HIGH SCHOOL ARITHMETIC

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ARITHMETIC

FOR

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AND

COLLEGIATE INSTITUTES.

BY

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PREFACE.

The following work was prepared for the use of pupils in High Schools and Collegiate Institutes. As all pupils in these schools are required to possess, before admission thereto, a sufficient knowledge of arithmetic to enable them to solve easy problems such as those in Exercise I and IV pp. 45 to 55 and 75 to 85 of the present work, the author has taken for granted the possession of such knowledge by those who will use this book. In other words, he has sought to supplement and to continue without any unnecessary repetition the course of arithmetic begun in the *Public School Arithmetic.* Furthermore, as the book is not intended for private study but for class-instruction under the guidance and with the intelligent assistance of competent mathematical masters, the Author has endeavored to avoid encroaching on the province proper to the instructor and has in general given only the main outlines of proofs and investigations leaving it to the teacher to fill in the details and to supply preliminary illustrations.

The work consists of three distinct parts. The first part, forming chapters i to iv, treats of Notation and Computation ; the second part, chapter v, treats of Mensuration or Metrical Geometry ; and the third part, chapters vi, vii and viii, deals with Commercial Arithmetic.

Chapter I treats of numbers and notation and of units of measurement. The student will already be well enough acquainted with Arabic and Roman notation and with various compound systems to be able to use them more or less freely, but to know a subject is one thing, to know it in words, *i. e.*, to know it so clearly and distinctly as to be able to put that knowledge into words, is quite another thing ;—this chapter will it is hoped, help the student to put into words his knowledge of arithmetical notation and of our ordinary units of value, mass, space and time.

Chapter I with §§ 42, 62 and 63 of Chapter II and §§ 119 and 120 of Chapter IV lay a foundation on which may be built the theory of numbers and the rationale of the art of calculation. True the "Fundamental Theorems" of chapters ii and iv are, strictly speaking, *Postulates* defining and determining the particular kinds of addition, multiplication and involution here considered, but this is a distinction which only those who have advanced some way in their studies can understand, and the history of mathematics teaches that the method of presenting the subject here adopted is the easiest and the best for beginners.

The greater part of Chapter II consists of descriptions of some of the methods of computation employed by experts. The proper place for these is a manual on the art of teaching, but as they are not to be found in their proper place and as many of the pupils in our High Schools purpose becoming teachers, it has been thought better to insert these descriptions in the present work than to leave it a matter of chance whether teachers shall know and practice any other than the traditional school-room methods. Here as elsewhere throughout the book, the Author makes no pretension to originality; he has selected for description the best methods and processes with which a somewhat extensive acquaintance with the literature of elementary mathematics has made him acquainted.

Approximation is a part of arithmetic which has until lately been adequately discussed in only the higher classes of text-books, being, one might be led to conclude from their neglect of it, an unknown subject to the writers of the average school-book. But the great practical importance of the subject is at length compelling its fuller recognition in school-work, and it will receive more and more attention in proportion as arithmetical instruction ceases to be impractical and as teachers become better acquainted with the requirements of the counting-house, the workshop and the laboratory. In Chapter III, two methods of approximation are described; the first, Approximation by Continued Fractions; the second, Approximation by Abridgment of Decimal Computations. The former of these takes precedence in historical order and also on account of its theoretical simplicity and of the wide range of subjects to which it is applicable,—from the purely speculative questions of Farey's series and the partitions of numbers to the laboratory problem of determining the formula of an organic compound from its percentage composition;—but the latter method is superior in facility of adaptation to all ordinary computations.

Approximation by continued fractions was well known to the ancient Greek and Indian arithmeticians, so much so that in the oldest of their writings now extant it is introduced abruptly and used without explanation as an elementary subject with which their readers are assumed to to be already familiar. The whole theory of the subject is contained in the single theorem,—

$\frac{a+h}{b+k}$ *lies in value between* $\frac{a}{b}$ *and* $\frac{h}{k}$, *being greater than one and less than the other,*

and the immediate corollary therefrom,—

$\frac{a}{b}$, $\frac{ma+nh}{mb+nk}$ *and* $\frac{h}{k}$ *are in order of magnitude,*

a, *b*, *h*, *k*, *m* and *n* denoting (absolute) numbers. But in the calculation and use of continued fractions, no proof is needed of the theorem in the general form in which it is here stated, its truth being tested in each separate instance of its application. Hence no reference to the general theorem is required in Chap. III and no such reference is made therein.

In the arrangement of the factors in the contracted multiplication due to Oughtred and known by his name, the figures of the multiplier are written in reverse order, but the arrangement adopted in *Example 1*, p. 68 obviates the awkwardness of this reversal and is as simple as Oughtred's in every other respect. Teachers who prefer to discuss abridged computation before convergents and those who prefer to omit all discussion of the latter will find that the method of treatment which has been adopted will permit of their following their preference. Those who seek for a fuller treatment of the theory of contracted calculations will find it in the Arithmetics of Munn, Cox, Sang, Serret and Beynac, in Ruchonnet's Elements de Calcul Approximatif, Lionnet's Approximations Numeriques and Vieille's Theorie Generale des Approximations Numeriques.

Chapter IV contains an elementary discussion of Involution, Evolution and Logarithms. Special attention has been given to 'Horner's Method' of Involution and Evolution not only on account of its simplicity, it being merely an extension of the ordinary rule of 'Reduction', but also because of its power and generality as a process and of its great and varied utility. The chief value of logarithms, at least in elementary mathematics, lies in their usefulness as an instrument of calculation, but the surest way to enable pupils to remember how to use tables of logarithms is to require them to compute a portion of such a table considered as a table of exponents to base 10. Teachers who prefer to have their pupils learn at first the mere mechanical use of the tables and defer the theory of logarithms until logarithmic series is reached will omit §§ 114 to 122 and 124 to 136 and Exercises X to XV, but it might be well if they should note that the development of the theory of logarithms preceded that of the logarithmic and exponential series, preceded even the invention of generalized exponents, and that no large table of logarithms was ever computed by the immediate use of logarithmic series.

In connection with the subject of chapters iii and iv, the following extract is a sign of the progress now making in England:—

"The Council notices with pleasure, as an example of what may be done by an examining body in the way of encouraging sound mathematical teaching, the following "Remarks" in the prospectus of the Technical College, Finsbury, with reference to the Entrance Examination:—'In Arithmetic, marks will be deducted on those answers in which bad and antiquated methods are used; for example, if the Italian method in division is not followed; if decimal workings are not properly contracted; if remainders are given in fractions instead of in decimals; if logarithms are not used where their use would save time. [Logarithm Books containing Four-figure tables, are provided at the examination.] . . . .
(Candidates) should be able to work square root and cube root by Horner's method.'" *Extract from the Report of the Council of the Association for the Improvement of Geometrical Teaching; England, January, 1889.*

Chapter IV closes the subject of pure calculation with the exception of the short Appendix on pp. 315 to 317, in which the notation of circulating decimals is explained without the usual hidden reference to infinite series and the method of limits. The curious and those who care to spend time on a subject of no practical and of but little speculative importance may consult the Arithmetics of Sangster, Brook-Smith, Cox, Lock, and Sonnenschein and Nesbitt, or the Traite d' Arithmetique et d' Arithmologie of P. Gallez.

Chapter V consists of a short treatise on elementary metrical geometry. Much of the text, especially in the stereometry consists of proofs of important geometrical theorems which are not to be found in the authorized text-book of geometry. A few propositions not properly belonging to elementary mensuration, *e. g.*, § 201, p. 237, are given without proofs.

The Author begs to acknowledge his obligations in this part of his subject to Die Elemente der Mathematik of R. Baltzer, the Planimetrie and Stereometrie of F. Reidt in Schloemilch's Handbuch der Mathematik, the Traite de Geometrie Elementaire of Rouche and Comberousse and the Theoremes et Problemes de Geometrie Elementaire of M. Eugene Catalan.

Chapters VI, VII and VIII complete the course of elementary commercial arithmetic begun in the *Public School Arithmetic.* In selecting questions for Exercises I and IV, it was assumed that pupils could solve by the so-called Unitary Method, the simpler problems in commercial arithmetic, including simple interest and discount; but that method, however excellent as a mere answer-obtaining process, has an almost irresistible tendency to withdraw the attention of those who make exclusive use of it, from the general principles upon which all methods are founded. Numerous easy problems have therefore been proposed in the earlier exercises of Chap. VII, which it is hoped the teacher will take advantage of to endeavor to raise his pupils at this stage of their studies from infantile dependence on the unitary crawling-on-hands-and-knees method, and lead them to make direct application of general principles, including that widest of all principles, The Substitution of Equivalents. For further information concerning promissory notes and bills of exchange, teachers should consult the Bills of Exchange Act of 1890. Data for an unlimited number of problems on stocks, bonds and debentures will be found in the Stock Exchange Year-Book by Th. Skinner. Students who wish to advance to the higher questions on interest, annuities and life insurance, will find an elaborate discussion of these subjects in the Institute of Actuaries' Text-Book; Part I, Interest, including Annuities-Certain, by Wm. Sutton; Part II, Life Contingencies, including Life Annuities and Assurances, by Geo. King; to these two volumes may be added Ackland and Hardy's Graduated Exercises and Examples.

The Author desires to express his general indebtedness to the writings of De Morgan, Duhamel, Grassmann, Schlegel and Houel. He also takes pleasure in acknowledging his special indebtedness to Professor R. R. Cochrane, of Wesley University, Winnipeg, and Mr. Robert Gill, Manager of the Ottawa Branch of the Canadian Bank of Commerce, for much valuable advice and assistance, and for many practical problems, and he tenders these gentlemen his grateful thanks for their kindness.

CONTENTS.

The following selected course may be taken by candidates preparing for the primary examination and by all who have no special aptitude for mathematics:—Pages 1 to 55, 66 to 73, 75 to 93, problems 1 to 7 of Ex. vi, pp. 97 to 105 omitting last six lines, problems 1 to 12 of Ex. ix, pp. 251 to 298 and 302 to 314, and the portions of chapter v covering the requirements in mensuration required for the primary examination, but substituting verification by inspection of models for purely logical demonstration of the geometrical theorems quoted.

ARITHMETIC.

CHAPTER I.

OF NUMBERS AND NOTATION.

1. The simplest expression of a Quantity consists of two factors or components. One of these factors is the name of a magnitude which has been selected as a standard of reference and which is necessarily of the same kind as the quantity to be expressed. The other factor expresses how many magnitudes, each equal to the standard magnitude, must be taken to make up the required quantity. The standard magnitude is termed a Unit and its cofactor, the other component of the expression, is termed the Numerical Value of the quantity. Hence,—

A **Unit** is any standard of reference employed in counting any collection of objects, or in measuring any magnitude.

A **Number** is that which is applied to a unit to express the comparative magnitude of a quantity of the same kind as the unit.

A Number is the direct answer to the question, 'How many?'

Thus, when it is said that a certain slate is ten inches long, the number ten applied to the unit-length, inch, indicates the magnitude of a certain length, that of the slate, compared with the unit-length, an inch.

2. The number and the unit together indicate the **absolute magnitude** of the quantity; the number indicates the **relative magnitude**, or, as it is termed, the *Ratio of the quantity to the unit.*

3. **Numeration** is counting, or the expressing of numbers in words.

The ordinary system of numeration is the **Decimal System** (Latin, *decem* ten), so called because it is based on the number ten.

4. The names of the first group of numbers in regular succession are one, two, three, four, five, six, seven, eight, nine. Other number-names are ten, hundred, thousand, million, billion, trillion, quadrillion, quintillion, sextillion, ... , tenth, hundredth, thousandth, millionth, billionth, and so on, forming names from the Latin numerals.

5. The number one applied to any unit denotes a quantity which consists of a single unit of the kind named.

The number two applied to any unit denotes a quantity which consists of one such unit and one unit more.

The number three applied to any unit denotes a quantity which consists of two such units and one unit more.

And so on with the numbers four, five, six, seven, eight, nine; applied to any unit they denote quantities increasing regularly by one such unit with each successive number.

6. The number next following nine is ten, which applied to any unit denotes a quantity consisting of nine such units and one unit more.

Counting now by ten units at a time, as before we counted by single units, we get the numbers ten, twenty (*twen-tig*, twain-ten), thirty (*three-tig*, three-ten), forty (four-ten), ... ninety (nine-ten).

The names of the numbers between ten and twenty are, in order, eleven (*endlufon* from *en*, *án*, one and *lif* ten), twelve (*twá* two and *lif* ten), thirteen (three and ten), fourteen (four and ten), ... nineteen (nine and ten).

The names of the numbers between twenty and thirty, thirty and forty, ... are formed by placing the names of the numbers one, two, three, ... nine, in order after twenty, thirty, ... ninety.

7. The number hundred applied to any unit denotes a quantity which consists of ten ten-units.

Counting now by a hundred units at a time, as before we counted by single units, we get the numbers one hundred, two hundred, ... nine hundred.

The names of the numbers between one hundred and two hundred, two hundred and three hundred, ... are formed by placing the names of the numbers from one to ninety-nine in regular succession after one hundred, two hundred, ... nine hundred.

8. The number thousand applied to any unit denotes a quantity which consists of ten hundred-units.

Counting now by a thousand units at a time as before we counted by single units, we get the numbers one thousand, two thousand, ... nine thousand, ten thousand, eleven thousand, twelve thousand, ... twenty thousand . . . . . one hundred thousand . . . . . two hundred thousand, ... nine hundred and ninety-nine thousand.

The names of the numbers between one thousand and two thousand, two thousand and three thousand, ... are formed by placing in order the names of the numbers from one to nine hundred and ninety-nine,—the numbers preceding a thousand,—after one thousand, two thousand, ... nine hundred and ninety-nine thousand.

9. The number million applied to any unit denotes a quantity which consists of a thousand thousand-units.

The number billion applied to any unit denotes a quantity which consists of a thousand million-units.

The number trillion applied to any unit denotes a quantity which consists of a thousand billion-units.

And so on with the numbers quadrillion, quintillion, sextillion, septillion, ... ; applied to any unit they denote quantities increasing regularly one-thousand fold with each successive number.

Counting by a million units at a time, as before we counted by single units from one to nine hundred and ninety-nine, we get the numbers

one million, two million, ... ten million, ... one hundred million, ... nine hundred and ninety-nine million.

Counting by a billion units at a time, we get the numbers

one billion, two billion, ... ten billion, ... one hundred billion, ... nine hundred and ninety-nine billion.

This system is continued with the numbers trillion, quadrillion, quintillion, &c., counting from one of each to nine hundred and ninety-nine of the same.

The names of the numbers between one million and two million, two million and three million . . . . . nine hundred and ninety-nine million and one billion are formed by placing in order after one million, two million, ... nine hundred and ninety-nine million, the names of the numbers preceding a million.

The names of the numbers between one billion and two billion, two billion and three billion, ... nine hundred and ninety-nine billion and one trillion are formed by placing in order after one billion, two billion, ... nine hundred and ninety-nine billion, the names of the numbers preceding one billion.

This system is continued with the numbers trillion, quadrillion, quintillion, ... , by placing in order after one of each, two of each, three of each, &c., the names of all the numbers that precede one of the same.

10. English arithmeticians, following the example of Locke, the inventor of these names (An Essay concerning Human Understanding, Bk. II, Chap. xvi, § 6), employ billion as the name not of a thousand millions but of a million of millions; trillion then signifies million of billions, quadrillion means million of trillions, and so on. This gives what is known as the ENGLISH SYSTEM OF NUMERATION. These names are however of little practical importance, being seldom or never required in the ordinary affairs of life, while for scientific purposes another system, to be explained hereafter, is generally employed.

11. The number tenth applied to any unit denotes that quantity of which ten make up the unit.

The number hundredth applied to any unit denotes that quantity of which ten make up one tenth of the unit.

Consequently, one hundred of the hundredths of any unit make up that unit.

The number thousandth applied to any unit denotes that quantity of which ten make up one hundredth of the unit.

Consequently, one thousand of the thousandths of any unit make up that unit.

The number millionth applied to any unit denotes that quantity of which a thousand make up one thousandth of the unit.

The number billionth applied to any unit denotes that quantity of which a thousand make up one millionth of the unit.

The numbers trillionth, quadrillionth, &c.; applied to any unit denote quantities decreasing regularly one-thousandfold with each successive number.

12. We count by the tenth of a unit at a time, as before we counted from one to nine by a single unit each time, thus,—

one tenth, two tenths, ... nine tenths.

We count by a hundredth of a unit at a time, as before we counted from one to ninety-nine by a single unit each time, thus,—

one hundredth, two hundredths, ... ninety-nine hundredths.

We count the thousandths of a unit, the millionths of a unit, the billionths of a unit, &c., from one of each to nine hundred and ninety-nine of the same in like manner as we count thousands of the unit, millions of the unit, billions of the unit, &c.

13. Notation is the art of expressing numbers by means of certain marks or characters called numerals.

The system of notation in general use is the **Arabic Notation,** so named because it was introduced into Europe by the Arabs. Another system now employed for only a few special purposes such as numbering the chapters in books and marking the hours on clock-faces, is the **Roman Notation,** so called because it was the system in use among the ancient Romans.

14. The **Arabic Numerals,** styled also **Figures,** are

	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,
denoting	nought,	one,	two,	three,	four,	five,	six,	seven,	eight,	nine

respectively. The first of these is named NOUGHT, cipher or *zero;* the remaining nine are called **digits.** By means of these numerals and a dot called a **decimal point** we can write down any number expressed decimally. The method of doing so may be described as follows :—

A figure immediately to the left of the decimal point denotes so many single units.

A figure immediately to the left of the single-units figure denotes so many tens of the units, while a figure immediately to the right of the single-units figure denotes so many tenths of the unit.

Figures to the left of the tens-figure, taking them in order from right to left, denote so many hundreds of the unit, so many thousands of the unit, so many ten-thousands of the unit, so many hundred-thousands of the unit, so many millions of the unit, &c.

Figures to the right of the tenths-figure, taking them in order from left to right, denote so many hundredths of the unit, so many thousandths of the unit, so many ten-thousandths of the unit, so many hundred-thousandths of the unit, so many millionths of the unit, &c.

15. Since the number expressed by a digit depends not only on the particular digit made use of but also on the place the digit occupies relative to the place of the single-units figure, the several places *within* any number must be distinctly marked off. This is done by leaving no such place vacant ; places occupied by digits are sufficiently marked off by these digits,—one digit, one place ; *places unoccupied by digits are filled up with noughts,—one place, one nought.* In determining the place occupied by any of the figures of a number expressed in Arabic notation, count right or left, as the case may require, *from the ones or single-units figure*, not from the decimal point.

16. The sole use of the decimal point is to distinguish or point out the figure expressing single units, which figure is always the first to the left of the point. If in any number there are no figures to the right of the decimal point, that point is omitted and the right-hand figure then expresses the number of single units.

17. According to our system of numeration, the figures to the left of the decimal point are necessarily read in groups of three figures each—ones, thousands, millions, &c. The same system of reading may be conveniently applied to the figures to the right of the decimal point by counting and reading each tenth of the unit as a hundred of the thousandths of the unit, each hundredth of the unit as ten of the thousandths of the unit ; each ten-thousandth of the unit as a hundred of the millionths of the unit, each hundred-thousandth of the unit as ten of the millionths of the unit ; and so on.

18. The Arabic system of Notation may be exhibited in tabular form, thus :—

of Billions.			of Millions.			of Thousands.							of Thousandths.			of Millionths.			of Billionths.		
Hundreds.	Tens.	Ones.	Hundreds.	Tens.	Ones.	Hundreds.	Tens.	Ones.	Hundreds.	Tens.	Ones.	(Decimal Point).	Hundreds.	Tens.	Ones.	Hundreds.	Tens.	Ones.	Hundreds.	Tens.	Ones.
7	3	2	4	6	8	9	0	8	5	1	9	·	4	3	1	1	3	0	0	7	2

The number here placed as an example beneath the table is Seven hundred and thirty-two billion, four hundred and sixty-eight million, nine hundred and eight thousand, five hundred and nineteen, *and* four hundred and thirty-one thousandths, one hundred and thirty millonths and seventy-two billionths.

19. Previous to the introduction of the Arabic system of notation, the Roman system was in general use in Western Europe; but it was employed only for recording numbers, not in performing calculations. These were made by means of lines drawn on a sand-strewn tablet, or by the movement of counters arranged on a reckoning-board called an abacus.

20. The **Roman Numerals** and their equivalents in Arabic notation are

Early Forms.	Later Forms.	Arabic Equivalents.
I	I	1
Λ or V	V	5
X	X	10
ↆ or ⊥	L	50
Φ or ⊕	C	100
ᑎ	D	500
ↀ or ∞	M	1,000
ↁ		5,000
ↂ		10,000
ↇ		50,000
ↈ		100,000

After the invention of printing ᑎ and ↀ and the forms derived from them, were, for convenience in type-setting, modified to IƆ and CIƆ and forms correspondingly derived from these.

When letter-forms were used as numerals, it was a very common practice to distinguish a numeral from an ordinary letter by drawing a short horizontal stroke through the numeral, thus Đ, or over it. thus C̄. The former method was employed in early times, but the latter method superseded it in later times.

Quite recently it has been proposed to employ a short stroke over a Roman numeral to denote a thousandfold increase in the value of the numeral; thus V denoting 5, V̄ would then denote 5,000, V̿ would denote 5,000,000, &c. As no numeral higher than M is ever now made use of, this innovation is not needed; and as the bar over a letter has, for more than two thousand years past, been used *solely* to mark that the letter is a *numeral*, the innovation is worse than useless.

21. In Roman notation a numeral standing alone has the value assigned it in the preceding section. If the numeral be followed by another or by others of equal or of less value, the sum of their values is indicated. If a numeral be preceded by another of less value, the difference of their values is to be taken. Thus

II	expresses	two.	IV	expresses	four.
III	"	three.	IX	"	nine.
VI	"	six.	XL	"	40.
XXIII	"	23.	XC	"	90.
MDCCCLXXXVIII		1888.	CMXCIX	"	999.

22. The ordinary system of numeration is based on the number ten, but any other number might be adopted as the basis of a system, and in fact a system founded on the number twelve is employed to a large extent in counting small articles that are bought and sold by number. In this, **the duodecimal system,** the number twelve receives the name *dozen*, a dozen dozen is termed *a gross*, and a dozen gross is called *a great gross*. The transactions in which this system of counting is adopted, do not often involve numbers of higher order than a great gross, consequently, no names have been coined for these higher numbers, the gross gross or dozen great gross, the gross great gross, &c.

A similar state of affairs long prevailed in the decimal system of counting which, until the comparatively recent introduction of the word million, had no single-word name for any number greater than a thousand; thus a million was called ten hundred thousand. Even now the names above a hundred million are not fixed, billion meaning with some writers a thousand million, with other writers a million million, and a thousand million being named indifferently a billion and a milliard.

23. In the notation of the duodecimal system, Arabic numerals are made use of, but to distinguish a number expressed in this system from one expressed decimally, the names dozen, gross and great gross are inserted in the expression under the forms doz., gro., gr. gro. This also avoids the necessity for special symbols for ten and eleven. Thus 5 gro. 3 doz. 7 denotes five gross three dozen and seven, 11 doz. 5 denotes eleven dozen and five.

These numbers might otherwise be written 537_{xii} and $e5_{xii}$ and their product would be $505te_{xii}$ in which t and e denote ten and eleven respectively, and the subscript $_{xii}$ indicates that the numbers are expressed duodecimally. Expressed decimally these numbers are 763, 137 and 104531 respectively.

24. Sometimes more than one unit is employed in expressing a magnitude, as when it is said that the height of a certain doorway is seven feet three inches, or that a certain book weighs two pounds five ounces. In such case one of the units is taken as the principal or **Prime Unit**, and the other units, termed **Auxiliary Units**, are derived from it either by repeating it a given number of times, the resulting multiple forming *a unit of a higher order*, or by dividing it into a given number of equal parts, one of these parts forming *a unit of a lower order*.

Thus, if a gallon be taken as the prime unit in the quantity five bushels three pecks and one gallon, a peck, which is equal to two gallons, will be the unit of the first order higher than a gallon; and a bushel, which is equal to four pecks and therefore to eight gallons, will be the unit of the second order higher than a gallon. If a yard be taken as the prime unit in the length, three yards two feet and seven inches, a foot, which is the third part of a yard, will be the unit of the first order lower than a yard; and an inch, which is the twelfth part of a foot and consequently the thirty-sixth part of a yard, will be the unit of the second order lower than a yard.

25. A **Simple Quantity** is a quantity expressed in terms of a single unit.

A **Compound Quantity** is a quantity expressed in terms of two or more units. Compound quantities are often called, though not with strict accuracy, **Compound Numbers.**

26. The **Prime Units** of the quantities commonly treated of in the ordinary arithmetic are :—

VALUE,	Canadian. United States.	DOLLAR.
	British, - -	POUND STERLING.
Weight and **Mass,** -		**Pound Avoirdupois.**
Length, - - - -		**Yard.**
AREA, - - - -		SQUARE YARD.
VOLUME, - - - -		CUBIC YARD. GALLON.
Time, - - - -		**Mean Solar Day.**
Angle, - - - -		**Complete Revolution.**

27. A tabular statement of the numerical relations or ratios of any set of auxiliary units to their prime unit and to each other is called a Table of Values, of Weights or of Measures, the special designation depending upon the nature of the units.

TABLES OF VALUES, WEIGHTS AND MEASURES.

Canadian Money.

1,000 mills = 100 cents (ct.) = 1 dollar. $.

The mill is defined by statute but is not recognized in ordinary commercial transactions. Its use is practically confined to stating rates of local taxation which are generally described as so many mills on the dollar of assessed value; thus, a rate of ·015 is described as 15 mills on the dollar.

The dollar is defined by statute to be of such value that four dollars and eighty-six cents and two-thirds of a cent shall be equal in value to one pound sterling. ($\$4{\cdot}86\frac{2}{3} = £1.$)

United States money is practically the same as Canadian money both in values and in names, although in Canada the United States' silver coins are subject to a discount from their nominal values, and in the United States Canadian silver coins are similarly subject to a discount. The cause of this is that the market value of the silver in the coins,—the amount of gold for which the silver will exchange,—is less than the nominal values of the coins,—the values stamped on them; and while a coin passes current for its nominal value in the country of issue, it is worth only its market value as silver in any other country.

The one-dollar gold piece which is the prime unit, or standard of value in the United States, weighs 25·8 grains; nine-tenths of it is pure gold and the remaining one-tenth is an alloy of copper and silver.

British, or Sterling Money.

4 farthings = 1 penny, (d.)
12 pence = 1 shilling, (s. or /-.)
20 shillings = 1 pound, (£.)

1, 2 and 3 farthings are denoted by $\frac{1}{4}$d., $\frac{1}{2}$d., and $\frac{3}{4}$d. respectively.

Sterling Money is the money of account employed in Great Britain and Ireland. The prime unit is the pound which is the value of the coin named a sovereign. The sovereign is coined of standard gold which is composed of 11 parts of pure gold to 1 part of alloy. 1869 standard sovereigns weigh 480 ounces Troy of 480 grains each. Hence a sovereign should contain 123·27447 grains of standard gold of $\frac{11}{12}$ fineness, but a "remedy," or allowance for error is permitted of ·2 of a grain in weight, and of 2 parts in 1,000 in fineness. The least current weight is 122·5 grains ; below this the sovereign is "light," and is not legal tender, *i. e.*, it need not be received as of full value.

Avoirdupois Weight.

7,000 grains (gr.) = 16 ounces (oz.) = 1 pound		(℔.)
2,000 pounds = 1 ton		(T.)
480 grains = 1 ounce Troy		(oz. Tr.)

The one-sixteenth part of an ounce avoirdupois is named by statute a *dram*, but the term is not used in commerce, fractions of an ounce being employed instead. 100 pounds is called a *cental* or *hundredweight*, denoted by *cwt.* 2240 pounds is called a *long ton.*

The Dominion Weights and Measures Act declares that "All articles sold by weight shall be sold by avoirdupois weight, except that gold and silver, platinum and precious stones, and articles made thereof, may be sold by the ounce troy, or by any decimal part of such ounce."

The prime unit or standard of weight is the avoirdupois pound which is determined by the weight of a certain piece of platinum-iridium, called the Dominion standard, deposited in the Department of Inland Revenue at Ottawa. The weight of this standard is declared to be "6999·98387 grains when it is weighed in air at the temperature of 62 degrees of Fahrenheit's thermometer, the barometer being at 30 inches," and 7,000 such grains make one pound avoirdupois.

Avoirdupois Weight is used in Great Britain and Ireland, the grain, the ounce and the pound being the same as the Canadian weights bearing these names but the hundredweight is equal to 112 lb. and the ton, equal to 20 cwt., is equal to 2240 lb. The Table of British or Imperial Avoirdupois Weight is :—

7000 grains (gr.) = 16 ounces (oz.)	= 1	pound,	(lb.)
14 pounds	= 1	stone,	(st.)
8 stone	= 1	hundredweight,	(cwt.)
20 hundredweight	= 1	ton,	(T.)

Linear Measure.

12 inches (in) = 1 foot, (ft.)
3 feet = 1 yard, (yd.)
1,760 yards = 1 mile, (mi.)

In measuring land, surveyors use a **chain** 22 yards long, divided into 100 equal parts called **links.** Hence

100 links = 1 chain, (ch.)
80 chains = 1 mile.

In calculations the links are written as decimals of a chain.

The following measures are used only occasionally, or for special purposes :—

The *line* = $\frac{1}{12}$ inch.

The *size* = $\frac{1}{3}$ inch, used by shoemakers.

The *nail* = $2\frac{1}{4}$ inches = $\frac{1}{16}$ yard, formerly used in cloth measure. *The word is now obsolete as a term of measurement.*

The *hand* = 4 inches, used in measuring the height of horses.

The *fathom* = 6 feet and
The *cable-length* = 120 fathoms, } used by sailors.

The *rod, pole,* or *perch* = $5\frac{1}{2}$ yards, used in measuring land, *but not by surveyors.*

The *furlong* = 220 yards = $\frac{1}{8}$ mile.

The *league,* not a fixed length, but in England commonly = 3 miles.

The *geographical* or *nautical mile,* called also a minute of mean latitude, is $\frac{1}{10800}$ of the earth's semicircumference from pole to pole. Its length is 6,077 feet, but for rough approximations it is taken as = 6,000 feet = 1,000 fathoms.

The *Paris foot* = 12·79 inches.

The *French perch* = 18 Paris feet = 6·395 yards.

The *arpent,* or "acre" = 180 Paris feet = 64 yards *nearly.*

The three measures last-named are used under authority of the Dominion Weights and Measures Act for measuring lands in certain parts of the Province of Quebec. Distances less than a mile are often stated in that Province in "acres."

The prime unit, or standard of length, is the distance in a straight line between the centres of two gold plugs or pins in a certain bronze bar deposited in the Department of Inland Revenue at Ottawa, measured when the bar is at a temperature of 61·91 degrees of Fahrenheit's thermometer.

Surface Measure.

144 square inches (sq. in.)	=1 square foot,	(sq. ft.)
9 square feet	=1 square yard,	(sq. yd.)
4,840 square yards	=1 acre,	(A.)
640 acres	=1 square mile,	(sq. mi.)
10,000 square links	=1 square chain.	
10 square chains	=1 acre.	

Square links and *square chains* are used by land-surveyors in describing land-areas ; in calculations they are written as decimals of an acre.

In old deeds and descriptions of property the square rod pole or perch=$30\frac{1}{4}$ sq. yd., and the rood=$\frac{1}{4}$ acre are sometimes used, but these terms are now practically obsolete.

The prime unit or standard of surface measurement is the square yard, that is, a square surface whose sides are each one yard in length. *Hence the prime unit of surface measurement is derived from, and is determined by, the prime unit of linear measurement.*

Cubic, or Volume Measure.

1,728 cubic inches (c. in.)	=1 cubic foot,	(c. ft.)
27 cubic feet	=1 cubic yard,	(c. yd.)

Firewood and rough stone are measured by the *cord* of 128 cubic feet, which is equal to a pile of the material 8 feet long, 4 feet wide, and 4 feet high. The cord is not a statutory measure, that is, it is not defined by statute.

The prime unit of volume measure is the cubic yard, that is, a cube whose edges are each one yard in length. *Hence the prime unit of volume measurement is derived from, and is determined by, the prime unit of linear measurement.*

Measure of Capacity.

2 pints (pt.)	=1 quart,	(qt.)
4 quarts	=1 gallon,	(gal.)
2 gallons	=1 peck,	(pk.)
4 pecks	=1 bushel,	(bu.)

The *gill* is one-quarter of a pint; it is not defined by statute, but the term is used in the second schedule to the Dominion Weights and Measures Act of 1879.

The capacity of cisterns, reservoirs and the like is often expressed in *barrels* (bbl.) of $31\frac{1}{2}$ gallons each, or in *hogsheads* (hhd.) of 63 gallons each.

The legal bushel of certain substances is determined not by measure, but by weight. These weights are given in the following table :—

Blue Grass Seed -	14 ℔.	Indian Corn - - -	56 ℔.
Oats - - - - - - -	34 ℔.	Rye - - - - - - -	56 ℔.
Malt - - - - - - -	36 ℔.	Wheat, Beans, Peas, and Red Clover Seed - - - - -	60 ℔.
Castor Beans - - -	40 ℔.		
Hemp Seed - - -	44 ℔.		
Barley - - . - - -	48 ℔.	Potatoes, Turnips, Carrots, Parsnips, Beets and Onions	60 ℔.
Buckwheat - - -	48 ℔.		
Timothy Seed - -	48 ℔.		
Flax Seed - - -	50 ℔.	Bituminous Coal -	70 ℔.

A *barrel of flour* contains 196 ℔.

A *barrel of pork* or *of beef* contains 200 ℔.

A *quarter of wheat*, or of other grain = 8 bushels = 480 pounds. This measure is very commonly used in England but not in Canada.

A *chaldron* = 36 bushels, used in measuring coal, coke, and a few other articles.

Apothecaries subdivide the pint as follows :—

60 fluid minims (♏)	= 1 fluid drachm - - -	(fl. ʒ)
8 fluid drachms	= 1 fluid ounce - - -	(fl. ℥)
20 fluid ounces	= 1 pint - - - - - - -	(O.)

The prime unit or standard measure of capacity is the gallon containing ten Dominion standard pounds of distilled water weighed in air against brass weights with the water and the air at the temperature of sixty-two degrees of Fahrenheit's thermometer, and with the barometer at thirty inches. The weight of a cubic foot of water under these conditions is 62·356 lb., consequently a gallon contains, or is equal to, 277·118 cubic inches.

The Imperial gallon was formerly declared by statute to be of 277·274 cubic inches capacity, which is the volume of 10 lb. of pure water at 67·5 °F., but this part of the statute of weights and measures has been repealed.

A cubic foot of pure water at 52 ° F. weighs 62·4 lb. = 998·4 oz., and this is the weight usually adopted in calculations aiming at a high degree of accuracy, but where great accuracy is not required, 62·5 lb. = 1,000 oz. is taken as the weight of water per cubic foot.

This approximation is close enough for ordinary purposes, the more so as natural waters contain mineral matter in solution and consequently are somewhat denser than pure water.

Measures of Time.

60 seconds (sec.)	= 1 minute - - - - - - -	(min.)
60 minutes	= 1 hour - - - - - - - -	(hr.)
24 hours	= 1 day - - - - - - - -	(da.)
7 days	= 1 week - - - - - - -	(wk.)
365 days	= 1 common year - - -	(yr.)
366 days	= 1 leap year - - - - -	

The calendar year is divided into twelve parts called months; seven of these consist of 31 days each, four consist of 30 days each, and one (February) consists of 28 days—in leap years of 29 days. The lengths of the several months may be remembered from the following rhymes:—

Thirty days have September,
April, June, and November;
February has twenty-eight alone,
All the rest have thirty-one;
But leap year coming once in four,
Gives February one day more.

The civil day begins and ends at 12 o'clock midnight. A.M. denotes time before noon; M., at noon; P.M., after noon.

The prime unit of time is the day, or, strictly speaking, the mean solar day. A solar day is the time-interval between two successive transits of the sun's centre over the meridian; but as these intervals are of unequal length, we take the mean or average of all the solar days in the year, and to this mean solar day we give, in ordinary speech, the name DAY.

A year is the period of the earth's revolution about the sun, from some determinate position back again to the same position. If the starting point be the vernal equinox, the interval is called a tropical year and has been found to consist of 365·242216 mean solar days = 365 da. 5 hr. 48 min. 47½ sec. The tropical year determines the recurrence of the seasons, and of all the important phenomena of vegetation and life depending thereon, but to adopt it as the civil or calendar year, the year of ordinary business affairs, would involve having one part of a day belonging to one year and the remainder of the day belonging to the following year. This partition of a day is avoided by having civil years of two different lengths, the one of 365 days which is less than a tropical year, the other, called bissextile or leap year, of 366 days, which is greater than a tropical year. Now 400 tropical years would be greater than 400 years of 365 days each by ·242216 da. × 400 = 96·8864 da., or nearly 97 days, hence if every 400 years consist of 303 years of 365 days each and 97 years of 366 days each, the *average* civil year will be practically of the length of a tropical year, and the seasons will recur at the same times by the calendar. This is accomplished by making every year whose date-number is exactly divisible by 4, a leap-year, except in the case of the years whose dates are even hundreds, the date-numbers of these must be exactly divisible by 400. Thus the years 1880, 1884, 1888 were leap-years ; 1881, 1882, 1886, 1887 were not leap-years ; 1600 was and 2000 will be a leap-year ; 1800 was not and 1900 will not be a leap-year.

Neither the period of the earth's revolution about the sun nor the period of its rotation on its axis is absolutely constant. The latter is lengthening by the $\frac{1}{3927}$ part of itself per hundred years.

Angular Measure.

60 seconds (″)	= 1 minute - - - - - - -	(′)
60 minutes	= 1 degree - - - - - - -	(°)
90 degrees	= 1 quadrant or right angle.	
4 quadrants, or 360 degrees	= 1 circle or whole circuit.	

The prime Unit of angular measure is one complete revolution. Angles less than seconds are expressed as decimals of a second. Angles are always measured in practice by Angular Measure, but in many theoretical investigations another system of measurement, called Circular Measure, is adopted.

THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

28. The French or **Metric System** of Weights and Measures which is a decimal system, or system based on ten as the common scale of relation among each set of units of the same kind, is used in scientific treatises. Its use is permissive in Canada, the British Islands, and the United States, and it has been adopted absolutely as the sole system throughout great part of Europe and South America.

29. The prime units in this system and their ratios to the prime units of the Dominion or Imperial system are

Length	**Metre**	=1·09362311 yards.
Area	**Are**	=119·601150 square yards.
Volume or Capacity	**Litre**	= ·00130798582 cubic yards. = ·22021444 gallons.
Weight and Mass.	**Gramme**	=15·43234874 grains.

30. The fundamental unit of this system is the **metre** which was intended to be the ten-millionth, (·000,000,1), part of a quadrant of latitude, *i. e.*, of the distance of the pole of the earth from the equator, measured at the level of the sea. In this respect the legal metre is not quite exact, but this is of no consequence as practically the length of the metre is fixed in each country adopting the metric system, by means of Standard Metres marked on metal rods, just as the Standard Yard is determined. The original of these rods is the French Standard Metre,* a platinum rod deposited in the state archives at Paris.

From the metre are derived the are, the litre and the gramme. The are is equal to 100 square metres; the litre to the ·001 of a cubic metre; and the gramme to the ·000,001 of the weight in vacuo of a cubic metre of distilled water at its temperature of greatest density. In measuring wood, a STERE=1 cubic metre=1,000 litres is used, and in weighing heavy articles a MILLIER or *Metric Ton*= 1,000,000 grammes is employed.

31. The names of the auxiliary units in this system are formed by attaching certain prefixes to the names metre, are, litre and gramme respectively; thus:—

---

* The Canadian Standard Metre is defined by statute as equal to $1·0939\frac{4}{9}$ standard yards. It therefore differs appreciably from the French Standard Metre which is equal to 1·09362311 standard yards, the difference amounting to rather more than a yard in two miles.

micro –			·000,001		
milli –			·001		
centi –			·01		
deci –	– metre		·1		metre
	– are	denoting		of a	are
deka –	– litre		10		litre
hecto –	– gramme		100		gramme respectively.
kilo –			1000		
myria –			10000		
mega –			1,000,000		

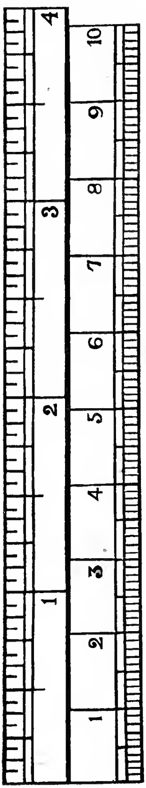
The name *micrometre* is shortened to *micron*, and *kilogramme* frequently to *kilog.* 100 kilogrammes is named a *quintal*, and the megagramme is the *Millier*. The ·001 part of a micrometre is termed a *micromillimetre.*

32. The metric system of Linear Measure may be tabulated as an example, thus :—

1,000 micromillimetres, ($\mu\mu$.)
=1 micron, - - - - - - (μ)
1,000 microns=1 millimetre, - (mm.)
10 millimetres=1 centimetre, (cm.)
10 centimetres=1 decimetre, (dm.
10 decimetres=1 metre, - - - (m.)
10 metres=1 dekametre, - - (Dm.)
10 dekametres=1 hectometre, (Hm.)
10 hectometres=1 kilometre, (Km.)
1,000 kilometres=1 megametre. (Mgm.)

33. The accompanying scales and diagram may perhaps assist those accustomed to Imperial or Dominion measures alone, in the realization of the actual magnitudes of the metric units. The upper of the two scales is 4 inches in length and is divided into inches and subdivided into sixteenths of an inch. The lower scale is 1 decimetre in length and is divided into 10 centimetres and subdivided into 100 millimetres.

Four Inches in Sixteenths of an Inch.



One Decimeter in Millimetres.

1 sq. cm.

1 Square Decimetre.

Each side of this square measures
1 decimetre, or
$3\frac{15}{16}$ inches, very nearly.

A litre is a cube each face of which has the dimensions of this square.

A gramme is the weight of a cubic centimetre (see small square above) of distilled water, weighed *in vacuo* at temperature of maximum density, 39·1 F. A litre or cubic decimetre of such water weighs 1 kilogramme or $2\frac{1}{5}$ lb. nearly.

34. The following approximations may be noticed :—
5 inches is very nearly 127 millimetres.
8 kilometres is somewhat less than 5 miles.
10 metres is nearly 11 yards.
64 metres is very nearly 70 yards.
64 miles is very nearly 103 kilometres.
43 square feet is nearly 4 centiares.
61 centiares is nearly 55 square yards.
2 hectares is nearly 5 acres.
22 gallons is nearly 100 litres or 1 hectolitre.
22 pounds is nearly 10 kilogrammes.

35. It is evident that whenever a quantity is expressed in the decimal notation in terms of a single unit, a decimal system of values, weights or measures is employed. Thus 23·75 lb. expresses 23 lb. 12 oz. decimally, and 3·375 yd. is the decimal equivalent of 3 yd. 1 ft. $1\frac{1}{2}$ in.

The metric system being a decimal system, it is not necessary to employ more than a single unit in expressing any quantity metrically. Thus, 3 dekametres 7 metres 4 decimetres 5 centimetres and 7 millimetres is written 37·457m. which is read 37 metres 457 millimetres. If it becomes necessary to change the unit of the expression, such change is accomplished by shifting the decimal point, at the same time changing the unit-denomination. Thus, 37·457m = 374·57dm. = 3745·7cm. = 37457mm. = 3·7457Dm. = ·037457Km.

WORD SYMBOLS.

36. Certain words and phrases recur so often in Arithmetic that it is found convenient to represent them by easily made symbols. These are

$=$, read *is equal to, will be equal to*, &c., thus $\frac{8}{12}=\frac{2}{3}$;
$\equiv$, read *is, is the same as, represents, denotes*, thus V$\equiv$5, D$\equiv$500.
$>$, read *is greater than*, thus $\frac{6}{7}>\frac{4}{5}$;
$<$, read *is less than*, thus $\frac{2}{3}<\frac{4}{5}$;
$\therefore$, read *therefore, consequently, hence;*
$\because$, read *because, since*, thus $\because$ $\frac{6}{7}>\frac{4}{5}$ and $\frac{2}{3}<\frac{4}{5}$, $\therefore$ $\frac{6}{7}>\frac{2}{3}$.

CHAPTER II.

THE FOUR ELEMENTARY OPERATIONS.

ADDITION AND SUBTRACTION.

37. Addition is the operation of finding that quantity which is made up as a whole of two or more given quantities as its parts.

The quantities to be added together are called **addends,**—or *addenda.*

The result of the addition is termed the sum of the addends.

Since the sum is the whole of which the addends are the parts,—

Addends and sum must all be quantities of the same kind, i. e., they must all have the same unit.

38. The sign of addition is +, read *plus,* meaning *increased by.* The sign + placed before any quantity indicates that the quantity is an addend. Thus 8 + 3 is read 'eight plus three' and denotes that 3 is to be added to 8. In like manner 24+9+5 is read 'twenty-four plus nine plus five,' and denotes that 9 is to be added to 24 and then 5 added to the sum.

The sum of any number of given quantities is expressed by writing the quantities in a row in the order in which they are to be added, with the sign + between every adjacent pair.

39. Subtraction is the operation of finding the part of a given quantity which remains after a given part of the quantity has been taken away.

The quantity from which a part is to be taken away is called the **minuend.**

The part of the minuend which is to be taken away is called the **subtrahend.**

The result of the subtraction is called the **remainder** and also the **difference** between the minuend and the subtrahend.

Since the minuend is the whole of which the subtrahend and the remainder are the parts,—

Minuend, subtrahend and remainder must all have the same unit; and,—*If the subtrahend be added to the remainder the sum will be the minuend.*

40. The sign of subtraction is $-$, read *minus*, meaning *diminished by*. The sign $-$ placed before any quantity indicates that the quantity is a subtrahend. Thus $8-3$ is read 'eight minus three' and denotes that 3 is to be subtracted from 8. In like manner $24-9-5$ is read 'twenty-four minus nine minus five' and denotes that 9 is to be subtracted from 24 and then 5 subtracted from the remainder. So $24+9-5$ denotes that 5 is to be subtracted from $24+9$ while $24-9+5$ denotes that 5 is to be added to $24-9$.

41. An expression consisting of a succession of addends and subtrahends, such as $8+5-3+6-3-4$, is called an **aggregate.**

The several parts, the addends and the subtrahends, as 8, $+5$, -3, $+6$, -3, -4, are called the **terms** of the aggregate; and

The quantity which results from collecting the terms by performing the indicated additions and subtractions is called the **total** or sum of the aggregate.

42. The **Fundamental Theorems** of Addition and Subtraction are ;—

I. *If equals be added to equals, the wholes are equal.*

II. *If equals be subtracted from equals, the remainders are equal.*

III. *The sum of two addends will be the same whether the second be added to the first or the first be added to the second.*

IV. *Adding to an addend adds an equal quantity to the sum.*

V. *Subtracting from an addend subtracts an equal quantity from the sum.*

VI. *Adding to the minuend adds an equal quantity to the remainder.*

VII. *Subtracting from the minuend subtracts an equal quantity from the remainder.*

VIII. *Adding to the subtrahend subtracts an equal quantity from the remainder.*

IX. *Subtracting from the subtrahend adds an equal quantity to the remainder.*

X. *Adding zero to any quantity leaves the quantity unchanged.*

Theorms III to IX may be stated in a single theorem, thus,—

Changing the order of collecting the terms of any aggregate, does not change the total or sum of the aggregate.

43. To **prove** any calculation is to perform another calculation that will *test* or **put to proof** the correctness of the results of the first calculation.

44. The simplest and best way **to prove a result in addition** is to repeat the addition, adding downwards the columns that were added upwards on the first addition and upwards the columns that were then added downwards.

45. In the additions of tabulated numbers which are to be added both vertically and horizontally the agreement of the grand total of the row of partial sums with the grand total of the columns of partial sums is, in general, a sufficient test of mere correctness, but if a mistake has been made, it is not enough to detect its existence, the mistake must be located in the partial sums and there corrected. This location and correction is often greatly facilitated by what is known as **Computers' Addition.** In this method *the sum of each column is set down separately*, the right hand figure of each partial sum being placed under the column from which it is derived, and the other figures in their order diagonally downwards to the left. These partial sums are then added together to obtain *the* sum. By this arrangement the addition of any column can be tested independently of that of the preceding column, no knowledge of the 'carried' number being required. Thus if it be known that an error has been committed in the addition of the *hundreds*, it can be discovered and corrected without adding the *tens* to ascertain the 'carriage.'

Example.

8784	27
3295	19
2133	9
8594	26
6272	17
9585	27
7986	30
9286	25
5993	26
7298	6
5463	20
61928	

In this example, the sum of the first column is 38. The 8 is placed under the first column and the 3 under the second column but in the line next below that of the 8. The sum of the second column is 69. The 9 is placed under the second column immediately on the left of the 8 and above the 3 of 38, and the 6 is placed on the left of the 3. The sum of the third column is 42. The 2 is placed under the third column immediately on the left of the 9 of 69 and the 4 diagonally below to the left. The sum of the fourth column is 57, of which the 7 is written beneath the fourth column from which it was obtained, and the 5 is placed diagonally below it to the left. These partial sums are now added to obtain *the* sum, 61928.

46. Since in this method the columns are added independently, the result *may* be tested by adding together the digits in each horizontal row as shown in the example. The total of these sums, —in the example, 206,—should be the same as the total of the column-sums,—in the example, 38, 69, 42 and 57,—treated as a row of mutually independent numbers.

47. Some computers prefer to arrange the figures of the column-sums from right diagonally upwards to left and to add in the carried numbers as is done in the ordinary method. Taking the preceding example, the lowest addend and the result would by this arrangement appear as in the margin, the upper addends being here omitted merely to save space. The column-sums would be 38, 72, 49 and 61. The 6 of 61, the last column-sum, is not written in the carriage-line but is placed at once in the sum-line.

```
 ::::
 5993
  473
-----
61928
```

48. Computers' Subtraction. The best way to perform subtraction is the method based on the fundamental theorem that the sum of the subtrahend and the remainder is equal to the minuend.

Example. From 435,846 take 259,784.

It is required to find what number added to 259,784 will make 435,846.

Write the subtraction under the minuend so that the figures of the same decimal order in each shall be in the same vertical column as in the margin.

```
435846
------
259784
176062
```

To 4, the right-handed figure of the subtrahend, 2 must be added to make up 6 the right-hand figure of the minuend; put down this 2 as the right-hand figure of the remainder. The 8 (ten) of the subtrahend cannot be *made up* to the 4 (ten) of the minuend, so make it up to 14 (ten), this requires that 6 (ten) be added; put down this 6 (ten) in the remainder. To the 7 (hundred) of the subtrahend add 1 (hundred) carried from the 14 (ten), thus making it 8 (hundred), and 0 (hundred) is required to make this 8 (hundred) up to the 8 (hundred) of the minuend; put 0 (hundred) in the remainder. To the 9 (thousand) of the subtrahend add 6 (thousand) to make up 15 (thousand) which will give the 5 (thousand) of the minuend; and put down this 6 (thousand) in the remainder.

To the 5 (ten thousand) of the subtrahend add 1 (ten thousand) from the 15 (thousand) already made up and then add **7** (ten thousand) more to make up 13 (ten thousand) in the minuend, putting down this 7 (ten thousand) in the remainder. To the 2 (hundred thousand) of the subtrahend carry 1 (hundred thousand) from the 13 (ten thousand) and add **1** (hundred thousand) more to make up the 4 (hundred thousand) of the minuend, putting this 1 (hundred thousand) in the remainder.

Fancy you are doing addition with the sum at the top of the columns of addends and work thus setting down, as you pronounce them, the figures here printed in thick-faced type :—

4 and **2**, six ; 8 and **6**, fourteen ; 8 and **0**, eight ; 9 and **6**, fifteen ; 6 and **7** thirteen ; 3 and **1** four.

After a little practice the minuend-sums need not be pronounced.

The actual character of the process will perhaps be better comprehended by working a few examples with the subtrahend written as the lower of two addends, and the minuend written as their sum, the problem being to find the other addend. Arrange the preceding example in this way, (see margin), and repeat the working given above.

```
176062
259784
------
435846
```

49. This method is nearly always adopted in "making change" and so lends itself to calculations involving both additions and subtractions that it is almost universally employed by professional computers, and is generally known as Computers' Subtraction.

Example. From 9564 take 1357 + 498 + 1976 + 83 + 3758.

Arrange the subtrahends in column under the minuend as addends are arranged in addition ; —see margin.

Add the subtrahends together and 'make up' to the minuend, setting down the 'making up' number. Thus

```
9564
----
1357
 498
1976
  83
3758
----
1892
```

1st. Column ;	11,	17,	25,	32 & **2** ;	34 ;	*carry* 3 :		
2nd. "	8,	16,	23,	32,	37 & **9** ;	46 ;	" 4 :	
3rd. "	11,		20,	24,	27 & **8** ;	35 ;	" 3 :	
4th. "	6,		7,		8 & **1** ;	9.		

50. To prove any subtraction add the subtrahend to the remainder, the sum should be the same as the minuend.

MULTIPLICATION AND DIVISION.

51. The simplest expression of a quantity consists of two components, one naming the unit, the other stating the number of such units in the quantity. But since the unit is a magnitude it may itself be considered as a quantity and expressed in terms of another unit which relative to it is called a **primary unit.** Thus *the expression of a given quantity may consist of* THREE *components, one* NAMING A PRIMARY UNIT, *a second stating* THE NUMBER *of these primary units composing a standard quantity or* DERIVED UNIT, *and a third stating* THE NUMBER *of these derived units in the* GIVEN QUANTITY.

52. The number of primary units in such a quantity is called the **product** of the number of primary units in the derived unit multiplied by the number of derived units in the quantity.

Thus 35 marbles is the same quantity as 5 counts of 7 marbles each, therefore 35 is the product of 7 multiplied by 5. In this case the primary unit is a marble and the derived unit is a count of 7 marbles.

Again $\frac{1}{2}$ yd. is the same quantity as $\frac{2}{3}$ of $\frac{3}{4}$ yd., hence $\frac{1}{2}$ is the product of $\frac{3}{4}$ multiplied by $\frac{2}{3}$. In this case the primary unit is a yard and the derived unit is $\frac{3}{4}$ of a yard.

53. Multiplication is the operation of finding the product of two numbers ; in other words,

Multiplication is the process of finding the number of units of a given kind in a quantity which contains a given number of standard quantities each consisting of a given number of units of the given kind.

The numbers to be multiplied together are called the **factors** of the product.

The factor which is to be multiplied by the other is called the **multiplicand.**

The factor by which the other is to be multiplied is called the **multiplier.**

54. A boy who has to read 18 pages of 38 lines each wishes to know how many **lines** he has to read. Here it is required to find the number of lines in the quantity 18 pages-of-38-lines, a quantity whose unit, a page-of-38-lines, is expressed in terms of the primary unit, a line. The required number may be found by counting, or by addition, or by multiplication. In this case the product, that of 38 and 18, *may* be obtained by addition.

A man is required to weigh out $\frac{3}{14}$ of an article the whole weight of which is $\frac{4}{15}$ of a pound. What part of a pound must he weigh out? Here it is required to find the number of pounds in the quantity $\frac{3}{14}$ of $\frac{4}{15}$ lb., a quantity whose unit, $\frac{4}{15}$ lb., is expressed in terms of the primary unit, a pound. The required number may be found by counting, or by a series of additions and subtractions, or by multiplication. In this case the product, that of $\frac{4}{15}$ and $\frac{3}{14}$ *may* be obtained by a series of additions and subtractions.

There are, however, cases—to be treated of hereafter, (see § 120), in which the product of two factors cannot be obtained by mere counting and in which, in consequence, multiplication cannot be replaced by or be resolved into any number, however great, of additions and subtractions.

Thus certain calculations may be performed by addition or by multiplication indifferently; other calculations, as is known, belong to addition exclusive of multiplication, and still other calculations belong to multiplication exclusive of addition.

55. In arithmetical multiplication, the multiplier must be simply a number, for it states the number of multiplicands in the product; but for the purely numerical multiplicand there may be substituted the derived unit, the quantity whose absolute magnitude is expressed by taking as components the multiplicand proper and the primary unit. In such case the product is the quantity whose absolute magnitude is expressed by the purely numerical product as one component, and the primary unit as the other. But although the primary unit may thus appear in the multiplicand, *it is not itself operated on in any way, the multiplier operating on the numerical component of the multiplicand and on it alone.*

56. The sign of multiplication is ×, read "*multiplied by.*" The sign × placed before any number indicates that the number is a multiplier. Thus 5 lb. × 4 is read "5 lb. multiplied by 4" or "4 times 5 lb." and denotes a weight equal to 4 weights of 5 lb. each. In like manner $\frac{3}{4}$ yd. × $\frac{5}{6}$ is read "$\frac{3}{4}$ yd. multiplied by $\frac{5}{6}$" or "$\frac{5}{6}$ of $\frac{3}{4}$ yd." and denotes a length which is $\frac{5}{6}$ of the length, $\frac{3}{4}$ yd.

The product of two or more factors may be expressed by writing the factors in a row with the sign × between every adjacent pair. If there are more than two factors and if in none of the factors there appears a decimal point, the sign × may be replaced by a simple dot or period; thus $3 \times 5 \times 7 \times 11 \times 13$ may be written 3.5.7.11.13, but $3{\cdot}5 \times 7 \times 11{\cdot}13$ must not be written 3·5.7.11·13, as the difference in position between the decimal point and the period is not marked enough to prevent confusion.

57. Division is the operation of finding either of two factors, there being given the other factor of the two and also their product.

The factor found is called the **Quotient.**

The factor given is called the **Divisor.**

The given product of the Divisor and the Quotient is called the **Dividend.**

58. Division is the inverse of multiplication, for in multiplication two or more factors are given and it is required to find their product; in division, on the other hand, the product of two factors is given and also one of the two factors and it is required to find the other factor. This being the case, division may appear under either of two guises according as the factor to be found is the multiplicand or the multiplier, when the dividend is recalculated as the product of the divisor and the quotient.

In the first case, that in which the quotient is to the divisor as multiplicand to multiplier, the divisor is simply a number and the quotient is a quantity of the same kind as the dividend.

Examples. If 75 ct. be divided into 15 equal parts, what will be the value of one of these parts? Answer, 5 ct.; for 5 ct. × 15 = 75 ct. What is the weight of an iron rod if $\frac{35}{36}$ of it weigh $\frac{14}{15}$ lb.? Answer, $\frac{24}{25}$ lb.; for $\frac{24}{25}$ lb. $\times \frac{35}{36} = \frac{14}{15}$ lb.

In the second case, that in which the quotient is to the divisor as multiplier to multiplicand, the divisor is a quantity of the same kind as the dividend and the quotient is simply a number.

Examples. How many five-cent pieces will make up a sum of 75 ct.? Answer, 15; for 5 ct. × 15 = 75 ct. What part of an iron rod will weigh $\frac{14}{15}$ lb. if the whole rod weigh $\frac{24}{25}$ lb.? Answer, $\frac{35}{36}$; for $\frac{24}{25}$ lb. $\times \frac{35}{36} = \frac{14}{15}$ lb.

59. There are three signs of division, viz., :, ÷, and /, all read "*divided by.*" Any of these signs placed before a number indicates that the number is a divisor. Thus 75 ct. ÷ 15 is read "75 ct. divided by 15," and denotes that 75 ct. is to be divided by 15. In like manner 36 : 3 : 4 is read "36 divided by 3, divided by 4," and denotes that 36 is to be divided by 3 and the quotient then divided by 4. So $36 \times 3 : 4$ denotes that 36 is to be multiplied by 3 and the product divided by 4, while $36 : 3 \times 4$ denotes that 36 is to be divided by 3 and the quotient multiplied by 4.

60. In an expression containing a succession of multipliers and divisors, the operations are to be performed in order from left to right. Thus,

$$9 \times 5 \div 3 \times 6 \div 10 \div 4 = 45 \div 3 \times 6 \div 10 \div 4 = 15 \times 6 \div 10 \div 4$$
$$= 90 \div 10 \div 4 = 9 \div 4 = 2\tfrac{1}{4}.$$

Compare this with

$$9+5-3+6-10-4=14-3+6-10-4=11+6-10-4$$
$$=17-10-4=7-4=3.$$

In an aggregate whose terms contain multipliers and divisors, *the multiplications and the divisions are to be performed* BEFORE *the additions and the subtractions are made.* Thus,

$$6 \times 5 + 15 \times 4 \div 3 - 16 \div 2 \times 3 = 30 + 20 - 24 = 26.$$

61. The signs $\div$ and / are employed exclusively by English-speaking nations, all other nations denote division by the sign : alone. Furthermore, while the laws governing the use of the sign : are definite and invariable, the signs $\div$ and / are employed in one way by one writer and in another way by another. Thus $30 \div 5 \times 3$ would be interpreted by one author "30 divided by 5 and the quotient multiplied by 3," while another would interpret it "30 divided by 5×3." The first author would write $30 \div 5 \times 3 = 18$; the second author would write $30 \div 5 \times 3 = 2$.

In like manner English mathematicians are not united in their views regarding the employment of the sign $\times$. Many authors place the multiplier *before* the sign $\times$ which they then read "multiplied into," or simply "into"; their order of arrangement is thus multiplier, sign, multiplicand. Other authors, following the uniform practice of 'continental' mathematicians, adopt the arrangement 'multiplicand, sign, multiplier,' thus preserving the analogy in use between the signs $\times$ and $\div$ and the signs $+$ and $-$.

62. The **Fundamental Theorems** of Multiplication and Division are :—

XI. *If equals be multiplied by equals the products are equal.*

XII. *If equals be divided by equals the quotients are equal.*

XIII. *The product of two purely numerical factors will be the same whether the first factor be multiplied by the second or the second factor be multiplied by the first.*

XIV. *Multiplying a factor by any number multiplies the product by the same number.*

XV. *Dividing a factor by any number divides the product by the same number.*

XVI. *Multiplying the dividend by any number multiplies the quotient by the same number.*

XVII. *Dividing the dividend by any number divides the quotient by the same number.*

XVIII. *Multiplying the divisor by any number divides the quotient by the same number.*

XIX. *Dividing the divisor by any number multiplies the quotient by the same number.*

XX. *Multiplying any number by one leaves the number unchanged.*

XXI. *If one of the factors be zero, the product will be zero.*

Theorems XIII to XIX may be stated in a single theorem, thus :—

If an expression contain a succession of multipliers and divisors, changing the order of the multipliers and the divisors does not change the value of the expression.

Example. $10 \div 5 \times 12 \div 3 = 10 \div 5 \div 3 \times 12 = 10 \div 3 \times 12 \div 5$
$= 10 \times 12 \div 5 \div 3 = 8.$

63. The **Fundamental Theorems** connecting the operations of addition and subtraction with the operations of multiplication and division are,—

XXII. *Multiplying the several terms of an aggregate by any number multiplies the aggregate by that number.*

XXIII. *Dividing the several terms of an aggregate by any number divides the aggregate by that number.*

64. Scholars' Multiplication. Multiplications in which both multiplier and multiplicand require many digits to express them are generally best made by means of a *table of multiples* of the multiplicand. This table may be formed by successive additions of the multiplicand written on a slip of paper to be moved down the column of multiples as the successive additions are made. The multiples should extend from the first to the tenth, the last testing the accuracy of the work ; and, for convenience of reference, straight lines should be drawn under the first, fifth and ninth multiples.

Example. Find the product of 74,853,169 and 2968457.

Multiple Table.

1	2968457
2	5936914
3	8905371
4	11873828
5	14842285
6	17810742
7	20779199
8	23747656
9	26716113
10	29684570

```
        2968457
       74853169
       --------
       26716113
      17810742
      2968457
     8905371
   14842285
  23747656
 11873828
20779199
---------------
222198413490233
```

65. Scholars' Division. A table of multiples of the divisor may be employed in the case of division in which both divisor and dividend require many digits to express them.

Example. Divide 2808332109244 by 58679.

Multiple Table.

1	58679
2	117358
3	176037
4	234716
5	293395
6	352074
7	410753
8	469432
9	528111
10	586790

```
           47859236
58679)2808332109244
      234716
       461172
       410753
        504191
        469432
         347590
         293395
          541959
          528111
           138482
           117358
            211244
            176037
             352074
             352074
```

66. Computers' Multiplication. In multiplying by a number requiring several digits to express it, we may set down each partial product as it is calculated, and then sum the whole of them; or, as each partial product after the first is calculated, we may add to it the sum of all the previously calculated partial products.

Example. Multiply 56437 by 3852967.

```
      56437
    3852967
    -------
     395059
    378127
   545745
  167448
 298929
481388
-----------
217449898579
```

The first line of products is simply 7 times the multiplicand. The next line is formed thus:—

6 times 7 and 5. the tens of the first line of products, =47. Write the 7 beneath the 5 added in and carry 4. 6 times 3 and 4 carried=22. Write 2 on the left of the 7 last written and carry 2. 6 times 4 and 2 carried and 5 from the first line of products=31. Write 1 on the left of the 2 last written and carry 3. 6 times 6 and 3 carried and 9 from the first line of products=48. Write 8 on the left of the 1 last written and carry 4. 6 times 5 and 4 carried and 3 from the first line of products=37. Write 37 on the left of the 8 last written. The partial product thus formed with the 9 brought from the line above is 378127*9* which is 67 times 56437, the multiplicand.

The third line of partial products is formed by multiplying the multiplicand 56437 by 9 (hundred) and adding in successively the proper digits of the second partial product, thus:—

9 times 7 and 2 from the second partial product=65. Write 5 beneath the 2 added in and carry 6. 9 times 3 and 6 carried and 1 from the second partial product=34. Write 4 on the left of the 5 last written and carry 3. 9 times 4 and 3 carried and 8 from the second partial product=47. Write 7 on the left of the 4 last written and carry 4. Proceeding in this way we obtain as third partial product 545745*79* (the 79 being brought down from the lines above) which is 967 times 56437.

In like manner, multiplying by 2 (thousand) and adding in the third partial product we obtain 2967 times 56437.

Next multiplying by 5 (ten thousand), then by 8 (hundred thousand) and finally by 3 (million), each time adding in the last-obtained partial product, we obtain 217449898579 which is the product of 56437 multiplied by 3852967. The six figures on the right in this final product, viz. 898579, are the right hand figures of the six preceding partial products.

67. Computers' Division. In computers' multiplication the product is built up by successive additions of multiples of the multiplicand, these multiples being determined by the several digits of the multiplier. In computers' division this process is reversed; the dividend is broken up or resolved by successive subtractions of multiples of the divisor, these multiples determining the several digits of the quotient.

Example 1. Divide 217,449,898,579 by 56,437.

Here 56437 is contained 3 (million) times in 217449 (million). Write 3 in the quotient and proceed to obtain the 'remainder' by computers' subtraction, thus:—

```
            3852967
56437)217449898579
       481388
        298929
         167448
          545745
           378127
            395059
```

3 (million) times 7 and 8 (million) complement = 29 (million). Write the 8 (million) complement under the 9 (million) in the dividend and carry 2 from the 29. 3 times 3 and 2 carried and 3 complement = 14. Write the 3 complement on the left of the 8 last written and carry 1 from the 14. 3 times 4 and 1 carried and 1 complement = 14. Write the 1 complement on the left of the 3 last written and carry 1 from the 14. 3 times 6 and 1 carried and 8 complement = 27. Write the 8 complement on the left of the 1 last written and carry 2 from the 27. 3 times 5 and 2 carried and 4 complement = 21. Write the 4 complement on the left of the 8 last written. This completes the subtraction of 3 (million) times 56437 from the dividend. To the right of the partial remainder 48138 just found, bring down 8, the 'next figure' of the dividend; we thus obtain 481388 as the second partial dividend giving 8 as the 'next' figure of the quotient. Write 8 in the quotient on the right of the 3 formerly written therein, and from 481388 take by computers' subtraction 8 times 56437. There will remain 29892 to which 'bring down' 9 from the dividend to obtain a new partial dividend. Continue thus subtracting and 'bringing down' till the operation is finished, or is carried to a sufficient degree of accuracy.

Comparing this example with the example given in the preceding section, it will be found, if the whole of the work be written out, that *the one process is the exact reverse of the other.*

68. A slightly more convenient arrangement of the work may be obtained by 'carrying up' the figures of the successive complements or remainders, instead of 'bringing down' the successive figures of the dividend. This is merely a change in the arrangement, not in the working of the division, the wording of the process will remain the same with the exception of the omission of "bring down the next figure of the dividend." Arranged in this way the example just worked out will appear as in the margin.

56437	217449898579
	4813824425
	29894710
	167585
	5479
	33
	3852967

We give two other examples of this arrangement.

Example 2. Divide 372,956,483 by 7.

7	372956483
	21563654
	$53279497\frac{4}{7}$

$7\times\mathbf{5}+\mathit{2}=37$. Write 5 in the quotient-line and *2* in the remainder-line. The next partial dividend is thus 22. $7\times\mathbf{3}+\mathit{1}=22$.

$7\times\mathbf{2}+\mathit{5}=19$. $7\times\mathbf{7}+\mathit{6}=55$. $7\times\mathbf{9}+\mathit{3}=66$. $7\times\mathbf{4}+\mathit{6}=34$.
$7\times\mathbf{9}+\mathit{5}=68$. $7\times\mathbf{7}+\mathit{4}=53$.

Example 3. Divide 3,893,865,378 by 179.

179	3893865378
	314518705
	13967645
	11
	$21753437\frac{155}{179}$

The first partial dividend is 389, giving 2 as the first figure of the quotient and 31 as the first remainder. The second partial dividend is therefore 313 which gives 1 as the second figure of the quotient and 134 as the second remainder. The third partial dividend is 1348, the third figure of the quotient is 7 and the third remainder is 95. The fourth partial dividend is 956, the fourth figure of the quotient is 5 and the fourth remainder is 61. The remaining partial dividends are, in order, 615, 783, 677 and 1408. The final remainder is 155.

69. Special Cases. In the case of certain multipliers and certain divisors, special methods may be adopted with advantage. The following are examples of these.

i. To multiply by 5, multiply mentally by 10 and divide the product by 2. $(5=10\div2.)$

ii. To multiply by 25, multiply mentally by 100 and divide the product by 4. $(25=100\div4.)$

iii. To multiply by 125, multiply mentally by 1000 and divide the product by 8. (125=1000÷8.)

iv. To multiply by 75, multiply by 300 and divide the product by 4. (75=300÷4.)

v. To multiply by 375, multiply by 3000 and divide the product by 8. (375=3000÷8.)

vi. To multiply by 875, multiply by 7000 and divide the product by 8. (875=7000÷8.)

vii. To multiply by 11, add each figure of the multiplicand to the figure on its right hand beginning from 0 mentally pictured as written on both the right and the left of the multiplicand.

Example. 35725876×11
392984636

Calculation.

6=**6**. 7+6=1**3**. 1+8+7=1**6**. 1+5+8=1**4**. 1+2+5=**8**. 7+2=**9**. 5+7=1**2**. 1+3+5=**9**. 0+3=**3**.

Explanation. 11=10+1, hence $35725876\times11=\left\{\begin{array}{r}357258760\\+035725876\\\hline 392984636\end{array}\right.$

viii. To multiply by 101, or 1001, or 10001, ..., employ computers' multiplication, using the multiplicand as the first partial-product line.

ix. To multiply by 13, 14, ... 17, 21, 31, ... 91, 102, 103, ... 109, 201, 301, ... 901, or other number beginning with 1 or ending with 1, write the multiplier above the multiplicand and use the multiplicand itself as the partial-product line arising from the digit 1 in the multiplier.

Examples.

```
      17                 71
 4372965            4372965
30610755           30610755
--------          ---------
74340405          310480515
```

x. To multiply by 9, subtract the multiplicand from 10 times itself by making up each figure to that on its right hand beginning from 0 mentally pictured as written on both the right and the left of the multiplicand.

Example. 7285634×9.
65570706

Calculation. 4+6=10. 1+3+0=4. 6+7=13. 1+5+0=6. 8+7=15. 1+2+5=8. 7+5=12. 1+0+6=7.

Explanation. 9=10−1, hence $7285634 \times 9 = \begin{cases} 72856340 \\ -07285634 \end{cases}$
65570706

xi. To multiply by 99, 999, 9999, &c. subtract the multiplicand from 100, 1000, 10000, &c. times itself by making up each figure to the 2nd, 3rd, 4th, &c. on its right hand beginning from two, three, four, &c., zeros mentally pictured as written on both the right and the left of the multiplicand.

xii. To multiply by 97, 997, 9997, ... take 3 times the multiplicand from 100, 1000, 10000, ... times the multiplicand. To multiply by 974, 9974, 99974, ... take 26 times the multiplicand from 1000, 10000, 100000, ... times the multiplicand. Similarly resolve other multipliers expressed by one or more 9's followed by one or more figures other than 9. The subtractions should be made by computers' method, see the *Example* following §49, p. 25.

Examples.

953784 =20 times.	114572 =400 times.
476892 × 19	28643 × 399
9060940 = 19 times.	11428557 = 399 times.

xiii. If two or more consecutive figures in a multiplier constitute a number which is a multiple of another figure of the multiplier we may save a line of partial products.

Examples. (*1.*)

47289	
567	
331023 =	7 times.
2648184 = 7 times × 80 =	560 times.
26812863 =	567 times.

(*2.*)

2985643	
36872	
23885144	800
214966296	8 × 9 = 72
107483148	72000 ÷ 2 = 36000
110086628696	36872.

(*3.*)

84629	
5397	
592403	7
4146821	490
4146821	4900
456742713	5397.

xiv. If the multiplier is seen to be the product of two or more small factors, multiply the given multiplicand by any one of these factors, multiply the product so formed by a second factor, this second product by a third factor, and so continue till all the factors have been used. The final product is the product required.

xv. To divide by 5, multiply by 2 and divide the product by 10. $(5\times 2=10.)$

xvi. To divide by 25, multiply by 4 and divide the product by 100. $(25\times 4=100.)$

xvii. To divide by 125, multiply by 8 and divide the product by 1000. $(125\times 8=1000.)$

xviii. To divide by 75, or 175, or 225, or 275, ..., multiply by 4 and divide the product by 300, or 700, or 900, or 1100, ..., as the case may be.

xix. To divide 375, or 875, or 1375, ... multiply by 8 and divide the product by 3000, or 7000, or 11000, ... as the case may be.

xx. If the divisor is seen to be the product of two or more factors each less than 13, divide by these factors 'in succession,' the final quotient is the quotient required.

Example. $8765348 \div 462$.

$462=6\times 7\times 11.$

$$\begin{array}{r|l} 6 & 8765348 \\ 7 & 1460891\frac{2}{6} \\ 11 & 208698\frac{32}{42} \\ & 18972\frac{284}{462} \end{array}$$

The $\frac{2}{6}$ might have been written $\frac{1}{3}$ in which case $\frac{32}{42}$ would have become $\frac{16}{21}$ and $\frac{284}{462}$ become $\frac{142}{231}$. These are the forms in which the fractions would have appeared had the divisor 462 been resolved into $2\times 3\times 7\times 11$ and these four factors used as successive divisors.

xxi. Since

$$\begin{aligned} 100&=99\times 1+1 \\ 200&=99\times 2+2 \\ 300&=99\times 3+3 \\ 325&=300+25=99\times 3+3+25 \\ 894&=800+94=99\times 8+8+94 \\ \&c.&=\&c. \end{aligned}$$

therefore when any number is divided by 99 the remainder increased if necessary, by 99 or a multiple of 99, exceeds the remainder when that number is divided by 100, by the quotient when 100 is the divisor. Successive applications of this leads to a convenient method of dividing any number by 99. Thus :—

$$\begin{aligned} 297689 &= 297600+89 \\ &= 2976\times 99+2976+89 \\ &= 2976\times 99+2900+76+89 \\ &= 2976\times 99+29\times 99+29+76+89 \\ &= 3005\times 99+194 \\ &= 3005\times 99+100+94 \\ &= 3005\times 99+1\times 99+1+94 \\ &= 3006\times 99+95 \end{aligned}$$

$\therefore\ 297689 \div 99 = 3006\frac{95}{99}$.

This may be arranged for working as follows :—

2976	89
29	76
	29
1	94
	1
Quot. = 3006	95 = Rem.

Similarly for any number expressed by 9's only.

70. Tests of Exact Divisibility. The following tests of exact divisibility are often useful in a search for the factors of a number.

i. A number is exactly divisible by 2 if its right-hand figure is zero or a number exactly divisible by 2.

ii. A number is exactly divisible by 4 if its two right-hand figures are zeros or express a number exactly divisible by 4.

Examples. 173528 is exactly divisible by 4, for 28 is exactly divisible by 4; but 319378 is not a multiple of 4, for 78 is not exactly divisible by 4.

iii. A number is exactly divisible by 8 if its three right-hand figures are zeros or express a number exactly divisible by 8.

Examples. 536 is a multiple of 8, therefore 1397536 is exactly divisible by 8; but 356 is not a multiple of 8, consequently 4679356 is not exactly divisible by 8.

iv. A number is exactly divisible by 5, 25, 125, ... if the number expressed by the right-hand figure or the two, three, ... right-hand figures is exactly divisible by 5, 25, 125, ...

v. A number is exactly divisible by 3 if the sum of its digits is exactly divisible by 3.

vi. A number is exactly divisible by 9 if the sum of its digits is exactly divisible by 9.

Examples. Test whether 18637569 and 7385621 are divisible by 9.

$1+8+6+3+7+5+6+9=45=9\times5$, $\therefore$ 18637569 is exactly divisible by 9.

$7+3+8+5+6+2+1=32=9\times3+5$, $\therefore$ 7385621 is not exactly divisible by 9.

vii. A number is exactly divisible by 6 if it is exactly divisible by both 2 and 3.

viii. A number is exactly divisible by 12 if it is exactly divisible by both 4 and 3.

ix. A number is exactly divisible by 11 if the difference between the sum of its 1st, 3rd, 5th, 7th, &c. figures and the sum of its 2nd, 4th, 6th, 8th, &c. figures is zero or a number exactly divisible by 11.

Examples. Test whether 729583624 and 457983621 are exactly divisible by 11.

$4+6+8+9+7=34$; $2+3+5+2=12$; $34-12=22=11\times2$; $\therefore$ 729583624 is exactly divisible by 11.

$1+6+8+7+4=26$; $2+3+9+5=19$; $26-19=7$; $\therefore$ 457983621 is not exactly divisible by 11.

There are no easily applied tests for exact divisibility by 7 and by 13, but in the case of very large numbers the following may be applied.

x. Point off the number into periods of three figures each, beginning on the right; if the difference between the sum of the 1st., 3rd., 5th., &c. periods and the sum of the 2nd., 4th., 6th., &c. periods is zero or is exactly divisible by 7, by 11, or by 13, the number is exactly divisible by 7, by 11, or by 13, as the case may be.

Example. Test 6,576,353,693 for 7, 11, and 13 as factors.

$693+576-353-6=910=10\times7\times13$; $\therefore$ 7 and 13 are factors but 11 is not a factor.

xi. Any number less than 1000 will be exactly divisible by 7 if the sum of the ones figure, thrice the tens figure and twice the hundreds figure be exactly divisible by 7.

Examples. Is 7 a factor of 623 and of 685?

$3+6+12=21=7\times3$, ∴ 7 is a factor of 623.

$5+24+12=41=7\times5+6$, ∴ 7 is not a factor of 685.

xii. If a number is exactly divisible by each of two numbers prime to each other, it is exactly divisible by their product; and conversely,

xiii. If a number is exactly divisible by the product of two numbers, it is exactly divisible by each of the numbers.

71. Theorems xii and xiii follow immediately from Theorem XIX, § 62. The truth of the other theorems may be shown as follows:—

i. 2 is a measure of 10;

∴ 2 is a measure of every multiple of 10;

∴ 2 is a measure of the part of any number consisting of the tens, hundreds, thousands, &c.;

∴ in testing any number for exact divisibility by 2, the tens, hundreds, thousands, &c. may be neglected as certainly multiples.

ii. 4 is a measure of 100;

∴ 4 is a measure of every multiple of 100;

∴ 4 is a measure of the part of any number consisting of the hundreds, thousands, ten-thousands, &c.;

∴ in testing any number for exact divisibility by 4, the hundreds, thousands, ten-thousands, &c., may be neglected as certainly multiples.

iii. 8 is a measure of 1000;

∴ 8 a measure of every multiple of 1000;

∴ 8 is a measure of the part of any number consisting of the thousands, ten-thousands, hundred-thousands, &c.;

∴ in testing any number for exact divisibility by 8, the thousands, ten-thousands, hundred-thousands, &c., may be neglected as certainly multiples.

iv. The demonstration is similar to those for 2, 4 and 8.

v. and vi. Both 3 and 9 are measures of 9, 99, 999, 9999, &c., that is, of $10-1$, $100-1$, $1000-1$, $10000-1$, &c.

∴ if from any number there be deducted the ones and 1 from *each* 10, 1 from *each* 100, 1 from *each* 1000, &c., the remainders from the tens, the hundreds, the thousands, &c., constitute a number which will be a multiple of 9 and therefore also of 3 and which may be neglected in testing the number for exact divisibility by either 9 or 3. There will then remain to be tested the total of the deductions; these were the ones, the *number* of tens, the *number* of hundreds, the *number* of thousands, &c., and therefore their total is simply the sum of the digits of the given number.

Example. Is 78546 exactly divisible by either 9 or 3?

70000=7	times	10000=7	times	9999	and	7
8000=8	"	1000=8	"	999	"	8
500=5	"	100=5	"	99	"	5
40=4	"	10=4	"	9	"	4
6=6	"	1=6	"	0	"	6
Adding, 78546=			a multiple of 9 and			30

30 is exactly divisible by 3 but not by 9,

∴ 78546 is exactly divisible by 3 but not by 9.

It should be noticed that the theorem really proved is:—

The remainder in dividing any number by 9 is the same as the remainder in dividing the sum of the digits of the number by 9.

vii and viii are special cases of xii.

ix. 11 is a measure of 11, 1111, 111111, 11111111, &c.;

∴ 11 is a measure of 11, of 1111 − 110, of 111111 − 11110, of 11111111 − 1111110, &c.;

i. e., 11 is a measure of 11, 1001, 100001, 10000001, &c.;

∴ 11 is a measure of 11, 99, 1001, 9999, 100001, 999999, &c.; i. e., of 10+1, 100 − 1, 1000 + 1, 10000 − 1, 100000 + 1, 1000000 − 1, &c.; ∴ if from any number there be deducted the ones and 1 from *each* 100, 1 from *each* 10000, 1 from *each* 1000000, &c. and there be added 1 to *each* 10, 1 to *each* 1000, 1 to *each* 100000, &c., the resulting number will be a multiple of 11 and it may consequently be neglected in testing the number for exact divisibility by 11. There will then remain to be tested the difference between the deductions and the additions which were made, *i. e.*, the difference between the sum of the digits in the odd places, (numbering from the right,) and the sum of those in the even places.

Example. Is 8756439 exactly divisible by 11.

8000000=8	times	1000000=8	times	999999 and 8	
700000=7	"	100000=7	"	100001	less 7
50000=5	"	10000=5	"	9999 and 5	
6000=6	"	1000=6	"	1001	less 6
400=4	"	100=4	"	99 and 4	
30=3	"	10=3	"	11	less 3
9=9	"	1=9	"	0 and 9	

Adding, 8756439= a multiple of 11 and 26 less 16

26 less 16=10 which is not exactly divisible by 11

∴ 8756439 is not exactly divisible by 11.

Here also it should be noticed that the Theorem really proved is,

The remainder in dividing any number by 11 is the same as the remainder in dividing the number obtained by subtracting the sum of the digits in the even places, numbering from the right, from the sum of the digits in the odd places increased if necessary by 11 or a multiple of 11.

x. By substituting 1000 for 10 and 1001 for 11 and making corresponding changes in the other numbers employed in the preceding discussion of the test for exact divisibility by 11, it will be proved that,

If any number be pointed off into periods of three figures each, beginning from the right, the remainder in dividing the number by 1001 will be the same as the remainder in dividing the number obtained by subtracting the sum of the periods in the even places, numbering from the right, from the sum of the periods in the odd places, increased if necessary by 1001 or a multiple of 1001.

Now $1001=13\times11\times7$, hence if the remainder in dividing any number by 1001 be exactly divisible by 7, by 11 or by 13, the number itself will be exactly divisible by 7, by 11 or by 13 as the case may be.

xi. From $10=7+3$ and $100=98+2=14\times7+2$ comes at once test or rule xi.

72. To cast the nines out of any number is to find the remainder in dividing the number by 9. To do this, add together the digits of the given number, omitting any nines there may be among the digits, then add together the digits of that sum again omitting all nines, and so continue until a number of one digit is obtained. This last number, if it be less than 9, will be the remainder in dividing the given number by 9; if it be 9, the remainder will be zero.

Example 1. Cast the nines out of 73856942.

$7+3+8+5+6+4+2=35$, $3+5=8$, *remainder.*

Instead of adding all the digits together and casting the nines out of the sum, the nines may be cast out of the partial sums as fast as they rise above 8. Adopting this method the preceding example would appear

$7+3=10$, $(1+0=1)$, $1+8=9$; $5+6=11$, $(1+1=2)$, $2+4+2=8$.

Wording; ten, *one*, nine, eleven, *two*, six, eight.

Example 2. Cast the nines out of 3587968594.

8, 16, (7), 14, (5), 11, (2), 10, (1), 6, 10, 1 *remainder.*

The applications of the operation of casting out the nines depend upon two theorems:—

A. *The sum of two numbers has the same remainder to 9 as the sum of their remainders to 9.*

B. *The product of two numbers has the same remainder to 9 as the product of their remainders to 9.*

All numbers may be regarded as multiples of 9 + their remainders to 9. On adding or multiplying these numbers, all the multiples of 9 will yield multiples of 9 and all these will disappear in casting out the nines; the result will therefore be the same as if the numbers had been reduced from the first to their remainders to 9.

73. Proofs of Multiplication. Multiplication may be proved,—

(1.) By repeating the calculation with multiplier and multiplicand interchanged.

(2.) By dividing the product by the multiplicand; the quotient should be equal to the multiplier.

(3.) By casting the nines out of the multiplier and the multiplicand, then multiplying the remainders together and casting the nines out of their product; the remainder thus obtained should be the same as the remainder from casting the nines out of the product of multiplier and multiplicand.

In arranging the several remainders it is usual to write the remainder from the multiplicand on the left-hand of an oblique cross, the remainder from the multiplier on the right-hand of the cross, the remainder from the product below the cross and the remainder from the product of the remainders above the cross.

Example 1. Apply the test of casting out the nines to
$2968457 \times 74853169 = 222198413490233$. (*See* § 64.)

8
Multnd. 5 ✕ 7 *Multr.* $5 \times 7 = 35 = 9 \times 3 + 8$.
8
Product.

Example 2. Prove $56437 \times 3852967 = 217449898579$ by casting out the nines. (*See* § 66.)

1
7 ✕ 4
1

74. Of these three proofs the second possesses the advantage of locating any errors that may be detected but it doubles the labor of calculation. The third proof is by far the easiest of application but it is subject to the serious disadvantage of not pointing out an error of 9 or a multiple of 9 in the product. Thus if 0 has been written for 9 or 9 for 0, if a partial product has been set down in the wrong place, if one or more noughts have been inserted or omitted in any of the products, if two figures have been interchanged or, generally, if one figure set down is as much too great as another is too small, casting out the nines will fail to declare the presence of error, for in each case the remainder to 9 will remain unaffected by the error.

75. Proofs of Division. Division may be proved,—

(1.) By repeating the calculation with the integral part of the quotient for a divisor.

(2.) By multiplying the divisor by the complete quotient; or, as it is generally stated, by multiplying the divisor by (the integral part of) the quotient and adding the remainder to the product; the result should be equal to the dividend.

(3.) By casting the nines out of the divisor, the integral part of the quotient and the 'remainder' in the division, multiplying the first two of these remainders together and adding the third to their product and casting the nines out of this sum; the remainder to 9 thus obtained should be the same as the remainder from casting the nines out of the dividend.

Example. Prove $3893865378 \div 179 = 21753437\frac{155}{179}$ by casting out the nines. (*See* § 68.)

Divisor 8 ✕ 5, 2 *Quot.* (6 above, 6 below) $8 \times 5 + 2 = 42 = 9 \times 4 + 6$.

Dividend.

The proof of division by casting out the nines labors under disadvantages corresponding to those to which the proof of multiplication by casting out the nines is subject.

EXERCISE I.

MISCELLANEOUS PROBLEMS.

1. By what number must 2000 be divided that the quotient and the 'remainder' may be the same as the quotient and the 'remainder' in the division of 101 by 11?

2. What number contains 13·75 as often as 18·27 contains ·0693?

3. If a strip of carpet 27 in. wide and 50 yd. long make a roll weighing 135 lb., what area could be covered by 4 T. of such carpet?

4. A rectangular block of granite measures 7′ 1′ × 2′ 4″ × 1′ 3″; what must be the length of another rectangular block 2′ 1″ × 1′,

(i), if it is to weigh the same as the first block,

(ii), if it is to have the same surface-area as the first block,

(*a*) exclusive of end-surfaces,

(*b*) inclusive of end-surfaces?

5. A boat's crew rowed a distance of 4 mi. 800 yd. in 36 min. 45 sec. What was the speed per hour? What was the average time per mile?

6. A man who owns $\frac{10}{21}$ of a mill, sells $\frac{2}{5}$ of his share; what fraction of the mill does he still own? Had he sold $\frac{2}{5}$ of the mill, what fraction of the mill would he still have owned?

7. A grocer drew off 4 gal. from a full barrel of vinegar and filled the barrel up with water. Next day he drew off 4 gal. of the mixture and then filled up the barrel with water. On the third day he drew off 4 gal. of the mixture and filled up the barrel with water. If the barrel held just 32 gallons, how many gallons of the vinegar originally contained in the barrel remained in it after the third drawing off?

8. G can do as much work in 4 days as H can do in 5 days, or as much in 5 days as M can do in 9 days. The three undertake a contract and G and H work together on it for 18 days, then M takes G's place and H and M work together on it for 26 days and thus finish the contract. How long would it have taken G working all the time alone to have executed the contract?

9. A grocer buys two kinds of tea, one kind at 23ct. per lb., the other kind at 35ct. per lb., and mixes them in the proportion of 5lb. of the cheaper to 3 lb. of the dearer kind. At what price per pound (an integral number of cents) must he sell the mixture to gain *at least* 30% on the buying price?

10. Find the interest on $794·35 for 188 days at 5%.

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11. The product of 25 and 25 is 625. By how much must this product be increased to obtain the product of 26 and 25? By how much must the product of 26 and 25 be diminished to obtain the product of 26 and 24? Hence by how much must the product of 25 and 25 be diminished to obtain the product of 26 and 24? By how much must the product of 26 and 24 be diminished to obtain the product of 27 and 23? Hence by how much must the product of 25 and 25 be diminished to obtain the product of 27 and 23, *i.e.*, the product of $25+2$ and $25-2$? By how much must the product of 25 and 25 be diminished to obtain the product of (i) 28 and 22, (ii) 29 and 21, (iii) 30 and 20?

12. I take 344 steps in walking round a rectangular play-ground, keeping 3 ft. within the boundary fence. Find the area of the play-ground if 9 of my steps are equal to $7\frac{1}{2}$ yd. and a shorter side of my walk is 68 steps in length.

13. Find the value of a rectangular field 330 yd. by 156 yd. @ $36·50 per acre.

14. What must be the depth of a rectangular cistern to hold 350 gallons when filled to 6 in. from the top, if the horizontal section of the cistern is to be 3′ 6″ square?

15. How many miles will be travelled between 9,25 a.m. and 5,40 p.m. at an average of $22\frac{1}{4}$ mi. per hour for 3 hr. 35 min. and of $28\frac{3}{5}$ mi. per hour for the remainder of the time?

16. After drawing off 124 gal. of water from a cistern, $\frac{3}{11}$ of the water still remained. How many gallons did the cistern at first contain? How many gallons were left in it?

17. A block of maple weighed 35 lb. and a block of red pine of exactly the same size weighed 25 lb. Find the weight of a block of maple of the same size as a block of red pine weighing 164 lb. and the size of a block of red pine which will weigh the same as 23·75 cubic feet of maple, all three blocks of maple and likewise all three of red pine being of the same quality.

18. A laborer was engaged @ \$1·12 and his board for each day he worked, but was charged 38ct. for board for each day he was idle. At the end of 31 days he received \$25·72. How many days did he work?

19. A tradesman bought goods for \$1200 and sold one-third of them at a loss of 10%. For how much must he sell the remainder to gain 20% on the whole?

20. Find the interest on \$273·68 from 13th May to 7th Sept. at $7\frac{1}{2}$%.

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21. A rectangular lot 45 ft. front by 99 ft. deep was sold for \$3150. What was the price per foot frontage, and what the price per acre at the rate of the selling-price of the lot?

22. Find the area of a rectangle whose length is three times its width and whose perimeter is 143·76 in.

23. What must be the depth of a cylindrical cistern 3′ 6″ in diameter to hold 350 gal. when filled to 6 in. from the top?

