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ABSTRACT

In this paper an application of differential game theory in the area of microeconomics is presented. The problem considered is that of dynamic duopoly where two firms each limited by a maximum capacity of production, share the same market, and try simultaneously but independently to maximize their profits over a certain planning horizon. Necessary conditions for the Cournot solution in the general case are discussed and more specific results for the special case of linear demand and quadratic cost functions are developed.

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Introduction

Static and comparative static formulations of the duopoly game following the line of Cournot [1] and Stackelberg [2] flourished in the 1950's after the path-breaking work of von Neumann and Morgenstern [3], Theory of Games and Economic Behavior. Zero-sum or non-zero-sum, two (or n) person games, cooperative or noncooperative games, etc. are well documented (see Shubik [4], for instance). Methodologically, the static and traditional duopoly theory seems to have failed to actively interact with game theoretic approaches (see Osborne [5]).

It may be worthwhile to point out that logical developments of the traditional duopoly game in its comparative static sense, by taking advantage of a type of sequential decision-making procedures in reaching some reasonable solutions such as the models of Cyert and DeGroot [6-7] and Friedman [8], may have taken a step towards the dynamization of the traditional theory. However, these models are still not completely dynamic in nature. For instance, in the Cyert and DeGroot model it is assumed that decisions are made sequentially in alternating periods while profits are maximized over a certain planning horizon, and in the

Friedman model it is assumed that decisions are made simultaneously at each period but that profits are only maximized over that particular period.

In this paper, we propose a model where decisions are made simultaneously at each period of time and where the profits are maximized over the whole planning horizon. We formulate the model in continuous time as a non-zero-sum differential game problem. A similar discrete time version of this model can also be formulated. In this model we assume that the demand curve is described by a differential equation which gives at each instant of time the relationship between the price of the commodity, the rate of change of the price and the production outputs of both firms. We assume that each firm's objective is to maximize its total discounted profits over a prespecified time horizon. A general formulation and solution of this model, and a comparison between the Cournot and collusive behaviors of both firms has been presented in [9]. In this paper we give a complete characterization of the Cournot solution especially for the case where each firm has a maximum production capacity that cannot be exceeded. We show that the Cournot solution can be a combination of several possibilities where each firm may either stay out of the market, place its maximum supply in the market, act as a monopolist, or share the market with its rival firm and act as a duopolist. We treat in detail the case where the demand function is linear and the production cost functions are quadratic.

Even though differential game theory has recently received a great deal of attention in the control literature, very little has been done in applying it to microeconomic problems. The simple, single commodity,

model considered in this paper provides such an application and demonstrates that useful conceptual results can be obtained. Before going into the dynamic formulation, let us first briefly review the static duopoly model as formulated by Cournot.

1. Static Cournot Duopoly Model

Let x_1 and x_2 be the outputs of firms 1 and 2 respectively, and let the commodity price p be related to $x_1 + x_2$ by the following well behaved continuous and differentiable (in R^+) demand function:

(1) $$p = h(x_1+x_2)$$

Let the total production cost functions for Firms 1 and 2 be $g_1(x_1)$ and $g_2(x_2)$ respectively, then the profits to be maximized are:

(2)
$$\Pi_1(x_1,x_2) = x_1 h(x_1+x_2) - g_1(x_1) \qquad \text{for Firm 1 , and}$$
$$\Pi_2(x_1,x_2) = x_2 h(x_1+x_2) - g_2(x_2) \qquad \text{for Firm 2 .}$$

If x_1 and x_2 are not constrained then the solution of this problem as proposed by Cournot is determined in terms of the reaction functions which specify the output of one firm in terms of the output of the other firm (see Intrilligator [10]). These functions are obtained from:

(3)
$$\frac{\partial \Pi_1}{\partial x_1} = h(x_1+x_2) + x_1 \frac{\partial h}{\partial x_1} + x_1 \frac{\partial h}{\partial x_2} \frac{\partial x_2}{\partial x_1} - \frac{dg_1(x_1)}{dx_1} = 0 \text{ , and}$$
$$\frac{\partial \Pi_2}{\partial x_2} = h(x_1+x_2) + x_2 \frac{\partial h}{\partial x_2} + x_2 \frac{\partial h}{\partial x_1} \frac{\partial x_1}{\partial x_2} - \frac{dg_2(x_2)}{dx_2} = 0 \text{ .}$$

