A

SYSTEM

OF

MECHANICS,

BEING THE

SUBSTANCE OF LECTURES UPON THAT BRANCH OF NATURAL PHILOSOPHY,

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MDCCLXXXV.

TO THE

TUTORS, AND OTHER MEMBERS OF THE UNIVERSITY,

INTERESTED IN THE

ADVANCEMENT OF PHILOSOPHICAL KNOWLEDGE,

THE FOLLOWING WORK,

FOR THE USE OF THE ACADEMIC STUDENT,

IS,

WITH GREAT DEFERENCE AND RESPECT,

INSCRIBED

BY THEIR VERY HUMBLE SERVANT,

THOMAS PARKINSON.

CHRIST'S COLLEGE,
NOV. 25, 1784.

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P R E F A C E.

THE want of ſyſtematic treatiſes on mechanics and hydroſtatics hath long been conſidered as equally troubleſome to the tutor, and diſcouraging to his pupils, and firſt induced a deſire to facilitate the attainment of theſe branches of ſcience by a collection, and methodical arrangement, of their ſcattered parts. The preſent coincides, very nearly, with the propoſitions, or heads of lectures, upon this ſubject, adopted by the generality of tutors in the univerſity, and is recommended by an obvious connexion and regular order of dependancy. In all natural ſcience, the analytic method of reaſoning neceſſarily precedes the ſynthetic, and the converſe of this order would terminate in uncertainty and chimera. The exiſtence and delineation of thoſe properties of matter, which are conceived to generate the phenomena of preſſure and motion, are therefore conciſely and generally premiſed in the firſt ſix chapters, and ſupply data, derived from experience and uncontroverted facts, for ſynthetic demonſtrations. The general properties of motion are deſcribed in the chapter upon ſolidity, becauſe the cleareſt conceptions of motion are derived from impulſe, which ariſes from ſolidity; and the laws of motion are introduced in the chapter upon the inertia of matter, because they are inſeparably connected

with

with it. The compofition and refolution of forces are immediately fubfequent; becaufe every propofition upon this fubject is deducible from the fecond law of motion, and may be efteemed a corollary to it. The connection of the other parts of the treatife is, I truft, equally natural and obvious. To have divefted this elementary branch of philofophy of that dry, uninviting afpect, fo frequently and feelingly lamented by the young pupil, was certainly moft defirable; but the fame complaint is applicable to the elements of all fcience, and could not, I believe, be removed without an entire change of the formality of precife definitions and propofitions into a more undefined and lefs fcientific form of compofition, nor confequently without facrificing real advantage to lefs material amufement. The following performance, therefore, claims little more than the inferior merit of facilitating the progrefs of the ftudent by a felection, from the works of others, which may fuperfede the neceffity of applying to a multitude of books, and an arrangement coinciding with his lectures. The whole is written in the fame language for the fake of uniformity; demonftrations are dilated or contracted as was deemed expedient; and fometimes, though as feldom as poffible, new proofs are given. It is no eafy tafk to compofe a fyftem equally accommodated to every defcription of underftandings, and I dare not hope to have accomplifhed it: for the benefit of thofe, who may be diffatisfied with the difcuffion of any fubject here, accurate references to the page or chapter of the beft writers where it is treated, are always inferted. To be of fervice to the ignorant and uninformed was the chief motive for undertaking this work, and the fole object of attention in the execution of it: this laft confideration may ferve to obviate fome objections, which naturally will occur to more enlightened readers already converfant with the fubject. Left I appear to prefume upon the experience that may be fuppofed to attend my fituation, I beg leave to explain

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myself. When several demonstrations of the same proposition are exhibited, let it not be inferred that one was deemed insufficient to establish its truth and required an auxiliary: when any important subject is treated with prolixity, and extended beyond the limits usually prescribed to an elementary treatise, let it not be attributed to negligence or inadvertency. they are both the effects of design, and I am warranted, by my own experience at least, to assert their utility. It must be remembered, that clear and adequate ideas upon a new subject are not communicated by a transient impression, and that the capacity and comprehension of an uninformed mind are only expanded and improved by repeated exertions and assiduity. Science implies something more than a mere ability to go through a demonstration; and a clear comprehensive knowledge of a question, which may be competent for the solution of problems and dissipation of doubts and objections, is perhaps only to be attained by long reflection, and studiously contemplating it on every side; and these are certainly much assisted by different demonstrations, which place the subject in different points of view, and by tracing its affinity with other truths similar to, or deducible from, it. A single demonstration may easily be too long, and too anxious a desire to be perspicuous may occasion perplexity and confusion; but it is not easy to give too many demonstrations of a fundamental proposition, or too many problems and corollaries resulting from it. This species of prolixity appears to me to be extremely useful, as it may, at least, teach a young student the art of thinking, an art which can only be acquired, and is found indeed to be no easy acquisition. For these reasons, the reader will observe some leading subjects extended beyond the bounds absolutely necessary, and some parts will apparently be superfluous: I shall hold him justified in thinking so, when he shall be so informed on the subject as to pronounce them useless to him.

I cannot conclude this preface without expressing my sincere thanks to Mr. Fisher and Mr. Vince, of Caius College; to the first for his general theorem of the wedge; to the latter for his friendly revision of the whole work. Had I consulted these friends more frequently, many errors and imperfections, for I believe there are many, might possibly have been avoided. My thanks are also justly due to those gentlemen who have kindly honoured this undertaking with their approbation and support, and I willingly embrace this public opportunity of acknowledging my obligation.

INTRODUCTION

TO THE STUDY OF

NATURAL PHILOSOPHY.

1. NATURAL phenomena, in the wideſt acceptation of the terms, denote any effects in the material part of the creation addreſſed to one or more of the ſenſes; and natural philoſophy is the hiſtory of theſe phenomena, and an inveſtigation of the cauſes employed in their production. A phenomenon may itſelf be a natural cauſe productive of numberleſs effects, and each of theſe may alſo be a cauſe of others, &c.; for, of that infinite variety of events obſervable in the material world, none are induced per ſaltum, but effect is dependent upon effect in contiguous ſucceſſion. As matter is totally inactive, and incapable of communicating motion to itſelf, all its motions, and powers of producing a change of motion, in the various operations of nature, are derivative: but the inſtruments immediately directing the movements of the ſeveral parts of the ſyſtem elude the inquiſition of human ability, and whether any inexplicable effect be owing to the Creator's immediate fiat, or ſome ſecondary material power, cannot be known; for the action of a pure ſpirit upon matter cannot be comprehended: but many ſubordinate inſtruments in the government of nature are conſpicuous, matter being impreſſed by its great Creator with ſeveral attributes, which appear, and are conceived by ſome philoſophers, to reſide in it, miniſterial to the continuance of exiſtence and preſervation. The rules by which theſe attributes are directed in their operations, are called natural principles or laws, becauſe in ſimilar circumſtances, they are invariably the ſame; and the attributes themſelves are generally called powers or forces,

 from

from the ſimilitude of their effects to thoſe produced by animal exertions: ſuch are gravity, coheſion, elaſticity, magnetiſm, electricity. The uniform and regular action and utility of ſome of theſe natural powers are very obſervable; but the laſt powers ſeem to be ſtill in a ſtate of analyſis, and the laws, by which their influence is directed, very imperfectly aſcertained. The attraction of coheſion, or of that power by whoſe influence the minute particles of matter tend to each other at ſmall diſtances, is the cement which prevents the diſperſion of the component parts of matter, and, as far as we are competent to decide, adminiſters to the growth of bodies in the animal, vegetable, and foſſil kingdoms. The figures and motions of the great bodies compoſing the ſolar ſyſtem are preſerved by an uninterrupted exertion of the attraction of gravity. Nor is the oppoſite quality of elaſticity or repulſion leſs regular or important. The particles of air are endued with this repulſive power which is eſſential to the preſervation of all bodies contiguous to it, and a diminution of it would proportionably diminiſh its ſalutary influence, rendering it noxious to animal and vegetable life; and many hard bodies are obviouſly poſſeſſed of this quality probably to preſerve their ſpecific nature by protecting their conſtituent particles from the effects of attrition, &c. The operations of other attributes of matter, however deſultory and accidental apparently, are, upon the fulleſt information, diſcovered to be reſtrained within preſcribed limits, whoſe obſervance is productive of harmony, and violation of diſorder. Plants and animals are always produced from their proper ſeed; the reflection and refraction of light are effected according to inviolate laws; matter, by its vis inertiæ, reſiſts the action of any material power, and when in motion moves, according to determinate rules, &c. To deſcribe the various phenomena exhibited in the production and change of motion in the material world, whether explicable or not, promiſcuouſly, is the buſineſs of the natural hiſtorian; and to ſelect thoſe which are explicable, and inveſtigate the ſecondary powers, or qualities, conceived to be reſident in matter, by whoſe inſtrumentality they appear to be effected, is the province and proper occupation of the natural philoſopher.

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2. The hiſtorical part of philoſophy, or deſcription of natural phenomena, is immediately tranſcribed from the works of nature; and, from the inadequacy of our ideas of matter, an inveſtigation of the ſecondary material powers producing them muſt be derived from the ſame ſource. Some phenomena being obſerved to be invariably coexiſtent or ſucceſſive, a connexion is underſtood, or preſumed, to obtain between them; and, the object of philoſophy being an evolution of this connexion, the works of nature where only it is diſcoverable, muſt be attentively explored, the relation noted, and extended, by a cautious ſelection of all the circumſtances of ſimilitude in other phenomena, to a common principle or law of nature. The following proceſs is therefore adopted by the beſt philoſophers, and is the only certain method of philoſophizing.

Firſt, By repeated and accurate obſervations upon matter, or upon ſome general ſimilar phenomena in the material world, the exiſtence and qualities of any individual cauſe are demonſtrated from the phenomena evidently and invariably attached to it; the cauſe of this cauſe, and the cauſe of this laſt, &c. and the effects ſucceſſively more ſubordinate and particular reſulting from the phenomena, conſidered as mechanical cauſes, are then inveſtigated by again exploring the works of nature, and this proceſs is repeated to the limit of human ability. How far the chain of theſe ſecondary material powers, either progreſſive or regreſſive, from thoſe that are more general to their ſubordinate effects, and vice verſa, is extended, is undiſcoverable; but the philoſopher's reſearches are ſoon limited by the occurrence of bodies inconceivably minute, or removed to immeaſurable diſtances. An examination of the few powers that are known is, however, a rational and not unprofitable employment; for they exhibit proofs of unbounded power, conſummate wiſdom, and paternal benevolence in the great Creator; and by their connexion with many uſeful arts, are made ſubſervient to the wants and infirmities of his creatures. *Secondly*, The intenſity at a given diſtance, and law of variation at different diſtances, of any material power being aſcertained by the menſuration of its effects, it is aſſumed as a principle, and from its influence, all ſimilar phenomena are demonſtrated to reſult by mathematical or other ſcientific methods of reaſoning. For many effects are, upon examination, found to bear ſtrong marks of ſimilitude, which to a negligent obſerver exhibit no likeneſs; and, theſe being diveſted of all adventitious particularities, and ranged under the ſame cauſe, a few principles are diſcovered to pervade the whole ſyſtem of matter, producing innumerable phenomena. The firſt of theſe is called the analytic, and the ſecond the ſynthetic method of philoſophizing.

3. Examples illuſtrating the analytic method of reaſoning.

EXAMP. I. The ſpherical figure of the earth proved analytically.

PHENOMENA.

PHENOM. I. The altitude of either pole of the equator is always equal to the latitude of the obſerver.

PHENOM. II. In an eclipſe of the moon a ſection of the earth's ſhadow is always terminated by a circular arc.

Many other phenomena might be adduced in proof of the globular figure of the earth, but it is eſtabliſhed by theſe, becauſe no other afford a ſolution of them. This figure was impreſſed at the original formation of the earth, and is preſerved by an unremitted exertion of the attraction of gravity; and here the analyſis terminates, the cauſe of gravity being undiſcovered.

4. EXAMP. II. The diurnal revolution of the earth proved analytically.

PHENOMENA.

PHENOM. I. The ſun and moſt of the planets revolve round their axes, and a ſimilar motion of the earth is analogous to theſe.

PHENOM. II. The ſun, planets and fixed ſtars appear to revolve uniformly in circles, whoſe planes are perpendicular to, and centers in, a line paſſing through the center of the earth, and perform one revolution in twenty-four hours nearly.

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PHENOM. III. The gravity of bodies is leaſt under the equator, and increaſes as they recede from it.

If this motion of the heavenly bodies were real, they would be retained in their orbits by forces tending to their correſponding centers in the axis of the world (Newt. ſect. 2. prop. 2.), which are imaginary points in which no attractive powers reſide; and ſome of theſe motions could not be effected without the influence of forces infinitely greater than any mechanical powers yet diſcovered. They are therefore apparent only, and reſult from the rotation of the earth round an axis; and this is confirmed by phenom. 2. which is eaſily explicable by, and inexplicable without, ſuch rotation.

5. EXAMP. III. The properties of light inveſtigated analytically.

PHENOMENA.

PHENOM. I. If the ſun's rays be admitted, through a ſmall aperture, into a dark room, and refracted through a glaſs priſm, an oblong image conſiſting of ſeveral colours will be formed upon a ſheet of paper placed at a proper diſtance.

PHENOM. II. If another priſm be placed behind the former, ſo that their axes be perpendicular, the image will be refracted on one ſide, of the ſame length and breadth as before.

PHENOM. III. Any one colour refracted through any number of priſms, always exhibits a circular image of that colour.

If the ſun's rays were equally refrangible, its image would be circular, and, if differently refrangible, oblong; and conſequently (phenom. 1ſt and 2d) light is compoſed of heterogeneous parts differently refrangible. If the ſun's rays ſplitted or dilated by refraction, the image would be circular and increaſed by every refraction, therefore (phenom. 2d) they do not ſplit or dilate; and (phenom. 3d) the properties of homogeneous rays are innate and unalterable by refraction.

6. The

6. The conclusions thus derived by analysis are analogical, as they result from a comparison between present and past phenomena, and will be just only when these comparisons are repeatedly made, and the similitude of the phenomena perfectly established. In example the 1st. the altitude of the pole being found equal to the latitude of the observer, and a section of the earth's shadow to be a circular arc, and these phenomena being invariably the same in innumerable trials, an assurance that they result from some fixed causes and will never vary, arises and commands our assent.

Philosophy therefore, when real and not fantastical, has for its basis uniform and uncontroverted experience; and hence appears the necessity of adhering to the first rule of philosophizing.

RULE I.

No more causes of natural events ought to be admitted than are real, and sufficient to explain the phenomena.

7. Of those bodies which are neither concealed by their minuteness, nor remote distance, few can undergo an analytical discussion; and philosophy would be very limited, were it confined to those only which have actually been the objects of experiments; but it is rendered universal by observing the following rules.

RULE II.

Effects of the same kind are to be ascribed to the same cause.

The same cause occasions the descent of bodies in different places of the earth, respiration in different animals, the sensation of heat from a culinary fire and the sun, solutions of bodies in different menstruums, and of water in air, &c.

RULE III.

8. *Those qualities of matter which do not vary, and are found in all bodies that admit of experimental examination, ought to be considered as qualities of all bodies in general.*

Thus

Thus the attraction of gravity is discovered, by experiments and astronomical observations, to pervade all matter with which we are acquainted: all bodies near the earth's surface, and the moon, gravitate towards the earth; the waters of the sea gravitate towards the moon; the sun, planets, and comets, gravitate towards each other mutually; and of this principle no bodies accessible to experiments can be divested, and therefore, from this rule, it is inferred to be an universal attribute of matter.

9. As much false philosophy hath been disseminated by fanciful hypotheses and mere metaphysical considerations, unsupported by the reality of facts, the only sources of natural science, the utility of the following rule is apparent.

RULE IV.

In experimental philosophy, propositions collected from phenomena by induction are to be deemed, notwithstanding contrary hypotheses, either accurately true or very nearly so; until other phenomena occur by which they may be rendered either more accurate, or liable to exception.

10. The existence, quantity at a given distance, and law of variation at different distances, of any natural power being demonstrated by the analytic method of reasoning, the only way by which they can be known; they are assumed as established general principles, and their adequacy to the production of other phenomena, similar to those used in the analysis, is proved synthetically.

Examples illustrating the synthetic method of reasoning.

EXAMP. I. The influence of the attraction of gravity, its quantity and law of variation, being ascertained analytically, it is assumed as an established general principle, and demonstrated geometrically to be an adequate cause of the motions of pendulums, projectiles, precession of the equinoxes, irregularities of the moon's motion, rising and ebbing of the sea, and innumerable other phenomena.

EXAMP.

EXAMP. II. The weight of the air being aſcertained by experiments, it is ſuppoſed to be allowed, and affords an eaſy ſolution of the phenomena of pumps, ſyringes, barometers, &c.

EXAMP. III. The equality of the angles of incidence and reflection of light, of the conſtant ratio obtaining between the ſines of incidence and refraction, and the unequal refrangibility of different colours, are diſcovered by experiments; and, being ſuppoſed to be univerſally true, they afford eaſy explanations of the figure and magnitude of images formed by reflection and refraction, of the rainbow, &c.

