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DETERMINING THE ACHIEVEMENT OF PUPILS IN COMMON FRACTIONS

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In School Committee, Boston, June 26, 1918.

Ordered, That four thousand (4,000) copies of a bulletin on "Determining the Achievement of Pupils in Common Fractions," prepared by the Department of Educational Investigation and Measurement and approved for publication by the Board of Superintendents at its meeting on June 19, 1918, be printed as a school document.

Attest:

THORNTON D. APOLLONIO,
Secretary.

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INTRODUCTION.

As fast as practicable the Department of Educational Investigation and Measurement has undertaken to extend educational measurement in arithmetic beyond the four fundamental operations covered by the Courtis Tests. To that end from time to time the department has given tests in the four processes of common fractions, and in a more limited way in problem work. This bulletin, prepared by the Assistant Director, Mr. Arthur W. Kallom, covers the work done thus far with the addition, subtraction, multiplication, and division of common fractions. Particular attention is directed to Mr. Kallom's discussion of the various types of problems involved in computation with fractions, and also to the suggestions offered of ways and means of improving unsatisfactory achievements of pupils.

It is frequently argued in educational discussions that command over the tools of an education should be acquired by the end of the sixth grade. Obviously, ability to compute with common fractions is merely a means to a desired end and not an education in itself. This study shows that either (*a*) not as much ability to use common fractions is being developed before the end of the sixth grade as should be, or (*b*) the period of the first six years in the elementary school is too short to furnish pupils with all the tools of an education.

The manuscript for this bulletin was approved for publication by the Board of Superintendents at its meeting on June 19, 1918.

FRANK W. BALLOU,
Assistant Superintendent in Charge.

DETERMINING THE ACHIEVEMENT OF PUPILS IN COMMON FRACTIONS.

I. Results of Previous Tests in Fractions.

Tests in the four fundamental operations have been given in the City of Boston during the past five years. During this time the gain in the amount of work done has been from 12 per cent to 17 per cent. This gain in amount of work done has been accompanied by an actual increase in the accuracy with which the work was completed. This increase by which pupils are being graduated or are being promoted into the next grade with varying degrees of superiority up to 17.7 per cent over the results which were being obtained previous to the giving of the Courtis standard tests, is due directly or indirectly to the system of educational measurement as established in Boston.

The ability to handle integers does not constitute, however, the sum total of the tools necessary for the child in order that he may do arithmetic. Fractions, in one form or another, play a large part in the arithmetical work of the pupil. That we might know how well the pupils are doing their work in common fractions, a plan was organized in 1915 to give tests in addition, subtraction, multiplication, and division of fractions in successive years to a group of approximately 1,000 children in Grades VI, VII, and VIII in an experimental way. The tests were organized by the department in such a way as to determine not only what the ability was to do the various operations, but also if the pupils failed, in what type of examples in any given operation the pupil failed. Of the two phases of the work, the latter is of the greater importance. It is not enough to say pupils fail to do addition of fractions with a speed or an accuracy which seems desirable. One must go further and show the types of examples in which pupils fail.

Further, if a single test be given in a certain operation and a pupil fails, it becomes the work of the teacher to determine in what particular the pupil fails. This enables the teacher to place the emphasis upon the work where it belongs and not waste her efforts and those of the pupils in drilling on matter which needs no drilling.

Addition of fractions may be divided into fourteen types; * subtraction of fractions may be divided into nine types similar to those in addition, the difference in number being due to the fact that it is impossible to reduce any answer in subtraction to a mixed number. The types in multiplication and division will be analyzed in the succeeding pages of this bulletin.

Summary of Results in Addition of Fractions.

The results of the tests in addition of fractions were published in School Document No. 3, 1916. The data upon which the conclusions were drawn are shown in the following tables. Table I shows the type of examples used in the six tests together with the time allowance.

TABLE I.

Showing Examples Used in Tests in Addition of Fractions, December, 1915.

Addition of Fractions.— Test 1.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$\frac{1}{4}$	$\frac{3}{14}$	$\frac{5}{16}$	$\frac{1}{10}$
$\frac{1}{4}$	$\frac{1}{14}$	$\frac{7}{16}$	$\frac{7}{10}$

Addition of Fractions.— Test 2.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$\frac{1}{3}$	$\frac{2}{7}$	$\frac{2}{3}$	$\frac{1}{3}$
$\frac{1}{6}$	$\frac{3}{14}$	$\frac{1}{12}$	$\frac{7}{15}$

* See School Document No. 3, 1916. "Determining the Achievement of Pupils in Addition of Fractions."

Addition of Fractions.— Test 3.— Time, 2 Minutes.

(1) $\frac{3}{5}$	(2) $\frac{5}{6}$	(3) $\frac{5}{7}$	(4) $\frac{14}{15}$
$\frac{11}{15}$	$\frac{1}{2}$	$\frac{11}{14}$	$\frac{2}{3}$

Addition of Fractions.— Test 4.— Time, 2 Minutes.

(1) $\frac{1}{7}$	(2) $\frac{7}{9}$	(3) $\frac{3}{4}$	(4) $\frac{4}{9}$
$\frac{9}{10}$	$\frac{1}{4}$	$\frac{3}{7}$	$\frac{5}{8}$

Addition of Fractions.— Test 5.— Time, 2 Minutes.

(1) $\frac{1}{10}$	(2) $\frac{4}{9}$	(3) $\frac{1}{6}$	(4) $\frac{1}{12}$
$\frac{1}{6}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{1}{10}$

Addition of Fractions.— Test 6.— Time, 2 Minutes.

(1) $\frac{1}{6}$	(2) $\frac{5}{6}$	(3) $\frac{1}{8}$	(4) $\frac{7}{12}$
$\frac{9}{10}$	$\frac{3}{8}$	$\frac{9}{10}$	$\frac{7}{10}$

Table II shows the medians obtained as a result of the tests.

TABLE II.

Summary Sheet — City Medians.

Addition of Fractions, December, 1915.

Grade.	Pupils Tested.	Test 1.		Test 2.		Test 3.		Test 4.		Test 5.		Test 6.	
		Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy. Median.	Speed Median.	Accuracy Median.
VIII.......	1,130	20.7	88.0	11.6	74.0	8.4	47.0	6.0	68.0	6.9	52.0	6.4	47.0
VII........	1,243	16.5	87.0	10.1	73.0	7.3	46.0	5.3	69.0	6.3	55.0	5.7	48.0
VI.........	1,265	10.7	80.0	7.7	66.0	5.5	42.0	4.0	70.0	4.6	51.0	4.4	49.0

The following conclusions were drawn as a result of the analysis of the tests.*

1. The factors that enter into the problem of adding fractions are much more complex than those that enter into the problem of adding integers.

2. The errors were largely due to the failure of pupils to reduce consistently either to lowest terms or to mixed numbers. This failing on the part of many children to use the principle of reduction would seem to indicate that the method, now largely in use, of teaching such reductions by themselves, has failed to produce satisfactory results. In view of this fact, would it not be well to teach reductions as such, in connection with the subject of addition of fractions? This would at least make a closer connection between the two operations, and thereby tend to form the habit of writing the answer in its best form.