The terms $\partial x_2/\partial x_1$ and $\partial x_1/\partial x_2$ are called "conjectural variations" terms and they reflect the effect of variations in the output of one firm on the output of the other firm. In the Cournot analysis, these terms are assumed to be zero. The solution of the two simultaneous equations in (3) yields the Cournot equilibrium outputs x_1^* and x_2^*. In the case where $h(x_1+x_2)$ is linear in (x_1+x_2) and $g_i(x_i)$ is quadratic in x_i for $i = 1,2$:

(4) $$h(x_1+x_2) = c - b(x_1+x_2) \quad , \qquad c > 0 \quad b > 0$$

(5) $$g_i(x_i) = (1/2)\alpha_i x_i^2 \quad , \qquad \alpha_i > 0 \quad i = 1,2$$

then, the solution of (3) with zero conjectural variations is straightforward:

(6) $$x_i^* = \frac{(b+\alpha_i)c}{(2b+\alpha_1)(2b+\alpha_2)-b^2} \quad , \qquad i = 1,2$$

and it follows that

(7) $$p^* = \frac{(b+\alpha_1)(b+\alpha_2)c}{(2b+\alpha_1)(2b+\alpha_2)-b^2}$$

It is important to note that the optimal Cournot outputs x_1^* and x_2^* derived by the above procedure satisfy the following set of inequalities:

(8) $$\Pi_1(x_1^*,x_2^*) \geq \Pi_1(x_1,x_2^*)$$
$$\Pi_2(x_1^*,x_2^*) \geq \Pi(x_1^*,x_2)$$

2. Dynamic Cournot Duopoly Model

The model that we shall discuss in this section is essentially a "dynamization" of the static model discussed in the previous section. Let the demand function be described by the following differential equation:

(9) $$\frac{dp(t)}{dt} = \dot{p}\,(t) = G[h(x(t))-p(t)] \quad , \quad t \in [0,T],\ p(0) = p_o$$

where $[0,T]$ is the planning horizon, $x(t) = x_1(t)+x_2(t)$ is the sum of the outputs $x_1(t)$ and $x_2(t)$ of both firms respectively and $G[u]$ is a monotone increasing function of its argument satisying (see Samuelson [11]):

(10) $$G[0] = 0 \quad \text{and} \quad \frac{dG[u]}{du} > 0 \quad \forall\ u \in R.$$

This G function can be considered as the speed of adjustment function. If, at a certain time $t_1 \in [0,T]$, $h(x(t_1))-p(t_1) \neq 0$ for some reason such as the market price during its adjustment process was not high (low) enough, resulting in a smaller (larger) supply quantity appearing in the market than what the market actually desired at that price level, then the market price moves in the direction stipulated by equation (9). Equilibrium conditions are reached when $h(x(t))-p(t) = 0$ for all t. The rationale above is a genuinely dynamic price adjustment mechanism and is considered to be a natural extension of the static price response function (see Nikaido [12]). If we let

(11) $$f(p(t),x(t)) = G[h(x(t))-p(t)]$$

and if we assume[1] that $\frac{\partial h}{\partial x} < 0$ for all $t \in [0,T]$, then it follows that $\frac{\partial f}{\partial p} < 0$ and $\frac{\partial f}{\partial x} < 0$ (note that $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x}$, $i = 1,2$). It is interesting to note that equation (9) relates the price at time t to the price at a previous time t_1 and to the entire history of total supply $x_{[t_1,t]}$ over the interval $[t_1,t]$. Functionally this can be written as

$$(12) \qquad p(t) = \Phi(t,p(t_1),x_{[t_1,t]})$$

where Φ is the trajectory of the solution of equation (9) for a given $p(t_1)$ and $x_{[t_1,t]}$. Thus, in contrast to the static market theory which does not address itself to the question of the process by which changes in the price are brought about, but only compares the prices before and after the change takes place, the dynamic market theory investigates how the price changes with time and what trajectory it follows.