EXAMP. IV. The ſpherical figure of the earth, its diurnal motion, and obliquity of the ecliptic are diſcovered by repeated obſervation, and, being aſſumed as known principles, they afford an eaſy ſolution of the doctrine of the ſphere, of the art of dialling, of day and night and their inequalities, of heat and cold, and many other phenomena.

11. Of theſe two methods of reaſoning, the latter may ſtrictly be denominated ſcience; for the exiſtence of any natural power, its magnitude and variation being preſuppoſed, it is demonſtrated geometrically to be competent to the production of various effects, and, were this power changed or annihilated, the truth of this reaſoning would remain unaltered. Whether this ſcience be chimerical, or accord with actual exiſtence, depends upon the truth of the preſuppoſed data, or analytic concluſions, which admit of various degrees of conviction from bare preſumption to certainty. The exiſtence of ſome undiſcovered and ſtill unknown quality of matter, collected only from a few experiments, amounts only to a preſumption; but the degree of probability increaſes with their number, and, by repeated and uniformly concurring trials, eſtabliſhes at length an evidence as unqueſtioned as the exiſtence of matter or the ſenſes. That the qualities of matter will remain unchanged and continue always to produce the ſame effects, depends upon the will of that divine Agent, who ordained that the various

various events in the material world ſhould be effected by the intervention of ſimple and general material cauſes, acting with invariable conſtancy and uſefulneſs, and only ſuffered to tranſgreſs their preſcribed bounds according to the dictates of infinite wiſdom, power and goodneſs. Preſuming therefore upon the identity of the ſenſes and the uniformity of natural operations, the ſtudy of philoſophy may be deemed ſcientific; for, being founded upon the baſis of uniform and uncontroverted experience, and geometric demonſtration, its certainty will continue with them, and conſequently can only be affected or ſubverted by a change or ſubverſion of the conſtitution of nature.

12. Of all magnitudes, numbers are the moſt ſimple, and their minute differences moſt eaſily diſcriminated, and therefore more ſuſceptible of clear ideas than any other magnitudes; and the relations ſubſiſting between powers or forces, and the changes or velocities produced by their exertions, are ſaid to be known, when reduced to numeral expreſſions, though the mode of producing theſe effects be unknown, and perhaps undiſcoverable. The next, in degree of ſimplicity and clearneſs of comprehenſion, are lines, ſurfaces and ſolids, being permanent and eaſily compared by juxta-poſition. The conception of other magnitudes is more remote and difficult. Forces, velocities, and times, having no permanent repreſentatives like numbers and geometric magnitudes, and diſappearing when not actually ſubſiſting, are beſt underſtood from their effects; and this ſtudy is therefore much ſimplified by finding magnitudes of eaſier conception, and more eaſily meaſurable and permanent, as numbers, lines and ſurfaces, whoſe relation is the ſame with that of times and velocities, and natural powers. Philoſophy may conſequently be reduced to an inveſtigation of the relation ſubſiſting between lines, ſurfaces, &c. which vary as, and may be denominated the repreſentatives of, natural powers and their effects; and a knowledge of ratios, or of the rules by which theſe relations are increaſed and diminiſhed, is a neceſſary lemma to this ſtudy and here premiſed.

RATIOS.

13. DEF. *WHATEVER is capable of increase or decrease, is called magnitude or quantity, as numbers, lines, velocities, forces, &c.*

14. DEF. *Ratio is the mutual relation of two magnitudes of the same kind in respect of their greatness or smallness; the first is called the antecedent and the second the consequent of the ratio.*

15. Ratio is therefore a comparison, which always implies a similitude, and is limited by the definition to the relative greatness or smallness of the quantities compared. The magnitude of the ratio of equal quantities is equal to nothing; for the existence of ratios results from the inequality of the quantities compared, though not measured by it; and it is evident, that the magnitude of a ratio may increase or decrease through every stage of assignable quantity.

EXAMPLES.

FIG. I. EXAMP. I. If the distance of *E* from *B* be equal to nothing, the diameter *QB*, and chord *QE* coincide, and their ratio is a ratio of equality, and its magnitude equal to nothing; but as the arc *BE* increases from nothing to a semicircle, the magnitude of this ratio increases from nothing through every stage of assignable magnitude, and, when *E* and *Q* coincide, is unassignably great.

EXAMP. II. The ratio of the tangent *TA* to the sine *SN* of an arc *SA*, or of the secant *CT* to the radius *CS*, is a ratio of equality, and its magnitude equal to nothing, when the arc vanishes, and *S* and *A* coincide. But as the arc *SA* increases from nothing to a quadrant, the magnitude of this ratio increases from nothing through

through every ſtage of aſſignable magnitude, and, when S and Q coincide, is unaſſignably great. If the magnitude of this ratio (when e. g. $SA = 30°$) be expounded by any finite line L, and twice this ratio by twice this line, &c. when S and A coincide, L is equal to nothing; and as S recedes from A, L increaſes and becomes infinitely great when S arrives at Q.

Ratios therefore are magnitudes, and like all other magnitudes the object of ratios, and capable of addition, ſubtraction, multiplication and diviſion. They are poſitive or negative, according to their different effects in addition. The ſum of two poſitive, or of two negative ratios, will conſtitute a poſitive, or negative ratio, greater than either of them; and, according as a poſitive ratio is greater or leſs than a negative one, their ſum will be poſitive or negative; and, if they be equal, their ſum is nothing. The magnitudes of the ratio of $L:M$ and of $M:L$, or of $3:2$ and of $2:3$, are clearly equal, but of different denominations; and any third ratio, of the ſame denomination with the ratio of $L:M$, will be equally encreaſed by the addition of the ratio of $L:M$ and ſubtraction of the ratio of $M:L$. If the ratio obtaining between L and M, of which L is the greateſt, be called poſitive, and repreſented by any line or number a, twice this ratio will be repreſented by twice a, and n times this ratio by $n \times a$. The ratio alſo of $M:L$ will be repreſented by $-a$, and $n \times$ the ratio of $M:L$ by $n \times -a$; and a line or number, repreſenting any third affirmative ratio, will be equally encreaſed by the addition of $n \times a$ and ſubtraction of $n \times -a$. As magnitudes are only meaſurable by magnitudes, ſui generis, a line by a line, a ſurface by a ſurface, and a ratio by a ratio: to aſcertain the quantity of any ratio, ſome more ſimple ratio may be uſed as a criterion. Thus logarithms are a ſeries of numbers expreſſing the relation which ſubſiſts between any given ratio, conſidered as a criterion, and all other ratios with which it is compared.

16. DEF. *In a ſeries of magnitudes of the ſame kind, either increaſing or decreaſing, the ratio of the extremes is ſaid to be compounded of the ratios of the intermediate terms.*

In the series of magnitudes A, B, C, D, &c. the ratio of the first to the last, or of $A:D$, is said to be compounded of the ratios of $A:B$, $B:C$, $C:D$; and any ratio, as that of $PQ:PR$, is said to be resolvable into, and equal to, the ratios of $PQ:Pa$, $Pa:Pb$, $Pb:Pc \ldots PL:PR$; for the ratio of $PQ:PR$ is a real magnitude, and like other magnitudes divisible into its component parts, which are the ratios of $PQ:Pa$, $Pa:Pb$, $Pb:Pc$, &c.: therefore the sum of any number of continued ratios, where the antecedent of any ratio is the consequent of the preceding, is equal to the ratio of the first and last terms.

FIG. II.

FIG. II. 17. Cor. When any ratio, as that of $PQ:PR$, is to be divided into any other ratios by an arbitrary insertion of other quantities Pa, Pb, &c.; these are not necessarily intermediate, or contained between PQ and PR, the ratio of $PQ:PR$ being equal to the ratios of $PQ:Pv$ and of $Pv:PR$. For the ratio of $PQ:Pv$ is equal to the ratios of $PQ:PR$ and of $PR:Pv$ (16), and consequently the ratio of $PQ:PR$ is equal to the ratio of $PQ:Pv$ diminished by the ratio of $PR:Pv$, or added to the ratio of $Pv:PR$ (15).

ADDITION OF RATIOS.

18. PROP. *To add the ratios of* A : B, C : D, E : F, *&c. together.*

When the two ratios of $A:B$ and $C:D$ are to be added, let A and C, B and D be respectively the sides of two rectangular parallelograms X and Z; and (Euclid. VI. 23.) the ratios of $A:B$ and $C:D$, when compounded, are equal to the ratio of $X:Z$, or of $A \times C : B \times D$, these quantities being respectively equal to X and Z. When the ratios of $A:B$, $C:D$, $E:F$ are to be added, the sum of the two first is equal to the ratio of $AC:BD$ by the process above; and, by making $A \times C$ and E, $B \times D$ and F, respectively the sides of two rectangles, it appears, by the same process, that the sum of the ratios of $AC:BD$ and of $E:F$ is equal to

FIG. III.

the

the ratio of $ACE : BDF$. Whatever be the number of ratios to be added the procefs is fimilar, and gives the following

RULE.

Multiply the antecedents together for a new antecedent; and the confequents for a new confequent.

19. EXAMP. I. The fum of the ratios of $1:2$, $3:4$, $5:6$, $7:8$ is equal to the ratio of $1\times3\times5\times7 : 2\times4\times6\times8$, or of $105:384$. This alfo appears from the following analogies and article (16); for (38)

$$\begin{array}{l} 1 : 2 :: 105 : 210 \\ 3 : 4 :: 210 : 280 \\ 5 : 6 :: 280 : 336 \\ 7 : 8 :: 336 : 384. \end{array}$$

The fum of the ratios of $1:2$, $3:4$, $5:6$, $7:8$, is equal to the fum of the ratios of $105:210$, $210:280$, $280:336$, $336:384$, or (16) to the ratio of $105:384$, as above.

20. EXAMP. II. When two bodies move with uniform motions, philofophical writers fay, that the fpaces defcribed S and s are to each other in a ratio compounded of the velocities and times; the meaning of which is, that the ratio of the fpaces is equal to the fum of the ratio of the velocities when changed, and the ratio of the times when changed, or that the ratio of $S:s$ is equal to the two ratios of $V:v$ and of $T:t$, or to the ratio of $V\times T : v\times t$, fuppofing v and t to reprefent any correfponding values of the velocity and time.

21. Cor. 1. If there be any number of quantities A, B, C, D, &c. of which $A:B::R:r$ and $B:C::S:s$ and $C:D::T:t$, A will be to D as $RST:rst$; for $A:D$ in a ratio compounded of the ratios of $A:B$, $B:C$, $C:D$ (16), or their equals $R:r$, $S:s$, $T:t$, or as $RST:rst$ (18).

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22. When the terms of any ratios are forces, times, velocities, &c. lines or numbers are ſuppoſed to be taken whoſe ratios are the ſame with them, and rectangular parallelograms whoſe ſides are theſe lines, or the products of theſe numbers actually multiplied together, are always implied in the multiplication of ſuch magnitudes. The ſum of the ratios of $L:M$ (denoting forces), of $N:P$ (denoting velocities), of $Q:R$ (denoting ſpaces), and of $S:T$ (denoting times), is equal to the ratio of $L\times N\times Q\times S:M\times P\times R\times T$, which ſignify two products of numbers whoſe factors are as $L:M$, $N:P$, $Q:R$, $S:T$.

SUBTRACTION OF RATIOS.

23. Prop. *To ſubtract the ratio of* C : D *from that of* A : B.

Let the ratio of $A:B$ be equal to the ratios of $C:D$ and $x:y$; and (18) $A:B::xC:yD$, and $x:y::\frac{A}{C}:\frac{B}{D}$ (Euc. v. 4.) $::AD:BC$ (Euc. v. 10.); from hence we have the following

RULES.

Rule I. *Divide the antecedent of the ſubtrahend by the antecedent of the ratio to be ſubtracted for a new antecedent, and the conſequent by the conſequent for a new conſequent.*

Or, II. *Invert the terms of the ratio to be ſubtracted, and proceed as in addition.*

24. Examp. I. The ratio of $6:5$ ſubtracted from the ratio of $3:2$ is equal to the ratio of $\frac{3}{6}:\frac{2}{5}$ or of $15:12$, which is thus confirmed. The ratio of $3:2$ is equal to the ratio of $6:4$, which is equal to the ratio of $6:5$ and of $5:4$ (16); and if the ratio of

6 : 5

$6:5$ be taken away, the remainder is the ratio of $5:4$ or of $15:12$ (Euc. V. 15.).

25. EXAMP. II. The ratio of $2:3$ diminished by the ratio of $4:5$ is equal to the ratio of $\frac{2}{4}:\frac{3}{5}$ or of $10:12$, which is thus confirmed. The ratio of $2:3$ is equal to the ratio of $4:6$ (Euc. V. 15.), that is, of $4:5$ and $5:6$ (16); and, if the ratio of $4:5$ be taken away, the remainder is the ratio of $5:6$ or of $10:12$ (Euc. V. 15.).

MULTIPLICATION OF RATIOS.

26. PROP. *To multiply the ratio of* A : B *by any number* m.

The ratio of $A:B$ added to itself is equal to the ratio of $A^2:B^2$, and this added to the ratio of $A:B$ is equal to the ratio of $A^3:B^3$, and this repeated m times will clearly give the ratio of $A^m:B^m$ (18), and we have this rule.

RULE.

Involve the terms of the ratio to a dimension equal to the multiplier.

27. EXAMP. I. Four times the ratio of $A:B$ is equal to the ratio of $A^4:B^4$, which is confirmed by the following process; for (38)

$$\begin{aligned} A:B &:: A^4 : A^3B \\ A:B &:: A^3B : A^2B^2 \\ A:B &:: A^2B^2 : AB^3 \\ A:B &:: AB^3 : B^4. \end{aligned}$$

Therefore four times the ratio of $A:B$ is equal to the ratios of $A^4:A^3B$, $A^3B:A^2B^2$, $A^2B^2:AB^3$, $AB^3:B^4$, or the ratio of $A^4:B^4$ (16).

28. EXAMP.

28. EXAMP. II. Five times the ratio of $2:3$ is equal to the ratio of $2^5:3^5$ or $32:243$, which also appears from the process above and (16). For (38)

$$\begin{array}{l} 2:3::32:48 \\ 2:3::48:72 \\ 2:3::72:108 \\ 2:3::108:162 \\ 2:3::162:243. \end{array}$$

Therefore the ratio of $2:3$ multiplied into five is equal to the ratios of $32:48$, $48:72$, $72:108$, $108:162$, $162:243$; or to the ratio of $32:243$ (16).

DIVISION OF RATIOS.

29. PROP. *To divide the ratio of* A : B *by any number* m.

Let $A=x^m$ and $B=y^m$; and the ratio of $A:B$ is equal to the ratio of $x^m:y^m$ or to m times the ratio of $x:y$ (18), which is therefore $\frac{1}{m}$ part of the ratio of $A:B$, and equal to the ratio of $A^{\frac{1}{m}}:B^{\frac{1}{m}}$; because $x=A^{\frac{1}{m}}$ and $y=B^{\frac{1}{m}}$. Whence we have the following

RULE.

Extract that root of the terms of the ratio which is expressed by the divisor.

30. EXAMP. The ratio of $32:162$ divided by four is equal to the ratio of $\overline{32}|^{\frac{1}{4}}:\overline{162}|^{\frac{1}{4}}$ or of $2:3$. And this is confirmed by the process used in example (28), where the ratio of $32:162$ is equal to the ratio of $\overline{2:3}\times 4$, and consequently the ratio of $2:3$ is $\frac{1}{4}$th of the ratio of $32:162$.

31. PROP. *When the difference of two magnitudes is very small compared with the magnitudes themselves, their ratio is multiplied or divided*

divided by any number m, *by increasing or diminishing their difference* m *times.*

DEM. Let the two magnitudes be A and $A \pm y$, whose difference y is very small compared with A; and m times the ratio of $A : A \pm y$ is equal to the ratio of $A^m : \overline{A \pm y}|^m$, or of $A^m : A^m \pm mA^{m-1}y + m \cdot \frac{m-1}{2} A^{m-2}y^2$, &c. or (dividing the antecedent and consequent by A^{m-1}) to the ratio of $A : A \pm my$; because the terms involving $\frac{y^2}{A}, \frac{y^3}{A^2}$, &c. are evanescent compared with the two first, and may be neglected.

By a similar process $\frac{1}{m}$th part of the ratio of $A : A \pm y$ appears to be equal to the ratio of $A : A \pm \frac{y}{m}$. Q. E. D.

32. EXAMP. Twice the ratio of 11 : 10 is equal to the ratio of $\overline{11}|^2 : \overline{10}|^2$ or of 121 : 100; and, according to this proposition, it is the ratio of 11 : 9, or of 121 : 99.

Twice the ratio of 101 : 100 is equal to the ratio of $\overline{101}|^2 : \overline{100}|^2$ or of 10201 : 10000; and according to this proposition it is equal to the ratio of 101 : 99 or of 10201 : 9999, which is nearly equal to the former.

33. EXAMP. II. A half of the ratio of 101 : 100 is equal to the ratio of $\overline{101}|^{\frac{1}{2}} : \overline{100}|^{\frac{1}{2}}$, or of 10.049 : 10 nearly; and by this proposition it is equal to the ratio of $100\frac{1}{2}$: 100, or of 201 : 200, or of 10.05 : 10.

METHODS OF COMPARING RATIOS.

