the equation of a conic circumscribing the quadrilateral of which t and u, v and w, are opposite sides.

First, being of the second degree, (a) represents some conic.

Secondly, (a) is satisfied when $t=0$ and $v=0$ are satisfied.

But these two equations are satisfied by the coordinates of their cross. Therefore (a) is satisfied by the coordinates of the point $t \cdot v$. Similarly, it is satisfied by the coordinates of the points $t \cdot w$, $u \cdot v$, $u \cdot w$. Therefore (a) is the equation of the conic circumscribing the quadrilateral of which t and u, v and w are the opposite sides.

Now let w approach and finally coalesce with v. Then (a) becomes

$$tu+kv^2=0. \quad\text{.........................}(b)$$

In this case t intersects the two coincident straight lines represented by $v^2=0$, in two coincident points whose coordinates satisfy $t=0$ and $v=0$, and consequently satisfy (b). Therefore $t=0$ is a tangent to the curve at the cross of t and v^2. Similarly, $u=0$ is a tangent at the cross of u and v^2. Therefore the equation of a conic in terms of two tangents and the chord of contact is of the form

$$0=tu+kv^2, \quad\text{or}\quad v^2+\frac{1}{k}tu=0, \quad\text{.............}(28)$$

where $t=0$ and $u=0$ are tangents to the conic and $v=0$ is the chord of contact.

23°. $\Lambda=(pqr)$ is a tangent to $\phi(xyz)$. To find the coordinates of the tangent parallel to Λ.

Every parallel to $px+qy+rz=0$
must be of the form

$$(tp+l)x+(tq+m)y+(tr+n)z=0.$$

The condition that a parallel line should be itself a tangent is obtained by substituting $tp+l$ for p, $tq+m$ for q, and $tr+n$ for r in the matrix of 6°, and the result is

$$F(pqr)t^2+2(Ap+Bq+Cr)t+D=0. \quad \text{.........(29)}$$

Now $F(pqr)=0$, this being the condition that Λ should be a tangent to the curve. One of the roots of (29) is consequently infinite. But when $t=\infty$, the distance between the parallels is zero, IV, (6); or every straight line is parallel to itself.

If $D=0$, as in the case of the parabola, the other root of (29) is zero, and the distance between the two tangents is infinite.

In other words, if an arbitrary tangent be drawn to a parabola, the only other tangent parallel to it is Λ_∞.

When D has an actual value, as in the case of central conics, the second root of (29) is

$$t=\frac{-D}{2(Ap+Bq+Cr)},$$

and the coordinates of the tangent parallel to Λ are

$$\{2l(Ap+Bq+Cr)-Dp, \quad 2m(Ap+Bq+Cr)-Dq,$$
$$2n(Ap+Bq+Cr)-Dr\}. \quad \text{..................(30)}$$

CHAPTER VI

SPECIAL CONICS

1°. The locus of the term of the variable vector

$$\rho = \frac{t^2 l\alpha + u^2 m\beta + v^2 n\gamma}{t^2 l + u^2 m + v^2 n},$$

with the condition $\qquad t+u+v=0.$

Comparing this expression with the standard form,

$$t^2 = x, \quad u^2 = y, \quad v^2 = z.$$

Eliminating t, u, v from these three equations and $t+u+v=0$, we get

$$x^2+y^2+z^2-2yz-2zx-2xy=0. \quad \text{............(1)}$$

2°. $\phi_x = x-y-z, \quad \phi_y = -x+y-z, \quad \phi_z = -x-y+z,$
and the general equation of a tangent at $(x'y'z')$ is

$$(x'-y'-z')x+(-x'+y'-z')y+(-x'-y'+z')z=0,$$

which is satisfied by the coordinates of the points $A'=(011)$, $B'=(101)$ and $C'=(110)$. The conic consequently touches the sides of the given triangle in A', B', C', whatever be the position of O with respect to the given triangle.

3°. If O is inside the triangle, the ratios $l:m:n$ are all positive and $D=4\Sigma lm$ is always positive. The conic therefore is either an inscribed ellipse or circle.

4°. If O is outside the triangle, two of the ratios $l:m:n$ are negative. Σlm may consequently be negative, positive or null, and the curve may be a hyperbola, ellipse or parabola—in this case escribed to the triangle. Let

$$l>0, \quad m>0, \quad n<0, \quad m+n>0, \quad n+l>0;$$

on which suppositions Σlm may be $\gtreqless 0$. For n (a negative number) must be either greater, equal to, or less than $\frac{-lm}{l+m}$.

If $n > \frac{-lm}{l+m}$, $mn+nl > -lm$, and $mn+nl+lm > 0$.

„ $=$ „ , $mn+nl+lm=0$.

„ $<$ „ , $mn+nl<-lm$, and $mn+nl+lm<0$.

In the first case the conic is an ellipse or circle; in the second a parabola; and in the third a hyperbola.

Let the same condition be written, $\frac{-n}{l} \lesseqgtr \frac{m+n}{m}$. Draw AE parallel to $B'A'$ (fig. 13), and complete the parallelogram $BCAD$. Then

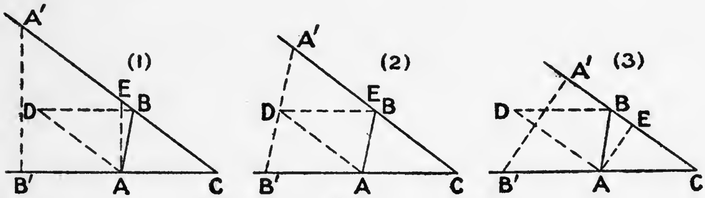

Fig. 13.

$$\frac{-n}{l}=\frac{AB'}{CB'}=\frac{EA'}{CA'};\quad \frac{n}{m}=\frac{BA'}{A'C}$$

and
$$\frac{m+n}{m}=\frac{BA'+A'C}{A'C}=\frac{BC}{A'C}=\frac{CB}{CA'}=\frac{AD}{CA'}.$$

Therefore $\frac{-n}{l} \lesseqgtr \frac{m+n}{m}$ according as $EA' \lesseqgtr AD$; that is, the conic is an ellipse, a parabola or a hyperbola according as the point D lies within $A'B'C$, fig. (1), or on the line $A'B'$, fig. (2), or without $A'B'C$, fig. (3). Hamilton.

5°. The locus of the term of the vector

$$\rho=\frac{t^{-1}l\alpha+u^{-1}m\beta+v^{-1}n\gamma}{lt^{-1}+mu^{-1}+nv^{-1}}$$

with the condition $\quad t+u+v=0.$

Here $x=t^{-1}$, $y=u^{-1}$, $z=v^{-1}$, and we evidently have

$$yz+zx+xy=0, \quad \text{.....................(2)}$$

a curve which passes through A, B, C since $u=v=w=0$.

Writing the equation to avoid fractions,

$$2yz+2zx+2xy=0.$$

$\Delta=2$ and $D=4lm-(l+m-n)^2$, the curve being a hyperbola, a parabola or an ellipse according as D is negative, null or positive.

The centre is

$$\{m+n-l,\ n+l-m,\ l+m-n\}, \qquad\text{.............(3)}$$

and its vector, with the help of the equation,

$$l\alpha+m\beta+n\gamma=0,$$

can be reduced to

$$OK=\frac{l^2\alpha+m^2\beta+n^2\gamma}{l^2+m^2+n^2-2\Sigma lm}.$$

The centre of (1) is

$$\{m+n,\ n+l,\ l+m\}, \qquad\text{....................(4)}$$

and its vector can be similarly reduced to the form

$$OK'=\frac{-(l^2\alpha+m^2\beta+n^2\gamma)}{2\Sigma lm}.$$

Therefore $\dfrac{OK}{OK'}=$ a scalar and K, O, K' are collinear.

The pole of a line (pqr) in respect to (1) is

$$P_1=(q+r,\ r+p,\ p+q);$$

and the pole of the same line in respect to (2) is

$$P_2=(q+r-p,\ r+p-q,\ p+q-r).$$

Putting $p+q+r=2v$, we get

$$P_1=\{2v-p,\ 2v-q,\ 2v-r\},$$

$$P_2=\{2v-2p,\ 2v-2q,\ 2v-2r\}.$$

Therefore $OP_1=-\dfrac{pl\alpha+qm\beta+rn\gamma}{2v\Sigma l-\Sigma lp}$,

$$OP_2=-\frac{2(pl\alpha+qm\beta+rn\gamma)}{2v\Sigma l-2\Sigma lp}$$

and $\dfrac{OP_1}{OP_2}=$ a scalar. Therefore P_1, O, P_2 are collinear.

Hence KK' and P_1P_2 intersect in O.

6°. The term 'circumconic' is used to denote the family of conics, of whatever species, represented by the equation

$$yz+zx+xy=0.$$

Similarly, the term 'inconic' denotes here the family of conics represented by the equation

$$x^2+y^2+z^2-2yz-2zx-2xy=0, \qquad \text{.............}(a)$$

all of which touch the given triangle in A', B', C'. An indefinite number of conics of all species, inscribed and escribed, touch the sides of the triangle in other points, but such conics are not represented by (a). For instance, the ellipse

$$x^2+9y^2+4z^2-12yz-4zx-6xy=0 \qquad \text{............}(b)$$

touches the sides,

BC in $D=(023)$, CA in $E=(201)$, AB in $F=(310)$.

By taking the point (623), in which the lines AD, BE, CF concur, for origin, (b) may be transformed into (a). In this case, IX, (3),

$$x':y':z'=f^{-1}x:g^{-1}y:h^{-1}z=x:3y:2z;$$

and $$x=x', \quad y=\tfrac{1}{3}y', \quad z=\tfrac{1}{2}z'.$$

Substituting the values for x, y, z in (b), we get

$$x'^2+y'^2+z'^2-2y'z'-2z'x'-2x'y'=0.$$

But at the same time the conic (a), which does not touch the sides in the new A', B', C' (namely, D, E, F), becomes

$$36x'^2+4y'^2+9z'^2-12y'z'-36z'x'-24x'y'=0.$$

In a word, any conic represented by (a) is here called 'the inconic,' while any conic such as (b) is called 'an inconic.'

7°. It was pointed out in V, 21°, that when u', v', w' do not appear in the general equation, each side of the given triangle is harmonically cut by the two other sides and the conic. This may be illustrated by the curve

$$x^2-y^2-z^2=0. \qquad \text{........................}(5)$$

This equation represents a conic because $\Delta=1$.

$$D=l^2-m^2-n^2,$$

and the conic is a hyperbola, parabola or ellipse according as D is negative, null or positive. The ellipse is shown in fig. 14.

It is easy to show that the curve cuts AB in C' and C'', and since

$$OC'=\frac{l\alpha+m\beta}{l+m}, \quad OC''=\frac{l\alpha-m\beta}{l-m},$$

C' and C'' are harmonic conjugates of A and B, VIII, (5).

Similarly, the curve cuts CA in B' and B'', which are harmonic conjugates of C and A, and it cuts BC harmonically in two imaginary points

$$(0, \sqrt{-1}, 1) \quad \text{and} \quad (0, -\sqrt{-1}, 1).$$

The tangents to the conic at these two imaginary points,

$$y\sqrt{-1}+z=0; \quad -y\sqrt{-1}+z=0,$$

intersect in $(2\sqrt{-1}, 0, 0)=(100)=A$.

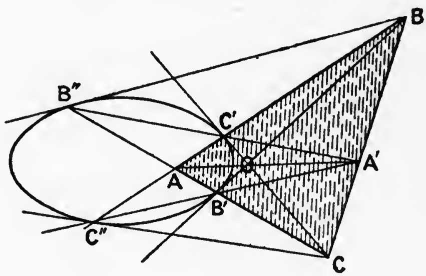

FIG. 14.

Since the tangents from A to the curve are imaginary, A lies within the curve; and since BC cuts the curve in imaginary points, it lies wholly without the curve.

The lines BB', BB'', CC', CC'' are tangents to the conic; hence B is the pole of CA, C the pole of AB. Therefore, V, 9°, (c), A is the pole of BC.

The triangle is consequently self-conjugate, or autopolar, in respect to (5).

The coordinates of the centre are $(-l, m, n)$.

If $l-m-n=0$, $B''C''$ will be a diameter of the ellipse (fig. 14), and if OA be produced to meet $B''C''$ in X, OX will be trisected in A.

When a parabola, the curve touches the lines M_2M_3, M_3M_1, M_1M_2 drawn through the middle points of the sides of ABC.

8°. In general, if $PQRS$ be any quadrilateral whose internal diagonals meet in Y and whose opposite sides meet PS and QR in X, and PQ and SR in Z; then the triangle XYZ is self-conjugate to any conic whatever which passes through P, Q, R and S.

As an illustration, let $BA'OC'$ be the quadrilateral (fig. 1). Then X is A, Y is B''' (121) and Z is C. The equations of

OA' and OC' being respectively $y-z=0$ and $x-y=0$, we have for the equation of any conic passing through B, A', O and C',

$$z(y-z)+kx(x-y)=0=kx^2-z^2+yz-kxy.$$

The polar of A with respect to this conic is $2x-y=0$, or $B'''C$; the polar of B''' is $y=0$, or CA; and the polar of C is $y-2z=0$, or AB'''.

CHAPTER VII

TANGENTIAL EQUATIONS

1°. By the principle of duality the equation,

$$px+qy+rz=0,$$

admits of a double interpretation. When the set p, q, r are constant and the set x, y, z are variable, as in the preceding chapters, the equation means that a variable point (xyz) lies somewhere on the fixed straight line (pqr). When the set p, q, r are variable and the set xyz are constant, the equation means that a variable line (pqr) passes in some direction through the fixed point (xyz). The hypothesis of a variable point and a locus are discarded here and replaced by the hypothesis of a variable line and an envelope.

2°. A straight line $\Lambda=px+qy+rz=0$ (fig. 15) cuts the sides of the given triangle in

$$D=(q\bar{p}o), \quad E=(or\bar{q}), \quad F=(\bar{r}op).$$

Then

$$(ADBC'')=\frac{p}{q},$$

$$(BECA'')=\frac{q}{r},$$

$$(CFAB'')=\frac{r}{p}.$$

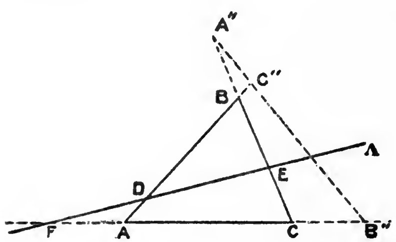

Fig. 15.

The product of these three anharmonic functions is unity, and any two of them suffice to determine the position of Λ with respect to the given triangle. The tangential coordinates of Λ are

$$(pqr),$$

which are the coordinates of its local equation. A line is fully represented by this symbol and it has no equation. The coordinates of BC are (100); of $A''B''$, (111); of Λ_∞, (lmn).

3°. As the local equation of a straight line is a relation between the coordinates of a variable point which in every position lies on the line, so the tangential equation of a point is a relation between the coordinates of a variable line which in every position passes through the point. Thus the tangential equation

$$xp+yq+zr=0, \qquad \text{.........................(1)}$$

where x, y, z are constant and p, q, r are variable, is the equation of a fixed point whose anharmonic coordinates are (xyz). To obtain the tangential equation of any particular point, we have merely to substitute its anharmonic coordinates for x, y, z in (1). Thus, for

$A,\ p=0$ $A'',\ q-r=0$ $O,\ p+q+r=0.$
$B,\ q=0$ $B'',\ r-p=0$ Mean Point, $mnp+nlq+lmr=0.$
$C,\ r=0$ $C'',\ p-q=0$ Incentre, $mnap+nlbq+lmcr=0.$

4°. The tangential equations of the cyclic points are, IX, (3),

$$I,\quad mnap+nl(ce^{iB}-a)q-lmcre^{iB}=0.$$
$$J,\quad mnap+nl(ce^{-iB}-a)q-lmcre^{-iB}=0.$$

Multiplying these two equations together, we get for the two points

$$\left.\begin{aligned}\Omega^2 &= m^2n^2a^2p^2+n^2l^2b^2q^2+l^2m^2c^2r^2\\ &\quad -2lmn(lbc\cos Aqr+mca\cos Brp+nab\cos Cpq)=0,\\ ,, &= mna^2(mp-lq)(np-lr)+nlb^2(nq-mr)(lq-mp)\\ &\qquad +lmc^2(lr-np)(mr-nq)=0.\end{aligned}\right\}\ (2)$$

These equations are identical in form with Z^2, IV, (2); but in the latter p, q, r are constant, while in the present case they are variable.

When the triangle is equilateral and its mean point the origin, (2) becomes

$$0=p^2+q^2+r^2-qr-rp-pq=(p+\omega q+\omega^2 r)(p+\omega^2 q+\omega r). \ (3)$$

5°. Let $(p'q'r')$ be a line which passes through the two given points

$$x'p+y'q+z'r=0 \quad \text{and} \quad x''p+y''q+z''r=0.$$

Since the coordinates of the line must satisfy the equations of both points

$$x'p'+y'q'+z'r'=0,$$
$$x''p'+y''q'+z''r'=0.$$

Therefore $\dfrac{p'}{y'z''-y''z'}=\dfrac{q'}{y'x''-z''x'}=\dfrac{r'}{x'y''-x''y'}$;

or the coordinates of the join of two points are

$$(y'z''-y''z',\ z'x''-z''x',\ x'y''-x''y'). \quad\text{............(4)}$$

6°. Let the equation of the cross of two given lines, $(p'q'r')$ and $(p''q''r'')$, be

$$x'p+y'q+z'r=0.$$

Since this equation must be satisfied by the coordinates of both lines

$$x'p'+y'q'+z'r'=0,$$
$$x'p''+y'q''+z'r''=0.$$

Therefore $\dfrac{x'}{q'r''-q''r'}=\dfrac{y'}{r'p''-r''p'}=\dfrac{z'}{p'q''-p''q'}$,

or the coordinates of the cross of two straight lines are

$$(q'r''-q''r',\ r'p''-r''p',\ p'q''-p''q'). \quad\text{............(5)}$$

7°. Let $$x'p+y'q+z'r=0$$

be the equation of the point in which two parallel lines, $(p'q'r')$ and $(p''q''r'')$, concur with Λ_∞. Then, since this equation must be satisfied by the coordinates of the three lines,

$$x'p'+y'q'+z'r'=0,$$
$$x'p''+y'q''+z'r''=0,$$
$$x'l+y'm+z'n=0,$$

and

$$\begin{vmatrix} p' & p'' & l \\ q' & q'' & m \\ r' & r'' & n \end{vmatrix}=0.$$

Therefore the coordinates of the parallel $(p''q''r'')$ are of the form $(tp'+l,\ tq'+m,\ tr'+n)$, which satisfy this condition.

8°. The distance between a point (fgh) and a line (pqr) is, IV, (7),

$$d^2=\frac{\Sigma^2 fp}{\Sigma^2 fl}\,\frac{l^2m^2n^2b^2c^2\sin^2 A}{Z^2}.$$

If we suppose d to remain constant while p, q, r vary under the condition of this equation, Z becomes Ω, and

$$d^2=\frac{\Sigma^2 fp}{\Sigma^2 fl}\,\frac{l^2m^2n^2b^2c^2\sin^2 A}{\Omega^2}, \quad\text{..................(6)}$$

which, being a relation between the coordinates of a variable straight line (pqr) at a constant distance from a fixed point, is a tangential equation of a circle.

If $d=0$, we get $\quad fp+gq+hr=0,$

the equation of the centre.

If $d=\infty$, we get $\quad \Omega^2=0.$

Thus when the radius of a circle becomes infinite, its equation is resolved into the equations of the cyclic points, through which all circles pass, IX, 4°.

9°. The tangential equation of a curve is a relation between the coordinates of a tangent to the curve. Consequently, the equation of the curve is satified by the coordinates of any tangent to it, and any straight line whose coordinates satisfy the equation of the curve is a tangent to the curve.

"Whatever the order of a plane curve may be, or whatever be the degree of $f(xyz)$, the tangent to the curve at the point $P=(xyz)$ is the right line

$$\Lambda=(lmn), \text{ if } l=D_xf, \; m=D_yf, \; n=D_zf;$$

expressions which, by the supposed homogeneity of f, give the relation, $lx+my+nz=0$, and therefore enable us to establish the system of the two following differential equations

$$ldx+mdy+ndz=0; \quad xdl+ydm+zdn=0.$$

If, then, by elimination of the ratios of x, y, z, we arrive at a new homogeneous equation of the form

$$0=F(D_xf, \; D_yf, \; D_zf),$$

as one true for all values of x, y, z which render the function $f=0, \ldots$, we shall have the equation

$$F(lmn)=0$$

as a condition that must be satisfied by the tangent Λ to the curve, in all the positions which can be assumed by that right line. And, by comparing the two differential equations, $\quad dF(lmn)=0, \quad xdl+ydm+zdn=0,$

we see that we may write the proportion

$$x:y:z=D_lF:D_mF:D_nF,$$

and the symbol, $P=(D_lF, \; D_mF, \; D_nF)$, if (xyz) be as above the point of contact P of the variable tangent line (lmn), in any of its positions, with the curve which is its envelope. Hence we can pass from the tangential equation $F=0$ of a curve considered as the envelope of a right line Λ, to the local equation $f=0$ of the same curve considered as the locus of a point P: since, if we obtain, by elimination of the ratios l, m, n, an equation of the form

$$0=f(D_lF,\ D_mF,\ D_nF)$$

as a consequence of the homogeneous equation $F=0$, we have only to substitute for these partial derivatives, D_lF, etc., the anharmonic coordinates x, y, z, to which they are proportional. And when the functions f and F are not only homogeneous, but also rational and integral; then, while the degree of the function f, or of the local equation, marks the *order* of the curve, the degree of the other homogeneous function F, or of the tangential equation $F=0$, is easily seen to denote . . . the *class* of the curve to which that equation belongs."*

(a) To transform $\phi(xyz)$ into $F(pqr)$, we have, as explained by Hamilton, to eliminate x, y, z from the equations

$$\begin{aligned} D_xf=\phi_x=p&=ux+w'y+v'z,\\ D_yf=\phi_y=q&=w'x+vy+u'z,\\ D_zf=\phi_z=r&=v'x+u'y+wz,\\ 0&=px+qy+rz, \end{aligned}$$

where p, q, r are used instead of Hamilton's l, m, n, which are otherwise required.

Hence

$$0=\begin{vmatrix} p & q & r & 0\\ u & w' & v' & p\\ w' & v & u' & q\\ v' & u' & w & r \end{vmatrix}\begin{aligned}&=Up^2+Vq^2+Wr^2+2U'qr+2V'rp\\ &\qquad\qquad\qquad\qquad\qquad +2W'pq\\ &=F(pqr).\end{aligned}$$

(b) To transform $F(pqr)$ into $\phi(xyz)$, we have to eliminate p, q, r from the equations

$$\begin{aligned} D_pF=F_p=x&=Up+W'q+V'r,\\ D_qF=F_q=y&=W'p+Vq+U'r,\\ D_rF=F_r=z&=V'p+U'q+Wr,\\ 0&=xp+yq+zr. \end{aligned}$$

* Sir W. R. Hamilton's *Elements of Quaternions*, 1866, pp. 43-4.

