Faculty Working Papers

ASSEMBLY OPERATIONS

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Abstract

ASSEMBLY OPERATIONS

This paper considers some of the queueing problems of interest in the study of a single sewer who assembles two items to produce a single unit of output. This system is a simplified prototype of many industrial operations. Some characteristics of optimal assembly operations are developed.

The system to be considered consists of an operator who assembles two parts to make one item of finished product. The traditional single server queue can provide the analysis of the congestion levels of one part for several special cases of this system. One instance of this occurs when one of the parts is always available. In another case, one part is ordered or produced after each assembly. In this situation, the time to produce the first part can be added to the time to assemble the item to give an occupation time, i.e., time until the server is ready to begin the next assembly. In another special situation, one of the parts for the next assembly is ordered or produced at the start of each assembly. Using an occupation time, which is the maximum of the assembly operation time and the time to obtain the part which has just been ordered for the next assembly, allows the standard queueing results to be used.

There are many questions about this general type of system which can be raised. This suggests that the first consideration should be to determine which systems are in some sense good. From a management science point of view, the study of optimal designs and controls should be the goal of the study of congestion systems. For the system considered here it is almost necessary to begin with these questions. The reason is that if the difference between the expected number of arrivals of the two parts becomes arbitrarily large as time goes to infinity, the system must be unstable. Most real systems of this type do not exhibit

this marked instability. Often such systems are components of larger systems in which it is possible to control the arrival processes at least partially. The function of queues in any system is to decrease the dependence among the production or service operations performed in the system. Although the limiting conditions of unbounded queues in networks lead to simple results [1], this extreme is not likely to be desirable when there are costs of providing and maintaining waiting lines. Unfortunately, most of the analyses of queueing theory depend on the assumption of only one reflecting barrier at the system empty condition. The combination of the need to work in two or more dimensions and a bounded state space require major developments in analysis.

A Model

A simple model can be developed for the two dimensional process $N(t) = [N_A(t), N_B(t)]$ with $N_A(t)$ the number of parts of type A in the system and $N_B(t)$ the number of type B parts. Assume that the process operates in discrete time with the time interval small. Assume that if there is a part of type A and also one of type B in the system at the start of the period, there is a probability of μ of completing the assembly in the period. Assume that there is a probability $\lambda^A_{ij}(t)$ of an arrival of a part of type A in the period $(t, t+\Delta t)$ if $N(t) = (i,j)$. Furthermore, assume that $\lambda^A_{ij}(t)$ may be chosen to have any value between 0 and λ^A for any state and any time. Similarly, $\lambda^B_{ij}(t)$ may be chosen between 0 and λ^B. Moreover, assume that at most, one event occurs

in a period. The transition probabilities for the process are

$$t_{ij\,rs} = \begin{cases} \mu\epsilon(i,j) & (r,s) = (i-j,\ j-1) \\ 1 - \mu^{\epsilon}(i,j) - \lambda_{ij}^{A} - \lambda_{ij}^{B} & (r,s) = (i,j) \\ \lambda_{ij}^{A} & (r,s) = (i+1,j) \\ \lambda_{ij}^{B} & (r,s) = (i,j+1) \\ 0 & \text{otherwise} \end{cases}$$

$$\text{where } \epsilon(ij) = \begin{cases} 0 & i \text{ or } j = 0 \\ 1 & i \text{ and } j \geq 1 \end{cases}$$

The analysis of this process, even for reasonable assumptions on $\lambda_{ij}^{A}(t)$, $\lambda_{ij}^{B}(t)$ is a formidable challenge.

<u>A Profit Structure</u>

In order to focus on optimal systems, it is necessary to assume some sort of a profit structure. A simple but interesting structure is provided by assuming a cost for holding parts and a revenue for each item produced. Specifically, for each item of type A stored in the system for a period, assume the cost is h_A. For each part of type B, charge h_B per item per period. For each item completed, assume that there is a gain of g. Let $V_{ij}(\lambda_{ij}^{A}, \lambda_{ij}^{B})$ be the expected one-period profit starting form state ij. The assumptions imply that:

$$V_{ij}(\lambda_{ij}^{A}, \lambda_{ij}^{B}) = - h_A\, i - h_B\, j - \lambda_{ij}^{A} h_A - \lambda_{ij}^{B} h_B + \mu(g + h_A + h_B)\ \epsilon(i,j)$$

Dynamic Program

The obvious analysis for the problem of selecting values for $\lambda^A_{ij}(t)$ and $\lambda^B_{ij}(t)$ is dynamic programming. Since the one period expected profit is linear in λ^A_{ij} and λ^B_{ij} with negative coefficients, the optimal policy for one period is to use zero arrival probabilities for all states. The recursive problem has the form

$$W_{0i,j} = 0$$

$$W_{ni,j} = \max_{\lambda^A_{ij},\, \lambda^B_{ij}} \{V(\lambda^A_{ij}, \lambda^B_{ij})\} + \beta T(\{\lambda^A_{ij}, \lambda^B_{ij}\})\, W_{n-1}\}$$

where $W_{ni,j}$ is the maximum discounted expected profit from n periods starting from state ij, and W_n is the matrix of these quantities, β is the discount factor, and T is the tensor whose elements are the transition probabilities. Clearly, the objective function is always linear in the decision variables, and thus they must be either 0 or their maximum values. The same is true for the infinite horizon problem assuming that it is meaningful. Thus the decision problem really is equivalent to one in which there are four choices available in each state.

Properties of n Period Optimal Policy

The obvious properties to try to establish for the n period optimal policy are that the optimal λ^A_{ij} is λ^A for low values of i and 0 for high values of i for each j and the symetric result for λ^B_{ij}. This property

defines two single-valued boundry functions, $i_n^*(j)$ for which the optimal λ_{ij}^A is 0 for $i \geq i_n^*(j)$ and $j_n^*(i)$ for which the optimal λ_{ij}^B is 0 for $j \geq j_n^*(i)$. It should not be surprising that these functions exhibit some smoothness properties. First, they are monotone non-decreasing. The somewhat surprising property is that when the functions increase, the magnitude of the increase is one. Thus, $i_n^*(j) \leq i_n^*(j+1)$ and $i_n^*(j) + 1 \geq i_n^*(j+1)$. Similarly, $j_n^*(i) \leq j_n^*(i+1)$ and $j_n^*(i) + 1 \geq j_n^*(i+1)$. This last characteristic does have an intuitive explanation. If the optimal policy defines a transition operate which makes it possible to enter state i,j from one state k,m < i, j, then it makes it possible to enter either from i-1,j or i,j-1, or both, which is clearly the most direct possibility. Thus, the optimal n period policy has the form shown in Figure I.

