Arch and Bilinearity as Competing Models for Nonlinear Dependence

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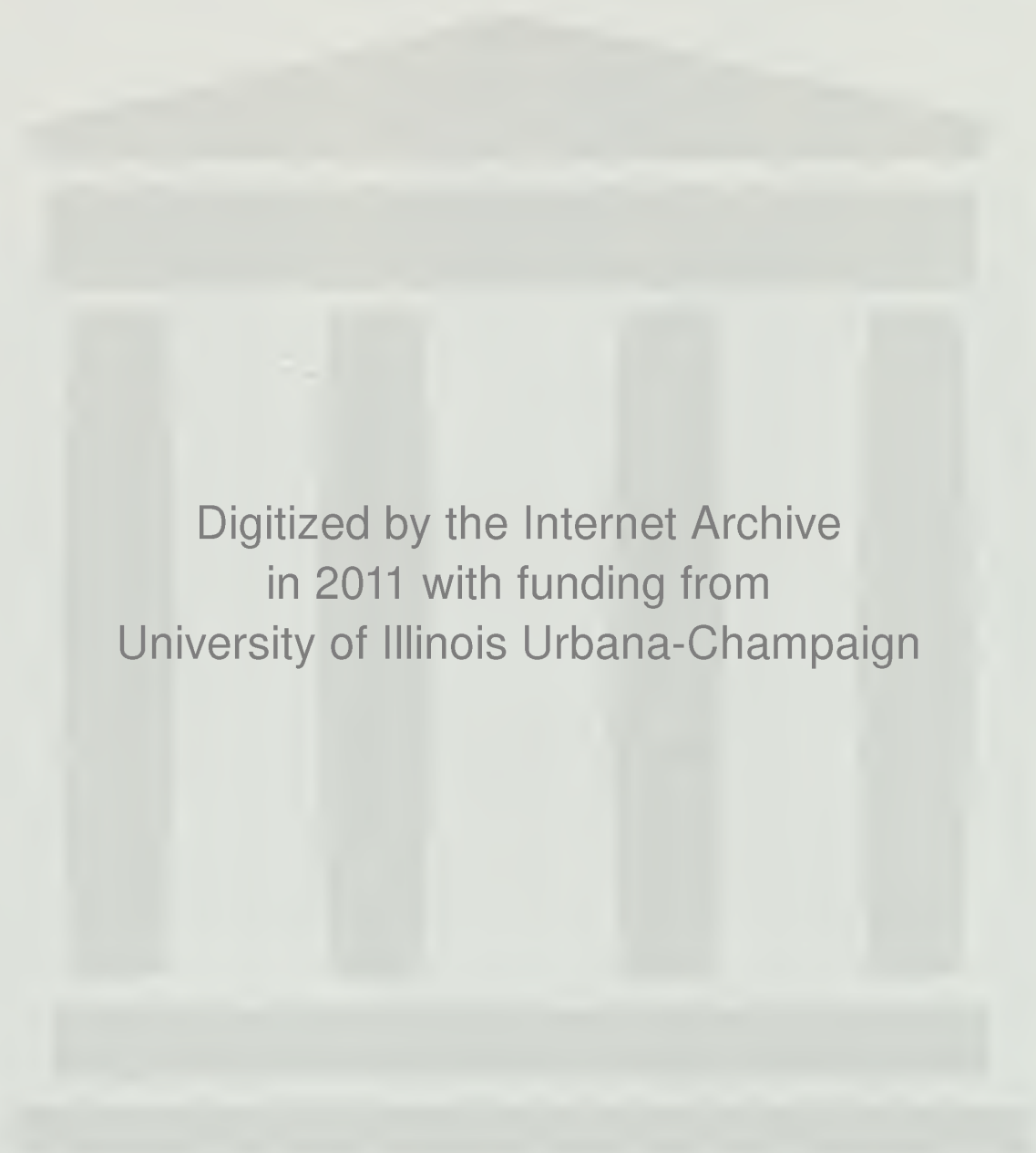College of Commerce and Business Administration

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SUMMARY

In this paper we consider whether the wide acceptance of the ARCH process may be at the expense of other nonlinear processes which can offer an improvement in the predictive power of econometric models. We first propose a test which should have good power against the simultaneous presence of ARCH and bilinearity. A non-nested test is then suggested to determine whether nonlinear dependence should be attributed to ARCH or bilinearity. The tests are applied to three U.S. spot exchange rates. Results indicate that nonlinear dependence may not be fully attributable to ARCH.

1. INTRODUCTION

Of all nonlinear stochastic processes, the ARCH model of Engle (1982), and its generalization to GARCH by Bollerslev (1986), has been the most widely applied in economics. The ARCH model provides a parsimonious and easily interpreted parameterization of the conditional variance of an economic time series. It is ideally suited for modeling changes in volatility that are so prevalent in financial and monetary variables. The ARCH model has also lead to an enormous research effort in detecting and interpreting nonlinear dependence in economic data. To comprehend the size of this literature, the reader needs only to consult the recent survey papers by Bollerslev, Chou and Kroner (1992) and Bera and Higgins (1992).

The apparent success of the ARCH model, however, suggests that the incorporation of nonlinear dynamics may not increase the explanatory power and forecasting performance of econometric models. In an ARCH model, past observations only provide information about future variances of the process, not future means. Hence, the existence of ARCH cannot increase the predictive power of an econometric model. The ARCH literature indicates that once ARCH effects are accounted for, little in the way of nonlinearity remains in economic data. A study which exemplifies this conclusion is Hsieh (1989). Hsieh sought to determine whether nonlinear dependence exist in daily exchange rates, and if so, to determine the appropriate model for this nonlinearity. Working with the log first differences of the daily U.S. spot rate against five major currencies, he applied a battery of tests for

nonlinearity including Brock, Dechert and Scheinkman's (1987) correlation dimension test, Tsay's (1986) test, McLeod and Li's (1978) portmanteau test and Engle's (1982) LM test for ARCH. All tests convey strong evidence of nonlinearity. Hsieh then fitted GARCH models to each of the series and applied the same tests to the standardized residuals. He found that the standardized residuals displayed little nonlinear dependence, and hence, concluded that most of the nonlinearity in exchange rates can be accounted for by GARCH.