3. Eight per cent of the pupils in Grade VI, 11 per cent in Grade VII, and 5 per cent in Grade VIII were unable to do the simplest problems in the addition of fractions.

4. Drill and individual work given the children in Grade V of selected schools in the spring at the suggestion of the department showed its effect in the work of Grade VI in the late fall. This was evidenced by an increase in both speed and accuracy over that obtained in the entire city and in two cases over that shown by the whole number of pupils in the grade in which the selected groups were enrolled.

Summary of Results in Subtraction of Fractions.

The results of the tests in subtraction of fractions were not published because they were not materially different from the results of the tests in addition. The following table shows the types of examples used in the five tests together with the time allowance.

* School Document No. 3.

TABLE III.

Showing Examples Used in Tests in Subtraction of Fractions, December, 1916.

Subtraction of Fractions.— Test 1.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$\frac{1}{4}$	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{9}{16}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{3}{16}$

Subtraction of Fractions.— Test 2.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$\frac{1}{2}$	$\frac{6}{7}$	$\frac{2}{3}$	$\frac{3}{4}$
$\frac{1}{9}$	$\frac{3}{5}$	$\frac{3}{11}$	$\frac{5}{9}$

Subtraction of Fractions.— Test 3.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$\frac{5}{6}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{7}{10}$
$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{12}$	$\frac{8}{15}$

Subtraction of Fractions.— Test 4.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
4	6	6	6
$2\frac{1}{2}$	$5\frac{3}{5}$	$2\frac{3}{5}$	$3\frac{1}{6}$

Subtraction of Fractions.— Test 5.— Time, 2 Minutes.

(1)	(2)	(3)	(4)
$9\frac{1}{6}$	$7\frac{3}{14}$	$7\frac{1}{12}$	$7\frac{1}{3}$
$1\frac{1}{3}$	$6\frac{2}{7}$	$4\frac{2}{3}$	$2\frac{7}{15}$

In the tests in addition the addition of mixed numbers was not included although it is very probable that there would have been some difficulty in disposing of the sum of the fractions, especially if the sum were more than an integer. This same phase occurs in the multiplication of mixed numbers by an integer and will be pointed out in its proper place. The subtraction of mixed numbers, however, is a vital problem especially when the fraction in the subtrahend is larger than the fraction in the minuend. Because of this, Tests 4 and 5 were given upon this type of example. Table IV shows the medians in speed and accuracy in the subtraction of fractions.

TABLE IV.

Summary Sheet — City Medians.

Subtraction of Fractions, December, 1916.

Grade.	Pupils.	Test 1. Speed Median.	Test 1. Accuracy Median.	Test 2. Speed Median.	Test 2. Accuracy Median.	Test 3. Speed Median.	Test 3. Accuracy Median.	Test 4. Speed Median.	Test 4. Accuracy Median.	Test 5. Speed Median.	Test 5. Accuracy Median.
VIII	1,239	22.5	91.0	7.3	86.0	6.1	65.0	18.0	99.0	6.4	81.0
VII	1,283	19.7	84.0	6.0	85.0	5.6	61.0	14.2	97.0	5.2	66.0
VI	1,499	15.1	73.0	4.9	76.0	4.6	51.0	11.9	85.0	4.6	64.0

The following summary shows the number of districts, the number of grade classes, the grades tested, and the total number of pupils included in the test in addition and subtraction of fractions.

ADDITION OF FRACTIONS.

December, 1915.

Number of elementary districts	12
Number of grade classes	91
Given in grades	VIII, VII, VI
Number of pupils	3,638

SUBTRACTION OF FRACTIONS.

December, 1916.

Number of elementary districts	10
Number of grade classes	102
Given in grades	VIII, VII, VI
Number of pupils	4,021

II. MULTIPLICATION AND DIVISION OF FRACTIONS.

Extent of Tests.

The following summary shows the number of districts, the number of grade classes, the grades tested, and the number of pupils included in the test in multiplication and division of fractions, given in December, 1917.

Number of elementary districts	10
Number of grade classes	95
Given in grades	VIII, VII, VI
Number of pupils	3,513

Types of Fractions.

An extended analysis was made in addition and subtraction of fractions to determine the various types with which the pupil came in contact. An analysis on a similar basis of the processes of multiplication and division of fractions was not believed necessary. In these two processes the separation into types depended upon the character of the multiplier and multiplicand or the dividend and divisor.

The process of division in fractions is either one of two procedures. In one case one proceeds to find how many times one number of a certain denomination is contained in a larger number of the same denomination. This is to determine how many measures of a certain length there are in a measure of a different length. This type of procedure has been called *division by measuring.* In the other case one proceeds to separate a number into a certain number of parts. This type of division is called *division by parting.* These two types, measuring and parting, became the basis upon which the three tests in division of fractions were formulated. In view of these conditions the following types were selected:

Multiplication.

Integer multiplied by fraction.
Fraction multiplied by integer.
Mixed number multiplied by integer.
Integer multiplied by mixed number.
Mixed number multiplied by fraction.
Fraction multiplied by mixed number.
Mixed number multiplied by mixed number.
Fraction multiplied by fraction.

Division.

Integer divided by fraction (measuring).
Fraction divided by integer (parting).
Mixed number divided by integer (parting).
Integer divided by mixed number (measuring).
Fraction divided by fraction (measuring).
Mixed number divided by fraction (measuring).

The two types, fraction divided by mixed number and mixed number divided by mixed number, are not included because they do not conform to either the *parting* or *measuring* criterion. In practical work we neither have to perform such examples as dividing a fraction into $3\frac{1}{2}$ parts nor finding how many $3\frac{1}{2}$ inches there are in $\frac{1}{4}$ of an inch. Neither are we required to perform such examples as dividing a mixed number into $3\frac{1}{2}$ parts nor finding how many $3\frac{1}{2}$ inches there are in $4\frac{1}{4}$ inches.

Of course, it is recognized that common fractions are taking less and less place in our practical life, the process giving way more and more to the use of the decimal fraction. However, there is still use for the common fraction having a small denominator, and it is still a part of the required work in our courses of study. This being true, it is pertinent to ascertain what results we are achieving.

Construction of the Tests.

In constructing the tests the department decided in the light of previous experience with addition and subtraction of fractions that multiplication and division might be given at one time. In order to decrease the number of tests, two types were placed in a test. For example, multiplication of an integer by a fraction and multiplication of a fraction by an integer comprised Test 1. As will be seen in Table V, the other tests were made in a similar way. In the analysis of the results the two types will be discussed separately. There was an effort to keep the tests within the realm of the practical. In all cases the terms of the fractions involved were kept small. A fraction multiplied by a fraction is not in any test but is included in the process of multiplication of mixed number by a fraction. The latter type was used because it was considered more difficult. If this be true, a pupil might be able to do the former type but unable to do the latter. However, ability to do the latter would include ability to do the former. Table V shows the types of examples and the time allowance for each test.