24. A man starts at 8,10 a.m. on a journey of 18 miles and travels for $3\frac{1}{4}$ hours at the rate of $3\frac{1}{4}$ miles per hour. If he then quicken his speed by $\frac{3}{8}$ of a mile per hour, at what hour of the day will he arrive at the end of his journey? How much sooner will this be than would have been the hour of his arrival had he not quickened his pace?

25. Herbert's age is just $\frac{2}{3}$ of Maud's. Four years ago, his father, who is now 36 years old, was just $5\frac{1}{3}$ times as old as Herbert then was. How old is Maud?

26. A man pays out $\frac{2}{21}$ of his income for rent and $\frac{1}{60}$ for taxes. What fraction of his income do these two sums form? If the two sums amount together to \$220·90, what must be the amount of the man's income?

27. Two blocks of exactly the same size, the one of birch, the other of willow, weighed 454 lb. and 249·7 lb. respectively. The block of birch floated in water with only $\frac{8}{11}$ of its volume immersed. How much of the volume of the willow-block would be immersed were the block to float in water?

28. *A B* and *C* can do a piece of work in 10 days, all three working together. The three undertake the job and work on it for 4 days, then *C* leaves off work, but *A* and *B* continue and finish the piece of work in 10 days. If *A* could have done the whole work by himself in 30 days, in what time could *B*, and in what time could *C* have done it?

29. A tradesman sold $\frac{1}{3}$ of a certain lot of goods at a loss of 10%, at what per cent. advance on the cost must he sell the remainder of the lot in order to gain 20% on the whole?

30. To what sum would \$87·68 amount in 97 days @ $6\frac{1}{2}$% interest?

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31. Express the following distances in kilometres:—

(i), From Montreal to Toronto, 333 miles; (ii), from Toronto to Hamilton, 38·72 miles; (iii), from Toronto to Stratford, 88·34 miles; (iv), from Hamilton to London, 75·90 miles; (v), from Stratford to London, 32·68 miles; (vi), from Montreal to London *via* Hamilton; (vii), from Montreal to London *via* Stratford.

32. In front and on one side of a rectangular lot 66 ft. by 132 ft. and 2 ft. out from the line of the lot, a sidewalk 40 in. wide is laid. How many square feet of ground does the side-walk cover?

33. Into a rectangular cistern 3′ 4″ by 2′ 9″ in horizontal section, water is flowing at the rate of 25 gal. per minute. How long will it take at that rate of flow to increase the depth of the water in the cistern by 4′ 6″?

34. Find the area of a rectangle of 6·006 ft. perimeter, if its ength is (i) equal to, (ii) double, (iii) thrice, (iv) four times, (v) ive times, (vi) eight times, (vii) ten times its width.

35. *A* can run a mile in 5 min. 55 sec. and *B* can run a mile in 5 min. 2 sec. By how many yards would *A* win in a mile race run at these rates?

36. Having paid an income-tax of 19·2 mills on the $1, I have an income of $5735·92 left. What amount of income-tax did I pay?

37. Goods which cost $2756·13 for 17 T. 1335 lb. are sold at an advance of $\frac{3}{13}$ on cost. Find the selling price per cwt.

38. In a hundred-yard race, *A* can beat *B* by 17 yd. and *C* by 3 yd. At these rates of running how many yards start ought *C* to give *B* in a 200 yd. race that they may run a dead heat?

39. (*a*). *A*'s age is greater than *B*'s by 12 yr. which is 25% of *A*'s age. Determine *B*'s age.

(*b*). *M*'s age which is 69 yr. is greater than *N*'s age by 15% of *N*'s age. Determine *N*'s age.

(*c*). *V*'s age is less than *W*'s age by 10% of *W*'s age and the sum of their ages is 76 yr. Determine *V*'s age.

40. At what rate per cent. per annum would $183·40 yield $2·78 interest in 123 days?

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41. What number is the same part of 95·9 that 18·27 is of 29, and what number is the same multiple of ·119 that 57057 is of 15·96?

42. The standard of fineness of British gold coins is $\frac{1}{12}$ of alloy, and 480 oz. Troy of standard gold is coined into 1869 sovereigns equal in value to $4·86$\frac{2}{3}$ each. Find the value of (i) an oz. Troy, (ii) an oz. avoirdupois, of pure gold.

43. Find the area of the outer surface of a cylindrical stove-drum 16″ in diameter and 24″ in height, deducting two circles, the pipe-holes, of 7″ diameter each.

44. The depth of water in a rectangular cistern of 3′ by 2′ 9″ horizontal section increases at the rate of 5′ 4″ in 12 min. What is the rate of inflow in gallons per minute?

45. A horse trotted a mile in 2 min. 12 sec. Taking his stride at 16 ft., how many times per second did his feet touch the ground?

46. The municipal rates being reduced from $19\frac{3}{8}$ mills to $17\frac{7}{8}$ mills on the $1, my taxes are lowered by $4·05. For how much am I assessed?

47. A boat's crew that can row at the rate of 264 yd. per min. in still water, rowed 3 miles down a stream in 16 min. Find the velocity of the stream.

48. Sold 19 yd. of silk @ $1·86 a yard, thus gaining the cost price of 12 yd. Find the cost price per yard.

49. *A*'s age which is 49 yr. is less than *B*'s age by $12\frac{1}{2}\%$ of *B*'s age, and *B*'s age is less than *C*'s age by $12\frac{1}{2}\%$ of *C*'s age. What is *C*'s age?

50. The interest on $270·25 for 93 days is $4·82; to what sum would $725 amount in 125 days at the same rate?

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51. The sum of two numbers is 106 and one exceeds the other by 28·62. What fraction is the smaller of the larger number?

52. Find the area of a circular field enclosed by a ring fence 440 yd. long.

53. A circular pond 17′ 6″ in diameter and 5′ deep is to be filled by means of a pipe which discharges 100 gal. per min. How long will it take to fill the pond?

54. A train runs the first 120 miles of a trip of 280 miles at a speed of 32 miles per hour. At what speed must the remainder of the trip be run, if the whole trip is to be accomplished in 8 hours?

55. If during the day I pay out $\frac{1}{2}$, then $\frac{1}{3}$, next $\frac{1}{12}$, and lastly $\frac{1}{15}$ of the money I had in the morning, what fraction of it have I left? If the sum left amounts to $1·54 what sum had I at first?

56. A train 220 ft. in length is running at the rate of 25 mi. per hour. How long will it take to pass another train 330 ft. long if the second train be (i), standing on a parallel track; (ii), moving in the opposite direction at the rate of 15 mi. per hour; (iii), moving in the same direction at the rate of 15 mi. per hour?

57. *A* has $480 which is less than what *B* has by 20% of what *B* has, and the sum *B* has is greater than what *C* has by 20% of what *C* has. What sum does *C* possess?

58. *A* has more money than *B* by 10% of *B*'s money. By what per cent. of *A*'s money is *B*'s money less than *A*'s?

59. At what rate of interest would \$379·45 amount to \$396 in 245 days?

60. The owner of a house offered an agent \$500 commission if the agent could sell the house for \$10,500. What rate per cent. commission was the owner offering? Had the owner offered 5% commission, what would have been the commission on \$10,500?

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61. A sum of money was divided between *A* and *B*, *A* receiving \$5 for every \$4 received by *B*, and it was found that *A* had received \$12·60 less than double of what *B* had received. How much did each receive?

62. The area of Europe is 3,823,400 sq. mi. and its average elevation above the level of the sea is 974 ft. Find the volume in cubic miles of the portion of Europe above sea-level.

63. If a cubic foot of gold weigh 1208 lb., what must be the thickness of a gold ribbon $1\frac{1}{2}$ in. wide and 10 ft. long, weighing 480 grains? (*Avoirdupois Weight.*)

64. If the telegraph poles beside a certain railway are placed at intervals of 50 yd., at what speed is a train running which traverses two of these intervals in seven seconds?

65. In a certain subscription list $\frac{1}{8}$ of the number of subscriptions are for \$5 each, $\frac{1}{6}$ are for \$4 each, $\frac{1}{5}$ are for \$2 each, $\frac{1}{3}$ are for \$1 each, and the remaining subscriptions, amounting to \$10·50, are for 50ct. each. Find the whole number of subscribers and the total amount of their subscriptions.

66. A train 80 yd. long crossed a bridge 140 yd. long in $22\frac{1}{2}$ sec. Find the average speed of the train while crossing.

67. Find the gain per \$100 on a cargo of raw-sugar bought at \$53 per ton of 2240 lb., refined at a cost of \$1·35 per 100 lb. of refined sugar and sold at $6\frac{1}{4}$ct. per lb., if 7 lb. of raw sugar yields 5 lb. of refined sugar.

68. *A* and *B* insure their houses against fire and *A* has to pay \$7·50 more than *B* who pays \$28·75. Find the value of their houses, the rate of insurance being $\frac{5}{8}$ %, and express the value of *B*'s house as a decimal of the value of *A's* house.

69. In what time would the interest on \$182·50 amount to \$5 at 5 %?

70. A man bought 50 shares in a company at $40 per share. Next year the price was $45 per share but each year thereafter there was a fall of $4 per share. Each year from the date of his purchase he sold out 10 shares and found at the end of five years that including his dividends with the amounts realized by the sales of his shares he had neither gained nor lost. What dividend per share did the company pay?

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71. Prove that if 12 be added to the product of the first 11 integers, 13 will be a factor of the sum.

72. Prove that if 16 be added to the product of the first 15 integers, 17 will be a factor of the sum. (See *Public School Arithmetic*, Ex. xxx, Probs. 26 and 27.)

73. Make out a bill dated Feb. 1st, 1889, for the following transactions and receipt it on behalf of Messrs. Kent & Sons.

L. D. Walker bought of Messrs. Kent & Sons, Hamilton:—

Dec. 1st, 1888, Am't of Acc't rendered, $30·07; Dec. 14th, $1\frac{1}{4}$ yd. Lawn @ 28 ct., 2 Spools @ 5 ct., 1 Cloud $1·25, $2\frac{1}{2}$ yd. Lace @ 80 ct., 1 Towel 27 ct., 2 yd. Ribbon @ 11 ct., $1\frac{1}{8}$ yd. Embroidery @ 15 ct.; Dec. 22nd, Silk Handerchief 85 ct., do. do. $1·20, 6 Linen do. @ 22 ct.; Dec. 28th, 1 pr. Cashmere Hose 57ct., 4 Sk. Wool @ 10 ct., $\frac{3}{4}$ yd. Frilling @ 15 ct., $\frac{3}{4}$ yd. do. @ 20 ct., 1 pr. Silk Gloves 75 ct., $3\frac{1}{4}$ yd. Pink Flannel @ 32 ct.; Jan'y 25th, 1889, $1\frac{1}{2}$ yd. Lining @ 22 ct., $4\frac{1}{2}$ yd. Silesia @ 13 ct., $4\frac{1}{2}$ yd. Jet Trimming @ 22 ct., 1 Spool Silk 15ct., 2 Twist @ 3 ct., $1\frac{1}{2}$ doz. Buttons @ 10ct., 3 yd. Braid @ 2 ct.; Jan. 29th, $\frac{1}{2}$ doz. Table-Nap. @ $2·10, $\frac{1}{2}$ doz. do. @ $2·50 *(per doz.)*, $\frac{3}{4}$ yd. Veiling 25 ct., 1 yd. Frilling 20 ct. Jan. 3rd, 1889, L. D. Walker paid Cash on Acc't., $25·00; Feb'y 4th, paid Acc't in full.

74. If the telegraph poles beside a certain railway are placed at intervals of 66 yd., at what speed can a train be running if it traverse three of these intervals in between 11 and 12 seconds? At what speed can the train be running if it traverse 10 intervals in a time between 38 and 39 seconds in length?

75. *(a)* How much must be added to the numerator of $\frac{7}{12}$ that the resulting fraction may be equal to $\frac{5}{7}$?

(b) How much must be subtracted from the numerator of $\frac{5}{7}$ that the resulting fraction may be equal te $\frac{7}{12}$?

76. A man distributed a bag of marbles among 4 classes consisting of 7 boys each, giving the same number of marbles to each boy in a class. Among the boys in the first class he distributed half the marbles; among those of the second class, $\frac{1}{3}$ of them; among those of the third class, $\frac{1}{8}$ of them; and among those of the fourth class, the remaining marbles which allowed the boys just one apiece. How many did each boy in the other three classes receive and how many marbles were there altogether?

77. A train 54 yd. long running at the rate of 31 mi. per hr., passed another train 78 yd. long running on a parallel track; the two trains completely clearing each other (i) in 5·4 sec. from the time of meeting, (ii) in 22·5 sec. from the time of the former train overtaking the latter. Find the speed of the slower train?

78. A house assessed at \$2200 was rented for \$23 a month, the tenant to pay taxes and water-rates. The taxes were $17\frac{3}{4}$ mills on the \$1 and the water-rates were \$5 per quarter year. How much altogether did the tenant pay per year for the house. If the property had cost the landlord \$2500, what rate per cent. per year was he receiving on his investment?

79. In what time would \$143 amount to \$150 at 7 % interest?

80. The manufacturer of an article makes a profit of 25 %, the wholesale dealer makes a profit of 20 %, and the retail dealer makes a profit of 30 %. What is the cost to the manufacturer of an article that retails at \$15·60.

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81. Prove that if the number of integers less than 9 and prime to it be multiplied by the number of integers less than 16 and prime to it, the product will be the number of integers less than 144 $(=16\times 9)$ and prime to it.

82. How many times must a man walk round a rectangular play-ground 165 ft. by 132 ft. in order to travel $4\frac{1}{2}$ miles?

83. How many cubic feet of air will a rectangular room 27′ 8″ × 18′ 3″ × 12′ 4″, contain, and how much will the air in the room weigh if a cubic foot of the air weigh 565 grains?

84. *A* can run a mile in 4 min. 56 sec., *B* can run a mile in 5 min. 23 sec. If *A* give *B* 27 yd. start, in what distance will he overtake him?

85. Make out and receipt for K. Dewar & Son the following account :—

K. Dewar & Son of Stratford sold to Edwin Reesor on 2nd Sept., 1885, 28 lb. Furnace Cement @ 20 ct., 7 ft. of 12-in. Hot-air Pipe @ 46 ct., 14 lengths of 8-in. Smoke-pipe @ 18 ct., 1 Chimney Ring, 25 ct., 1 8-in. Elbow, 50 ct., 2 8-in Rings for Lawson Regulator @ 35 ct., 21 $1\frac{1}{4}$-in. Bolts @ 2 ct., 8 2-in. Bolts @ 3 ct., 16 $2\frac{1}{2}$-in. Bolts @ 4 ct. A man and an assistant from K. Dewar & Son's worked 49 hours cleaning and repairing E. Reesor's furnace, rate for the two together, 35 ct. per hour. E. Reesor paid $15 on this account on the 3rd Oct. and the balance on the 28th Nov., 1885.

86. A man sold $\frac{1}{2}$ of his farm, then $\frac{1}{3}$ of the remainder, then $\frac{1}{4}$ of what remained, then $\frac{1}{5}$ of what still remained, and he then found that he had sold altogether 72 acres more than he had remaining. How many acres had he at first?

87. *(a).* How much must be added to the denominator of $\frac{5}{7}$ that the resulting fraction may be equal to $\frac{7}{12}$?

(b). How much must be subtracted from the denominator of $\frac{7}{12}$ that the resulting fraction may be equal to $\frac{5}{7}$?

88. *A* and *B* run a mile race; *A* runs the whole course at a uniform speed of 320 yd. per min.; *B* runs the first half mile at a speed of 300 yd. per min. and the second half mile at a speed of 340 yd. per min. Which wins the race and by how many yards?

89. What principal would at 7% interest amount to $450 in 213 days?

90. An agent receives $7850 to be invested. What sum should he invest if he pay $12·30 expenses and charge $1\frac{1}{8}$% commission on the amount of the investment?

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91. Two wheels in gear with one another have 30 and 128 teeth respectively; how many revolutions will the smaller wheel make while the larger revolves 675 times? If two marked teeth, one on each wheel, are in contact at a certain moment, how many revolutions will each wheel make before the same two teeth are in contact again?

92. Find the volume in cu. in. of 10 lb. of *(a)* lead, *(b)* cast-iron, *(c)* marble, *(d)* brickwork, *(e)* oak, *(f)* birch, if a cubic foot of lead weigh 712 lb., of cast-iron 444 lb., of marble 172 lb., of brickwork 112 lb., of oak 54 lb., and of birch 44·4 lb.

93. Taking the weight of a cubic foot of water to be 997·7 oz., what weight of water would fill a rectangular bath 35′ 6″ by 13′ 3″ by 5′ $7\frac{1}{2}$″?

94. Make out an invoice of the following, supplying names and dates :—

A. B. bought of *C. D.* 15 doz. First Readers Pt. I @ \$1·20, 18 doz. First Readers Pt. II @ \$1·80, 27 doz. Second Readers @ \$3·00, 24 doz. Third Readers @ \$4·20, 9 doz. Fourth Readers @ \$6·00, 30 doz. Public School Grammars @ \$3·00, 30 doz. Public School Arithmetics @ \$3·00, 6 doz. Public School Geographies @ \$9·00; the whole subject to 20 and 5 off.

95. *A* can run a mile in 4 min. 56 sec., *B* can run a mile at the rate of 110 yards per 18 seconds. If they start together and run uniformly at these rates which will be the first and by how many yards (i) when *A* has run half a mile, (ii) when *B* has run a mile?

96. If when $\frac{7}{12}$ of a certain time has elapsed, then 1 hr., and then $\frac{3}{5}$ of the remainder of the time, it is found that 16 min. of the time still remain, what was the whole time?

97. *(a).* What number added to both terms of $\frac{7}{12}$ will give a fraction equal to $\frac{5}{7}$?

(b). What number subtracted from both terms of $\frac{5}{7}$ will give a fraction equal to $\frac{7}{12}$?

98. Find the length of a bridge which a train 100 yd. long required 1 min. 15 sec. to cross, running at a speed of 15 mi. per hour.

99. An account bearing interest at 6 % amounted at the end of 93 days to \$117·45. What was the original amount of the account?

100. A grocer professes to retail a certain tea at 20 % profit but mixes with it $\frac{1}{4}$ of its weight of an inferior tea which costs him only $\frac{2}{3}$ of the price he pays for the better article. What rate per cent. of profit does he make?

CHAPTER III.

APPROXIMATION.

CONVERGENT FRACTIONS.

76. In the calculations which arise in the ordinary course of affairs, fractions with large terms are not unfrequently met with. For these large-termed fractions, other fractions nearly equal to them in value but with smaller terms, may often be substituted without impairing the accuracy of the result for practical purposes, but with the advantage of very decidedly lessening the labor of computation. If a small-termed fraction is in this way to replace a fraction with large terms, the difference in value between the two fractions should be the least possible consistent with the condition that the terms of the replacing fraction shall be small. This requires us to be able to find all the fractions whose values approach so near the value of any given fraction that it is impossible to insert between the given fraction and any of the fractions found another fraction intermediate in value but with terms less than those of the fractions found. The fractions which fulfil this condition are termed **Convergents** to the given fraction.

77. The following examples exhibit the simplest method of computing the convergents to a given fraction.

Example 1. Find the convergents to $\frac{48}{155}$.

They are

$\frac{1}{0}$, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{16}$, $\frac{9}{29}$, $\frac{22}{71}$, $\frac{35}{113}$, $\frac{48}{155}$,

$\frac{0}{1}$, $\frac{1}{4}$, $\frac{2}{7}$, $\frac{3}{10}$, $\frac{4}{13}$, $\frac{13}{42}$, $\frac{61}{197}$, $\frac{109}{352}$, &c.

The method of calculation is as follows :—

Write $\frac{1}{0}$ and below it $\frac{0}{1}$ as initials.

From these initials treated as if they were fractions form another fraction with the sum of their numerators as its numerator and the sum of their denominators as its denominator; $\frac{0+1}{1+0}=\frac{1}{1}$. This newly formed fraction, $\frac{1}{1}$, being greater than the given fraction $\frac{48}{155}$, write it in the upper line, the line of $\frac{1}{0}$.

From $\frac{1}{1}$ and $\frac{0}{1}$ form a fraction with the sum of their numerators as its numerator and the sum of their denominators as its denominator; $\frac{1+0}{1+1}=\frac{1}{2}$. This $\frac{1}{2}$ being greater than $\frac{48}{155}$, write it in the upper line.

From $\frac{1}{2}$ and $\frac{0}{1}$ form the fraction $\frac{1+0}{2+1}=\frac{1}{3}$.

$\frac{1}{3}>\frac{48}{155}$, $\therefore$ write $\frac{1}{3}$ in the upper line.

From $\frac{1}{3}$ and $\frac{0}{1}$ form the intermediate fraction $\frac{1+0}{3+1}=\frac{1}{4}$.

$\frac{1}{4}<\frac{48}{155}$, $\therefore$ write $\frac{1}{4}$ in the lower line.

From $\frac{1}{4}$ and $\frac{1}{3}$ form the intermediate fraction $\frac{1+1}{4+3}=\frac{2}{7}$.

$\frac{2}{7}<\frac{48}{155}$, $\therefore$ write $\frac{2}{7}$ in the lower line.

From $\frac{2}{7}$ and $\frac{1}{3}$ form the intermediate fraction $\frac{2+1}{7+3}=\frac{3}{10}$.

$\frac{3}{10}<\frac{48}{155}$, $\therefore$ write $\frac{3}{10}$ in the lower line.

From $\frac{3}{10}$ and $\frac{1}{3}$ form the intermediate fraction $\frac{3+1}{10+3}=\frac{4}{13}$.

$\frac{4}{13}<\frac{48}{155}$, $\therefore$ write $\frac{4}{13}$ in the lower line.

From $\frac{4}{13}$ and $\frac{1}{3}$ form the intermediate fraction $\frac{4+1}{13+3}=\frac{5}{16}$.

$\frac{5}{16}>\frac{48}{155}$, $\therefore$ write $\frac{5}{16}$ in the upper line.

From $\frac{5}{16}$ and $\frac{4}{13}$ form the intermediate fraction $\frac{5+4}{16+13}=\frac{9}{29}$.

$\frac{9}{29}>\frac{48}{155}$, $\therefore$ write $\frac{19}{29}$ in the upper line.

From $\frac{9}{29}$ and $\frac{4}{13}$ form the intermediate fraction $\frac{9+4}{29+13}=\frac{13}{42}$.

$\frac{13}{42}<\frac{48}{155}$, $\therefore$ write $\frac{13}{42}$ in the lower line.

From $\frac{13}{42}$ and $\frac{9}{29}$ form the intermediate fraction $\frac{13+9}{42+29}=\frac{22}{71}$.

$\frac{22}{71}>\frac{48}{155}$, $\therefore$ write $\frac{22}{71}$ in the upper line.

Continuing this process we arrive at length at the given fraction $\frac{48}{155}$ which may be placed in *either* line. If the calculation be continued beyond this, the succeeding convergents will be all less or all greater than the given fraction accoıding as the latter was written in the upper or in the lower line. In the preceding list we have placed $\frac{48}{155}$ in the upper line and have given the next two lower-line convergents, viz. $\frac{61}{197}$ and $\frac{109}{352}$. Had we written $\frac{48}{155}$ in the lower line, we would have obtained as the next two upper-line convergents $\frac{48+35}{155+113}=\frac{83}{268}$ and $\frac{83+48}{268+155}=\frac{131}{423}$. Both lines being endless may be continued indefinitely, but as the terms of all the convergents following the given fraction are larger than the terms of the latter, these succeeding convergents are useless for purposes of approximation and need not here be considered.

78. The convergents $\frac{0}{1}$, $\frac{1}{3}$, $\frac{4}{13}$, $\frac{9}{29}$ and $\frac{13}{42}$ are called **Principal Convergents** to $\frac{48}{155}$; the others are named *Secondary* or **Intermediate Convergents.**

Example 2. Find the convergents to $\frac{25}{134}$.

Proceeding as in *Example 1* we obtain

$$\begin{array}{lllllllllllllll} \frac{1}{0}, & \frac{1}{1}, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{4}, & \frac{1}{5}, & & & \frac{3}{16}, & & & & \frac{14}{75}, & \frac{25}{134}. \\ & \frac{0}{1} & & & & & \frac{1}{6}, & \frac{2}{11}, & & \frac{5}{27}, & \frac{8}{43}, & \frac{11}{59}, & & \end{array}$$

Computation.

$$\frac{0+1}{1+0}=\frac{1}{1}.\quad \frac{1+0}{1+1}=\frac{1}{2}.\quad \frac{1+0}{2+1}=\frac{1}{3}.\quad \frac{1+0}{3+1}=\frac{1}{4}.\quad \frac{1+0}{4+1}=\frac{1}{5}.$$

$$\frac{1+0}{5+1}=\frac{1}{6}.\quad \frac{1+1}{6+5}=\frac{2}{11}.$$

$$\frac{2+1}{11+5}=\frac{3}{16}.$$

$$\frac{3+2}{16+11}=\frac{5}{27}.\quad \frac{5+3}{27+16}=\frac{8}{43}.\quad \frac{8+3}{43+16}=\frac{11}{59}.$$

$$\frac{11+3}{59+16}=\frac{14}{75}.\quad \frac{14+11}{75+59}=\frac{25}{134}.$$

The convergents following $\frac{25}{134}$ are omitted.

Here the Principal Convergents are $\frac{0}{1}$, $\frac{1}{5}$, $\frac{2}{11}$, $\frac{3}{16}$ and $\frac{11}{59}$.

Example 3. Find convergents to $\frac{43}{13}$.

Computing as before we obtain

$$\begin{array}{lllllllllll} \frac{1}{0}, & & & & & \frac{4}{1}, & \frac{7}{2}, & \frac{10}{3}, & & & & \frac{43}{13}. \\ & \frac{0}{1}, & \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & & & & \frac{13}{4}, & \frac{23}{7}, & \frac{33}{10}, & \end{array}$$

Computation. $\frac{0+1}{1+0}=\frac{1}{1}.\quad \frac{1+1}{1+0}=\frac{2}{1}.\quad \frac{2+1}{1+0}=\frac{3}{1}.$

$$\frac{3+1}{1+0}=\frac{4}{1}.\quad \frac{4+3}{1+1}=\frac{7}{2}.\quad \frac{7+3}{2+1}=\frac{10}{3}.$$

$$\frac{10+3}{3+1}=\frac{13}{4}.\quad \frac{13+10}{4+3}=\frac{23}{7}.\quad \frac{23+10}{7+3}=\frac{33}{10}.\quad \frac{33+10}{10+3}=\frac{43}{13}.$$

The Principal Convergents are $\frac{3}{1}$ and $\frac{10}{3}$.

79. From the method of formation of these convergents it is apparent that ;—

(a) The difference between any two consecutive convergents, whether in the same line or in different lines, is a fraction with 1 for numerator and with the product of the denominators of the two convergents for denominator.

(b) Each convergent to a given fraction approaches nearer in value to the given fraction than do any of the preceding convergents *in the same line.*

Thus $\frac{1}{2}-\frac{48}{155}>\frac{1}{3}-\frac{48}{155}>\frac{5}{16}-\frac{48}{155}>\frac{9}{29}-\frac{48}{155}>\frac{22}{71}-\frac{48}{155}$, &c., and $\frac{48}{155}-\frac{1}{4}>\frac{48}{155}-\frac{2}{7}>\frac{48}{155}>\frac{3}{10}>\frac{48}{155}-\frac{4}{13}>\frac{48}{155}-\frac{13}{42}$, &c.

80. From these two fundamental laws, four others follow as immediate consequences. These are :—

1°. All convergents are in their lowest terms.

2°. Between a given fraction and any convergent to it there cannot be inserted a fraction of intermediate value with terms less than those of the next succeeding convergent *in the same line.*

Thus, between $\frac{1}{3}$ and $\frac{48}{155}$ there cannot be inserted a fraction $<\frac{1}{3}$ but $>\frac{48}{155}$, with terms less than those of $\frac{5}{16}$. Between $\frac{1}{4}$ and $\frac{48}{155}$ there cannot be inserted a fraction $>\frac{1}{4}$ but $<\frac{48}{155}$, with terms less than those of $\frac{2}{7}$.

Corollary. The terms of all fractions intermediate in value between a given fraction and any *principal convergent* to it are *greater* than the terms of the next succeeding *principal convergent.*

3°. The difference between any two consecutive *principal convergents* is a fraction with 1 for numerator and with the product of the denominators of the two convergents for denominator.

4°. The difference between a given fraction and any *principal convergent* to it is less than the difference between the given fraction and any fraction with terms smaller than those of the principal convergent.

81. The Corollary to the Second Law applies to principal convergents only and distinguishes them from intermediate convergents, it being noted that the principal convergents to any given fraction are alternately greater and less than the given fraction.

82. The Fourth Law holds for principal convergents but does not necessarily hold for intermediate convergents. Cases occur in which a convergent in one line differs less from the given fraction than does a succeeding and therefore larger-termed intermediate fraction *in the other line*. Thus, in *Example 1* page 56, $\frac{48}{155}-\frac{1}{4}>\frac{1}{3}-\frac{48}{155}$, so also $\frac{48}{155}-\frac{2}{7}>\frac{1}{3}-\frac{48}{155}$ and $\frac{5}{16}-\frac{48}{155}>\frac{48}{155}-\frac{4}{13}$. Hence both $\frac{1}{4}$ and $\frac{2}{7}$ are inferior to $\frac{1}{3}$ and $\frac{5}{16}$ is inferior to $\frac{4}{13}$, if we consider these fractions solely as approximations to $\frac{48}{455}$, regardless of whether they are approximations in excess or in defect. This Fourth Law, therefore, marks out the Principal Convergents to any large-termed fraction as, in general, the best small-termed substitutes for such large-termed fraction, in approximate calculations. It is consequently important to have an expeditious method of calculating the principal convergents to any given fraction. Such a method is exhibited in the following examples.

Example 1. Find the principal convergents to $\frac{48}{155}$.

A. Divide both terms of the given fraction by the numerator.

$$\frac{48}{155}=\frac{1}{155\div 48}=\frac{1}{3+\frac{11}{48}}$$

Now $3<3+\frac{11}{48}$,

$$\therefore \quad \frac{1}{3}>\frac{1}{3+\frac{11}{48}} \qquad \text{i. e.,} \ \frac{1}{3}>\frac{48}{155}. \qquad \text{(i)}$$

B. Divide both terms of $\frac{11}{48}$ by the numerator.

$$\frac{11}{48}=\frac{1}{48\div 11}=\frac{1}{4+\frac{4}{11}}, \quad \therefore \quad \frac{48}{155}=\frac{1}{3+\cfrac{1}{4+\frac{4}{11}}}$$

Now $4<4+\frac{4}{11}$,

$$\therefore \quad \frac{1}{4}>\frac{1}{4+\frac{4}{11}},$$

$$\therefore \quad \frac{1}{3+\frac{1}{4}}<\frac{1}{3+\cfrac{1}{4+\frac{4}{11}}} \qquad \text{i. e.,} \quad \frac{1}{3+\frac{1}{4}}<\frac{48}{155}. \qquad \text{(ii)}$$

C. Divide both terms of $\frac{4}{11}$ by the numerator.

$$\frac{4}{11}=\frac{1}{11\div 4}=\frac{1}{2+\frac{3}{4}}, \quad \therefore \frac{48}{155}=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\frac{3}{4}}}}$$

Now $2<2+\frac{3}{4}$,

$$\therefore \quad \frac{1}{2}>\frac{1}{2+\frac{3}{4}},$$

$$\therefore \quad \frac{1}{4+\frac{1}{2}}<\cfrac{1}{4+\cfrac{1}{2+\frac{3}{4}}}$$

$$\therefore \quad \cfrac{1}{3+\cfrac{1}{4+\frac{1}{2}}}>\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\frac{3}{4}}}}, \text{ i. e. } \cfrac{1}{3+\cfrac{1}{4+\frac{1}{2}}}>\frac{48}{155}. \qquad \text{(iii)}$$

D. Divide both terms of $\frac{3}{4}$ by the numerator.

$$\frac{3}{4}=\frac{1}{4\div 3}=\frac{1}{1+\frac{1}{3}}, \quad \therefore \frac{48}{155}=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}} \qquad \text{(v)}$$

Now $1<1+\frac{1}{3}$,

$$\therefore \quad \frac{1}{1}>\frac{1}{1+\frac{1}{3}},$$

$$\therefore \quad \frac{1}{2+\frac{1}{1}}<\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}},$$

$$\therefore \quad \cfrac{1}{4+\cfrac{1}{2+\frac{1}{1}}}>\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}$$

$$\therefore \quad \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\frac{1}{1}}}}<\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}}, \text{ i.e. } \cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\frac{1}{1}}}}<\frac{48}{155} \qquad \text{(iv)}$$

We thus find that $\frac{48}{155}$

is less than $\frac{1}{3}$ but greater than $\frac{1}{3+\frac{1}{4}}$, (i) & (ii)

is less than $\dfrac{1}{3+\dfrac{1}{4+\frac{1}{2}}}$ but greater than $\dfrac{1}{3+\dfrac{1}{4+\dfrac{1}{2+\frac{1}{1}}}}$, (iii) & (iv)

and is equal to $\dfrac{1}{3+\dfrac{1}{4+\dfrac{1}{2+\dfrac{1}{1+\frac{1}{3}}}}}$. (v)

83. That these fractions are the principal convergents to $\frac{48}{155}$ may be shown thus ;—

Reducing all to simple fractions they are $\frac{1}{3}$, $\frac{4}{13}$, $\frac{9}{29}$, $\frac{13}{42}$, $\frac{48}{155}$.

1°. $\frac{48}{155}-\frac{13}{42}=\frac{1}{155\times 42}$,

∴ the terms of all fractions $<\frac{48}{155}$ but $>\frac{13}{42}$ are greater than the terms of $\frac{48}{155}$;

$\frac{48}{155}$ is the principal convergent to itself,

∴ $\frac{13}{42}$ is the principal convergent next preceding $\frac{48}{155}$.

2°. $\frac{9}{29}>\frac{48}{155}>\frac{13}{42}$ and $\frac{9}{29}-\frac{13}{42}=\frac{1}{29\times 42}$,

∴ the terms of all fractions $>\frac{48}{155}$ but $<\frac{9}{29}$ are greater than the terms of $\frac{13}{42}$;

$\frac{13}{42}$ is a principal convergent to $\frac{48}{155}$,

∴ $\frac{9}{29}$ is the principal convergent next preceding $\frac{13}{42}$.

3°. Similarly it may be proved that $\frac{4}{13}$ and $\frac{1}{3}$ are the other principal convergents to $\frac{48}{155}$.

84. The operations A, B, C and D may be summarized as follows :—

Divide 155 by 48 ; divide 48 by 11, the remainder in the preceding division ; divide 11, the first remainder, by 4, the second remainder ; divide 4, the second remainder, by 3, the third remainder ; divide 3, the third remainder, by 1, the fourth remainder.

Now this is nothing else than the series of operations for finding the G. C. M. of the two numbers 48 and 155. Arranging the work as in the *Public School Arithmetic*, page 100, it appears thus ;—

	3	4	2	1	3	*Quotients.*
155	48	11	4	3	1	
144	44	8	3	3		
11	4	3	1			

The convergents may now be written down from the line of quotients, thus ;—

3, 4, 2, 1, 3.

$$\frac{1}{3},\quad \frac{1}{3+\frac{1}{4}},\quad \frac{1}{3+\cfrac{1}{4+\frac{1}{2}}},\quad \frac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\frac{1}{1}}}},\quad \frac{1}{3+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{3}}}}}.$$

The simple fractions equivalent to these convergents may be calculated by the ordinary method of reducing complex fractions to simple forms, or otherwise thus ;—

Quotients.	*Calculation.*	*Convergents.*	
		$\frac{1}{0}$, $\frac{0}{1}$,	*(Initial.)*
3	$\frac{1+0\times3}{0+1\times3}=$	$\frac{1}{3}$,	(i)
4	$\frac{0+1\times4}{1+3\times4}=$	$\frac{4}{13}$,	(ii)
2	$\frac{1+4\times2}{3+13\times2}=$	$\frac{9}{29}$,	(iii)
1	$\frac{4+9\times1}{13+29\times1}=$	$\frac{13}{42}$,	(iv)
3	$\frac{9+13\times3}{29+42\times3}=$	$\frac{48}{155}$,	(v)

85. Limits for the errors arising from the substitution of $\frac{1}{3}$, $\frac{4}{13}$, &c. for $\frac{48}{155}$ may be obtained as follows :—

$\frac{1}{3} > \frac{48}{155} > \frac{4}{13}$, $\quad \therefore \frac{1}{3} - \frac{48}{155} < \frac{1}{3} - \frac{4}{13} = \frac{1}{3\times13}$,

i. e. the error arising from the use of $\frac{1}{3}$ for $\frac{48}{155}$ is less than $\frac{1}{3\times13}$.

$\frac{4}{13} < \frac{48}{155} < \frac{9}{29}$, $\quad \therefore \frac{48}{155} - \frac{4}{13} < \frac{9}{29} - \frac{4}{13} = \frac{1}{13\times29}$,

i. e. the error arising from using $\frac{4}{13}$ for $\frac{48}{155}$ is less than $\frac{1}{13\times29}$,

Similarly it may be shown that $\frac{1}{29\times41}$ is a superior limit of error in the substitution of $\frac{9}{29}$ for $\frac{48}{155}$.

$\frac{1}{42\times155}$ is the error in the substitution of $\frac{13}{42}$ for $\frac{48}{155}$.

Example 2. Find the principal convergents to $\frac{23478}{33593}$,

	1	2	3	8	1	3	13
33593	23478	10115	3248	371	280	91	7=G. C. M. of terms.
23478	20230	9744	2968	280	273	91	
10115	3248	371	280	91	7		

Quotients, 1, 2, 3, 8, 1, 3, 13.

Convergents, $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{58}{83}$, $\frac{65}{93}$, $\frac{253}{362}$, $\frac{3354}{4799}$.

Limits of error, $\frac{1}{1\times3}$, $\frac{1}{3\times10}$, $\frac{1}{10\times83}$, $\frac{1}{83\times93}$, $\frac{1}{93\times362}$, $\frac{1}{362\times4799}$, 0.

86. If the given fraction be improper, reduce it to a mixed number and use the integral part of the mixed number as numerator in place of 0 in the initial $\frac{0}{1}$.

Example 3. Find a series of convergents to 3·14159265 which is approximately the ratio of the circumference of a circle to its diameter, *i. e.*, approximately the measure of the circumference in terms of the diameter as unit.

	7	15	1	288
100000000	14159265	885145	882090	3055
99114855	885145	882090	6110	
885145	5307815	3055	27109	
	4425725		24440	
	882090		26690	
			24440	

Quotients, 7, 15, 1.

Convergents, $\frac{1}{0}$, $\frac{3}{1}$, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$.

Hence the circumference of a circle is longer than 3 diameters of the circle, is shorter than $\frac{22}{7}$ diameters, is longer than $\frac{333}{106}$ diameters and again is shorter than $\frac{355}{113}$ diameters.

The limits of error are $\frac{1}{7}$, $\frac{1}{7\times106}$, $\frac{1}{106\times113}$, and $\frac{1}{113\times32650}$ respectively.

87 From this example it is evident that *those convergents which immediately precede large quotients are the best approximations to employ as substitutes for exact values.*

Example 4. Find a series of convergent comparisons of the metre = 39·370432 in. and the yard = 36 in.

The quotients of 36/39·370432 are

1, 10, 1, 2, 7, 2, 1, 5;

and the corresponding convergents, omitting initials, are

$\frac{1}{1}$, $\frac{10}{11}$, $\frac{11}{12}$, $\frac{32}{35}$, $\frac{235}{257}$, $\frac{502}{549}$, $\frac{737}{806}$, $\frac{4187}{4579}$.

Hence 10 m. < 11 yd. but 11 m. > 12 yd.;
32 m. < 35 yd. but 235 m. > 257 yd.; &c.

EXERCISE II.

Find the principal convergents to ;—

1. $\frac{20}{29}$. **6.** $\frac{12056}{100309}$. **11.** 1·4142. **16.** ·0498756.
2. $\frac{117}{326}$. **7.** $\frac{9131}{13128}$. **12.** 1·73205. **17.** ·2439.
3. $\frac{252}{365}$. **8.** $\frac{2801}{14972}$, **13.** 2·44949. **18.** 1·41844.
4. $\frac{90}{37}$. **9.** $\frac{493699}{1147355}$. **14.** ·43589. **19.** 2·71828.
5. $\frac{1363}{740}$. **10.** $\frac{488459}{375813}$. **15.** ·55744. **20.** 2·302585.

Find a series of convergent comparisons of :—

21. The kilometre = 1093·62311 yd. and the mile = 1760 yd.

22. The hectare and the acre.

23. The kilogramme and the pound.

24. The millier and the ton.

25. The kilolitre and the cubic yard.

26. The litre and the quart.

27. The Canadian standard metre = 39·382 in. and the French standard metre = 39·37043 in.

28. The earth's polar diameter = 41708954 ft. and its longest equatorial diameter = 41853258 ft.

29. The tenacity of steel and the tenacity of copper wire the former being $\frac{793}{414}$ times the latter.

30. The excess of the mean solar year of 365 da. 5 hr. 48 m. 47·46 sec. over the ordinary civil year of 365 da., and one day. Hence show that if there were 8 leap-years in every 33 years, this system would not be wrong by so much as 1 day in 4224 years, and compare this with the Gregorian system of 97 leap years in every 400 years.

APPROXIMATE CALCULATIONS.

88. The greater part of the labor of computation in calculations in which fractions occur arises in general from the several fractions having different denominators. For example, if two or more fractions are to be added together, they must all be brought to the same denominator, if one fraction is to be divided by one or more others all of different denominators, the terms of the quotient are in most cases much larger than the terms of the dividend. The labor of computation may be lessened by using convergents instead of exact values; it may often be lessened and the calculations may always be simplified by replacing the fractions by approximately equal decimal numbers. If we adopt either of these ways of lessening the labor of computation, we deliberately incur an error in calculation which we know will give a result sufficiently near the truth for all practical purposes.

89. In calculations concerning quantities which presuppose measurements, it should be remembered that these measurements cannot be made with absolute accuracy. In the measurements of every-day life we are satisfied if we do not err by more than one part in a thousand; in the most careful scientific work it is rarely possible to reduce the error below one part in a million. The results of calculations based on such measurements are necessarily affected by the errors of measurement and it is therefore a mere waste of time and labor to carry any calculation beyond the degree of accuracy with which measurements can be made. It is moreover misleading, for the results then present an appearance of exactness where exactness does not and cannot exist.

90. The first significant figure in any number is the first digit,—the first figure other than zero,—on the left of the number.

Examples. In 980·61 min., 9 is the first significant figure and in ·000122 da., 1 is the first significant figure.

91. A number is said to be correct to two, three, four, ... significant figures if it does not differ from the number that would express the exact value by more than 5 in the second, third, fourth, ... place on the right of the first significant figure.

Example 1. If it is said that the length of a certain line is 3·9 in. correct to *two* significant figures, it is meant that the actual length is between 3·85 in. and 3·95 in.

If the length is given as 3·94 in. correct to *three* significant figures, it is meant that the actual length lies between 3·935 in. and 3·945 in.

If the length is said to be 3·937 in. correct to *four* significant figures, the actual length may be any between 3·9365 in. and 3·9375 in.

Example 2. If the length of the greatest equatorial diameter of the earth be given as 41,852,000 ft. and the length of the polar diameter as 41,710,000 ft., correct in both cases to *five* significant figures, it is meant that the actual length of that particular equatorial diameter is not less than 41,851,500 ft. but is less than 41,852,500 ft., and that the actual length of the polar diameter is not less than 41,709,500 ft. but is less than 41,710,500 ft.

92. The degree of any approximation is measured by the fraction which the total error is of the exact value, *i. e.*, by the quotient of the difference between the exact and the approximate value divided by the exact value. *The degree of approximation is* therefore *independent of the unit of measurement.*

Example. If the length of the polar diameter of the earth is 41,710,000 ft. correct to five figures, the difference between this length and the exact length is *at most* 500 ft. and the actual length of the polar diameter is greater than 41,709,500 ft. Hence the greatest possible rate of error is 500 ft. in 41,709,500 ft. $\doteq$ 1 part in 83,419. The degree of approximation is therefore at worst $\frac{1}{83419}$ of the whole.

Had we used the mile instead of the foot, as the unit of measurement in the foregoing, the degree of approximation would have been found to be at least as close as $\frac{500}{5280}$ mi. in $\frac{41709500}{5280}$ mi. $=1$ part in $83{,}419=\frac{1}{83419}$ of the whole.

93. Different degrees of approximation may be roughly compared by comparing together the significant figures known to be correct in each case.

Thus if the first three significant figures are known to be correct the approximation is about ten times as close as it would be if only the first two were known to be correct. "Correct to six significant figures" means an approximation about 1000 times as close as that of "correct to three significant figures."

94. In expressing mixed numbers and fractions by approximately equal decimal numbers, it is in general sufficient if the calculations are correct to four or at most to seven significant figures. *Beyond seven figures we very seldom need go.*

So also if one approximate number is to be multiplied by another or to be divided by another, the result need not be calculated to a greater number of significant figures than are correct in the given numbers.

Example 1. Find the product of 678·233 multiplied by 47·9583 correct to six significant figures.

Uncontracted Form.

678·233	
47·9583	
27129·32	
4747·63	1
610·40	97
33·91	165
5.42	5864
·20	34699
32526·90	16839

Contracted Form.

678·233	
47·9583	
27129·32	
4747·63	(*a*).
610·41	(*b*).
33·91	(*c*).
5·42	(*d*).
·20	(*e*).
32526·9	

We begin by multiplying by 4, the first significant figure in the multiplier. The product contains 7 significant figures; this is one more than the number required to be correct, but we retain all seven that we may determine the 'carriage' to the sixth significant figure when adding together the partial products. We contract the subsequently calculated partial products thus;—

(*a*). Strike the right hand 3 from the multiplicand and multiply by 7, carrying 2 from the 3×7 struck out.

(*b*). Strike the second 3 from the already contracted multiplicand, and multiply by 9 carrying 3 from the 3×9 struck out.

(*c*). Strike 2 from the multiplicand as contracted in (*b*) and multiply by 5 carrying 1 from 2×5 struck out.

(*d*). Strike 8 from the multiplicand as contracted in (*c*) and multiply by 8 carrying 6 from the 8×8 struck out.

(*e*). Strike 7 from the multiplicand as contracted in (*d*) and multiply by 3 carrying 2 from the 7×3 struck out.

The sum of the right-hand figures of the partial products is 9. This would be the seventh significant figure of the product, but as the product is to be correct to only six significant figures, we change 9 to the nearest multiple of 10 which in this case is 10 itself. We now complete the addition of the partial products as in the ordinary uncontracted form.

The approximation in line (*d*) would have been closer had we carried 7 from 8~~2~~ × 8 struck out instead of carrying 6 from 8 × 8 struck out; but as we are working to one figure more than the number required to be correct in the result, the carried 6 is practically as good an approximation as the carried 7 would be and is more easily and quickly obtained requiring us to take account of only one figure, the last figure struck out. In line (*e*), the carriage should have been from 8 × 3 instead of from 7 × 3, the 7~~8~~ struck out being nearer 80 than 70.

[For the position of the multiplier and of the decimal point in the product, see *Public School Arithmetic* p. 155 and the examples on p. 156.]

Example 2. Find the product of 15876 multiplied by 15876 multiplied by 15876, correct to 5 significant figures.

1	15876
2	31752
3	47628
4	63504
5	79380
6	95256
7	111132
8	127008
9	142884
10	158760

```
         15876
         15876
     ---------
     15876
       79380
       12701
        1111
          95
     ---------
     252047000 . . . . Multiplier.
         15876 . . . . Multiplicand.
 -------------
 31752
  7938
   318
     6
     1
 -------------
 4001500000000
```

[For a condensed notation applicable to examples like this, see § 122.]

Example 3. Divide 32526·9 by 678·233, obtaining the quotient correct to 6 significant figures.

```
Uncontracted Form.                  Contracted Form.
            47·9583                             47·9583
678233)32526900                     678233)32526900
        2712932                             2712932
         539758|0                            539758
         474763|1                            474763 . . . . (a).
          64994|9·0                           64995
          61040|9·7                           61041 . . . . (b).
           3953|9 30                           3954
           3391|1 65                           3391 . . . . (c).
            562|7 650                           563
            542|5 864                           542 . . . . (d).
             20|1 7860                           21
             20|3 4699                           20 . . . . (e).
              -|1·6839                            1
```

The sign – before 1·6839 denotes that the quotient 47·9583 is *too great;* it is however nearer the exact quotient than 47·9582 would be.

For the method of obtaining lines (*a*), (*b*), (*c*), (*d*) and (*e*) see *Example 1*, page 68.

Computers' Contracted Form.

```
3|32526900·
3|  539758
2|   64995
8|    3954
7|     563
6|      21
 |       1
 ------------
        47·9583
```

Example 4. Find the weight (in Imperial tons of 2240 lb. each) of the carbon in the carbonic acid gas in the atmosphere resting on a square mile of land when the pressure of the atmosphere is 14·73 lb. to the square inch, given (i) that each cubic foot of air contains ·00035 of a cubic foot of carbonic acid gas, correct to 2 significant figures; (ii) that the weight of any volume of carbonic acid gas is, to 3 significant figures, 1·52 times the weight of an equal volume of air under the same pressure and at the same temperature; (iii) that $\frac{3}{11}$ by weight of all carbonic acid gas is carbon, correct to 4 significant figures. (*See Huxley's Physiography, Chap. VI.*)

Our answer will be correct to only 2 significant figures, for datum (ii) is correct to only 2 significant figures, and it is the datum with the least number of figures correct that determines the number of figures correct in the result of any calculation. We compute at first to 4 significant figures, reducing this number to 3 and finally to 2 as the number of operations to be performed become fewer.

Wt. of air on sq. in.	$=14{\cdot}73$ lb.
1 mi.	$=$ 63360 in.
Wt. of air on sq. mi.	$=14{\cdot}73$ lb. $\times 63360 \times 63360$
	$=59{,}130{,}000{,}000$ lb.
Wt. of carb. acid gas in this air	$=59{,}130{,}000{,}000$ lb. $\times {\cdot}00035 \times 1{\cdot}52$
	$=31{,}500{,}000$ lb.
Wt. of carbon in this gas	$=31{,}500{,}000$ lb. $\times \frac{3}{11}$
	$=$ 8,600,000 lb.
	$=$ 3,800 T. *Imperial.*

95. These methods of contraction are easily adapted to calculations in which the result is required to be correct to a given number of decimal places.

Example. Find the interest on \$79·27 for 93 days at $7\frac{1}{2}\%$.

```
 $79·27  (9, 2, 7 struck out)
     ·075
 ---------
   5·55  . . . . . . . . . . . . . . . . . . . (a).
    ·40  . . . . . . . . . . . . . . . . . . . (b).
 ---------
   5·95  . . . . . . . . . . . . . . . . . . . (c).
 ---------
 5|5 53·35 (3, 5 struck out) . . . . . . . . . (d).
 6|1 88
 3|   5
  |   1
 ---------
   $1·51
```

(*a*) The 7 in the multiplier stands above the 4th decimal place, but only 2 decimal places are required in the result, therefore strike ·27, the two right-hand figures, out of the multiplicand, and then multiply 79, the uncancelled part, by 7, carrying 2 from ·27 × 7 struck out.

(*b*) Strike 9 from the multiplicand as contracted in (*a*), and multiply by 5 carrying 5 from 9 × 5 struck out.

(*c*) \$5·95 is the interest, to the nearest cent, on \$79·27 for 1 year at $7\frac{1}{2}\%$.

(*d*) To multiply 5·95 by 93, multiply 5·95 by 7 and 'make up' the product, figure by figure as computed, to 595·00, *i.e.*, to 5·95 × 100 setting down the 'making up' numbers, thus,—

7 times 5			and	**5**	(set down)		= 4,0′
7 "	9 and 4	(carried)	"	**3**	"	"	= 7,0′
7 "	5 " 7	"	"	**3**	"	"	= 4,5′
	4	"	"	**5**	"	"	= 9′
				5	"	"	= 5′

The accented figures are those of 595·00. (*See* § 69, *Case* xii.)

96. In the preceding calculation, the sole influence of the 27 cents in the principal is the addition to the *annual* interest, of the 2 cents 'carried' in line (*a*). Even this small increment disappears from the interest for 93 days, \$1·51 being practically the interest on \$79 for 93 days at $7\frac{1}{2}\%$. The omission from the principal or the addition to it of any number of cents less than 50, will not in general change by more than one cent the computed amount of the interest for a short-term loan, but the retention of the cents in the calculation will considerably increase the labor of computation. For this reason, business men compute on the nearest number of dollars, when reckoning short-term interest and when determining the equated time of an account. (See *Public School Arithmetic*, p. 168.)

EXERCISE III.

1. Find the sum of 143·035472, 29·680037, ·089173, 4·99876 and 2923·937958, correct to 4 decimal places.

2. Find the value of 379·28056 + 29·68043 + 6·8409207 − 44·398642 − 3·7984061 + ·2368592 − 300·790797, correct to 6 significant figures.

3. Find the product of 478·593 and 3·14159 correct to 3 decimal places.

4. Find the value of 427·803 × ·00749 correct to 5 decimal places.

5. Find the value of 3·1416 × 3·1416 × 3·1416 to the nearest integer.

6. Find the product of 2·9957323 and ·4342945 correct to 6 decimal places.

7. Find the value of 5·7037825 × ·4342945 correct to 6 decimal places.

8. Find the value of 3·14159265 × ·96 × ·995 × ·9998 × ·99992 × ·999997, correct to 6 decimal places.

9. Find the value of 2·7182818 × ·8 × ·992 × ·9993 × ·99998 × ·999993 × ·9999994 correct to 7 decimal places.

10. Find the value of 2·3025851 × ·9 × ·97 × ·995 × ·99995 × ·999997 correct to 7 decimal places.

11. Find the product of 1·0000127 × 1·004 and ·99898 × ·99898 correct to 7 decimal places.

12. Find the value of 10000127 × 999987, correct to 8 significant figures.

13. Find the value of 16·934 × 16·934 × 16·934 correct to 5 significant figures.

14. Find the value of 4·8784 × 4·8784 × 4·8784 correct to 5 significant figures.

15. Find the value of 9·0708324 × 9·0708324 × 9·0708324 × 9·0708324, correct to 6 significant figures.

16. Find the value of 2·0188223 × 2·0188223 × 2·0188223 × 2·0188223 × 2·0188223 × 2·0188223 correct to 6 significant figures.

Find the values of the following quotients, correct to 6 significant figures :—

17. 100 ÷ 1·414214.
18. 25000 ÷ 3·141593.
19. ·07 ÷ 2·64575.
20. 1·95 ÷ 139·6424.
21. ·6931472 ÷ 2·302585.
22. 1·098612 ÷ 2·302585.
23. 1 ÷ 3·14159265.
24. 1 ÷ ·43429448.
25. 11 ÷ 2·22398 ÷ 2·22398.
26. 4517 ÷ 16·5304 ÷ 16·5304.
27. 19·5 ÷ 2·236068 ÷ 6·244998.

Find the values of the following, to 5 significant figures :—

28. $1+\frac{1}{1}+\frac{1}{1\times 2}+\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}+\frac{1}{1\times 2\times 3\times 4\times 5}+$ &c.

29. $1-\frac{1}{1}+\frac{1}{1\times 2}-\frac{1}{1\times 2\times 3}+\frac{1}{1\times 2\times 3\times 4}-\frac{1}{1\times 2\times 3\times 4\times 5}+$ &c.

30. $\frac{1}{1}+\frac{1}{1\times 3}+\frac{1}{1\times 3\times 5}+\frac{1}{1\times 3\times 5\times 7}+\frac{1}{1\times 3\times 5\times 7\times 9}+$ &c.

31. $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+$ &c.

32. $\frac{1}{1}+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+$ &c.

33. $\frac{1}{1}+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}+$ &c.

34. Prove that the answer to problem 29 is the reciprocal to 5 significant figures of the answer to problem 28.

35. Express 8749 yd. in metres correct to 4 significant figures.

36. Express 1760 metres in yards correct to 4 significant figures.

37. Express 4840 sq. yd. in centiares correct to 4 significant figures.

38. Express 4840 centiares in sq. yd. correct to 4 significant figures.

39. Express 100 acres in hectares correct to 4 significant figures.

40. Express 100 hectares in acres correct to 4 significant figures.

41. Express 600 litres in gallons correct to 4 significant figures.

42. Express 132 gallons in litres correct to 4 significant figures.

43. The mean distance of the moon from the earth is 238800 miles; express this in kilometres to 4 significant figures.

44. The mean distance of the sun from the earth is 91,430,000 miles; express this in kilometres to 4 significant figures.

45. The mean distance of Saturn from the sun is 872,140,000 miles, and of the earth from the sun 91,430,000 miles; form a series of convergent comparisons of these distances.

46. Form a series of convergent comparisons of 346·619 da. and 29·5306 da., and hence show that 19 times the former period is nearly equal to 223 times the latter. Express these products in terms of a year of 365·25 days.

47. Taking the length of the sidereal year as 365·25636 days and that of the lunar month as 29·53059 days find a series of convergent comparisons of the lunar month and the sidereal year.

48. Mars revolves about the sun in 686·9797 days and the earth revolves about the sun in 365·2564 days; find a series of convergent comparisons of the length of the Martian year with that of the earth.

49. Jupiter rotates on its axis once every 9 hr. 55 min. 26 sec., and the earth once every 23 hr. 56 min. 4 sec.; find a series of convergent comparisons of these times of rotation.

50. Mercury revolves about the sun in 87·9693 da. at a mean distance of 35,392,000 miles; and the earth revolves about the sun in 365·2564 da. at a mean distance of 91,430,000 miles. Find convergent comparisons of the speed of Mercury and the earth in their orbits.

EXERCISE IV.

MISCELLANEOUS PROBLEMS.

1. The mercury in a barometer rose ·121 in., ·073 in. and ·019 in. in three successive days, it fell ·054 in. and ·065 in. during the two following days, rose ·053 in. on the sixth day and fell ·028 in. on the seventh day. If its height at the beginning of the first day was 30·078 in., what was its height at the close of the seventh day?

2. Find the weight of a rectangular beam of oak $18' \times 13'' \times 13''$, weighing 47·375 lb. per cu. ft. How many cubic feet of water would be of the same weight as the beam?

3. A clock gains $\frac{11}{15}$ of $3\frac{1}{2}$ sec. in 2 hr. 30 min. If allowed to run at this rate how much will the clock gain in 8 da. 8 hr. correct time? How much will it gain at this rate, if it run for 8 da. 8 hr. by its own time?

4. A man sold $\frac{5}{9}$ of his wheat and then $\frac{7}{22}$ of the remainder and next $\frac{11}{20}$ of what then remained and had 18 bushels more than ·12 of his wheat left. How many bushels had he at first?

5. An india-rubber band $8''$ long $\frac{3}{8}''$ wide and $\frac{1}{16}''$ thick is stretched until it is $18''$ long and $\frac{1}{4}''$ wide. What must be its thickness, the volume of the india-rubber remaining unchanged?

6. If 1 lb. of brass consisting of 84 parts of copper and 16 of zinc, be mixed with 2 lb. of brass consisting of 75 parts of copper and 25 of zinc, find the percentage of copper and of zinc in the mixture.

7. In 1881 the silver mines of Austria yielded 12,383 metric tons of silver ore from which 31,359 kilogrammes of silver were extracted. What percentage of the ore was silver? Express the weight of the ore in Imperial tons and the weight of the silver in Troy ounces, and employing these expressions of the weights, recalculate the percentage which the silver constitutes of the ore.

8. *A. B.* bought goods amounting to \$7460 subject to 25 and 5 off, \$3730 subject to 30 off and \$1492 subject to 20 and 10 off, find the net cost of the goods. Were the invoice-clerk to bill *A. B.* with goods amounting to \$12682 subject to 30 off, what would be the amount of the error in the net cost of the goods?

9. Find the equated time of payment of a bill for \$748 of which \$225 is at 30 days, \$245 is at 60 days and the balance is at 90 days all from 31st Aug. 1889.

10. The proceeds of a draft for \$628·60 drawn at 90 days, amounted to \$615·79. What was the rate of discount ?

---

11. A train is due at a certain station at 42 min. past 2 p.m. The actual times of its arrival at the station for a certain week were :— Monday, 2,38 p.m. ; Tuesday, 2,47 p.m. ; Wednesday, 3,07 p.m. ; Thursday, 2,39 p.m. ; Friday, 2,42 p.m. ; Saturday, 3,11 p.m. By how many minutes on an average was the train late that week, (i) not counting 'minutes ahead of time,' (ii) including 'minutes ahead of time' in the averaging ?

12. How often is the circumference of a circle 1′ 9″ radius contained in the diameter of a circle whose circumference is 100 feet ?

13. What will be the weight of a rectangular sheet of glass 6′ $3\frac{1}{2}$″ long by 4′ $4\frac{1}{2}$″ wide and $\frac{5}{16}$ in. thick, the glass weighing 168 lb. per cubic foot ?

14. How many days were there from 13th Nov. 1887 to 9th June 1888 ? Express the interval from noon on the former day to noon on the latter day as a fraction of the year 1887 and also as a fraction of the year 1888.

15. A watch is set right on Monday at 9,15 a.m. and it gains $3\frac{1}{3}$ sec. per hour. On what day and at what hour will it have gained exactly 5 min. and what time will it then indicate ? What will be the correct time when the watch indicates 9,15 on the following Monday morning ?

16. Out of a certain sum of money one-half was spent, then one-third of the remainder, next one-twelfth of what still remained and lastly one-fifteenth of what then remained, leaving 39ct. less than one-half of what was spent. What was the original sum ?

17. A man buys milk at 5ct, a quart and having mixed it with water, sells the mixture at 6ct. a quart. His profits are equal to 40% of the cost of the milk. How much water is mixed with each quart of milk ? What proportion of the mixture is water ?

18. If an investment of \$7483·50 yield a net profit of \$483·67, what rate per cent. of profit is returned by the investment ? If this profit is reinvested along with the original investment, and the whole yield a second profit at the same rate per cent. as the first, what will be the amount of this second profit ?

19. James King & Co. of Brantford sold to Henry Adams of Paris bills of merchandise as follows :—12th Dec. 1888, $1174·80 at 90 da. ; 3rd Jan. 1889, $729·65 at 90 da. ; 21st Jan. 1889, $106·20 at 75 da. ; 12th Feb. 1889, $1485·45 at 60 da. ; 7th March 1889, $973·28 at 30 da. Find the equated time and make out a statement of account on the average date.

20. A note for $355 drawn on 3rd April 1889 was discounted on 11th April ; the proceeds amounted to $348·03. What was the rate of discount, the rate of exchange being $\frac{1}{5}$%, reckoned to nearest cent.

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21. A bicyclist rode 50 mi. in 3 hr. 6 min. 40 sec. ; what was his rate in feet per second, in yards per minute, and in miles per hour ?

22. The leading wheels of a locomotive are 3′ 2″ in diameter and the driving wheels 5′ 6″ ; how many revolutions will the former make while the latter make 2166 ? What distance will have been run ? If the distance is run in 20 min. at what rate in miles per hour will the run be made ?

23. Find the weight of a slate blackboard measuring [19′ 6′ × 3′ 6″ × $\frac{5}{8}$″] if a cubic foot of the slate weigh 178 lb.

24. In a certain gold mine, 11 tons of ore yielded $7\frac{1}{2}$ oz. (Troy) of pure gold, what fraction of the ore was gold ? Express the proportion of gold to ore in grammes per metric ton.

25. A clock which gains 9 sec. per 1 hr. 11 min., is set right at 10 a.m. on 1st March, when will it denote correct time again ?

26. After drawing off 15 gal. of the contents of a certain cask and then $\frac{5}{11}$ of what was left, the remainder sold at $5\frac{1}{2}$ ct. a pint brought $3·96. How many gallons were there originally in the cask ?

27. A mixture of coffee and chicory in the proportion of 8 parts of coffee to 1 part of chicory is sold at 35 ct. a pound, being an advance of 40% on the cost. The chicory cost 9 ct. a pound, find the cost of the coffee per pound.

28. A man bought a house and lot for $4750. After spending $1143 on repairs and improvements and paying $128 for taxes and other expenses, he sold the property for $6800. What rate per cent. of profit did his investment yield him ?

29. On 18th June 1888, a merchant purchased goods amounting per catalogue prices to $647·80, subject to 25 and 5 off. He was allowed 3 months credit after which he was charged interest at 8%. Find the amount of the account on 21st February 1889.

30. Find the difference between the discount taken off a draft for $500 drawn at 90 days and discounted at 7% and the interest on the proceeds for 93 days at 7%. Find the interest for 93 days at 7% on the amount of the discount taken off the draft.

---

31. A man skated 10 miles in 36 min. 37·2 sec.; what was his speed in yards per minute, in miles per hour, in metres per min., in kilometres per hour?

32. A circular race-track 24 ft. wide encloses a circle of 50 yd. radius. How long would it take a man to run round the outer edge of the track at a speed which would take him round the inner edge in one minute?

33. Find the weight of slate per cubic foot if a rectangular slate blackboard 16′ 8″ long, 3′ 6″ wide and $\frac{5}{8}$ in. thick weigh 547 lb.

34. A road 44 ft. wide is made directly across a field 210 yd. square. What fraction of the field does the road occupy? What would be the value of the part taken for the road, at $144 an acre?

35. A cubic foot of pure water at 62° F. weighs 62·356 lb. and a cubic foot of sea-water at the same temperature weighs 64·05 lb.; find the weight of 25 gal. of sea-water.

36. A wheel makes 72 revolutions per minute. If its speed be increased by $\frac{7}{150}$ of itself, how many revolutions will it make in 6 working days of 10 hours each? Had the time of making a revolution been increased by $\frac{7}{150}$ of itself, how many revolutions would the wheel have made in 6 days of 10 hours each?

37. A grocer buys 80 lb. of tea at 21 ct. a lb. and mixes it with some dearer tea he has on hand. Selling the mixture for $43·75, this being at the rate of 35ct. a lb., he clears $15·25 on the whole. How many pounds of the higher priced tea did he mix with the other and how much per pound did this higher priced tea cost him?

38. The population of a certain city was 27,413 at the date of taking one census and at the time of taking the next census the population had risen to 44,229; find the increase per cent. correct to 4 significant figures. Express this as an increase per thousand.

39. What rate of interest is equal to 8% discount for one year?

40. On 19th April 1889, a merchant purchased goods amounting per catalogue prices to $1239·35, subject to 30 and 5 off; terms 3 months credit or 5 off for cash, $\frac{1}{2}$% per month on accounts overdue. Find the amount of this account on 19th Oct. 1889. What would have been the amount had the account been paid on 19th April 1889? What rate of interest will the merchant be paying if he settle on the 19th Oct. instead of on 19th April?

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41. Find to the nearest 100 sec. and also to the nearest minute the time occupied by light in passing from the sun to the planet Neptune, the velocity of light being 187,200 miles per second and the distance of Neptune from the sun being 2,746,000,000 miles.

42. How many yards of carpet 27″ wide will be required to carpet a room 25′ 8″ by 15′ 8″ allowing 9″ per width for matching? How many rolls of wall-paper and how many yards of bordering will be required for the same room, allowing on the wall-paper a width of 42″ each for 3 windows and 2 doors?

43. Find the value of a pile of cordwood 13′ 4″ long by 3′ 9″ high at $4.50 the cord?

44. Find the weight of a circular copper plate $\frac{5}{8}$ in. thick and 11″ in diameter, copper weighing 549 lb. per cubic foot.

45. If an express run at 30 mi. an hour and an accommodation train at 22 miles an hour, what is a man's time worth if he would lose 45ct. in travelling a journey of 270 miles by accommodation instead of by express?

46. Find the number of cubic inches which 10 lb. of (*a*) water, (*b*) hard-coal, (*c*) silver, (*d*) oak will occupy if a cubic foot of water weigh 62 lb. 6·8 oz. and if hard-coal be 1·6 times and silver be 10·5 times heavier, volume for volume, than water, and if a cubic foot of oak weigh $\frac{7}{8}$ as much as a cubic foot of water.

47. A publisher sells a certain book at 78ct. per copy. He pays the printer 17$\frac{1}{2}$ct., the binder 15ct., and for other expenses 9ct. on every copy *printed*. He also pays the author 12$\frac{1}{2}$ct. on every copy *sold*. Of one edition of 1000 copies he sells 879 and the rest are left on his hands. Does he gain or does he lose on the transaction? How much? At what rate per cent.?

48. A did $\frac{1}{3}$ of a piece of work, B did $\frac{5}{9}$ of the remainder, C did $\frac{5}{8}$ of what was left undone by B, and D then finished the work. How much should D get for his work if A receive \$7·00 for his?

49. Find the proceeds of the following joint note discounted in St. Thomas on 18th Dec. 1888, at $7\frac{1}{2}$%.

$\$347\frac{50}{100}$. St. Thomas 18th Dec., 1888.

Ninety days after date we jointly and severally promise to pay to the order of Jno. Locke & Co., Three hundred and forty-seven $\frac{50}{100}$ dollars, at the Standard Bank here. Value received.

Isaac Harper.
H. H. Friedlaender.

50. What rate of discount is equal to 8% interest reckoning (*a*) for a year, (*b*) for 93 days, (*c*) for 63 days?

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51. The British ship Egeria found a depth of ocean of 4430 fathoms at a certain place off the Friendly Islands and the U. S. ship Tuscarora found a depth of 4655 fathoms off the north-east coast of Japan. What must be the pressure per square inch due to the superincumbent water at these depths, sea-water weighing 64·05 lb. per cubic foot? Express the pressure in kilogrammes per square centimetre.

52. What will be the cost of 1000 yards of side-walk 8 ft. wide, made of 3 in. plank laid on three lines of cedar stringers, if the planks cost \$12·00 per M., the cedars $4\frac{1}{2}$ ct. per running-foot and preparing and laying the sidewalk \$3·50 per yard?

53. Out of a circle 18″ in diameter there is cut a circle 13·5′ in diameter. What fraction of the original circle is left?

54. Find the weight of a cast-iron cylinder 8′ in length and 7″ in diameter, if a cubic foot of cast-iron weigh 444 lb.

55. A vessel holds $2\frac{4}{13}$ qt., how many times can it be filled from a barrel containing $31\frac{1}{2}$ gal. of oil? After filling the vessel as often as possible how much oil will remain in the barrel? What fraction of a vesselful will this remaining quantity be?

56. If 9 lb. of rice cost as much as $6\frac{1}{2}$ lb. of sugar and $10\frac{1}{4}$ lb. of sugar cost as much as 1 lb. 10 oz. of tea and 1·25 lb. of tea cost as much as $2\frac{2}{3}$ lb. of coffee, find the cost of 100 lb. of coffee if rice is worth 7 ct. a pound.

57. If a lamp burn ·08 of a pint of oil per hour and 6 lamps are used every night and 30 gal. of oil are consumed from 27th Sept. to 4th Jan. next following, both nights included, how many hours per night are the lamps alight?

58. In an examination *A* obtained 78% of the full number of marks beating *B* by 16% of the full number. If *A* received 975 marks, how many did *B* receive? What percentage of *A*'s number was *B*'s number? What percentage of *B*'s number was *A*'s number? It was afterwards decided to deduct 20% from the total number of marks and also from the numbers obtained by *A* and *B*, what effect would this change have on the answers to the preceding three questions?

59. Find the proceeds of the following note discounted in Toronto on 17th Oct. 1888 at $7\frac{1}{2}$%, exchange $\frac{1}{4}$% reckoned to nearest cent.

$\$211\frac{00}{100}$. HAMILTON, 12th Oct., 1888.

Three months after date I promise to pay to the order of A. J. Wilson & Co., Two hundred and eleven Dollars at the Bank of Commerce here. Value received.

HENRY TOMLINSON.

60. For how much must a ninety-day note be drawn to realize $190 when discounted at 6°/ₒ?

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61. If 8 metres of silk cost 76 francs what will be the price of 10 yd. at the same rate, reckoning 10 francs equal to $1·93?

62. If 2 horses are worth as much as 7 oxen and 3 oxen as much as 17 sheep, find the value of 5 horses that of 9 sheep being $60.

63. Find the price of a rectangular slate blackboard 23′ 4″ long by 3′ 6″ wide @ 44 ct. per square foot.

64. Find the weight of a cast-iron pipe 7′ 6″ long and of $5\frac{1}{4}$″ external and 4″ internal diameter, a cubic foot of cast-iron weighing 444 lb.

65. A train is running at the rate of 20 miles per hour and a second train starts after it at the rate of $27\frac{1}{2}$ miles per hour and overtakes it in 3 hr. 25 min. How many miles an hour did the second train gain on the first? How far ahead was the first train when the second train started?

66. A man who has had his wages increased by $\frac{2}{23}$ is in receipt of \$12·50 per week. What fraction of itself must be taken off this weekly sum to reduce his wages to the original rate?

67. How many boys each doing ·6 of the work of a man must be engaged with 51 men to do in 20 days as much work as 28 men could do in 45 days?

68. The average rainfall at Toronto is less than the average rainfall at St. John, N. B., by $45\frac{5}{8}$°/。 of the latter, and the average rainfall at Windsor, Ont., which is 30 in. per annum, is greater than the average rainfall at Toronto by 8·1°/。 of the latter. Find the weight per acre of the average annual rainfall at St. John, N.B.

69. Find the proceeds of the following draft discounted on 15th Feb., 1889 at 6 °/。, exchange $\frac{1}{8}$ °/。:

\$791$\frac{85}{100}$. GUELPH, 12th Feb., 1889.

Sixty days after date pay to the order of Henry Meadows & Co. of Belleville, Seven hundred and ninety-one $\frac{85}{100}$ dollars. Value received.

STUART & GEE.

To J. J. NEWCOMB,
Belleville.

70. The proceeds of a note payable in 3 months from 1st Feb. 1889 and discounted on the 6th Feb. 1889, amounted to \$847·18. For what sum was the note drawn?

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71. How many tiles 6″ square would pave a hallway $\frac{2}{9}$ the size of a courtyard which required 9360 bricks to pave it, at the rate of $8\frac{1}{2}$′ by $4\frac{1}{2}$″ per brick? Find the length of the courtyard and the width of the hallway given that the length of the hallway and the width of the courtyard are each 42 ft. 6in.

72. Find the weight of 5 miles of steel wire of ·147″ diameter, the steel weighing 492 lb. per cubic foot.

73. Sound travels at the rate of 1120 ft. per second more slowly than light; at what distance is a lightning-flash the thunder of which is heard $7\frac{11}{21}$ sec. after the lightning is seen?

74. *A* by working on piece-work $\frac{2}{9}$ as fast again as *B* is able to earn \$2·09 per day. How much does *B* earn per day?

75. A man asks to have his working hours decreased from 10 hr. to 8 hr. per day without any decrease in his daily pay. By what fraction of his wages per hour does he ask them to be increased?

76. A contractor undertakes a contract to be completed in 120 days. He employs 48 men and at the end of 25 days finds that he has ·2 of the work finished. How many additional men must he now put on in order to have the contract completed 15 days sooner than the time specified in his agreement? *(Only working-days are counted in this statement.)*

77. *A* can do a certain piece of work in 10 days working 8 hr. per day. *B* can do the same work in 9 days working 12 hr. per day. They decide to work together and to finish the work in 6 days. How many hours a day must they work?

78. A man travels 360 miles in 12 days travelling 8 hours per day. If he increase his speed by 20 °/ₒ, how many hours per day less than before need he travel in order to accomplish 450 miles in 20 days?

79. A market-woman bought a certain number of eggs @ 11 for 9 ct. and sold them, all but 3 which were broken and thrown away, at 9 for 11 ct., thus clearing $2·63 on the transaction. How many eggs did she buy and what rate per cent. of profit did she make?

80. On 23rd July 1889, Messrs. Ingram, Hughes, Leighton & Co., of Toronto, take to the Bank of Commerce, to be discounted and the proceeds placed to their credit, drafts as follows :—One at 60 days from date on S. Cassidy & Co., Paris, for $372·85; one at 90 days from date on Th. Moore & Co., Owen Sound, for $629·30; one at 10 days from date on Gregg & Weir, Belleville, for $125; one at 45 days from date on Brock & Eaton, St. Thomas, for $748·50; one at 4 mo. from date on Colby & Masson, Chatham, for $917·60; one at 2 mo. from date on Bowles & Co., Guelph, for $322·10.

Draw up and fill in a discount sheet for these drafts arranging them in the order of maturing; discount 7%, exchange (reckoned to nearest cent on each bill) $\frac{1}{5}$ % on drafts up to $400, $\frac{1}{8}$ % on drafts for more than $400.

81. A dealer bought eggs at 10 for 14ct. and sold them at 14 for 24ct. On a certain day he received $6·60; how much of this was profit? Had he bought the eggs at 14 for 24ct. and sold them at 10 for 14ct., how much would he have lost on the day's sales?

82. Find the cost of painting the walls and ceiling of a hall $62' \times 34'\ 6'' \times 15'\ 6''$ at 27ct. per square yard,—no deductions for openings.

83. A rectangular box made of boards $1\frac{1}{4}''$ thick, measures on the outside $3'\ 7''$ by $2'\ 5''$ by $1'\ 10''$. Find its internal content, *(a)* the measurements, including the lid; *(b)* the measurements being of the box without the lid.

84. If a boat 36 ft. long travel $\frac{5}{8}$ of its length at each stroke of the oars, how many strokes will be required in rowing a distance of $2\frac{37}{44}$ miles? How many strokes per minute will the rowers require to make in order to row the distance in 26 min. 40 sec.?

85. If a man earns $\frac{1}{4}$ as much as 7 women and a boy earns $\frac{4}{9}$ of $\frac{3}{5}$ of the wages of 2 women, what fraction of a man's wages does a boy earn; the time of earning being in all cases the same?

86. A town council was offered gravel unscreened at \$4·50 a cord, screened at \$5·50 a cord. Allowing 25ct. as the cost of screening a cord of unscreened gravel; at what fraction of the unscreened gravel do the above prices estimate the loss by screening?

87. Divide \$40·71 among 7 men, 16 women and 25 children, so that 5 men may get as much as 6 women, and 5 women as much as 6 children.

88. By selling a certain book for \$3·96 I would lose 12 % of the cost; what advance on this proposed selling price would give a profit of 12 % of the cost? What rate per cent. on the proposed selling price would this advance be?

89. On 28th Aug. 1888, a merchant purchased goods amounting per catalogue prices to \$987·50 subject to 20 and 5 off; terms 3 months credit or 5 % off for cash. To what rate of interest is this 5 % off for cash equal? If the merchant were to discount at 7 % a note drawn at 3 months for the *credit* amount of the above account, by how much would the proceeds of the note exceed the *cash* amount of the account?

90. A merchant buys goods amounting per catalogue prices to \$1573·45, subject to 20 and 10 off; terms 90 days credit or 5% off for cash. For how much must the merchant make a note payable in 90 days, that the note discounted at 7 % may realize the *cash* amount of the above bill? For how much must the note be drawn to allow $\frac{1}{5}$ % off for exchange?

91. Four foremen A, B, C, D, are placed over 260 men. For every 4 men under A there are 5 under C, for every 9 under B there are 10 under D, and for every 2 under A there are 3 under B. How many are under each?

92. Find the cost of plastering the walls and ceiling of a room 27′ 8″×13′ 4″×9′ 2″ at 22ct. per square yard, there being 3 windows 6′ 9″×4′ 3″ and 2 doors 7′ 3″×4′ 3″. How many cubic feet of plaster would be required to plaster the room, the average thickness of the plaster being half an inch?

93. Find the surface-area and the volume of a rectangular block 3′ 9″×2′ 4″×1′ 3″. What fraction of the block would be cut away and by what fraction of itself would its surface be diminished were 2″ each to be taken off its length, its breadth and its thickness?

94. How long will it take to travel $13\frac{1}{13}$ kilometres at the rate of 11·9 miles in 1 hr. 45 min.? How long will it require to travel $13\frac{1}{13}$ miles at the rate of 11·9 kilometres in 1 hr. 45 min.?

95. A line A is half as long again as B and B is one quarter as long again as C. What fraction of the length of A is equal to $\frac{1}{3}$ of the length of C?

96. A merchant sold $\frac{1}{5}$ of his stock for $\frac{3}{4}$ of the cost of the whole stock; $\frac{1}{2}$ of the remainder at a gain of \$80; $\frac{1}{4}$ of what still remained for its cost, \$150; and the rest at a reduction of $\frac{3}{8}$ of the cost. What was his total gain?

97. If 7 men, 15 women and 9 boys earn \$8701·40 in a year (313 working-days) and if a woman's earnings are ·6 of a man's and a boy's are $\frac{5}{7}$ of a woman's, what are the weekly earnings of a man, of a woman and of a boy respectively?

98. Goods are sold at a loss of 15 % on the cost. By what percentage of itself should the selling price be advanced to yield a profit of 15 % on the cost?

99. What rate of discount is equal to 5% off for cash on a purchase on 90 days credit, reckoning $\frac{1}{4}$% for exchange with the discount?

100. What must a merchant charge for goods that cost him \$976·50 cash, in order that after giving 6 months credit, thus involving the discount @ 7% of a note drawn at 90 days to yield the cash price of the goods and a renewal note also drawn at 90 days and discounted at 7%, he may obtain a profit of 15% on the cash price paid by him for the goods?

CHAPTER IV.

THE THREE HIGHER OPERATIONS.

INVOLUTION.

97. An Integral Power of any number is the product or the quotient resulting from successive multiplications or successive divisions by the number, the initial multiplicand or initial dividend being in every case ONE. The power is said to be **positive,** if it be formed by multiplications; **negative,** if formed by divisions. In naming positive powers, the term *positive* is usually omitted. The second positive power of a number is commonly called the **square** of the number; the third positive power, its **cube**; and the initial 1, neither multiplied nor divided, the **zeroth** power.

The first (positive) power of 5 is	1×5	$=5$
The second power or square of 5 is	$1\times 5\times 5$	$=5\times 5=25$
The third power or cube of 5 is	$1\times 5\times 5\times 5$	$=25\times 5=125$
The fourth power of 5 is	$1\times 5\times 5\times 5\times 5$	$=125\times 5=625$
The fifth power of 5 is	$1\times 5\times 5\times 5\times 5\times 5$	$=625\times 5=3125$
The first negative power of 5 is	$1\div 5$	$=\frac{1}{5}$
The second negative power of 5 is	$1\div 5\div 5$	$=\frac{1}{5}\div 5=\frac{1}{25}$
The third negative power of 5 is	$1\div 5\div 5\div 5$	$=\frac{1}{25}\div 5=\frac{1}{125}$
The zeroth power of 5 is	1	$=1$

98. The **base** of a power is the number used as multiplier (or as divisor) in forming the power.

99. The **Exponent** or **Index** of a power is the number which expresses how often the base occurs as factor (multiplier or divisor) in forming the power. The figures of an exponent are usually made somewhat smaller than those of its base and are placed on the right of the base and a little above it. The sign *minus* is employed as a negative sign and is written before the exponents of negative powers.

Instead of $1\times 5\times 5$ or 5×5 we write 5^2 which is read "5 square." Here 5 is the base and 2 is the exponent.

Instead of $1\times 7\times 7\times 7\times 7$ or $7\times 7\times 7\times 7$, we write 7^4 which is read "7 to the fourth," *power* being understood after *fourth*. In this example, 7 is the base and 4 is the exponent.

Similarly 3^7 which is read "3 to the seventh" (*power*), represents $1\times3\times3\times3\times3\times3\times3\times3$; and 3^{-5}, read "3 to the negative fifth," represents $1\div3\div3\div3\div3\div3$. In 3^7, the base is 3 and the exponent is 7; in 3^{-5} the base is 3 and the exponent is -5.

The exponent 1 is not usually expressed, the first positive power of any number being simply the number itself. Hence *when no exponent is expressed, the exponent 1 is to be understood.*

100. The **Degree** of a power is the number of successive multiplications (or successive divisions) by the base. The exponent of any power is, therefore, the *Index of the Degree* of the power. The greater the number of multiplications, or the less the number of divisions by the base, the higher is the degree of the power; the fewer the multiplications, or the more numerous the divisions, the lower is the degree. The phrases 'higher power,' 'lower power' are very frequently used instead of the full phrases 'power of higher degree,' 'power of lower degree.'

101. Involution is the operation of raising a given base to a power of given degree. In other words, Involution is the operation of finding the product of a given number of factors each equal to a given number.

Example 1. Find the first six positive powers of 1·4678 correct in each case to five figures.

1	1·4678
2	2·9356
3	4·4034
4	5·8712
5	7·3390
6	8·8068
7	10·2746
8	11·7424
9	13·2162

1·4678 . . . 1st power.
58712
8807
1027
117
2·15443 . . . 2nd power.

2·9356
14678
7339
587
59
4
3·16227 . . . 3rd power.

3·16227 . . . 3rd power.

4·4034
14678
8807
294
29
10
4·64158 . . . 4th power.

5·8712
88068
5871
147
73
12
6·81291 . . . 5th power.

8·8068
1 17424
1468
294
132
1
9·99999 . . . 6th power.

We have made 1·4678 the multiplicand in each multiplication, because by so doing, only a single table of multiples is required. (See *Example 2*, page 69.) The computations have been carried to six figures in order to ensure accuracy in the fifth. The six powers, each correct to five figures, are 1·4678, 2·1544, 3·1623, 4·6416, 6·8129 and 10 respectively.

Example 2. Find the value of

$$1+\frac{\cdot 47}{1}+\frac{\cdot 47^2}{1\times 2}+\frac{\cdot 47^3}{1\times 2\times 3}+\frac{\cdot 47^4}{1\times 2\times 3\times 4}+\frac{\cdot 47^5}{1\times 2\times 3\times 4\times 5}.$$

correct to 4 decimal places. (Work to 5 decimals.)

1	47
2	94
3	1·41
4	1·88
5	2·35
6	2·82
7	3·29
8	3·76
9	4·23

```
      ·47 . . . . . (a)
     ─────
     ·188
      329
   2)·2209
     ·11045 . . . . . (b)
     ·047
       47
        19
         2
   3)·05191
     ·01730 . . . . . (c)
     ·0047
       329
        14
   4)·00813
     ·00203 . . . . . (d)
     ·00094
          1
   5)·00095
     .00019 . . . . . (e)
```

```
1
 ·47    . . . . . (a)
 ·1105  . . . . . (b)
   173  . . . . . (c)
    20  . . . . . (d)
     2  . . . . . (e)
──────
1·6
```

We square ·47 and divide by 2 and thus obtain (*b*). We next multiply ·47 by (*b*), this gives one-half of the cube of ·47; we divide by 3 and obtain (*c*) the sixth part of the cube. We then multiply ·47 by (*c*) and divide by 4 to obtain (*d*); and multiply ·47 by (*d*) and divide by 5 to obtain (*e*). Finally we find the sum of 1, (*a*), (*b*), (*c*), (*d*) and (*e*) correct to the fourth decimal.

EXERCISE V.

Write the following products as powers :—

1. $2 \times 2 \times 2$.

2. 3×3.

3. $5 \times 5 \times 5 \times 5$.

4. $10 \times 10 \times 10 \times 10 \times 10$.

5. $\cdot 1 \times \cdot 1 \times \cdot 1 \times \cdot 1$.

6. $2\cdot 3 \times 2\cdot 3 \times 2\cdot 3 \times 2\cdot 3 \times 2\cdot 3$.

7. $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$.

8. $\frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5} \times \frac{4}{5}$.

Write the following powers as products :—

9. 3^4.

10. 12^3.

11. 15^2.

12. 25^5.

13. $2\cdot 5^5$.

14. $\cdot 25^5$.

15. $(\frac{2}{3})^4$.

16. $(\frac{11}{15})^3$.

17. $2^3 \times 3^2 \times 5 \times 7^2$.

Find the value of :—

18. 2^6.

19. 6^2.

20. 5^4.

21. 4^5.

22. 2375^2.

23. 5873^3.

24. 273^4.

25. 27^5.

26. $1\cdot 1^3$.

27. $\cdot 1^5$.

28. $\cdot 02^6$

29. $1\cdot 02^6$.

30. 49^2.

31. $4\cdot 9^2$.

32. $\cdot 49^2$.

33. $23\cdot 6^3$.

34. $2\cdot 36^3$.

35. $\cdot 0236^3$.

36, $(\frac{2}{3})^2$.

37. $(\frac{3}{4})^3$.

38. $(\frac{1}{2})^5$.

39. $(\frac{4}{5})^4$.

40. $\dfrac{4^4}{5^4}$

41. $2^3 \times 3^4$.

42. $2^4 \times 3^3 \times 5^3$.

43. $2^2 \times 3^3 \times 7^2$.

44. $2^5 \times 5^3 \times 7 \times 11^3 \times 13^2$.

45. $(7^2)^3$.

46. $5^6 \div 5^2$.

47. $2^7 \times 3^4 \times 5^3 \div 2^4 \div 3^2 \div 5^3$.

Resolve the following numbers into their prime factors, expressing the repetition of a factor by an index :—

48. 2520. **49.** 70200. **50.** 1024. **51.** 11368. **52.** 530712.

Find the value, correct to four significant figures, of :—

53. $\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \frac{1}{5^6} + \frac{1}{5^7} + \frac{1}{5^8} + \frac{1}{5^9}$.

54. $\frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \frac{1}{6^5} + \frac{1}{6^6} + \frac{1}{6^7} + \frac{1}{6^8} + \frac{1}{6^9}$.

55. $\frac{1}{9} - \frac{1}{9^2} + \frac{1}{9^3} - \frac{1}{9^4} + \frac{1}{9^5} - \frac{1}{9^6} + \frac{1}{9^7}$.

56. $\frac{1}{10^2} - \frac{3}{1} \times \frac{1}{10^4} + \frac{3 \times 4}{1 \times 2} \times \frac{1}{10^6} - \frac{3 \times 4 \times 5}{1 \times 2 \times 3} \times \frac{1}{10^8}$.

57. $\frac{1}{2} - \frac{1}{3} \times \frac{1}{2^3} + \frac{1}{5} \times \frac{1}{2^5} - \frac{1}{7} \times \frac{1}{2^7} + \frac{1}{9} \times \frac{1}{2^9} - \frac{1}{11} \times \frac{1}{2^{11}}$

$+ \frac{1}{3} - \frac{1}{3} \times \frac{1}{3^3} + \frac{1}{5} \times \frac{1}{3^5} - \frac{1}{7} \times \frac{1}{3^7} + \frac{1}{9} \times \frac{1}{3^9}$.

58. $\frac{1}{2}-\frac{1}{3}\times\frac{1}{2^3}+\frac{1}{5}\times\frac{1}{2^5}-\frac{1}{7}\times\frac{1}{2^7}+\frac{1}{9}\times\frac{1}{2^9}-\frac{1}{11}\times\frac{1}{2^{11}}$

$+\frac{1}{5}-\frac{1}{3}\times\frac{1}{5^3}+\frac{1}{5}\times\frac{1}{5^5}-\frac{1}{7}\times\frac{1}{5^7}+\frac{1}{9}\times\frac{1}{5^9}$

$+\frac{1}{8}-\frac{1}{3}\times\frac{1}{8^3}+\frac{1}{5}\times\frac{1}{8^5}.$

59. $2\left\{\frac{1}{3}-\frac{1}{3}\times\frac{1}{3^3}+\frac{1}{5}\times\frac{1}{3^5}-\frac{1}{7}\times\frac{1}{3^7}+\frac{1}{9}\times\frac{1}{3^9}\right\}$

$+\frac{1}{7}-\frac{1}{3}\times\frac{1}{7^3}+\frac{1}{5}\times\frac{1}{7^5}.$

60. $4\left\{\frac{1}{5}-\frac{1}{3}\times\frac{1}{5^3}+\frac{1}{5}\times\frac{1}{5^5}-\frac{1}{7}\times\frac{1}{5^7}+\frac{1}{9}\times\frac{1}{5^9}\right\}$

$-\frac{1}{70}+\frac{1}{3}\times\frac{1}{70^3}+\frac{1}{99}-\frac{1}{3}\times\frac{1}{99^3}.$

61. $4\left\{\frac{1}{5}-\frac{1}{3}\times\frac{1}{5^3}+\frac{1}{5}\times\frac{1}{5^5}-\frac{1}{7}\times\frac{1}{5^7}\right\}-\frac{1}{239}.$

62. $1+\frac{\cdot 7}{1}+\frac{\cdot 7^2}{1\times 2}+\frac{\cdot 7^3}{1\times 2\times 3}+\frac{\cdot 7^4}{1\times 2\times 3\times 4}+\frac{\cdot 7^5}{1\times 2\times 3\times 4\times 5}$

$+\frac{\cdot 7^6}{1\times 3\times 3\times 4\times 5\times 6}+\&c.$

63. $1-\frac{\cdot 7}{1}+\frac{\cdot 7^2}{1\times 2}-\frac{\cdot 7^3}{1\times 2\times 3}+\frac{\cdot 7^4}{1\times 2\times 3\times 4}-\frac{\cdot 7^5}{1\times 2\times 3\times 4\times 5}+\&c.$

64. $2\left\{7\times\left(\frac{1}{31}+\frac{1}{3\times 31^3}+\frac{1}{5\times 31^5}+\frac{1}{7\times 31^7}\right)\right.$

$\left.+5\times\left(\frac{1}{49}+\frac{1}{3\times 49^3}+\frac{1}{5\times 49^5}\right)+3\times\left(\frac{1}{161}+\frac{1}{3\times 161^3}\right)\right\}.$

65. $2\times\left\{23\times\left(\frac{1}{31}+\frac{1}{3\times 31^3}+\frac{1}{5\times 31^5}+\frac{1}{7\times 31^7}\right)\right.$

$\left.+17\times\left(\frac{1}{49}+\frac{1}{3\times 49^3}+\frac{1}{5\times 49^5}\right)+10\times\left(\frac{1}{161}+\frac{1}{3\times 161^3}\right)\right\}.$

102. Horner's Method.—The simplest and easiest method of raising a given base to a power of given positive integral degree, is that which was adopted in *Example 1*, p 87. In that system the successive positive integral powers were calculated one after another, all the figures of the base being used in the multiplicand in each multiplication. The calculations may however be conducted on a different plan. We may begin with a single figure of the base, (by preference the first on the left-hand) and having raised this single-digit number to the assigned degree, we may then proceed to build up the required power step by step as we add figure by figure to the base. This way of computing the positive integral powers of numbers is known as the Method of Differences, and the best arrangement of the process, that exhibited in the following examples, is named **Horner's Method.** In ordinary cases of involution, Horner's Method is neither so easy nor so simple as that employed in *Example 1*, p. 87, but it has the advantage of being applicable to whole classes of problems for which the other method is of little or no use.

