LITTLE BLUE BOOK NO. 856
Edited by E. Haldeman-Julius

Arithmetic Self Taught

Part I

Lloyd E. Smith

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Part I

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FOREWORD

Arithmetic is the beginning of all mathematics. The child who counts his fingers and toes is on the way to differential calculus—may the benevolent deities of digits have mercy on him! Practical arithmetic confronts all of us every day of our lives. We cannot buy the necessities of life or earn a means of sustenance without using arithmetic.

The subject, in its elementals, is comparatively simple. It belongs with the three "R's"—Reading, 'Riting, and 'Rithmetic. Part I, the present booklet, deals with all the elemental stages of the subject. Part II, naturally, goes on from where Part I stops, but it lays particular emphasis on the practical side of arithmetic, by far the greater portion of the book being devoted to problems and their solution. In offering the elements of arithmetic, the extreme and empty simplicity of the texts used in the lower grades of elementary schools has been strenuously avoided. It is assumed that the "self-teachers" of these Little Blue Textbooks are students with adult and alert minds—but, at the same time, those who are more perspicacious than this text assumes should be a little patient if the explanation is in many cases obvious and seemingly unnecessary.

The student of arithmetic will find *A Handbook of Useful Tables* (Little Blue Book No. 835), of great value to him in many of his practical calculations.

ARITHMETIC SELF TAUGHT

I. NUMERATION AND NOTATION

Before any subject can be undertaken, its material and terminology (the names it uses) must be understood. *Arithmetic*, according to the New Standard Dictionary, is the "science of numbers and the art of reaching results by their use." It may be divided into *abstract* arithmetic, which deals with pure number or quantity (just 4, and not 4 apples or 4 cents or 4 "anything"), and *practical* arithmetic, which applies the science of pure number to the problems of everyday life. It is almost entirely with practical arithmetic that this book is to deal.

Numeration is the reading or naming of *notation*, which is merely to say that notation is the expression of figures by numbers or letters, and numeration is the naming of those numbers or letters. Thus:

Notation: 1 2 3 4 5 6 7 8 9 0

Numeration: One, two, three, four, five, six, seven, eight, nine, zero (cipher, naught, or "nothing").

The zero (0) may be placed first or last, since it has no value. It is used in notation to show the absence of a number or quantity. This system of notation is commonly called the Arabic system, but its true origin is claimed to be Hindu. The Greeks and Romans had no zero, and consequently did not progress very far in arithmetic. The value of the zero is clearly seen when it is understood that in the Arabic system numbers take their value by their position in relation to their component *digits* (each of the nine *figures*, or *numerals*, above, is a digit; the *digits* of the *number* 485 are 4, 8, and 5). Thus, if 7 stands alone, it signifies seven *units* (or "ones"; seven single sticks, for example); but if a zero is added, making 70, its value is multiplied by ten, and it stands for *seventy* units. The other numerals increase the value of their associated digits in the same way, thus: In the number 765, the nearest digit to the right indicates the units (here five units), the next to the left indicates the *tens* (here six tens or *sixty*), and the next to the left the *hundreds* (here seven hundreds), so that the number is read "seven hundred and sixty-five."

Proceeding in numeration from the nine, last in the notation above, the next number is ten (10), and the next two are eleven (11) and (12). So from one to twelve the names are distinct. But going up from twelve the rest of the names are compounds. Thus, three and ten make thirteen (13), the number following twelve—the name *thirteen* being a compound of *three* (contracted to *thir-*) and *ten* (expanded to *-teen*). Beginning with this first compound, thirteen, the next seven numbers end in *-teen* (so that girls are said to be "in their teens," when between the ages of twelve and twenty) —thirteen (13), fourteen (14), fifteen (15), sixteen (16), seventeen (17), eighteen (18), nineteen (19). The manner in which each *units* numeral is combined (in name) with the *tens* is clearly shown.

Going up from nineteen, the next number is twice ten, or twenty (20). The series started with 1, a single digit, a unit, *one*. By adding a zero (10) this became ten times as much, because in the Arabic system, proceeding from right to left in a number, each digit to the left is multiplied by ten once for each digit to its right (thus, in 387, 8 is multiplied by 10 once, for it has one digit, 7, to its right; but 3 is multiplied by 10 twice (10 times 3 is thirty, and 10 times 30 is 300) because it has two digits, 8 and 7, to its right). And now twenty is twice ten (the name being compounded of *twen-* for *two*, and *-ty* for *ten*). If one is added (21), the name *one* is added to the name also, making *twenty-one*, and, going up, twenty-two (22), twenty-three (23), twenty-four (24), and so on. (Notice that in these compound numbers there is always a hyphen.)

Numeration is now simply a matter of combining the fundamental forms. Counting by tens, the names are ten (10), twenty (20), thirty (30), forty (40), fifty (50), sixty (60), seventy (70), eighty (80), ninety (90), and (ten times ten, for 1 is followed by two digits) one hundred (100). The intermediate numbers are simple compounds: sixty-four (64), eighty-nine (89), fifty-five (55), one hundred and three (103), two hundred and twenty-four (224), etc. Ten hundreds (the units digit is multiplied by ten three times) make one thousand (1000); ten thousands make ten thousand

(10,000); ten ten-thousands make one hundred thousand (100,000); ten hundred-thousands make one million (1,000,000), which is also a thousand thousands; a thousand millions make one billion (1,000,000,000) in America, but in England this is only a thousand millions, an English billion being a million millions (12 ciphers); and a thousand billions make a trillion (12 ciphers), or the English billion, and so on. In large numbers (four digits or more) it is customary to divide the numerals into groups of three by commas, counting from right to left, as is shown in the preceding numbers—which makes them easier to read at a glance.

All of the numbers thus far considered have been *whole* numbers, or *integers*. They represent values increasing from 1 up to as high as one may wish to go. But it is possible to have only half of a whole—for half an apple is certainly not a whole apple. This may be expressed in arithmetic by what is called a *fraction*, in this case one-half, written ½—one over two, with a horizontal line separating the top number from the bottom number. Fractions will be explained in detail a little later.

Or the *part* of a whole number may be expressed in arithmetic by what is called a *decimal*, a name which means specifically a *tenth* part. The Arabic system, as has been seen from the preceding explanation, has the number *ten* as its basis—the values of the digits increasing by ten times according to their position. Just as the numerals *increased* by ten times when counting from *right to left*, according to the number of digits on their *right*, so the numerals may be *decreased* by ten times when counting from *left to right*, according to the number of digits on their *left*. The starting point for the counting is a period, or decimal point—which was not written for the previous numbers because they were all whole numbers. When the numbers being used are all integers, the decimal point is customarily omitted. But it would be perfectly correct to write it, thus: 758. represents a whole number, seven hundred and fifty-eight, counting its value from right to left from the decimal point. If the period were omitted it would be assumed that the figure was a whole number. But suppose the position of the period is changed, and the number is .758 instead of 758. The counting must now proceed (always from the decimal point) from left to right, and the 7, instead of being a unit-digit, is a tenth-digit, and, if it stood alone (.7) it would be read *seven tenths.* Similarly, the 5 is in, not the tens place, but the hundredths place, and if it stood alone (.05—the zero being necessary to distinguish it from tenths) it would be read *five hundredths.* The 8 is in the thousandths place, and .008 would be read *eight thousandths.* The number .758 is read *seven hundred and fifty-eight thousandths.*

The range of the Arabic system is thus indicated by columns from the right and left of the decimal point:

5	4	3	2	1	.	1	2	3	4	5
ten thou-sands	thou-sands	hun-dreds	tens	units	deci-mal point	ten-ths	hun-dred-ths	thou-san-dths	ten thou-san-dths	hun-dred thou-san-dths

Notice that ciphers standing between the decimal point and a digit on the right or left are very important, and change the value of the number (400. vs. .400). But ciphers preceding a whole number (without themselves being preceded by any other digit), or following a decimal (without themselves being followed by any other digit) have no significance and are not usually written (00400. is still four hundred, and .00400 is still four thousandths).

To summarize notation and numeration, the notation of a number would be: 48,562,791,006; and the numeration of the same number is: forty-eight billion five hundred sixty-two million seven hundred ninety-one thousand and six. The two ciphers indicate that there are no hundreds and no tens.

The Roman System: The Roman system of notation uses letters instead of figures, but is used nowadays only on clock dials and in "dignified" places generally. A complete table of Roman numerals is given in *A Handbook of Useful Tables* (Little Blue Book No. 835, Table I). To indicate the general nature of the system, some of the numbers are: I (one), II (two), III (three), IIII or IV (four), V (five), VI (six), VII (seven), VIII (eight), IX (nine), X (ten), XI (eleven), XII (twelve), XX (twenty), XXX (thirty), XL (forty), L (fifty), LX (sixty), LXX (seventy), LXXX (eighty), XC (ninety), C (one hundred), D (five hundred), and M (one thousand).

II. ADDITION

The fundamental processes of arithmetic are four: addition, multiplication, subtraction, and division. Addition and subtraction are habitually associated, but addition logically belongs with multiplication, and subtraction with division. On these four processes all arithmetical, even all mathematical, processes are based.

Addition, defined, is the process of finding the *sum* or *total* of two or more numbers. The sign of addition is +, and is read *plus*. If you have two books and put with these two books three other books, you then have how many? Five. This is addition, and put in abstract form the process is $2+3$ (two plus three) make 5. Or, to put the problem in strict arithmetical form, $2+3$ *equals* 5, or

$$2+3=5$$

The sign of equality (=) may also be read *is equal to*.

