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# NAVAL POSTGRADUATE SCHOOL

## Monterey, California

ASYMPTOTIC REPRESENTATION OF

STIRLING NUMBERS OF THE SECOND KIND

by

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9 February 1977

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ABSTRACT:

The distribution of the Stirling numbers S(n,k) of the second kind with respect to k has been shown by Harper [Ann. Math. Statist., 38 (1967), 410-414] to be asymptotically normal near the mode. A new single-term asymptotic representation of S(n,k), more effective for large k, is given here. It is based on Hermite's formula for a divided difference and the use of sectional areas normal to the body diagonal of a unit hypercube in k-space. A proof is given that the distribution of these areas is asymptotically normal. A numerical comparison is made with the Harper representation for n=200.

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## 1. Introduction.

Previous asymptotic representations of Stirling numbers $S(n,k)$ of the second kind have been of two types. One type has been a complete infinite series expansion as given by Hsu [1], and by Bleick and Wang [2] and [3]. A second type has been the single-term representation of $S(n,k)$ given by Harper [4] as the normal distribution approximation

(1) $$S(n,k) \sim \frac{B_n}{\sigma\sqrt{2\pi}} \exp[-(k-\mu)^2/2\sigma^2]$$

where the mean $\mu$ and the variance $\sigma^2$ are expressed in terms of the Bell numbers $B_n$ by

(2) $$\mu = B_{n+1}/B_n - 1$$

and

(3) $$\sigma^2 = B_{n+2}/B_n - (B_{n+1}/B_n)^2 - 1 .$$

The purpose of this note is to give a new single-term asymptotic representation based on Hermite's formula for a divided difference, and to compare it with that of Harper.

## 2. Use of Hermite's formula.

A Stirling number $S(n,k)$ of the second kind is defined as the kth difference of $z^n$ at $z=0$ divided by $k!$. By [5,p.10] we find that this divided difference can be represented by a formula of Hermite as the repeated definite integral

(4) $$S(n,k) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} (d^k u_1{}^n/du_1{}^k)\, dt_k$$

where $u_1=t_1+t_2+..+t_k$. We imagine that $t_1$, $t_2$, .., $t_k$ constitute a set of rectangular Cartesian coordiantes and impose an orthogonal transformation of coordinates to $u_1$, $u_2$, .., $u_k$. The volume of the space over which the integration in (4) is performed is a portion of a unit hypercube in k-space. If we allow the coordinate $u_1$ to vary along the body diagonal of the hypercube from 0 at one vertex to k at the opposite vertex, the sectional areas normal to the diagonal cut by the hyperplane $u_1=t_1+t_2+..+t_k$ from the domain of integration define a positive function $g(u_1,k)$ even with respect to the argument $u_1-k/2$. We take the integral of $g(u_1,k)$ to be

$$(5) \qquad \int_0^k g(u_1,k)du_1 = 1/k!$$

to agree with the volume of the space over which the integration in (1) is performed. We drop the $u_1$ subscript henceforth. Noting that $g(u,k)=0$ for $k<u<0$, we find that

$$(6) \qquad g(u,1) = 1 \quad \text{for } 0 \leq u \leq 1 \quad ,$$

$$(7) \qquad 2!g(u,2) = (1 - |u-1|) \quad \text{for } 0 \leq u \leq 2 \ ,$$

and

$$(8) \qquad 3!g(u,3) = \begin{cases} (3/2-|u-3/2|)^2/2 & \text{for } 1/2 \leq |u-3/2| \leq 3/2 \\ 3/4 - (u-3/2)^2 & \text{for } 1 \leq u \leq 2 \ . \end{cases}$$