In static market theory it is well-known that the demand function has the property that "an increase (decrease) in the total market supply will cause a decrease (increase) in the market price of the commodity." We shall show next that this property also holds true, locally, for our dynamic market; that is if $(\hat{p}(t),\hat{x}(t))$ is a trajectory satisfying $\dot{\hat{p}}(t) = f(\hat{p}(t),\hat{x}(t))$, $t \in [0,T]$ for a given p_o, then a positive (negative) perturbation in $\hat{x}(t)$ will cause a negative (positive) first order perturbation in $\hat{p}(t)$.

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[1]The economic significance of this assumption is shown in proposition 1.

Proposition 1:

The condition $\frac{\partial f}{\partial x} < 0$ implies that for any pair $(\hat{p}(t),\hat{x}(t))$ satisfying (9), if $x(t) = \hat{x}(t) + \delta x(t)$ where $\delta x(t) \geq 0 \ \forall \ t \in [0,T]$; then $p(t) = \hat{p}(t) + \delta p(t) +$ Higher order terms, where $\delta p(t) \leq 0 \ \forall \ t \in [0,T]$.

Proof

(Note that the same proof holds if the sign of $\delta x(t)$ and $\delta p(t)$ are reversed). The Condition $\frac{\partial f}{\partial x} < 0$ follows from (11). Linearize equation (9) around the trajectory $(\hat{p}(t),\hat{x}(t))$ by expanding the RHS in a Taylor series expansion. The first order terms give the linear differential equation

$$\dot{\delta p} = \left(\frac{\partial f}{\partial p}\bigg|_{\wedge}\right) \delta p + \left(\frac{\partial f}{\partial x}\bigg|_{\wedge}\right) \delta x \quad , \qquad \delta p(0) = 0 \ .$$

The solution of this equation is

$$\delta p(t) = \int_0^t e^{\left(\frac{\partial f}{\partial p}\big|\right)(t-\tau)} \left(\frac{\partial f}{\partial x}\bigg|\right) \delta x(\tau) d\tau$$

and clearly $\frac{\partial f}{\partial x} < 0$ and $\delta x(\tau) \geq 0$ imply that $\delta p(t) \leq 0$.

In the price adjustment process described above, the supply quantities $x_1(t)$ and $x_2(t)$ are treated as decision variables chosen independently by each firm in order to maximize its total profits. If we assume that $g_i(x_i(t))$, a convex function in $x_i(t)$, is the cost of production at time t per unit time for firm i, then the profits accumulated by firm i over the horizon [0,T] can be written as:

$$(13)\qquad \Pi_i(x_1,x_2) = \int_0^T e^{-r_i t}\,[p(t)x_i(t)-g_i(x_i(t))]dt \qquad , \qquad i = 1,2$$

where r_i is an appropriate discount rate for firm i.

Thus, the dynamic duopoly problem we are considering (9)-(13) is a direct application of non-zero-sum differential game theory. While there are several approaches for solving non-zero sum differential game problems (see [13-16] for instance), in this paper we shall consider only the Cournot solution (sometimes referred to as Nash). Furthermore, we shall assume that the production functions need not necessarily be functions of time only, but that they are allowed to be "feedback" functions of the current price p(t) in the market; i.e. $x_1 = x_1(t,p(t))$ and $x_2 = x_2(t,p(t))$. Furthermore, as required by practical economic considerations, we assume that each firm has a maximum capacity of production X_i. Thus, the admissible strategy spaces χ_1 and χ_2 for firms 1 and 2 respectively are defined by:

$$(14)\qquad \chi_i = \{x_i(t,p(t)) : 0 \le x_i(t,p(t)) \le X_i\} \qquad \text{for } i = 1,2$$

The Cournot solution for this problem is therefore defined as a pair (x_1^*,x_2^*) with $x_1^* \in \chi_1$ and $x_2^* \in \chi_2$ such that the set of inequalities in (8) is satisfied with Π_1 and Π_2 as given by (13).