Abstract numbers (numbers which stand alone without referring to quantities of any substance or thing) may be added indiscriminately. But when dealing with *concrete* numbers (numbers used to name the quantity of something, as 9 miles, 8 apples, 3 books), only *like* concrete numbers can be added. Thus, 2 books may be added to 3 books, but 2 books cannot be added to 3 cows or 3 buckets of water. Not in arithmetic!

The child is taught addition by example, by placing two balls with one ball, and making three balls; by placing four sticks with five sticks and making nine sticks; by adding the five fingers on one hand to the five fingers on the other hand, and making ten fingers; and so on. This process is clearly indicated in arithmetic by $2+1=3$; $4+5=9$; $5+5=10$; and so on.

Familiarity with numbers soon enables anyone to add such simple sums as these "in one's head," that is, without the aid of sticks, or fingers, or figures written on paper. Anyone can add any two, or any three, simple digits together without trouble. But as the numbers become larger the problem grows complicated, until at last, with many large numbers, only human wizards, or "lightning calculators," can perform such sums "in their heads."

Only simple sums are written with plus and equals signs. Longer sums are written in columns, each number being placed directly beneath the one above it, care being taken by the writer to place each units figure in the units column, each tens figure in the tens column, and so on. Thus, when written properly, a sum in addition has all the figures at the extreme right directly under one another in a straight vertical column, and the next column just as straight, and so on. A simple example (say $15+25+34$) would thus be written:

$$\begin{array}{r} 15 \\ 25 \\ +34 \\ \hline 74 \end{array}$$

The plus sign is usually placed, as shown, to indicate the arithmetical process being performed. A line is drawn at the foot of the column, and the sum is written beneath it. To add such a column one begins at the right-hand single column of digits, and adds upward (preferably), combining in one's head the units digits (4 and 5 make 9, and 5 make 14; or simply, when used to figures, 4, 9, 14). The units digits of the sum (which is here 4, for the sum is 14), is then written down in the units column of the total. The other digits (other than the units digit, here only one, 1) are "carried over" to the next column, and serve as a beginning (1 and 3 make 4, and 2 make 6, and 1 make 7), and the tens digit is placed in the answer.

A longer problem may be made up of "uneven" figures, so that the column, while even on the right, will be "ragged" on the left. Thus, to add 45, 672, 8, 59, 782, and 5,282:

$$\begin{array}{r} 45 \\ 672 \\ 8 \\ 59 \\ 782 \\ +5282 \\ \hline 6848 \end{array}$$

The mental process for this example is, in detail: 2 and 2 make 4, and 9 make 13, and 8 make 21, and 2 make 23, and 5 make 28, putting 8 down, and 2 "to carry"; 2 and 8 make 10, and 8 make 18, and 5 make 23, and 7 make 30, and 4 make 34, putting 4 down, and 3 to carry; 3 and 2 make 5, and 7 make 12, and 6 make 18, putting 8 down, and 1 to carry; 1 and 5 make 6.

The figures "to carry" from one column to another may be written down by beginners, in small numerals, at the tops of each column (as shown in the written example, Page 16, Plate I, No. 1), and added to the others when they are reached instead of being added at the bottom, so that the example in the illustration is added: 8 and 4 make 12, and 0 make 12, and 9 make 21, and 6 make 27, and 8 make 35, putting 5 down, with 3 to carry, and writing the 3 at the top of the second column; 7 and 3 make 10, and 0 make 10, and 2 make 12, and 5 make 17, and 7 make 24, and 3 make 27.

The addition of *decimals* is just the same as the addition of whole numbers. The decimal points are placed carefully one below another, and the columns of figures correspondingly (as in Plate I, No. 2). The decimal point of the answer falls directly beneath the decimal points above. The addition begins at the extreme right, as always, the only difference being that in a sum of decimals the extreme right column is not necessarily complete or "even."

To add rapidly the student must practise.

Facility in addition, or in any of the subsequent arithmetical processes, comes only through practise. The following exercises will serve to aid the student, the correct answers being given at the end of this book. When he has done these the student may make up his own exercises. He can check or verify his answers by adding his columns first *up*, and then *down* (covering the answer during the second process so that he won't be unwarily led into the same answer)—if he gets the same result both times, without trouble, his answer is likely to be correct. (Zeros may be written in, if desired, to fill out decimal columns.)

(A1)	(A2)	(A3)	(A4)	(A5)
4285	2766	22,764	22.756	.00015
2	421	4,877	.82	2.6123
464	80	7	3081.	4567.89
10	9	51,006	52.366	3.1416
555	8988	91	481.2	282.
6000	11	53	5271.008	3.771
987	908	711	32.66	491.876

PLATE I

(1)
$$\begin{array}{r} {\scriptstyle 1\,2\,3} \\ 5478 \\ 256 \\ 729 \\ 100 \\ 34 \\ +\ 2078 \\ \hline 8675 \end{array}$$

(2)
$$\begin{array}{r} {\scriptstyle 1\,2\ \,1} \\ 278.9 \\ .46 \\ 32.547 \\ +\ \ 2.8 \\ \hline 314.707 \end{array}$$

(3)
$$\begin{array}{r} 87612 \\ \times\ \ 428 \\ \hline 700896 \\ 175224 \\ 350448 \\ \hline 37,497,936 \end{array}$$

(4)
$$\begin{array}{r} 92.827 \\ \times\ \ 4.8 \\ \hline 742616 \\ 371308 \\ \hline 445.5696 \end{array}$$

(5)
$$\begin{array}{r} 5478 \\ -\ 1832 \\ \hline 3646 \end{array}$$

(6)
$$\begin{array}{r} 453.00 \\ -\ 127.88 \\ \hline 325.12 \end{array}$$

III. MULTIPLICATION

Multiplication is an expanded and somewhat special form of *addition*—it deals with adding a given number *to itself* a certain number of times. Thus, if 1 is added to 1, we have 2, or: $1+1=2$; and, similarly, if we add 1 and 1 and 1, we have 3, or: $1+1+1=3$. This is addition, certainly. But $1+1$ may also be expressed as 2 times 1, or, in arithmetical symbols: $2\times1=2$; and similarly with 1 and 1 and 1, which may be expressed as 3 times 1, or: $3\times1=3$. The times symbol ($\times$) therefore shows that a number is to be taken and added to itself a certain number of times. If two and two make four, it is equally true that two times two makes four also. And $2\times5=10$; $3\times4=12$; $4\times8=32$; $7\times5=35$; etc. Expressed as addition, these appear:

$$\begin{array}{r}5\\+5\\\hline10\end{array}\qquad\begin{array}{r}4\\4\\+4\\\hline12\end{array}\qquad\begin{array}{r}8\\8\\8\\+8\\\hline32\end{array}\qquad\begin{array}{r}7\\7\\7\\7\\+7\\\hline35\end{array}$$

Since the numbers from 1 to 12, inclusive, are often multiplied by each other in the various arithmetical processes and problems, it is very important to know at once, by memory, the *product* of any two of these low numbers. The *product* is the answer obtained when one number is multiplied by another (or, explained in terms of addition, when one number is added to itself a certain number of times, the number of times in each particular case being specified by another number). The complete list of the products of all the combinations of these numbers (1 to 12) is called the *multiplication tables.* Any number multiplied by one, as $1\times2=2$, is not changed in value. So the tables begin with the 2's:

$1\times2=2$	$1\times3=3$	$1\times4=4$
$2\times2=4$	$2\times3=6$	$2\times4=8$
$3\times2=6$	$3\times3=9$	$3\times4=12$
$4\times2=8$	$4\times3=12$	$4\times4=16$
$5\times2=10$	$5\times3=15$	$5\times4=20$
$6\times2=12$	$6\times3=18$	$6\times4=24$
$7\times2=14$	$7\times3=21$	$7\times4=28$
$8\times2=16$	$8\times3=24$	$8\times4=32$
$9\times2=18$	$9\times3=27$	$9\times4=36$
$10\times2=20$	$10\times3=30$	$10\times4=40$
$11\times2=22$	$11\times3=33$	$11\times4=44$
$12\times2=24$	$12\times3=36$	$12\times4=48$

Notice that either 9×4 or 4×9 are the same thing. In any expressed multiplication it makes no difference which number is put first, for the answer will always be the same. It is customary with larger numbers (each two digits or more), however, to place the smaller number first, as 42×526.