Example 1. Find the square of 3472.

1	0	0	
	3	900	$= 30^2$,
	60		
1	64	115600	$= 340^2$,
	680		
1	687	12040900	$= 3470^2$,
	6940		
1	6942	12054784	$= 3472^2$.

Mark off into columns the space set apart for the calculation, the number of columns being greater by one than the exponent of the required power. At the top of the left-hand column write 1, this 1 is to be understood as repeated in every line down this *initial* column. The other columns contain the actual calculations and may be called the working-columns and numbered from the left. At the top of each of these write a zero. This forms the first or initial line of the calculation.

Multiply 1 in the initial column by 3, the left-hand digit of the base 3472, and add the product to the zero in the first working-column. Set the result which is 3, in the first working-column. Multiply the 3 just set in the first working-column by the base-digit 3 and adding the product to the 0 in the second working-column, set the result which is 9, in the second working-column. Begin again with the initial 1, multiply 1 by the base-digit 3 and add the product to the 3 in the first working-column. Set the result which is 6, in the first working-column.

We have now instead of the initial line 1, 0, 0, the new line 1, 6, 9 ; the 6 being double the base-digit 3, and the nine being the square of this 3. Prepare this line for the next step by placing *one* zero after the 6 and *two* zeros after the 9, thus converting them into 60, the double of 30, and 900 the square of 30.

We now repeat the system of operations just described using 4, the next figure of the base after 3 instead of 3 and the line 1, 60, 900 instead of the line 1, 0, 0; thus :—

$1\times 4+60=64$, which is to be placed in first working-column.

$64\times 4+900=1156$, to be placed in the second working-column.

$1\times 4+64=68$ to be placed in the first working-column.

We thus obtain a third line of calculation, 1, 68, 1156, the 68 being double the base 34 and 1156 being 34^2. Prepare this line for the next step by placing *one* zero after 68 and *two* zeros after 1156, thus converting them into $680=340\times 2$, and $115600=340^2$.

Repeat this system of operations using 7, the next figure of the base, as multiplier and 1, 680, 115600 as line of calculation, thus :—

$1\times 7+680=687$, in 1st working-column ;

$687\times 7+115600=120409$, in 2nd working-column ;

$1\times 7+687=694$, in 1st working-column.

This gives as fourth line of calculation, 1, 694, 120409, which, preparatory for the next step, is converted into 1, 6940, 12040900.

Repeat the first course in this system of operations using as multiplier, 2, the last figure of the base, and as line of calculation, 1, 6940, 12040900.

$1\times 2+6940=6942$, in 1st working-column.

$6942\times 2+12040900=12054784$, in 2nd working-column.

This completes the calculation of 3472^2.

Example 2. Find the cube of 2574.

		I.	II.	III.	
	1	0	0	0	
(*a*).		2	4	8000	$=20^3$
(*b*).		4	1200		
(*c*).		60			
(*d*).		65	1525	15625000	$=250^3$
(*e*).		70	187500		
(*f*).		750			
(*g*).		757	192799	16974593000	$=2570^3$
(*h*).		764	19814700		
(*k*.).		7710			
(*l*).		7714	19845556	17053975224	$=2574^3$
(*m*).		(7718)	(19876428)		
(*n*).		(7722)			

Here we are required to find a third power, we must therefore have three working-columns. In the first set of operations we take 2, the left-hand digit of the base 2574, as multiplier and we have 1, 0, 0, 0 as initial line,

(*a*) $1\times2+0=2$, in column I.
$2\times2+0=4$, in col. II.
$4\times2+0=8$, in col. III. Change 8 to 8000.

(*b*) $1\times2+2=4$, in col. I.
$4\times2+4=12$, in col. II. Change 12 to 1200.

(*c*) $1\times2+4=6$, in col. I. Change 6 to 60.

We have now a new line of calculation, 1, 60, 1200, 8000. In this line, $60=20\times3$, $1200=20^2\times3$ and $8000=20^3$.

Repeat the system of operations starting from this new line of calculation and using as multiplier 5, the second figure of the base.

(*d*) $1\times5+60=65$. Col. I.
$65\times5+1200=1525$. Col. II.
$1525\times5+8000=$
15625. Col. III. 15625000.

(*e*) $1\times5+65=70$. Col. I.
$70\times5+1525=1875$. Col. II. 187500.

(*f*) $1\times5+70=75$. Col. I. 750.

We thus obtain a third line of calculation, 1, 750, 187500, 15625000, in which $750=250\times3$, $187500=250^2\times3$, $15625000=250^3$.

Repeat the system of operations, starting from the third line of calculation and using the third figure of the base as multiplier.

(*g*) $1\times7+750=757$. Col. I.
$757\times7+187500=192799$. Col. II.
$192799\times7+15625000=16974593$. Col. III.

(*h*) $1\times7+757=764$. Col. I.
$764\times7+192799=198147$. Col. II.

(*k*) $1\times7+764=771$. Col. I.

We thus obtain a fourth line of calculation, 1, 7710, 19814700, 16974593000, in which $7710=2570\times3$, $19814700=2570^2\times3$, $16974593000=2570^3$.

Starting with this fourth line of calculation, repeat the first course of the system of operations, employing as multiplier the fourth figure of the base.

(*l*) $1\times4+7710=7714$. Col. I.
$7714\times4+19814700=19845556$. Col. II.
$19845556\times4+16974593000=$
17053975224. Col. III.

17053975224 being the cube of 2574, we need go no farther in this system of operations unless we wish to prepare for another step in advance. This we have done in the example, having calculated and recorded (within parentheses) the lines marked (m) and (n) respectively.

Example 3. Find $1·584893193^5$ correct to 9 significant figures.

The required power being the fifth, five working-columns will be needed. Nine figures are required to be correct, the computation must therefore be carried to at least eleven figures in the fifth working-column. The decimal point is omitted as unnecessary, except in the last working-column.

1	0	0	0	0	0	
	1	1	1	1	1·00000	,1
	2	3	4	50000		
	3	6	10000			
	4	1000				
	50					
	55	1275	16375	131875	7·5937500000	,·5
	60	1575	24250	2531250000		
	65	1900	33750000			
	70	225000				
	750					
	758	231064	35598512	2816038096	9·8465804768	,8
	766	237192	37496048	3116006480		
	774	243384	39443120			
	782	249640				
	790					
		250	395431	313182372	9·9718534256	,4
			396431	314768096		
			397431			
			3976	31508618	9·9970603200	,8
			3978	31540442		
			3980			
			40	3154404	9·9998992836	,9
				3154764	9·9999939264	,3
					9·9999970812	,1
					9·9999999207	,9
					10·0000000152	,3

Hence $1·584893193^5 = 10·00000002$, correct to the last figure.

In the first set of operations, we begin with 1, 0, 0, 0, 0, 0 as the initial line of calculation and we take as multiplier 1, the left-hand digit of the base, 1·584893193. We obtain therefrom, the new line of calculation 1, 5·*0*, 10·*00*, 10·*000*, 5·*0000*, 1·*00000*.

In the second set of operations, we begin with this new line of calculation and we take as multiplier ·5, the second figure of the base. We obtain therefrom as third line of calculation,

1, 7·5*0*, 22·50*00*, 33·750*000*, 25·3125*0000*, 7·59375*00000*,

in which it is worthy of notice that

$$7{\cdot}5 = 1{\cdot}5 \times 5$$
$$22{\cdot}50 = 1{\cdot}5^2 \times 10,$$
$$33{\cdot}750 = 1{\cdot}5^3 \times 10$$
$$25{\cdot}3125 = 1{\cdot}5^4 \times 5,$$
and $$7{\cdot}59375 = 1{\cdot}5^5.$$

In the third set of operations, we begin with the line of calculation last obtained and we take as multiplier ·08, the third figure of the base. We obtain therefrom as fourth line of calculation

1, 7·90*0*, 24·9640*00*, 39·443120*000*, 31·16006480*0000*,
9·8465804768*00000*;

in which it should be noticed that

$$7{\cdot}90 = 1{\cdot}58 \times 5,$$
$$24{\cdot}9640 = 1{\cdot}58^2 \times 10,$$
$$39{\cdot}443120 = 1{\cdot}58^3 \times 10,$$
$$31{\cdot}16006480 = 1{\cdot}58^4 \times 5,$$
and $$9{\cdot}8465804768 = 1{\cdot}58^5.$$

The contracting begins at the figure 4 of the base; the uncontracted fifth working-column would on passing from 8 to 4 of the base, receive an extension of *five* figures, these are all omitted and as a consequence the other working-columns must also be contracted by five figures each. Allowing for their "extensions" this will require the cancelling of the right-hand figure in the fourth working-column, of two figures on the right in the third, of three figures on the right in the second and of four figures on the right in the first working-column. In like manner, on proceeding from 4 of the base to the 8 following it, from 8 to 9, from 9 to 3, from 3 to 1, &c., the first four working-columns are contracted at each step by cancelling 1, 2, 3, and 4 figures respectively.

Example 4. Find the value of $3658^3 + 2574^3$.

In *Example 2* p. 92, we have the value of 2574^3 and the working-columns of the calculations prepared for any addition to the base. Now $3658 - 2574 = 1084$, therefore we may take advantage of the calculation of 2574^3 to obtain the value of 3658^3 by giving the base 2574 the successive increments 4, 80 and 1000.

1	7722	19876428	$17053975224 = 2574^3$
	7726	19907332	$17133604552 = 2578^3$
	7730	19938252	
	7734		
	7814	20563372	$18778674312 = 2658^3$
	7894	21194892	
	7974		
	8974	30168892	$48947566312 = 3658^3$

$\therefore\ 3658^3 + 2574^3 = 66001541536.$

Example 5. Find the value of $4{\cdot}877^3 - 116$.

Instead of the initial line 1, 0, 0, 0 employed in finding the value of $4{\cdot}877^3$, use the initial line 1, 0, 0, -116, the sign $-$ before 116 denoting that the difference is to be taken between 116 and the number carried from the second working-column to the third.

1	0	0	-116	
	4	16	$-\ 52{\cdot}000$	, 4
	8	4800		
	120			
	128	5824	$-5{\cdot}408000$	, ·8
	136	691200		
	1440			
	1447	701329	$-{\cdot}498697000$	, 7
	1454	7115070 0		
	14610			
	14617	71253019	$+{\cdot}000074133$	, 7

Hence $4{\cdot}877^3 - 116 = {\cdot}000074133$.

EXERCISE VI.

Find the value of :—

1. 2^2, 23^2, 235^2, 2357^2, 23578^2, 235781^2.
2. 4^3, 43^3, 437^3, 4375^3, 43759^3.
3. 12^2, 12^3 ; 127^2, 127^3 ; 1278^2, 1278^3 ; 12786^2, 12786^3.
4. $51{\cdot}449^2$, $51{\cdot}449^3$, $51{\cdot}449^4$.
5. ${\cdot}136^2$, ${\cdot}136^3$, ${\cdot}136^4$, ${\cdot}136^5$, ${\cdot}136^6$.
6. $205{\cdot}389 - 5{\cdot}9^3$.
7. $170 - 5{\cdot}5391^3$.
8. $3{\cdot}14159^5 - 306$.
9. $8{\cdot}241^3 - 8{\cdot}241^2 + 8{\cdot}241 - 500$.
(*Take* 1, -1, 1, -500 *as initial line.*)
10. $11{\cdot}48^3 + 11{\cdot}48^2 - 1554$.
(*Take* 1, 1, 0, -1554 *as initial line.*)

EVOLUTION.

103. The *square root* of a given number is that number whose square is the given number.

Examples. 4 is the square of 2, ∴ 2 is the square root of 4; 9 is the square of 3, ∴ 3 is the square root of 9; 100 is the square of 10, ∴ 10 is the square root of 100.

The *cube root* of a given number is that number whose cube is the given number.

Examples. 8 is the cube of 2, ∴ 2 is the cube root of 8; 125 is the cube of 5, ∴ 5 is the cube root of 125; 1000 is the cube of 10, ∴ 10 is the cube root of 1000.

The *fourth root, fifth root, sixth root* . . . . of a given number is that number whose fourth power, fifth power, sixth power . . . . is the given number.

Examples. 81 is the fourth power of 3, ∴ 3 is the fourth root of 81; ·00032 is the fifth power of ·2, ∴ ·2 is the fifth root of ·00032.

The square root, cube root, fourth root, fifth root, ... of a given number is therefore the base whose square, cube, fourth power, fifth power, ... is the given number.

104. Evolution is the operation of finding any root of a given number. It is therefore the operation of finding the base of which a given number is the power of given degree.

105. In Involution, the base and the exponent (the index of the degree of the power) are given and the power is to be determined therefrom. In Evolution, on the other hand, the base is to be determined, the power itself being given and also the exponent or index of its degree. Evolution is therefore an inverse of Involution.

106. There are two ways of denoting Evolution. In the first or older notation, the square root of a given number is denoted by prefixing the symbol $\sqrt{\ }$ to the given number; the cube root is denoted by prefixing $\sqrt[3]{\ }$, the fourth root by prefixing $\sqrt[4]{\ }$, the fifth root by prefixing $\sqrt[5]{\ }$, and all other roots are similarly denoted, viz., by prefixing to the given number the root-symbol $\sqrt{\ }$ combined with an index number indicating which root is to be taken.

Examples. $\sqrt{64}$ denotes the square root of 64; $\sqrt[3]{64}$ denotes the cube root of 64; $\sqrt[4]{81}$ denotes the fourth root of 81; and $\sqrt[5]{\frac{1}{32}}$ denotes the fifth root of $\frac{1}{32}$.

The second or modern notation for evolution employs fractional exponents to denote the roots of numbers. The exponent of the square root is $\frac{1}{2}$, that of the cube root is $\frac{1}{3}$, that of the fourth root is $\frac{1}{4}$, and, generally, the exponent of any root is the reciprocal of the exponent of the corresponding power.

Examples. $49^{\frac{1}{2}}$ denotes the square root of 49; $\cdot125^{\frac{1}{3}}$ denotes the cube root of ·125; $\left(\frac{1}{1024}\right)^{\cdot 1}$ denotes the tenth root of $\frac{1}{1024}$; and $81^{-\frac{1}{4}}$ denotes the reciprocal of the fourth root of 81.

[107. The root-symbol $\surd$ is merely a variant form of the letter *r*. The employment of an index number with $\surd$ is of comparatively recent date, the old notation was $\surd$q for the square root, $\surd$c for the cube root, $\surd$qq for the fourth root, $\surd$cq for the fifth root, $\surd$cc for the sixth root and so on for other roots. After this came the notation $\surd$[6] for the sixth root, $\surd$[7] for the seventh root,, and a similar notation for other roots. Later still came the notation $\surd^6$, $\surd^7$, &c.; from this form our present notation is derived.

The exponential notation is as much superior to the root-symbol notation as Arabic is to Roman notation and excels it very much in the same respects. As a notation merely of record, the root-symbol notation is perhaps quite equal to the exponential but the latter notation by its very forms suggests calculation by exponents, (see §143,) and the index laws and the many theorems following therefrom; of these the root-symbol notation gives not the slightest hint, tending rather to hide them from sight or make them obscure.]

EXERCISE VII.

Prove the following statements of equality :—

1. $25^{\frac{1}{2}}=5.$	**7.** $6\cdot25^{\frac{1}{2}}=2\cdot5.$	**13.** $\left(\frac{16}{81}\right)^{\frac{1}{4}}=\frac{2}{3}.$
2. $125^{\frac{1}{3}}=5.$	**8.** $4900^{\frac{1}{2}}=70.$	**14.** $13^{\frac{1}{3}}>3\cdot6.$
3. $16^{\frac{1}{4}}=2.$	**9.** $1\cdot728^{\frac{1}{3}}=1\cdot2.$	**15.** $11^{\frac{1}{3}}<2\cdot224.$
4. $81^{\frac{1}{4}}=3.$	**10.** $\cdot008^{\frac{1}{3}}=\cdot2.$	**16.** $1\cdot16^{\frac{1}{3}}>1\cdot05.$
5. $1000^{\frac{1}{3}}=19.$	**11.** $0\cdot001^{\frac{1}{3}}=0\cdot1.$	**17.** $\cdot41^{\frac{1}{4}}>\cdot8.$
6. $100000^{\frac{1}{5}}=10.$	**12.** $0\cdot00001^{\frac{1}{5}}=0\cdot1$	**18.** $2^{\frac{1}{5}}<1\cdot1487.$

108. Evolution being an inverse of Involution a calculation in the former will be merely the reversal or undoing of a calculation in the latter. We require therefore a reversible process of involution and such a reversible process we have in Horner's Method. In it the required power is built up by successive increments as additions are made to the base or as it is enlarged figure by figure. To reverse this process we must withdraw the successive increments of the direct process, and since the increments may be added in any order (compare *Example 2*, p. 92 and *Example 4*, p. 95), *they may also be withdrawn in any order.* At the beginning of the calculation, the only digit of the root, the unknown base, of which we can be sure, is the first digit on the left, therefore we commence by raising this digit to the degree of the power which the given number is to be of the required root, and subtracting this power from the given number.

In determining this first digit of the root, it must be remembered that each figure subsequently added to the root or base gives two additional figures in the square of that root, three in the cube, four in the fourth power, five in the fifth power, ... and that for each figure to the right of the decimal point in the root there will be two to the right in the square of that root, three to the right in the cube, four to the right in the fourth power, five to the right in the fifth power, ... Hence, in preparing to extract any root of a number, *we begin at the decimal point and mark off the figures left and right in pairs in case of the square root, in sets of three in the case of the cube root, in sets of four in the case of the fourth root, in sets of five in the case of the fifth root,* . . . . This done, the set or period on the left will determine the first figure on the left of the root.

109. The following Table will assist in determining the first root-digit in cases of square root and cube root:—

Root.	1,	2,	3,	4,	5,	6,	7,	8,	9.
Square.	1,	4,	9,	16,	25,	36,	49,	64,	81.
Cube.	1,	8,	27,	64,	125,	216,	343,	512,	729.

Root.	·1,	·2,	·3,	·4,	·5,	·6,	·7,	·8,	·9.
Square.	·01,	·04,	·09,	·16,	·25,	·36,	·49,	·64,	·81.
Cube.	·001,	·008,	·027,	·064,	·125,	·216,	·343,	·512,	·729.

Example 1. Find the square root of 5476.

The root is to be squared, hence *two* working columns will be required. As the root is found its square is to be withdrawn or subtracted from 5476, therefore we begin the second working column with – 5476, the prefixed – indicating the subtraction of the square of the root. The initial line will thus be 1, 0, –5476.

1	0	$-5\dot{4}76$(74 *sq. rt.*
	7	49
1	140	–576
	144	576

Mark off the figures of 5476 in pairs counting in this case from the right-hand figure, there being no digits on the right of the decimal point in the given number. The marking off may be done by placing a point or dot over the right-hand figure of each period except in the case of the period immediately on the left of the decimal point, in which the decimal point serves as the marking off or distinguishing point. This period is named the zeroth period and the others are numbered from it as first, second, third, ... , positive or negative, (left or right,) as the case may be.

The first or left-hand period is 54. By the table of squares given above $54 > 7^2$ but $< 8^2$,

$\therefore$ the square root of $54 > 7$ but < 8,

$\therefore$ the square root of $5476 > 70$ but < 80,

$\therefore$ the first figure of the root is 7.

Write 7 in the place set apart for the root and then proceed with the calculation exactly as if the problem were to subtract 5476 from the square of a given base whose first digit is 7. This gives as the second line of calculation 1, 140, –576.

To obtain a "trial digit" for the second figure of the root, divide the 576 in the last column by the 140 in the next preceding column. The 'quotient' is 4. Write 4 as second figure in the root and proceed as in involution to find the value of $74^2 - 5476$. There is no 'remainder' therefore 74 is the square root of 5476.

Example 2. Find the square root of 12054784.

(Compare the calculation with that of *Example 1*, page 91, noting that there the square is built up, but that here the process is virtually the opposite.)

1	(0)	$-1\dot{2}0\dot{5}4\dot{7}84$(3472, *sq. rt.*
	3	9
1	(60)	-305
	64	256
1	(680)	-4947
	687	4809
1	(6940)	-13884
	6942	13884

The first period, 12, determines 3 as the first digit of the root. From 12 substract 3^2 and to the remainder, 3, 'bringing down' 05, the next period of the given number, and complete the formation of the second line of calculation 1, 60, -305. Divide 305 by 60. The quotient 5 is found on trial to be too large but 4 on trial proves to be the right digit. Proceed as in involution to form the third line of calculation which will be found to be 1, 680, -4947. Dividing 4947 by 680 gives 7 for trial as next digit of the root. On trial 7 is found to be the right digit. Continuing this process it will be found that 2 is the fourth digit of the root and that $3472 = 12054784^{\frac{1}{2}}$.

In subsequent calculations we shall omit the initial column and, in general, the minus signs in the last column and the lines in the working-column corresponding to those enclosed in parentheses in column two of the above example. If computers' subtraction be employed the subtrahends need not be recorded in the last column. Had all these omissions been made in the preceding example it would have appeared thus :—

3	$1\dot{2}0\dot{5}4\dot{7}84^{\frac{1}{2}} = 3472.$
64	305
687	4947
6942	13884

Example 3. Find the value of $204{\cdot}08163^{\frac{1}{2}}$ correct to six figures.

In the square of any number, if there be figures on the right of the decimal point, the number of such figures is even, but in 204·08163 the number of figures on the right of the decimal point is odd viz. 5. The number of 'decimal figures' must be made even and this is done by affixing a zero to the given number making it 204·081630.

1	$\dot{2}04\cdot0\dot{8}1\dot{6}3\dot{0}^{\frac{1}{2}} = 14\cdot2857+$
	1
24	104
	96
282	808
	564
2848	24416
	22784
28565	163230
	142825
285707	2040500
	1999949
	40551

After 'bringing down' all the periods in 204·081630, we find we have only five figures in the root and six figures are required. To obtain the additional root-figure we imagine a period of zeros, in this case two zeros, affixed to the given number and 'bring them down,' we thus virtually extract the square root of 204·08163000.

Example 4. Find the value of $10^{\frac{1}{2}}$ correct to ten figures.

3	$10^{\frac{1}{2}}$	$=3\cdot162277660,2$
	9	
61	1·00	
	61	
626	3900	
	3756	
6322	14400	
	12644	
63242	175600	
	126484	
632447	4911600	
	4427129	
632454	484471	
	442715	
	41756	
	37950	
	3806	
	3792	
	14	

Having found six figures of the root by the ordinary uncontracted process we may find four or five figures more by contracting the process in exactly the same way as we contract in involution. In this example we divide 484471 by 63245 by contracted division, knowing that the figures rejected from the divisor will not affect the quotient figures, here root-figures, till the divisor is reduced to one or at most to two figures. The root thus found is correct to eleven figures.

The general rule is that when the number of figures obtained by the uncontracted process is one more than half the number of figures required in the square root, than a third of the number required in the cube root, than a quarter of the number required in the fourth root, than a fifth of the number required in the fifth root, ... the rest of the figures may be obtained by contracted operations.

Example 5. Find the cube root of 1·25 correct to ten figures.

0	0	1·250	1·077217345
1	1	1	
2	3	·250000	
307	32149	225043	
314	34347	24957000	
3217	3457219	24200533	
3224	3479787	756467	
,32,31	3480433	696087	
	348108	60380	
		34811	
		25569	
		24368	
		1201	
		1044	
		157	
		139	
		18	

Example 5 page 95 is virtually an example of the extraction of the cube root of 116 correct to six figures and if each line in the fifth working-column of *Example 3* page 94 be subtracted from 10, the example will exhibit the operation of extracting the fifth root of 10 to ten figures.

EXERCISE VIII.

Find the square root of :—

1. 576. **2.** 1849. **3.** 103041. **4.** 10·3041. **5.** 2321·3124. **6.** ·0050367409.

Find the cube root of :—

7. 389017. **8.** 814780504. **9.** 700227072. **10.** 700227·072. **11.** 6·199083253. **12.** ·000160103007.

Find, correct to six significant figures, the value of :—

13. $2^{\frac{1}{2}}$. **14.** $20^{\frac{1}{2}}$. **15.** $200^{\frac{1}{2}}$. **16.** $2000^{\frac{1}{2}}$. **17.** $\cdot2^{\frac{1}{2}}$. **18.** $\cdot02^{\frac{1}{2}}$.

19. $40^{\frac{1}{2}}$. **20.** $4000^{\frac{1}{2}}$. **21.** $\cdot4^{\frac{1}{2}}$. **22.** $7449^{\frac{1}{2}}$. **23.** $1000^{\frac{1}{2}}$. **24.** $609800\cdot192^{\frac{1}{2}}$.

25. $123456^{\frac{1}{3}}$. **26.** $123\cdot456^{\frac{1}{3}}$. **27.** $2^{\frac{1}{3}}$. **28.** $20^{\frac{1}{3}}$. **29.** $200^{\frac{1}{3}}$. **30.** $\cdot2401^{\frac{1}{3}}$.

31. Find $\sqrt{3}$ correct to six significant figures and hence prove that $2-\sqrt{3}$ is the reciprocal of $2+\sqrt{3}$ to six figures.

32. Prove to six significant figures that $\sqrt{3}\times\sqrt{5}=\sqrt{15}$ and that the product of $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$ is 2.

110. The square of a given fraction has for numerator the square of the numerator of the given fraction and for denominator the square of the denominator of the given fraction. Hence inversely the square root of a given fraction has for numerator the square root of the numerator of the given fraction and for denominator the square root of the denominator of the given fraction.

The cube, fourth power, fifth power . . . . of a given fraction has for numerator the cube, fourth power, fifth power . . . . of the numerator of the given fraction and for denominator the cube, fourth power, fifth power, ... of the denominator of the given power. Hence inversely the cube root, fourth root, fifth root, ... of a given fraction has for numerator the cube root, fourth root, fifth root, ... of the numerator of the given fraction, and for denominator the cube root, fourth root, fifth root, ... of the denominator of the given fraction.

Examples. $\left(\frac{2}{3}\right)^2=\frac{2^2}{3^2}=\frac{4}{9}$, $\therefore$ $\left(\frac{4}{9}\right)^{\frac{1}{2}}=\frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}}=\frac{2}{3}$.

$\left(\frac{7}{11}\right)^2=\frac{7^2}{11^2}=\frac{49}{121}$, $\therefore$ $\left(\frac{49}{121}\right)^{\frac{1}{2}}=\frac{49^{\frac{1}{2}}}{121^{\frac{1}{2}}}=\frac{7}{11}$.

$\left(\frac{3}{8}\right)^3=\frac{3^3}{8^3}=\frac{27}{512}$, $\therefore$ $\left(\frac{27}{512}\right)^{\frac{1}{3}}=\frac{27^{\frac{1}{3}}}{512^{\frac{1}{3}}}=\frac{3}{8}$.

111. If in extracting any root of a given fraction, it is found that the root of the denominator cannot be obtained exactly, the fraction may be reduced to decimal form and the root extracted to any required degree of accuracy. Another method is to multiply both terms of the given fraction by *any* factor that will make the denominator an exact power of degree the reciprocal of the degree of the root; the root of the resulting fraction is then extracted. This process is called rationalizing the denominator. It can often be used with advantage to obtain a rapid approximation to a required root of a small number.

Example 1. Extract the square root of $\frac{4}{11}$ correct to four figures.

1° $\left(\frac{4}{11}\right)^{\frac{1}{2}}=\cdot36363636^{\frac{1}{2}}=\cdot6030+$.

2° $\left(\frac{4}{11}\right)^{\frac{1}{2}}=\left(\frac{4\times11}{11\times11}\right)^{\frac{1}{2}}=\frac{44^{\frac{1}{2}}}{11}=\frac{6\cdot6333-}{11}=\cdot6030+$.

Example 2. Find the value of $\left(\frac{5}{12}\right)^{\frac{1}{2}}$ correct to six figures.

1° $\left(\frac{5}{12}\right)^{\frac{1}{2}}=\cdot41666667^{\frac{1}{2}}=\cdot645497+$.

2° $\left(\frac{5}{12}\right)^{\frac{1}{2}}=\left(\frac{5\times3}{12\times3}\right)^{\frac{1}{2}}=\frac{15^{\frac{1}{2}}}{6}=\frac{3\cdot872983+}{6}=\cdot645497+$.

3° $\left(\frac{5}{12}\right)^{\frac{1}{2}}=\left(\frac{15}{36}\right)^{\frac{1}{2}}=\left(\frac{4^2-1}{6^2}\right)^{\frac{1}{2}}$,

$\therefore$ $\left(\frac{5}{12}\right)^{\frac{1}{2}}<\frac{4}{6}$, and $\frac{1}{6\times4\times2}<\textit{error}<\frac{4}{6}\times\frac{1}{4^2\times2-1}$; (i).

$\therefore$ $\left(\frac{5}{12}\right)^{\frac{1}{2}}<\frac{4^2\times2-1}{6\times4\times2}=\frac{31}{6\times8}$, and

$$\frac{1}{6\times8\times31\times2} < error < \frac{31}{6\times8}\times\frac{1}{31^2\times2-1}; \quad \text{(ii)}.$$

$$\therefore \left(\frac{5}{12}\right)^{\frac{1}{2}} < \frac{31^2\times2-1}{6\times8\times31\times2} = \frac{1921}{6\times8\times62}, \text{ and}$$

$$\frac{1}{6\times8\times62\times1921\times2} < error < \frac{1921}{6\times8\times62}\times\frac{1}{1921^2\times2-1}; \quad \text{(iii)}.$$

$$\therefore \left(\frac{5}{12}\right)^{\frac{1}{2}} < \frac{1921^2\times2-1}{6\times8\times62\times1921\times2} = \frac{7380481}{6\times8\times62\times3842} = \cdot645497224368,$$

$$\text{and } \frac{1}{6\times8\times62\times3842\times7380481\times2}$$

$$< error < \frac{7380481}{6\times8\times62\times3842}\times\frac{1}{7380481^2\times2-1}. \quad \text{(iv)}.$$

In this 3° method we first rationalize the denominator and then we try successively 15×1^2, 15×2^2, 15×3^2, 15×4^2, ... till we find a product that differs but little from a square number. Such a product is 15×1^2 which differs from 4^2 by 1. We therefore write $\left(\frac{15}{36}\right)^{\frac{1}{2}}$ in the form $\left(\frac{4^2-1}{6^2}\right)^{\frac{1}{2}}$ from which we obtain at once $\frac{4}{6}$ as a first approximation to the required root with an error in excess somewhat greater than $\frac{1}{6\times4\times2}$, as may be proved by squaring $\frac{4}{6}-\frac{1}{6\times4\times2}$. We next take $\frac{4}{6}-\frac{1}{6\times4\times2}=\frac{4^2\times2-1}{6\times4\times2}$ $=\frac{31}{6\times8}$ as a second approximation in excess with an error somewhat greater than $\frac{1}{6\times8\times31\times2}$ as may be proved by squaring $\frac{31}{6\times8}-\frac{1}{6\times8\times31\times2}$. This gives $\frac{31}{6\times8}-\frac{1}{6\times8\times31\times2}$ $=\frac{31^2\times2-1}{6\times8\times31\times2}=\frac{1921}{6\times8\times62}$ as a third approximation in excess with an error somewhat greater than $\frac{1}{6\times8\times62\times1921\times2}$, as may be proved by squaring $\frac{1921}{6\times8\times62}-\frac{1}{6\times8\times62\times1921\times2}$.

This gives $\frac{1921}{6\times 8\times 62}-\frac{1}{6\times 8\times 62\times 1921\times 2}=\frac{1921^2\times 2-1}{6\times 8\times 62\times 1921\times 2}$

$=\frac{7380481}{6\times 8\times 62\times 3842}$ as a fourth approximation to the required root. The first approximation is correct to one decimal place, the second is correct to three decimal places; the third, to six decimal places; and the fourth, to fourteen decimal places.

It should be noticed that

(i.) comes from $\frac{15}{36}=\frac{4^2-1}{6^2}$,

(ii.) comes from $\frac{15}{36}=\frac{15\times 8^2}{36\times 8^2}=\frac{31^2-1}{6^2\times 8^2}$,

(iii.) comes from $\frac{15}{36}=\frac{15\times 8^2\times 62^2}{36\times 8^2\times 62^2}=\frac{1921^2-1}{6^2\times 8^2\times 62^2}$ and

(iv.) comes from $\frac{15}{36}=\frac{15\times 8^2\times 62^2\times 3842^2}{36\times 8^2\times 62^2\times 3842^2}=\frac{7380481^2-1}{6^2\times 8^2\times 62^2\times 3842^2}$

Example 3. Find an approximate value of $\left(\frac{2}{11}\right)^{\frac{1}{2}}$.

$$\frac{2}{11}=\frac{22}{11\times 11}=\frac{5^2-3}{11^2}$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{5}{11} \text{ and } \frac{3}{11\times 5\times 2}<\textit{ error }<\frac{5}{11}\times\frac{3}{5^2\times 2-3}.$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{5^2\times 2-3}{11\times 5\times 2}=\frac{47}{11\times 10}, \text{ and}$$

$$\frac{9}{11\times 10\times 47\times 2}<\textit{ error }<\frac{47}{11\times 10}\times\frac{9}{47^2\times 2-9};$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{47^2\times 2-9}{11\times 10\times 47\times 2}=\frac{4409}{11\times 10\times 94}, \text{ and}$$

$$\frac{81}{11\times 10\times 94\times 4409\times 2}<\textit{ error }<\frac{4409}{11\times 10\times 94}\times\frac{81}{4409^2\times 2-81};$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{4409^2\times 2-81}{11\times 10\times 94\times 4409\times 2}=\frac{38878481}{11\times 10\times 94\times 8818}$$

The next *correction* would be $\frac{6561}{11\times 10\times 94\times 8818\times 38878481\times 2}$, the numerator 6561 being 81^2. From this example we may see that if possible a numerator should be found that differs from a square number by but 1 or 2. This might easily have been done in this case by selecting 22×3^2 as numerator to work from. The calculation would then have appeared as follows :—

$$\frac{2}{11}=\frac{22}{11^2}=\frac{22}{11^2}\times\frac{3^2}{3^2}=\frac{198}{33^2}=\frac{14^2+2}{33^2}$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}>\frac{14}{33},\text{ and}$$

$$\frac{2}{33\times14\times2}=\frac{1}{33\times14}>\textit{ error }>\frac{14}{33}\times\frac{2}{14^2\times2+2}=\frac{14}{33}\times\frac{1}{14^2+1}$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{14^2+1}{33\times14}=\frac{197}{33\times14},\text{ and}$$

$$\frac{1}{33\times14\times197\times2}<\textit{error}<\frac{197}{33\times14}\times\frac{1}{197^2\times2-1};$$

$$\therefore \left(\frac{2}{11}\right)^{\frac{1}{2}}<\frac{197^2\times2-1}{33\times14\times197\times2}=\frac{77617}{33\times14\times394},\text{ \&c.}$$

Here the first approximation is in defect, we therefore add the first correction. This correction is in excess, hence the second approximation is in defect, further since the numerator of the first correction was reduced to 1, the numerators of all subsequent corrections will also be 1. In fact the second approximation is obtainable from the equality

$$\frac{2}{11}=\frac{198}{33^2}=\frac{198\times14^2}{33^2\times14^2}=\frac{197^2-1}{33^2\times14^2}.$$

Example 4. Find approximately the square root of 45.

$$45=7^2-4,$$

$$\therefore\ 45^{\frac{1}{2}}<7,\text{ and }\frac{2}{7}<\textit{ error }<7\times\frac{2}{7^2-2};$$

$$\therefore\ 45^{\frac{1}{2}}<\frac{7^2-2}{7}=\frac{47}{7},\text{ and }\frac{2}{7\times47}<\textit{ error }<\frac{47}{7}\times\frac{2}{47^2-2};$$

$$\therefore\ 45^{\frac{1}{2}}<\frac{47^2-2}{7\times47}=\frac{2207}{7\times47},\text{ and}$$

$$\frac{2}{7\times47\times2207}<\textit{error}<\frac{2207}{7\times47}\times\frac{2}{2207^2-2};$$

$$\therefore\ 45^{\frac{1}{2}}<\frac{2207^2-2}{7\times47\times2207}=\frac{4870847}{7\times47\times2207},\text{ and}$$

$$\frac{2}{7\times47\times2207\times4870847}<\textit{ error }<\frac{4870847}{7\times47\times2207}\times\frac{2}{4870847^2-2}.$$

The terms of the errors and the corrections are reduced each time by division by the common factor 2.

In any case in which only the result of the computation is required, the limits of error need not be calculated for each successive approximation; it will be sufficient to examine the superior limit to the error of the last approximation.

Example 5. Find the square root of 111 correct to six figures.

$111 = 11^2 - 10$, $\therefore$ the first three approximations to $111^{\frac{1}{2}}$ are (i), 11; (ii), $\frac{11^2 - 5}{11} = \frac{116}{11}$; (iii), $\frac{116^2 \times 2 - 25}{11 \times 116 \times 2} = \frac{26887}{11 \times 232} = 10{\cdot}5357 -$.

The error is less than $10{\cdot}54 \times \frac{625}{26887^2 \times 2 - 625} < 10 \times \frac{625}{25000^2 \times 2}$ $= {\cdot}000005$. The error being thus less than 5 in the sixth decimal place, the division of 26887 by 11×232 might have been carried one step further; the quotient is $10{\cdot}535658+$, and allowing for the error we obtain $111^{\frac{1}{2}} = 10{\cdot}53565+$, correct to seven figures.

Example 6. Find the cube root of $\frac{7}{12}$ correct to five figures.

$$1^\circ \quad \left(\frac{7}{12}\right)^{\frac{1}{3}} = {\cdot}583333333^{\frac{1}{3}} = {\cdot}83555-$$

$$2^\circ \quad \left\{\frac{7}{12}\right\}^{\frac{1}{3}} = \left\{\frac{7}{2^2 \times 3}\right\}^{\frac{1}{3}} = \left\{\frac{7 \times 2 \times 3^2}{2^3 \times 3^3}\right\}^{\frac{1}{3}} = \frac{126^{\frac{1}{3}}}{6} = \frac{5{\cdot}01330}{6}$$
$$= {\cdot}83555-$$

$$3^\circ \quad \frac{7}{12} = \frac{7}{2^2 \times 3} = \frac{7 \times 2 \times 3^2}{2^3 \times 3^3} = \frac{126}{6^3} = \frac{5^3 + 1}{6^3}$$

$$\therefore \quad \left\{\frac{7}{12}\right\}^{\frac{1}{3}} > \frac{5}{6} \text{ and } \frac{1}{6 \times 5^2 \times 3} > \textit{error}. \qquad \text{(i)}.$$

$$\therefore \quad \left\{\frac{7}{12}\right\}^{\frac{1}{3}} < \frac{5}{6} + \frac{1}{6 \times 5^2 \times 3} = \frac{5^3 \times 3 + 1}{6 \times 5^2 \times 3} = \frac{376}{6 \times 75} = {\cdot}835556 - \cdot \qquad \text{(ii)}.$$

$$\frac{7}{12} = \frac{126 \times 75^3}{6^3 \times 75^3} = \frac{53156250}{6^3 \times 75^3} = \frac{376^3 - 1126}{6^3 \times 75^3}$$

$$\therefore \quad \left\{\frac{7}{12}\right\}^{\frac{1}{3}} < \frac{376}{6 \times 75} \text{ and } \frac{1126}{6 \times 75 \times 376^2 \times 3} < \textit{error}. \qquad (a).$$

$$\therefore \quad \left\{\frac{7}{12}\right\}^{\frac{1}{3}} < \frac{376}{6 \times 75} - \frac{1126}{6 \times 75 \times 376^2 \times 3} = \frac{376^3 \times 3 - 1126}{6 \times 75 \times 376^2 \times 3}$$
$$= \frac{159471002}{6 \times 75 \times 424128} = {\cdot}83554965+. \qquad \text{(iii)}.$$

(i) is correct to two figures; (ii) is correct to five figures, the final 6 being rejected without augmenting the preceding 5, on account of the sign < and the correction (*a*).

112. The process of forming a series of convergents to a given fraction which was exemplified in §77 may be applied to obtain a series of convergents to any root of a number.

Example 1. Find convergents to the square root of 6.

$2^2<6<3^2$, $\therefore$ we take $\frac{2}{1}$ as the inferior and $\frac{3}{1}$ as the superior initial convergent of the series. The next convergent is $\frac{3+2}{1+1}=\frac{5}{2}$; and since $\left\{\frac{5}{2}\right\}^2>6$, $\frac{5}{2}$ is written in a line beside $\frac{3}{1}$ higher than the line of $\frac{2}{1}$. The next convergent is $\frac{5+2}{2+1}=\frac{7}{3}$; $\left\{\frac{7}{3}\right\}^2<6$, $\therefore\ \frac{7}{3}$ is written in the lower line, the line of $\frac{2}{1}$. The next convergent is $\frac{7+5}{3+2}=\frac{12}{5}$; $\left\{\frac{12}{5}\right\}^2<6$, $\therefore\frac{12}{5}$ is written in the lower line. The process thus far followed is continued until there is obtained a sufficiently close approximation to the required root.

$$\begin{array}{ccccccccccccccc} & \frac{3}{1}, & \frac{5}{2} & & & & & \frac{27}{11}, & \frac{49}{20}, & & & & & \frac{267}{109}, & \frac{485}{198}. \\ \frac{2}{1}, & & & \frac{7}{3}, & \frac{12}{5}, & \frac{17}{7}, & \frac{22}{9}, & & & \frac{71}{29}, & \frac{120}{49}, & \frac{169}{69}, & \frac{218}{89}, & & \end{array}$$

The principal convergents, as far as the series has been formed are therefore $\frac{2}{1}$, $\frac{5}{2}$, $\frac{22}{9}$, $\frac{49}{20}$, $\frac{218}{89}$, $\frac{485}{198}$, these being alternately less and greater than $6^{\frac{1}{2}}$. It is worthy of notice that beginning with the superior initial $\frac{3}{1}$, there are throughout the whole series two superior convergents followed by four inferior convergents, followed in their turn by two superior convergents. This enables us to form with great ease and rapidity any required number of principal convergents, after the first two are known. Thus, keeping to numerators alone, $5\times4+2=22$, $22\times2+5=49$, $49\times4+22=218$, $218\times2+49=485$. The denominators may be similarly computed, thus the denominator following next after 198 is $198\times4+89=881$. The error committed in taking

$\frac{485}{198} = 2 \cdot 449495 -$ for $6^{\frac{1}{2}}$ is less than $\frac{1}{198 \times 881} < \frac{1}{160000} < \cdot 000007$,

hence $6^{\frac{1}{2}} = 2 \cdot 44949 -$, correct to six figures.

Example 2. Form a series of convergents to the cube root of 6.

$1^3 < 6 < 2^3$, $\therefore$ we take $\frac{1}{1}$ and $\frac{2}{1}$ as initial convergents, and form from them a series in the usual way, cubing each term to test whether it is a superior or an inferior convergent.

$\frac{2}{1}$, $\frac{11}{6}$, $\frac{20}{11}$, $\frac{169}{93}$, $\frac{318}{175}$, $\frac{467}{257}$.

$\frac{1}{1}$, $\frac{3}{2}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{9}{5}$, $\frac{29}{16}$, $\frac{49}{27}$, $\frac{69}{38}$, $\frac{89}{49}$, $\frac{109}{60}$, $\frac{129}{71}$, $\frac{149}{82}$,

$\frac{467}{257} = 1 \cdot 8171206 +$, which is the cube root of 6 to eight figures.

The next two principal convergents to $6^{\frac{1}{3}}$ are

$$\frac{467 \times 508 + 149}{257 \times 508 + 82} \text{ and } \frac{467 \times 509 + 149}{257 \times 509 + 82},$$

113. This is the oldest and perhaps the simplest systematic process for obtaining a series of approximations converging to the value of any required root of a given number. It is subject however to the disadvantage of being extremely tedious and laborious except where the law of immediate formation of the successive principal convergents is known, in which case it becomes an easy and rapid method of evolution. The following examples exhibit one method of directly computing the successive principal convergents to the square root of a given number.

Example 1. Find approximately the square root of 31.

$31^{\frac{1}{2}} = 5 +$

0	5	1	4	5	5	4	1	5	5	1	4	&c.
1	6	5	3	2	3	5	6	1	6	5	3	&c.
5	1	1	3	5	3	1	1	10	1	1	3	&c.

Quotients. 5 1 1 3 5 3 1 1 10 &c.

Convergents. $\frac{0}{1}$, $\frac{1}{0}$, $\frac{5}{1}$, $\frac{6}{1}$, $\frac{11}{2}$, $\frac{39}{7}$, $\frac{206}{37}$, $\frac{657}{118}$, $\frac{863}{125}$, $\frac{1520}{273}$, $\frac{16063}{2885}$.

The first column always consists of 0, 1 and the greatest integer whose square is less than the given number. In this example the first column will therefore consist of 0, 1 and 5.

Let a, b and c denote the numbers in any column; a denoting the number in the 1st row; b, the number in the 2nd row; and c, the number in the 3rd row. Let A, B and C denote the corresponding numbers in the next following column. The successive columns are formed each from the column next before it, thus :—

$$A = bc - a\ ;\quad B = \frac{N - A^2}{b},\quad C = \text{integral part of } \frac{I + A}{B},$$

in which N denotes the number whose square root is required, *in this example 31*, and I denotes the integral part of the square root of N, *in this example 5.*

Thus in the first column a = 0, b = 1, c = 5;

$$\therefore \text{ the second column is } \begin{cases} A = & 1 \times 5 - 0 = 5. \\ B = & \dfrac{31 - 5^2}{1} = 6. \\ C = & \text{Int.} \left(\dfrac{5+5}{6}\right) = 1. \end{cases}$$

In the second column a = 5, b = 6 and c = 1;

$$\therefore \text{ the third column is } \begin{cases} A = & 6 \times 1 - 5 = 1. \\ B = & \dfrac{31 - 1^2}{6} = 5. \\ C = & \text{Int.} \left(\dfrac{5+1}{6}\right) = 1. \end{cases}$$

This process of forming each column from the preceding column is continued until the second column occurs again, after which the several columns are repeated in the same order.

The principal convergents to $31^{\frac{1}{2}}$ are obtained from the initials $\frac{0}{1}$ and $\frac{1}{0}$, by employing as 'quotients' the numbers in the third row, viz., 5, 1, 1, 3, 5, 3, 1, 1, 10, 1, 1, 3, 5, 3, 1, 1, 10, 1, 1, 3, &c.

Example 2. Find a series of principal convergents to $6^{\frac{1}{2}}$.

The greatest integer whose square is less than 6 is 2, ∴ the first column is 0, 1, 2. The succeeding columns are formed each from the immediately preceding column, thus :—

$$A = bc - a,\qquad B = \frac{6 - A^2}{b},\qquad C = \text{integral part of } \frac{2 + A}{B}.$$

$6^{\frac{1}{2}} = 2 +$

0	2	2	2				
1	2	1	2				
2	2	4	2	4	2	4	&c.

Quotients. 2, 2, 4, 2, 4, 2, 4, &c.

Convergents. $\frac{0}{1}, \frac{1}{0}, \frac{2}{1}, \frac{5}{2}, \frac{22}{9}, \frac{49}{20}, \frac{218}{89}, \frac{485}{198}, \frac{2158}{881}$, &c.

Compare with *Example 1*, § 112, p. 110.

EXERCISE IX.

Find, correct to six significant figures, the value of :—

1. $\left(\frac{1}{4}\right)^{\frac{1}{2}}$. **10.** $\left(\frac{3}{8}\right)^{\frac{1}{2}}$. **19.** $35^{\frac{1}{2}}$.

2. $\left(\frac{4}{9}\right)^{\frac{1}{2}}$. **11.** $\left(\frac{12}{175}\right)^{\frac{1}{2}}$. **20.** $37^{\frac{1}{2}}$.

3. $\left(\frac{256}{2401}\right)^{\frac{1}{2}}$. **12.** $\left(\frac{28}{45}\right)^{\frac{1}{2}}$. **21.** $7^{\frac{1}{2}}$.

4. $\left(\frac{5}{9}\right)^{\frac{1}{2}}$. **13.** $3^{\frac{1}{2}}$. **22.** $11^{\frac{1}{2}}$.

5. $\left(\frac{17}{25}\right)^{\frac{1}{2}}$. **14.** $5^{\frac{1}{2}}$. **23.** $53^{\frac{1}{2}}$.

6. $\left(\frac{9}{32}\right)^{\frac{1}{2}}$. **15.** $15^{\frac{1}{2}}$. **24.** $77^{\frac{1}{2}}$.

7. $\left(\frac{16}{27}\right)^{\frac{1}{2}}$. **16.** $17^{\frac{1}{2}}$. **25.** $97^{\frac{1}{2}}$.

8. $\left(\frac{3}{10}\right)^{\frac{1}{2}}$. **17.** $24^{\frac{1}{2}}$. **26.** $1601^{\frac{1}{2}}$.

9. $\left(\frac{5}{7}\right)^{\frac{1}{2}}$. **18.** $26^{\frac{1}{2}}$. **27.** $2400^{\frac{1}{2}}$.

28. $\left(\frac{7}{11}\right)^{\frac{1}{2}}$, given $\frac{7}{11}=\frac{77}{121}=\frac{77\times 40^2}{11^2\times 40^2}=\frac{351^2-1}{440^2}$.

29. $\left(\frac{9}{20}\right)^{\frac{1}{2}}$, given $\frac{9}{20}=\frac{45}{100}=\frac{45\times 24^2}{10^2\times 24^2}=\frac{161^2-1}{240^2}$.

30. $\left(\frac{13}{24}\right)^{\frac{1}{2}}$, given $\frac{13}{24}=\frac{78}{144}=\frac{78\times 6^2}{12^2\times 6^2}=\frac{53^2-1}{72^2}$.

31. $11^{\frac{1}{2}}$, given $11=\frac{11\times 3^2}{3^2}=\frac{10^2-1}{3^2}$.

32. $6^{\frac{1}{2}}$, given $6=\frac{6\times 20^2}{20^2}=\frac{49^2-1}{20^2}$.

33. $5^{\frac{1}{2}}$, given $5=\frac{5\times 4^2}{4^2}=\frac{9^2-1}{4^2}$.

34. $2^{\frac{1}{2}}$, given $2=\frac{2\times 5^2}{5^2}=\frac{7^2+1}{5^2}$,

also $2=\frac{2\times 12^2}{12^2}=\frac{17^2-1}{12^2}$, also $2=\frac{2\times 29^2}{29^2}=\frac{41^2+1}{29^2}$.

35. $\left(\frac{125}{343}\right)^{\frac{1}{3}}$. **37.** $\left(\frac{49}{216}\right)^{\frac{1}{3}}$. **39.** $\left(\frac{2}{3}\right)^{\frac{1}{3}}$.

36. $\left(\frac{729}{4913}\right)^{\frac{1}{3}}$. **38.** $\left(\frac{841}{8000}\right)^{\frac{1}{3}}$. **40.** $\left(\frac{5}{7}\right)^{\frac{1}{3}}$.

114. Two given quantities are commensurable if there be an integral multiple of one of them which is also an integral multiple of the other.

For example, let there be two lines *A* and *B* of lengths such that a third line which is five times the length of the line *A* is twelve times the length of the line *B*. Divide this third line into $5\times 12=60$ equal parts, then the length of any one of these parts will be $\frac{1}{60}$ of five times the length of the line *A*, i. e., the sixtieth part of the third line will be $\frac{1}{12}$ of the line *A* or be contained twelve times in the line *A*. But the length of the same part will be $\frac{1}{60}$ of twelve times the length of the line *B*, i. e., the sixtieth part of the third line will be $\frac{1}{5}$ of the line *B* or be contained five times in the line *B*. Hence a sixtieth part of the third line will measure both the line *A* and the line *B*, i. e., the lines *A* and *B* have a common measure or are commensurable.

______________ *A*.
__________ *B*.

Expressed in symbols the preceding example is:—

If $$5\text{ A}=12\text{ B}$$

$\therefore$ $$\frac{5\text{ A}}{5\times 12}=\frac{12\text{ B}}{5\times 12},$$

$\therefore$ $$\frac{\text{A}}{12}=\frac{\text{B}}{5},$$

$\therefore$ $$\text{A}=12\left(\frac{\text{B}}{5}\right);$$

and $$\text{B}=5\left(\frac{\text{B}}{5}\right),$$

$\therefore$ *A* and *B* are commensurable, $\frac{1}{5}$ of B being a common measure or common unit.

115. If either of two commensurable quantities be expressed in terms of the other as unit, the number expressing their ratio or relative magnitude will be an integer, a fraction with integral terms or with terms reducible to integers, or a mixed number consisting in part of an integer and in part of an integral-termed fraction. For this reason integers, integral-termed fractions and integral-termed mixed numbers, whether decimally expressed or otherwise, are called commensurable or rational numbers.

116. Two given quantities are incommensurable if no integral multiple of one of them is an integral multiple of the other.

If either of two incommensurable quantities of the same kind be expressed in terms of the other as unit, the number expressing their ratio or relative magnitude will not be expressible exactly by any integer, integral-termed fraction or integral-termed mixed number whatever. For this reason a number which cannot be expressed exactly by any integer or any fraction or mixed number with integral terms is called an incommensurable or irrational number.

If the length of the diagonal of a square be expressed in terms of the length of a side of the square as unit, the number expressing their ratio or relative magnitude will be the square root of 2. Now, in extracting the square root of 2, whether as a decimal number or as a fraction, there is always a remainder i. e., it is impossible to find a rational or commensurable number of which the square is exactly 2, Hence $2^{\frac{1}{2}}$ is an incommensurable number, and the lengths of the diagonal and the side of the same square are *relatively* incommensurable quantities.

Other examples of incommensurable numbers are $3^{\frac{1}{2}}$, $5^{\frac{1}{2}}$, $10^{\frac{1}{2}}$, $2^{\frac{1}{3}}$, $5^{\frac{1}{3}}$, $9^{\frac{1}{3}}$, $100^{\frac{1}{3}}$, $2^{\frac{1}{4}}$, $4^{\frac{1}{4}}$, $100^{\frac{1}{4}}$, $3^{\frac{1}{5}}$.

117. Every number formed by combining a *definite* number of ones (or of integers) by means of the operations of addition, subtraction, multiplication and division, and of these only, is reducible to an integer or to a fraction, proper or improper, with integral terms, i. e., every number so formed is a commensurable number, hence *no incommensurable number can be expressed by combining a definite number of commensurable numbers by additions, subtractions, multiplications and divisions and these only.*

118. Incommensurable numbers which can be formed from a definite number of commensurable numbers combined by means of the operations of addition, subtraction, multiplication, division, involution and evolution, are sometimes called surd numbers or surds to distinguish them from incommensurable numbers which cannot be so formed. The latter are called transcendental numbers.

Examples. $2^{\frac{1}{2}}$, $3^{\frac{1}{4}}$, $1+2^{\frac{1}{2}}$, $3+2^{\frac{1}{3}}-4^{\frac{1}{3}}$, $5^{\frac{1}{2}}\times 6^{\frac{1}{4}}$, $8^{\frac{1}{5}}\div 4^{\frac{1}{3}}$, are surds.

The ratio of the circumference of a circle to its diameter is a transcendental number as also is the exponent which expresses the degree of the power which 20 is of 10. (See *Logarithms.*)

119. Involution is the operation of raising a given base to a power of given degree. In the examples of this operation hitherto considered, the exponent or index of degree of the power has been either an integer or, in the case of roots, the reciprocal of an integer. But no such restriction need be laid on the values of exponents; these may be integral or fractional, commensurable or incommensurable, positive or negative, provided that the terms degree and power be interpreted in accordance with this extension and provided that the laws laid down for operating upon and with these generalized powers are consistent with each other and include as particular or special cases, the laws governing operations upon and with powers of integral degrees and their corresponding roots. These laws which thus constitute the Fundamental Theorems of Involution and Evolution are ;—

XXIV. *If equals be raised to equal degrees, (have equal exponents), the powers are equal.*

(Equal-degreed roots of equals are equal.)

XXV. *Equal powers of equals are of equal degree, (have equal exponents.)*

(Equal roots of equals are of equal degree.)

XXVI. *Raising the base to any degree raises the power to the power of itself of that degree.*

(Extracting any root of the base extracts the equal-degreed root of the power.)

XXVII. *Multiplying the exponent by any number raises the power to a power of itself of degree denoted by the multiplier.*

(Dividing the exponent by any number reduces the power to its root of degree denoted by the reciprocal of the divisor.)

XXVIII. *The product of two or more powers of the same base is that power of the base which has for exponent the aggregate of the exponents of the factors.*

(The quotient of two powers of the same base is that power of the base which has for exponent, the remainder obtained by subtracting the exponent of the divisor from the exponent of the dividend.)

XXIX. *To multiply by a negative power of any base divide by the reciprocal of the power, i. e., divide by the power of corresponding positive degree.*

(To divide by a negative power of any base, multiply by the reciprocal of the power.)

120. The Fundamental Theorem connecting the operations of multiplication and division with the operations of involution and evolution is,—

XXX. *Raising the several factors of a product to any degree raises the product to that degree.*

(Reducing the several factors of a product to their roots of a given degree reduces the product to its root of the same degree.)

Examples of Theorem XXVI.

1. Let 2 be the base and 2^3 be the power, and let the base be squared, then will

$$(2^2)^3 = (2^3)^2\,;$$

for $(2^2)^3 = (2\times 2)\times(2\times 2)\times(2\times 2) = (2\times 2\times 2)\times(2\times 2\times 2) = (2^3)^2$.

2. Let 729 be the base and $729^{\frac{1}{3}}$ be the power, and let the square root of the base be extracted, then will

$$(729^{\frac{1}{2}})^{\frac{1}{3}} = (729^{\frac{1}{3}})^{\frac{1}{2}}$$

for $729^{\frac{1}{2}} = 27$ and $729^{\frac{1}{3}} = 9$

and $27^{\frac{1}{3}} = 3 = 9^{\frac{1}{2}}$.

3. $(10^{\frac{1}{2}})^{\frac{1}{5}} = (10^{\frac{1}{5}})^{\frac{1}{2}}$

for $10^{\frac{1}{2}} = 3{\cdot}16228-$ and $10^{\frac{1}{5}} = 1{\cdot}58489+$,

and $(3{\cdot}16228-)^{\frac{1}{5}} = 1{\cdot}25893- = (1{\cdot}58489+)^{\frac{1}{2}}$.

4. $(8^{\frac{1}{3}})^2 = (8^2)^{\frac{1}{3}}$

for $8^{\frac{1}{3}} = 2$ and $8^2 = 64$

and $2^2 = 4 = 64^{\frac{1}{3}}$.

5. $(12^{\frac{1}{4}})^{-3} = (12^{-3})^{\frac{1}{4}}$.

6. $\left(24^{-\frac{2}{3}}\right)^{-\frac{3}{4}}=\left(24^{-\frac{3}{4}}\right)^{-\frac{2}{3}}$.

Examples of Theorem XXVII.

1. Let 2^3 be the power and let the exponent be multiplied by 2, then will

$$2^{3\times 2}=(2^3)^2 \text{ or } 2^6=(2^3)^2$$

for $2^6=2\times2\times2\times2\times2\times2=(2\times2\times2)\times(2\times2\times2)=(2^3)^2$.

2. Let the exponent of $729^{\frac{1}{3}}$ be multiplied by $\frac{1}{2}$, then will

$$729^{\frac{1}{3}\times\frac{1}{2}}=(729^{\frac{1}{3}})^{\frac{1}{2}}, \quad \text{or } 729^{\frac{1}{6}}=(729^{\frac{1}{3}})^{\frac{1}{2}}$$

3. $10^{\frac{1}{2}\times\frac{1}{5}}=(10^{\frac{1}{2}})^{\frac{1}{5}}$ or $10^{\frac{1}{10}}=(10^{\frac{1}{2}})^{\frac{1}{5}}$.

4. $8^{\frac{1}{3}\times 2}=(8^{\frac{1}{3}})^2$ or $8^{\frac{2}{3}}=(8^{\frac{1}{3}})^2$.

5. $6^{-3\times\frac{1}{4}}=(6^{-3})^{\frac{1}{4}}$ or $6^{-\frac{3}{4}}=(6^{-3})^{\frac{1}{4}}$.

Examples of Theorems XXVIII and XXIX.

1. $7^2\times7^3=7^{2+3}=7^5$,

for $7^2\times7^3=(7\times7)\times(7\times7\times7)=7^5$.

2. $64^{\frac{1}{2}}\times64^{\frac{1}{3}}=64^{\frac{1}{2}+\frac{1}{3}}=64^{\frac{5}{6}}$,

for $64^{\frac{1}{2}}=8$ and $64^{\frac{1}{3}}=4$.

and $8\times4=32=2^5=(64^{\frac{1}{6}})^5=64^{\frac{5}{6}}$.

3. $7^{\frac{1}{3}}\times7^{\frac{3}{4}}=7^{\frac{1}{3}+\frac{3}{4}}=7^{\frac{13}{12}}=7\times7^{\frac{1}{12}}$.

4. $3^5\div3^2=3^{5-2}=3^3$,

for $3^5\div3^2=(3\times3\times3\times3\times3)\div(3\times3)=3\times3\times3=3^3$.

5. $6^{\frac{2}{3}}\div6^{\frac{1}{5}}=6^{\frac{2}{3}-\frac{1}{5}}=6^{\frac{7}{15}}$.

6. $5^5\times5^{-3}=5^{-5}\div5^3=5^{5-3}=5^2$.

for $5^5\times5^{-3}=(5\times5\times5\times5\times5)\times(1\div5\div5\div5)$

$=(5\times5\times5\times5\times5)\div(5\times5\times5)=5^5\div5^3$.

7. $64^{\frac{1}{2}}\times64^{-\frac{1}{3}}=64^{\frac{1}{2}}\div64^{\frac{1}{3}}=64^{\frac{1}{2}-\frac{1}{3}}=64^{\frac{1}{6}}$.

for $64^{\frac{1}{2}}\times64^{-\frac{1}{3}}=8\times\frac{1}{4}=8\div4=64^{\frac{1}{2}}\div64^{\frac{1}{3}}$.

8. $11^{\frac{4}{5}}\times11^{-\frac{2}{3}}=11^{\frac{4}{5}}\div11^{\frac{2}{3}}=11^{\frac{4}{5}-\frac{2}{3}}=11^{\frac{2}{15}}$.

9. $2^2\times2^{-5}=2^2\div2^5=1\div2^3=2^{-3}, \quad =2^{2-5}$.

10. $3^{-3}\times3^{-2}=1\div3^3\div3^2=1\div(3^3\times3^2)=1\div3^{3+2}=3^{-5}$.

11. $3^3\div3^{-4}=3^3\times3^4=3^{3+4}=3^7$.

12. $3^{-3}\div3^{-4}=3^{-3}\times3^4=3^4\div3^3=3^{4-3}=3$.

Examples of Theorem XXX.

1. $3^2 \times 5^2 = (3\times 5)^2 = 15^2$

for $3^2 \times 5^2 = (3\times 3)\times(5\times 5) = (3\times 5)\times(3\times 5) = (3\times 5)^2$.

2. $4^{\frac{1}{2}} \times 9^{\frac{1}{2}} = (4\times 9)^{\frac{1}{2}} = 36^{\frac{1}{2}}$

for $4^{\frac{1}{2}} = 2,\ 9^{\frac{1}{2}} = 3$ and $6 = 36^{\frac{1}{2}}$

$\therefore\ 4^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 2\times 3 = 6 = 36^{\frac{1}{2}}$.

3. $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (2\times 3)^{\frac{1}{2}} = 6^{\frac{1}{2}}$;

$2^{\frac{1}{2}} = 1{\cdot}414214\times,\ ,\ 3^{\frac{1}{2}} = 1{\cdot}73205+,\ 6^{\frac{1}{2}} = 2{\cdot}44949-$,

and $1{\cdot}414214 \times 1{\cdot}73205 = 2{\cdot}44949-$.

4. $7^{\cdot 3} \times 11^{\cdot 3} = (7\times 11)^{\cdot 3} = 77^{\cdot 3}$.

5. $8^3 \div 27^3 = (8\div 27)^3 = \left(\frac{8}{27}\right)^3$

6. $8^{\frac{1}{3}} \div 27^{\frac{1}{3}} = (8\div 27)^{\frac{1}{3}} = \left(\frac{8}{27}\right)^{\frac{1}{3}}$

7. $7^{-2} \div 11^{-2} = 11^2 \div 7^2 = \left(\frac{11}{7}\right)^2 = \left(\frac{7}{11}\right)^{-2}$.

8. $\left(\frac{8}{15}\right)^{\frac{2}{3}} \times \left(\frac{5}{12}\right)^{\frac{2}{3}} = \left(\frac{8}{15}\times\frac{5}{12}\right)^{\frac{2}{3}} = \left(\frac{2}{9}\right)^{\frac{2}{3}}$.

[121. The fundamental theorems of addition and subtraction set forth in §42, those of multiplication and division set forth in §§62 and 63 and those of involution and evolution set forth in §§118 and 119, may by mere counting be proved to be true in every instance in which the numbers to be combined are all commensurable, but they cannot be thus proved if the numbers to be combined or operated upon are incommensurable. In the latter case we practically assume or postulate the truth of these theorems which thus contain implicitly, or rather actually become the definitions of, the generalized operations of addition, subtraction, multiplication, division, involution and evolution. For instance, we may prove by mere counting that twice three is equal to thrice two, that one-half of one-third is equal to one-third of one-half, that the square root of four multiplied by the square root of nine is equal to the square root of nine multiplied by the square root of four, but we cannot by such method prove absolutely and completely that the square root of two multiplied by the square root of three is equal to the square root of three multiplied by the square root of two or even that twice the square root of three is equal to two multiplied by the square root of three. So also we may prove by mere counting that $2 \times 3 = 6$, that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ and that $4^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 36^{\frac{1}{2}}$, but we cannot by counting and solely by counting prove absolutely and completely that $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$, or that, if the index of the power which 3 is of 10 be added to the index of the power which 2 is of 10 the sum will be the index of the power which 6 is of 10.]

EXERCISE X.

Prove the truth of the following statements:—

1. $(2^3)^4 = (2^4)^3 = 2^{12}$.

2. $(3^3)^4 = (3^4)^3 = 3^{12}$.

3. $(5^3)^4 = (5^4)^3 = 5^{12}$.

4. $(2^3)^5 = (2^5)^3 = 2^{15}$.

5. $(2^4)^5 = (2^5)^4 = 2^{20}$.

6. $(7^3)^6 = (7^6)^3 = 7^{18}$.

7. $(10^3)^7 = (10^7)^3 = 10^{21}$.

8. $(8^{\frac{1}{3}})^6 = (8^6)^{\frac{1}{3}} = 8^2$.

9. $(64^{\frac{1}{2}})^{\frac{1}{3}} = (64^{\frac{1}{3}})^{\frac{1}{2}} = 64^{\frac{1}{6}}$.

10. $\left\{\left(\frac{1}{16}\right)^{\frac{1}{4}}\right\}^2 = \left\{\left(\frac{1}{16}\right)^2\right\}^{\frac{1}{4}} = \left(\frac{1}{16}\right)^{\frac{1}{2}}$.

11. $(2^3)^{-4} = (2^{-4})^3 = 2^{-12} = (2^{12})^{-1} = (2^{-1})^{12}$.

12. $(3^3)^{-4} = (3^{-4})^3 = 3^{-12} = (3^{12})^{-1} = (3^{-1})^{12}$.

13. $(5^3)^{-4} = (5^{-4})^3 = 5^{-12} = (5^{12})^{-1} = (5^{-1})^{12}$.

14. $(2^{-3})^5 = (2^5)^{-3} = 2^{-15} = (2^{15})^{-1} = (2^{-1})^{15}$.

15. $(2^4)^{-5} = (2^{-5})^4 = 2^{-20} = (2^{20})^{-1} = (2^{-1})^{20}$.

16. $(7^3)^{-6} = (7^{-6})^3 = 7^{-18} = (7^{18})^{-1} = (7^{18})^{-1}$.

17. $(10^{-3})^7 = (10^7)^{-3} = 10^{-21} = (10^{21})^{-1} = (10^{-1})^{21}$.

18. $(8^{\frac{1}{3}})^{-6} = (8^{-6})^{\frac{1}{3}} = (8^{-\frac{1}{3}})^6 = (8^6)^{-\frac{1}{3}} = 8^{-2}$.

19. $(64^{-\frac{1}{2}})^{\frac{1}{3}} = (64^{\frac{1}{3}})^{-\frac{1}{2}} = (64^{\frac{1}{2}})^{-\frac{1}{3}} = (64^{-\frac{1}{3}})^{\frac{1}{2}} = 64^{-\frac{1}{6}}$.

20. $\left\{\left(\frac{1}{16}\right)^{\frac{1}{4}}\right\}^{-2} = \left\{\left(\frac{1}{16}\right)^{-2}\right\}^{\frac{1}{4}} = \left\{\left(\frac{1}{16}\right)^{\frac{1}{2}}\right\}^{-1} = \left\{\left(\frac{1}{16}\right)^{-1}\right\}^{\frac{1}{2}} = \left(\frac{1}{16}\right)^{-\frac{1}{2}}$

21. $(2^{-3})^{-4} = (2^{-3})^{-4} = 2^{12}$.

22. $(3^{-3})^{-4} = (3^{-4})^{-3} = 3^{12}$.

23. $(5^{-3})^{-4} = (5^{-4})^{-3} = 5^{12}$.

24. $(2^{-3})^{-5} = (2^{-5})^{-3} = 2^{15}$.

25. $(2^{-4})^{-5} = (2^{-5})^{-4} = 2^{20}$.

26. $(7^{-3})^{-6} = (7^{-6})^{-3} = 7^{18}$.

27. $(10^{-3})^{-7}=(10^{-7})^{-3}=10^{21}$.

29. $(64^{-\frac{1}{2}})^{-\frac{1}{3}}=(64^{-\frac{1}{3}})^{-\frac{1}{2}}=64^{\frac{1}{6}}$.

28. $(8^{-\frac{1}{3}})^{-6}=(8^{-6})^{-\frac{1}{3}}=8^{2}$.

30. $\left\{\left(\frac{1}{16}\right)^{-\frac{1}{4}}\right\}^{-2}=\left\{\left(\frac{1}{16}\right)^{-2}\right\}^{-\frac{1}{4}}=\left(\frac{1}{16}\right)^{\frac{1}{2}}$.

31. $2^{3}\times2^{4}=2^{3+4}=2^{7}$.

32. $3^{3}\times3^{4}=3^{3+4}=3^{7}$.

33. $5^{3}\times5^{4}=5^{3+4}=5^{7}$.

34. $2^{3}\times2^{5}=2^{3+5}=2^{8}$.

35. $2^{4}\times2^{5}=2^{4+5}=2^{9}$.

36. $7^{3}\times7^{6}=7^{3+6}=7^{9}$.

37. $10^{3}\times10^{7}=10^{3+7}=10^{10}$.

38. $8^{\frac{1}{3}}\times8^{6}=8^{6+\frac{1}{3}}=8^{\frac{19}{3}}$.

39. $64^{\frac{1}{2}}\times64^{\frac{1}{3}}=64^{\frac{1}{2}+\frac{4}{3}}=64^{\frac{5}{6}}$.

40. $\left(\frac{1}{16}\right)^{\frac{1}{4}}\times\left(\frac{1}{16}\right)^{2}=\left(\frac{1}{16}\right)^{2\frac{1}{4}}=\left(\frac{1}{16}\right)^{\frac{9}{4}}$.

41. $2^{3}\div2^{4}=2^{3-4}=2^{-1}$.

42. $3^{3}\div3^{4}=3^{3-4}=3^{-1}$.

43. $5^{3}\div5^{4}=5^{3-4}=5^{-1}$.

44. $2^{5}\div2^{3}=2^{5-3}=2^{2}$.

45 $2^{5}\div2^{4}=2^{5-4}=2$.

46. $7^{3}\div7^{6}=7^{3-6}=7^{-3}$.

47. $10^{7}\div10^{3}=10^{7-3}=10^{4}$.

48. $8^{6}\div8^{\frac{1}{3}}=8^{6-\frac{1}{3}}=8^{\frac{17}{3}}$.

49. $64^{\frac{1}{2}}\div64^{\frac{1}{3}}=64^{\frac{1}{2}-\frac{1}{3}}=64^{\frac{1}{6}}$.

50. $\left(\frac{1}{16}\right)^{\frac{1}{4}}\div\left(\frac{1}{16}\right)^{2}=\left(\frac{1}{16}\right)^{\frac{1}{4}-2}=\left(\frac{1}{16}\right)^{-\frac{7}{4}}$.

51. $2^{3}\times2^{-4}=2^{3-4}=2^{-1}$.

52. $3^{3}\times3^{-4}=3^{3-4}=3^{-1}$.

53. $5^{3}\times5^{-4}=5^{3-4}=5^{-1}$.

54. $2^{5}\times2^{-3}=2^{5-3}=2^{2}$.

55. $2^{5}\times2^{-4}=2^{5-4}=2$.

56. $7^{3}\times7^{-6}=7^{3-6}=7^{-3}$.

57. $10^{7}\times10^{-3}=10^{7-3}=10^{4}$.

58. $8^{-\frac{1}{3}}\times8^{6}=8^{6-\frac{1}{3}}=8^{\frac{17}{3}}$.

59. $64^{\frac{1}{2}}\times64^{-\frac{1}{3}}=64^{\frac{1}{2}-\frac{1}{3}}=64^{\frac{1}{6}}$.

60. $\left(\frac{1}{16}\right)^{-2}\times\left(\frac{1}{16}\right)^{\frac{1}{4}}=\left(\frac{1}{16}\right)^{-2+\frac{1}{4}}=\left(\frac{1}{16}\right)^{-\frac{7}{4}}$.

61. $2^{3}\div2^{-4}=2^{3}\times2^{4}=2^{7}$.

62. $3^{3}\div3^{-4}=3^{3}\times3^{4}=3^{7}$.

63. $5^{3}\div5^{-4}=5^{3}\times5^{4}=5^{7}$.

64. $2^{3}\div2^{-5}=2^{3}\times2^{5}=2^{8}$.

65. $2^{4}\div2^{-5}=2^{4}\times2^{5}=2^{9}$.

66. $7^{3}\div7^{-6}=7^{3}\times7^{6}=7^{9}$.

67. $10^{3}\div10^{-7}=10^{3}\times10^{7}=10^{10}$.

68. $8^{6}\div8^{-\frac{1}{3}}=8^{6}\times8^{\frac{1}{3}}=8^{\frac{19}{3}}$.

69. $64^{\frac{1}{2}}\div64^{-\frac{1}{3}}=64^{\frac{1}{2}}\times64^{\frac{1}{3}}=64^{\frac{5}{6}}$.

70. $\left(\frac{1}{16}\right)^{\frac{1}{4}}\div\left(\frac{1}{16}\right)^{-2}=\left(\frac{1}{16}\right)^{\frac{1}{4}}\times\left(\frac{1}{16}\right)^{2}=\left(\frac{1}{16}\right)^{\frac{9}{4}}$.

71. $2^{-3}\times 2^{-4}=2^{-3-4}=2^{-7}$.

72. $3^{-3}\times 3^{-4}=3^{-3-4}=3^{-7}$.

73. $5^{-3}\times 5^{-4}=5^{-3-4}=5^{-7}$.

74. $2^{-5}\times 2^{-3}=2^{-5-3}=2^{-8}$.

75. $2^{-5}\times 2^{-4}=2^{-5-4}=2^{-9}$.

76. $7^{-3}\times 7^{-6}=7^{-3-6}=7^{-9}$.

77. $10^{-3}\times 10^{-7}=10^{-3-7}=10^{-10}$.

78. $8^{-\frac{1}{3}}\times 8^{-6}=8^{-\frac{1}{3}-6}=8^{-\frac{19}{3}}$.

79. $64^{-\frac{1}{2}}\times 64^{-\frac{1}{3}}=64^{-\frac{1}{2}-\frac{1}{3}}=64^{-\frac{5}{6}}$.

80. $\left(\frac{1}{16}\right)^{-\frac{1}{4}}\times\left(\frac{1}{16}\right)^{-2}=\left(\frac{1}{16}\right)^{-\frac{1}{4}-2}=\left(\frac{1}{16}\right)^{-\frac{9}{4}}$.

81. $2^{-3}\div 2^{-4}=2^{-3}\times 2^{4}=2^{4-3}=2$.

82. $3^{-3}\div 3^{-4}=3^{-3}\times 3^{4}=3^{4-3}=3$.

83. $5^{-3}\div 5^{-4}=5^{-3}\times 5^{4}=5^{-3+4}=5$.

84. $2^{-5}\div 2^{-3}=2^{-5}\times 2^{3}=2^{-5+3}=2^{-2}$.

85. $2^{-5}\div 2^{-4}=2^{-5}\times 2^{4}=2^{4}\div 2^{5}=2^{-1}$.

86. $7^{3}\div 7^{-6}=7^{-3}\times 7^{6}=7^{6-3}=7^{3}$.

87. $10^{-3}\div 10^{-7}=10^{-3}\times 10^{7}=10^{7-3}=10^{4}$.

88. $8^{-\frac{1}{3}}\div 8^{-6}=8^{-\frac{1}{3}}\times 8^{6}=8^{6}\div 8^{\frac{1}{3}}=8^{\frac{17}{3}}$.

89. $64^{-\frac{1}{2}}\div 64^{-\frac{1}{3}}=64^{-\frac{1}{2}}\times 64^{\frac{1}{3}}=64^{\frac{1}{3}}\div 64^{\frac{1}{2}}=64^{-\frac{1}{6}}$

90. $\left(\frac{1}{16}\right)^{-\frac{1}{4}}\div\left(\frac{1}{16}\right)^{-2}=\left(\frac{1}{16}\right)^{-\frac{1}{4}}\times\left(\frac{1}{16}\right)^{2}=\left(\frac{1}{16}\right)^{\frac{7}{4}}$.

91. $3^{2}\times 4^{2}=(3\times 4)^{2}=12^{2}$.

92. $3^{3}\times 4^{3}=(3\times 4)^{3}=12^{3}$.

93. $3^{5}\times 4^{5}=(3\times 4)^{5}=12^{5}$.

94. $5^{2}\times 3^{2}=(5\times 3)^{2}=15^{2}$.

95. $4^{2}\times 5^{2}=(4\times 5)^{2}=20^{2}$.

96. $3^{7}\times 6^{7}=(3\times 6)^{7}=18^{7}$.

97. $7^{10}\times 3^{10}=(7\times 3)^{10}=21^{10}$.

98. $\left(\frac{1}{3}\right)^{8}\times 6^{8}=\left(\frac{1}{3}\times 6\right)^{8}=2^{8}$.

99. $\left(\frac{1}{2}\right)^{5}\times\left(\frac{1}{3}\right)^{5}=\left(\frac{1}{2}\times\frac{1}{3}\right)^{5}=\left(\frac{1}{6}\right)^{5}$.

100. $2^{7}\times\left(\frac{1}{4}\right)^{7}=\left(2\times\frac{1}{4}\right)^{7}=\left(\frac{1}{2}\right)^{7}$.

101. $3^{2}\div 4^{2}=(3\div 4)^{2}=\left(\frac{3}{4}\right)^{2}$.

102. $3^{3}\div 4^{3}=(3\div 4)^{3}=\left(\frac{3}{4}\right)^{3}$.

103. $3^{5}\div 4^{5}=(3\div 4)^{5}=\left(\frac{3}{4}\right)^{5}$.

104. $5^{2}\div 3^{2}=\left(\frac{5}{3}\right)^{2}$.

105. $4^{2}\div 5^{2}=\left(\frac{4}{5}\right)^{2}$.

106. $3^{7}\div 6^{7}=(3\div 6)^{7}=\left(\frac{1}{2}\right)^{7}$.

107. $7^{10}\div 3^{10}=\left(\frac{7}{3}\right)^{10}$.

108. $\left(\frac{1}{3}\right)^{8}\div 6^{8}=\left(\frac{1}{3}\div 6\right)^{8}=\left(\frac{1}{18}\right)^{8}$

109. $\left(\frac{1}{2}\right)^{5}\div\left(\frac{1}{3}\right)^{5}=\left(\frac{1}{2}\div\frac{1}{3}\right)^{5}=\left(\frac{3}{2}\right)^{5}$

110. $2^{7}\div\left(\frac{1}{4}\right)^{7}=\left(2\div\frac{1}{4}\right)^{7}=8^{7}$.

111. $3^{-2}\times 4^{-2}=(3\times 4)^{-2}=12^{-2}$.

112. $3^{-3}\times 4^{-3}=12^{-3}$.

113. $3^{-5}\times 4^{-5}=12^{-5}$.

114. $5^{-2}\div 3^{-2}=\left(\frac{5}{3}\right)^{-2}$.

115. $4^{-2}\div 5^{-2}=\left(\frac{4}{5}\right)^{-2}$.

116. $3^{-7}\times 6^{-7}=(3\times 6)^{-7}=18^{-7}$.

117. $7^{10}\div 3^{-10}=7^{10}\times 3^{10}=21^{10}$.

118. $\left(\frac{1}{3}\right)^8 \times 6^{-8} = \left(\frac{1}{3}\right)^8 \div 6^8 = \left(\frac{1}{18}\right)^8 = 18^{-8}$.

119. $\left(\frac{1}{2}\right)^5 \times \left(\frac{1}{3}\right)^{-5} = \left(\frac{1}{2}\right)^5 \div \left(\frac{1}{3}\right)^5 = \left(\frac{3}{2}\right)^5 = \left(\frac{2}{3}\right)^{-5}$.

120. $2^{-7} \div \left(\frac{1}{4}\right)^7 = 2^{-7} \times \left(\frac{1}{4}\right)^{-7} = \left(\frac{2}{4}\right)^{-7} = \left(\frac{4}{2}\right)^7 = 2^7$.

Find, correct to six figures, the square roots of

2, 3, 5, 6, 7, 8, 10, 12, 15, 18, 20, 24, 27, 30, 35, 50

and the cube roots of

2, 3, 4, 5, 6, 10, 12, 15, 16, 24, 25, 135, 256,

and employing these roots and actually performing the multiplications indicated, prove that, (to five significant figures) :—

121. $\left(2^{\frac{1}{2}}\right)^3 = \left(2^3\right)^{\frac{1}{2}}, \equiv 2^{\frac{3}{2}}$

122. $\left(3^{\frac{1}{2}}\right)^3 = \left(3^3\right)^{\frac{1}{2}}, \equiv 3^{\frac{3}{2}}$.

123. $\left(5^{\frac{1}{2}}\right)^4 = \left(5^4\right)^{\frac{1}{2}}, \equiv 5^{\frac{4}{2}} = 5^2$.

124. $\left(7^{\frac{1}{2}}\right)^4 = \left(7^4\right)^{\frac{1}{2}}, \equiv 7^{\frac{4}{2}} = 7^2$.

125. $\left(3^{\frac{1}{3}}\right)^2 = \left(3^2\right)^{\frac{1}{3}}, \equiv 3^{\frac{2}{3}}$.

126. $\left(4^{\frac{1}{3}}\right)^2 = \left(4^2\right)^{\frac{1}{3}}, \equiv 4^{\frac{2}{3}}$.

127. $\left(5^{\frac{1}{3}}\right)^2 = \left(5^2\right)^{\frac{1}{3}}, \equiv 5^{\frac{2}{3}}$.

128. $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$.

129. $2^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 10^{\frac{1}{2}}$.

130. $3^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 15^{\frac{1}{2}}$.

131. $5^{\frac{1}{2}} \times 7^{\frac{1}{2}} = 35^{\frac{1}{2}}$.

132. $5^{\frac{1}{2}} \times 10^{\frac{1}{2}} = 50^{\frac{1}{2}}$.

133. $2^{\frac{1}{2}} \times 3^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 30^{\frac{1}{2}}$.

134. $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} = 4^{\frac{1}{3}}, \equiv 2^{\frac{2}{3}}$.

135. $2^{\frac{1}{3}} \times 3^{\frac{1}{3}} = 6^{\frac{1}{3}}$.

136. $2^{\frac{1}{3}} \times 4^{\frac{1}{3}} = 8^{\frac{1}{3}}$, *i. e.*, $2^{\frac{1}{3}} \times 2^{\frac{2}{3}} = 2^{\frac{3}{3}} = 2$.

137. $2^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 10^{\frac{1}{3}}$.

138. $3^{\frac{1}{3}} \times 4^{\frac{1}{3}} = 12^{\frac{1}{3}}$.

139. $3^{\frac{1}{3}} \times 5^{\frac{1}{3}} = 15^{\frac{1}{3}}$.

140. $8^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 2, \equiv 2\sqrt{2}$.

141. $12^{\frac{1}{2}} = 3^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 3^{\frac{1}{2}} \times 2, \equiv 2\sqrt{2}$

142. $18^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 9^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 3, \equiv 3\sqrt{2}$

143. $20^{\frac{1}{2}} = 5^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 5^{\frac{1}{2}} \times 2, \equiv 2\sqrt{5}$

144. $50^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 25^{\frac{1}{2}} = 2^{\frac{1}{2}} \times 5, \equiv 5\sqrt{2}$.

145. $24^{\frac{1}{2}} = 6^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 6^{\frac{1}{2}} \times 2, \equiv 2\sqrt{6}$

146. $27^{\frac{1}{2}} = 3^{\frac{3}{2}} = 3^{\frac{1}{2}} \times 3, \equiv 3\sqrt{3}$.

147. $16^{\frac{1}{3}} = \left(2^4\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} \times 2, \equiv 2\sqrt[3]{2}$

148. $24^{\frac{1}{3}} = 3^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 3^{\frac{1}{3}} \times 2, \equiv 2\sqrt[3]{3}$

149. $135^{\frac{1}{3}} = 5^{\frac{1}{3}} \times 27^{\frac{1}{3}} = 5^{\frac{1}{3}} \times 3, \equiv 3\sqrt[3]{5}$.

150. $256^{\frac{1}{3}} = \left(2^8\right)^{\frac{1}{3}} = 2^2 \times 2^{\frac{2}{3}}, \equiv 4\sqrt[3]{4}$.

122. If a number which is correct to but a few significant figures, be either very large or very small, it may in general be most conveniently written as the product of two factors, one factor being the number expressed by the significant figures with the decimal point between the first and second of them, the other factor being the power of 10 required to yield the proposed number as the product of the two factors. The exponent of 10 in the second factor is called the **characteristic** of the number to 10 as base.

Example 1. The sun's mass is 330,000 times that of the earth and its distance from the earth is about 91,400,000 miles; these numbers might be written $3{\cdot}3\times10^5$ and $9{\cdot}14\times10^7$ respectively. The characteristic of the first number is 5, that of the second is 7.

Example 2. The velocity of light is about 186,300 miles per second and the wave-length of green light is about ·0000208 of an inch. These quantities may be written $1{\cdot}863\times10^5$ miles per second and $2{\cdot}08\times10^{-5}$ inch, respectively. The characteristic of the wave-length number is *negative.*

Example 3. Find the cube of 15876 correct to five significant figures.

1	15876	1·5876	$\times10^4$
2	31752	1·5876	$\times10^4$
3	47628	1·5876	
4	63504	79380	
5	79380	12701	
6	95256	1111	
7	111132	95	
8	127008	2·52047	$\times10^8$
9	142884	1·5876	$\times10^4$
10	158760	3·1752	
		7938	
		318	
		6	
		1	
		4·0015	$\times10^{12}$

(See *Example 2*, § 94.)

Example 4. Find the weight (in Imperial tons of 2240lb. each) of the carbon in the carbonic acid gas in the atmosphere resting on a square mile of land when the pressure of the atmosphere is 14·73 lb. to the square inch, given (i), that each cubic foot of air contains ·00035 of a cubic foot of carbonic acid gas, correct to 2 significant figures; (ii), that the weight of any volume of carbonic acid gas is, to 3 significant figures, 1·52 times the weight of an equal volume of air under the same pressure and at the same temperature; (iii), that $\frac{3}{11}$ by weight of all carbonic acid gas is carbon, correct to 4 significant figures. (See *Example 4*, § 94.)

Wt. of air on sq. in. $= 1{\cdot}473\text{ lb.} \times 10.$

1 mi. $= 6{\cdot}336\text{ in.} \times 10^4.$

Wt. of air on sq. mi. $= 1{\cdot}473\text{ lb.} \times 6{\cdot}336^2 \times 10^9$

$= 5{\cdot}913\text{ lb.} \times 10^{10}.$

Wt. of carb. acid gas in this air $= 5.913\text{ lb.} \times 10^{10} \times 3{\cdot}5 \times 10^{-4} \times 1{\cdot}52$

$= 3{\cdot}15\text{ lb.} \times 10^7.$

Wt. of carbon in this gas $= 3{\cdot}15\text{ lb.} \times 10^7 \times 3 \div 11.$

$= 8{\cdot}6\text{ lb.} \times 10^6.$

$= 3{,}800$ T. *Imperial.*

EXERCISE XI.

What is the characteristic factor of :

1. 33240.

2. 7890000.

3. 29986000000.

4. ·0000335.

5. ·000000081.

6. 12756·78

Write in ordinary notation :

7. $1{\cdot}00074 \times 10^7$.

8. $1{\cdot}27418 \times 10^4$.

9. $2{\cdot}26 \times 10^{-4}$.

10. 6×10^{-9}.

11. $1{\cdot}0832 \times 10^{21}$.

12. $3{\cdot}04763 \times 10^{-5}$.

Find to five significant figures the value of :—

13. $9{\cdot}14 \times 10^7 \times 1{\cdot}60933 \times 10^3$.

14. $4{\cdot}73 \times 10^5 \times 1{\cdot}0089 \times 10^{-8}$.

15. $10^9 \div 1{\cdot}2759$.

16. $3{\cdot}98 \times 10^{20} \div (4.374 \times 10^{16})$.

17. $1{\cdot}863 \times 10^5 \times 6{\cdot}336 \times 10^4 \div (2{\cdot}08 \times 10^{-5})$.

18. $10^6 \div (981 \times 8{\cdot}837 \times 10^{-5})$.

19. $\cdot 33092 \times (6{\cdot}37 \times 10^6)^3$.

20. $(1{\cdot}27418 \times 10^4)^3 \times 3{\cdot}1416 \div 6$.

21. $1{\cdot}96 \times 10^{12} \times 283 \times (1{\cdot}22 \times 10^{-5})^2$.

22. $(1{\cdot}6 \times 10^6 \times 6{\cdot}3709 \times 10^4 \times 2)^{\frac{1}{2}}$.

23. $(2{\cdot}37 \times 10^{-8})^{\frac{1}{2}} \times 1{\cdot}4707 \times 10^8$.

24. $(6{\cdot}25 \times 10^{-11})^{\frac{1}{2}} \div (3{\cdot}1416 \times 3{\cdot}956 \times 10^3)$.

25. $(3{\cdot}003 \times 10^{-10})^{\frac{1}{3}}$.

26. $(4 \times 10^7)^{\frac{1}{3}} \div (4 \times 10^{-2})$.

27. $(1{\cdot}275678 \times 10^7 \times 1{\cdot}275584 \times 10^7 \times 1{\cdot}271278 \times 10^7)^{\frac{1}{3}}$.

LOGARITHMATION.

123. The **Logarithm** of a given number to a given base is the exponent of the power which the given number is of the given base. *The terms logarithm and exponent are* therefore *merely different names for the same thing.* Thus, instead of saying "the exponent of 100 to base 10 is 2" we say "the logarithm of 100 to base 10 is 2;" instead of saying "the exponent of 32 to base 2 is 5" we say "the logarithm of 32 to base 2 is 5;" and instead of writing $100=10^2$ and $32=2^5$ we may write $\log_{10} 100=2$ and $\log_2 32=5$. *If the base is 10 it is usually omitted both in writing and in reading logarithms.*

Examples.

$81=3^4$, $\log_3 81=4$. $10=10^1$, $\log 10=1$.

