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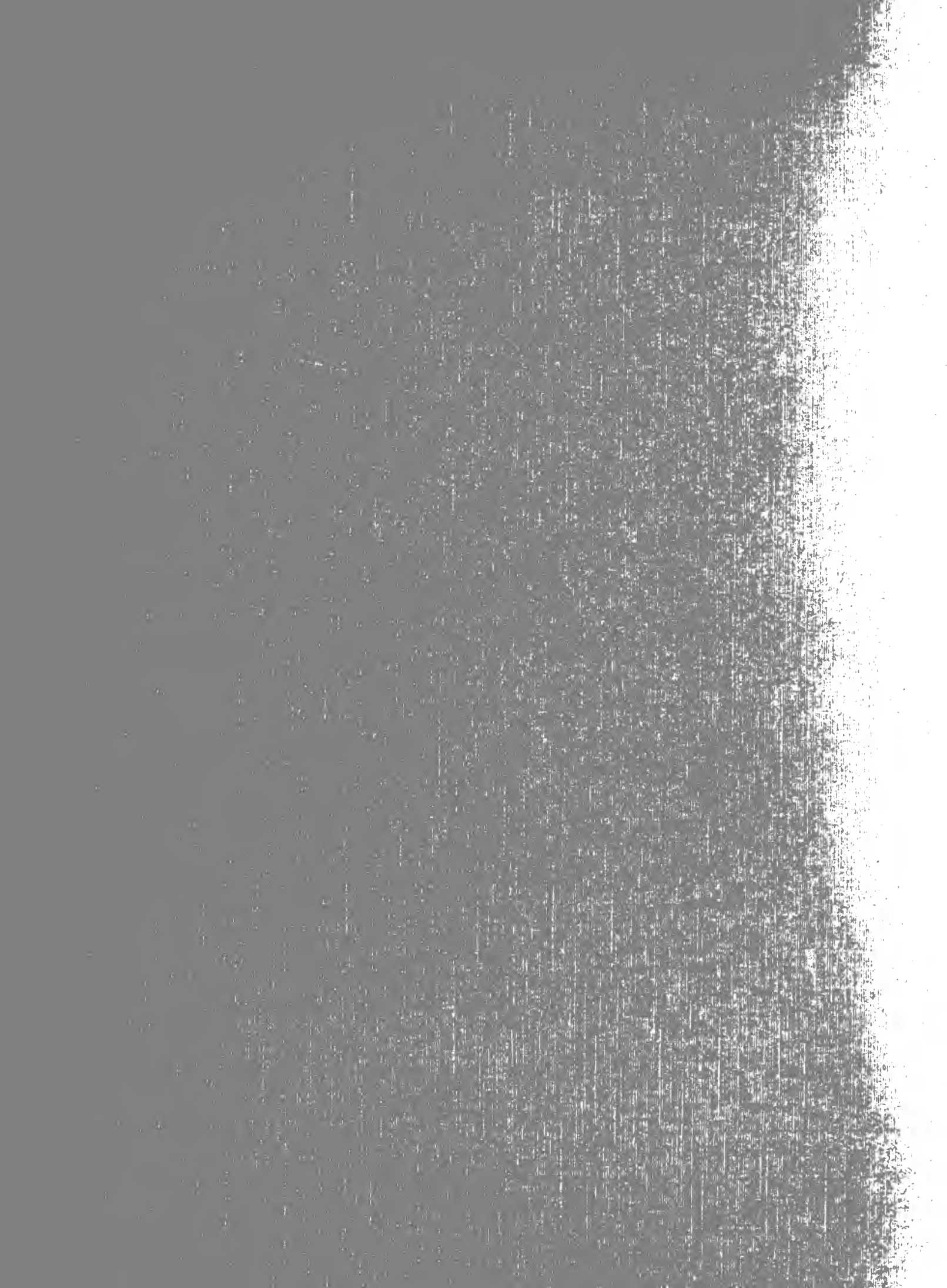
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Asset Pricing, Higher Moments and the
Market Risk Premium: A Note

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Abstract

The purpose of this note is to examine, theoretically, why the market risk premium $(\overline{R}_M - R_f)$ may influence tests of asset pricing models with higher moments. When moments of higher order than the variance are added to a pricing model developed within the usual two-fund separation assumptions, the market risk premium enters the pricing equation in a nonlinear fashion and is implicit in the estimation of _each_ moment's coefficient. Unless this nonlinearity is recognized, incorrect conclusions regarding the tests of such models may result.