Only three of the multiplication tables have been given (preceding). The student can discover the others for himself from the following device—the product of 5 and 8 is secured by locating 5 in the extreme left vertical column, and running across on that line of numbers until the column headed by 8 is reached. The answer is thus seen to be 40. Or, going down the second column, the 2's table is there complete, for, at 4, the top number, 2, is multiplied by the left number, 2; and at 6, 2 is multiplied by 3; and at 8, 2 is multiplied by 4; and so on. These tables must be memorized.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

When a product is expressed in arithmetic, each portion of it has a name:

37	×	548	=	20,276
multi-plier	times sign	multi-pli-cand	equals sign	product (answer)

The *multiplier* tells the number of times that the *multiplicand* is to be added to itself. This could be expressed in addition by placing 548, the multiplicand in the preceding example, in a column 37 times, and then adding the column. But, in multiplication, this is simplified:

$$\begin{array}{r} 548 \\ \times\ 7 \\ \hline 3,836 \end{array} \qquad \begin{array}{r} 548 \\ \times\ 30 \\ \hline 16,440 \end{array} \qquad \begin{array}{r} 3,836 \\ +16,440 \\ \hline 20,276 \end{array} \qquad \begin{array}{r} 548 \\ \times\ 37 \\ \hline 3836 \\ 1644\ \\ \hline 20,276 \end{array}$$

Since the multiplier tells us to take 548 and add it to itself 37 times, suppose we do it first 7 times. Expressing it as in the first example above, we now use our knowledge of the multiplication tables (which *must* be learned by heart by every student of arithmetic), and perform the operation something like this: 7 times 8 is 56, putting down 6, with 5 to carry; 7 times 4 is 28, with 5 carried over from the previous multiplying, making 33, putting down 3 and having 3 to carry; 7 times 5 is 35, with 3 to carry, making 38, and, since this is the last operation, the whole 38 is put down. The product of 7 times 548 is thus 3836. But we must take 548, not 7, but 37 times, so we now multiply it by 30 (having already done 7 of the 37).

In multiplying by 30, we first multiply 548 by 0, and since the product of any number and 0 is always 0 (if you have nothing, and add it to itself you still have nothing, and if you multiply it any number of times, you still and always have nothing!), we write the 0 down at once, directly under the 0 multiplier. Multiplying 548 by 3 is a simple matter (3 times 8 is 24, 4 down, 2 to carry; 3 times 4 is 12, adding the 2 carried, is 14, 4 down, 1 to carry; 3 times 5 is 15, adding the 1 carried is 16, and 16 is put down since it is the last amount). and the result of 548 times

30 is obtained as 16440. To find the product of 548 and 37 we now have only to add these two smaller products. Doing so, we find the answer is 20,276.

This is simple multiplication. As the student grows more familiar with his material, however, he will do his multiplying and adding all at once. This is shown in the operation of 548 times 37 directly—548 is first multiplied by 7, as before, the first figure of the answer being placed directly under the multiplier (7) in the units column, and the second figure in the tens column, and so on; 548 is then multiplied by the next figure of the multiplier (3), the first figure of the answer being placed *directly under the multiplier* (3). (It is thus unnecessary to multiply by 0 here, for the same thing is accomplished by the indentation secured when the second product is written directly under its multiplier.)

Longer problems are performed in the same way. Write down the multiplicand, and, directly beneath it, as though for addition, the multiplier. Multiply the multiplicand by the first (righthand) figure of the multiplier, placing the first (righthand or units) figure of the product directly under the first multiplier. Then multiply the multiplicand by the second (from right to left) figure of the multiplier, writing the first (righthand) figure of the product directly under the second multiplier. And so on. The final product is obtained by adding the partial products. A problem in multiplication, when correctly performed by hand, will appear as No. 3, in Plate I, Page 16.

When decimals are multiplied, the process is exactly the same. The position of the decimal point in the answer is determined by adding together the "number of places" (counting from left to right from the decimal point) in the multiplicand and multiplier, and counting off this total (from right to left) in the final product. See No. 4, Plate I, Page 16.

Examples for practise (correct answers at end of book):

(A6)	(A7)	(A8)	(A9)	(A10)
135	678	5712	5278	20037
×2	×34	×49	×307	×5074

(A11) 28.909 × 66
(A12) 2,378,001 × 800
(A13) 234.506 × .32
(A14) 2.11807 × 3.6
(A15) $54.76 × 89

IV. SUBTRACTION

Subtraction is the reverse of addition. If you have five cents, and lose two of them, you have how many left? Three. Two is thus taken from five, leaving three. This is subtraction. Or, expressed arithmetically, the problem is:

$$5-2=3$$

The sign of subtraction (—) is read *minus*. All of the possible subtractions between the numbers from 1 to 12 inclusive are probably familiar to the student. If not, he should practise until he knows them. For the sums (addition), products (multiplication), and differences (subtraction) of or between any two of these numbers must be known as well as the student knows his own name.

In subtraction there can never be more than two numbers. They are arranged as though they were to be added:

```
 5478 minuend
—4329 subtrahend
─────
 1149 difference (remainder)
```

The greater of the two numbers is the *minuend*, and is always placed "on top." The lesser of the two numbers is the *subtrahend*. The answer is the *difference*, or *remainder*. A minus sign is usually placed at the left, as shown, to indicate the operation being performed.

Performing the operation, we begin always at the righthand or units column. The first thing to do is to take 9 from 8, but to do this we must "borrow one" from the tens column to make the 8 into 18, for 9 is smaller than 8, and we cannot subtract unless the minuend is greater than the subtrahend. Taking 9, then, from 18 (and remembering we have borrowed one from 7, making the 7 virtually a 6), we write down the difference, 9. Proceeding, we now take 2 from 6 (one was borrowed from the 7), and write down the difference, 4. Then 3 from 4 is 1, and 4 from 5 is 1. The complete difference is then 1149.

The way a problem in subtraction should look is shown by No. 5, Plate I, Page 16. No. 6, Plate I, shows the subtraction of decimals. The point keeps its position as in addition. But it should be noted that the subtrahend (as in the example shown on the plate) may sometimes have more decimal digits than the minuend. When this happens the minuend is filled out to the required number of places with zeros (as shown). The subtraction is then performed as usual, 1 being borrowed from the lefthand digit for each cipher, making the operation proceed: 8 from 10 leaves 2, and 8 from 9 leaves 1, and 7 from 12 leaves 5, and 2 from 4 leaves 2, and 1 from 4 leaves 3.

An example in subtraction may be *proved* (that is, the answer may be tested for correctness) by adding the subtrahend and the difference. If this gives the minuend, the example has been done correctly. Thus, in No.

5, Plate I, 3646 plus 1832 is 5478, so the answer 3646 is correct.

Exercises (correct answers at end of book):

(A16)	(A17)	(A18) Take 32 from 566.
5411	245.006	(A19) Take 54.83 fom 465.
—391	—1.999	(A20) Take 2,800 from 3,009.

V. DIVISION

Division is the process of finding how many times one number is contained in another. Thus, if you have six apples and wish to divide them evenly among three boys, what do you do? You find out how many times 3 is contained in 6 by "dealing out" the apples to the boys—each receiving 2, so 3 is contained 2 times in 6. Therefore, 6 divided by 3 equals 2, or $6 \div 3 = 2$. The sign of division ($\div$) is read *divided by*.

The number (here 6) that contains the other is called the *dividend;* the number (here 3) contained in the dividend is the *divisor;* and the number of times (here 2) the divisor is contained in the dividend is the answer or *quotient.*

Division may also be expressed in arithmetic in the form of a *fraction*:

$$\frac{6}{3} = 2 \text{ OR } 6 \div 3 = 2$$

The horizontal line then has the same meaning and force as the division sign.

Division is seen to be a form of subtraction, for it consists in seeing how many times one number can be subtracted from another, with or without a remainder. Thus, in dividing 6 by 3 it can be said that 3 is taken from 6 once, leaving 3, so it can be taken away again, leaving nothing—or 3 can be subtracted from 6 twice, leaving no remainder, so 3 is contained in 6 two times. Division is also the reverse of multiplication, for if $6 \div 3 = 2$, it is also true, reversing the expression, that $2 \times 3 = 6$. So in performing the division the student must memorize the reverse of the multiplication tables, and not only know that $4 \times 5 = 20$ and $6 \times 7 = 42$, but also that $20 \div 5 = 4$ and $42 \div 7 = 6$.

Since multiplication and division are opposite processes, one operation can be used to prove the other. Thus, after obtaining a product of a multiplier and a multiplicand, the product can be verified by dividing it by the multiplier. If this gives the multiplicand, the answer is correct. Similarly, to prove a quotient, the quotient may be multiplied by the divisor. If this gives the dividend, the answer is correct.

Problems in division between any of the numbers from 1 to 12 inclusive may, after the tables have been duly learned, be performed in one's head. These are comparatively simple, and should be practised until the student has them all at his tongue's end.

But there is what is known as *long division*, which involves numbers beyond 12, often of many digits. Short division is simply:

$$\begin{array}{rrl} & 208104 & \text{quotient} \\ \text{divisor } 4 & \overline{)832416} & \text{dividend} \end{array}$$

The divisor (here 4) is "put into" the first digit (lefthand) of the dividend (here 8), and the number of times it is contained (here 2) is placed above for the first figure of the quotient. Then 4 is put into 3, and goes no times, so 0 is placed above. The 3 (or whatever remainder there may be) is now carried over to the next digit, making it 32, so that 4 is now put into 32 and goes 8 times. Then 4 into 4 is 1, and 4 into 1 is 0, and 4 into 16 is 4. This problem happens to "come out even," that is, without an odd remainder. If the last figure of the dividend were 7, the last operation would then be 4 into 17, which goes 4 times with a remainder of 1. To express 1 divided by 4, the 1 is simply placed over the 4, making a fraction, ¼, which is placed at the end of the quotient, which would then be 208,104¼.