Hence

$$0=\begin{vmatrix} x & y & z & 0 \\ U & W' & V' & x \\ W' & V & U' & y \\ V' & U' & W & z \end{vmatrix} \begin{aligned}&=(VW-U'^2)x^2+(WU-V'^2)y^2 \\ &\quad +(UV-W'^2)z^2+2(V'W'-UU')yz \\ &\quad +2(W'U'-VV')zx \\ &\quad +2(U'V'-WW')xy \\ &=\Delta(ux^2+vy^2\ldots+2w'xy) \\ &=\phi(xyz).\end{aligned}$$

(c) The tangential equation of the incircle is

$$s_1 lqr+s_2 mrp+s_3 npq=0.$$

To find its local equation we have the equations

$$\begin{aligned} F_p&=x=s_3 nq+s_2 mr, \\ F_q&=y=s_3 np+s_1 lr, \\ F_r&=z=s_2 mp+s_1 lq, \\ 0&=xp+yq+zr. \end{aligned}$$

Hence

$$0=\begin{vmatrix} x & y & z & 0 \\ 0 & s_3 n & s_2 m & x \\ s_3 n & 0 & s_1 l & y \\ s_2 m & s_1 l & 0 & z \end{vmatrix} \begin{aligned}&=s_1{}^2 l^2 x^2+s_2{}^2 m^2 y^2+s_3{}^2 n^2 z^2 \\ &\quad -2s_2 s_3 mnyz-2s_3 s_1 nlzx-2s_1 s_2 lmxy,\end{aligned}$$

the local equation of the incircle.

The utility of the method depends mainly, as shown above, on the equation, $px+qy+rz=0$, "which may at pleasure be considered as expressing, either that the variable point (xyz) is situated somewhere on the given right line (pqr), or else that the variable line (pqr) passes in some direction through the given point (xyz)" (Hamilton).

10°. As a local equation of the second degree may be the product of the equations of two straight lines, so a tangential equation of the second degree may be the product of the equations of two points. The criterion in the latter case is strictly analogous to that in the former. The equation

$$Up^2+Vq^2+Wr^2+2U'qr+2V'rp+2W'pq=0$$

will be the product of two equations of the first degree if the discriminant

$$\begin{vmatrix} U & W' & V' \\ W' & V & U' \\ V' & U' & W \end{vmatrix}=0. \qquad\qquad (7)$$

11°. Since the coordinates of a tangent $(p'q'r')$ drawn from a given point to the curve, $F(pqr)=0$, must satisfy the equations of both the point and the curve, we can determine the ratios of the coordinates of the tangent from these two equations. If we solve for $\frac{q'}{r'}$, we obtain a quadratic equation. Therefore the ratios $p':q':r'$ have two and only two sets of values, or, only two tangents can be drawn from the given point to the curve.

12°. Let $t_1=(p'q'r')$ be a tangent to $F(pqr)=0$. Then, $p'F_{p'}+q'F_{q'}+r'F_{r'}=0$, and t_1 evidently passes through some point, $P=pF_{p'}+qF_{q'}+rF_{r'}=0$.

Let the second tangent from P to the curve be

$$t_2=(p''q''r'').$$

Since t_2 passes through P,

$$p''F_{p'}+q''F_{q'}+r''F_{r'}=0=p'F_{p''}+q'F_{q''}+r'F_{r''},$$

and t_1 passes through some point $Q=pF_{p''}+qF_{q''}+rF_{r''}=0$. But since t_2 is a tangent, $p''F_{p''}+q''F_{q''}+r''F_{r''}=0$, and t_2 also passes through Q.

Since then t_1 and t_2 both pass through P and Q, these two points must be identical and

$$\frac{F_{p'}}{F_{p''}}=\frac{F_{q'}}{F_{q''}}=\frac{F_{r'}}{F_{r''}}.$$

Consequently $$\frac{p'}{p''}=\frac{q'}{q''}=\frac{r'}{r''},$$

that is, the two tangents are identical, and

$$pF_{p'}+qF_{q'}+rF_{r'}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is the equation of the point of contact of the tangent $(p'q'r')$.

13°. Let $(p_1q_1r_1)$ and $(p_2q_2r_2)$ be tangents to $F(pqr)$ at the points in which it is cut by any line $(p'q'r')$. The points of contact of the two tangents are, (8),

$$pF_{p_1}+qF_{q_1}+rF_{r_1}=0 \quad \text{and} \quad pF_{p_2}+qF_{q_2}+rF_{r_2}=0;$$

and since both these points lie on the line $(p'q'r')$,

$$p_1F_{p'}+q_1F_{q'}+r_1F_{r'}=0 \quad \text{and} \quad p_2F_{p'}+q_2F_{q'}+r_2F_{r'}=0.$$

Therefore both tangents, $(p_1q_1r_1)$ and $(p_2q_2r_2)$, pass through the point
$$pF_{p'}+qF_{q'}+rF_{r'}=0, \qquad\qquad (9)$$
which is consequently the pole of $(p'q'r')$.

It is immaterial whether $(p'q'r')$ cuts the conic in real or imaginary points. For example, the line (011) lies altogether outside the inconic, $qr+rp+pq=0$. Here
$$F_{p'}=q'+r'=2, \quad F_{q'}=r'+p'=1, \quad F_{r'}=p'+q'=1;$$
and the pole of (011) is $2p+q+r=0$, a point which lies inside the conic since the tangents from it to the curve are imaginary.

14°. It follows from (9) that the pole of (lmn), or Λ_∞, is
$$\begin{aligned} 0 &= pF_l+qF_m+rF_n \\ ,, &= (Ul+W'm+V'n)p+(W'l+Vm+U'n)q \\ &\qquad\qquad\qquad\qquad +(V'l+U'm+Wn)r \\ ,, &= Ap+Bq+Cr, \qquad\qquad (10) \end{aligned}$$
the tangential equation of the centre of the conic.

15°. Let $(p'q'r')$ be the polar of $x'p+y'q+z'r=0$.

For the pole of $(p'q'r')$ we have the two equations
$$pF_{p'}+qF_{q'}+rF_{r'}=0,$$
$$x'p+y'q+z'r=0.$$

Therefore,
$$\frac{F_{p'}}{x'}=\frac{F_{q'}}{y'}=\frac{F_{r'}}{z'}=-k;$$
or
$$\begin{aligned} Up'+W'q'+V'r'+x'k&=0, \\ W'p'+Vq'+U'r'+y'k&=0, \\ V'p'+U'q'+Wr'+z'k&=0. \end{aligned}$$

Therefore
$$\frac{p'}{\begin{vmatrix} W' & V' & x' \\ V & U' & y' \\ U' & W & z' \end{vmatrix}}=\frac{-q'}{\begin{vmatrix} U & V' & x' \\ W' & U' & y' \\ V' & W & z' \end{vmatrix}}=\frac{r'}{\begin{vmatrix} U & W' & x' \\ W' & V & y' \\ V' & U' & z' \end{vmatrix}}$$
or, treating the constants x', y', z' as variables, the tangential coordinates of the polar of $x'p+y'q+z'r=0$ are
$$(\phi_{x'},\ \phi_{y'},\ \phi_{z'}), \qquad\qquad (11)$$

just as the local coordinates of the pole of $px+qy+rz=0$ are, V, 8°,

$$(F_p, F_q, F_r).$$

16°. In the preceding sections the following correspondences have been established:

Local.	**Tangential.**
The symbol of a point.	The symbol of a line.
„ equation of a line.	„ equation of a point.
„ „ „ tangent.	„ „ „ the point of contact of a tangent.
The polar of a point.	The pole of a line.
„ pole of a line.	„ polar of a point.

We may therefore, when convenient, transform expressions in one system into corresponding expressions in the other directly, without calculation. Take for example the local equation of a pair of tangents drawn to a conic from a point $F=(fgh)$, V, (25),

$$\phi(fgh)\phi(xyz)-(f\phi_x+g\phi_y+h\phi_z)^2=0.$$

Let Λ be the chord of contact of the tangents.

Then $\phi(xyz)$ becomes $F(pqr)$, the tangential equation of the conic. $\phi(fgh)$ becomes $F(fgh)$; the local function of the point (fgh) becoming the tangential function of the line (fgh). $f\phi_x+g\phi_y+h\phi_z$, the local expression for the polar of F, becomes the tangential expression for the pole of Λ. Finally, the equation

$$F(fgh)F(pqr)-(fF_p=gF_q+hF_r)^2=0 \quad\text{.........(12)}$$

is the equation of the two points in which a conic is cut by any straight line (fgh); for since the tangential equation of the point of contact of one tangent corresponds to the local equation of the tangent, the tangential equation of the points of contact of a pair of tangents, *i.e.* the points in which the conic is cut by the chord of contact, must correspond to the local equation of a pair of tangents. For the discriminant of (12), see XII, 7°.

Ex. 1. For (lmn), (12) becomes

$$(lF_p+mF_q+nF_r)^2-F(lmn)F(pqr)=0.$$

The equation of the incircle is,

$$s_1lqr+s_2mrp+s_3npq=0,$$

and $lF_p+mF_q+nF_r=mnap+nlbq+lmcr$; $F(lmn)=2lms.$

Therefore

$$0=m^2n^2a^2p^2+n^2l^2b^2q^2+l^2m^2c^2r^2$$
$$-2lmn(lbc\cos Aqr+mca\cos Brp+nab\cos Cpq)=\Omega^2;$$

i.e. Λ_∞ cuts the incircle in the cyclic points.

Ex. 2.

Local.

The equation of the pair of tangents drawn to the inconic,

$$x^2+y^2+z^2-2yz-2zx-2xy=0,$$

from the point $(22\bar{1})$, *i.e.* the pole of $(11\bar{5})$, is

$$0=9(x^2\ldots-2xy)-(x+y-5z)^2$$
$$,,\ =2x^2+2y^2-4z^2-2yz-2zx-5xy$$
$$,,\ =(x-2y-2z)(2x-y+2z),$$

two tangents which touch the conic in (411) and (141).

Tangential.

The equation of the pair of points in which the inconic,

$$2qr+2rp+2pq=0,$$

is cut by the line $(11\bar{5})$, *i.e.* the polar of $(22\bar{1})$, is

$$0=18(2qr+2rp+2pq)$$
$$+(-4p-4q+2r)^2$$
$$=4p^2+4q^2+r^2+5qr+5rp+17pq$$
$$=(4p+q+r)(p+4q+r),$$

two points which locally are (411) and (141).

17°. Since by definition the coordinates of a tangent must satisfy the equation of a conic; to obtain the coordinates of the two tangents which can be drawn from a point to a conic, we have merely to determine the ratios $p:q:r$ from the equations of the point and the curve.

Let $x'p+y'q+z'r=0$ be the point and $F(pqr)=0$ the conic. Then solving for $\frac{q}{r}$, we get

$$(Uy'^2+Vx'^2-2W'x'y')\frac{q^2}{r^2}+2(Uy'z'+U'x'^2-V'x'y'-W'z'x')\frac{q}{r}$$
$$+(Uz'^2+Wx'^2-2V'z'x')=0;\ \ldots\ldots\ldots\ldots\ldots\ldots(13)$$

and ultimately, writing

$$\delta=\begin{vmatrix} x' & y' & z' & 0\\ U & W' & V' & x'\\ W' & V & U' & y'\\ V' & U' & W & z'\end{vmatrix},\ \ldots\ldots\ldots\ldots\ldots(14)$$

the ratios of the coordinates are

$$\left.\begin{aligned} p&=-Vz'x'+U'x'y'-V'y'^2+W'y'z'\pm y'\sqrt{-\delta},\\ q&=-Uy'z'-U'x'^2+V'x'y'+W'z'x'\mp x'\sqrt{-\delta},\\ r&=Uy'^2+Vx'^2-2W'x'y'.\end{aligned}\right\}\ldots.(15)$$

Ex. The two tangents from $q-r=0$ (A'') to

$$p^2+q^2+\ldots-2pq=0.$$

Here $$x'=0,\quad y'=1,\quad z'=-1;$$

$$U=V=W=1,\quad U'=V'=W'=-1;\quad \delta=-4.$$

Therefore $p=4$ or 0; $q=1$; $r=1$, and the coordinates of the two tangents are

$$(411) \quad\text{and}\quad (011).$$

18°. The value of δ in the last section determines whether a given point lies on a given conic or not. If the point lies on the curve, the two tangents become one and the same, and the roots of equation (13), which may be written

$$at^2+2bt+c=0,$$

must be equal. Therefore

$$0=b^2-ca=-x^2\delta=\delta.$$

If then $\delta=0$, the point lies on the curve.

Obviously, A'' does not lie on the circumconic in the preceding section, for $\delta=-4$. Does the point $p=0$? It will be found that $\delta=1-1=0$, and $p=0$ lies on the curve.

19°. The coordinates of the tangent at a given point on the conic are given by (13), $at^2+2bt+c=0$. For in this case the roots are equal and

$$t=\frac{q}{r}=\frac{-b}{a}.$$

Ex. The tangent at the point

$$p+q+4r=0 \quad\text{to}\quad qr+rp+pq=0.$$

Here $$x'=1,\quad y'=1,\quad z'=4;$$

$$U=V=W=0;\quad U'=V'=W'=1;\quad a=-2,\quad b=-4.$$

$$\frac{q}{r}=-2,\quad \frac{p}{r}=-2,$$

and the coordinates of the tangent at the given point are $(22\bar{1})$.

20°. If the point from which the two tangents are drawn is the centre, (13) gives the coordinates of the asymptotes.

In this case, if δ be positive the asymptotes will be imaginary and the conic will be an ellipse or circle. If

$\delta=0$, (13) will have equal roots and the conic will be a parabola; and if δ be negative the conic will be a hyperbola.

Ex. The centre of the hyperbola

$$7p^2+7r^2+32qr+2rp+32pq=0,$$

with the condition, $l:m:n=2:1:2$,

is $$p+2q+r=0,$$

and it will be found that the coordinates of its asymptotes are $$(11\bar{3}) \quad \text{and} \quad (\bar{3}11).$$

To calculate the equations of the asymptotes from the local equation of this curve,

$$16x^2-3y^2+16z^2+12yz-32zx+12xy=0;$$

by V, (25),

$$36(16x^2-3y^2+16z^2+12yz-32zx+12xy)$$
$$-\{(16x+6y-16z)+2(6x-3y+6z)+(-16x+6y+16z)\}^2=0,$$

which gives $\quad 0=-3x^2+y^2-3z^2-2yz+10zx-2xy$

$$,, =(x+y-3z)(-3x+y+3).$$

CHAPTER VIII

CROSS RATIO

1°. Let $OL=\lambda$ and $ON=\nu$ be two vectors (fig. 16). If a third vector $OX=\rho$ cut LN so that

$$LX:XN=y:x,$$

then $$\rho=\frac{x\lambda+y\nu}{x+y}. \qquad\qquad (1)$$

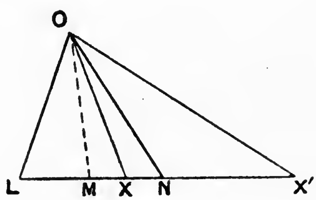

FIG. 16.

If X' be another point on LN such that

$$LX':X'N=y':x',$$

$$OX'=\rho'=\frac{x'\lambda+y'\nu}{x'+y'}. \qquad\qquad (2)$$

The ratio $\frac{y'}{x'}$ will of course be positive or negative according as the definite line LN is cut internally or externally (as in fig. 16) by the point X'.

Since $$LX=\frac{y(\nu-\lambda)}{x+y}, \quad XN=\frac{x(\nu-\lambda)}{x+y},$$

$$LX'=\frac{y'(\nu-\lambda)}{x'+y'}, \quad X'N=\frac{x'(\nu-\lambda)}{x'+y'};$$

$$\frac{LX}{XN}:\frac{LX'}{X'N}=\frac{LX\,.\,NX'}{XN\,.\,X'L}=\frac{yx'}{xy'}.$$

If we define the anharmonic function of any four collinear points A, B, C, D to be, II, 1°,

$$(ABCD)=\frac{AB}{BC}:\frac{DA}{CD}=\frac{AB.CD}{BC.DA}, \qquad \text{.............(3)}$$

where the cyclical order of the letters—A, B, C, D and B, C, D, A—is preserved above and below, we have for fig. 16,

$$(LXNX')=\frac{LX.NX'}{XN.X'L}=\frac{yx'}{xy'}. \qquad \text{.............(4)}$$

If $(LXNX')=-1$, then $\frac{-y'}{x'}=\frac{y}{x}$, and (1) and (2) become

$$\rho=\frac{x\lambda+y\nu}{x+y}, \quad \rho'=\frac{x\lambda-y\nu}{x-y}, \qquad \text{.............(5)}$$

the general expression for a pair of harmonic conjugates to λ and ν.

2°. $$\frac{AB.CD}{BC.DA}=\frac{BA.DC}{AD.CB}=\frac{CD.AB}{DA.BC}=\frac{DC.BA}{CB.AD};$$

that is, $(ABCD)=(BADC)=(CDAB)=(DCBA)=k$,

$$(BCDA)=\frac{BC.DA}{CD.AB}=\frac{BC.DA}{AB.CD}=\frac{1}{(ABCD)}=\frac{1}{k}. \qquad \text{.....}(a)$$

$$(ACBD)=\frac{AC.BD}{CB.DA}=\frac{(AB+BC)(BC+CD)}{CB.DA}$$

$$\text{,,} \quad =\frac{BC(AB+BC+CD)}{CB.DA}+\frac{AB.CD}{CB.DA}$$

$$\text{,,} \quad =\frac{BC.AD}{CB.DA}+\frac{AB.CD}{CB.DA}=1-k. \qquad \text{.............}(b)$$

The reciprocal of a function, (a), is obtained by continuing the cyclic progression one stage: $(BCDA)$ is the reciprocal of $(ABCD)$.

By reversing the order of the two central letters, (b), we obtain a function which is unity *minus* the original function:

$$(ACBD)=1-(ABCD).$$

$$\frac{1-k}{k}=(ACBD)(BCDA)=\frac{AC.BD}{CB.DA}\frac{BC.DA}{CD.AB}$$

$$=-\frac{AC.DB}{CD.BA}=-(ACDB).$$

Therefore $(ACDB)=\dfrac{k-1}{k}$ and $\dfrac{k}{k-1}=(CDBA)$.

If $(ABCD)$ is harmonic, $(DCBA)$ is harmonic; and all the cyclic permutations of both are harmonic:

$$-1=(ABCD)=(BCDA)=(CDAB)=(DABC)$$
$$=(DCBA)=(CBAD)=(BADC)=(ADCB).$$

The foregoing results are collected for convenience.

	When $(ABCD)=-1$
1. $(ABCD)=(BADC)=(CDAB)=(DCBA)=k.$	$=-1$
2. $(ADCB)=(BCDA)=(CBAD)=(DABC)=\frac{1}{k}.$	$=-1$
3. $(ACBD)=(BDAC)=(CADB)=(DBCA)=1-k.$	$=2$
4. $(ADBC)=(BCAD)=(CBDA)=(DACB)=\frac{1}{1-k}.$	$=\frac{1}{2}$
5. $(ACDB)=(BDCA)=(CABD)=(DBAC)=\frac{k-1}{k}.$	$=2$
6. $(ABDC)=(BACD)=(CDBA)=(DCAB)=\frac{k}{k-1}.$	$=\frac{1}{2}$

3°. (*a*) If A, B, C, D are any four collinear points, and if $A=(x_1y_1z_1)$ and $C=(x_3y_3z_3)$; then, III, (3), the coordinates of B and D must be of the form

$$(tx_1+ux_3,\ ty_1+uy_3,\ tz_1+uz_3)$$

and $$(t'x_1+u'x_3,\ t'y_1+u'y_3,\ t'z_1+u'z_3),$$

or, for shortness, (t, u) and (t', u'). Let

$$lx_1+my_1+nz_1=\Sigma lx_1=\sigma_1,\quad \Sigma lx_3=\sigma_3.$$

Then

$$AB=OB-OA$$
$$=\frac{(tx_1+ux_3)l\alpha+(ty_1+uy_3)m\beta+\text{etc.}}{t\sigma_1+u\sigma_3}-\frac{x_1l\alpha+y_1m\beta+\text{etc.}}{\sigma_1}$$
$$,,\quad =\frac{u\{(x_3\sigma_1-x_1\sigma_3)l\alpha+(y_3\sigma_1-y_1\sigma_3)m\beta+(z_3\sigma_1-z_1\sigma_3)n\gamma\}}{\sigma_1(t\sigma_1+u\sigma_3)}$$
$$,,\quad =\frac{u\theta}{\sigma_1(t\sigma_1+u\sigma_3)}.$$

Similarly,

$$BC=\frac{t\theta}{\sigma_3(t\sigma_1+u\sigma_3)};\quad CD=\frac{-t'\theta}{\sigma_3(t\sigma_1+u\sigma_3)};\quad DA=\frac{-u'\theta}{\sigma_1(t\sigma_1+u\sigma_3)}.$$

Therefore $$\frac{AB.CD}{BC.DA}=(ABCD)=\frac{ut'}{tu'}. \qquad\qquad (6)$$

Ex. Let the row be $(\bar{1}11)$, (100), (211), (322). Calculating from the coordinates of the first and third points the values of t and u for the second and of t' and u' for the fourth, we get

$$t=\frac{-1}{3},\quad u=\frac{1}{3},\quad t'=\frac{1}{3},\quad u'=\frac{5}{3},$$

and $$(ABCD)=\frac{-1}{5}.$$

(*b*) The cross ratio of pencils is strictly analogous. If two rays, VA and VC, be $(p_1q_1r_1)$ and $(p_3q_3r_3)$, the coordinates of the second and fourth, VB and VD, must be of the form (t, u) and (t', u'), III, 8°.

Let (pqr) be any transversal. Its intersections with the rays are:

$$\begin{aligned} &\text{for } VA-\{qr_1-q_1r,\ \ rp_1-r_1p,\ \ pq_1-p_1q\}=(a_1b_1c_1),\\ &\text{,, } \ VC-\{qr_3-q_3r,\ \ rp_3-r_3p,\ \ pq_3-p_3q\}=(a_3b_3c_3),\\ &\text{,, } \ VB-\{ta_1+ua_3,\ \ tb_1+ub_3,\ \ tc_1+uc_3\},\\ &\text{,, } \ VD-\{t'a_1+u'a_3,\ t'b_1+u'b_3,\ t'c_1+u'c_3\}.\end{aligned}$$

Hence for the four points of intersection, K, L, M, N,

$$V\cdot ABCD=(KLMN)=\frac{ut'}{tu'}=\frac{|pq_1r_2|\,|pq_3r_4|}{|pq_2r_3|\,|pq_4r_1|}. \qquad (7)$$

(*c*) If the four lines cut by the transversal are not concurrent, equation (7) still holds true.