Asset Pricing, Higher Moments and the Market Risk Premium: A Note

I. Introduction

Following the work of Markowitz [15], Sharpe [22], Lintner [14] and

Mossin [17] developed the first formulations of the mean-variance capi-

tal asset pricing model (CAPM). Subsequent modifications to the theory

were made by Fama [5], Brennan [4] and Black [2] as well as others.

Proponents of the CAPM note its simplicity and potential for testability;

however, the model has not been empirically validated in the tests of

Black, Jensen and Scholes [3], Miller and Scholes [16], Fama and MacBeth

[6] and many others. Furthermore, Roll [18] has warned us of the ambi-

guous nature of such tests because of a number of measurement difficul-

ties and joint hypotheses present in the model.

Efforts to respecify the pricing equation have gone in several

directions. The direction that is of interest in this note is the re-

search that has expanded the utility function beyond the second moment

to examine the importance of higher moments. There has been recent

interest in the importance of higher moments as evidenced in a paper by

Scott and Horvath [20] which develops a utility theory of preference for

all moments under rather general conditions. The third moment (skewness)

has already received some attention in the literature [1, 8, 9, 10, 11,

12, 20].[1] Following the work of Rubinstein [19], Kraus and Litzenberger

(KL) [13] derived and tested a linear three moment pricing model,

finding the additional variable (co-skewness) to explain the empirical

anomalies of the two moment CAPM. The three moment model was re-examined

by Friend and Westerfield (FW) [7] with mixed results. The FW study

found some, but not conclusive evidence of the importance of skewness in

the pricing of assets. In particular, FW found empirical tests of the
three moment model to be "...especially sensitive to the relationship
between the market rate of returns (R_M) and the risk-free rate (R_f)..."
[7, p. 899] and concluded "...there is no obvious reason to expect the
sign of the co-skewness coefficient to depend on the relationship
between R_M and R_f" [7, p. 908].

The purpose of this note is to examine, theoretically, why the market
risk premium $(\overline{R}_M - R_f)$ may influence tests of asset pricing models with
higher moments. When moments of higher order than the variance are added
to a pricing model developed within the usual two-fund separation assump-
tions, the market risk premium enters the pricing equation in a nonlinear
fashion and is implicit in the estimation of each moment's coefficient.
Unless this nonlinearity is recognized, incorrect conclusions regarding
the tests of such models may result.

In Section II, the three moment model is re-examined to demonstrate
the presence of the market risk premium in each moment's coefficient.
Because the market risk premium introduces non-linearities in the model,
empirical tests should be redesigned to distinguish between the effects
of $(\overline{R}_M - R_f)$ and skewness. Furthermore, expressing the model in this
manner provides a clearer understanding of the conditions which are
necessary if skewness is to be useful in explaining the two moment CAPM
empirical results. A brief summary is contained in Section III.

II. Higher Moments and the Market Risk Premium

The Three Moment Model

Using the framework and notation developed in KL [13], the theore-
tical market equilibrium relationship between security excess returns

$(\overline{R}_i - R_f)$, the market risk premium $(\overline{R}_M - R_f)$, systematic risk (β_i) and systematic skewness (γ_i) is:

$$\overline{R}_i - R_f = [(\overline{R}_M - R_f)/(1+K_3)]\beta_i + [K_3(\overline{R}_M - R_f)/(1+K_3)]\gamma_i \qquad (1)$$

where: $K_3 = [(d\overline{W}/dm_W)/(d\overline{W}/d\sigma_W)](m_M/\sigma_M)$, the market's marginal rate of substitution between skewness and risk times the risk-adjusted skewness of the market portfolio

   σ_M, m_M = second and third central moments about the market portfolio's return

   $\overline{W}$, σ_W, m_W = first, second and third central moments about end of period wealth.

