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RECTANGULAR ARRAYS

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Approximate Fixed Points in Rectangular Arrays

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Abstract

An extension of Kuhn's "Cubical Sperner Lemma" is used to present two algorithms for locating approximate fixed points.

Introduction

Let f: $[0, 1] \to [0, 1]$ be continuous. The natural way to find a point x such that f(x) is close to x would be something like this: compute f(1/2). If it is greater than 1/2, confine your search to [1/2, 1]. Otherwise search in [0, 1/2]. Next compute f(3/4) in the first case or f(1/4) in the second case... Each computation cuts the size of the interval under consideration in half.

The same idea applies in n dimensions. Let f: $[0,1]^n \to [0, 1]^n$ be continuous. By the Brouwer theorem, there is an x such that f(x) = x. Divide the unit cube into two rectangles R_1, R_2 according to whether $x_1 \geq 1/2$ or $x_1 \leq 1/2$. Let $B(R_1)$ be the boundary of R_1 and assume f has been computed there. It may be the case that every continuous function g: $R_1 \to [0, 1]^n$ for which g(x) = f(x) on $B(R_1)$ has a fixed point. In that case f must have a fixed point in R_1 and R_2 may be ignored. If this is not the case then f must have a fixed point in R_2 and R_1 may be ignored. In general:

Proposition: Let $R \subset [0, 1]^n$ be an n-dimensional rectangle and f: $R \to [0, 1]^n$ be such that every h: $R \to [0, 1]^n$ for which h(x) = f(x) on B(R) has a fixed point in R. Let $R = R_1 \cup R_2$ where R_1 and R_2 are two n-dimensional rectangles whose interiors are disjoint. Then either (i) every g: $R_1 \to [0, 1]^n$ for which g(x) = f(x) on $B(R_1)$ has a fixed point in R_1, or (ii) every g: $R_2 \to [0, 1]^n$ for which g(x) = f(x) on $B(R_2)$ has a fixed point in R_2.

Proof: If (i) and (ii) fail there are g_1: $R_1 \to [0, 1]^n$ and g_2: $R_2 \to [0, 1]^n$ without fixed points, which agree with f on $B(R_1) \cup B(R_2)$. But then we could define h: $R \to [0, 1]^n$ by $h(x) = g_1(x)$ for x in R_1 and $h(x) = g_2(x)$ for x in R_2. h would not have a fixed point, contradicting our hypothesis. Q.E.D.

The key point here is that knowing the values of f on the "small" subset $B(R_1) \cup B(R_2)$ enables us to cut our search space in half by confining our subsequent hunt for a fixed point to R_1 in case (i) and to R_2 in case (ii).

In practice one computes approximate fixed points, i.e., points x for which f(x) is close to x. The information at hand is the values of f(x) on a rectangular grid, a finite subset of $[0,1]^n$, e.g.

$$A_k = \left\{ \left(\frac{a_1}{k}, \frac{a_2}{k}, \ldots \frac{a_n}{k} \right) \middle| \, 0 \le a_i \le k \; a_i \text{ integer} \right\}$$

The main purpose of this paper is to show how the "bifurcation" idea described above may be used to compute approximate fixed points.

Approximate Fixed Points in Rectangular Arrays

Rectangular grids were discussed by Kuhn [2] and we essentially follow his treatment.

By uniform continuity there is an integer k such that $|x-y| \le \frac{n}{k}$* implies $|f(x) - f(y)| \le \varepsilon$. Approximate fixed points for f are obtained by locating a subcube of edge length $\frac{1}{k}$ with vertices in A_k such that, for each $1 \le i \le n$, there are vertices of the cube x_i, y_i such that the ith component of x_i (respectively y_i) is $\ge$ (resp. $\le$) the ith component of $f(x_i)$ (respectively $f(y_i)$). If x is any point of such a cube then |(ith component of x) - (ith component of f(x))| $\le \varepsilon + \frac{1}{k}$* so x may be considered an approximate fixed point.

An Extension of the "Cubical Sperner Lemma"

Fix $f: [0,1]^n \to [0, 1]^n$. As before $A_k = \{ (\frac{a_1}{k}, \ldots, \frac{a_n}{k}) |$ $0 \le a_i \le k$, a_i integer$\}$ for some fixed k. $S_1, \ldots S_n$ are fixed subsets of A_k.