34. DEF. *Proportion is an equality of ratios. When the ratios of* A : B *and* C : D *are equal, they are said to be proportional, and usually written thus* A : B :: C : D, *or* A *is to* B *as* C *to* D.

35. Cor.

35. Cor. If A, B, C, D be proportional, and $A = \frac{1}{4}B$, or $\frac{m}{n} \times B$, C will be equal to $\frac{1}{4}D$, or to $\frac{m}{n} \times D$; or if thefe magnitudes be incommenfurate, and A be greater, or lefs, than any part or parts of B, C will be greater, or lefs, than the fame part or parts of D. If A be contained between $\frac{m}{n} \times B$ and $\frac{m+1}{n} \times B$, (m and n being any numbers whatever,) C will alfo be contained between $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$. This is evident from the definitions of ratios and proportion; for if A were contained between $\frac{18}{17} \times B$ and $\frac{19}{17} \times B$, or $\frac{m}{n} \times B$ and $\frac{m+1}{n} \times B$, and C were not contained between $\frac{18}{17} \times D$ and $\frac{19}{17} \times D$, or $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$, A's magnitude compared with B's, would not be equal to C's magnitude compared with D's, or the ratios of $A : B$ and $C : D$ would not be equal, and they would not be proportional.

36. Prop. *If* A *cannot be greater than, equal to, or lefs than, any part or parts of* B, *but at the fame time* C *is greater than, equal to, or lefs than, the fame part or parts of* D, *they will be proportional, or* A : B :: C : D.

Dem. Let $A : B :: E : D$; and if $A = \frac{m}{n} \times B$, $E = \frac{m}{n} \times D$ (35), and, from the hypothefis, $C = \frac{m}{n} \times D$: therefore $E = C$ and $A : B :: C : D$. Let thefe magnitudes be incommenfurate, and if A be contained between $\frac{m}{n} \times B$ and $\frac{m+1}{n} \times B$, E will be contained between $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$ (35), and, from the fuppofition,

C will

C will be contained between $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$. The difference therefore between C and E is between $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$, and consequently is not greater than $\frac{D}{n}$; and because this is true whatever be the magnitnde of the number n, which may be unassignably great and $\frac{D}{n} = 0$, C will be equal to E, and $A : B :: C : D$. Q. E. D.

37. Cor. 1. If four magnitudes A, B, C, D be proportional, the products of the extreme, and middle, terms are equal, or $A \times D = B \times C$. For let $A = \frac{m}{n} \times B$, and therefore $C = \frac{m}{n} \times D$ (35): consequently $A \times D = \frac{m}{n} \times B \times D$, and $B \times C = B \times \frac{m}{n} \times D = A \times D$. If they be incommensurate, let A be greater than $\frac{m}{n} \times B$, and less than $\frac{m+1}{n} \times B$, and C will be contained between $\frac{m}{n} \times D$ and $\frac{m+1}{n} \times D$ (35); therefore $A \times D$ is contained between $D \times \frac{m}{n} \times B$ and $D \times \frac{m+1}{n} \times B$, and $C \times B$ is contained between $B \times \frac{m}{n} \times D$, and $B \times \frac{m+1}{n} \times D$. The difference of these products is therefore not greater than $\frac{DB}{n}$, which, because n may be taken unassignably great, $= 0$; and consequently $A \times D = B \times C$.

38. Cor. 2. If two products be equal, their factors are proportional. Let $A \times D = B \times C$; and if $A = \frac{m}{n} \times B$, then $C = \frac{A \times D}{B}$

= (by

$=$ (by fubftituting A's value) $\frac{m}{n} \times D$; or if A be lefs than $\frac{\overline{m+1}}{n} \times B$, but greater than $\frac{m}{n} \times B$; then C, being equal to $\frac{A \times D}{B}$, will be lefs than $\frac{\overline{m+1}}{n} \times D$, and greater than $\frac{m}{n} \times B$; confequently (36) $A:B::C:D$. In the fame manner it may be proved, that $A:C::B:D$, if they be fimilar; or that $A:BC::1:D$, and $B:A::D:C$, or $B:A \times D::1:C$, or $A:B::C:\frac{BC}{A}$. Any equation is therefore refolvable into a proportion by fo arranging the terms, that the rectangle of the extreme terms may be one fide of the equation, and that of the mean terms the other.

39. Cor. 3. Ratios, which are equal to any ratio, are equal to each other. Let $A:B::C:D$ and $C:D::E:F$; and if A be equal to $\frac{m}{n} \times B$, or contained between $\frac{m}{n} \times B$ and $\frac{\overline{m+1}}{n} \times B$, C will be equal to $\frac{m}{n} \times D$, or contained between $\frac{m}{n} \times D$ and $\frac{\overline{m+1}}{n} \times D$ (35), and alfo E will be equal to $\frac{m}{n} \times F$, or contained between $\frac{m}{n} \times F$ and $\frac{\overline{m+1}}{n} \times F$ (35); therefore A and E are either equal to, or contained between, the fame parts of B and F refpectively, and $A:B::E:F$ (36).

40. Cor. 4. If $A:B::C:D$ they will be proportional inverfely, or $B:A::D:C$. Let A be equal to $\frac{m}{n} \times B$ or contained between $\frac{m}{n} \times B$ and $\frac{\overline{m+1}}{n} \times B$, and C will be equal to $\frac{m}{n} \times D$, or contained between $\frac{m}{n} \times D$ and $\frac{\overline{m+1}}{n} \times D$ (35); therefore $\frac{n}{m} \times A$ is equal to or

or greater than, and $\frac{n}{m+1} \times A$ is lefs, than B, or B is contained between $\frac{n}{m+1} \times A$ and $\frac{n}{m} \times A$, or equal to $\frac{n}{m} \times A$; and for the fame reafon D is contained between $\frac{n}{m+1} \times D$ and $\frac{n}{m} \times D$, or equal to $\frac{n}{m} \times C$; therefore (36) $B : A :: D : C$.

41. Cor. 5. Magnitudes are proportional to their equimultiples or equal parts. Let $A = \frac{m}{n} \times B$, and $2A$ will be equal to $\frac{m}{n} \times 2B$, and $\frac{1}{2}A = \frac{m}{n} \times \frac{1}{2}B$; therefore (36) $A : B :: 2A : 2B :: \frac{1}{2}A : \frac{1}{2}B$; or let A be contained between $\frac{m}{n} \times B$ and $\frac{\overline{m+1}}{n} \times B$, and $2A$ will be contained between $\frac{m}{n} \times 2B$ and $\frac{\overline{m+1}}{n} \times 2B$; and $\frac{1}{2}A$ between $\frac{m}{n} \times \frac{1}{2}B$ and $\frac{\overline{m+1}}{n} \times \frac{1}{2}B$; therefore (36) $A : B :: 2A : 2B :: \frac{1}{2}A : \frac{1}{2}B$, &c.

42. Cor. 6. If $A : B :: C : D$, $A^m : B^m :: C^m : D^m$ and $A^{\frac{1}{m}} : B^{\frac{1}{m}} :: C^{\frac{1}{m}} : D^{\frac{1}{m}}$. For m times, or an m^{th} part of, the ratio of $A : B$ is equal to m times, or an m^{th} part of, that of $C : D$, becaufe thefe ratios are equal to each other; therefore (34) $A^m : B^m :: C^m : D^m$; and $A^{\frac{1}{m}} : B^{\frac{1}{m}} :: C^{\frac{1}{m}} : D^{\frac{1}{m}}$.

43. Cor. 7. If $A : B :: C : D$; then $A+B : B :: C+D : D$. Let A be equal to $\frac{m}{n} \times B$, or contained between $\frac{m}{n} \times B$ and $\frac{\overline{m+1}}{n} \times B$, and (35) C will be equal to $\frac{m}{n} \times D$, or contained between $\frac{m}{n} \times D$

and

and $\overline{\frac{m+1}{n}} \times D$; therefore adding B to both, $A+B$ will be equal to $\overline{\frac{m+n}{n}} \times B$, or greater than $\overline{\frac{m+n}{n}} \times B$ and lefs than $\overline{\frac{m+1+n}{n}} \times B$; and $C+D$ will be equal to $\frac{m+n}{n} \times D$, or greater than this, and lefs than $\overline{\frac{m+1+n}{n}} \times D$; and confequently (36) $A+B : B :: C+D : D$. In the fame manner it may be proved, that $A-B : B :: C-D : D$; and $A : B :: A \pm C : B \pm D$.

44. DEF. *That ratio is faid to be greater or lefs than another, whofe antecedent has a greater proportion to its confequent than the antecedent of the other to its confequent.*

Two ratios, whofe antecedents, or confequents, are the fame, are eafily compared: for it is evident that the ratio of $6:3$ is greater, and the ratio of $4:3$ lefs, than the ratio of $5:3$; and the negative ratio of $3:6$ is greater, and of $3:4$ lefs, than the ratio of $3:5$.

45. *The firft method of comparing unequal ratios is to reduce them to other ratios equal to them with a common antecedent, or confequent.*

EXAMP. To compare the ratios of $A:B$ and of $C:D$. Find a ratio equal to either of them, whofe antecedent or confequent is the antecedent or confequent of the other; thus, $A:B :: C:\frac{BC}{A}$ (38), and confequently the ratio of $A:B$ being equal to that of $C:\frac{B\times C}{A}$, is greater or lefs than the ratio of $C:D$, according as $\frac{BC}{A}$ is lefs or greater than D.

Let

Let the ratios of $3:5$ and of $6:9$ be compared; and $3:5::6:\frac{30}{3}(10)$, therefore the ratio of $3:5$, being the same with that of $6:10$, is a greater negative ratio than that of $6:9$.

46. *The second method of comparing ratios, is to divide the antecedents by their respective consequents, and that ratio will be the greatest, the quotient of whose terms is the greatest.*

EXAMP. The ratio of $A:B$, being equal to that of $C:\frac{BC}{A}$ (38), is greater than, equal to, or less than the ratio of $C:D$, according as $\frac{B\times C}{A}$ is less than, equal to, or greater than D, or $\frac{C}{A}$ less than, equal to, or greater than $\frac{D}{B}$, or $\frac{A}{C}$ greater than, equal to, or less than $\frac{B}{D}$. This method is expeditious and true, but let it be remembered that $\frac{A}{B}$ is a number as every quotient is, and does not measure the ratio of $A:B$, nor is the relative magnitude or ratio of the ratios of $A:B$ and $C:D$ equal to the ratio of $\frac{A}{B}:\frac{C}{D}$.

47. *The third method of comparing ratios. If* A *be contained between* $\frac{m}{n}\times B$ *and* $\frac{m+1}{n}\times B$, *and* C *be contained between* $\frac{m}{n}\times D$ *and* $\frac{m+1}{n}\times D$, *the ratios of* A : B *and* C : D *are equal: if therefore* A *be greater than* $\frac{m}{n}\times B$ *and* C *be not greater than* $\frac{m}{n}\times D$, *or if* nA *be greater than* mB, *but* nC *not greater than* mD, *the ratio of* A : B *is greater than that of* C : D.

EXAMP.

EXAMP. Let the ratios to be compared be $7:5$ and $4:3$. Multiply the first and third by 3, and the second and fourth by 4, and the resulting numbers are 21, 20, 12, 12; and the first multiple being greater than the second, but the third not greater than the fourth, 7 has to 5 a greater ratio than $4:3$.

48. DEF. *Any magnitude* A *is said to be, or vary, directly as another* B, *when* A *is to any new value of* A *as* B *to a corresponding new value of* B, *that is, when* A : a :: B : b, a *and* b *being corresponding new values of* A *and* B.

FIG. IV. EXAMP. I. If the line mn move parallel to itself, and its extremities m and n be always in the lines LP, LQ; Lm varies as mn. For $LA : La$ (a new value of Lm) :: $mn : AB$ (a corresponding new value of mn).

EXAMP. II. The area Lmn varies directly as the square of Lm or mn; for $Lmn : LAB$ (a new value of Lmn) :: $mn^2 : AB^2$ (the square of a new value of mn).

49. DEF. *Any magnitude* A *is said to be, or vary, inversely as another* B, *when* A *is to a new value of* A, *as a cotemporary new value of* B *is to* B, *or when* A : a :: b : B.

EXAMP. I. In the lever the power is inversely as the perpendicular let fall from the center of motion upon its direction; for let F and f be two powers in equilibrio, P and p the perpendiculars let fall from the center of motion upon their directions; and $F : f$ (a new value of the power) :: p (the perpendicular let fall upon f's direction) : P.

EXAMP.

EXAMP. II. If a given ſpace be deſcribed in different times T and t, with unequal uniform velocities V and v, the velocities and times are inverſely as each other; for $V:v$ (a new value of the velocity) :: t (a new value of the time correſponding to v) : T.

EXAMP. III. If two variable right lines AB, AC form the equal rectangles AD, EF, they are inverſely as each other; for (Euc. VI. 14.) $AB : CF$ (a new value of AB) :: CE (a new value of AC) : AC. FIG. V.

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50. When any quantity A varies directly, or inverſely as any other B, an analogy is always implied, and A and B are not neceſſarily homogeneous.

51. Cor. 1. If A be directly as B, and B directly as C, A will be directly as C; for (48) $A:a::B:b$ and $B:b::C:c$; therefore (39) $A:a::C:c$, and A is directly as C (48).

52. Cor. 2. If A be as B and B as C; A will be as $B \pm C$, or as $\sqrt{BC}$; for $A:a::B:b::C:c$, and (43) $B:b::B \pm C:b \pm c$; therefore (39) $A:a::B \pm C:b \pm c$, and A is as $B \pm C$ (48): and becauſe the ratios of $B:b$ and $C:c$ are equal, $B^2:b^2::BC:bc$ and $B:b::\sqrt{BC}:\sqrt{bc}$. B therefore, and conſequently A (51), is as $\sqrt{BC}$.

53. Cor. 3. A varies as $m \times A$, or as $\frac{A}{m}$; for $A:a::m \times A:m \times a$, or as $\frac{A}{m}:\frac{a}{m}$ (41); therefore (48) A is as $m \times A$, or as $\frac{A}{m}$.

54. Cor. 4. If A be as B, A^2 will be as B^2, and A^m as B^m, and $A^{\frac{1}{m}}$ as $B^{\frac{1}{m}}$; for since $A : a :: B : b$ (48) $A^m : a^m :: B^m : b^m$, or $A^{\frac{1}{m}} : a^{\frac{1}{m}} :: B^{\frac{1}{m}} : b^{\frac{1}{m}}$ (42); therefore A^m is as B^m and $A^{\frac{1}{m}}$ as $B^{\frac{1}{m}}$ (48).

55. Cor. 5. If S be as $V \times T$, $\frac{S}{T}$ will be as V, $\frac{S}{V}$ as T, and $\frac{S}{V \times T}$ a given quantity: for (48) $S : s :: VT : vt$ and $\frac{S}{V} : \frac{s}{v} :: T : t$ and $\frac{S}{T} : \frac{s}{t} :: V : v$, and $\frac{V}{VT} : \frac{s}{vt} :: 1 : 1$ (41); therefore $\frac{S}{V}$ is as T, $\frac{S}{T}$ as V, and $\frac{S}{VT}$ does not vary (48).

56. Cor. 6. If A be as B, and C as D, $A \times C$ will be as $B \times D$; for $A : a :: B : b$ and $C : c :: D : d$; therefore the ratios of $A : a$ and $C : c$ are equal to the ratios of $B : b$ and $D : d$, and the sums of these ratios are equal, or the ratio of $AC : ac$ is equal to that of $BD : bd$ (18), and $AC : ac :: BD : bd$ (34), and AC is as BD (48).

57. Cor. 7. If A be inversely as B, then B will be inversely as A; for $A : a :: B : b$ (49) and $b : B :: a : A$; and if B be to b as $4 : 5$, $A : a :: 5 : 4$.

58. Cor. 8. If A be directly as B, and B inversely as C, A will be inversely as C; for $A : a :: B : b$ (48) and $B : b :: c : C$; therefore $A : a :: c : C$ (39), and A is inversely as C (49).

59. Cor. 9. If A be inversely as B, and B inversely as C; A will be directly as C: for $A : a :: b : B$ and $b : B :: C : c$, therefore $A : a :: C : c$ and A is as C.

60. Cor. 10. If $A \times B$ be always the same, A is inversely as B; for $A \times B : a \times b :: 1 : 1$ and $A : a :: b : B$; or otherwise, since $A \times B = a \times b$, $A : a :: b : B$ (38).

61. Cor. 11. If A be inversely as B, it will be as $\frac{1}{B}$ and v, v; for $A : a :: b : B$ (49) $:: \frac{1}{B} : \frac{1}{b}$ (41), therefore A is as $\frac{1}{B}$ (48). If A be as $\frac{1}{B}$, $A : a :: \frac{1}{B} : \frac{1}{b}$ (48) $:: b : B$ (41); and A is inversely as B (49).

62. Cor. 12. If A be as $\frac{L}{M}$, A will be inversely as $\frac{M}{L}$; for $A : a :: \frac{L}{M} : \frac{l}{m}$ (48) $:: \frac{m}{l} : \frac{M}{L}$ (41); and A is inversely as $\frac{M}{L}$ (49).