TABLE V.

Showing Examples Used in Tests in Multiplication and Division of Fractions, December, 1917.

Multiplication of Fractions.— Test 1.— Time, 2 Minutes.

(1) $\frac{1}{8} \times 6$ (2) $\frac{7}{9} \times 8$ (3) $\frac{5}{6} \times 12$ (4) $12 \times \frac{5}{16}$

Multiplication of Fractions.— Test 2.— Time, 4 Minutes.

(1)	(2)	(3)	(4)	(5)
$246\frac{1}{5}$	$573\frac{4}{5}$	275	$456\frac{1}{3}$	189
5	5	$8\frac{3}{4}$	2	$5\frac{1}{5}$

Multiplication of Fractions.— Test 3.— Time, 2 Minutes.

(1) $4\frac{7}{8} \times \frac{1}{8}$ (2) $7\frac{1}{2} \times \frac{2}{3}$ (3) $5\frac{1}{2} \times \frac{3}{4}$ (4) $\frac{5}{6} \times 2\frac{2}{3}$

Multiplication of Fractions.— Test 4.— Time, 5 Minutes.

(1)	(2)	(3)	(4)	(5)
$32\frac{1}{3}$	$84\frac{1}{3}$	$29\frac{3}{4}$	$25\frac{3}{4}$	$19\frac{1}{8}$
$69\frac{1}{2}$	$79\frac{1}{5}$	$28\frac{1}{3}$	$17\frac{2}{3}$	$97\frac{1}{2}$

Division of Fractions.— Test 5.— Time, 2 Minutes.

(1) $\frac{3}{4} \div 8$ (2) $9 \div \frac{3}{8}$ (3) $6 \div \frac{4}{5}$ (4) $8 \div \frac{3}{5}$

Division of Fractions.— Test 6.— Time, 4 Minutes.

(1) $5678\frac{1}{3} \div 5$ (2) $2789\frac{2}{3} \div 4$ (3) $2467 \div 8\frac{1}{4}$

(4) $6752 \div 12\frac{1}{3}$

Division of Fractions.— Test 7.— Time, 3 Minutes.

(1) $\frac{3}{5} \div \frac{1}{3}$ (2) $3\frac{3}{4} \div \frac{1}{5}$ (3) $5\frac{4}{5} \div \frac{2}{3}$ (4) $6\frac{2}{5} \div \frac{4}{5}$

Giving of the Tests and Correction of Results.

Following the plan developed in 1912 and continued since the department was organized,* twenty-five Normal School seniors were trained to give the tests in a uniform manner. The tests were given to 1,290

* Ballou, F. W., "Training Normal School Seniors in Educational Measurement," School and Society, Volume V., No. 108, pages 61–70, January 20, 1917.

pupils in Grade VI, 1,196 pupils in Grade VII, and to 1,027 pupils in Grade VIII in December, 1917.

The old course of study for Grade V requires:

> Multiplication of fractions and mixed numbers and integers; finding fractional parts of integers including the cases where the parts so obtained are mixed numbers.

Thus the sixth grade may begin their work without a knowledge of division of fractions, and it is possible that division of fractions may not have been taught during the first three months of the school year. In spite of this knowledge, it was decided to test in Grade VI for two reasons. First, that it might be known just what the status of Grade VI actually is on a city-wide basis in multiplication and division of fractions; and second, to find out what is done by those schools which did more work than was actually required by the course of study.

After completing the work, the examiners brought the tests to the office of the department and all the work of correction and tabulation was done by members of the department. Certain rules were formulated for the correction of results.

(*a*) All results which were not reduced to lowest terms or to mixed numbers were called wrong.

(*b*) The papers on which children added or subtracted the fractions were counted as I. N. F. papers. (Instructions Not Followed.)

(*c*) All other papers, regardless of how the child did the examples, were scored as right or wrong.

(*d*) The form of doing the work did not count against the child if his answer was correct.

(*e*) Some children did not do Test 1, but started upon Test 2, owing to confusion in understanding the directions. Any paper showing no work at all in Test 1 was marked I. N. F. in all tests. (Instructions Not Followed.) These were very few.

(*f*) If in Tests 5, 6, and 7 (division of fractions) pupils *multiplied*, the papers were not marked I. N. F. (Instructions Not Followed.) This was because there is much confusion between the two processes and many pupils really multiply when they *think* they are dividing. If any other process was used the test was marked I. N. F. (Instructions Not Followed.)

III. Analysis of Results.

Achievement.

Table VI shows the results for the entire number of pupils tested. In the first column is shown the grade, followed by a column showing the number of pupils tested in each grade. Under each test is given the speed median and the accuracy median for each test and grade. The table is to be interpreted as follows: In Grade VIII, 1,027 pupils were tested. These pupils attained a speed median of 11.1 examples with an accuracy median of 93 per cent in Test 1. In Test 2 the speed median was 8.8 and the accuracy median was 63 per cent. Thus, reading across the page on the first line one will find the medians in speed and accuracy for each test for Grade VIII. The table shows the same facts for Grades VII and VI.

TABLE VI.

Summary Sheet — City Medians.

Multiplication and Division of Fractions.

Grade.	Pupils Tested.	Multiplication.								Division.					
		Test 1.		Test 2.		Test 3.		Test 4.		Test 5.		Test 6.		Test 7.	
		Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.	Speed Median.	Accuracy Median.
VIII......	1,027	11.1	93	8.8	63	7.6	85	4.7	0	10.1	75	3.3	29	10.3	79
VII.......	1,196	8.4	88	7.7	38	6.4	81	4.2	0	8.2	59	2.9	0	8.5	68
VI........	1,290	6.2	13	8.2	0	4.7	0	5.6	0	5.4	0	3.2	0	4.9	0

It will be noticed that the accuracy medians for Test 2, multiplication of mixed number by integer or integer by mixed number, Test 4, multiplication of mixed number by a mixed number, and Test 6, division of mixed number by an integer, are especially low.

It is probably true that there is no great use for the type of work shown in these three tests in practical life, but the business world does require it to some extent; business courses in our high schools require the processes, and the new course of study requires this work. In view of these three conditions, it was thought best to include these three tests in order that we might have some facts on which to base the development of our work in multiplication and division of fractions.

Analysis of Results in Grades VII and VIII.

The analysis of results which is given in this bulletin is based wholly upon a study of the wrong examples in the work performed by pupils in the test given in December. It is perfectly possible that a pupil who does the work and reaches the right result may be doing it in an inefficient and round-about manner. When correcting large numbers of papers, his work does not attract the attention that is attracted by a pupil who does many examples and gets none or only a few *right*.

This study is based, then, upon those papers which showed low scores in accuracy. Furthermore, owing to the low degree of accuracy in Grade VI, due largely to lack of knowledge, the analysis is based upon work in Grades VII and VIII. In a study like the present one a piece of work done by a person ignorant of the process has little or no value. The value of a study of this kind comes from studying results of pupils who are supposed to have been taught the process. An analysis of Grade VI will be made in a later part of the bulletin.