Consideration of the Laplace transforms of (6), (7) and (8) suggests that we conjecture the Laplace transform of $k!g(u,k)$ to be

$$(9) \qquad (1-e^{-s})^k/s^k = e^{-ks/2}\left(\frac{\sinh s/2}{s/2}\right)^k$$

for all k. We demonstrate the truth of this conjecture later. On performing the integration in (4) over the variables $u_2$, $u_3$, .., $u_k$ we find

$$(10) \qquad S(n,k) = k!\binom{n}{k}\int_0^\infty u^{n-k}g(u,k)du \ .$$

Using operation 82 of [6,p.10] on the Laplace transform of

(11) $$k! \int_0^u u^m g(u,k)\,du$$

we find the mth moment of the k!g(u,k) distribution about u=0 to be

(12) $$\lim_{s\to 0} (-1)^m (d/ds)^m (1-e^{-s})^k/s^k \ .$$

It is now easy to demonstrate the truth of the conjecture (9) by showing, with the aid of the multinomial theorem, that (12) is the same as the repeated integral

(13) $$\int_0^1 dt_1 \int_0^1 dt_2 \ldots \int_0^1 (t_1+t_2+..+t_k)^m dt_k$$

over the volume of the hypercube.

Use of (12) and (5) shows the variance of the k!g(u,k) distribution to be

(14) $$\sigma^2 = k/12 \ .$$

Using (14) the series

(15) $$\exp(\sigma^2 s^2/2) = 1 + \frac{ks^2/24}{1!} + \frac{(ks^2/24)^2}{2!} + ..$$

is the bilateral, but not s multiplied, Laplace transform of the normal distribution

(16) $$(1/\sigma\sqrt{2\pi})\exp(-t^2/2\sigma^2)$$

according to [7,p.21]. The corresponding series for (9) multiplied by $e^{ks/2}$, or the bilateral Laplace transform of k!g(u,k) shifted left by k/2, is

(17) $$(2/s)^k \sinh^k s/2 = [1 + \frac{s^2/4}{3!} + \frac{(s^2/4)^2}{5!} + ..]^k \ .$$

The dominant k power term in the coefficient of $(s^2/4)^n$ in (15) is $k^n/6^n n!$, and may be shown to be the same in the expansion of (17) by the use of the recurrence formula 6.361 of [8,p.119]. This proves that the k!g(u,k) distribution is asymptotically normal as $k\to\infty$. It is remarkable that the normal distribution should arise in the purely geometrical context of sectional areas normal to the body diagonal of a hypercube of high dimension.

On replacing $k!g(u,k)$ in (10) by its Gaussian normal approximation of mean $\mu=k/2$ and variance $\sigma^2=k/12$ we find

$$(18) \qquad S(n,k) \sim \frac{1}{\sigma\sqrt{2\pi}}\binom{n}{k}\int_0^\infty u^{n-k}\exp[-(u-k/2)^2/2\sigma^2]du$$

$$\sim \frac{1}{\sqrt{2\pi}}\binom{n}{k}\int_{-\infty}^{\sqrt{3k}}(k/2-\sigma t)^{n-k}e^{-t^2/2}dt \; .$$

3. Numerical example.

Table 1 compares the exact values of S(200,k) with the asymptotic approximations computed from the single-term representations (1) and (18). Harper's representation (1), which uses $B_{200}=.62475\ 10^{276}$, $\mu=49.975$ and $\sigma=3.0551$, gives an excellent fit near the mode (k=50), but (18) gives a much better fit for large values of k.

Table 1. Values of S(200,k)

| k | Asymptotic from (1) | Exact | Asymptotic from (18) |
|---|---|---|---|
| 2 | $.23135\ 10^{222}$ | $.80347\ 10^{6}$ | $.69244\ 10^{126}$ |
| 40 | $.39504\ 10^{273}$ | $.24458\ 10^{273}$ | $.42658\ 10^{273}$ |
| 50 | $.81579\ 10^{275}$ | $.81493\ 10^{275}$ | $.15285\ 10^{277}$ |
| 60 | $.37452\ 10^{273}$ | $.53533\ 10^{273}$ | $.29658\ 10^{274}$ |
| 100 | $.49065\ 10^{217}$ | $.22839\ 10^{235}$ | $.27994\ 10^{235}$ |
| 150 | $.13938\ 10^{43}$ | $.30251\ 10^{143}$ | $.30441\ 10^{143}$ |
| 199 | $.16955\ 10^{-241}$ | $.19900\ 10^{5}$ | $.19900\ 10^{5}$ |

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