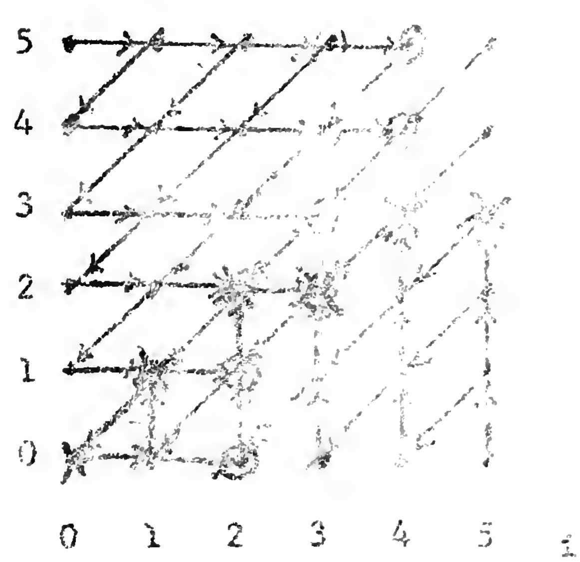

Figure I Optimal 20 period policy for

$\lambda^A = .1$, $\lambda^B = .2$, $\mu = .3$, $h_A = 1$, $h_B = 2$, $\mu = 60$, $\beta = .9$

states $i, j_{20}^*(i)$ are in 0's, states $i^*(j), j$ are marked by x's. The arrows show possible transitions of the optimal operator. Ergodic states (0,0), (1,0), (2,0), (1,1), (2,1), (2,2), (3,2).

Properties of W_n

To prove the properties of the optimal n period policy requires a discussion of the properties of the first differences of the functions W_n. The optimal policy will have the required form if

a) $W_{ni,j} - W_{ni-1,j}$ is non-increasing in i and non-decreasing in j.

b) $W_{ni,j} - W_{ni,j-1}$ is non-increasing in j and non-decreasing in i.

c) $(W_{ni,j} - W_{ni-1,j}) \leq (W_{ni-1,j-1} - W_{ni-2,j-1})$

d) $(W_{ni,j} - W_{ni,j-1}) \leq (W_{ni-1,j-1} - W_{ni-1,j-2})$

If these properties hold, then they will also hold for

$-h_A + \beta(W_{ni,j} - W_{ni-1,j})$ and $-h_B + \beta(W_{ni,j} - W_{ni,j-1})$

which are the two test criteria which determine the optimal λ_{ij}^A and λ_{ij}^B, respectively. Clearly, property a) implies that $i^*_{n+1}(j)$ will be single valued and non-decreasing in j. Property c) quarantees that when the optimal λ_{ij}^A is λ^A, then

$0 \leq -h_A + \beta(W_{ni,j} - W_{ni-1,j}) \leq -h_A + \beta(W_{ni-1,j-1} - W_{ni,2,j-1})$

and the optimal $\lambda_{i-1,j-1}^A = \lambda^A$. Property d) provides this same property for the optimal λ_{ij}^B.

Reformulation

To prove these properties of the functions W_n requires examination of the TW_{n-1}. If W_{n-1} were a function of one variable, it might be possible to argue that T is a totally positive operator [2], and thus preserves the required unimodal properties. Unfortunately, for linear operators applied to functions of two variables, very few general results are available. One class of operators which does preserve the montonicity of first differences is the class of positive translation operators. T is positive translation operator on functions of two integer variables if the coordinate i,j of TW is $(TW)_{i,j} = \sum_{r\in S_1}\sum_{s\in S_2} t_{r,s} W_{i-r,j-s}$ for some positive numbers t_{rs}, $r\in S_1$ and $s\in S_2$. For such an operator, $(TW)_{i,j} - (TW)_{i-n,j-m} = \Sigma\Sigma\, t_{r,s} [W_{i-r,j-s} - W_{i-n-r,j-m-s}]$. Thus, if $W_{i,j} - W_{i-n,j-m} \geq 0$ for al i and j, then TW will have this same property. Similarly, if $W_{ij} - W_{i-n,j-m}$ is monotone increasing or decreasing in i and j, TW will also have this same property.

Unfortunately, the operators in this problem, even with the regularity of the optimal policy, are not translation operators. The fact that there are four regions in the optimal policy, corresponding to all four choices of the two arrival probabilities is one difficulty. The second departure from the requirements of a translation operator occurs along the boundary of the state space. When i or j is zero, then there is no assembly possible, and the term $\mu W_{i-1,j-1}$ is not present, nor is $-\mu W_{i,j}$. To permit an orderly discussion of all the special cases which are inherently possible in TW, a reformulation of the problem can be developed which avoids making the boundary state special.