It is impossible for the *report* of a study to be as helpful to a teacher as if the individual teacher had *made* the study for herself. It is only when the teacher will take the work of her class room and make some similar analysis, seeking to find out *why* the pupil makes the failure and just what the pupil does in making the failure, that we are going to make great gains in the treatment of individual pupils. The analysis is given here rather in detail in the hopes that it may act as a guide and stimulate some teachers to undertake this rather laborious but extremely interesting work for the good of the individual who is having trouble with his fractions and school work in general.

TABLE VII.

Showing Percentage of Pupils Who Fail in Multiplication and Division of Fractions.

Grade.	Test 1.		Test 2.		Test 3.		Test 4.		Test 5.		Test 6.				Test 7.	
							MIXED NUMBER × A MIXED NUMBER.				MIXED NUMBER ÷ AN INTEGER.		INTEGER ÷ A MIXED NUMBER.			
	Integer × a Fraction.	Fraction × an Integer.	Mixed Number × an Integer.	Integer × a Mixed Number.	Fraction × a Mixed Number.	Mixed Number × a Fraction.	Vertical.	Horizontal.	Fraction ÷ an Integer.	Integer ÷ a Fraction.	Inversion.	Long Division.	Inversion.	Long Division.	Fraction ÷ a Fraction.	Mixed Number ÷ a Fraction.
VIII....................	3.1	2.6	17.9	30.6	8.9	8.6	90.4	4.6	42.5	25.0	18.3	83.4	20.0	97.3	24.9	20.4
VII....................	13.2	13.5	34.3	53.4	15.5	19.7	91.6	6.9	49.5	35.4	30.7	87.2	40.5	99.2	39.0	32.5

Table VII shows the general situation in regard to failures in Grades VII and VIII. In the compiling of this table, it was considered (1) that a pupil had failed to do a certain type if he did not get at least one example right among those he attempted, (2) that a pupil did not fail if he used the correct method even though he did not get at least one right answer. The table is to be interpreted as follows. In Test 1, 3.1 per cent of Grade VIII failed in the multiplication of an integer by a fraction and 2.6 per cent failed in the multiplication of a fraction by an integer. In Test 2, 17.9 per cent failed in the multiplication of a mixed number by an integer and 30.6 per cent failed in the multiplication of an integer by a mixed number and so on.

In multiplying a mixed number by a mixed number, Test 4, two forms were used as illustrated below.

(*a*) Vertical method:

$$\begin{array}{r} 32\frac{1}{3} \\ 69\frac{1}{2} \\ \hline 16\frac{1}{6} \\ 23 \\ 288 \\ 192 \\ \hline 2247\frac{1}{6} \end{array}$$

(*b*) Horizontal method:

$$32\tfrac{1}{3} \times 69\tfrac{1}{2} =$$

$$\frac{97}{3} \times \frac{139}{2} = \frac{13483}{6} = 2247\tfrac{1}{6}$$

In this paper whenever the example was done similarly to illustration (*a*), it has been termed the vertical method, and when (*b*) the mixed numbers were reduced to improper fractions, it has been termed the horizontal method. When the example was done by the vertical method, the eighth grade failed in 90.4 per cent of the cases; when done by the horizontal method the same grade failed in 4.6 per cent of the cases.

In Test 6 the pupils were required to divide a mixed number by an integer (Examples 1 and 2) or an integer by a mixed number (Examples 3 and 4). Two possibilities of doing the work present themselves. First, pupils may reduce the mixed numbers to improper fractions and follow the general rule for division of fractions.

For example (*a*): $5678\frac{1}{3} \div 5 = \frac{17035}{3} \div 5 =$

$$\frac{17035}{3} \times \frac{1}{5} = \frac{3407}{3} = 1135\frac{2}{3}$$

Second, they may, when the fraction is in the dividend, do the example by either short or long division as it stands.

For example (*b*): $5678\frac{1}{3} \div 5$

$5)\underline{5678\frac{1}{3}}$

1135 Rem. $= 3\frac{1}{3}$

$$3\frac{1}{3} \div 5 = \frac{10}{3} \times \frac{1}{5} = \frac{2}{3}$$

$1135\frac{2}{3}$ Ans.

When the mixed number is in the divisor, they may place the example on the paper as though they were doing an example in long division, multiply both dividend and divisor by the denominator of the fraction and proceed as in long division.

For example (*c*): $2467 \div 8\frac{1}{4}$

$$8\frac{1}{4})\,2467$$
$$\underline{4 \qquad 4}$$
$$33)\,9868$$

$$\begin{array}{r} 299\frac{1}{33} \\ 33)\overline{9868} \\ \underline{66} \\ 326 \\ \underline{297} \\ 298 \\ \underline{297} \\ 1 \end{array}$$ Ans.

These two processes will be termed (*a*) process of inversion and (*b* and *c*) process of long division.

Diagnosis of Results in Each Test.

TEST 1.

Type of Examples Used in Test 1.

(1) $\frac{1}{8} \times 6$ (2) $\frac{7}{9} \times 8$ (3) $\frac{5}{6} \times 12$ (4) $12 \times \frac{5}{16}$

In this test the pupils were required to multiply a fraction by an integer (Examples 1, 2 and 3) or an integer by a fraction (Example 4). About 13 per cent failed in the seventh grade and about 3 per cent failed in the eighth grade in each type. In such a simple test the chances of making errors are limited; so they fall very largely into two groups. In one group the pupils find the answer by multiplying the integer by one of the terms of the fraction and adding the other.

For example: $\frac{1}{8} \times 6 = 14 \quad (6 \times 1 + 8)$ *or*
$\frac{1}{8} \times 6 = 49 \quad (6 \times 8 + 1)$

The pupils in the second group multiply both numerator and denominator by the integer.

For example: $\frac{7}{9} \times 8 = \frac{56\ (8 \times 7)}{72\ (8 \times 9)}$

Cancellation gives little trouble because comparatively few pupils use this method of shortening the procedure. In some cases there was evidence of cancelling by dividing the integer by the numerator, but these cases were few. There seemed to be a mixture of processes in the minds of some pupils because a few inverted one or the other of the factors.

TEST 2.

Type of Examples Used in Test 2.

(1)	(2)	(3)	(4)	(5)
$246\frac{1}{5}$	$273\frac{4}{5}$	275	$456\frac{1}{3}$	189
5	5	$8\frac{3}{4}$	2	$5\frac{1}{5}$

In this test the pupils were required to multiply a mixed number by an integer (Examples 1, 2 and 4) or an integer by a mixed number (Examples 3 and 5). The percentage of failure for the first type for Grade VII was 34.3 per cent and for Grade VIII, 17.9 per cent. For the second type the percentage was nearly twice as much, being 53.4 per cent and 34.3 per cent respectively. This great difference was due very largely to the construction of two examples, the fourth, $456\frac{1}{3} \times 2$, and the thirteenth, $379\frac{1}{4} \times 3$. In many cases pupils had the fourth example right and also the thirteenth, if they reached this example, and no others. In these two examples the multiplication of the fraction by the integer gives the fractional part of the product without further reduction.