Looking for nonlinear dynamics in the conditional mean of a series by examining the residuals of a fitted linear model with GARCH errors, however, will tend to favor the conclusion that GARCH accounts for the nonlinearity in the data. Granger (1991, p. 142) emphasized this point and stressed the need to place nonlinearity in the conditional mean on equal footing with nonlinearity in the conditional variance. Despite the wide acceptance of ARCH, a few authors have considered whether the ARCH phenomenon may be largely an artifact of misspecified nonlinear dynamics in the conditional mean of a model. For example, Pagan and Hong (1991) re-examined Engle, Lillien and Robins' (1987) ARCH in the mean (ARCH-M) model for the yield on three month Treasury bills and Hodrick's (1987) ARCH model for the index of industrial production. Pagan and Hong estimated the conditional means of the models using non-parametric techniques and subsequently found that the evidence for the presence of ARCH is greatly reduced.

In this paper, we pursue the question of whether a process with a nonlinear conditional mean can provide an alternative to ARCH for modeling

nonlinear dependence in economic time series. As an alternative to ARCH, we focus on the bilinear process of Granger and Andersen (1978). We chose bilinearity because it has an unconditional moment structure very similar to ARCH, and hence, may be easily mistaken for ARCH [see the discussion in Bera and Higgins (1992)]. Yet, a bilinear process is forecastable, and therefore its detection may increase the predictive power of a model. Weiss (1986) noted the similarity between ARCH and bilinear. He treated them as complementary processes and built simultaneous ARCH and bilinear models. We will treat ARCH and bilinearity as competing models for nonlinear dependence. We directly test ARCH models against bilinear models, and vice-versus, using a stochasticly simulated Cox test as suggested by Pesaran and Pesaran (1992).

The outline of the paper is as follows. In section 2, we compare and contrast the conditional and unconditional moment structures of ARCH and bilinear models. In section 3, we propose a test which should have good power against both ARCH and bilinear alternatives. We summarize in section 4 the Pesaran and Pesaran (1992) procedure for the stochasticly simulated Cox test and describe how it can be implemented for conducting a non-nested test of ARCH against bilinearity and vice-versa. To determine if bilinearity might provide an alternative representation of nonlinearity, in section 5 we apply the above tests to three Dollar spot exchange rates. Concluding remarks are presented in section 6.

2. MOMENT STRUCTURES OF ARCH AND BILINEARITY

ARCH and bilinear processes can be defined in terms of univariate or multivariate time series or in terms of the innovations of single or multiple equation structural models. In this paper we characterize them in the context of the errors of a dynamic single equation model. The dependent variable y_t is assumed to be generated by

$$y_t = x'_t \beta + u_t, \quad t = 1, \ldots, T, \tag{2.1}$$

where x_t is a $k \times 1$ vector of exogenous variables and lagged values of the dependent variable, β is a $k \times 1$ vector of parameters and u_t is a stochastic equation error. Let Φ_t denote the information set at time t. Then u_t is said to be generated by a GARCH(p,q) process if

$$u_t | \Phi_{t-1} \sim N(0, h_t), \tag{2.2}$$

where

$$h_t = \alpha_0 + \alpha_1 u^2_{t-1} + \ldots + \alpha_q u^2_{t-q} + \beta_1 h_{t-1} + \ldots + \beta_p h_{t-p}. \tag{2.3}$$

A bilinear process for u_t is defined by the stochastic difference equation

$$u_t = \sum_{i=1}^{r} \sum_{j=1}^{s} b_{ij} u_{t-i} \epsilon_{t-j} + \epsilon_t, \tag{2.4}$$

where b_{ij} $i = 1, \ldots, r$, $j = 1, \ldots, s$, are parameters and ϵ_t is an i.i.d.

$N(0,\sigma_\epsilon^2)$ innovation which drives the bilinear process. The bilinear process can be justified as a second-order approximation of the Wold representation of a stationary process.

A fundamental qualitative distinction between a GARCH process and a bilinear process is evident when one considers the conditional distribution of the dependent variable y_t under the two specifications. For the ARCH model, with the conditional distribution of the error specified as (2.2), it is immediate that

$$E(y_t|\Phi_{t-1}) = x_t'\beta$$

and

$$Var(y_t|\Phi_{t-1}) = h_t$$

In the presence of GARCH the conditional variance becomes a function of the information set, however, the conditional mean of the dependent variable is the same as it would be if the errors where Gaussian white noise. Hence, the presence of GARCH does not provide additional forecastablity of the dependent variable. When the equation error u_t is generated by the bilinear process (2.4), the conditional mean and the conditional variance are

$$E(y_t|\Phi_{t-1}) = x_t'\beta + \sum_{i=1}^{r} \sum_{j=1}^{s} b_{ij} u_{t-i} \epsilon_{t-j}$$

and

$$Var(y_t|\Phi_{t-1}) = \sigma_\epsilon^2.$$

In the presence of bilinearity, while the conditional variance is constant, the conditional mean of the dependent variable is augmented with interaction terms between past equation errors and the innovations. Incorporating these nonlinearities can increase the predictability of the dependent variable.