Upon applying the differential game results developed in [13], the necessary conditions for the Cournot solution of the above problem can be written as follows: First define the Hamiltonian functions:

$$H_i(t,p,x_1,x_2,\lambda_i) = p(t)x_i(t,p(t)) - g_i(t,x_i(t,p(t))) + \lambda_i(t)f(p(t),x_1(t,p(t)) + x_2(t,p(t))) \quad \text{for } i = 1,2 \tag{15}$$

where $\lambda_i(t)$ is the adjoint variable for the i^{th} firm, then (x_1^*,x_2^*) must satisfy the conditions (see Appendix 1).

$$\dot{p}(t) = f(p(t),x_1(t,p(t)) + x_2(t,p(t))) \quad , \ p(0) = p_o \tag{16a}$$

$$\dot{\lambda}_1(t) = r_1\lambda_1(t) - x_1(t,p(t)) - \lambda_1(t)\left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial p}\right) \tag{16b}$$

$$\dot{\lambda}_2(t) = r_2\lambda_2(t) - x_2(t,p(t)) - \lambda_2(t)\left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial p}\right) \tag{16c}$$

$$H_1(t,p,x_1^*,x_2^*,\lambda_1) = \max_{x_1 \varepsilon X_1} H(t,p,x_1,x_2^*,\lambda_1) \tag{16d}$$

$$H_2(t,p,x_1^*,x_2^*,\lambda_2) = \max_{x_2 \varepsilon X_2} H(t,p,x_1^*,x_2,\lambda_2) \tag{16e}$$

with

$$\lambda_1(T) = 0 \quad \text{and} \quad \lambda_2(T) = 0 \tag{16f}$$

as boundary conditions for (16b) and (16c). In the above conditions, the discounting factors $e^{-r_i t}$ have been eliminated by redefining the adjoint variables as described in [17].

Because of the capacity constraints (14), conditions (16d) and (16e) can be written as:

$$p(t) - \frac{dg_i}{dx_i} + \lambda_i(t)\frac{\partial f}{\partial x_i} \begin{cases} = 0 \text{ for } 0 \leq x_i(t,p(t)) \leq X_i & (17a) \\ < 0 \text{ for } \quad x_i(t,p(t)) = 0 & (17b) \\ > 0 \text{ for } \quad x_i(t,p(t)) = 0 & (17c) \end{cases}$$

for $i = 1,2$. Thus, the solution may exhibit any combination of the nine possibilities of (17a)-(17c) for $i = 1,2$ each holding over a sub-interval of time in $[0,T]$. These possibilities are tabulated in Table 1; and whether a possibility is a part of the solution or not is determined by the parameters of the problem and the initial price p_o.

Since in general it is almost impossible to obtain an analytic solution for the coupled partial differential equations (15)-(17), we shall in the next section discuss the solution for the special case of linear demand and quadratic cost functions. It is important to mention that this special case can be considered as a first order approximation of the local behavior of corresponding non-linear duopoly problems.

3. Linear-Quadratic Dynamic Duopoly Model

Let the demand function (9) be linear of the form:

$$\dot{p}(t) = c - ap(t) - b(x_1(t)+x_2(t)) \tag{18}$$

$$= a\left[\frac{c}{a} - \frac{b}{a}(x_1(t) + x_2(t)) - p(t)\right] \text{ (see (9))}$$

and the cost functions be quadratic of the form:

$$g_i(x_i(t)) = (1/2)\alpha_i x_i^2(t) \qquad i = 1,2 \tag{19}$$

TABLE 1

POSSIBILITIES IN THE SOLUTION OF THE DUOPOLY MARKET

Possibility	$x_1(t,p(t))$	$x_2(t,p(t))$	Firm 1	Firm 2
P1	0	0	Out	Out
P2	0	≥ 0	Out	Monopolist
P3	≥ 0	0	Monopolist	Out
P4	0	X_2	Out	Max. Capacity
P5	X_1	0	Max. Capacity	Out
P6	≥ 0	X_2	Monopolist	Max. Capacity
P7	X_1	≥ 0	Max. Capacity	Monopolist
P8	X_1	X_2	Max. Capacity	Max. Capacity
P9	≥ 0	≥ 0	Duopolist	Duopolist