To do this for $i,j \geq -1,-1$, define

$$V^*_{i,j} = \begin{cases} -(1-\mu)g - h_B(j+1) & i=-1 \\ -(1-\mu)g - h_A(j+1) & j=-1 \\ \mu g = -h_A i - h_B j & i,j \geq 0,0 \end{cases}$$

and

$$W^*_{n i,j} = \begin{cases} W_{n\ 0,\ j+1} & i=-1 \\ W_{n\ i+1,0} & j=-1 \\ W_{n\ i,j} & i,j \geq 0,0 \end{cases}$$

Further, redefine T as

$$T^*_{i,j,k,m} = \begin{cases} \mu & k,m = i-1,j-1 \\ 1-\mu-\lambda^A_{ij} - \lambda^B_{ij} & k,m = i,j \\ \lambda^A_{ij} & k,m = i+1,j \\ \lambda^B_{ij} & k,m = i,j+1 \end{cases}$$

Using these definitions,

$$V = T^* V^*$$

and $V + \beta T W_n = T^*(V^* + \beta W^*_n)$

The function $V^* + \beta W^*_n$ will have the required first difference properties if W_n has them and in addition

$$(V^*_{i,j} + \beta W^*_{n\ i,j}) - (V^*_{0,j} + \beta W_{n\ 0,j}) \leq (V^*_{0,j} + \beta W_{n\ 0,j}) - (V^*_{-1,j} + \beta W_{n\ -1,j})$$

and the corresponding inequality

$$\beta\,[W_{n\ 1,j} - 2W_{n\ 0,j} + W_{n\ 0,j+1}] \leq g + h_A + h_B.$$

This inequality will certainly hold if it holds for $\beta=1$. An alternative form of this inequality can be found, assuming the properties of W_n previously mentioned are true. In particular, since

$$(W_{n\ 0,j+1} - W_{n\ 0,j}) \leq (W_{n\ 0,j} - W_{n\ 0,j-1})$$

the previous inequality will hold if

$$W_{n\ 1,j} - W_{n\ 0,j-1} \leq g + h_A + h_B.$$

A more general result is

e) $W_{n\ i,j} - W_{n\ i-1,j-1} \leq g + h_A + h_B$,

and it is this one which will be established inductively. This inequality will also supply the symetic condition required for the boundary on which $j=0$.

Induction

The function $W_{oi,j} = 0$ clearly satisfies all of the properties, a to e. Within each of the four regions of the optimal policy, the optimal T^* is a translation operator, and thus, properties a to d, hold within these regions. Property e holds since the definition of V^* and W^*_{n-1} gives

$$(V^*_{i,j} + \beta W^*_{n,ij}) - (V^*_{i-1,j-1} - \beta W^*_{n-1\ i-1,j-1}) \leq \begin{cases} \beta(W_{n-1\ i,j} - W_{n-1\ i-1,\ j-1}) & \text{for } i,j \geq 1, \\ g + h_A + h_A & \text{for } i \text{ or } j = 1 \end{cases}$$

Using the induction hypothesis $T^*(V^* + \beta W^*_{n-1}) \leq \sum\sum_{km} t^*_{ij\ km}(g+h_A+h_B) = g+h_A+h_B$ since T^* is a Markov operator. This result does not depend the translation operator characteristics of T^* and thus holds everywhere.

Crossing from one region of the optimal policy to another requires further analysis. In crossing the boundary at which the optimal λ^A_{ij} changes from λ^A to 0, $W_{n\ i,j} - W_{n\ i-1,j}$ drops the term $\lambda^A\ (V^*_{i+j} + \beta\ W^*_{n-1\ \ i+1,j}) - \lambda^A(V^*_{i\ j} + \beta\ W^*_{n-1\ i,j})$ drops out of the expression for W_{nij}. At this point the term becomes negative since the optimal λ^A_{ij} is zero, but its replacement by zero does not cause an increase since previous terms of this kind have been non-negative. All of the other terms in this difference are positive multiples of terms which are not greater than their counterparts in the previous difference. The next difference looses the negative of this term, but since the term itself has become negative, this is the loss of a positive contribution to the difference. When increasing the first variable causes crossing of the boundary $i,j^*_n(t)$, the optimal λ^B_{ij} changes from 0 to λ^B. This adds the term $\lambda^B[(V^*_{ij+1} + \beta\ W_{n-1\ \ i,j+1}) - V^*_{ij} + \beta\ W^*_{n-1\ \ ij}]$ to the first difference. Since the previous test criterion for λ^B_{ij} is negative since the optimal value is $\lambda^B_{ij} = 0$, $-\ \lambda^B[(V^*_{i-1\ \ j+1} + \beta\ W_{n-1\ \ i-1,\ j+1}) - (V^*_{i\,j+1} + \beta\ W^*_{n-1\ \ i-1,j})]$ may be added to the first difference producing something larger. Rearranging terms, one has $\lambda^B[(V_{i\,j+1} + \beta\ W^*_{n-1\ \ i\,j+1}) - (V^*_{i-1,j+1} + \beta\ W_{n-1\ \ i-1,\ j+1})]$ $+\ (1-\mu-\lambda^B)\ [(V^*_{i,j} + \beta\ W^*_{n-1\ \ i,j}) - (V_{i-1,j} + \beta\ W^*_{n-1\ \ i-1,j})]$ plus other terms which are non-increasing either by the induction argument or because of the previous discussion of crossing the $i^*_n(j),j$ boundary.

In place of these two special terms, the previous difference one has only the term $(1-\mu)\,((V^*_{i-1,j} + \beta W^*_{n-1\ i-1,j}) - (V^*_{i-2\ j} + \beta W^*_{n-1\ i-2,j}))$. This may be written as $(1-\mu-\lambda^B) + \lambda^B$ times the bracketed term. By the induction $(1-\mu-\lambda)$ times the bracket term is not less than its counter part in the previous expression. Moreover, also by induction, $\lambda^B\,((V^*_{i-1,j} + \beta W^*_{n-1\ i-1,j}) - (V^*_{i-2,j} + \beta W^*_{n-1\ i-2,j}))$ is not less than $\lambda^B((V^*_{i\ j+1} + \beta W^*_{n-1\ i,j}) - [V^*_{i-1,\ j+1} + \beta W^*_{n-1\ i-1,\ j+1}])$. Thus, this first change cannot increase the first difference. Further increasing the first variable means that the first difference will have both the terms

$\lambda^B((V^*_{i+1,j+1} + \beta W^*_{n-1\ i+1,j+1}) - (V^*_{i+1,j} + \beta W^*_{ni+1,j}))$ and

$-\lambda^B((V^*_{i\ j+1} + \beta W^*_{n-1\ i,j+1}) - (V^*_{i,j} + \beta W^*_{n-1\ i,j}))$ while the previous first difference will have only the term $\lambda^B((V^*_{i\ j+1} + \beta W^*_{n-1\ i,j+1}) - (V^*_{ij} + \beta W_{n-1\ i,j}))$. Eliminating the special term from the previous first difference can only decrease its value, but if this is done, the argument just completed showsthat what is left is still not smaller than the new first difference. From this point on, the positive translation operator agreement is again valid.