Let the lines $(p_1q_1r_1)\dots(p_4q_4r_4)$ be cut by (pqr). Then the coordinates of the cross of (pqr) and $(p_1q_1r_1)$ are $(|qr_1|, |rp_1|, |pq_1|)$, with corresponding results for the remainder. Calculating the values of t and u for the second point, and of t' and u' for the fourth, from the coordinates of the first and third points, we get

$$\begin{aligned} t&=|qr_3|\,|rp_2|-|qr_2|\,|rp_3|=r\,|pq_3r_2|,\\ u&=|qr_2|\,|rp_1|-|qr_1|\,|rp_2|=r\,|pq_2r_1|,\\ t'&=|qr_3|\,|rp_4|-|qr_4|\,|rp_3|=r\,|pq_3r_4|,\\ u'&=|qr_4|\,|rp_1|-|qr_1|\,|rp_4|=r\,|pq_4r_1|,\end{aligned}$$

and $$\frac{ut'}{tu'}=\frac{|pq_1r_2|\,|pq_3r_4|}{|pq_2r_3|\,|pq_4r_1|}.$$

Ex. The four lines,

$$z=0, \quad x+2y-3z=0, \quad -3x+2y+z=0, \quad x=0,$$

no three of which are concurrent, are cut by $y=0$; to find the cross ratio of the intersections.

Here $(pqr)=(010)$, $(p_1q_1r_1)=(001)$, $(p_2q_2r_2)=(12\bar{3})$, $(p_3q_3r_3)=(\bar{3}21)$, $(p_4q_4r_4)=(100)$;

and $|pq_1r_2|=1$; $|pq_2r_3|=8$; $|pq_3r_4|=1$; $|pq_4r_1|=-1$.

Consequently, the cross ratio is $\dfrac{-1}{8}$.

4°. (*a*) The cross ratio of a pencil in terms of the vertex, $V=(x_0y_0z_0)$, and the points in which the rays are cut by a transversal, $P_1=(x_1y_1z_1) \dots P_4=(x_4y_4z_4)$.

$$V\cdot P_1P_2P_3P_4=(P_1P_2P_3P_4)=\frac{P_1P_2 \,.\, P_3P_4}{P_2P_3 \,.\, P_4P_1}$$

$$\text{,,} \qquad =\frac{\sin P_1VP_2 \,.\, \sin P_3VP_4}{\sin P_2VP_3 \,.\, \sin P_4VP_1}=\text{constant}$$

$$\text{,,} \qquad =(\text{IV}, (9))\,\frac{|x_0y_1z_2|\,|x_0y_3z_4|}{|x_0y_2z_3|\,|x_0y_4z_1|}. \quad \dots\dots\dots(8)$$

(*b*) The cross ratio of a pencil in terms of its vertex V and any four points upon its rays, P_1, P_2, P_3, P_4.

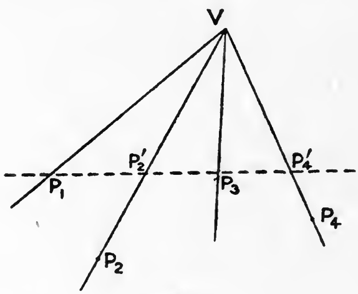

FIG. 17.

Let the pencil be $V\cdot P_1P_2P_3P_4$ (fig. 17); $V=(x_0y_0z_0)$; $P_1=(x_1y_1z_1) \dots P_4=(x_4y_4z_4)$.

The transversal P_1P_3 cuts VP_2 and VP_4 in $P'_2=(t, u)$ and $P'_4=(t', u')$.

Then 4°, (a), $\quad V\cdot P_1P'_2P_3P'_4=\dfrac{ut'}{tu'}$.

Now, since V, P_2, P'_2 and also V, P_4, P'_4 are collinear,

$$0=\begin{vmatrix} x_0 & x_2 & tx_1+ux_3 \\ y_0 & y_2 & ty_1+uy_3 \\ z_0 & z_2 & tz_1+uz_3 \end{vmatrix}=-t\,|x_0y_1z_2|+u\,|x_0y_2z_3|$$

and $$0=\begin{vmatrix} x_0 & x_4 & t'x_1+u'x_3 \\ y_0 & y_4 & t'y_1+u'y_3 \\ z_0 & z_4 & t'z_1+u'z_3 \end{vmatrix}=t'\,|x_0y_4z_1|-u'\,|x_0y_3z_4|.$$

Therefore $$\frac{ut'}{tu'}=V\cdot P_1P_2P_3P_4=\frac{|x_0y_1z_2|\,|x_0y_3z_4|}{|x_0y_2z_3|\,|x_0y_4z_1|}. \quad \text{.........(9)}$$

(Hamilton.)

It may be observed that if the points $(x_1y_1z_1)$ and $(x_2y_2z_2)$, or $(x_3y_3z_3)$ and $(x_4y_4z_4)$ coincide, the anharmonic function vanishes. If $(x_2y_2z_2)$ and $(x_3y_3z_3)$, or $(x_4y_4z_4)$ and $(x_1y_1z_1)$ coincide, the function becomes infinite; and if $(x_1y_1z_1)$ and $(x_3y_3z_3)$, or $(x_2y_2z_2)$ and $(x_4y_4z_4)$ coincide, it becomes unity.

5°. (a) If two homographic pencils, $V\cdot ABCD$ and $V'\cdot A'B'C'D$ (fig. 18) (a), have different vertices and a corresponding ray in common, the crosses of the remaining rays are collinear.

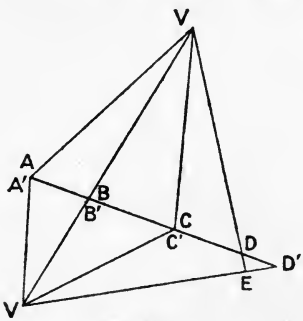

FIG. 18 (a).

Let the common ray be VBV'; let the first and third rays meet in A and C; and let the two remaining rays meet the line AC in D and D', their cross E not lying on the line AC. Then, by hypothesis,

$$\frac{AB\,.\,CD}{BC\,.\,DA}=\frac{A'B'\,.\,C'D'}{B'C'\,.\,D'A'}$$

and $$\text{,,} \quad =\frac{A'B'\,.\,C'D}{B'C'\,.\,DA'}$$

Therefore

$$\frac{D'A}{D'C}=\frac{DA}{DC};\quad \frac{CA}{D'C}=\frac{CA}{DC};\quad D'C=DC;\quad D'C-DC=D'D=0.$$

Therefore D' is D, a point on the line AC.

(*b*) If two homographic rows have a corresponding point in common, the joins of the remaining points are concurrent (fig. 18) (*b*).

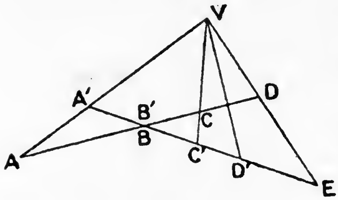

FIG. 18 (*b*).

Let B be the common point, and let AA' and $C'C$ meet in V, through which $D'D$ does not pass.

By hypothesis,

$$(ABCD)=(A'B'C'D'),$$

and since AC and $A'C'$ are transversals of $V\cdot ABCD$, by 4° (*a*),

$$(ABCD)=(A'B'C'E).$$

Therefore $$\frac{EA'}{EC'}=\frac{D'A'}{D'C'};\quad \frac{C'A'}{EC'}=\frac{C'A'}{D'C'};$$

$$EC'=D'C';\quad EC'-D'C'=ED'=0.$$

Therefore $D'=E$, and AA', $C'C$ and $D'D$ are concurrent.

6°. (*a*) When $x'=kx$ and $y'=k'y$, k and k' being constants, equation (4) of 1° becomes

$$(LXNX')=\frac{k}{k'}. \qquad\text{.........................(10)}$$

When, therefore, $\frac{y}{x}$ varies under the conditions of equations (1) and (2) of 1°,

$$\rho=\frac{x\lambda+y\nu}{x+y},\quad \rho'=\frac{kx\lambda+k'y\nu}{kx+k'y},$$

the points X and X' form two homographic divisions on the indefinite line LN, L and N being the double points of the system.

For let the successive positions of X and X' be A and A', B and B', etc.; the successive values of $\frac{y}{x}$ being $\frac{y_1}{x_1}$ for A, $\frac{y_2}{x_2}$ for B, etc. Then

$$\frac{k}{k'}=(LANA')=(LBNB')=(LCNC')=\text{etc.} \qquad\text{......(11)}$$

Now

$$AB = OB - OA = \frac{x_2\lambda + y_2\nu}{x_2 + y_2} - \frac{x_1\lambda + y_1\nu}{x_1 + y_1} = \frac{|x_1y_2|(\nu - \lambda)}{(x_1+y_1)(x_2+y_2)}.$$

Writing out the values of the four segments,

$$AB = \frac{|x_1y_2|(\nu-\lambda)}{(x_1+y_1)(x_2+y_2)}; \qquad CD = \frac{|x_3y_4|(\nu-\lambda)}{(x_3+y_3)(x_4+y_4)}.$$

$$BC = \frac{|x_2y_3|(\nu-\lambda)}{(x_2+y_2)(x_3+y_3)}; \qquad DA = \frac{|x_4y_1|(\nu-\lambda)}{(x_4+y_4)(x_1+y_1)}.$$

$$A'B' = \frac{kk'|x_1y_2|(\nu-\lambda)}{(kx_1+k'y_1)(kx_2+k'y_2)}; \quad C'D' = \frac{(kx_3+k'y_3)(kx_4+k'y_4)}{kk'|x_3y_4|(\nu-\lambda)}.$$

$$B'C' = \frac{(kx_2+k'y_2)(kx_3+k'y_3)}{kk'|x_2y_2|(\nu-\lambda)}; \quad D'A' = \frac{kk'|x_4y_1|(\nu-\lambda)}{(kx_4+k'y_4)(kx_1+k'y_1)}.$$

Therefore $(ABCD) = \dfrac{|x_1y_2||x_3y_4|}{|x_2y_3||x_4y_1|} = (A'B'C'D')$. ....(12)

(*b*) Let the given equations be

$$\rho = \frac{x\lambda + y\nu}{x+y}, \quad \rho' = \frac{kx\lambda' + k'y\nu'}{kx + k'y},$$

where $\lambda' = OL'$, $\nu' = ON'$ (fig. 19); the variable points X and X' now moving on different lines LN and $L'N'$.

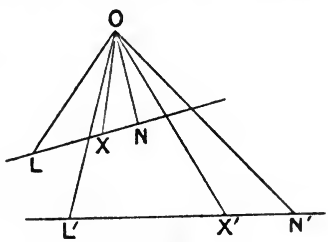

Fig. 19.

As $\dfrac{y}{x}$ varies, X will assume on LN successive positions A, B, etc., such that

$$(ABCD) = \frac{|x_1y_2||x_3y_4|}{|x_2y_3||x_4y_1|};$$

and at the same time X' will assume on $L'N'$ successive positions A', B', etc., such that

$$(A'B'C'D') = \frac{|x_1y_2||x_3y_4|}{|x_2y_3||x_4y_1|} = (ABCD).$$

7°. From the ratios given in 6° (a),

$$LA=\frac{y_1(\nu-\lambda)}{x_1+y_1}, \quad CL=\frac{-y_3(\nu-\lambda)}{x_3+y_3},$$

$$LA'=\frac{k'y_1(\nu-\lambda)}{kx_1+k'y_1}, \quad C'L=\frac{-k'y_3(\nu-\lambda)}{kx_3+k'y_3}.$$

Combining these values with those of AB and BC, $A'B'$ and $B'C'$, given in the same section,

$$\left.\begin{aligned}(LABC)&=\frac{-y_1\,|x_2y_3|}{y_3\,|x_1y_2|}=(LA'B'C').\\ \text{Similarly,}\quad (NABC)&=\frac{-x_1\,|x_2y_3|}{x_3\,|x_1y_2|}=(NA'B'C').\end{aligned}\right\}\quad\ldots\ldots\text{(13)}$$

Since $(LACB)=(LA'C'B')$ and $(NACB)=(NA'C'B')$,

$$\frac{LA\,.\,CB}{AC\,.\,BL}=\frac{LA'.\,C'B'}{A'C'.\,B'L'},$$

$$\frac{NA\,.\,CB}{AC\,.\,BN}=\frac{NA'.\,C'B'}{A'C'.\,B'N}.$$

Dividing, $\dfrac{LA\,.\,BN}{NA\,.\,BL}=\dfrac{LA'.\,B'N}{NA'.\,B'L}$; $\dfrac{LA\,.\,NB}{AN\,.\,BL}=\dfrac{LA'.\,NB'}{A'N\,.\,B'L}$;

$$(LANB)=(LA'NB'). \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(14)}$$

8°. If two homographic rows have no common point, and if all the points which do not correspond are joined—A and

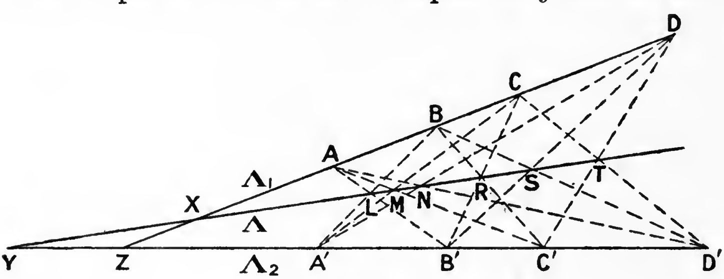

FIG. 20.

B', A' and B, and so on—the joining lines intersect on a straight line, the directive axis, Λ (fig. 20).

Let the points A, B, C on Λ be

$$A=(x_1y_1z_1),\ B=(tx_1+ux_3,\ ty_1+uy_3,\ tz_1+uz_3),\ C=(x_3y_3z_3),$$

and let the corresponding points on Λ_2 be

$$A'=(x'_1y'_1z'_1),\quad B'=(vx'_1+wx'_3,\ vy'_1+w'y_3,\ vz'_1+wz_3),$$
$$C'=(x'_3y'_3z'_3).$$

Let
$$\begin{array}{llll} |y_1z'_1|=a_1 & |y_1z'_3|=b_1 & |y_3z'_1|=c_1 & |y_3z'_3|=d_1. \\ |z_1x'_1|=a_2 & |z_1x'_3|=b_2 & |z_3x'_1|=c_2 & |z_3x'_3|=d_2. \\ |x_1y'_1|=a_3 & |x_1y'_3|=b_3 & |x_3y'_1|=c_3 & |x_3y'_3|=d_3. \end{array}$$

It will be found in the usual way that the equation of the line through $L=AB'\cdot A'B$ and $M=AC'\cdot A'C$ is

$$(twb_1-uvc_1)x+(twb_2-uvc_2)y+(twb_3-uvc_3)z=0. \quad \text{....(15)}$$

This is the directive axis, Λ. If any fourth arbitrary point D be taken on Λ_1 and joined to any one of the three points on Λ_2, say C', cutting Λ in T; the point corresponding to D on Λ_2 is found by drawing a line from C (corresponding to C') through T. The point D' in which it cuts Λ_2 corresponds to D.

For let D be $(t'x_1+u'x_3,\ t'y_1+u'y_3,\ t'z_1+u'z_3)$, and let any point whatever on Λ_2 be

$$P=(v'x'_1+w'x'_3,\ v'y'_1+w'y'_3,\ v'z'_1+w'z'_3).$$

It will be found that the intersections of $A'D$ and AP with Λ are

$$N=\left\{\begin{array}{l} tw\,|a_2b_3|-tw\dfrac{u'}{t'}|b_2c_3|+uv\,|c_2a_3|, \\ tw\,|a_3b_1|-tw\dfrac{u'}{t'}|b_3c_1|+uv\,|c_3a_1|, \\ tw\,|a_1b_2|-tw\dfrac{u'}{t'}|b_1c_2|+uv\,|c_1a_2| \end{array}\right\}. \quad \text{......}(a)$$

$$Q=\left\{\begin{array}{l} tw\,|a_2b_3|-uv\dfrac{w'}{v'}|b_2c_3|+uv\,|c_2a_3|, \\ tw\,|a_3b_1|-uv\dfrac{w'}{v'}|b_3c_1|+uv\,|c_3a_1|, \\ tw\,|a_1b_2|-uv\dfrac{w'}{v'}|b_1c_2|+uv\,|c_1a_2| \end{array}\right\}. \quad \text{......}(b)$$

Since $\dfrac{w'}{v'}$ may have any value whatever, let it be $\dfrac{twu'}{uvt'}$. On substituting this value for $\dfrac{w'}{v'}$ in (b), it becomes (a).

Therefore the lines $A'D$, AP and Λ are concurrent when $\frac{ut'}{tu'}=\frac{wv'}{vw'}$, that is when $(ABCD)=(A'B'C'D')$. This proves the proposition.

It will be observed in fig. 21 that

X on Λ_1 corresponds to Z on Λ_2,

and Y „ Λ_2 „ „ Z „ Λ_1.

9°. It may be shown in a precisely similar manner that if two homographic (flat) pencils have no common ray, $V{\cdot}ABCD$ and $V'{\cdot}A'B'C'D'$ (fig. 21), and if the intersections of rays which do not correspond are joined, $VA{\cdot}V'B'$ and $V'A'{\cdot}VB$, $VB{\cdot}V'C'$ and $V'B'{\cdot}VC$, and so on; all the connecting lines concur in a point

$$L=\{twf_1-uvg_1,\ twf_2-uvg_2,\ twf_3-uvg_3\}. \quad\ldots\ldots(16)$$

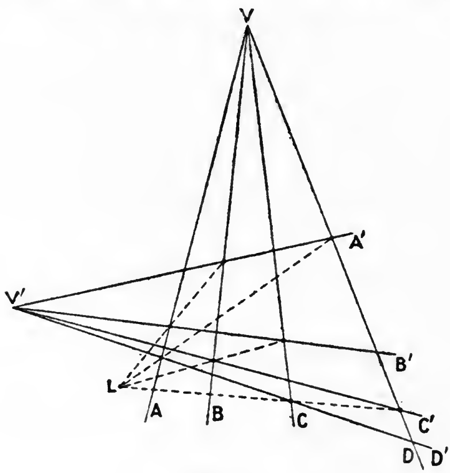

Fig. 21.

The rays of the two pencils are:

for V, $(p_1q_1r_1)$, (t, u), $(p_3q_3r_3)$, (t', u');

„ V', $(p'_1q'_1r'_1)$, (v, w), $(p'_3q'_3r'_3)$, (v', w');

and

$$f_1=|q_1r'_3| \qquad g_1=|q_3r'_1|$$
$$f_2=|r_1p'_3| \qquad g_2=|r_3p'_1|$$
$$f_3=|p_1q'_3| \qquad g_3=|p_3q'_1|$$

L is the directive centre.

If VA, VB, VC are given rays of the V-pencil and the corresponding rays of the V'-pencil are $V'A'$, $V'B'$, $V'C'$, and if an arbitrary fourth ray of the first, VD, be drawn; the corresponding ray of the second is found by joining the point $V'C' \cdot VD$ to the directive centre. The point in which this join cuts VC is the cross of VC with the sought ray $V'D'$.

Ex. Let there be 2 pencils of 3 rays each in which

$$\left.\begin{aligned} VA&=(\bar{2}11)\\ VB&=(\bar{1}11)\\ VC&=(\bar{1}99)\end{aligned}\right\} \text{ correspond to } \left\{\begin{aligned} V'A'&=(1\bar{1}1)\\ V'B'&=(1\bar{2}1)\\ V'C'&=(1\bar{3}1)\end{aligned}\right.$$

From these data we find that

$$t=\tfrac{8}{17},\quad u=\tfrac{1}{17},\quad v=\tfrac{1}{2},\quad w=\tfrac{1}{2};$$
$$f_1=4,\quad f_2=3,\quad f_3=5;$$
$$g_1=18,\quad g_2=10,\quad g_3=-8;$$

and the directive centre is (7, 7, 24).

Now let a fourth arbitrary ray, $VD=(011)$, be drawn to the first pencil. VD cuts $V'C'$ in $(41\bar{1})$; the join of $(41\bar{1})$ and the directive centre cuts VC in $(279, 75, -44)$; and the fourth ray of the second pencil, $V'D'$, is $(15, -47, 15)$. The two pencils will be homographic; for

$$t'=\frac{-1}{17},\quad u'=\frac{2}{17},\quad v'=-1,\quad w'=16$$

and
$$\frac{ut'}{tu'}=\frac{w}{v}\frac{v'}{w'}.$$

Given three corresponding pairs of points or rays, if we select a fourth point or ray in one system we are enabled to draw the corresponding point or ray of the other system by means of the directive axis or centre. But we can calculate the coordinates of the fourth corresponding point or ray, without the assistance of either, by the equation

$$(ABCD)=\frac{ut'}{t'u'}.$$

For since $(ABCD)=(A'B'C'D')$, $\dfrac{ut'}{tu'}=\dfrac{wv'}{vw'}$ and $\dfrac{v'}{w'}=\dfrac{uvt'}{twu'}$, which gives the sought point or ray. Let the two rows be

$$(\bar{1}11),\ (100),\ (211),\ (322) \quad \text{and} \quad (011),\ (01\bar{1}),\ (0\bar{3}1).$$

Then for the first row, $t=\frac{-1}{3}$, $u=\frac{1}{3}$, $t'=\frac{1}{3}$, $u'=\frac{5}{3}$; and for the second, $v=\frac{-2}{3}$, $w=\frac{-1}{3}$. Consequently

$$\frac{v'}{w'}=\frac{uvt'}{twu'}=\frac{-2}{5},$$

and the fourth point of the second row is $(0, \overline{15}, 3)=(0\bar{5}1)$.

The directive axis, however, enables us to find easily the point on one axis which corresponds to infinity on the other.

10°. The point on Λ_2 (fig. 20), corresponding to the point at infinity on Λ_1, is obtained in the same way as any other corresponding point. Let I and J (not used in this connexion as symbols of the circular points) be the points at infinity on Λ_1 and Λ_2 respectively. Draw a line from A' to I (that is, parallel to Λ_1), cutting Λ in (say) X, and the line AX will cut Λ_2 in I', the point corresponding to I on Λ_1.

Ex. Let the two axes be the sides AB, AC of the given triangle, and let the points C', B, C'' on AB correspond to B', C, B'' on AC (fig. 22).