Despite this fundamental difference between the conditional moments of GARCH and bilinear processes, the unconditional moments of the two processes can be very similar. This is important because specification tests for these processes and the identification of the orders of these processes is based on their unconditional moment structures. As a result, it is easy to mistake one process for the other.

To illustrate the similarity of the unconditional moments, we consider the autocorrelations of the errors and the squared errors. First assume that u_t is generated by an GARCH process (2.2). Since the conditional mean of u_t is zero for any realization of Φ_{t-1}, using the law of iterated expectations it is immediate that

$$E(u_t u_{t-m}) = E[E(u_t u_{t-m}|\Phi_{t-1})] = 0$$

for all $m \geq 1$. Hence, the GARCH errors are serially uncorrelated. The dependence in GARCH errors is revealed through higher order moments. To derive the form of the autocorrelation function of the squared errors u_t^2, it is convenient to represent a GARCH process by the generating equation

$$u_t = z_t \cdot h_t^{1/2} \tag{2.5}$$

where z_t is i.i.d. $N(0,1)$ and h_t is the same as (2.3). As shown by Bollerslev (1986), squaring both sides of (2.5), produces a generating equation for u_t^2 of

$$u_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j u_{t-j}^2 - \sum_{j=1}^{p} \beta_j \nu_{t-j} + \nu_t,$$

where $\nu_t = (z_t^2 - 1)h_t$ is a serially uncorrelated mean zero disturbance term. Therefore, the squared error u_t^2 is generated by an ARMA(r,q) process, with $r = \max(p,q)$, and its autocorrelation function will have the familiar form. For comparison, now assume that u_t is generated by the bilinear process (2.4). For the general bilinear process, it is difficult to derive expressions for the autocorrelations of u_t and u_t^2. However, when only a single bilinear term is allowed, expression for these autocorrelations can be obtained. Unlike the general GARCH(p,q) process, the general bilinear process is not serially uncorrelated. An important sub-class of the bilinear process, however, has a moment structure similar to a GARCH process. Any single-term bilinear model of the form

$$u_t = b_{ij} u_{t-i} \epsilon_{t-j} + \epsilon_t, \tag{2.6}$$

where $i > j$, has conditional mean zero since

$$E(u_t) = E[E(u_t | \Phi_{t-i})]$$

$$= E[b_{ij}u_{t-i}E(\epsilon_{t-j}|\Phi_{t-i}) + E(\epsilon_t|\Phi_{t-i})]$$

$$= 0.$$

In this class of bilinear models, where the conditional mean is zero for all realizations of Φ_{t-i}, the autocorrelations of u_t are zero since

$$\rho_m = E(u_t u_{t-m}) = E[E(u_t u_{t-m}|\Phi_{t-1})] = 0$$

for all $m \geq 1$. Also like the GARCH(p,q) process, the autocorrelations of the squares of the process (2.6) are non-zero. Let ρ_m^2 be the m'th order autocovariance of u_t^2. Granger and Andersen (1978, p. 45) showed that

$$\rho_m^2 \neq 0 \quad \text{for } m \leq j$$

and

$$\rho_m^2 = (b_{ij}^2 \sigma_\epsilon^2)\rho_{m-i}^2 \quad \text{for } m > j.$$

Hence the autocorrelation function of u_t^2 has the form of the autocorrelation function of an ARMA(i,j) process. Since the process is autocorrelated in the squares, it will also display the temporal clustering of large and small deviations of the GARCH model.

The fact that both ARCH and certain bilinear processes are serially uncorrelated in the levels but display autocorrelation in the squares, indicates that one process may easily be mistaken for the other. Granger and Andersen (1978) recommended that bilinearity be detected by looking for

significant autocorrelations in the squares of least squares residuals. Engle (1982) recommended that ARCH be detected by means of a Lagrange multiplier (LM) test and showed that the LM test for ARCH can be computed as $T \cdot R^2$ where R^2 is the coefficient of multiple correlation from the artificial regression of $\tilde{u}_t^2$ on an intercept and $\tilde{u}_{t-1}^2$, ..., $\tilde{u}_{t-q}^2$, $\tilde{u}_t$ being the least squares residuals from (2.1). As pointed out by Luukkonen, Saikkonen and Teräsvirta (1988), this statistic is asymptotically equivalent to the McLeod and Li (1978) portmanteau test using squared residuals. Therefore, finding autocorrelations in the square OLS residuals may indicate nonlinearity, but it is not constructive as to what the alternative nonlinear model should be.

3. A JOINT REGRESSION TEST FOR ARCH AND BILINEARITY

Since we are interested in the presence of GARCH or bilinearity, prior to considering testing one specification against another, we will derive a test that should have good power against both of these alternatives when the null hypothesis is Gaussian white noise. Higgins and Bera (1989) showed that a joint test for ARCH and bilinearity could be constructed by adding the standard LM tests for each individual alternative. In this section, we propose a similar, but computationally easier test which can be computed as the sum of two F-tests of the significance of regressors in a single artificial linear model.