Long division is expressed in a similar way, but it cannot be performed in the head of an ordinary mortal, so more figures are needed. A problem in long division is worked on Plate II (No. 1), Page 29. Here 272 is the divisor, 4,765,932 is the dividend, and 17,521 55/68 is the quotient. Since the divisor contains more than one digit, it must first be put into a portion of the dividend which will contain it. So 272 is first put into 476, the first three figures of the dividend. A little examination shows that it won't go twice, so it must go once (since 476 is greater than 272, although not twice as great). The first figure of the quotient is therefore 1, which is placed above the dividend, directly over the *last* figure of the portion of the dividend into which the divisor was put to obtain the 1 (here 1 is placed over the 6). The divisor (272) is now multiplied by this first figure of the quotient (1), and the product (272) is placed directly beneath the

PLATE II

(1)

$$
\begin{array}{r}
17521\frac{55}{68} \\
272\,\overline{)4765932} \\
\underline{272} \\
2045 \\
\underline{1904} \\
1419 \\
\underline{1360} \\
593 \\
\underline{544} \\
492 \\
\underline{272} \\
220
\end{array}
\qquad
\frac{220 \div 4}{272 \div 4} = \frac{55}{68}
$$

(2)

$$
\begin{array}{r}
171.271+ \\
52.38\,\overline{)8971.20000} \\
\underline{5238} \\
37332 \\
\underline{36666} \\
6660 \\
\underline{5238} \\
14220 \\
\underline{10476} \\
37440 \\
\underline{36666} \\
7740 \\
\underline{5238} \\
2502
\end{array}
$$

(3)

$$
\frac{\cancel{2} \times \overset{4}{\cancel{12}} \times \overset{7}{\cancel{49}} \times \overset{5}{\cancel{50}}}{\cancel{3} \times \underset{7}{\cancel{14}} \times \underset{7}{\cancel{35}} \times \underset{10}{\cancel{40}}} = \frac{1}{1}
$$

(4)

$$
\frac{77 \times \frac{5}{11} \times \frac{10}{13} \times 26}{\frac{4}{7} \times 30 \times 8\frac{1}{3} \times 9} =
$$

$$
\frac{\overset{7}{\cancel{77}} \times \cancel{5} \times \cancel{10} \times \overset{2}{\cancel{26}} \times 7 \times \cancel{3}}{\underset{2}{\cancel{4}} \times \underset{3}{\cancel{30}} \times \underset{5}{\cancel{25}} \times \underset{3}{\cancel{9}} \times \cancel{11} \times \cancel{13}} = \frac{49}{90}
$$

476, units under units, tens under tens, as in addition or subtraction. Now 272 is subtracted from 476, the remainder (204) being placed directly beneath as shown.—The first step is now completed, and we have the first figure of the quotient. To proceed it is necessary to bring down the next figure of the dividend (here 5), and place it with the 204, making it 2045. The divisor (272) is now put into this figure (2045), and until the student is experienced in division he may have to try one or two possibilities before he gets the right number of times that it is contained. (He may try 8, but if 272 is multiplied by 8 he will find that the product is greater than 2045. He may try 6, but if 272 is multiplied by 6, he will find that when the product is subtracted from 2045 the remainder is greater than 272, so it will go more than 6 times. It must then go 7 times.) It is found that 272 goes 7 times into 2045, so 7 is the next figure of the quotient. The 7 is placed directly above the figure (5) which was brought down, and the divisor (272) is then multiplied by 7, the product (1904) being placed under 2045. The 1904 is then subtracted from 2045, the difference (141) being written below.—The second step is now completed. The next figure (9) is brought down to the 141, making it 1419, and 272 is put into 1419. Finding that it goes at least 5 times, 5 is placed in the quotient, 272 is multiplied by 5 as before, and the work goes on until all the figures of the dividend have been brought down in turn. After the last figure of the dividend is brought down, the division performed, and the subtraction completed, there may be, as in this case, a remainder. This remainder is then placed over the divisor (220 over 272) in the form of a fraction, as shown. It is possible to "reduce" this big fraction by dividing both top and bottom parts of it by 4—a process which will be explained more at length under Fractions. The remainder is thus 55/68 and is placed in the quotient.

To simplify the process above, the student, when dividing 272 into 476, may consider the operation step by step. The first figure of the divisor (2) goes into the first figure of the dividend (4) twice. So 272 may go into 476 twice. Trying it, however, shows that this is not so, for the other figures (72) won't go into 76 twice. But this preliminary test serves to show approximately how many times the divisor may go into a particular portion of the quotient.

It might be that a three-figure divisor (as 272) wouldn't go into the the first three figures of the quotient even once (suppose 476 were here 176). Since 272 won't go into 176, then, it must be put into the first four figures, or into 1765. The first figure of the quotint is then written over the 5, and the first figure to be brought down will be 9, and the process goes on as before. Similarly, in one of the remainders, say the second one (204 with the 5 brought down, 2045), if the figures are smaller than 272 (suppose the remainder was only 4, which with the 5 brought down became 45), the divisor won't go at all. The next figure of the quotient is then 0 (placed in this case over the 5), and another figure of the dividend must be brought down (here 9, making 459), when the process goes on as before.

To divide decimals, the divisor must be treated as a whole number. If the divisor contains a decimal, the point must be moved to the right of the last digit, making the divisor in effect a whole number. But this must be compensated for in the dividend by moving the decimal point of the dividend the same number of places to the right as the decimal point of the divisor was moved. Thus, in example No. 2, Plate II, Page 29, there were two decimal places in the divisor (52.38), so the point is moved to the right of the 8 (5238.), or two places to the right. The decimal point in the dividend (8971.2) must therefore be moved two places to the right also (897120.), a zero being added to make the required second digit. If there are no decimal digits expressed in the dividend, the point is moved just the same, beginning at the end of the whole number, zeros being added for each placed marked off. The process of division then goes on as usual, the decimal point of the quotient being placed directly above the decimal point of the dividend.

Fractional remainders do not occur in the division of decimals, for the quotient can always be "carried out" to any desired number of decimal places. In the example shown, zeros are added after the decimal point in the dividend, and brought down as required, the quotient being carried out to three decimal places (171.271). There is, finally, a remainder of 2502 which is not accounted for. This is not usually put in the quotient, but is discarded for all practical purposes. If the divisor would go into this remainder 5 times or more, it would be proper to change the last figure of the decimal quotient to one more than it is, making the quotient 171.272. To show (when desired) that the quotient is a little more than the actual division, a minus sign would be added (in this case 171.272—); and, similarly, to show that it is a little less than the actual division, a plus sign may be added (171.271+).

Exercises (correct answers at end of book):

(A21) Divide 962721 by 3.
(A22) Divide 243612144 by 12.
(A23) Divide 4170 by 1.5 (until it comes out even).
(A24) Divide 172788 by 308.
(A25) Divide 3712598 by 891.
(A26) 23418 ÷ 32.
(A27) 241.4 ÷ 28 (to three places).
(A28) 3098.22 ÷ 2.96 (to four places).
(A29) .0065 ÷ 2.3 (to five places).
(A30) 1 ÷ 2 (decimally, to one place).

VI. FACTORING AND CANCELATION

The *factors* of a number are those numbers which, if multiplied together, will give the given number. Thus, 5 and 2 are the factors of 10, for if 5 and 2 are multiplied together (5×2) they give 10. Similarly, 8 and 7 are the factors of 56, for $8\times7=56$. Also 2 and 2 and 2 and 7 are factors of 56, for $2\times2\times2\times7=56$.

A *prime factor* is a number which cannot be divided by any other number (except itself and 1). Thus 3 is a prime factor, for it cannot be divided (without a remainder) by any numbers except 3 (itself) and 1. The prime factors proceed from 1 as follows: 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, etc. Any number which is not a prime number or prime factor may be divided into two or more prime factors. Thus, 4 is not a prime factor, for its prime factors are 2 and 2.

To find the prime factors of a given number, it is only necessary to divide, beginning with the smallest prime factor which will go evenly into the number. Thus, to find the prime factors of 13860:

```
2)13860
 2)6930
 3)3465
 3)1155
   5)385
    7)77
      11
```

The prime factors are therefore:

$$2\times2\times3\times3\times5\times7\times11=13860.$$

This process of factoring may be used in division to simplify the divisor and dividend. When so used it is called *cancelation*, for it consists in canceling out certain *common* factors. When two numbers have among their prime factors one or more of the same prime factors, these duplicated factors are *common* to both numbers.

Thus, if it is desired to divide the product of $2\times12\times49\times50$ by the product of $3\times14\times35\times40$, the problem may be written:

$$\frac{2\times12\times49\times50}{3\times14\times35\times40}=\frac{1}{1}=1$$

The process of canceling out the prime factors, as shown on Plate II, No. 3, Page 29, thus brings us to the answer, 1, in very short order. This is certainly easier than multiplying out to find that the divisor and dividend are equal. For division is rapid: 3 goes into 12 4 times, and 4 goes into 40 10 times, and 10 goes into 50 5 times, and 5 goes into 35 7 times, and 7 into 49 7 times, and 7 into 14 twice, and 2 into 2 once. When the factors go once, the 1 is usually not written. Since all the factors of the dividend become 1, we have $1\times1\times1\times1=1$, and the same for the divisor. And $1\div1$ is certainly 1, so that's the answer.