63. Def. *If any quantity* A *be dependent upon several others* P, Q, R, S, T, *all independent of each other, so that none of the quantities* P, Q, R *can vary, but* A *varies in the same ratio, nor either* S *or* T *can vary, but* A *varies in a contrary ratio;* A *is said to be as* P *and* Q *and* R *directly, and* S *and* T *inversely.*

Thus, the fraction $\frac{LMN}{XZ}$ varies as L and M and N directly and X and Z inversely; because none of the factors in the numerator can vary, but the value of the fraction varies in the same ratio; and none of the factors in the denominator can vary, but the value of the fraction varies in a contrary ratio. If any one of the factors in the numerator be multiplied by 2, 3, &c. the value of the fraction is 2, 3, &c. times its former value; and if any of the factors in the denominator be multiplied by 2, 3, &c. the value of the fraction becomes $\frac{1}{2}$, $\frac{1}{3}$, &c. of its former value.

 64. Cor.

64. Cor. 1. If any magnitude A vary as P and Q and R directly, and as X and Z inverſely, theſe magnitudes being independent of each other, and any one may vary without affecting the reſt, A will vary as $\frac{P \times Q \times R}{X \times Z}$. For let any new cotemporary values of P, Q, R, X, Z, be reſpectively equal to p, q, r, x, z, and let A, when changed in the ratios of $P : p$, $Q : q$, $R : r$, $\frac{1}{X} : \frac{1}{x}$, and $\frac{1}{Z} : \frac{1}{z}$, become a; that is,

Let $P : p :: A : B =$ A's value after the change of P.

$Q : q :: B : C =$ A's value after the change of P, Q,

$R : r :: C : D =$ A's value after the change of P, Q, R.

$\frac{1}{X} : \frac{1}{x} :: D : E =$ A's value after the change of P, Q, R, X.

$\frac{1}{Z} : \frac{1}{z} :: E : a =$ A's value after the change of P, Q, R, X, Z.

The ſum of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, $E : a$, or the ratio of $A : a$ (16) is equal to the ſum of the ratios of $P : p$, $Q : q$, $R : r$, $\frac{1}{X} : \frac{1}{x}$, $\frac{1}{Z} : \frac{1}{z}$, or to the ratios of $\frac{PQR}{XZ} : \frac{pqr}{xz}$ (18); and A varies as $\frac{PQR}{XZ}$ (48).

65. Cor. 2. If any of the quantities P, Q, R, X, Z be given or remain invariable, they are to be rejected, and A will vary as the reſt. Let P be given or $P = p$, and, becauſe $P : p :: A : B$, $A = B$ (38), and the ratios of $B : C$, $C : D$, $D : E$, $E : a$, or the ratio of $B = A : a$, is equal to the ratio of $\frac{QR}{XZ} : \frac{qr}{xz}$ (18), and A is as $\frac{QR}{XZ}$ (48). Let Q be alſo given or $Q = q$, and A $(= B = C)$: $a :: \frac{R}{XZ} : \frac{r}{xz}$, or A is as $\frac{R}{XZ}$ (48). Hence may be underſtood what is meant by philoſophical writers, when they ſay the velocity (V) varies

varies as the ſpace (S) directly, and the time (T) inverſely, or as $\frac{S}{T}$; and, if T be given, V is as S, and, if S be given, V varies as $\frac{1}{T}$.

66. EXAMP. I. The number of feet (S) deſcribed in any time (T), with an uniform velocity (V), encreaſes or decreaſes directly with V and T, both of which are independent of each other, and therefore varies as $V \times T$. If V be doubled or encreaſed in the ratio of $2:1$, and T be encreaſed in the ratio of $3:1$, S will be encreaſed in the ratios of $2:1$ and $3:1$, or of $6:1$.

67. EXAMP. II. The quantity of matter (Q) in different bodies varies as the magnitude (M) multiplied into the denſity (D). For if the magnitude vary in the ratio of $M:m$, and the denſity or cloſeneſs of the conſtituent parts, in the ratio of $D:d$, the quantity of matter will be changed according to both theſe ratios, or $Q:q::M:m$ and $D:d::M \times D:m \times d$, or Q varies as $M \times D$ (48).

68. EXAMP. III. The velocity (V) of a body moving uniformly in the peripheries of different circles, varies as the radius (R) directly, and periodic time (P) inverſely. For, if P remain conſtant whilſt the ſpace deſcribed or periphery varies, the velocity will encreaſe or decreaſe directly as the ſpace, or as R, becauſe the peripheries of circles are as their diameters; and if the periodic time be encreaſed in the ratio of 2 or $3:1$, or univerſally in the ratio of $P:p$, the velocity will be changed in the ratio of $1:2$ or 3, or of $p:P$. If therefore the radius be changed in the ratio of $R:r$, and periodic time in the ratio of $P:p$, and V become v, $V:v$ as $R:r$, and $\frac{1}{P}:\frac{1}{p}$, or as $\frac{R}{P}:\frac{r}{p}$, and the velocity is as $\frac{R}{P}$ (48).

69. Ex-

69. EXAMP. IV. If V be the velocity communicated, by the action of a force F, to a quantity of matter reprefented by Q, and thefe quantities be fuppofed to vary, V will be as $\frac{F}{Q}$; for if the force become $2F$, $3F$, &c. the velocity will become $2V$, $3V$, &c. and if a quantity of matter become $2Q$, $3Q$, &c. the velocity communicated will be $\frac{V}{2}$, $\frac{v}{3}$, &c. and V therefore is as $\frac{F}{Q}$ (64).

A SYSTEM

A SYSTEM OF NATURAL PHILOSOPHY.

MECHANICS.

MECHANICAL philoſophy acquired the name from its utility in the conſtruction of machines; but it is now, in a more general ſenſe, underſtood to comprehend two branches of ſcience, cultivated at different periods of time, denominated Statics, or the ſcience of the equilibrium and relation of powers, and Dynamics, or the ſcience of actual motion. The firſt profeſſes to deſcribe the conſtruction, properties, and mechanical advantages of machines, and their various combinations, calculated to ſuſtain the preſſure of heavy bodies and facilitate their motion; and to inveſtigate the equilibrium of powers acting upon them, or their relative magnitudes, when by oppoſite exertions they deſtroy each other's effect or remain quieſcent. This branch comprehends that part called the mechanical powers, and is ſometimes called practical mechanics. The object of dynamics is the nature, geneſis, and change of actual motion, or an inveſtigation of the direction,

rection, quantity, and law of variation, of a force or power capable of generating any motion or change of motion; and vice versâ. This comprehends the laws by which all motions are regulated, the motions refulting from collifions, the theory of ofcillations, projectiles, and centripetal forces, and is fometimes called rational mechanics. The principles of ftatics were calculated and eftablifhed by Archimedes, and have, fince that period, been almoft exhaufted by the labours of fucceeding writers. Galileo demonftrated the laws of defcent of heavy bodies, and from him originated the fcience of dynamics, which has fince been profecuted to an amazing extent in Euler's Mechanica, and Newton's Principia. As the objects of mechanical philofophy are the equilibrium and motions of bodies; a defcription of the different qualities, or mechanical affections of matter, producing preffure, motion, and other phenomena, ought to be premifed. Mechanics therefore might with great propriety be divided into two parts; of which the firft would contain the properties of matter, or the exiftence, intenfity at a given diftance, and laws of variation at different diftances, of thofe general principles which are the apparent origin of motion, and the continuation of motion in the material world, inveftigated by analyfis: and the fecond would be an application of thefe principles, containing fynthetic demonftrations of their effects upon machines, commonly called the mechanical powers, when in equilibrio, and their relation to actual motions, rectilineal and curvilineal, uniform and variable. This is the moft obvious divifion of the fubject, and every deviation from it muft be attributed to the defign of this felection, which is folely the utility of the academic ftudent.

CHAP.

MATTER, AND ITS PROPERTIES.

CHAP. I.

OF MATTER.

70. MATTER is the ſubſtance employed in the formation of that part of the creation, whoſe exiſtence is evidenced by the teſtimony of the ſenſes; and the moſt characteriſtic and prominent marks of it ſeem to be extenſion and ſolidity. Indeed ſolidity is the moſt diſcriminating mark of matter, as it diſtinguiſhes it from every thing elſe; but taſtes, ſmells, ſounds equally indicate the exiſtence, though whatever is ſolid and extended is the common and moſt general deſcription, of matter. Many other qualities invariably adhere to all matter with which we are acquainted, whether hard, ſoft, or fluid; and from our ignorance of its internal conſtitution, or that latent principle, by whoſe influence its qualities are connected, and from which they neceſſarily derive their origin, theſe, and many others inacceſſible to the ſenſes, may be ingredients in its eſſence. All our knowledge of matter, as far as it is related to the preſent ſubject, is either geometrical, or philoſophical: the firſt conſiders matter as being of ſome magnitude, or circumſcribing ſpace and having ſome figure, then called body, and is uſually denominated ſtereometry, or the menſuration of magnitudes of three dimenſions, length, breadth, and thickneſs; and the ſecond comprehends all the properties of matter addreſſed to the ſenſes, which may be ſtiled phyſical or philoſophical, becauſe all the phenomena of nature are conceived to reſult imme-

diately from them; as extension, ſolidity, inertia, and thoſe apparently more active qualities, gravity, magnetiſm, electricity, coheſion, elaſticity. Theſe laſt are called mechanical affections of matter, or mechanical cauſes, becauſe they are conceived to reſide in matter, and all mechanical phenomena, or changes of motion obſervable in the material world, appear to be directed by their neceſſary and uniform agency. The philoſophical properties of matter are diſtinguiſhable into two kinds, general, and ſpecific: the firſt are ſuch as univerſally adhere to every ſpecies of matter, and of which no art hath been able to diveſt them, as *extenſion, ſolidity, mobility, quieſcibility, inertia, figure, attractions and repulſions*, and probably many more which are too ſubtle for the obſervation of ſenſe, or have yet eluded the inquiſition of the philoſopher; and the ſecond are ſuch as are attached only to particular ſpecies of matter, as opacity, tranſparency, hardneſs, fluidity, colour, magnetiſm, elaſticity, &c. Another diſcrimination of theſe qualities is, that ſome are incapable of intenſion or remiſſion, as extenſion, ſolidity, inertia, mobility, figure, which are inſeparable from a body and its component parts; and others are relative and capable of encreaſe and decreaſe, as attractions and repulſions, whoſe intenſity depends upon the magnitude and diſtance of the attracting or repelling bodies, and as that magnitude and diſtance may vary without limit, the influence of theſe powers may, by remoteneſs, decreaſe without limit, and become evaneſcent. A knowledge of the exiſtence, and diſcrimination of theſe qualities, is entirely derived from experiment and obſervation, and conſtitutes the ſum of all that is known concerning matter. The internal conſtitution of it is unknown, and its effects cannot be inveſtigated without the aſſiſtance of experience; and whether any more, beſides thoſe eight general properties enumerated, belong to it, and what; or whether any connection ſubſiſt between thoſe already diſcovered, and to what particular conſtitution ſpecific qualities are owing, muſt be derived from the ſame ſource, experiment, and if ever diſcovered, will probably reſult from the labour of the chymiſt.

CHAP.

CHAP. II.

OF EXTENSION.

71.* THERE are three kinds of extenſion, lineal, as the line *AB*; ſuperficial, having length and breadth, as *ABCD*; and ſolid, having length, breadth and thickneſs, as *AH*. Ideas of theſe three kinds of extenſion are undefinable, and only to be acquired by the ſenſes of ſeeing and feeling. The perceptions introduced to the mind by ſeeing or feeling two diſtant parts of the line *AB*, a ſurface *ABCD*, and material body *AH*, convey reſpectively the meaning of lineal, ſuperficial and ſolid extenſion; which are diſſimilar magnitudes, and incapable of compariſon with each other. A line, having no breadth, cannot, however repeated, conſtitute a ſurface; and a ſurface, having no thickneſs, cannot conſtitute a ſolid. The parts of a line, though divided without limit, are ſtill lineal extenſion, and the parts of a ſurface, or ſolid, though divided into a number of parts unaſſignably great, are ſtill ſuperficial and ſolid extenſion. This quality is ſo far eſſential to matter that it cannot be diveſted of it, or conceived to exiſt without it. FIG. VIII.

72. DEF. *A magnitude is ſaid to be finite when an equal to it can be aſſigned, or when its encreaſe and decreaſe are limited within aſſignable bounds.*

73. DEF. *A magnitude is ſaid to be infinitely great or ſmall, when no finite ratio obtains between it and a finite magnitude, or when its encreaſe or decreaſe is not limited by any aſſignable boundary.*

74. PROP.

* Keil's Phyſics. Muſchenbroek.

FIG. VI. 74. PROP. *A finite right line* EX *is divisible in infinitum.*

DEM. A mathematical point A may be taken between X and E, and another B between A and E, and this process may be continued without limit, because the intermediate points A, B, &c. having no length, can never coincide with X or E, or with each other. Q. E. D.

FIG. VII. Otherwise: Through E and X draw EP, XC parallel to each other, and, because they never meet, a number of points a, b, c, d, &c. greater than any assignable number may be taken in XC; and if right lines be drawn from them to any point P on the other side of EX, they will divide it into a number of parts equal to the number of points a, b, c, &c.: for if they did not, two lines must pass through the same part, that is, either intersect each other in EX, or coincide till they arrive at it, and then diverge, both of which are impossible. There is therefore no assignable limit to the divisibility of EX. Q. E. D.

FIG. VIII. 75. Cor. 1. The finite surface $ABCD$ is infinitely divisible; for it is equally divided with AB, by drawing mathematical right lines pf, qg, rh, &c. from the points p, q, r, &c. parallel to AD, which having no breadth cannot coincide. If $ABCD$ be a boundary of the solid AH, and mathematical planes be drawn through pf, qg, rh, &c. parallel to AF, they cannot coincide, having no thickness, and consequently will divide the solid into the same number of parts with the surface $ABCD$, or line AB.

76. Cor. 2. Every finite line, surface and solid, is therefore composed of an unlimited number of parts; and each of these parts
FIG. IX. is composed of an unlimited number, &c. For let LM be infinitely greater than Ln, and take $LM : Ln :: Ln : Lo :: Lo : Lp$, &c.; and drawing any finite line MA, making an angle with LM, and

and nB, oC, pD, &c. parallel to MA and terminated by LA; MA is infinitely greater than nB; nB than oC; oC than pD, &c. But MA is infinitely divifible, and right lines, drawn from L to the points of divifion, will cut nB, oC, &c. into the fame number of parts with it.

77. Cor. 3. Matter is therefore infinitely divifible, becaufe extended. And this propofition and corollaries are applicable to magnitudes of every kind, as velocities, forces, times, &c.

SCHOLIUM.

78. The terms infinitely great and fmall are relative, and imply a comparifon with an affignable magnitude, and compared with it, all infinitely great or fmall magnitudes are to each other in a ratio of equality; but, compared with each other, they admit of the fame inequality of ratios with finite magnitudes. The line AM is infinitely divifible, and if the points of divifion be equidiftant, any one part xy, multiplied into their number, is equal to AM; and $xy : AM ::$ unity: a number unaffignably great, or xy is an infinitefimal of the firft order. The infinitefimal xy is infinitely greater than wt, and wt than rs; for $xy : wt :: wt : xs$ ∵ $LM : Ln$. But infinitefimals of the fame order may be to each other in any affignable ratio; for, let LM be to LF as 3, 4, 5, or or $m : 1$, and FG, being drawn parallel to MA, will be equally divided with it, and the infinitefimal $vz : xy :: LM : LF :: 1 : 3, 4, 5$, or any number m. This is called the mathematical divifibility of matter, and is equally applicable to fpace and all other magnitudes, which may be reprefented by extenfion; but that an actual divifibility, or actual feparation of the parts of matter, is limited by certain inviolate bounds, is inferred from the identity of natural fubftances.

PHENOM. I. Salt diffolved in a menftruum, becomes the fame falt when the menftruum is fuppofed to dry gradually. Salt converted

verted, by a chymical procefs, to an acid fpirit, is, by reverfing the procefs, regenerated into the fame falt. Metals liquidated by fire, are reftored to their priftine ftate by the attraction of cohefion. But were the decompounded particles of the falt, or metal, attenuated by the action of the menftruum or fire, and again to coalefce by their cohefive force, fubftances of a different texture and appearance, and of different fpecific gravities, would refult. The moft powerful natural agent, with which we are acquainted, is fire; which, whether artificial or collected in the focus of a burning glafs, only attenuates bodies to a limited degree by converting them into a thick fmoke, glafs, &c. Every fpecies of matter fluctuates and decays by the feparation of its component parts, and is renovated by their acceffion; but were the nutritious particles, adminiftering to the encreafe of natural fubftances, capable of divifibility or diminution by attrition, &c. new fpecies of fubftances would refult with new properties and characters. Water and earth, compofed of old particles, and fragments of particles produced by attrition, would have a different fpecific gravity, from water and earth compofed of entire and unbroken particles; and the nature and textures of thefe and all other fubftances would by repeated attritions, be perpetually changing, which is contrary to experience. Bodies therefore break, not in the midft of folid particles, but where thofe particles cohere in a few points; and the divifibility of matter is only a feparation of its conftituent parts, effected by a diffolution of that cohefive force which unites them, and is limited by ultimate elementary particles which are a perfect repletion of fpace, without pores and indivifible.

CHAP.