The test is based on an alternative model which incorporates both ARCH and bilinearity. We consider the dynamic linear regression model

$$y_t = x_t'\beta + u_t \tag{3.1}$$

in which the equation error u_t is generated by the bilinear process

$$u_t = \sum_{i=1}^{r} \sum_{j=1}^{s} b_{ij} u_{t-i} \epsilon_{t-j} + \epsilon_t, \tag{3.2}$$

where $b_{ij} = 0$ for $i < j$, and the innovations to the bilinear process ϵ_t are generated by the ARCH(q) process

$$\epsilon_t | \Phi_{t-1} \sim N(0, h_t) \tag{3.3}$$

where

$$h_t = \sigma^2 + \alpha_1 \epsilon_{t-1}^2 + \ldots + \alpha_q \epsilon_{t-q}^2.$$

In the specification of the bilinear process (3.2), we impose $b_{ij} = 0$ for $i < j$ because a LM test cannot be constructed against an alternative which includes both $u_{t-i} \epsilon_{t-j}$ and $u_{t-j} \epsilon_{t-i}$ [see Saikkonen and Luukkonen (1988)]. Similarly, for the conditional variance we consider ARCH rather than GARCH since a LM test can not be constructed which tests simultaneously for lagged ϵ_t's and lagged h_t's [see Bollerslev (1986)]. Let $m = \min(r,s)$ and $d = r \cdot s - m(m - 1)/2$. Then let $b = (b_{11}, \ldots, b_{rs})'$ denote the $d \times 1$ vector of bilinear parameters and let $\alpha = (\alpha_1, \ldots, \alpha_q)'$ denote the $q \times 1$ vector of ARCH parameters. We consider the joint null hypotheses $H_A: \alpha = 0$ and $H_B: b = 0$. When H_A is true the model reduces to a linear regression model with bilinear

errors and when H_B is true the model reduces to a linear model with ARCH

errors. When both H_A and H_B are true, the model becomes the standard linear

model.

We derive the computationally convenient test for the joint hypothesis

by employing the concept of a locally equivalent alternative (LEA) model

suggested by Godfrey and Wickens (1982). Consider the artificial linear

model

$$y_t = x_t'\beta + \sum_{i=1}^{q} \alpha_i \tilde{\epsilon}_t (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2) + \sum_{i=1}^{r} \sum_{j=1}^{s} b_{ij} \tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j} + \epsilon_t \qquad (3.4)$$

where $\tilde{\epsilon}_t$ is the least squares residual of (3.1), $\tilde{\sigma}^2 = \frac{1}{T} \sum_t \tilde{\epsilon}_t^2$ and $b_{ij} = 0$ for

$i < j$. Note that under the joint null hypothesis $u_t = \epsilon_t$. Let $\theta = (\theta_1', \theta_2')'$,

where $\theta_1 = (\beta', \sigma^2)'$ and $\theta_2 = (\alpha', b')'$. Let $\ell(\theta)$ denote the log-likelihood

for model (3.1)-(3.3) and let $\ell^*(\theta)$ denote the log-likelihood for the model

(3.4). Model (3.4) is said to be a LEA of model (3.1)-(3.3) if under H_A and

H_B

(i) the models are identical

(ii) $\dfrac{\partial \ell_t}{\partial \theta_2} = \dfrac{\partial \ell_t^*}{\partial \theta_2}$.

Condition (i) requires that both models are equivalent under the joint null

hypothesis. Condition (ii) ensures that the LM test for the null hypothesis

based on the LEA model is asymptotically equivalent to the LM test based on

the original model. It is obvious that (i) is true. To demonstrate that

(ii) is true, we follow Godfrey and Wickens (1982, p. 85) and first show that the Jacobian term in the log-likelihood of (3.4) can be neglected for local alternatives. Denote

$$f_t^+ = y_t - x_t'\beta - \sum_{i=1}^{q} \alpha_i \tilde{\epsilon}_t (\tilde{\epsilon}_{t-i} - \tilde{\sigma}^2) - \sum_{i=1}^{r} \sum_{j=1}^{s} b_{ij} \tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j}.$$

The Jacobian term in $\ell^*(\theta)$ is

$$\sum_{t=1}^{T} \log \left| \frac{\partial f_t^+}{\partial y_t} \right| = \sum_{t=1}^{T} \log \left(1 - \sum_{i=1}^{q} \alpha_i (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2) \right)$$

$$\simeq - \sum_{t=1}^{T} \sum_{i=1}^{q} \alpha_i (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)$$

$$= - \sum_{i=1}^{q} \alpha_i \sum_{t=1}^{T} (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)$$

$$= 0,$$

where the approximate equality follows from $\log(1 - x) \simeq -x$ for small x. Then

$$\frac{\partial \ell_t}{\partial \alpha_i} = \frac{\epsilon_{t-i}^2}{2\sigma^2} \left(\frac{\epsilon_t^2}{\sigma^2} - 1 \right) = \frac{\epsilon_{t-i}^2 (\epsilon_t^2 - \sigma^2)^2}{2\sigma^4}$$

and it is easily shown that

$$\frac{\partial \ell_t^*}{\partial \alpha_i} = \frac{\epsilon_t \tilde{\epsilon}_t (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)}{\sigma^2}.$$

Therefore, the LM for H_A in model (3.1)-(3.3) is essentially based on the quantity

$$\sum_{t=1}^{T} \epsilon_{t-i}^2 (\epsilon_t^2 - \sigma^2)$$

and the LM test for H_A in model (3.4) utilizes

$$\sum_{t=1}^{T} \epsilon_t \tilde{\epsilon}_t (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2).$$

However, since $\tilde{\epsilon}_t \xrightarrow{p} \epsilon_t$ these two test must be asymptotically equivalent. In a similar fashion, one can show that the LM tests for H_B using models (3.1)-(3.3) and (3.4) are asymptotically equivalent. Therefore model (3.4) represents a LEA for model (3.1)-(3.3).