It is often possible to greatly simplify a problem in division—even to perform the entire division—by cancelation:

	210,576		
3	631,728		
7	4,422,096		
2	8.844,192	=	210,576
2	184,254		4,837
7	92,127		
3	13,161		
	4,387		

Thus 210,576 divided by 4,387 is far simpler than the original problem. It so happens that cancelation can be carried further if it is seen that 41 is a prime factor of both numbers—but to test factors all the way up to 41 is often as arduous as it is to perform the indicated division, so, in canceling, factors beyond 11 are not usually considered (unless, of course, they are evident for some reason). The factors used in the above cancelation are indicated by the figures at the left.

Exercises (correct answers at end of book):

(A31) Find the prime factors of (a) 3,080; (b) 735; (c) 1,188; (d) 7,812; (e) 33,096.

(A32) Divide $37 \times 64 \times 210$ by $74 \times 16 \times 56 \times 6$. (Simplify first by cancelation.)

(A33) Divide $108 \times 1000 \times 49$ by $24 \times 81 \times 625 \times 56$. (Simplify first by cancelation.)

(A34) Divide $169 \times 42 \times 121 \times 150$ by $26 \times 39 \times 77 \times 33$. (Simplify first by cancelation.)

(A35) Divide 194,040 by 166,320. (Simplify by cancelation.)

VII. FRACTIONS

Fractions, as the term is now applied in arithmetic, include only what were once distinguished as *common fractions*, and are exclusive of *decimal fractions*, which are now called simply *decimals*. That decimals and fractions are closely related will soon be seen. As before explained, a *whole number* or *integer* may be subdivided into several equal parts—these parts being *fractions* of the whole. Thus, if a pie is cut in half, each half is ½ of the whole pie—and ½ is a fraction. Similarly, if the pie is cut in six pieces, each piece is 1/6 of the whole pie—and 1/6 is another fraction. If someone takes one piece, five pieces are left, making up 5/6 of the whole pie—and 5/6 is another fraction. As seen, a fraction (as said, exclusive of decimals) is written by placing one number over another, the top number (or *numerator*) telling how many times the bottom number (or *denominator*) is to be taken Thus, when speaking of 5/6 of a pie, the numerator (5) names the number of pieces, so to speak, and the denominator (6) tells what part each piece is as related to the whole—tells us, indeed, that the whole pie has been divided into six pieces, and that we are here considering five of them.

Although we certainly will not always be dealing with pies, we cannot get along in arithmetic without using fractions. We must be familiar enough with them to add, subtract, multiply, and divide them, just as we can do with whole numbers.

The more familiar fractions range from ½ to 1/16—½ being a great deal larger than 1/16, for the larger the denominator the smaller the part that the fraction represents. It is clear that something divided into two parts, so that each is one-half, has larger parts than something divided into sixteen parts, so that each is one-sixteenth. Similarly, ½ is larger than 1/3 and 1/3 is larger than 1/4. Tabulating, the everyday fractions are:

1/2 (one-half)

1/3 (one-third)
2/3 (two-thirds)

1/4 (one-fourth)
3/4 (three-fourths)

1/5 (one-fifth)
2/5 (two-fifths)
3/5 (three-fifths)
4/5 (four-fifths)

1/6 (one-sixth)
5/6 (five-sixths)

1/7 (one-seventh)
2/7 (two-sevenths)
3/7 (three-sevenths)
4/7 (four-sevenths)
5/7 (five-sevenths)
6/7 (six-sevenths)

1/8 (one-eighth)
3/8 (three-eighths)
5/8 (five-eighths)
7/8 (seven-eighths)

1/9 (one-ninth)
2/9 (two-ninths)
4/9 (four-ninths)
5/9 (five-ninths)
7/9 (seven-ninths)
8/9 (eight-ninths)

1/10 (one-tenth)
3/10 (three-tenths)
7/10 (seven-tenths)
9/10 (nine-tenths)

1/16 (one-sixteenth)
3/16 (three-sixteenths)
5/16 (five-sixteenths)
7/16 (seven-sixteenths)
11/16 (eleven-sixteenths)
13/16 (thirteen-sixteenths)
15/16 (fifteen-sixteenths)

The above are all what are known as *proper fractions* because the numerators are all smaller than their denominators. An *improper fraction* is a fraction in which the numerator is larger than the denominator, as 5/4 or 7/3. Every improper fraction is *more than* a whole number, just as every proper fraction is *less than* a whole number. Thus, 7/3 is one whole number (3/3), and another (3/3), making two whole numbers, with 1/3 left over—or 2 1/3. A whole number with a fraction, as this 2 1/3, is called a *mixed number*. It is read *two and one-third.*

A fraction in which the numerator and denominator are equal is equa to 1. Thus $3/3=1$, because both numerator and denominator can be divided by 3 (cancelation), and we have 1/1, or 1.

The *terms* (numerator and denominator) of a fraction may be large or small. Thus,

$$\begin{array}{ll} \text{numerator} & 2891 \\ \hline \text{denominator} & 3276 \end{array}$$

is just as much a fraction as 1/2. But we learned before that this was another way to express division, and so it is. For a fraction is nothing more than an expressed division—1/2 really signifies $1\div2$, for it is one (whole one) divided into two parts, of which one part is here taken. So, if we say 2/3, we mean $2\div3$; one whole one is divided into three parts, of which two are here taken. To read 2/3 as *two-thirds* is really a short way of saying *two one-thirds.*

REDUCTION OF FRACTIONS. — Fractions may be changed in form without being changed in value. Thus, by cancelation, it is seen that $4/12=1/3$. The fraction 4/12 is thus said to be *reduced* to 1/3, for its value has not been changed. The process of cancelation as applied to fractions is sometimes called *simplifying.*

By cancelation it is seen that dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction. Similarly, if both numerator and denominator are multiplied by the same number, the value is not changed either. Turning the preceding example about, it can be said that both numerator and denominator of 1/3 are multiplied by 4, giving 4/12, and, as we have seen, the value remains the same.

A fraction may be changed from one denominator to another, providing that one denomiator is a factor or multiple of the other. (A *multiple* of a number is a product of the number and some other number. Thus, 25 is a multiple of 5—a product of 5 and 5.) Thus, 5/6 may be reduced to thirty-sixths (that is, to a fraction with the denominator 36), since 36 is a multiple of 6. Reversing, 30/36 may be reduced to sixths, for 6 is a factor of 36. To reduce sixths to thirty-sixths, the new denominator, 36, is first divided by the old (if it is larger, as here), and the quotient (here 6) is then used as a multiplier for both numerator and denominator of the fraction to be reduced:

$$\frac{5 \times 6}{6 \times 6} = \frac{30}{36}$$

And 30/36 can be reduced to 5/6 by the already familiar process of cancelation, or simplifying. When 30/36 is thus simplified, it is *reduced to lower terms,* for 5 and 6, the terms, are both lower than 30 and 36, the former terms.

A fraction is said to be reduced to its *lowest* *terms* when both numerator and denominator are such that they do not possess any common factors—or when numerator and denominator cannot both be divided by the same number without a remainder. Thus 5/8 is in its lowest terms, for 5 and 8 have no common factors.

A whole number or a mixed number may be readily changed to an improper fraction of the same value. Suppose you wish to change 9 to thirds. How many thirds are there in 9 whole units? There are 3 thirds in every unit, for each whole can be divided into three equal parts, so there must be 9 times as many in 9 units, or 27. Expressing this as a fraction (improper), we have $9 = 27/3$. (Turning about, 27/3 may be reduced to 9.)

Similarly, 15 1/4 may be changed to an improper fraction by changing the 15 to fourths, and adding the extra fourth. If there are 4 fourths in 1, there are 15 times as many in 15, so, to change a mixed number to an improper fraction, multiply the whole number by the denominator of the fraction, add the numerator of the fraction to the product, and place the final result as the new numerator over the old denominator:

$$15\tfrac{1}{4} = \frac{4 \times 15 + 1}{4} = \frac{61}{4}$$

(Turning about, 61/4 may be reduced to 15 1/4 by dividing 4 into 61, and placing the remainder 1, over the denominator, 4.)

ADDITION OF FRACTIONS.—Fractions, to be added, must have the same denominators—that is, they must have *common denominators*. It is possible to add 3/4, 1/4, and 5/4 at once:

$$\frac{3}{4}+\frac{1}{4}+\frac{5}{4}=\frac{9}{4}$$

The numerators only are added, the resultant fraction having the same denominator. For, in principle, 3/4 signifies three one-fourths, and 5/4 five one-fourths, so if you add three to one to five, you have nine one-fourths.

But to add 5/9, 7/16, 9/10, and 5/8, a *common denominator* must be found. It is reasonable to perceive at once that the product of all four denominators will provide a common denominator, but this may be a great deal larger than necessary. If the fractions are added by using a common denominator larger than necessary, the resultant sum will have to be simplified. In arithmetic every process is made as simple and labor-saving as possible, so it is necessary to find not only the common denominator, but the *least common denominator* (often abbreviated l. c. d.).