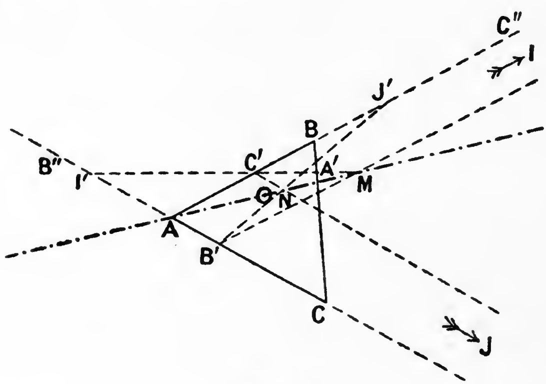

FIG. 22.

Since BB' and CC' cross in O and $B'C''$ and $C'B''$ in A', the directive axis is OA',

$$y-z=0.$$

$$AB \text{ cuts } \Lambda_\infty \text{ in } (m, -l, o)=I;$$

$$AC \text{ cuts } \Lambda_\infty \text{ in } (-n, o, l)=J;$$

Therefore

$$B'I=(l,\ m,\ -l) \quad\text{and}\quad B'I\cdot OA'=(l-m,\ l,\ l)=M.$$

Similarly,

$$C'J=(l,\ -l,\ n) \quad\text{and}\quad C'J\cdot OA'=(l-n,\ l,\ l)=N,$$

$$C'M=(l,\ -l,\ m) \quad\text{and}\quad C'M\cdot AC=(-m,\ o,\ l)=I',$$

$$B'N=(l,\ n,\ -l) \quad\text{and}\quad B'N\cdot AB=(n,\ -l,\ o)=J'.$$

Since $(C'BJ'I)=(B'CJI')$, $\dfrac{C'B}{BJ'}=\dfrac{B'C}{I'B'}=\dfrac{CB'}{B'I'}$. ....(17)

To verify this:

$$C'B=\frac{l(\beta-\alpha)}{l+m}; \quad BJ'=\frac{n(\beta-\alpha)}{m-n};$$

$$CB'=\frac{l(\alpha-\gamma)}{n+l}; \quad B'I'=\frac{n(l+m)(\alpha-\gamma)}{(n+l)(m-n)}.$$

Therefore $$\frac{C'B}{BJ'}=\frac{l(m-n)}{n(l+m)}=\frac{CB'}{B'I'}.$$

11°. Given two homographic rows on an axis, to determine the double points, L and N.

By (13), $$(LBCD)=(LB'C'D'),$$

and $$B'C'.CD.LB.LD'=BC.C'D'.LB'.LD.$$

Let $AL=x$, and assuming that L lies to the left of A (fig. 16),

$$LB=AB-x,\ LD'=AD'-x,\ LB'=AB'-x,\ LD=AD-x.$$

Hence

$$B'C'.CD(AB-x)(AD'-x)=BC.C'D'(AB'-x)(AD-x). \quad (20)$$

This quadratic will give two values for AL. One will be the value of AL, the other the value of AN; for not only is $(LBCD)=(LB'C'D')$, but $(NBCD)=(NB'C'D')$, (13°).

Ex. Let $AB=1$, $BC=2$, $CD=4$, $DD'=10$, $D'C'=4$, $C'B'=8$, $B'A'=16$.

Then $8\times4(1-x)(17-x)=2\times4(29-x)(7-x)$,

and $$x^2-12x-45=0.$$

Therefore $$x=-3 \quad\text{or}\quad 15;$$

L being 3 units to the left, N 15 units to the right of A.

Verification.

$$(LANB)=\tfrac{1}{5}\,\tfrac{7}{2}=\tfrac{7}{10},\quad (LA'NB')=\tfrac{8}{5}\,\tfrac{7}{16}=\tfrac{7}{10},$$

and $$(LANB)=(LA'NB').$$

12°. Given four fixed points, no three of which are collinear, P_1, P_2, P_3, P_4 (fig. 23); to find the locus of a fifth point $P=(xyz)$, subject to the condition

$$P\cdot P_1P_2P_3P_4=\frac{-h}{f},$$

a constant.

Let $$P_1=(x_1y_1z_1)\ldots P_4=(x_4y_4z_4).$$

Then, by (8), $$-\frac{h}{f}=\frac{|xy_1z_2||xy_3z_4|}{|xy_2z_3||xy_4z_1|}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

$$f|xy_1z_2||xy_3z_4|+h|xy_2z_3||xy_4z_1|=0,$$

which is evidently an equation of the second degree. The locus of P therefore is a conic, and it passes through P_1, P_2, P_3, P_4; for if we substitute for the variables in this equation the coordinates of any one of the four points, say P_3, the second and third matrices vanish and the equation becomes identically, $0=0$.

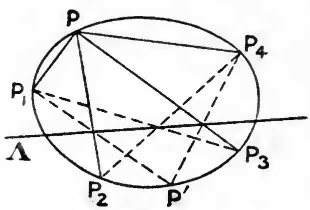

Fig. 23.

This theorem shows that a conic must pass through any five arbitrary points, no three of which are collinear. A sixth point, P', will only lie upon it if equation (a) remains true when the coordinates of P' are substituted in it for those of P. From a geometric point of view, P' will lie on the curve if the intersections

$$PP_2\cdot P_1P',\quad P_1P_3\cdot P_2P_4,\quad PP_3\cdot P_4P'$$

are collinear, as shown in fig. 23.

It follows that the cross ratio of any four points on a conic is constant.

13°. Given four fixed lines, Λ_1, Λ_2, Λ_3, Λ_4, no three of which are concurrent; to find the envelope of a fifth line $\Lambda=(pqr)$, such that the cross ratios of its intersections with the four fixed lines is constant, $\dfrac{-g}{f}$.

Let $\Lambda_1=(p_1q_1r_1)\ldots\Lambda_4=(p_4q_4r_4)$. Then, by 3° (c), the cross ratio of the intersections for some fixed position of Λ is

$$\frac{|pq_1r_2||pq_3r_4|}{|pq_2r_3||pq_4r_1|}=-\frac{g}{f},$$

or
$$f|pq_1r_2||pq_3r_4|+g|pq_2r_3||pq_4r_1|=0.$$

Now let p, q, r vary under the condition of this equation, and we have a tangential equation of the second order, which consequently represents a conic. The four fixed lines touch the curve; for if the coordinates of any one of them, say $(p_4q_4r_4)$, are substituted for those of the variable, the equation becomes identically zero.

Ex. Let the four given lines be $(2\bar{1}2)$, (001), (010), (100), and let $\frac{g}{f}=1$. Substituting these values for the constants above, the equation becomes

$$qr+rp+pq=0,$$

the tangential equation of the inconic.

It follows that the cross ratio of the intersections of a variable tangent to a conic with four fixed tangents is constant.

14°. Every triangle which circumscribes the inconic

$$x^2+y^2+z^2-2yz-2zx-2xy=0$$

is inscribed in the circumconic $yz+zx+xy=0$ (fig. 24).

From any point D on the circumconic draw two tangents to the inconic cutting BC in d, CA in c and AB in b. From E, the point in which Db cuts the circumconic, draw a tangent cutting BC in e and meeting Dd in F. Then F lies on the circumconic.

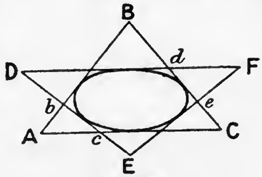

Fig. 24.

It will be observed that the four fixed tangents, AB, AC, FD, FE, cut the tangent BC in d and e, and the tangent DE in b and c. Therefore, 13°,

$$(BCde)=(bcDE)=\text{const.}$$
$$F\cdot BCDE=A\cdot BCDE=\text{const.}$$

Therefore, F lies on the circumconic, 12°.

This proposition only holds good for conics which are represented by the equation $x^2 \ldots -2xy=0$, *i.e.* inscribed or escribed conics which touch the sides in A', B', C'. For example, the ellipse

$$x^2+9y^2+4z^2-12yz-4zx-6xy=0$$

touches the sides of the given triangle, and on this curve lie the points (24, 2, 3), (683) and (316). If tangents be drawn at these points, it will be found that they cross in the points $(12, 4, \bar{3})$, $(\bar{3}23)$ and $(6\bar{1}3)$; and the coordinates of the two latter points do not satisfy the equation of the circumconic. Consequently the triangle of which these three points are the corners is not inscribed in the circumconic.

15°. The points in which a circle is cut by conjugate chords form a harmonic group.

Let L and N be any two points upon a circle (fig. 25), and let the tangents at these points meet in M. Then LN and any secant MAA' are conjugate chords, V, 9° (*b*).

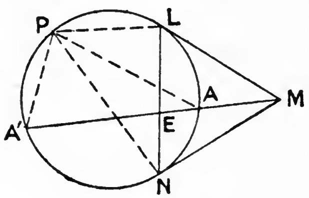

FIG. 25.

Let LN and MA cross in E. Let P be any point on the circle and join it to L, A, N, A'. Since M is the pole of LN, $(MAEA')=-1$, V, 11°, and consequently

$$L\cdot MANA'=-1.$$

But $\quad \angle MLA=\angle LPA, \quad \angle NLA'=\angle NPA',$

$$\angle ALN=\angle APN, \quad \angle A'LM=\angle A'PL.$$

Therefore $\quad L\cdot MANA'=P\cdot LANA'=-1,$

and $\quad (LANA')=-1.$

The points in which a circle is cut by any diameter form a harmonic group with I and J. For I and J lie on every circle and IJ and any diameter are conjugate chords, either passing through the pole of the other.

If a number of secants be drawn through M cutting the circle in B, B'; C, C'; D, D', etc.; we have

$$(LANA')=(LBNB')=(LCNC')=\text{etc.}=-1,$$

and $\quad MA\,.\,MA'=MB\,.\,MB'=MC\,.\,MC'=\text{etc.}=LM^2.$

Pairs of points thus related form a system in involution.

16°. It was shown in 1° that the harmonic conjugates of λ and ν are

$$\rho=\frac{x\lambda+y\nu}{x+y}, \quad \rho'=\frac{x\lambda-y\nu}{x-y}.$$

Let M be the centre of LN (fig. 16). Then

$$OM=\mu=\frac{\lambda+\nu}{2};$$

$$MN=\nu-\mu=\frac{\nu-\lambda}{2}; \quad MX=\frac{x\lambda+y\nu}{x+y}-\frac{\lambda+\nu}{2}=\frac{y-x}{y+x}\frac{\nu-\lambda}{2};$$

$$MX'=\frac{x\lambda-y\nu}{x-y}-\frac{\lambda+\nu}{2}=\frac{y+x}{y-x}\frac{\nu-\lambda}{2}.$$

Therefore $$\overline{MX}\,.\,\overline{MX'}=\left(\frac{\nu-\lambda}{2}\right)^2=\overline{MN}^2.$$

If $\frac{y}{x}>1$, X and X' will lie to the right of M; if <1, both points will lie to the left of M.

Let the successive positions of X and X', as $\frac{y}{x}$ varies, be A, A'; B, B', etc., and

$$\overline{MA}\,.\,\overline{MA'}=\overline{MB}\,.\,\overline{MB'}=\text{etc.}=\overline{MN}^2. \quad \text{..........(21)}$$

Thus the variable points form divisions in involution on the indefinite line LN. The points L and N are the foci of the involution, and M (the conjugate of the point at infinity) is the centre.

17°. Given two pairs of points, A, A' and B, B' on a straight line; to find a point M such that

$$MA\,.\,MA'=MB\,.\,MB'=k^2.$$

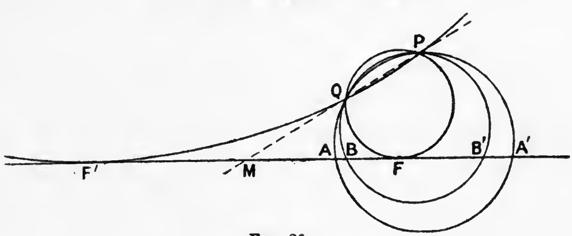

FIG. 26.

Draw a circle through AA', as in fig. 26, and draw another circle through BB' and any point P on the first circle. The cross of PQ (the radical axis of the two circles) and the axis is the sought point M; for

$$MP \, . \, MQ = MA \, . \, MA' = MB \, . \, MB'.$$

To find the conjugate of any fifth point C; draw a circle through P, Q and C and it will cut the axis in C', the conjugate of C.

18°. The position of the radical axis, PQ (fig. 26), with respect to the axis, or the position of M on the axis, depends upon the relative position of the points A, A'; B, B', etc. M may lie (a) outside the circles; or (b) it may coincide with Q on the axis; or (c) it may lie within the circles.

(a) When M lies without the circles we have the hyperbolic involution of 16° and fig. 26, where k^2 is positive. Since the points and their conjugates lie on the same side of M, and since A' moves towards M as A moves from M, one pair of conjugates must ultimately meet in a point F such that, $MP \, . \, MQ = OF^2$. OF is therefore equal in length to a tangent from M to any of the circles of fig. 26; all such tangents being equal because M is a point on their common radical axis PQ. To find F, we draw a circle through P and Q, touching the axis. Two such circles can in general be drawn, one of which touches the axis in F, the other in F' (fig. 26). Obviously, $MF = F'M$.

In a hyperbolic involution we have, 16°,

$$(F'AFA') = (F'BFB') = \text{etc.} = -1,$$

in addition to the general equation of involution, (21).

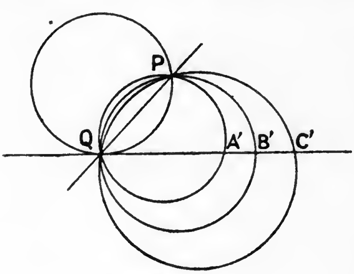

Fig. 27.

(b) If Q happens to lie on the axis, M must coincide with it, as also must F', F and A, B, C, and in this case (fig. 27)

$$MP \, . \, MQ = MA \, . \, MA' \\ = MB \, . \, MB' = k^2 = 0.$$

This is parabolic involution.

(c) When M lies within the circles the foci are imaginary; algebraically, because k^2 is negative, the points and their conjugates lying on opposite sides of M (fig. 28); geometrically, because the foci are the points of contact in which circles through P and Q touch the axis, and in the present case these circles are imaginary.

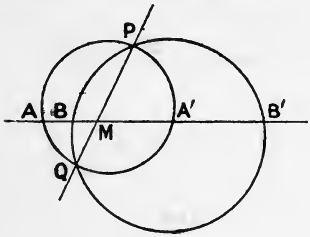

FIG. 28.

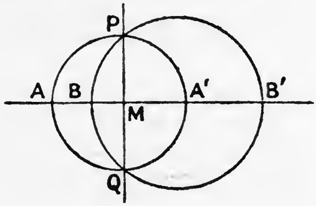

FIG. 29.

This is elliptic involution and

$$MP \,.\, MQ = MA \,.\, MA' = MB \,.\, MB^2 = \text{etc.} = -k^2.$$

(d) When the radical axis is bisected at right angles by the axis we have circular involution (fig. 29), in which the segments AA', BB', etc., subtend right angles at P and Q. As in (c),

$$MP \,.\, MQ = MA \,.\, MA' = MB \,.\, MB' = \text{etc.} = -k^2.$$

The peculiar property of this species is, that each ray of the pencil $P \cdot ABA'B'$ is at right angles to its conjugate, whatever the number of points. It also enables us to introduce the imaginary focal rays; for, 15°,

$$P \cdot IAJA' = P \cdot IBJB' = \text{etc.} = -1.$$

It will be observed that in elliptic involution the segments AA', BB', etc., overlap, that in parabolic involution they have one point in common, and that in hyperbolic involution they lie wholly within or without one another.

An involution may have, (a), two real and distinct foci, or, (b), two real and coincident foci, or, (c) and (d), two imaginary foci. Every involution has a centre, and in all cases the product of the distances of any two conjugate points from the centre is constant.

This distance is

$$\left.\begin{array}{llll} MF^2 = MA \,.\, MA' = k^2, & \text{for hyperbolic involution.} \\ \;\;\text{,,} \;=\; \text{,,} \;\; = 0, & \text{,, parabolic} \quad \text{,,} \\ \;\;\text{,,} \;=\; \text{,,} \;\; = -k^2, & \text{,, elliptic} \quad\;\; \text{,,} \end{array}\right\}(22)$$

19°. To calculate the position of the centre of involution M, given the distances between four collinear points (*a*) and (*b*) (fig. 30).

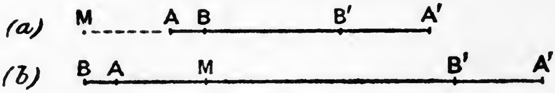

FIG. 30.

Since the segment BB' lies wholly within AA', (*a*) is a case of hyperbolic evolution; while (*b*) is elliptic, the segments overlapping one another.

Let $AM = x$. Then, since $MA \,.\, MA' = MB \,.\, MB'$,

$$-x(AA' - x) = (AB - x)(AB' - x)$$

and
$$x = \frac{AB \,.\, AB'}{AB + AB' - AA'}. \quad \text{..............(23)}$$

In (*a*) let $AB = 1$, $AB' = 6$, $AA' = 9$ and $x = -3$.

In (*b*) let $AB = -1$, $AB' = 12$, $AA' = 15$ and $x = 3$.

$$MA \,.\, MA' = MB \,.\, MB' = \pm 36.$$

If y be the distance from the centre to either focus in (*a*), we have, 16°,

$$y^2 = MA \,.\, MA' = 36 \quad \text{and} \quad y = \pm 6.$$

20°. We have now to draw certain deductions from the general equation

$$MA \,.\, MA' = MB \,.\, MB' = MC \,.\, MC' = \text{etc.} = \text{constant}.$$

Since

$$\frac{MA}{MB} = \frac{MB'}{MA'}; \quad \frac{MA - MB}{MB} = \frac{MB' - MA'}{MA'} \quad \text{and} \quad \frac{AB}{A'B'} = \frac{BM}{MA'};$$

with corresponding expressions for $\frac{BC}{B'C'}$, etc.

Again,

$$\frac{MA}{MB'} = \frac{MB}{MA'}; \quad \frac{MA - MB'}{MB'} = \frac{MB - MA'}{MA'} \quad \text{and} \quad \frac{AB'}{BA'} = \frac{MB'}{MA'};$$

with corresponding expressions for $\frac{BC'}{CB'}$, etc.

Writing out the two series for clearness and convenience:

$$\left.\begin{array}{ll} (1) & \dfrac{AB}{A'B'}=\dfrac{BM}{MA'}. \\ (2) & \dfrac{BC}{B'C'}=\dfrac{CM}{MB'}. \\ (3) & \dfrac{CD}{C'D'}=\dfrac{DM}{MC'}. \\ (4) & \dfrac{DA}{D'A'}=\dfrac{AM}{MD'}. \end{array}\right\} \ldots\ldots(a) \qquad \left.\begin{array}{ll} (5) & \dfrac{AB'}{BA'}=\dfrac{MB'}{MA'}. \\ (6) & \dfrac{BC'}{CB'}=\dfrac{MC'}{MB'}. \\ (7) & \dfrac{CD'}{DC'}=\dfrac{MD'}{MC'}. \\ (8) & \dfrac{DA'}{AD'}=\dfrac{MA'}{MD'}. \end{array}\right\} \ldots\ldots(b)$$

The product of the first and third expressions of (a) divided by the product of the second and fourth is

$$\frac{AB.CD.B'C'.D'A'}{BC.DA.A'B'.C'D'}=\frac{MB.MB'.MD.MD'}{MC.MC'.MA.MA'}=1.$$

Therefore $$(ABCD=(A'B'C'D'). \qquad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

Multiplying together the left-hand expressions of (b), and also the right-hand expressions,

$$\frac{AB'.BC'.CD'.DA'}{A'B.B'C.C'D.D'A}=\frac{MB'.MC'.MD'.MA'}{MA'.MB'.MC'.MD'}=1$$

and $$AB'.BC'.CD'.DA'=A'B.B'C.C'D.D'A. \qquad \ldots\ldots(25)$$

By (1) and (3) of (a),

$$\frac{AB.C'D'}{CD.A'B'}=\frac{MB.MC'}{MD.MA'}.$$

By (6) and (8) of (b),

$$\frac{BC'.D'A}{B'C.DA'}=\frac{-MC'.-MD'}{MA'.MB'}=\frac{MB.MC'}{MD.MA'}.$$

Therefore

$$\frac{AB.C'D'}{BC'.D'A}=\frac{A'B'.CD}{B'C.DA'}, \text{ or } (ABC'D')=(A'B'CD). \ldots(26)$$

By (1) and (4) of (a),

$$\frac{AB}{DA}=\frac{A'B'.MB.MD'}{D'A'.MA.MA'}.$$

By (6) and (7) of (b),

$$\frac{C'D}{BC'}=\frac{CD'.MB'}{B'C.MD'}.$$

Therefore $\dfrac{AB.C'D}{BC'.DA}=\dfrac{A'B'.CD'.MB.MB'}{B'C.D'A'.MA.MA'}$,

and $(ABC'D)=(A'B'CD')$. ....(27)

21°. The connexion between the coordinates of a system of points in involution.

Let $(ABCD)=(A'B'C'D')$ and $(AB'CD)=(A'BC'D')$.

$$\frac{AB.CD}{BC.DA}=\frac{A'B'.C'D'}{B'C'.D'A'},$$

$$\frac{AB'.CD}{B'C.DA}=\frac{A'B.C'D'}{BC'.D'A'}.$$

By division,

$$\frac{AB.B'C}{BC.AB'}=\frac{A'B'.BC'}{B'C'.A'B'};\quad \frac{A'B.CB'}{BC.B'A'}=\frac{AB'.C'B}{B'C'.BA},$$

and $(A'BCB')=(AB'C'B)$.

Therefore $(A'CBB')=(AC'B'B)$,

and $\dfrac{A'C.BB'}{CB.B'A'}=-\dfrac{AC'.BB'}{C'B'.BA}$.

Therefore

$$-1=\frac{AB.CA'.B'C'}{BC.A'B'.C'A}=(ABCA'B'C'). \quad ....(28)$$

Let $A=(x_1y_1z_1)$ and $A'=(x_2y_2z_2)$;

" $B=(t_1x_1+u_1x_2,\ t_1y_1+u_1y_2,\ t_1z_1+u_1z_2)$
$=(t_1,\ u_1)$; $C=(t_2,\ u_2)$;

" $B'=(t_3,\ u_3)$; $C'=(t_4,\ u_4)$. Then if we calculate the values of the various vectors $AB \ldots C'A$, we get, (28),

$$\frac{u_1t_2\,|\,t_3u_4\,|}{t_3u_4\,|\,t_1u_2\,|}=-1,$$

which may be more conveniently written

$$\frac{u_1t_2u_3t_4}{t_1u_2t_3u_4}=1, \quad ....(29)$$

as the condition that the six points shall be in involution.