where a, b, c, α_1 and α_2 are positive constants. The profits over [0,T] can be written as:

$$\Pi_i(x_1,x_2) = \int_0^T e^{-rt}\,[p(t)x_i(t)-(1/2)\alpha_i x_i^2(t)]dt \tag{20}$$

for i = 1,2. In (20), for the sake of simplicity we have assumed that $r_1 = r_2 = r$. The production functions $x_i(t,p(t))$ are to be selected from the admissible sets of controls and we assume that capacity constraints of the form:

$$0 \leq x_i(t,p(t)) \leq X_i \qquad i = 1,2 \tag{21}$$

are imposed. In (21) the function p(t) is the price trajectroy satisfying equation (18). Furthermore, in order to insure that p(t) does not cross to the negative region, we shall assume that $X_1 + X_2 \leq \frac{c}{b}$. Thus, if we define $q(t) = \frac{c}{a} - p(t)$, then it follows from (18) that

$$\dot{q}(t) \geq -aq(t) \tag{22}$$

and hence $q(t) \geq 0$ for all t. This implies that $p(t) \leq \frac{c}{a}$ for all t. The region of interest is therefore a rectangle in the p - x plane defined by $0 \leq p \leq \frac{c}{a}$ and $0 \leq x \leq \frac{c}{b}$. Applying the necessary conditions (15)-(17) we get:

$$\dot{p}(t) = c - ap(t) - b(x_1(t,p(t))+x_2(t,p(t))) \;,\; p(0) = p_o \tag{23a}$$

$$\dot{\lambda}_1(t) = (r+a+b\,\frac{\partial x_2(t,p(t))}{\partial p})\,\lambda_1(t) - x_1(t,p(t)) \tag{23b}$$

$$\dot{\lambda}_2(t) = (r+a+b\,\frac{\partial x_1(t,p(t))}{\partial p})\,\lambda_2(t) - x_2(t,p(t)) \tag{23c}$$

and

$$
p(t) - \alpha_i x_i(t,p(t)) - b\lambda_i(t) \begin{cases} = 0 \text{ for } 0 \leq x_i(t,p(t)) \leq X_i & (23d) \\ < 0 \text{ for } \quad x_i(t,p(t)) = 0 & (23e) \\ > 0 \text{ for } \quad x_i(t,p(t)) = X_i & (23f) \end{cases}
$$

for $i = 1,2$ and with $\lambda_1(T) = \lambda_2(T) = 0$. The solution of the above system of equations can be represented by the following nine possibilities as summarized in Table 1.

Case 1 (Possibility P1)

Condition (23e) holds for both firms. That is $x_1^*(t,p(t)) = 0$ and $x_2^*(t,p(t)) = 0$. This implies that $p(t) < b\lambda_i$ for $i = 1,2$. But [13] $\lambda_i = \partial V_i/\partial p$ where

$$
(24) \qquad V_i(p,t,x_1 x_2) = \int_t^T e^{-r\tau}(p(\tau)x(\tau)-(1/2)\alpha_i x_i^2(\tau))d\tau
$$

Then, this condition means that when the price becomes less than $b\partial V_i/\partial p$ (the marginal revenues) for $i = 1,2$, both firms will not be accumulating any profits and hence their best policy is to stay out of the market. This naturally causes the price to increase since $\dot{p}(t) = c - ap(t) > 0$.

Case 2a (P2)

Condition (23e) holds for firm 1 and Condition (23d) for firm 2. That is, $x_1^*(t,p(t)) = 0$ and $x_2^*(t,p(t)) \geq 0$. This implies that

$p(t) < b\partial V_1/\partial p$ for firm 1 and $p(t) > b\partial V_2/\partial p$ for firm 2; and as a result it is only profitable for firm 2 to enter the market. Thus, firm 2 is now acting as a monopolist with no influence from its rival firm. The necessary conditions which maximize its profits follow directly from (23a), (23b) and (23e). The monopolistic behavior of firm 2 has been separately studied in [18] and the results can be directly applied in this case. We only mention however that the optimum supply rule for firm 2 is an affine function in $p(t)$, of the form $x_2^*(t,p(t)) = K_2(t)p(t) + E_2(t)$ where $K_2(t)$ and $E_2(t)$ are functions of time, determined, and with properties as described in [18].