To find the least common denominator of 5/9, 7/16, 9/10 and 5/8, we must examine the denominators (9, 16, 10, 8) for a multiple that will be the smallest multiple of each. We can hit upon this by guessing, but this may be an arduous method. So we do it arithmetically:

$$\begin{array}{r|rrrr}
2 & 9, & 16, & 10, & 8 \\ \hline
2 & 9, & 8, & 5, & 4 \\ \hline
2 & 9, & 4, & 5, & 2 \\ \hline
2 & 9, & 2, & 5, & 1 \\ \hline
3 & 9, & 1, & 5, & 1 \\ \hline
3 & 3, & 1, & 5, & 1 \\ \hline
5 & 1, & 1, & 5, & 1 \\ \hline
 & 1, & 1, & 1, & 1 \\ \hline
\end{array}$$

The least common denominator is therefore $2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5$, or 720—not a very small one, to be sure. (*Explanation*: To find the least common denominator of several unlike denominators, place the denominators in a row, separating one from another by commas. Then divide by factors that will go into at least one of the denominators without remainder, usually proceeding with the prime factors from 2 up, in each division bringing down unchanged any denominators into which the factor-divisor will not go evenly. Continue this until the last row is entirely 1's. The product of the resultant factors will be the least common denominator.)

Using this least common denominator, we now reduce each fraction to a fraction with this denominator:

$$\frac{5 \times 80}{9 \times 80} = \frac{400}{720} \qquad \frac{7 \times 45}{16 \times 45} = \frac{315}{720}$$

$$\frac{9 \times 72}{10 \times 72} = \frac{648}{720} \qquad \frac{5 \times 90}{8 \times 90} = \frac{450}{720}$$

And, adding:

$$\frac{400}{720} + \frac{315}{720} + \frac{648}{720} + \frac{450}{720} = \frac{1813}{720}$$

$$\begin{array}{r} 400 \\ 315 \\ 648 \\ +\ 450 \\ \hline 1813 \end{array}$$

Thus, fractions with large or small denominators may be added. To add mixed numbers, you must first change the mixed numbers to improper fractions, and then find the least common denominator of those fractions. Or, in more simple form, find the least common denominator of the fractional portions of the mixed numbers. Thus, to add 15 5/6 and 18 1/15 and 24 7/30—find first the least common denominator of 5/6, 1/15, and 7/30. A little examination shows that this is 30, so these fractions are reduced to fractions each having 30 as a denominator: 25/30, 2/30, 7/30. And, adding:

$$\begin{array}{r} 15\frac{25}{30} \\ 18\frac{2}{30} \\ +\ 24\frac{7}{30} \\ \hline \end{array}$$

$$57\frac{34}{30} = 58\frac{4}{30} = 58\frac{2}{15}$$

Explanation: Adding the fractions first, the total is found to be 34/30. This is an improper fraction which may be simplified to 1 2/15. The whole unit is therefore added to the sum of the whole numbers, making the final result 58 2/15.

SUBTRACTION OF FRACTIONS. — Before fractions with unlike denominators can be subtracted one from another they must be reduced to fractions with the same denominators. So, as in addition, the least common denominator must first be found. The numerators are then subtracted, and the difference is placed over the common denominator to form a new fraction—the answer. Thus, 8/15 minus 9/60 is (l. c. d. = 60):

$$\frac{32}{60} - \frac{9}{60} = \frac{23}{60}$$

Mixed numbers are subtracted in a way very similar to that used in adding them:

$$9\tfrac{2}{3} - 5\tfrac{5}{8} \quad (\text{l. c. d.} = 24)$$

$$\begin{array}{r} 9\dfrac{16}{24} \\ -\ 5\dfrac{15}{24} \\ \hline 4\dfrac{1}{24} \end{array}$$

But the lower fraction, in subtraction, may sometimes be greater than the upper:

$$\begin{array}{r} 9\dfrac{16}{24} \\ -\ 3\dfrac{18}{24} \\ \hline 5\dfrac{22}{24} \end{array} = 5\ 11/12$$

In this case, a whole unit, equal to 24/24, is borrowed from the 9, making the 16/24 into 40/24. 18/24 is then subtracted from 40/24, leaving 22/24, or 11/12; and 3 is then subtracted from 8, for 1 has been borrowed from the 9. Similarly, if there is no fraction at all in the minuend, a whole unit is borrowed from the whole number, and considered as a fraction with the denominator of the fraction of the subtrahend.

MULTIPLICATION OF FRACTIONS.—To multiply fractions it is not necessary to change them to fractions with a least common denominator. The numerators are multiplied together, forming by their product the numerator of the answer; and the denominators are multiplied together, and their product is the denominator of the answer. Thus:

$$\frac{3}{5} \times \frac{7}{16} = \frac{3 \times 7}{5 \times 16} = \frac{21}{80}$$

Or, if a fraction is multiplied by a whole number, the whole number is regarded as a fraction with 1 for a denominator (*of* may stand for the times sign, as "⅝ of 6"=⅝×6):

$$\frac{5}{8} \times 6 = \frac{5 \times 6}{8 \times 1} = \frac{30}{8} = \frac{15}{4} = 3\tfrac{3}{4}$$

If the product can be simplified, it is always best to do so, as shown.

If several fractions are multiplied together, it may be possible to simplify by cancelation:

$$\frac{\overset{1}{15}}{\underset{2}{16}} \times \frac{\overset{1}{11}}{\underset{2}{30}} \times \frac{\overset{\overset{1}{8}}{32}}{\underset{\underset{1}{4}}{44}} = \frac{1}{4}$$

(In this example the 1's are printed because the numbers canceled are not crossed out.)

Mixed numbers are multiplied by first changing them to improper fractions:

$$7\frac{3}{5} \times 2\frac{5}{6} = \frac{\overset{19}{38}}{5} \times \frac{17}{\underset{3}{6}} = \frac{323}{15} = 21\frac{8}{15}$$

The example was simplified a little, as shown, by canceling 2 from 38 and 6.

To multiply a mixed number by a whole number, the fractional part of the mixed number is treated separately, and, if it is an improper fraction, any units derived therefrom are added to the product of the whole numbers:

$$\begin{array}{r} 38\frac{7}{8} \\ \times\ 4 \\ \hline \frac{28}{8} \\ 152 \\ \hline 155\frac{1}{2} \end{array} \qquad\qquad \begin{array}{r} 29 \\ \times\ 5\frac{5}{9} \\ \hline 9\ )\ \underline{145} \\ 16\frac{1}{9} \\ 145 \\ \hline 161\frac{1}{9} \end{array}$$

The principle is the same in both examples, but when the fraction is in the multiplier instead of in the multiplicand, the simplification of the improper fraction is accomplished at once in the manner shown. Thus, $\frac{5}{9} \times 29 = \frac{145}{9}$, and this is simplified by dividing the numerator, 145, by the denominator, 9, without more ado. This may always be done where preferred.

DIVISION OF FRACTIONS.—To divide one fraction by another it is only necessary to invert the divisor and proceed as in multiplication. To *invert* a fraction is to turn it "upside down," so that what was the denominator becomes the numerator, and what was the numerator becomes the denominator. Thus, 3/4 inverted becomes 4/3.

The principle of thus dividing is shown when it is seen that fractions may be divided by dividing the numerators or multiplying the denominators:

$$\frac{8}{9} \div 4 = \frac{8 \div 4}{9} = \frac{2}{9}$$

$$\text{or, } \frac{8}{9} \div 4 = \frac{8}{9 \times 4} = \frac{8}{36} = \frac{2}{9}$$

Or, what has really been done in the second case, the whole number is regarded as a fraction with the denominator 1, and is inverted, and the resultant fractions are multiplied:

$$\frac{8}{9} \div 4 = \frac{8}{9} \div \frac{4}{1} = \frac{8}{9} \times \frac{1}{4} = \frac{8}{36} = \frac{2}{9}$$

Similarly, with fraction divided by fraction:

$$\frac{3}{5} \div \frac{7}{8} = \frac{3}{5} \times \frac{8}{7} = \frac{24}{35}$$

It should be remembered that 4 over 9, or 4/9, signifies 4÷9, the horizontal line taking the place of the division sign. Thus, a long problem in division, where fractional factors are involved, may be expressed as in No. 4, Plate II, Page 29, and be correspondingly simplified by inversion of the fractions, and canceling. Thus, considering each fraction separately, it can be said that 5/11 is to be divided by all below the long horizontal line. Since this is so, we can invert and multiply, so we put 5 in the numerator and 11 in the denominator of the "big" fraction—considering all above the long line as the numerator and all below it as the denominator. Similarly with 4/7 in the denominator: we put 7 in the numerator and leave 4 in the denominator. Which amounts to transferring only the denominators of the fractions. Notice that the mixed number, 8 1/3, is first changed to an improper fraction, 25/3.

Important Note: This cancelation cannot be done if a plus or minus sign occurs above or below the line. Only when the numerator and denominator are continuous (expressed) *products* is cancelation permissible. When cancelation, for any reason, cannot be used, the result must be obtained by multiplying out the numerator and denominator and performing the expressed division.

Exercises in fractions (correct answers at end of book):

(A36) Reduce (a) ⅜ to fortieths; (b) ⅔ to fifteenths; (c) 25/100 to fourths.

(A37) Reduce the following fractions to their lowest terms: 54/81; 27/9; 10/200; 55/99.

(A38) Change the following mixed numbers to improper fractions: 24⅓; 10½; 52¾; 19⅞.

(A39) Add ⅛, ⅜, ⅝ and ⅞.

(A40) Find the l. c. d. of 5/6, 7/12, ⅝, ⅔, ¾, and ½. Then add the fractions together.

(A41) Add 3/15, 9/20, and 11/30; add 11/16, ⅜, ¼, and 3/2.

(A42) Add 24 5/9 and 56 7/18 and 12⅔.