In this test there are three chief sources of error.

(*a*) If the fraction be in the multiplicand, the multiplication of the fraction by the integer in the multiplicand and if the fraction be in the multiplier, the multiplication of the fraction by the integer in the multiplier.

For example:

$$\begin{array}{rl} 246\frac{1}{5} & \\ 5 & \\ \hline 49\frac{1}{5} & = \frac{1}{5} \times 246 \\ 1230 & = 5 \times 246 \\ \hline 1279\frac{1}{5} & \text{Ans.} \end{array} \qquad \begin{array}{rl} 275 & \\ 8\frac{3}{4} & \\ \hline 6 & = \frac{3}{4} \times 8 \\ 2200 & = 8 \times 275 \\ \hline 2206 & \text{Ans.} \end{array}$$

(*b*) Placing of the second partial product one place to the left of the first partial product.

For example:

$$\begin{array}{rl} 275 & \\ 8\frac{3}{4} & \\ \hline 206\frac{1}{4} & = \frac{3}{4} \times 275 \\ 2200 & = 8 \times 275 \\ \hline 22206\frac{1}{4} & \text{Ans.} \end{array}$$

(*c*) Multiplication of the denominator of the fraction by the integer and adding the numerator.

For example:

$$\begin{array}{rl} 246\frac{1}{5} & \\ 5 & \\ \hline 26 & = 5 \times 5 + 1 \\ 1330 & \\ \hline 1356 & \end{array}$$

The kind of error in (*a*) develops because it is not clear in the minds of many pupils which integer is to be multiplied by the fraction. If the fraction be in the multiplier, the integer of the multiplier is multiplied by the fraction and this product given as one of the partial products. It is not strange that this should be done on account of the drill which has been given in reduction of mixed numbers to improper fractions. In other words an old habit is at work, for the pupil has not yet appreciated that it is not the same thing but something entirely different. For those pupils who persist in doing this work, individual attention is probably the only method of eradicating the error.

The kind of error noted in (*b*) is due probably to the same cause, viz., the following of an old habit. In multiplication of integers the pupil was taught that he must place the second partial product one step to the left of the first partial product. When the process in fractions is performed, the pupil follows the same habit unless he is led to see the difference *and a large amount of practice in the correct method of doing the work is given.* Some pupils will need more of this practice than others before the new habit is fixed.

The third source of error noted in (*c*) develops through the multiplication of the fraction by the integer. For example, it was a fairly common error in such an example as $246\frac{1}{5} \times 5$ to call $5 \times \frac{1}{5} = 26$. That is, apparently the example was done exactly as though it was reduced to an improper fraction and then the denominator, 5, thrown away.

Another habit is probably at work in this case which is not generally taken into consideration. In teaching multiplication of integers emphasis is placed upon the fact that the product of one number by another is larger than either of the factors. To have a pupil realize that a number may be multiplied by another such that the product is smaller than one of the factors and that when both factors are fractions the product is smaller than either of the fractions, means that the pupil must break old habits and form new ones. The ability to meet this new experience and use it means a large amount of drill before the old habit can be modified to meet the new conditions. Unless this drill is adequate such errors as those just pointed out are likely to occur.

Emphasis is being placed upon estimating the answer in many schools, but even though a pupil is able to make an approximate estimate of the product in these examples, he will not be able to trace the error until he appreciates the possibilities of these three types of error.

Many strange methods were used which do not fall into the foregoing groups. The following examples are illustrations of these:

$$\begin{array}{rl} 341 & \\ 7\frac{2}{3} & \\ \hline 113\frac{2}{3} & = 341 \div 3 \\ 56\frac{1}{2} & = 113 \div 2 \\ 2387 & = 7 \times 341 \\ \hline 2556\frac{7}{6} & \text{Ans.} \end{array}$$

$$\begin{array}{rl} 706\frac{3}{8} & \\ 7 & \\ \hline 5651 & = 8 \times 706 + 3 \\ 4942 & = 7 \times 706 \\ \hline 10593 & \text{Ans.} \end{array}$$

If teachers will give a test similar to this one to their classes, it is more than likely that some of these strange ways of doing the work will manifest themselves. It should be clear that class work does not reach these individuals and if the pupil is *to learn the correct method it is only through individual help.*

TEST 3.

Type of Examples Used in Test 3.

(1) $4\frac{7}{8} \times \frac{1}{8}$ (2) $7\frac{1}{2} \times \frac{2}{3}$ (3) $5\frac{1}{2} \times \frac{3}{4}$ (4) $\frac{5}{6} \times 2\frac{2}{3}$

In this test the pupils were required to multiply a mixed number by a fraction (Examples 1, 2 and 3), or a fraction by a mixed number (Example 4). The percentage of failure was about 17 per cent for Grade VII and 8.7 per cent for Grade VIII in each test.

The greatest difficulties in this test are shown in (1) the reduction of the mixed number to an improper fraction and (2) in the process of cancellation. The first type of failure shows itself in many ways. For example, some pupils multiply the two fractions and add the integer.

For example: $4\frac{7}{8} \times \frac{1}{8} = 4\frac{7}{64}$.

Some pupils consider the integer as another factor instead of a part of one of the factors.

For example: $\frac{5}{6} \times 2\frac{2}{3} = \frac{20}{18} \quad \left(\frac{5 \times 2 \times 2}{6 \times 3} = \frac{20}{18}\right)$.

Another common method was to multiply the integer by the numerator of the fraction and add the numerator of the fraction which is a part of the mixed number.

For example: $\frac{3}{8} \times 3\frac{1}{5} = \frac{10}{40} \quad \left(\frac{3 \times 3 + 1}{8 \times 5} = \frac{10}{40}\right)$.

There were many other improper methods of finding the product of a mixed number and a fraction.

In cancellation the difficulty came in canceling before the reduction of the mixed number to the improper fraction and also in canceling the integer of the mixed number. Examples of this type of error seem unnecessary.

TEST 4.

Type of Examples Used in Test 4.

(1)	(2)	(3)	(4)	(5)
$32\frac{1}{3}$	$84\frac{1}{3}$	$29\frac{3}{4}$	$25\frac{3}{4}$	$19\frac{1}{8}$
$69\frac{1}{2}$	$79\frac{1}{5}$	$28\frac{1}{3}$	$17\frac{2}{3}$	$97\frac{1}{2}$

In this test the pupils were required to multiply a mixed number by a mixed number.

There were 78 per cent of the pupils who attempted to do the work vertically and 22 per cent who did it horizontally. Of those who did the work vertically 92 per cent failed to do the work correctly, and 1 per cent had the method correct but made errors in the work, while the remaining 7 per cent had the correct answer. Of those who did the work horizontally 7 per cent failed to do the work correctly, 45 per cent had the method correct but made errors in the work, while 48 per cent had the correct answer. The two methods will be considered separately.