Ex. The transversal, $x-4y+2z=0$, cuts the sides and internal diagonals of the quadrilateral $AC'A'C$ (fig. 1) as follows:

$C'A'$ in $(\bar{2}13)$, CA in $(20\bar{1})$, AC' in (410),
CC' in (211), AA' in (223), $A'C$ in (012),

and connecting these coordinates with the letters of (28),

$$\begin{array}{cccccc} A, & B, & C, & C', & B', & A'. \\ (\bar{2}13) & (20\bar{1}) & (410) & (211) & (223) & (012) \end{array}$$

Calculating the values of t and u for B, C, B', C', from the coordinates of A and A', we get,

$$\begin{array}{llll} \text{for } B, \; t_1=-1, & u_1=1; & \text{for } B', \; t_3=-1, & u_3=3. \\ \;\;\text{,,} \;\; C, \; t_2=-2, & u_2=3; & \;\;\text{,,} \;\; C', \; t_4=-1, & u_4=2. \end{array}$$

Here $$\frac{u_1t_2u_3t_4}{t_1u_2t_3u_4}=\frac{6}{6}=1,$$

and the system is in involution.

22°. It follows from V, 9°, (d), that if a variable point P forms a row on a line q, p, the polar of P, will form a pencil with Q, the pole of q, for vertex. And the converse. What is the connexion between the row formed by P and the pencil formed by p?

Let four of the positions occupied by P on q be

$$A=(x_1y_1z_1), \quad B=(tu), \quad C=(x_3y_3z_3), \quad D=(t'u').$$

Then $(ABCD)=\dfrac{ut'}{tu'}$. Let the polars of A and C be respectively,

$$ax+by+cz=0, \text{ and } a'x+b'y+c'z=0.$$

On forming the equations of the polars of B and D in the usual way, it will be found that the polar of

$$B=(ta+ua', \; tb+ub', \; tc+uc'),$$
$$D=(t'a+u'a', \; t'b+u'b', \; t'c+u'c').$$

Consequently, the cross ratio of the pencil of polars is, 5°, (b),

$$\frac{ut'}{tu'}.$$

Therefore the cross ratio of any four collinear points is the cross ratio of the pencil formed by their polars.

Let A', B', C', D' be the points in which the polars of A, B, C, D cut the axis. Then the two homographic rows, A, B, C, D and A', B', C', D', form a system in involution. For since the polar of A passes through A', the polar of A' passes through A; V, 9°, (a); and $(A'BCD)=(AB'C'D')$

for the same reason that $(ABCD)=(A'B'C'D')$, namely, because A, B', C', D' are the points in which the polars of A', B, C, D cut the axis. Since, then, $(ABCD)=(A'B'C'D')$ and $(A'BCD)=(AB'C'D')$, the system is in involution.

If the	axis	cuts	the conic	the involution is	hyperbolic.
,,	,,	touches	,,	,,	parabolic.
,,	,,	lies without	,,	,,	elliptic.

In the last case, when the involution happens to be circular, the axis becomes the directrix and its pole (the point Q) is a focus of the conic.

CHAPTER IX

TRANSFORMATION OF COORDINATES

1°. The coordinates of the various points of a net with ABC for the given triangle and any new origin O' are obtained in exactly the same way as those of the corresponding points of the old net, and the symbols of corresponding points are identical. Thus $(01\bar{1})$ is the symbol of the new A'', as it was of the old A'', but in general old A'' and new A'' are not the same point in the plane. In both cases A'' is the cross of the lines BC and $B'C'$; but in changing the origin from O to O' we shift the position of $B'C'$, and the new $B'C'$ will not cut BC in the same point as the old $B'C'$. In changing the origin from O to O', then, we generally change the position of every point in the net except A, B, C, although the symbols of corresponding points remain unaltered. At the same time α, β, γ become α', β', γ', and l, m, n become l', m', n', scalars such that $l'\alpha'+m'\beta'+n'\gamma'=0$.

To save space, the lines drawn from the corners of the triangle through any point to the opposite sides will be called the rays of the point.

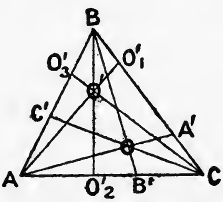

FIG. 31.

2°. The relation between the ratios in which the rays of two points cut the sides of the given triangle (fig. 31).

Let the old origin be O as usual, and let the new origin be a rational point of the O net, $O'=(fgh)$, whose rays cut the sides as $l':m':n'$. Then,

$$\text{for } O, \quad \frac{BO'_3}{O'_3A}=\frac{fl}{gm}; \quad \frac{CO'_1}{O'_1B}=\frac{gm}{hn}; \quad \frac{AO'_2}{O'_2C}=\frac{hn}{fl};$$

$$\text{,, } O', \quad \text{,,} =\frac{l'}{m'}; \quad \text{,,} =\frac{m'}{n'}; \quad \text{,,} =\frac{n'}{l'}.$$

Therefore $\quad l' : m' : n' = fl : gm : hn, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$

and $\quad l : m : n = f^{-1}l' : g^{-1}m' : h^{-1}n'. \quad \ldots\ldots\ldots(2)$

3°. Let $P=(xyz)$ be any rational point of the O net (fig. 32). To find its coordinates $(x'y'z')$ to a new rational origin, $O'=(fgh)$.

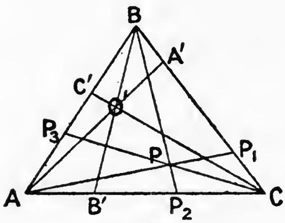

Fig. 32.

$$(AC'BP_3)=\frac{m'lx}{l'my}=x',$$

$$(BA'CP_1)=\frac{n'my}{m'nz}=y',$$

$$(CB'AP_2)=\frac{l'nz}{n'lx}=z'.$$

Therefore $\quad x' : y' : z' = m'n'lx : n'l'my : l'm'nz,$

and (1), $\quad$ „ $\quad = f^{-1}x : g^{-1}y : h^{-1}z. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$

Conversely, $\quad x : y : z = fx' : gy' : hz'. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

4°. The commonest case is that in which the new origin is irrational.

Let $O'=\left(\frac{f}{l}, \frac{g}{m}, \frac{h}{n}\right)$, its rays cutting the sides as $f:g:h$.

Then $\quad (AC'BP_3)=\dfrac{glx}{fmy}=x'; \quad (BA'CP_1)=\dfrac{hmy}{gnz}=y';$

$$(CB'AP_2)=\frac{fnz}{hlx}=z'.$$

$$x' : y' : z' = f^{-1}lx : g^{-1}my : h^{-1}nz, \quad \ldots\ldots\ldots\ldots(5)$$

$$x : y : z = l^{-1}fx' : m^{-1}gy' : n^{-1}hz'. \quad \ldots\ldots\ldots\ldots(6)$$

Ex. 1. To transform the equation of the circumcircle

$$mna^2yz + nlb^2zx + lmc^2xy = 0 \quad \ldots\ldots\ldots\ldots\ldots(a)$$

from origin O to origin S, the symmedian point, $\left(\frac{a^2}{l}, \frac{b^2}{m}, \frac{c^2}{n}\right)$.

Here $\left(\frac{f}{l}, \frac{g}{m}, \frac{h}{n}\right)$ is represented by $\left(\frac{a^2}{l}, \frac{b^2}{m}, \frac{c^2}{n}\right)$, and by (6),

$$x : y : z = \frac{a^2}{l}x' : \frac{b^2}{m}y' : \frac{c^2}{n}z'.$$

Consequently,

$$0 = mna^2yz + nlb^2zx + lmc^2xy = mna^2\frac{b^2c^2}{mn}y'z' + nlb^2\frac{c^2a^2}{nl}z'x' + lmc^2\frac{a^2b^2}{lm}x'y',$$

$$\text{,,} = \quad \text{,,} \quad \text{,,} \quad = y'z' + z'x' + x'y'.$$

In fact we have merely to substitute a^2, b^2, c^2 for l, m, n in (a).

Ex. 2. Let the converse problem be considered: to transform to origin O the equation

$$yz + zx + xy = 0,$$

which represents the circumcircle when S is origin.

As S is irrational in respect to O, the symbol of O when S is origin is $\left(\frac{l}{a^2}, \frac{m}{b^2}, \frac{n}{c^2}\right)$. These coordinates now represent $\left(\frac{f}{l}, \frac{g}{m}, \frac{h}{n}\right)$ above, and we have, (6),

$$x : y : z = \frac{lx'}{a^2} : \frac{my'}{b^2} : \frac{nz'}{c^2}.$$

Therefore

$$0 = yz + zx + xy = \frac{mn}{b^2c^2}y'z' + \frac{nl}{c^2a^2}z'x' + \frac{lm}{a^2b^2}x'y',$$

$$\text{,,} = \quad \text{,,} \quad = mna^2y'z' + nlb^2z'x' + lmc^2x'y'.$$

5°. In the foregoing sections the origin only was changed, the triangle remaining the same. We have now to consider the case in which both the origin and the triangle are changed.

Let any point O' be chosen for the new origin and any three points A_1, B_1, C_1 for the corners of the new triangle; and let their old coordinates (for the triangle ABC and origin O) be

$$O' = (x_0y_0z_0), \quad A_1 = (x_1y_1z_1), \quad B_1 = (x_2y_2z_2), \quad C_1 = (x_3y_3z_3).$$

Let P be any point whose old coordinates are (xyz): it is required to find its new coordinates $(x'y'z')$ with respect to the new triangle $A_1B_1C_1$ and the new origin O'.

By II, 1°, $\dfrac{x'}{y'}=(C_1\cdot A_1O'B_1P)$, and by VIII, (9), the value of this pencil is

$$(C_1\cdot A_1O'B_1P)=\frac{\begin{vmatrix} x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \\ x_0 & y_0 & z_0 \end{vmatrix}\begin{vmatrix} x_3 & y_3 & z_3 \\ x_2 & y_2 & z_2 \\ x & y & z \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 & z_3 \\ x_0 & y_0 & z_0 \\ x_2 & y_2 & z_2 \end{vmatrix}\begin{vmatrix} x_3 & y_3 & z_3 \\ x & y & z \\ x_1 & y_1 & z_1 \end{vmatrix}}=\frac{\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} x_0 & y_0 & z_0 \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}}{\begin{vmatrix} x & y & z \\ x_3 & y_3 & z_3 \\ x_1 & y_1 & z_1 \end{vmatrix}\begin{vmatrix} x_0 & y_0 & z_0 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}}$$

or,

$$\frac{x'}{y'}=\frac{|xy_2z_3|\,|x_0y_3z_1|}{|xy_3z_1|\,|x_0y_2z_3|}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Similarly,

$$\frac{y'}{z'}=\frac{|xy_3z_1|\,|x_0y_1z_2|}{|xy_2z_2|\,|x_0y_3z_1|};\quad \frac{z'}{x'}=\frac{|xy_1z_2|\,|x_0y_2z_3|}{|xy_2z_3|\,|x_0y_1z_2|}.$$

From these three ratios we have

$$\left.\begin{aligned} x'=|xy_2z_3|\,|x_0y_3z_1|\,|x_0y_1z_2|;\quad y'&=|xy_3z_1|\,|x_0y_1z_2|\,|x_0y_2z_3|; \\ z'&=|xy_1z_2|\,|x_0y_2z_3|\,|x_0y_3z_1|. \end{aligned}\right\}\ (8)$$

(Hamilton.)

6°. Let the point (xyz) be $C'''=(112)$ (fig. 1); let the points chosen for the corners of the new triangle be $(12\bar{1})$, $(01\bar{1})$, $(1\bar{1}0)$; and let the new origin be $(1\bar{3}1)$. Here

$$\begin{array}{lllll} x=1 & x_0=1 & x_1=1 & x_2=0 & x_3=1 \\ y=1 & y_0=-3 & y_1=2 & y_2=1 & y_3=-1 \\ z=2 & z_0=1 & z_1=-1 & z_2=-1 & z_3=0 \end{array}$$

Substituting these values in (8) we get, $x'=-4$, $y'=8$, $z'=\dfrac{-2}{3}$; and putting these numbers in continued proportion,

$$x':y':z'=6:-12:1,$$

or the new coordinates of P are $(6,\ -12,\ 1)$.

We may verify this result without any reference to the equations of (8).

The points taken for the new triangle are

$$D=B'C'\cdot CB'''=(12\bar{1}),\quad A''=(01\bar{1})\ \text{and}\ C''=(1\bar{1}0)$$

(fig. 33), an extension of fig. 1. For the new origin, an auxiliary point $H = BC' \cdot AD = (02\bar{1})$ was determined, and the point $BB' \cdot C''H = (1\bar{3}1)$ was chosen for O'.

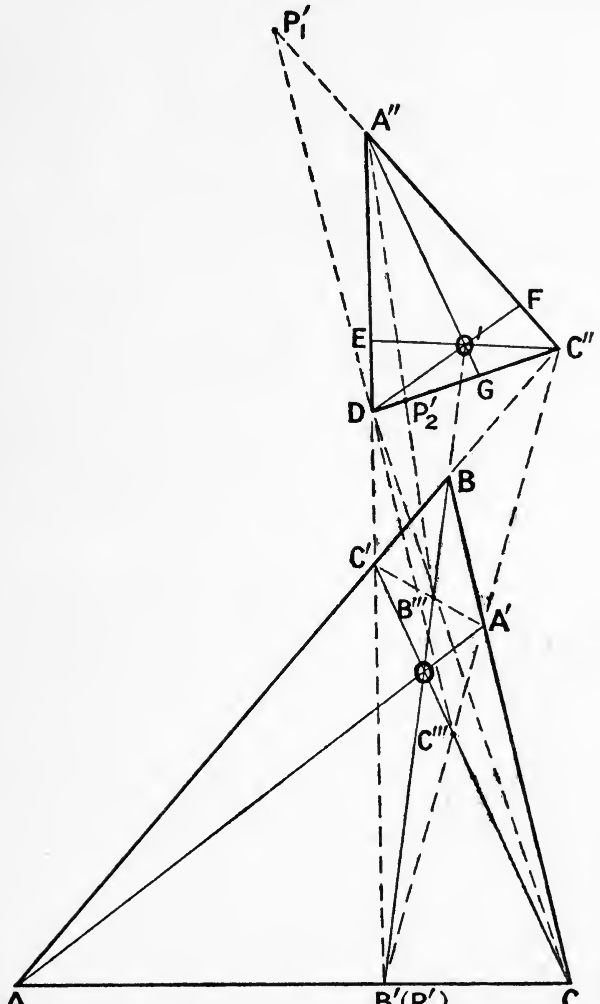

Fig. 33.

The old O is chosen so that

$$l : m : n = 1 : 5 : 2.$$

It will be found that $O'D$, $O'A''$, $O'C''$ cut the sides of $DA''C''$ in the ratios

$$\frac{A''E}{ED}=\frac{l+2m-n}{m-n}=\frac{3}{1};$$

$$\frac{C''F}{FA''}=\frac{m-n}{-3(l-m)}=\frac{1}{4};\quad \frac{DG}{GC''}=\frac{4}{3}.$$

Hence $$l':m':n'=3:1:4.$$

Lines drawn from C''' through D, A'' and C'' meet the opposite sides of

$$DA''C'' \text{ in } P'_1=(13\bar{4}),\ P'_2=(15\bar{2}),\ P'_3=B'=(101).$$

Hence $$\frac{A''P'_3}{P'_3D}=\frac{l'x'}{m'y'}=\frac{l+2m-n}{-2(m-n)}=\frac{3}{-2};$$

$$\frac{C''P'_1}{P'_1A''}=\frac{m'y'}{n'z'}=\frac{4(m-n)}{l-m}=\frac{-3}{1};$$

$$\frac{DP'_2}{P'_2C''}=\frac{n'z'}{l'x'}=\frac{2}{9};$$

or $$l'x':m'y':n'z'=9:-6:2.$$

But $$l':m':n'=3:1:4.$$

Therefore, $$x':y':z'=6:-12:1, \text{ as before,}$$

or $$P'=(6,\ -12,\ 1).$$

7°. We may somewhat simplify Hamilton's equations.

Let the nine minors of $|x_1y_2z_3|$, in the usual terminology, be

$$L_1 \dots M_2 \dots N_3,$$

and let
$$|x_0y_3z_1|\,|x_0y_1z_2|=k_1,$$
$$|x_0y_1z_2|\,|x_0y_2z_3|=k_2,$$
$$|x_0y_2z_3|\,|x_0y_3z_1|=k_3.$$

Then the equations of (8) become

$$\left.\begin{aligned} x'&=k_1(L_1x+M_1y+N_1z),\\ y'&=k_2(L_2x+M_2y+N_2z),\\ z'&=k_3(L_3x+M_3y+N_3z).\end{aligned}\right\} \qquad \dots\dots\dots\dots\dots(9)$$

Therefore,

$$x=\begin{vmatrix} M_1 & N_1 & \frac{x'}{k_1} \\ M_2 & N_2 & \frac{y'}{k_2} \\ M_3 & N_3 & \frac{z'}{k_3} \end{vmatrix}; \quad y=\begin{vmatrix} N_1 & L_1 & \frac{x'}{k_1} \\ N_2 & L_2 & \frac{y'}{k_2} \\ N_3 & L_3 & \frac{z'}{k_3} \end{vmatrix}; \quad z=\begin{vmatrix} L_1 & M_1 & \frac{x'}{k_1} \\ L_2 & M_2 & \frac{y'}{k_2} \\ L_3 & M_3 & \frac{z'}{k_3} \end{vmatrix}.$$

Evidently,

$$x=|M_2N_3|\frac{x'}{k_1}+|M_3N_1|\frac{y'}{k_2}+|M_1N_2|\frac{z'}{k_3}.$$

But

$$|M_2N_3|=x_1|x_1y_2z_3|;\ |M_3N_1|=x_2|x_1y_2z_3|;\ |M_1N_2|=x_3|x_1y_2z_3|.$$

Consequently, Similarly,

$$\left.\begin{aligned} x&=\frac{x_1}{k_1}x'+\frac{x_2}{k_2}y'+\frac{x_3}{k_3}z'.\\ y&=\frac{y_1}{k_1}x'+\frac{y_2}{k_2}y'+\frac{y_3}{k_3}z',\\ z&=\frac{z_1}{k_1}x'+\frac{z_2}{k_2}y'+\frac{z_3}{k_3}z'.\end{aligned}\right\} \qquad \text{...............(10)}$$

For the triangles of fig. 28,

$k_1=\bar{3}$; $x_1=1$; $y_1=2$; $z_1=\bar{1}$; $L_1=\bar{1}$; $M_1=\bar{1}$; $N_1=\bar{1}$.
$k_2=\bar{3}$; $x_2=0$; $y_2=1$; $z_2=\bar{1}$; $L_2=1$; $M_2=1$; $N_2=3$.
$k_3=1$; $x_3=1$; $y_3=\bar{1}$; $z_3=0$; $L_3=\bar{1}$; $M_3=1$; $N_3=1$.

Equations (9) and (10) consequently become

$$\left.\begin{aligned} x'&=3(x+y+z); & x&=-\tfrac{1}{3}x'+z'=x'-3z'.\\ y'&=-3(1+y+3z); & y&=-\tfrac{2}{3}x'-\tfrac{1}{3}y'-z'=2x'+y'+3z'.\\ z'&=-x+y+z; & z&=\tfrac{1}{3}x'+\tfrac{1}{3}y'=-x'-y'.\end{aligned}\right\}(11)$$

From these equations we get the new coordinates of the corners of the old triangle

$$A=(\bar{3}31),\quad B=(3\bar{3}1),\quad C=(3\bar{9}1),\quad O=(9,\ \overline{15},\ 1).$$

From the value of x', it is clear that the equation of the side $C''A''$ of the new triangle, $x'=0$, is in the old coordinates

$$x+y+z=0,$$

the axis of perspective of the old triangle, according to construction.

The axis of perspective of the new triangle in the old coordinates is

$$x-y+5z=0.$$

A circumconic of the old triangle, $yz+zx+xy=0$, in the new coordinates is

$$x^2+y^2+9z^2+3yz+3zx+3xy=0.$$

8°. The matrix formed from the coefficients of the transformed values of x, y, z, (11),

$$\begin{vmatrix} 1 & 0 & \bar{3} \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = -6$$

is the modulus of transformation, and the invariants of two conics are calculated in the same way as in other systems of coordinates. Let the equations of two conics be

$$-8y^2+2yz+4zx+2xy=0,$$
$$2yz+2zx+2xy=0.$$

The invariants of these equations are

$$\Delta=36, \quad \Theta=42, \quad \Theta'=16, \quad \Delta'=2.$$

The transformed equations are

$$36x^2+10y^2+90z^2+48yz+96zx+40xy=0,$$
$$2x^2+2y^2+18z^2+6yz+6zx+6xy=0;$$

their invariants are

$$\Delta_1=1296, \quad \Theta_1=1512, \quad \Theta'_1=576, \quad \Delta'_1=72;$$

and $$\frac{\Delta_1}{\Delta}=\frac{\Theta_1}{\Theta}=\frac{\Theta'_1}{\Theta'}=\frac{\Delta'_1}{\Delta'}=36=\begin{vmatrix} 1 & 0 & \bar{3} \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix}^2.$$

CHAPTER X

THE CIRCLE

1°. The condition that the circumconic, $yz+zx+xy=0$, shall be a circle (fig. 34).

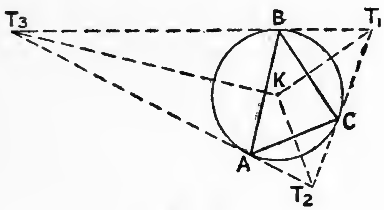

FIG. 34.

The tangents to the curve at A, B and C meet in $T_1=(\bar{1}11)$, $T_2=(1\bar{1}1)$, $T_3=(11\bar{1})$, and K the centre of the curve is $(m+n-l,\ n+l-m,\ l+m-n)$.

Therefore the equations of KT_1, KT_2, KT_3 are

$$(m-n)x+my-nz=0,$$
$$-lx+(n-l)y+nz=0,$$
$$lx-my+(l-m)z=0.$$

These lines cut the triangle respectively in (onm), (nol), (mlo), the middle points of the sides, and when the conic is a circle they are perpendicular, KT_1 to BC, KT_2 to CA, KT_3 to AB.

Applying the condition of perpendicularity, III, 10°, (c), we get

$$l(b^2-c^2)-ma^2+na^2=0,$$
$$lb^2+m(c^3-a^2)-nb^2=0,$$
$$-lc^2+mc^2+n(a^2-b^2)=0;$$

whence $$l:m:n=a^2:b^2:c^2. \qquad \text{(1)}$$

Therefore the equation $yz+zx+xy=0$ represents a circle when the symmedian point is origin.

Changing the origin from the symmedian point to O, we get, IX, 4°,

$$mna^2yz+nlb^2yz+lmc^2zx=0 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

as the equation of the circumcircle.