The KL version of the model is given by (KL 3):

$$\overline{R}_i - R_f = [(d\overline{W}/d\sigma_W)\sigma_M]\beta_i + [(d\overline{W}/dm_W)m_M]\gamma_i \qquad (KL\ 3)$$

Kraus and Litzenberger recognize that, in equilibrium, (KL 3) also implies the following condition:

$$\overline{R}_M - R_f = [(d\overline{W}/d\sigma_W)\sigma_M] + [(d\overline{W}/dm_W)m_M] \qquad (2)$$

since $\beta_M = \gamma_M = 1$. Thus, (2) produces one empirical hypothesis of the model--that the sum of the estimated coefficients should equal $(\overline{R}_M - R_f)$. However, the (KL 3) specification does not reveal all of the information in the theoretical model since it does not specify the effects of the market risk premium on the individual coefficients of β_i and γ_i. That is, the development of (KL 3) and (2) also impose restrictions between $(\overline{R}_M - R_f)$ and each of the coefficients individually. These

effects can be seen by dividing (KL 3) by (2) which produces equation
(1). Equation (1) indicates that $(\overline{R}_M - R_f)$ is implicit in each of the
model's coefficients on β_i and γ_i. Although (KL 3) and (1) both come
from the same theoretical model, the derivation of (1) is consistent
with the risk premium formulation of the two moment CAPM[2] and brings to
light some insights regarding the three moment model and why empirical
tests of the model can be sensitive to the market risk premium.

Consider the linear empirical version of (1):

$$\overline{R}_i - R_f = b_0 + b_1\beta_i + b_2\gamma_i \tag{3}$$

where: b_0 = intercept, hypothesized to equal 0

$$b_1 = [(\overline{R}_M - R_f)/(1+K_3)]$$

$$b_2 = [K_3(\overline{R}_M - R_f)/(1+K_3)]$$

Previous studies have focused upon the entire coefficients of β_i and
γ_i, b_1 and b_2, in the examination of risk and skewness. Such examina-
tions, however, measure the joint effects of $(\overline{R}_M - R_f)$ and K_3. Failure
to separate $(\overline{R}_M - R_f)$ from K_3 may result in incorrect inferences
regarding the importance as well as the sign of risk and skewness.

Consistent with the two moment model (a special case of (1)), the
importance of risk is more properly measured by $\widehat{(\overline{R}_M - R_f)} = \hat{b}_1 + \hat{b}_2$,
rather than $\hat{b}_1$; likewise, the importance of skewness, K_3, is gauged by
$\hat{b}_2/\hat{b}_1$, rather than $\hat{b}_2$. Thus, skewness is evaluated on a relative basis
as measured by the market's tradeoff between skewness and risk.[3] Since
the nonlinear parameters of (1) are identified in terms of the linear
parameters of (3), time series (non-stationary), cross-sectional tests
such as those performed in [7, 13] can still be used to test separate
hypotheses about K_3 and $(\overline{R}_M - R_f)$.

In addition, because of the interaction between $(\overline{R}_M - R_f)$ and K_3 in the determination of b_1 and b_2, under certain conditions b_1 and b_2 can also give misleading signals regarding the signs of risk and skewness in the model. When the empirical model estimates a negative market premium $(\hat{b}_1 + \hat{b}_2 < 0)$, $\hat{b}_2$ attaches the wrong sign to skewness. When $(\overline{R}_M - R_f)$ < 0 and $\hat{K}_3 < 0$, $\hat{b}_2 > 0$. Similarly, when $(\overline{R}_M - R_f) < 0$ and $\hat{K}_3 > 0$, $\hat{b}_2 < 0$.[4] Since the linear model focuses on $\hat{b}_2$ rather than $\hat{K}_3$, its use will lead to incorrect conclusions regarding the sign of skewness when $(\overline{R}_M - R_f) < 0$. This is especially troublesome in a study such as FW in which 26 of the 68 regressions result in $(\overline{R}_M - R_f) < 0$. In 24 of these cases, the use of $\hat{b}_2$ rather than $\hat{K}_3$ leads to incorrect inferences regarding the sign of skewness.[5] In similar fashion, when $\hat{K}_3 < -1$, the use of $\hat{b}_1$, rather than $\hat{b}_1 + \hat{b}_2$, results an incorrect inconclusion regarding the sign of risk.