Having proved the first part of property a and by symetry, the first part of property b now consider $W_{n\ i,j} - W_{n\ i-1,j}$ as a function of j. Again, although not necessary, it is possible that increasing j may cause the crossing of both boundaries. Again, consider each one seperately. In crossing the boundary from the region in which the optimal $\lambda^B_{ij} = \lambda^B$ to the one in which they are zero the first term in

the first difference to be dropped is

$$-\lambda^{B}((V^{*}_{i-1,j+1} + \beta W^{*}_{n-1\ i-1,j+1}) - (V^{*}_{i\ 1,j} + \beta W^{*}_{n\ i-1,j})).$$

The term $\lambda^{B}[(V^{*}_{i,j+1} + \beta W^{*}_{n-1\ i,j-1}) - (V^{*}_{i,j} + \beta W^{*}_{n\ i\ j})]$ is left by itself and cannot be balanced against other terms using positive multiples of non-decreasing first differences argument. Since this extra term is positive, just ignore it and what remains is not greater than the true first ference and is not less than the previous one. Increasing j by 1, again use the device of writing $(1-\mu-\lambda^{B}+\lambda^{B})$ $((V^{*}_{i,j} + \beta W^{*}_{n\ i,j}) - (V^{*}_{i-1,j} + \beta W^{*}_{ni-1,j}))$. The $(1-\mu-\lambda^{B})$ balances the term in the previous difference corresponding those in the bracket, i.e., the same term with $j=j-1$. The λ^{B} term equals a term in the previous first difference. If it were possible to increase j further without dropping the term $\lambda^{B}((V^{*}_{i,j+1} + \beta W^{*}_{ni,j+1})$ $- (V^{*}_{i\ j} + \beta W^{*}_{n\ i,j}))$ the argument could be in difficulty for, although other terms are non-decreasing, this one has been shown to be non-increasing. Fortunately, this term must drop out with the next increase in j. Again, everything is not symetric so that the positive multiplier argument can work immediately. Adding $\lambda^{B}((V^{*}_{i-1,j} + \beta W^{*}_{n\ i-1,j})$ $- (V^{*}_{i-1\ j-1} + \beta W^{*}_{n\ i-1,j-1}))$, which must be positive since the optimal $\lambda^{B}_{i-1,j-1}$ is 0, to the previous difference merely increases its value. This allows the splitting of $(1-\mu)$ into $(1-\mu-\lambda^{B}) + \lambda^{B}$ to balance terms. Thus, the new first difference is not less than a quantity which is not less than the previous first difference.

Next, it is necessary to explain the effect on $W_{n\,i\,j} - W_{n\,i-1,j}$ of crossing into the region in which the optimal $\lambda^A_{ij} = \lambda^A$ because of an increase in j. The first effect is the introduction of $-\lambda^A((V_{i,j} + \beta W^*_{n-1\,i,j}) - (V^*_{i-1,j} + \beta W^*_{n-1\,i-1,j}))$. In the first difference for the previous j value, split the $1-\mu$ term into $(1-\mu-\lambda^A)$ and λ^A. The $(1-\mu-\lambda^A)$ multiplies a term which is non-decreasing in the successive differences.

$\lambda^A((V^*_{i\,j-1} + \beta W^*_{n+i\,j-1}) - (V^*_{i-1,j-1} + \beta W_{n+i-1,j-1}))$ is negative since the optimal $\lambda^A_{i-1,j}$ is zero and its loss in the next difference only serves to increase the difference. Increasing j by 1 means that the two differences will both have $(1-\mu-\lambda^A)\ ((V^*_{ij} + \beta W^*_{n-1\,i\,j}) - (V^*_{i-1j} + \beta W^*_{n-1\,i-1,j}))$ terms and the positive multiplier argument works. At some point, j increases, the term $\lambda^A((V^*_{i+1,j} + \beta W^*_{n-1\,i+1,j}) - (V^*_{ij} + \beta W_{n-1\,i,j}))$ appears and it causes an increase. Further increases in j are permissible under the positive translation operator argument. Thus, property a is justified completely, and by symmetry, so is property b.

Next, properties c and d need to be verified. As i increases and $(W_{n\,i,j} - W_{n\,i-1,j}) - (W_{n\,i-1,j-1} - W_{n\,i-2,j-1})$ begins to cross the boundary at which the optimal λ^A_{ij} changes from λ^A to 0, the first change is the loss of a term of the four $\lambda^A((V^*_{i+1,j} + W^*_{n-1i+1,j}) - (V^*_{i,j} + W^*_{n-1\,i,j}))$. When this happens, the corresponding term $-\lambda^A((V^*_{i\,j-1} + \beta W^*_{n-1\,ij-1}) - (V^*_{i-1,j-1} + \beta W^*_{n-1\,i-1,j-1}))$ may or may not disappear, but since it is negative, it cannot make the difference of differences positive. In what remains, there are only positive multiples of negative differences of first differences. The second term must disappear with an increase in i, since when the optimal $\lambda^A_{ij} = 0$ so must $\lambda^A_{ij-1} = 0$. The positive

multiple agreement requires combining terms so that one has

$(1-\mu-\lambda^A)\ [((V^*_{i,j} - \beta W^*_{n-1\ i,j}) - (V^*_{i-1} + \beta W^*_{n+i-1,j}))]$

$- [(V^*_{i-2,j-1} + \beta W_{n-1\ i-2,j-1})]$ with $1-\mu-\lambda^A > 0$ since it is, of course, a probability. The next term which is dropped is $-\lambda^A((V^*_{i,j} - \beta W^*_{n-1\ i,j}) - (V^*_{i-1,j} + \beta W^*_{n-1\ i-1,s}))$ because it becomes positive. This change serves to reduce the difference of differences below the already negative result that the positive multiplier argument gives.