(*a*) In a study of the vertical method it is noticeable that only a small percentage of the pupils (8 per cent) even get the right method. There were three common erroneous ways of doing the work.

1. The product or sum of the fractions added to product of the integers.

For example:

$$\begin{array}{rl} 32\frac{1}{3} & \\ \underline{69\frac{1}{2}} & \\ 288 & = 9 \times 32 \\ 192 & = 6 \times 32 \\ \underline{\frac{1}{6}} & = \frac{1}{3} \times \frac{1}{2} \\ 2208\frac{1}{6} & \end{array} \quad \textit{or} \quad \begin{array}{rl} 32\frac{1}{3} & \\ \underline{69\frac{1}{2}} & \\ 288 & = 9 \times 32 \\ 192 & = 6 \times 32 \\ \underline{\frac{5}{6}} & = \frac{1}{3} + \frac{1}{2} \\ 2208\frac{5}{6} & \end{array}$$

By far the largest proportion of the pupils who did the work vertically found their answers in this manner.

2. A disregard of one or both of the fractions.

For example:

$$\begin{array}{rl} 32\frac{1}{3} & \\ \underline{69\frac{1}{2}} & \\ 16 & = \frac{1}{2} \times 32 \\ 288 & = 9 \times 32 \\ 192 & \\ \hline 2224 & \end{array} \quad \textit{or} \quad \begin{array}{rl} 32\frac{1}{3} & \\ \underline{69\frac{1}{2}} & \\ 288 & = 9 \times 32 \\ 192 & = 6 \times 32 \\ \hline 2208 & \end{array}$$

3. Process correct except that the product of the fractions is omitted.

For example:

$$\begin{array}{rl} 32\frac{1}{3} & \\ \underline{69\frac{1}{2}} & \\ 23 & = \frac{1}{3} \times 69 \\ 16 & = \frac{1}{2} \times 32 \\ 288 & = 9 \times 32 \\ 192 & = 6 \times 32 \\ \hline 2247 & \end{array}$$

(*b*) The chief error when the work is done horizontally consists of inverting one or both of the fractions after the mixed numbers have been reduced to improper fractions.

It is not the intention of this bulletin to point out the method to be followed in multiplication of mixed numbers. There are three facts, however, that are significant.

1. The larger proportion, 78 per cent, of the pupils did their work vertically either through habit, or because the examples were placed in a vertical position in the test, or because of ignorance, and therefore followed the line of least resistance.

2. Of those who did the examples vertically and used the right method, only a small percentage made errors.

3. There was a large percentage, 45 per cent, who used the correct method when doing the work horizontally but made errors in the work. It is not fair, however, to draw the conclusion that the better method is the vertical method, because of the high percentage of accuracy. The number of pupils concerned is not large enough to make this conclusive.

In multiplying mixed numbers horizontally the pupil must take the following steps to do the work:

Reduction of one mixed number to improper fraction.
1. Integer multiplied by the denominator.
2. Add the product to the numerator.
3. Write the improper fraction.

Reduction of second mixed number to improper fraction.
4. Same as 1 (above) for second fraction.
5. Same as 2 (above) for second fraction.
6. Same as 3 (above) for second fraction.

Multiplication of improper fractions.
7. Numerator of first fraction multiplied by numerator of second fraction.
8. Denominator of first fraction multiplied by denominator of second fraction.
9. Reduction of improper fraction to a mixed number to find the answer (7 ÷ 8).

Thus in doing this test multiplication and division of integers play an important part. When one considers that the median accuracy in multiplication of integers is 80 per cent and the median accuracy in division of integers is 90 per cent, such a large percentage of error as shown in doing this type of examples by the horizontal method leads one to suspect that there must be some factor present which is not being considered but has an important influence upon the result.

The error may occur in any one of the nine steps noted in the foregoing. An error in the early steps may result in an increased error in the answer. Which method should be used in multiplication of mixed numbers depends on two important questions to neither of which do we have an answer.

1. By which method is the pupil most likely to obtain the correct answer?

2. How long does it take pupils to learn the method?

With sixth grade pupils it may be possible that the time used to teach the vertical method would be out of proportion to the results achieved through a greater percentage of accuracy providing this method is more likely to produce accurate results. It may be that pupils of sixth grade ability are not mature enough to learn the vertical method without an unreasonable expenditure of time. If this be true, the teaching of this process according to this method should be left to the high school classes. Whatever may be the truth in the case, if we are going to teach multiplication of mixed numbers, 50 per cent of the eighth grade class should obtain a higher accuracy than 0.

TEST 5.

Type of Examples Used in Test 5.

(1) $\frac{3}{4} \div 8$ (2) $9 \div \frac{3}{8}$ (3) $6 \div \frac{4}{5}$ (4) $8 \div \frac{3}{5}$

Tests 5, 6, and 7 consisted of examples in division of fractions. In Test 5 the pupils were required to divide an integer by a fraction (Examples 2, 3, and 4) or a fraction by an integer (Example 1). The percentage of error was very large, being 35.4 per cent for the first type and 49.5 per cent for the second type in Grade VII and 25.3 per cent and 42.9 per cent for Grade VIII in the respective types.

The chief cause for failure to get at least one example right in such a large percentage of cases is due to failure to invert the divisor. Either the pupil does not invert the divisor or, not knowing which is the dividend and which the divisor, inverts the dividend. A notable fact is that when the divisor is an integer the chance of failure is nearly doubled. Evidently the pupil does not know the possibilities in this case.

As pointed out under the discussion of the fourth test, habit plays a very important part. The particular habit which probably influences the results in this instance is one formed in work with integers. Here the pupil learned that the answer in division must be smaller than the dividend. There certainly comes a new experience into the life of the pupil when he sees for the first time a division example in which the answer is larger than the dividend. Unless it is made very clear it must be a difficult thing for pupils to understand how one can divide 4 by $\frac{1}{4}$ and get an answer of 16. To get such a large answer seems to violate all their previous conception of the meaning of division. It may be possible that the teaching of the idea of partition and measuring as pointed out in the earlier part of the bulletin would be a material help in conquering this difficulty.

The working of old habits may then be a partial explanation of the cause of such a large percentage of error in such simple examples as those given in Test 5.

The method of doing the examples was very largely the method of inversion. There were a few cases of the longer and more laborious method of reduction to a common denominator and then dividing one numerator by the other. These, however, were isolated cases and need only be mentioned in passing.

TEST 6.

Type of Examples Used in Test 6.

(1) $5678\frac{1}{3} \div 5$ (2) $2789\frac{2}{3} \div 4$ (3) $2467 \div 8\frac{1}{4}$

(4) $6752 \div 12\frac{1}{3}$

Table VIII shows the results in Test 6. The table is to be read as follows: in doing the type mixed number divided by an integer, 18.3 per cent of the eighth grade failed to do correctly, even in method, any of the examples, 60.7 per cent did the work correctly in at least one example, and 21 per cent used the correct method but did not have a single example correct. The rest of the table is read in a similar way.