Skewness Preference and the Two Moment CAPM

Empirical tests of the two moment CAPM have found a positive intercept and a slope value less than its theoretical value, $(\overline{R}_M - R_f)$. If the three moment model is the correct pricing mechanism, then the omission of γ_i from the two moment model should explain, empirically, the two moment model's results. Explicit consideration of $(\overline{R}_M - R_f)$ in each coefficient in the three moment model (1) provides a linkage between the two models and enables an examination of the theoretical conditions under which the omission of γ_i is consistent with the two moment empirical results.

The two moment CAPM is given by (4):

$$\overline{R}_i - R_f = b_0^* + b_1^* \beta_i \tag{4}$$

Under the hypothesis that the three moment model is correct:

$$b_1{}^* = \text{cov}[(\overline{R}_i - R_f), \beta_i]/\text{var}(\beta_i)$$

$$= \text{cov}[(b_0 + b_1\beta_i + b_2\gamma_i), \beta_i]/\text{var}(\beta_i)$$

$$= (\overline{R}_M - R_f)[(1 + \alpha K_3)/(1 + K_3)] \qquad (5)$$

where $\alpha = \text{cov}(\beta_i, \gamma_i)/\text{var}(\beta_i)$, the slope of the regression of

γ_i against β_i

Equation (5) provides theoretical support for KL's "heuristic rationale" [13, p. 1098] and their empirical results since if $\alpha > 1$ when $K_3 < 0$ ($m_M > 0$), $b_1^* < (R_M - R_f)$ and $b_0^* > 0$. The empirical evidence provided by KL and FW indicates considerable correlation between β_i and γ_i when $m_M > 0$ as well as when $m_M < 0$.[6] Furthermore, it seems reasonable that $\text{var}(\gamma_i) > \text{var}(\beta_i)$. Together, these imply that $\alpha > 1$ and the empirical results of the two moment CAPM are consistent with a market preference for positive skewness when $m_M > 0$. However, note that $b_1^* < (\overline{R}_M - R_f)$ and $b_0^* > 0$ when $K_3 > 0$ ($m_M < 0$) only if $\alpha < 1$. Thus, a preference for positive skewness when $m_M < 0$ requires higher β_i's to be associated with proportionately smaller γ_i's.

Extension of the Pricing Model to N Moments

With the recent interests in asset prices and in higher moments, some researchers may be tempted to expand the asset pricing model beyond three moments. The interaction of $(\overline{R}_M - R_f)$ with higher moments becomes compounded when moments higher than skewness are included. The theoretical N moment pricing model is:

$$\overline{R}_i - R_f = (\overline{R}_M - R_f) \sum_{n=2}^{N} [(K_n \gamma_i^n) / \sum_{n=2}^{N} K_n] \tag{6}$$

where: $\qquad K_n = [(d\overline{W}/dm_{n,W})/(d\overline{W}/dm_{2,W})](m_{n,M}/m_{2,M})$

$m_{n,M}$ = the nth central moment about the market portfolio's

rate of return, where $m_{2,M} = \sigma_M$ and $m_{3,M} = m_M$ as in (1)

$m_{n,W}$ = the nth central moment about the investor's end of

period wealth, where $m_{2,W} = \sigma_W$ and $m_{2,W} = m_W$ as in (1)

γ_i^n = the systematic portion of the nth moment for asset i,

where $\gamma_i^2 = \beta_i$ and $\gamma_i^3 = \gamma_i$ as in (1)

The two (N=2) and three (N=3) moment models are simply special cases of (6). As seen in (6), $(\overline{R}_M - R_f)$ appears in each of the N moments' coefficients and the importance of the nth moment is an assessment of the preference tradeoffs in the market between the nth moment and the second moment (risk).

Conclusion

Recent research has examined the importance of skewness in the pricing of risky assets, finding the results of such tests to be influenced by the market risk premium. The purpose of this note has been to explore a not so obvious theoretical relationship within such models-- namely, that such models are intrinsically non-linear in the market risk premium. Failure to account for this interaction may lead to erroneous conclusions regarding the empirical results of such models.