Continuing the discussion of this difference of differences, it is necessary to examine what happens as the boundary at which the optimal λ^B_{ij} changes from λ^B to 0. The first effect on $(W_{n\ ij} - W_{n\ i-1,j}) - (W_{n\ i-1,j-1} - W_{n\ i-2,j-1})$ is that the term $-\lambda^B\ ((V^*_{i-1,j+1} + \beta W^*_{n-1\ i-1,j+1}) - V^*_{i-1,j} + \beta W^*_{ni-1,j}))$ is eliminated. This gives no difficulty since this happens only when the term is positive. Dropping a positive term merely makes the result more negative than the already negative result of the positive multiplier argument. The next loss is not unique and might happen simultaneously. First, the term considers $\lambda^B((V^*_{i-2,j} + \beta W^*_{n-1\ i-2,j}) - (V^*_{i-2,j-1} + \beta W^*_{n-1\ i-2,j-1}))$. Either this drops out either simultaneously with the first loss or on the next increase of j because the form of the boundary $i^*_n(j)$. If this drops out with the first loss, then the only remaining λ^B terms can be collected by giving $\lambda^B[((V^*_{i,j+1} + \beta W^*_{n-1\ i,j+1}) - (V^*_{i,j} + \beta W^*_{n-1\ i,j})) - ((V^*_{i-1,j} + \beta W^*_{n-1\ i-1,j}) - (V^*_{i-1,j-1} + \beta W^*_{n-1\ i-1,j-1}))]$.
This is negative as a result of property d of the induction hypothesis.
The term $\lambda^B((V^*_{i,j+1} + \beta W^*_{n-1\ i,j+1}) - (V^*_{i,j} + \beta W^*_{n-1\ i,j}))$ must drop out with the next increase of j. If it drops out simultaneously with the

loss of the first term, one can have left $-\lambda^B[((V^*_{i-1,j} + \beta W^*_{n-1\ i-1,j}) - (V^*_{i-1,j-1} + \beta W^*_{n-1\ i-1,j-1})) - ((V^*_{i-2,j} + \beta W^*_{n-1\ i-2,j}) - (V^*_{i-2,j-1} + \beta W^*_{n-1\ i-2,j-1}))]$. The term in the bracket is positive since it is the difference of differences which are now decreasing. Thus, multipled by $-\lambda^B$ makes the product non-positive. The remaining terms are negative since they are positive multiples of non-positive differences of differences by hypothesis c. If the first and the next two terms drop out simultaneously, the last remaining λ^B term is $-\lambda^B[(V^*_{i-1,j} + \beta W^*_{n-1\ i-1,j}) - (V^*_{i-1,j-1} + \beta W^*_{n-1\ i-1,j-1})]$, but this is negative since the term in the bracket is positive. This term also drops out at the next increase in j and none of the λ^B terms are left. This completes the proof of property c and the symmetric counterpart property d.

This completes the induction. In the analysis the only terms involving the arrival rate which was known to be changing at each boundary were discussed. It is possible that in some cases the two boundaries are crossed simultaneously. This presents no problem. The only possible problem comes when $1-\mu-\lambda^A$ and $1-\mu-\lambda^B$ are used. If necessary, however, $k_1-\lambda^A \geq 0$ and $k_2-\lambda^B \geq 0$ can be used with $k_1+k_2=1-\mu$, since $1-\mu-\lambda^A-\lambda^B$ is a probability.

Dynamics

Dynamic properties of this analysis seem somewhat more difficult to establish than the results on the form of the optimal n period policy.

One result, which is relatively easy, is that there exist bounds i* and j* such that for all n the optimal arrival rates are $\lambda^A_{ij} = 0$ for $i \geq i^*$ and $\lambda^B_{ij} = 0$ for $j \geq j^*$. The argument for this is that the n period profits can be written as a sum of profits in each of the n periods. For large i the first period profit will h_A less in state

i + 1 than it is in state i. For at least the next i periods, all sample paths starting from i + 1 will have at least as large an inventory of A parts as will those from state i, and thus, the profits starting from i + 1 will not exceed those starting from i for this interval. Equality can occur if at some time the history generated starting from i + 1 is in a state k,m in which the optimal $\lambda^A_{k,m}$ is 0, but the optimal $\lambda^A_{k-1,m}$ is λ^A. In this case, some of the histories starting from i will have the same level of A inventory as the corresponding ones starting from i + 1. From this time on, there will be a one-to-one match of the remaining possible history of the process. On the other hand, there will be histories in which the number of A items on hand is always 1 more starting from i + 1 than is the cast starting from i until the level 0 is reached. At this time, the histories starting from i + 1 have a profit which is $\mu g - h_A$ higher than those starting from i. For the remaining time until the horizon, the possible histories may separate, but the difference in profits is bounded by $\sup_j |W_{n1,j} - W_{n0,j-1}| \leq g+h_A+h_B$ by induction. Thus, the difference in the worth of starting in i + 1 and starting in i is not greater than $-h_A + \beta^i [g+h_A+h_B]$ which can be made negative for i sufficiently large. Clearly, when $W_{ni+1,j} - W_{ni,j} \leq 0$, the optimal $\lambda^A_{n+1\ i,j} = 0$. This also easily establishes that if $n < i$, $\lambda^A_{ij} = 0$. In n periods, no more than n type A items can be used, and thus profits can only be decreased by adding a type A item at any time during the n periods.

The bounds i^*, j^* just discussed can be used to say that there is a finite state finite action problem for which the recursive optimal discounted expected profit functions $W_{ni,j}$ are equal to the infinite state $W_{ni,j}$ when $(i,j) \leq (i^*,j^*)$. This follows immediately from the transition operator, which guarantees that if $\lambda^A_{ij} = 0$ for $i > i^*$, and $\lambda^\beta = 0$ for $j \leq j^*$ and all n, then $W_{ni,j}$ depends only on $W_{mi',j'}$ for $i' < i^*$ and $j' \leq j^*$ and $m \leq n$. This dependence is the same for W_n and W'_n.