CHAP. III.

OF SOLIDITY.

79. THE fecond philofophical quality of matter is that by which it occupies fpace to the exclufion of all other matter, and is called its folidity; but the meaning of it cannot be conveyed by words, being undefinable and only to be acquired by the fenfe of touching. The perception introduced to the mind by the infuperable refiftance felt in a body is an idea of folidity. This refiftance, and total exclufion of matter from fpace already occupied, is infeparable from all matter with which we are acquainted, whether hard, foft, or fluid; and the little refiftance experienced in fome bodies, is not owing to a want of folidity, but to their fluidity and foftnefs, by which qualities their component parts are eafily difplaced. That hard and foft bodies are folid, and cannot occupy the fame part of fpace, appears from uncontroverted experience; for the oppofite fides of the fubftance compreffing them are never found to meet, but by removing the intervening parts. And the fame is difcovered to obtain in all fluids within the reach of obfervation; for no portion of water, air, mercury, &c. can occupy the fame place, becaufe they afford an infuperable refiftance preventing the coincidence of the oppofite fides of the fubftance compreffing them; and their dimenfions are never found to be diminifhed by preffure without an adequate caufe, viz. compreffibility, and tranfmiffion through the vacuities of the veffel containing them. Numberlefs facts demonftrate electric and magnetic effluvia, whofe parts are immeafurably minute, to be capable of impulfe and refiftance like other fluids; and it is inferred from analogy, that thefe fluids, circumfcribed and compreffed by plane furfaces, would invariably oppofe their junction, were the vacuities in thofe furfaces lefs than the particles of

the fluids. Solidity therefore is an univerſal attribute of matter, but its cauſe is unknown and probably undiſcoverable by human faculties. The approach of one body to another is apparently limited by the actual contact of their neareſt parts, which abſolutely fill the ſpaces occupied by them, and conſequently a nearer acceſs and greater ſurfaces of contact, only reſult from their diſlocation; but this is not allowed without controverſion. Compreſſibility, contraction by cold, elaſticity, &c. prove that the minute parts of ſome bodies are not in mathematical contact, and their reſiſtance is an indication of repulſion; and the nearer acceſs therefore of two bodies is imagined to be impeded by a ſtrong repulſive power, which being overcome, a mutual penetration of parts enſues, and the bodies occupy the ſame part of ſpace. But this doctrine is ſtill only hypothetical; and though the minute parts of ſome bodies do exert an influence at a diſtance, and a repulſive power be confeſſed to obtain between them, it cannot be concluded, generally, to be the only cauſe preventing their nearer approach, nor admitted as a general principle in nature, contrary to the common apprehenſion of mankind, till eſtabliſhed by ſatiſfactory and uncontroverted experiments. The tranſmiſſion of one body through another, and apparent penetration of parts, ſeem, and may be conceived, to reſult from empty unoccupied ſpace between thoſe parts; and the quantity of theſe vacuities is collected from the following phenomena.

80. PHENOM. Many vacuities or pores are actually viſible, through a microſcope, in every ſpecies of animals, vegetables and foſſils. The bottom of the ſea is viſible at a greater depth than ſixty feet. A man's finger, placed before the aperture of a dark chamber, is tranſparent by the paſſage of light through its pores: and light is tranſmitted in almoſt every direction through glaſs and water, and when condenſed three thouſand times in the focus of a burning glaſs, it ſeems to be admitted into water and glaſs without obſtruction. Electric effluvia paſs through gold with a velocity unaſſignably great, and the magnetic power is tranſmitted through every ſpecies of matter, except iron, without diminution.

The

The volatile ſpirit of ſulphur tinges, with a brown colour, ſilver ſurrounded with repeated coverings of cloth or paper; the ſcents of muſk, civet, &c. paſs through wood. Air and water imbibe each other; oils penetrate the vacuities of ſulphur and ſome ſtones; mercury penetrates the pores of gold, braſs, and is tranſmitted through human ſkin, leather, &c.; water is tranſmitted through the membranes of animals, and the fine tubes of vegetables, and may, by compreſſion, be forced through gold, ſilver, &c.* From numberleſs chymical experiments it appears, that all animals, vegetables, and foſſils, yield water plentifully by the force of heat; and all bodies, whether hard or fluid, admit the particles of fire into their pores, through which it paſſes and is diſſipated.

81. DEF. *The magnitude of a body is the magnitude of ſolid extententenſion, that is, length, breadth and thickneſs, or the number of cubical inches contained in it.*

82. DEF. *The quantity of matter in a body is the number of equal particles contained in it. If the matter compoſing different bodies be reduced to equal particles without pores, the quantity of matter in each will be equal to one particle multiplied reſpectively into their number.*

83. DEF. *Denſity of a body is the contiguity, or cloſe adheſion, of its particles; but this term uſually implies the ratio obtaining between the number of equal particles, or quantities of matter, in bodies of the ſame magnitude.*

84. DEF. *Homogeneous bodies are thoſe which have the ſame denſity in every part; heterogeneous are thoſe which have not.*

85. DEF.

* In the celebrated Florentine experiment, a quantity of water was encloſed in a hollow ſphere of ſilver, and then forcibly compreſſed by ſcrews, till the fluid was ſeen to ooze through the pores of the metal and cover its ſurface like a dew.

85. DEF. *The porosity of a body is the remoteness of its elementary particles, or the ratio of the quantities of vacuity in different bodies of equal bulk.*

86. PROP. *If* Q, D, B *represent respectively the quantity of matter, density, and magnitude, of a body, and be supposed to vary;* Q *will vary as* D × B.

DEM. If D, or B, be encreased or diminished in the ratio of 2, 3, &c. to 1, Q will evidently be encreased or diminished in the same ratio, and D and B are unconnected; therefore (64) Q is as $D \times B$. Q. E. D.

87. Cor. 1. Since Q is as $D \times B$, if q, d, b be cotemporary new values of Q, D, B; $Q : q :: DB : db$, and $\frac{Q}{D} : \frac{q}{d} :: B : b$, & $\frac{Q}{B} : \frac{q}{b} :: D : d$ (41); or if D be given, Q is as B; and if B be given, Q is as D; and if Q be given, D is $\frac{1}{B}$ and B as $\frac{1}{D}$. This cor. follows also from (65).

88. Cor. 2. If numbers, or lines, whose ratio is the same with that of D and B, be substituted for them, Q is properly expressed by the product of these numbers, or a rectangle whose sides are these lines.

89. Cor. 3. If the density of a body be encreased in the ratio of 2, 3, &c. to 1, the porosity (P) will be diminished in the ratio of 1 to 2, 3, &c.; and consequently P is as $\frac{1}{D}$, or (87) as $\frac{B}{Q}$. If B be given, P is as $\frac{1}{Q}$; and if Q be given, P is as B; and P being given, B is as Q.

EXAMP.

EXAMP. I. The denſity of gold is to that of water as $19\frac{1}{2}:1$, and conſequently the relative quantity of pores in gold and water is as $1:19\frac{1}{2}$. The relative denſities of gold and cork are as $81\frac{1}{2}:1$, and their quantities of pores therefore as $1:81\frac{1}{2}$. If one half of the magnitude of gold were vacuous, the relative quantities of pore and ſolid parts in the water and cork, would be reſpectively as $39:1$ and $163:1$.

EXAMP. II. If a body whoſe magnitude is A were conſtructed of particles cohering in ſuch a manner as to have $\frac{1}{2}$ of its magnitude vacuous, and theſe particles were ſimilarly conſtructed of other leſs particles having $\frac{1}{2}$ of their magnitude vacuous, and the third order of particles were elementary, and a perfect repletion of ſpace, the quantity of vacuity would be equal to $A\times\frac{1}{2}+A\times\frac{1}{2}\times\frac{1}{2}+A\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=A\times\overline{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}=A\times\frac{7}{8}$. And the magnitude of vacuity is to the magnitude of matter as $7:1$.

SCHOLIUM.

90. What the figure and magnitude of the elementary particles of matter are, cannot be known from the ſenſes; which, with every microſcopical aſſiſtance, are unable to diſcern them. An elementary particle of matter hath probably never yet been ſeen. A number of elementary particles, uniting by the power of coheſion, form greater particles; and theſe, uniting again by the ſame power, form greater ſtill; and, this proceſs being made repeatedly, a corpuſcle is at length formed of a ſenſible bulk. All bodies ſeem to be compoſed of theſe derivative corpuſcles, which, formed of more or fewer repeated unions, compoſe bodies more or leſs denſe. Theſe derivative corpuſcles ſeem ſometimes to be ſimilar: if a beam of light be ſeparated by a priſm into ſmall coloured rays, and any ſlender ray of the ſame colour be minutely examined, its component parts ſeem to be ſimilar, becauſe they affect the ſight exactly in the ſame manner. Pure mercury ſqueezed through the pores of leather, or raiſed into fume, and received upon clean glaſs,

exhibit globules ſimilar and undiſtinguiſhable. This is alſo obſervable of the vapours of pure water. And, in other ſpecies of matter, the derivative particles are combined from others exactly ſimilar to them; a red globe of blood is obſerved, through a microſcope, to be compoſed of ſix yellowiſh ſerous globes, and every one of theſe is compoſed of ſix lymphatic globes; but farther the microſcope does not enable us to proceed. Every ſpecies of matter, however different in denſity, may be conceived to be formed of equal and ſimilar elementary particles. Thus, the particles
FIG. X. *A*, *B*, *C*, compoſed each of ſix equal elementary particles, are different; and theſe particles may, by a different combination, form different ſtill; and theſe repeated coalitions may, by changing the original and ſucceeding numbers and their poſitions, form maſſes of matter differing in endleſs variety.

UNIFORM MOTION.

91. From the ſolidity of matter, whether owing to repulſion or actual contact, reſults its capacity of impulſe or of being protruded, and conſequent mobility of body, as far as the cauſe of mobility ſeems to be known. If the body *A*, in motion, ſtrikes another body *B*, not retained in its place by any force, *B* will be protruded and move; becauſe, from their ſolidity, they cannot penetrate each other's dimenſions; and the communication of motion in this caſe is the neceſſary conſequence of ſolidity. Motion ſuppoſes the ſucceſſive exiſtence of the body moving in different parts of ſpace, and therefore cannot be underſtood without preſuppoſing a knowledge of ſpace and time; which therefore muſt be premiſed.

92. DEF. *Space is that which contains the whole ſenſible creation, is unlimited, and its parts are homogeneous, inſeparable, co-exiſting and unreſiſting.*

93. Cor. Since ſpace is unlimited in every direction, and its parts ſimilar and undiſtinguiſhable, any portion of it can only be aſcer-

aſcertained by its relation to ſome aſſumed ſenſible mark or object. Hence of place, which is a part of ſpace, there are two kinds, abſolute and relative.

94. DEF. *The abſolute place of a body is that portion of this unlimited ſpace, which is occupied by it when ſixed and immoveable; and the relative place of a body is its ſituation with reſpect to ſome aſſumed mark or object, which itſelf may be moveable.*

95. Cor. The abſolute and relative place of a body coincide, when it and the aſſumed object are immoveable, or remain in the ſame part of fixed ſpace. But we cannot perceive the abſolute place of any object.

96. DEF. *Duration is that which flows uniformly, is unbounded, continuous, whoſe parts are ſimilar, and no two exiſt together.* An idea of duration is obtained by obſerving the interval between our ideas, or between the ſucceſſive appearances of any external object; and, as it is, ſtrictly, only meaſurable by a portion of duration, and no two parts exiſt together, conſequently cannot be compared by juxta poſition, time, which is a part of duration, is of two kinds, abſolute and relative.

97. DEF. *Abſolute time is a portion of duration, whoſe quantity is only known by a compariſon with another portion; and conſequently the relation between any two parts of abſolute time is undiſcoverable. Relative time is a part of duration which elapſes during any motion of body, or any ſucceſſion of external appearances.*

98. DEF. *An inſtant is the boundary between any two contiguous portions of time, as a point is the boundary of any contiguous lines; and a moment is any ſmall portion of time.*

99. DEF-

99. DEF. *Abſolute motion is a change of abſolute place, and abſolute reſt is a permanence in the ſame abſolute place.*

100. DEF. *The direction of motion is the poſition of the line upon which it is made. When the motion is rectilineal, its direction is this right line; when curvilineal, the direction is the tangent to that point of the curve where the moving point is.*

101. DEF. *Velocity is the quickneſs or ſlowneſs of motion, or the rate at which a body moves.*

102. DEF. *A body is ſaid to move with an uniform, accelerated, or retarded velocity, when it continues the ſame, encreaſes, or decreaſes. When the encreaſe, or decreaſe, of velocity is the ſame in any equal times, the acceleration, or retardation, is ſaid to be uniform; and when this encreaſe or decreaſe of velocity, encreaſes or decreaſes in any equal times, the acceleration, or retardation, encreaſes or decreaſes in the ſame ratio.*

103. DEF. *The line joining all the ſucceſſive places, through which a moving body paſſes, is called the ſpace deſcribed.*

104. PROP. *If* S *repreſent the ſpace uniformly deſcribed with the velocity* V *in the time* T, *and theſe magnitudes be ſuppoſed to vary,* S *will vary as* V × T.*

DEM.

* This propoſition is proved, not inelegantly, by the following proceſs: Let S and s be reſpectively deſcribed uniformly, with the velocities V and v, in the times T and t; and $T : t :: S : \frac{S \times t}{T}$ (38) = the ſpace deſcribed in the time t with an uniform velocity equal to V; and $\frac{S \times t}{T} : s :: V : v$ and $S \times t \times v = s \times V \times T$ (37), and (38) $S : s :: V \times T : v \times t$. Q. E. D.

DEM. If either V or T be encreaſed or diminiſhed in the ratio of 2, 3, 4 .... n : 1, the ſpace will evidently be encreaſed or diminiſhed in the ſame ratio; and becauſe V and T are unconnected, or either of them may be changed without affecting the other, S will vary as $V \times T$ (64). Q. E. D.

105. Cor. 1. Becauſe S is as $V \times T$, V will be as $\frac{S}{T}$, T as $\frac{S}{V}$, and, if S be given, T as $\frac{1}{V}$ and V as $\frac{1}{T}$ (65).

106. Cor. 2. If V be given, S is as T, or the ſpaces deſcribed, with the ſame uniform velocity, are true meaſures of time, and may be ſubſtituted for it. And, T being given, S is as V, or the ſpaces deſcribed in the ſame time uniformly, are meaſures of the velocity: if V, or any other ſymbol, be called velocity, it denotes the length or number of feet deſcribed uniformly in one ſecond, or any other time. Let any values of the velocity be to each other as $A : B$, or as 3 : 2, and any cotemporary values of the time be as $C : D$, or as 5 : 4, and the ſpaces deſcribed will be to each other as $X : Z$, or as $3 \times 5 : 2 \times 4$. PLATE I. FIG. III.

107. Cor. 3. If lines therefore be ſubſtituted for V and T, the ſpace S will be as the rectangle, whoſe ſides are theſe lines; and, if numbers be ſubſtituted for them, the ſpace will be equal to the product of theſe numbers, the time equal to the quotient of the ſpace divided by the velocity, and the velocity equal to the quotient of the ſpace by the time. Let V be equal to n feet in one ſecond, and $T = p$ ſeconds; and S is actually equal to $n \times p$ feet, T actually equal to $\frac{S}{V}$ ſeconds, and $V = \frac{S}{T}$ feet in one ſecond.

Keil's Phyſics, Lect. IX.

VARIABLE MOTION.

108. PROP. *In variable finite velocities, the velocity during an infinitely ſmall time, is uniform.*

DEM. The encreaſe, or decreaſe of velocity produced in any finite time, is finite, and, if the whole encreaſe or decreaſe be divided into any infinite number of increments or decrements, each will be infinitely ſmall and vaniſh, compared with the whole velocity, which therefore is uniform. Q. E. D.

109. Cor. If therefore S', V', T', repreſent correſponding increments of S, V, T, reſpectively; V' will be evaneſcent compared with V; S' will vary as $V \times T'$ (104), V as $\frac{S'}{T'}$, and T' as $\frac{S'}{V}$ (105).

PLATE I. FIG. XI. 110. PROP. *If the abſciſſa* AS *of any curve* DEF *repreſent the whole time, and the ordinates* AD, BE, CF, *&c. be as the velocities at the inſtants* A, B, C, *&c.. the ſpaces deſcribed in the times* AB, BC, *will vary as the areas* AE, BF, *&c.*

DEM. For let Am, mn, np, &c. be moments of time, and the ſpaces deſcribed in theſe moments, are as the rectangular parallelograms Aq, mr, nv, (107) &c.; and the whole ſpaces, deſcribed in the ſums of theſe moments, i.e. in the times AB, BC, are as the ſums of theſe parallelograms, or as the areas AE, BF. Q. E. D.

111. Cor. 1. If the relation between V and T, or the ordinate and abſciſſa, be given, the relation between the ſpaces, or the areas AE, BF may be found.

112. Cor.

112. Cor. 2. If the area AE be always as the ſpace deſcribed (*S*) in the time AB or T, the velocity is as the ordinate BE; for according to this ſuppoſition, S' is as $AD \times Am$, or $AD \times T'$, and (109) as $V \times T'$; therefore V is as AD. PLATE I. FIG. XI.