Case 2b (P3)

Condition (23e) holds for firm 2 and Condition (23d) for firm 1. This is the dual situation of case 2a, and firm 1 will now be the monopolist.

Case 3a (P4)

Condition (23e) holds for firm 1 and condition (23f) for firm 2. That is $x_1^*(t,p(t)) = 0$ and $x_2^*(t,p(t)) = X_2$. For firm 1 this implies that $p(t) < b\partial V_1/\partial p$ and for firm 2, $p(t) > \alpha_2 X_2 + b\partial V_2/\partial p$. Thus firm 2 is a monopolist who saturates the market with his product collecting maximum profits, while firm 1 cannot afford to sell the product at the prevailing price, and therefore stays out of the market.

Case 3b (P5)

Condition (23e) holds for firm 2 and Condition (23f) for firm 1. This is the dual of case 3a and firm 1 will now place its maximum output in the market.

Case 4a (P6)

Condition (23d) holds for firm 1 and Condition (23f) for firm 2. That is $x_1^*(t,p(t)) \geq 0$ and $x_2^*(t,p(t)) = X_2$. This implies that $p(t) > b\partial V_i/\partial p$ for both firms; however, for firm 2 the price is high enough for it to place its maximum output X_1 in the market. Firm 1 supplies the remaining need of the market and this is also done in an optimal monopolistic way. Thus firm 1 now acts as a monopolist in a market described by

$$(25) \qquad \dot{p}(t) = c_1 - ap(t) - bx_1$$

where $c_1 = c - bX_2$. The solution of this case is also as discussed in [18] where the optimum supply rule is shown to be an affine function in $p(t)$ of the form $x_1(t,p(t)) = K_1(t)p(t) + E_1(t)$.

Case 4b (P7)

Condition (23f) holds for firm 1 and Condition (23d) for firm 2. That is $x_1^*(t,p(t)) = X_1$ and $x_2^*(t,p(t)) \geq 0$. This is the dual of case 4a.

Case 5 (P8)

Condition (23f) holds for both firms. That is $x_1^*(t,p(t)) = X_1$ and $x_2^*(t,p(t)) = X_2$ and $p(t) > b\partial V_i/\partial p$ for both firms. Both firms place their maximum outputs in the market and realize maximum profits without any competitive effort.

Case 6 (P9)

Condition (23d) holds for both firms and this represents a true duopoly situation where both firms are actively engaged in a competitive market. While in all previous cases we did not differentiate between closed-loop (feedback) and open-loop supply curves since they both lead to the same solution: in this case we must differentiate between them because (see [13]) their corresponding solutions are different. We discuss each case separately.

(a) Closed-loop (feedback) Supply Curves

In this case x_i^* are strictly functions of t and p(t). It can easily be checked that the Two Point Boundary Value problem (23a), (23b), (23c) and (23e) admits affine supply curves as a solution. After some simple algebraic manipulations, it can be shown [9] that

$$(26) \qquad x_i^*(t.p(t)) = \frac{1}{\alpha_i}\,[1-bK_i(t)]p(t) - \frac{b}{\alpha_i}\,E_i(t)$$

where $K_i(t)$ and $E_i(t)$ satisfy:

(27) $\dot{K}_i = (r+2a+\frac{b}{\alpha_i}(1-bK_i) + 2\frac{b}{\alpha_j}(1-bK_j))K_i - \frac{1}{\alpha_i}(1-bK_i)$

(28) $\dot{E}_i = (r+a+\frac{b}{\alpha_i}(1-bK_i) + \frac{b}{\alpha_j}(1-bK_j))E_i - \frac{b^2}{\alpha_j}K_iE_j - K_ic$

for $i = 1,2$, $j = 1,2$ and $i \neq j$. Equations (27) for $i = 1,2$ are quadratic of the Riccati type and equations (28) are linear and can be solved after (27) has been solved for K_1 and K_2. If this duopoly case holds over a subinterval $[t_1,t_2] \subset [0,T]$, then the boundary conditions for (27) and (28) are obtained from:

(29) $\lambda i(t_2) = \frac{\partial V_i}{\partial p}\Big|_{t = t_2}$

where, $\lambda_i(t)$, $i = 1,2$ are expressed in terms of $K_i(t)$ and $E_i(t)$ as:

(30) $\lambda_i(t) = K_i(t)p(t) + E_i(t)$ $\qquad i = 1,2$.