2°. The coordinates of the points in which Λ_∞ cuts (2) have three forms:

$$\left.\begin{array}{ll}(1) & x:y:z=mna:nl(ce^{\pm iB}-a):-lmce^{\pm iB}.\\ (2) & ,,:,,:,,=-mnae^{\pm iC}:nlb:lm(ae^{\pm iC}-b).\\ (3) & ,,:,,:,,=mn(be^{\pm iA}-c):-nlbe^{\pm iA}:lmc.\end{array}\right\} \ldots(3)$$

These are the Cyclic Points at infinity, I and J.

If an angle of the given triangle, say A, happens to be 90°, the lines AI and AJ will be found to be the harmonic conjugates of AB and AC.

If $a=b=c=1$ and $l:m:n:1$, the 1st form becomes

$$\left\{1,\ -\frac{1}{2}\pm\frac{\sqrt{-3}}{2},\ -\frac{1}{2}\mp\frac{\sqrt{-3}}{2}\right\}, \quad \ldots\ldots\ldots\ldots(4)$$

three of the cube roots of unity, which will be as usual written $(1\omega\omega^2)$ and $(1\omega^2\omega)$. It will be observed that

$$\omega+\omega^2=-1,\quad \omega^3=1,\quad \omega^4=\omega.$$

3°. A metric equation of the circle may be obtained as follows.

Let d be the constant distance of a variable point (xyz) from a fixed point $F=(fgh)$. Then the given triangle being equilateral and its mean point the origin, the distance between the points is

$$\begin{aligned} d^2\Sigma^2 f\,.\,\Sigma^2 x &= p^2+q^2+r^2-qr-rp-pq \\ ,, \quad &= (p+\omega q+\omega^2 r)(p+\omega^2 q+\omega r) \\ ,, \quad &= FI\,.\,FJ, \end{aligned}$$

because $p=hy-yz,\ q=fz-hx,\ r=gx-fy$.

Consequently, $FI\,.\,FJ-d^2\Sigma^2 f\,.\,\Sigma^2 x=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$

is the equation of a circle with (fgh) for centre and d for radius.

The tangential equation, VII, (6),

$$d^2 = \frac{\Sigma^2 fp}{\Sigma^2 fl} \frac{(lmnbc \sin A)^2}{\Omega^2}$$

corresponds to the local equation (5),

$$d^2 = \frac{FI \,.\, FJ}{\Sigma^2 f \,.\, \Sigma^2 x}.$$

If $d=0$, the local equation becomes $FI \,.\, FJ=0$, the product of the equations of two imaginary lines; the tangential equation becomes $\Sigma fp=0$, the equation of the centre, which is the cross of these two imaginary lines. If $d=\infty$, the tangential equation becomes $\Omega^2=0$, the product of the equations of two imaginary points; the local equation becomes $\Sigma x=0$, the equation of the (analytically) real line Λ_∞, the join of these two imaginary points.

4°. Equation (5) represents a circle, and its form shows it is in terms of two tangents, $FI=0$, and $FJ=0$, which touch the curve at I and J respectively, and the chord of contact $x+y+z=\Lambda_\infty=0$, V, 22°. Since the pair of tangents are drawn from the centre and touch the curve at infinity, they are the (imaginary) asymptotes of the circle.

The value of d and the position of F being arbitrary, the general conclusion is that all circles pass through the two cyclic points I and J.

5°. The circle is the only conic which passes through both I and J. Every parabola meets Λ_∞ in two real and coincident points: every hyperbola is cut by it in two real and distinct points. No ellipse can pass through both. The coordinates of the points of intersection of two conics are derived from two quadratic equations, and consequently have four, and only four, sets of values; or, two conics intersect in four points only. Suppose a certain ellipse to pass through I and J. Let any three points P, Q, R be taken on the curve and let a circle be drawn through them. Then the two conics intersect in P, Q, R and also in I and J, that is, in five points; which is impossible. Therefore no ellipse can pass through both I and J.*

6°. It follows from the foregoing that if Λ represent any straight line, S any circle, and if k be an arbitrary constant,

$$\Lambda\Lambda_\infty + kS = 0 \quad \text{......................(6)}$$

* Whitworth, *Modern Analytic Geometry*, p. 289.

represents some other circle, S'. First, being of the second degree, (6) represents some conic. Secondly, S' passes through the two points in which S is cut by Λ_∞ and the two points in which it is cut by Ω. But S is cut by Λ_∞ in I and J. Therefore S' is a circle, the only conic that passes through these two points. Since S and S' are cut by Λ in the same two points, Λ is the radical axis of S and S'.

7°. The condition that $\phi(x, y, z)=0$ shall represent a circle.

Let $\Lambda=px+qy+rz=0$, and let S represent the circum-circle in (6). Then,

$$(px+qy+rz)(lx+my+nz)+k(mna^2yz+nlb^2zx+lmc^2xy)=0$$

is the general (graphic) equation of the circle, and to this form the general equation must be reducible when it represents a circle. Equating the coefficients of the squares of the variables in the two equations,

$$u=pl, \quad v=qm, \quad w=rn, \quad \text{and} \quad p=\frac{u}{l}, \quad q=\frac{v}{m}, \quad r=\frac{w}{n}.$$

The general equation must therefore be reducible to the form

$$\left(\frac{u}{l}x+\frac{v}{m}y+\frac{w}{n}z\right)(lx+my+nz) \\ +k(mna^2yz+nlb^2zx+lmc^2xy)=0. \quad \ldots(7)$$

Expanding this equation and equating the coefficients of yz, zx, xy to those of the same quantities in the general equation, we get

$$\left.\begin{aligned} 2u'&=\frac{n}{m}v+\frac{m}{n}w+kmna^2, \\ 2v'&=\frac{n}{l}w+\frac{n}{l}u+knlb^2, \\ 2w'&=\frac{m}{l}u+\frac{l}{m}v+klmc^2. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(8)$$

Eliminating k from these three equations, we get

$$l^2b^2c^2(n^2v+m^2w-2mnu')=m^2c^2a^2(l^2w+n^2u-2nlv') \\ =n^2a^2b^2(m^2u+l^2v-2lmw'), \quad \ldots(9)$$

the condition that $\phi(x, y, z)=0$ shall represent a circle.

8°. Equation (7) may be written

$$\frac{u}{l}x\Lambda_\infty+\frac{v}{m}y\Lambda_\infty+\frac{w}{n}z\Lambda_\infty$$
$$+k(mna^2yz+nlb^2zx+lmc^2xy)=0, \quad \text{......(10)}$$

which enables us to find the equation of a circle passing through any three given points. For substituting the coordinates of the given points in (10), we get three equations of the form

$$\left.\begin{array}{l} a_1u+b_1v+c_1w+d_1k=0, \\ a_2u+b_2v+c_2w+d_2k=0, \\ a_3u+b_3v+c_3w+d_3k=0. \end{array}\right\} \quad \text{..............(11)}$$

Therefore

$$\frac{u}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}}=\frac{-v}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}}=\frac{w}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}=\frac{-k}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \quad (12)$$

Having obtained the proportional values of u, v, w and k from these three matrices, u', v' and w' are obtained from (8).

Ex. The Brocard circle passes through the Brocard points Ω_1 and Ω_2 and the symmedian point,

$$\left(\frac{c^2a^2}{l},\ \frac{a^2b^2}{m},\ \frac{b^2c^2}{n}\right),\quad \left(\frac{a^2b^2}{l},\ \frac{b^2c^2}{m},\ \frac{c^2a^2}{n}\right),\quad \left(\frac{a^2}{l},\ \frac{b^2}{m},\ \frac{c^2}{n}\right).$$

Substituting successively these values of the variables in (10), we get

$$c^2a^2\frac{u}{l^2}+a^2b^2\frac{v}{m^2}+b^2c^2\frac{w}{n^2}+a^2b^2c^2k=0,$$

$$a^2b^2\frac{u}{l^2}+b^2c^2\frac{v}{m^2}+c^2a^2\frac{w}{n^2}+a^2b^2c^2k=0,$$

$$a^2\frac{u}{l^2}\Sigma a^2+b^2\frac{v}{m^2}\Sigma a^2+c^2\frac{w}{n^2}\Sigma a^2+3a^2b^2c^2k=0.$$

Consequently,

$$\frac{u}{l^2}=a^2b^4c^4\begin{vmatrix} a^2 & b^2 & 1 \\ c^2 & a^2 & 1 \\ \Sigma a^2 & \Sigma a^2 & 3 \end{vmatrix}=a^2b^4c^4(\Sigma a^4-b^2c^2-c^2a^2-a^2b^2).$$

Similarly,

$$\frac{v}{m^2}=a^4b^2c^4(\Sigma a^4-b^2c^2-c^2a^2-a^2b^2);$$

$$\frac{w}{n^2}=a^4b^4c^2(\Sigma a^4-b^2c^2-c^2a^2-a^2b^2);$$

$$-k=a^2b^2c^2\Sigma a^2(\Sigma a^4-b^2c^2-c^2a^2-a^2b^2).$$

Hence, suppressing common factors, we have

$$u=b^2c^2l^2;\quad v=c^2a^2m^2;\quad w=a^2b^2n^2;\quad k=-(a^2+b^2+c^2).$$

Substituting these values of u, v, w and k in (8),

$$2u'=-a^4mn;\quad 2v'=-b^4nl;\quad 2w'=-2c^4lm.$$

Consequently the equation of the Brocard circle is

$$b^2c^2l^2x^2+c^2a^2m^2y^2+a^2b^2n^2z^2-a^4mnyz-b^4nlzx$$
$$-c^4lmxy=0, \quad \text{......(13)}$$

or,

$$(b^2c^2lx+c^2a^2my+a^2b^2nz)(lx+my+nz)$$
$$-(a^2+b^2+c^2)(mna^2yz+nlb^2zx+lmc^2xy)=0;$$

this second form showing that the line

$$b^2c^2lx+c^2a^2my+a^2b^2nz=0$$

is the radical axis of this circle with respect to the circumcircle.

When S, the symmedian point, is taken for origin, the equation of the Brocard circle assumes the simple form

$$a^2x^2+b^2y^2+c^2z^2-a^2yz-b^2zx-c^2xy=0. \quad \text{......(14)}$$

9°. The inconic touches the sides of the given triangle in the points A', B', C'. When the conic is a circle, we know that

$$\frac{BC'}{C'A}=\frac{s_2}{s_1},\quad \frac{CA'}{A'B}=\frac{s_3}{s_2},\quad \frac{AB'}{B'C}=\frac{s_1}{s_3}.$$

In words, the equation $x^2+y^2+z^2-2yz-2zx-2xy=0$ represents a circle when

$$l:m:n=s_2s_3:s_3s_1:s_1s_2,$$

that is, when the Gergonne point is the origin.

Transforming the equation to the general origin O,

$$0=s_1{}^2l^2x^2+s_2{}^2m^2y^2+s_3{}^2n^2z^2-2s_2s_3mnyz-2s_3s_1nlzx$$
$$-2s_1s_2lmxy. \quad \text{...(15)}$$

10°. The IX or nine points circle passes through the middle points of the sides of a triangle (onm), (nol), (mlo). Applying the principles of 8°, its equation is found to be

$$0 = bc\cos A\, l^2x^2 + ca\cos B\, m^2y^2 + ab\cos l\, n^2z^2 - a^2mnyz - b^2nlzx - c^2lmxy. \quad \text{.........(16)}$$

11°. The condition that

$$-x^2 + y^2 + z^2 = 0$$

shall represent the polar circle (fig. 35).

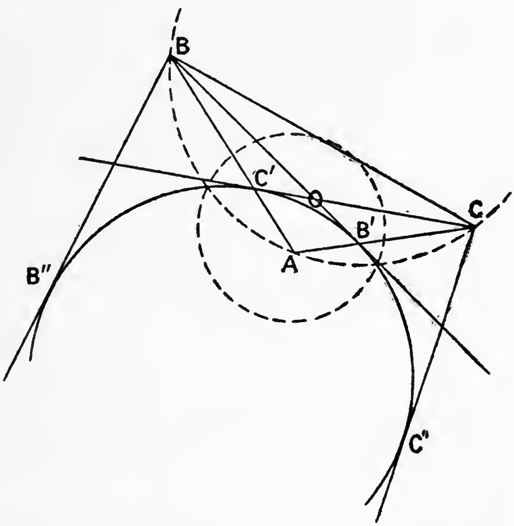

FIG. 35.

By (9),

$$a^2 : b^2 : c^2 = l^2(m^2 + n^2) : m^2(l^2 - n^2) : n^2(l^2 - m^2),$$

whence $\quad l^2 : m^2 : n^2 = -\tan A : \tan B : \tan C.$

Consequently the conic is a circle when the point

$$P = (\pm\sqrt{-\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C})$$

is the origin. In order that this point shall be real, the angle A must be obtuse; and in this case, taking the square roots as all positive or all negative, the origin lies within the triangle.

To transform the equation to the general origin O. Since O is irrational to P, its symbol is

$$\left(\frac{l}{\sqrt{-\tan A}}, \frac{m}{\sqrt{\tan B}}, \frac{n}{\sqrt{\tan C}}\right).$$

Therefore, IX, (6),

$$x:y:z=\frac{lx'}{\sqrt{-\tan A}}:\frac{my'}{\sqrt{\tan B}}:\frac{nz'}{\sqrt{\tan C}}.$$

Consequently,

$$-x^2+y^2+z^2=\cot Al^2x'^2+\cot Bm^2y'^2+\cot Cn^2z'^2=0, \quad (17)$$

which is real when A is obtuse.

The centre of the circle is $\left(\frac{\tan A}{l}, \frac{\tan B}{m}, \frac{\tan C}{n}\right)$, the orthocentre, which lies outside the triangle because A is obtuse and $\tan A$ negative.

Equation (17) may be written, $bc\cos Al^2x^2+\text{etc.}=0$, and can be thrown into the form

$$\Lambda_\infty(bc\cos Alx+ca\cos Bmy+ab\cos Cnz)-S=0,$$

where S represents the circumcircle. Now the equation of the IX circle, (16), may be written,

$$\Lambda_\infty(bc\cos Alx+ca\cos Bmy+ab\cos Cnz)-2S=0.$$

Therefore the polar and the IX circles have the same radical axis in respect to the circumcircle; or the three circles intersect each other in the same two points.

12°. Since

$a=s_2+s_3$, $b=s_2+s_1$, $c=s_1+s_2$, and $\cos A=\cos^2\frac{1}{2}A-\sin^2\frac{1}{2}A$,

the equation (a) of the IX circle may be written,

$$\Lambda_\infty\{(ss_1-s_2s_3)lx+(ss_2-s_3s_1)my+(ss_3-s_1s_2)nz\}-2S=0.$$

The equation of the incircle in the same form is

$$\Lambda_\infty\{s_1{}^2lx+s_2{}^2my+s_3{}^2nz\}-S=0.$$

Now, if two circles are given,

$$C=\Lambda_\infty\Lambda+kS=0 \text{ and } C'=\Lambda_\infty\Lambda'+k'S=0,$$

then
$$C=\Lambda_\infty\left(\Lambda-\frac{k}{k'}\Lambda'\right)+\frac{k}{k'}C',$$

where $\Lambda - \frac{k}{k'}\Lambda' = 0$ is the radical axis of C and C'. Applying this result to these equations of the in- and IX circles, we get

$$\frac{lx}{b-c} + \frac{my}{c-a} + \frac{nz}{a-b} = 0 \quad \text{.................}(18)$$

as their radical axis.

Let the Gergonne point, $(s_2 s_3,\ s_3 s_1,\ s_1 s_2)$, be taken for origin. Then this equation of the radical axis becomes

$$\frac{x}{s_1(b-c)} + \frac{y}{s_2(c-a)} + \frac{z}{s_3(a-b)} = 0\,; \quad \text{...........}(a)$$

and at the same time the equation of the incircle becomes

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0.$$

Now, for this equation, $U = V = W = 0$; $U' = V' = W' = 2$. Therefore the condition that (a) shall be a tangent to the incircle is, V, 6°,

$$s_1(b-c) + s_2(c-a) + s_3(a-b) = 0.$$

But this equation is identically zero. Therefore the radical axis (18) is a tangent to the incircle, and consequently to the IX circle. The point of contact is

$$\{s_1{}^2(b-c)^2,\ s_2{}^2(c-a)^2,\ s_3{}^2(a-b)^2\} \quad \text{...........}(19)$$

to the Gergonne point as origin, and to origin O,

$$\left\{\frac{s_1(b-c)^2}{l},\ \frac{s_2(c-a)^2}{m},\ \frac{s_3(a-b)^2}{n}\right\}. \quad \text{...........}(20)$$

These results may be reached more easily as follows.

The tangential coordinates of the radical axis of the IX and incircles are, (18),

$$\left(\frac{l}{b-c},\ \frac{m}{c-a},\ \frac{n}{a-b}\right),$$

and the tangential equation of the incircle is

$$ls_1 qr + ms_2 rp + ns_3 pq = 0.$$

For the Gergonne point as origin, these expressions become

$$\left(\frac{s_2 s_3}{b-c},\ \frac{s_3 s_1}{c-a},\ \frac{s_1 s_2}{a-b}\right)$$

and
$$qr + rp + pq = 0.$$

Substituting the coordinates of the radical axis in the equation of the circle, we get

$$s_1(b-c)+s_2(c-a)+s_3(a-b)=0,$$

which is identically zero. Therefore the radical axis is a tangent to the incircle and consequently to the IX circle.

13°. If $X'=(x'y'z')$ be any point without a circle with centre $F=(fgh)$ and radius r, the length of a tangent from X' is, (5), $t^2=FX'^2-r^2$. If we regard FX' as the radius of another circle with F for centre, $FX'^2=d^2=\dfrac{FI\,.\,FJ}{\Sigma^2f\,.\,\Sigma^2x}$.

Therefore $$t^2=\frac{FI\,.\,FJ}{\Sigma^2f\,.\,\Sigma^2x}-r^2, \quad \text{.....................}(21)$$

the length of a tangent from any external point to a circle, with the conditions, $a=b=c$, and $l:m:n=1$.

CHAPTER XI

THE FOCI OF A CONIC

1°. Let $\Lambda=0$ be the equation of a fixed line and $S=0$ the local equation of a conic. Then the equation, $S-\Lambda^2=0$, represents a conic S' which has double contact with S in the two points in which it is cut by the chord of contact Λ, whether Λ cuts S in real or imaginary points. Consequently S' and S have two common tangents, which are real in the first case and imaginary in the second. Let S be a circle with a fixed point $F=(fgh)$ for centre and an arbitrary radius r_0; and let Λ be (pqr), the function of its coordinates being Z^2 as usual, IV, (2).

Let $a=b=c$ and $l:m:n=1$. Then the equation,

$$S-\Lambda^2=0,$$

may be written, X, (5),

$$0=(FI\,.\,FJ-r_0{}^2\Sigma^2f\,\Sigma^2x)-\Sigma^2px=\left(\frac{FI\,.\,FJ}{\Sigma^2f\,\Sigma^2x}-r_0{}^2\right)-\frac{\Sigma^2px}{\Sigma^2f\,\Sigma^2x}$$

$$,, =\left(\frac{FI\,.\,FJ}{\Sigma^2f\,\Sigma^2x}-r_0{}^2\right)-\frac{4Z^2}{3\Sigma^2f}\left(\frac{3\Sigma^2px}{4Z^2\Sigma^2x}\right). \qquad \text{.........................(1)}$$

Now by X, (21), the first term in brackets is the length squared of the tangent drawn from any point, $X=(xyz)$ on S' to the circle S, say τ^2; $\dfrac{4Z^2}{3\Sigma^2f}$ is a constant, say e^2; and the second term in brackets is, IV, (7), the perpendicular squared from X to (pqr), say σ^2. Hence, τ and σ being variables, for every point on S',

$$\tau^2=e^2\sigma^2. \qquad \text{..........................(2)}$$

In words, if a circle S have double contact with a conic S', the tangent drawn to the circle from any point X on the conic is in a constant ratio to the perpendicular from the point to the chord of contact.

2°. Let Λ cut S in imaginary points and let r_0 approach zero. At the limit (1) becomes

$$0=\frac{FI\,.\,FJ}{\Sigma^2 f\,\Sigma^2 x}-e^2\frac{3\Sigma^2 px}{4Z^2\Sigma^2 x}. \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The first term now represents the distance squared from X to F, say ρ^2, and the equation may be written

$$0=\rho^2-e^2\sigma^2, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

the known equation of a conic section, ρ being the distance of the variable point from a fixed point (fgh) and σ its perpendicular distance from a fixed line (pqr). According as $e \lesseqqgtr 1$, the curve is an ellipse, parabola or hyperbola.

The focus (fgh), then, may be considered as an infinitely small circle which touches the conic in the two imaginary points in which it is cut by the directrix.

3°. Substituting for e^2 its value $\dfrac{4Z^2}{3\Sigma^2 f}$, equation (3) becomes

$$0=FI\,.\,FJ-\Sigma^2 px. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Now the form of this expression shows that it is the equation of a conic in terms of two (imaginary) tangents, FI and FJ, and their (real) chord of contact. Consequently a focus, still regarded as an evanescent circle, is the cross of two imaginary tangents to the conic, the one from I, the other from J. But four such tangents may be imagined as drawn, two from I and two from J, intersecting in four points which form a quadrangle. Therefore a conic has four foci. In the case of the parabola two of these imaginary tangents, one from I and one from J, coincide with Λ_∞, which is itself a tangent to the curve.

Since FI and FJ remain the same in (5) whatever (pqr) may be, it follows that all conics which have a common focus have two common imaginary tangents; and if they have two (real) foci in common, they have four common imaginary tangents.

4°. We have now to enquire into the nature and position of the four foci.

In V, 18°, (b), were given the separate coordinates of the two tangents from a point (fgh) to a conic. If we suppose that (fgh) becomes successively $I=(1\omega\omega^2)$ and $J=(1\omega^2\omega)$, the first set will contain the quantity $\sqrt{-\Delta\phi(1\omega\omega^2)}$ and the second $\sqrt{-\Delta\phi(1\omega^2\omega)}$, which it is necessary to expand.

Let $\quad u+2u'=a, \quad v+2v'=b, \quad w+2w'=c.$

Then

$$\phi(1\omega\omega^2)=a+b\omega^2+c\omega=\frac{2a-b-c}{2}-i\frac{(b-c)\sqrt{3}}{2},$$

$$\phi(1\omega^2\omega)=a+b\omega+c\omega^2=\frac{2a-b-c}{2}+i\frac{(b-c)\sqrt{3}}{2}.$$

Therefore

$$\sqrt{\phi(1\omega\omega^2)}=\sqrt{\frac{\sqrt{(a^2+b^2+c^2-bc-ca-ab)}+(2a-b-c)}{2}}$$
$$-i\sqrt{\frac{\sqrt{(a^2+b^2+c^2-bc-ca-ab)}-(2a-b-c)}{2}}$$

" $\quad =\sqrt{P}-i\sqrt{Q}$. Similarly, $\sqrt{\phi(1\omega^2\omega)}=\sqrt{P}+i\sqrt{Q}$.