For $1 \le j \le n$ a j-dimensional rectangle will be defined as a non-degenerate j-dimensional rectangular subset of $[0, 1]^n$ each of whose

* These figures can be sharpened.

vertices is in A_k. A j-dimensional rectangle is not a (j+1)-dimensional rectangle. A j-dimensional cube will be a j-dimensional rectangle with each edge of length $\frac{1}{k}$.

The boundary B(R) of a j-dimensional rectangle R is the union of the (j-1)-dimensional faces of R.

For each j, we define* two families of j-dimensional cubes, the j-fixed cubes and the special j-fixed cubes. We proceed by induction:

A 1-fixed cube is a 1-dimensional cube (i.e. a line segment of length $\frac{1}{k}$) such that exactly one of the two vertices is a member of S_1. A special 1-fixed cube is a 1-fixed cube both of whose vertices are members of S_2.

For $j > 1$ a j-fixed cube is a j-dimensional cube which contains an odd number of special (j-1)-fixed cubes. A special j-fixed cube is a j-fixed cube all of whose vertices are members of S_{j+1}.

<u>Example</u>: for n = k = 2. If $S_1 = \{(0, 0), (1/2, 1/2)\}$ and $S_2 = \{(0, 0), (0, 1/2), (1/2, 1/2)\}$ then $\{(0,0), (1/2, 0)\}$ is a 1-fixed cube while $\{(0, 0), (0, 1/2)\}$ is a special 1-fixed cube. The 2-dimensional cube $\{(0, 0), (0, 1/2), (1/2, 0), (1/2, 1/2)\}$ is <u>not</u> a 2-fixed cube since it also contains the special 1-fixed cube $\{(0, 1/2), (1/2, 1/2)\}$. If S_2 were changed to $\{(0, 0), (0, 1/2)\}$ we would have a 2-fixed cube.

<u>Theorem</u> 1: If R is any j-dimensional rectangle the boundary B(R) contains an even number of (j-1) fixed cubes. Proof: Each (j-2) dimensional cube in B(R) is contained in exactly two (j-1) dimensional cubes in B(R). Therefore:

$$\sum_{\substack{Q \text{ is } (j-1) \text{ dim.} \\ \text{cube in } B(R)}} \begin{pmatrix}\text{no. of special } (j-2) \text{ fixed} \\ \text{cubes in } Q\end{pmatrix} = 2\begin{pmatrix}\text{no. of special } (j-2) \text{ -} \\ \text{fixed cubes in } B(R)\end{pmatrix}$$

* A similar definition is used in Wolsey [4].

So the number of odd terms, which correspond to (j-1)-fixed cubes, must be even. Q.E.D.

Corollary: For each $1 \leq i \leq j$ a j-fixed cube contains vertices x_i, y_i such that $x_i \in S_i$ and $y_i \notin S_i$.

Proof: We argue by induction on j. For j=1 this is immediate. For j > 1 a j-fixed cube contains at least one special (j-1)-fixed cube, so we have members and non-members of $S_1, \ldots S_{j-1}$, by induction hypothesis, as well as members of S_j. By Theorem 1, the j-fixed cube contains an even number of (j-1)-fixed cubes, so it must contain at least one (j-1)-fixed cube that is not special. This implies there is a vertex which is not in S_j. Q.E.D.

If $S_i = \{x \in A_k \mid \text{ith component of } x \geq \text{ith component of } f(x)\}$ then an n-fixed cube qualifies, by our corollary, as an approximate fixed point in the sense of the preceding section.

Theorem 2: For any j-dimensional rectangle R the parity of the number of j-fixed cubes in R is the parity of the number of special (j-1)-fixed cubes in B(R).

Proof: Each (j-1)-dim. cube in B(R) is in exactly one j-dim. cube in R. Each (j-1)-dim. cube not in B(R) is in exactly two j-dim. cubes in R. Therefore:

$$\sum_{\substack{Q \text{ is a j-fixed} \\ \text{cube in } R}} \begin{pmatrix}\text{no. of special} \\ \text{(j-1)-fixed cubes in } Q\end{pmatrix} = \begin{pmatrix}\text{no. of special (j-1)-fixed} \\ \text{cubes in } B(R)\end{pmatrix} + 2\begin{pmatrix}\text{no. of special (j-1)-} \\ \text{fixed cubes not in } B(R)\end{pmatrix}$$

Each odd term on the left corresponds to a j-fixed cube. Q.E.D.

The original "cubical Sperner lemma" [2] dealt with the sets S_i described previously. These had the special property that $S_i \supset$

{x|ith component of x = 1} and $S_i \cap$ {x|ith component of x = 0} = $\emptyset$.