As far as the behavior of W_n as n approaches infinity is concerned, the fact that the system evolves according to a Markov chain having positive probability of only finite changes in state, and the fact profits are discounted by $\beta < 1$, combine to guarantee convergence for all finite states. A standard contraction argument easily establishes

$$|W_{n+1\ i,j} - W_{ni,j}| \leq \beta\ |W_{ni,j} - W_{n-1\ i,j}|$$

This unfortunately does not provide a characterization of the sequence of optimal policies. There is one obvious result which can be supported. If $\lambda^A_{ij} = 0$ and $\lambda^A_{i-1,j-1} = 0$ in the optimal policy for period n, then $\lambda^A_{ij} = 0$ for period n + 1. In this case, the test criterion is strictly negative.

$$-h_A + \beta\, W_{n+1\ i+1,j} - W_{n+1\ i,j} = -h_A + \beta\ \{(1-\mu)[-h_A + W_{ni+1,j} - W_{ni,j}]$$

$$+\mu[-h_A + W_{n\ i,j-1} - W_{n\ i-1,\ j-1}]\} \leq -h_A < 0.$$

This implies that the region in which λ^A_{ij} should be λ^A cannot expand by more than 1 in each period. If i is sufficiently large that the optimal $\lambda^A_{ij} = 0$ for all periods up to n and states which can be reached from i,

then this criterion behaves like $-h_A\ (1-\beta^{n+2})/(1-\beta)$ which is monotonic decreasing in n.

What is both more interesting and more difficult is the behavior in the regions in which λ^A_{ij} should be λ^A. So far no counter example has been found for the hypothesis that the optimal λ^A_{ij} are non-decreasing functions of n. A proof that this must be so involves showing that

$$(-h_A + \beta W_{n+1\ i+1,j} - \beta W_{n+1\ i,j}) \geq (-h_A + \beta W_{n\ i+1,j} - \beta W_{ni,j})$$

whenever the right-hand-side is positive. This is equivalent to

$$(W_{n+1\ i+1,j} - W_{n+1\ i,j}) \geq (W_{n\ i+1,j} - W_{ni,j})$$

when the right-hand-side is $\geq h_A$. All that is easy to argue is that

$$(-h_A + W_{n\ i+1,j} - W_{ni,j}) \geq 0 \longrightarrow (W_{n+1\ i+1,j} - W_{n+1\ i,j}) \geq 0$$

This is an immediate consequence of the recursive definition of W_{n+1} which permits this difference to be written as a sum of positive terms. To say more than this requires a stronger inequality than

$$(W_{ni,j-1} - W_{n\ i-1,\ j-1}) \geq (W_{n\ i+1,j} - W_{ni,j})$$

which was shown earlier. A stronger version does not hold everywhere. When the optimal $\lambda^A_{i-1,\ j-1} = 0$, there can be equality especially for small n.

Small Intervals

The natural interest in differential equations for queueing systems resulting from the M/M/1 analysis raises the question of what happens if the interval size is reduced. To examine this, one needs to reformulate the problem somewhat. Following the usual approach, the probabilities of the three possible state change events are re-defined as $\lambda^A \Delta t$, $\lambda^B \Delta t$, and $\mu \Delta t$. The holding cost can be re-defined as h_A and h_B per unit time respectively. The cost of holding one type A item for Δt is now $h_A \Delta t$. Finally, the discount factor has to be modified to be $1 - \sigma \Delta t$. The criterion for choosing the n + 1st λ^A_{ij} in these terms is

$$-h_A \Delta t + (1- \sigma\Delta t) (W_{n\ i+1,j} - W_{n\ i,j})$$

Since this is an affine function of Δt, it changes sign at most once as Δt goes to zero. This guarantees that shrinking Δt will not produce oscilations in the transition probabilities which would make the limiting differential equation meaningless. At the moment, this differential equation is not of computational interest. Perhaps the most interesting aspect of the formulation leading to the differential equation occurs in the limit as the time parameter gets large. To the accuracy of Δt

$$V(\lambda^A_{ij}, \lambda^B_{ij})_{ij} = V'_{ij} \Delta t \{-h_A i - h_B j + \mu g\ \varepsilon_{ij}\} \Delta t$$

and the limiting W for any Δt satisfies

$$0 = \max_{\lambda^A_{ij},\ \lambda^B_{ij}} (V_{ij}\ \Delta t\ (T(\lambda^A_{ij},\ \lambda^B_{ij}) - I)\ \Delta t\ W)$$

As usual, in queueing processes, the Δt can be factored out, both the limiting W and the optimal policy do not depend on the size of Δt. This also is the limit to which the solution to the differential equation must converge. In M/M/1, an interpretation of the unimportance of Δt is that the limiting probabilities depend only on ratios of transition probabilities, not their absolute values. This is, of course, also true here for the probabilities. It is not true that an optimal stationary policy depends only on ratios of the cost parameters. The scaling possibility here is among the holding costs, h_A and h_B, and the expected revenue per unit time given that the operation is working μg. Thus, the absolute value of μ does become important. To eliminate this one must factor from the function to be optimized $\mu \Delta t$ in which case h_A/μ and h_B/μ the expected holding cost during an assembly will be the holding cost parameters.

<u>Dependence on μ</u>

In many situations, not only is it possible to control the inputs, but also it is necessary to choose the man and/or equipment to perform the assembly. In this model, this is equivalent to choosing the completion probability μ. This choice is made once when the system begins operation. In general, this decision depends on both the sequence of arrival probabilities and the initial conditions. The combined decision problem has the form

$$\sup_{\mu \in \theta} \left(\left(Po, \max_{D} W_{\mu}(D) \right) - C(\mu) \right)$$

In this expression D is an infinite sequence of decision functions D_n, each one of which specifies λ_{ij}^A and λ_{ij}^B for each state ij. The function $C(\mu)$ is a cost for choosing μ as the assembly probability. The first bracket represents the scalar product and is the expectation with respect to the initial conditions of the discounted expected return over an infinite horizon using policy D for a fixed value μ. In many situations the set of values θ from which μ must be chosen is a finite set and a maximum will exist. If the number of values in θ is small, it will be feasible to solve the problem by enumeration of the μ values. If $C(\mu)$ is quite irregular, such a procedure may be the only possibility. From a modeling point of view, even if θ is interval $[0,\mu^m]$, it is probably permissible to approximate θ by a finite set containing only a few values since the entire model is an approximation anyhow. From any point of view other than immediate implementation, this resort to crude numerical methods has little, if any, appeal. The problem is to develop properties of $\max_D W_\mu(D)$ as a function of μ and assumptions on the form of $C(\mu)$ which will insure that a more efficient sequential search procedure than enumeration can be used to solve the problem. The situation of being able to partially characterize the optimal policy of a sequential decision process and possesion of relatively efficient computational means for finding it is typical of the analysis of many queueing systems. As in this example, this should not be the end of the analysis, for real problems have both design (selection of μ) and control (selection of λ_{ij}^A and λ_{ij}^B for each decision time) aspects.