$125=5^3$, $\log_5 125=3$. $100=10^2$, $\log 100=2$.

$1024=2^{10}$, $\log_2 1024=10$. $1000=10^3$, $\log 1000=3$.

$2401=7^4$, $\log_7 2401=4$. $2=8^{\frac{1}{3}}$, $\log_8 2=\frac{1}{3}$.

$1331=11^3$, $\log_{11} 1331=3$. $4=8^{\frac{2}{3}}$, $\log_8 4=\frac{2}{3}$.

$27=9^{1\cdot5}$, $\log_9 27=1\cdot5$

$\log_5 1\cdot7<\frac{1}{3}$ but $\log_5 1\cdot71>\frac{1}{3}$

for $1\cdot7<5^{\frac{1}{3}}$ but $1\cdot71>5^{\frac{1}{3}}$

i. e., $1\cdot7^3<5$ but $1\cdot71^3>5$

for $1\cdot7^3=4\cdot913$ and $1\cdot71^3=5\cdot000211$.

EXERCISE XII.

Prove the truth of the following statements, and express them in logarithmic notation:—

1. $128=2^7$.
2. $256=4^4$.
3. $729=3^6$.
4. $729=9^3$.
5. $3,125=5^5$.
6. $7,776=6^5$.
7. $14,641=11^4$.
8. $1,000,000=10^6$.
9. $2=16^{\cdot25}$.
10. $4=16^{\cdot5}$.
11. $8=16^{\cdot75}$.
12. $32=16^{1\cdot25}$

13. $64 = 16^{1 \cdot 5}$

14. $1024 = 16^{2 \cdot 5}$

15. $125 = 25^{1 \cdot 5}$

16. $279936 = 36^{3 \cdot 5}$

17. $\frac{1}{64} = 2^{-6}$.

18. $\frac{1}{64} = 4^{-3}$.

19. $\frac{1}{64} = 8^{-2}$.

20. $\frac{1}{64} = 16^{-1 \cdot 5}$.

21. $\frac{1}{243} = 3^{-5}$.

22. $\frac{1}{243} = 9^{-2 \cdot 5}$.

23. $0.1 = 10^{-1}$.

24. $0.0001 = 10^{-4}$.

Prove the truth of the following statements and express them in exponential notation :—

25. $\log_2 8 = 3$.

26. $\log_4 64 = 3$.

27. $\log_8 512 = 3$.

28. $\log_7 343 = 3$.

29. $\log_3 2187 = 7$.

30. $\log_6 10077696 = 9$.

31. $\log_{12} 20736 = 4$.

32. $\log_{1 \cdot 6} 16.777216 = 6$.

33. $\log_8 1024 = 3\frac{1}{3}$.

34. $\log_{25} 5 = \cdot 5$.

35. $\log_{27} 9 = \frac{2}{3}$.

36. $\log_2 (\frac{1}{8}) = -3$.

37. $\log_2 (\frac{1}{32}) = -5$.

38. $\log_3 (\frac{1}{81}) = -4$.

39. $\log_7 (\frac{1}{2401}) = -4$.

40. $\log_8 (\frac{1}{1024}) = -3\frac{1}{3}$.

41. $\log 10 = 1$.

42. $\log 1000 = 3$.

43. $\log 100000 = 5$.

44. $\log 1 = 0$.

45. $\log 0 \cdot 1 = -1$.

46. $\log 0.01 = -2$.

47. $\log 0.001 = -3$.

48. $\log 0.00001 = -5$.

49. Prove that $\log 2 \cdot 154 < \frac{1}{3}$ but that $\log 2 \cdot 155 > \frac{1}{3}$.

50. Prove that log 2 is somewhat greater than ·3.

[**124.** The word logarithm means *ratio-number*, and logarithms were so named because they record the number of successive multiplications (or successive divisions) by a fixed base, a common ratio or rate of progression as it was at first called, the initial multiplicand (or initial dividend) being in every case 1.

Thus, 2 is the fixed base, the common rate of progression by multiplication, of the series of numbers

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024

and 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

are the corresponding logarithms recording the number of successive multiplications by the ratio 2.

The fixed base or common ratio of progression by multiplication is 3 in the series of numbers.

$\frac{1}{27}$, $\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9, 27, 81, 243, 729

and −3, −2, −1, 0, 1, 2, 3, 4, 5, 6

are the corresponding logarithms. The sign − preceding the first three of these logarithms denotes that successive divisions, not multiplications, are recorded.

The common rate of progression by multiplication is 10 in the series

	0·0001,	0·001,	0·01,	0·1,	1,	10,	100,	1000,	10000
and	−4,	−3,	−2,	−1,	0,	1,	2,	3,	4

are the corresponding logarithms.

If the fixed base or common rate of progression by multiplication be 16, and if the series of numbers be

	1,	2,	4,	8,	16,	32,	64,	128,	256,	512,	1024,	2048,	4096
then will	0,	·25,	·5,	·75,	1,	1·25,	1·5,	1·75,	2,	2·25,	2·5,	2·75,	3

be the corresponding logarithms. In this example, the numbers 2, 4, 8 have been interpolated between 1 and 16 the zeroth and first terms of the series to base 16, and the numbers 32, 64, 128 have been interpolated between 16 and 256, the first and second terms of the series to base 16.]

125. A logarithm being simply an exponent, the term logarithm may be substituted for the term exponent in Theorems XXVII and XXVIII, §119, which may then be expressed as follows :—

XXVIII (*a*). *The logarithm of a product is the aggregate of the logarithms of the factors.*

(*The logarithm of a quotient is the remainder resulting from subtracting the logarithm of the divisor from the logarithm of the dividend.*)

XXVII (*a*). *The logarithm of a power* (*or of a root*) *of a number is the product of the logarithm of the number and the exponent of the power* (*or of the root.*)

Example 1. $\log_2 8=3, \quad \log_2 32=5$;

$$\log_2 (8\times 32)=\log_2 256=8=3+5=\log_2 8+\log_2 32.$$

i.e. $8=2^3, \qquad 32=2^5$;

$$8\times 32=256=2^8=2^{3+5}=2^3\times 2^5.$$

Example 2. $\log_2 4=2$;

$$5\log_2 4=10=\log_2 1024=\log_2 (4^5).$$

i.e. $4=2^2$;

$$2^{10}=1024=2^{2\times 5}=4^5.$$

EXERCISE XIII.

Prove the truth of the following statements :—

1. $\log_2(16\times 128)=\log_2 16+\log_2 128.$

2. $\log_4(16\times 128)=\log_4 16+\log_4 128.$

3. $\log_8(16\times 128)=\log_8 16+\log_8 128.$

4. $\log_2(512\div 64)=\log_2 512-\log_2 64.$

5. $\log_4(512\div 64)=\log_4 512-\log_4 64.$

6. $\log_8(64\div 256)=\log_8 64-\log_8 256.$

7. $\log_8(2\times 32\div 4)=\log_8 2+\log_8 32-\log_8 4.$

8. $\log_3(27\times 243)=\log_3 27+\log_3 243.$

9. $\log_9(27\times 243)=\log_9 27+\log_9 243.$

10. $\log_9(27\div 243)=\log_9 27-\log_9 243.$

11. $\log_2 8^3=3\log_2 8.$

12. $\log_4 8^3=3\log_4 8.$

13. $\log_8 8^3=3\log_8 8.$

14. $\log_{16} 8^3=3\log_{16} 8.$

15. $\log_4 8^{\frac{1}{2}}=\frac{1}{2}\log_4 8.$

16. $\log_4 8^{\frac{1}{3}}=\frac{1}{3}\log_4 8.$

17. $\log_9 3^{\cdot 25}=\cdot 25\log_9 3.$

18. $\log_9 3^{\cdot 2}=\cdot 2\log_9 3.$

19. $\log_9 27^{\cdot 4}=\cdot 4\log_9 27.$

20. $\log_8 \cdot 25^{\cdot 7}=\cdot 7\log_8 \cdot 25.$

126. Logarithmation is the operation of finding the logarithm of a given number to a given base. It is, therefore, an inverse both of involution and of evolution; for in involution a base and an exponent are given and the power of the base denoted by the exponent is required, and in evolution a power of an unknown base and the exponent of that power are given and the unknown base is to be found, but in logarithmation there are given a base and a number considered as a power of that base and the exponent which denotes that power is to be determined. For example, involution and evolution would furnish answers to the questions 'What is the fourth power of 3?', 'What is the cube of the tenth root of 10?'; but logarithmation is required to answer the questions 'What power of 3 is 81?', 'What power of 10 is 2?'.

127. Thus of the seven fundamental operations of Arithmetic, addition and subtraction are each the inverse of the other; so also are multiplication and division inverse to each other, but the three remaining operations, viz., involution, evolution and logarithmation, are so related to one another that each has the other two operations as its inverses.

128. There are several methods of computing logarithms, but we shall give examples of only two of them. Of these, the first was one of the methods proposed by Napier the inventor of logarithms, and was the method by which the first published tables of logarithms to base 10 were calculated.

129. First or Napier's Method. Extract the square-root of the base correct to three figures more than the number of decimal places to which the logarithms are to be correct. Next extract the square-root of the root just found, then extract the square-root of this last-found root, and so continue until there has been formed a table similar to Table I which follows. In forming this table, 10 having been selected as the base, the roots were extracted to ten decimal places and eight decimal places retained.

TABLE A.

$10^{\frac{1}{2}}$	$=10^{\cdot 5}$	$=3{\cdot}16227766.$
$3{\cdot}16227766^{\frac{1}{2}}$	$=10^{\cdot 25}$	$=1{\cdot}77827941.$
$1{\cdot}77827941^{\frac{1}{2}}$	$=10^{\cdot 125}$	$=1{\cdot}33352143.$
$1{\cdot}33352143^{\frac{1}{2}}$	$=10^{\cdot 0625}$	$=1{\cdot}15478198.$
$1{\cdot}15478198^{\frac{1}{2}}$	$=10^{\cdot 03125}$	$=1{\cdot}07460783.$
$1{\cdot}07460783^{\frac{1}{2}}$	$=10^{\cdot 015625}$	$=1{\cdot}03663293.$
$1{\cdot}03663293^{\frac{1}{2}}$	$=10^{\cdot 0078125}$	$=1{\cdot}01815172.$
$1{\cdot}01815172^{\frac{1}{2}}$	$=10^{\cdot 00390625}$	$=1{\cdot}00903505.$
$1{\cdot}00903505^{\frac{1}{2}}$	$=10^{\cdot 001953125}$	$=1{\cdot}00450736.$
$1{\cdot}00450736^{\frac{1}{2}}$	$=10^{\cdot 000976563}$	$=1{\cdot}00225115.$
$1{\cdot}00225115^{\frac{1}{2}}$	$=10^{\cdot 000488281}$	$=1{\cdot}00112494.$
$1{\cdot}00112494^{\frac{1}{2}}$	$=10^{\cdot 000244141}$	$=1{\cdot}00056231.$
$1{\cdot}00056231^{\frac{1}{2}}$	$=10^{\cdot 000122070}$	$=1{\cdot}00028112.$
$1{\cdot}00028112^{\frac{1}{2}}$	$=10^{\cdot 000061035}$	$=1{\cdot}00014055.$

130. The exponents which would follow ·000061035 in order in the preceding Table, are obtained by taking $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c. of ·000061035, and the decimal parts of the corresponding powers, correct to eight decimal places, by taking $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, &c., of ·00014055, the decimal part of 1·00014055, the power of which ·000061035 is the exponent. Hence the logarithm to base 10 of any number greater than 1 but less than 1·00014055 is, to nine decimal places, $\frac{\cdot 000061035}{\cdot 00014055}$ or, to a closer approximation, $\frac{\cdot 000061035}{\cdot 000140545}$ of the decimal or fractional part of the number.

131. The fraction $\frac{\cdot 000061035}{\cdot 000140545}$ which is equal to ·434273+ is an approximation, correct to four decimal places, to a number called the **Modulus** of logarithms to base 10. If any number other than 10 had been made the base in Table A, a different number would have been obtained as the modulus; *e. g.*, had the base been 2·718281828459, the modulus would have been 1, *i. e.*, the logarithm to this base of any number greater than 1 but less than 1·0001 is simply the decimal part of the number, correct to eight or more decimal places. Had the roots in Table A been calculated to 32 decimal places, it would have been necessary to extend the columns to fifty-five terms before the decimal parts of the roots would be proportional to the exponents,* but in such case, the modulus would have been obtained correct to some eighteen decimal places. It has been computed to 136 decimal places †; to twelve places it is ·434294481903.

132. If the powers in the third column of Table A be considered as given numbers, the exponents in the second column of the Table will be their logarithms to base 10. In Table B which follows, the

* Such a table was actually computed by Henry Briggs, Savilian Professor of Geometry at Oxford. The fifty-fifth exponent or 2^{-54} he found to be

0·000,000,000,000,000,055,511,151,231,257,827

and the corresponding root, the result of fifty-four successive extractions of the square root, to be

1·000,000,000,000,000,127,819,149,320,032,35.

Briggs was the first to compute and publish logarithms to the base 10.

† By means of the series the earlier terms of which are given in problem 65, page 90. The modulus is the reciprocal of that series.

powers are tabulated as numbers and the exponents as the logarithms of these numbers.

TABLE B.

NUMBER.	LOGARITHM.	NUMBER.	LOGARITHM.
10	1	1·00903505	·00390625
3·16227766	·5	1·00450736	·001953125
1·77827941	·25	1·00225115	·000976563
1·33352143	·125	1·00112494	·000488281
1·15478198	·0625	1·00056231	·000244141
1·07460783	·03125	1·00028112	·000122070
1·03663293	·015625	1·00014055	·000061035
1·01815172	·0078125	1·0001	·000043427

TABLE C.

Multiples of the Modulus $\frac{61035}{140545}$

1. ·43427.	4. 1·73709.	7. 3·03992.
2. ·86855.	5. 2·17137.	8. 3·47419.
3. 1·30282.	6. 2·60564.	9. 3·90846.

Example 1. Find the logarithm of 2 correct to eight decimal places ; *i. e.*, find the exponent of the power to which 10 must be raised so that the result may be 2.

From the columns of Numbers in Table B, (Col. III, the column of powers in Table A,) select the largest number less than the given number 2, and divide 2 by the number thus selected. From the columns of Numbers in Table B select the largest number less than the quotient just obtained, and divide that quotient by this second selected number. From the columns of Numbers in Table B select the largest number less than the last obtained quotient and divide that quotient by this third selected number. Continue thus to select and divide until there is obtained a quotient less than 1·00014055. These operations resolve 2 into a series of factors all of which, except the last, are numbers in Table B. Consequently the logarithms of these factors, except that of the last factor, are given in Table B and the logarithm of the last factor can be obtained by multiplying the decimal part of the factor by the modulus ·43427. The logarithms of the factors being known, the logarithm of 2, their product, may be found, being the sum of the logarithms of the factors.

$$2 \div 1{\cdot}77827941 = 1{\cdot}12468265$$
$$1{\cdot}12468265 \div 1{\cdot}07460783 = 1{\cdot}04659823$$
$$1{\cdot}04659823 \div 1{\cdot}03663293 = 1{\cdot}00961314$$
$$1{\cdot}00961314 \div 1{\cdot}00903505 = 1{\cdot}00057292$$
$$1{\cdot}00057292 \div 1{\cdot}00056231 = 1{\cdot}00001060$$

$$\therefore 2 = 1{\cdot}77827941 \times 1{\cdot}07460783 \times 1{\cdot}03663293 \times 1{\cdot}00903505 \times 1{\cdot}00056231 \times 1{\cdot}00001060$$
$$= 10^{\cdot 25} \times 10^{\cdot 03125} \times 10^{\cdot 015625} \times 10^{\cdot 00390625} \times 10^{\cdot 000244141} \times 10^{\cdot 0000106 \times \cdot 43427}$$
$$= 10^{\cdot 25 + \cdot 03125 + \cdot 015625 + \cdot 00390625 + \cdot 000244141 + \cdot 000004604}$$
$$= 10^{\cdot 301029995}$$

or log 2 = ·30103000, correct to eight decimal places.

Written in logarithmic instead of in exponential notation, the latter part of the preceding calculation would be

$$\log 2 = \log 1{\cdot}77827941 + \log 1{\cdot}07460783 + \log 1{\cdot}03663293 + \log 1{\cdot}00903505 + \log 1{\cdot}00056231 + \log 1{\cdot}00001060$$
$$= \cdot 25 + \cdot 03125 + \cdot 015625 + \cdot 00390625 + \cdot 000244141 + \cdot 0000106 \times \cdot 43427$$
$$= \cdot 301029995,$$

∴ log 2 = 3·0103000, correct to eight decimal places.

[log 2 = ·301029995663981, correct to fifteen decimal places.]

Example 2. Find log 48847, correct to eight decimal places.

Write 48847 in the form $4{\cdot}8847 \times 10^4$ and resolve 4·8847 into factors selected from the columns of Numbers in Table B.

$$4{\cdot}8847 \div 3{\cdot}16227766 = 1{\cdot}54467777,$$
$$1{\cdot}54467777 \div 1{\cdot}33352143 = 1{\cdot}15834491$$

$$1\cdot15834491 \div 1\cdot15478198 = 1\cdot00308537$$
$$1\cdot00308537 \div 1\cdot00225115 = 1\cdot00083235$$
$$1\cdot00083235 \div 1\cdot00056231 = 1\cdot00026988$$
$$1\cdot00026988 \div 1\cdot00014055 = 1\cdot00012931\,;$$

$$\therefore\ 48847 = 10^4 \times 3\cdot16227766 \times 1\cdot33352143 \times 1\cdot15478198 \times 1\cdot00225115 \times 1\cdot00056231 \times 1\cdot00014055 \times 1\cdot00012931\,;$$

$$\therefore\ \log 48847 = \log 10^4 + \log 3\cdot16227766 + \log 1\cdot33352143 + \log 1\cdot15478198 + \log 1\cdot00225115 + \log 1\cdot00056231 + \log 1\cdot00014055 + \log 1\cdot00012931.$$
$$= 4 + \cdot5 + \cdot125 + \cdot0625 + \cdot000976563 + \cdot000244141 + \cdot000061035 + \cdot434273 \times \cdot00012931$$
$$= 4\cdot688837895,$$ correct to within 1 in the last figure.

$\therefore$ log 48847 = 4·68883790, correct to eight places of decimals.

133. The logarithm of any number may be found by this method independently of finding the logarithm of any other number, but in forming a table of logarithms, the logarithms of prime numbers alone need be computed, the logarithm of any composite number being the sum of the logarithms of the factors of such composite number and the logarithm of a power being the product of the logarithm of the base of the power and the exponent of the power. Thus knowing log 2 = ·3010300, we obtain $\log 4 = \log 2^2 = 2 \log 2 = \cdot6020600$, $\log 8 = \log 2^3 = 3 \log 2 = \cdot9030900$, &c.

134. The knowledge of the logarithm of one number will often greatly aid in computing the logarithm of another number which differs but little from the number whose logarithm is known.

Example 3. Find log 81 correct to eight decimal places, given log 80 = 1·903089987.

$$81 = 80 + 1 = 80 \times (1 + \tfrac{1}{80}) = 80 \times 1\cdot0125.$$

Resolve 1·0125 into factors selected from the columns of numbers in Table B.

$$1\cdot0125 \div 1\cdot00903505 = 1\cdot00343393$$
$$1\cdot00343393 \div 1\cdot00225115 = 1\cdot00118012$$
$$1\cdot00118012 \div 1\cdot00112494 = 1\cdot00005512$$

$$\therefore\ 81 = 80 \times 1\cdot00903505 \times 1\cdot00225115 \times 1\cdot00112494 \times 1\cdot00005512$$

$$\therefore\ \log 81 = \log 80 + \log 1\cdot00903505 + \log 1\cdot00225115 + \log 1\cdot00112494 + \log 1\cdot00005512$$
$$= 1\cdot903089987 + \cdot00390625 + \cdot000976563 + \cdot000488281 + \cdot43427 \times \cdot00005512$$
$$= 1\cdot908485018,$$ correct to within 1 in the last figure.

$\therefore$ log 81 = 1·90848502, correct to eight places of decimals.

From log 81 we may obtain log 3 for

$81 = 3^4, \therefore \log 81 = \log 3^4 = 4 \log 3.$

$\therefore$ 4 log 3 = 1·90848502,

$\therefore$ log 3 = ·47712125, correct to eight decimal places.

[log 3 = ·477121254719662, correct to 15 decimal places.]

This problem is virtually,—Find log 3, given log 2. We proceed thus ;—

$3-1=2$	and log 2 is given,
$3+1=4$	and log 4 = 2 log 2,
$\therefore\ 3^2-1=2\times4=8$	and log 8 = log 2 + log 4 ;
$3^2+1=10$	and log 10 = 1,
$\therefore\ 3^4-1=8\times10=80$	and log 80 = log 8 + log 10;

$\therefore\ 3^4=80+1=80\times(1+\frac{1}{80})=80\times1{\cdot}0125.$

The remainder of the calculation is that already given.

Example 4. Find log 7 given log 2 and log 3.

$7-1=6$	and log 6 = log 2 + log 3
$7+1=8$	and log 8 = 3 log 2.
$\therefore\ 7^2-1=48$	and log 48 = log 6 + log 8
$7^2+1=50$	and log 50 = log 100 − log 2
$\therefore\ 7^4-1=2400$	and log 2400 = log 48 + log 50

$\therefore\ 7^4=2400+1=2400\times(1+\frac{1}{2400})=2400\times1{\cdot}00041667.$

$1{\cdot}00041667\div1{\cdot}00028112=1{\cdot}00013551$

$\therefore\ 7^4=2400\times1{\cdot}00028112\times1{\cdot}00013551$

$\therefore\ \log 7^4=\log 2400+\log 1{\cdot}00028112+\log 1{\cdot}00013551.$

$\therefore\ 4\log 7=3{\cdot}380211242+{\cdot}000122070+{\cdot}43427\times{\cdot}00013551$

$=3{\cdot}380392160$

$\therefore\ \log 7={\cdot}845098040.$

[log 7 = ·845098040014257, correct to 15 decimal places.]

Example 5. Find log 11, given log 2, log 3 and log 7.

$99=11\times3^2 \quad \therefore\ \log 99=\log 11+2\log 3=\log 11+{\cdot}954242509.$

$99-1=98,$	$\log 98=\log 2+\log 49=\log 2+2\log 7,$
$99+1=100,$	$\log 100=2;$

$\therefore\quad 99^2-1=9800, \qquad \log 9800=2+\log 2+2\log 7;$

$\therefore\quad 99^2=9800+1=9800\times(1+\frac{1}{9800})=9800\times 1{\cdot}00010204;$

$\therefore\quad \log 99^2=\log 9800+\log 1{\cdot}00010204$

$\therefore\ 2\log 99=2+\log 2+2\log 7+{\cdot}434273\times{\cdot}00010204$

$=3{\cdot}991270389.$

$\therefore\quad \log 99=1{\cdot}995635195$

$\therefore\quad \log 11+{\cdot}954242509=1{\cdot}995635195$

$\therefore\quad \log 11=1{\cdot}995635195-{\cdot}954242509$

$=1{\cdot}041392686$, correct to eight places of decimals.

[$\log 11=1{\cdot}041392685158225$, correct to the 15th decimal.]

135. If the number to be resolved into factors selected from the columns of Numbers in Table B or any quotient arising in the course of its resolution be but very little less than one of the tabular factors, it will in general be better to use such number or such quotient as next divisor and the tabular factor next greater than it as dividend. The tabular factor then becomes a divisor, not a multiplier, in the resolved form of the given number.

Example 6. Find log 3·14159265, correct to eight decimal places.

$3{\cdot}16227766\div 3{\cdot}14159265=1{\cdot}00658424,$

$1{\cdot}00658424\div 1{\cdot}00450736=1{\cdot}00206756$

$1{\cdot}00225115\div 1{\cdot}00206756=1{\cdot}00018321$

$1{\cdot}00018321\div 1{\cdot}00014055=1{\cdot}00004266$

$\therefore 3{\cdot}14159265=3{\cdot}16227766\div 1{\cdot}00450736\div 1{\cdot}00225115\times 1{\cdot}00014055\times 1{\cdot}00004266$

$\therefore \log 3{\cdot}14159265=\log 3{\cdot}16227766-\log 1{\cdot}00450736-\log 1{\cdot}00225115+\log 1{\cdot}00014055+\log 1{\cdot}00004266$

$={\cdot}5-{\cdot}001953125-{\cdot}000976563+{\cdot}000061035+{\cdot}43427\times{\cdot}00004266$

$={\cdot}497149873$, correct to the last figure.

[$\log 3{\cdot}14159265={\cdot}497149872694134-$.]

Had 3·14159265 been resolved into a *product* of factors, as 2 was resolved in *Example 1* and 48847 in *Example 2*, no less than nine divisions would have been required to effect the resolution instead of the four divisions required in the resolution just given.

EXERCISE XIV.

Find, correct to 7 decimal places :—

1. log 1·00001.

2. log 1·00002.

3. log 1·00003.

4. log 1·00007.

5. log 1·000135.

6. log 1·0002497.

Find, correct to 4 decimal places :—

7. log 1·001.

8. log 1·0012.

9. log 1·0029.

10. log 3.

11. log 7.

12. log 2·718.

13. log 31, given log 32 = 1·50515.

14. log 13, given log 7 and log 11 and that $7\times11\times13=1001$.

15. log 17, given log 3 and log 7 and $3^5\times7=1701$.

Find, correct to 6 decimal places :—

16. log 7, given log 2 and log 3 and that $2^2\times3^6\times7^3=1000188$.

17. log 17, given log 2 and log 7 and that $7^6\times17=2000033$.

18. log 13, given log 2, log 3, log 7 and log 11 and that $123200=2^4\times7\times11\times10^2$ and $123201=3^6\times13^2$.

19. log 19, given log 2 and log 3 and that $19^2-1=2^2\times3^2\times10$.

20. log 19, given log 2 and log 3 and that $2\times3^6\times19^3=10000422$.

21. log 23, given log 2 and log 19 and that $23^3=190\cdot109375\times2^6$.

22. log 29, given log 2, log 3, log 7, log 11 and log 13 and that $96059600=2^2\times7^2\times13^2\times10^2\times29$, and $96059601=3^8\times11^4$.

23. log 41, given log 2, log 3 and log 13 and that $2^8\times3^6\times13^3=410012928$.

24. log 23, given log 2, log 3, log 7, log 11, log 13, and log 17 and that $23=\frac{1000000}{999999}\times\frac{2893400}{2893401}$.

25. log 2, given that $2^{196}=10^{59}$

$$\times\left(\frac{1025}{1024}\right)^5\times\left(\frac{1048576}{1048575}\right)^8\times\left(\frac{6560}{6561}\right)^3\times\left(\frac{15624}{15625}\right)^8\times\left(\frac{9801}{9800}\right)^4.$$

136. Second or Taylor's Method. This is a method of finding the convergent fractions to the logarithm of a given number. The following examples which are self-explanatory will easily enable one to understand the mode of procedure.

Example 1. Find log 2.

				1	<10	A
				2	>1	B
1	$\times 2$	$=2$	=	2	<10	$A\times B$
2	$\times 2$	$=2^{2}$	=	4	<10	$A\times B^{2}$
2^{2}	$\times 2$	$=2^{3}$	=	8	<10	$C=A\times B^{3}$
2^{3}	$\times 2$	$=2^{4}$	=	16	>10	$B\times C$
2^{4}	$\times 2^{3}$	$=2^{7}$	=	128	$>10^{2}$	$B\times C^{2}$
2^{7}	$\times 2^{3}$	$=2^{10}$	=	1024	$>10^{3}$	$D=B\times C^{3}$
2^{10}	$\times 2^{3}$	$=2^{13}$	=	8192	$<10^{4}$	$C\times D$
2^{13}	$\times 2^{10}$	$=2^{23}$	=	8388608	$<10^{7}$	$C\times D^{2}$
2^{23}	$\times 2^{10}$	$=2^{33}$	=	85898346. .	$<10^{10}$	$C\times D^{3}$
2^{33}	$\times 2^{10}$	$=2^{43}$	=	87960930.	$<10^{13}$	$C\times D^{4}$
2^{43}	$\times 2^{10}$	$=2^{53}$	=	90071992.	$<10^{16}$	$C\times D^{5}$
2^{53}	$\times 2^{10}$	$=2^{63}$	=	91209720.	$<10^{19}$	$C\times D^{6}$
2^{63}	$\times 2^{10}$	$=2^{73}$	=	93398754.	$<10^{22}$	$C\times D^{7}$
2^{73}	$\times 2^{10}$	$=2^{83}$	=	96714066.	$<10^{25}$	$C\times D^{8}$
2^{83}	$\times 2^{10}$	$=2^{93}$	=	99035203.	$<10^{28}$	$E=C\times D^{9}$
2^{93}	$\times 2^{10}$	$=2^{103}$	=	101412048.	$>10^{31}$	$D\times E$
2^{103}	$\times 2^{93}$	$=2^{196}$	=	100433628.	$>10^{59}$	$F=D\times E^{2}$
2^{196}	$\times 2^{93}$	$=2^{289}$	=	99464647.	$<10^{87}$	$E\times F$
2^{289}	$\times 2^{196}$	$=2^{485}$	=	99895954.	$<10^{146}$	$G=E\times F^{2}$
2^{485}	$\times 2^{196}$	$=2^{681}$	=	100329130.	$>10^{205}$	$F\times G$
2^{1651}	$\times 2^{485}$	$=2^{2136}$	=	100016289.	$>10^{643}$	$H=F\times G^{4}$
2^{11165}	$\times 2^{2136}$	$=2^{13301}$	=	99993628.	$<10^{4004}$	$J=G\times H^{6}$
2^{15437}	$\times 2^{13301}$	$=2^{28738}$	=	100003544.	$>10^{8651}$	$K=H\times J^{2}$
2^{28738}	$\times 2^{13301}$	$=2^{42039}$	=	99997172.	$<10^{12655}$	$L=J\times K$
2^{42029}	$\times 2^{28738}$	$=2^{70777}$	=	100000716.	$>10^{21306}$	$M=K\times L$
2^{183593}	$\times 2^{70777}$	$=2^{254370}$	=	99999320.	$<10^{76573}$	$N=L\times M^{2}$
2^{254370}	$\times 2^{70777}$	$=2^{325147}$	=	100000036	$>10^{97879}$	$P=M\times N$

Writing A in the form $2^0 < 10^1$ and B in the form $2^1 > 10^0$, we have

A	$=2^{0}$	$<10^{1}$	say $2<10^{\frac{1}{0}}$	or $\log 2<\frac{1}{0}$
B	$=2^{1}$	$>10^{0}$	$\therefore\ 2>10^{\frac{0}{1}}$	.. $\log 2>\frac{0}{1}$
$C=A\times B^{3}$	$=2^{3}$	$<10^{1}$	$\therefore\ 2<10^{\frac{1}{3}}$	.. $\log 2<\frac{1}{3}$
$D=B\times C^{3}$	$=2^{10}$	$>10^{3}$	$\therefore\ 2>10^{\frac{3}{10}}$	.. $\log 2>\frac{3}{10}$
$E=C\times D^{9}$	$=2^{93}$	$<10^{28}$	$\therefore\ 2<10^{\frac{28}{93}}$	.. $\log 2<\frac{28}{93}$
$F=D\times E^{2}$	$=2^{196}$	$>10^{59}$	$\therefore\ 2>10^{\frac{51}{196}}$	.. $\log 2>\frac{59}{196}$
$G=E\times F^{2}$	$=2^{485}$	$<10^{146}$	$\therefore\ 2<10^{\frac{146}{485}}$	.. $\log 2<\frac{146}{485}$
$H=F\times G^{4}$	$=2^{2136}$	$>10^{643}$	$\therefore\ 2>10^{\frac{643}{2136}}$	.. $\log 2>\frac{643}{2136}$
$J=G\times H^{6}$	$=2^{13301}$	$<10^{4004}$	$\therefore\ 2<10^{\frac{4004}{13301}}$	.. $\log 2<\frac{4004}{13301}$
$K=H\times J^{2}$	$=2^{28738}$	$>10^{8651}$	$\therefore\ 2>10^{\frac{8651}{28738}}$	.. $\log 2>\frac{8651}{28738}$
$L=J\times K$	$=2^{42039}$	$<10^{12655}$	$\therefore\ 2<10^{\frac{12655}{42039}}$	.. $\log 2<\frac{12655}{42039}$
$M=K\times L$	$=2^{70777}$	$>10^{21306}$	$\therefore\ 2>10^{\frac{21306}{70777}}$	.. $\log 2>\frac{21306}{70777}$
$N=L\times M^{3}$	$=2^{254370}$	$<10^{76573}$	$\therefore\ 2<10^{\frac{76573}{254370}}$	.. $\log 2<\frac{76573}{254370}$
$P=M\times N$	$=2^{325147}$	$>10^{97879}$	$\therefore\ 2>10^{\frac{97879}{325147}}$	.. $\log 2>\frac{97879}{325147}$

We have obtained the first twelve principal convergents to log 2 by keeping a record of the exponents of the powers of 2 and of 10 which are of approximately equal values, but there is no absolute necessity for the keeping of such a record. The convergents may be computed by assuming $\frac{1}{0}$ and $\frac{0}{1}$ as initial convergents, the second of these initials being the characteristic of 2 the given number to 10 the given base, and then taking as the convergent-quotients the exponents of the multipliers B, C, D, &c. in the second column of the above calculation. These exponents are, each of them, less by 1 than the number of successive multiplications required in the several cases to pass from > through < to > again or *vice versa;* thus they record without repetitions the number of such multiplications.

Quotients, 3, 3, 9, 2, 2, 4, 6, 2, 1, 1, 3, 1;

Convergents, $\frac{1}{0}$, $\frac{0}{1}$, $\frac{1}{3}$, $\frac{3}{10}$, $\frac{28}{93}$, $\frac{59}{196}$, $\frac{146}{485}$, $\frac{643}{2136}$, $\frac{4004}{13301}$, $\frac{8651}{28738}$, $\frac{12655}{42039}$, $\frac{21306}{70777}$, $\frac{76573}{254370}$, $\frac{97879}{325147}$.

The next quotient, the 13th. cannot be less than 1, and for 1 as 13th. quotient, the upper limit of error of the 12th. convergent is $\frac{1}{325147 \times (325147 + 254370)}$ which is $< \frac{1}{300000 \times 600000} < 6 \times 10^{-12}$; hence $\frac{97879}{325147}$ does not differ from log 2 by so much as 6 in the twelfth decimal place.

But the 11th. and 12th. convergents being close approximations to log 2, the required number, it is not necessary, in order to determine the 13th. quotient, to actually perform the multiplications which that quotient records. Consider for example how the fourth convergent quotient may be determined by the powers of 2 denoted by D and E, page 138. The fourth convergent-quotient is simply the number of successive multiplications of 1024, the D-power of 2, by 99035 . . . . ., the E-power of 2, which are required to produce the F-power of 2, and 1024 is approximately 10^3, 99035 . . . . . approximately 10^{28} and the F-power of 2 approximately an integral power of 10; the number of these multiplications will therefore be less than the quotient of $1024 \div 10^3 - 1$ divided by $1 - 99035 . . . \div 10^{28}$, *i. e.*, than $\cdot 024 \div \cdot 00965$, but will be approximately equal to this quotient. We may therefore use the integral part of $\cdot 024 \div \cdot 00965$ as a convergent quotient to form the fourth or F-convergent; and in point of fact the integral part of $\cdot 024 \div \cdot 00965$ is 2, the fourth convergent quotient. The correctness of the foregoing argument may be seen at once, if the proper method of multiplying by 99035 . . . . be adopted, viz., that described in § 69, xii, page 36. It should however be noticed that if the terms of the division, here $\cdot 024 \div \cdot 00965$, are not both very small the convergent-quotient sought may be *greater* than the quotient arising from the division. For example had we sought to determine the third convergent from $(1 - \cdot 8) \div (1 \cdot 024 - 1)$ we would have obtained 8 as the third convergent-quotient instead of 9 the correct value.

In like manner from the powers of 2 and 10 yielding any two consecutive convergents after the fourth, the quotient determining the third consecutive convergent may be obtained, and consequently the 13th. convergent may be computed from the powers of 2 and 10 yielding the 11th. and 12th. convergents. Thus the dividend obtained from N is $1-\cdot 99999320\ldots = \cdot 00000680$

the divisor obtained from P is $1\cdot 00000036\ldots - 1 = \cdot 00000036$

$\therefore$ the quotient is $680 \div 36 = 18+$

$\therefore$ the 13th. convergent to log 2 is $\dfrac{97879 \times 18 + 76573}{325147 \times 18 + 254370} = \dfrac{1838335}{6107016}$

An upper limit of error for this convergent is

$$\frac{1}{6107016 \times (6107016 + 325147)}$$

which is $< \dfrac{1}{6000000 \times 6000000} < 3 \times 10^{-14}$.

$\therefore \log 2 = \dfrac{1838335}{6107016} = \cdot 3010299956634$, correct to 13 decimal places.

Example 2. Find log 3.

Powers of 3.	Multipliers producing the next power.
1	
3	
3	3
9	3
27	9
243	9
2187	9
19683	9
177147	9
1594323	9
1434891.	9
1291402..	9
1162261...	9
1046035...	9
941432...	1046035,..
984771...	1046035...
1030105...	984771...
1014418...	984771...
998969...	1014418...
1013...	

The multiplier 3 occurs twice, 9 occurs ten times, the others twice, twice and once respectively; hence the first five convergent quotients to log 3 are 2, 10, 2, 2 and 1, and the sixth quotient will be the integral part of $(1\cdot 014418 - 1) \div (1 - \cdot 998969) = 14418 \div 1031 = 13\cdot 9+$, which is 13. The characteristic of 3 to base 10 is 0, therefore the initial convergents are $\dfrac{1}{0}$ and $\dfrac{0}{1}$; hence we have for log 3

Quotients; $\quad 2, \quad 10, \quad 2, \quad 2, \quad 1, \quad 13$

Convergents; $\frac{1}{0}, \frac{0}{1}, \frac{1}{2}, \frac{10}{21}, \frac{21}{44}, \frac{52}{109}, \frac{73}{153}, \frac{1001}{2098};$

or $\log 3 = \frac{1001}{2098} = {\cdot}477121$, correct to six decimal places, for the error of this convergent is $< \frac{1}{2098 \times (2098+153)} < \frac{1}{4000000} < 3 \times 10^{-7}$

Had 13·9 been used instead of 13 as sixth convergent quotient, the resulting convergent would have been

$$\frac{73 \times 13{\cdot}9 + 52}{153 \times 13{\cdot}9 + 109} = \frac{10667}{22357} = {\cdot}47712126-$$

which is a closer approximation than even $\frac{1001}{2098}$.

Example 3. Find log 48847.

Powers of 48847.	Multipliers producing the next power.
1	
48847	
48847	48847
2386029 . . .	48847
1165504 . . .	48847
569314 . . .	1165504 . . .
663537 . . .	. .
773355 . . .	. .
901348 . . .	. .
1050525 . . .	901348 . . .
946889 . . .	1050525 . . .
994730 . . .	. .
104	

The first five convergent quotients are 1, 2, 4, 1, 2, and the sixth is 9, the integral part of $(1{\cdot}050525 - 1) \div (1 - {\cdot}994730) = 50525 \div 5270$. The characteristic of 48847 to base 10 is 4, and therefore the initial convergents are $\frac{1}{0}$ and $\frac{4}{1}$.

Quotients; $\quad 1, \quad 2, \quad 4, \quad 1, \quad 2, \quad 9$

Convergents; $\frac{1}{0}, \frac{4}{1}, \frac{5}{1}, \frac{14}{3}, \frac{61}{13}, \frac{75}{16}, \frac{211}{45}, \frac{1974}{421}$.

The error of the last of these convergents is

$$< \frac{1}{421 \times (421+45)} < \frac{1}{180000} < 6 \times 10^{-6}$$

$\therefore \quad \log 48847 = \frac{1974}{421} = 4{\cdot}68884-$, correct to five decimal places.

EXERCISE XV.

Obtain, correct to 4 decimal places :—

1. log 7. **3.** log 31. **5.** log 2·72.
2. log 6. **4.** log 6·6. **6.** log 1·371.

137. Many other methods of calculating logarithms have been proposed, the greater number of them being merely variations of one or other of the two processes already described, but all of these methods are so tedious and involve so much labor in their application that were it necessary to calculate a logarithm anew every time it was required, computation by the aid of logarithms would be a useless curiosity. To overcome this objection to their employment, the logarithms of all integral numbers from 1 to 200,000 have been calculated and recorded to seven places of decimals, once for all. A small part of this record, being a Table of Logarithms correct to six decimal places, is given at the end of this volume. In Table I are entered the logarithms to base 10 of all numbers from 1 to 100, in Table II are given the logarithms to base 10 of all numbers from 1·000 to 9·999 by increments of ·001, and Table III contains the logarithms to base 10 of all numbers from 1 to 1·0999 by increments of ·0001. The logarithms entered in Tables II and III are all decimals, but in printing the tables the decimal point has been omitted as unnecessary. *The decimal part of a logarithm is termed the* **mantissa** *of the logarithm, and the integral part, the* **characteristic** *of the logarithm.* (See § 122.)

138. The following examples will show how to use Tables II and III either to find the logarithm of a given number or to find the number corresponding to a given logarithm.

Example 1. Find log 4·884, log 48840 and log ·04884.

We glance along the columns marked **N°.** until we find 488, the first three digits of the given number; we then pass horizontally along the line of 488 to the column headed **4**, the fourth digit of the given number; in that column we find 8776, these are the last four figures of the mantissa of the required logarithm. The first two or leading figures are 68, they will be found standing over the blank space which appears in the line of 488 in the column headed **O**, Hence $\log 4{\cdot}884 = {\cdot}688776.$ (*A.*)

$$48840 = 4{\cdot}884 \times 10^4$$

$$\therefore \quad \log 48840 = \log 4{\cdot}884 + \log 10^4$$

$$= {\cdot}668776 + 4 = 4{\cdot}688776. \qquad (B.)$$

$$\cdot04884 = 4{\cdot}884 \times 10^{-2}$$

$$\therefore \quad \log {\cdot}04884 = \log 4{\cdot}884 + \log 10^{-2}$$

$$= {\cdot}688776 - 2 = \bar{2}{\cdot}688776. \qquad (C.)$$

It will be noticed that when the characteristic is negative, as it is in (*C*), the minus sign is written *above* the characteristic, not in front of it. The mantissa in (*C*) is positive, being the logarithm of the factor 4·884.

Example 2. Find log 4·076, log 407·6, log 40760, and log·0004076.

We first find 407 in the columns marked **N°.** and then run horizontally across to the column headed **6** in which we find *0234, the last four figures of the logarithm sought. The * in front of these figures indicates that the two leading figures of the logarithm are *at the foot* of the blank space in the column headed **O.** Looking there we find the leading figures to be 61, hence

$$\log 4{\cdot}076 = {\cdot}610234.$$

$$407{\cdot}6 = 4{\cdot}076 \times 10^2,$$

$$\therefore \quad \log 407{\cdot}6 = {\cdot}610234 + 2 = 2{\cdot}610234.$$

$$40760 = 4{\cdot}076 \times 10^4,$$

$$\therefore \quad \log 40760 = {\cdot}610234 + 4 = 4{\cdot}610234.$$

$$\cdot0004076 = 4{\cdot}076 \times 10^{-4},$$

$$\therefore \quad \log {\cdot}0004076 = {\cdot}610234 - 4 = \bar{4}{\cdot}610234.$$

139. It may be seen from these examples that changing the position of the decimal point in a number changes the characteristic but does not change the mantissa of the logarithm of the number.

The characteristic of the logarithm of a given number may and should be written down before the mantissa is found in the Table of Logs., for the characteristic to base 10 is simply the number of places which the first significant figure of the given number is from the ones' figure of the number, the ones' figure itself not being counted, *i. e.*, it is considered as standing in the zeroth place. If the first significant figure of the given number stands to the left of the decimal point, the characteristic will be positive or zero; if it stands to the right of the decimal point, the characteristic will be negative.

EXERCISE XVI.

Find from Table I the logarithm of :—

1. 7. **2.** 37. **3.** 59. **4.** 79. **5.** 97.

Write down the characteristic of :—

6. 723. **7.** ·0723. **8.** 200000. **9.** 0·00002. **10.** 372·58.

Find the logarithm of :—

11. 23.	**16.** 4·321.	**21.** 7246.	**26.** $1·178 \times 10^{14}$.
12. 230.	**17.** 4321.	**22.** ·007246.	**27.** $1·63 \times 10^{42}$.
13. 2300.	**18.** ·04321.	**23.** 676700.	**28.** $6·48 \times 10^{-8}$.
14. 2·3.	**19.** 6789.	**24.** 67·67.	**29.** $4·496 \times 10^{-6}$.
15. ·023.	**20.** 7810.	**25.** 200·2.	**30.** $7·604 \times 10^{-5}$.

140. *Example 3.* Find log 4·8847.

Log 4·8847 is not given in the Tables but evidently it lies in value between log 4·884 and log 4·885 both of which are given and it may be computed from these tabular logarithms as follows :

$$\begin{aligned} \log 4·885 &= ·688865 \\ \log 4·884 &= ·688776 \\ \therefore \quad \log 4·885 - \log 4·884 &= ·000089 \\ ·7 \text{ of } ·000089 &= ·000062 \\ \therefore \quad \log 4·8847 &= ·688776 + ·000062 = ·688838 \end{aligned}$$

We need not have actually subtracted log 4·884 from log 4·885 to obtain the difference ·000089, for this difference is recorded in the right-hand column of the Table, the column headed **D**. Thus if we glance along the horizontal line in which we find log 4·884, we shall find the 'difference' 89 in the column **D**, and the computation of log 4·8847 will then appear as follows :—

$$\begin{aligned} \log 4·884 \quad &= ·688776 \qquad D = 89· \\ 7, \quad &\quad\;\; 62{,}3 \\ \hline \log 4·8847 &= 688838 \end{aligned}$$

141. This method of computing the logarithm of a number intermediate in value to two tabular numbers is merely a variation of the method of calculating logarithms exhibited in *Examples 3, 4,* and *5* of § 134, for

$$4·885 = 4·884 + ·001 = 4·884 \times \left(1 + \tfrac{·001}{4·884}\right)$$

$$\begin{aligned} \therefore \quad \log 4·885 &= \log 4·884 + \log\left(1 + \tfrac{1}{4884}\right) \\ &= ·688776 + \tfrac{1}{4884} \text{ of } ·4343 \\ &= ·688776 + ·000089 \qquad (A.) \\ &= ·688865, \quad \text{as given in Table II.} \end{aligned}$$

$$4{\cdot}8847 = 4{\cdot}8840 + {\cdot}0007 = 4{\cdot}884 \times (1 + \tfrac{\cdot 0007}{4\cdot 884})$$

$$\therefore \quad \log 4{\cdot}8847 = \log 4{\cdot}884 + \log (1 + \tfrac{\cdot 7}{4884})$$
$$= {\cdot}688776 + \tfrac{\cdot 7}{4884} \text{ of } {\cdot}4343$$
$$= {\cdot}688776 + {\cdot}7 \text{ of } \tfrac{1}{4884} \text{ of } {\cdot}4343$$
$$= {\cdot}688776 + {\cdot}7 \text{ of } {\cdot}000089. \qquad (B.)$$
$$= {\cdot}688776 + {\cdot}000062$$
$$= {\cdot}688838, \text{ as found in } \textit{Example 3}.$$

Now the 'difference' ·000089 obtained above in (*A*) is given in the Table of Logarithms, and knowing this difference and log 4·884, we may at once write down the line marked (*B*).

Example 4. Find log 2·718282.

log 2·718	= ·434249	D = 160
2	32,0	
8	12,80	
2	,320	

∴ log 2·718282 = ·434294.

(See § 13.)

EXERCISE XVII.

Find the logarithm of :—

1. 7·3254.	**6.** 676767.	**11.** $6{\cdot}37839 \times 10^{8}$.
2. 595·12.	**7.** ·186825.	**12.** $6{\cdot}35639 \times 10^{8}$.
3. 47763.	**8.** 80008.	**13.** $1{\cdot}0832 \times 10^{27}$.
4. ·0049056.	**9.** ·00457009.	**14.** $4{\cdot}30725 \times 10^{-7}$.
5. 295·947.	**10.** 30033000.	**15.** $3{\cdot}04763 \times 10^{-5}$.

142. To find the number corresponding to a given logarithm, we simply reverse the process of finding the logarithm of a given number.

Example 1. Find the number of which ·656769 is the logarithm.

We look in Table II along the columns headed O till we find 65, the two leading figures of ·656769, the mantissa of the given logarithm, and in the columns between the line led by 65 and that led by 66 we look for 6769, the remaining figures of the given mantissa. We find these four figures in the line of the number 453 and in the column headed 7, hence

·656769 = log 4·537.

Had the given logarithm been 3·650769, we should have found the number 4·537 by means of the mantissa and then have moved the decimal point three places farther to the right as indicated by the characteristic 3, thus obtaining

3·656769 = log 4537.

In like manner may be found

5·656769 = log 453700.
$\bar{4}$·656769 = log ·0004537.

Example 2. Of what number is ·497150 the logarithm?

On looking for ·497150 among the logarithms of Table II we cannot find this mantissa, we therefore take out the logarithm next smaller than ·497150 the given mantissa, and also the number corresponding to the logarithm taken out. This gives us

·497068 = log 3·141.

Then subtracting ·497068, the tabular logarithm from ·497150, the given mantissa, we obtain ·000082 as difference. From the column of differences we find that log 3·142 − log 3·141 = ·000138, *i. e.*, a difference of ·000138 in the logarithms makes a difference of ·001 in the corresponding numbers, hence a difference of ·000082 in the logarithms will make a difference of $\frac{\cdot 000082}{\cdot 000138}$ of ·001 = $\frac{82}{138}$ of ·001 = ·00059 in the corresponding numbers, hence ·497068 + ·000082 = log (3·141 + ·00059), *i.e.*, ·497150 = log 3·14159.

The actual calculation will appear as follows:

$$\begin{array}{rl}
\cdot 497150 & \\
068 & = \log 3\cdot 141 \\
\hline
138)820 & 5 \\
1300 & 9 \\
58 & \\
\hline
\cdot 497150 & = \log 3\cdot 14159
\end{array}$$

The division is performed by the method exhibited in *Example 1*, §, 67, page 33. It is not carried farther than the quotient 9 because the 'remainder' 58 is practically within the limit of error of the tabular logarithm ·497068, which is correct to but 6 figures and which represents all logarithms from ·49706750 to ·49706849; consequently the 'remainder' 58 may be too small by 50 or too large by 49. In the former case, the figure next following 9 in 3·14159, would be 6, in the latter case it would, to nearest approximation, be 1; it is therefore indeterminate with the tables at our command and consequently we omit it, ending our computation with 9.

143. The process of computing the logarithm of a number intermediate in value to two tabular numbers or inversely of computing the number corresponding to a logarithm intermediate in value to two tabular logarithms is termed **Interpolation** of logarithms or of numbers as the case may be.

144. In the early part of Table II, the differences between consecutive logarithms are comparatively large and they change rapidly; as a consequence interpolation will not in this part of the Table yield accurate results. This difficulty may however be avoided by employing Table III for all numbers and logarithms within its range. The method of using this Table is the same as that of using Table II.

EXERCISE XVIII.

Find the numbers corresponding to the following logarithms :—

1. ·480007.	**6.** ·817342.	**11.** 2·830083.	**16.** ·000000.
2. ·734960.	**7.** 1·817342.	**12.** 4·830457.	**17.** 2·000204.
3. ·740047.	**8.** 5·817342.	**13.** $\bar{3}$·900000.	**18.** ·301030.
4. 2·477121.	**9.** $\bar{1}$·817342.	**14.** 3·301000.	**19.** 3·010300.
5. ·937700.	**10.** $\bar{5}$·817342.	**15.** $\bar{1}$·500005.	**20.** ·030103.

Find the characteristic-factor and cofactor of the numbers corresponding to the following logarithms :—

21. 11·716671. **22.** $\bar{5}$·534626. **23.** $\overline{10}$·817037. **24.** 14·660000.
25. $\overline{100}$·000123.

COMPUTATION BY HELP OF LOGARITHMS.

145. The use of logarithms in lessening the labor of computation depends on the theorems numbered xxvii (a) and xxviii (a) of § 125. These may be restated as follows :—

A. *The logarithm of a product is the aggregate of the logarithms of the factors of the product.*

B. *The logarithm of a quotient is the remainder resulting from the subtraction of the logarithm of the divisor from the logarithm of the dividend.*

C. *The logarithm of a power is the product of the exponent of the power and the logarithm of the base of the power.*

D. *The logarithm of a root of a given number is the quotient arising from the division of the logarithm of the given number by the root-index of the required root.*

As the root-index is the reciprocal of the exponent of the root considered as a fractional power, dividing by the root-index is equivalent to multiplying by the exponent, hence **D.** is comprehended under **C.**

146. The following examples will show how these theorems are applied to facilitate computation.

Example 1. Find the weight in tons of a rectangular block of stone measuring $7{\cdot}413' \times 5{\cdot}822' \times 3{\cdot}224'$ and weighing ·09722 lb. per cubic inch.

Volume of block $= 7{\cdot}413 \times 5{\cdot}822 \times 3{\cdot}224$ cu. ft.
$= 7{\cdot}413 \times 5{\cdot}822 \times 3{\cdot}224 \times 1728$ cu. in.

∴ Weight of block $= 7{\cdot}413 \times 5{\cdot}822 \times 3{\cdot}224 \times 1728 \times .09722$ lb.
$= (7{\cdot}413 \times 5{\cdot}822 \times 3{\cdot}224 \times 1728 \times {\cdot}09722 \div 2000)$ T.

log 7·413	= ·869994		
log 5·822	= ·765072		
log 3·224	= ·508395		
log 1728	= 3·237544		
log·09722	= $\bar{2}{\cdot}987756$		
	4·368761		
log 2000	3·301030		
log 11·6877	= 1·067731		
	443	= log 11·68	
	372)288		7
	276		7
	156		

∴ Wt. of block $= 11{\cdot}6877$ Tons.

It will be observed that *adding* the $\bar{2}$ (the negative 2) of $\bar{2}{\cdot}987756$ is equivalent to *subtracting* 2. This is merely another way of saying "*Multiplying* by ·01 is equivalent to *dividing* by 100." The mantissa ·987756 is not negative and is therefore added like the other mantissæ.

Example 2. Find the continued product of 3·9497, 3·9634 and 3·1416, correct to five significant figures.

$$\begin{array}{lrl} \log 3\cdot949 & = & \cdot596487 \\ 7 & & 77 \\ \log 3\cdot963 & = & \cdot598024 \\ 4 & & 44 \\ \log 3\cdot141 & = & \cdot497068 \\ 6 & & 83 \\ \hline \log 49\cdot1793 & & 1\cdot691783 \\ & & \underline{\ldots.700} \quad = \log 49\cdot17 \\ & & 89)83 \qquad\qquad 9 \\ & & \quad 29 \qquad\qquad 3 \\ & & \quad 23 \end{array}$$

$\therefore\ 3\cdot9497 \times 3\cdot9634 \times 3\cdot1416 = 49\cdot179+.$

Example 3. Divide $3\cdot728 \times \cdot4837$ by $\cdot02383 \times \cdot09769$.

$$\begin{array}{rl} \log 3\cdot728 = & \cdot571476 \\ \log \cdot4837 = & \bar{1}\cdot684576 \\ \hline & \cdot256052 \\ & \bar{3}\cdot366974 \\ \hline & 2\cdot889078 = \log 774\cdot6 \end{array} \qquad \begin{array}{rl} \log \cdot02385 = & \bar{2}\cdot377124 \\ \log \cdot09769 = & \bar{2}\cdot989850 \\ \hline & \bar{3}\cdot366974 \end{array}$$

$\therefore\ 3\cdot728 \times \cdot4837 \div (\cdot02383 \times \cdot09769) = 774\cdot6.$

$\cdot571476 + \bar{1}\cdot684576 = \cdot571476 + \cdot684576 - 1 = 1\cdot256052 - 1 = \cdot256052.$

$\bar{2}\cdot377124 + \bar{2}\cdot989850 = 1\cdot366974 - 4 = \cdot366974 - 3 = \bar{3}\cdot366974.$

$\cdot256052 - \bar{3}\cdot366974 = \cdot256052 + 3 - \cdot366974 = 3\cdot256052 - \cdot366974 = 2\cdot889078.$

Notice that *adding* $\bar{1}$ is changed into *subtracting* 1, and that *subtracting* $\bar{3}$ is changed into *adding* 3.

Example 4. Find (*a*) the 77th. power of 74·13 and (*b*) the 110th. power of $\frac{4205}{7413}$.

(*a*) $\log 74\cdot13 = 1\cdot869994$ (Multiply log 74·13 by 77.)

$$\begin{array}{r} 1\cdot869994 \\ 77 \\ \hline 13\cdot089958 \end{array}$$

$\log 74\cdot13^{77} = 143\cdot989538 = 143 + \log 9\cdot762,$

$\therefore\ 74\cdot13^{77} = 9\cdot762 \times 10^{143}.$

(*b*)
$$\begin{array}{lcr} \log 4205 & = & 3\cdot623766 \\ \log 7413 & = & 3\cdot869994 \\ \hline & & \bar{1}\cdot753772 \\ & & 110 \\ \hline & & \overline{28}\cdot914920 = \log 8\cdot2209 - 28 \end{array}$$

$$\therefore\quad \left(\frac{4502}{7413}\right)^{110} = 8{\cdot}2209\times10^{-28};\qquad 8{\cdot}2209\div10^{28}.$$

(log 4205 – log 7413) × 110 = (·753772 – 1) × 110 = 82·914920 – 110 = $\overline{28}$ 914920.

Example 5. Find (*a*) the 11th. root of 35480, (*b*) the 7th. root of ·00075367, and (*c*) the 7th. power of the 11th. root of ·7576.

(*a*) $\log 35480^{\frac{1}{11}} = \frac{1}{11}$ of $\log 35480 = \frac{1}{11}$ of $4{\cdot}549984 = {\cdot}413635$
$= \log 2{\cdot}592,$

$\therefore\quad 35480^{\frac{1}{11}} = 2{\cdot}592.$

(*b*) $\log{\cdot}00075367^{\frac{1}{7}} = \frac{1}{7}$ of $\log{\cdot}00075367 = \frac{1}{7}$ of $\overline{4}{\cdot}877181$
$= \frac{1}{7}$ of $(\overline{7} + 3{\cdot}877181) = \overline{1} + {\cdot}553883 = \overline{1}{\cdot}553883 = \log{\cdot}358,$

$\therefore\quad {\cdot}00075367^{\frac{1}{7}} = {\cdot}358.$

(*c*) $\log{\cdot}7576^{\frac{7}{11}} = \frac{7}{11}$ of $\log{\cdot}7576 = \frac{7}{11}$ of $\overline{1}{\cdot}879440 = \frac{7}{11}$ of
$(\overline{11} + 10{\cdot}879440) = \overline{7} + 6{\cdot}923280 = \overline{1}\,{\cdot}923280 = \log{\cdot}83807.$

$\therefore\quad {\cdot}7576^{\frac{7}{11}} = {\cdot}83807.$

In (*b*) we do not at once divide the characteristic $\overline{4}$ by 7 the root-index, for this would introduce a negative fraction into the quotient, in addition to the (positive) fraction arising from the division of the mantissa ·877181, and a negative *fraction* in the quotient must be avoided if the required root is to be expressed decimally. To overcome the difficulty of a negative fraction we add $\overline{3}$ to the characteristic $\overline{4}$, thus making the negative part of the dividend an exact multiple of the divisor 7 and to counterbalance the addition of $\overline{3}$ we add 3 to the mantissa ·877181. The corresponding operation on the number ·00075367 of which $\overline{4}$·877181 is the logarithm, is the change of $\frac{7{\cdot}5367}{10000}$ into $\frac{7536{\cdot}7}{10000000}$.

Had we divided $\overline{4}$ and ·877181 separately by 7, the required root would have been obtained in the form of a fraction, the numerator being the 7th root of 7·5367 and the denominator the 7th root of 10000.

In (*c*) we add $\overline{10}$ to the characteristic $\overline{1}$, to make the negative part of the logarithm exactly divisible by the root-index 11, and we counterbalance this addition of $\overline{10}$ to the characteristic by adding 10 to the mantissa; *i.e.*, we change

$$\left(7{\cdot}576\div10^{-1}\right)^{\frac{7}{11}} \text{ into } \left(7{\cdot}576\times10^{10}\div10^{11}\right)^{\frac{7}{11}} = \left(7{\cdot}576\times10^{10}\right)^{\frac{7}{11}}\div10^{7}$$

$$= 8{\cdot}3807\times10^{6}\div10^{7} = 8{\cdot}3807\div10^{-1} = {\cdot}83807.$$

Example 6. What power of 1·05 is 2?

log 1·05 × exponent = log 2.

$\therefore\quad$ exponent = log 2 ÷ log 1·05 = ·301030 ÷ ·021189 = 14·207

$\therefore\quad 1{\cdot}05^{14{\cdot}207} = 2.$

EXERCISE XIX.

Apply logarithms to obtain approximate values of the following indicated products, quotients, powers and roots :—

1. $3·7485 \times 42·396 \times 3·14159$. **2.** $2·96374 \times 4·83625 \times ·284639$.

3. $·372856 \times ·129745 \times ·386429 \times ·47638$.

4. $43·8629 \times ·0048579 \times ·27846 \times 1·49635$.

5. $78549 \times ·0029638 \times 43·7865 \times ·0085247$.

6. $493764 \div 879·63$.

7. $2·98573 \div 4·76845$.

8. $·379648 \div 57·6483$.

9. $1·62964 \div ·047285$.

10. $·029683 \div ·0023867$.

11. $39·6452 \times ·084763 \div ·427859$.

12. $·27634 \times ·0028463 \div ·058496$.

13. $4·3785 \div 4986·43 \times ·29739$.

14. $8·976 \times 10^{11} \times 2·8648 \times 10^{-5} \div (7·293 \times 10^{9})$

15. $1·4783 \times 10^{-7} \times 2·9653 \times 10^{-9} \div (3·4965 \times 10^{-10})$.

16. $48·739^{3}$.

17. $1·4786^{25}$.

18. $·4763^{5}$.

19. $1·045^{35}$.

20. $·0999^{20}$.

21. $10^{\frac{1}{10}}$.

22. $2·7486^{\frac{1}{12}}$.

23. $·08754^{\frac{1}{2}}$.

24. $·08754^{\frac{1}{3}}$.

25. $·08754^{\frac{1}{5}}$.

26. $·002^{\frac{1}{8}}$.

27. $·001^{\frac{1}{24}}$.

28. $1·8476^{\frac{9}{25}}$.

29. $·8643^{\frac{3}{13}}$.

30. $·008643^{\frac{7}{15}}$.

31. $·1^{·1}$.

32. $·02496^{·7}$.

33. $·00478^{·365}$.

34. $\left(3·954 \times 10^{-8}\right)^{\frac{22}{7}}$

35. $\left(4·658 \times 10^{-10}\right)^{·35}$.

36. $(\frac{1}{3})^{\frac{1}{5}}$.

37. $(\frac{1}{5})^{\frac{1}{3}}$.

38. $\left(\frac{7}{11}\right)^{\frac{11}{7}}$.

39. $\left(\frac{365}{366}\right)^{\frac{100}{97}}$.

40. $\left(\frac{3928}{3927}\right)^{·01}$

41. $248·7^{2} \times 3·14159^{3}$.

42. $248·7^{\frac{1}{2}} \times 3·14159^{\frac{1}{3}}$

43. $·3762^{\frac{1}{4}} \times ·02785^{\frac{3}{7}}$.

44. $·03762^{\frac{3}{4}} \div ·2785^{\frac{5}{7}}$.

45. $·17458^{\frac{3}{11}} \times ·03965^{-·3965}$.

From log 2, log 3, log 7, log 11, and log 13, taken from Table I obtain,—

46. log 32.

47. log 48.

48. log 49.

49. log·625.

50. log 1024.

51. log 2401.

52. log 1·701.

53. log 676·676.

54. log $\frac{9801}{9800}$.

55. log $\frac{123200}{123201}$.

56. Show that $\log_7 10 \times \log_{10} 7 = 1$.

57. Express 10 as a power of 2, *i. e.*, from $\log_{10} 2$ obtain $\log_2 10$.

From log 347 to base 10 given in Table II obtain log 10 to the base,—

58. 3·47.
59. 34·7
60. 3470.
61. ·347.
62. ·0347.

Prove that,—

63. $\log_{10} 12 \times \log_{12} 3 = \log_{10} 3$.
64. $\log_{10} 12 \times \log_{12} 1{\cdot}37 = \log_{10} 1{\cdot}37$.
65. $\log_{10} 12 \times \log_{12} {\cdot}0478 = \log_{10} {\cdot}0478$.

66. Hence show how, from a table of logarithms to base 10 a table of logarithms to any other base may be computed.

67. Prove that $\log_{10} 23 \times \log_{23} 14 \times \log_{14} 9 = \log_{10} 9$.

68. What power of 2 is 7?

69. What power of 7 is 2?

70. What power of 7·386 is 94·853?

71. What power of 94·853 is 7·386?

72. What power of 29·84 is 4738?

73. What power of 4·768 is ·04768?

74. What power of ·02837 is 1·05?

75. What power of ·0476 is ·000476?

Find the exponent of the power which 2 is of,—

76. 1·035.
77. 1·04.
78. 1·06.
79. 1·07.
80. $\frac{43}{40}$.
81. $\frac{3928}{3927}$.

Find the number of figures in the developed value of,—

82. 2^{1000}. **83.** 3^{100}. **84.** 47^{123}. **85.** 2378^{13}.

Find the number of figures in the integral part of the developed value of,—

86. $4{\cdot}7^{123}$. **87.** $1{\cdot}1^{156}$. **88.** $23{\cdot}78^{\frac{147}{11}}$. **89.** $3{\cdot}576^{48{\cdot}32}$.

How many zeros are there between the decimal point and the left-hand digit in the developed value of,—

90. ${\cdot}104^{7}$.
91. ${\cdot}2^{100}$.
92. ${\cdot}047^{123}$.
93. ${\cdot}00976^{\frac{3}{17}}$.
94. ${\cdot}000856^{{\cdot}074}$.
95. ${\cdot}0477^{24{\cdot}7}$

What is the decimal order of the first digit on the left in the developed value of,—

96. $2{\cdot}843^{-37}$. **98.** $\cdot 47683^{-5}$. **100.** $\cdot 000674^{-\cdot 0473}$.

97. $4768{\cdot}3^{-\frac{5}{24}}$. **99.** $\cdot 04862^{-\frac{23}{3}}$.

147. It is now necessary to examine the degree of precision attainable in calculations made by means of Tables I, II and III. The mantissæ entered in these tables are not absolutely correct, they are merely the nearest representations of the correct values attainable with six decimal places, and they are in some cases in excess, in other cases in defect, but the excess or the defect is never greater than 5 in the *seventh* decimal place, *i. e.*, **the error in a tabular logarithm never exceeds ·0000005.** Now by means of Table **B** p. 132 we find that ·0000005 is the logarithm of 1·00000115, hence any logarithm actually entered in Table I, Table II or Table III may be the logarithm of its corresponding tabular number divided by 1·00000115, or of the tabular number multiplied by 1·00000115, or of any number between these limits. Hence **the error in a tabular number never exceeds the ·00000115 of the tabular number itself.** If a logarithm be obtained by interpolation, the operation of interpolation may itself introduce an error not greater than ·0000005, and this error may be on the same side as the tabular error and consequently added to it, so that a logarithm obtained by interpolation may be in error by ·000001.

If then we perform any calculation by help of logarithms, the result is liable to an error of the ·00000115 part (say the one nine-hundred thousandth part) of itself for every logarithm employed and for every interpolation made in the process of calculation. If a logarithm be multiplied by any number, we must multiply the possible error from that logarithm by the multiplier of the logarithm. This is assuming that the errors lie all on one side, *i. e.* are all in excess or all in defect, and that each error is nearly at its limit. The cases in which this will occur will be comparatively rare, yet rare as they may be, we must take them into account in estimating *the limit beyond which our result cannot err.*

In ordinary computations by the help of 6-figure logarithms, we may count on the result as almost certainly correct to 5 significant figures and as probably correct to 6 figures. We exclude cases of involution to high powers.

EXERCISE XX.

1. The squares of the times of revolution of the planets round the sun are as the cubes of their mean distances from the sun, *i.e.*, A and B being two planets, if a fraction be formed having A's time of revolution round the sun as numerator and B's time of revolution as denominator and a second fraction be formed having A's mean distance from the sun as numerator and B's mean distance as denominator, the square of the former fraction will be equal to the cube of the latter. Mercury performs a revolution about the sun in 87·969 days; Venus performs a revolution in 224·701 days; the Earth, in 365·256 days; Mars, in 686·98 days; Jupiter, in 4332·585 days, and Saturn, in 10759·22 days; determine the mean distance from the sun of Mercury, Venus, Mars, Jupiter and Saturn respectively, taking the mean distance of the earth from the sun as the unit of length. Express these distances in miles, assuming the mean distance of the earth from the sun to be (a) 91,430,000, (b) 92,780,000.

2. A pupil who was "strong at figures" undertook to multiply 15 by itself on the first day of his holidays, to multiply the product by itself on the second day, to multiply the second product by itself on the third day, to multiply the third product by itself on the fourth day, and so to continue to do each day (Sundays and Saturdays excepted) to the end of his holidays which were to last four weeks. How many figures would there be in the twentieth product thus formed? Determine the first five and the last ten figures of this product. How long would it take the boy to write down this product at the rate of three figures per second? How many figures would there be in the partial product formed in computing the twentieth product from the nineteenth, assuming that in the nineteenth product the several figures 0, 1, 2, ... 9, occur each an equal number of times except that 5 occurs once oftener than any of the others? Find to the nearest number of days how long it would take 100 men to compute these partial products, working at the rate of two figures per second for six hours per day for 313 days per year.

[Obtain log 15 to fifteen places of decimals from log 2 and log 3 which are given correct to fifteen places, the former on p. 133, the latter on p. 135.]

3. In § 131, it is asserted that "had the base [in Table A] been 2·718281828459, the modulus would have been 1." Test the truth of this assertion by forming a table with exponents the same as those in Table A but with 2·71828 as base instead of 10, and with the calculations carried to six decimal places instead of to eight. (Extract the roots with the aid of Tables of Logarithms II and III.)

Show how to employ the table thus formed to calculate logarithms to base 2·71828.

4. Form a six-decimal-place table similar to Table A but with 12 as base instead of 10 and show therefrom and from the Tables of Logarithms that the modulus of logarithms to base 12 is equal to log 2·71828 ÷ log 12 which is equal to $\log_{12} 2{\cdot}71828$.

Show how to employ this table to calculate logarithms to base 12.

5. A seven-figure table similar to Table A but with 2·71828 as base instead of 10, having been formed, show that if the exponents in the second column be all multiplied by ·4342945, the modulus of logarithms to base 10, the numbers in the first and third columns remaining meanwhile unchanged, the common base of the second column will be changed from 2·71828 to 10.

6. A seven-figure table similar to Table A but with 2·71828 as base instead of 10, having been formed, show that if the numbers in the first and third columns be retained unchanged but the exponents in the second column be all multiplied by the modulus of logarithms to base 12, the common base of the second column will be changed from 2·71828 to 12.

7. State and prove the general theorem of which the theorems of Probs. 4 and 5 are particular cases, and thence show that the modulus of the logarithms to any given base may be used as a constant multiplier to convert logarithms to 2·71828 as base into the corresponding logarithms to the given base.

8. Hence show that the modulus of the logarithms to a given base is the logarithm of 2·71828 to the given base. *(See Exercise XIX, Prob. 66.)* Example, ·434294 = log 2·71828.

9. If the difference between the logarithms of any two numbers be divided by the difference between the numbers and the quotient be multiplied by each of the two numbers, the products will be one greater the other less than the modulus of the logarithms. Test the accuracy of this theorem in the case of logarithms to base 10 by applying it to numbers and their logarithms selected from the Tables of Logarithms I, II and III. This theorem seems to fail in application to many pairs of numbers selected from Table III and from the latter part of Table II, show that these may be cases of seeming and not of real failure of the theorem. *Example* ;—Table II gives $\log 7{\cdot}001 - \log 7 = {\cdot}000062$ which is correct to six decimal places, but to ten decimal places the difference is ·0000620376 ; the theorem fails if ·000062 is taken as the difference between log 7·001 and log 7, but it does not fail if ·0000620376 is taken as the difference.

10. Show that the theorem of Prob. 8 may be deduced from the last theorem of § 130 in all cases in which the difference between the numbers is less than the ten-thousandth part of the smaller number. (In the case of six-figure logarithms, it will be sufficient if the difference between the numbers is not greater than the thousandth part of the smaller number as will at once appear if the numbers in the third column of Table A be reduced to six decimals.)

11. Show that the theorem of Problem 8 enables us to calculate the differences of the logarithms in Tables II and III directly from the modulus ·4342945, without any previous calculation of the logarithms themselves and that consequently a table of differences having been thus computed, Tables II and III may be formed by mere additions.

[This "method of differences" is the method which is now employed whenever it is found desirable to extend a table of logarithms or, for the purposes of verification, to recalculate any part of such a table. In actual practice, the differences of the logarithms are not obtained directly by division of the modulus as here proposed, but are themselves computed from second differences. The number of divisions which must be made, is thus greatly reduced.]

12. The modulus of logarithms to base 10 is ·43429448 and log 49 is 1·69019608 each correct to eight decimals, determine therefrom the logarithms of 4901, 4902, 4903, 4904, 4905, correct in each case to six decimal places.

13. Show that the exponents of the powers to which the bases 1·01, 1·02, 1·03, 1·04, 1·05, 1·06, 1·07 must severally be raised to produce 2 are approximately equal to 70 divided by 1, 2, 3, 4, 5, 6 and 7 respectively, and to produce 3 the several exponents are approximately the quotients of 110 divided by the same seven numbers.

In the following problems, the values of the logarithms which are stated to be 'given' are to be taken from the Tables of Logarithms, and the values of the logarithms to be computed are to be found correct to six places of decimals.

14. Given $\log 2$ and $\log 3$ and $2^2 \times 3^6 \times 7^3 = 1000188$, find $\log 7$.

15. Given $\log 3$ and $3^{11} = 177147$, $11^6 = 1771561$, find $\log 11$.

16. Given $\log 2$, $\log 3$, $\log 7$ and $\log 11$ and $2^4 \times 7 \times 11 = 1232$, $3^6 \times 13^2 = 123201$, find $\log 13$.

17. Given $\log 2$ and $\log 7$ and $7^6 \times 17 = 2000033$, find $\log 17$.

18. Given $\log 2$ and $\log 3$ and $2 \times 3^6 \times 19^3 = 10000422$, find $\log 19$.

19. Given $\log 2$, $\log 3$ and $\log 11$ and $2 \times 3^3 \times 11^3 = 71874$, $5^5 \times 23 = 71875$, find $\log 23$.

20. Given $\log 2$, $\log 3$, $\log 7$, $\log 11$ and $\log 13$ and $2^3 \times 5^2 \times 7^2 = 9800$, $3^4 \times 11^2 = 9801$, $2 \times 13^2 \times 29 = 9802$, find $\log 29$.

21. Given $\log 2$, $\log 3$, $\log 7$, $\log 11$ and $\log 13$ and $2 \times 3 \times 7 \times 11 \times 13 = 6006$, $5^4 \times 31^2 = 600625$, find $\log 31$.

22. Given $\log 3$, $\log 7$, $\log 11$ and $\log 13$ and $3^3 \times 7 \times 11 \times 13 \times 37 = 999999$, find $\log 37$.

23. Given $\log 17$, $\log 19$ and $\log 23$ and $17^3 \times 19^3 \times 23^3 = 410006814589$, find $\log 41$.

24. Given $\log 2$, $\log 3$, $\log 7$, $\log 11$ and $\log 13$ and $2^6 \times 3^2 \times 43 = 24768$, $7 \times 11^5 \times 13^3 = 2476803329$, find $\log 43$.

25. Given $\log 3$ and $\log 17$ and $17^2 \times 47^3 = 30004847$, find $\log 47$.

CHAPTER V.

MENSURATION OR METRICAL GEOMETRY.

148. To Measure any magnitude is to determine what multiple or part or multiple of a part the magnitude is of a specified magnitude of the same kind selected as a standard or unit of measurement.

The number which expresses what multiple or part or multiple of a part the measured magnitude is of the unit, is termed the **Measure** of the magnitude.

The relation which is determined or *sought to be determined* by such measurement is called the **Ratio** of the magnitude measured to the unit of measurement.

149. If the first of two quantities of the same kind be divided by the second, the quotient will be the measure of the first quantity in terms of the second quantity as unit.

Thus 4 is the measure of 12 ft. in terms of 3 ft. as unit, for 12 ft. ÷ 3 ft. = 4 or, as it may otherwise be expressed, 12 ft. = 4 (3 ft.)

The measure of 3 oz. in terms of 8 oz. as unit is $\frac{3}{8}$ or ·375 for 3 oz. ÷ 8 oz. = $\frac{3}{8}$ = ·375 or, otherwise expressed, 3 oz. = $\frac{3}{8}$ (8 oz.) = ·375 (8 oz.)

150. Four magnitudes are said to be **proportional,** to be **in proportion** or *to form a proportion,* if the ratio of the first magnitude to the second is the same as the ratio of the third magnitude to the fourth.

151. *Hence if four magnitudes be in proportion and if the first magnitude be a multiple of the second, the third magnitude will be the same multiple of the fourth; if the first magnitude be a part of the second, the third magnitude will be the same part of the fourth; if the first magnitude be a multiple of a part of the second the third magnitude will be the same multiple of the same part of the fourth.*

152. A, B, C and D denoting four magnitudes of which A and B are of the same kind and C and D also of the same kind, but not necessarily of the same kind as A and B, the expression $A:B::C:D$, read "A is to B as C is to D," denotes that the magnitudes A, B, C and D are in proportion in the order named, *i.e.*, that if A is a multiple of B, C is the same multiple of D; if A is a part of B, C is the same part of D; if A is a multiple of a part of B, C is the same multiple of the same part of D; and generally that the ratio of A to B is the same as the ratio of C to D.

Thus 12 in. = 4 (3 in.) and 20 lb. = 4 (5 lb.)

∴ 12 in. : 3 in. :: 20 lb. : 4 lb., read "12 in. is to 3 in. as 20 lb. is to 4 lb."

So also, 15 gal. = $\frac{3}{7}$ (35 gal.) and $1\frac{1}{2}$ min. = $\frac{3}{7}$ ($3\frac{1}{2}$ min.)

∴ 15 gal. : 35 gal. :: $1\frac{1}{2}$ min. : $3\frac{1}{2}$ min.,

read "15 gal. is to 35 gal. as $1\frac{1}{2}$ min. is to $3\frac{1}{2}$ min.

153. The measure of the length of a line is the number which expresses the ratio which the measured line bears to a line selected as the unit of length.

The unit of length or linear unit is usually either

(a), a fundamental unit, or

(b), a multiple or a fraction of some fundamental linear unit.

The yard and the metre which are both defined by physical standards (see pp. 12 and 17,) are examples of fundamental linear units. The mile and the kilometre are examples of units which are multiples of these fundamental units; the inch, the foot and the centimetre are examples of units which are definite parts or determinate fractions of fundamental units.

Example 1. A certain rope is stated to be 37 yd. long. Here the unit of measurement is the linear unit, a yard, and the measure of the declared length of rope is the number 37.

Example 2. The length of the circumference of a certain circle is found to be 47·85 in. Here the number 47·85 is the measure of the length of the circumference, and the linear unit, an inch, is the unit of measurement.

154. The measure of the area of a surface-figure is the number which expresses the ratio which the measured figure bears to some determinate surface-figure chosen as the unit of area.

The unit of area generally selected is either

(a), a square whose side is some specified unit of length, or

(b), a multiple of such a square.

Example 1. The area of the floor of a certain hall is 240 sq. yd. Here the measure of the area of the floor is 240 and the unit of measurement is the areal unit, a square yard, *i. e.*, a square whose sides are each a yard in length.

Example 2. The area of a certain field is found to be $7\frac{1}{6}$ ac. Here the measure of the area of the field is $7\frac{1}{6}$ and the unit of measurement is the areal unit an acre which is equal to 10 square chains or 4840 square yards.

155. The measure of the volume of any solid or space-figure is the number which expresses the ratio which the measured figure bears to some determinate space-figure chosen as the unit of volume.

The unit of volume is either

(*a*), a cube whose edge is some specified unit of length, or

(*b*), the volume of a given mass of some specified substance under stated conditions, or

(*c*), a multiple or a fraction of this volume.

Example 1. The volume of air in a certain school-room is 560 cu. yd. Here the measure of the volume of air is 560 and the unit of measurement is the volume-unit a cubic yard, *i. e.* a cube whose edges are each a yard long.

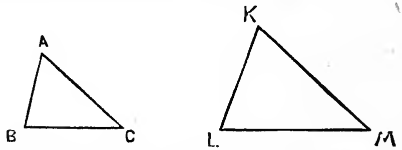
Example 2. A certain pitcher will hold $\frac{5}{8}$ of a gallon of water. Here the measure of the capacity of the pitcher is $\frac{5}{8}$ and the unit of measurement is a gallon, *i.e.*, the volume of ten Dominion standard pounds of distilled water weighed in air against brass weights with the water and the air at the temperature of sixty-two degrees of Fahrenheit's thermometer and with the barometer at thirty inches.

156. Two plane rectilineal figures are **similar** if to every angle in one of the figures there is a corresponding equal angle in the other, and if also the sides about the angles in one figure are proportional to the sides about the corresponding angles in the other. The sides extending between corresponding angular points are termed *corresponding* or *homologous* sides.

157. Hence if the lengths of two sides of a triangle be given and also the length of one of the corresponding sides of a triangle similar to the former, the length of the second corresponding side of the latter triangle can be determined.

158. *If two triangles have two angles of the one equal to two angles of the other, each to each, the triangles will be similar.* (Euclid, vi, 4.)

Example. Let the triangles A B C and K L M have the angle B equal to the angle L and the angle C equal to the angle M; also let the sides A B, B C and C A be respectively 6, 8 and 9 sixteenths of an inch in length and the side L M be 12 sixteenths of an inch long. Find the lengths of the sides K L and K M.



By § 158 the triangles are similar and the corresponding sides are A B and K L, B C and L M, C A and M K;

$\therefore$ K L : L M : : A B : B C.

The length of A B is 6 and that of B C is 8 sixteenths of an inch,

$\therefore$ A B $= \frac{6}{8}$ of B C

$\therefore$ K L $= \frac{6}{8}$ of L M

$= \frac{6}{8}$ of 12 sixteenths of an inch

$=$ 9 " " " "

Similarly, $\because$ K M : M L : : A C : C B

and A C $= \frac{9}{8}$ of C B

$\therefore$ K M $= \frac{9}{8}$ of M L

$= \frac{9}{8}$ of 12 sixteenths of an inch

$=$ $13\frac{1}{2}$ " " " "

EXERCISE XXI.

1. A B C and K L M are similar triangles, the angles A and K being equal to one another and the angles B and L also equal to one another; the side A B is 9″ long, the side B C is 10″ long and the side K L is 22·5″ long, find the length of the side L M.

2. A B C and G H K are similar triangles, A and G being corresponding angles and B and H also corresponding angles; the lengths of the sides A B, A C, G H and H K being 7″, 15″, 5·25″ and 15″ respectively, find the lengths of B C and ~~H K~~. GK

3. A B C and G H K are similar triangles, the angles A and G being equal to one another and the angles B and K also equal to one another; the measures of the sides are A C $=$ 25, G H $=$ 44, H K $=$ 35 and K G $=$ 75. Find the measures of the sides A B and B C.

4. In A B, a side of the triangle A B C, a point D is taken, and the straight line D E is drawn parallel to the side B C; find the length of D E, the length of A B being 35′, that of B C 24′ and that of A D 11′ 2″.

5. The construction being the same as in problem 4, find the measure of D E, given that the measures of A D, D B and B C are 7, 23 and 18 respectively.

6. The construction being the same as in problem 4, find the measure of A D, given A B = 45, B C = 20 and D E = 8.

7. The construction being the same as in problem 4, determine the length of B C, given that A D is 24 yd. long, D B, 30 yd. long, and D E, 18 yd. long.

8. The construction being the same as in problem 4, determine the length of B C, A B being 104 ft. long, B D being 44 ft. long, and D E being 80 ft. long.

9. The construction being the same as in problem 4, what will be the length of A D if B D be 36 chains long, B C, 36 chains long, and D E, 15 chains long.

10. A stick 3′ in length placed upright on the ground is found to cast a shadow 2′ 6″ long, what must be the height of a flagpole which casts a shadow 28′ in length?

11. A gas-jet is 12 ft. above the pavement, how far from the ground-point directly beneath the jet must a man 5 ft. 8 in. in height stand that his shadow may be 6 ft. long.

12. The vertical line through a gas-jet 9′ 4″ above the sidewalk is 10′ 6″ from a man 5′ 10″ in height, find the length of his shadow.

13. An electric light is 15 ft. above the pavement, what will be the length of the shadow of a man 5 ft. 10 in. in height if he stand 30 ft. from the vertical line through the light?

14. The parallel sides of a trapezoid are respectively 27 ft. and 35 ft. in length and the non-parallel sides are respectively 18 ft. 7 in. and 23 ft. 11 in. long. The latter sides are produced to meet; find the respective lengths of the produced sides between the point of meeting and the shorter of the parallel sides of the trapezoid.

15. The lengths of the parallel sides of a trapezoid are 10·75 and 12·35 chains respectively; four straight lines are run across the trapezoid parallel to these sides so that the six lines are at equidistant intervals; find the lengths of these four lines.

16. The lengths of the parallel sides of a trapezoid are 15 and 28 inches respectively, and of the non-parallel sides 12 and 20 inches respectively; through the intersection of the diagonals of the trapezoid a straight line is drawn parallel to the parallel sides. Find the lengths of the sections into which this line divides the non-parallel sides.

17. Taking the diameter of the sun to be 880,000 miles and the sun's distance from the earth to be 92 400,000 miles, what must be the diameter of a circular disk that it may just hide the sun when held between the eye and the sun and 21 inches in front of the eye?

18. Three men *A*, *B* and *C* stand in a row on a level pavement, *A*'s height is 5′ $3\frac{1}{2}$″, *B*'s is 5′ 9″ and *C*'s is 6′ $1\frac{1}{2}$″; if *A* stand 10′ to the right of *B*, how far to the left of *B* must *C* stand that the tops of the heads of the three men may range in a straight line?

19. The lengths of the sides of a triangle are 7 yd., 11 yd. and 12 yd. respectively and the perimeter of a similar triangle is 25 ft.; find the lengths of the sides of the latter.

20. The perimeters of two similar triangles are 25 ft. 6 in. and 56 yd. 2 ft. respectively. A side of the smaller triangle is 7 ft. long and a non-corresponding side of the larger triangle is 17 yd. 1 ft. in length. Find the lengths of the other sides of the triangles.

159. Any one of the sides of a parallelogram having been selected as the base of the figure, **the altitude of the parallelogram** is the *perpendicular distance* between the base and the side parallel to the base.

One of the sides of a triangle having been selected as the base of the figure, the opposite angle becomes the **vertex**, and the **altitude of the triangle** is the *length of the perpendicular* from the vertex on the base, or the base produced.

160. A **polyhedron** is a solid-figure enclosed by plane polygons.

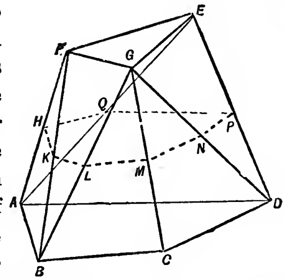
A polyhedron enclosed by four polygons, (in this case, triangles) is called a tetrahedron; by six, a hexahedron; by eight, an octahedron; by twelve, a dodecahedron; by twenty, an icosahedron.

The faces of a polyhedron are the enclosing polygons. If the faces are all equal and regular, the polyhedron is regular.

The edges of a polyhedron are the lines in which its faces meet.

The summits of a polyhedron are the points in which its edges meet.

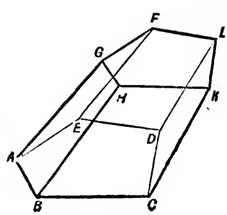
A Prismatoid is a polyhedron two of whose faces are polygons situated in parallel planes and whose other faces are triangles having the sides of the polygons as bases and having their vertices at the angular points of the polygons. The polygons situated in parallel planes are called the **ends** of the prismatoid, and if one of them be taken as the *base* of the solid, the other becomes the *opposite parallel face.* The other faces are called the *lateral faces* and their common edges are named the *lateral edges.* (A B C D E F G is a prismatoid on the quadrilateral base A B C D, the opposite parallel face is the triangle E F G.)

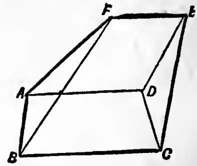


The midcross-section of a prismatoid is its section by a plane parallel to the planes in which are situated the end polygons and midway between these planes. *The midcross-section therefore bisects all the lateral edges of the prismatoid.* (H K L M N P Q is the midcross-section of the prismatoid A B C D E F G. The angular points H, K, L, M, N, P, Q are the mid-points of A F, B F, B G, C G, D G, D E, and A E respectively.)

If the bases of two adjacent lateral faces of a prismatoid are parallel the two faces lie in one plane and together form a trapezoid.

A Prismoid is a prismatoid whose lateral faces are all trapezoids. The end polygons must therefore have the same number of sides and each corresponding pair must be co-parallel. (A B C D E F G H K L is a prismoid with pentagonal ends A B C D E and F G H K L.)





A Wedge is a solid enclosed by five plane figures, the base is a trapezoid, two of the lateral faces are trapezoids and the other two lateral faces are triangles. A wedge is therefore a prismatoid on a trapezoidal base, in which the face opposite the base has become reduced to a straight line parallel to the two co-parallel sides of the base. (ABCDEF is a wedge; the base ABCD is a trapezoid, the sides BC and AD being parallel to each other; EF is parallel to both BC and AD, hence BCEF and ADEF are both trapezoids.)

A Prism is a polyhedron two of whose faces are parallel polygons, and the other faces, parallelograms.

The bases or **ends of a prism** are the parallel polygons.

The altitude of a prism is the *perpendicular distance* between the planes of its bases.

A right prism is one whose lateral edges are perpendicular to its bases.

A parallelepiped is a prism whose bases are parallelograms. A parallelepiped is therefore a solid contained by six parallelograms of which every opposite pair are parallel.

A quad or QUADRATE SOLID is a right parallelepiped with rectangular bases. It is therefore contained by six rectangles. **A cube** is a quad whose faces are all squares.

161. A cylindric surface is a surface generated by a straight line so moving that it is always parallel to a fixed straight line.

A cylinder is a solid enclosed by a cylindric surface and two parallel planes.

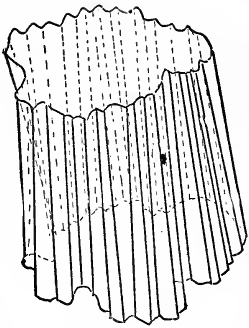
The bases of a cylinder are the parallel plane faces.

The altitude of a cylinder is *the perpendicular distance* between the planes of its bases.

A right cylinder is one in which the generating lines of the cylindric surfaces are perpendicular to the bases of the cylinder.

A right circular cylinder is a right cylinder whose bases are circles.

A Cylindroid is a solid bounded by two parallel planes and the surface described by a straight line which simultaneously describes two closed curves, one in each of the parallel planes. The plane figures enclosed by the curves in the parallel planes are called the ends of the cylindroid.



A Sphenoid is a prismatoid or a cylindroid, one of whose ends has become reduced to a line.

162. A pyramid is a polyhedron one of whose faces, called **the base,** is a polygon and whose other faces are triangles whose bases form the sides of the polygon and whose vertices meet in a point called **the vertex of the pyramid.**

A pyramid is therefore a prismatoid one of whose parallel ends has become reduced to a point.

A regular pyramid is one whose base is a regular polygon and whose other faces are equal isosceles triangles

The altitude of a pyramid is the *length of the perpendicular* let fall from the vertex on the plane of the base.

163. A conical surface is a surface generated by a straight line which so moves that it always passes through a fixed point called the **vertex** of the surface.

A **cone** is a solid enclosed by a conical surface and a plane. It is therefore a cylindroid one of whose parallel ends has become reduced to a point.

The base of a cone is the plane face opposite the vertex.

The altitude of a cone is the *length of the perpendicular* let fall from the vertex on the plane of the base.

A right circular cone has a circle for its base, and the straight line joining the vertex of the cone and the centre of the base is perpendicular to the plane of the base.

The frustum of a pyramid or **of a cone** is the portion included between the base and a plane cutting the pyramid or the cone parallel to the base.

164. Two polyhedra are **similar** if to every solid angle in one of them there is a corresponding equal solid angle in the other, and to every face of one of them there is a corresponding similar face in the other.

The corresponding edges of similar polyhedra are those which are corresponding sides of corresponding faces.