(A43) Subtract: 3/7 from 13/14; 4/5 from 9/10; 49/50 from 199/200; 39⅞ from 42; 14 5/9 from 27⅔; 4/11 from 18 7/11.

(A44) Multiply 15/16 by ½; ⅗ by 2/7; 24 1/6 by 2¼; 58 by 13 5/7.

(A45) Divide 12 by ¾; 7/15 by ⅔; 12¼ by 5/6; ⅞ by 3.

(A46) Simplify 22 × 14 5/6 × 11/16 × 72 over 33 9/11 × 175 × 11/12.

VIII. DECIMALS

A decimal, as has been explained, is a fraction expressed in some multiple of ten—tenths, hundredths, thousandths, ten-thousandths, etc. The nature of decimal fractions is very clearly shown in the monetary system of the United States, which is known as a decimal system. The unit is the dollar, which is equal to 100 cents. Ten cents is then one-tenth of a dollar, and is expressed decimally by .10, or, with the dollar sign, $.10. Read decimally, this is ten-hundredths, but one-hundredth is one cent, so this is ten cents. (*Cent* comes from the Latin *centum*, meaning "hundred.") Similarly, $.25 = 25c or twenty-five cents, and may be read decimally as twenty-five hundredths. This is one-quarter of a dollar, so that the coin of this value is colloquially called a *quarter*.

Decimals are equivalents of common fractions, which they largely replace because it is much easier to perform operations with decimals than with common fractions, especially when the fractions are large ones. The equivalent relation between decimals and common fractions has already been seen in the above example of twenty-five cents—.25 = ¼. Any decimal may be readily written as a common fraction by placing the decimal (without the decimal point) over whatever denominator it needs to express it. Thus, .25 becomes 25/100 (twenty-five *hundredths*), which, reduced to its lowest terms, becomes ¼.

It is possible also to change any common fraction to a decimal by dividing (decimally) the numerator by the denominator, thus:

$$4\,\overline{)\,1.00}\quad .25$$

Since 1 is smaller than 4, a decimal point is placed immediately after it to separate it from its decimal portion, which is entirely ciphers. The division then proceeds normally.

There are a number of common fractions with decimal equivalents which ought to be learned by heart by the student, for their occurrence is frequent and the ability to use readily the fractional or decimal equivalent, as may be desired, will be found of great assistance in performing rapid calculations. A table of the more important of these follows:

$\frac{1}{2} = .5$	$\frac{1}{6} = .16\frac{2}{3}$
$\frac{1}{4} = .25$	$\frac{1}{8} = .12\frac{1}{2}$
$\frac{3}{4} = .75$	$\frac{3}{8} = .37\frac{1}{2}$
$\frac{1}{3} = .33\frac{1}{3}$	$\frac{5}{8} = .62\frac{1}{2}$
$\frac{2}{3} = .66\frac{2}{3}$	$\frac{7}{8} = .87\frac{1}{2}$
$\frac{1}{5} = .2$	$\frac{1}{10} = .1$

Supposing, knowing the preceding values very well, the student desires to find the cost of 28 yards of cloth at $37\frac{1}{2}$c a yard. He knows at once that $.37\frac{1}{2}$ is equivalent to 3/8 of a dollar, so he takes 3/8 of 28 ($3/8 \times 28$), and finds the cost to be $10.50—a much simpler process than multiplying 28 by .375 or $.37\frac{1}{2}$. Similarly, if a wholesale discount is 33 1/3% (see Percentage), he can find it very readily by figuring it as "1/3 off."

The use of decimals is entirely a matter of thinking. The student should be able, for instance, to reduce inches to the decimal part of a foot. Suppose he is asked to change 8 inches into a decimal fraction of a foot. He thinks about it for a moment, and decides that 8 inches is 8/12 of a foot, since a foot contains 12 inches. Reducing 8/12, he finds it to equal 2/3, and this, he knows, is equivalent to .66 2/3—so 8 inches is equal to .66 2/3 of a foot. If he did not know at once the decimal equivalent of the fraction, he could easily find it by dividing the numerator by the denominator:

$$\begin{array}{r} .66\tfrac{2}{3} \\ 3\overline{)\,2.00} \end{array}$$

If it is desired to express a decimal as a common fraction with a given denominator, this may readily be done by multiplying the decimal by a fraction with numerator and denominator the same as the denominator desired (a whole unit reduced to a fraction with the given denominator). Thus, to change .2814 to fifteenths:

$$.2814 \times \frac{15}{15} = \frac{4.221}{15}$$

The fraction (15/15) is 1 expressed as a fraction with the denominator 15, for when the numerator and denominator are equal the fraction is always equal to 1. The result may be considered as approximately 4/15. This is probably accurate enough for all practical purposes.

Exercises (correct answers at end of book):

(A47) Change to common fractions: .6; .98; 83⅓; .39856; and reduce to lowest terms.

(A48) Change to decimals: 4/5; 9/10; 11/16; 7/12; 583/797.

(A49) Express .5614 in twelfths and .2674 in sixteenths.

(A50) What decimal part of a yard is 2⅝ feet?

IX. PERCENTAGE

Percentage is the rate per hundred, or the proportion of something in a hundred parts—as the percentage of alloy in a gold coin would be the number of parts of alloy in 100 parts of gold and alloy together, the total of both equalling 100. That is, in percentage the basis is 100, which is the whole. Every whole is valued at 100, and 1 percent (the sign of percent is %, so 1%=1 percent) is therefore one part in a hundred. (NOTE: *Percent* is etymologically an abbreviation of the Latin *per centum*, "by the hundred" or "per hundred"—but the custom of writing it with a period—per cent.—is slowly giving way to the adoption of a single word, *percent*.)

Since percentage is based on 100 as a whole one, it is very closely related to decimals. So that since 6% signifies six parts in 100, 6% may be written and thought of as six-hundredths, or .06—a decimal. (Notice carefully the difference between 6%=.06, and 60%=.60: the percent sign, %, takes the place of the decimal point, so that .06% would *not* be 6%, but 6/100%.)

The United States monetary system may also be used to illustrate percentage, since its basis is a dollar divided into 100 equal parts. One cent is, as we have seen, one-hundredth of a dollar, and is written decimally as $.01. Since it is one part in one hundred, one cent is also 1% of a dollar. Twenty-five cents ($.25) is 25% of a dollar.

Something else is thus seen very clearly. Twenty-five cents is twenty-five hundredths of a dollar, twenty-five percent of a dollar, and one-quarter of a dollar. Therefore $25\% = .25 = \frac{1}{4}$. And so on through the table given under Decimals:

$50\% = \frac{1}{2}$ $\quad 16\frac{2}{3}\% = 1/6 \quad$ $87\frac{1}{2}\% = \frac{7}{8}$

To find a certain percent of any number it is only necessary to multiply the number by the amount of percent desired (expressed as a decimal). Thus, to find 45% of 780, simply multiply 780 by .45:

$$\begin{array}{r} 780 \\ \times .45 \\ \hline 3900 \\ 3120 \\ \hline 351.00 \end{array}$$

And 45% of 780 is found to be exactly 351. Sometimes, when the percent is equivalent to some very common fraction (as in the table of equivalents under Decimals), it is easier to find the fractional part. Thus, it is easier to think of 50% as $\frac{1}{2}$—and 50% of 244 is obtained at once as $\frac{1}{2}$ of 244, or 122.

It is frequently desired to know what percent of a certain number another number is. This is exactly the same thing as finding the decimal part that one number is of another. For if we want to find what percent 43 is of 983, we must first know what decimal part of 983 the number 43 is. To find this we divide as in decimals:

$$\begin{array}{r} .043 \\ 983\overline{)43.000} \\ 39\,32 \\ \hline 3680 \\ \underline{2949} \end{array}$$

Therefore 43 is about 4% of 983, for .04 is equal to 4% (the extra figure 3 being dropped, since two places is enough for all practical purposes).

If it is known that 43 is 4% of some number, the number may be found by dividing the number by the percent:

```
                1075
 ______   = ________
.04 )43.    4 )4300.
```

Accordingly, 1075 is exactly the number of which 43 is 4%. This can be proved:

```
 1075
  .04
 _____
43.00
```

Above, 43 was only approximately 4% of 983. If the percent is figured as nearer .044, and 43 is divided by .044, the result is about 977—which is near enough to show the correctness of the process. (If the results in such odd calculations are carried out several places, the answers will be more accurate—but two or three places are usually considered enough when working with decimals, unless, of course, some problem specifically requests or requires a certain number of decimal places.)

The preceding paragraph is only a reverse of finding the percentage, for 4% of 1075 is found, as shown, by multiplying 1075 by .04. The principle is therefore nothing new, for it was explained under Multiplication.

In problems it is usually desired to find the net amount after an increase or decrease of a certain percent of the whole. Thus, if a town's population was 3,500 and has increased 18%—the present population may be found by adding 18% to 3,500. Or it may all be done in one operation by multiplying 3,500 by 1.18 instead of by .18 only, thus:

$$\begin{array}{r} 3500 \\ \times .18 \\ \hline 280\ 00 \\ 350\ 0 \\ \hline 630.00 \\ +3500 \\ \hline 4130 \end{array} \qquad\qquad \begin{array}{r} 3500 \\ \times 1.18 \\ \hline 280\ 00 \\ 350\ 0 \\ 3500 \\ \hline 4130.00 \end{array}$$

And the present population is 4,130.