We´ can now construct a computationally easy test for ARCH and bilinearity by testing the joint hypotheses $H_A:\alpha = 0$ and $H_B:b = 0$ in the LEA model (3.4). One approach is to consider the standard LM test which can be computed as $T \cdot R^2$ from the regression of a vector of ones on the elements of $\partial \ell_t^* / \partial \theta$. From (3.4), these elements are seen to be

$$\frac{\partial \ell_t^*}{\partial \beta} = \frac{1}{\tilde{\sigma}^2} x_t' \tilde{\epsilon}_t$$

$$\frac{\partial \ell_t^*}{\partial \sigma^2} = \frac{\tilde{\epsilon}_t^2 - \tilde{\sigma}^2}{2\tilde{\sigma}^4}$$

$$\frac{\partial \ell_t^*}{\partial b_{ij}} = \frac{1}{\tilde{\sigma}^2} \tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j} \quad \text{for } i = 1,\ldots,r, \; j = 1,\ldots,s, \; i \geq j$$

$$\frac{\partial \ell_t^*}{\partial \alpha_i} = -(\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2) + \frac{\tilde{\epsilon}_t^2(\tilde{\epsilon}_{t-i}^2 - \sigma^2)}{\tilde{\sigma}^2} \qquad \text{for } i = 1, \ldots, q.$$

Note the first term in $\partial \ell_t^*/\partial \alpha_i$ comes from the Jacobian. Recall that $\sum_t \log|\partial f_t^+/\partial y_t|$ vanishes, however, the contribution of the Jacobian to the score function for each t does not vanish.

A second approach is to estimate (3.4) by least squares and then compute two separate F-tests for the hypotheses $H_A: \alpha = 0$ and $H_B: b = 0$. From the additivity result of Higgins and Bera (1989), the two sets of regressors $\tilde{\epsilon}_t(\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)$ and $\tilde{\epsilon}_{t-i}\tilde{\epsilon}_{t-j}$ are asymptotically orthogonal. Therefore a joint test for ARCH and bilinearity can be obtained by just adding these two F-statistics.

As pointed out by Godfrey and Wickens (1982, p. 86), a slight complication arises with the second approach. For the q regressors $\tilde{\epsilon}_t(\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)$, the standard least squares formula will not produce the correct covariance matrix. To see this, note that for the normalized inner product of these regressors with the equation error

$$T^{-1/2} \sum_{t=1}^{T} \epsilon_t \tilde{\epsilon}_t(\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2) \overset{a}{=} T^{-1/2} \sum_{t=1}^{T} (\epsilon_t^2 - \sigma^2)(\epsilon_{t-i}^2 - \sigma^2),$$

where "$\overset{a}{=}$" denotes asymptotically equivalent. Since the sequences $\epsilon_t^2 - \sigma^2$ and $\epsilon_{t-i}^2 - \sigma^2$ are i.i.d. $(0, 2\sigma^4)$, we have that

$$T^{-1/2} \sum_{t=1}^{T} \epsilon_t \tilde{\epsilon}_t(\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2) \overset{\mathcal{D}}{\longrightarrow} N(0, 4\sigma^8).$$

The OLS estimate of the variance, however, is

$$\sigma^2 \frac{1}{T} \sum_{t=1}^{T} \tilde{\epsilon}_t^2 (\tilde{\epsilon}_{t-i}^2 - \tilde{\sigma}^2)^2 \overset{a}{=} 2\frac{\sigma^6}{T} \sum_{t=1}^{T} \epsilon_t^2 \overset{p}{\longrightarrow} 2\sigma^8,$$

which omits the factor 2 from the true variance. This can be corrected by appropriately normalizing the F-statistic. The other d regressors $\tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j}$ present no problem. The normalized inner product of these regressors with the equation error is

$$T^{-1/2} \sum_{t=1}^{T} \epsilon_t \tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j} \overset{a}{=} T^{-1/2} \sum_{t=1}^{T} \epsilon_t \epsilon_{t-i} \epsilon_{t-j}$$

and since the sequence $\epsilon_t \epsilon_{t-i} \epsilon_{t-j}$ is i.i.d. $(0,\sigma^6)$, we have that

$$T^{-1/2} \sum_{t=1}^{T} \epsilon_t \tilde{\epsilon}_{t-i} \tilde{\epsilon}_{t-j} \overset{\mathcal{D}}{\longrightarrow} N(0,\sigma^6).$$

Least squares will compute the variance as

$$\frac{\sigma^2}{T} \sum_{t=1}^{T} \tilde{\epsilon}_{t-i}^2 \tilde{\epsilon}_{t-j}^2 \overset{a}{=} \frac{\sigma^2}{T} \sum_{t=1}^{T} \epsilon_{t-i}^2 \epsilon_{t-j}^2 \overset{p}{\longrightarrow} \sigma^6,$$

and hence the problem does not arise. If we let F_1 denote the F-statistic for $\alpha = 0$ and F_2 denote the F-statistic for $b = 0$ in (3.4), a joint test can be based on the statistic

$$\frac{q}{2} \cdot F_1 + d \cdot F_2 \overset{\mathcal{D}}{\longrightarrow} \chi^2_{q+d}.$$

This joint test for ARCH and bilinearity is very easy to compute. It requires estimating (3.1) by least squares and then running the artificial regression (3.4). The joint test can then be constructed with standard output from the the artificial regression.

4. NON-NESTED TEST BETWEEN ARCH AND BILINEARITY

The above test is designed to have power against both GARCH and bilinearity. Therefore, if a joint null hypothesis is rejected, it does not provide any guidance as to whether the nonlinearity should be attributed to GARCH or bilinearity. The challenging problem is to directly test one nonlinear specification against another. Since neither the GARCH nor the bilinear process is a special case of the other, such a test must be based on a non-nested testing principle. The highly nonlinear nature of the likelihood functions of the GARCH and bilinear processes, however, make it difficult to analytically derive standard non-nested tests. Recently, Pesaran and Pesaran (1992) have suggested a simulation method for implementing the Cox test which circumvents these analytic difficulties. In this section, we describe how their simulation technique can be employed to test GARCH and bilinearity against one another.