113. Cor. 3. If AS repreſent the ſpace deſcribed, and the ordinates AD, BE, CF, &c. be always inverſely as the velocities at thoſe points; the times of deſcribing AB, BC, &c. will be as the correſponding areas AE, BF, &c. For the time of deſcribing any ſmall ſpace Am is as $\frac{S'}{v}$, or as $Am \times AD$; and conſequently the ſum of the moments, or whole time of deſcribing AB, is as AE.

114. Prop. *The acceleration and retardation of velocity, vary as the change uniformly produced directly, and the moment of time in which they are produced inverſely.*

If the changes of velocity, uniformly produced in the ſame time, be as 2:1, the acceleration is as 2:1 (102); and if the times, in which the ſame change of velocity is effected, be as 1 : 3, the law of acceleration is as 3 : 1, or inverſely as the times; becauſe if the times were as 3 : 3, the changes of velocity uniformly produced in theſe equal times, which meaſure the ratio of acceleration, would evidently be as 3 : 1, therefore the acceleration is as $\frac{V'}{T'}$, and in the ſame manner the retardation is as $-\frac{V'}{T'}$. Q. E. D.

115. Cor. If the velocity vary as the time, the acceleration and retardation are conſtant; for $V : v :: T : t$, and, ſuppoſing $n =$ the number of moments of time, or changes of velocity, $\frac{V}{n}(V') : \frac{v}{n}(v') :: \frac{T}{n}(T') : \frac{t}{n}(t')$; or V' is as T' and $\frac{V'}{T'}$ is given.

 NOTE.

NOTE.

PLATE I. FIG. XII.

*116. PROP. *In variable motions, if the velocity* (V) *in any point* L, *be as any power of the space described, the time* (T) *of describing that space may be found.*

Let the body begin to move from A, and $AL = z$, and V be as z^n, and T the time of describing AL. $\dot{T}$ is as $\frac{\dot{z}}{V}$ (109) as $\frac{\dot{z}}{z^n}$, and $T = \frac{z^{1-n}}{1-n} + C$ (correction). Q. E. D.

117. Cor. 1. If V be constant, or $n = o$; T will be as (z) the space described.

118. Cor. 2. If V be as the space described, or $n = 1$; $T = \frac{1}{o}$ or is infinite, and the body will never move from the point A. There are therefore no motions in nature, whose velocities are not in a less ratio than that of the spaces described, in the beginning of motion.

119. Cor. 3. If V be as the power of the space described, whose exponent is $\frac{1}{2}$, $-\frac{1}{2}$, -1 -2, &c. the time will be as that power of the space described, whose exponent is $\frac{1}{2}$, $\frac{3}{2}$, 2, 3, &c.

FIG. XIII.

120. Cor. 4. Let the body move in AL, and the velocity (V) be as the ordinates of the curve line AM, which meets LA in A, and consequently at A, $V = o$. It is evident, that if the time of describing AL be finite, or (118) V be as some power of AL, whose exponent is less than unity, or fractional, the tangent at A will be perpendicular to AL. Let V be as AL^n and $\dot{V}$ as $n \times AL^{n-1} \times \dot{AL}$, and $\dot{V} : \dot{AL} :: n \times AL^{n-1} : 1$; and, if $\dot{V}$ be as any power of AL, whose exponent is less than unity, viz. $\frac{1}{2}$, $\dot{V} : \dot{AL} :: \frac{1}{2} \times AL^{-\frac{1}{2}} : 1 :: \frac{1}{2AL^{\frac{1}{2}}} : 1 :: \frac{1}{o} : 1$, when AL vanishes, and consequently the fluxion of the ordinate LM is infinitely greater than that of the absciss AL.

121. EXAMP. If AMB be a semicircle, $AB = 2a$, $AL = s$, and the velocity at any point L, as LM, or such as would describe $n \times LM$ feet in 1″; $T =$ the fluent of $\frac{\dot{s}}{v}$, or of $\frac{\dot{s}}{n \times LM}$ or $\frac{\dot{s}}{n \times \sqrt{2as - s^2}}$, or $\frac{a\dot{s}}{na \times \sqrt{2as - s^2}}$, or $\frac{\dot{AM}}{n \times AC}$ seconds. Therefore the whole time of describing $AB = \frac{AMB}{n \times AC}$ seconds, which is a given quantity, and whatever be the length of AB, it will be described in the same time.

PLATE II. FIG. XIV.

122. PROP. *The times of describing* AL *and* al (T *and* t) *with velocities which are always to each other as the ordinates* LM, lm *of similar curves* AM, am, *are equal.*

* Euler's Mechanica.

DEM.

DEM. Let AL and al be homologous ſpaces, whoſe ratio is that of $p:q$, and if $AL = s$, and $LM = v$; $al = \frac{AL \times q}{p}$, and $lm = LM \times \frac{q}{p}$. The time of deſcribing AL or $(T) =$ flu. $\frac{\dot{s}}{v}$, and $t =$ flu. $\frac{\dot{al}}{lm} =$ flu. $\frac{\dot{AL}}{LM} =$ flu. $\frac{\dot{s}}{v} = T$. Q. E. D.

123. DEF. *The ſcale of velocity is a line whoſe ordinates are as the velocities at thoſe points from whence they are drawn. If the body move in* AL, *as in the laſt propoſition, &c.* AM *is the ſcale of velocity.*

124. DEF. *The ſcale of time is a line whoſe ordinates are as the time. If the ordinate* MQ *be always as the time of deſcribing* AM, *the line* AQ *is called the ſcale of time.* PLATE II. FIG. XV.

125. PROP. *If the ſcale of velocity* (V) *be given, the ſcale of the time* (T) *may be found.*

For $\dot{T}$ is as $\frac{\dot{S}}{V}$ and $T =$ flu. of $\frac{\dot{S}}{V} + C$, and if the relation of the ordinate and abſciſs, or V and S be known, the fluent of $\frac{\dot{S}}{V}$ may be found. Q. E. D.

126. Cor. If the ſcale of velocity AN be a circular arc, the time is that arc; for $\dot{T}$ is as $\frac{\dot{S}}{V}$ or $\frac{\dot{S}}{MN}$, or $A\dot{N}$, and T is as AN.

127. PROP. *The ſcale of time* AQ *being given to conſtruct the ſcale of velocity.*

$\dot{T}$ is as $\frac{\dot{S}}{V}$, and V as $\frac{\dot{S}}{\dot{T}}$ or $\frac{MQ}{MO}$, ſuppoſing QO to be perpendicular to the curve. Take therefore $MN = \frac{MQ}{MO}$, which is $= \frac{\dot{S}}{\dot{T}}$ or as V, and AN is the ſcale of velocity.

128. Cor. 1. If AQ be a right line, and T as $m \times S$; $\dot{T}$ is as $m \times \dot{S}$, and V is as $\frac{\dot{S}}{\dot{T}}$ as $\frac{\dot{S}}{m \times \dot{S}}$ as $\frac{1}{m}$, and AN is a right line.

129. Cor. 2. If T be as S^m and $\dot{T}$ as $m \times S^{m-1}\dot{S}$; V will be as $\frac{\dot{S}}{m \times S^{m-1}\dot{S}}$, or as $\frac{1}{m \times S^{m-1}}$. If AQ be the common parabola, or $m = \frac{1}{2}$; $\frac{1}{m \times s^{m-1}}$ will be equal to $2 s^{\frac{1}{2}}$, and AN is alſo a parabola; and the relation of T (MQ) and the abſciſs AM or S being known, the relation of V (MN) to S may always be found.

RELATIVE MOTION.

130. DEF. *Relative motion is a change of relative place, and relative reſt is a permanence in the ſame relative place.* Relative and apparent motion are ſometimes diſtinguiſhed from each other, the firſt being defined to be that which is attributed to a moving object by an obſerver in motion, and the ſecond that attributed to an object really quieſcent by an obſerver in motion.

131. DEF. *The relative velocity of two bodies is the velocity with which they accede to, or recede from, each other.*

132. Cor. 1. Relative motion and reſt of a body coincide with abſolute, when the aſſumed object, by which its ſituation is determined, remains in the ſame part of fixed ſpace. If the earth be quieſcent, every ſhip which moves, or is quieſcent, with reſpect to a fixed object upon the ſhore, moves alſo abſolutely, or is abſolutely quieſcent.

PLATE II. FIG. XVI. 133. Cor. 2. If the object, by which the ſituation of a body is determined, move, abſolute and relative motion and reſt do not coincide. If the body *B*, fixed at the point *B* in the line *AB*, be the aſſumed object with which the ſituation of *A*, placed in the ſame line and partaking of its motion, be compared, its abſolute velocity will be the ſum or difference of its own velocity and that of the line, according as they move in the ſame, or oppoſite, directions. If the line *AB* move in the direction *YX*, with an uniform velocity of 100 feet in 1″, and *A* alſo move in the ſame direction, with an uniform velocity of 50 feet in 1″; *A*'s whole abſolute velocity in the direction *YX* is 150 feet in 1″, and its relative velocity, or uniform receſs from *B*, is 50 feet in 1″. If the line *AB* move uniformly in the direction *YX* with a velocity of 100 feet in 1″, and *A* move in the oppoſite direction *XY*, with a velocity of 100 feet in 1″, it will be abſolutely quieſcent, and its relative velocity, or B's accefs to it, will be 100 feet in $1''$ uniformly: if A's velocity, in the laft fuppofition, be 50 feet in $1''$ uniformly, its relative velocity, or approach to B will be 50 feet in $1''$, and its abfolute velocity will alfo be 50 feet in $1''$ in an oppofite direction.

134. Cor. 3. A fpectator therefore, placed upon the moving line AB at the point B, only perceives A's relative motion, which may be always different from the true.

*135. PROP. *If two bodies* A *and* B *move at the fame time from* A *in the directions* AM *and* AL, *with uniform velocities* a *and* b *refpectively, their relative velocity will be uniform, and to* a *as the fine of* $\angle$ BAD : $\angle$ BDA, *and to* b *as fin.* $\angle$ BAD : *fin.* $\angle$ DBA *or* $\angle$ BAF, *fuppofing* B *and* D *to be cotemporary pofitions of* A *and* B, *and* AF *to be parallel to* BD. PLATE II. FIG. XVII.

DEM. Let B, D; C, E; M, L; be cotemporary pofitions of A and B; and (106) $a : b :: AB : AD :: AC : AE :: AM : AL$; and confequently BD, CE, LM, which meafure the relative velocity, are parallel to each other, and encreafe uniformly becaufe AM does. But the relative velocity is to $a :: LM : AM ::$ fin. $\angle MAL$: fin. $\angle ALM$, and it is to $b :: LM : AL ::$ fin. $\angle LAM$: fin. $\angle LMA$ or $\angle MAH$. Q. E. D.

136. Cor. 1. The only fenfible motion of B to a fpectator placed at A, and fuppofing himfelf to be quiefcent, will be along Ab parallel and equal to ML; and when A hath really defcribed AM, B will appear to have defcribed $Ab = ML$ uniformly.

137. Cor. 2. If A and B move uniformly with any equal velocities in parallel directions AM, BL, their relative fituation is not changed, being always equal, and parallel, to AB; and if any FIG. XVIII.

fpace

* Euler's Mechanica.

ſpace $AMBL$, containing any number of bodies, move, their relative ſituations are not affected by it. To a ſpectator therefore placed in A, in this moving ſpace, and ſuppoſing himſelf to be quieſcent, any body B will appear to be quieſcent.

138. Cor. 3. It is evident that the abſolute motion of a body may be changed into innumerable relative motions uniform and rectilineal, if its abſolute motion and that of the body, by which its ſituation is determined, be ſo.

PLATE II. FIG. XIX.

139. Prop. *If two bodies* A *and* B *move, at the ſame time, from the point* A, *in the directions* AM, AL, *with uniform velocities, as* AM, AL *reſpectively, and* B *be conſidered as quieſcent;* A's *apparent motion will be uniform and its direction and velocity as the diagonal of a parallelogram whoſe ſides are* AM, *and* AD *equal and oppoſite to* AL.

Dem. Let m, l be cotemporary poſitions of A and B, or let $Am : Al :: AM : AL$ and lm, Lm are parallel. But when B is at l and L, A's diſtance is equal to lm, LM; or, ſuppoſing B to be quieſcent at the point A, and AE, AD be reſpectively equal and oppoſite to Al, AL, and En, DN be drawn parallel to AM, A will appear at n and N. But An varies as Am, and therefore encreaſes uniformly, and AN is the diagonal of a parallelogram, whoſe ſides are $AD = AL$, and AM. Q.E.D.

140. Cor. 1. The apparent and real velocities of A are to each other as $AN : AM ::$ ſin. $\angle MAD$: ſin. $\angle NAD$.

141. Cor. 2. Suppoſing A and B to move as before, and A to be quieſcent, the diſtances and directions of A from B will be Ab, and AH, ſuppoſing them to be reſpectively equal and parallel to ml, ML. Whilſt B therefore really deſcribes AL, it appears to deſcribe $AH = AN$, and in the ſame right line with it; and the apparent

apparent motion of A and B, ſeen from each other, are equal and oppoſite.

142. Cor. 3. An uniform relative motion along the diagonal of any parallelogram $AMND$ may be conſidered as reſulting from, and equivalent to, two uniform abſolute motions in the ſides AM and $AL = AD$; and theſe motions are in the ſame plane, and their velocities are to each other as the diagonal and ſides. FIG. XIX.

143. Cor. 4. If the motions of A and B be oppoſite, their relative motion is equal to the ſum of their real motions, AN being, in that ſuppoſition, equal to $AM + AD$. If the motions of A and B conſpire, their relative motion is equal to the difference of their real motions, or of AM and AD. And if the real velocities of A and B be variable, according to the ſame law, their relative velocity will vary according to that law.

144. Cor. 5. The real motions in AM and AL are ſaid to be equivalent to a motion in AN, becauſe they produce the ſame effect in the ſame time, as if B were quieſcent at A, and A were to deſcribe AN uniformly in that time. And in the ſame manner two motions in AN and AD are equivalent to a motion in AM, or its oppoſite and equal AP, according as the body deſcribing AD, or AN, is ſuppoſed to be quieſcent.

145. Cor. 6. If the real motion of A and B, or AM and AL, and their inclination be known, the relative motion of either of them may be found; for AM, MN and $\angle MAD$, and conſequently its ſupplement AMN, being known, the baſe AN, and the $\angle MAN$, or the inclination of A's apparent to its real path, may be found.

146. Cor.

146. Cor. 7. The relative and real motion of A, or AM, AN, and the angle MAN, being known, the real motion of B, which is ſuppoſed to be quieſcent, or $AL = AD$ may be found.

PLATE II. FIG. XX. 147. PROP. *If two bodies* A *and* B *move at the ſame time uniformly in the directions* AM, BL, *with velocities equal to* a *and* b, *the relative motion of* B *will be uniform and rectilineal upon the line* BN, *ſuppoſing* AN *to be always equal, and parallel to right lines joining the cotemporary poſitions of* A *and* B.

DEM. Let m, l; M, L, be cotemporary poſitions of A and B, or let $a : b :: Am : Bl :: AM : BL$, and ml, ML will be the diſtances of B from A in the points m and M. And if A be conſidered as quieſcent at A, taking An, AN, reſpectively parallel and equal to ml, ML, B will appear at n and N in the line BnN, which is therefore the apparent path of B. But $a : b :: Am\ (nl) : Bl :: AM\ (NL) : BL$; therefore BnN is a right line, and encreaſes uniformly, becauſe it varies as BL. Q. E. D.

148. Cor. If A and B move as in this propoſition, and B be conſidered as quieſcent, A's apparent path will be in the right line ADE, parallel and equal to BnN, as is evident by taking BD and BE reſpectively parallel and equal to lm and LM. A's velocity will therefore be uniform and equal to B's velocity, when A was conſidered as quieſcent.

149. The abſolute and relative motions of B, and their directions being known, the direction and velocity of A's motion, which occaſioned B's relative motion, may be found; for, let A and B be cotemporary poſitions of A and B, and let Bl, Bn, be the ſpaces deſcribed in the ſame time by B's abſolute and apparent motion, and if ln be joined, AM drawn parallel to it, will be A's path, and A's velocity : B's velocity :: $nl : Bl$.

150. Cor.

150. Cor. 1. An absolute uniform and rectilineal motion BL may be changed into any infinite number of uniform relative rectilineal motions, by changing the direction and velocity of A's motion; for BN may be drawn of any length, and in any direction, and the direction and velocity of A, moving uniformly so as to produce this relative motion, may be found by drawing AM parallel and equal to NL. FIG. XX.

151. Cor. 2. When two bodies move uniformly in right lines, and one of them is considered as quiescent, the relative motion of the other is therefore uniform and rectilineal; and the spaces, described by it relatively, vary as the velocity multiplied into the time (104), or S varies as $V \times T$, T as $\frac{S}{V}$, and V as $\frac{S}{T}$.

152. Cor. 3. If the relative motion of B be variable, the absolute motion of either A or B is variable.

153. Prop. *If a body* A *move in any curve line* A m M, *and another body* B *move in any other curve* B l L, B'*s apparent motion may be found.* PLATE II. FIG. XXI.