The value functions $V_i, i = 1,2$ under any of the possibilities of the previous five cases are clearly quadratic in $p(t)$, and hence the boundary conditions $K_i(t_2)$ and $E_i(t_2)$ are readily available from (29) and (30).

(b) Open-loop Supply Curves

In addition to the above solution, the Two Point Boundary Value problem (23a), (23b), (23c) and (23e) also admits supply curves which are strictly functions of time only as solution. In this case the terms $\partial x_i/\partial p$ in (23b) and (23c) will vanish and the solution can be obtained by directly solving the resulting coupled system of linear ordinary differential equations. Another way which transforms the system into a single

point boundary problem is to follow a procedure similar to the closed-loop case. If we assume that $\lambda_i(t)$ $i = 1,2$ can be written in the form:[2]

(31) $$\lambda_i(t) = D_i(t)p(t) + F_i(t)$$

then (23a) and (23d) will give:

(32) $$\dot{p}(t) = -(a + \frac{b}{\alpha_1}(1-bD_1) + \frac{b}{\alpha_2}(1-bD_2))p(t) + (c + \frac{b^2}{\alpha_1}F_1 + \frac{b^2}{\alpha_2}F_2)$$

$$= -\sigma(t)p(t) + \rho(t)$$

whose solution gives the open-loop Cournot price trajectory

(33) $$p^*(t) = p(t_1)\, e^{-\int_{t_1}^{t}\sigma(\tau)d\tau} + \int_{t_1}^{t} e^{-\int_{\tau}^{t}\sigma(\beta)d\beta}\rho(\tau)d\tau$$

The open-loop Cournot supply functions are then obtained from (23d) as:

(34) $$x_i^*(t) = \frac{1}{\alpha 1}(1-bD_i(t))\left[p(t_1)e^{-\int_{t_1}^{t}\sigma(\tau)d\tau} + \int_{t_1}^{t} e^{-\int_{\tau}^{t}\sigma(\beta)d\beta}\rho(\tau)d\tau\right] - \frac{b}{\alpha_i}F_i(t)\ .$$

The functions $D_i(t)$ and $F_i(t)$ can be shown to be solutions of the differential equations

(35) $$\dot{D}_i(t) = (r+2a+\frac{b}{\alpha_i}(1-bD_i) + \frac{b}{\alpha_j}(1-bD_j))D_i - \frac{1}{\alpha_i}(1-bD_i)$$

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[2]These λ_i's are different from the closed-loop λ_i's of eq. (30).

(36) $\dot{F}_i(t) = (r+a+ \frac{b}{\alpha_i}(1-bD_i))F_i - \frac{b^2}{\alpha_j}D_iF_j-D_ic$

for $i = 1,2$, $j = 1,2$ and $i \neq j$.

The difference between the feedback and open-loop solutions is due to the terms $\partial x_i(t,p(t))/\partial p$ in equations (23b) and (23c) which are set equal to zero in the open-loop case. These terms express the awareness of one firm of the dependency of the other firm's supply policy on the current price in the market. Ideally, both solutions guarantee a Cournot-type equilibrium in which no firm has an incentive to cheat. However, in the feedback case, if one firm cheats the other firm automatically adjusts its supply, after detecting the resulting change in the prevailing price in the market. In the open-loop case this cannot be done since each firm is committed to a production program that specifies at each t the quantity to be supplied to the market. However, in both cases, the firm that attempts to cheat will suffer a loss in its profits.