Consequently,

$$\sqrt{-\Delta\phi(1\omega\omega^2)}=\sqrt{Q\Delta}+i\sqrt{P\Delta},$$
$$\sqrt{-\Delta\phi(1\omega^2\omega)}=-\sqrt{Q\Delta}+i\sqrt{P\Delta}.$$

The coordinates of the two tangents from I, when reduced, are,

$$p=6(-U'+V'+W')\mp2\sqrt{3P\Delta}$$
$$+i\sqrt{3}\{4V-4W+2V'-2W'\pm2\sqrt{Q\Delta}\},$$

$$q=6(W-U-U'+V')\pm(\sqrt{3P\Delta}-3\sqrt{Q\Delta})$$
$$+i\sqrt{3}\{2U-2W-2U'+6V'-4W'\mp(\sqrt{3P\Delta}+\sqrt{Q\Delta})\},$$

$$r=6(-U+V-U'+W')\pm(\sqrt{3P\Delta}-3\sqrt{Q\Delta})$$
$$+i\sqrt{3}\{-2U+2V+2U'+4V'-6W'\pm(\sqrt{3P\Delta}-\sqrt{Q\Delta})\}.$$

The two tangents from J are

$$p'=6(-U'+V'+W')\pm2\sqrt{3P\Delta}$$
$$-i\sqrt{3}\{4V-4W+2V'-2W'\mp2\sqrt{Q\Delta}\},$$

$$q'=6(W-U-U'+V')\mp(\sqrt{3P\Delta}-3\sqrt{Q\Delta})$$
$$-i\sqrt{3}\{2U-2W-2U'+6V'-4W'\pm(\sqrt{3P\Delta}+\sqrt{Q\Delta})\},$$

$$r'=6(-U+V-U'+W')\mp(\sqrt{3P\Delta}-3\sqrt{Q\Delta})$$
$$-i\sqrt{3}\{-2U+2V+2U'+4V'-6W'\mp(\sqrt{3P\Delta}-\sqrt{Q\Delta})\}.$$

The coordinates of the four tangents may be written:

$$\left.\begin{aligned}&\text{From } I, \quad \begin{cases} t_1=(d+ei,\ f+gi,\ h+ki), \\ t_2=(d'+e'i,\ f'+g'i,\ h'+k'i). \end{cases} \\ &\text{From } J, \quad \begin{cases} t_3=(d'-e'i,\ f'-g'i,\ h'-k'i), \\ t_4=(d-ei,\ f-gi,\ h-ki). \end{cases}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(6)$$

It is evident from these expressions, that

$$\left.\begin{aligned} F&=t_1\cdot t_4=(gh-fk,\ dk-eh,\ ef-dg), \\ F'&=t_2\cdot t_3=(g'h'-f'k',\ d'k'-e'h',\ e'f'-d'g'.\end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} Y=t_1\cdot t_3&=\begin{cases} fh'-f'h+gk'-g'k+i(gh'+g'h-fk'-f'k), \\ hd'-h'd+ke'-k'e+i(dk'+d'k-eh'-e'h), \\ df'-d'f+eg'-e'g+i(ef'+e'f-dg'-d'g); \end{cases} \\ Y'=t_2\cdot t_4&=\begin{cases} fh'-f'h+gk'-g'k-i(gh'+g'h-fk'-f'k), \\ hd'-h'd+ke'-k'e-i(dk'+d'k+eh'-e'h), \\ df'-d'f+eg'-e'g-i(ef'+e'f-dg'-d'g). \end{cases}\end{aligned}\right\} \quad (8)$$

It appears from the equations, (7), that F and F' are real points, and from (8) that Y and Y' are imaginary. But the line YY' is real, as is evident from the form of the coordinates of Y and Y',

$$(p+qi,\ r+si,\ t+vi),$$
$$(p-qi,\ r-si,\ t-vi).$$

These are general conclusions, and F, F' may be any two points in the plane. Let, then, $F=(x_1y_1z_1)$, $F'=(x_2y_2z_2)$; the equation of FF' being as usual,

$$px+qy+rz=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(a)$$

where $p=(y_1z_2-y_2z_1)$, $q=(z_1x_2-z_2x_1)$, $r=(x_1y_2-x_2y_1)$.

Forming the equations of FI, $F'J$, etc., we get

$$\begin{aligned} FI&=(z_1\omega-y_1\omega^2,\ x_1\omega^2-z_1,\ y_1-x_1\omega), \\ FJ&=(z_1\omega^2-y_1\omega,\ x_1\omega-z_1,\ y_1-x_1\omega^2), \\ F'J&=(z_2\omega^2-y_2\omega,\ x_2\omega-z_2,\ y_2-x_2\omega^2), \\ F'I&=(z_2\omega-y_2\omega^2,\ x_2\omega^2-z_2,\ y_2-x_2\omega). \end{aligned}$$

From these equations,

$$\begin{aligned} Y=FI\cdot F'J=\{&(-2p+q+r)+i(x_1\sigma_2+x_2\sigma_1)\sqrt{3}, \\ &(p-2q+r)+i(y_1\sigma_2+y_2\sigma_1)\sqrt{3}, \\ &(p+q-2r+i(z_1\sigma_2+z_2\sigma_1)\sqrt{3}\}. \end{aligned}$$

$$Y' = FJ \cdot F'I = \{(-2p+q+r) - i(x_1\sigma_2 + x_2\sigma_1)\sqrt{3},$$
$$(p-2q+r) - i(y_1\sigma_2 + y_2\sigma_1)\sqrt{3},$$
$$(p+q-2r) - i(z_1\sigma_2 + z_2\sigma_1)\sqrt{3}\},$$

where $\sigma_1 = x_1 + y_1 + z_1$ and $\sigma_2 = x_2 + y_2 + z_2$.

Let $-2p+q+r = a, \quad p-2q+r = b, \quad p+q-2r = c,$

$$x_1\sigma_2 + x_2\sigma_1 = M_1, \quad y_1\sigma_2 + y_2\sigma_1 = M_2, \quad z_1\sigma_2 + z_2\sigma_1 = M_3;$$

and the equation of YY' will be the real line

$$(bM_3 - cM_2)x + (cM_1 - aM_3)y + (aM_2 - bM_1)z = 0. \quad \ldots(b)$$

Now III, 1°, the middle point of the line FF' is (M_1, M_2, M_3), and

$$(bM_3 - cM_2)M_1 + (cM_1 - aM_3)M_2 + (aM_2 - bM_1)M_3 = 0.$$

Therefore the real line YY' bisects FF'.

Again, since the given triangle is equilateral and its mean point the origin, the condition of perpendicularity, III, 10°, is

$$2pp' + 2qq' + 2rr' - (qr' + q'r + rp' + r'p + pq' + p'q) = 0.$$

Applying this test to (a) and (b), the equations of FF' and YY', we get

$$0 = 2p(bM_3 - cM_2) + 2q(cM_1 - aM_3) + 2r(aM_2 - bM_1)$$
$$-q(aM_2 - bM_1) - r(cM_1 - aM_3) - r(bM_3 - cM_2)$$
$$-p(aM_2 - bM_1) - p(cM_1 - xM_3) - q(bM_3 - cM_2)$$
$$,, = (bM_3 - cM_2)(2p - q - r) + (cM_1 - aM_3)(-p + 2q - r)$$
$$+ (aM_2 - bM_1)(-p - q + 2r)$$
$$,, = -a(bM_3 - cM_2) - b(cM_1 - aM_3) - c(aM_2 - bM_1) = 0,$$

identically.

The two imaginary foci, then, are situated on a real line which bisects at right angles the line joining the two real foci.

5°. The coordinates of the foci given in 4° are ill-adapted for calculation, and we have now to consider other methods which will give these coordinates in a less complicated form

The following method of finding the coordinates of the foci, in the case when the given triangle is equilateral and its mean point the origin, is given by Sir William Hamilton.

Writing ϕ for $\phi(xyz)$, l for ϕ_x, m for ϕ_y, n for ϕ_z, and $(1\theta\theta^2)$ and $(1\theta^2\theta)$ for the cyclic points: the equation for a pair of tangents to a conic from a point (fgh) is, V, (25),

$$\phi(fgh)\phi-(fl+gm+hn)^2=0.$$

If I and J be chosen successively for (fgh), we have

$$\left.\begin{aligned}\phi(1,\,\theta,\,\theta^2)\phi-(l+m\theta+n\theta^2)^2=0,\\ \phi(1,\,\theta^2,\,\theta)\phi-(l+m\theta^2+n\theta)^2=0.\end{aligned}\right\}$$

Now
$$\begin{aligned}\phi(1,\,\theta,\,\theta^2)&=u+2u'+(v+2v')\theta^2+(w+2w')\theta,\\ \phi(1,\,\theta^2,\,\theta)&=u+2u'+(v+2v')\theta+(w+2w')\theta^2,\\ (l+m\theta+n\theta^2)^2&=l^2+2mn+(m^2+2nl)\theta^2+(n^2+2lm)\theta,\\ (l+m\theta^2+n\theta)^2&=l^2+2mn+(m^2+2nl)\theta+(n^2+2lm)\theta^2.\end{aligned}$$

Let
$$u+2u'=a,\quad v+2v'=b,\quad w+2w'=c,$$
$$l^2+2mn=\lambda,\quad m^2+2nl=\mu,\quad n^2+2lm=\nu,$$

and we have
$$\left.\begin{aligned}(a+b\theta^2+c\theta)\phi-\lambda-\mu\theta^2-\nu\theta=0,\\ (a+b\theta+c\theta^2)\phi-\lambda-\mu\theta-\nu\theta^2=0,\end{aligned}\right\}$$

whence
$$a\phi-\lambda=b\phi-\mu=c\phi-\nu. \qquad\text{.....................(9)}$$

By means of these three equations we can determine the four points in which the two pair of tangents from I and J intersect.

Let p, q, r be any three constants such that $p+q+r=0$. Then
$$p(a\phi-\lambda)+q(b\phi-\mu)+r(c\phi-\nu)=0 \qquad\text{..........(10)}$$

represents a conic passing through the four foci.

Let
$$p=b-c,\quad q=c-a,\quad r=(a-b),$$

and this equation becomes
$$(b-c)(l^2+2mn)+(c-a)(m^2+2nl)$$
$$+(a-b)(n^2+2lm)=0, \qquad\text{......(11)}$$

where a, b, c are known and real constants (the given conic being real by hypothesis), and l, m, n represent real and homogeneous functions of $\phi(x,\,y,\,z)$.

This equation breaks up into two real straight lines. For let $h^2=a^2+b^2+c^2-bc-ca-ab$, which is real since the conic is real, and (11) is equivalent to

$$0=\{(b-c)l+(a+b)m+(c-a)n+h(m-n)\}$$
$$\times\{(b-c)l+(a-b)m+(c-a)n+h(n-m)\}, \qquad\text{...(12)}$$

the product of two real and distinct straight lines on which the foci are situated. These two lines are consequently the axes of the conic; their cross is the centre; their intersections with (10) are the four foci; and their intersections with $\phi(x, y, z)$ are the four vertices of the conic (Hamilton).

Ex. Let the conic be

$$8y^2 - 2yz - 4zx - 2xy = 0.$$

Here $a = u + 2u' = -2$; $b = v + 2v' = 4$; $c = w + 2w' = -2$,

$$h^2 = 36, \text{ and } h \text{ may be taken as } 6.$$

$$l = \phi_x = -y - 2z;\ m = \phi_y = -x + 8y - z;\ n = \phi_z = -2x - y,$$

$$\left.\begin{aligned} \lambda &= 4x^2 - 15y^2 + 4z^2 + 6yz + 4zx - 30xy, \\ \mu &= x^2 + 66y^2 + z^2 - 12yz + 10zx - 12xy, \\ \nu &= 4x^2 - 14y^2 + 4z^2 - 30yz + 4zx + 6xy. \end{aligned}\right\}$$

Substituting the above values of a, b, c in (12), we get

$$0 = (l - n)(l - 2m + n) = y(x - z).$$

The axes of the conic are therefore $y = 0$ and $x - z = 0$.
The cross of these lines (101) is the centre.
The foci are the intersections of the axes with (10).
Let $p = 2$, $q = -1$, $r = -1$, and we ultimately get for this equation

$$x^2 - 11y^2 + z^2 + 14yz - 10zx - 22xy = 0. \quad \ldots\ldots(13)$$

The intersections of $y = 0$ with this conic will be found to be $(1 + \sqrt{\tfrac{2}{3}}, 0, 1 - \sqrt{\tfrac{2}{3}})$ and $(1 - \sqrt{\tfrac{2}{3}}, 0, 1 + \sqrt{\tfrac{2}{3}})$.

These are the real foci. The imaginary foci are the intersections of $x - z = 0$ with (13), namely

$$(11, 6\sqrt{-2} - 4, 11) \text{ and } (11, -(6\sqrt{-2} + 4), 11).$$

The intersections of $y = 0$ with the conic give two of the vertices, (100) and (001). The intersections of $x - z = 0$ with the conic give the other two, (111) and $(2\bar{1}2)$. The conic is an ellipse since it has four real vertices.

Since C and A are two opposite vertices and $CA = 1$ by hypothesis, the length of one semiaxis is $\frac{1}{2}$. The distance of the centre (101) from either of the other vertices is $\dfrac{1}{2\sqrt{3}}$, the length of the minor semiaxis. From the lengths of the semiaxes, the eccentricity is found to be $\sqrt{\tfrac{2}{3}}$.

6°. When the given triangle is scalene and $l:m:n \neq 1$, we may transform the coordinates by selecting an equilateral triangle for the new triangle and taking its mean point for the new origin. If, as generally happens, the figure under consideration contains no equilateral triangle, we may construct on the base of the given triangle an equilateral triangle ADC, the points B and D lying on the same side of CA. The reader will find little difficulty in proving that the coordinates of D are given by

$$x:y:z$$
$$=mna(\sin C - \cos C\sqrt{3}) : nlb\sqrt{3} : lmc(\sin A - \cos A\sqrt{3}), \quad (14)$$

and that the coordinates of M, the mean point of ADC, are

$$x:y:z$$
$$=mna(\sin C\sqrt{3} - \cos C) : nlb : lmc(\sin A\sqrt{3} - \cos A). \quad (15)$$

We then proceed to Hamilton's equations, 5°.

This transformation of coordinates, owing to the form of the coordinates of D and M, is tedious, and the following method is in general preferable.

7°. If $T=0$ be the tangential equation of a conic and $u=0$ and $v=0$ the equations of any two points, then

$$T+kuv=0 \quad \text{......................(16)}$$

is the equation of a conic T' so related to T that two of their common tangents pass through u and the other two through v. For if (pqr) be either of the tangents from u to T, its coordinates must satisfy the equations of both u and T, and consequently satisfy (16), the equation of T'. Therefore (pqr) is a tangent to T', VII, 9°. In like manner the coordinates of the other tangent from u and those of both the tangents from v to T satisfy (16), and all three of them are consequently tangents to T'. Therefore T and T' are both inscribed in the quadrilateral formed by the intersections of their common tangents from u and v. Now the equation of T' possesses this property, that when, for certain values of the constant k, it breaks up into the equations of pairs of points, these points are the opposite corners of the quadrilateral in which T and T' are inscribed.

This may be illustrated simply. The lines BA and BC

of the given triangle (fig. 36) are tangents to the inconic, $qr+rp+pq=0$, and $B''A$ is also a tangent. The second tangent from B'' cuts BC in $D=2q+r=0$ and BA in $E=p+2q=0$. For the conic which has four tangents passing through B and B'' in common with the inconic, we have

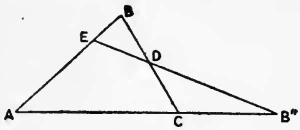

Fig. 36.

$$0=qr+rp+pq+kq(r-p)=(1+k)qr+rp+(1-k)pq.$$

To obtain the values of k for which this equation breaks up into pairs of points, we must equate its discriminant to zero and solve for k.

$$\begin{vmatrix} 0\ , & 1-k, & 0 \\ 1-k, & 0\ , & 1+k \\ 1\ , & 1+k, & 0 \end{vmatrix} = 2(1-k^2)=0, \text{ the roots of which are } \pm 1.$$

For $k=1$, the given equation becomes

$$0=2qr+rp=r(p+2q)=C\times E.$$

For $k=-1$, $\quad 0=rp+2pq=p(2q+r)=A\times D.$

8°. To obtain the equations of the foci, I and J are taken for u and v, and we deduce the values of k from the discriminant of the equation

$$P+k\Omega^2=0,$$

equated to zero. This discriminant is

$$\begin{vmatrix} U+km^2n^2a^2\ , & W'-klmn^2ab\cos C, & V'-klm^2nca\cos B \\ W'-klmn^2ab\cos C, & V+kn^2l^2b^2\ , & U'-kl^2mnbc\cos A \\ V'-klm^2nca\cos B, & U'-kl^2mnbc\cos A\ , & W+kl^2m^2c^2 \end{vmatrix}=0. \quad (17)$$

(a) (b) (c) (d) (e) (f)

This matrix can be resolved into eight matrices of the third order, formed from the columns of the above which have been lettered for ease of reference.

The matrix $(ace)=\Delta^2$, Δ being the discriminant of the local equation of the conic, $\phi(xyz)=0$.

The matrix (bdf), which involves the third power of k, vanishes.

The matrices (adf), (bcf) and (bde) involve k^2, and their determinants are

$$(adf)=l^3m^2n^2Ab^2c^2\sin^2 Ak^2,$$
$$(bcf)=l^2m^3n^2Bb^2c^2\sin^2 Ak^2,$$
$$(bde)=l^2m^2n^3Cb^2c^2\sin^2 Ak^2.$$

Consequently the coefficient of k^2 is

$$l^2m^2n^2b^2c^2\sin^2 A\,(lA+mB+nC)=l^2m^2n^2Db^2c^2\sin^2 A. \quad (18)$$

The matrices (bce), (ade) and (aef) involve k.

$$(bce)=\Delta k(um^2n^2a^2-v'lm^2nca\cos B-w'lmn^2ab\cos C),$$
$$(ade)=\Delta k(vn^2l^2b^2-w'lmn^2ab\cos C-u'l^2mnbc\cos A),$$
$$(acf)=\Delta k(wl^2m^2c^2-u'l^2mnbc\cos A-v'lm^2nca\cos B).$$

Therefore the coefficient of k is

$$\Delta\{um^2n^2a^2+vn^2l^2b^2+wl^2m^2c^2$$
$$-2lmn(u'lbc\cos A+v'mca\cos B+w'nab\cos C)\} \quad (19)$$
$$=\Delta[mna^2\{umn-l(-u'l+v'm+w'm)\}$$
$$+nlb^2\{vnl-m(u'l-v'm+w'n)\}$$
$$+lmc^2\{wlm-n(u'l+v'm-w'm)\}]$$
$$=\Delta\Theta', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

Θ' being the ordinary invariant symbol.

Consequently the complete determinant of the original matrix is

$$l^2m^2n^2Db^2c^2\sin^2 Ak^2+\Delta\Theta'k+\Delta^2=0. \quad \ldots\ldots\ldots(21)$$

In this equation Δ is the discriminant of $\phi(xyz)=0$, the local form of the tangential equation $T=F(pqr)=0$. D is the bordered discriminant of $\phi(xyz)=0$, and Θ' is the ordinary invariant symbol whose value is given at length in (19).

9°. *Ex.* 1. The foci of the ellipse

$$8y^2-2yz-4zx-2xy=0, \quad \ldots\ldots\ldots\ldots\ldots(a)$$
$$a=b=c; \quad l:m:n=1$$

have been already calculated from this, its local equation, example of 5°.

They will now be calculated from its tangential form,

$$T = -p^2 - 4q^2 - r^2 + 4qr + 34rp + 4pq = 0.$$

Since $\Omega^2 = m^2n^2a^2p^2 \ldots - 2lmn^2ab\cos Cpq$

$$\text{,,} \quad = p^2 + q^2 + r^2 - qr - rp - pq;$$

$$T + k\Omega^2 = (k-1)p^2 + (k-4)q^2 + (k-1)r^2 - (k-4)qr \\ -(k-34)rp - (k-4)pq = 0. \quad (b)$$

From (a), $\Delta = -36$, $D = 36$, $\sin A = \frac{\sqrt{3}}{2}$.

$$\Theta' = um^2n^2a^2 \ldots - 2w'lmn^2ab\cos C$$

$$\text{,,} \quad = u + v + w - u' - v' - w' = 12.$$

Therefore the equation for k is, (21),

$$0 = 36 \times \tfrac{3}{4}k^2 - 12 \times 36k + 36 = k^2 - 16k + 48;$$

and $k = 4$ or 12.

Substituting 4 for k in (b),

$$0 = p^2 + r^2 + 10rp = \{(1 + \sqrt{\tfrac{2}{3}})p + (1 - \sqrt{\tfrac{2}{3}})r\} \\ \{(1 - \sqrt{\tfrac{2}{3}})p + (1 + \sqrt{\tfrac{2}{3}})r\}.$$

The two real foci are therefore

$(1 + \sqrt{\tfrac{2}{3}})p + (1 - \sqrt{\tfrac{2}{3}})r = 0$ and $(1 - \sqrt{\tfrac{2}{3}})p + (1 + \sqrt{\tfrac{2}{3}})r = 0$,
or locally, $(1 + \sqrt{\tfrac{2}{3}}, 0, 1 - \sqrt{\tfrac{2}{3}})$ and $(1 - \sqrt{\tfrac{2}{3}}, 0, 1 + \sqrt{\tfrac{2}{3}})$.

Putting 12 for k in (b),

$$0 = 11p^2 + 8q^2 + 11r^2 - 8qr - 22rp - 8pq,$$

$$\text{,,} = \left\{p + \frac{6\sqrt{-2} - 4}{11}q + r\right\}\left\{p - \frac{6\sqrt{-2} + 4}{11}q + r\right\},$$

or locally, $\{11, 6\sqrt{-2} - 4, 11\}\{11, -6\sqrt{-2} - 4, 11\}$,
the focoids.

Ex. 2. The foci of the conic

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0;$$

the triangle being scalene and $l : m : n = a^2 : b^2 : c^2$, or the origin being the symmedian point S.