Corollary [2, 4]: If the special property on S_i holds, then $[0, 1]^n$ contains an odd number of n-fixed cubes.

Proof: We argue by induction on n. The case n = 1 is immediate. For n > 1 we show that the boundary of $[0, 1]^n$ has an odd number of special (n-1)-fixed cubes and apply theorem 1. A direct consequence of the definitions is that the only special (n-1)-fixed cubes in $B([0, 1]^n)$ are the (n-1)-fixed cubes in {x|nth components of x = 1}. By induction hypothesis there are an odd number of these. Q.E.D.

Applications of Theorem 2 to Fixed-Point Algorithms

Theorem 2 suggests several methods of locating approximate fixed points. We describe two in detail: 1. Bifurcation. This is the method described in the introduction. Starting with $[0, 1]^n$ and the S_i defined previously, we have shown that the boundary of $[0, 1]^n$ has an odd number of special (n-1)-fixed points. Take any (n-1)-dimensional rectangle which divides $[0, 1]^n$ into n-dimensional rectangles R_1, R_2. One of R_1, R_2 will have an odd number of special (n-1)-fixed cubes in its boundary, so theorem 2 says that rectangle has an approximate fixed point. We then divide that rectangle into two rectangles, one of which must contain an odd number of special (n-1)-fixed cubes in its boundary, etc. If at each step we divide the rectangle into halves as nearly equal as possible, we will obtain an n-fixed cube after at most $n(\log_2 k + 1)$ divisions.

Note that finding the parity of the number of special (n-1)-fixed cubes in B(R) never requires evaluation of f in the interior of R. It is not always necessary that f be evaluated at every point of B(R) either. For example, continuity considerations may enable one to locate an (n-1) dimensional rectangle T in B(R) such that $T \cap A_k \subset S_i$ for some $i \leq n-1$.

In this case, T contains no (n-1)-fixed cubes by the corollary to theorem 1. If $T \cap A_k \subset S_n$ then theorem 2 says we need only compute the parity of the number of special (n-2)-fixed cubes in B(T). Undoubtedly other refinements are also possible.

2. "Wandering Willie"* The proof of theorem 2 says things about the number of n-fixed cubes by looking at special (n-1)-fixed cubes and associating two n-dimensional cubes if they have a special (n-1)-fixed cube in common. This suggests that a n-fixed cube may be located by wandering through a sequence of special (n-1)-fixed cubes in some manner. One first has to locate an (n-1)-fixed cube and there are other complications.

For convenience, a point in A_k is a 0-dimensional cube and a special 0-fixed cube is a member of S_1. Also define $D_i = \{x \varepsilon [0, 1] \mid$ last n-i-1 components of x are 1$\}$. Thus $D_i \subset S_j$ for $j \geq i + 2$. $D_{n-1} = D_n = [0, 1]^n$.

The algorithm generates a sequence of cubes C_i of dimension d_i as follows:

Step 1 [initialize]: $C_1 = \{(1,\ldots 1)\}$. $d_1 = 0$. $i = 1$.

Step 2 [wandering attempt]: Let $r = d_i$. Try to find a $(r + 1)$-dimensional cube $E \subset D_r$ which has C_i as a face but such that $C_{i-1} \neq E$ and C_{i-1} is not a r-dimensional face of E. If there is no such E go to step 5, otherwise go to step 3.

Step 3 [successful wander]: Try to find a special r-fixed face of E that is not among $C_1,\ldots,C_i$. If there is no such face go to step 4. Otherwise let C_{i+1}=any such face. d_{i+1}=r. Increase i by 1 and return to step 2.

Step 4 [discovery of special $(r + 1)$-fixed cube]: Let C_{i+1}= E. d_{i+1} = r+1. If d_{i+1}=n stop. Otherwise increase i by 1 and return to step 2.

*A similar algorithm has been developed by Wolsey [4].

Step 5 [smashing into the boundary of D_r]: Try to locate a special $(r-1)$-fixed face of C_i that is not among $C_1, \ldots C_i$. If there is none, stop. Otherwise let C_{i+1} = any such face. $d_{i+1} = r-1$. Increase i by 1 and return to step 2.