For applications, it will generally be much more difficult and expensive to rectify errors in design than to improve control.

An important question to answer is whether an optimal μ exists. Unfortunately, the dependence of the functions $W_{ni,j}$ on μ is not as easy to describe as one might hope. Consider the derivatives $d\,W_{ni,j}/d\mu$. First

$$\frac{dW_{1i,j}}{d\mu} = \frac{dV_{i,j}}{d\mu} = (g+h_A+h_B)\ \epsilon_{ij}$$

$$\frac{d^2W_{10j}}{d\mu^2} = \frac{d^2V_{i,j}}{d\mu^2} = 0$$

These one period functions are extremely well behaved, but examination of $dW_{2i,j}/d\mu$ gives a more complicated picture. For the second period only $\lambda^A_{o\,j}$ for $j > 0$ and can be positive. The function is

$$W_{2i,j}\begin{cases} [-h_A i-h_B j+\mu(g+h_A+h_B)]\ (1+\beta)\ +\mu\beta\ (h_A+h_B) & i>1,\ j>1 \\ [-h_A-h_B j+\mu(g+h_A+h_B)]\ \ (1+\beta(1-\mu))+\ \mu\beta\ (-h_B(j-1)) & i=1\ \ j\geq 1 \\ [-h_A i-h_B+\mu(g+h_A+h_B)]\ (1+\beta(1-\mu))\ +\mu\beta\ (-h_A(i-1)) & j=1\ \ i\geq 1 \\ -h_A\lambda^A_{0j}\ -h_B j\ \ (1+(1-\lambda^A_{0j})\beta)\ +\beta\lambda^A_{0j}\ \ (-h_A-h_B j+\mu(g+h_A+h_B)) & i=0\ \ j>0 \\ -h_B\lambda^B_{i0}-h_A i\ \ (1+(1-\lambda^B_{i0})\beta)\ +\beta\lambda^B_{i0}\ \ (-h_A i-h_B+\mu(g+h_A+h_B)) & i>0\ \ j=0 \\ 0 & i,j=0,0 \end{cases}$$

The first derivatives are

$$\frac{dW_{2i,j}}{d\mu} = \begin{cases} (g+h_A+h_B0\ (1+\beta) + \beta(h_A+h_B) & i>1\ \ j>1 \\ (g+h_A+h_B)\ (1+\beta-2\beta\mu) + (h_A+h_B) & \left\{\begin{array}{l} i=1\ \ j\geq 1 \\ i\geq 1\ \ j=1 \end{array}\right. \\ \beta\ \lambda^A_{0j}\ (g+h_A+h_B) & i=0\ \ j>0 \\ \beta\ \lambda^B_{i0}\ (g+h_A+h_B) & i>0\ \ j=0 \\ 0 & i=0\ \ j=0 \end{cases}$$

At the points at which λ^A_{0j} and λ^B_{i0} change values the derivatives are undefined. The test criterion for λ^A_{0j} is

$$-h_A+\beta(W_{1\ 1,j} - W_{10,j}) = -h_A+\beta((\mu g+h_A+h_B)\ -h_A)$$

which is negative until

$$\mu \geq \frac{h_A-\beta\ h_B}{g}$$

Thus, for small μ, λ^A_{0j} should be zero, while for large values it becomes λ^A. This makes $W_{20,j}$ piecewise linear and convex. $W_{2i,0}$ is similar. $W_{2i,j}$ and $W_{2i,1}$ are concave increasing; while the remaining functions $W_{2i,j}$ for i>1 j>1 are increasing and linear in μ.

For the general case, one has, by induction, that the functions $W_{n+1\ i,j}$ are continuous in μ since, for any continuous function, $G(\mu)$, $\lambda^A(\max(0,G(\mu)))$ is continuous. For the intervals in which the derivatives are defined

$$\frac{dW_{n+1\ i,j}}{d\mu} = \frac{dV(\lambda^A_{ij},\lambda^B_{ij})}{\mu} + \beta\ \{\mu\varepsilon(i,j)\ \frac{dW_{n\ i-1,\ j-1}}{d\mu}$$

$$+[1-\lambda^A_{ij}-\lambda^B_{ij}-\mu\varepsilon(i,j)]\ \frac{dW_{nij}}{d\mu} +$$

$$\lambda^A_{ij}\ \frac{dW_{n\ i+1,j}}{d\mu} + \lambda^B_{ij}\ \frac{dW_{n\ i,j+1}}{d\mu} +$$

$$\mu\varepsilon(ij)\ W_{n\ i-1,j-1} - \mu\varepsilon(i,j)\ W_{ni,j}\ \}$$

As in the case of $W_{20,j}$ when λ^A_{ij} changes from 0 to λ^A, the derivative of $W_{n+1\ i,j}$ experiences a positive jump. Since $W_{ni+1,j} - W_{n\ i,j}$ must be increasing. The same holds for λ^B_{ij}. When the derivatives exist, they must be non-negative. Under the induction hypothesis,

$$\frac{dW_{n+1\ i,j}}{d\mu} \geq \frac{dV(\lambda^A_{ij},\ \lambda^B_{ij})}{d\mu} + \beta\mu\varepsilon(ij)\ \ (W_{n\ i-1,j-1} - W_{nij})$$

Since $W_{ni,j} - W_{n\ i-1,j-1}$ has already been shown not greater than $g+h_A+h_B$, the right-hand-side must be non-negative. These derivatives inherit discontinuities from those of W_n. All discontinuities give positive jumps, if for any n λ^A_{ij} changes only once from 0 to λ^A and similarly λ^B_{ij} changes at most once.