Let m, l, M, L, be cotemporary positions of A and B; and B's distance and the right lines in which it appears at m and M, are ml and ML. Or, if A be considered as quiescent, and An, AN, be taken respectively parallel and equal to ml, ML, B will appear at n and N, and its path will appear to be BnN. Q. E. I.

154. Cor. It is evident that the curve BnN, the relative path of B, may be described by the motion of A in different curves; but all the curves described by A will be equal and similar, and have their corresponding parts parallel, because they are always subtended by right lines equal and parallel to nl, NL, &c.

FIG. XXII. 155. PROP. *If* B *be absolutely quiescent, and* A *move in any curve line* A m M *uniformly,* B *will appear to move uniformly in a curve equal and similar to* A m M, *and these curves are similarly situated with respect to the points* A *and* B.

Let m, M be any contiguous positions of A, which being considered as quiescent at the point A, the cotemporary positions of B are n and N, supposing An, AN to be respectively parallel and equal to Bm, BM. And because $An = Bm$ and $AN = BM$, and the angle $NAn =$ angle mBM, the small arc $Nn = mM$, and they make equal angles with the distances AN, BM, and An, Bm. Q. E. D.

156. Cor. 1. If B move in the curve BnN, and A be quiescent at the point A, it will appear to describe the curve AmM similar and equal to BnN.

PLATE III. FIG. XXIII. 157. Cor. 2. If A and B, placed in the same right line ABC, describe the circles AmM, BlL in the same time uniformly, and A be considered as quiescent, B's apparent path is a circle whose center is A and radius AB. For let m, l, and M, L, be cotemporary positions of A and B, and B will appear at l and L; or, since A is considered as quiescent, if An, AN, be drawn parallel and equal to ml, ML, it will appear at n and N, and its apparent distance from A always $= AB$, and the direction of B's apparent motion is the same with that of A and B.

158. Cor. 3. If B be considered as quiescent, and they move as in the last corollary, A's apparent path will be a circle whose center is B and radius BA, and the direction of its motion is the same with that of B in that corollary. For taking any cotemporary positions m, l, and M, L; and, drawing Bn, BN respectively parallel and equal to ml, mL, A will appear at n and N, and its apparent distance from B always $= AB$.

159. Cor.

159. Cor. 4. The relative motions of *A* and *B* seen from each other, are equal and in similar curves. For let *m* and *M* be any contiguous positions of *A*, and *l*, *L* corresponding positions of *B*; and joining *m l*, *ML*, *B*'s relative motion, whilst it really describes *lL*, will make it describe *nN*, supposing *A* to be quiescent, and *An*, *AN* to be respectively equal and parallel to *lm*, *LM*. Supposing *B* to be quiescent, and *Bb*, *BH* to be equal and parallel to *ml*, *ML*, or *An*, *AN*, *A* will appear to describe *bH*, which is equal and similar to *Nn*. FIG. XXIV.

160. Cor. 5. If *B* be immoveable and not placed in the plane of *A*'s motion, it will appear to describe a line equal and similar to the line described by *A*, and they will be in parallel planes; for taking any two contiguous portions of *A*'s path, the right lines joining the real and apparent places of *A* and *B* are equal and parallel, and consequently the small right lines joining them are equal and parallel: and this and the preceding corollaries are true where-ever the eye's imaginary place is.

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161. To distinguish real and absolute, from relative motion, is an important, but very difficult problem; because the same apparent motions may result from real motions combined in endless variety. Every apparent motion of a body results from, and may be explained by, its real motion and that of the spectator and *v*,*v*; and therefore its real motion, and the real motion of the spectator, or this, and the apparent motion of the body, must be presupposed, and, with the assistance of mathematics, the other may then be detected. A spectator ignorant of the earth's annual and diurnal motion, and supposing himself to be quiescent, must draw erroneous conclusions, in all his reasonings, concerning the absolute motion of any body, because these motions of the earth will communicate an apparent motion to a body absolutely quiescent, and affect the absolute motion of a body moving (134). The relative motions of bodies must therefore be ascertained from observation, and their real motions, when dis-

difcoverable, are deduced from thefe, the properties of real motion, and the quantity of mechanical powers, or qualities in matter, that are found to generate motion. If the properties, invariably attached to all real motions, obtain in any obferved motion; or if any mechanical caufe be proved to exert an influence, which is exactly competent for the production of any obferved motion, it may fafely be inferred that this motion is not imaginary, but real.

EXAMP. I. The variation of diftance between a fhip and a remote object is owing to the motion of the fhip, when the wind and current act and are competent for its production.

FIG. XXV. EXAMP. II. A fhip failing from fouth to north at the rate of five miles in an hour, obferves another fhip which failed from the fame point *E* going to the fouth-weft at the rate of $7\frac{1}{10}$ miles in an hour nearly, and from hence it appears, that the laft fhip failed from eaft to weft at the rate of five miles in an hour nearly (139).

EXAMP. III. The magnitude of the earth's attractive power, or any other mechanical caufe, is incompetent for the production of the diurnal revolution of the heavenly bodies, which is therefore probably apparent.

EXAMP. IV. The aberrations of the fixed ftars are proved geometrically to refult neceffarily from the progreffive motions of light and of the fpectator combined, and are confequently only apparent.

EXAMP. V. A real circular motion is infeparably attended with a centrifugal force, and diftinguifhable therefore by this effect. Thus the diminution of gravity, which is obferved in acceding towards the equator, affords a proof of the diurnal rotation of the earth.

CHAP. IV.

INERTIA OF MATTER.

162. DEF. *POWER and force are uſed ſynonimouſly to ſignify any action upon a body, which produces motion, or a tendency to motion, as animal exertions and the influence of phyſical cauſes, gravity, elaſticity, magnetiſm, &c. When a force acts always with the ſame intenſity, or produces the ſame effect in a given time, it is ſaid to be uniform or conſtant, and variable when it does not.*

163. DEF. *Any momentary impreſſion upon a body producing motion, or a tendency to motion, is called an impulſe, or percuſſion, of a power or force.*

164. DEF. *Whatever reſiſts the action of a force is called an obſtacle.*

165. DEF. *Inertia, or vis inertiæ, of matter, is that quality by whoſe influence a change, in the direction or quantity of its motion, cannot be produced without the application of a force.* The cauſe of a body's continuing in a ſtate of reſt, or of uniform rectilineal motion, is not any external force, but the nature and conſtitution of matter; and this internal cauſe or principle is called its inertia.

166. This property of matter, or inertia, is eſſential in the conſtitution of a ſyſtem, whoſe preſervation is neceſſarily dependent upon the regular obſervance of preſcribed, determinate, laws; but its

its exiſtence can only be collected from experience. *Firſt*, Every change in the ſituation of a body, from reſt to motion, and from motion to reſt and to an encreaſe or decreaſe of motion, indicates an inertia, which is found to pervade every ſpecies of matter acceſſible to obſervation, the particular facts, adduced in proof of it, being innumerable and concurring to eſtabliſh its univerſal exiſtence. A quieſcent body is never diſcovered to move, and motion, or change of motion, is never induced, without the actual impreſſion of a cauſe able to produce theſe effects; for matter not only continues in a ſtate of reſt, by an incapacity to give motion to itſelf, but never ceaſes to move, or changes the quantity and direction of its motion, without the application of ſome philoſophic influence adequate to their production, as magnetiſm, elaſticity, gravity, &c. or animal exertion, as percuſſion, protruſion, &c. The time of the motion of a wheel upon its axis, or braſs topp upon a poliſhed ſurface, is encreaſed with the diminution of friction upon the axis, or ſurface; and the motion of a body, placed upon the deck of a ſhip, and partaking of its motion, continues in the ſame direction after the ſhip has ceaſed to move; and theſe motions are evidently diſcontinued by the operation of a cauſe, i. e. friction, competent to deſtroy them. New motions are obſerved without any ſenſible material impulſe, reſulting apparently from an innate tendency to motion; thus, a body not ſupported deſcends towards the earth, and, projected in a direction not perpendicular to the earth's ſurface, deviates from the line of projection with a velocity perpetually variable; and new motions ariſe amongſt the minute particles of bodies; but theſe are the neceſſary reſult of eſtabliſhed natural powers, gravity, elaſticity, &c. *Secondly*, This quality is ſometimes called the vis inſita, or vis inertiæ, of matter, from the ſimilitude of its effects to thoſe of animal powers exerciſed upon a body, both producing a change of motion. If a body *A*, placed upon a poliſhed table, or ſuſpended by a rope, be quieſcent, it will continue in that ſtate till urged by ſome external cauſe; and if another body *B* be projected with any velocity, and impinge upon *A*, it will communicate motion to it, and be itſelf retarded. When the effect of *B*'s

impact

impact is considered in relation to the change produced in *A*, *B* is said to act upon *A*; and, when considered in relation to the change of motion produced in itself, it is said to be resisted by *A*. Action is usually ascribed to a moving, and resistance to a quiescent, body; but they may both be considered as actions, because they produce similar effects, that is, a change of motion. *A*, by its vis inertiæ, destroys a part of *B*'s motion, and *B*, by endeavouring to retain its present state, that is, by its vis inertiæ, protrudes, and communicates motion to, *A*. *Thirdly*, Every change in the situation of a body is therefore universally allowed to indicate some philosophical influence, or animal exertion; and whether a body be quiescent, or moving, the quantity of its inertia is invariably the same; for it is discovered from experiments, the only source of information, that a force, communicating a velocity equal to a to the quiescent body *A*, will communicate an encrease of velocity, equal to a, to that body moving. The force of gravity, which communicates a velocity nearly equal to thirty-two feet in one second of time, to a body falling from a state of rest, communicates an encrease or decrease of velocity equal to this, in one second, to a body already moving, according as it conspires with, or is opposite to, the direction of the body's motion. And if the body *B*, moving with a velocity equal to b, impinge upon *A* at rest, and communicate a velocity to it equal to a, and lose a velocity equal to l, it appears from experiments, that *B* moving with the velocities $b + m$, $b + 2m$, $b + 3m$, &c. will communicate an encrease of velocity equal to a, and lose a velocity equal to l, by impinging on *A* moving in the same direction with velocities equal to m, $2m$, $3m$, &c. Consequently the vis inertiæ of *A*, estimated by the velocity, and encrease or decrease, communicated to it, is the same in a state of motion and rest. *Fourthly*, As the whole vis inertiæ of a body is composed of that of all its parts, and as we cannot conceive the vis inertiæ of the same or equal particles to be encreased or diminished, the whole vis inertiæ of different bodies will vary as the number of equal particles, or quantities of matter, contained in them. The invariable result of experiment is, that bodies equal

to

to A, $2A$, $3A$, &c. and heterogeneous bodies, whofe weights are as 1, 2, 3, &c. receive the fame velocity by the action of forces, whofe magnitudes are as 1, 2, 3, &c.

167. DEF.* *The moment, quantity of motion, vis infita, of a body, are ufed fynonimoufly to denote the impetus, or force, with which it moves, or its capacity to communicate motion or preffure.*

168. PROP. *If* M, Q, V, *reprefent moment, quantity of matter, and velocity, refpectively, and be fuppofed variable,* M *will vary as* Q × V.

DEM. Prefuming that equal particles of matter have an equal inertia, it is evident that the preffure of one particle upon an immoveable obftacle, or the force requifite to generate or deftroy its velocity, will vary as this velocity, or as V; and becaufe the impetus of any body is compofed of that of every particle or Q, and Q and V are independent, M will vary as $V \times Q$ (64). Q. E. D.

Another demonftration.

The moments of different bodies vary as the uniform forces capable of generating or deftroying them, in equal times, being their whole cotemporary effects; but forces equal to F, $2F$, $3F$, &c. will evidently communicate, in the fame time, the fame velocity to 1, 2, 3, &c. equal particles endued with an equal inertia, and velocities as 1, 2, 3, &c. to one particle. Therefore F, which is as M, is encreafed in the fame ratio, with the velocity or V, and number of equal particles or Q, and, thefe being independent, vary as $Q \times V$ (64). Q. E. D.

* Keil's Phyfics, Lect. IX.

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169. These, or similar demonstrations, are usually adduced in support of this proposition, which is perhaps more properly and with more conviction, demonstrated by experiments. The moments of different bodies are collected by measuring the magnitudes of the forces required to produce them, or the magnitudes of their cotemporary effects similarly produced, in the simplest and most intelligible instances, which may be deemed the common measure of their moments.

Exp. I. If two unequal spherical bodies, moving in opposite directions, meet and after impact be quiescent, their moments must be equal; but their quantities of matter are always found to be inversely as their velocities. Or, if they move in the same direction, and one overtakes the other, the velocity gained by the struck body, and lost by the striking body, are always found to be inversely as their quantities of matter.

Exp. II. If a body A be placed on the same side of the fulcrum of a straight lever, at the distance of 1, 2, 3, &c. feet from A, it will move with velocities as 1, 2, 3; and be restored to an equilibrium by the bodies A, $2A$, or $3A$, &c. placed, at the distance of one foot, on the other side. And, in general, a body, whose weight is $n \times A$ pounds, at the distance of one foot from the fulcrum, is found to equilibrate with bodies, whose weights are $\frac{n \times A}{2}$, $\frac{n \times A}{3}$, $\frac{n \times A}{4}$, &c. at the distance of 2, 3, 4, &c. feet from the fulcrum respectively. The pressures or moments, are justly inferred from these experiments, to be as their weights multiplied into their distances, or as the quantities of matter multiplied into their velocities. And the same conclusion results from experiments upon every other, however complicated, machine, when allowance is made for friction, and the velocity is properly estimated.

 170. Cor.

170. Cor. 1. Since M varies as $Q \times V$, V will vary as $\frac{M}{Q}$, and Q as $\frac{M}{V}$ (55). If lines be taken in the ratio of Q to V, M will vary as the area of a rectangle, whose sides are these lines; and if numbers be taken in that ratio, M will vary as their product.

171. Cor. 2. If S represent the space described, in the time T, with the velocity V, M, varying as $Q \times V$, will vary as $\frac{Q \times S}{T}$ (105), S, varying as $V \times T$, will vary as $\frac{M \times T}{Q}$, and T will vary as $\frac{S \times Q}{M}$.

172. Cor. 3. If two bodies therefore A and B, moving in opposite directions with velocities respectively equal to a and b, meet, and after impact be quiescent; or if their effects to produce motion upon any machine be opposite and equal, and their velocities, when properly estimated, be a and b; $A \times a = B \times b$, and $A : B :: b : a$ (38), or A and B are to each other inversely as their velocities. The converse of this is true, and if the bodies be inversely as their velocities, or $A : B :: b : a$, their moments are equal, or they are in equilibrio upon any machine, and $A \times a = B \times b$ (37).

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173. Forces are distinguished by some foreign writers into two kinds; 1st, of bodies at rest, and 2dly, of bodies in motion.

First, *The force of a quiescent body, such as is conceived to reside in one lying upon a table, suspended by a string, sustained by springs, &c. is called its pressure, tension, force, solicitation, vis mortua, &c. and is estimated by the weight with which the table is pressed, the string stretched, spring bent, &c. To this kind are referred centripetal and centrifugal forces, as their effects are similar to those of weights, pressures, tensions.*

Secondly,

Secondly, *The force of a moving body, arising from its inertia, by which it communicates motion, or a change of motion, to another body by impact, overcomes gravity, friction, and other pressures, and is only destroyed by an extinction of its motion, is called the vis motrix of that body, or its vis viva, to distinguish it from the vis mortua.*

Concerning the measure of the first of these forces no doubts are entertained, the pressure of one particle, upon the arm of a lever, being universally allowed to be as its velocity, and of a number of particles, as that number and velocity conjointly; but the measure of the vis viva hath long been a subject of warm contention between the adherents of Newton and Leibnitz, the former maintaining it to be the product of the quantity of matter and velocity, and the latter the product of the quantity of matter and square of the velocity. The term force being defined to be that, which, acting upon a body, communicates motion, or a tendency to motion, the magnitude of the force of a moving body, at any instant of time, may be best, and indeed can only be clearly, conceived by measuring the motion communicated by it; and in this mensuration three things are to be considered, viz. the magnitude of the body or quantity of matter to be moved, the velocity communicated, and the time in which it is communicated. Some of these circumstances may be, as they have been, omitted, and different measures of the same force have resulted. The followers of Newton conceive forces to act, for the same time, with their respective intensities continued uniformly, and estimate their magnitudes by their cotemporary effects. If two forces F and f be supposed to act, at the same instant, upon two bodies A and B, with uniform intensities, and communicate velocities to them, equal to V and v respectively, in the same time, the magnitudes of these forces are to each other as $A \times V : B \times v$. And in the communication of motion by the impact of one body upon another, this law is found to obtain, wherever the effects are distinctly understood, and can be ascertained, the quantity of motion, if measured by the product of the quantity of matter and velocity, remaining invariable, when estimated in the same direction. But the followers of Leibnitz, adopting a different definition of force, derive a different conclusion: they do not suppose the force to act with uniform intensity, as it may decrease gradually to evanescence, which happens in the collisions of bodies, and actions of springs; and the time of action, and sometimes the direction, is disregarded, and not deemed to affect the result. Though it be true in some particular cases of the communication of motion from one body to another, that the force, according to their acceptation of the term, varies as the product of the quantity of matter and square of the velocity, or that this product is the same before and after impact, this conclusion cannot be affirmed to obtain generally. The doctrine advanced by Newton is universally true, according to his meaning of the term force; but whether the opinion of Leibnitz be true or not, is best known from experiments, the result of which is generally repugnant to it: and it would therefore, perhaps, create less confusion to adhere to the old definition of force, include the time of action, and suppose the intensity of the force, during that time, to be the same. The distinction between these two kinds of forces hath often been urged to be superfluous, for the following reasons. *First*, The force of a moving body is a vague and undefinable term; for there is no force in a body considered absolutely, except its inertia, which is always the same, whether the body be quiescent or moving; but if a moving body impinge upon another body either moving or quiescent, its inertia exerts itself as a force, whose magnitude is relative and depends upon its velocity, and the magnitude and velocity of the struck body. If the body *A*, moving with the same velocity *a*, impinge upon the quiescent bodies *B*, 2 *B*, 3 *B*, &c. or upon the same body *B* quiescent, and moving with different velocities equal to *b*, 2 *b*, 3 *b*, &c. it will in every case produce a different effect, and the change of state, both in the impinging and struck body, will be different. The force of springs, and animal exertions, is also relative, and it seems therefore improper to talk of the absolute force of a moving body, springs, &c. because it is relative. *Secondly*, No idea can be formed of an instantaneous communication of motion; for, in the collision of bodies, the parts which come into contact first, are displaced and move, and some time elapses before motion be communicated to the whole body. This interval of time between the first contact of the nearest parts of the imping-

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ing bodies, and their motion, is the time of collision, in which the bodies exert a mutual, and perpetually variable pressure, by which their state is gradually changed. This is most observable in soft bodies, or those whose parts yield to percussion; and, as the parts of all bodies do yield, obtains universally, in a greater or less degree, according to the different contexture of their parts. The forces of percussion are therefore pressures, exerted with variable intensity for a short time; and, to comprehend their magnitude, this time ought to be ascertained, the quantity of pressure, for every instant of this time, computed, and the whole collected into one sum. This discrimination of forces seems therefore to be unnecessary, because the action of collision is nothing more than the operation of a continued pressure, and pressures and percussions are similar, and properly expressed by the same term, force. The arguments adduced, in support of their opinion, by the advocates of Leibnitz, are derived from experiments upon, the collision of bodies, the action of elastic springs, the composition and resolution of motion, the indentings or cavities formed in clay, tallow, &c. by falling bodies, and the velocity of fluids in hydraulics. I will only produce a few experiments, which may convey some useful knowledge, are most strongly urged in support of this hypothesis, and appear most to militate against the old opinion.