TABLE VIII.

Showing Results Attained in Test 6.

Division of Fractions.

Grade.	Mixed Number Divided by Whole Number.						Whole Number Divided by Mixed Number.					
	Inversion.			Long Division.			Inversion.			Long Division.		
	Per Cent of Failure.	Per Cent Right.	Per Cent of Error.	Per Cent of Failure.	Per Cent Right.	Per Cent of Error.	Per Cent of Failure.	Per Cent Right.	Per Cent of Error.	Per Cent of Failure.	Per Cent Right.	Per Cent of Error.
VIII.............	18.3	60.7	21	83.4	15.5	1.1	20.0	31.4	48.6	97.3	2.7	.0
VII..............	30.7	40.3	29	87.2	10.3	2.5	40.5	15.4	44.1	99.2	.5	.3

The failure, when the work was done by inversion, was in inverting the wrong fraction as pointed out in analysis of Test 5. When the work was done by long division, the failure was the inability to dispose of the remainder. The large percentage of failures in both types when done by long division does not necessarily show anything because there was considerable evidence that this type of division of fractions had not been taught by many teachers. It is indeed possible to come to the same conclusion as reached in the analysis of Test 4, viz., if this type is going to be taught (and it is required by the new course of study) it should be taught effectively enough so that the median accuracy should be more than 39 per cent in the eighth grade.

TEST 7.

Type of Examples Used in Test 7.

(1) $\frac{3}{5} \div \frac{1}{3}$ (2) $3\frac{3}{4} \div \frac{1}{5}$ (3) $5\frac{4}{5} \div \frac{2}{3}$ (4) $6\frac{2}{5} \div \frac{4}{5}$

In this test the pupils were required to divide a fraction by a fraction (Example 1) or a mixed number by a fraction (Examples 2, 3 and 4). In the eighth grade about 20 per cent and 24 per cent failed respectively in each type, and in the seventh grade 39 per cent and 33 per cent failed respectively. The general cause of failure was the same as in Test 5, viz., difficulty with inversion in one form or another.

TABLE IX.

Showing Results in Accuracy in Ten Schools in Multiplication and Division of Fractions.

School.	Test 1.			Test 2.			Test 3.			Test 4.			Test 5.			Test 6.			Test 7.		
	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.	Median.	75 Percentile.	Number Reaching 100 Per Cent.
12	0	20	7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
27	10	60	8	30	70	8	0	60	17	0	0	0	0	30	0	0	0	0	0	20	1
30	60	90	31	20	60	5	50	100	33	0	0	2	0	50	15	0	0	7	0	70	0
57	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
60	50	90	18	0	30	1	0	60	11	0	0	0	0	0	0	0	0	0	0	0	0
56	50	80	26	20	50	2	0	0	1	0	0	0	0	0	2	0	0	2	0	0	0
62	60	100	77	0	30	14	50	100	64	0	0	9	0	20	10	0	0	4	0	10	7
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
46	0	60	11	0	0	1	0	0	6	0	0	0	0	0	0	0	0	0	0	0	1
61	70	90	28	20	40	4	20	80	24	0	0	6	0	30	7	0	0	4	0	50	1

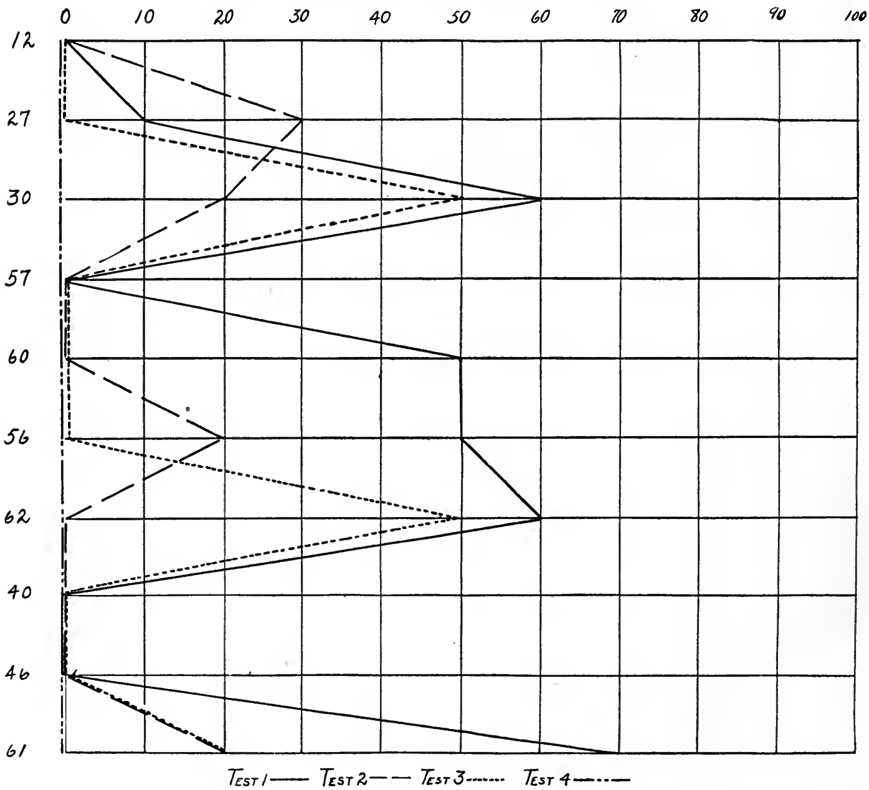
Analysis of Results in Grade VI.

Attention has already been called to the fact that the pupils of the sixth grade were apparently unable to do the work in multiplication and division of fractions. It seems worth while, however, to call attention to the

FIGURE 1.

Median Scores for Grade VI in Ten Schools in Multiplication of Fractions.

Median Scores for Tests 5, 6, and 7 are all 0.



great variation in the accuracy of the various schools in the work of multiplication and division of fractions.