Footnotes

[1] In this paper, the word "skewness" refers to the third moment of the return distribution. Many authors use the term "skewness" as the third moment divided by the standard deviation cubed.

[2] In the two-moment version of the model, the investor's problem is to maximize: $E[U(\widetilde{W})] = U[\overline{W}, \sigma_W]$ subject to: $\sum_i q_i + q_f = W_0$. The equilibrium conditions are:

$$(\overline{R}_i - R_f) = [(d\overline{W}/d\sigma_W)\sigma_M]\beta_i \tag{a}$$

$$(\overline{R}_M - R_f) = [(d\overline{W}/d\sigma_W)\sigma_M] \tag{b}$$

Dividing (a) by (b) produces the familiar two-moment CAPM in terms of the market risk premium, $\overline{R}_i - R_f = (\overline{R}_M - R_f)\beta_i$. In both the two-moment model and equation (1), the market's marginal rate of substitution between return and risk times market risk, $(d\overline{W}/d\sigma_W)\sigma_M$, is not present in the final equation. However, the contribution of skewness to the model is evaluated by the relative importance of the third moment vis a vis the second moment (K_3).

[3] When $|K_3| < 1 \,(> 1)$, risk is more (less) important than skewness in the pricing of assets. When $|K_3| = 1$, the market views risk and skewness as equally important. When $K_3 = 1$, equation (1) becomes:

$$\overline{R}_i - R_f = \frac{1}{2}(\overline{R}_M - R_f)(\beta_i + \gamma_i)$$

However, when $K_3 = -1$, the theoretical model specification becomes ambiguous since as $K_3 \to -1 \Rightarrow (\overline{R}_M - R_f) \to 0$ and:

$$\overline{R}_i - R_f = \left\{ \begin{matrix} \lim \\ K_3 \to -1 \\ \overline{R}_M - R_f \to 0 \end{matrix} \right\} [(\overline{R}_M - R_f)/(1+K_3)]\beta_i + [K_3(\overline{R}_M - R_f)/(1+K_3)]\gamma_i$$

$$= [d\overline{W}/d\sigma_W)\sigma_M](\beta_i - \gamma_i) \quad \underline{or}$$

$$= [d\overline{W}/dm_W)m_M](\gamma_i - \beta_i)$$

It is only in this case, when $K_3 = -1$, where $(\overline{R}_M - R_f)$ is not theoretically implicit in both coefficients on β_i and γ_i.

[4] This assumes that $|K_3| < 1$ when $K_3 < 0$. Exceptions to this in FW [7] correspond to Table IV: 1972-1976 and Table VII: 1952-1976.

[5] The two exceptions to this in FW [7] correspond to instances where $\hat{K}_3 < -1$ (see footnote 4). The most dramatic illustration of the effects of $(\overline{R}_M - R_f)$ on b_2 can be seen in the FW study where the periods are divided into cases where $\overline{R}_M > R_f$ and where $\overline{R}_M < R_f$ (e.g., Table VI). Since $(\overline{R}_M - R_f)$ is implicit in b_2, the sign of b_2 will be influenced by the sign of $(\overline{R}_M - R_f)$.

[6] For example, see [7, p. 902, fn. 15] and [13, p. 1098, Table III].

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Notes for the Reviewer

Derivation of the N Moment
Capital Asset Pricing Model

Extension of the KL framework to an N moment pricing model implies
that the investor seeks to:

maximize: $E[U(\tilde{W})] = U[\overline{W}, m_{2,W}, m_{3,W}, \ldots, m_{N,W}]$ (1)

subject to: $\sum_i q_i + q_f = W_0$ (2)

where:

$E[U(\tilde{W})]$ = expected value of the utility of terminal wealth W

$$\overline{W} = E(\tilde{W}) = \sum_i q_i \overline{R}_i + q_f R_f \tag{3}$$

$$m_{2,W} = [E(\tilde{W}-\overline{W})^2]^{1/2} = [\sum_{ij} q_i q_j m_{ij}]^{1/2} \tag{4}$$

$$m_{3,W} = [E(\tilde{W}-\overline{W})^3]^{1/3} = [\sum_{ijk} q_i q_j q_k m_{ijk}]^{1/3}$$

$$\vdots$$

$$m_{N,W} = [E(\tilde{W}-\overline{W})^N]^{1/N} = [\sum_{ij}\cdots\sum_N q_i q_j \cdots q_N m_{ij\ldots N}]^{1/N}$$

q_i, q_f = amount (in dollars) of initial wealth (W_0) invested
      in asset i and the riskless asset f

$\overline{R}_i, R_f$ = expected holding period return on i and the holding
      period return on f

$$m_{ij} = E[(R_i - \overline{R}_i)(R_j - \overline{R}_j)] \tag{5}$$

$$m_{ijk} = E[(R_i - \overline{R}_i)(R_j - \overline{R}_j)(R_k - \overline{R}_k)]$$

$$\vdots$$

$$m_{ij\cdots N} = E[(R_i - \overline{R}_i)(R_j - \overline{R}_j)\cdots(R_N - \overline{R}_N)]$$

At the end of the period, the investor receives $\widetilde{W} = \sum_i q_i \widetilde{R}_i + q_f R_f$ (6)

For the investor's portfolio, define the following terms:

$$\overline{R}_p = E(R_p) = \sum_i (q_i/W_0)\overline{R}_i + (q_f/W_0)R_f \tag{7}$$

$$\gamma_{ip}^2 = m_{ip}/m_{2,p}^2 = E[(R_i - \overline{R}_i)(R_p - \overline{R}_p)]/E(R_p - \overline{R}_p)^2 \tag{8}$$

$$= \sum_j (q_j/W_0)m_{ij}/\sum_{ij}\sum (q_iq_j/W_0^2)m_{ij} = \text{the systematic risk of}$$

asset i with the investor's portfolio p

$$\gamma_{ip}^3 = m_{ipp}/m_{3,p}^3 = E[(R_i - \overline{R}_i)(R_p - \overline{R}_p)^2]/E(R_p - \overline{R}_p)^3$$

$$= \sum_{jk}\sum (q_jq_k/W_0^2)m_{ijk}/\sum_{ijk}\sum\sum (q_iq_jq_k/W_0^3)m_{ijk} = \text{the systematic}$$

skewness of asset i with the investor's portfolio p

$$\vdots$$

$$\gamma_{ip}^N = m_{ip\cdots p}/m_{N,p}^N = E[(R_i - \overline{R}_i)(R_p - \overline{R}_p)^{N-1}]/E(R_p - \overline{R}_p)^N$$

$$= \sum_j\cdots\sum_N (q_j\cdots q_N/W_0^{N-1})m_{ij\cdots N}/\sum_{ij}\sum\cdots\sum_N (q_iq_j\cdots q_N/W_0^N)m_{ij\cdots N}$$

= the systematic portion of the nth moment for asset i

with portfolio p

The Lagrangian and first order conditions are:

$$L = U(\overline{W}, m_{2,W}, m_{3,W}, \ldots, m_{N,W}) - \lambda[\sum_i q_i + q_f - W_0] \qquad (9)$$

$$dL/dq_i = (dU/d\overline{W})(d\overline{W}/dq_i) + (dU/dm_{2,W})(dm_{2,W}/dq_i) + (dU/dm_{3,W})(dm_{3,W}/dq_i)$$

$$+ \ldots + (dU/dm_{N,W})(dm_{N,W}/dq_i) - \lambda = 0 \text{ for all } i \qquad (10)$$

$$dL/dq_f = (dU/d\overline{W})(d\overline{W}/dq_f) - \lambda \qquad \qquad = 0 \qquad (11)$$

$$dL/d\lambda = \sum_i q_i + q_f - W_0 \qquad \qquad = 0 \qquad (12)$$

In solving for the investor's portfolio equilibrium conditions, note that:

$$m_{2,W} = \sum_i q_i \gamma_{ip}^2 m_{2,p} \qquad (13)$$

$$m_{3,W} = \sum_i q_i \gamma_{ip}^3 m_{3,p}$$

$$m_{N,W} = \sum_i q_i \gamma_{ip}^N m_{N,p}$$

Conditions (3) and (13) imply:

$$d\overline{W}/dq_i = \overline{R}_i \qquad (14)$$

$$d\overline{W}/dq_f = R_f \qquad (15)$$

$$dm_{2,W}/dq_i = \gamma_{ip}^2 m_{2,p} \tag{16}$$

$$dm_{3,W}/dq_i = \gamma_{ip}^3 m_{3,p}$$

$$dm_{N,W}/dq_i = \gamma_{ip}^N m_{N,p}$$

$$dm_{n,W}/dq_f = 0 \quad \text{for } n = 2, \ldots, N$$

Conditions (11) and (15) imply:

$$\lambda = (dU/d\overline{W})R_f \tag{17}$$

Substituting (14), (16) and (17) into (10):

$$(dU/d\overline{W})(\overline{R}_i - R_f) = - (dU/dm_{2,W})\gamma_{ip}^2 m_{2,p} - (dU/dm_{3,W})\gamma_{ip}^3 m_{3,p}$$

$$- \ldots - (dU/dm_{N,W})\gamma_{ip}^N m_{N,p} \quad \text{for all } i \tag{18}$$

Moving from the investor's equilibrium condition (18) to a market equilibrium requires that (18) holds for all individuals and that markets clear. For markets to clear, all assets have to be held which requires the value weighted average of all individual's portfolios equal the market portfolio m. Summing (18) across all individuals gives:

$$(dU/d\overline{W})(\overline{R}_i - R_f) = - (dU/dm_{2,W})\gamma_i^2 m_{2,M} - (dU/dm_{3,W})\gamma_i^3 m_{3,M}$$

$$- \ldots - (dU/dm_{N,W})\gamma_i^N m_{N,M} \quad \text{for all } i \tag{19}$$

Since (19) holds for any security or portfolio, it also holds for the market portfolio:

$$(dU/d\overline{W})(\overline{R}_M - R_f) = - (dU/dm_{2,W})m_{2,M} - (dU/dm_{3,W})m_{3,M}$$

$$- \ldots - (dU/dm_{N,W})m_{N,M} \tag{20}$$

Dividing (19) by (20) gives the capital asset pricing model in terms of the N moments and the market risk premium $(\overline{R}_M - R_f)$

$$\overline{R}_i - R_f = (\overline{R}_M - R_f)[(K_2 \gamma_i^2 / \sum_{n=2}^{N} K_n) + (K_3 \gamma_i^3 / \sum_{n=2}^{N} K_n)$$

$$+ \ldots + (K_N \gamma_i^N / \sum_{n=2}^{N} K_n)]$$

$$\overline{R}_i - R_f = (\overline{R}_M - R_f) \sum_{n=2}^{N} [(K_n \gamma_i^n) / \sum_{n=2}^{N} K_n] \qquad (21)$$

where:

$$K_n = [(d\overline{W}/dm_{n,W})/(d\overline{W}/dm_{2,W})](m_{n,M}/m_{2,M})$$

In words, equation (21) says that in equilibrium the excess return on security i, $(\overline{R}_i - R_f)$, is a function of the excess return on the market $(\overline{R}_M - R_f)$, the market-related systematic risks of variance and the higher moments (γ_i^n), and the preference tradeoffs in the market between risk and all higher moments. This is equation (6) in the text.