165. Similar surface-figures need not be rectilinea., they need not even be plane surfaces. Thus all circles are similar to one another, parallel plane sections of a cone are similar figures, all spherical surfaces are similar to one another, and generally the complete surfaces of similar solids are themselves similar. If two plane surface-figures are similar, they are, or they may be so placed as to be, parallel plane sections of a pyramid or else of a cone.

Similar solid-figures are not necessarily bounded by plane surfaces; *e. g.*, all spheres are similar to one another, so also are the spheroids described by similar ellipses rotating about corresponding axes.

[Similar figures, whether surface or solid, may be described as figures which are alike in form but which are not necessarily equal in size.]

166. In the theorems which immediately follow, the areal unit is the square and the volume unit is the cube described on the linear unit as side and edge respectively.

167. *The measure of the area of a square is the square of the measure of the length of a side of the square.*

The measure of the length of a side of a square is the square root of the measure of the area of the square.

168. *The measure of the volume of a cube is the cube of the measure of the length of an edge of the cube.*

The measure of the length of an edge of a cube is the cube root of the measure of the volume of the cube.

Examples. If the length of a side of a square be 5 ft., the area of the square will be 5^2 sq. ft. If the area of a square be 4840 sq. yd., the length of a side of the square will be $4840^{\frac{1}{2}}$ yd.

If the length of an edge of a cube be 7·3 in., the volume of the cube will be $7{\cdot}3^3$ cu. in. If the volume of a cube be 10 cu. ft., the length of an edge of the cube will be $10^{\frac{1}{3}}$ ft., and the area of a face of the cube will be $10^{\frac{2}{3}}$ sq. ft.

169. In the formulæ which follow S, a and b denote the *measures* of the area, the altitude and the length of the base respectively and r, l, t and z subscribed to S are to be severally read rectangle, parallelogram, triangle and trapezoid.

i. *The measure of the area of a rectangle is the product of the measures of the lengths of two adjacent sides*, or

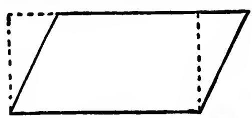
$$S_r = ab.$$

(Special case,—*square.*)

ii. *The measure of the area of a parallelogram is the product of the measures of the altitude and the length of the base*, or

$$S_l = ab.$$

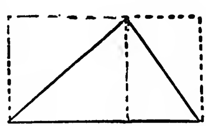
(Special case,—*rectangle.*)



iii. *The measure of the area of a triangle is ONE-HALF of the product of the measures of the altitude and the length of the base*, or

$$S_t = \tfrac{1}{2}ab.$$

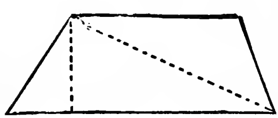
(Special case,—*sector of a circle, including circle itself.*)



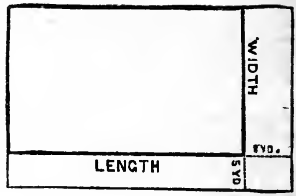
iv. *The measure of the area of a trapezoid is ONE-HALF of the product of the measure of the altitude and the sum of the measures of the parallel sides*, or

$$S_z = \tfrac{1}{2}a(b_1 + b_2).$$

(Special cases,—*parallelogram, triangle* and *sector of an annulus, including annulus itself.*)



Example 1. The width of a rectangular building-lot is to its



length as 3 to 5, and if the length of the lot be increased by 8 yd. and the width by 5 yd., its area will be increased by 481 sq. yd. Find the length of the lot.

The increment to the lot may be considered to consist of three parts, viz.,

1° A rectangle 8 yd. by the *width* of the lot;

2° A rectangle 5 yd. by the *length* of the lot;

3° A rectangle 8 yd. by 5 yd.

The area of these three rectangles taken together is 481 sq. yd. and the area of the 3° rectangle is 40 sq. yd. Subtracting 40 sq. yd. from 481 sq. yd., there will remain 441 sq. yd. as the area of the 1° and 2° rectangles taken together.

The 1° rectangle is 8 yd. by the width of the lot.

The width of the lot is $\frac{3}{5}$ of its length.

Therefore the 1° rectangle is 8 yd. by $\frac{3}{5}$ of the length of the lot, which is equivalent to a rectangle $\frac{8\times3}{5}$ yd. by the length of the lot,
$=\frac{24}{5}$ yd. by the length of the lot.

The 2° rectangle is 5 yd. by the length of the lot.

Therefore the two rectangles are together equivalent to a rectangle $(\frac{24}{5}+5)$ yd. by the length of the lot,
$=\frac{49}{5}$ yd. by the length of the lot.

The sum of the areas of these two rectangles is 441 sq. yd.;

$\therefore$ $\frac{49}{5}$ of the measure of the length of the lot $=441$;

$\therefore$ the measure of the length of the lot $=441\div\frac{49}{5}$,
$=45$;

$\therefore$ the length of the lot is 45 yd.

Example 2. A rectangular park is 400 yd. by 660 yd. It is surrounded by a road of uniform width the whole area of which is one-sixth of the area of the park. Determine the width of the road.

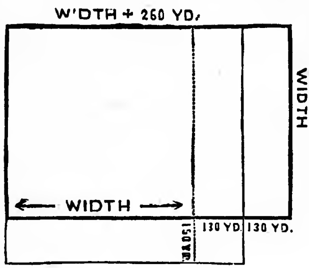
The area of the park is (400×660) sq. yd. $=264000$ sq. yd.

The area of the road is $\frac{1}{6}$ of 264000 sq. yd. $=44000$ sq. yd.

Therefore the area of the rectangle composed of both road and park $=264000$ sq. yd. $+44000$ sq. yd. $=308000$ sq. yd.

The park is a rectangle 260 yd. longer than it is wide.

When the road is included with the park, both the length and the width of the rectangle is increased by double the width of the road; the resulting block of land is therefore still a rectangle 260 yd. longer than it is wide.



Hence if the length of the block be reduced by 130 yd. and the width of the block when thus shortened be increased by 130 yd., the resulting rectangle will be a *SQUARE* whose sides will be each 130 yd. longer than the *width* of the original block.

Reducing the length of the block by 130 yd *takes from the block* a rectangle 130 yd. by the width of the block.

Increasing the width of the shortened block *adds to this block* a rectangle 130 yd. by 130 yd. more than the width of the original block, *i. e.*, it *adds* a rectangle 130 yd. by the width of the original block and a square 130 yd. square.

Hence the two operations of reducing the length of the original block and increasing the width of this shortened block increase the area of the resulting figure as compared with the area of the original block, by the area of the 'completing square' of 130 yd. square, *i. e.*, by an area of 130^2 sq. yd. $=16900$ sq. yd.

The area of the original block was found to be 308000 sq. yd.

The area of the completing square has been found to be 16900 sq. yd.

Therefore the area of the completed square or square block will be 308000 sq. yd. + 16900 sq. yd. = 324900 sq. yd.

Therefore the length of the side of the square block $=(324900)^{\frac{1}{2}}$ yd. $=570$ yd.

Therefore the length of the rectangular block = 570 yd. + 130 yd. = 700 yd.

The length of the park = 660 yd.

Therefore double the width of the road = 700 yd. − 660 yd. = 40 yd.

Therefore the width of the road = 20 yd.

EXERCISE XXII.

1. Find to the nearest inch the length of the side of a square whose area is an acre?

2. A square field contains exactly 8 acres. Determine the length of a side of the field, correct to the nearest link.

3. The area of a chess-board marked in 8 rows of 8 squares each, is 100 sq. in. Find the length of a side of a square.

4. On a certain map it is found that an area of 16000 acres is represented by an area of 6·25 sq. in. Give the scale of the map in miles to the inch and also in the form of a ratio.

5. A rectangle measures 18′ by 30′; find the difference between its area and that of a square of equal perimeter.

6. Six sheets of paper measuring 8 in. by 10 in. weigh an ounce; find the weight of 120 sheets of the same kind of paper, each sheet measuring 11 in. by 17 in.

7. Two rectangular fields are of equal area, one field measures 15 chains by 20 chains, the other is square. Find the length of a side of the latter field, correct to the nearest link.

8. How many stalks of wheat could grow on an acre of ground, allowing each stalk a rectangular space of 2″ by 3″?

9. How many pieces of turf 3′ 6″ by 1′ 3″ will be required to sod a rectangular lawn 28′ by 60′?

10. Sidewalks 12 ft. wide are laid on both sides of a street 440 yd. long. Find the cost of the sidewalks at $1·35 per square yard for the pavement and 75 cents per lineal yard for curbing; deducting three crossings of 54 ft. each on both sides of the street.

11. The area of a rectangular field is 15 acres; the length of the field is double the width, find the length of the field.

12. How many yards of fencing-wire will be required to enclose a rectangular field thrice as long as it is wide, if the field contain 10 acres and the fence be made 5 wires high?

13. The lengths of the sides of a rectangular piece of land are as 3 to 8, and its area is 60 acres. Find the lengths of the sides.

14. The perimeter of a rectangle is 154 in., and the difference in length of two adjacent sides is 11 in. Find the area of the rectangle.

15. The length of a rectangle is 88 ft.; if the width were increased by 8 ft., the area of the rectangle would in such case be 616 sq. yd. Find the width of the original rectangle.

16. The area of a certain rectangle is 1980 sq. yd. If the length of the rectangle were increased by 12 ft., the area would be 2100 sq. yd. Determine the lengths of the sides of the rectangle.

17. Find the difference between the perimeter of a square field containing 22·5 acres and the perimeter of a rectangular field of equal area, the length of the latter field being to its width as 5 to 2.

18. A rectangular block of building-lots is 660 ft. long by 198 ft. wide. Find the area covered by an eight-foot sidewalk around the block just outside of it.

19. A six-foot sidewalk of 3 in. planks is to be laid around a rectangle 266 ft. 8 in. by 480 ft., the inner edge of the sidewalk to be twelve inches out from the sides of the rectangle. Find the value at $14 the M, board-measure, of the planking for the sidewalk.

20. Find the areas of the outer and the inner surface of a hollow iron cube measuring 8 in. on the outside edge, the iron being $\frac{3}{8}$ in. thick.

21. Find the area of the inside surface of a hollow quad measuring 3′ 2″ by 2′ 8″ by 2′ 1″ externally, the enclosing walls being $1\frac{1}{4}$″ thick.

22. The length of the base of a parallelogram is 45 ft.; the length of the perpendicular on the base from the opposite side is 28 ft.; the length of a side adjacent to the base is 35 ft.; find the length of the perpendicular on this side from the side opposite to it.

23. The adjacent sides of a parallelogram measure 132 ft. and 84 ft. respectively and the area of the parallelogram is two-thirds of that of a square of equal perimeter. Find the perpendicular distance between each pair of parallel sides.

24. Find the cost of painting the gable-end of a house @ 22 ct. per sq. yd., the width of the house being 32 ft.; the height of the eaves above the ground, 36 ft.; and the perpendicular height of the ridge of the roof above the eaves, 15 ft.

25. Find the area of a field in the form of an isosceles right-angled triangle, the length of the perpendicular on the hypothenuse being 7·50 chains.

26. The length of one of the diagonals of a quadrilateral is 27·7 ft. and the lengths of the perpendiculars on this diagonal from opposite angles of the quadrilateral are 18·5 ft. and 11·3 ft. respectively. Find the area of the quadrilateral, 1°, if the diagonal lies wholly within the quadrilateral; 2°, if the diagonal lies wholly without the quadrilateral.

27. The lengths of the diagonals of a courtyard in the form of a rhombus are 40 ft. and 25 ft. How many bricks 9″ by $4\frac{1}{2}$″ will be required to pave the courtyard; adding 5 % to the area to allow for broken bricks and for waste at the sides of the courtyard?

28. One of the diagonals of a parallelogram measures 819 ft. and the perpendicular on it from an opposite angle of the parallelogram measures 237 ft. Find the area of the parallelogram.

29. The area of a quadrilateral is 7956 sq. yd., the length of one of the diagonals is 416 ft. and the length of the perpendicular on this diagonal from an opposite angle of the quadrilateral is 192 ft. Find the length of the perpendicular from the other opposite angle, 1°, if the diagonal is internal; 2°, if it is external.

30. The area of a quadrilateral is 12·48 acres and the length of one of the internal diagonals is 19·50 chains. Find the sum of the lengths of the perpendiculars on this diagonal from the two opposite angles.

31. The area of a quadrilateral is 906·5 sq. yd.; the length of one of the internal diagonals is 147 ft.; and the difference between the lengths of the perpendiculars on this diagonal from the opposite angles of the quadrilateral is 33 ft. Find the lengths of these perpendiculars.

32. A B C D is a quadrilateral, A B = 400 ft., B C = 203 ft., C D = 396 ft., and D A = 195 ft.; the angles at A and C are right angles. Find the area of the quadrilateral.

33. Find the area of a trapezoid whose parallel sides measure 12′ 7″ and 19′ 3″ respectively, the perpendicular distance between them being 8′ 5″.

34. A B C D is a quadrilateral; A B = 37·48 chains, B C = 21·85 chains and C D = 29·64 chains. A B is parallel to D C and the angle at C is a right angle. Determine the areas of the triangles ABD and ACD and of the quadrilateral.

35. Find the area of a quadrilateral one of whose sides measures 23·29 chains and the perpendiculars on this side from the opposite angles of the quadrilateral 17·75 chains and 13·45 chains respectively, the distances of the feet of these perpendiculars from the adjacent angles being 3·64 chains and 2·40 chains respectively.

36. The area of a trapezoidal field is $3\frac{1}{3}$ acres and the sum of the lengths of the parallel sides is 440 yd. Find the perpendicular distance between these sides. The lengths of the sides being in the ratio of 5 to 6, find these lengths.

37. The area of a trapezoid is 9750 sq. yd. and the perpendicular distance between the parallel sides is 234 ft. If the length of one of the parallel sides be 410 ft., what will be the length of the other parallel side?

38. The area of a trapezoid is 47·142 acres. One of the parallel sides is 6·12 chains longer than the other and the perpendicular distance between the parallel sides is 11·64 chains. Determine the lengths of the two parallel sides.

39. The lengths of the parallel sides of a trapezoid are 12 ft. and 17 ft. and the perpendicular distance between these sides is 8 ft. A straight line is drawn across the trapezoid parallel to the parallel sides and midway between them. Find the areas of the two parts into which the trapezoid is thus divided.

40. The lengths of the parallel sides of a trapezoidal field are 15·80 chains and 18·70 chains respectively and the perpendicular distance between these parallel sides is 14·40 chains. Four straight lines are drawn across the field parallel to the two parallel sides and dividing the distance between these sides into five equal parts. Find the areas of these five parts of the field.

41. The area of a triangle is 551 sq. yd. and the length of its base is 95 ft. Two straight lines are drawn across the triangle parallel to the base and dividing into three equal parts the perpendicular from the vertex on the base. Find the areas of the parts into which the triangle is divided by these lines.

42. A trapezoid with parallel sides whose lengths are to be as 4 to 3 is to be cut from a rectangular board 14 ft. long. Find the lengths of the parallel sides that the trapezoid may be one-third of the board, the trapezoid to be of the same width as the board.

43. The length of a rectangle is to its width as 7 to 4, and if its length be diminished by 3 ft. while its width is increased by 3 ft., its area will be increased by 198 sq. ft. Find the length of the rectangle.

44. The length of a rectangular piece of land is to its breadth as 9 to 5; if its length be increased by 4 ft. and its breadth be diminished by 3 ft., its area will be diminished by 355 sq. ft. Find the length and the breadth of the piece of land.

45. The length of a rectangle is to its width as 16 to 9; if its length be diminished by 2 ft. and its width diminished by 3 ft., its area will be diminished by 720 sq. ft. Find the area of the rectangle.

46. A rectangular field 200 yd. long is surrounded by a road of the uniform width of 60 ft. The total area of both field and road is 9 A. 1240 sq. yd. Find the width of the field.

47. A rectangular field 780 ft. in length is surrounded by a road of the uniform width of 50 ft., the area of the whole road being 15000 sq. yd. Find the area of the field.

48. A rectangular field 180 yd. by 150 yd. is surrounded by a walk of uniform width, the whole area of the walk being 10000 sq. ft. Find the width of the walk.

49. Around a rectangular park runs a path of uniform width; paths of the same width cross the park dividing it into four equal rectangles. The total length of the park, including paths is 330 yd.; its area, including paths, is 15 A.; exclusive of paths the area is 13·775 A. Find the width of the paths.

50. The areas of two squares differ by 64 sq. yd. and the lengths of their sides differ by 2 yd. Find their areas.

51. The sum of the perimeters of two squares is 200 ft. and the difference of their areas is 400 sq. ft. Find their areas.

52. The area of a rectangle is 945 sq. ft. and that of a square of equal perimeter is 961 sq. ft. Find the lengths of the sides of the rectangle.

53. The area of a rectangle is 37249 sq. ft. and its length exceeds its breadth by 40 ft. Find its length.

54. The area of a triangle is 2 A. 2152 sq. yd. and the length of the base exceeds the altitude of the triangle by 38 yd. Find the length of the base.

55. A certain rectangular field of area $3\frac{2}{3}$ A. is surrounded by a road of the uniform width of 55 ft., the total area of the road being $2\frac{1}{3}$ A. Find the length and the width of the field.

170. In the formulæ which follow V, a, B and M denote the measures of the volume, the altitude, the area of the base and the area of the midcross-section respectively, and q, p, c, y, k, f, d and w subscribed to V are to be read severally quad, prism, cylinder, pyramid, cone, frustum of pyramid or of cone, prismatoid (or prismoid) and wedge.

I. *The measure of the volume of a quad* (rectangular parallelepiped,) *is the product of the measures of the lengths of three adjacent edges*, *i. e.*, of three edges meeting in a summit, or

$$V_q = a\, b_1\, b_2 = a\, B,$$

in which b_1 and b_2 denote the measures of adjacent edges of the base and consequently $b_1\, b_2 = B$.

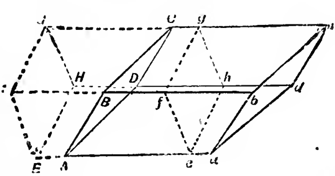
(Special case,—*cube.*)

II. *The measure of the volume of a prism is the product of the measures of the altitude and the area of the base*, or

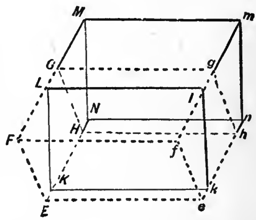
$$V_p = a\, B.$$

(Special cases.—*quad*, and *cylinder*.)

1°. This theorem is true of the oblique parallelepiped for every oblique parallelepiped can be transformed into a rectangular parallelepiped with base equal to and altitude the same as that of the oblique parallelepiped.



For example, let ABCD*abcd* be an oblique parallelepiped. Through *e*, a point in the edge A*a*, pass a plane at right angles to the edges A*a*, B*b*, C*c*, D*d* and cutting these edges in the points *e*, *f*, *g* and *h* respectively. Transfer the solid *efghabcd* from end to end of ABCD*abcd* thus transforming this parallelepiped into the parallelepiped EFGH*efgh* on the rectangular base EH*he* which is equal to the base AD*da*.

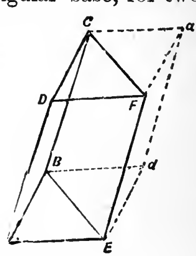


Through *k*, a point in the edge *eh*, pass a plane at right angles to the edges *eh*, *fg*, EH, FG, and cutting these edges in the points *k*, *l*, K, and L respectively. Transfer the solid EFLK*eflk* from side to side of EFGH*efgh*, thus transforming this parallelepiped into the rectangular parallelepiped KLMN*klmn*.

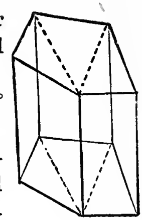
The measure of the volume of AC*bd* is the same as the measure of the volume of KM*ln*, the two parallelepipeds being made up of the same parts differently arranged. The measure of the volume of KM*ln* is, by Theorem I, the product of the measures of its altitude and its base-area. Hence the measure of the volume of AC*bd* is the product of the measures of the altitude and base-area of KM*ln*, which is the same as the product of the measures of the altitude and base-area of AC*bd* itself, for the altitude of the parallelepipeds remains unchanged during the transfers and the base KN*nk* is merely the base AD*da* with its parts transposed.

The theorem is therefore true of parallelepipeds.

2°. The theorem is true of a prism on a triangular base, for two similar and equal prisms on triangular bases may be so joined together as to form a parallelepiped with both volume and base double the volume and base of either prism. Hence double the measure of the volume of a prism on a triangular base is the product of the measure of the altitude and the measure of double the area of the base, and therefore the measure of the volume is the product of the measures of the altitude and the area of the base.



3°. The theorem is true of prisms whose bases have five or more sides for by passing planes through any one lateral edge and through all the other lateral edges except the two immediately adjacent to the first edge, any such prism will be resolved into an aggregate of triangular based prisms which have all the same altitude as the resolved prism and whose triangular bases together make up the base of the resolved prism.



All prisms are included under one or other of 1°, 2° or 3°, therefore the theorem is true generally.

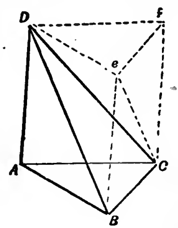
[The student should make models of the solid-figures here considered and also of those considered under theorems III and IV which follow. Solid-figures can very easily be cut out of potatoes or turnips.]

III. *The measure of the volume of a pyramid is ONE-THIRD of the product of the measures of the altitude and the area of the base*, or

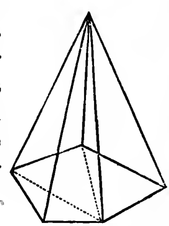
$V_y = \frac{1}{3} aB.$

(Special cases,—*cone* and *sector of a sphere, including sphere itself.*

This theorem is true of tetrahedra (triangular pyramids) for any triangular prism, *e.g.*, ABCDef, can be divided into three tetrahedra of which two, ABCD and DefC, will be of the same altitude as the prism and will have the triangular faces of the prism as their respective bases; the third tetrahedron BCDe, may be seen to have an altitude and a base equal to each of the other two by resting the prism first on the face Ae and next on the face Bf.



The theorem is true of pyramids with bases which have four or more sides; for, by passing planes through any one lateral edge and all the other lateral edges except the two adjacent to the first edge any such pyramid will be resolved into an aggregate of tetrahedra which have all the same altitude as the pyramid and whose bases together make the base of the pyramid.

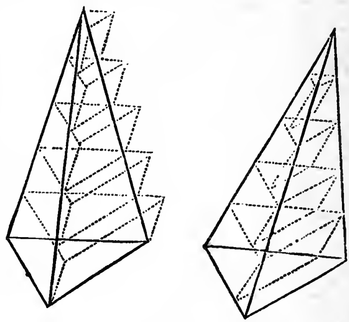


Hence the theorem is true of all pyramids.

[The preceding proof assumes that two tetrahedra on equal and similar bases and of the same altitude are of equal volume, a proposition which is a particular case of Euclid xii, 5. The proposition may also be proved as follows :—

Divide one of the lateral edges of each tetrahedron into any number of equal parts, the same number in both tetrahedra, and through the points of division pass planes parallel to the bases. All the sections of the first tetrahedron are triangles equal and similar to the corresponding sections of the second tetrahedron.

Beginning with the base of the first tetrahedron, construct on the base and on each section as base a prism with lateral edges parallel to one of the edges of the tetrahedron and with altitude equal to the perpendicular distance between the sections.



Beginning with the first section above the base of the second tetrahedron, construct on each section as anti-base or upper triangular surface, a prism with lateral edges parallel to one of the edges of the tetrahedron and with altitude equal to the perpendicular distance between the sections.

The aggregate of the first-constructed series of prisms is greater than the first tetrahedron and the aggregate of the second series is less than the second tetrahedron, therefore the difference in volume between the tetrahedra is less than the difference in volume between the prism-aggregates.

But, by II p. 178, each prism in the second tetrahedron is equal in volume to the prism in the first tetrahedron next above it in order numbering from the prisms on the bases of the tetrahedra. Therefore the difference between the prism-aggregates is the basal prism in the first aggregate.

Now the volume of this basal prism may be made less than any assignable volume, for the measure of its volume is the product of the measures of the altitude and the area of the base. The base is constant being the base of the tetrahedron, but the altitude being the perpendicular distance between the sections may be increased or diminished by changing the number of the sections. By doubling the number of the sections the altitude and with it the volume of the prism will be diminished by one-half of itself. If we again double the number of sections, we shall again diminish the volume of the prism by one-half of itself. Repeating the doubling we repeat the subdividing, and the process may be continued till the volume of the basal prism is less than that of any assigned solid however small.

Hence the tetrahedra can have no assignable difference of volume, and, both being constants, they cannot have a variable difference ; therefore they are of equal volume.]

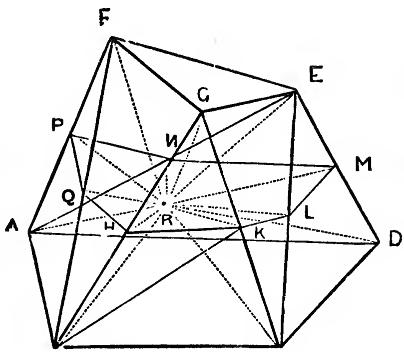
IV. *The measure of the volume of a prismatoid is ONE-SIXTH of the product obtained by multiplying the measure of the altitude by the sum formed by adding the measures of the areas of the parallel faces to four times the measure of the area of the midcross-section, or*

$$V_a = \tfrac{1}{6}a(B_1 + 4\,M + B_2).$$

(Special cases,—*prismoid* and *cylindroid*, *wedge* and *sphenoid*, *prism* and *cylinder*, *pyramid* and *cone* and *frusta of pyramid* and *cone*, *ellipsoid* and *frustum of ellipsoid by planes perpendicular to an axis.*)

Let ABCDEFG be a prismatoid and denote the measure of its altitude by a, the measure of the area of the base ABCD by B_1, and the measure of the area of the face EFG, the face parallel to the base, by B_2.

Bisect the lateral edge BG in the point H and through H pass a plane parallel to the base ABCD, cutting, and therefore bisecting, the other lateral edges in K, L, M, N, P and Q respectively. The polygon HKLMNPQ is the midcross-section of the prismatoid and its perpendicular distance both from the base ABCD and from the parallel face EFG is one-half of the altitude of the prismatoid. The measure of that distance is therefore $\tfrac{1}{2}a$. Let M denote the measure of the area of the midcross-section.



In the plane of the midcross-section take any point R and pass planes through R and each edge of the prismatoid thus resolving that solid into the nine pyramids R ABCD, R EFG, R BCG, R GCE, R CDE, R DAE, R EFA, R ABF, R FGB, a pyramid on each face of the prismatoid.

The measure of the volume of R ABCD is one-third of the product of $\frac{1}{2}a$ and B_1 *i.e.*, $\frac{1}{6}aB_1$.

The measure of the volume of R EFG is one-third of the product of $\frac{1}{2}a$ and B_2, *i.e.*, $\frac{1}{6}aB_2$.

To determine the measure of the volume of the other pyramids join RH, RK, RL, RM, RN, RP and RQ, also join BK. Let the measures of the areas of the triangles R HK, R KL, R LM, R MN, R NP, R PQ, R QH be denoted respectively by m_1, m_2, m_3, m_4, m_5, m_6 and m_7, therefore

$$m_1+m_2+m_3+m_4+m_5+m_6+m_7=M.$$

Because CG is bisected in K, the triangle BCG is double the triangle BKG. Because BG is bisected in H, the triangle BKG is double of the triangle HKG. Therefore the triangle BCG is four times the triangle HKG. Therefore the pyramid RBCG is four times the pyramid RHKG. But taking G as the apex and RHK as the base of RHKG, the altitude of this pyramid is one-half that of the prismatoid therefore the measure of the volume of RHKG is one-third of the product of $\frac{1}{2}$ a and m_1., *i. e.*, $\frac{1}{6}am_1$. Therefore

the measure of the volume of RBCG is $\frac{4}{6}am_1$.

In like manner it may be shown that

the measure of the volume of RGCE is $\frac{4}{6}am_2$,
" " " " " " RCDE is $\frac{4}{6}am_3$,
" " " " " " RDAE is $\frac{4}{6}am_4$,
" " " " " " REFA is $\frac{4}{6}am_5$,
" " " " " " RABF is $\frac{4}{6}am_6$,
" " " " " " RFGB is $\frac{4}{6}am_7$.

Hence the sum of the measures of the volumes of the pyramids on the lateral faces of the prismatoid is

$$\tfrac{4}{6}a(m_1+m_2+m_3+m_4+m_5+m_6+m_7)=\tfrac{4}{6}\,a\,M.$$

Adding to this sum the measure of the volume of the basal pyramids RABCD and REFG, the measure of the volume of the whole prismatoid is found to be

$$\tfrac{1}{6}aB_1+\tfrac{4}{6}aM+\tfrac{1}{6}aB_2,$$

or

$$V_d=\tfrac{1}{6}a(B_1+4M+B_2).$$

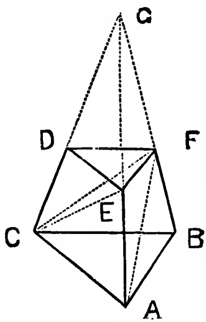
This is known as the **Prismoidal Formula.** It is of the very highest importance, nearly all the elementary formulæ in stereometry being but special cases of it.

IV, *a. The measure of the volume of a frustum of a pyramid is ONE-THIRD of the product formed by multiplying the measure of the altitude by the sum obtained by adding the measures of the areas of the two parallel faces to the square root of the product of these two measures;* or

$$V_f=\tfrac{1}{3}a\left\{B_1+\left(B_1B_2\right)^{\frac{1}{2}}+B_2\right\}.$$

This theorem may easily be deduced from the Prismoidal Formula, but it may also be proved independently as follows:—

All cases of frusta on bases having four or more sides can be reduced to the case of frusta on triangular bases; for, by passing planes through any one lateral edge and all the other lateral edges except the two adjacent to the first edge, any frustum whose base has more than three sides will be resolved by these planes into an aggregate of triangular-based frusta which have all the same altitude as the given frustum and whose triangular bases together make up the base of the given frustum. Hence it will be sufficient to consider only frusta on triangular bases.



Let ABCDEF be the frustum of a tetrahedron Let a denote the measure of its altitude; B_1, the measure of the area of the base ABC; and B_2, the measure of the area of the face DEF.

Pass a plane through AC and F and another plane through EF and C, thus resolving the frustum into the three tetrahedra ABCF, DEFC, ACEF. Let V_1, V_2 and V_3 denote the measures of the volumes of these several tetrahedra, therefore

$$V_1+V_2+V_3=V_f$$

Complete the pyramid of which ABCDEF is a frustum by producing the lateral faces,—and thereby producing the lateral edges,—to meet in a common point G.

Taking the triangle ABC as the base of ABCF, the tetrahedron and the frustum have the same altitude; therefore the measure of the volume of ABCF is $\frac{1}{3}aB_1$, or

$$V_1 = \tfrac{1}{3}aB_1.$$

Taking the triangle DEF as the base of DEFC, the tetrahedron and the frustum have the same altitude; therefore the measure of the volume of DEFC is $\frac{1}{3}aB_2$, or

$$V_2 = \tfrac{1}{3}aB_2.$$

Taking C as the common summit of the tetrahedra ABFC and AEFC, these two pyramids will have the same altitude, and therefore in determining the ratio of their volumes their common altitude may be omitted as being merely a common factor. The volumes of the tetrahedra will therefore have the same ratio as the areas of their bases have, or

$$\frac{\text{ABFC}}{\text{AEFC}} = \frac{\text{ABF}}{\text{AEF}}. \tag{1}$$

Taking F as the common summit of the tetrahedra CAEF and CDEF, these two pyramids will have the same altitude, and therefore in determining the ratio of their volumes, their common altitude may be omitted. The volumes of the tetrahedra will therefore have the same ratio as the areas of their bases have, or

$$\frac{\text{CAEF}}{\text{CDEF}} = \frac{\text{CAE}}{\text{CDE}}. \tag{2}$$

AB and EF being parallel, the triangles ABF and AEF have the same altitude, viz., the perpendicular distance of EF from AB; the areas of these triangles will therefore have the same ratio as the lengths of their bases AB and EF have, or

$$\frac{\text{ABF}}{\text{AEF}} = \frac{\text{AB}}{\text{EF}}. \tag{3}$$

Also $$\frac{\text{AB}}{\text{EF}} = \frac{\text{AG}}{\text{EG}}. \tag{4}$$

CA and DE being parallel, the triangles CAE and CDE have the same altitude ; the areas of these triangles will therefore have the same ratio as the lengths of their bases CA and DE have, or

$$\frac{CAE}{CDE}=\frac{CA}{DE}. \qquad (5)$$

Also
$$\frac{CA}{DE}=\frac{AG}{EG}. \qquad (6)$$

Collecting the equalities numbered (1), (3), (4), (6), (5) and (2) and arranging them in the order here indicated, we obtain

$$\frac{ABFC}{AEFC}=\frac{ABF}{AEF}=\frac{AB}{EF}=\frac{AG}{EG}=\frac{CA}{DE}=\frac{CAE}{CDE}=\frac{CAEF}{CDEF},$$

therefore
$$\frac{ABFC}{AEFC}=\frac{AEFC}{DEFC}. \qquad (7)$$

For the volumes of the three tetrahedra ABFC, DEFC, AEFC substitute the measures of these volumes in terms of a common unit and (7) becomes

$$\frac{V_1}{V_3}=\frac{V_3}{V_2}.$$

$$\therefore \quad V_3^2=V_1V_2$$

$$\therefore \quad V_3=(V_1V_2)^{\frac{1}{2}}=(\tfrac{1}{3}\,aB_1\times\tfrac{1}{3}\,aB_2)^{\frac{1}{2}}=\tfrac{1}{3}\,a(B_1B_2)^{\frac{1}{2}}.$$

$$\therefore \quad V_f=\tfrac{1}{3}\,aB_1+\tfrac{1}{3}\,aB_2+\tfrac{1}{3}\,a(B_1B_3)^{\frac{1}{2}}$$

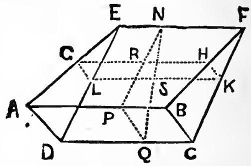
$$=\tfrac{1}{3}\,a\left\{B_1+(B_1B_2)^{\frac{1}{2}}+B_2\right\}.$$

IV, *b*. *The measure of the volume of a wedge is ONE-SIXTH of the continued product of the measure of the altitude of the wedge, the measure of the width of the base and the sum of the measures of the lengths of the three parallel edges, or*

$$V_w=\tfrac{1}{6}\,ab(b_1+b_2+b_3)$$

in which a denotes the measure of the altitude of the wedge, b denotes the measure of the width of the base and b_1, b_2 and b_3 denote the measures of the respective lengths of the three co-parallel edges.

Let ABCDEF be a wedge on the base ABCD. It may be treated as a prismatoid whose base is a trapezoid and whose face parallel to the base has become reduced to the straight line EF.



Let NPQ be a plane section of the wedge at right angles to the edge EF and therefore also at right angles to the edges AB and CD which are parallel to EF. The length of the line PQ is the width of the base and the length of the perpendicular from N on PQ is the altitude of the wedge. Hence the measure of the length of PQ is b, and the measure of the length of the perpendicular from N on PQ is a.

Let GHKL be the midcross-section of the prismatoid and let it cut the triangle NPQ in the straight line RS which will therefore be parallel to PQ. R is the mid-point of NP and S is the mid-point of NQ, therefore $RS=\frac{1}{2}PQ$, and therefore the measure of RS is $\frac{1}{2}b$.

The measure of the length of AB is b_1, that of the length of EF is b_3, and G and H are the respective mid-points of AE and BF, therefore the measure of the length of GH is $\frac{1}{2}(b_1+b_3)$.

The measure of the length of CD is b_2, that of the length of FE is b_3, and K and L are the respective mid-points of CF and DE, therefore the measure of the length of KL is $\frac{1}{2}(b_2+b_3)$.

Applying the Prismoidal Formula, the measure of the volume of the wedge is

$$\frac{1}{6}a(B_1+4M+B_2) \qquad (1)$$

B_1 is the measure of the area of the trapezoid ABCD. The measures of the lengths of the parallel sides of this trapezoid are b_1 and b_2 respectively and the measure of its width is b.

$$\therefore \quad B_1=\frac{1}{2}b(b_1+b_2). \qquad (2)$$

M is the measure of the area of the trapezoid GHKL. The measures of the lengths of the parallel sides of this trapezoid are $\frac{1}{2}(b_1+b_3)$ and $\frac{1}{2}(b_2+b_3)$ respectively and the measure of its width is $\frac{1}{2}b$,

$$\therefore \quad M=\frac{1}{4}b\left[\frac{1}{2}(b_1+b_3)+\frac{1}{2}(b_2+b_3)\right]$$

$$\therefore \quad 4M=b(\tfrac{1}{2}b_1+\tfrac{1}{2}b_2+b_3). \qquad (3)$$

B_2 is the measure of the *area* of the line EF,

$\therefore \quad B_2 = 0.$ (4)

Substitute in (1) the values of B_1, $4M$ and B_2 given in (2), (3) and (4),

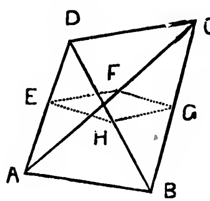
$$\therefore \quad V_w = \tfrac{1}{6} a \left[\tfrac{1}{2} b(b_1 + b_2) + b(\tfrac{1}{2} b_1 + \tfrac{1}{2} b_2 + b_3)\right]$$
$$= \tfrac{1}{6} ab(b_1 + b_2 + b_3).$$

In the case of the common wedge or wedge on a rectangular base, b and b_1 are the measures of the lengths of adjacent basal edges and $b_2 = b_1$

$$\therefore \quad V_w = ab(2\,b_1 + b_3).$$

IV, c. *The measure of the volume of a tetrahedron is TWO-THIRDS of the product of the measure of the perpendicular distance between any two opposite edges and the measure of the area of the parallelogram whose angular points are the mid-points of the other four edges of the tetrahedron,* or

$$V_t = \tfrac{2}{3} aM.$$



The tetrahedron is a prismatoid whose parallel faces are reduced to two straight lines, and the mid-parallelogram is its mid-cross-section, therefore by the Prismoidal Formula,

$$V_t = \tfrac{2}{3} aM.$$

Each side of the mid-parallelogram is equal to half of the edge of the tetrahedron parallel to the side, therefore, if the midcross-section of the tetrahedron be a rectangle

$$M = c_1 c_2$$

and $\quad V'_t = \tfrac{2}{3} ac_1c_2,$

in which c_1 and c_2 denote half the measures of the lengths of the two edges parallel to the midcross-section. In this case the midcross-section divides the tetrahedron into two wedges (hemitetrahedra) whose altitudes are equal as also are their volumes.

It is worthy of notice that if a prism, a hemitetrahedron and a pyramid are on equal bases and are of the same altitude, the volume of the prism is thrice and the volume of the hemitetrahedron is twice that of the pyramid, or

$$V_p = aB, \quad V_{ht} = \tfrac{2}{3} aB, \quad V_y = \tfrac{1}{3} aB.$$

Example 1. An iron tank in the form of a hollow cube whose sides, bottom and top are all and everywhere of the same thickness, has a capacity of 381 gallons. The length of an outside edge of the tank is 4 ft. Find the thickness of the sides.

The capacity of the tank is 381 gal.

$$= 277{\cdot}118 \text{ cu. in.} \times 381.$$
$$= (277{\cdot}118 \times 381) \text{ cu. in.}$$

The length of the edge of a cube of this capacity is

$$(277{\cdot}118 \times 381)^{\frac{1}{3}} \text{ in.}$$

The cube root of $277{\cdot}118 \times 381$ may be obtained directly by multiplication and evolution, or it may be computed by the aid of logarithms thus :—

$$\log (277{\cdot}118 \times 381)^{\frac{1}{3}} = \tfrac{1}{3}(\log 277{\cdot}118 + \log 381)$$
$$= \tfrac{1}{3}(2{\cdot}442665 + 2{\cdot}580925)$$
$$= 1{\cdot}674530 = \log 47{\cdot}264.$$

$\therefore$ $(277{\cdot}118 \times 381)^{\frac{1}{3}}$ in. $= 47{\cdot}264$ in.

$\therefore$ the length of an inside edge of the tank is 47·264 in.

The 〃 〃 〃 outside 〃 〃 〃 〃 〃 48 in.

The difference between the lengths of an outside and an inside edge is double the thickness of the sides ;

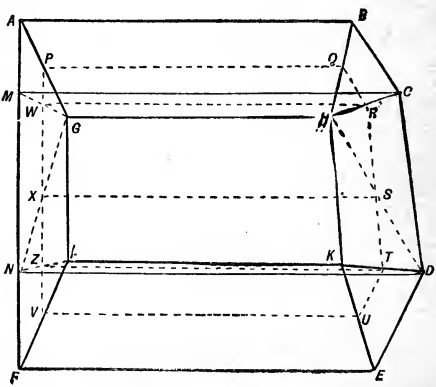
$\therefore$ double the thickness of the sides is 48 in. − 47·264 in.

$$= {\cdot}736 \text{ in.}$$

$\therefore$ the thickness of the sides is $\frac{1}{2}$ of ·736 in. = ·368 in. *which is very nearly three-eighths of an inch.*

Example 2. Find the air-capacity of an attic given the accompanying plan and the following dimensions :—

The floor of the attic is a hexagon ABCDEF ; the ceiling is a trapezoid GHKL ; AB, MC, ND, FE, GH and LK are all parallel to each other, AF and GL are also parallel to each other, AF is at right angles



to AB and therefore also to MC, ND and FE, and GL is at right angles to GH and LK. AB = 28 ft., MC = 32 ft., ND = 34 ft., FE = 30 ft.; GH = 22 ft., LK = 23 ft.; AM = 6 ft., MN = 15 ft., NF = 8 ft.; GL = 12 ft.; and the vertical height of the ceiling above the floor is 10 ft. 6 in.

The area of the floor is the sum of the areas of the three trapezoids ABCM, MCDN, NDEF.

The area of ABCM is $\frac{1}{2}$ of $6(28+32)$ sq. ft. $=180$ sq. ft.
" " " MCDN is $\frac{1}{2}$ of $15(32+34)$ sq. ft. $=495$ sq. ft.
" " " NDEF is $\frac{1}{2}$ of $8(34+30)$ sq. ft. $=256$ sq. ft.

$\therefore$ the area of the floor is $(180+495+256)$ sq. ft. $=931$ sq. ft. (1)

The area of the ceiling is $\frac{1}{2}$ of $12(22+23)$ sq. ft. $=270$ sq. ft. (2)

The area of the midcross-section is the sum of the areas of the *four* trapezoids PQRW, WRSX, XSTZ, ZTUV. To determine the areas of these trapezoids, the lengths of their parallel sides and of the normal distances between these sides must first be found.

$$PQ=\tfrac{1}{2}(AB+GH)=\tfrac{1}{2}(28+22)\text{ ft.}=25\text{ ft.}$$
$$WR=\tfrac{1}{2}(MC+GH)=\tfrac{1}{2}(32+22)\text{ ft.}=27\text{ ft.}$$
$$XS=\tfrac{1}{2}(ND+GH)=\tfrac{1}{2}(34+22)\text{ ft.}=28\text{ ft.}$$
$$ZT=\tfrac{1}{2}(ND+LK)=\tfrac{1}{2}(34+23)\text{ ft.}=28\tfrac{1}{2}\text{ ft.}$$
$$VU=\tfrac{1}{2}(FE+LK)=\tfrac{1}{2}(30+23)\text{ ft.}=26\tfrac{1}{2}\text{ ft.}$$
$$PW=\tfrac{1}{2}AM=\tfrac{1}{2}\text{ of } 6\text{ ft.}=3\text{ ft.}$$
$$WX=\tfrac{1}{2}MN=\tfrac{1}{2}\text{ of } 15\text{ ft.}=7\tfrac{1}{2}\text{ ft.}$$
$$XZ=\tfrac{1}{2}GL=\tfrac{1}{2}\text{ of } 12\text{ ft.}=6\text{ ft.}$$
$$ZV=\tfrac{1}{2}NF=\tfrac{1}{2}\text{ of } 8\text{ ft.}=4\text{ ft.}$$

The area of PQRW is $\frac{1}{2}$ of $3(25+27)$ sq. ft. $=78$ sq. ft.
" " " WRSX is $\frac{1}{2}$ of $7\frac{1}{2}(27+28)$ sq. ft. $=206\frac{1}{4}$ sq. ft.
" " " XSTZ is $\frac{1}{2}$ of $6(28+28\frac{1}{2})$ sq. ft. $=169\frac{1}{2}$ sq. ft.
" " " ZTUV is $\frac{1}{2}$ of $4(28\frac{1}{2}+26\frac{1}{2})$ sq. ft. $=110$ sq. ft.

The area of the midcross-section is $(78+206\frac{1}{4}+169\frac{1}{2}+110)$ sq. ft. $=563\frac{3}{4}$ sq. ft. (3)

$\therefore$ *the capacity of the attic is* $\frac{1}{6}$ of $10\frac{1}{2}\{931+4(563\frac{3}{4})+270\}$ cu. ft. $=6048$ cu. ft.

EXERCISE XXIII.

1. Find the number of cubic inches in the volume of a quad measuring a foot by a yard by a metre.

2. Find to the nearest gallon the volume of a quad measuring 75 in. by 87·5 in. by 126·875 in.

3. Find, correct to four significant figures, the length of the inside edge of a cubical vessel which will just hold 10 gallons.

4. Find, correct to four significant figures, the length of the inside edge of a cubical vessel which will just hold 100 gallons.

5. One acre of a certain wheat-field yielded 2100 lb. of wheat weighing 7 lb. $10\frac{1}{2}$ oz. per *measured* gallon. At this rate what was the yield in cubic inches per square yard of the field, and what would be the length of the edge of a cube equal to the yield of a square inch of the acre?

6. A quadrate reservoir is 147 ft. 8 in. long, 103 ft. 6 in. wide and 11 ft. 9 in. deep. When the reservoir is nearly full of water, how many cubic feet of water must be drawn off that the water-surface may sink 4 ft. 4 in.?

7. Find to the nearest gallon the capacity of an open quadrate tank measuring 7′ 6″ by 6′ 4″ by 5′ 8″ externally; the material of which the tank is made being $1\frac{1}{4}$ inches in thickness.

8. Three cubes of lead measuring respectively $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{5}{6}$ of an inch on the edge were melted together and cast into a single cube. Find the length of the edge of the cube thus formed, neglecting loss of lead in melting and casting.

9. Four cubes of lead measuring respectively 6, 7, 8 and 9 inches on the edge were melted together and cast into a single cube. Find the length of the edge of the cube thus formed, if 4 per cent. of the lead was lost in the melting and casting.

10. Three cubes of lead measuring respectively 3·1, 3·6 and 3·7 inches on the edge were melted and cast into a single quadrate lump 5 in. long by 4·1 in. wide. Find the height of the quad, neglecting loss of lead in melting and casting.

11. A cube of lead measuring 64·1 mm. on the edge was melted and cast in the form of a quad with square ends and with length three times the width. Find the dimensions of the quad, neglecting loss of lead in melting and casting.

12. The length of a quad is thrice its width and the width is double the height. Find the length of the quad, its volume being a cubic yard.

13. The three adjacent edges of a quad are to one another as 2 : 3 : 5 and its volume is a cubic metre. Find the length of the edges and the areas of the faces of the quad.

14. Find the volume of a cube the area of whose surface is 100·86 sq. in.

15. The surface of a cube measures 30 sq. in. Find the area of the surface of a cube of five times the volume of the former.

16. The volume of a cube is 30 cu. in. Find the volume of a cube whose surface has an area five times the area of the surface of the former cube.

17. A cube measures 5 in. on the edge. A second cube is of thrice the volume of the first. By how much does the length of an edge of the second cube exceed that of an edge of the first cube?

18. A cube measures 5 in. on the edge. Find the volume of a cube whose surface-area is thrice that of the former cube.

19. A quadrate cistern is 5 ft. wide by 6 ft. long by 4 ft. 2 in. deep. Its width and its length are each increased by 6 inches. How much deeper must it be made that the total increase of its capacity may be 250 gallons?

20. If a quad has its length, its breadth and its height respectively a twelfth, a thirteenth and a fourteenth as long again as the corresponding dimensions of another quad; show that the volume of the first quad will be a quarter as large again as the volume of the second quad.

21. By raising the temperature of a cube of iron, the length of each of its edges was increased by ·5 per cent. Find correct to four decimals the ratio of increase in the volume of the cube.

22. Each edge of a cube is diminished by a tenth of its length. By what fraction of itself is the volume diminished? By what fraction of itself is the area of the surface diminished?

23. By taking the decimeter as equal to 4 in. what percentage of error is introduced into (*a*), linear measurements; (*b*), areal measurements; (*c*), volume measurements?

24. The height of a solid six-inch cube of India-rubber is diminished by pressure to 5·85 in. If the volume of the solid remain the same and the lateral expansion be uniform throughout, what will be the dimensions of the new base?

25. The length of a quad is 7 in., its height is 3 in., and the total area of its surface is a square foot. Find the volume of the quad.

26. The length of a quad is 13·3 in., its width is 8·4 in., and the total area of its surface is 466·5 sq. in. Find its volume.

27. The width of a quad is 7·05 ft., its height is 3·13 ft. and the total area of its surface is 30 sq. yd. Find its volume.

28. The width of a quad is 371 mm., its height is 284 mm., its volume is a cubic metre. Find the area of its base.

29. Find the area of the surface of a quad 31·62 cm. wide by 38·73 cm. long and of ·03 cubic metre volume.

30. The area of the base of a quad is 71·288 sq. in., that of a side of the quad is 56·868 sq. in. and that of an end is 52·65 sq. in. Find the volume of the quad.

31. The length of the perimeter of the base of a quad is 20 in.; the area of the base is 20·16 sq. in.; and the total area of the surface of the quad is 90·32 sq. in. Find the volume of the quad.

32. The perimeter of the base of a quad measures 25·4 in., the area of one end of the quad is 23·1 sq. in., and the volume of the quad is 166·32 cu. in. Find the lengths of the edges of the quad.

33. Find the measure of the length of the edge of a cube the measure of whose volume is equal to the measure of the area of its surface.

34. The measure of the volume of a quad two of whose edges measure 3 in. and 4 in. respectively, is the same as the measure of the area of the whole surface of the quad. Find the length of the third edge.

35. The area of the surface of a quad on a square base is 192 sq. in. The area of the base is equal to the sum of the areas of the two sides and two ends. Find the volume of the quad.

36. The volume of a quad on a square base is 6572 cu. in., the height of the quad is 11·8 in. Find the length of an edge of the base.

37. Find the weight of the air in a rectangular room measuring 27′ 8″ by 23′ 5″ by 12′ 4″, the weight of the air being the ·001295 of the weight of an equal volume of water.

38. If a cubic foot of gold weigh 1200 lb., find the thickness of gold-leaf of which 1200 leaves $3\frac{1}{2}$ inches square weigh an ounce troy.

39. A quadrate block of stone measuring 5·297″ by 7·472″ by 9·57″ weighs 38·14 lb. Compare the weight of any volume of the stone with the weight of an equal volume of water at 62° F.

40. What will be the weight of 36 iron rods each 14 ft. long and of cross-section $\frac{3}{8}$ of an inch square, if the specific gravity of the iron be 7·7?

41. What length of a bar of iron will weigh 10 lb., the cross-section of the bar being a rectangle measuring $\frac{3}{8}$ in. by $1\frac{1}{4}$ in.?

42. What weight will just keep under water a stick of square-timber measuring 36 ft. by 10 inches square, the specific gravity of the wood being ·725?

43. What sized cube of iron placed on a quad of dry pine measuring 8 ft. by 5·6 ft. by 4 in. will just sink the quad in water, the specific gravity of the pine being ·472 and that of the iron 7·7?

44. An open quadrate tank is 5 ft. 6 in. long, 4 ft. 3 in. wide and 3 ft. 8 in. high, the sides and bottom of the tank are $\frac{3}{8}$ in. thick. Find the number of cubic inches of material in the vessel.

45. Find the weight of a hollow iron cube measuring 2·735 inches on the outer edge, the thickness of the iron being ·167 of an inch and its specific gravity 7·7.

46. Find the thickness of the sides of an iron box in the form of a hollow cube, which weighs 266 lb. when empty and 566 lb. when filled with water; the sides, bottom and top being all of the same thickness and the specific gravity of the iron 7·7.

47. The sides, bottom and lid of a quadrate box have a uniform thickness of $\frac{3}{8}$ in. The outside measurements of the box are 8 in. by 12·5 in. by 16·25 in. How many cubes each $\frac{3}{8}$ of an inch on the edge, will the box hold.

48. Find the thickness of the material of which a closed hollow iron cube is constructed, if the cube weigh 33 lb. 4 oz. and measure 10·5 in. on an outside edge, the specific gravity of iron being 7·7.

49. An iron cube is coated with a uniform thickness of gold. Find the thickness of the gold if the coated-cube is 3 inches long and weighs 7·525 lb., the specific gravity of the gold being 19·25 and that of the iron 7·7.

50. Find the volume of a right triangular prism 8 in. long, the terminal triangles being right-angled and the lengths of the sides containing the right angle being 1·2 in. and 2·1 in. respectively.

51. The normal length of a triangular prism is 79 mm., the length of an edge of one of the terminal triangles is 43 mm., and the length of the perpendicular on that edge from the opposite superficial angle is 29 mm. Find the volume of the prism in cubic centimetres.

(The normal length is the length measured at right angles to the parallel ends. If one of these ends be taken as the base of the prism, the normal length will be the altitude of the solid.)

52. The altitude of a prism is 17·3 in. and its base is a parallelogram of length 25·75 in. and normal width 9·7 in. Find the volume of the prism.

53. The normal length of a trapezoidal prism is 97 ft. 6 in., the lengths of the parallel edges of the trapezoidal ends are 37 ft. 5 in. and 23 ft. 4 in. respectively and the perpendicular distance between these edges is 9 ft. 6 in. Find the volume of the prism.

54. The length of a prism is 9 ft. 4 in. A right cross-section of the prism is a quadrilateral, one of whose diagonals measures 3 ft. 7 in. and the perpendiculars on that diagonal from the opposite angles of the section are respectively 1 ft. 5 in. and 1 ft. 7 in. long. Find the volume of the prism.

55. The fabled wall of China was said to be 25 ft. wide at the bottom, 15 ft. wide at the top, 20 ft. high and 1500 miles long. How many cubic yards of material would such a wall contain.

56. How many cubic yards of earth must be removed in the digging of a ditch 147 ft. long, 8 ft. wide at the top, 5 ft. wide at the bottom and 4 ft. 6 in. deep, the ends of the ditch being vertical?

57. How many gallons of water will fill a horse trough 7 ft. 6 in. long, 10 in. deep, 14 in. wide at the top and 11 in. wide at the bottom; the ends of the trough being at right angles to the bottom and sides?

58. How many prismatic bars of lead each 10·5 in. long must be melted down to make a cube 6·25 in. on the edge, a right cross-section of each bar being a trapezoid measuring 1·3 in. and ·6 in. respectively on the parallel sides and ·5 in. in perpendicular distance between these sides; ·5 per cent. of the lead being lost in the melting?

59. 7843 cu. yd. of earth were removed in digging a ditch 2 ft. 9 in. deep, 4 ft. 6 in. wide at the top and 3 ft. wide at the bottom. Find the length of the ditch assuming that the ends were vertical.

60. The cross-section of a canal is 36 ft. wide at the surface of the water and 20 ft. wide at the bottom; what must be the depth of the water if 100 yd. in length of the canal contain 589315 gallons of water?

61. A ditch 125 yd. long is filled to a depth of 1 ft. 9 in. by 10571 gal. of water. What must be the width of the ditch at the bottom if the width at the surface of the water be 3 ft. and the ends of the ditch be vertical?

62. A stream flows at the rate of 3 miles per hour through a trough whose cross-section is a trapezoid. The width of the bottom of the trough is 21 inches, the depth of the water is 4·5 inches and the width of the surface of the water in the trough is 25 inches. How many gallons flows through the trough per minute?

63. A prism of 21 inches altitude weighs one ton. Find the area of the base, the material of the prism weighing 524 lb. per cubic foot.

64. The volume of a prism is 6 cu. ft.; its height is 9 in., and its base is an isosceles right-angled triangle. Find the lengths of the edges of the base.

65. A shed with a single sloping roof is 22 ft. long by 12 ft. wide; the height of the roof above the floor is 12 ft. at the front and 8 ft. at the back. Find the total capacity of the shed.

66. A school-room with attic ceiling is 32 ft. long by 28 ft. wide. The ceiling at the side walls is 10 ft. above the floor and slopes upward until it attains a height of 14 ft. 6 in. and then becomes level, the width of the level part being 12 ft. The ceiling meets the end walls at right angles. Find the air-capacity of the school-room.

67. A parallelepiped is cut by two planes which neither meet the ends nor intersect. The area of a right cross-section is 96 sq. in. and the lengths between the cutting planes of the four parallel edges are respectively 6 in. 7·5 in. 10 in. and 8·5 in. Find the volume of the portion of the parallelepiped between the cutting planes.

68. The base of a pyramid is a triangle one side of which measures 15·3 in.; the length of the perpendicular on that side from the opposite angle of the base is 9·6 in. and the altitude of the pyramid is 12·5 in. Find the volume of the pyramid.

69. Find the volume of a tetrahedron whose base is a right-angled triangle, the sides of the base containing the right angle measuring 17 in. and 19 in. respectively and the altitude of the tetrahedron being 18 in.

70. Find the volume of a tetrahedron whose base is a right-angled isosceles triangle, the altitude of the tetrahedron being 7 ft. 5 in. and the length of the hypothenuse of the base being 5 ft. 7 in.

71. Find the weight of the pyramid formed by cutting off a corner of a cube of lead by a plane passing through three adjacent corners, the length of an edge of the cube being 2·5 in. and the specific gravity of lead being 11·4.

72. One of the corners of a quad of gold is cut off by a plane which meets the three conterminous edges, 2·7 inches, 4·3 inches and 3·6 inches respectively from their common point. Find the value of the piece cut off, the specific gravity of the gold being 17·66 and its value $18·95 per ounce troy.

73. The base of a pyramid is a square whose side is 3·45 ft. long. The altitude of the pyramid is 4·75 ft. Find the volume of the pyramid.

74. Find the volume of a pyramid whose altitude is 4 ft. 5 in. and whose base is a rectangle measuring 3 ft. 4 in. by 3 ft. 9 in.

75. The altitude of a pyramid is 2 ft. 3 in., its base is a trapezoid whose parallel sides measure 1 ft. 9 in. and 1 ft. 3 in. respectively, the perpendicular distance between these sides being 1 ft. 4 in. Find the volume of the pyramid.

76. The base of a pyramid is a square 2 ft. 7 in. long and its volume is 3·2 cu. ft. Find the altitude of the pyramid.

77. The volume of a pyramid on a rectangular base is half a cubic yard. The length of the base is 3 ft. 9 in. and the altitude of the pyramid is 3·2 ft. Find the width of the base.

78. The volume of a pyramid on a square base is a cubic yard and its altitude is a yard. Find the length of an edge of the base.

79. The volume of a pyramid on a square base is 30·87 cu. in. and the altitude of the pyramid is equal to the length of an edge of the base. Find the altitude.

80. The volume of a pyramid is 77 cu. in. The base of the pyramid is a quadrilateral ; the length of one of the diagonals of the base is 15 in. and the lengths of the perpendiculars on this diagonal from the opposite angles of the base are 10·6 in. and 9 in. Find the altitude of the pyramid.

81. The base of a pyramid is a square 15 in. long and the altitude of the pyramid is 16 in. The base of another pyramid is a rectangle 16 in. long by 12·5 in. wide. Find the altitude of the second pyramid, the volumes of the two pyramids being equal.

82. The Great Pyramid of Egypt when complete was 480 ft. 9 in. in height, and its base was a square 764 ft. in length ; in its present condition the pyramid is 450 ft. 9 in. high and its base is a square 746 ft. long and wide. Find to the nearest cubic yard the volume of the pyramid in its complete and also in its present state.

83. The representative gold pyramid in the International Exhibition of 1862 was 10 ft. square at the base and 44 ft. $9\frac{1}{4}$ in. high. Find the volume of the pyramid, and the weight and the value of the gold represented by it, taking the specific gravity of the gold at 19·25 and its value at $20·67 per ounce troy.

84. Since the construction of the pyramid mentioned in problem 83, about 25,000,000 ounces troy of gold have been mined ; how much higher would the pyramid require to be made to include this quantity ?

85. The base of a pyramid is a square whose sides are 25 in. long. The altitude is 16 in. A plane parallel to the base divides the pyramid into parts of equal volume. Find the perpendicular height of the plane above the base.

86. The base of a pyramid is a trapezoid whose parallel sides measure 19·5 in. and 13·7 in., the perpendicular distance between them being 12·6 in. The altitude of the pyramid is 14 in. At what height above the base must a plane parallel t the base be drawn, that it may bisect the pyramid ?

87. The base of a pyramid is a square whose area is 7 sq. ft. The altitude of the pyramid is one yard. A plane parallel to the base so divides the pyramid that the volume of the frustum between the base and the plane is double the volume of the pyramid abov the plane. Find the height of the frustum.

88. The altitude of a pyramid is 15 in. A plane parallel to the base divides the pyramid into two parts whose volumes are such that thrice the volume of the frustum between the plane and the base is equal to five times the volume of the pyramid above the plane. Find the height of the frustum.

89. Find the volume of a prismoid whose top and bottom are rectangles the corresponding dimensions of which are 3 ft. by 2 ft. and 5 ft. by 3·5 ft., the altitude of the prismoid being 3·5 ft.

90. Find the volume of a prismoid whose top and bottom are rectangles the corresponding dimensions of which are 3 ft. by 2 ft. and 3·5 ft. by 5 ft., the altitude of the prismoid being 3·5 ft.

91. Find the capacity of a cart the top of which measures 4′ 3″ by 3′ 8″; the bottom, 3′ 9″ by 3′ 2″; and the depth, 2′ 3″.

92. How many gallons of water will fill a ditch 2 ft. deep, the top and bottom of the ditch being rectangles whose corresponding dimensions are 148 ft. by 3 ft. 4 in. and 146 ft. 6 in. by 2 ft. 3 in.?

93. Find the weight of an iron shaft whose ends are rectangles, one end measuring 10·5 in. by 17 in., the other end measuring 7 in. by 12 in., the length of the shaft being 13 ft. 6 in. and the specific gravity of the iron, 7·7.

94. What weight will just sink a scow in the form of a hollow prismoid with rectangular base, the length of the scow over all being 14 ft. 1 in.; its width, 3 ft. 8 in.; its full depth, 2 ft. 11 in.; the length of the bottom outside, 12 ft.; the width of the bottom 3 ft. and its weight 920 lb.

95. Find the volume of a pile of broken stones, the base of the pile being a rectangle measuring 13 ft. 6 in. by 7 ft. 5 in.; the top of the pile a rectangle measuring 12 ft. 2 in. by 6 ft.; and the height of the pile being 2 ft. 10 in.

96. It is usual to take as the measure of the volume of a pile of broken stones the product of the measure of the altitude of the pile and the measure of the area of its midcross-section. By how much would the volume thus calculated be in defect of the actual volume in the case of the pile described in the problem immediately preceding.

97. Find the number of cubic yards in a railway cutting in the form of a prismoid with trapezoidal ends; the lengths of the parallel sides at one end being 124 ft. and 33 ft., and the distance between them 28 ft.; the corresponding dimensions of the other end being 104 ft., 33 ft. and 21 ft. respectively; and the distance between the ends being 235·5 yd.

98. A straight ditch with a fall of 1 ft. in 300 yd. is to be dug in level ground. The sides are to slope 1 in 1, the bottom is to be 4 ft. wide, and the depth at the upper end is to be 3 ft. 6 in. Find the number of cubic yards of earth that will require to be removed in digging the first 1000 yards of the ditch.

99. How many cubic yards of earth will be excavated in making a railway cutting through ground whose surface is an inclined plane rising in the same direction as the rails, the length of the cutting being 123 yd.; the width at the bottom 33 ft.; the width at the top at one end, 66 ft.; at the other end, 100 ft.; and the depths of these ends, 22 ft. and 48 ft. respectively?

100. A railway-embankment is made on ground which falls at 20 ft. per mile in the same direction of the rails, which themselves fall 1 in 800. The length of the embankment is 2100 yd.; its width at the top is 33 ft., the slope of the sides is 1 in 1 and the height at the upper end is 1 ft. 8 in. Find the number of cubic yards of earth in the embankment.

101. The ends of a prismoid are rectangles whose corresponding dimensions are 17·3 in. by 11·4 in. and 9·5 in. by 6·6 in.; the altitude of the prismoid is 21·6 in. The prismoid is divided in two parts by a plane parallel to the ends and midway between them. Find the volume of each part.

102. The ends of a prismoid are rectangles whose corresponding dimensions are 7 ft. by 5 ft. and 3 ft. by 2 ft. The prismoid is divided by planes parallel to the ends, into three prismoids each 1 ft. 8 in. in altitude. Find the volume of each of these three prismoids.

103. A prismoid, one of whose ends is a rectangle measuring 15 in. by 12·5 in., the opposite end measuring 9·6 in. by 8·4 in., and whose altitude is 2 ft. is cut into two wedges by a plane which passes through the longer edge of one end and the opposite longer edge of the other end. Find the volumes of the wedges.

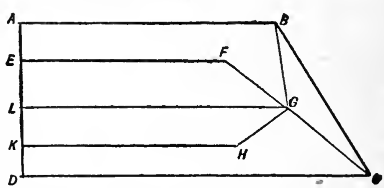
104. The height of a wedge is 18 in., the length of the edge is 16 in., and the dimensions of the base which is a rectangle, are 12 in. by 8 in. The wedge is divided into two parts by a plane parallel to the base and midway between the base and the edge. Find the volume of each part.

105. Had the wedge described in problem 104 been bisected by the plane parallel to the base, what would have been the height of the plane above the base?

106. The length of the edge of a wedge is 8·5 in., the length of the base which is a parallelogram is 6·3 in. and its normal width is 4·5 in., the height of the wedge is 15 in. The wedge is divided into three parts of equal height by planes parallel to the base. Find the volume of each part.

107. The ends of a prismoid are rectangles whose corresponding dimensions are 18 in. by 15 in., and 10 in. by 18 in.; the height of the prismoid is 7 ft. 4 in. The prismoid is cut by a plane parallel to the ends and at a distance of 2 ft. from the larger end. Show that the section is a square, and find the volumes of the parts into which the plane divides the prismoid.

108. Find the number of cubic yards of earth in an embankment from the accompanying plan and following data:—



The base ABCD is a quadrilateral and the top EFGHK is a pentagon. The edges AB, EF, KH and DC are all parallel to each other, AD and EK are in a plane at right-angles to AB, and LG is parallel to EF. AB=96 yd., DC=124 yd.; EF=84 yd., LG=98 yd., KH=90 yd.; AE=18 ft., EL=16 ft., LK=18 ft., KD=12 ft., the last four measurements being 'in plan', *i.e.* being the horizontal distances between verticals through the points A, E, L, K and D. The height of the embankment is 18 ft.

109. The lengths of two opposite edges of a tetrahedron are 7·2 in. and 5·6 in. respectively and the perpendicular distance between them is 6·4 in. The midcross-section is a rectangle. Find the volume of the tetrahedron.

110. A rectangular tank 3 ft. long by 2 ft. 4 in. wide by 2 ft. 6 in. deep, rested on props 3 in. high, a prop at each corner. By accident one of the props was knocked out of its place and the cistern was tilted on the adjacent two until the unsupported corner touched the ground. How much less water would the tank hold in that position than it would hold when level?

111. The base of a wedge is a rectangle measuring 3·6 in. by 2·4 in., the length of the opposite edge is 3 in., the height of the wedge is 8 in. Find the volume 1° if the three-inch edge is parallel to the longer side of the base; 2° if it is parallel to the shorter side of the base.

112. The base of a sphenoid is a square whose sides are 10 in. long; the opposite edge is parallel to the diagonal of the base, and of the same length as the diagonal; the altitude of the sphenoid is 15 in. Find the volume of the sphenoid.

113. The lengths of the three parallel edges of a wedge are 7·5 in., 5·7 in. and 6·9 in. respectively and the area of a section at right angles to these edges 76 sq. in. Find the volume of the wedge.

114. The base of a wedge is a rectangle measuring 13·5 in. by 11·2 in., the length of the opposite edge is 5·4 in., this edge being parallel to the longer side of the base; the perpendicular distance of this edge from the plane of the base is 18 in. The wedge is divided into two pieces by a plane which intersects the edge opposite the base at a point distant 7·5 in. from one end and which cuts the two edges parallel to this edge at points distant 5·25 in. and 7·5 in. from the ends corresponding to that from which the 7·5 inches was measured. Find the volume of each part.

115. The base of a wedge is a rectangle measuring 18 in. by 15 in.; the opposite edge is parallel to the longer side of the base and is 10 in. long; the length of the perpendicular from this edge on the base is 21 in. Find the volume of the parts into which the wedge is cut by a plane passing through one end of the edge opposite the base and which is parallel to the triangular face at the other end.

116. The base of a wedge is a trapezoid whose parallel edges are 3 ft. and 1 ft. 9 in. long respectively and whose width at right angles to these sides is 15 in., the length of the edge opposite the base is 18 in., and the volume of the wedge is 2 cu. ft. Find the altitude of the wedge.

117. The length of a side of the base of a frustum of a square pyramid is 3′ 9″, that of a side of the top is 1′ 8″, the altitude of the frustum is 2′ 6″. Find the volume of the frustum.

118. Find the number of cubic feet in a stick of square timber 18″ square at one end, 14″ square at the other end and 36′ long.

119. Find the weight of a frustum of a square pyramid of marble, the height of the frustum being 6 ft. 6 in. ; the length of an edge of the base, 4 ft. 4 in., and of an edge of the top, 2 ft. 8 in., the weight of a cubic foot of marble 172 lb.

120. In the frustum of a square pyramid whose base-area is 2 sq. yd. and whose altitude is 4 ft. 6 in., the lengths of the basal edges are to those of the top edges as 3 to 2. Find the volume of the frustum.

121. The areas of the base and top of a frustum of an iron pyramid are 1 sq. ft. 48 sq. in. and 1 sq. ft. 3 sq. in. respectively and the weight of the pyramid is 888 lb. Find the height of the pyramid, the specific gravity of the iron being 7·11.

122. The altitude of a frustum of a square pyramid is equal to the length of a side of the base and is double of the length of a side of the top. Find the altitude, the volume of the frustum being 4 cu. ft.

123. A frustum of a pyramid has the area of its base nine times the area of its top. Compare its volume with that of a prism whose altitude and base-area are respectively the same as the altitude and the base-area of the frustum.

124. A frustum of a pyramid has the area of its base four times the area of its top. Compare its volume with that of a pyramid whose altitude and base-area are respectively the same as the altitude and the base-area of the frustum.

125. Find the volume of the frustum of a pyramid on a rectangular base measuring 4 ft. by 2 ft. 8 in., the height of the frustum being 5 ft. 8 in. and the length of the top, 3 ft. 6 in.

126. The volume of the frustum of a pyramid on a rectangular base is 3·6 cu. ft. The length of one side of the base is 1 ft. 6 in., the length of the corresponding side of the top is 10 in., the height of the frustum is 1 ft. 4 in. Find the lengths of the other sides of the base and top.

127. The base of the frustum of a pyramid is a rectangle whose length is double its width; the area of the top is half the area of the base; the height of the frustum is 3 ft. 8 in. and its volume is a cubic yard. Find the length of the base.

128. The base of a frustum of a pyramid is a trapezoid the lengths of whose parallel sides are 275 cm. and 225 cm. respectively, the distance between them being 192 cm. The height of the frustum is 2375 mm. and the width of its top is 148 cm. Find the volume of the frustum in cubic metres.

129. Find the area of the surface of a square pyramid whose basal edges are each 3′ 4″ long, the slant height of each side being 3′ 6″.

130. Find the area of the surface of a frustum of a square pyramid, the length of a side of the base being 18 in.; that of a side of the top, 6 in.; and the slant height of each of the lateral faces being 27 in.

131. Find the height and the width of a quad whose length is 3 ft. whose volume is 9 cu. ft. and the area of whose surface is 28 sq. ft. 108 sq. in.

132. A horse-trough 9 ft. long, 15 in. wide at the top and 10 in. wide at the bottom, and 12 in. deep, is full of water. If 30 gallons of water be drawn off by how many inches will the surface of the water in the trough sink? (The ends of the trough are vertical; the calculation is to be made to the nearest tenth of an inch.)

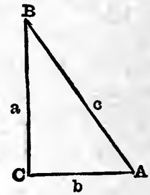
133. The cross-section of a canal is 33 ft. wide at the bottom and 58 ft. wide at a height of 10 ft. from the bottom. At what depth must the water in the canal stand that 1000 yd. in length of the canal may contain 4,545,725 gallons?

134. The volume of the frustum of a square pyramid is 172 cu. ft., the height of the frustum is 36 ft. and the length of a side of the base is 2 ft. 8 in. Find the length of each side of the top.

135. A covered rectangular tank whose dimensions are 3′ 6″ by 2′ 11″ by 1′ 9″ will hold just 81 gallons. What must be the thickness of the material of which the tank is made, the bottom, sides and top being all of the same thickness? (*This problem is of a type which is the inverse of the type to which problems 9 and 10 of Exercise VI belong. The calculations for these three problems will therefore follow parallel lines.*)

171. In any right-angled triangle, the squares on the sides containing the right angle are together equal to the square on the hypothenuse or side opposite the right angle. (Euclid I, 47.)

Let ABC be a triangle right-angled at C, and let a, b and c be the MEASURES of the lengths of the sides opposite the angles A, B and C respectively ; $\therefore$ a^2, b^2 and c^2 are the *measures* of the areas of the squares on these sides and it follows from this and the preceding proposition that



$$a^2+b^2=c^2, \qquad (1)$$

and $\therefore$ $$c=(a^2+b^2)^{\frac{1}{2}}, \qquad (A)$$

that is ;—*In any right-angled triangle, the measure of the length of the hypothenuse is the square root of the sum of the squares of the measures of the lengths of the sides containing the right angle.*

From (1) $$a^2=c^2-b^2=(c+b)(c-b), \qquad (2)$$

$\therefore$ $$a=\{(c+b)(c-b)\}^{\frac{1}{2}}, \qquad (B)$$

that is ;—*In any right-angled triangle, the measure of the length of either of the sides containing the right angle is the square root of the product of the sum and the difference of the measures of the lengths of the other two sides of the triangle.*

Example 1. The lengths of the sides containing the right angle of a right-angled triangle are 336 ft. and 527 ft. respectively ; find the length of the hypothenuse and of the perpendicular from the right angle on the hypothenuse.

Length of hypothenuse $=(336^2+527^2)^{\frac{1}{2}}$ ft.
$=(112896+277729)^{\frac{1}{2}}$ ft.
$=390625^{\frac{1}{2}}$ ft.
$=625$ ft.

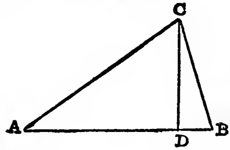
Measure of length of perpendicular on hypothenuse $\times 625$
$=$double of the measure of the area of the triangle
$=336\times 527$
$=177072$.

$\therefore$ the measure of the length of the perpendicular on the hypothenuse
$=177072\div 625$
$=283{\cdot}3152$;

$\therefore$ the length of the perpendicular on the hypothenuse $=283{\cdot}3152$ ft.

Example 2. The lengths of two of the sides of a triangle are 1590 mm. and 1037 mm. respectively, and the length of the perpendicular from the opposite angle on the longer of these sides is 988 mm. Find the length of the third side of the triangle.

Let ABC be the triangle, AB = 1590 mm., and BC = 1037 mm. From C let fall on AB the perpendicular CD then CD = 988 mm. (*The accompanying figure is drawn on a scale of* 1:64.)



$$DB = \{(1037 + 988) \times (1037 \quad 988)\}^{\frac{1}{2}} \text{ mm.}$$
$$= (2025 \times 49)^{\frac{1}{2}} \text{ mm.}$$
$$= 315 \text{ mm.}$$
$$\therefore \quad AD = 1590 \text{ mm.} - 315 \text{ mm.}$$
$$= 1275 \text{ mm.}$$
$$\therefore \quad AC = (1275^2 + 988^2)^{\frac{1}{2}} \text{ mm.}$$
$$= (1625625 + 976144)^{\frac{1}{2}} \text{ mm.}$$
$$= 2601769^{\frac{1}{2}} \text{ mm.}$$
$$= 1613 \text{ mm.}$$

172. If m and n be any two whole numbers then shall

$$m^2 - n^2, \ 2mn \text{ and } m^2 + n^2$$

be the measures of the lengths of the sides of a right-angled triangle.

For $(m^2 - n^2)^2 = m^4 - 2m^2n^2 + n^4,$

$(2mn)^2 = \quad 4m^2n^2$

and $(m^2 + n^2)^2 = m^4 + 2n^2n^2 + n^4.$

EXERCISE XXIV.

1. Find the length of the hypothenuse of a right-angled triangle whose other sides are 65 in. and 72 in. long respectively.

2. Find the length of the hypothenuse of a right-angled triangle, the lengths of the other sides being 777 mm. and 464 mm.

3. Find the length of the diagonal of a square whose side is one foot long.

4. Find the length of the diagonal of a cube whose edge is one foot long.

5. What is the length of the side of a square whose diagonal is one foot long?

6. What is the length of the edge of a cube whose diagonal is one foot long?

7. What is the length of the edge of the largest cube that can be cut out of a sphere 6 inches in diameter?

8. What is the length of the diagonal of a cube if the length of a diagonal of one of the faces of the cube is 3 ft.?

9. What will be the length of the diagonals of the faces of the largest cube that can be cut out of a sphere 3 inches in diameter?

10. A quad measures 24 in. by 11·7 in. by 4·4 in. Find the lengths of the diagonals of its faces.

11. A quad is 14 ft. long by 5 ft. wide by 2 ft. thick. Find the length of its diagonal.

12. A quad measures 6·325 m. by 5·796 m. by ·528 m. Determine the length of its diagonal and the lengths of the diagonals of its faces.

13. The lengths of the diagonals of the faces of a quad are 22 ft., 6 ft. and 3 ft. respectively. Find the length of the diagonal of the quad.

14. The diagonals of the faces of a quad are respectively 25 in., 23·79 in. and 9·79 in. long. Determine the lengths of the edges of the quad.

15. The lengths of the sides of the base of a triangular pyramid are 38·83 in., 30·92 in. and 25·95 in. respectively. The lateral edges meet at right angles at the vertex. Find the volume of the pyramid.

16. The lateral edges of a pyramid are all equal to one another. The base is a rectangle 4 ft. 8 in. long by 4 ft. wide. The height of the pyramid is 3 ft. 9 in. Find the area of the surface.

17. The hypothenuse of a right-angled triangle is 16·13 in. in length and one of the other sides is 12·75 in. long. Determine the length of the third side.

18. Two sides of a triangle are 218 ft. and 241 ft. in length and the perpendicular from the included angle on the third side is 120 ft. long. Find the length of the third side.

19. A ladder 25 ft. long stands vertically against a wall. How far must the foot of the ladder be drawn out horizontally from the wall that the top of the ladder may be drawn down one foot?

20. A rope hanging loose from a hook 26 ft. above level ground, just reaches the ground. How high above the ground will the lower end of the rope be when it is drawn 10 ft. aside from the vertical?

21. Two of the sides of a triangle are 1450 ft. and 1021 ft. long respectively. From the contained angle a perpendicular is let fall on the third side, and the segment of that third side between the foot of the perpendicular and the shorter of the first mentioned two sides is 779 ft. in length. Find the area of the triangle.

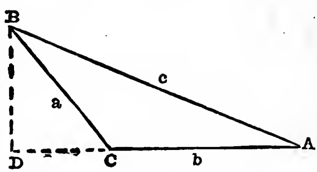
22. The base of a pyramid is a rectangle 12 in. long by 10 in. wide. The lateral edges are each 31 in. long. Find the volume of the pyramid and the area of its surface.

23. A flagpole 53 ft. 4 in. in height is broken by the wind and the top falling over strikes the ground 14 ft. 8 in. from the foot of the pole before the pieces part at the place of breaking. Find the length of the piece broken off, the ground being level.

24. *A B* and *C* are three houses standing at the angles of a right-angled triangle. *A* is 80 ch. east of *C*, and *B* is north of *C* and 51·20 ch. nearer to it than to *A*. Find the distance from *A* to *B*.

25. The lengths of the four sides of a trapezoid taken in order are 608 ft., 554 ft., 250 ft. and 520 ft. Find its area and the lengths of its diagonals.

173. In any obtuse-angled triangle, the squares on the sides containing the obtuse angle are together less than the square on the third side or side opposite the obtuse angle by twice the rectangle under either of the two sides containing the obtuse angle and the projection on it of the other of these two sides. (Euclid II, 12.)



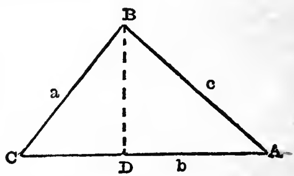
Let ABC be a triangle obtuse-angled at C, and let a, b and c be the MEASURES of the lengths of the sides opposite the angles A, B and C respectively, and let l_a be the *measure* of the length of CD, the projection of CB on AC produced, then will a^2, b^2 and c^2 be the *measures* of the areas of the squares on the sides and bl_a will be the *measure* of the area of the rectangle contained by CA and CD. It follows from this and the preceding proposition that

$$a^2 + b^2 + 2b\ell_{-a} = c^2 \qquad (3)$$

and ∴ $$(a^2 + b^2 + 2b\ell_{-a})^{\frac{1}{2}} = c, \qquad (C)$$

that is ;—*In any obtuse-angled triangle, if to the sum of the squares of the measures of the lengths of the sides containing the obtuse angle there be added twice the product of the measures of the lengths of either of these sides and the projection on it of the other of these sides, the square root of the sum will be the measure of the length of the side opposite the obtuse angle.*

174. In any triangle, the squares on the sides containing an acute angle are together greater than the square on the third side, or side opposite the acute angle, by twice the rectangle under either of the two sides containing the acute angle and the projection on it of the other of these two sides. (Euclid II, 13.)



Let ABC be a triangle acute-angled at C and let a, b and c be the MEASURES of the lengths of the sides opposite the angles A B and C respectively, and let ℓ_a be the *measure* of the length of CD, the projection of CB on CA (produced if necessary) ; then will a^2 b^2 and c^2 be the *measures* of the areas of the squares on the sides and $b\ell_a$ will be the *measure* of the area of the rectangle contained by CA and CD. It follows from this and the preceding proposition that

$$a^2 + b^2 - 2b\ell_a = c^2 \qquad (4)$$

and ∴ $$(a^2 + b^2 - 2b\ell_a)^{\frac{1}{2}} = c, \qquad (D)$$

that is ;—*In any triangle, if from the sum of the squares of the measures of the lengths of the sides containing an acute angle there be subtracted twice the product of the measures of the lengths of either of these sides and the projection on it of the other of these sides, the square root of the remainder will be the measure of the length of the side opposite the acute angle.*

175. If the angle BCD be *one-third* of two right angles, *i.e.*, if it be equal to the angle of an equilateral triangle, the line CD will be equal to half of the side CB; therefore twice the rectangle under CD and CA will be equal to the rectangle under CB and CA and consequently (C) of § 173 will become

$$c=(a^2+b^2+ab)^{\frac{1}{2}} \qquad \text{(Cc)}$$

and (D) of § 174 will become

$$c=(a^2+b^2-ab)^{\frac{1}{2}} \qquad \text{(Dd)}$$

It should be noticed that in the case of (Cc) the angle BCD is an *external* angle of the triangle ABC and the internal obtuse angle is *two-thirds* of two right angles.

EXERCISE XXV.

1. The lengths of the sides of a triangle are 125 ft., 244 ft. and 267 ft. respectively. Find the area of the rectangle under each side and the projection on it of either of the other sides. Find also the lengths of the sides of three squares equal in area to the three rectangles thus obtained.

2. The lengths of the sides of a triangle are 84 ft. 1 in., 158 ft. 2 in. and 188 ft. 3 in. respectively. Find the length of the projection of the shortest side on the longest.

3. The lengths of the sides of a triangle are 595 mm., 769 mm. and 965 mm. respectively. Find the length of the projection of the shortest side on each of the others.

4. The lengths of the sides of a triangle are 25 in., 39 in., and 40 in. respectively. Find the lengths of the projection of the shortest side on each of the others and the lengths of the perpendiculars on these sides from the opposite angles.

5. In a right-angled triangle, the lengths of the sides containing the right angle are 30 ft. 4 in. and 52 ft. 3 in. Find the lengths of the segments into which the hypothenuse is divided by the perpendicular on it from the right-angle, and also the length of that perpendicular, and prove that the product of the measures of the lengths of the segments of the hypothenuse is equal to the square of the measure of the length of the perpendicular.

6. The lengths of the sides of a triangle are 13 yd., 14 yd. and 15 yd. respectively. Find the lengths of the perpendiculars on the sides from the opposite angles.

7. Show that the triangle whose sides are respectively 25 ft., 39 ft. and 56 ft. long, is obtuse angled and find the lengths of the projections of each side on the other two.

8. From a point O, three lines OA, OB and OC whose lengths are respectively 195 ft., 264 ft. and 325 ft. are drawn making equal angles with one another in the same plane. Find the lengths of the lines AB, BC and CA.

9. From a point O, three lines OD, OB and OC whose lengths are respectively 440 ft., 264 ft. and 325 ft. are drawn making equal angles with one another in the same plane. Find the lengths of the sides of the triangle BCD.

10. The lengths of the sides of a triangle are 21 ft. 2 in., 21 ft. 10 in. and 26 ft. 4 in. respectively. Find the lengths of the medians of the triangle. (See Mackay's Euclid, Ap. II, Prop. 1.)

176. If the lengths of the sides of a triangle are known, the propositions of §§ 173 and 174 will enable us to determine the length of the perpendicular on any side from the opposite angle and consequently to find the area of the triangle. It is not however necessary to compute the length of the perpendicular on a side in order to find the area of the triangle, this may be determined directly from the lengths of the sides as follows :—

iii, *a*. *From the measure of the length of the semiperimeter of the triangle subtract the measure of the length of each side separately, multiply together the three remainders and the common minuend, the square root of the product will be the measure of the area of the triangle*; or,

$$S_t = \{ s(s-a)(s-b)(s-c) \}^{\frac{1}{2}}$$

in which S_t is the measure of the area of the triangle, a, b and c are the measures of the lengths of the sides and s is the measure of the semiperimeter ; *i.e.*,

$$2s = a + b + c.$$

Let x, y and h denote the measures of the lengths of AD, CD and BD respectively, see Figs. of §§ 173 and 174, then will

Fig. of § 173.	Fig. of § 174.
$x - y = b$	$x + y = b$
$x^2 \quad = c^2 + h^2$	$x^2 \quad = c^2 + h^2$
$y^2 \quad = a^2 + h^2$	$y^2 \quad = a^2 + h^2$

$\therefore$	$x^2 - y^2 = c^2 - a^2$	$x^2 - y^2 = c^2 - a^2$
$\therefore$	$(x-y)(x+y) = c^2 - a^2$	$(x+y)(x-y) = c^2 - a^2$
$\therefore$	$b(x+y) = c^2 - a^2$	$b(x-y) = c^2 - a^2$
and	$b(x-y) = b^2$	$b(x+y) = b^2$
$\therefore$	$2bx = b^2 + c^2 - a^2.$	$2bx = b^2 + c^2 - a^2.$

$$h^2 = c^2 - x^2 = (c+x)(c-x)$$

$$\therefore \quad 4b^2h^2 = (2bc+2bx)(2bc-2bx)$$
$$= 2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)$$
$$= \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$$
$$= (b+c+a)(b+c-a)(a+b-c)(a-b+c)$$
$$= 2s(2s-2a)(2s-2c)(2s-2b)$$
$$= 16s(s-a)(s-b)(s-c).$$

$$\therefore \quad \tfrac{1}{4}b^2h^2 = s(s-a)(s-b)(s-c);$$

$$\therefore \quad \tfrac{1}{2}bh = \{s(s-a)(s-a)(s-c)\}^{\frac{1}{2}}.$$

But
$$S_t = \tfrac{1}{2}bh$$

$$\therefore \quad S_t = \{s(s-a)(s-b)(s-c)\}^{\frac{1}{2}}.$$

An important advantage of this method of computing the area of a triangle is that it can be adapted to calculation by logarithms for it yields at once

$$\log S_t = \tfrac{1}{2}\{\log s + \log(s-a) + \log(s-b) + \log(s-c)\}.$$

Example. Find the area of a triangle the lengths of whose sides are 13·14 m., 14·15 m. and 15·13 m. respectively.

	$2s = 42{\cdot}42$	
$\therefore$	$s = 21{\cdot}21$	$\therefore \log s = 1{\cdot}326541$
	$s-a = 8{\cdot}07$	$\log(s-a) = {\cdot}906874$
	$s-b = 7{\cdot}06$	$\log(s-b) = {\cdot}848805$
	$s-c = 6{\cdot}08$	$\log(s-c) = {\cdot}783904$
		$2)3{\cdot}866124$
		$\therefore \log S = 1{\cdot}933062 = \log 85{\cdot}716$

$\therefore$ the area of the triangle is 85·716 square metres.

177. If the measures of the lengths of the sides of a triangle be

$$kl(m^2+n^2), \quad mn(k^2+l^2), \quad (kn+lm)(km-ln),$$

the measure of the area of the triangle will be

$$klmn(kn+lm)(km-ln)$$

k, l, m and n denoting any numbers whatsoever.

The triangle can be resolved into two right-angled triangles, the measures of the lengths of the sides of the first being

$$kl(m^2+n^2), \quad kl(m^2-n^2), \quad 2klmn,$$

and the measures of the lengths of the sides of the second being

$$mn(k^2+l^2), \quad mn(k^2-l^2), \quad 2klmn;$$

and $$kl(m^2-n^2)+mn(k^2-l^2)=(kn+lm)(km-ln).$$

EXERCISE XXVI.

Find the areas of the triangles the lengths of whose sides are respectively

1. 13 yd., 10 yd. and 13 yd.

2. 13 yd., 24 yd. and 13 yd.

3. 13 ft., 4 ft. and 15 ft.

4. 13 ft., 14 ft. and 15 ft.

5. 13 in., 11 in. and 20 in.

6. 13 in., 21 in. and 20 in.

7. 13 m., 37 m. and 40 m.

8. 13 m., 45 m. and 40 m.

9. 1.23 ch., 5·95 ch. and 6·76 ch.

10. 73·2 ch., 45·5 ch. and 87·6 ch.

11. What will be the value at $73 per acre of a triangular piece of land the lengths of whose sides are 478·5 chains, 329·6 chains and 237·4 chains respectively?

12. A triangular piece of land the lengths of whose sides were ·1234 miles, ·2345 miles and ·2086 miles respectively was sold for $975. What was the price per acre?

13. The lengths of the sides of a triangle are respectively 212 ft., 225 ft. and 247 ft. A straight line is drawn across the triangle joining the mid-points of two of the sides. Find the area of the trapezoid thus formed.

14. The lengths of the sides of a triangle are 126 m., 269 m. and 325 m. respectively. Straight lines are drawn across the triangle parallel to one of the sides and joining points of trisection of the other two sides. Find the areas of the parts into which the triangle is thus divided.

15. The length of the side of a square is 44 ft. A point is taken within the square distant 12·9 ft. and 37·7 ft. respectively from the ends of one side. Find the areas of the triangles formed by joining the point to the four corners of the square.

16. The lengths of two adjacent sides of a rectangle are 349 ft. and 247 ft. A point is taken within the rectangle distant 225 ft. and 164 ft. respectively from the ends of the longer side of the rectangle. Find the areas of the triangles into which the rectangle is divided by lines joining its angular points to the given point.

17. The lengths of two of the sides of a triangle are 55 ch. and 39 ch. respectively and the angle contained between these sides is two-thirds of a right-angle. Find the area of the triangle.

18. Find the area of the gable end of a barn 66·2 ft. wide, the height of the eaves being 19 ft. at the front of the barn and 8 ft. at the back, and the lengths of the rafters being 29 ft. on the front and 56·2 on the back, the barn standing on level ground.

19. The lengths of the sides of the triangle ABC are 6983 mm., 17079 mm. and 18574 mm. Find the area of a triangle whose sides are equal to the medians of the triangle ABC.

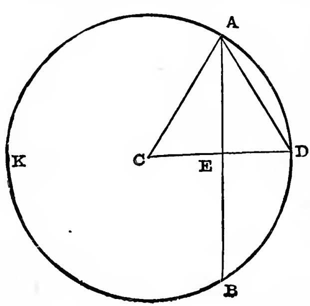
20. The lengths of the medians of a triangle are 16·45 ch., 47·77 ch. and 60·52 chains respectively; find the area of the triangle.