Pesaran and Pesaran (1992) considered two hypotheses which specify the conditional distribution of the dependent variable y_t as

$$H_f: \quad f(y_t, \theta \mid x_t)$$

$$H_g: \quad g(y_t, \gamma \mid z_t)$$

where x_t and z_t are conditioning variables which may be exogenous or predetermined, and θ and γ are unknown parameters. The conditional densities in H_f and H_g are assumed to be non-nested in the sense that one can not be obtained as a special case or as an approximation of the other. The standard Cox (1961, 1962) test for such non-nested hypotheses, assuming H_f is to be tested against H_g, is

$$T_f(\hat{\theta}, \hat{\gamma}) = \bar{d} - \hat{E}_f(\bar{d}),$$

where $\bar{d} = T^{-1} \Sigma_{t=1}^{T} d_t$, with $d_t = \log f(y_t, \hat{\theta} \mid x_t) - \log g(y_t, \hat{\gamma} \mid z_t)$, is the average maximized log-likelihood ratio and $\hat{\theta}$ and $\hat{\gamma}$ denote the MLE's of θ and γ. The term $\hat{E}_f(\bar{d})$ denotes a consistent estimate of the expected value of $\bar{d}$ assuming H_f is true. To implement the Cox test, $\bar{d}$ can be obtained by numerically maximizing the likelihood function under H_f and H_g. The term $\hat{E}_f(\bar{d})$ is usually taken to be the Kullback-Leibler information measure

$$C(\theta_0) = E_f \left[\log f(y, \theta_0) - \log g(y, \gamma_*(\theta_0)) \right]$$

evaluated at $\hat{\theta}$, where θ_0 is the true value of θ and $\gamma_*(\theta_0) = \text{plim}_f \hat{\gamma}$. In simple models, such as non-nested linear regression models, both $C(\theta_0)$ and $\gamma_*(\theta_0)$ can be obtained analytically. In more complex models, such as ours, this is usually not the case.

Pesaran and Pesaran (1992), proposed a method for evaluating the plim and expectation in $\gamma_*(\hat{\theta})$ and $C(\hat{\theta})$ by simulation. The technique is based on generating R artificial samples $Y_j = (y_{1j}, \ldots, y_{Tj})'$, $j = 1, \ldots, R$ from the density $f(y, \hat{\theta}|x_t)$. A consistent estimator of $\gamma_*(\hat{\theta})$ is then $\hat{\gamma}_*(R) = R^{-1}\Sigma_{j=1}^R \hat{\gamma}_j$, where $\hat{\gamma}_j$ is the MLE of γ using the j'th artificial sample Y_j. Similarly, a consistent estimator of $C(\hat{\theta})$ is

$$C_R(\hat{\theta}) = \frac{1}{R} \sum_{j=1}^R \left[L_f(Y_j, \hat{\theta}) - L_g(Y_j, \hat{\gamma}_*(R)) \right]$$

where $L_f(Y_j, \hat{\theta}) = T^{-1}\Sigma_{t=1}^T \log f(y_{tj}, \hat{\theta})$ and $L_g(Y_j, \hat{\gamma}_*(R)) = T^{-1}\Sigma_{t=1}^T \log g(y_{tj}, \hat{\gamma}_*(R))$. The consistency of these two estimators follows immediately from the law of large numbers since the sample Y_j are independently generated. In an empirical application of testing a logit and probit against on another, Pesaran and Pesaran (1992) found that $C_R(\hat{\theta})$ and $\hat{\gamma}_*(R)$ converge very rapidly. Beyond $R = 100$, the estimates change very little.

To implement the test, one needs a consistent estimate of the asymptotic variance of $T^{1/2} \cdot T_f(\hat{\theta}, \hat{\gamma})$. Pesaran and Pesaran (1992) demonstrated that

$$\hat{v}_f^2 = \frac{1}{T} \sum_{t=1}^T (d_t - \bar{d})^2 - \frac{1}{T}\left(\sum_{t=1}^T d_t \frac{\partial \log f(y_t, \hat{\theta})}{\partial \theta} \right)'$$

$$\times \left(\sum_{t=1}^T \frac{\partial \log f(y_t, \hat{\theta})}{\partial \theta} \frac{\partial \log f(y_t, \hat{\theta})}{\partial \theta'} \right)^{-1} \left(\sum_{t=1}^T d_t \frac{\partial \log f(y_t, \hat{\theta})}{\partial \theta} \right) \qquad (4.1)$$

is a consistent estimator of the asymptotic variance which can be computed as

the estimated standard error from the regression of d_t on an intercept and $\partial \log f(y_t, \hat{\theta})/\partial \theta$, not corrected for degrees of freedom. Pesaran and Pesaran then showed that

$$T^{1/2} (\bar{d} - C_R(\hat{\theta}))/\hat{v}_f \xrightarrow{\mathcal{D}} N(0,1),$$
(4.2)

under appropriate regularity conditions.

Once $C_R(\hat{\theta})$ has been computed, the statistic (4.2) can be readily computed from an artificial regression without explicitly computing the variance from (4.1) This can be done through a moment test interpretation of the Cox test as suggested by Tauchen (1985). The Cox test is based on the criterion function $c(y_t, \theta) = d_t - E_f(d_t)$, for which clearly $E_f[c(y_t, \theta)] = 0$. The moment $E_f(d_t)$ is estimated under the null model by $C_R(\hat{\theta})$. The test is then conducted by testing the significance of $\bar{m} = \frac{1}{T} \Sigma_{t=1}^{T} m_t$, where $m_t = d_t - C_R(\hat{\theta})$. Now let $m' = (m_1, \ldots, m_T)$, $f_t = \partial \log f(y_t, \hat{\theta})/\partial \theta$, $F' = (f_1, \ldots, f_T)$ and $i' = (1, \ldots, 1)$. Following Tauchen (1985), consider the t-statistic of β_0 in the artificial regression

$$m = i\beta_0 + F\beta_1 + \eta.$$
(4.3)