Again, supposing that the same town's population *decreased* 18%, the whole operation can be done at once by finding the net value of the present population in percent. If it decreased 18%, then its present population must be 1 minus .18, or:

$$\begin{array}{r} 1.00 \\ -.18 \\ \hline .82 \end{array}$$

Since the present population is 82% of what it was when it was 3,500:

$$\begin{array}{r} 3,500 \\ \times .82 \\ \hline 70\ 00 \\ 2800\ 0 \\ \hline 2870.00 \end{array} \qquad \begin{array}{r} 3,500 \\ \times .18 \\ \hline 280\ 00 \\ 350\ 0 \\ \hline 630.00 \end{array} \qquad \begin{array}{r} 3,500 \\ -630 \\ \hline 2,870 \end{array}$$

The present population is 2,870, the same result being obtained either way.

But supposing the town's present population is given as 4,130, and we are told that this is 18% more than it was ten years ago. How shall we find the population of ten years ago? By reversing our process—adding .18 to 1, and dividing cur amount (4,130) by 1.18:

```
        3500.
     ---------
118)413000.        (The decimal point has
    354             been moved in divisor
    ---             and dividend.)
     590
     590
     ---
```

The population ten years ago was therefore 3,500. Similarly, if we are given the present population as 2,870, and are told that this is 18% *less* than it was ten years ago, we can find the population of ten years ago by subtracting .18 from 1, and dividing 2,870 by .82 (the difference).

Exercises (correct answers at end of book):

(A51) Find 15% of 500; 32% of 784; 37½% of 240; 90% of 412; .2% of 1000.

(A52) What percent of 1000 is 50; of 2986 is 57; of 394 is 55.16?

(A53) 5 is 20% of what number? 32 is what percent of 256? 1,247 is what percent of 2,900?

(A54) If the daily output of a department has been 275 toys a day, and new machinery increases this by 24%, what is the present output?

(A55) If the value of a plot of land, of which the purchase price was $4,980, has decreased by 12%, what is the present value?

(A56) The apple harvest of a large orchard was 493 barrels five years ago. Due to lack of care, this is 44% more than the harvest for this year. How many barrels were picked this year?

(A57) The number of savings banks in the U. S. reached a high water mark in 1915 with 2.159 banks. The lowest number was in 1830, and the figure for 1915 represents an increase of about 5,897%. About how many banks were there in 1830?

X. AVERAGES

The computation of averages is an important function of arithmetic, and a comparatively simple one. Suppose that over a period of six days, a man succeeded in winning between five and ten sets of tennis each day. If his tabulated results were exactly:

Monday	5 sets
Tuesday	8 sets
Wednesday	6 sets
Thursday	10 sets
Friday	8 sets
Saturday	6 sets
	43 sets

He won in all 43 sets, during six days. What was his *average* winning score per day? This value called the *average* is obtained by dividing the total by the number of items, or here, by dividing 43 by 6. The average number of sets he won per day is therefore about 7 (exactly 7.16 2/3).

Exercises (correct answers at end of book):

(A58) If the attendance of a certain grammar school for four weeks of five school days each was for each of the twenty school days in succession: 252, 251, 248, 255, 256, 250, 253, 252, 252, 254, 256, 257, 258, 256, 253, 252, 249, 249, 248, 247. what was the average attendance per day? If a perfect attendance (total enrollment) was 258, what percentage of perfection does this average represent?

(A59) If a runner ran the 220-yard dash on three consecutive days in 28 5/6, 27½, and 25 1/5 seconds, respectively, what was his average time for the dash?

(A60) In New York state there were 144,469 marriages in 1920; 130,110 in 1921; and 138,242 in 1922. What was the average number of marriages per year for the three years?

XI. RATIO AND PROPORTION

The *ratio* between two numbers is their relation to each other as to size. Thus, the ratio of 3 to 6 is 2, for 6 is twice as large as 3. Ratio may be expressed as a fraction, but it is customary to use a colon:

$$\frac{3}{6} = \frac{4}{8} \text{ or } 3:6::4:8$$

The first form is merely an equality between fractions. The second form is an expression in ratio, and is read "three is to six as four is to eight." Ratio is thus a comparison between two numbers (two abstract numbers, or two concrete numbers of the same kind). The fractional form of expression receives great favor in higher mathematics, particularly in algebra, for it has all the advantages of being in a form readily used and simplified. It may be read the same as the form with the colons (note the double colon instead of an equals sign, though "=" may be substituted).

The *terms* of a ratio are the numbers compared (3 and 6, and 4 and 8, are the terms of the preceding ratios; the two ratios together form a *proportion*). *Proportion* is merely an equality of ratios. In a proportion the first term of the first ratio, and the last term of the last ratio, are called the *extremes*; and the inner terms (second term of the first ratio, and first term of the second ratio) are called the *means.* In every proportion the product of the means is always equal to the product of the extremes. (If expressed fractionally, the denominator of the first and numerator of the second form the means, the other two the extremes.) Thus, in the preceding example, $3 \times 8 = 6 \times 4$. It is thus possible, with any three terms of a proportion given, to find the missing term:

3:?::4:8	?:6::4:8	3:6::?:8	3:6::4:?
$\frac{3 \times 8}{4} = 6$	$\frac{6 \times 4}{8} = 3$	$\frac{3 \times 8}{6} = 4$	$\frac{6 \times 4}{3} = 8$

This is what is familiarly termed the *rule of three.*

Any ratio (since it is in effect a fraction, or an expressed division) may be raised to any power (see Powers and Roots, Part II), or its terms may be multiplied or divided by the same number, without being altered in value. Similarly, the same root of each of its terms may be taken without changing its value. If this holds true for ratios, it must hold true also for proportions, since proportions are composed of ratios.

The principle of all calculations in proportion is that, three terms being given, the fourth is to be found. This "rule of three" is one of the most important and valuable principles of arithmetic. Thus, if 3 men earn $15 in one week, how much will 6 mean earn? Why, simply (a simple example anyway!) a matter of proportion:

$15 : Amount 6 men earn : : 3 men : 6 men

Reading, "$15 is to the number of dollars 6 men will earn as 3 men are to 6 men." And the answer is obviously $30.

Exercises (correct answers at end of book):

(A61) What are the following ratios: 2:6; 48:144; 100:1000; 100:10?

(A62) Add a simple ratio in equality with each of the above ratios, making of each a proportion.

(A63) In the following proportions, find the missing terms: (a) 14:28:: ?: 56; (b) 23: ? :: 3:9; (c) 16:64::22: ? .

(A64) If 7 men can unload a carload of kegs of nails in two hours, how long will it take 9 men?

(A65) If a post 4½ feet high casts a shadow 3 feet long, at a certain hour of the day, how high is a flagpole that casts a shadow 38 feet long at the same time of day?

ANSWERS TO EXERCISES

(A1) 12,303
(A2) 13,183
(A3) 79,509
(A4) 8941.[illegible]
(A5) 5351.29105
(A6) 270
(A7) 23,052
(A8) 279,888
(A9) 1,620,346
(A10) 101,667,738
(A11) 1,907,994
(A12) 1,902,400,800
(A13) 75.04192
(A14) 7.625052
(A15) $4,873.64
(A16) 5.020
(A17) 243.007
(A18) 534
(A19) 410.17
(A20) 209
(A21) 320.907
(A22) 20,301,012
(A23) 2.780
(A24) 561
(A25) 4.166 7/9
(A26) 731 13/16
(A27) 8.621+
(A28) 1046.6959+
(A29) .00282
(A30) .5
(A31) (a) 2x2x2x5x7x11
(b) 3x5x7x7
(c) 2x2x3x3x3x11
(d) 2x2x3x3x7x31
(e) 2x2x2x3x7x197
(A32) 35÷28=1¼
(A33) 7÷90=.078—
(A34) 50
(A35) 7÷6=1 1/6
(A36) (a) 15/40
(b) 10/15
(c) ¼
(A37) 2/3; 3; 1/20; 5/9
(A38) 73/3; 21/2; 211/4; 159/8

(A39) 16/8=2
(A40) L.C.D.=24; sum= =95/24=3 23/24
(A41) 61/60=1 1/60; 45/16=2 13/16
(A42) 93 11/18
(A43) ½; 1/10; 3/200; 2⅛; 13 1/9; 18 3/11
(A44) 15/32; 6/35; 54⅜; 795 3/7
(A45) 16; 7/10; 14 7/10; 7/24
(A46) $\frac{11\times 11\times 3}{2\times 62}$
(A47) 3/5; 49/50; 250-3 $\frac{2491}{6250}$
(A48) .8; .9; .68¾; .58⅓; .7315
(A49) $\frac{6.7368}{12}$; $\frac{4.2784}{16}$
(A50) .875
(A51) 75; 250.88; 90; 370.8; 2
(A52) 5%; about 2%; 14%
(A53) 25; 12½%; 43%
(A54) 341 toys
(A55) $4,382.40
(A56) About 342 bbls.
(A57) About 36 banks
(A58) 252.4 per day=98%
(A59) 27 8/45 seconds
(A60) 137,607 marriages per year.
(A61) 3; 3; 10; 1/10
(A62) Possibly 2:6::1:3
48:144::2:6
100:1000::2:20
100:10::20:2
(A63) 28; 69; 88
(A64) 1 5/9 hours
(A65) 57 feet