178. *Given the length of the radius of a circle and the length of the chord of any arc of the circle, to find the length of the chord of half the arc.*

Let r, k and k_2 denote the measures of the lengths of the radius, the chord of the arc and the chord of half the arc respectively, then will

$$k_2 = \left\{ 2r^2 - r(4r^2 - k_1^2)^{\frac{1}{2}} \right\}^{\frac{1}{2}}.$$

Let ABK be the circle, C its centre, ADB the arc and D the mid-point of the arc. Join AD, AB, AC and CD; the radius CD will bisect the chord AB at right angles, say in E. The measures of the lengths of AC, AB and AD are respectively r, k_1 and k_2. Let q denote the measure of the length of CE.



$$CE^2 = CA^2 - AE^2$$

$$\therefore \quad q^2 = r^2 - (\tfrac{1}{2}k_1)^2$$

$$\therefore \quad 4q^2 = 4r^2 - k_1^2$$

$$\therefore \quad 2q=(4r^2-k_1^2)^{\frac{1}{2}} \tag{1}$$

$$AD^2=AC^2+CD^2-2\,CD\cdot CE$$

$$=2\,CD^2-2\,CD\cdot CE$$

$$\therefore \quad k_2^2=2r^2-2rq \tag{2}$$

by (1)
$$=2r^2-r(4r^2-k_1^2)^{\frac{1}{2}}$$

$$\therefore \quad k_2=\{2r^2-r(4r^2-k_1^2)^{\frac{1}{2}}\}^{\frac{1}{2}}. \tag{3}$$

Example 1. The side of a regular hexagon inscribed in a circle is equal to the radius of the circle, find the length of a side of the inscribed regular convex dodecagon.

In this case we are given $k_1=r$,

$$\therefore \quad k_2=\{2r^2-r(4r^2-r^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$$

$$=(2r^2-3^{\frac{1}{2}}r^2)^{\frac{1}{2}}$$

$$=(2-1\cdot73205081)^{\frac{1}{2}}r$$

$$=\cdot26794919^{\frac{1}{2}}r$$

$$=\cdot51763809r.$$

Therefore *the length of a side of a regular convex dodecagon inscribed in a circle is ·51763809 of the length of the radius of the circle.*

The length of the semiperimeter of the dodecagon is six times the length of a side and $\cdot51763809\,r\times 6=3\cdot1058285\,r$, therefore

The length of the semiperimeter of a regular convex dodecagon inscribed in a circle is 3·1058285 *times the length of the radius of the circle.*

Example 2. Find the length of a side of a regular convex polygon of 24 sides, inscribed in a circle.

In this case k_1 is the measure of the length of a side of regular convex polygon of twelve sides, inscribed in the circle, ∴ by *Example 1*, $k_1=\cdot51763809\,r$,

$$\therefore \quad k_2=\{2r^2-r(4r^2-\cdot51763809^2r^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$$

$$=(2r^2-3\cdot73205081^{\frac{1}{2}}r^2)^{\frac{1}{2}}$$

$$=\cdot26105238\,r.$$

Therefore the length of a side of a regular convex 24-gon inscribed in a circle is ·26105238 *of the length of the radius of the circle.*

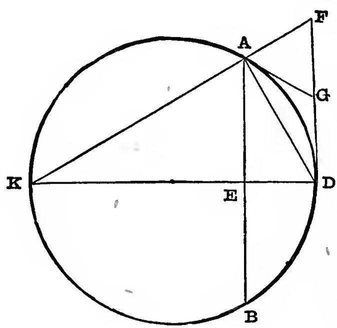
The length of the semiperimeter of the 24-gon is 12 times the length of a side and $\cdot 26105238\, r \times 12 = 3\cdot 1326286\, r$, therefore

The length of the semiperimeter of a regular convex 24-gon inscribed in a circle is 3·1326286 *times the length of the radius of the circle.*

179. *Given the lengths of the radius of a circle, of the chord of any arc of the circle and of the chord of half the arc, to find the sum of the lengths of the tangents from the ends of the half-arc to their point of intersection.*

Let r, k_1 and k_2 denote the measures of the lengths of the radius, the chord of the arc and the chord of half the arc respectively, and let t denote the sum of the measures of the lengths of the tangents from the ends of the half-arc to their point of intersection, then will

$$t = \frac{2k_2^2}{k_1}$$



Let ABK be the circle, ADB the arc and D the mid-point of this arc. Join AD and AB and draw tangents to the circle at A and D and let them meet in G. Draw the diameter DK bisecting the chord AB in E. Join KA and produce KA and DG to meet in F. Then because GA and GD are equal, being tangents from the point G, and DAF is a right angle, therefore the angle GAF is equal to the angle GFA, therefore GA is equal to GF and consequently the tangents AG and DG are together equal to DF. The measures of the lengths of AB and AD are k_1 and k_2, and the sum of the measures of the lengths of AG and DG which is equal to the measure of the length of DF, is t.

EA is parallel to DF, both being at right angles to DK, therefore the angle EAD is equal to the angle ADF, also the angle AED is equal to the angle DAF, both being right angles, therefore the triangle AED is similar to the triangle DAF

$$\therefore \quad FD : DA :: DA : AE$$

$$\therefore \quad t : k_2 :: k_2 : \tfrac{1}{2}k_1$$

$$\therefore \quad \frac{t}{k_2} = \frac{2k_2}{k_1}$$

$$\therefore \quad t = \frac{2k_2^2}{k_1}.$$

Example. Find the length of a side of a regular convex dodecagon circumscribed about a circle.

Let r be the measure of the length of the radius of the circle, then will $k_1 = r$ and $k_2 = \cdot 51763809\, r$. (See *Example 1*, p. 214.)

$$\therefore \quad t = 2(\cdot 51763809\, r)^2 \div r$$
$$= \cdot 53589838\, r.$$

Therefore *the length of a side of a regular convex dodecagon circumscribed about a circle is* ·53589838 *of the length of the radius of the circle.*

The length of the semiperimeter of the dodecagon is six times the length of a side, and $\cdot 53589838\, r \times 6 = 3\cdot 2153903\, r$; therefore

The length of the semiperimeter of a regular convex dodecagon circumscribed about a circle is 3·2153903 *times the length of the radius of the circle.*

EXERCISE XXVII.

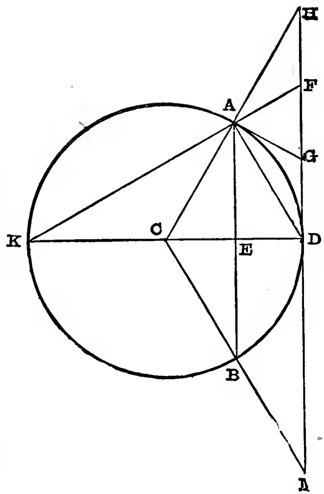
Find the length of a side and also the length of the semiperimeter of a regular convex polygon inscribed in a circle, the unit of measurement being the radius of the circle and the number of the sides of the polygon being

1. 48. **2.** 96.

Find the length of a side and also the length of the semiperimeter of a regular convex polygon circumscribed about a circle, the unit of measurement being the radius of the circle and the number of the sides of the polygon being

3. 24. **4.** 48. **5.** 96.

180. RECTIFICATION OF THE CIRCLE. Let ABK be a circle, C its centre, ADB an arc of the circle and D the mid-point of the arc. Join AD, AB, AC and CB. Draw the diameter DCK bisecting the chord AB at right angles in E. Draw AG and DG, tangents to the circle at A and D. Produce CA and DG to meet in H, and CB and GD to meet in M. Draw KA and produce it to meet DH in F.



If the chord AB be a side of a regular convex polygon of n sides, say a regular n-gon, inscribed in the circle ABK, the chord AD will be a side of a regular $2n$-gon inscribed in the circle, HM will be a side of a regular n-gon circumscribed about the circle and FD which is equal to AG+GD, will be equal to a side of a regular $2n$-gon circumscribed about the circle.

The perimeter of the inscribed n-gon will be n times AB which is equal to $2n$ times AE. Let $2n$ (AE) denote and be read "$2n$ times AE."

The perimeter of the inscribed $2\,n$-gon will be $2n$ (AD).

The perimeter of the circumscribed n-gon will be n (HM) which is equal to $2n$ (HD).

The perimeter of the circumscribed $2n$-gon will be $2n$ (FD).

Now AED being a right angle, AD is greater than AE,

$$\therefore \qquad 2n\,(AD) > 2n\,(AE),$$

i.e., the perimeter of the inscribed regular $2n$-gon is greater than the perimeter of the inscribed regular n-gon.

Because FD is less than HD

$$\therefore \qquad 2n\,(FD) < 2n\,(HD),$$

i.e., the perimeter of the circumscribed regular $2n$-gon is less than the perimeter of the circumscribed regular n-gon.

The angle DAF being a right angle, FD is greater than AD

$\therefore \quad 2n(\text{FD}) > 2n(\text{AD})$

i.e., the perimeter of the circumscribed regular $2n$-gon is greater than the perimeter of the inscribed regular $2n$-gon.

If then a regular hexagon be inscribed in a circle, and a similar hexagon be circumscribed about the circle, the perimeter of the circumscribed hexagon will be greater than the perimeter of the inscribed hexagon.

If next a regular convex dodecagon be inscribed in the circle in which the hexagon was inscribed and a similar dodecagon be circumscribed about the same circle, the perimeter of the inscribed dodecagon will be greater than the perimeter of the inscribed hexagon, and the perimeter of the circumscribed dodecagon will be less than the perimeter of the circumscribed hexagon but will be greater than the perimeter of the inscribed dodecagon. Hence the difference in length between the circumscribed and inscribed dodecagons is less than the difference in length between the circumscribed and inscribed hexagons.

If next a regular 24-gon be inscribed in the circle and a similar 24-gon be circumscribed about the circle, the perimeter of the inscribed 24-gon will be greater than the perimeter of the inscribed 12-gon, and the perimeter of the circumscribed 24-gon will be less than the perimeter of the circumscribed 12-gon but greater than the perimeter of the inscribed 24-gon. Hence the difference in length between the perimeters of the circumscribed and inscribed 24-gons is less than the difference in length between the perimeters of the circumscribed and inscribed 12-gons.

If next a regular 48-gon be inscribed in the circle and a similar 48-gon be circumscribed about the circle, the difference in length between their perimeters will be less than the difference in length between the perimeters of the circumscribed and inscribed 24-gons.

By continuing this process we shall obtain a series of pairs of polygons whose perimeters become more and more nearly equal at each doubling of the number of their sides.

Now as the circumference of a circle is greater than the perimeter of any regular convex polygon inscribed in the circle but is less than the perimeter of any similar polygon circumscribed about the circle, the lengths of the perimeters of these polygons may be taken as limits between which the length of the circumference must lie. But it has been shown that, beginning with a regular hexagon, as the number of sides of the inscribed and circumscribed polygons is successively doubled the difference between the lengths of their perimeters becomes less and less. In other words, by repeatedly doubling the number of the sides of similar inscribed and circumscribed polygons, the limits between which the length of the circumference lies, are made continually to approach each other * and therefore a nearer and nearer approach may be made to the exact length of the circumference.

As an example let us take the measures of the lengths of the semiperimeters of the inscribed and circumscribed regular convex polygons of 12, 24, 48, and 96 sides respectively, which are given in the *Examples* of §§ 178 and 179 and in the answers to the problems in Exercise XXVII, and, π denoting the measure of the length of the semicircumference when the radius is the unit of measurement, we shall obtain

from the 12-gons	$3{\cdot}10 < \pi < 3{\cdot}22$
from the 24-gons	$3{\cdot}13 < \pi < 3{\cdot}16$
from the 48-gons	$3{\cdot}139 < \pi < 3{\cdot}1461$
from the 96-gons	$3{\cdot}141 < \pi < 3{\cdot}1428.$

Since $3\frac{10}{71} < 3{\cdot}141$ and $3{\cdot}1428 < 3\frac{10}{70}$

$$\therefore \quad 3\tfrac{10}{71} < \pi < 3\tfrac{10}{70}.$$

These are known as Archimedes' Limits of the ratio of the semicircumference of a circle to its radius, or of the circumference to its diameter.

[181. Had we carried our calculations beyond the 96-gons to the 12,288-gons we should have obtained

$$3{\cdot}1415926 < \pi < 3{\cdot}1415927.$$

Vieta, doubling the number of sides 16 times successively computed to 10 places of decimals the lengths of the perimeters of the inscribed and circumscribed regular 393,216-gons and found that

$$3{\cdot}1415926535 < \pi < 3{\cdot}1415926537.$$

* We do not here enquire whether the limits thus found may be made approach each other indefinitely, nor is it necessary to ascertain whether they do so, for we seek not an exact but only an approximate rectification of the circle.

Ludolph van Ceulen starting from squares and successively doubling the number of sides 60 times, determined π to 35 decimal places.

182. The method which has been described of approximating to the value of π depends on the proposition that the *arc* AD (see Fig. on p. 215) is greater than the *chord* AD but is less than the sum of the tangents AG and GD, *i.e.*, less than FD. This method is extremely tedious if π is to be computed to more than three or four decimal places, but the following theorems afford a means of greatly reducing the labor of calculation.

1°. The *arc* AD > *chord* AD + $\frac{1}{3}$(AD − AE);

2°. The *arc* AD < *chord* AD + $\frac{1}{3}$(DF − AD).

If for these magnitudes we substitute the measures of their lengths in terms of the radius as unit of measurement, and multiply throughout by $2n$, we shall obtain

$$2\pi > 2nk_2 + \tfrac{1}{3}(2nk_2 - nk_1),$$
$$2\pi < 2nk_2 + \tfrac{1}{3}(2nt - 2nk_2).$$

If P_n and P_{2n} denote the measures of the lengths of the perimeters of the inscribed regular n-gon and $2n$-gon respectively and Q_{2n} denote the measure of the length of the perimeter of the circumscribed regular $2n$-gon, in terms of the radius as unit of measurement, then will

$$P_n = nk_1,\ P_{2n} = 2nk_2 \text{ and } Q_{2n} = 2nt$$

and the preceding limits may be written

$$2\pi > P_{2n} + \tfrac{1}{3}(P_{2n} - P_n),$$
$$2\pi < P_{2n} + \tfrac{1}{3}(Q_{2n} - P_{2n}).$$

As an example of the closeness of these limits let us take the case in which P_{2n} is the measure of the length of the perimeter of the inscribed regular 96-gon, then will

$$\tfrac{1}{2}P_n = 3{\cdot}1393502,$$
$$\tfrac{1}{2}P_{2n} = 3{\cdot}1410319, \text{ and } \tfrac{1}{2}Q_{2n} = 3{\cdot}1427146$$

and ∴ $\pi > 3{\cdot}1410319 + \frac{1}{3}(3{\cdot}1410319 - 3{\cdot}1393502)$

but $\pi < 3{\cdot}1410319 + \frac{1}{3}(3{\cdot}1427146 - 3{\cdot}1410319)$

i.e., $\pi > 3{\cdot}1415925$

but $\pi < 3{\cdot}1415928.$

183. Numerous other geometrical constructions of the approximate length of an arc have been proposed for the evaluation of π, the best being one which yields

$$30\pi < 8\,Q_{4n} + 8\,P_{2n} - P_n$$

This was published in 1670 by James Gregory who at the same time laid the foundation of the modern methods of computing π by proving that

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \dots\dots,$$

the series to be continued endlessly.

Twenty-nine years later, Machin announced that

$$\tfrac{1}{4}\pi = 4\left(\frac{1}{5} - \frac{1}{3\times 5^3} + \frac{1}{5\times 5^5} - \frac{1}{7\times 5^7} + \dots\right)$$
$$-\left(\frac{1}{239} - \frac{1}{3\times 239^3} + \frac{1}{5\times 239^5} - \frac{1}{7\times 239^7} + \dots\right)$$

and computed π thereby to 100 places of decimals. Recently W. Shanks, employing this series, has calculated π to 707 places of decimals.

The rapidity of convergence of Machin's series gives it a great advantage over Gregory's for purposes of calculation, but it and the many other series which have been proposed and used for the evoluation of π, can be easily deduced from Gregory's series.

It may here be mentioned that IT HAS BEEN PROVED that π is a transcendental number, *i.e.*, *π cannot be exactly expressed by a definite number of integers combined by the operations of addition, subtraction, multiplication, division, involution and evolution.*]

184. Let P_c be the length of the circumference of a circle and R be the length of the radius, therefore, since 2π is the *measure* of P_c in terms of R as unit of measurement, P_c will be equal to 2π radii, which we shall denote by $P_c = 2\pi(R)$. If we now adopt any other unit than R, say U, and if p_c be the *measure* of P_c and r be the *measure* of R both in terms of U, then will

$$P_c = p_c(U) \text{ and } R = r(U),$$

and $\therefore$ $\quad p_c(U) = 2\pi\{r(U)\}$
$$= (2\pi r)(U),$$

$\therefore$ $\quad p_c = 2\pi r.$

that is, *the measure of the length of the circumference of a circle is the product of π and twice the measure of the length of the radius,*

$$\pi \text{ being } \begin{cases} \frac{22}{7} & \textit{correct to 3 significant figures,} \\ 3{\cdot}1416 & \text{" " 5 " "} \\ \frac{355}{113} & \text{" " 7 " "} \end{cases}$$

185. If $2\,a$, $2\,b$ and p_o denote the measures of the lengths of the major and minor axes and of the perimeter of an ellipse and if a^2-b^2 be small compared to a^2, then will

$$p_o > \pi a \left\{ 1 + \frac{a^2+3\,b^2}{3\,a^2+b^2} \right\}$$

but $$p_o < \pi \left\{ 2(a^2+b^2) \right\}^{\frac{1}{2}}.$$

[**186.** If a conical spiral beginning with a radius of r_o units, advance in n revolutions through a distance of h units measured on the axis of the cone, and have then a radius of r_n units, the measure of the length of the spiral will be roughly approximate to

$$\left\{ n^2\pi^2(r_o+r_n)^2+h^2 \right\}^{\frac{1}{2}}.$$

If $r_o=r_n$, the curve is a cylindric spiral or helix, (the edge of the thread of a screw) and the rectification is exact.

If $h=o$, the curve is the common spiral or spiral of Archimedes.]

EXERCISE XXVIII.

1. The inner diameter of a circular drive is 210 ft. in length and the width of the drive is 28 ft. Find the length of the inner and of the outer edge of the drive.

2. What will be the cost of the wire at $1.25 per 100 yd. for a barbed-wire fence five wires high around a circular fish-pond 60 ft. in diameter?

3. The minute hand of a clock measures 1 ft. $3\frac{3}{4}$ in. from the centre of its arbor to the tip of the hand. Find the distance travelled by the tip of the hand during the course of 365 days.

4. Find the length of the radius of a wheel which made 1600 revolutions in rolling 3·25 miles.

5. A circular path is 400 yd. in length on its inner edge. What will be its length 5 ft out from that edge all around?

6. The length of the hypothenuse of a right-angled triangle is 2·9 in. and that of one of the other sides is 2·1 in. Find the length of the radius of a circle whose circumference is equal to the sum of the lengths of the circumferences of circles described on the three sides of the triangle as diameter.

7. The difference in length between the diameter and the circumference of a circle is 2 ft. 6 in.; find the length of the diameter.

8. If Mercury describe round the sun in 87·97 days a circle whose radius is 35,700,000 miles in length and Saturn describe in 10759·22 days a circle whose radius is 882,000,000 miles long, what will be the orbital speed in miles per minute of each of these planets ?

9. Find the length of the arc which subtends an angle of 60° at the centre of a circle of 10 in. radius.

10. Find the length of the arc which subtends an angle of 36° at the centre of a circle of 25 in. radius.

11. Of how many degrees will the angle be which an arc whose length is 1 ft., subtends at the centre of a circle of 2 ft. radius.

12. How many degrees will there be in the angle subtended at the centre of a circle of 1 ft. radius, by an arc whose length is 2 ft. ?

13. How many degrees will there be in the angle subtended at the centre of a circle by an arc whose length is equal to the length of the radius, if the length of the radius be (*a*) 1 ft., (*b*) 2 ft., (*c*) 3 ft., (*d*) 7 ft., (*e*) 27·3 in.

14. The length of the radius of a circle is 17·5 in.; find the length of the perimeter of a sector of which the angle is (*a*) 90^c, (*b*) 270°.

15. What will be the length of the perimeter of the segment of a circle of 18 in. radius, if the arc of the segment subtend an angle of 45° at the centre of the circle ?

16. The length of the perimeter of a semicircle is 5 ft.; find the length of the diameter.

17. The length of the perimeter of a sector of a circle if 7·2 ft.; find the length of the radius the angle of the sector being 30°.

18. The length of the perimeter of the segment of a circle is 7·2 ft.; find the length of the radius if the arc of the segment subtend an angle of 30° at the centre of the circle.

19. Find the length of the perimeter of an ellipse the lengths of whose axes are 12 in. and 10 in. respectively.

20. Find the length of the quadrantal arc of an ellipse whose semiaxes measure 11·9 in. and 7·9 in. respectively.

21. Find the length of the quadrantal arc of an ellipse whose semiaxes are 10·199 m. and 9·799 m. respectively in length.

22. Find the length of the radius of a circle whose circumference is of the same length as the perimeter of an ellipse whose semiaxes are 40·399 yd. and 39·599 yd. long respectively.

23. Find the length of the equator; 1°, assuming it to be an ellipse the lengths of whose semiaxes are 20,926,629 feet and 20,925,105 feet respectively; 2°, assuming it to be a circle of 20,926, 202 feet radius.

24. The French metre was originally defined to be the 10,000,000th part of the length of a meridian quadrant taken from the equator to the pole. Had this definition been retained what would be the length of a metre in inches, if the length of the polar axis of the earth be 41,709,790 ft. and the length of the equatorial diameter be 41,852,404 ft.

25. Mars revolves around the sun in an ellipse, the centre of the sun being one of the foci of the ellipse. Find the lengths of the semiaxes and of the perimeter of Mars' orbit if the greatest and the least distance of the planet from the sun be respectively 154,000,000 miles and 128,000,000 miles.

26. Find the average speed in miles per minute of Mars in his orbit, given the data in problem 25 and that his periodic time is 687 days.

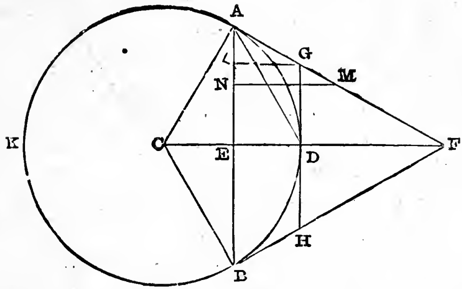
27. Assuming the earth to be a sphere of 7913 miles diameter, find the length of a degree of longitude in latitude 60°.

28. Assuming the earth to be a sphere 7913 miles in diameter, find the length of a degree of longitude in 45° north latitude.

29. The length of the perimeter of an ellipse is 383 in. and the length of the axes are as 10 to 7; find the lengths of the axes.

30. The difference between the lengths of the radii of a front and a hind wheel of a carriage is 7 in. What must be the lengths of these radii if the front wheel make 70 revolutions more than the hind wheel makes in rolling a mile.

187. Let ABK be a circle; C, its centre; AF, half of a side of a regular n-gon circumscribed about the circle; BF, half of an adjoining side of the n-gon.



Join AB and CF. CF will bisect the *chord* AB at right angles, say at E, and will bisect the *arc* AB, say at D. Draw GDH, tangent to the circle at D and meeting AF in G and BF in H. Then GH is equal to a side of a regular $2n$-gon circumscribed about the circle ABK. Also $AG = GD = DH = HB$. Bisect AF in M and draw GL and MN parallel to FC and meeting AE in L and N respectively.

Because the angle GDF is a right angle

$\therefore \quad GD < GF,$

$\therefore \quad AG < GF$

$\therefore$ M lies between G and F,

and $\therefore \quad 2\,AG + 2\,GM = AF.$

But $\quad 2\,AG = 2\,GD = 2\,LE = 2\,LN + 2\,NE$

$\qquad\qquad = 2\,LN + AE.$

$\therefore \quad 4\,AG + 2\,GM = AF + 2\,LN + AE.$

But $\quad GM > LN$

$\therefore \quad 4\,AG < AF + AE$

$\quad AE < \text{arc } AD$

$\therefore \quad AF + AE < AF + \text{arc } AD$

$\therefore \quad 4\,AG < AF + \text{arc } AD,$

and $\therefore\ 4\,AG - 2\,\text{arc } AD < AF - \text{arc } AD.$

$\therefore \quad (AG + GH + HB) - \text{arc } AB < \frac{1}{2}(AF + FB) - \frac{1}{2}\,\text{arc } AB,$

$\therefore \quad (AG + GH + HB) - \text{arc } AB < \frac{1}{2}\{(AF + FB) - \text{arc } AB\};$

i.e., the excess of the length of the broken line AGHB over the length of the arc AB is less than half the excess of the length of the broken line AFB over the length of the arc AB.

Applying this theorem to all the other pairs of adjoining half-sides of the regular n-gon circumscribed to ABK, and taking the aggregates of the excesses, we find that

The excess of the length of the perimeter of a regular 2n-gon circumscribed about a circle over the length of the circumference of the circle is less than half *of the excess of the length of the perimeter of the regular n-gon circumscribed about the same circle over the length of the circumference of the circle.*

188. Join AD, AC and CB in the figure in the preceding section.

$$AG < GF,$$

$\therefore$ the triangle AGD < the triangle GFD,

$\therefore$ double the triangle AGD < the triangle AFD,

$\therefore$ 2(triangle AGD) − 2(segment AD) < triangle AFD − segment AD;

$\therefore$ triangle AGD − segment AD < $\frac{1}{2}$ (triangle AFD − segment AD);

$\therefore$ figure AGDC − sector ADC < $\frac{1}{2}$ (triangle AFC − sector ADC;)

$\therefore$ figure AGHBC − sector ADBC < $\frac{1}{2}$(figure AFBC − sector ADBC);

i.e., the excess of a sector of the circumscribed regular $2n$-gon over the corresponding sector of the circle is less than half of the excess of the corresponding sector of the circumscribed regular n-gon over the sector of the circle.

Applying this theorem to all the sectors of the circumscribed polygons and taking the aggregates of the excesses, we find that

The excess of the area of a regular 2n-gon circumscribed about a circle over the area of the circle is less than half *of the excess of the area of the regular n-gon circumscribed about the same circle over the area of the circle.*

189. Every n-gon circumscribed about a circle is the aggregate of the triangles whose bases are the sides of the n-gon and whose vertices all meet at the centre of the circle. Now the radius of the circle is the altitude of each of these triangles, therefore the area of the n-gon is the sum of the areas of the triangles whose bases are the sides of the n-gon and whose common altitude is the radius of the circle about which the n-gon is circumscribed. In terms of the measures of the areas,

$$S_n = \tfrac{1}{2} r q_n$$

in which r is the measure of the length of the radius, and q_n is the measure of the length of the perimeter and S_n the measure of the area of the n-gon.

190. Quadrature of the Circle. Describe a circle and circumscribe a regular hexagon and a regular convex dodecagon about it. The excess of the length of the perimeter of the dodecagon over the length of the circumference of the circle is less than half of the excess of the length of the perimeter of the hexagon over the length of the circumference of the circle. Circumscribe a regular 24-gon about the circle. The excess of the length of the perimeter of the 24-gon over the length of the circumference of the circle is less than half of the excess of the length of the perimeter of the dodecagon over the length of the circumference of the circle. So also the excess of the length of the perimeter of a circumscribed regular 48-gon, over the length of the circumference of the circle is less than half of the excess of the length of the perimeter of the circumscribed regular 24-gon over the length of the circumference of the circle.

Thus, every time the number of the sides of the circumscribed regular convex polygon is doubled, the excess of the length of the perimeter of the polygon over the length of the circumference of the circle is reduced to less than half of what it was before the doubling took place.

Hence by repeating the doubling a sufficient number of times, the excess of the length of the perimeter of the circumscribed regular n-gon over the length of the circumference of the circle can be made less than any explicitly assigned length however small.

Had we begun with any other circumscribed regular n-gon than the hexagon, the reasoning would have advanced step by step with the preceding reasoning, and we should have arrived at the same result.

Expressing that result in terms of the measures of the lengths of the perimeters and circumference, it becomes

$q_n - p_c$ can, by sufficiently increasing n, be made less than any proposed number however small.

Therefore, r being constant, $\frac{1}{2} r (q_n - p_c)$ can, by sufficiently increasing n, be made less than any proposed number however small.

But $\frac{1}{2} r (q_n - p_c) = \frac{1}{2} rq_n - \frac{1}{2} rp_c$
and $S_n = \frac{1}{2} rq_n$

$\therefore$ $S_n - \frac{1}{2} rp_c$ can, by sufficiently increasing n, be made less than any proposed number however small.

191. By a train of reasoning similar to that in the preceding section, but applied to areas instead of to lengths of perimeters, it may be proved that by doubling the number of the sides of a regular n-gon circumscribed about a circle the excess of the area of the n-gon over the area of the circle is reduced to less than half of what it was before the doubling took place, and that by repeating the doubling a sufficient number of times, the excess of the area of the circumscribed n-gon over the area of the circle can be made less than any explicitly assigned area however small. This result expressed in terms of the measures of the areas instead of in terms of the areas themselves is,—

$S_n - S_c$ can, by sufficiently increasing n, be made less than any proposed number however small.

But it was shown in the preceding section that

$S_n - \frac{1}{2} rp_c$ can, by sufficiently increasing n, be made less than any proposed number however small,

$\therefore$ $(S_n - \frac{1}{2} rp_c) - (S_n - S_c)$ can, by sufficiently increasing n, be made less than any proposed number however small.

But $(S_n - \frac{1}{2} rp_c) - (S_n - S_c) = S_c - \frac{1}{2} rp_c$

Now S_c, r and p_c are constants, and increasing n can have no effect on them,

$\therefore$ $S_c - \frac{1}{2} rp_c$ must be less than any proposed number however small; and it cannot be variable,

$\therefore$ $S_c - \frac{1}{2} rp_c = 0$,

$\therefore$ $S_c = \frac{1}{2} rp_c$,

that is;—*The measure of the area of a circle is one-half of the product of the measures of the lengths of the radius and the circumference of the circle.*

192. Hence, by Euclid, VI, 33,

iii, *b*. *The measure of the area of* A SECTOR OF A CIRCLE *is one-half of the product of the measures of the lengths of the radius and the arc of the sector.*

193. Substitute $2\pi r$ for p_c in the equation

$$S_c = \tfrac{1}{2} r p_c$$

and it becomes

$$S_c = \pi r^2.$$

v. *The measure of the area of a* CIRCLE *is the product of* π *and the square of the measure of the length of the radius of the circle.*

194. Let $2a$ denote the measure of the length of the major axis and $2b$ denote the measure of the length of the minor axis of an ellipse and let S_e denote the area of the ellipse.

The ratio of the area of an ellipse to the area of the circle described on the major axis as diameter is the same as the ratio of the length of the minor axis to the length of the major axis. But the length of the minor axis is b/a of the length of the major axis, therefore the area of the ellipse is b/a of the area of the circle described on the major axis as diameter.

$$\therefore \quad S_e = \frac{b}{a} \text{ of } \pi a^2,$$

$$\therefore \quad S_e = \pi ab.$$

vi. *The measure of the area of an ellipse is the continued product of* π *and the measures of the lengths of the semiaxes of the ellipse.*

EXERCISE XXIX.

[In the following problems π may be taken equal to 3·1416 and $\log \pi =$ ·497150.]

1. Find the area of a circle the length of whose radius is 3·75 in.

2. Find the area of a circle of 7 ft. diameter.

3. Find the area of a circle whose circumference is 13·09 cm. in length.

4. Find the length of the radius of a circle whose area is an acre.

5. Find the length of the diameter of a circle whose area is a square mile.

6. Find the length of the circumference of a circle whose area is 18·7 acres.

7. How much will it cost to gravel a circular piece of ground 51 ft. in diameter, at 7 cents per square yard?

8. Find the length of the radius of a circle whose area is equal to the sum of the areas of four circles of 10 in., 15 in., 18 in. and 24 in. radius respectively.

9. Find the total pressure on a plate 25 inches in diameter, the pressure per square inch being 65 lb.

10. The circumference of the circular basin of a fountain measures 117·81 ft. on the outside of the masonry and the thickness of the masonry is 30 in. Find the area of the surface of the water within the basin.

11. A circular hole is cut in a circular metal plate of 7 in. radius, so that the weight of the plate is reduced by 40 per cent. Find the length of the radius of the hole.

12. A rectangular room, 27′ 6″ by 13′ 6″, has a semicircular bow-window 8′ 4″ in diameter, thrown out at the side. Find the area of the floor of the whole room.

13. The area of a semicircle is 13·1 sq. in. Find the length of its perimeter.

14. The lengths of the sides of a triangle are 13 ft. 14 ft. and 15 ft. respectively. Find the difference between the area of the triangle and that of a circle of equal perimeter.

15. The perimeters of a circle, a square and an equilateral triangle are each 6 ft. in length. Find by how much the area of the circle exceeds the area of each of the other figures.

16. Find the difference between the area of a circle of 5 m. radius and that of a regular hexagon of equal perimeter.

17. Find the length of the diameter of a circle whose area is equal to that of a square whose sides are each 12 ft. long.

18. The length of the diameter of a circle is 18·7 yd. Find the length of the side of a square whose area is equal to that of the circle.

19. A circle is inscribed in a square whose sides are each 17 in. long. Find the area between the sides of the square and the circumference of the circle.

20. A square is inscribed in a circle of 11 ft. radius. Find the area between the circumference of the circle and the sides of the square.

21. Find the difference between the area of a circle of 7·7 m. radius and that of a regular inscribed hexagon.

22. Find the area of the semicircle described on the hypothenuse of a right-angled triangle as diameter, the lengths of the other sides of the triangle being 7 ft. and 17 ft. respectively.

23. Show that in any right-angled triangle, the area of the semicircle described on the hypothenuse as diameter is equal to the sum of the areas of the semicircles described on the other two sides as diameters.

24. The lengths of the radii of an annulus or plane ring are 23·4 cm. and 36·6 cm. respectively. Find the area of the annulus.

25. Out of a circle of radius 3 ft. is taken a circle of radius 2 ft. Find the area of the remainder.

26. The length of the radius of the inner boundary of an annulus is 25 ft. and the area of the annulus is 100 sq. yd. Find the length of the outer boundary.

27. The length of the chord touching the inner boundary of an annulus is 6 ft. Find the area of the annulus.

28. A circular fish-pond whose area is 2·5 acres is surrounded by a walk 3 yd. wide. Find the cost at 9 ct. per square yard of gravelling the walk from its outer boundary to within one foot of the edge of the pond.

29. Around a circular lawn containing 2·36 acres runs a walk of uniform width containing a quarter of an acre. Find the width of the walk.

30. A circular lawn 98 yards in diameter has a drive of uniform width around it. Find that width, if the area of the drive is just half that of the lawn.

31. What will it cost to pave a circular courtyard 55 ft. in diameter, at 60c. per square foot, leaving in the centre unpaved a hexagonal space whose sides are each 3 ft. long.

32. A circle of 54 in. radius is divided into three equal parts by two concentric circles. Find the lengths of the radii of these circles.

33. Find the area of a sector of 45°, the length of the radius being 10·5 in.

34. Find the area of a sector of 36° the length of the circumference of the whole circle being 1309 mm.

35. The area of a sector is 11·9 sq. ft. and the angle of the sector is 30°. Find the length of the radius.

36. The area of a sector is equal to the area of the square on the radius of the sector. Find the number of degrees in the angle of the sector.

37. A sector of an annulus is 12 inches broad and the lengths of its bounding arcs are 35 in. and 28 in. respectively. Find the area of the sector, its angle and the lengths of its radii.

The length of the radius of a circle being one foot find the area of a segment which subtends at the centre of its circle an angle of

38. 60°. **39.** 120° **40.** 90°.

41. The length of the radius of a circle is 24 in. Two parallel chords are drawn both on the same side of the centre, one subtending an angle of 60° at the centre, the other subtending there an angle of 90°. Find the area of the zone between the chords.

42. Show that the chord of a quadrant divides the circle into parts whose areas are very nearly in the same ratio of 10 to 1.

43. Three circles so intersect that the circumference of each passes through the centres of the other two. Find the area of the figure common to the three circles, the length of the radius of each circle being 15 in.

44. Three circles of 2 ft. radius each, touch each other. Find the area of the figure enclosed by them.

45. The length of the chord of a sector is 5·73 in. and the length of the radius is 10 in. Find the area of the sector. (Apply 1° theorem, § 182, p. 220.)

46. An elliptic flower-bed is described by means of a string 16 ft. long passing over two pegs 6 ft. apart. What is the area of the bed?

47. The area of the circle circumscribed about an ellipse is 12 sq. ft., and that of the circle inscribed in the ellipse is 7·5 sq. ft. Find the area of the ellipse.

48. In a rectangular plot of land measuring 100 yd. by 70 yd. there is dug a fish-pond in the shape of an ellipse the lengths of whose axes are 90 yd. by 60 yd. Find the cost of gravelling the remainder of the plot at 7·5 ct. per square yard.

49. A lawn in the shape of an ellipse the lengths of whose axes are 98 ft. and 58 ft. is surrounded by a walk 2 yards wide. Find the area of the walk.

50. The clear span of a semielliptic arch is 72 ft. and the clear height is 24 ft. The thickness of the arch at the crown is 6 ft. and at the springing it is 7 ft. 6 in, Find the area of the face.

195. The **mantel** of a cylinder or of a cone is the lateral or curved surface of the cylinder or the cone.

S denoting the measure of the area of a surface, cy, mcy, rk, mrk, s and z subscribed to S, are to be read of a cylinder, of the mantel of a cylinder, of a right circular cone, of the mantel of a right circular cone, of a sphere and of a zone of a sphere, respectively.

196. vii. *The measure of the area of the mantel of a* CYLINDER *is the product of the measure of the length of the mantel and the measure of the length of the perimeter of a right cross-section of the cylinder*, or

$$S_{mcy} = lp.$$

The truth of this theorem will appear at once on developing or unwrapping the mantel by rolling the cylinder on a plane surface. The developed mantel can by a single transposition of parts be transformed into a parallelogram whose base is a generating line of the mantel and whose width at right angles to the base is the length of the perimeter of a right cross-section of the cylinder. In the case of the right cylinder the mantel develops into a rectangle.

vii, *a*. In the case of the right circular cylinder

$$p = 2\pi r,$$

$$\therefore \quad S_{mcy} = 2\pi ra$$

in which r is the measure of the length of the radius of the base and a is the measure of the altitude of the cylinder.

vii, *b*. Adding the areas of the ends to the area of the mantel, gives for the area of the whole surface of a right circular cylinder

$$S_{cy} = 2\pi r(a + r).$$

197. viii. *The measure of the area of the mantel of a* RIGHT CIRCULAR CONE *is the continued product of π the measure of the slant height of the cone and the measure of the length of the radius of the base*, or

$$S_{mrk} = \pi rl.$$

If the cone be rolled on a plane surface, the mantel will develop into the sector of a circle whose radius is the slant height of the cone and whose arc is equal in length to the circumference of the base of the cone; and the measure of the length of that circumference is $2\pi r$.

viii, *a*. In the case of a frustum of a right circular cone, the mantel develops into the sector of an annulus, therefore

$$S_{mf} = \pi l (r_1 + r_2)$$
$$= 2\pi l r_m$$

in which r_1, r_2 and r_m are the measures of the lengths of the radii of the ends and of the midcross-section of the frustum.

viii, *b*. Adding the area of the base to the area of the mantel

$$S_{rk} = \pi r (l + r).$$

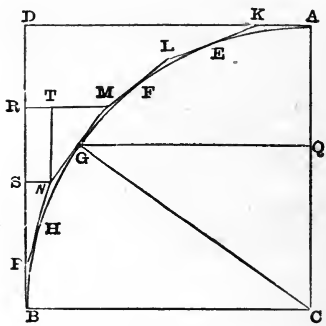
198. ix. *The measure of the area of the* SURFACE OF A SPHERE *is four times the product of* π *and the square of the measure of the length of the radius of the sphere*, or,

$$S_s = 4\pi r^2.$$

ix, *a*. *The measure of the area of a CAP or a ZONE of a sphere is twice the continued product of* π *the measure of the length of the radius of the sphere and the measure of the altitude of the segment whose curved surface is the cap or the zone to be measured*, or,

$$S_z = 2\pi r h.$$

199. Let ABC be the quadrant of the circle, C being its centre. Draw the tangents AD and BD. Divide the arc AB into any number of equal parts, *say AE, EF, FG, GH and HB, and draw KEL, LFM, MGN and NHP tangents to the arc AB at the points of division which will therefore be the mid-points of the tangents. The broken line AKLMNPB is the quarter of the perimeter of a regular convex polygon which would circumscribe the circle of which ABC is the quadrant.



*In the figure as drawn, the arc AB is divided into five equal arcs, but it might have been divided into any other number and the method of proof would have applied equally well.

If now the whole figure revolve about AC as axis, the arc AB will generate the surface of a hemisphere, the tangents KL, LM, MN, NP will generate the mantels of a series of frusta of right circular cones, and BD will generate the mantel of a right circular cylinder circumscribed about the hemisphere. We proceed to prove that each mantel generated by a tangent is equal in area to the mantel generated by the projection of that tangent on BD, the generating line of the mantel of the cylinder.

Consider the mantel generated by the tangent MGN join GC and let fall GQ perpendicular to AC, and MR and NS perpendicular to BD, and NT perpendicular to MR.

By viii *a*, p. 234, the mantel generated by MN revolving round AC as axis is equal to 2π rectangles each equal to the rectangle under GQ and MN, which quantity we shall denote by 2π (GQ.MN).

The triangle MNT is similar to the triangle CGQ,

$\therefore$ CG : GQ : : MN : NT

$\therefore$ GQ.MN = CG.NT = CB.SR

$\therefore$ 2π(GQ.MN) = 2π(CB.SR)

$\therefore$ the mantel generated by MN rotating about AC as axis is equal to 2π (CB.SR).

But 2π (CB.SR) is equal to the mantel generated by SR rotating about AC as axis, and SR is the projection of NM on BD

$\therefore$ the mantel generated by MN rotating about AC is equal to the mantel generated by the projection of MN on BD, rotating about AC.

In a similar manner it may be proved that the mantels generated by the other tangents, KL, LM, NP are each equal in area to the mantels generated by their projections on BD. Hence the aggregate of the mantels generated by the tangents will be equal to the mantel generated by the aggregate of the projections of the tangents on BD, *i.e.*, the mantel generated by PD.

Therefore, the aggregate of the mantels generated by the broken line KLMNPB revolving around AC will be equal to the mantel generated by the line BD revolving around AC.

If now the number of equal parts into which the arc AB is divided, be doubled, the number of mantels of frusta will be doubled (including in each case the mantel generated by the 'final' tangent, PB), but each mantel generated by a tangent being still equal to the mantel generated by the projection of that tangent on BD, and the aggregate of the projections being still BD, the aggregate of the mantels generated by the tangents will still be the mantel generated by BD. We may therefore double the number of tangents as often as we please and the aggregate of the mantels generated by them will remain equal to the mantel generated by BD.

By doubling often enough the number of equal parts into which the arc AB is divided, the point K can be brought as near to A as we please, and therefore the aggregate length of the tangents, KL, LM, MN, ... can be made differ from the length of the broken line AKL . . . . . B by less than any explicitly assigned length however small. Hence the surface generated by the broken line revolving about AC as axis can be made to differ in area from the mantel generated by BD revolving about AC as axis, by less than any explicitly assigned area however small.

But, § 187, by doubling the number of tangents often enough, the broken line AKL . . . . . B can be made differ in length from the arc AB by less than any explicitly assigned length however small. Hence the surface generated by the broken line AKL . . . . . B revolving about AC as axis can be made differ in area from the hemisphere-surface generated by the arc AB revolving about AC as axis, by less than any explicitly assigned area however small.

Hence the hemisphere-surface generated by the arc AB and the cylindric mantel generated by BD differ in area by less than any explicitly assigned area. Therefore the difference in area of these surfaces cannot be constant.

Neither can their difference in area be variable, for the surfaces themselves are constant and two constants cannot have a variable difference.

Therefore if the figure ACBD revolve about AC as axis, the area of the curved surface of the hemisphere generated by the quadrant ABC will be equal to the area of the mantel of the cylinder generated by the square ADBC, *i.e.*, the mantel of the cylinder circumscribing the hemisphere.

But by vii *a*, p. 233, the mantel of this circumscribing cylinder is equal to

$$2\pi\,(BC.BD) = 2\pi\,(\text{sq. on } BC)$$

Hence the surface of the sphere whose radius is BC is equal to 4π (sq. on BC),

$\therefore \qquad S_s = 4\pi r^2.$

200. From the preceding investigation, it is evident that if a right circular cylinder be circumscribed about a sphere and two planes parallel to the ends of the cylinder cut both sphere and cylinder, the area of the zone between the planes is equal to the area of the cylindric mantel between the planes. If one of the planes coincide with an end of the cylinder, the zone will become a spherical cap. But r being the measure of the length of the radius of the sphere and h being the measure of the normal distance between the planes of section, the measure of the area of the mantel between these planes will be $2\pi rh$,

$\therefore \qquad S_z = 2\pi rh.$

201. If $2l$ and $2k$ denote the measures of the lengths respectively of the polar axis and of an equatorial diameter of a spheroid, and if l and k are *very* nearly equal, the measure of the area of the surface of the spheroid will be nearly

$$2\pi k^{\frac{2}{3}}(k^{\frac{4}{3}} + l^{\frac{4}{3}}).$$

In the case of the oblate spheroid, in which $k > l$, the measure of the area of the surface will be

$$> 4\pi k^2 \left\{\frac{k^2 + 2l^2}{2k^2 + l^2}\right\}$$

but $$< 2\pi k^{\frac{2}{3}} (k^{\frac{4}{3}} + l^{\frac{4}{3}}).$$

EXERCISE XXX.

1. Find the area of the mantel of a right cylinder of 3 ft. altitude and 15 in. perimeter of base.

2. The slant height of a cylinder is 39 in. and the length of the perimeter of a right cross-section is 40 in. Find the area of the mantel.

3. The length of a cylinder is 22 ft. and its least girth is 22 in. Find the area of the mantel.

4. Find the area of a right circular cylinder of 25 in. altitude and 12 in. radius of base.

5. The axes of the base of a right elliptic cylinder are 15 in. and 12 in. long respectively and the length of the cylinder is 7 ft. 6 in. Find the area of the mantel.

6. Find the area of the whole surface of a right circular cylinder of 15 in. radius and 5 ft. altitude.

7. Find the area of the *whole* surface of a cylindric pipe 8 ft. 6 in. long and an inch and a quarter thick, the length of the internal diameter being $10\frac{1}{2}$ in.

8. Find the area of the whole surface of a right elliptic cylinder 6 ft. long, the lengths of the axes of the base being 12 in. and 10 in. respectively.

9. The area of the mantel of a cylinder is 8 sq. ft. and the length of the perimeter of a right cross-section is 3 ft. Find the length of the cylinder.

10. The area of the mantel of a right circular cylinder is 2 sq. ft. 117 sq. in. and the length of the radius of the base is 6·75 in. Find the length of the cylinder.

11. The area of the whole surface of a right circular cylinder is 21 sq. ft. and the height of the cylinder is equal to the length of the diameter of the base. Find the length of the diameter.

12. The area of the whole surface of a right circular cylinder is 27 sq. ft. and the length of the cylinder is thrice the length of the radius. Find the length of the radius.

Find the area of the mantel of a right circular cone whose dimensions are

13. Slant height 3 ft. 6 in., length of circumference of base 4 ft. 9 in.

14. Slant height 4 ft. 6 in., length of radius of base 1 ft. 3 in.

15. Altitude 3 ft. 9 in., length of radius of base 2 ft. 4 in.

16. Altitude 8 ft. 3 in., length of circumference of base 5 ft. 3 in.

Find the area of the whole surface of a right circular cone whose dimensions are

17. Slant height 2 ft. 6 in., length of radius of base $10\frac{1}{2}$ in.

18. Slant height 7 ft. 5 in., length of circumference of base 7 ft. 1 in.

19. Altitude 2 ft., length of radius of base 10 in.

20. Altitude 5 ft., length of circumference of base 9 ft. 11 in.

21. The area of the mantel of a right circular cone is 5 sq. ft. and the length of the circumference of the base is 45 in. Find the slant height of the cone.

22. The area of the mantel of a right circular cone is 7 sq. ft. 72 sq. in. and the length of the circumference of the base is 5 ft. Find the altitude of the cone.

23. Find the slant height of a right circular cone whose mantel has an area of 15 sq. in. and whose base-radius has a length of 1·5 in.

24. Find the altitude of a right circular cone, given that the area of its mantel is 5 sq. ft. and the length of the radius of its base is 6 in.

25. The area of the mantel of a right circular cone is 2·5 sq. ft. and its slant height is 25 in. Find the length of the circumference of the base.

26. The area of the mantel of a right circular cone is 15 sq. ft. and the slant height is 2 ft. Find the length of the radius of the base.

27. The area of the whole surface of a right circular cone is 2 sq. yd. and the slant height is twice the length of the diameter of the base. Find the length of the diameter of the base.

28. How many yards of canvas 45 in. wide will be required to make a conical tent 10 ft. wide and 9 ft. high?

29. How many yards of canvas 32 in. wide will be required to make a conical tent 15 ft. wide and 10 ft. high, if 10 % of the canvass is cut away or turned in, in the making of the tent.

30. The area of the mantel of a right circular cone is twice the area of the base. Find the vertical angle.

31. A right circular cylinder and a right circular cone stand on equal bases and are of the same altitude, the altitude being equal to the length of a diameter of either base. Find the ratio of (a) the mantels, (b) the whole surfaces of the cone and cylinder.

32. Find the area of the mantel of the frustum of a right circular cone whose slant height is 7 in., the lengths of the circumferences of the ends of the frustum being 15 in. and 2 ft. respectively.

33. The radii of the ends of the frustum of a right circular cone are 15 in. and 5 in. long respectively and the slant height of the frustum is 12 in. Find the area of its mantel.

34. The altitude of a frustum of a right circular cone is 12 in. and the lengths of the end-radii are 9 in. and 16 in. respectively. Find the area of the mantel.

35. Find the area of the whole surface of the frustum of a right circular cone, the lengths of the circumferences of the ends being 11 in. and 17 in. respectively and the slant height of the frustum being 7 in.

36. The lengths of the end-radii of a frustum of a right circular cone are 3·3 ft. and 1·7 ft. respectively, and the slant height of the frustum is 27 in. Find the area of the whole surface of the frustum.

37. The altitude of a frustum of a right circular cone is 7·7 in. and the lengths of the end-radii are 6·4 in. and 10 in. respectively. Find the area of the whole surface of the frustum.

38. The altitude of a frustum of a right circular cone is 20·8 in. and the lengths of the end-radii are 7·5 in. and 18 in. respectively. If the frustum be divided into two frusta whose mantels are of equal area, what will be the altitude of each?

39. The lengths of the sides containing the right angle of a right-angled triangle are 1·248 and 1·265 metres respectively. If the triangle revolve about an axis parallel to and 1·25 metres distant from its shortest side, what will be the area of the whole surface described by the sides of the triangle?

40. Find the area of the surface of a sphere of 3 in. radius.

41. Find the area of the surface of a sphere 12 inches in circumference.

42. The area of the surface of a sphere is a square foot. Find the length of the radius to the nearest hundredth of an inch.

43. A cylindric tube 8 ft. long and 2 ft. 6 in. in diameter is closed at each end by a hemisphere. Find the area of the whole external surface.

44. The length of the radius of a sphere is 15 in. Find the area of a cap on the sphere, 5 inches in height.

45. Find the area of the whole surface of a segment of a sphere of 21 inches radius, the height of the segment being 10 inches, and the distance of its base from the centre of the sphere, 11 inches.

46. Find the area of the whole surface of a zonal segment of a sphere of 12 in. radius, the distances from the centre of the sphere of the terminal circles of the zone being 5 in. and 9 in. both on the same side of the centre.

✓ **47.** The length of the diameter of a sphere is 30 in. and the length of the radius of the base of a cap-segment of the sphere is 5 in. Find the height of the cap at right angles to its base.

48. A sphere is 30 inches in diameter. What fraction of the whole surface will be visible to an eye placed at a distance of 10 ft. from the centre of the sphere ?

✓ **49.** At what distance from the centre of a sphere of 9 in. radius must a luminous point be placed to light up one-third of the surface of the sphere ?

50. Find in square miles the area of the surface of the earth assuming it to be practically an oblate spheroid the lengths of whose semiaxes are 20,926,202 feet and 20,854,895 feet respectively.

202. *If two solids on equal bases and of equal altitudes are such that all plane sections of the solids parallel to and at equal distances from their bases are equal to one another, the section of one solid at each and every distance from its base equal to the section of the other solid at the same distance from its base, then will the solids be equal in volume.*

This proposition may be shown to follow from Theorem II p. 177, by applying a method of demonstration similar to that employed on pp. 180 and 181 to prove that tetrahedra on equal and similar bases and of the same altitude are of equal volumes, and on pp. 225 to 228 to obtain the quadrature of the circle.

203. II, *a.* *The measure of the volume of a cylinder is the product of the measures of the altitude of the cylinder and the area of its base*, or

$$V_{cy} = aB.$$

This proposition follows immediately from Theorem II, p. 177, and § 202.

In the case of the right circular cylinder,

by § 193 $\quad B = \pi r^2$,

and ∴ $\quad V_{rcy} = \pi a r^2$.

204. III, *a.* *The measure of the volume of a cone is ONE-THIRD of the product of the measures of the altitude of the cone and the area of its base*, or

$$V_k = \tfrac{1}{3} aB.$$

This proposition follows immediately from Theorem III, p. 179, and § 202.

In the case of the right circular cone,

by § 193 $B=\pi r^2$

and ∴ $V_{rk}=\frac{1}{3}\pi a r^2$.

For the measure of the volume of a frustum of a right circular cone, IV *a*, p. 183, gives

$$V_{rkf}=\frac{1}{3}\pi a\,(r_1^2+r_1r_2+r_2^2).$$

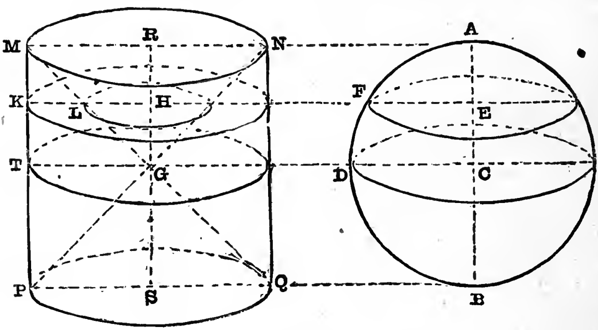
205. V. *The measure of the volume of an ellipsoid is FOUR-THIRDS of the continued product of π and the measures of the lengths of the semiaxes of the ellipsoid*, or

$$V_e=\frac{4}{3}\pi abc.$$

V, *a*. *The measure of the volume of the sphere is four-thirds of the product of π and the cube of the measure of the length of the radius of the sphere*, or

$$V_s=\frac{4}{3}\pi r^3.$$

These theorems may be obtained at once from the Prismoidal Formula but they may also be proved independently as follows :—



Let AFDB be a sphere, C its centre and ACB a diameter.

Let MNPQ be a right circular cylinder whose diameter and altitude are both equal to the diameter of the sphere. Let there be hollowed out of the cylinder two right circular cones MGN and PGQ whose bases are the ends of the cylinder and whose vertices meet at G the mid-point of RS the axis of the cylinder.

In CA take any point E, draw EF at right angles to CA and meeting the surface of the sphere in F, and join CF. EF is the radius of the small circle which is the plane section of the sphere at the distance CE from the centre.

Let the measures of the lengths of CF and CE be r and x respectively then will r^2-x^2 be the square of the measure of the length of EF, and therefore the measure of the area of the small circle at the distance CE from C will be $\pi(r^2-x^2.)$

In GR take GH equal to CE and draw HLK at right angles to GR and cutting GM in L and PM in K. The section of the hollowed cylinder by a plane through H parallel to the base of the cylinder is the annulus whose centre is H and whose radii are HL and HK.

Because RM is equal to GR, therefore HL is equal to GH. But GH is equal to CE and HK is equal to CF therefore the measure of the length of HL is x and that of the length of HK is r. Therefore, the measure of the area of the annulus whose centre is H and radii HL and HK, is $\pi(r^2-x^2)$. But this is the measure of the area of the small circle which is the plane section of the sphere at distance CE, equal to GH, from the centre.

Hence the area of a plane section of the sphere at any distance from its centre, C, is equal to the area of the right cross-section of the hollowed cylinder at the same distance from its *centre*, G.

Hence by § 202, the volume of the sphere is equal to the volume of the hollowed cylinder, and if the cylinder be constituted between planes tangent to the sphere (as it is in the figure), the volume of the spherical segment between any two planes parallel to the tangent planes is equal to the volume of the part of the hollowed cylinder between the two parallel planes.

By § 203, the measure of the volume of a right circular cylinder is πar^2 and here $a=2r$, therefore the volume of the unhollowed cylinder is $2\pi r^3$.

But by § 204, the measure of the volume of each of the two cones hollowed out of the cylinder is $\frac{1}{3}\pi r^3$.

Therefore the measure of the volume of the hollowed cylinder is $2\pi r^3-\frac{2}{3}\pi r^3=\frac{4}{3}\pi r^3$.

But the measure of the volume of the sphere is equal to the measure of the volume of the hollowed cylinder,

∴ $$V_s = \frac{4}{3}\pi r^3.$$

For the hollowed cylinder in the preceding proof, there may be substituted a tetrahedron whose altitude (distance between a pair of opposite edges) is equal to a diameter of the sphere and whose midcross-section is equal in area to a midcross-section of the sphere.

206. The proof of V follows step by step the preceding proof of V,*a*, employing, however, an ellipsoid instead of a sphere, a right elliptic hollowed cylinder instead of a right circular hollowed cylinder and ellipses instead of circles. CA and CD of the figure should be semiaxes of the ellipsoid.

207. It should be noticed that if the sphere be inscribed in the hollowed cylinder and two planes parallel to the ends of the cylinder be drawn cutting the figures, not only will the volumes of the sphere-segment and hollowed cylinder between the cutting planes be equal, the areas of the zone of the sphere and the mantel of the cylinder between the cutting planes will also be equal.

208. If a right circular cylinder, a hemisphere and a right circular cone be on equal bases and of the same altitude, the volume of the cylinder will be thrice and the volume of the hemisphere will be twice the volume of the cone, or

$$V_{rcy} = \pi r^3, \quad V_{hs} = \frac{2}{3}\pi r^3, \quad V_{rk} = \frac{1}{3}\pi r^3.$$

Compare these relations in volume with those of the prism, the hemitetrahedron and the pyramid, given on page 187.

209. X. *The measure of the area of a* TORE *or* RING *is the product of the measures of the length of the perimeter of a right cross-section and the length of the axis of the tore.*

VI. *The measure of the volume of a tore is the product of the measure of the area of a right cross-section and the measure of the length of the axis of the tore.*

210. *The areas of* SIMILAR PLANE FIGURES *or of* SIMILAR SURFACES *are to one another as the squares of the measures of the lengths of their corresponding linear dimensions.*

The volumes of SIMILAR SOLIDS *are to one another as the cubes of the measures of the lengths of their corresponding linear dimensions.*

EXERCISE XXXI.

1. The length of the radius of the base of a right circular cylinder is 5 in. and the altitude of the cylinder is 8 in. Find its volume.

2. The lengths of the axes of the base of an elliptic cylinder are 6 in. and 4 in. respectively and the altitude of the cylinder is 12 in. Find its volume.

3. Find the length of the radius of the base of a cylinder whose volume is a cubic foot and whose altitude is a linear foot.

4. The area of the mantel of a right circular cylinder is 6 sq. ft. and the volume of the cylinder is 6 cu. ft. Find the length of the radius of the base.

5. The area of the mantel of a right circular cylinder is a square yard and the volume of the cylinder is a cubic foot. Find the length of the cylinder.

6. The area of the base of a right circular cylinder is 5 sq. ft. and the volume of the cylinder is 5 cu. ft. Find the area of the mantel.

7. A vessel, in the form of a right circular cylinder is to have a capacity of one gallon and the depth of the vessel is to be equal to the length of the diameter of a right cross-section of it. Find the depth and the whole internal area, the vessel being without a lid.

8. The French and German liquid measures are right circular cylinders whose depth is in each case equal to twice the length of its diameter. Find the diameter of a measure holding 10 litres.

9. The French dry measures are right circular cylinders whose depth is in each case equal to the length of its diameter. Find the depth of the hectolitre.

10. The German dry measures are right circular cylinders whose depth is in each case equal to two-thirds of the length of its diameter. Find the depth of the hectolitre.

11. A cubic foot of brass is drawn into wire the twentieth of an inch in diameter. Find the length of the wire.

12. Mr. C. V. Boys has drawn quartz fibres which have been estimated to be only the millionth of an inch in diameter. How many miles of such a fibre would a grain of sand make. the grain being a right circular cylinder one-hundredth of an inch long by one-hundredth of an inch in diameter?

13. Find the volume of a hollow right circular cylinder, the length of the radius of the inner surface being 3·5 in.; of the radius of the outer surface, 4·125 in.; and of the cylinder, 7 ft. 6 in.

14. Find the thickness of the lead in a pipe of three-quarter inch bore, if 10 ft. of the pipe weigh 21 lb. and a cubic foot of lead weigh 712 lb.

15. A hollow right circular cylinder of cast iron 15 feet in length and 4 feet in diameter of outer surface, is set upright and bears on the top a weight of 250 tons. Determine the thickness of the metal so that the pressure on the base may be 1500 lb. per square inch, the weight of a cubic foot of cast iron being 444 lb.

16. Find the volume of a hollow-elliptic cylinder 75 ft. in length, the lengths of the axes of the inner surface being 5 ft. and 3 ft. respectively and the thickness of the walls being 8 in.

17. Find the volume of a cone whose altitude is 15 in. and whose base is a circle 10 in. in diameter.

18. The volume of a cone is 3·5 cubit feet and its altitude is 5 feet. Find the length of the radius of the base which is a circle.

19. Find the volume of a cone whose slant height is 65 in. and whose base is a circle 32 in. in diameter.

20. Find the volume of a cone whose altitude is 35 in. and whose slant height all round is 37 in.

21. Find the volume of a cone on a circular base of 5 in. radius, the area of the mantel of the cone being a square foot.

22. Find the volume of a cone on a circular base, the altitude of the cone being 10 in. and the area of the mantel being a square foot.

23. Find the volume of the frustum of a cone on a circular base, the height of the frustum being 10·5 in. and the lengths of the radii of the ends being 5 in. and 2 in.

24. The slant height of a frustum of a right circular cone is 10 in. and the lengths of the radii of the ends are 16 in. and 10 in. respectively. Find the volume of the frustum.

25. Find the volume of the cone from which the frustum in problem 24 was cut.

26. The lengths of the radii of the ends of a frustum of a right circular cone are 6 ft. and 9 ft. respectively and the altitude of the frustum is 4 ft. Find the volumes of the two frusta formed by cutting the frustum by a plane parallel to the ends and midway between them.

27. The lengths of the radii of the ends of a frustum of a right circular cone are 4 ft and 6 ft. respectively and the altitude of the frustum is 3 ft. Find the volumes of the three pieces produced by cutting the frustum by two planes parallel to the ends and trisecting the height of the frustum.

28. A pyramid 15 inches in altitude is divided into three parts of equal volumes by planes parallel to the base. Find the altitudes of the three parts.

29. The lower portion of a haystack is in the form of a frustum of a right circular cone with the end of shorter diameter below, the upper part of the stack is in the form of a cone. The total height of the stack is 25 ft., the length of its greatest circumference is 54 ft., the height of the frustum is 15 ft. and the length of the diameter of the base is 15 ft. How many cubic yards are there in the stack?

30. The area of the whole surface of a right circular cone is 25 sq. ft. Find the volume of the cone, the slant height being five times the length of the radius of the base.

31. The volume of a right circular cone is 7854 cubic inches. Find the area of the whole surface of the cone, the altitude being thrice the length of the radius of the base.

32. A vessel in the form of a right circular cone whose slant depth is equal to the length of the diameter of its mouth, just holds a gallon. Find the slant depth.

33. Find the volume of a sphere 12 inches in diameter.

34. Find the volume of a sphere a great circle of which is 33 in. in circumference.

35. The area of the surface of a sphere is a square yard. Find the volume of the sphere.

36. How many gallons will a hemispherical bowl 18 inches in diameter hold?

37. What will be the weight of a spherical shot of cast iron 5·5 inches in diameter if a cubic foot of iron weigh 444 lb. ?

38. Find the weight of a sphere of lead 3·75 inches in diameter, the lead weighing 712 lb. per cubic foot.

39. What weight of gunpowder will fill a spherical shell of 7 in. internal diameter, if 30 cubic inches of the gunpowder weigh a pound ?

40. Find the volume 1° of the greatest sphere, 2° of the greatest hemisphere, that can be cut out of a cube of wood measuring 7·5 inches on the edge.

41. The largest possible cube is cut out of a sphere one foot in diameter. Find the length of an edge of the cube and the volume of material cut away in making the cube.

42. Find the weight of a spherical shell 1·75 in. thick and of 8 inches external radius, the material composing the shell weighing 490 lb. per cubic foot.

43. The length of the greatest circumference of a spherical shell is 25 in. and the length of the internal diameter is 5·75 in. Find the weight of the shell, the substance of which it is composed weighing 500 lb. per cubic foot.

44. A spherical shell weighs 13 lb. and the lengths of the external and internal diameters are 6 in. and 4 in. respectively. Find the weight of a shell of the same substance but of 8 in. external and 5 in. internal diameter.

45. Find the volume of a solid in the form of a right circular cylinder with hemispherical ends, the length of the diameter of the cylinder being 3 ft. 6 in. and the extreme length of the solid being 25 feet.

46. A cylindrical pontoon with hemispherical ends is constructed of sheet-iron ·125 in. thick, the extreme length of the pontoon is 22 ft. and the length of its outside diameter is 2 ft. 6 in. Find the weight which the pontoon will support when half immersed and also the greatest load it will bear assuming the specific gravity of sheet-iron to be 7·75 and taking the weight of water at 62·5 lb. per cubic foot.

47. Find the thickness of an 8-inch shell if it weigh half as much as a solid ball of the same diameter and of like material.

48. A spherical shell 10 in. in diameter weighs ·9 as much as a solid ball of the same diameter and substance. Find the length of the internal diameter.

49. A cast iron shell 8 in. in diameter, is filled with gunpowder and plugged with iron; the whole then weighs 75·5 lb. Find the thickness of the shell, supposing the iron to weigh 444 lb. per cubic foot and the gunpowder to weigh 57·6 per cubic foot.

50. If the nature of the earth's crust be known to an average depth of 5 miles, what proportion of the whole volume of the earth is known, assuming the earth to be a sphere 7912 miles in diameter?

51. If the ocean cover 73·5 per cent. of the earth's surface and its average depth be 2 miles, what proportion will its volume bear to the volume of the whole earth considered as a sphere 7912 miles in diameter?

52. If the atmosphere extend to a height of 45 miles above the earth's surface what proportion will its volume bear to that of the earth assumed to be a sphere of 7912 miles diameter?

53. The radius of the base of a right circular cone is 2 inches and the volume of the cone is equal to that of a spherical shell of 4 in. external and 2 in. internal diameter. Find the altitude of the cone.

54. A Stilton cheese is in the form of a cylinder, a Dutch cheese is in the form of a sphere. Find the length of the diameter of a Dutch cheese weighing 9 lb., a Stilton cheese 8 inches in diameter and 7 inches high weighing 6 lb.

55. The length of the radius of the base of a segment of a sphere is 2 in. and the length of the radius of the sphere is 6 in. Find the volume of the segment.

56. The height of a segment of a sphere is 6 in. and the length of the radius of the base is 8 in. Find the volume of the segment.

57. The lengths of the radii of the ends of a zonal segment of a sphere are 5 in. and 8 in. respectively, and the height of the segment is 3 in. Find the volume of the segment.

58. Find the volume of a zonal segment of a sphere, the ends of the segment being on opposite sides of the centre of the sphere and distant from it 10 in. and 15 in. respectively, the length of the radius of the sphere being 20 inches.

59. A section parallel to the base of a hemisphere bisects its altitude. Find the ratio of the volumes of the segments.

60. A sphere whose volume is a cubic yard is divided by a plane into segments whose altitudes are as 2 to 3. Find the volumes of the segments.

61. How much water will run over if a heavy globe of 2 in. diameter be dropped into a conical glass full of water, the diameter of the mouth of the glass being 2·5 in. and its depth 3 in.?

62. Find the volume of the prolate spheroid generated by an ellipse of 12 in. major and 10 in. minor axis.

63. Find the volume of the earth assuming it to be an oblate spheroid of 41,709,790 ft. polar axis and 41,852,404 ft. equatorial diameter.

64. Find the volume of the earth assuming it to be an ellipsoid the lengths of whose semiaxes are 20,926,629 ft., 20,925,105 ft. and 20,854,477 ft. respectively. Find also the length of the mean-radius or radius of a sphere of the same volume as the earth.

65. Find the length of 100 complete coils of a wire one-tenth of an inch in diameter coiled closely upon a cylinder of 5 in. radius.

66. On examining and taking the dimensions of a steep cistern, which was supposed to be perfectly cylindrical, I found the bottom cross diameters to be 70 inches each, but the top diameters were 68 and 72 inches respectively. The depth of the vessel was 65 inches. What is the difference in the capacities of the true cylinder at 70 inches diameter and the one examined?

67. A steep cistern in the form of a frustum of an elliptic cone, the cross diameters at the bottom being 84 and 64 inches, and the diameters at the top 72 and 57 inches, is 50 inches deep, and is filled to the depth of 25 inches with dry barley. How many cubic inches does it contain?

68. A cylindrical iron tank, 20 feet long and 4 feet 6 inches in diameter, was placed horizontally on a flat car and filled with oil at Petrolia. When it arrived at Toronto, it was found upon being dipped from the top, to be 10 inches to the surface of the oil. What was the wantage in gallons?

CHAPTER VI.

PROPORTIONAL AND IRREGULAR DISTRIBUTION AND PARTNERSHIP.

211. If four magnitudes be in proportion and if the first magnitude be a multiple of the second, the third magnitude will be the same multiple of the fourth ; if the first magnitude be a part of the second, the third magnitude will be the same part of the fourth ; if the first magnitude be a multiple of a part of the second, the third magnitude will be the same multiple of the same part of the fourth ; and, generally, according as the first magnitude is greater than, equal to or less than any multiple or part or multiple of a part of the second, the third magnitude is also greater than, equal to or less than the same multiple or the same part or the same multiple of the same part of the fourth ; and, conversely ; *if these conditions are satisfied the four magnitudes are in proportion.* (See §§ 148 to 152, pp. 159 and 160.)

212. Hence if four quantities be in proportion the first and second quantities will also be proportional to any equimultiples of the third and fourth quantities or to any equifractional parts of these quantities, *i.e.* the third and fourth quantities may both be multiplied or both divided by the same number without affecting the proportion.

Example. \$12 = $\frac{2}{3}$ of \$18 and 64 lb. = $\frac{2}{3}$ of 96 lb.,

$\therefore$ \$12 : \$18 : : 64 lb : 96 lb.

Dividing both 64 lb. and 96 lb. by 7 will not affect the $\frac{2}{3}$ in the statement 64 lb. = $\frac{2}{3}$ of 96 lb., nor will multiplying the two quotients by 4 affect the $\frac{2}{3}$,

$\therefore$ \$12 : \$18 : : $9\frac{1}{7}$ lb. : $13\frac{5}{7}$ lb.

and \$12 : \$18 : : $36\frac{4}{7}$ lb. : $54\frac{6}{7}$ lb.

So also if four quantities be in proportion, the first and second quantities may both be multiplied or both divided by the same number without affecting the proportion.

Thus in the preceding example, multiplying both \$12 and \$18 by 2 and dividing the products by 5 will not affect the $\frac{2}{3}$ in the statement $\$12 = \frac{2}{3}$ of \$18,

$\therefore$ $\frac{2}{5}$ of \$12 : $\frac{2}{5}$ of \$18 : : 64 lb. : 96 lb.,

i.e. \$4·80 : \$7·20 : : 64 lb. : 96 lb.,

and $\therefore$ \$4·80 : \$7·20 : : $36\frac{1}{4}$ lb : $54\frac{6}{7}$ lb.

Hence, generally, if four quantities be in proportion any equimultiples or equifractional parts of the first and second quantities will also be proportional to any equimultiples or equifractional parts of the third and fourth quantities.

213. If four quantities be in proportion and if any equimultiples or equifractional parts of the first and third quantities be taken and also any equimultiples or equifractional parts of the second and fourth quantities, these multiples or fractional parts taken in the order of the quantities are in proportion.

Example. 15 in. $= \frac{3}{5}$ of 25 in. and 57 gal. $= \frac{3}{5}$ of 95 gal.

$\therefore$ 15 in. : 25 in. : : 57 gal. : 95 gal.

Multiplying both 15 in. and 57 gal. by 6 will multiply the $\frac{3}{5}$ by 6 in both the statements,

15 in. $= \frac{3}{5}$ of 25 in. and 57 gal. $= \frac{3}{5}$ of 95 gal.

which thus become

15 in. $\times 6 = \frac{3}{5} \times 6$ of 25 in. and 57 gal. $\times 6 = \frac{3}{5} \times 6$ of 95 gal.

Multiplying both 25 in. and 95 gal. by 7 will divide the $\frac{3}{5} \times 6$ by 7 in both these statements which thus become

15 in. $\times 6 = \frac{3}{5} \times \frac{6}{7}$ of (25 in. $\times$ 7) and 57 gal. $\times 6 = \frac{3}{5} \times \frac{6}{7}$ of (95 gal. $\times$ 7)

$\therefore$ (15 in. $\times$ 6) : (25 in. $\times$ 7) : : (57 gal. $\times$ 6) : (95 gal. $\times$ 7),

i.e. 90 in. : 175 in. : : 342 gal. : 665 gal.

214. If four quantities be in proportion and if the first and second quantities be expressed in terms of one and the same unit and the third and fourth quantities be also expressed in terms of one and the same unit, the unit of the first and second quantities not being necessarily the same as the unit of the third and fourth quantities, it follows from the preceding section that the product of the measures of the first and fourth quantities is equal to the product of the measures of the second and third quantities. For, if the first and third quantities both be multiplied by the measure of the fourth quantity, and the second and fourth quantities both be multiplied by the measure of the third quantity, in the proportion formed by these multiples the third and fourth quantities will be equal to one another and therefore the first and second quantities will be equal to one another. But the measure of the first quantity in this proportion formed by the multiples, is the product of the measures of the first and fourth quantities of the original proportion, and the measure of the second quantity in the new proportion is the product of the measures of the second and third quantities of the original proportion. Hence the product of the measures of the first and fourth quantities of the original proportion is equal to the product of the measures of the second and third quantities of the original proportion.

Example. $35 = $\frac{5}{8}$ of $56 and 55 yd. = $\frac{5}{8}$ of 88 yd.

∴ $35 : $56 : : 55 yd. : 88 yd.

Multiply $35 and 55 yd. both by 88, the measure of 88 yd., the fourth quantity or term of the proportion.

Also multiply $56 and 88 yd. both by 55, the measure of 55 yd., the third quantity or term of the proportion. Then by § 213

$35 × 88 : $56 × 55 : : 55 yd. × 88 : 88 yd. × 55

But 55 yd. × 88 = 88 yd. × 55

∴ $35 × 88 = $56 × 55.

215. If four quantities form a proportion, the quantities are called **the terms** of the proportion; the first and fourth quantities are called *the extreme terms* or **the extremes** of the proportion and the second and third quantities are called *the mean terms* or **the means** of the proportion.

Employing this phraseology and with the implication of the conditions regarding the units of the terms, the theorem of § 214 may be briefly stated under the form

The product of the measures of the extremes of a proportion is equal to the product of the measures of the means of the proportion.

the mean terms of a proportion be equal to one another, *i. e.*, if the first of three quantities of the same kind be to the second as the second is to the third, the third quantity is said to be a *third proportional* to the first and second quantities, and the second quantity is said to be a **mean proportional** between the first and third quantities.

216. If the first of four quantities be to the second as the third is to the fourth ;

i. *The second quantity will be to the first as the fourth quantity is to the third ;*

ii. *The sum of the first and second quantities will be to the second as the sum of the third and fourth quantities is to the fourth ;* and

iii. *The difference between the first and the second quantity will be to the second quantity as the difference between the third and the fourth quantity is to the fourth quantity.*

These theorems follow immediately from § 211 but in the case of a proportion with commensurable terms ii and iii are merely special cases of the theorem of § 213.

Examples. If A's money : B's money : : \$3 : \$5
then will B's money : A's money : : \$5 : \$3,
A's money + B's money : B's money : : \$8 : \$5,
and B's money : A's money + B's money : : \$5 : \$8.

So also if N's weight : M's weight + N's weight : : 4 lb. : 11 lb.
then will M's weight + N's weight : N's weight : : 11 lb. : 4 lb.
and M's weight : N's weight : : 7 lb. : 4 lb.

217. If either the first and second quantities or the third and fourth quantities of a proportion be replaced by their measures in terms of a common unit, the other pair of quantities are then said to be proportional to the numbers which constitute these measures. Thus, if A's money is to B's money as \$3 to \$5, we may say that A's money is to B's money as 3 to 5.

EXERCISE XXXII.

Prove that

1. \$12 : \$18 : : 42 yd. : 63 yd.

2. 26537 gal. : 56865 gal. : : 54992 min. : 117840 min.

3. 2·6 A. : 26·6 A : : $27\frac{6}{7}$ bu. : 285 bu.

4. 1 yd. 8 in. : 1 mi. 256 yd. 2 ft. : : 36 min. : 5 wk. 6 da. 6 hr.

5. 8 ft. : $12^{\frac{1}{2}}$ ft. : : $48^{\frac{1}{2}}$: 3

6. $2^{\frac{1}{2}}$ sec. : $3^{\frac{1}{3}}$ sec. : : $648^{\frac{1}{6}}$ mi. : 3 mi.

Supply the missing term in

7. \$12 : \$15 : : 20 gal. : ( ).

8. 1 yd. : 2 yd. : : 3 da. : ( ).

9. $3\frac{1}{3}$ yd. : $3\frac{3}{4}$ yd. : : ( ) : $2\frac{6}{7}$ wk.

10. ( ) ; 7·5 A : : 3332 T. : 5236 T.

11. ( ) : $\frac{1}{4}$ oz. $-$ $\frac{1}{9}$ oz. : : $\frac{1}{2}+\frac{1}{3} : \frac{1}{2}-\frac{1}{3}$.

12. $2^{\frac{1}{3}}+3^{\frac{1}{2}}$: ( ) : : $3+108^{\frac{1}{6}}$: 3.

13. What sum is to \$1·25 as 25 ft. to 4 ft.?

14. One waterpipe discharges 141 gal. per hour, another discharges 235 gal. per hour. Compare their rates of discharge (*a*), per hour; (*b*), per minute; (*c*), per second; (*d*), per day; (*e*), per seventh of a day. Also compare the times in which the pipes would each discharge (*a*), 705 gal.; (*b*), 705 qt.; (*c*), 705 pt.; (*d*), 1000 gal.; (*e*), 1 gal.

15. Two taps when both open discharge water at the rate of 481 gal. per hour; the discharge of the smaller of the two being at the rate of 148 gal. per hour. Compare the volume discharged by the larger tap in any given time with the volume discharged by the smaller tap in the same time. Compare also the time in which the larger tap will discharge a given number of gallons with the time required by the smaller to discharge the same number of gallons.

16. One train travels $8\frac{1}{2}$ mi. in 20 min., and a second train 9 mi. in 15 min.; compare their rates per hour.

17. A person walks from his house to his office at the rate of 4 mi. per hr.; but finding he has forgotten something returns at the rate of 5 mi. per hour.; compare the time spent in going with that spent in returning.

18. A man can row 6 mi. an hour in still water; compare his rate of rowing down a stream which flows at the rate of $2\frac{1}{2}$ mi. an hour with his rate of rowing up.

19. A greyhound pursuing a hare takes 3 leaps to every 4 the hare takes; but 2 leaps of the hound are equal in length to 3 leaps of the hare; compare the speed of the hound with that of the hare.

20. *A*'s money is to *B*'s as 3 to 4, and *B*'s money to *C*'s as 4 to 5. How much money has *A* compared to *C*?

21. A grocer has 84 lb. of a mixture of green and black teas, the weight of green tea in the mixture being to the weight of black tea in it as 5 to 1; how many pounds of black tea must be added to make the weight of green to that of black as 4 to 1?

22. Milk is worth 20 cents a gallon, but by watering it the value is reduced to 15 cents a gallon. Find the proportion of water to milk in the mixture.

23. Divide $4500 between two persons in proportion to their ages which are 21 and 24 years.

24. Two men receive $15 for doing a certain piece of work. Now one man had worked but 3 days while the other had worked 5 days on the job. If the money is to be divided in proportion to the lengths of time the men worked, how much should each receive?

25. A farm is divided into two parts whose areas are as 9 to 13, and the area of the larger part exceeds that of the smaller by 18 A. 880 sq. yd. Find the area of the farm.

218. Let there be any number of quantities, say A, B, C, D, ..., all of one kind and an equal number of quantities, say **a**, **b**, **c**, **d**, ..., also all of one kind but not necessarily of the same kind as the quantities of the first set, then if

$$A : B :: \mathbf{a} : \mathbf{b},$$
$$B : C :: \mathbf{b} : \mathbf{c},$$
$$C : D :: \mathbf{c} : \mathbf{d},$$

and so on throughout the two sets, the quantities A, B, C, D, ... are said to be proportional to the quantities **a**, **b**, **c**, **d**, ... A and **a**, B and **b**, C and **c**, D and **d**, ... are called corresponding or *homologous* terms.

If the quantities of either set be replaced by their measures in terms of a common unit, the quantities of the other set are then said to be proportional to the numbers which constitute these measures.

The expression

$$A : B : C : D :: \mathbf{a} : \mathbf{b} : \mathbf{c} : \mathbf{d}$$

denotes that A, B, C and D are proportional to **a**, **b**, **c** and **d**.

Example 1. Divide \$720 into parts proportional to 4, 5 and 6.

$$4+5+6=15,$$

$\therefore$ if 15 be divided into parts proportional to 4, 5 and 6, these parts will be 4, 5 and 6;

$\therefore$ if 1 be divided into parts proportional to 4, 5 and 6, these parts will be $\frac{4}{15}$, $\frac{5}{15}$, $\frac{6}{15}$,

$\therefore$ if \$720 be divided into parts proportional to 4, 5 and 6, these parts will be $\frac{4}{15}$ of \$720, $\frac{5}{15}$ of \$720, and $\frac{6}{15}$ of \$720.

$$\tfrac{4}{15} \text{ of } \$720=\$192.$$
$$\tfrac{5}{15} \text{ of } \$720=\$240$$
$$\tfrac{6}{15} \text{ of } \$720=\$288$$

Proof. $\$192+\$240+\$288=\$720,$

Also $\$192=\frac{4}{5}$ of \$240 *i.e.*, \$192 : \$240 : : 4 : 5

and $\$240=\frac{5}{6}$ of \$288 *i.e.*, \$240 : \$288 : : 5 : 6.

$\therefore$ \$192 : \$240 : \$288 : : 4 : 5 : 6.

Example 2. Divide 316 lb. into parts proportional to $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{8}$.

$$\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{8}=\frac{40}{120}+\frac{24}{120}+\frac{15}{120}=\frac{79}{120},$$

$\therefore$ if $\frac{79}{120}$ be divided into parts proportional to $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{8}$, these parts will be $\frac{40}{120}$, $\frac{24}{120}$ and $\frac{15}{120}$,

$\therefore$ if 79 be divided into parts proportional to $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{8}$ these parts will be 40, 24 and 15;

$\therefore$ if 1 be divided into parts proportional to $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{8}$ these parts will be $\frac{40}{79}$, $\frac{24}{79}$ and $\frac{15}{79}$.

$\therefore$ if 316 lb. be divided into parts proportional to $\frac{1}{3}$, $\frac{1}{5}$ and $\frac{1}{8}$ these parts will be $\frac{40}{79}$ of 316 lb., $\frac{24}{79}$ of 316 lb. and $\frac{15}{79}$ of 316 lb.

$$\tfrac{40}{79} \text{ of } 316 \text{ lb.}=160 \text{ lb.},$$
$$\tfrac{24}{79} \text{ of } 316 \text{ lb.}=96 \text{ lb.},$$
$$\tfrac{15}{79} \text{ of } 316 \text{ lb.}=60 \text{ lb.}$$

Proof. 160 lb. + 96 lb. + 60 lb. = 316 lb.

160 lb. ÷ 96 lb. $=5\div3=\frac{1}{3}\div\frac{1}{5}$, *i.e.*, 160 lb. : 96 lb. : : $\frac{1}{3}$: $\frac{1}{5}$

96 lb. ÷ 60 lb. $=8\div5=\frac{1}{5}\div\frac{1}{8}$, *i.e.*, 96 lb. : 60 lb. : : $\frac{1}{5}$: $\frac{1}{8}$

$\therefore$ 160 lb. : 96 lb. : 60 lb. : : $\frac{1}{3}$: $\frac{1}{5}$: $\frac{1}{8}$.

EXERCISE XXXIII.

Divide—

1. 1331 into parts proportional to 2, 4, 5.

2. 19 T. 1120 lb. into parts proportional to $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

3. $57 into parts proportional to $\frac{1}{4}$, $\frac{2}{5}$, $\frac{1}{6}$.

4. $169·65 into parts proportional to 1, 2, 3, 3, 4.

5. $1064 into parts proportional to 2, $2\frac{1}{4}$, $2\frac{2}{5}$.

6. $1720 into parts proportional to 10, $2\frac{1}{2}$, 1, $\frac{1}{2}$, $\frac{1}{3}$.

7. 180 lb. into parts proportional to 3·3, ·7, ·5.

8. $253 in the proportion of 6, 7, and 10.

9. $6336 in the proportion of $\frac{1}{2}$, $\frac{3}{5}$, and ·7.

10. 15223 in the proportion of $\frac{5}{8}$, $\frac{7}{9}$, $\frac{9}{10}$, $\frac{5}{12}$, $\frac{8}{15}$.

11. Sugar is composed of 49·856 parts oxygen, 43·265 carbon, and 6·879 hydrogen; how many pounds of each is there in 1300 lb. of sugar?

12. Gunpowder is composed of nitre, charcoal and sulphur in the proportion of 33, 7 and 5.

(1.) How many lb. of sulphur are there in 180. lb. of powder?

(2.) How many lb. of powder can be made with 30 lb. of sulphur?

(3.) How much nitre and sulphur must be mixed with 112 lb. of charcoal to form gunpowder?

13. A man divides $3300 amongst his three sons, whose ages are 16, 19, and 25 years, in sums proportional to their ages: two years afterwards he similarly divides an equal sum, and again after three years more; how much does each receive in all?

14. Two sums of money are to be divided among three persons, one sum equally and the other in the proportion of 3, 5, and 8. The shares of the first two amount to $64·56 and $81·36 respectively. Determine the sums.

15. I want an alloy consisting of 19 parts by weight of nickel, 17 of lead, and 41 of tin. The only nickel I can obtain is 10 lb. of an alloy containing 11 parts of nickel to 7 parts of tin and 5 of lead. How much lead and tin must I add to make up the alloy I want?

16. Two persons travelling together agree to pay expenses in the ratio of \$7 to \$5. The first (who contributes the greater sum) pays on the whole \$103·20, the second \$63·40. What must one pay the other to settle their expenses according to agreement?

17. Capital originally invested so as to yield an income of \$22500, at the rate of 9%, is reinvested at 10%, and then divided among three persons in the proportion of 4, 7 and 9. Find the yearly income of each.

18. Three persons, A, B, C, agree to pay their hotel bill in the proportion of 4, 5, 6. A pays the first day's bill which amounts to \$6·10; B the second, which amounts to \$8·66; and C the third, which amounts to \$9·24. How must they settle accounts?

19. A founder is required to supply a ton (2240 lb.) of fusible metal consisting of 8 parts by weight of bismuth, 5 of lead, and 3 of tin. The only bismuth he has in stock is in an alloy consisting of 9 parts bismuth, 4 lead and 3 tin. How much of the alloy must he take, and how much lead and tin must he add to make up the order?

Example 3. Divide 53·5A. among three men so that the first man may receive 7A. as often as the second receives 8A., and the second may receive 5A. as often as the third receives 4A.

Share of 1st : share of 2nd : : 7A. : 8A.

∴ share of 1st $=\frac{7}{8}$ of share of 2nd

Share of 2nd : share of 3rd : : 5A. · 4A.

∴ share of 2nd $=\frac{5}{4}$ of share of 3rd,

and ∴ share of 1st $=\frac{7}{8}$ of $\frac{5}{4}$ of share of 3rd.

∴ share of 1st + share of 2nd + plus share of 3rd

$=(\frac{7}{8}$ of $\frac{5}{4}+\frac{5}{4}+1)$ share of 3rd,

$=(\frac{35}{32}+\frac{40}{32}+\frac{32}{32})$ share of 3rd,

$=\frac{107}{32}$ of share of 3rd.

∴ 53·5A $=\frac{107}{32}$ of share of 3rd

∴ $\frac{32}{107}$ of 53·5A = share of 3rd

∴ share of 3rd = 16A.

and share of 2nd $=\frac{5}{4}$ of share of 3rd = 20A.

and share of 1st $=\frac{7}{8}$ of $\frac{5}{4}$ of share of 3rd = 17·5A.

EXERCISE XXXIV.

1. Divide \$1050 among A, B, C and D so that A's share may be to B's as 2 to 3, B's share to C's as 4 to 5, and C's to D's as 6 to 7.

2. Divide £28. 13s. 8d. among A, B and C, so that for every shilling given to A, B gets 10s., and C a half-guinea. (21s. = 1 guinea.)

3. Divide 32 gal. 3 qt. $1\frac{1}{2}$ pt. into four measures so that the first shall be to the second as 9 to 14, the second to the third as 21 to 25, the third to the fourth as 20 to 23.

4. An assemblage of 700 persons consists of 5 men for every 2 children, and 3 children for every 7 women. How many of each ?

5. The joint capital of four partners, A, B, C, D, is \$12600 ; A's investment is \$10 for every \$17 of B's, C's is \$34 for every \$65 of D's, and B's is half as much again as C's. Required the amount of the investment of each

6. Divide \$3274·70 among A, B and C, giving A five per cent more than B, and six per cent. less than C.

7. A's rate of working is to B's as 7 to 5, B's to C's as 4 to 3, C's to D's as 5 to 6 ; time A works per day is to time B works per day as 9 to 10, time B works to that C works at 10 to 11, that of C to that of D as 10 to 7; number of days A works to number B works as 15 to 7, number B works to number C works as 11 to 20, and number C works to number D works as 7 to 5. How should \$1220, the sum paid for the work, be divided among them ?

Example 4. Divide the number 429 into three parts such that five times the first part may be equal to seven times the second and to nine times the third.

First $\times$ 5 = second $\times$ 7 = third $\times$ 9

and first + second + third = 429

∴ first $\times 7 \times 9$ + second $\times 7 \times 9$ + third $\times 7 \times 9 = 429 \times 7 \times 9 = 27027$

∴ first $\times 7 \times 9$ + first $\times 5 \times 9$ + first $\times 7 \times 5 = 27027$

∴ first $\times 143 = 27027$

∴ first $= 27027 \div 143 = 189$,

and second = first $\times 5 \div 7 = 135$,

and third = first $\times 5 \div 9 = 105$.

EXERCISE XXXV.

1. Divide \$9·60 between A and B so that 3 times A's share may be equal to 5 times B's.

2. A, B, and C have together \$1740; if $\frac{21}{200}$ of A's $=\frac{3}{40}$ of B's $=\frac{21}{100}$ of C's, find the share of each.

3. A pound of tea, a pound of coffee, and a pound of sugar together cost \$1·37; find the price of each having given that 7 lb. of tea cost as much as 16 lb. of coffee, and 3 lb. of coffee as much as 11 lb. of sugar.

4. Divide \$1650 into two parts, such that the simple interest on one of them at $4\frac{1}{2}$ % for 3 years would be equal to the simple interest on the other at 5 % for $2\frac{1}{4}$ years.

5. Divide \$1560·50 into three such parts that the amount of the first for $2\frac{1}{3}$ years at 5 % may be equal to the amount of the second for $2\frac{1}{2}$ years at $3\frac{1}{2}$ % and also to the amount of the third for 4 years at 4 %, simple interest.

6. A father leaves \$15000 to be divided among his three sons, aged respectively 16, 18, and 20 years so that if their respective shares be put to simple interest at 6 %, they may have equal shares on coming of age. How is the money to be divided?

7. Divide 365 into three parts, such that twice the first, 5 times the second, and 24 % of the third, may be equal to one another.

8. Three coal wagons contain 195 cwt. of coal in such proportions that 10 times the load in the first, 12 times that in the second, and 15 times that in the third, are equal quantities. What weight does each wagon carry?

9. A man, a woman, and a boy finish in a day a piece of work for which \$4·65 is paid. Find the share of each on the supposition that 2 men do as much as 3 women or 5 boys, and that the pay is proportional to the work done by each.

10. Divide the number 80 into four such parts that the first increased by 3 the second diminished by 3, the third multiplied by 3 and the fourth divided by 3, may give equal results.

Example 5. The daily wages of 9 men, 11 women and 12 boys is \$53·40. Find the daily wages of each man, on the supposition that 3 men do as much work as 5 women, and 4 women as much as 5 boys.