Using the partition inverse formula, and noting $F'i = 0$ and $\bar{m} = \bar{d} - C_R(\hat{\theta})$, we have

$$t_{\hat{\beta}_0} = \frac{i'M_F m}{\sigma_\eta (i'M_F i)^{1/2}}$$

$$= \sqrt{T} \ \frac{\bar{m}}{\hat{\sigma}_\eta}$$

$$= \sqrt{T} \ \frac{\bar{d} - C_R(\hat{\theta})}{\hat{\sigma}_\eta}.$$

where $M_F = I - F(F'F)^{-1}F'$ and $\hat{\sigma}_\eta$ is the standard error of the regression (4.3). But

$$\hat{\sigma}_\eta^2 = \frac{1}{T} \ m' \left(I - [\iota \ \ F] \begin{bmatrix} m'm & 0 \\ 0 & F'F \end{bmatrix}^{-1} \begin{bmatrix} \iota' \\ F' \end{bmatrix} \right) m$$

$$= \frac{1}{T} \sum_{t=1}^{T} (m_t - \bar{m})^2 - \frac{1}{T} \ m' F(F'F)^{-1}F'm$$

$$= \frac{1}{T} \sum_{t=1}^{T} (d_t - \bar{d})^2 - \frac{1}{T} \ d'F(F'F)^{-1}F'd,$$

which is exactly the the variance estimator $\hat{v}_f$ in (4.1). Therefore, the t-statistic of β_0 in (4.3) is identical to the statistic (4.2). This provides a convenient method for computing the test. Although computationally intensive, it is straight forward to implement this procedure to test GARCH and bilinearity against each other. In the following section, we conduct the tests for three Dollar spot exchange rates.

5. Nonlinear Dependence in U.S. Exchange Rates

One of the most successful areas in which ARCH models have been applied

is in the study of short run movements in daily and weekly foreign exchange rates [see, for example, MaCurdy and Morgan (1987), Milhøj (1987), Diebold (1988), Ballie and Bollerslev (1989), Diebold and Nerlove (1989) and Hsieh (1989)]. In this section we use data similar to these studies. We estimate both ARCH and bilinear models for each spot rate and then compute the stochasticly simulated Cox test of each specification against the other.

The three currencies we consider are the British Pound, the Canadian Dollar and the Japanese Yen To avoid the complications of day of the week and holiday effects, we employ weekly data. The original data are the interbank closing spot prices on Wednesday denominated in U.S. cents per unit of foreign currency. The data run from the first week of January 1973 through the first week of June 1985. The data are analyzed in the first differences of logarithms. Therefore, the analyzed series represent the percentage rates of return from holding a unit of foreign currency one week. After making the logarithmic transformation, there are 651 observations.

As a first stage in building a GARCH and bilinear model for each series, we specify autoregressive (AR) models to account for any linearity in the data. The order of the AR processes were chosen through the standard identification techniques. The Canadian Dollar and the Japanese Yen can be adequately modeled by an AR(1) process. The Pound required an AR(4) process. After estimating the AR models, we computed the joint regression F-test for ARCH and bilinearity as described in section 3. The results for various orders of q, r and s are shown in Table 1. Not surprisingly, all three series show significant evidence of nonlinearity. We also computed the joint

test for ARCH and bilinearity described in Higgins and Bera (1989) and the results were very similar to those reported in Table 1.

Next, we fit AR models with GARCH and bilinear errors to each of the series. To identify the order of the GARCH processes, models of various order were estimated. Each series seemed adequately represented by a GARCH(0,2) or GARCH(1,1). The maximized value of the likelihoods tended to favor the later, therefore we specified each series as GARCH(1,1). The estimated model are shown in Table 2. For the Japanese Yen, the sum of $\hat{\alpha}_1$ and $\hat{\beta}_1$ are close to one. This is characteristic of the integrated GARCH process of Engle and Bollerslev (1986). Similarly, the order of the bilinear models were selected by considering t-statistics and likelihoods for models of various orders. The estimated models are shown in Table 3. Comparing the conditional means of the AR models with GARCH errors and the AR models with bilinear errors, there is little difference in the estimated AR parameters and their computed standard errors. This suggests that the bilinear error structure captures dependence in the conditional means of the series which cannot be represented by a pure AR process.

We then proceed to compute the stochasticly simulated Cox test of each model against the other. The tests are implemented as described in section 4 and computed from the artificial regression (4.3). Both $\hat{\gamma}_*(R)$ and $\hat{C}_R(\theta)$ were computed with R = 50 and R = 100 replications. Since the values of the test statistic changed little as the number of replications increased, we only report the results based on 100 replications. The results for the three exchange rates are shown in Table 4.

Considering the results in Table 4, it is evident that GARCH may not totally account for the nonlinearities in the data. For the British Pound, GARCH cannot be rejected in favor of bilinearity, but bilinearity can be rejected in favor of GARCH. Considering the Canadian Dollar, however, GARCH is clearly rejected against bilinearity. Although bilinearity can not be rejected against GARCH at the 1% level using a two-tailed test, it can be rejected at the 5% level. This indicates that any nonlinear dependence in the Canadian Dollar may marginally be better represented through a conditional mean model rather than by a GARCH model for the conditional variance. For the Japanese Yen, both GARCH and bilinearity are rejected at any reasonable level of significance, indicating that neither model appropriately represents the nonlinear dynamics in the series. However, we interpret the results for the Yen cautiously. From the estimated GARCH model in Table 2, the Japanese Yen appears to be very close to an I-GARCH process for which the unconditional variance and higher moments do not exist. The lack of higher order moments may invalidate the asymptotic distribution theory of the Cox test, although the results of Lee and Hansen (1991) and Lumsdaine (1991) show that the standard asymptotic theory of the MLE goes through for the I-GARCH model.