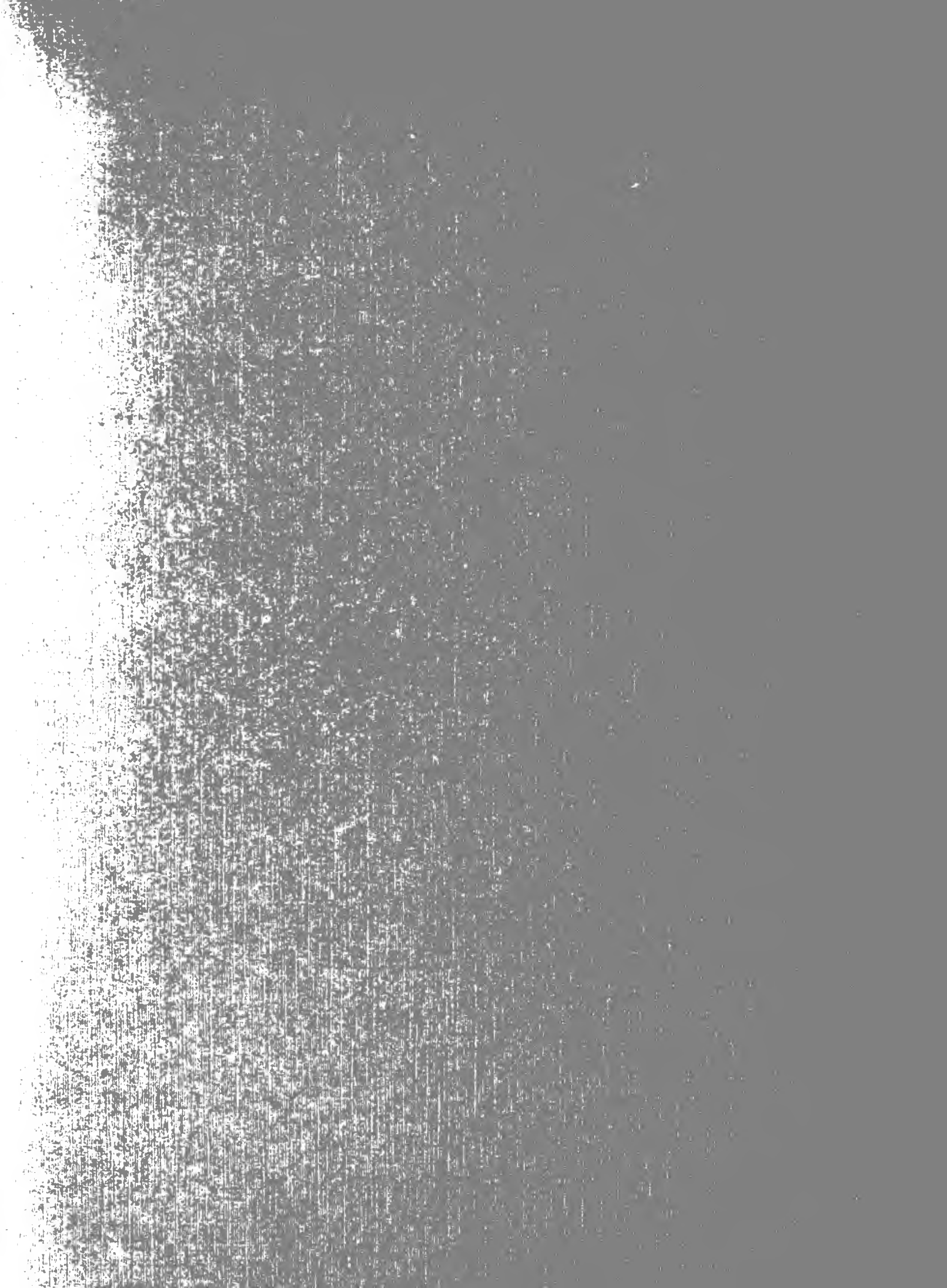## Special Cases: Mean-Variance and Three Moment Pricing Models

An investor who makes investment decisions solely upon the mean and variance of wealth seeks to maximize $E[U(\tilde{W})] = U[\overline{W}, m_{2,W}]$. Similarly, an investor who considers only the first three moments will maximize $E[U(\tilde{W})] = U[\overline{W}, m_{2,W}, m_{3,W}]$. These two versions are special cases of (21) where $N = 2$ and $N = 3$. When $N = 2$, we have the two moment CAPM model:

$$\overline{R}_i - R_f = (\overline{R}_M - R_f)\gamma_i^2 \qquad\qquad (22)$$

and when $N = 3$, we obtain the three moment version (equation (1) in text):

$$\overline{R}_i - R_f = [(\overline{R}_M - R_f)/(1+K_3)]\gamma_i^2 + [(\overline{R}_M - R_f)K_3/(1+K_3)]\gamma_i^3 \qquad (23)$$