The feedback supplies also have the desirable property that if an external disturbance that causes the price to deviate from its Cournot optimal trajectory takes place, then a Cournot equilibrium, with profits as defined in (24), will still hold for the remaining part of the trajectory. This is not so in the open-loop case.

Over the interval of time where duopoly prevails (case 6) the firms may negotiate what supply form to use. Naturally the one that leads to more profits is more desirable. However, from the consumer's point of view, it may seem that allowing the firms to adjust their supplies according to the prevailing price of the commodity in the market,

will result in increased profits for the firms. While this may be so in some cases of monopolies or collusive duopolies; the competitive, non-cooperative nature of the Cournot solution may actually result in a decrease in the profits of the firms.

Conclusion

In this paper, a dynamic duopoly problem where each firm is limited by a maximum production capacity has been formulated and solved within the framework of differential game theory. It was shown that the Cournot solution may be a combination of various possibilities which may include each firm either staying out of the market, or competing with its rival for a share of the market, or placing its maximum capacity in the market. Thus a Cournot solution of a dynamic duopoly problem may include sub-intervals of time where one firm acts independently as a monopolist. When a duopoly situation prevails, however, the Cournot supply curves can be either in feedback or in open-loop forms, each leading to a different price trajectory. Necessary conditions for each case have been derived and it was shown that the closed-loop Cournot supply curves are always affine functions of the price. The marginal supplies (i.e. $\partial x_i/\partial p$) are shown to be functions of time that satisfy a set of Riccati-like differential equations.

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Appendix 1

We will show how the necessary conditions (15)-(17) are derived for i=1 (firm 1). An identical derivation follows for i=2. Define the Hamiltonian for firm 1:

$$\tilde{H}_1(t,p,x_1,x_2,\gamma_1) = e^{-r_1 t}\,[p(t)x_1(t,p(t))-g_1(x_1(t,p(t))]$$

$$+\ \gamma_1(t)\ f(p(t),\ x_1(t,p(t)) + x_2(t,p(t))$$

then the necessary conditions are:

$$\text{(A1)}\qquad \dot{\gamma}_1(t) = -\frac{\partial \tilde{H}_1}{\partial p} = -e^{-r_1 t}\,[x_1 + p\frac{\partial x_1}{\partial p} - \frac{dg_1}{dx_1}\frac{\partial x_1}{\partial p}]$$

$$-\gamma_1\ (\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial p} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial p}) \quad ,\ \gamma(T) = 0$$

and

$$\text{(A2)}\qquad \tilde{H}_1(t,p,\overset{*}{x}_1,\overset{*}{x}_2,\gamma_1) = \underset{x_1 \varepsilon X_1}{\text{Max}}\ \tilde{H}_1(t,p,x_1,\overset{*}{x}_2,\gamma_1)$$

Now (as done in Arrow [17]) let $\lambda_1 = e^{r_1 t}\gamma_1$ and define

$$H_1(t,p,x_1,x_2,\lambda_1) = e^{r_1 t}\,\tilde{H}_1(t,p,x_1,x_2,\gamma_1)$$

Condition (A1), then reduces to:

$$(A3) \qquad \dot{\lambda}_1 = r_1\lambda_1 - x_1 - \left(p - \frac{dg_1}{dx_1} + \lambda_1 \frac{\partial f}{\partial x_1}\right)\frac{\partial x_1}{\partial p}$$

$$-\lambda_1\left(\frac{\partial f}{\partial p} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial p}\right), \qquad \lambda_1(T) = 0$$

and (A2) can be written as:

$$(A4) \quad p - \frac{dg_1}{dx_1} + \lambda_1 \frac{\partial f}{\partial x_1} \quad \begin{cases} = 0 & \text{for } 0 \leq x_1(t,p(t)) \leq X_1 \\ < 0 & \text{for } x_1(t,p(t)) = 0 \\ > 0 & \text{for } x_1(t,p(t)) = X_1 \end{cases}$$

Furthermore we see that condition (A4) implies that:

$$\left(p - \frac{dg_1}{dx_1} + \lambda_1 \frac{\partial f}{\partial x_1}\right)\frac{\partial x_1}{\partial p} = 0$$

and this reduces (A3) to condition (16b).