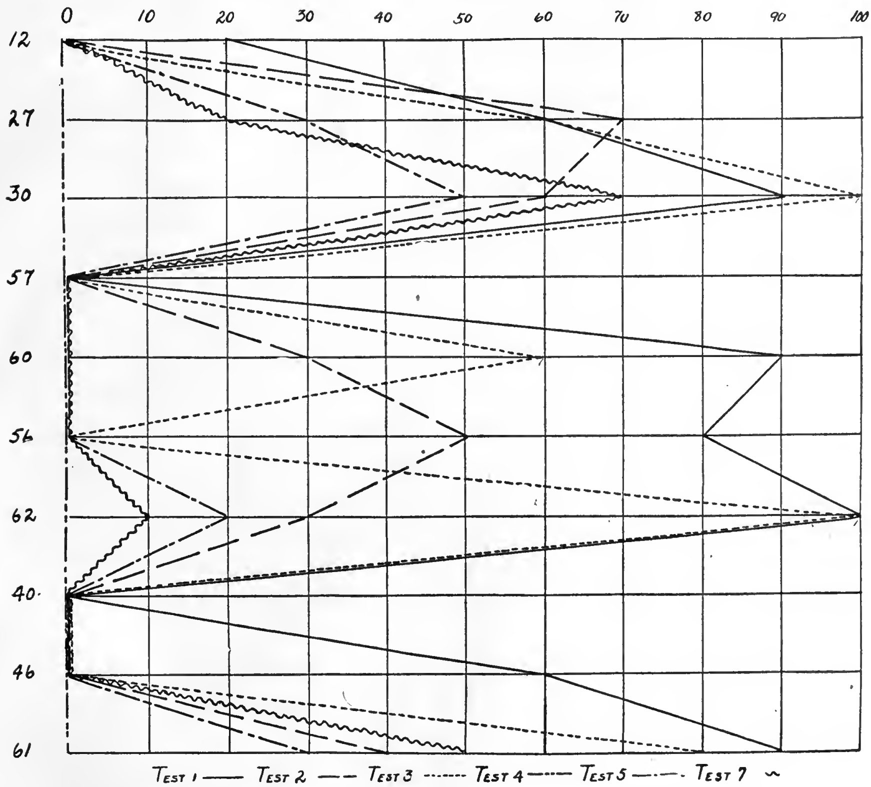
Table IX shows the results in accuracy in the ten schools which were tested. In the first column is shown the school designated by a number. Under each test is shown the median or the 50 percentile score, the

75 percentile score, which is the score above which are 25 per cent of the cases and below which are 75 per cent of the cases, and also the number of pupils who obtain a rank of 100 per cent accuracy. That is, the first line is read as follows: in school No. 12 the pupils in Test 1

FIGURE 2.

Seventy-five Percentile Scores for Grade VI in Ten Schools in Multiplication and Division of Fractions.

Seventy-five Percentile Score for Test 6 is 0.



have a median of 0. The 75 percentile point is at 20 and 7 pupils reach 100 per cent. In Test 2 the median and 75 percentile is 0 and there are none who reach 100 per cent. The results in the other tests are in succeeding columns. The record of the other schools is read in a similar way. Attention is called to the scores attained by Schools 30, 60, 62, 61, in contrast with Schools 12, 57, 40.

Figure 1 and Figure 2 show the data of Table IX in graphic form. Figure 1 shows the variation in the medians. The records for Tests 5, 6, and 7 are not given because they are all 0 and would be drawn over the line showing the results for Test 4. Figure 2 shows the variation in the 75 percentile scores for the ten schools. The record for Test 6 is not drawn because all the schools had a record of 0 the same as in Test 4. These two graphs emphasize the great variation in the results much more strongly than Table IX.

As before pointed out, it was not expected that the results of Grade VI would be high because of lack of preparation in Grade V, but neither was so great a variation expected among the different schools. This difference in variation is probably due to a difference in procedure in the different schools.

IV. Plan of Diagnosis for Teacher.

As a result of this study of common fractions, how may a teacher effectively check the work in her grade? How may a teacher determine and keep in some permanent form a record of the pupils' ability in common fractions? The following form of record (page 37) has been used and proved to be an effective method of doing this work. This sheet may be duplicated so that each pupil may have a copy. It was planned to be used as follows:

The teacher may give examples in addition similar to the types illustrated in the sixth grade course of study, School Document No. 19, 1917, and if the pupils get the answers right the type may be checked in column marked "R", if wrong it may be checked in column marked "W". These checks in the "wrong" column should be changed as fast as the pupil has mastered the type. After giving a series of lessons covering the various types and recording results, the teacher has a record of the ability of each individual in the room showing his strength and weakness. If a pupil fails in a problem, it can be determined immediately by referring to the record and by asking a few judicious questions, whether the difficulty is in the mechanics of the problem or in the problem itself, or both. In any case the teacher can easily tell to what extent she is required to give help to the pupil.

RECORD IN COMMON FRACTIONS OF

Name........................... School.........................

Age............ Room............

Grade............

	R.	W.		R.	W.
ADDITION.			MULTIPLICATION.		
Type 1..................			Fraction by an integer.....		
2..................			Integer by a fraction.......		
3..................			Integer by a mixed number,		
4..................			Mixed number by an integer,		
5..................			Fraction by a mixed number,		
6..................			Mixed number by a fraction,		
7..................			Fraction by a fraction......		
8..................			Mixed number by a mixed number................		
9..................					
10..................					
11..................					
12..................					
13..................					
14..................					
Mixed numbers............					
SUBTRACTION. (Without "Borrowing.")			DIVISION.		
Type 1..................			Fraction by an integer.....		
2..................			Integer by a fraction.......		
3..................			Integer by a mixed number,		
4..................			Mixed number by an integer,		
5..................			Mixed number by a fraction,		
6..................			Fraction by a fraction......		
7..................					
8..................					
9..................					
Fraction from mixed number,					
Mixed number from mixed number (with "borrowing"),					
Fraction from integer.......					
Fraction from mixed number,					
Mixed number from integer..					
Mixed number from mixed number................					

Summary and Conclusions.

1. The median accuracy in all but the simplest tests in multiplication is strikingly low in some schools and high in others. The range of variation in the medians of the ten school districts tested extends from 0 to 92 per cent.

2. Analysis of results in multiplication of mixed numbers and division of a mixed number by an integer and of an integer by a mixed number seems to indicate a lack of drill in these types commensurate with their difficulty. A large percentage of the pupils show an utter lack of knowledge of the process.

3. In the tests in division of fractions, the chief source of error is in the apparent inability of the individual pupil to distinguish between the dividend and the divisor. This results in an inversion of either dividend or divisor and sometimes both.

4. The low percentage of accuracy in Tests 4 and 6 where the process consists of a number of steps leads one to think that some factors are influencing the results which are not usually considered as important.

5. The ineffectiveness of the instruction as indicated by the large variation within the class is again shown in these tests in multiplication and division of fractions. Class room drills tend to increase the difference between the individuals of the class by increasing the ability of the bright pupil and not reaching the slow pupil. The difficulty of the individual can only be reached by individual instruction whether that pupil be advanced or retarded. The waste through nonpromotion, poor attendance, and other causes may be eliminated. It is highly important that we find out the reasons for failure through the analysis of results and apply the remedy needed in each individual case.

ANNOUNCEMENT.

Bulletins published by the Department are distributed by the Secretary of the School Committee, who will, so far as the supply on hand permits, fill mail applications for copies when such requests are accompanied by the price indicated.

No. I. Provisional Minimum and Supplementary Lists of Spelling. Words for Pupils in Grades I to VIII.
School Document No. 8. 1914. *Out of Print.*

No. II. Provisional Miminum Standards in Addition, Subtraction Multiplication and Division for Pupils in Grades IV to VIII.
School Document No. 9. 1914. *Out of Print.*

No. III. Educational Standards and Educational Measurement.
School Document No. 10. 1914. *Out of Print.*

No. IV. Spelling, Determining the Degree of Difficulty of Spelling Words.
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No. VI. English. Determining a Standard in Accurate Copying.
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