For this curve,

$$\Delta = -4, \text{ and } D = 4(a^2b^2 + b^2c^2 + c^2a^2) = 4\Sigma a^2b^2.$$

Since D is positive the conic is an ellipse. Its tangential equation is

$$T=4qr+4rp+4pq=0.$$

$$\Theta'=a^2b^2c^2(\Sigma a^2b^2+4a^2bc\sin B\sin C)$$
$$=a^2b^2c^2(\Sigma a^2b^2+4b^2c^2\sin^2 A).$$

Let $\Sigma a^2b^2=\sigma$, and $b^2c^2\sin^2 A=4(\text{triangle})^2=4t^2$. Then the equation for k, (21), is

$$4a^4b^4c^4t^2\sigma k^2-a^2b^2c^2(\sigma+16t^2)k+4=0,$$

and
$$k=\frac{1}{4a^2b^2c^2t^2} \text{ or } \frac{4}{a^2b^2c^2\sigma}.$$

$$\Omega^2=a^2b^4c^4p^2+a^4b^2c^4q^2+a^4b^4c^2r^2-2a^4b^3c^3\cos A\,qr$$
$$-2a^3b^4c^3\cos B\,rp-2a^3b^3c^4\cos C\,pq;$$

and taking the second value of k, $T+k\Omega^2$ is

$$b^2c^2p^2+c^2a^2q^2+a^2b^2r^2+(b^2c^2+a^4)qr+(c^2a^2+b^4)rp$$
$$+(a^2b^2+c^4)pq=0,$$

that is, $$(b^2p+c^2q+a^2r)(c^2p+a^2q+b^2r)=0.$$

The tangential equations of the foci therefore are

$$b^2p+c^2q+a^2r=0 \text{ and } c^2p+a^2q+b^2r=0,$$

and their local coordinates are (to origin S)

$$(b^2,\ c^2,\ a^2) \text{ and } (c^2,\ a^2,\ b^2).$$

To origin O these coordinates are

$$\left(\frac{a^2b^2}{l},\ \frac{b^2c^2}{m},\ \frac{c^2a^2}{n}\right) \text{ and } \left(\frac{c^2a^2}{l},\ \frac{a^2b^2}{m},\ \frac{b^2c^2}{n}\right),$$

which are the Brocard points. The given conic is the Brocard ellipse; an inconic which touches the sides of the triangle in the points in which they are cut by lines drawn from the corners through the symmedian point.

Ex. 3. $$x^2+y^2+z^2-2yz-2zx-2xy=0,$$
$$l:m:n=3:-2:6;\quad a=b=c=1.$$

This equation represents a conic because $\Delta=-4$ is actual and a parabola because $D=0$. It is a b-escribed conic, and its tangential form is

$$T=4qr+4rp+4pq=0.$$

$$\Omega^2=m^2n^2a^2p^2\ \ldots\ -2lmn^2ab\cos C$$
$$,,\ =144p^2+324q^2+36r^2+108qr+72rp+216pq=0.$$

Therefore

$$T+k\Omega^2=k(144p^2+324q^2+36r^2)+(4+108k)qr+(4-72k)rp$$
$$+(4+216k)pq. \quad \ldots\ldots\ldots(a)$$

Since $D=0$, one root of (21) is infinite, and for the other

$$k=\frac{-\Delta}{\Theta'},$$

$$\Theta'=um^2n^2\ldots-w'lmn^2=7\times 36, \text{ and } k=\tfrac{1}{63}.$$

Therefore (a) becomes

$$\begin{aligned}0&=4p^2+9q^2+r^2+10qr+5rp+13pq\\&=(p+q+r)(4p+9q+r).\end{aligned}$$

One focus therefore is $p+q+r=0$, the origin.

The local symbol for the other is (491), and its vector is

$$\rho=\frac{4l\alpha+9m\beta+n\gamma}{4l+9m+n}=\frac{2\alpha-3\beta+\gamma}{2-3+1}.$$

Since the denominator is zero, the second focus is at an infinite distance, although real.

The axis of the parabola, which passes through the two foci, is $(\bar{8}53)$ and its vertex is (16, 1, 25). The directrix, the polar of the focus (111), is the axis of perspective of the given triangle, $x+y+z=0$.

Since the given triangle is a triangle of tangents to the curve, its orthocentre $(2\bar{3}1)$ lies on the directrix, and the circumcircle passes through the focus.

The axis cuts the directrix in $P=(2, \overline{13}, 11)$. Consequently the distance from the focus, $F=0=(111)$ to P ought to be twice the distance from O to the vertex, $V=(16, 1, 25)$ or $2\overline{OV}=\overline{OP}$.

To ascertain the coordinates of $2\overline{OV}$ we may employ the method III, 1°, (2),

$x:y:z$

$$=\left(\frac{v}{t}-1\right)\Sigma fl+f\Sigma l:\left(\frac{v}{t}-1\right)\Sigma fl+g\Sigma l:\left(\frac{v}{t}-1\right)\Sigma fl+h\Sigma l.$$

Here

$$\frac{t}{v}=2 \text{ and } \frac{v}{t}-1=-\frac{1}{2};\ \Sigma l=3-2+6=7;\ f=16, g=1, h=25;$$

$$\Sigma fl=196.$$

Therefore $\qquad x:y:z=2:-13:11$
and

$$2\overline{OV}=2\frac{16la+m\beta+25n\gamma}{16l+m+25n}=\frac{2la-13m\beta+11n\gamma}{2l-13m+11n}=\overline{OP}.$$

Ex. 4. $\qquad x^2+9z^2-20yz-10zx+4xy=0,$
with $\qquad l:m:n=1:2:3$ and $a=5,\ b=4,\ c=3.$

This equation represents a conic because $\Delta=64$ is actual and a hyperbola because $D=-3\times 64$ is negative. Its tangential form is

$$T=-100p^2-16q^2-4r^2-40rp+64pq=0,$$
$$\Omega^2=25\times 36p^2+9\times 16q^2+4\times 9r^2-24\times 9rp-36\times 16pq=0,$$
$$T+k\Omega^2=25(9k-1)p^2+4(9k-1)q^2+(9k-1)r^2$$
$$-(54k+10)rp-16(9k-1)pq=0. \quad (a)$$

$\Theta'=32\times 36.$ Therefore the equation for k is

$$0=k^2-\frac{2}{27}k-\frac{1}{9\times 27}, \text{ and } k=\frac{1}{9} \text{ or } \frac{-1}{27}.$$

For $k=\frac{1}{9}$, equation (a) becomes

$$0=rp, \text{ or } r=0 \text{ and } p=0.$$

Therefore the real foci are the corners C and A of the given triangle.

For $k=\frac{-1}{27}$,

$$0=25p^2+4q^2+r^2+6rp-16pq,$$
$$,,=\{(3-4\sqrt{-1})p+2q\sqrt{-1}+r\}$$
$$\{(3+4\sqrt{-1})p-2q\sqrt{-1}+r\}.$$

The hyperbola cuts CA in $M=(901)$ and $M'=(101)$, the vertices of the curve.

Since $CA=4$ and $l:m:n=1:2:3$, the length of the transverse axis, MM', is 2. It will be found that the eccentricity is 2.

The equation for the asymptotes,

$$\Delta(lx+my+nz)^2-D\phi(xyz)=0,$$

gives $\quad 0=x^2+y^2+9z^2-12yz-6zx+4xy,$

$$,,=\{x+(2+\sqrt{3})y-3z\}\{x+(2-\sqrt{3})y-3z\}.$$

Ex. 5. The conic, $x^2-y^2-z^2=0$, with the conditions

$$l:m:n=-1:2:2;\quad a^2=b^2+c^2;\quad b=c=2.$$

For this conic $\Delta=1$, $D=-7$, and consequently the curve is a hyperbola.

$\Theta=24$, and k is $\frac{1}{32}$ or $\frac{-1}{56}$; the first being the value for the real foci.

$$\Omega^2=16\times 8p^2+16q^2+16r^2+16\times 4rp+16\times 4pq=0;$$

and $$T=p^2-q^2-r^2=0.$$

Therefore

$$T+k\Omega^2=9p^2+4rp+4pq=0=p(9p+4q+4r);$$

and the equations of the foci are

$$p=0;\quad 9p+4q+4r=0;$$

or locally $$(100);\quad (944).$$

The centre is (122) and the asymptotes are

$\{(2+2\sqrt{7}),\ -(4+\sqrt{7}),\ 3\}$ and $\{(2-2\sqrt{7}),\ -(4-\sqrt{7}),\ 3\}$,

or locally $\quad -8x^2+3y^2+3z^2-8yz-4zx-4xy=0.$

The given triangle being self-conjugate to the conic, the side BC is the polar of the focus A, and is consequently a directrix.

CHAPTER XII

MISCELLANEOUS THEOREMS

1°. The harmonic properties of a plane net (fig. 1).

By

$$\overline{OA'}=\frac{m\beta+n\gamma}{m+n};\quad \overline{OB'}=\frac{n\gamma+l\alpha}{n+l};\quad \overline{OC'}=\frac{l\alpha+m\beta}{l+m},\qquad (1)$$

$$\overline{OA''}=\frac{m\beta-n\gamma}{m-n};\quad \overline{OB''}=\frac{n\gamma-l\alpha}{n-l};\quad \overline{OC''}=\frac{l\alpha-m\beta}{l-m}.$$

Therefore

A' and A'' are the harmonic conjugates of B and C,
B' „ B'' „ „ C „ A,
C' „ C'' „ „ A „ B;

or
$$(AC'BC'')=(BA'CA'')=(CB'AB'')=-1. \qquad \text{......(2)}$$

Again, let $\overline{OA'}=\alpha'$, $\overline{OB'}=\beta'$, $\overline{OC'}=\gamma'$. Then

$$\overline{OA'''}=\frac{2l\alpha+m\beta+n\gamma}{2l+m+n}=\frac{(l\alpha+m\beta)+(n\gamma+l\alpha)}{(l+m)+(n+l)}$$
$$=\frac{(l+m)\gamma'+(n+l)\beta'}{(l+m)+(n+l)},$$

$$(\overline{OA''})=\frac{m\beta-n\gamma}{m-n}=\frac{(l\alpha+m\beta)-(n\gamma+l\alpha)}{(l+m)-(n+l)}$$
$$=\frac{(l+m)\gamma'-(n+l)\beta'}{(l+m)-(n+l)}.$$

Therefore

A''' and A'' are the harmonic conjugates of B' and C'.

Similarly,

B''' and B'' are the harmonic conjugates of C' and A',
C''' „ C'' „ „ A' „ B';

or
$$(A'C'''B'C'')=(B'A'''C'A'')=(C'B'''A'B'')=-1. \qquad \text{...(3)}$$

Since $(B' \cdot AC'BC'') = (B' \cdot AA'''OA)$, AO is cut harmonically in A''' and A'; while $B'C'$ is cut harmonically in A''' and A'' (3), and CB is cut harmonically in A' and A'' (2). Therefore each of the three diagonals of the complete quadrilateral $AC'OB'$ is cut harmonically by its two other diagonals.

Let $B'B$ be produced to meet $A''C''$ in D. Then $A''C''B''$ is the harmonic triangle, and $B''B'''D$ the diagonal triangle, of the quadrilateral $A'B'C'B$.

2°. A theorem by Roger Cotes.

If a straight line revolve in the plane round a fixed point O, cutting the sides of a given triangle in R_1, R_2, R_3, and if a point R be taken on this transversal such that

$$\frac{3}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \frac{1}{OR_3};$$

then the locus of R is a straight line.

Let O be the origin, let the triangle be the given triangle ABC, and let the transversal be $px + qy + rz = 0$. Since the line passes through the origin

$$p + q + r = 0. \quad \text{......................(1)}$$

It will be found that $R_1 = (or\bar{q})$, $R_2 = (\bar{r}op)$, $R_3 = (q\bar{p}o)$. Then by the aid of (1) and $la + m\beta + n\gamma = 0$, we get

$$OR_1 = \frac{rm\beta - qn\gamma}{rm - qn},$$

$$OR_2 = \frac{-rla + pn\gamma}{pn - rl} = \frac{rm\beta - qn\gamma}{pn - rl},$$

$$OR_3 = \frac{qla - pm\beta}{ql - pm} = \frac{rm\beta - qn\gamma}{ql - pm}.$$

Let $rm\beta - qn\gamma = \theta$. Then

$$\frac{3}{OR} = \frac{rm - qn}{\theta} + \frac{pn - rl}{\theta} + \frac{ql - pm}{\theta}$$

$$= \frac{(q-r)l + (r-p)m + (p-q)n}{\theta},$$

and $$OR = \frac{3(rm\beta - qn\gamma)}{(q-r)l + (r-p)m + (p-q)n}$$

$$\text{,,} \quad = \frac{(q-r)la + (r-p)m\beta + (p-q)n\gamma}{(q-r)l + (r-p)m + (p-q)n}.$$

Comparing this expression with the standard form, we have

$$q-r=x, \quad r-p=y, \quad p-q=z,$$

and since

$$(q-r)+(r-p)+(p-q)=0,$$

$$x+y+z=0,$$

the equation of the axis of perspective, or polar, of the given triangle.

3°. Let Q and R (fig. 37) be the isogonal and isotomic conjugates of the rational point $P=(fgh)$. Then the ratios of the various segments of the sides of the triangle are

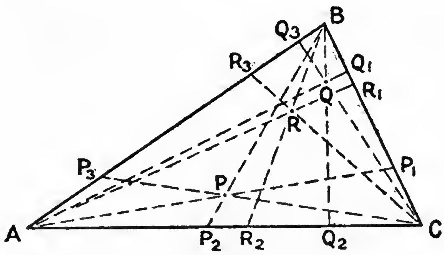

FIG. 37.

$$\frac{BP_3}{P_3A}=\frac{lf}{mg} \qquad \frac{CP_1}{P_1B}=\frac{mg}{nh} \qquad \frac{AP_2}{P_2C}=\frac{nh}{lf}$$

$$\frac{BQ_3}{Q_3A}=\frac{mna^2gh}{nlb^2hf} \qquad \frac{CQ_1}{Q_1B}=\frac{nlb^2hf}{lmc^2fy} \qquad \frac{AQ_2}{Q_2C}=\frac{lmc^2fg}{mna^2gh}$$

$$\frac{BR_3}{R_3A}=\frac{mngh}{nlhf} \qquad \frac{CR_1}{R_1B}=\frac{nlhf}{lmfg} \qquad \frac{AR_2}{R_2C}=\frac{lmfg}{mngh}.$$

Consequently,

$$\left.\begin{array}{ll} Q, \text{ the isogonal conjugate of } P, \text{ is} & \left(\dfrac{a^2gh}{l^2}, \dfrac{b^2hf}{m^2}, \dfrac{c^2fg}{n^2}\right), \\ R, \text{ ,, isotomic ,, ,,} & \left(\dfrac{gh}{l^2}, \dfrac{hf}{m^2}, \dfrac{fg}{n^2}\right). \end{array}\right\} \quad (1)$$

If P be an irrational point, $\left(\dfrac{f}{l}, \dfrac{g}{m}, \dfrac{h}{n}\right)$,

$$\left.\begin{array}{l} Q=\left(\dfrac{a^2gh}{l}, \dfrac{b^2hf}{m}, \dfrac{c^2fg}{n}\right), \\ R=\left(\dfrac{gh}{l}, \dfrac{hf}{m}, \dfrac{fg}{n}\right). \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots(2)$$

Ex. 1. The isogonal conjugate of the symmedian point, $\left(\frac{a^2}{l}, \frac{b^2}{m}, \frac{c^2}{n}\right)$, is $\left(\frac{a^2b^2c^2}{l}, \frac{b^2c^2a^2}{m}, \frac{c^2a^2b^2}{n}\right)=(mn, nl, lm)$, the mean point.

Ex. 2. The Gergonne point of the triangle is

$$\left(\frac{s_2s_3}{l}, \frac{s_3s_1}{m}, \frac{s_1s_2}{n}\right);*$$

and its isotomic conjugate is $\left(\frac{s_1}{l}, \frac{s_2}{m}, \frac{s_3}{n}\right)$, the point in which concur the lines drawn from the points of contact of the three escribed circles to the opposite corners of the triangle.

Ex. 3. Any two lines whose equations are of the form $px+qy+rz=0$, and $p^{-1}x+q^{-1}y+r^{-1}z=0$, cut the sides of the triangle isotomically.

Ex. 4. The Brocard points, II, (4), are isogonal conjugates, as also are the orthocentre and circumcentre.

4°. The isogonal conjugate of every point upon the circumcircle is at infinity. Let the point be $P=(pqr)$.

Since P is on the circumcircle,

$$mna^2qr+nlb^2rp+lmc^2pq=0 \text{ and } p=\frac{l(nb^2r+mc^2q)}{-mna^2qr}.$$

The point P may therefore be written

$$\left(\frac{-mna^2qr}{l(nb^2r+mc^2q)}, q, r\right),$$

the isogonal conjugate of which is, (1),

$$Q=\left(\frac{-(nb^2r+mc^2q)}{l}, \frac{nb^2r}{m}, \frac{mc^2q}{n}\right).$$

The vector of Q consequently is

$$\overline{OQ}=\frac{-(nb^2r+mc^2q)\alpha+nb^2r\beta+mc^2q\gamma}{-(nb^2r+mc^2q)+nb^2r+mc^2q},$$

which is infinitely long because its denominator is zero. Therefore the isogonal conjugate of every point on the circumcircle is at infinity.

* See 'Conventional Signs' at the beginning of the book.

5°. Pascal's Theorem.

The crosses of the opposite sides of a hexagon inscribed in a conic are collinear (fig. 38).

6°. Brianchon's theorem.

The joins of the opposite corners of a hexagon circumscribed to a conic are concurrent (fig. 38).

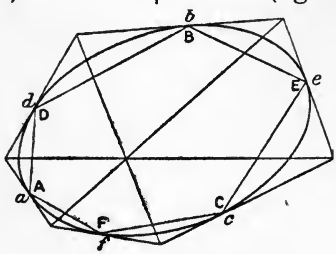

Fig. 38.

Let ABC be the given triangle, and

let $D=(x_1y_1z_1)$,
„ $E=(x_2y_2z_2)$,
„ $F=(x_3y_3z_3)$.

The equation of the conic is
$$yz+zx+xy=0.$$

The condition that the points D, E, F shall lie on the conic is

$$\begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & \frac{1}{z_1} \\ \frac{1}{x_2} & \frac{1}{y_2} & \frac{1}{z_2} \\ \frac{1}{x_3} & \frac{1}{y_3} & \frac{1}{z_3} \end{vmatrix}=0.$$

Let ABC be the given triangle, and

let $D=x_1p+y_1q+z_1r=0$,
„ $E=x_2p+y_2q+z_2r=0$,
„ $F=x_3p+y_3q+z_3r=0$.

The equation of the conic is
$$p^2+q^2+r^2-2qr-2rp-2pq=0.$$

The six points, A ... F are the points of contact of six tangents, the sides of the circumhexagon. The coordinates of these six tangents are

$a=(011)$, $b=(101)$, $c=(110)$;
$d=(y_1+z_1,\ z_1+x_1,\ x_1+y_1)$,
$e=(y_2+z_2,\ z_2+x_2,\ x_2+y_2)$,
$f=(y_3+z_3,\ z_3+x_3,\ x_3+y_3)$.

The condition that the lines d, e, f shall touch the conic is

If the three crosses of the opposite sides be calculated, it will be ultimately found that the condition that they shall be collinear is

If the three joins of the opposite corners be calculated, it will be ultimately found that the condition that they shall be concurrent is

$$\begin{vmatrix} \dfrac{1}{x_1} & \dfrac{1}{y_1} & \dfrac{1}{z_1} \\ \dfrac{1}{x_2} & \dfrac{1}{y_2} & \dfrac{1}{z_2} \\ \dfrac{1}{x_3} & \dfrac{1}{y_3} & \dfrac{1}{z_3} \end{vmatrix} = 0,$$

which is the condition that the hexagon shall be inscribed in the conic.

which is the condition that the hexagon shall be circumscribed to the conic.

7°. To express a homogeneous equation of the second degree,

$$F(fgh) = Uf^2 + \ldots + 2W'fg = 0,$$

in terms of its derived functions, F_f, F_g, F_h.

$$\left.\begin{aligned} F(fgh) &= fF_f + gF_g + hF_h, \\ F_f &= Uf + W'g + V'h, \\ F_g &= W'f + Vg + U'h, \\ F_h &= V'f + U'g + Wh. \end{aligned}\right\}$$

Eliminating f, g and h from these four equations, we get

$$0 = \begin{vmatrix} F_f, & F_g, & F_h, & F(fgh) \\ U, & W', & V', & F_f \\ W', & V, & U', & F_g \\ V', & U', & W, & F_h \end{vmatrix} \begin{aligned} &= (VW - U'^2)F_f^2 \ldots \\ &+ 2(U'V' - WW')F_fF_g - \Delta^2 F(fgh); \\ &\Delta^2 F(fgh) = \Delta(uF_f^2 \ldots + 2w'F_fF_g); \end{aligned}$$

$$\Delta F(fgh) = \phi(F_f, F_g, F_h).$$

Similarly, $$\Delta\phi(fgh) = F(\phi_f, \phi_g, \phi_h).$$

The first of these two equations is met with in calculating the discriminant of the equation

$$F(fgh)F(pqr) - (pF_f + qF_g + rF_h)^2 = 0,$$

in order to verify the conclusion drawn in VII, 16°, that this is the equation of two points, not of a conic.

Putting $F(fgh)=k$, $F_f=a$, $F_g=b$, $F_h=c$, we have

$$k(Up^2\ldots+2W'pq)-(a^2p^2\ldots+2abpq)=0,$$

the discriminant of which is

$$\Delta=\begin{vmatrix} kU-a^2, & kW'-ab, & kV'-ca \\ kW'-ab, & kV-b^2, & kU'-bc \\ kV'-ca, & kU'-bc, & kW-c^2 \end{vmatrix}.$$

Four of the matrices of the third order into which this matrix resolves are zero. The determinants of the remaining four give

$$\begin{aligned} \Delta &= k^3\Delta^2-ak^2\Delta(ua+w'b+v'c)-bk^2\Delta(w'a+vb+u'c) \\ &\qquad\qquad -ck^2\Delta(v'a+u'b+wc) \\ \text{,,} &= k^2\Delta\{k\Delta-\phi(abc)\}=\Delta F^2(fgh) \\ &\qquad\qquad \times\{\Delta F(fgh)-\phi(F_f, F_g, F_h)\}=0. \end{aligned}$$

In conclusion may be quoted the opinion of M. Laissant about the Quaternion method, which seems to be applicable to Anharmonic Coordinates: "la méthode d'Hamilton n'est pas d'une application universelle, non plus qu'aucune autre; mais elle me semble présenter dans des cas nombreux de réels avantages. . . . Ce serait un tort, à mon sens, de se priver de ressources nouvelles, sous prétexte que ces ressources ne sont pas d'un usage constant." *

* *Applications Mecaniques du Calcul des Quaternions*, Paris, 1877.

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