Assume the work done by one woman in one day as the unit of work. Then

the 11 women do 11 units of work
the 9 men do $\frac{9}{3}$ of $5 = 15$ units of work
the 12 boys do $\frac{12}{5}$ of $4 = 9\frac{3}{5}$ units of work

Hence the money must be divided in the proportion of 15, 11, and $9\frac{3}{5}$,
which is in the proportion of 75, 55 and 48.

$\therefore$ the 9 men's daily wages $= \frac{75}{178}$ of \$53·40

$\therefore$ each man's daily wages $= \frac{1}{9}$ of $\frac{75}{178}$ of \$53·40 $=$ \$2·50.

EXERCISE XXXVI.

1. Divide \$490 among 2 men, 8 women, and 10 children for work done, on the supposition that 1 man does as much as 3 women or 5 children.

2. A, B, C, rent a pasture for \$92; A puts in 6 horses for 8 weeks, B, 12 oxen for 10 weeks, C, 50 cows for 12 weeks. If 5 cows are reckoned equivalent to 3 oxen, and 4 oxen to 3 horses, what shall each pay?

3. Three workmen, A, B, C, did a certain piece of work and were paid daily wages according to their several degrees of skill. A's efficiency was to B's as 4 to 3, and B's to C's as 6 to 5; A worked 5 days, B, 6 days, and C, 8 days. The whole amount paid for the work was \$36·25. Find each man's daily wages.

4. Three men, working respectively 8, 9, 10 hours a day, receive the same daily wages. After working thus for 3 days, each works one hour a day longer, and the work is finished in 3 days more. If \$114·05 is paid for the work, how much should each man receive?

5. Three mechanics, A, B, C, are to divide among them the proceeds of a job valued at \$125·50, and finished in 9 weeks, the share of each being proportional to the work done by him. B can do half as much again in the same time as C, and A twice as much.

C works steadily 8 hours a day; B works 7 hours a day for the first 2 weeks, 5 for the next 2, 3 for the next 4, and 11 for the last. During the first 7 weeks, A works only 2 hours a day for 4 days of the week, and during the last 2 he works 14 hours a day, but finds that in the last 4 hours of each day he can get through no more work than C could. How much should each receive?

Example 6. A drover bought oxen at \$40, cows at \$30, and sheep at \$10 a head, paying for all \$1440. There were $2\frac{1}{2}$ times as many cows as oxen, and 5 times as many sheep as cows, how many did he buy of each?

$$\text{No. oxen} : \text{No. cows} :: 1 : 2\tfrac{1}{2} :: 2 : 5,$$

$$\text{No. cows} : \text{No. sheep} :: 1 : 5 :: 5 : 25,$$

$$\therefore \quad \text{No. oxen} : \text{No. cows} : \text{No. sheep} :: 2 : 5 : 25;$$

∴ as often as he expends \$80 in purchasing oxen he will expend \$150 in purchasing cows, and \$250 in sheep;

Hence the money must be divided in the proportion of 80, 150, 250, which is in the proportion of 8, 15, 25;

$$\therefore \quad \text{cost of oxen} = \tfrac{8}{48} \text{ of } \$1440 = \$240$$

$$\therefore \quad \text{No. oxen} = \$240 \div \$40 = 6.$$

EXERCISE XXXVII.

1. A person bought wheat at 80c, barley at 75c, and oats at 40c a bushel, expending for barley half as much again as for wheat, and for oats twice as much as for wheat. He sold the wheat at a gain of 5 %, the barley at a gain of 8 %, and the oats at a gain of 10 %, and received altogether \$9740. How many bushels of each did he buy?

2. Suppose that \$95·10 is to be divided among a certain number of men, women and boys; that there are 10 boys for every 3 men, and 16 men for every 39 women, that each boy receives 5 cents, each woman 10 cents, and each man 25 cents; find the number of men, of women, and of boys.

3. A debt of \$176 is paid in \$5 bills, \$2 bills, and \$1 bills, the number of each denomination being proportional to 4, 7 and 10; how many were there of each?

4. A debt of \$350 is paid in \$10 bills, \$5 bills, and \$2 bills, there are $\frac{3}{4}$ as many ten's as five's and $2\frac{1}{2}$ times as many two's as five's. How many were there of each denomination?

5. A merchant paid \$84 for 100 yd. of cloth of three different kinds. For every 4 yd. of the first kind he had $3\frac{1}{2}$ of the second and for every $1\frac{3}{4}$ yd. of the second he had $1\frac{1}{4}$ yd. of the third; if 2 yd. of the first cost as much as 3 yd. of the second, and 5 yd. of the second as much as 4 yd. of the third; find the price per yard of each kind of cloth.

Example 7. Divide \$7840·70 among A, B, C and D, giving A \$77·74 more than 40 % of what B and D receive; B \$88 less than $\frac{3}{5}$ of what C and D receive; and C \$99 more than $33\frac{1}{3}$ % of what D receives.

Assume D's share as the *unit*, that is, express the shares of the others in terms of D's share and known quantities. Then, since

D's share = D's share,

∴ C's " $= \frac{1}{3}$ D's " + \$99,

B's share $= \frac{3}{5}$ (C's + D's) − \$88 $= \frac{4}{5}$ D's " − \$28·60,

A's " $= \frac{2}{5}$ (B's + D's) + \$77.74 $= \frac{18}{25}$ D's " + \$66·30.

∴ sum of shares $= \frac{214}{75}$ D's share + \$136·70.

∴ $\frac{214}{75}$ D's share + \$136·70 = \$7840·70;

∴ D's share = (\$7840·70 − \$136·70) × $\frac{75}{214}$
= \$2700.

EXERCISE XXXVIII.

1. Divide \$3000 among A, B, C and D so that A may receive \$40 more than $33\frac{1}{3}$ % of what B, C and D receive; B \$50 less than 60 % of the united shares of C and D; and C $\frac{2}{5}$ of D's share and \$30 besides.

2. Two men A and B, make a bet on the result of a walking match, the total sum staked being \$105. A's stake is to B's as B's original money is to A's. If A win he will have $2\frac{1}{2}$ times as much money as B will have left, but if he lose he will have left $\frac{13}{15}$ of the sum B will then have; how much had each at first?

3. Four men own a timber limit, which they sell for $7200; the first receives $900 more than $\frac{2}{5}$ of what the other three get; the second $600 less than 70% of the joint shares of the third and fourth; and the third $400 more than $\frac{2}{5}$ of a sum which exceeds the share of the fourth by $2300. How much do each receive, after paying their proportionate share of the expenses of the sale which amount to $360?

4. Divide $52·50 among A, B and C so that B's share may be half as much again as A's, and C's one-third as much again as A's and B's together.

5. Divide $252·50 among A, B, C and D so that the sum of the shares of A and B may be $\frac{5}{8}$ of the sum of the shares of C and D, and that B's share may be $\frac{3}{14}$ of A's, and C's $\frac{7}{10}$ of B's.

6. In a certain factory the number of men is $\frac{3}{10}$ the number of boys, and the number of women 36% of the whole number of persons employed. If to give each boy 6d., each woman 1s., and each man 2s. 6d. requires £47. 11s., find the number of men, women, and boys.

7. A, B and C engage to hoe an acre of corn for $4·68. A alone could hoe it in 48 hours; B, in 36 hours; and C, in 24 hours. A begins first and works alone 10 hours; then B commences and A and B work together 6 hours, when C begins and all work together till the job is finished. How much should each receive?

8. Two men, A and B, hired a span of horses and a carriage for $7 to go from M to R, a distance of 42 miles. At N, 12 miles from M, they took in C, agreeing to carry him to R and back to N for his proportionate share of the expenses. At P, 24 miles from M, they took in D, agreeing to take him to R and back to P for his proportionate share of the expenses. What should each person pay?

(Give briefly the arguments for and those against each of the two commonly presented solutions of this problem.)

9. $1200 is to be distributed among A, B and C. From part of it they are to receive equal amounts, and of the rest B's shares is to be 10% more than A's, and C's 10% more than B's. Altogether B's share is $8\frac{16}{123}$% more than A's and $7\frac{23}{36}$% less than C's. Find the part of the $1200 that was equally divided.

PARTNERSHIP.

219. A **Partnership** is a voluntary association of two or more persons who combine their money, goods or other property, their labor or their skill, any or all of these, for the transaction of business or the joint prosecution of any occupation or calling, such as the carrying on of any manufacture or trade or the practice of any profession, upon an agreement that all gains and losses shall be shared in certain specified proportions among the persons constituting the partnership.

Such an association is styled a *Firm*, a *Company*, or a *House* and the persons uniting to constitute the association are called the *Partners of the Firm*.

The **Investment** of a partner in a firm is the money or property contributed by him to the firm.

The **Capital** of a firm is the total of the investments of the partners.

The **Net Gain** within a certain period is the excess of the total gains of a firm over its total losses within the period.

The **Net Loss** within a certain period is the excess of the total losses of a firm over its total gains within the period.

A **Dividend** is the share of the net gain or of any sum divided among the members of a firm or a company, which belongs to any partner. The dividends to the several partners are generally in proportion to their investments.

220. In a partnership in which the gains and losses are to be divided among the partners in proportion to their investments, to find each partner's share of any net gain or net loss :—

i. *If the investments are contributed for equal times, divide the net gain or the net loss in proportion to the investments.*

ii. *If the investments are contributed for unequal times, multiply each investment by the measure of the length of time during which it was invested and divide the net gain or the net loss in proportion to the products.*

EXERCISE XXXIX.

1. R. Stuart and G. Armstrong enter into partnership and agree to share all gains and losses in proportion to their investments. Stuart contributes $4500 to the partnership and Armstrong contributes $7500. Their net gain at the end of the year is $1750. How much of this sum should each partner receive?

2. Three partners invest respectively $7800, $5750 and $9450 in business. At the end of the first year they find their net gain to be $3156. What is the amount of each partner's share of this gain?

3. Two contractors, G. Rose and W. Crerar, undertake to build a bridge for the sum of $31,500. Crerar supplies the material at a cost of $11,727 and Rose pays the wages of the mechanics and laborers and all other expenses connected with the contract, amounting altogether to $15,645·80. If the profit on the contract is to be divided in proportion to investment, how much of the $31,500 should each partner receive?

4. A. Jones and D. Smith enter into partnership, the former investing $13,500 and the latter investing $22,800, and they agree that Jones shall receive a salary of $2000 for managing the business, and that all gains over and above this sum and all losses shall be shared in proportion to their respective investments. At the end of a year their resources are $74,850 and their liabilities are $17,943·86. Find the amount of the interest of each partner at the end of the year.

5. Th. Sinclair, C. Harvey and H. Stevens enter into partnership, Sinclair investing $37,500, Harvey $28,600, and Stevens $24,000, and they agree to share all gains and all losses in proportion to their investments. At the end of the year the resources of the firm are $124,368·50 and the liabilities are $37,429·50. Stevens now wishes to withdraw from the firm and sells to his partners his interest in the business in shares proportional to their interests in it. How much should he receive from each?

6. T. Allan and E. Jamieson engage in business with a joint capital of $19,200 and agree to share gains and losses in proportion to their investments. At the end of a year Allan receives a dividend of $1100 and Jamieson a dividend of $1300. What was the amount of the investment of each?

7. D. Rowan, F. Galbraith and J. Munro enter into partnership and agree to share all gains and all losses in proportion to their several investments. They gain $7500 of which Rowan receives $2100, Galbraith $3100, and Munro the balance. How much did Rowan and Galbraith respectively invest if the amount of Munro's investment was $18,000?

8. Three merchants enter into partnership, the first invests $1855 for 7 months, the second invests $887·50 for 10 months and the third invests $770 for 11 months; and they gain $434. What should be each partner's share of the gain?

9. L, M and N entered into partnership and invested respectively $19,200, $22,500 and $28,300. At the end of 5 months L invested $3800 additional; M, $2500; and N, $3700. At the end of a year the net gain of the firm was found to be $7850. What was each partner's share of this, if all gains and all losses were shared among the partners in proportion to their average investments?

10. Graves and Barr form a partnership, Graves investing $7000 and Barr $8000. At the end of 3 months Graves increases his investment to $9000 but at the end of 5 months more he withdraws $4000 from the business. Barr, 4 months after the formation of the partnership, withdraws $2000 of his investment but 5 months later increases it by $4000. At the end of the year the resources of the firm are $27,850 and its liabilities are $8460. What is the amount of each partner's interest in the business now, the net gain being divided between the partners in proportion to their average investments?

11. Stuart and Moss enter into partnership, Stuart contributing $5000 more capital than Moss. At the end of 5 months Stuart withdraws $2500 of his capital and 2 months later Moss increases his investment by $2500. At the end of their first year of partnership, their assets exceed their liabilities by $24,800 and on dividing their net gain in the ratio of their average investments, Stuart's interest in the business is found to exceed that of Moss by $461·54. Find the amount of the original investment of each.

CHAPTER VII.

I. PERCENTAGE.

221. The phrase **per cent.** which is a shortened form of the Latin *per centum*, is equivalent to the English word **hundredths.** Hence a **rate per cent.** is a rate or ratio per hundred and a number expressing a rate per cent of any quantity expresses simply so many hundredths of the quantity. Thus 5 per cent. of any sum of money is 5 hundredths of the sum ; $7\frac{1}{2}$ per cent. of a given length is $7\frac{1}{2}$ hundredths of the length ; and 225 per cent. is 225 hundredths.

222. The symbol % is frequently employed to denote the words per cent., and may therefore be read either *per cent.* or *hundredths.* Thus $5\% = \cdot 05$, $25\% = \cdot 25$, $\frac{1}{2}\% = \cdot 005$, $133\frac{1}{3}\% = 1 \cdot 33\frac{1}{3}$, $7\frac{1}{2}\%$ of $840 = \cdot 075$ of $840 = 63$, 145% of $\$640 = 1 \cdot 45$ of $\$640 = \$928$.

EXERCISE XL.

1. A lawyer collected \$287·50 and charged 5% for his services ; how much did he retain, and how much did he pay over ? What per cent. is the amount paid over of the amount collected ?

2. On Jan. 10, a merchant buys goods, invoiced at \$876·40 on the following terms : 4 mos., or less 6% if paid in 10 days. What sum will pay the debt on Jan. 15 ?

3. A house is sold for \$16,400, and 25% of the purchase money is paid down, the balance to remain on mortgage. How much remains on mortgage ?

4. A man invests 42 % of his capital in real estate and has \$53,070 left ; what is his capital ?

5. A horse was sold for \$658 which was $16\frac{2}{3}\%$ more than its cost ; how much did it cost ?

6. A bankrupt's assets are \$23,625, and he pays 40% of his liabilities ; what are his liabilities ?

7. A paymaster receives $150,000 from the treasury but fails to account for $2250; what is the percentage of loss to the government?

8. $640 increased by a certain per cent. of itself equals $720; required the rate per cent.

9. A tea merchant mixes 40 lb. of tea at 45ct. per lb. with 50 lb. at 27ct. per lb. and sells the mixture at 42ct. per lb. What per cent. profit does he make?

10. A merchant buys a bill of dry goods, Apl. 16, amounting to $6377·84, on the following terms: 4 mos., or less 5% if paid within 30 days. How much would settle the account on May 16? The amount paid May 16 is what % of the full amount of the bill?

11. On Aug. 16, a merchant buys a bill of goods amounting to $2475 on the following terms: 4 mos., or less 5% if paid in 30 days. Sept. 15, he makes a payment of $1000, with the understanding that he is to have the benefit of the discount of 5%. With what amount should he be credited on the books of the seller? How much would be due at the expiration of the 4 mos.?

12. Paid $664·25 for transportation on an invoice of goods amounting to $8866. What per cent. must be added to the invoice price to make a profit of 20% on the full cost?

13. A business firm's resources consist of notes, merchandise, personal accounts, &c., to the amount of $9117·61, and a balance, which is 44% of their entire capital, on deposit in bank. How much is on deposit?

14. At a forced sale a bankrupt's house was sold for $8000, which was 20% less than its real value. If the house had been sold for $12,000 what per cent. of its real value would it have brought?

15. The population of a town of 64,000 inhabitants increases at the rate of $2\frac{1}{2}$% in each year, find its population (i) 1, (ii) 2, (iii) 3 years hence.

16. The population of a city increases at the rate of 2% yearly. It now has 132,651 inhabitants; how many had it (i) 1, (ii) 2, and (iii) 3 years ago?

17. A ship depreciates in value each year at the rate of 10% of its value at the beginning of the year, and its value at the end of 3 years is $14,580; what was its original value?

18. A man in business loses in his first year 5% of his capital, but in his second year he gains 6% of what he had at the end of the first year, and his capital is now \$14 more than at first; what was his original capital?

19. Wine which contains $7\frac{1}{2}\%$ of spirit is frozen, and the ice which contains no spirit being removed, the proportion of spirit in the wine is increased to $8\frac{3}{4}\%$. How much water in the state of ice was removed from 504 gal. of the original wine?

20. The stuff out of a lead mine contains at first 15·9% of lead. After washing, by which process the amount of lead ore is not diminished, the stuff contains 87·45% of lead. How much rock was washed away out of 216 tons 5 cwt. of the original stuff?

21. The money deposited in a savings bank during the year 1885 was 5% greater than that deposited in 1884. In 1886 the deposits were $33\frac{1}{3}\%$ greater than in 1885, while the amount deposited in 1887 exceeded the average of the three previous years by 20%. The aggregate of the four years was \$150,937·50. Find the amount deposited in each year.

22. In 1871 the populations of Toronto, Hamilton and St. Thomas were severally 56091, 26716 and 2197. In the next ten years they increased 54%, 34·6%, and 280·8% respectively. Determine the increase per cent. of their united population.

23. The cattle on a stock-farm increase at the rate of $18\frac{3}{4}\%$ per annum. In 1889 there were 6859 head of cattle on the farm; how many were there in 1886?

24. In a certain election A polled 88% of the votes promised him, and B polled 90% of those promised him, and B was elected by a majority of 3 votes. Had each candidate received the full number of votes promised him, A would have been elected by a majority of 25. How many votes did each candidate receive?

25. The delivery of letters in a certain town is carried on by four postmen, two of whom deliver on 14 streets and two on 17 streets, but the work of the latter two is 20% less per street than that of the former two. A fifth man is put on to help them. In what ratio should he help the two pairs of men so that all five shall have equal work?

II. PROFIT AND LOSS.

223. The **Prime Cost** of merchandise or other property is the net sum paid by the purchaser thereof to the seller thereof.

The **Gross Cost** of merchandise or other property is the sum of the prime cost, all charges for purchasing, and all expenses for freight, storage, handling, and such like.

224. Profit is the amount by which the selling price exceeds the cost price. *Net Profit* or *Gain* is the amount by which the selling price exceeds the gross cost.

The **Rate of Profit** is usually expressed as a percentage of the prime cost.

225. Loss is the amount by which the selling price falls short of the cost price. *Net loss* is the amount by which the selling price falls short of the gross cost.

The **Rate of Loss** is usually expressed as a percentage of the prime cost.

EXERCISE XLI.

1. A lot of dry goods was sold at an advance of 18%. If the gain was $436·50, what was the cost?

2. I made a mixture of wine consisting of one gallon at 50 cents, 3 at 90 cents, 4 at $1·20, and 12 at 40 cents. I sell the mixture at $1·60 a gallon; find my gain %.

3. A merchant's price is 25% above cost; if he allow a customer a discount of 12% on his bill, what % profit does he make?

4. If cloth, when sold at a loss of 25%, brings $5 a yard, what would be the gain or loss % if sold at $6·40 a yard?

5. Eggs are bought at 27 cents a dozen, and sold at the rate of 8 for 25 cents; find rate of profit.

6. A merchant sells goods to a customer at a profit of 60%, but the buyer becomes bankrupt and pays only 70 cents on the dollar; what % does the merchant gain or lose on the sale?

7. A man sells an article at 5% profit; if he had bought it at 5% less and sold it for \$12 less he would have gained 10%. Find cost price.

8. A man bought a horse which he sold again at a loss of 10%. If he had received \$45 more for him he would have gained $12\frac{1}{2}$%; find cost of horse.

9. A merchant buys wine at 16s. a gal.; 20% of it is wasted; at what price per gal. must he sell the remainder to gain 20% on his outlay?

10. A tradesman proposes to retail his goods at 10% profit; but adulterates them by adding $\frac{1}{4}$ of their weight of an inferior article which costs him $\frac{4}{5}$ of the price of the better; what % profit does he make?

11. I purchase 2276 lb. of coffee at 21ct. per lb. and mix it with chicory at $4\frac{1}{2}$ct per lb. in the ratio of 3 parts by weight of the former to 2 of the latter; at what price per lb. must I sell it to gain 25%?

12. I buy oranges at the rate of 3 for 2d., and a third as many at the rate of 2 for 1d.; at what rate per doz. must I sell them to gain 20% on my outlay? Supposing my total profit to be 5s. 4d., how many did I buy?

13. A merchant buys 3150 yd. of cloth. He sells $\frac{1}{3}$ of it at a gain of 6%, $\frac{1}{5}$ at a gain of 8%, $\frac{1}{7}$ at a gain of 12%, and the remainder at a loss of 3%. Had he sold the whole at a gain of 5% he would have received \$28·98 more than he did. Find the prime cost of one yard.

14. Sold steel at \$25·44 a ton, making thereby a profit of 6%, and a total profit of \$103·32. Find the quantity sold.

15. A baker's outlay for flour is 70% of his gross receipts, and his other trade expenses amount to $\frac{1}{5}$ of his receipts. The price of flour falls 50% and the other trade expenses are thereby reduced 25%; to make the same amount of profit, by how much should he now reduce the price of the 5 cent loaf?

16. A man having bought a certain quantity of goods for \$150, sells $\frac{1}{3}$ of them at a loss of 4%, by what increase % must he raise that selling price that by selling the whole at that increased rate he may gain 4% on his entire outlay?

17. 4 horses and 7 cows cost \$390; but, if the price of the horses were to rise 25% and that of the cows 15% they would cost \$466·50; find the cost of a horse and of a cow.

18. The cost of freight and insurance on a certain quantity of goods was 15% and that of duty 10% on the original outlay. The goods were sold at a loss of 5%, but had they brought \$3 more there would have been a gain of 1%. How much did they cost?

19. A bookseller sold a book at 17% below cost, but had he charged 50 cents more for it, he would have gained 7%. Find the cost of the book to the bookseller, and the price at which he sold it.

20. A man buys pears at 35ct. a score, and after selling 7 dozen at 45ct. a dozen (giving 13 to the dozen) he finds he has cleared his original outlay. If he then sell the remainder at the rate of 2 for a cent, what will he gain % on the whole transaction?

21. I buy two cows for \$55; if I sell the first at a loss of 5% and the second at a gain of 5%, I should gain $\frac{5}{11}$%; what was the price of each cow?

22. I bought a lot of coffee at 12ct. per lb. Allowing that the coffee will fall short about 5% in roasting and weighing it out, and that 10% of the sales will be bad debts, for how much per pound must I sell it so as to gain 14% on the cost?

23. A grocer mixed together two kinds of tea and sold the mixture, 144 lb., at an advance of 20% on cost, receiving for it \$62·10. Had he sold each kind of tea at the same price per pound as he sold the mixture he would have gained 15% on the one and 25% on the other. How many pounds of each were there in the mixture, and what was the cost of each per pound?

24. The manufacturer of an article charged 20% profit, the wholesale dealer charged 25% of an advance on the manufacturer's price and the retail dealer charged 30% of an advance on the wholesale price. Find the cost to the manufacturer of an article for which the retail dealer charged \$23·40.

25. I sold for \$296, two horses which had cost me \$280. The gain per \$100 on one of them was equal to the loss per \$100 on the other and also equal to the difference in cost of the two horses. Find the cost of each.

III. INSURANCE.

226. Insurance is a contract by which one party, *the insurer*, in consideration of a sum of money received from another party, *the insured*, engages to pay a stipulated sum on the happening of a particular event or undertakes to indemnify the insured or his representatives for loss or damage arising from certain specified causes, if sustained within a stated time.

The instrument or document setting forth the contract is termed an **Insurance Policy.**

The sum paid by the insured to the insurer is styled the **Premium.** It is generally a fixed percentage of the amount insured.

The **Term** of an insurance is the period for which the contract is made and the risk assumed.

227. The ordinary kinds of insurance are **Fire Insurance, Marine Insurance** and **Life Insurance.**

228. In **Fire Insurance**, the insurer undertakes to indemnify the insured up to a specified sum, for loss or damage that may occur to certain property described in the policy, if caused by fire, within a stated time, generally one, two or three years.

229. In **Marine Insurance,** the insurers contract to indemnify the insured up to a stipulated sum for any loss or damage that may occur to a certain ship, cargo or freight, any or all of them, by storms or other perils of navigation during a particular voyage or within a specified period not usually exceeding twelve months.

230. In *Life Insurance*, the insurer engages to pay on the death of the insured, a sum specified in the policy. In an *Endowment Policy*, the stipulated sum is payable to the insured if he should survive a specified number of years, but should he die before the expiration of the period named, the sum assured is to be paid to the representatives of the insured or to a person named in the policy.

231. Fire and life insurances are usually undertaken by companies or corporations organized to carry on such business;

marine insurance is undertaken both by companies and by private persons. A marine insurance by private individuals is generally undertaken by several parties and each of them writes his name under or at the foot of the policy, and engages on his own account to indemnify the insured to the amount set opposite his name: on this account individual marine insurers are called *underwriters.*

232. In an ordinary fire policy, if the loss is only partial, the insurer undertakes to pay the full value of the property destroyed or the full amount of the depreciation of the property damaged, provided it does not exceed the sum covered by the insurance. In marine policies there is commonly an average clause which declares that the indemnity for a partial loss of property not insured to its full value will be the same part of the loss as the sum covered by the insurance is of the full value of the property.

233. If a property is insured in two or more companies or by two or more underwriters, the insurers are liable for the indemnity for a partial loss, in sums proportionate to the amounts of the risks severally assumed by them.

EXERCISE XLII.

1. A factory valued at \$35,000 was insured for $\frac{3}{5}$ of its value, the rate of insurance being $\frac{5}{8}$% for one year. What was the amount of the premium?.

2. A warehouse valued at \$62,500 was insured for $\frac{3}{5}$ of its value, the rate of insurance was $1\frac{1}{4}$% for three years, and the cost of the policy and the agent's expenses were \$2·50. What was the amount paid for the insurance?

3. What will be the cost of insuring a cargo of 24,000 bushels of wheat valued at \$1·05 per bushel, the insurance covering $\frac{4}{5}$ of the value of the cargo, the premium rate being $1\frac{1}{8}$% and the other expenses of the insurance being $2\frac{1}{2}$% of the premium?

4. A merchant's stock was insured for \$42,000, $\frac{1}{2}$ of this amount being at $\frac{7}{8}$%, $\frac{2}{3}$ of the remainder at $\frac{3}{4}$% and the remainder at $\frac{5}{8}$%. Find the total amount of premium paid.

5. A building and contents are insured as follows:—$12,000 in the Imperial, $8000 in the National and $5000 in the Lancashire Insurance Company. Were a loss to the extent of $3500 to occur through fire, what portion of the loss should each company bear?

6. Merchandise valued at $63,000 was insured in the Phœnix Insurance Co. for $15,000, in the North British and Mercantile Insurance Co. for $12,000 and in the Norwich Union Fire Insurance Society for $8000; if the merchandise is damaged by fire to the extent of $10,500, how much of the damage should each company pay?

7. A merchant insured his stock for $33,000 for one year at $\frac{7}{8}$%. Six months thereafter the policy was cancelled at the request of the insured. Find the amount of premium returned, the short rate for six months being $\frac{5}{8}$%.

8. A factory and the machinery therein is insured for $65,000; $\frac{3}{5}$ of this sum is at $\frac{3}{4}$% premium and the remainder is at $\frac{7}{8}$%. What is the average rate per cent. of premium paid on the whole?

9. A fire insurance company insured a building for $60,000 at $\frac{7}{8}$% premium and reinsured one-half of the risk in another company at $\frac{5}{8}$% and one-third of the risk in a third company at $\frac{3}{4}$%. What amount and what rate of premium did the company net on the remainder of their risk?

10. A steamboat worth $60,000 is insured in three companies, in two to the amount of $15,000 each and in the third to the amount of $20,000. For what sum would each company be liable if the vessel were to sustain damage to the extent of $6600?

11. A ship worth $56,000 was insured for $15,000 in one insurance company at $\frac{7}{8}$% premium and for $32,000 in another company at $\frac{3}{4}$%. The vessel received damage in a storm to the extent of $7500. What amount had each company to pay to the owners of the vessel and by how much did each amount exceed the premium received by the company paying that amount?

12. A fire insurance company charged $196·88 for insuring a house for $17,500. What was the rate per cent. of insurance?

13. A merchant's stock was worth $120,000; he insured it at $\frac{2}{3}$ its value paying $700 premium. What was the rate per cent. of insurance? What was the rate in cents per $100?

14. A shipment of goods is insured for \$7500 and \$18·75 is paid as premium. At that rate, what would be the amount of the premium on \$18,750?

15. The sum of \$285 was paid for the insurance at $\frac{3}{4}$ of its value of a ship worth \$50,000. What was the rate per cent. of premium, if \$3·75 was charged for the policy and the preliminary survey?

16. For what sum was a house insured if the premium paid was \$17·50 and the rate of insurance $\frac{7}{8}$%?

17. For what sum was a shop insured if the rate of insurance was 65 cents per \$100 and the premium paid was \$81·25?

18. A fire insurance company received \$350 for insuring a factory at $1\frac{1}{4}$% premium, and charged $\frac{7}{8}$% for insuring a less hazardous property of the same valuation as the factory. What was the amount of the premium paid on the second property?

19. A merchant owns $\frac{2}{3}$ of a steamship and insures $\frac{3}{5}$ of his interest at $\frac{5}{8}$%, paying \$337·50 premium. What was the value of his interest in the steamer? If during the continuance of the policy, the vessel be damaged in a collision to the extent of \$35,000, what sum will the merchant be entitled to receive from the insurance company?

20. The invoice price of a shipment of goods is \$1845. The shipper wishes to insure the goods for such a sum as will, in case of loss, cover both invoice price and amount of premium. For what sum should the shipment be insured if the rate of insurance is $\frac{3}{8}$%?

21. The value of a consignment is \$4250. For what sum should it be insured that the owner may receive both the value of the consignment and the amount of the premium in case of total loss, the rate of insurance being 55 cents per \$100?

22. For what sum should a cargo worth \$18,750 be insured to cover the value of the cargo, the cost of insurance at $\frac{7}{8}$% and \$2·50 for the policy and broker's charges?

23. A cargo of wheat invoiced at \$9930 is insured for \$10,000 which sum covers not only the invoice value of the wheat but also the premium paid and \$5 for expenses. What was the rate per cent. of the insurance?

24. A shipment of goods is insured for \$6000, which sum covers the value of the goods, the premium at $1\frac{1}{8}$% and \$2·50 for expenses. What was the value of the goods?

IV. COMMISSION AND BROKERAGE.

234. An **Agent** is a person authorized to transact business for another. The person for whom the agent transacts business is called his *Principal.*

235. A **Commission Merchant** is one who buys or sells goods for other persons by their authority. Commission merchants are usually placed in possession of the goods bought or sold.

236. A **Broker** is a person who, in the name of his principal, effects contracts to buy or to sell. The broker is not in general placed in possession of the goods bought or sold.

The title Broker is also applied to persons who deal in stocks, bonds, bills of exchange, promissory notes, &c., and to mercantile agents who transact the business for a ship when in port.

237. **Commission** is the charge made by an agent for transacting business.

238. The **Gross Proceeds** of a sale or of a collection is the total amount received by an agent for his principal.

239. The **Net Proceeds** of a sale or of a collection is the sum due the principal from the agent, after deducting his commission and all other charges. These charges include freight, handling, storage, advertising, and such like.

240. The **Prime Cost** of a purchase is the net sum paid by an agent for merchandise or other property and does not include his commission or other charges.

241. *Commission is usually reckoned at a rate per cent. on the gross proceeds of sales and collections, on the prime cost of purchases, and on the net amount of investments.*

EXERCISE XLIII.

1. A commission merchant sold 270 barrels of flour at $6 a barrel, and received 5% commission. What was his commission? How much did he remit to his employer

2. A commission of $242·58 was charged for selling $3772 worth of goods. What was the rate of commission?

3. A grain-dealer charged $3\frac{1}{2}$% for selling a quantity of wheat, and received for his commission $218·40; for how much did he sell the wheat?

4. A real-estate broker sold a house on $6\frac{1}{4}$% commission, and sent to the owner $3060. What was the broker's commission, and what sum did he receive for the house?

5. A merchant sent $3238·30 to New Orleans to be expended in cotton. The broker in New Orleans charged 6% commission. What sum was paid for the cotton?

6. If $512·50 include the price paid for certain goods and $2\frac{1}{2}$% commission to the agent, how much money does the agent expend in purchasing the goods?

7. An agent sold 210 bush. of oats at 60ct. a bush. and charged $3·78 for doing so. Find his rate of commission.

8. How many yards of cloth at 90ct. a yd. can an agent buy with the commission received from the sale of 360 bush. of potatoes at 50ct. a bush., his rate of commission being $1\frac{1}{2}$%?

9. A man bought a horse and carriage for $450, which sum was his commission at $2\frac{1}{2}$% on the sale of a farm. For how much was the farm sold?

10. A broker is offered a commission of $5\frac{1}{2}$% for selling wool and guaranteeing payment, or a commission of $3\frac{3}{4}$% without guaranteeing payment. He accepts the $5\frac{1}{2}$% and guarantees payment. The sales amount to $17,000, and the bad debts to $295.50. How much did he gain by choosing the $5\frac{1}{2}$%?

11. Sent to a commission merchant in Guelph $2080·80 to invest in flour, his commission being 2% on the amount expended; how many barrels of flour could be purchased at $4·25 a barrel?

12. An agent sold 6 mowing-machines at $120 each, and 12 at $140 each. He paid for transportation $72, and, after deducting his commission, remitted $2208 to his employer. What was the rate of commission?

13. A man allows his agent 5% of his gross rentals, and receives a net rental of $3488·40. If the gross rental is 6% of the value of the property, what is the value of the property?

14. On a debt of \$1725 a creditor receives a dividend of 60%, on which he allows his attorney 5%. He receives a further dividend of 25%, on which he allows his attorney 6%. What is the net amount that he receives?

15. An agent sold a quantity of cotton amounting to \$7317·83, and charged a commission of $2\frac{1}{2}$%. He was instructed to invest the proceeds in dry goods, after deducting a commission of $1\frac{1}{2}$% on the amount so expended. What was his total commission?

16. An agent sold 300 bales of cotton, averaging 462 lb. to the bale, at 15·7ct. per lb., his commission being 25ct. per bale, and the charges being \$161. He purchased for the consignor dry goods amounting to \$2576·37, charging a commission of $1\frac{1}{2}$%. How much was still due the consignor?

17. A commission merchant sold a consignment of bacon at $11\frac{1}{2}$ ct. per pound and invested the proceeds, less his commission, in tea at 38 ct. per pound. His commission on the two transactions at the rate of 5 % on the sale of the bacon and 2% on the purchase of the tea amounted altogether to \$52·50. How many pounds of bacon did he sell and how many pounds of tea did he buy?

18. A miller sends 4000 bbl. of flour to a commission merchant with instructions to sell the flour and remit the net proceeds by draft. The consignee pays \$462·40 for freight and other expenses, sells the flour at \$6·75 per barrel, charges 3 % commission and pays $\frac{1}{4}$ % premium for draft. Find the amount of the draft.

19. The owner of certain property pays his agent $2\frac{1}{2}$% for collecting his rents, insurance and repairs cost him $6\frac{2}{3}$ % of his *net* income but on this sum he pays no income tax, his income tax at $17\frac{1}{2}$ mills on the dollar amounts to \$153·73. Find the gross rents from his property.

20. An agent sold a consignment of boots and shoes for \$3825 and invested the proceeds, less his commission, in leather. His total commission on the two transactions amounted to \$150. What rate did he charge, the rates on both sale and purchase being the same?

21. An agent sold a consignment of fish for \$2460 and invested the proceeds, less his commission, in flour. The commission on the sale exceeded the commission on the purchase by \$3. What rate did he charge, the rates being the same on the two transactions?

V. DISCOUNT.

242. Discount *is an abatement or reduction from the nominal price or value of anything*; as, for example, from the catalogue or list price of an article, from the amount of a bill or invoice of goods or of a debt, or from the face value of a promissory note.

243. The **Rate of Discount** is usually stated as a rate per cent. of *the amount from which the discount is made.*

244. Trade Discounts *are reductions made from the catalogue or list prices of goods.*

In some branches of business the manufacturers and the wholesale dealers catalogue their goods at fixed prices, usually the retail selling price, and then allow retail dealers reductions or discounts from these catalogue prices. These discounts generally depend on the amount of the purchase and the terms of payment, whether cash or credit. By varying the rate of discount, the manufacturer can raise or lower the price of his goods without issuing a new catalogue.

245. Very often two or even more successive trade discounts are to be deducted. In such cases the *first* rate denotes a percentage of *the catalogue price*; the *second* rate denotes a percentage of *the remainder after the first discount has been made*; the *third* rate, a percentage of *the remainder after the second discount has been made*; and so on.

Thus, discounts of 20% and 5% in succession off any amount, or, as it is generally expressed in business, *20 and 5 off*, means that ·20 of the amount is to be deducted from it, and then from the remainder ·05 of that remainder is to be taken.

EXERCISE XLIV.

1. What is the difference between discounting a bill of $3000 at 40%, and then taking a discount off the remainder of 5% for cash, and discounting the whole at 45%?

2. An invoice of crockery, amounting to $1473·20, was sold Jan. 3, at 90 days, subject to 40% and 10% discount, with an additional discount of 6% if paid within 20 days. How much will be required to pay the bill on Jan. 21?

3. What must be the marking price so that a merchant, in closing out a sale, may sell broadcloth costing $3·60 a yard at 10% below cost, and yet be able to allow 40% off the marking price?

4. A cabinet dealer directed his salesman to mark a set of furniture so that, by allowing 20% off the marked price he may realize a gain of 25%. The salesman marked the set by mistake at $200, or at a loss to the dealer of 20% of the sale. How much less than the required marking price was the set marked?

5. What single discount is equivalent to successive discounts of 20% and 10%?

6. A merchant buys goods at 40 and 20 off the list price and sells them at 30 and 10 off the list price. What is his gain per cent.?

7. A manufacturer sells certain goods at 30 and 10 off, and gains thereby $12\frac{1}{2}$%. What is the list price, if the goods cost $28?

8. I purchase books at $2 each, less $33\frac{1}{3}$%, and 5% for cash. What is the net cost? What % discount may be given off the list price so that I may sell them at a net profit of 10%?

9. Show that successive discounts of specified rates may be taken off a list price in any order without affecting the net price. Thus 20 and 10 off is equivalent to 10 and 20 off, so also 30 and 10 and 5 off, 10 and 30 and 5 off, and 5 and 30 and 10 off are all equivalent.

10. 20 and what rate off are equivalent to 40% off?

11. 25 and what rate off are equivalent to 40% off?

12. 30 and what rate off are equivalent to 40% off?

13. 20 and what rate off are equivalent to $33\frac{1}{3}$% off?

14. What rate taken off twice in succession is equivalent to 36% off?

15. What rate taken off twice in succession is equivalent to 44% off?

16. What rate taken off thrice in succession is equivalent to 48·8% off?

17. What rate taken off thrice in succession is equivalent to 34% off?

18. What rate put on a list price and then taken off the increased price is equivalent to 4% off the list price?

246. A **Promissory Note** (often called briefly a NOTE) is a written promise to pay, unconditionally, on demand or at a fixed or a determinable future time, a specified sum of money, to a particular person named in the note, or to a person named or his order, or to bearer.

A note which is, or on the face of it purports to be, both made and payable within Canada, is an inland note: any other note is a foreign note.

247. The **Maker** of a note is the person who signs the promise.

The **Payee** is the person to whom or to whose order the note is made payable.

The **Holder** or Bearer of a note is the person who lawfully possesses it.

The **Face Value** (or simply the FACE) of a note is the sum of money (exclusive of interest) which the maker promises to pay.

248. A promissory note may be made by two or more makers, and they may be liable thereon jointly, or jointly and severally according to its tenor. If a note runs "I promise to pay," and is signed by two or more persons, it is deemed to be their joint and several note.

249. An **Indorser** of a note is a person who writes his name on the back of the note. By so doing he guarantees its payment and becomes responsible therefor, unless when indorsing he writes above his signature the words "without recourse." A note payable to *order* must be indorsed by the payee when transferred to anyone else, but a note payable to *bearer* need not be indorsed.

A special indorsement specifies the person, called the *indorsee*, to whom, or to whose order, the note is to be payable.

An indorsement in blank specifies no indorsee, and a note so indorsed becomes payable to bearer. When a note has been indorsed in blank, any holder may convert the blank indorsement into a special indorsement by writing above the indorser's signature a direction to pay the note to or to the order of himself or some other person.

An indorsement is restrictive which prohibits the further negotiation of the note or which expresses that it is a mere authority to deal with the note as thereby directed, and not a transfer of the ownership thereof, as, for example, if a note be indorsed "Pay D only," or "Pay D for the account of X," or "Pay D or order for collection." A restrictive indorsement gives the indorsee the right to receive payment of the note and to sue any party thereto that his indorser could have sued, but gives him no power to transfer his rights as indorsee unless it expressly authorise him to do so. Where a restrictive indorsement authorises further transfer, all subsequent indorsees take the note with the same rights and subject to the same liabilities as the first indorsee under the restrictive indorsement.

250. A **Negotiable Note** is one which may be sold or transferred by the payee to anyone else ; and a note is negotiated when it is transferred from one person to another in such a manner as to constitute the transferee the holder of the note. A negotiable note may be payable either to order or to bearer. A note is payable to bearer which is expressed to be so payable, or on which the only or last indorsement is an indorsement in blank. A note is payable to order which is expressed to be so payable, or which is expressed to be payable to a particular person, and does not contain words prohibiting transfer or indicating an intention that it should not be transferable. Where a note either originally or by indorsement, is expressed to be payable to the order of a specified person, and not to him or his order, it is nevertheless payable to him or his order, at his option.

A note payable to bearer is negotiated by delivery. A note payable to order is negotiated by the endorsement of the holder completed by delivery.

Where the holder of a note payable to bearer negotiates it by delivery without indorsing it, he is called a *transferor by delivery*. A transferor by delivery is not liable on the instrument. A transferor by delivery who negotiates a note thereby warrants to his immediate transferee, being a holder for value, that the note is what it purports to be, that he has a right to transfer it, and that at the time of transfer he is not aware of any fact which renders it valueless.

When a note contains words prohibiting transfer, or indicating an intention that it should not be transferable, it is valid as between the parties thereto, but it is not negotiable.

251. Where a promissory note is in the body of it made payable at a particular place, it must be presented for payment at that place in order to render the maker liable: in any other case, presentment for payment is not necessary in order to render the maker liable. Presentment for payment is necessary in order to render the indorser of a note liable. Where a note is in the body of it made payable at a particular place, presentment at that place is necessary in order to render an indorser liable; but when a place of payment is indicated by way of memorandum only, presentment at that place is sufficient to render the indorser liable, but a presentment to the maker elsewhere, if sufficient in other respects, will also suffice.

252. Maturity (properly DATE OF MATURITY) is the day on which the note becomes legally due. Where a note is not payable on demand, the day on which it falls due is determined as follows:—

Three days called days of grace, are, in every case where the note itself does not otherwise provide, added to the time of payment as fixed by the note, and the note is due and payable on the last day of grace. Whenever the last day of grace falls on a legal holiday or non-juridical day in the Province where any such note is payable, then the day next following, not being a legal holiday or non-juridical day in such Province, is the last day of grace.

A note is payable on demand, which is expressed to be payable on demand, or on presentation, or in which no time for payment is expressed.

253. Where a bill is payable at a fixed period after date, after sight, or after the happening of a specified event, the time of payment is determined by excluding the day from which the time is to begin to run and by including the day of payment. The term "Month" in a note means the calendar month. Every note which is made payable at a month or months after date becomes due on the same numbered day of the month in which it is made payable as the day on which it is dated—unless there is no such day in the month in which it is made payable, in which case it becomes due on the last day of that month—with the addition, in all cases, of the days of grace.

254. A note is not invalid by reason only that it is antedated or post-dated, or that it bears date on a Sunday.

255. A **Draft** or **Bill of Exchange** is a written order by one person (called the DRAWER) directing a second person (called the DRAWEE) to pay, unconditionally, on demand or at a fixed or determinable future time, a specified sum of money (called the FACE or PAR) to a third person (called the PAYEE) or to the payee's order, or to bearer.

Sections 249 to 254 apply to bills of exchange as well as to promissory notes.

256. Bank Discount *is a deduction made from the face value of a note or a draft for cashing it or buying it before maturity.*

257. The **Term of Discount** is the time between the date of the discounting and the date of maturity.

258. The **Rate of Discount** is the percentage of the FACE VALUE which would be deducted if the term of discount were ONE YEAR.

259. Exchange is a charge made for collection in cases in which the place of payment of the note or the draft is not the place of discount. The rate of exchange is generally from $\frac{1}{8}$ to $\frac{1}{4}$ of 1% of the face value, but if the face value is less than \$100, the full exchange on \$100 is usually charged.

260. The **Proceeds** of a note is the sum of money received for it on discounting it. It is equal to the sum due at maturity less the discount and the exchange.

Example. A note for \$572·80 drawn on 13th June and payable 4 months after date, was discounted at 7% on 27th June. Find the proceeds.

Maturity is 4 mo. 3 da. from 13th June = 16th Oct.

Term of discount is from 27th June to 16th Oct. = 111da. = $\frac{111}{365}$ yr.

Face of note = \$572·80. Rate of discount = ·07.

Discount = \$572·80 × ·07 × 111 ÷ 365 = \$12·20.

Proceeds = \$572·80 − \$12·20 = \$560·60.

[*Calculation of the discount.*

$$\log 572{\cdot}8 + \log {\cdot}07 + \log 111 - \log 365$$
$$= 2{\cdot}758003 + {\cdot}845098 - 2 + 2{\cdot}045323 - 2{\cdot}562293$$
$$= 1{\cdot}086131 = \log 12{\cdot}194.]$$

EXERCISE XLV.

Find the date of maturity, the term of discount and the proceeds in the following cases.

	Face of Note.	Date of Note.	Time.	Date of Discount.	Rate of Discount.
1.	\$312·80.	13th May, 1890.	90 da.	13th May.	6 %.
2.	\$975·65.	5th Sept. 1892.	3 mo.	16th Sept.	7 %.
3.	\$450.	28th Aug. 1891.	60 da.	4th Sept.	7 %.
4.	\$79·50.	17th Dec. 1889.	2 mo.	23rd Dec.	$7\frac{1}{2}$ %.
5.	\$586·67.	28th Dec. 1891.	4 mo.	15th Jan.	8 %.

6. Find the proceeds of the following note discounted in Toronto on 1st May 1890, at 7 %, exchange $\frac{1}{8}$ %.

\$390$\frac{50}{100}$ OTTAWA, *1st May, 1890.*

Three months after date I promise to pay to the order of Thomas A Stuart, Three Hundred and Ninety $\frac{50}{100}$ Dollars, at the Bank of Commerce here. Value received.

JAMES HENDERSON.

7. A note for \$250 was discounted 40 days before maturity and the proceeds were \$247·80. What was the rate of discount, there being no exchange?

8. A note for \$742·76 was discounted 93 days before maturity and the proceeds were \$730·47. What was the rate of discount, the rate of exchange being $\frac{1}{8}$ %?

9. For what sum must a note be drawn in order that if discounted 89 days before maturity, the proceeds may be \$425; the rate of discount being 7 % and there being no exchange?

10. For what sum must a draft payable thirty days after sight be drawn in order that if discounted on day of drawing the proceeds may be \$745·25; the rate of discount being $7\frac{1}{2}$ % and that of exchange $\frac{1}{8}$ %?

11. A promissory note for \$385·20 was discounted on 1st March, 1890, at 7 % discount and $\frac{1}{8}$ % exchange and the proceeds were \$377·70. Determine the date of maturity of the note.

VI. INTEREST.

261. Interest is the sum which the lender of money charges the borrower for the use of the sum borrowed, or which a creditor charges a debtor for allowing his debt to remain unpaid after it has become due.

262. The **Principal** is the sum borrowed or due.

263. The **Amount** is the sum total of principal and interest.

264. The **Rate of Interest** is always expressed as the rate per cent. of the principal which would be charged for its use for ONE YEAR.

265. The **Time** is the period, *expressed in years*, for which interest is reckoned.

266. Simple Interest is interest reckoned on the original principal and on it alone for the whole term during which that principal bears interest.

267. Compound Interest is interest which is reckoned for stated periods and added at the end of each period to the principal on which it was reckoned, the amount or sum total of principal and interest at the end of each period becoming the principal for the succeeding period.

It is as if the original principal had been loaned at simple interest for the first period, then the amount from that period loaned for the next period, the second amount loaned for the third period and so on, a new loan of the sum total of principal and all accrued interest being entered upon at the beginning of each period. Thus compound interest reckons interest upon interest.

268. The names **Annual, Semiannual, Quarterly** and **Monthly Interest** are applied to interest which is payable at the end of each year, half-year, quarter-year or month, as the case may be, throughout the time during which the principal bears interest.

269. Annual or other periodically payable interest differs from simple interest in that it is to be paid at stated intervals while simple interest is not due and collectible until the principal matures.

270. Annual or other periodically payable interest differs from compound interest in that it like simple interest is reckoned on the original principal and on it alone while compound interest is computed for each period on the original principal increased by all accrued interest. In effect, however, periodically paid interest is equivalent to compound interest, for the borrower loses the use of the money he pays as interest at the end of each period and the lender gains the use of it, and the value of this use is assumed to be interest at the rate paid on the principal. Hence, *in calculations concerning periodic payments, the methods, not of simple but of compound interest, should be employed.* For example it is usual with savings banks which pay annual interest to credit each depositor at the end of every interest year with all interest on his deposit accrued but undrawn, treating such interest as a new deposit, the net result being that the banks pay compound interest.

271. If interest which is by agreement to be paid at specified intervals, is not so paid, and the lender has to collect it by process of law, the courts have authority to grant at their discretion simple interest on the accrued periodic interest. The interest upon interest, if thus granted, is styled *damages* and its maximum rate is the *legal rate* of six per cent. per annum.

Simple Interest.

272. Problems in simple interest involve the consideration of principal, rate, time, interest and amount ; and any three of these being known the other two may be determined, for by definition ;—

1°. *The interest is the continued product of the principal, the rate per* unit *and the measure of the time.*

2°. *The amount is the sum of the principal and the interest.*

273. Expressed in general symbols these statements are

1°. $I = Prt$,

2°. $A = P + I$,

the letters I, P, t and A denoting severally the measures of the interest, principal, time and amount, and r denoting the rate per *unit*.

EXERCISE XLVI.

Find the simple interest on and the amount of

1. \$473·28 for 3 years at 6%.

2. \$385·35 for $1\frac{1}{2}$ years at 5%.

3. \$628·25 for 185 days at $4\frac{1}{2}$%.

4. \$935·68 for 66 days at $6\frac{1}{2}$%.

5. \$147·50 for 3 years 93 days at 7%.

6. \$250 from 9th July to 18th Aug. at 8%.

7. What principal will yield \$43·25 interest in $2\frac{1}{4}$ years at $5\frac{1}{2}$%?

8. What principal will in 95 days yield \$9·20 interest at 7%?

9. What principal will yield \$10 as interest at 6% from 1st. May to 31st. Oct. of the same year?

10. What principal will amount to \$1000 in $4\frac{1}{2}$ years at $4\frac{1}{2}$%?

11. What principal will amount to \$73·56 in 66 days at 8%?

12. A debt due on 3rd March was not paid and interest at $6\frac{1}{4}$% was charged on it from that date. On 6th June following, the debt amounted to \$100. What was the sum due on 3rd March?

13. At what rate will \$375·50 amount at simple interest to \$441·21 in $2\frac{1}{2}$ years?

14. At what rate will \$222·66 yield \$21 simple interest in 1 year and 94 days?

15. At what rate will \$438·88 borrowed on 17th Ap. amount at simple interest to \$446·93 on 29th July next following?

16. At what rate will a sum of money at simple interest double itself in 20 years?

17. At what rate will a sum of money at simple interest quadruple itself in 50 years?

18. In what time will \$273·85 yield \$28·86 simple interest at 6%?

19. In how many days will \$733·65 amount to \$743·70 at 5% simple interest?

20. A debt of \$175 became due on 13th June after which date interest was charged at the rate of 7%. When the debt was paid the interest accrued on it was \$4·10. When was the debt paid?

21. In what time will a sum of money double itself at 5% simple interest?

22. In what time will a sum of money triple itself at 8% simple interest?

23. The proceeds of a note for $137·50 discounted 40 days before maturity, were $136·30. What was the rate of discount charged on the face of the note and what was the rate of interest paid on the proceeds?

24. Find the discount off $385·77 due 86 days hence, (i) at 8% *discount*, (ii) at 8% *interest*. Show that the difference between amounts (i) and (ii) is the interest at 8% on (i) or the discount at 8% off (ii).

25. What rate of interest is equivalent to 10% discount, the term of discount being one year?

26. What rate of interest is equivalent to 10% discount, the term of discount being 95 days?

[274. The **Present Worth** *at a specified rate of interest* of a bill or a promissory note is the sum of money which put out at interest at the specified rate will when the bill is due or the note matures amount to the sum due on bill or note.

The difference between the present worth at a specified rate of interest of a bill or a promissory note and the amount of the bill or the note when due, is by some writers termed the *True Discount*, and the specified rate of interest is called the *Rate of Discount*. Considered as an abatement or deduction made from the amount of the bill or the note, the so-called True Discount is certainly a discount, *but so would be any other abatement*, but to call the rate at which the present worth *increases* by interest, a rate of *discount*, *i.e.*, *a rate of counting off*, is a perversion of the term which is not sanctioned by commercial usage and which leads to needless confusion when pupils go from the class-room to the counting-house.

The problems which are commonly given under the head of True Discount are properly problems on Interest and were they correctly worded and proposed as problems on Interest they would be perfectly legitimate and unexceptionable. Thus Prob. 10, Ex. xlvi, p. 291, may be put under the form :—What is the present worth at 4·5% *interest* of $1000 due 4·5 years hence?]

Averaging Accounts.

275. When one person owes another several amounts due at different times, the date on which all these debts may be discharged by payment of their sum, without loss of interest to either the debtor or the creditor is called the AVERAGE DATE or **Equated Time.**

Example. *A* bought goods of *B* as follows:—May 17, $200 at 30 days' credit; June 3, $250 at 60 days' credit; June 12, $210 at 90 days' credit. On July 5, *A* paid *B* $300 on account. Find the equated time for paying the balance.

Had *A* paid *B* $200 + $250 + $210 = $660 on May 17, *B* would have gained the interest on $200 for 30 days, the interest on $250 for 77 days and the interest on $210 for 116 days.

But if *A* delay from May 17 to July 5 to pay $300 of the $660, *B*'s gains will be reduced by the interest on the $300 for 49 days, the number of days of delay.

And if *A* defer the payment of the $360, balance of the $660, until the equated date, *B* will lose the balance of the interest he would have gained had all the payments been made on May 17.

Interest on $200 for	30 da.	= Int. on	($200 × 30 = $	6000)	for 1 da.
" " 250 "	77 "	= " "	(250 × 77 =	19250)	" "
" " 210 "	116 "	= " "	(210 × 117 =	24360)	" "
$660				$49610	for 1 da.
Interest on $300 for	49 da.	= Int. on	($300 × 49 =	14700)	" "
$360			$360	)$34910(97	

Interest on $34910 for 1 da. = Int. on $360 for (34910 ÷ 360) days
= Int. on $360 for 97 days.

Equated time = 97 days after May 17 = Aug. 22.

276. Should any of the items include cents, omit the cents in the calculation, and take the nearest number of dollars to the amounts of the items.

277. The method of determining the equated time of an account, which is exhibited in the preceding solution, is based on the assumption that what the debtor gains by retaining certain sums after they become due he loses by paying other sums before these become due, but as both gains and losses are computed on the full amounts of the items, while the actual gain is the interest on the amounts of the deferred payments and the actual loss is the interest on the present worth of the anticipated payments, it is evident that the solution is not absolutely exact. However, in ordinary business transactions, the error is too small to materially affect the result.

EXERCISE XLVII.

Find the equated date of payment of

1. Sep. 3,	\$350 @ 60 da.	**2.** Aug. 27,	\$325 @ 60 da.
"	\$520 @ 90 da.	Sept. 20,	\$280 @ 30 da.
"	\$175 @ 30 da.	Oct. 31,	\$785 @ 90 da.

3. On May 2, goods amounting to \$1250 were purchased on the following terms ; \$400 payable in 30 days, \$500 payable in 60 days and the balance payable in 90 days. Find the equated date for the payment of the whole bill.

4. On Sep. 19, a commission merchant received a consignment of 600 barrels of apples. He sold 120 barrels at \$2·25 on Sept. 24 ; 75 barrels at \$2·30 on Sep. 27 ; 150 barrels at \$2·40 on Oct. 7th ; 150 barrels at \$2·35 on Oct. 22 ; and the balance at \$2·20 on Nov. 18. Find the equated date of the total sales.

5. Henry Simpson sold A. Thomson & Co. merchandise as follows : Sep. 1, 225 bbl. flour @ \$6, on 30 days' credit ; Sep. 9, 180 bbl. of pork averaging 208 lb. @ $11\frac{1}{4}$ ct., on 60 days' credit ; Sep. 17, 150 doz. eggs @ 16 ct. per dozen on 2 months' credit ; Oct. 7, 572 lb. bacon @ $13\frac{1}{2}$ ct. on 3 months' credit ; Nov. 10, 460 lb. butter @ $21\frac{1}{2}$ ct. on 90 days' credit. Find the equated date for the payment of the sum-total of the several bills.

6. *A* holds three promissory notes made by *B*, one is for \$245·60 payable in 3 months from Feb. 13, 1889 ; another is for \$425 payable 60 days after date of Mar. 5, 1889 ; and the third is for \$186·25 and is dated Ap. 3, 1889, and payable 90 days after date. On Ap. 17, 1889, *B* offers to pay \$500 on the notes, and give in exchange for them a single note for the balance on them unpaid. When should the single note be payable ?

Partial Payments.

278. A **Partial Payment** is a payment of only a part of a debt and its accrued interest.

279. A **Receipt Indorsement** is an acknowledgment of the receipt of a partial payment written on the back of a note, mortgage or other documentary evidence of debt, stating the amount and the date of the payment.

280. When partial payments have been made on an interest-bearing note or other obligation, the balance unpaid and due at any given date may be found as follows :—

Find the interest on the principal from the date of the note or other obligation to the date of the first partial payment.

(*a*) *If the first partial payment is equal to or exceeds the interest thus found, subtract the first payment from the sum of the principal and its accrued interest, and consider the remainder as a new principal.*

(*b*) *If the first partial payment is less than the interest thus found, find the interest on the principal to the date of the next or of the earliest subsequent partial payment at which the sum of the payments equals or exceeds the interest due at such date, and subtract the sum of the payments to that date from the sum of the principal and its accrued interest to that date, and consider the remainder as a new principal.*

Similarly find the interest on the new principal to the date of the next partial payment. If that payment be equal to the interest thus found or if it be greater than the interest, proceed as in (*a*); *but, if the payment be less than the interest, proceed as in* (*b*). *So continue to the date of settlement.*

281. A partial payment in excess of the accrued interest will have the effect of reducing the principal, since, after discharging such accrued interest, there will remain a surplus to be so applied. A partial payment less than the accrued interest will not reduce the principal since such payment is not sufficient to discharge the accrued interest *which must first be paid.* No new principal should exceed the preceding principal, for such excess could arise only by the addition of interest to that preceding principal, and the effect would be to compute interest on interest, in computing the interest on the new principal.

282. In open accounts merchants generally charge interest upon all debts from the time they become due to the time of balancing accounts and allow interest to the same time upon all partial payments from the time they are made; they then deduct the sum of the partial payments and their accrued interest from the sum of the debts and their accrued interest, the remainder being the balance due. Upon this balance, if the account be not meanwhile paid or closed by note, interest is charged to the time when the accounts are again balanced, and is allowed to the same time upon all partial payments from the time they are made; and this process is continued until the account is either paid or closed by note.

EXERCISE XLVIII.

1. On a note for $620 on demand, dated Oct. 18, 1888, and drawing 6 % interest are indorsed the following payments: Nov. 26, 1888, $47·50; Dec. 28, 1888, $108·93; Feb. 11, 1889, $216·18; June 6, 1889, $60·10; Sep. 2, 1889, $183·25. How much was due on the note on Nov. 11, 1889?

2. On a mortgage for $3750 dated May 16, 1887, and bearing interest at 6 %, there were paid May 16, 1888, $350; Sept. 18, 1888, $280; Jan. 22, 1889, $750; May 16, 1889, $925; Oct. 31, 1889, $500. What sum was due on the mortgage on Jan. 2, 1890?

3. How much was due on the following note, on Oct. 31, 1889?

$850. Toronto, Oct. 31, 1887.

For value received, I promise to pay Alex. Thompson or order, on demand, Eight hundred and fifty Dollars, with interest from date at six per centum. John Stuart.

On this note the following payments were indorsed.

April 20, 1888, $125.	Jan. 21, 1889, $75.
Nov. 20, 1888, $125.	July 20, 1889, $425.

Compound Interest.

283. Compound Interest is interest which is computed for stated periods and added at the end of each period to the principal on which it was computed, the sum-total of principal and accrued interest at the end of each period becoming a new principal on which interest is computed for the next succeeding period.

284. The interest is said to be compounded annually, semi-annually, quarterly, monthly, ... according as the addition of interest to principal is made every year, half-year, quarter-year, month, or other interval.

285. In stating the rate of interest, one year is taken as the unit of time but is not expressed, and the rate is reduced to an annual rate as if it were for simple interest. Thus 4 % compounded semi-annually does not mean 4 % per half-year but 2 % per six months, the full phrase being,—'4 % per annum but compounded semi-annually.' A rate expressed in this way as if it were a simple interest rate is called a **nominal rate** to distinguish it from the actual or effective rate. A nominal rate of 6 % compounded quarterly is an actual rate of $1\frac{1}{2}$ % per quarter-year, and a nominal rate of 12 % compounded monthly is an actual rate of 1 % per month.

Example. If $1250 deposited in a savings-bank, draw interest at 4 % payable semi-annually, the interest accrued and due at the end of the first half-year will be ·02 of $1250 which is $25·00. If this $25·00 be not drawn it will be placed to the credit of the depositor, making his deposit $1275.

The interest for the second half-year will be computed on the increased deposit and will therefore be ·02 of $1275 which is $25·50. If this $25·50 be not drawn it will be placed to the credit of the depositor, making his deposit $1300·50 at the beginning of the third period of six months.

The interest for the third half year will be computed on the $1300·50 deposit and will therefore be ·02 of $1300·50 which is $26·01. This sum, if it be not drawn, will be added to the $1300·50 making a total of $1326·51 at the credit of the depositor at the end of 18 months.

Thus \$1250 at 4 % interest compounded semi-annually will in a year and a half amount to \$1326·51 ; and the compound interest at the specified rate and for the stated time will be \$1326·51 – \$1250 = \$76·51.

Computation.

```
$1250      = original amount or principal.
    1·02   = rate of increase in amount.
 ------
   25·00
 1250
 ------
$1275      = amount at end of 1st period.
    1·02
 ------
   25·50
 1275
 ------
$1300·50   = amount at end of 2nd period.
      1·02
 --------
    26·0100
  1300·50
 --------
 $1326·51  = amount at end of 3rd period.
```

EXERCISE XLIX.

Find the amount and the compound interest of :—

1. \$800 for 3 years at 5 % compounded annually.

2. \$425 for 4 years at 4 % compounded annually.

3. \$250 for 2 years at 6 % compounded semi-annually.

4. \$366·67 for $2\frac{1}{2}$ years at 4 % compounded semi-annually.

5. \$722·50 for $1\frac{1}{2}$ years at 4 % compounded quarterly.

Find correct to six significant figures the amount of \$1 at compound interest at 6 % for one year, interest compounded.

6. annually. **7.** semi-annually. **8.** quarterly,

Find correct to six significant figures the rate of increase in the amount of \$1 at 5 % interest compounded annually for

9. three years. **10.** five years. **11.** seven years.

Find correct to six significant figures the rate of increase in the amount of \$1 at 4 % interest compounded quarterly for

12. one year. **13.** two years. **14.** three years.

286. Problems in Compound Interest involve the consideration of *original amount* or principal, *rate*, *number of compoundings*, *final amount* and *interest*, and any three of these being known the other two may be determined.

Let r denote the nominal rate of interest PER UNIT; t the measure in years of the length of time between two successive compoundings; n the number of compoundings; A_0 the measure of the original amount, the principal; A_n the measure of the amount after n compoundings; and I_n the measure of the interest after n compoundings; then will

$$A_1 = A_0\,(1+rt),$$
$$A_2 = A_1\,(1+rt) = A_0\,(1+rt)^2$$
$$A_3 = A_2\,(1+rt) = A_0\,(1+rt)^3$$
$$A_4 = A_3\,(1+rt) = A_0\,(1+rt)^4$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$A_n = A_{n-1}\,(1+rt) = A_0\,(1+rt)^n, \qquad (A.)$$

and ∴ $$\log A_n = \log A_0 + n \log(1+rt); \qquad (Aa.)$$

and $$I_n = A_n - A_0. \qquad (B.)$$

287. If there should occur a *broken* period whose measure in years is t_1, t_1 being $< t$, the rate of increase for t_1 is by commercial usage taken to be $1+rt_1$.

Example 1. What will be the amount of $437·50 in 10 years at 5 % payable and compounded half-yearly?

The nominal rate of interest is ·05 per unit and the periods or terms are $\frac{1}{2}$ yr. each,

∴ the actual rate of interest is $\frac{1}{2}$ of ·05 per unit = ·025 per unit.

∴ the rate of increase by compounding is 1·025 per half year;

∴ the rate of increase for 10 years is $1{\cdot}025^{20}$

∴ the amount sought to be known is $\$437{\cdot}50 \times 1{\cdot}025^{20}$.

$$\log 1{\cdot}025 = {\cdot}010724$$
$$\underline{\qquad\qquad 20}$$
∴ $$20 \log 1{\cdot}025 = {\cdot}21448$$
$$\log 437{\cdot}5 = 2{\cdot}640978$$
$$\overline{2{\cdot}855458} = \log 716{\cdot}9$$

∴ amount at end of 10 years = $716·90.

Example 2. What will be the discount off \$100 for 6 years at 6 % *interest*, compounded quarterly?

The actual rate of interest is $(\cdot 06 \times \frac{1}{4})$ per unit $= \cdot 015$ per unit.

The interest is compounded $(6 \div \frac{1}{4})$ times $= 24$ times.

∴ \$100 present value will in 6 yr. amount to $\$100 \times 1 \cdot 015^{24}$

∴ $(\$100 \div 1 \cdot 015^{24})$ " " " " " " " \$100.

$\log 100 - 24 \log 1 \cdot 015 = 2 - \cdot 155184 = \cdot 844816 = \log 69 \cdot 95$

∴ \$69·95 present value will in 6 yr. amount to \$100

∴ discount = \$100 − \$69·95 = \$30·05.

[*The student should note the distinction between discounting at 6 %* INTEREST *(whether simple or compound) and discounting at 6 % of* DISCOUNT. See § 274, p. 292.]

EXERCISE L.

Find the amount and the compound interest of :—

1. \$750 for 15 years at 5 % compounded annually.

2. \$365 for 10 years at 6 % compounded semi-annually.

3. \$1250 for 20 years at 4 % compounded quarterly.

4. \$36·25 for 5 years at 6 % compounded monthly.

5. \$427·50 for 15 years at 5 % compounded triennially.

6. \$125 for 100 years at 4 % compounded quinquennially.

7. Find, correct to five significant figures, the sum to which one cent would amount in 1890 years at (*a*) 1 %, (*b*) 2 %, (*c*) 3 % interest compounded annually, given $\log 1 \cdot 01 = \cdot 0043213738$,

$\log 1 \cdot 02 = \cdot 0086001718$, $\log 1 \cdot 03 = \cdot 0128372247$.

Find the present worth of :—

8. \$1000 payable 20 yr. hence, at 4 % interest compounded annually.

9. \$372·50 payable $7\frac{1}{2}$ yr. hence, at 5 % interest compounded semi-annually.

10. \$372·50 payable $7\frac{1}{2}$ yr. hence, at 5 % interest compounded quarterly.

Find the discount off \$125 payable 10 years hence at

11. 5 % *discount.*

12. 5 % *simple interest.*

13. 5 % *interest* compounded annually.

14. 5 % *interest* compounded semi-annually.

15. 5 % *interest* compounded quarterly.

16. In what time will \$300 amount to \$426·63 at $4\frac{1}{2}$ % compounded annually?

17. In what time will \$250 amount to \$376·20 at 6 % compounded quarterly?

18. Show that a sum of money will about double itself in $(70\div 2)$ compoundings at 2 %, $(70\div 3)$ compoundings at 3 %, $(70\div 3\frac{1}{2})$ compoundings at $3\frac{1}{2}$ %, and in correspondingly obtained numbers of compoundings for 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, 7, 8, 9 and 10% respectively. (See problem 13, Exercise xx, p. 158).

In what time will a sum of money drawing 8 % interest increase to 10 times the original sum

19. if the interest be compounded annually?

20. if the interest be compounded semi-annually?

21. if the interest be compounded quarterly?

22. At what rate will \$225 amount to \$302·60 in 12 years, interest compounded annually?

23. At what rate will \$133 amount to \$456·15 in 14 years, interest compounded semi-annually?

24. In 1871 the population of a certain city was 27512, in 1881 it was 44653; what was the annual rate of increase of the city's population?

25. Two equal sums of money are placed at interest, one sum at 6 % the other sum at $1\frac{1}{2}$ %, the interest in both case being compounded annually. In what time will the amount at the higher rate be 10 times that at the lower rate?

26. Two equal sums of money are placed at interest, both at a nominal rate of 12 %, but in one case the interest is compounded monthly while in the other case it is compounded annually. In what time will the amount at the higher effective rate be double that at the lower?

27. What will be the effective rate *per annum* if the nominal rate be 6 % and the interest be compounded (*a*), monthly; (*b*), daily; (*c*), hourly; (*d*), per minute?

28. At what rate will a sum of money treble itself in 30 years, interest compounded quarterly? To what multiple of the original sum will it amount in 100 years at this rate?

VI. Stocks and Bonds.

288. A **Corporation** or **Incorporated Company** is an association of persons authorized by law to transact business as a single individual. The powers, rights, duties and obligations of a corporation, as such, are distinct from those of the members forming it.

289. The capital of a corporation or of a public company is usually divided into a definite number of equal parts called **Shares.** A share commonly represents \$100 (or £100) of the original capital of the corporation, but in some cases it represents as low as \$1 (or £1) of it and in other cases as high as \$1000 (or £1000) of it.

290. Any number of shares in a corporation or any amount of its capital is called **Stock,** but in the United States this term is also used distinctively for shares of \$100 each, shares of \$50 and of \$25 being called *half-stock* and *quarter-stock* respectively.

291. The proprietors of shares in a corporation or in a public company are called **shareholders** or **stockholders.** Each owner of stock may sell his shares or otherwise transfer them to another person without the consent of the other shareholders.

292. A **Stock Certificate** is an instrument issued by a corporation, certifying that the holder thereof owns a stated number of shares of the capital stock of the corporation.

293. **The Par Value** of a share is the value which is specified upon the face of the certificate for the share, and represents the amount of capital stock for which it was originally issued.

294. The **Market Value** of a share is the sum for which it can be sold.

Stock is said to be *above par* or *at a premium* when the market value of the shares is greater than their par value ; it is said to be *below par* or *at a discount* when the market value of the shares is less than their par value.

295. A **Dividend** is the part of the net earnings or profits of a corporation or a public company, which is divided among the stockholders thereof. Dividends are usually declared annually, semi-annually, or quarterly at a specified rate per cent. of the par value of the stock.

296. Preferred Stock is that part of the capital stock of a corporation on which a specified percentage is payable annually out of the net earnings, before any dividend can be declared on the ordinary stock.

297. A **Bond** or **Debenture** is a written obligation to pay the holder thereof a certain sum of money at the expiry of a certain term of years, and interest thereon at a specified rate per cent. at stated intervals. Bonds and debentures are issued for money borrowed by the General and Local Governments and by municipal and business corporations. Debentures frequently charge certain specified property with the repayment of the money borrowed on them ; in such cases the debentures are practically mortgages on the property.

298. An **Interest Coupon** is an interest certificate payable to bearer, printed at the bottom of bonds and debentures given for a term of years. There are as many coupons attached to each bond as there are instalments of interest to be paid on it, a coupon for each instalment. Each coupon is cut off and presented for payment when the interest for the period mentioned in it becomes due.

299. Consols, *i.e.*, Consolidated Annuities are British Government securities bearing 3 % interest. These with the other British Government securities for which permanent provision has been made, the most important of which are the Reduced Annuities and New three per cent. Annuities, are in England termed the *Public Funds.*

300. Rentes (*i.e.* Annuities) are French Government securities bearing various rates of interest.

301. Stock Brokers are persons who deal in stocks, bonds and similar securities. When a stock broker buys or sells for a principal he charges a commission, technically termed *brokerage,* which ranges, according to circumstances and previous agreement, from $\frac{1}{16}$ of 1 % to $\frac{1}{2}$ of 1 % of the par value of the securities bought or sold, the most common rate being $\frac{1}{8}$ of 1 %. Occasionally special rates are agreed upon and paid.

In England, stock brokers do not deal directly with each other but sell to or buy from stock-jobbers who act for themselves and make their profits out of the turn of the market.

302. In Canada and the United States, stock quotations usually state only the rates per cent. which the market values of the stocks and bonds quoted bear to their par values; but in England quotations of other than government securities generally give the price per share or per bond.

The following is an illustrative example of a stock report and quotations :—

The closing prices on the Toronto Stock Exchange to-day (6 Dec., 1889), were as follows :

STOCKS.	Par Value of Shares	Last $\frac{1}{2}$ year dividend.	1 P. M.		4 P. M.	
			Sellers per 100.	Buyers per 100.	Sellers per 100.	Buyers per 100.
Banks.	$	%				
Montreal	200	5	$225\frac{1}{4}$	$224\frac{1}{2}$	$225\frac{1}{4}$	$224\frac{3}{4}$
Ontario	100	$3\frac{1}{2}$	132	$131\frac{1}{2}$	$131\frac{1}{2}$	131
Molsons	50	......	158	......	156	......
Toronto	200	6	219	213	220	212
Merchants'	100	$3\frac{1}{2}$	142	139	140	139
Commerce	50	$3\frac{1}{2}$	$121\frac{1}{4}$	121	$121\frac{1}{4}$.	121
Imperial	100	4	$152\frac{3}{4}$	150	153	$150\frac{3}{4}$
Dominion	50	5	$222\frac{1}{2}$	221	223	222
Standard	50	$3\frac{1}{2}$	138	$137\frac{1}{2}$	138	$137\frac{1}{2}$
Hamilton	100	4	......	146	......	147

It will be seen from this report that in Toronto, on 6 Dec., 1889, sellers of Bank of Montreal stock were offering it at the rate of $225·25 *cash* per $100 *stock* and as each share represents $200 of stock, sellers were really asking $450·50 per share. The report also shows that buyers of Bank of Montreal stock were offering for it $224\frac{1}{2}$ % to $224\frac{3}{4}$ % of its par value, *i.e.*, were offering $449 to $449·50 per share for it.

Other forms of report may be seen in the 'Financial Columns' of the Toronto and Montreal daily newspapers and in the 'Share Lists' of any Stock Exchange.

Example 1. Find the price at $140\frac{3}{4}$ of 40 shares (\$40 each) of Western Assurance Co. stock, brokerage $\frac{1}{8}$ %.

Par value of stock $= \$40 \times 40 = \$1600$.

Rate paid $= 140\frac{3}{4}\,\% + \frac{1}{8}\,\% = 1 \cdot 40\frac{7}{8}$.

Cost of stock $= \$1600 \times 1 \cdot 40\frac{7}{8} = \$2254$.

Example 2. I sold 500 shares of Bank of Montreal stock at $224\frac{3}{4}$ and invested the proceeds in Bank of Commerce stock at $124\frac{1}{4}$, paying $\frac{1}{8}$ % brokerage on each transaction. Find the increase in my annual income, the Bank of Montreal paying a half-yearly dividend of 5 %, the Bank of Commerce a half-yearly dividend of $3\frac{1}{2}$ %.

Par value of B. of M. stock $= \$200 \times 500 = \$100,000$.

Rate received $= 224\frac{3}{4}\,\% - \frac{1}{8}\,\% = 2 \cdot 24\frac{5}{8}$.

Amount to be invested $= \$100,000 \times 2 \cdot 24\frac{5}{8} = \$224,625$.

Rate paid for B. of C. stock $= 124\frac{1}{4}\,\% + \frac{1}{8}\,\% = 1 \cdot 24\frac{3}{8}$.

Price of 1 share of B. of C. stock $= \$50 \times 1 \cdot 24\frac{3}{8} = \$62 \cdot 1875$.

Number of shares bought is the integral part of

$$\$224625 \div \$62 \cdot 1875$$

which is 3612

and there is \$3·75 of cash over.

Par value of 3612 shares of B. of C. stock $= \$180,600$.

2 dividends at 5 % each on \$100,000 of B. of M. stock

$$= \$100000 \times \cdot 10 = \$10000.$$

2 dividends at $3\frac{1}{2}$ % each on \$180600 of B. of C. stock

$$= \$180600 \times \cdot 07 = \$12642.$$

Increase of annual income $= \$12642 - \$10000 = \$2642$.

EXERCISE LI.

Find the cash value of

1. 25 shares Ontario Bank at 131.

2. 18 " Standard Bank at $137\frac{1}{2}$.

3. 75 " Bank of Toronto at 218.

4. 250 " (\$50) Dominion Telegraph Co. at $83\frac{3}{4}$.

5. 950 " (\$100) Canadian Pacific R.R. at $72\frac{7}{8}$.

6. 350 " (\$24·33$\frac{1}{3}$) North West Land Co. at $79\frac{3}{4}$.

7. Sold through a broker 1500 shares (\$100) of Jersey Central R.R. stock at $121\frac{1}{2}$, brokerage $\frac{1}{8}$%. What were the net proceeds of the sale ?

8. Bought through a broker 1600 shares (\$100) St. Paul R.R. stock at $69\frac{1}{4}$, brokerage $\frac{1}{8}$%. What was the gross cost of the stock ?

9. A speculator bought 36500 shares (\$100) Reading R.R. stock at $39\frac{3}{4}$ and sold them at $40\frac{3}{8}$. What was his gain on the transaction ?

10. A man bought through a broker 1900 shares (\$100) Canada Southern R.R. stock at $54\frac{3}{4}$ and sold them at $55\frac{3}{8}$. What was his net profit on the transaction, brokerage each way $\frac{1}{8}$% ?

11. A man bought through a broker 7600 shares (\$100) of Lake Shore R.R. stock at $107\frac{5}{8}$ and sold 2400 shares at $107\frac{3}{4}$ and the remainder at $107\frac{1}{8}$. What was the amount of his losses on the transactions, brokerage being $\frac{1}{8}$% each way ?

12. A bank declared a dividend of $3\frac{1}{2}$%. How much should a stockholder owning 120 shares (\$50) receive ?

13. An insurance company declared a dividend of 6%. What rate is that on the market value of the shares which are at 185

14. Compare the rates on the cash values of 6% on stock at 216 and $3\frac{1}{2}$% on stock at 125.

15. Sold 37 shares (\$25) B. and L. Association stock, receiving therefor \$1019·81. At what rate was the stock sold ?

16. Bought through a broker 750 shares (\$50) in the Farmers' Loan and Savings Society paying therefor \$43968·75. At what quotation were they bought, brokerage $\frac{1}{8}$% ?

17. Sold through a broker 215 shares (\$50) in the Dominion Savings and Loan Society receiving from him for them \$9728·75. At what quotation did the broker sell them, brokerage $\frac{1}{8}$% ?

18. Bought stock at $197\frac{5}{8}$ and sold it at $194\frac{7}{8}$, having meanwhile received a dividend of 6 % on it. My net gain by the transaction after paying $\frac{1}{8}$ % brokerage each way, is $336. How many shares ($40) did I buy ?

19. A man received $495 as dividend at $4\frac{1}{2}$ % on his bank stock. He sold 40 shares ($100) at $143\frac{1}{4}$ and the remainder at $144\frac{1}{2}$, paying $\frac{1}{8}$ % brokerage. What were the net proceeds of the sale ?

20. A capitalist had $20000 to invest. He purchased $8700, par value, of Canadian 4 % bonds at 103 and $7300, par value, of Canadian $3\frac{1}{2}$ % bonds at $93\frac{1}{2}$ and invested the balance as far as he could in bank stock (shares $100) at $149\frac{1}{4}$, paying half-yearly dividends of 4 % each. What was the gross amount of his investment he paying $\frac{1}{8}$ % brokerage for buying each class of securities ? What was his annual income from these investments ? What average rate per cent. per annum did he receive on these investments ?

21. The difference between the annual income derived from a certain sum invested in 7 % stock at 150 and that from an equal sum invested in 9 % stock at $202\frac{1}{2}$, is $40. What is the amount invested in the 7 % stock and what is the annual income therefrom ?

22. A shareholder receives a dividend of 6 % on his stock and pays thereon an income-tax of $16\frac{2}{3}$ mills on the dollar. Next year he receives a dividend of $6\frac{1}{2}$ % and pays an income-tax of $12\frac{1}{2}$ mills on the dollar. He finds that his income is $830 more in the latter year than it was in the former. How much stock does he hold ?

23. A man invests a certain sum in 3 % stock at 90 and an equal sum in 4 % at 95. Each stock rises 5 % in price ; the investor then sells out and invests the proceeds of each stock in the other. The stocks fall to their former value and he again sells out at a total loss of $1943.90. Find the sum he originally invested.

24. What sum invested in the three per cents at 95 will in $17\frac{1}{2}$ years amount to £10000, the price of the funds having risen meanwhile to $100\frac{1}{2}$; interest to be payable and compounded half yearly ?

25. If money be worth 5 %, what should be the price of 6 % bonds which are to be paid off at par 3 years after the date of purchase, the interest on the bonds being payable half-yearly.

CHAPTER VIII.

EXCHANGE.

303. Exchange is the system by which accounts between persons in distant places are settled without the necessity of sending large sums of money or large quantities of gold or silver from one place to the other, thus avoiding the risk and expense of transportation.

For example, suppose that *A* of Halifax owes *B* of Toronto $7500 for wheat and that *X* of Toronto owes *Y* of Halifax $7500 for dried fish. In such case, *B* in Toronto can draw on *A* in Halifax for $7500 and sell the draft to *X* who transmits it to *Y* who in turn presents it to *A* who thereupon pays *Y*. Thus instead of *A* sending $7500 from Halifax to Toronto to pay *B*, and *X* sending $7500 from Toronto to Halifax to pay *Y*, *X* of Toronto pays *B* in Toronto and *A* of Halifax pays *Y* in Halifax, the debts being as it were *exchanged*.

Domestic or *Inland Exchange* is exchange carried on between two cities in the same country.

Foreign Exchange is exchange carried on between two cities in different countries.

304. A **Draft** or **Bill of Exchange** is a written order by one person, called *the drawer*, directing a second person, called *the drawee*, to pay a specified sum of money to a third person, called *the payee*, or to the payee's order.

A *Domestic* or *Inland Bill of Exchange*, usually called a *Draft*, is one of which drawer and drawee reside in the same country.

A *Foreign Bill of Exchange* is one of which drawer and drawee reside in different countries. Foreign bills of exchange are usually drawn in sets of three, called respectively the *First*, the *Second* and the *Third of Exchange*, and are of the same tenor and date and so worded that when one of the set is paid, the others become void. The object of thus drawing the bills in sets of three is to provide against loss in transmission. The bills or two of them are sent either by different routes or by the same route at different dates.

305. An **Acceptance** is an agreement by the drawee to pay the sum specified in the draft or bill of exchange. The usual mode of accepting a bill of exchange is for the drawee to sign his name under the word "accepted" written across the face of the bill. If the bill be payable a specified number of days *after sight*, the date of acceptance should be inserted.

306. If the drawee of a bill refuses acceptance or if, having accepted, he fails to make payment when it is due, the bill is immediately *protested*, *i.e.*, a written declaration is made by a public officer called a Notary Public, at the request of the holder or person in legal possession of the bill, notifying the drawer and the indorsers of its non-acceptance or non-payment.

307. Bills of exchange are negotiable or non-negotiable upon the same conditions and are subject to the same indorsements as promissory notes. The date of maturity of bills of exchange is ascertained in the same manner as that of notes ; see § 252, p. 286.

308. The **Face** or **Par** of a bill of exchange is the sum specified in the bill, exclusive of interest, premiums, discount, or commission.

When bills of exchange on a given place sell for more than their par value, exchange on that place is said to be *above par* or *at a premium;* when they sell for less than their face value, exchange on that place is said to be *below par* or *at a discount.*

309. Exchange is usually conducted through bankers or brokers who buy commercial bills on distant cities and mail them for collection to their correspondents or agents in those cities. Drafts or bills of exchange are then drawn on the correspondents for the whole or for any required part of the sums thus placed to the credit of the principals and sold to persons who wish to use money in those cities. Bankers and their correspondents also draw on each other for sums required by persons dealing with them and at stated periods strike a balance of the sums thus drawn.

310. The **Par of Exchange** between two countries is the value of the monetary unit of one of the countries expressed in terms of the currency of the other.

The *intrinsic par of exchange* is the real or intrinsic value of coins estimated by the weight and purity of the metals of which they are composed.

The *legal par of exchange* is the par established under authority of statute.

The dollar of Canada is defined by statute to be of such value that four dollars and eighty-six cents and two-thirds of a cent shall be equal in value to one pound sterling; thus $4·86$\frac{2}{3}$ per £1 is the legal par of exchange between Canada and Great Britain. There being no Canadian gold coinage and the silver and bronze coins of Canada being only a token coinage, there is no intrinsic par of exchange between Canada and Great Britain.

The intrinsic value of the sovereign, the coin which determines the value of the pound sterling of Great Britain, in terms of the gold dollar, the monetary unit of the United States of North America, is $4·866564 – ; for, 1869 sovereigns contain 211200 grains of pure gold and the United States gold eagle contains 232·2 grains of pure gold and 211200 ÷ 1869 ÷ 23·22 = 4·866564 –. The value determined at the United States Mint and proclaimed by the Secretary of the Treasury is $4·8665, a sum which approaches the intrinsic value far within the 'remedy' allowed on the sovereign.

The intrinsic value of the ten-franc gold pieces of France, Belgium and Switzerland is $1·93. The intrinsic value of the ten-mark gold piece of the German Empire is $2·38.

311. The **Rate of Foreign Exchange** is the market or commercial value of the monetary unit of one country expressed in terms of the currency of another.

The following quotations were given by the New York Agents of the Canadian Bank of Commerce as indicating the rates for actual business in sterling exchange on 7 Dec., 1889.

Prime Bankers,	60 days	4·80$\frac{1}{4}$
do.	Demand	4·84$\frac{1}{4}$
do.	Cables	4·84$\frac{3}{4}$
Commercial	60 days	4·79$\frac{3}{4}$ – $\frac{1}{2}$
Documentary	do.	4·78$\frac{3}{4}$ @ 4·79.

Prime Bankers' Sterling Bills are those drawn by first-class banking houses in New York on first class banking-houses in London, England.

Commercial Bills are those drawn by merchants or commercial houses of good standing in America on their correspondents abroad.

A Documentary Commercial Bill is a bill drawn by a shipper upon his consignee for merchandise shipped. It is accompanied by a Bill of Lading and a Letter of Hypothecation giving control of the merchandise to the holder of the bill, with recourse to the drawer for the deficiency, if any should arise.

312. The New York quotations for bills on London are always given in dollars per pound sterling.

The quotations for bills on Paris, Antwerp or Geneva are given in *francs per dollar*. The quotations for bills on Hamburg, Bremen, Berlin and Frankfort are given in *cents per four marks*.

In Canada the legal par of sterling exchange was formerly $\$4{\cdot}44\frac{4}{9}$ per £1 and Canadian quotations are still usually given as a percentage premium on this *old par*. Thus when sterling exchange is quoted at $9\frac{1}{2}$ it is meant that the rate of exchange is $\$4{\cdot}44\frac{4}{9} \times 1{\cdot}095$ per £1, *i.e.*, $\$4{\cdot}86\frac{2}{3}$ per £1 which is the *new par*. So also sterling exchange at 9 means $\$4{\cdot}44\frac{4}{9} \times 1{\cdot}09$ per £1, *i.e.*, $\$4{\cdot}84\frac{4}{9}$ per £1.

313. The usage of Canadian bankers is to draw bills of exchange on London payable either at 60 days after sight or on demand, but as the greater part of the business is done in the former class of bills, quotations are assumed to be for sixty-day bills unless it is specifically stated to be otherwise at the time of making them.

314. **A Circular Letter of Credit** is a letter issued by a banking-house to a person who purposes to travel abroad and addressed to bankers generally and to the agents and correspondents of the banking-house in particular in the several countries which the traveller is about to visit, requesting them to supply the traveller with money as he requires it until a total amount has been paid him not exceeding the sum specified in the letter. The sums paid to the traveller from time to time are indorsed on the letter. A letter of credit is not transferable from one person to another.

Example 1. What will a bill of exchange on London for £6000 realise in Toronto exchange at $8\frac{1}{2}$?

The $8\frac{1}{2}$ here means $8\frac{1}{2}$ % premium on the old par of exchange of $\$4{\cdot}44\frac{4}{9}$ which gives $\$\frac{40}{9} \times 1{\cdot}085$ as the rate of exchange for the transaction.

$\therefore$ £6000 is equivalent to $\$\frac{40}{9} \times 1{\cdot}085 \times 6000 = \$28933{\cdot}33$.

Example 2. Exchange at New York on London is $4{\cdot}84\frac{3}{4}$, and at London on Paris it is 25·25 francs per £1. What sum remitted from New York through London to Paris will pay a debt in Paris of 12500 francs ?

$$25{\cdot}25 \text{ fr.} = £1 = \$4{\cdot}84\tfrac{3}{4}$$

$$\therefore \quad 12500 \text{ fr.} = \quad \$4{\cdot}84\tfrac{3}{4} \times 12500 \div 25{\cdot}25 = \$2399{\cdot}75.$$

EXERCISE LII.

1. What will a bill on London for £75 cost, exchange at $9\frac{1}{2}$?

2. What will a bill on London for £225 cost at $9\frac{3}{4}$?

3. What will be the value of a bill for £60 at 8 ?

4. What must be paid for a bill on London for £15 7s. 6d. at 10 ?

5. What sum sterling will be equal to \$100 Canadian, exchange $9\frac{1}{2}$?

6. What sum sterling should I receive for \$5500 Canadian, exchange $9\frac{3}{4}$?

7. The Government of Canada purchased the following sterling exchanges : For transmission to Messrs. Glyn, Mills & Co., £50,000 at $8\frac{1}{2}$ and £10,000 at $8\frac{7}{8}$; for transmission to Messrs. Baring Brothers & Co., £20,000 at 9 and £40,000 at \$4·846 per £1 stg. ; and for transmission to the Bank of Montreal, London, £20,000 at $8\frac{15}{16}$, £20,000 at $8\frac{13}{16}$, £20,000 at $8\frac{27}{32}$, and £20,000 at $8\frac{29}{32}$. Calculate the cost of each of the eight purchases and find what amount in dollars and cents should be charged to Glyn, Barings and the Bank of Montreal, London, respectively, that they may be charged at the par value $9\frac{1}{2}$, in their accounts.

8. Find the cost of a bill of exchange on Paris for 2400 francs at $5{\cdot}16\frac{1}{4}$ fr. per \$1.

9. A merchant wishes to transmit 2400 francs from Toronto to Paris, through London. For what sum (sterling) should the bill on London be drawn and how much will the merchant have to pay for it, sterling exchange being $9\frac{5}{8}$ and exchange between London and Paris 25·20 francs per £1 ?

10. What will be the cost of a bill of exchange on Berlin for 2400 marks, rate of exchange $95\frac{1}{8}$ cents per 4 marks ?

11. I bought in Ottawa a bill of exchange on London, England, for £60 at $9\frac{3}{4}$ and forwarded it to Calvary & Co. of Berlin who sold it for 1224 marks and gave me credit for the proceeds. What rate of exchange on Berlin did I thus obtain ?

12. Immediate payments to the extent of £200,000 stg. are required to be made in England on behalf of the Canadian Government, and in response to calls the following tenders have been received ;—for 60 days sight drafts £200,000 at $8\frac{15}{16}$, and for demand drafts the same sum at $9\frac{5}{8}$. Which tender would be the more profitable to the Government, taking the rate of discount in England at $3\frac{1}{4}$ per cent. and the time 63 days ? How much would the Government gain by accepting the more profitable tender ?