Inductive analysis can show that the test criterion is non-decreasing or equivalently it is continuous and $d(W_{n+1\ i+1,j}/d\mu - dW_{n+1\ i,j})/d\mu$ is non-negative when it is defined. Continuity follows the continuity of W_{n+1}. For one period, the derivative is zero except for i=0 j>0 when it is positive. The general case is

$$\frac{dW_{n+1\ i+1,j}}{d\mu} - \frac{dW_{n+1\ i,j}}{d\mu} = dV(\lambda^A_{i+1,j},\lambda^B_{i+1,j}) - \frac{dV(\lambda^A_{i,j},\lambda^B_{i,j})}{d\mu}$$

$$+\beta\ \mu\epsilon(i+1,j)\ \frac{dW_{ni\ j-1}}{d\mu} - \beta\mu\epsilon(i,j)\ \frac{dW_{n\ i-1,j-1}}{d\mu}$$

$$+\beta[1-\lambda^A_{i+1,j}-\lambda^B_{i+1,j}-\mu\epsilon(i+1,j)]\ \frac{dW_{n\ i+1,j}}{d\mu}$$

$$-\beta[1-\lambda^A_{i,j} - \lambda^B_{i,j} - \mu\epsilon I i,j)]\ \frac{dW_{n\ i,j}}{d\mu}$$

$$+\beta\ \lambda^A_{i+1,j}\ \frac{dW_{n\ i+2,j}}{d\mu} - \beta\lambda^A_{i,j}\ \frac{dW_{n\ i+1,j}}{d\mu}$$

$$+\beta\ \lambda^B_{i+1,j}\ \frac{dW_{\ i+1,j+1}}{d\mu} - \beta\ \lambda^B_{i,j}\ \frac{dW_{ni,j+1}}{d\mu}$$

$$-\beta\epsilon(i+1,j)\ W_{ni+1,j} + \beta\epsilon(i+1,j)\ W_{ni,j-1} + \beta\epsilon(i,j)\ W_{ni,j}$$

$$-\beta\epsilon(i,j)\ W_{ni-1,j-1}$$

The terms involving derivatives of W_n make no positive contribution due to the induction hypothesis when $\epsilon(i,i)=1$ and the decisions are the same in $i+1,j$ and ij. Similarly, if $\epsilon(i+1,j)$ and $\epsilon(i,j)$ are positive, the last four terms must have a non-negative sum because of the properties of W_n previously shown. There are, of course, other special cases to be considered. First, it may happen that $\lambda^B_{i+1,j} = \lambda^B$ and $\lambda^B_{i,j} = 0$, but this does not lead to a negative contribution. When $\lambda^B_{i+1,j}$ changes from 0 to λ^B, there is a positive jump, and when λ^B_{ij} becomes λ^B, a smaller negative jump. When λ^A_{ij} becomes λ^A and $\lambda^A_{i+1,j} = 0$, then there is a negative jump of $-\lambda^A(dW_{ni+1,j}/d\mu - dW_{ni,j}/d\mu)$, but this is nullified by the positive term $(1-\lambda^B_{i+1,j} - \mu\epsilon(i+1,j))\, dW_{ni+1,j}/d\mu - (1-\lambda^B_{i,j} - \mu\epsilon(i,j))\, dW_{ni,j}/d\mu$. When $\lambda^A_{i+1,j}$ becomes λ^A, there is a positive jump. The possibility of $\epsilon(i,j) = 0$, $\epsilon(i+1,j) = 1$ only contributes a positive addition to the derivative terms, for $-\mu dW_{ni,j}/d\mu$ is replaced by the larger value zero. In the last four terms, this possibility produces $-W_{ni+1j} + W_{ni,j-1}$, but by the previous induction this must be not less than $-g-h_A-h_B$, which is the negative of $dV(\lambda^A_{i+1,j},\lambda^B_{i+1,j})/d\mu - dV(\lambda^A_{i,j},\lambda^B_{i,j})/d\mu$ in this situation. Thus, inductively $dW_{n+1\ i+1,j}/d\mu - dW_{n+1\ i,j}/d\mu$ is non-negative when it is defined. This means that there is at most one value of μ at which λ^A_{ij} will change from 0 to λ^A and it will remain λ^A for all higher values of μ. A symmetric argument applies to λ^B_{ij}.

Optimal μ

The result just obtained guarantees that any stage the optimal λ^A_{ij} and λ^B_{ij} are well behaved. The monotonicity of $W_{ni,j}$ in μ is not a strong enough property to guarantee that there will be a unique

optimum μ for the design problem posed in the previous section. Much more is necessary if one wishes $(P_0, W_{nij}) - C(\mu)$ to be unimodel. The obvious disirable property of concavity does not hold for W_{2ij}. The difficulties of the analysis of this system are typical of problems of design and control of Markov systems, especially when the natural state space is two or higher in dimension.

Even if it were possible to show that under reasonable conditions there is an optimal μ, there still remains the problem of finding this value. An iterative procedure which approximately solves the dynamic programming problem for the optimal control for each value of μ and searches among these solutions for an optimum has little to offer other than its feasibility. What is needed is an iterative procedure which will pick a sequence $\mu(k)$, $\lambda^A_{ij}(k)$, $\lambda^B_{ij}(k) \mid k=1\ldots$ which will converge to an optimum if one exists, without the necessity of $\mu(k)$ being constant for large intervals of k values.

In the study of these systems, the author has engaged in some rather extensive numerical work. Unfortunately, the results of this work are not in a form that they can be presented as yet. Perhaps the most striking result so far is the very small number of states which are ergodic in these systems. In most cases so far, optimal queue sizes have been under 10, and, moreover, many fewer than the corresponding maximum of 121 states have been ergodic. Although it is easy to introduce further complexities, which will cause any numerical analysis to tax the power of a computer, it is striking how much of the imagined

difficulties disappear in calculation. The results presented here really constitute an "academic" exercise, for they all had strong support from calculations before the inductions had been completed. Only an "academic" could afford to ask are these properties always true before considering what happens when the structure of the problem is changed.

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