6. Conclusion

We consider the above results preliminary. The rejection of the models may be due to the specification of the conditional distribution rather than

misspecified nonlinear dynamics. As indicated by Bollerslev (1987), a conditionally normal GARCH process may not be appropriate for asset returns. The conditional distributions frequently have heavier tails than the normal distribution. A conditional normality assumption may also be inappropriate for a bilinear model. The estimator of the variance of the Cox test (4.1) is highly dependent upon the parametric density specified in the null hypothesis. If the parametric model is misspecified, the variance estimator is likely to be inconsistent. In future work, we plan to adopt a quasi-maximum likelihood approach and employ a robust variance estimator in the manner of White (1982).

Also, finding evidence of nonlinearity in the mean of a series does not necessarily indicate that incorporating this nonlinear structure will lead to improved forecasts. Recent papers by Diebold and Nason (1990), Mizrach (1992), and Prescott and Stengos (1990) conducted forecasting exercises which employed nonparametric techniques to allow for nonlinearities in the conditional means. These papers indicate that improved forecasting can be achieved with in the sample period, but not necessarily out of sample. The focus of our paper is restricted to the choice between bilinear and GARCH models. By employing our procedure the investigator should at least be able to decide whether bilinearity or GARCH, or neither of the two models, should be adopted to capture the nonlinearity in the data.

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Table 1. Joint Regression Tests for ARCH and Bilinearity.

order	Currency		
q,r,s	British Pound	Canadian Dollar	Japanese Yen
1,1,1	24.17*	8.63	16.69*
1,2,2	28.52*	10.53	16.88*
1,3,3	45.09*	24.24*	23.37*
2,1,1	39.06*	10.89	20.84*
2,2,2	45.83*	10.85	23.28*
2,3,3	56.28*	21.53*	26.90*
3,1,1	47.97*	13.01	28.51*
3,2,2	55.08*	11.87	31.71*
3,3,3	49.11*	16.97	31.65*
4,1,1	48.42*	13.02	35.99*
4,2,2	53.68*	11.87	39.14*
4,3,3	47.63*	17.10	40.02*

*Indicates significant at the 1% level.

Table 2. Estimated GARCH Models.

British Pound:

$$y_t = -.054 + .264\ y_{t-1} - .003\ y_{t-2} + .060\ y_{t-3} + .080\ y_{t-4}$$
$$(.033)\quad (.046)\qquad (.037)\qquad (.036)\qquad (.035)$$

$$h_t = .092 + .175\ \epsilon^2_{t-1} + .758\ h_{t-1}$$
$$(.036)\quad (.044)\qquad (.063)$$

Canadian Dollar:

$$y_t = -.006 + .278\ y_{t-1} \qquad h_t = .048 + .457\ \epsilon^2_{t-1} + .277\ h_{t-1}$$
$$(.013)\quad (.047)\qquad\qquad\ \ (.013)\quad (.087)\qquad (.102)$$

Japanese Yen:

$$y_t = .031 + .327\ y_{t-1} \qquad h_t = .003 + .052\ \epsilon^2_{t-1} + .945\ h_{t-1}$$
$$(.031)\quad (.041)\qquad\qquad\ \ (.002)\quad (.008)\qquad (.007)$$

Table 3. Estimated Bilinear Models.

British Pound:

$$y_t = -\,.079 + .257\,y_{t-1} - .054\,y_{t-2} + .072\,y_{t-3} + .058\,y_{t-4}$$
$$(.045)\quad(.040)\qquad(.034)\qquad(.029)\qquad(.029)$$

$$+\,.035\,u_{t-1}\epsilon_{t-1} - .138\,u_{t-1}\epsilon_{t-2} + .139\,u_{t-2}\epsilon_{t-1} - .020\,u_{t-2}\epsilon_{t-2}$$
$$(.013)\qquad\qquad(.068)\qquad\qquad(.050)\qquad\qquad(.017)$$

$$\hat{\sigma}^2_\epsilon = 1.149$$
$$\phantom{\hat{\sigma}^2_\epsilon = }(.064)$$

Canadian Dollar:

$$y_t = -\,.044 + .418\,y_{t-1} + .094\,u_{t-1}\epsilon_{t-1} + .225\,u_{t-2}\epsilon_{t-1}$$
$$(.009)\quad(.027)\qquad(.022)\qquad\qquad(.058)$$

$$+\,.225\,u_{t-2}\epsilon_{t-1} - .102\,u_{t-2}\epsilon_{t-2}\qquad \hat{\sigma}^2_\epsilon = .173$$
$$(.058)\qquad\qquad(.038)\qquad\qquad\qquad(.009)$$

Japanese Yen:

$$y_t = -\,.021 + .301\,y_{t-1} + .038\,u_{t-1}\epsilon_{t-1} - .020\,u_{t-1}\epsilon_{t-2}$$
$$(.015)\quad(.035)\qquad(.013)\qquad\qquad(.010)$$

$$\hat{\sigma}^2_\epsilon = 1.154$$
$$\phantom{\hat{\sigma}^2_\epsilon = }(.062)$$

Table 4   Stochasticly Simulated Cox Tests Between GARCH and Bilinearity.[a]

	H_0:GARCH H_1:Bilinear	H_0:Bilinear H_1:GARCH
British Pound	-1.17	-3.01[*]
Canadian Dollar	-6.94[*]	-2.03
Japanese Yen	-6.62[*]	-4.73[*]

[a]The simulations are based on R = 100 replications.

[*]Indicates significant at the 1% level for a two-sided test.