Example: We illustrate "Wandering Willie" for n = 2, k = 3. The points of A_k will be abbreviated by capital letters as shown in the diagram:

A	B	C	D
E	F	G	H
J	K	L	M
N	P	Q	R

Thus D corresponds to (1, 1), N corresponds to (0, 0), L corresponds to (2/3, 1/3) and so forth. Let f be such that S_1 = {B, D, F, H, L, M, R} and S_2 = {A, B, C, D, E, F, G, H, L}. Then the sequence of cubes C_i generated by "Wandering Willie" is:

D(1); CD(4); GH(3); GL(3); FG(3); BC(3); B(5); AB(4); EF(3); EFJK(4). EFJK is a 2-fixed cube.

Here we indicate a cube by its vertices and the numbers in parentheses refer to the steps generating the cubes.

Theorem 3: Wandering Willie halts after a finite number of steps at a n-fixed cube.

Proof (outline): First one shows by induction on L that if $C_1, \ldots C_L$ are the first L cubes generated by W. W. then (a) $C_1, \ldots C_L$ are all different (b) for $1 \leq i \leq L$ C_i is a special d_i-fixed cube contained in D_{d_i}. (c) if E is a j-dimensional cube in D_j which does not have C_L as a (j-1)-dimensional face, then $C_1, \ldots C_L$ contains an odd number of (j-1)-dimensional faces of E iff. $E = C_i$ and either C_i was introduced by step 4 or C_{i+1} was introduced at step 5. The induction involves many individually

simple cases and we omit the details.

Once (a), (b), (c) are established it follows that w. w. can never halt in step 5 (we only included this possibility for expository reasons). If C_L is such that step 2 fails when $i = L$, C_L cannot have been introduced by step 4. So, by (c), an even number of special $(d_L - 1)$-fixed faces of C_L appear in $C_1, \ldots C_L$. By (b) C_L has an odd number of special $(d_L - 1)$-fixed faces, so there must be at least one fresh face waiting at step 5.

Since w. w. must halt eventually, by (a), and it does not halt in step 5 it must wander into an n-fixed cube. Q.E.D.

We believe that the "worst case" behavior of bifurcation is superior to Wandering Willie in this sense: For any f, bifurcation (where we divide the cube into nearly equal halves at each stage) will locate a n-fixed cube after evaluating f at $\leq (1 + 4/(1 - (1/2)^{n-1}) \ (k + 1)^{n-1}$* of the $(k + 1)^n$ points of A_k. We conjecture that there is a $0 < \theta < 1$ independent of k such that functions f can be constructed for which Wandering Willie (or similar algorithms on simplexes) will evaluate f at $\theta(k + 1)^n$ points before finding an approximate fixed point. [specifically, we conjecture that we may take $\theta \geq (1/3)^n$].

However, we believe the "typical case" behavior of Wandering Willie will be reasonable. It is clear that unrefined bifurcation is not practical, since even a single division requires computation of f at $(k + 1)^{n-1}$ points.

Hybrids are also a possibility. One idea would be a bifurcation search on A_k for small k. This would locate a "large" n-fixed cube J. We could then introduce a finer rectangular subdivision only on J. If fortunate, B(J) would have an odd number of special (n-1)-fixed points with respect to the fine subdivision. If this happens, bifurcation can

*This figure can be sharpened.

be performed on J. If unfortunate, we would have to hunt for another large cube (which we hope is close to J) suitable for a second bifurcation.

Another possible algorithm would add (n-1) dimensional dividing planes on $[0, 1]^n$ one at a time, presumably having the next dividing plane pass through the n-fixed cube* currently located. This would be in the spirit of the "eccentric barycentric" adaptive algorithm of Zangwill [5]. Both of the hybrids sacrifice the "worst case" bound of pure bifurcation.

Concluding Remarks

A bifurcation algorithm is possible on a simplex. The result analogous to theorem 2 is:

Theorem 4: Let a subdivided n-dimensional simplex be arbitrarily labelled with {1, 2,...n+1}. The parity of the number of completely labelled subsimplices is the parity of the number of (n-1)-dimensional simplices in the boundary which have labels {1, 2,...,n}. Proof as for theorems 1 and 2. However, the subdivision has to be chosen rather carefully.

The similarity between the proofs of theorems 1, 2, and 4 and the "standard" proofs of Sperner's Lemma in [1] and [3] should be mentioned. It must be grudgingly admitted that the definition of n-fixed cubes was chosen to make these proofs work.

All the theorems and algorithms discussed here depended essentially on an arbitrary ordering of the co-ordinates of the cube or, equivalently, on a specific order for $S_1, \ldots S_n$. This is also the case for many of the algorithms in the literature, and we believe this may be more than an aesthetic defect.

*In this case, we are relaxing the requirement that the edges of the cube are all of equal length.

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