13. I purchased through a broker in New York a bill of exchange on London for £432 12s. 6d. at $4·84\frac{3}{8}$. What was the total cost, brokerage $\frac{1}{8}$ % ?

14. I sold through a New York broker a bill of exchange on Hamburg for 1260 marks at $95\frac{1}{8}$. What were the net proceeds due me, brokerage $\frac{1}{4}$ % ?

15. I bought through a broker in Boston a bill of exchange on Liverpool for £300 paying the broker $1457·64 for it. At what quotation was the bill purchased, allowing $\frac{1}{4}$ % for brokerage ?

16. I paid a broker $1511·90 for a bill of exchange on Bremen for 6400 marks. At what quotation was the bill purchased allowing $\frac{1}{8}$ % for brokerage ?

17. I sold through a broker a bill of exchange on Manchester for £600 and received $2912·35 as the net proceeds. At what rate of exchange was the bill sold allowing $\frac{1}{8}$ % for brokerage ?

18. I sold a bill of exchange on Paris for 8330 francs and received $1606·10 as the net proceeds. What was the rate of exchange on Paris, a brokerage of $\frac{1}{8}$ % having been charged me for selling the bill ?

19. I paid $2·40 as brokerage at $\frac{1}{8}$ % on a bill of exchange on Hamburg for 8040 marks. What was the rate of exchange ?

20. I sold through a broker a bill of exchange on London at 4·85 and received $4773·37 as net proceeds. What was the face of the bill, brokerage $\frac{1}{4}$ % ?

21. The cost including brokerage at $\frac{1}{8}$ %, of a bill of exchange on Geneva bought at 5·20 was \$3764·70. What was the face of the bill?

22. Find the cost of 120 marks paid in Berlin on a letter of credit, the rate of exchange being $95\frac{1}{4}$ and 28 cents being charged for commission and interest.

23. Complete the following :—

NEW YORK, *December 10, 1889.*

Canadian Bank of Commerce,

To Lazard Freres,

10 WALL STREET.

Acct. Letter of Credit 7520, paid Berlin, Nov. 25, to F. G. 160 marks, receipt enclosed ;

@ $95\frac{1}{4}$	- - - -
Com. $\frac{1}{2}$ %	- -
Int. 30 days @ 6 %.	- -
	- - - -

24. The Government of Canada procured silver coinage to the extent of \$200,000, for which the following quantities of bar silver were purchased, viz. :

50,341·80	ounces Troy	at	$51\frac{3}{4}$d.	per oz.
50,046·27	"	"	$51\frac{13}{16}$d.	"
49,055·26	"	"	52 d.	"

On the value of the silver so purchased brokerage was charged at $\frac{1}{8}$ per cent. ; the carriage and insurance from England to Canada, calculated at the par of $9\frac{1}{2}$, was, on \$60,000 at 18s. 6d. per £100 ; on \$80,000 at 16s. per £100 ; and on \$60,000 at 13s. per £100, and the cost of coinage £2,166 17s. 6d. What profit accrued to the Government in dollars and cents, taking the rate at $9\frac{1}{2}$ per cent. on the transaction ; and what weight of silver in grains is contained in a dollar?

APPENDIX.

CIRCULATING DECIMALS.

By an extension of the ordinary or Arabic system of notation the decimal fractions $\frac{7}{10}$, $\frac{89}{100}$, $\frac{541}{1000}$ are sêverally written $\cdot 7$, $\cdot 89$ and $\cdot 541$, and conversely $\cdot 3$, $\cdot 47$ and $\cdot 293$ denote $\frac{3}{10}$, $\frac{47}{100}$ and $\frac{293}{1000}$ respectively. Still further extending this system to the expression of complex decimal fractions $\dfrac{7\frac{1}{4}}{10}$, $\dfrac{89\frac{2}{3}}{100}$ and $\dfrac{541\frac{5}{8}}{1000}$ may be written $\cdot 7\frac{1}{4}$, $\cdot 89\frac{2}{3}$ and $\cdot 541\frac{5}{8}$ respectively, and conversely $\cdot 3\frac{1}{3}$, $\cdot 27\frac{3}{7}$ and $\cdot 001\frac{1}{9}$ will severally denote $\dfrac{3\frac{1}{3}}{10}$, $\dfrac{27\frac{3}{7}}{100}$ and $\dfrac{1\frac{1}{9}}{1000}$, with analogous expressions for all other fractions whose numerators are mixed numbers and denominators powers of 10.

A system of notation similar to this notation for decimal fractions is sometimes employed in the writing of fractions whose denominators are one less than a power of 10. For example $\frac{7}{9}$ is written $\cdot\dot{7}$, $\frac{85}{99}$ is written $\cdot\dot{8}\dot{5}$ and $\frac{3069}{9999}$ is written $\cdot\dot{3}06\dot{9}$, with corresponding expressions for all other fractions whose denominators are expressed by 9's only. Conversely, when this system is employed $\cdot\dot{5}$ denotes $\frac{5}{9}$, $\cdot\dot{2}1\dot{6}$ denotes $\frac{216}{999}$ and $\cdot\dot{2}3076\dot{9}$ denotes $\frac{230769}{999999}$, with a corresponding interpretation of all similar expressions.

This notation may be combined with that for decimal fractions; *e. g.*, $\cdot 4\frac{7}{9}$, $\cdot 35\frac{2}{9}$ and $4\cdot 27\frac{583}{999}$ may be written $\cdot 4\dot{7}$, $\cdot 35\dot{2}$ and $4\cdot 27\dot{5}8\dot{3}$ respectively, and $4\cdot 86\dot{6}$, $\cdot 3\dot{8}\dot{5}$ and $70\cdot 02\dot{4}3\dot{7}$ will severally denote $4\cdot 86\frac{6}{9}$, $\cdot 3\frac{85}{99}$ and $70\cdot 02\frac{437}{999}$.

When it is necessary to reduce such complex fractions as $\cdot 4\dot{7}$, $\cdot 35\dot{2}$ and $\cdot 27\dot{5}8\dot{3}$ to simple form advantage should be taken of the relations $9=10-1$, $99=100-1$, $999=1000-1$, &c. Thus,—

$$\cdot 4\dot{7}=\cdot 4\tfrac{7}{9}=\frac{4(10-1)+7}{90}=\frac{47-4}{90}=\frac{43}{90};$$

$$\cdot 35\dot{2}=\cdot 35\tfrac{2}{9}=\frac{35(10-1)+2}{900}=\frac{352-35}{900}=\frac{317}{900};$$

$$\cdot 27\dot{5}8\dot{3}=\cdot 27\tfrac{583}{999}=\frac{27(1000-1)+583}{99900}=\frac{27583-27}{99900}=\frac{27556}{99900}.$$

Example 1. Prove that $\cdot\dot{5} = \cdot 5\dot{5} = \cdot 55\dot{5} = \cdot 5555 = \cdot\dot{5}\dot{5} = \cdot 5\dot{5}\dot{5} = \cdot 55\dot{5}\dot{5}$.

$\frac{5}{9} = \cdot 5\frac{5}{9} = \cdot 55\frac{5}{9} = \cdot 555\frac{5}{9} = - - - - - - - - - - - - -,$

i. e. $\cdot\dot{5} = \cdot 5\dot{5} = \cdot 55\dot{5} = \cdot 555\dot{5} = - - - - - - - - - - - - .$

Also, $\frac{5}{9} = \frac{55}{99} = \cdot 5\frac{55}{99} = \cdot 55\frac{55}{99} = - - - - - - - - - - - - -,$

i. e. $\cdot\dot{5} = \cdot\dot{5}\dot{5} = \cdot 5\dot{5}\dot{5} = \cdot 55\dot{5}\dot{5} = - - - - - - - - - - - - - .$

Example 2. Prove that $\cdot\dot{2}3\dot{7} = \cdot 2\dot{3}7\dot{2} = \cdot 23\dot{7}2\dot{3} = 237\dot{2}3\dot{7} = \cdot\dot{2}3723\dot{7}$.

$\frac{237}{999} = \cdot 2\frac{372}{999} = \cdot 23\frac{723}{999} = \cdot 237\frac{237}{999} = \cdot 2372\frac{372}{999} = - - - - - - - - - - -,$

i. e., $\cdot\dot{2}3\dot{7} = \cdot 2\dot{3}7\dot{2} = \cdot 23\dot{7}2\dot{3} = \cdot 237\dot{2}3\dot{7} = \cdot 2372\dot{3}7\dot{2} = - - - - - - - - - - -,$

Also, $\frac{237}{999} = \frac{237237}{999999} = \cdot 2\frac{372372}{999999} = \cdot 23\frac{723723}{999999} = - - = \frac{237237237}{999999999} = - -,$

i. e., $\cdot\dot{2}3\dot{7} = \cdot\dot{2}3723\dot{7} = \cdot 2\dot{3}7237\dot{2} = \cdot 23\dot{7}2372\dot{3} = - - = \cdot\dot{2}3723723\dot{7} = - - .$

These two examples exhibit a property of fractions whose denominators are expressed by 9's only or by one or more 9's followed by one or more 0's, which has led to this class of fractions receiving the name of **repeating** or **circulating decimals.** If the circle of recurring figures includes all the figures to the right of the decimal point, the fraction is termed *a pure circulating decimal. Examples;* $\cdot\dot{7}\dot{4}$, $\cdot\dot{8}5\dot{3}$, $14\cdot\dot{3}25\dot{7}$. If there are one or more figures between the decimal point and the circle of recurring figures, the fraction is called *a mixed circulating decimal. Examples;* $\cdot 5\dot{7}\dot{4}$, $\cdot 85\dot{3}$, $14\cdot 3\dot{2}5\dot{7}$, $3\cdot 00\dot{2}$.

Example 3. Express $\frac{7}{11}$ in decimal notation.

$\frac{7}{11} = \cdot 6\frac{4}{11} = \cdot 63\frac{7}{11},$

$\therefore \frac{7}{11} \times 100 = 63\frac{7}{11},$

$\therefore \frac{7}{11} \times 99 = 63,$

$\therefore \frac{7}{11} = \frac{63}{99} = \cdot\dot{6}\dot{3}.$

Work.

```
11|7·00
  | 47
  -----
   ·63
```
(with dots over 6 and 3 of ·63)

Example 4. Express $\frac{13}{37}$ in decimal notation.

$\frac{13}{37} = \cdot 3\frac{19}{37} = \cdot 35\frac{5}{37} = \cdot 351\frac{13}{37},$

$\therefore \frac{13}{37} \times 1000 = 351\frac{13}{37},$

$\therefore \frac{13}{37} \times 999 = 351,$

$\therefore \frac{13}{37} = \frac{351}{999} = \cdot\dot{3}5\dot{1}.$

Work.

```
37|13·000
  | 1953
  |    1
  -------
  | ·351
```
(with dots over 3 and 1 of ·351)

The work of division in problems such as *Examples 3* and *4* is to be continued until a remainder occurs which is the same as either the original dividend or a preceding remainder ; if the division be carried beyond the second of these equal remainders the quotient-figures from the former remainder to the latter will all recur in the same order thus showing that they form a 'circle.' As the remainders must all be less than the divisor, the number of different remainders and therefore the number of figures in the circle cannot exceed $n-1$, in which n is the divisor. *i.e.*, the denominator of the fraction to be expressed in decimal notation.

Example 5. Express $\frac{47}{56}$ in decimal notation.

Work.

$$\begin{array}{r|l} 56 & 47\cdot000000000 \\ & \;\;2\;226820846 \\ & \quad 51434021 \\ \hline & \cdot839\dot{2}8571\dot{4}=\frac{47}{56} \end{array}$$

Explanation. $\frac{47}{56}=\cdot839\frac{16}{56}=\cdot839285714\frac{16}{56}$

$$\therefore\quad \frac{47}{56}\times10^9=839\tfrac{16}{56}\times1000000=839285714\tfrac{16}{56}$$

$$\therefore\quad 839\tfrac{16}{56}\times999999=839285714-839$$

$$\therefore\quad 839\tfrac{16}{56}=\frac{839285714-839}{999999}=839\tfrac{285714}{999999}=839\cdot\dot{2}8571\dot{4}$$

$$\therefore\quad \cdot839\tfrac{16}{56}=\cdot839\dot{2}8571\dot{4},$$

$$\therefore\quad \frac{47}{56}=\cdot839\dot{2}8571\dot{4}.$$

TABLES

OF

LOGARITHMS OF NUMBERS.

TABLE I.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1.778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806180	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

No	0	1	2	3	4	5	6	7	8	9	D.
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102	8600	9026	9451	9876	*0300	*0724	*1147	*1 70	*1993	*2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	*0361	*0775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*0195	*0600	*1004	*1408	*1812	*2216	*2619	*3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*0207	*0602	*0998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	0105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	*0380	*0766	*1153	*1538	*1924	*2309	*2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	0320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	*0038	*0407	*0776	*1145	*1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	9181	9543	9904	*0266	*0626	*0987	*1347	*1707	*2067	*2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	*0258	*0611	*0963	*1315	*1667	*2018	*2370	*2721	*3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*0026	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*0253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	3943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*0245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*0012	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	*0194	*0508	*0822	*1136	*1450	*1763	*2076	*2389	*2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
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141	9219	9527	9835	*0142	*0449	*0756	*1063	*1370	*1676	*1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*0168	*0469	*0769	*1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	*0126	*0413	*0699	*0985	*1272	*1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	*0051	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	*0029	*0303	*0577	*0850	*1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	4120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	*0051	*0319	*0586	*0853	*1121	*1388	*1654	*1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*0193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	*0050	*0300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*0176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
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181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	*0213	*0446	*0679	*0912	*1144	*1377	*1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	8754	8982	9211	9439	9667	9895	*0123	*0351	*0578	*0806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	*0161	*0378	*0595	*0813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	*0056	*0268	*0481	*0693	*0906	*1118	*1330	*1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	2219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	*0008	*0211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5453	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	*0047	*0246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
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220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	5549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	*0025	*0215	*0404	*0593	*0783	*0972	*1161	*1350	*1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	*0143	*0328	*0513	*0698	*0883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	*0030	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	*0051	*0228	*0405	*0582	*0759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	7940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	*0020	*0192	*0365	*0538	*0711	*0883	*1056	*1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	*0102	*0271	*0440	*0609	*0777	*0946	*1114	*1283	*1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
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261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	6129	9295	9460	9625	9791	165
263	9956	*0121	*0286	*0451	*0616	*0781	*0945	*1110	*1275	*1439	165
264	421604	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	*0075	*0236	*0398	*0559	*0720	*0881	*1042	*1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7116	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	*0122	*0279	*0437	*0594	*0752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	7158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	*0095	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	153
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	*0146	*0296	*0447	*0597	*0743	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	2398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	*0116	*0263	*0410	*0557	*0704	*0851	*0998	*1145	147
296	471292	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
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302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	*0099	*0239	*0380	*0520	*0661	*0801	*0941	*1081	*1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4294	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	*0099	*0236	*0374	*0511	*0648	*0785	*0922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	5150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	*0009	*0143	*0277	*0411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8382	132
330	8514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	*0090	*0221	*0353	*0484	*0615	*0745	*0876	*1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8783	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	*0072	128
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
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341	2754	2882	3009	3136	3204	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4034	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9462	9578	9703	9829	9954	*0079	*0204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2820	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
350	4068	4192	4316	4440	4564	4688	4812	4936	5060	5183	124
351	5307	5431	5555	5078	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	0106	123
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	6303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	*0026	*0146	*0265	*0385	*0504	*0624	*0743	*0863	*0982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3950	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5848	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	8202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	*0076	*0193	*0309	*0426	117
372	570543	0660	0776	0893	1010	1126	1243	1309	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3402	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5600	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8639	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
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381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
394	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
286	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	**61	*0173	*0284	*0396	*0507	*0619	*0730	*0842	*0953	112
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	*0101	*0210	*0319	*0428	*0537	*0646	*0755	*0864	109
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
400	2060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	**21	*0128	*0234	*0341	*0447	*0554	107
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	2784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4792	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	0032	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
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422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	**21	*0123	*0224	*0326	102
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2366	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	3468	3569	3670	3771	3872	3973	4074	4175	4276	4376	100
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	99
435	8489	8589	8639	8789	8888	8988	9088	9188	9287	9387	99
436	9486	9586	9686	9785	9885	9984	**84	*0183	*0283	*0382	99
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	3453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	**16	*0113	*0210	97
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
450	3213	3309	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	**11	*0106	*0201	*0296	*0391	*0486	*0581	*0676	*0771	95
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
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460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5581	5675	5769	5862	5956	6050	6143	6237	6331	6424	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	**60	*0153	93
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1266	1358	1451	1543	1636	1728	1821	1913	2005	93
470	2098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	**63	*0154	*0245	91
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	1241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	**19	*0107	89
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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501	9838	9924	**11	**98	*0184	*0271	*0358	*0444	*0531	*0617	87
502	700704	0790	0877	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3893	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7050	7144	7229	7315	7400	7485	85
510	7570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	0033	85
513	710117	0202	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2565	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4330	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	6003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	**77	83
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	4276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	**55	*0136	*0217	*0298	*0378	*0459	*0540	*0621	*0702	81
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
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541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	**47	*0126	*0205	*0284	79
550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2647	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	8188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	**45	*0123	*0200	*0277	*0354	*0431	77
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	5875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	9668	9743	9819	9894	9970	**45	*0121	*0196	*0272	*0347	75
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75

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581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	0042	74
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	0852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
600	8151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	*0029	*0101	*0173	*0245	72
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	5330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	***4	**74	*0144	*0215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0988	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
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621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	9341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	6180	6248	6316	6384	6451	6519	6587	6655	6723	6790	89
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	89
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	**31	**98	*0165	67
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
650	2913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7806	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
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661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	6075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7365	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8333	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	**11	**75	*0139	*0204	*0268	*0332	*0396	*0460	*0525	64
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	2509	2573	2637	2700	2764	2828	2892	2956	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	6881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7588	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	8849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	**43	63
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
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701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7943	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	1258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2175	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	7332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
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724	9739	9799	9859	9918	9978	**38	**98	*0158	*0218	*0278	60
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	3323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
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742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
750	5061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9898	9956	**13	**70	*0127	*0185	57
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
760	0814	0871	0928	0985	1042	1099	1156	1213	1271	1328	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	6491	6547	6604	6660	6716	6773	6829	6885	6942	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9134	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	**30	**86	*0141	*0197	*0253	*0309	*0365	56
777	890421	0477	0533	0589	0645	0700	0756	0812	0868	0924	56
778	0980	1035	1091	1147	1203	1259	1314	1370	1426	1482	56
779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
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781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	7627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	**39	**94	*0149	*0203	*0258	*0312	55
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	3090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	8485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	**37	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
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821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	9078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	**19	**71	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	4279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	9419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	**32	**83	*0134	*0185	*0236	*0287	*0338	*0389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
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861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	9519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	4483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	9390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	**24	**73	*0121	*0170	*0219	*0267	*0316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
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901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	9041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9864	9852	9900	9947	48
912	9995	**42	**90	*0138	*0185	*0233	*0280	*0328	*0376	*0423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	3788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	8483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	**21	**68	*0114	*0161	*0207	*0254	*0300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1481	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1975	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2338	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
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942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	7724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
957	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	2271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	6772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	**28	**72	*0117	*0161	*0206	*0250	*0294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
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981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	5635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7385	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
1000	000000	0043	0087	0130	0174	0217	0260	0304	0347	0391	43
1001	0434	0477	0521	0564	0608	0651	0694	0738	0781	0824	43
1002	0868	0911	0954	0998	1041	1084	1128	1171	1214	1258	43
1003	1301	1344	1388	1431	1474	1517	1561	1604	1647	1690	43
1004	1734	1777	1820	1863	1907	1950	1993	2036	2080	2123	43
1005	2166	2209	2252	2296	2339	2382	2425	2468	2512	2555	43
1006	2598	2641	2684	2727	2771	2814	2857	2900	2943	2986	43
1007	3029	3073	3116	3159	3202	3245	3288	3331	3374	3417	43
1008	3461	3504	3547	3590	3633	3676	3719	3762	3805	3848	43
1009	3891	3934	3977	4020	4063	4106	4149	4192	4235	4278	43
1010	4321	4364	4407	4450	4493	4536	4579	4622	4665	4708	43
1011	4751	4794	4837	4880	4923	4966	5009	5052	5095	5138	43
1012	5181	5223	5266	5309	5352	5395	5438	5481	5524	5567	43
1013	5609	5652	5695	5738	5781	5824	5867	5909	5952	5995	43
1014	6038	6081	6124	6166	6209	6252	6295	6338	6380	6423	43
1015	6466	6509	6552	6594	6637	6680	6723	6765	6808	6851	43
1016	6894	6936	6979	7022	7065	7107	7150	7193	7236	7278	43
1017	7321	7364	7406	7449	7492	7534	7577	7620	7662	7705	43
1018	7748	7790	7833	7876	7918	7961	8004	8046	8089	8132	43
1019	8174	8217	8259	8302	8345	8387	8430	8472	8515	8558	43
No	0	1	2	3	4	5	6	7	8	9	D.

No	0	1	2	3	4	5	6	7	8	9	D.
1020	008600	8643	8685	8728	8770	8813	8850	8898	8941	8983	43
1021	9026	9068	9111	9153	9196	9238	9281	9323	9366	9408	42
1022	9451	9493	9536	9578	9621	9663	9706	9748	9791	9833	42
1023	9876	9918	9961	*0003	*0045	*0088	*0130	*0173	*0215	*0258	42
1024	010300	0342	0385	0427	0470	0512	0554	0597	0639	0681	42
1025	0724	0766	0809	0851	0893	0936	0978	1020	1063	1105	42
1026	1147	1190	1232	1274	1317	1359	1401	1444	1486	1528	42
1027	1570	1613	1655	1697	1740	1782	1824	1866	1909	1951	42
1028	1993	2035	2078	2120	2162	2204	2247	2289	2331	2373	42
1029	2415	2458	2500	2542	2584	2626	2669	2711	2753	2795	42
1030	2837	2879	2922	2964	3006	3048	3090	3132	3174	3217	42
1031	3259	3301	3343	3385	3427	3469	3511	3553	3596	3638	42
1032	3680	3722	3764	3806	3848	3890	3932	3974	4016	4058	42
1033	4100	4142	4184	4226	4268	4310	4353	4395	4437	4479	42
1034	4521	4563	4605	4647	4689	4730	4772	4814	4856	4898	42
1035	4940	4982	5024	5066	5108	5150	5192	5234	5276	5318	42
1036	5360	5402	5444	5485	5527	5569	5611	5653	5695	5737	42
1037	5779	5821	5863	5904	5946	5988	6030	6072	6114	6156	42
1038	6197	6239	6281	6323	6365	6407	6448	6490	6532	6574	42
1039	6616	6657	6699	6741	6783	6824	6866	6908	6950	6992	42
1040	7033	7075	7117	7159	7200	7242	7284	7326	7367	7409	42
1041	7451	7492	7534	7576	7618	7659	7701	7743	7784	2826	42
1042	7868	7909	7951	7993	8034	8076	8118	8159	8201	8243	42
1043	8284	8326	8368	8409	8451	8492	8534	8576	8617	8659	42
1044	8700	8742	8784	8825	8867	8908	8950	8992	9033	9075	42
1045	9116	9158	9199	9241	9282	9324	9366	9407	9449	9490	42
1046	9532	9573	9615	9656	9698	9739	9781	9822	9864	9905	42
1047	9947	9988	*0030	*0071	*0113	*0154	*0195	*0237	*0278	*0320	41
1048	020361	0403	0444	0486	0527	0568	0610	0651	0693	0734	41
1049	0775	0817	0858	0900	0941	0982	1024	1065	1107	1148	41
1050	1189	1231	1272	1313	1355	1396	1437	1479	1520	1561	41
1051	1603	1644	1685	1727	1768	1809	1851	1892	1933	1974	41
1052	2016	2057	2098	2140	2181	2222	2263	2305	2346	2387	41
1053	2428	2470	2511	2552	2593	2635	2676	2717	2758	2799	41
1054	2841	2882	2923	2964	3005	3047	3088	3129	3170	3211	41
1055	3252	3294	3335	3376	3417	3458	3499	3541	3582	3623	41
1056	3664	3705	3746	3787	3828	3870	3911	3952	3993	4034	41
1057	4075	4116	4157	4198	4239	4280	4321	4363	4404	4445	41
1058	4486	4527	4568	4609	4650	4691	4732	4773	4814	4855	41
1059	4896	4937	4978	5019	5060	5101	5142	5183	5224	5265	41
No	0	1	2	3	4	5	6	7	8	9	D.

No	0	1	2	3	4	5	6	7	8	9	D.
1060	025306	5347	5388	5429	5470	5511	5552	5593	5634	5674	41
1061	5715	5756	5797	5838	5879	5920	5961	6002	6043	6084	41
1062	6125	6165	6206	6247	6288	6329	6370	6411	6452	6492	41
1063	6533	6574	6615	6656	6697	6737	6778	6819	6860	6901	41
1064	6942	6982	7023	7064	7105	7146	7186	7227	7268	7309	41
1065	7350	7390	7431	7472	7513	7553	7594	7635	7676	7716	41
1066	7757	7798	7839	7879	7920	7961	8002	8042	8083	8124	41
1067	8164	8205	8246	8287	8327	8368	8409	8449	8490	8531	41
1068	8571	8612	8653	8693	8734	8775	8815	8856	8896	8937	41
1069	8978	9018	9059	9100	9140	9181	9221	9262	9303	9343	41
1070	9384	9424	9465	9506	9546	9587	9627	9668	9708	9749	41
1071	9789	9830	9871	9911	9952	9992	*0033	*0073	*0114	*0154	41
1072	030195	0235	0276	0316	0357	0397	0438	0478	0519	0559	40
1073	0600	0640	0681	0721	0762	0802	0843	0883	0923	0964	40
1074	1004	1045	1085	1126	1166	1206	1247	1287	1328	1368	40
1075	1408	1449	1489	1530	1570	1610	1651	1691	1732	1772	40
1076	1812	1853	1893	1933	1974	2014	2054	2095	2135	2175	40
1077	2216	2256	2296	2337	2377	2417	2458	2498	2538	2578	40
1078	2619	2659	2699	2740	2780	2820	2860	2901	2941	2981	40
1079	3021	3062	3102	3142	3182	3223	3263	3303	3343	3384	40
1080	3424	3464	3504	3544	3585	3625	3665	3705	3745	3786	40
1081	3826	3866	3906	3946	3986	4027	4067	4107	4147	4187	40
1082	4227	4267	4308	4348	4388	4428	4468	4508	4548	4588	40
1083	4628	4669	4709	4749	4789	4829	4869	4909	4949	4989	40
1084	5029	5069	5109	5149	5190	5230	5270	5310	5350	5390	40
1085	5430	5470	5510	5550	5590	5630	5670	5710	5750	5790	40
1086	5830	5870	5910	5950	5990	6030	6070	6110	6150	6190	40
1087	6230	6269	6309	6349	6389	6429	6469	6509	6549	6589	40
1088	6629	6669	6709	6749	6789	6828	6868	6908	6948	6988	40
1089	7028	7068	7108	7148	7187	7227	7267	7307	7347	7387	40
1090	7426	7466	7506	7546	7586	7626	7665	7705	7745	7785	40
1091	7825	7865	7904	7944	7984	8024	8064	8103	8143	8183	40
1092	8223	8262	8302	8342	8382	8421	8461	8501	8541	8580	40
1093	8620	8660	8700	8739	8779	8819	8859	8898	8938	8978	40
1094	9017	9057	9097	9136	9176	9216	9255	9295	9335	9374	40
1095	9414	9454	9493	9533	9573	9612	9652	9692	9731	9771	40
1096	9811	9850	9890	9929	9969	*0009	*0048	*0088	*0127	*0167	40
1097	040207	0246	0286	0325	0365	0405	0444	0484	0523	0563	40
1098	0602	0642	0681	0721	0761	0800	0840	0879	0919	0958	40
1099	0998	1037	1077	1116	1156	1195	1235	1274	1314	1353	39
No	0	1	2	3	4	5	6	7	8	9	D.

ANSWERS.

(The answers of Exercises I, II and IV are due to Mr. Thomas McJanet of Ottawa; those of Exercises XXXII to XLVI, to Mr. Thomas Kirkconnell, Mathematical Master of Port Hope High School; the latter gentleman also tested the answers of Exercises I and III.)

Exercise I. **1.** 222. **2.** 3625. **3.** 2222 sq. yd. 2 sq. ft. **4.** (i), 9′ 11″; (ii), (*a*) 8′ $2\frac{29}{37}$″, (*b*) 8′ $6\frac{1}{37}$″. **5.** $7\frac{3}{11}$ mi. per hr.; $8\frac{1}{4}$ min. per mile. **6.** (i), $\frac{2}{7}$; (ii), $\frac{8}{105}$. **7.** $21\frac{7}{16}$ gal. **8.** $67\frac{29}{45}$ da. **9.** 36 ct. **10.** \$20·46. **12.** 1 A. 361 sq. yd. 7 sq. ft. **13.** \$388·23. **14.** 5′ 1″. **15.** $213\frac{47}{240}$. mi. **16.** (i), $170\frac{1}{2}$ gal.; (ii), $46\frac{1}{2}$ gal. **17.** 229·6 lb.; 33·25 c. ft. **18.** 25 da. **19.** \$1080. **20.** \$6·58. **21.** \$70; \$30800. **22.** 968·7627 sq. in. **23.** 6′ 4″. **24.** (i), 1:$28\frac{3}{29}$ p.m.; (ii), $14\frac{77}{377}$ min. **25.** 15 years. **26.** $\frac{47}{420}$; \$1974. **27.** 0·4. **28.** *B* in $37\frac{1}{2}$ da., *C* in 25 da. **29.** 35%. **30.** \$89·20. **31.** (i), 535·90 Km.; (ii), 62·31 Km.; (iii), 142·17 Km.; (iv), 122·15 Km.; (v), 52·59 Km.; (vi), 720·36 Km.; (vii), 730·66 Km. **32.** $684\frac{4}{9}$ sq. ft. **33.** $10\frac{5}{16}$ min. **34.** (i), 2·2545 sq. ft.; (ii), 2·004 sq. ft.; (iii), 1·6908 sq. ft.; (iv), 1·4428 sq. ft; (v), 1·2525 sq. ft.; (vi), 0·8906 sq. ft.; (vii), 0·7452 sq. ft. **35.** $34\frac{6}{181}$ yd. **36.** \$112·29. **37.** \$9·60. **38.** $19\frac{13}{23}$ yd. **39.** (*a*), 36 yr.; (*b*), 60 yr.; (*c*), 36 yr. **40.** $4\frac{1}{2}$ %. **41.** 60·417 and 425·425. **42.** (i), \$20·67; (ii), \$18·84. **43.** 10 sq. ft. $92\frac{1}{7}$ sq. in. **44.** 22·86 gal. **45.** 5 times per 2 sec. **46.** \$2700. **47.** 66 yd. per min. **48.** \$1·14. **49.** 64 yr. **50.** \$742·38. **51.** $\frac{73}{127}$. **52.** $3\frac{2}{11}$ A. **53.** 1 hr. 15 min. **54.** $37\frac{1}{17}$ mi. per hr. **55.** $\frac{1}{60}$; \$92·40. **56.** (i), 15 sec.; (ii), $9\frac{3}{8}$ sec.; (iii), $37\frac{1}{2}$ sec. **57.** \$500. **58.** $9\frac{1}{11}$%. **59.** $6\frac{1}{2}$%. **60.** (*i*), $4\frac{16}{21}$%; (ii), \$525. **61.** *A*. \$21, *B*, \$16·80. **62.** $705301\frac{29}{66}$ cubic miles. **63.** 0·000545 in. **64.** $29\frac{17}{77}$ mi. per hour. **65.** 120 subscribers, \$253.50. **66.** 20 mi. per hour. **67.** \$34·05. **68.** *A*'s \$5800;

B's $4600 ; 0·7931. **69.** 200 da. **70.** $1·00. **73.** Feb'y 4th, paid $20·88. **74.** Between 33 and 37 miles per hour. Between 34 and 36 miles per hour. **75.** (*a*) $1\frac{4}{7}$. (*b*) $\frac{11}{12}$. **76.** i, 12 ; ii, 8 ; iii, 3. 168 marbles. **77.** 19 mi. per hour. **78.** $335·05 ; 11·04 %. **79.** 255 days. **80.** $8·00. **82.** 40 times. **83.** $6227\frac{11}{36}$ c. ft. ; 502 lb. $4427\frac{23}{36}$ gr. **84.** 323 yd. **85.** Nov. 28th, paid $16·24. **86.** 120 acres. **87.** (*a*) $1\frac{4}{7}$. (*b*) $2\frac{1}{5}$. **88.** *A*, by $7\frac{1}{3}$ yd. **89.** $432·34. **90.** $7750·51. **91.** 2880 revolutions, 64 and 15 revolutions. **92.** (*a*) 24·27 c. in., (*b*) 38·92 c. in., (*c*) 100·465 c. in., (*d*) 154·286 c. in., (*e*) 320 c. in., (*f*) 389·2 c. in. **93.** 164985 lb. $13\frac{115}{128}$ oz. **94.** $372·55. **95.** (i), *B*, by $24\frac{4}{9}$ yd., (ii), *B*, by $47\frac{21}{37}$ yd. **96.** 4 hr. **97.** (*a*) $5\frac{1}{2}$. (*b*) $2\frac{1}{5}$. **98.** 450 yd. **99.** $115·68. **100.** $28\frac{4}{7}$ %.

Exercise II. **1.** $1\ \ \frac{2}{3}\ \ \frac{9}{13}\ \ \frac{20}{29}$. **2.** $\frac{1}{2}\ \ \frac{1}{3}\ \ \frac{4}{11}\ \ \frac{5}{14}\ \ \frac{14}{39}\ \ \frac{117}{326}$. **3.** $1\ \ \frac{2}{3}\ \ \frac{9}{13}\ \ \frac{20}{29}\ \ \frac{29}{42}\ \ \frac{252}{365}$. **4.** $\frac{5}{2}\ \ \frac{17}{7}\ \ \frac{90}{37}$. **5.** $2\ \ \frac{11}{6}\ \ \frac{35}{19}\ \ \frac{431}{234}\ \ \frac{466}{253}\ \ \frac{1363}{740}$. **6.** $\frac{1}{8}\ \ \frac{3}{25}\ \ \frac{25}{108}\ \ \frac{153}{1273}\ \ \frac{1096}{9119}$. **7.** $1\ \ \frac{2}{3}\ \ \frac{7}{10}\ \ \frac{9}{13}\ \ \frac{16}{23}\ \ \frac{249}{358}\ \ \frac{265}{381}\ \ \frac{779}{1120}\ \ \frac{1044}{1501}\ \ \frac{9131}{13128}$. **8.** $\frac{1}{5}\ \ \frac{2}{11}\ \ \frac{3}{16}\ \ \frac{26}{139}\ \ \frac{29}{155}\ \ \frac{84}{449}\ \ \frac{2801}{14972}$. **9.** $\frac{1}{2}\ \ \frac{3}{7}\ \ \frac{34}{79}\ \ \frac{37}{86}\ \ \frac{71}{165}\ \ \frac{250}{581}\ \ \frac{1321}{3070}\ \ \frac{18744}{43561}\ \ \frac{20065}{46631}\ \ \frac{78939}{183454}\ \ \frac{493699}{1147355}$. **10.** $\frac{4}{3}\ \ \frac{9}{7}\ \ \frac{13}{10}\ \ \frac{490}{377}\ \ \frac{503}{387}\ \ \frac{1496}{1151}\ \ \frac{7983}{6142}\ \ \frac{488459}{375813}$. **11.** $\frac{3}{2}\ \ \frac{7}{5}\ \ \frac{17}{12}\ \ \frac{41}{29}\ \ \frac{99}{70}\ \ \frac{140}{99}\ \ \frac{239}{169}\ \ \frac{7071}{5000}$. **12.** $2\ \ \frac{5}{3}\ \ \frac{7}{4}\ \ \frac{19}{11}\ \ \frac{26}{15}\ \ \frac{71}{41}\ \ \frac{97}{56}\ \ \frac{265}{153}\ \ \frac{362}{109}\ \ \frac{989}{571}\ \ \frac{2340}{1351}\ \ \frac{3329}{1922}\ \ \frac{15656}{9039}\ \ \frac{34641}{20000}$. **13.** $\frac{5}{2}\ \ \frac{22}{9}\ \ \frac{49}{20}\ \ \frac{218}{89}\ \ \frac{485}{198}\ \ \frac{2158}{881}\ \ \frac{2643}{1079}\ \ \frac{7444}{3039}\ \ \frac{17531}{7157}\ \ \frac{24975}{10196}\ \ \frac{42506}{17353}\ \ \frac{67481}{27549}\ \ \frac{244949}{100000}$. **14.** $\frac{1}{2}\ \ \frac{3}{7}\ \ \frac{7}{16}\ \ \frac{10}{23}\ \ \frac{17}{39}\ \ \frac{1489}{3416}\ \ \frac{1506}{3455}\ \ \frac{7513}{17236}\ \ \frac{9019}{20691}\ \ \frac{43589}{100000}$. **15.** $1\ \ \frac{1}{2}\ \ \frac{4}{7}\ \ \frac{5}{9}\ \ \frac{29}{52}\ \ \frac{34}{61}\ \ \frac{131}{235}\ \ \frac{296}{531}\ \ \frac{723}{1297}\ \ \frac{1742}{3125}$. **16.** $\frac{1}{20}\ \ \frac{20}{401}\ \ \frac{421}{8441}\ \ \frac{441}{8842}\ \ \frac{862}{17283}\ \ \frac{9061}{181672}\ \ \frac{9923}{198955}\ \ \frac{18984}{380627}\ \ \frac{28907}{579582}\ \ \frac{47891}{960209}\ \ \frac{124689}{2500000}$. **17.** $\frac{1}{4}\ \ \frac{9}{37}\ \ \frac{10}{41}\ \ \frac{2439}{10000}$. **18.** $\frac{3}{2}\ \ \frac{7}{5}\ \ \frac{10}{7}\ \ \frac{17}{12}\ \ \frac{61}{43}\ \ \frac{200}{141}\ \ \frac{35461}{25000}$. **19.** $\frac{3}{1}\ \ \frac{8}{3}\ \ \frac{11}{4}\ \ \frac{19}{7}\ \ \frac{87}{32}\ \ \frac{106}{39}\ \ \frac{193}{71}\ \ \frac{1264}{465}\ \ \frac{12833}{4721}\ \ \frac{14097}{5186}\ \ \frac{26930}{9907}\ \ \frac{67957}{25000}$. **20.** $\frac{7}{3}\ \ \frac{23}{10}\ \ \frac{76}{33}\ \ \frac{99}{43}\ \ \frac{175}{76}\ \ \frac{624}{271}\ \ \frac{3919}{1702}\ \ \frac{4543}{1973}\ \ \frac{8462}{3675}\ \ \frac{13005}{5648}\ \ \frac{21467}{9323}\ \ \frac{34472}{14971}\ \ \frac{55939}{24294}\ \ \frac{202289}{87853}\ \ \frac{460517}{200000}$. **21.** $1\ \ \frac{1}{2}\ \ \frac{2}{3}\ \ \frac{3}{5}\ \ \frac{5}{8}\ \ \frac{18}{29}\ \ \frac{23}{37}\ \ \frac{64}{103}\ \ \frac{343}{552}\ \ \frac{25103}{40399}\ \ \frac{50549}{81350}\ \ \frac{682240}{1097949}\ \ \frac{732789}{1179299}\ \ \frac{9475708}{15249537}\ \ \frac{10208497}{16428836}\ \ \frac{19684205}{31678373}\ \ \frac{29892702}{48107209}\ \ \frac{109362311}{176000000}$. **22.** $\frac{5}{2}\ \ \frac{42}{17}\ \ \frac{257}{104}\ \ \frac{299}{121}\ \ \frac{855}{346}\ \ \frac{56729}{22957}\ \ \frac{171042}{69217}\ \ \frac{227771}{92174}\ \ \frac{1082126}{437913}\ \ \frac{2392023}{968000}$. **23.** $\frac{9}{4}\ \ \frac{11}{5}\ \ \frac{86}{39}\ \ \frac{97}{44}\ \ \frac{571}{259}\ \ \frac{668}{303}\ \ \frac{8587}{3895}\ \ \frac{9255}{4198}\ \ \frac{1461622}{662981}\ \ \frac{1470877}{667179}\ \ \frac{2932499}{1330160}\ \ \frac{7335875}{3327499}\ \ \frac{10268374}{4657659}\ \ \frac{38140997}{17300476}\ \ \frac{48409371}{21958135}\ \ \frac{134959739}{61216746}\ \ \frac{318328849}{144391627}\ \ \frac{771617437}{350000000}$. **24.** $\frac{10}{9}\ \ \frac{11}{10}\ \ \frac{43}{39}\ \ \frac{97}{88}\ \ \frac{237}{215}\ \ \frac{334}{303}\ \ \frac{8921}{8093}\ \ \frac{9255}{8396}\ \ \frac{730811}{662981}\ \ \frac{2201688}{1997339}\ \ \frac{5134187}{4657659}\ \ \frac{43275184}{39258611}\ \ \frac{91684555}{83174881}\ \ \frac{226644294}{205608373}\ \ \frac{771617437}{700000000}$. **25.** $\frac{4}{3}\ \ \frac{17}{13}\ \ \frac{327}{250}\ \ \frac{344}{263}\ \ \frac{4799}{3669}\ \ \frac{48334}{36953}\ \ \frac{198135}{151481}$; $\frac{2227819}{1703244}\ \ \frac{15792868}{12074189}\ \ \frac{65399291}{50000000}$. **26.** $1\ \ \frac{7}{8}\ \ \frac{15}{17}\ \ \frac{22}{25}\ \ \frac{37}{42}\ \ \frac{207}{235}\ \ \frac{451}{512}\ \ \frac{658}{747}\ \ \frac{1109}{1259}\ \ \frac{10639}{12078}\ \ \frac{33026}{37493}\ \ \frac{175769}{199543}\ \ \frac{208795}{237036}$

$\frac{1219744}{1384723}$ $\frac{1428539}{1621759}$ $\frac{5505361}{6250000}$. **27.** $\frac{3403}{3402}$ $\frac{3404}{3403}$ $\frac{17019}{17014}$ $\frac{224651}{224585}$ $\frac{466321}{466184}$ $\frac{1157293}{1156953}$ $\frac{3938200}{3937043}$. **28.** 1 $\frac{289}{290}$ $\frac{8093}{8121}$ $\frac{24568}{24653}$ $\frac{81797}{82080}$ $\frac{106365}{106733}$ $\frac{294527}{295546}$ $\frac{400892}{402279}$ $\frac{1096311}{1100104}$ $\frac{1497203}{1502383}$ $\frac{4090717}{4104870}$ $\frac{5587920}{5607253}$ $\frac{20854477}{20926629}$. **29.** 2 $\frac{21}{11}$ $\frac{23}{12}$ $\frac{113}{59}$ $\frac{136}{71}$ $\frac{793}{414}$. **30.** $\frac{1}{4}$ $\frac{7}{29}$ $\frac{8}{33}$ $\frac{31}{128}$ $\frac{39}{161}$ $\frac{70}{289}$ $\frac{529}{2184}$ $\frac{599}{2473}$ $\frac{2326}{9603}$ $\frac{12229}{50488}$ $\frac{39013}{161067}$ $\frac{90255}{372622}$ $\frac{129268}{533689}$ $\frac{348791}{1440000}$.

Exercise III. **1.** 3101·7414. **2.** 67·0509. **3.** 1503·543. **4.** 3·20424. **5.** 31. **6.** 1·301030. **7.** 2·477121. **8.** 3. **9.** 2·1556589. **10.** 2. **11.** 1·0019656. **12.** 99999970. **13.** 4856. **14.** 116·1. **15.** 6770. **16.** 67·7. **17.** 70·7107. **18.** 795·775. **19.** 0·0264575. **20.** 0·013964. **21.** 0·301030. **22.** 0·477121. **23.** 0·318310. **24.** 2·30259. **25.** 2·22398. **26.** 16·5304. **27.** 1·39642. **28.** 2·7183. **29.** 0·36788. **30.** 1·4107. **31.** 2. **32** 1·5. **33.** 1·25. **35.** 8000 m. **36.** 1925 yd. **37.** 4047 centiares. **38.** 5789 sq. yd. **39.** 40·47 hectares. **40.** 247·1 A. **41.** 132·1 gal. **42.** 599·4 litres. **43.** 384,300 Km. **44.** 147,100,000 Km. **45.** $\frac{9}{1}$, $\frac{10}{1}$, $\frac{19}{2}$, $\frac{105}{11}$, $\frac{124}{13}$. **46.** $\frac{11}{1}$, $\frac{12}{1}$, $\frac{35}{3}$, $\frac{47}{4}$, $\frac{223}{19}$, $\frac{716}{61}$, $\frac{1655}{141}$. 18·03 yr. **47.** $\frac{1}{12}$, $\frac{2}{25}$, $\frac{3}{37}$, $\frac{8}{99}$, $\frac{19}{235}$, $\frac{160}{1979}$. **48.** $\frac{1}{1}$, $\frac{2}{1}$, $\frac{15}{8}$, $\frac{32}{17}$, $\frac{47}{25}$, $\frac{79}{42}$, $\frac{284}{151}$. **49.** $\frac{1}{2}$, $\frac{2}{5}$, $\frac{5}{12}$, $\frac{12}{22}$, $\frac{17}{41}$. **50.** $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{2}$. $\frac{5}{3}$, $\frac{8}{5}$, $\frac{37}{23}$, $\frac{45}{28}$.

Exercise IV. **1.** 30·197 in. **2.** 1000·796875 lb. 16·051 c. ft. **3.** 3 min. $25\frac{1}{3}$ sec. ; 3 min. 25 $\frac{74215}{270077}$ sec. **4.** 1100 bu. **5.** $\frac{1}{24}$ in. **6.** 78 % of copper, 22 % of zinc. **7.** 0·2532 %. **8.** \$9000·49, \$123·09. **9.** 2nd Nov., 1889. **10.** 8 %. **11.** (i), $9\frac{5}{6}$ min. ; (ii), $8\frac{2}{3}$ min. **12.** 2·89 times. **13.** 120·4264 lb. **14.** 209 da. $\frac{209}{365}$; $\frac{209}{366}$. **15.** Friday at 3,15 a.m. ; 3,20 a.m. 9,05 $\frac{730}{1081}$ a.m. **16.** \$5·40. **17.** $\frac{1}{6}$ qt. ; $\frac{1}{7}$. **18.** $6\frac{6932}{14967}$ %. \$514·93. **19.** 2nd Ap., 1889. **20.** 7%. **21.** $23\frac{4}{7}$ ft. per sec. ; $471\frac{3}{7}$ yd. per min. ; $16\frac{1}{14}$ mi. per hr. **22.** 3762 revolutions ; 7 mi. $160\frac{2}{7}$ yd. ; $21\frac{153}{560}$ mi. per hr. **23.** $632\frac{47}{64}$ lb. **24.** $\frac{9}{385000}$; $23\frac{29}{77}$ grammes per millier. **25.** Oct. 24 at 2 a.m. **26.** $31\frac{1}{2}$ gal. **27.** 27 ct. **28.** $12\frac{5648}{6021}$ %. **29.** \$477·34. **30.** 16 ct. ; 16 ct. **31.** $480\frac{1120}{1831}$ yd. per min. ; $16\frac{704}{1831}$ mi. per hr. ; 439·467 m. per min. ; 26·368 Km. per hr. **32.** 1 min. 9·6 sec. **33.** 180·041 lb. **34.** $\frac{22}{315}$; \$91·64. **35.** 256·791 lb. **36.** 271296 ; $247643\frac{49}{157}$. **37.** 45 lb. @ 26 ct. **38.** 61·35 %. 613·5 per 1000. **39.** $8\frac{16}{23}$ %. **40.** \$836·53 ; \$783·09 ; 13·65 %. **41.** 4 hr. 5 min. ;

4 hr. 4 min. **42.** $61\frac{1}{2}$ yd. 16 rolls 28 yd. **43.** $7·04. **44.** 8·878 lb. **45.** 13·75 ct. per hr. **46.** (*a*) 276·812 c. in.; (*b*) 173 lb.; (*c*) 26·363 lb.; (*d*) 316·357 lb. **47.** Gains $160·75; 30·63 %. **48.** $2·34. **49.** $340·86. **50.** (*a*) $7\frac{11}{27}$ %; (*b*) 7·834 %; (*c*) 7·91 %. **51.** 5·911 T., 6·2115 T., per sq. in. 831·2 kilog.; 873·4 kilog., per sq. cm. **52.** $4769. **53.** $\frac{7}{16}$. **54.** $949\frac{2}{3}$ lb. **55.** 54 times; $1\frac{5}{13}$ qt.; $\frac{3}{5}$. **56.** $28·66. **57.** 5 hr. **58.** 775 marks; $79\frac{19}{39}$ %; $125\frac{25}{31}$ %. It would reduce the 775 marks to 620 marks but would have no effect on the percentages. **59.** $206·56. **60.** $192·95. **61.** $16·77. **62.** $661·12. **63.** $35·94. **64.** 210·07 lb. **65.** $7\frac{1}{2}$ mi. per. hr. $25\frac{5}{8}$ mi. **66.** $\frac{2}{25}$. **67.** 20 boys. **68.** 5789·658 T. **69.** $783·04. **70.** $863·65. **71.** 2210 tiles; 58′ 6″; 13′. **72.** 1531·46 lb. **73.** 1·596 mi. **74.** $1·71. **75.** $\frac{1}{4}$. **76.** 12 men. **77.** $7\frac{31}{47}$ hr. **78.** 3 hr. **79.** 55 doz. $48\frac{19}{27}$ %. **80.** Net proceeds $3061·71. **81.** $1·21; $1·48. **82.** $153·92. **83.** (*a*) 12 c. ft. 1534 c. in.; (*b*) 12 c. ft. 192·4 c. in. **84.** 667 strokes; 25 strokes. **85.** $\frac{32}{105}$. **86.** $\frac{3}{22}$. **87.** $7·56; $14·40; $18·75. **88.** $1·08; $27\frac{3}{11}$ %. **89.** $21\frac{1}{19}$ %; $23·84. **90.** $1095·78; $1097·97. **91.** *A*, 48 men; *B*, 72 men; *C*, 60 men; *D*, 80 men. **92.** $25·59; 40·536 c. ft. **93.** 32 sq. ft. 102 sq. in.; 10 c. ft. 1620 c. in.; $\frac{2183}{9450}$; $\frac{68}{471}$. **94.** 1 hr. 11 min. 41·84 sec.; 3 hr. 5 min. 41·4 sec. **95.** $\frac{8}{45}$. **96.** $511·25. **97.** $8·40; $5·04; $3·60. **98.** 35·3 %. **99.** 19·264 %. **100.** $1164·14.

Exercise V. **1.** 2^3. **2.** 3^2. **3.** 5^4. **4.** 10^5. **5.** $0·1^4$. **6.** $2·3^5$. **7.** $(\frac{1}{2})^3$. **8.** $(\frac{4}{5})^6$. **9.** 3×3×3×3. **10.** 12×12×12. **11.** 15×15. **12.** 25×25×25×25×25. **13.** 2·5×2·5×2·5×2·5×2·5 **14.** 0·25×0·25×0·25×0·25×0·25. **15.** $\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}\times\frac{2}{3}$. **16.** $\frac{11}{15}\times\frac{11}{15}\times\frac{11}{15}$. **17.** 2×2×2×3×3×5×7×7. **18.** 64. **19.** 36. **20.** 625. **21.** 1024. **22.** 5640625. **23.** 202572273617. **24.** 5554571841. **25.** 14348907. **26.** 1·331. **27.** 0·00001. **28.** 0·000000000064. **29.** 1·126162419264. **30.** 2401. **31.** 24·01. **32.** 0·2401. **33.** 13144·256. **34.** 13·144256. **35.** 0·000013144256. **36.** $\frac{4}{9}$. **37.** $\frac{27}{64}$. **38.** $\frac{1}{32}$. **39.** $\frac{256}{625}$. **40.** $\frac{256}{625}$. **41.** 648. **42.** 54000. **43.** 5292. **44.** 6298292000. **45.** 117649. **46.** 625. **47.** 72.

48. $2^3 \times 3^2 \times 5 \times 7$. 49. $2^3 \times 3^3 \times 5^2 \times 13$. 50. 2^{10}. 51. $2^3 \times 7^2 \times 29$. 52. $2^3 \times 3^6 \times 7 \times 13$. 53. 0·25. 54. 0·2. 55. 0·1. 56. 0·009706. 57 to 61. 0·7854. 62. 2·014. 63. 0·4966. 64. 0·6931. 65. 2·303.

Exercise VI. 1. 4, 529, 55225, 5555449, 555922084, 55592679961. 2. 64, 79507, 83453453, 83740234375, 83791924694479. 3. 144, 1728; 16129, 2048383; 1633284, 2087336952; 163481796, 2090278243656. 4. 2646·999601, 136185·482471849, 7006606·887694149201. 5. 0·018496, 0·002515456, 0·000342102016, 0·000046525874176, 0·0000063275188887936. 6. 0·01. 7. 0·05139. 8. 0·02. 9. 0·00686. 10. 90·744.

Exercise VIII. 1. 24. 2. 43. 3. 321. 4. 3·21. 5. 48·18. 6. 0·07097. 7. 73. 8. 934. 9. 888. 10. 88·8. 11. 1·837. 12. 0·0543. 13. 1·41421. 14. 4·47214. 15. 14·1421. 16. 44·7214. 17. 0·447214. 18. 0·141421. 19. 6·32456. 20. 63·2456. 21. 0·632456. 22. 86·3076. 23. 31·6228. 24. 780·897. 25. 49·7933. 26. 4·97933. 27. 1·25992. 28. 2·71442. 29. 5·84804. 30. 0·621455.

Exercise IX. 1. $\frac{1}{2}$. 2. $\frac{2}{3}$. 3. $\frac{16}{49}$. 4. 0·745356. 5. 0·824621. 6. 0·530330. 7. 0·769800. 8. 0·547723. 9. 0·845154. 10. 0·612372. 11. 0·261861. 12. 0·788811. 13. 1·73205. 14. 2·23607. 15. 3·87298. 16. 4·12311. 17. 4·89898. 18. 5·09902. 19. 5·91608. 20. 6·08276. 21. 2·64575. 22. 3·31662. 23. 7·28011. 24. 8·77496. 25. 9·84886. 26. 40·0125. 27. 48·9898. 28. 0·797724. 29. 0·670820. 30. 0·73598. 31. 3·31662. 32. 2·44949. 33. 2.23607. 34. 1·41421. 35. $\frac{5}{7}$. 36. $\frac{9}{17}$. 37. 0·609884. 38. 0·471957. 39. 0·87358. 40. 0·893904.

Exercise XI. 1. 10^4. 2. 10^6. 3. 10^{10}. 4. 10^{-5}. 5. 10^{-8}. 6. 10^4. 7. 10,007,400. 8. 12741·8. 9. 0·000226. 10. 0·000,000,006. 11. 1,083,200,000,000,000,000,000. 12. 0·000,030,476,3. 13. $1·4709 \times 10^{11}$. 14. $4·8721 \times 10^{-3}$. 15. $7·8376 \times 10^8$. 16. $9·0992 \times 10^3$. 17. $5·675 \times 10^{14}$. 18. $1·1535 \times 10^7$.

19. $8{\cdot}5534 \times 10^{19}$. 20. $1{\cdot}0832 \times 10^{12}$. 21. $3{\cdot}7267 \times 10^{4}$. 22. $4{\cdot}5152 \times 10^{5}$, 23. $2{\cdot}2641 \times 10^{4}$. 24. $6{\cdot}3611 \times 10^{-10}$. 25. $6{\cdot}6966 \times 10^{-4}$. 26. $8{\cdot}5499 \times 10^{-1}$. 27. $1{\cdot}27418 \times 10^{7}$.

Exercise XIV. 1. 0·0000043. 2. 0·0000087. 3. 0·0000130. 4. 0·0000304. 5. 0·0000586. 6. 0·0001084. 7. 0·0004341. 8. 0·0005208. 9. 0·0012576. 10. 0·4771213. 11. 0·8450980. 12. 0·4342495. 13. 1·4913617. 14. 1·1139434. 15. 1·2304489. 16. 0·845098. 17. 1·230449. 18. 1·113943. 19. 1·278754. 20. 1·278754. 21. 1·361728. 22. 1·462398. 23. 1·612784. 24. 1·361728. 25. 0·301030.

Exercise XV. 1. 0·8451. 2. 0·7782. 3. 0·4914. 4. 0·8195. 5. 0·4346. 6. 0·1370.

Exercise XVI. 6. 10^{2}. 7. 10^{-2}. 8. 10^{5}. 9. 10^{-5}. 10. 10^{2}. 11. 1·361728. 12. 2·361728. 13. 3·361728. 14. 0·361728. 15. $\bar{2}{\cdot}361728$. 16. 0·635584. 17. 3·635584. 18. $\bar{2}{\cdot}635584$. 19. 3·831806. 20. 3·892651. 21. 3·860098. 22. $\bar{3}{\cdot}860098$. 23. 5·830396. 24. 1·830396. 25. 2·301464. 26. 14·071145. 27. 42·212188. 28. $\bar{8}{\cdot}811575$. 29. $\bar{6}{\cdot}652826$. 30. $\bar{5}{\cdot}881042$.

Exercise XVII. 1. 0·864831. 2. 2·774604. 3. 4·679092. 4. $\bar{3}{\cdot}690692$. 5. 2·471214. 6. 5·830439. 7. $\bar{1}{\cdot}271435$. 8. 4·903133. 9. $\bar{3}{\cdot}659925$. 10. 7·477599. 11. 8·804711. 12. 8·803211. 13. 27·034709. 14. $\bar{7}{\cdot}634200$. 15. $\bar{5}{\cdot}483962$.

Exercise XVIII. 1. 3·02. 2. 5·432. 3. 5·496. 4. 300. 5. 8·6636. 6. 6·5666. 7. 65·666. 8. 656660. 9. 0·65666. 10. 0·000065666. 11. 676·21. 12. 67680. 13. 0·0079433. 14. 1999·9. 15. 0·31623. 16. 1. 17. 100·047. 18. 2. 19. 1024. 20. 1·0718. 21. $5{\cdot}208 \times 10^{11}$. 22. $3{\cdot}4247 \times 10^{-5}$. 23. $6{\cdot}562 \times 10^{-10}$. 24. $4{\cdot}5709 \times 10^{14}$. 25. $1{\cdot}00028 \times 10^{-100}$.

Exercise XIX. 1. 499·27. 2. 4·0798. 3. 0·0089054. 4. 0·088785. 5. 86·898. 6. 561·33. 7. 0·62614. 8. 0·0065856. 9. 34·464. 10. 12·4368. 11. 7·8541. 12. 0·013446. 13. 0·00026113. 14. 0·0035259. 15. $1{\cdot}2537 \times 10^{-6}$. 16. 115779.

17. 1730·6. **18.** 0·024513. **19.** 4·66735. **20.** $9{\cdot}8019 \times 10^{-21}$. **21.** 1·2589. **22.** 1·08791. **23.** 0·29587. **24.** 0·44402. **25.** 0·61439. **26.** 0·45986. **27.** 0·74989. **28.** 1·24732. **29.** 0·96691. **30.** 0·108921. **31.** 0·79433. **32.** 0·075522. **33.** 0·1423. **34.** $5{\cdot}4143 \times 10^{-24}$. **35.** $5{\cdot}4184 \times 10^{-4}$. **36.** 0·80274. **37.** 0·5848. **38.** 0·491515. **39.** 0·99718. **40.** 1·0000025. **41.** 1917791. **42.** 23·097. **43.** 0·168792. **44.** 0·034278. **45.** 2·2339. **46.** 5 log 2. **47.** 4 log 2 + log 3. **48.** 2 log 7. **49.** (2 − 4 log 2) − 1. **50.** 10 log 2. **51.** 4 log 7. **52.** 5 log 3 + log 7 − 3. **53.** 2 log 2 + log 7 + log 11 + 3 log 13 − 3. **54.** 4 log 3 + 2 log 11 − log 2 − 2 log 7 − 2. **55.** (4 log 2 + log 7 + log 11 + 3 − 6 log 3 − 2 log 13) − 1. **57.** $2^{3{\cdot}322} = 10$; 1 ÷ log 2. **58.** 1·8507. **59.** 0·64921. **60.** 0·28246. **61.** −2·1755. **62.** −0·68512. **68.** 2·80736. **69.** 0·35621. **70.** 2·2766. **71.** 0·43924. **72.** 2·4923. **73.** −1·94843. **74.** −0·0136958. **75.** 2·5124. **76.** 20·149. **77.** 17.673. **78.** 11·8956. **79.** 10·2448. **80.** 9·58435. **81.** 27267. **82.** 302. **83.** 48. **84.** 206. **85.** 44. **86.** 83. **87.** 7. **88.** 46. **89.** 27. **90.** 6. **91.** 69. **92.** 164. **93.** None. **94.** None. **95.** 32. **96.** −17. **97.** −1. **98.** 1. **99.** 10. **100.** 0.

Exercise XX. **1.** Mer., 0·3871; Ven., 0·72333; Mars, 1·52369; Jup., 5·2012; Sat., 9·538. (*a*), 35,915,000; 66,134,000; 139,310,000; 475,540,000; 872,060,000. (*b*), 35,915,000; 67,111,000; 141,370,000; 482,560,000; 884,930,000. **2.** 1233222 figures; 1,169,649------18,212,890,625; 114 hr. 11 min. 14 sec. 342,188,706,078 figures; 253 yr. 21 da.

Exercise XXI. **1.** 25″. **2.** 20″; 11·25″. **3.** 19·886″; 42·614″. **4.** 7′ 7·886″. **5.** 4·2. **6.** 18. **7.** 40·5 yd. **8.** 138′ 8″. **9.** 25·714 ch. **10.** 33′ 7·2″. **11.** 6′ 8·5″. **12.** 17′ 6″. **13.** 19′ 1·091″. **14.** 62′ 8·625″, 80′ 8·625″. **15.** 11·07 ch., 11·39 ch., 11·71 ch., 1203 ch. **16.** 4·186″, 7·814″; 6·977″, 13·023″. **17.** 0·2″. **18.** 8′ 2·182″. **19.** 5′ 10″, 9′ 2″, 10′. **20.** 7 ft. 9·6 in., 10 ft. 8·4 in.; 15 yd. 1 ft. 8 in., 23 yd. 2 ft. 4 in.

Exercise XXII. **1.** 69 yd. 1 ft. 9 in. **2.** 8·944 ch. **3.** 1·25 in. **4.** 2 mi. to 1 in. 1 : 3520. **5.** 36 sq. ft. **6.** 46·75 oz, **7.** 17·32 ch.

8. 1045440 stalks. 9. 384 pieces. 10. \$4747·80. 11. 17·32 ch. 12. 5081 yd. 13. 15 ch., 40 ch. 14. 1452 sq. in. 15. 55 ft. 16. 30 yd. by 66 yd. 17. 6·41 ch. 18. 13984 sq. ft. 19. \$384·38. 20. 315·375 sq. in. 21. 37 sq. ft. 91·5 sq. in. 22. 36 ft. 23. 58·9 ft.; 92·57 ft. 24. \$34·03. 25. 5·625 A. 26. 1° · 402·73 sq. ft.; 2°, 99·72 sq. ft. 27. 1867 bricks. 28. 4 A. 2207 sq. yd. 29. 1°, 152·25 ft.; 2°, 536·25 ft. 30. 12·80 ch. 31. 127·5 ft., 94·5 ft. 32. 79194 sq. ft. 33. 133 sq. ft. 139 sq. in. 34. 40·9469 A.; 32·3817 A. 35. 31·7495 A. 36. 200 yd., 240 yd. 37. 340 ft. 38. 43·56 ch., 37·44 ch. 39. 53 sq. ft., 63 sq. ft. 40. 4·63392 A., 4·80096 A., 4·968 A., 5·13504 A., 5·30208 A. 41. 551 sq. ft., 1653 sq. ft., 2755 sq. ft. 42. 5′ 4″, 4′. 43. 161 ft. 44. 441 ft., 245 ft. 45. 17424 sq. ft. 46. 146 yd. 2 ft. 47. 413600 sq. ft. 48. 5 ft. 49. 11 ft. 50. 289 sq. yd., 225 sq. yd. 51. 841 sq ft., 441 sq. ft.. 52. 35 ft., 27 ft. 53. 194·0335 ft. 54. 174 yd. 55. 484 ft., 330 ft.

Exercise XXIII. 1. 17008 c. in. 2. 3005 gal. 3. 14·05 in. 4. 30·26 in. 5. 15·704 c. in.; 0·2297 in. 6. 66228·5 c. ft. 7. 1549 gal. 8. 1 in. 9. 12 in. 10. 2·014 in. 11. 133·3334 mm. by 44·4445 mm. by 44·4445 mm. 12. 2·2894 yd. 13. 643·66 mm. by 965·49 mm. by 1609·15 mm.; 0·6214, 1·0357 and 1·5536 centiares. 14. 68·921 c. in. 15. 87·72 sq. in. 16. 335·41 c. in. 17. 2·211 in. 18. 649·52 c. in. 19. 5·4 in. 20. 0·0151 to 1. 21. (*a*) 0·271; (*b*) 0·19. 22. In reductions *from* metric expressions a 'calculated length' will be in excess by 1·599 % of the actual length and should therefore be decreased by 1·576 % of itself; a 'calculated area' will be in excess by 3·224 % of the actual area and should be decreased by 3·123 % itself; and a 'calculated' volume will be in excess by 4·874 % of the actual volume and should be decreased by 4·648 % of itself. In reductions *to* metric expressions a 'calculated length' will be in defect by 1·574 % of the actual length and should be increased by 1·599 % of itself; &c. 23. 6·8457 in. square. 24. 107·1 c. in. 25. 625·683 c. in. 26. 244·798 c. ft. 27. 3·5211 ca. 28. 0·3447 ca. 29. 461·468 c. in. 30. 50·4 c. in. 31. 7·2 in.;

5·5 in. ; 4·2 in. **32.** 6. **33.** 12 in. **34.** 128 c. in. **35.** 23·6 in. **36.** 645·225 lb. **37.** 0·0000067. **38.** 27904 to 10000. **39.** 236·32 lb. **40.** 76·777 in. **41.** 584·59 lb. **42.** 12·095 in. **43.** 5066 c. in. **44.** 1·84 lb. **45.** $\frac{1}{12}$ in. **46.** 24149 cubes. **47.** $\frac{3}{16}$ in. **48.** 0·00102 in. **49.** 10·08 c. in. **50.** 49·2565 millilitres. **51.** 4321·1 c. in. **52.** 28134 c. ft., 1458 c. in. **53.** 50 c. ft., 288 c. in. **54.** 117,333,-333$\frac{1}{3}$ c. yd. **55.** 159·25 c. yd. **56.** 406 gal. **57.** 49·2 bars. **58.** 6844·8 yd. **59.** 11 ft. 3 in. **60.** 2 ft. 2 in. **61.** 394·4 gal. **62.** 2·181 sq. ft. **63.** 4 ft. by 4 ft. by 5·6578 ft. **64.** 2640 c. ft. **65.** 11840 c. ft. **66.** 768 c. in. **67.** 306 c. in. **68.** 969 c. in. **69.** 33293·4 c. in. **70.** 1·07 lb. **71.** $1235·31. **72.** 18·8456 c. ft. **73.** 18 c. ft., 696 c. in. **74.** 1·5 c. ft. **75.** 16·846 in. **76.** 3 ft. 4·5 in. **77.** 5·196 ft. **78.** 4·5243 in. **79.** 1·57 in. **80.** 18 in. **81.** 3464344 c. yd. ; 3096908 c. yd. **82.** 1492·36 c. ft. ; 895 T. 1363 lb. 1722 gr. ; $539984058·47. **83.** 42 ft. 10$\frac{1}{8}$ in. **84.** 3·3 in. **85.** 2·888 in. **86.** 11·04 in. **87** 2·175 in. **88.** 39·375 c. ft. **89.** 40·25 c. ft. **90.** 30·484 c. ft. **91.** 5128·27 gal. **92.** 5776·71 lb. **93.** 7008 lb. **94.** 244·369 c. ft. **95.** 771 c. in. **96.** 47272·264 c. yd. **97.** 5365·226 c. yd. **98.** 28246·722 c. yd. **99.** 101138·343 c. yd. **100.** 1699·38 c. in. ; 972·972 c. in. **101.** 47·685 c. ft. ; 29·352 c. ft. ; 15 c. ft. **102.** 165 c. in. ; 95·76 c. in. **103.** 696 c. in. ; 264 c. in. **104.** 5·654 in. **105.** 124·542 c. in. ; 82·791 c. in. ; 30·042 c. in. **106.** 3·618 c. ft. ; 8·045 c. ft. **107.** 10138 c. yd. 6 c. ft. **108.** 43·008 c. in. **109.** 11 gal. **110.** 32·64 c. in. ; 37·44 c. in. **111.** 1000 c. in. **112.** 509·2 c. in. **113.** 647·234 c. in. ; 441·406 c. in. **114.** 1575 c. in. ; 840 c. in. **115.** 18·432 in. **116.** 19 c. ft. 418 c. in. **117.** 64$\frac{1}{3}$ c. ft. **118.** 13954·3 lb. **119.** 57 c. ft. **120.** 20·48 in. **121.** 1·8998 ft. **122.** 13 to 9. **123.** 7 to 4. **124.** 53 c. ft. 352 c. in. **125.** 2·9 ft. **126.** 4·474 ft. **127.** 8987 litres. **128.** 34 sq. ft. 64 sq. in. **129.** 11·5 sq. ft. **130.** 2′ 8″; 1′ 1$\frac{1}{2}$″. **131.** 5·6 in. **132.** 6 ft. **133.** 1′ 8″. **134.** 1$\frac{1}{2}$ in.

Exercise XXIV. **1.** 97 in. **2.** 905 mm. **3.** 1 ft. 5 in. **4.** 1 ft. 8$\frac{3}{4}$ in. **5.** 8$\frac{1}{2}$ in. **6.** 6·928 in. **7.** 3·464 in. **8.** 3·674 ft. **9.** 2·45 in. **10.** 26·7 in., 24·4 in., 12·5 in. **11.** 15 ft. **12.** 8·595 m. ;

8·579 m., 6·347 m., 5·82 m. **13.** ------ **14.** 4·29 in., 8·8 in., 23·4 in. **15.** 962·676 c. in. **16.** 56 sq. ft. 40 sq. in. **17.** 9·88 in. **18.** 391 ft. **19.** 7 ft. **20.** 2 ft. **21.** 683128 sq. ft. or 168988 sq. ft. **22.** 1200 c. in.; 790·9 sq. in. **23.** 28 ft. 8·2 in. **24.** 36·9 ch. **25.** 23166 sq. ft.; 696 ft., 630 ft.

Exercise XXV. **1.** 1936 sq. ft., 57600 sq. ft., 13689 sq. ft.; 44 ft., 240 ft., 117 ft. **2.** 46 ft. 5·494 in. **3.** 9·2 mm., 359·5 mm. **4.** 7 in., 8·8 in.; 24 in., 23·4 in. **5.** 15 ft. 2·753 in., 45 ft. 2·247 in.; 26 ft. 2·797 in. **6.** 12·923 yd., 12 yd., 11·2 yd. **7.** 19·8 ft., 12·692 ft., 20 ft., 44·8 ft., 36 ft., 51·692 ft. **8.** 399 ft., 455 ft., 511 ft. **9.** 616 ft., 665 ft., 511 ft. **10.** 17 ft., 21 ft. 3. in., 21 ft. 9 in.

Exercise XXVI. **1.** 60 sq. yd. **2.** 60 sq. yd. **3.** 24 sq. ft. **4.** 84 sq. ft. **5.** 66 sq. ft. **6.** 126 sq. in. **7.** 240 sq. in. **8.** 252 sq. in. **9.** 2·9274 sq. ch. **10.** 166·417 A. **11.** $260653. **12.** $118·68. **13.** 16672·5 sq. ft. **14.** 18·2 Ares, 54·6 Ares, 91 Ares. **15.** 227·04 sq. ft., 804·32 sq. ft., 740·96 sq. ft., 163·68 sq. ft. **16.** 14760 sq. ft., 17352·28 sq. ft., 28341·5 sq. ft., 25749·22 sq. ft. **17.** 92·8812 A. **18.** 1698·8 sq. ft. **19.** 44·7154 sq. metres. **20.** 37·0843 A.

Exercise XXVII. **1.** 0·130806; 3·13935. **2.** 0·065438; 3·14103. **3.** 0·263305; 3·15966. **4.** 0·131087; 3·146086. **5.** 0·0654732; 3·142715.

Exercise XXVIII. **1.** 659·734 ft.; 835·664 ft. **2.** $3·93. **3.** 2 mi. 1710 yd. 2 ft. 3 in. **4.** 1 ft. $8\frac{1}{2}$ in. **5.** 410 yd. 1 ft. 5 in. **6.** $3\frac{1}{2}$ in. **7.** 14 in. **8.** 1770·7 mi. per min.; 357.7 mi. per min. **9.** 10·472 in. **10.** 15·708 in. **11.** 28° 38′ 52·4″. **12.** 114° 35′ 29·6″. **13.** 57° 17′ 44·8″. **14.** 62·489 in.; 82·467 in. **15.** 27·914 in. **16.** 1·945 ft. **17.** 2·853 ft. **18.** 6·915 ft. **19.** 34·6 in. **20.** 15·8 in. **21.** 15·7 m. **22.** 40 yd. **23.** 43,827,033 yd.; 43,827,735 yd. **24.** 1·093827 yd. **25.** 141,000,000 mi. and 140,400,000 mi. **26.** 899 mi. per min. **27.** 34·527 mi. **28.** 48·83 mi. **29.** 141·244 in. and 98·87 in. **30.** 28·45 in. and 35·45 in.

Exercise XXIX. **1.** 44·18 sq. in. **2.** 153·94 sq. ft. **3.** 13·636 sq. cm. **4.** 117·75 ft. **5.** 5957·84 ft. **6.** 3199·41 ft. **7.** $15·89. **8.** 35 in. **9.** 31907 lb. **10.** 829·58 sq. ft. **11.** 4·427 in. **12.** 386·146 sq. ft. **13.** 14·848 in. **14.** 56·37 sq. ft. **15.** 0·615 sq. ft.; 1·133 sq. ft. **16.** 7·31 sq. m. **17.** 13·54 ft. **18.** 16·572 yd. **19.** 62·02 sq. in. **20.** 138·13 sq. ft. **21.** 32·225 sq. m. **22.** 93·46 sq. ft. **24.** 2488·14 sq. cm. **25.** 15·708 sq. ft. **26.** 189·69 ft. **27.** 28·274 sq. ft. **28.** $96·10. **29.** 3·1 yd. **30.** 11 yd. **31.** $1411·47. **32.** 44·1 in. **33.** 40·15 sq. in. **34.** 136·35 sq. cm. **35.** 6·742 ft. **36.** 57° 17′ 44·8″. **37.** 33° 25′ 21″. **38.** 0·0906 sq. ft. **39.** 0·6142 sq. ft. **40.** 0·2854 sq. ft. **41.** 112·2 sq. in. **42.** 158·57 sq. in. **44.** 92·88 sq. in. **45.** 29·05 sq. in. **46.** 62·832 sq. ft. **47.** 9·487 sq. ft. **48.** $206·91. **49.** 1269·21 sq. ft. **50.** 692·72 sq. ft.

Exercise XXX. **1.** 3 sq. ft. 108 sq. in. **2.** 10 sq. ft. 120 sq. in. **3.** 40 sq. ft. 48 sq. in. **4.** 19·37 sq. in. **5.** 384 sq in. **6.** 49 sq. ft. 120·6 sq. in. **7.** 52 sq. ft. 134·7 sq. in. **8.** 18 sq. ft. 94·89 sq. in. **9.** 3 ft. 8 in. **10.** 9·55 in. **11.** 25·33 in. **12.** 12·44 in. **13.** 8 sq. ft. 45 sq. in. **14.** 17 sq. ft. 96·69 sq. in. **15.** 64 sq. ft. 104·66 sq. in. **16.** 21 sq. ft. 111·2 sq. in. **17.** 9·2775 sq. ft. **18.** 30·26 sq. ft. **19.** 7 sq. ft. 122·97 sq. in. **20.** 33 sq. ft. 118·55 sq. in. **21.** 2 ft. 8 in. **22.** 2 ft. 10·7 in. **23.** 3·183 in. **24.** 3 ft. 1·72 in. **25.** 2 ft. 4·8 in. **26.** 2·387 ft. **27.** 25·69 in. **28.** $14\frac{3}{8}$ yd. **29.** 41 yd. **30.** 60°. **31.** 2 to 1. **32.** 136·5 sq. in. **33.** 753·98 sq. in. **34.** 1021·02 sq. in. **35.** 130·627 sq. in. **36.** 78·63 sq. ft. **37.** 880·78 sq. in. **38** 8·343 in. and 12·457 in. **39.** 45·783 sq. m. **40.** 113·1 sq. in. **41.** 45·837 sq. in. **42.** 3·39 in. **43.** 82·467 sq. ft. **44.** 47·124 sq. in. **45.** 16·144 sq. ft. **46.** 6·065 sq. ft. **47.** 0·858 in. or 29·142 in. **48.** $\frac{7}{16}$. **49.** 27 in. **50.** 196,940,000 sq. miles.

Exercise XXXI. **1.** 628·32 c. in. **2.** 226·194 c. in. **3.** 6·77 in. **4.** 2 ft. **5.** $2\frac{2}{3}$ in. **6.** 7·927 ft. **7.** 7·0663 in.; 196·0844 sq. in. **8.** 185·336 mm. **9.** 503·08 mm. **10.** 383·92 mm. **11.** 13 mi. 1566·2 yd. **12.** 15·783 mi. **13.** 1347·45 c. in. **14.** 0·1502 in. **15.** 2·4 in. **16.** 733·037 c. ft. **17.** 392·7 c. in. **18.** 9·81 in. **19.** 16889·24 c. in. **20.** 5579·47 c. in. **21.** 201·16 c. in.

22. 186·7 c.in. **23.** 428·83 c.in. **24.** 4322·84 c.in. **25.** 5719·108 c.in **26.** 428·828 c. ft. **27.** 101 c. ft.; 78·66 c. ft.; 59·1 c. ft. **28.** 1·8963 in.; 2·7033 in.; 10·4004 in. **29.** 141·87 c. yd. **30.** 7·836 c. ft. **31.** 2408·66 sq. in. **32.** 10·8573 in. **33.** 904·78 c. in. **34.** 606·863 c. in. **35.** 4387·14 c. in. **36.** ·5096 gal. **37.** 22·283 lb. **38.** 11·377 lb. **39.** 5·9865 lb. **40.** 110·446 c. in. **41.** 6·928 in.; 574·226 c. in. **42.** 101·274 lb. **43.** 47·545 lb. **44.** 33·1 lb. **45.** 229·303 c. ft. **46.** 2378·9 lb.; 5625·9 lb. **47.** 0·825 in. **48.** 4·64 in. **49.** 1·42 in. **50.** 0·003787. **51.** 1 to 900. **52.** 103 to 1000. **53.** 7 in. **54.** 10·01 in. **55.** 20·123 c. in. **56.** 51·662 c. in. **57.** 433·541 c. in. **58.** 25525·4 c. in. **59.** 13 to 5. **60.** 0·36 c. yd.; 0·64 c.yd. **61.** 3·28392 c.in. **62.** 928·32 c.in. **63.** 2·598817 c.mi. $\times 10^{11}$. **64.** 2·598682 c.mi. $\times 10^{11}$; 20902046 ft. **65.** 3173 in. **66.** 68·068 c.in. **67.** 99041 c. in. **68.** 253 gal.

Exercise XXXII. **7.** 25 gal. **8.** 6 da. **9.** $2\frac{34}{63}$ wk. **10.** 4 A. 3740 sq. yd. **11.** $\frac{25}{36}$ oz. **12.** $3^{\frac{1}{2}}$. **13.** $7·8125. **14.** 3:5. 5:3. **15.** 9:4. 4:9. **16.** 17:24. **17.** 5:4. **18.** 17:7. **19.** 9:8. **20.** 5:8. **21.** $3\frac{1}{2}$ lb. **22.** 1 to 3. **23.** $2100, $2400. **24.** $5·62½, $9·37½. **25.** 100 A.

Exercise XXXIII. **1.** 242, 484, 605. **2.** 18055·4 lb., 12036·9 lb., 9027·7 lb. **3.** $17·45, $27·92, $11·43. **4.** $13·05, $26·10, $39·15, $39·15, $52·20. **5.** $320, $360, $384. **6.** $1200, $300, $120, $60, $40. **7.** 132 lb., 28 lb., 20 lb. **8.** $66, $77, $110. **9.** $1860, $2112, $2464. **10.** 2925, 3640, 4212, 1950, 2496. **11.** 648·128 lb. of oxygen, 562·445 lb. of carbon, 89·427 of hydrogen. **12.** 20 lb. 270 lb. 528 lb. of nitre and 80 lb. of sulphur. **13.** $2704, $3151, $4045. **14.** $134·40, $118·08. **15.** $2\frac{46}{437}$ lb. of lead, $17\frac{121}{437}$ lb. of tin. **16.** $6·02. **17.** $5000, $8750, $11250. **18.** A to C, 30 ct.; B to C, 36 ct. **19.** $1991\frac{1}{9}$ lb.; $202\frac{2}{9}$ lb., $46\frac{2}{3}$ lb.

Exercise XXXIV. **1.** $160, $240, $300, $350. **2.** £1 6s. 8¼d., £13 6s. 9¾d., £14 0s. 2d. **3.** 40·3088 pt., 62·7025 pt., 74·6459 pt., 85·8428 pt. **4.** 300 *m.*, 120 *ch.*, 280 *w.* **5.** $2100, $3570, $2380, $4550. **6.** A, $1085·70; B, $1034; C, $1155. **7.** A, $540; B, $200; C, $300; D, $180.

Exercise XXXV. 1. \$6, \$3·60. 2. \$600, \$840, \$300. 3. 88 ct., $38\frac{1}{2}$ ct., $10\frac{1}{2}$ ct. 4. \$900, \$750. 5. \$522, \$536, \$502·50. 6. \$4507·06; \$4965·41; \$5527·53. 7. 37·5, 15, 312·5. 8. 7800 lb., 6500 lb., 5200 lb. 9. \$2·25, \$1·50, 90 ct. 10. 12, 18, 5, 45.

Exercise XXXVI. 1. \$147, \$196, \$147. 2. \$10·82, \$20·29, \$60·89. 3. \$2·50, \$1·$87\frac{1}{2}$, \$1·$56\frac{1}{4}$. 4. \$38·25, \$38, \$37·80. 5. \$40, \$42·30, \$43·20.

Exercise XXXVII. 1. 2500 bu., 4000 bu., 10000 bu. 2. 144 *m.* 351 *w.*, 480 *b.* 3. 16, 28, 40. 4. 15, 20, 50. 5. 99·8 ct., 66·53 ct., 83·17 ct.

Exercise XXXVIII. 1. \$780, \$801·25, \$426·79, \$991·96. 2. A, \$240; B, \$180. 3. \$2565, \$1425, \$1710, \$1140. 4. \$9, \$13·50, \$30. 5. \$79·98, \$17·14, \$11·99, \$143·38. 6. 144 *m.*, 480 *b.*, 351 *w.* 7. \$2·10, \$1·50, \$1·08. 8. \$2·23, \$2·23, \$1·59, 95 ct. *or* \$2·42, \$2·42, \$1·41, 75 ct. 9. \$369, \$399, \$432.

Exercise XXXIX. 1. \$656·25, \$1093·75. 2. \$1070·30, \$789, \$1296·70. 3. \$18004·83, \$13495·17. 4. \$20419·64, \$34486·50. 5. \$13138·05, \$10019·95. 6. \$8800, \$10400. 7. \$16434·78, \$24260·87. 8. \$185·81, \$126·99, \$121·40. 9. \$2216·98, \$2480·08, \$3152·94. 10. \$9368·20, \$10021·80. 11. \$13000, \$8000.

Exercise XL. 1. \$14·38. 2. \$823·82. 3. \$12300. 4. \$91500. 5. \$564. 6. \$59062·50. 7. $1\frac{1}{2}$ %. 8. $12\frac{1}{2}$ %. 9. 20 %. 10. \$6058·95, 95 %. 11. \$1052·63, \$1422·37. 12. 21·5 %. 13. \$7163·84. 14. 120 %. 15. 65600, 67240, 68921. 16. 130050, 127500, 125000. 17. \$20000. 18. \$2000. 19. 72 gallons. 20. 353864 lb. 21. \$31250, \$32812·50, \$43750, \$43125. 22. 53·7 %. 23. 4096. 24. 1122:1125. 25. 37:32 in *work*.

Exercise XLI. 1. \$2425. 2. 150 %. 3. 10 %. 4. 4 % loss. 5. $38\frac{8}{9}$ %. 6. 12 % gain. 7. \$2400. 8. \$200. 9. 24 s. 10. $14\frac{7}{12}$ %. 11. 18 ct. 12. 9d.; 512. 13. \$1·40. 14. 71·75 T. 15. 2 ct. 16. $12\frac{1}{2}$ %. 17. \$45, \$30. 18. \$40. 19. \$2·08. 20. 14·13 %. 21. \$25, \$30. 22. 16 ct. 23. 75 lb. at $37\frac{1}{2}$ ct., 69 lb. at $34\frac{1}{2}$ ct. 24. \$12. 25. \$160, \$120.

Exercise XLII. 1. \$131·25. 2. \$471·25. 3. \$232·47. 4. \$332·50. 5. \$1680, \$1120, \$700. 6. \$4500, \$3600, \$2400. 7. \$82·50. 8. 0·8 %. 9. \$187·50; $1\frac{7}{8}$ %. 10. \$1980, \$2640. 11. \$2393·62, \$5106·38. 12. $1\frac{1}{8}$ %. 13. $\frac{7}{8}$ %; $87\frac{1}{2}$ ct. per \$100. 14. \$46·$87\frac{1}{2}$. 15. $\frac{3}{4}$ %. 16. \$2000. 17. \$12500. 18. \$245. 19. \$88333·33, \$14000. 20. \$1851·94. 21. \$4273·50. 22. \$18918 23. 65 ct. per \$100. 24. \$5930.

Exercise XLIII. 1. \$81, \$1539. 2. \$6·43. 3. \$6240. 4. \$3264. 5. \$3055. 6. \$500. 7. 3 %. 8. 3 yd. 9. \$18000. 10. \$2. 11. \$480. 12. 5 %. 13. \$61200. 14. \$1388·62. 15. \$288·39. 16. \$18909·18. 17. 6652 lb., 1901 lb. 18. \$25663·44. 19. \$9653·38. 20. 2 %. 21. $2\frac{1}{2}$ %.

Exercise XLIV. 1. \$60. 2. \$747·80. 3. \$5·40. 4. \$175. 5. 28 %. 6. $31\frac{1}{4}$ %. 7. \$50. 8. \$1·27; $30\frac{1}{3}$ %. 10. 25 %. 11. 20 %. 12. $14\frac{2}{7}$ %. 13. $16\frac{2}{3}$ %. 14. 20 %. 15. 25·17 % 16. 20 %. 17. 13 %. 18. 20 %.

Exercise XLV. 1. \$308·02. 2. \$960·12. 3. \$445·17. 4. \$78·54. 5. \$573·04. 6. \$382·90. 7. 8%. 8. 6%. 9. \$432·38. 10. \$751·28. 11. June 4.

Exercise XLVI. 1. \$85·19. 2. \$28·90. 3. \$14·33. 4. \$11. 5. \$33·61. 6. \$2·19. 7. \$349·49. 8. \$504·96. 9. \$333·33. 10. \$831·60. 11. \$72·51. 12. \$98·40. 13. 7 %. 14. $7\frac{1}{2}$ %. 15. $6\frac{1}{2}$ %. 16. 5 %. 17. 6 %. 18. 1 yr. 276 da. 19. 100 da. 20. Oct. 13. 21. 20 yr. 22. 25 yr. 23. 8 %. 8·034 %. 24. \$7·27; \$7·14. 25. $11\frac{1}{9}$ %. 26. 10·267 %.

Exercise XLVII. 1. 12 Nov. 2. 17 Dec. 3. 30 June. 4. 10 Oct. 5. 2 Nov. 6. 4 July *or* 8 July, 1889.

Exercise XLVIII. 1. \$22·58. 2. \$2364·33. 3. \$71·41.

Exercise XLIX. 1. \$926·10; \$126·10. 2. \$497·19; \$72·19. 3. \$281·38; \$31·38. 4. \$404·83; \$38·16. 5. \$766·95; \$44·45. 6. \$1·06. 7. \$1·0609. 8. \$1·061364. 9. 1·157625. 10. 1·276281. 11. 1·4071. 12. 1·040604. 13. 1·082857. 14. 1·126825.

Exercise L. 1. $1559·20; $809·20. 2. $659·23; $294·23. 3. $2770·89; $1520·89. 4. $48·90; $12·65. 5. $650·17; $222·67. 6. $4792·20; $4667·20. 7. $1470268; $1470268; $1·796076 $\times 10^{14}$; $1·829594 $\times 10^{22}$. 8. $456·39. 9. $257·20. 10. $256·61. 11. $62·50. 12. $41·67. 13. $48·26. 14. $48·72. 15. $48·95. 16. 8 yr. 17. 6 yr. 314 da. 19. 29 yr. 325 da. 20. 29 yr. 129 da. 21. 29 yr. 25 da. 22. $2\frac{1}{2}$%. 23. 9%. 24. Nearly 5%. 25. 53 yr. 29 da. 26. 114 yr. 34 da. 27. 6·167%; 6·183%; 6·184%. 28. 3·67%. 38·94.

Exercise LI. 1. $3275. 2. $1237·50. 3. $32700. 4. $10468·75. 5. $69231·25. 6. $6792·04. 7. $182062·50. 8. $111000. 9. $22812·50. 10. $712·50. 11. $4200. 12. $210. 13. 3·24%. 14. 125:126. 15. $110\frac{1}{4}$. 16. $117\frac{1}{8}$. 17. $90\frac{5}{8}$. 18. 280. 19. $15807·63. 20. $19989; $827·50; 4·14%. 21. $18000; 840. 22. $160000. 23. $35938·44. 24. £5625. 25. $102\frac{3}{4}$.

Exercise LII. 1. $365·00. 2. $1097·50. 3. $288. 4. $75·17. 5. £20 10s. $11\frac{1}{2}$d. 6. £1127 11s. 3d. 7. $241111·11, $48388·89; $96888·89, $193840; $96833·33, $96722·22, $96750, $96805·56, $292000, $292000, $389333·33. 8. $464·89. 9. $464·02. 10. $570·75. 11. $95\frac{5}{8}$. 12. The drafts at 60 days' sight. $648·51. 13. $2098·05. 14. $298·89. 15. $4·84$\frac{2}{3}$. 16. $94\frac{3}{8}$. 17. $9\frac{1}{2}$. 18. 5·18. 19. $95\frac{1}{2}$. 20. £986 13s. 4d. 21. 19552 fr. 22. $28·86. 23. $38·48. 24. $30538·93; 358·664 gr.

CORRECTIONS.

Page 12, line 5 up; *after* length *insert* is the yard which.

Page 77, Prob. 20; *after* 1889 *insert* and payable 9 July, 1889.

Page 82, Prob. 70; *after* discounted *insert* at 8 %.

Page 83, Prob. 81; *for* \$6·60 *read* \$29·40. *The answers will then be* \$5·39 *and* \$6·60.

Page 84, Prob. 90; *for* payable in *read* drawn at.

Page 98, line 7 up; *omit* of equality.

Page 123, Prob. 141; *for* $\sqrt{2}$ read $\sqrt{3}$.

Page 137, Prob. 24; *insert* $\times 3^{13} \times 7^3 \times 11 \times 13 \div 2^9 \div 5^8 \div 17$.

Page 151, line 13 up; *in second denominator*, *for* 6 *read* 0.

Page 151, line 5 up; *for* $\div 10^{-1}$ *read* $\times 10^{-1}$.

Page 155, line 9 up; *for* partial product *read* partial products.

Page 162, line 5 up; *for* HK *read* GK.

Page 187, line 10; *for* $ab(2b_1 + b_3)$ *read* $\frac{1}{6}ab(2b_1 + b_3)$.

Page 192, Prob. 34; *for* 3 in. read 6 in.

Page 200, Prob. 108, figure; join FB and HC.

Page 206, Prob. 13; *for* 22 ft., 6 ft. and 3 ft. *read* 53 ft., 48 ft. and 43 ft.

Page 213, line 15 up; for k read k_1.

Page 223, Prob. 6; *for* diameter *read* diameters.

Page 249, Prob. 49; *for* 75·5lb. *read* 45·5lb. *and insert* lb. *after* 57·6.

Page 307, Prob. 23; *for* at a total loss of \$1943·90 *read* at a loss of \$3514·75 on the amount realized by his former sales. *The answer will then be* \$64980.

Page 316, line 12 up; for $\cdot\dot{5}\dot{7}4$ *read* $\cdot 5\dot{7}\dot{4}$.