Exp. I. If one spring be requisite to destroy the motion of a body, whose velocity is 1, four springs, equal to it, are requisite to destroy its motion when its velocity is 2, nine springs when its velocity is 3; and generally the number springs varies as the square of the velocity of the moving body.

Exp. II. The number of men, horses, &c. of equal strength requisite to communicate velocities, as 1, 2, 3, &c. to any body, are as 1, 4, 9, &c. or as the square of the velocity.

These similar experiments may be true, and indeed seem to be established

established by Gravesand, Desaguliers, &c.; but they prove nothing against the common opinion of the force of bodies, which is not measured by effects produced in any unequal times. And besides, the effect of any number of agents, as springs or animal exertions, cannot properly be said to measure their force; because their whole force is not exerted, that of each, considered individually, being greater than when acting in conjunction with the rest, and will be greater or less according as the number of the rest is less or greater.

EXP. III. If the velocity of water, striking upon the float-board, or ladle-board, of a mill, be as 1, 2, 3, &c. the effect of the wheel is found to be as 1, 4, 9, &c. or as the square of the velocity of the stream; and the force of the water, being measured by its effect, varies therefore as the square of the velocity. In this experiment the number of particles of water striking the board in a given time, encreases as the velocity, and, the force of each being encreased in the same ratio, the effect ought to be as the square of the velocity. Theory is very seldom confirmed by practice without allowances for the operation of collateral causes, which, perhaps, cannot be made with demonstrative accuracy. Such experiments as this are too imperfectly understood to justify any general conclusions against experimental proofs clearly comprehended, or theoretic demonstrations, resulting from data, which, perhaps, do not take place in the experiment.

EXP. IV. Mr. Smeaton has proved, by some very ingenious experiments (Phil. Transf. Vol. LXVI. pag. 469.), that the mechanic power to be expended, varies as the square of the velocity to be generated, and the velocity as the impelling power multiplied into the time of action. But the mechanic power, according to Mr. Smeaton, is measured by the weight multiplied into the space descended, and from this definition, his conclusion is inevitable: but this power is different from what is usually meant by moment, force, &c. In N°. 2, 3, the weight *S* descends through spaces

ſpaces equal to 10, and $2\frac{1}{2}$, turns, in $28\frac{1}{4}''$ and $14\frac{1}{4}''$ reſpectively, and communicates velocities nearly as 2 : 1; and in this experiment the weight S acts with the ſame intenſity for times, whoſe ratio is 2 : 1, but its capacity to communicate motion, if meaſured by the ſpace, muſt be as 4 : 1.*

Exp. V. Equal cavities are formed in clay, tallow, &c. by equal bodies falling through ſpaces inverſely as their quantities of matter; but the cavities being equal, the cauſes, or forces of the bodies, are equal, or as $\frac{S}{S}$ (ſuppoſing S to repreſent the ſpace) or as $S \times Q$, from the ſuppoſition, or as $Q \times V^2$.

Exp. VI. A ball ſhot with velocities as 1, 2, 3, &c. is found to form cavities in wood, clay, &c. whoſe depths are as 1, 4, 9, &c. or as the ſquares of the velocities.

The two laſt experiments prove nothing againſt the common doctrine; and though they are of practical utility, yet admitting the propriety of meaſuring forces by their whole effects produced in any unequal times, they do not vary, in theſe experiments, as the ſquare of the velocity: becauſe the cavities formed are not the whole effect, a greater or leſs motion being communicated to the parts contiguous to the cavity in the ſtruck body, according to the magnitude and velocity of the ſtriking body. A ſmall body, moving with a great velocity, may by impact overcome the coheſive force of the particles of the ſtruck body, and effect a ſeparation between them; and from its great velocity, or ſhort time of action, little motion will be communicated to the contiguous parts; but

* Theſe experiments are immediately deducible from mechanical principles. Let $F =$ the radius of the barrel M; $V =$ velocity generated in K in the time T; X and $Z =$ the reſpective ſpaces paſſed through by S and K in the time T; and $1 = K$'s diſtance from the axis Be. And from eſtabliſhed principles, the force impelling K is .8 oz. $\times F$, or as F, and, becauſe F is conſtant, $F \times T$ is as V. $F \times T^2$ is alſo as Z; but $X : Z :: F : 1$, therefore X (or the ſpace deſcended by S, which is as the mechanic power) $= Z \times F$, and is as $F^2 \times T^2$, or as V^2. (Phil. Tranſ. Vol. LXVI. p. 469.)

but a great body, moving ſlowly, will form no cavity in the obſtacle; becauſe the moment of thoſe parts of its ſurface in contact, is leſs than the coheſive force of the parts of the obſtacle, and the parts ſtruck will, by their unknown connection with the parts adjoining, communicate motion to them, and theſe to the next, till the whole obſtacle move. Two bodies therefore moving with the ſame moment, may produce quite diſſimilar effects; and as lines, ſurfaces, &c. are meaſured by the application of a common meaſure, the magnitudes of forces are to be meaſured by their effects, which ought to be ſimilar in every reſpect, and this ſimilitude is moſt obſervable and beſt underſtood in experiments upon the lever and other ſimple machines.*

LAWS of MOTION.†

174. The laws of motion are general rules, or conſequences, reſulting from the nature and conſtitution of matter, and its relation to mechanical forces, the cauſes of motion, by which it is governed, and from which it cannot deviate without ſuffering a total change of its nature. Their truth is eſtabliſhed by a number of concurring and uncontroverted obſervations, all information upon natural powers and motions derived from them being only deducible from thence; and, when thus eſtabliſhed, they are eſteemed to be uniform characteriſtic marks obtaining in all material motions.

FIRST

* Let s repreſent the effect, or ſpace through which any obſtacle is impelled by a moving body, v be the velocity of the body: and if s be as v^2, $\dot{s}$ is as $2\,v\,\dot{v}$, or as $v\,\dot{v}$, or as $\frac{\dot{v} \times s}{\dot{t}}$ (if t be the time), and $\dot{v}$ therefore is as $\dot{t}$. The decrement of velocity is therefore as the time in which it is produced, and the reſiſtance muſt be conſtant; which ſeems to be a neceſſary property of the vis viva; but as this ſeldom happens, and the vis viva is really a preſſure, and may be meaſured in the ſame manner with the vis mortua, the term force may expreſs them both.

See Deſagulier, Vol. II. p. 51. Maclaurin's Account of Sir I. Newton's Diſcoveries, p. 178. Diſcours ſur les Loix de la comm. du Movement, Oper. Tom. III. and Diſſertat. de vera notione virium vivarum. Act. Petropol. Tom. I. p. 131. Muſchen. Int. ad Phil. Nat. Vol. I. p. 83, &c. Phil. Tranſ. No. 371, 375, 376, 396, 400, 401, 459.

† Maclaurin, Book II. Chap. II. Newt. Prin. p. 13. Keil's Phyſics, Lect. XI, XII. Helſham, Lect. III. IV.

FIRST LAW OF MOTION.

175. *Every body perſeveres in a ſtate of reſt, or of uniform rectilineal motion, unleſs affected by ſome external cauſe, as animal exertion or phyſical power.*

The truth of this law is collected by obſervation. It is nearly ſynonimous to the inertia, or that quality, of matter, by which it continues in every new ſtate; for, this being eſtabliſhed, it is a neceſſary reſult that it cannot begin to move from a ſtate of reſt, nor, if in motion, can it accelerate, retard, or change the direction of that motion.

176. Cor. Every body, moving in a curve line, or in a right line with a perpetually varying velocity, is acted upon without intermiſſion during its motion, by ſome external force; and every body, moving in a curve line, has a perpetual tendency to move in a right line, and to leave the curve in the direction of a tangent to that point of the curve where the body is.

SECOND LAW OF MOTION.

177. *Motion, and change of motion, are proportional to, and in the direction of, the force impreſſed.*

The motion communicated to a quieſcent body is proportional to the cauſe producing it; and when a body is accelerated or retarded, during its motion, the acceleration and retardation are proportional to the force producing them. If a body move upon an inclined plane *AB*, or in any curve line *LM*, by the action of FIG. XXVI.
a force tending to any point *S*, the change of motion is not proportional to the whole force directed to that point, but to that part which acts in the direction *AB*, or *NT* touching the curve where the body is; for this part, only, accelerates or retards the

body's motion. When water or air act upon the vanes of a mill, or sails of a ship, the change of motion is not proportional to the whole force of the water or air, but to that part which is actually impressed upon them; for, if the velocity of the water or air be equal to that of the vanes or sails, they will move on together without any acceleration; and, if the direction of the air or water be oblique to the motion of the vanes or ship, some force will be ineffectual, and the change of motion will be proportional to the remainder, acting in the direction of their motion.

The truth of this law is demonstrated by numberless experiments upon the motion, and change of direction and motion, communicated by material impulse. Every experiment upon the effects of conspiring and opposite forces, and the effects of forces, acting obliquely, or resulting from the composition and resolution of forces, demonstrated experimentally, exhibit concurring proofs of its truth; for were motion, and change of motion, not proportional to, and in the direction of, the force impressed, it may be proved geometrically that these effects would not take place. The conclusion is therefore inevitable, and, being thus established by uniform experience, may be deemed a fixed principle in nature, and applied not only to the explication of those particular experiments from whence it was deduced, but of all other natural phenomena. This law is also a proof of the inertia of matter, or, that being established, a corollary from it; for if matter can produce no change in itself, it must move in the direction of the force impressed; and, since the vis inertiæ of a body is the same, whether moving or quiescent, it is evident that the genesis, encrease, and decrease, of velocity equal to V in any body, require exactly the same force. If the force of collision, or impulse of any force such as gravity, be capable of generating a velocity, equal to V, in the body A, it will in the same time generate an encrease, or decrease, of velocity equal to V in that body, according as it conspires with, or opposes its motion.

178. Cor. 1. The velocity generated by one single impulse equal to nF, n being a number, is equivalent to that generated by n successive

ceſſive conſpiring impulſes, each of which is equal to F; and conſequently the velocity communicated to a body by any number of forces acting in the ſame direction is the ſame, whether they act together or ſeparately; becauſe the geneſis, and encreaſe, of a velocity $= V$, require exactly the ſame force.

179. Cor. 2. The velocity generated, in any body, by any number of unequal conſpiring impulſes X, Y, Z, &c. is as their ſum; for let x, y, z, &c. be the velocities reſpectively generated by them in any body A, and (177) $X : Y :: x : y$, and $X : X + Y :: x : x + y$, and $X : X + Y + Z :: x : x + y + z$, &c. And the velocity, communicated to any body by a force $= F$, in a given time, is as the magnitude of that force; for ſuppoſing this force to act by impulſes, whoſe magnitudes are X, Y, Z, &c., $F = X + Y + Z$, &c.

180. Cor. 3. The magnitudes of any forces are therefore as the ſpaces uniformly deſcribed in a given time, by the velocity which they communicate to the ſame body (106).

181. Cor. 4. The velocity generated in a body in the ſame time by two forces X, Y, acting in oppoſite directions, is as their difference; for (177) $X : Y :: x : y$ and $X : X - Y :: x : x - y$.

THIRD LAW OF MOTION.

182. *Action and reaction are always equal and oppoſite; or the mutual actions of two bodies are always equal, and in oppoſite directions.*

This is another rule obſervable in all the motions of nature, reſulting, like the two firſt, from the inertia of matter. Were matter diveſted of this property, motion would be communicable without reſiſtance, and conſequently without effecting any change in the force communicating it. The action of all forces, whether operating by the viſible impact of one body upon another, or the

 inviſible

invifible agency of gravity, magnetifm, &c. confifts in producing preffure and motion; and reaction in fupporting this preffure, or refifting the production of motion. And thefe this law afferts to be equal, when eftimated in oppofite directions; which, being proved experimentally, like the two firft, in all communications of motion with which we are acquainted, may be efteemed a general principle pervading the whole material fyftem.

In the communication of preffure upon any immoveable plane, whether arifing from the protrufion, gravity, or force of impact, of a body; the meaning of the law is, that the refiftance of the plane, and an oppofite force equal to that producing the preffure, have precifely the fame effect, as they only deftroy the force of protrufion, weight, or impact. In the communication of motion by vifible impact, the meaning of the law is, that action is mutual, equal and oppofite, or that the quantities of motion loft and gained, which meafure this action and reaction, are equal when eftimated in oppofite directions. It is always fuppofed, that the quantity of motion is eftimated by the product of the quantity of matter and velocity. No idea can be formed of the lofs of motion, except by communication; and that the quantity, loft by the impinging body, is gained by the ftruck body, appears from innumerable experiments upon the collifion of bodies. If a perfectly hard, or a foft body, A, moving with the velocity V, impinge upon an equal unelaftic body B, they will move together, after impact, with a velocity equal to $\frac{V}{2}$; and if A, moving with the velocity $V+u$, impinge upon B, moving in an oppofite direction with the velocity u, they will move together, after impact, with a velocity equal to $\frac{V}{2}$. And whatever be the magnitudes of A and B, it is the invariable refult of experiments, that the quantity of matter in A, multiplied into the velocity loft by it, is equal to the quantity of matter in B, multiplied into the velocity which it gains by impact; or if l and g reprefent the velocities loft by A and gained by B refpectively, $A \times l = B \times g$. When A and B are perfectly elaftic, the velocities loft and gained are doubled, and the law ftill obtains, $A \times 2l$ being equal to $B \times 2g$. If motion be communicated to any body A, by any

any force *X* protruding, pulling, &c. the meaning of the law is, that the reaction, or resistance, of *A* destroys such a part of *X*'s force, as would be destroyed by a similar force, capable of generating that motion in *A*, acting in opposition to *X*. In the communication of motion by unknown means, as by gravity, magnetism, repulsion, &c. the law asserts that the body, attracting or repelling, moves in an opposite direction to that of the body attracted or repelled, and with an equal quantity of motion. The attraction between the earth, and any body upon its surface, is mutual and equal; for, were it not, a rectilineal motion would ensue from the stronger attraction, which is contrary to experience. And, since gravity is an innate principle, it, and its effects will remain the same when that body is detached from the earth, and consequently their attractions continue to be mutual and equal, and they will meet with equal moments. If two magnets *A* and *B*, whose weights are unequal, be placed upon two pieces of wood, floating in water within the reach of attraction, they will meet with velocities inversely as their quantities of matter; and, if a reed be inserted between them to prevent their junction, they will be quiescent, which they would not be, were their attractions unequal. When the weight of *A* is equal to 2, 3, 4, &c. times that of *B*, its velocity, in an opposite direction, is equal to $\frac{1}{2}$, $\frac{1}{3}$, &c. of *B*'s, the products of their weights and velocities being always found to be equal. And if *A* is repelled, *B* is also found to be repelled in an opposite direction, and their velocities are always inversely as their weights. This law, being found to obtain in all actions of bodies within the reach of experiments, is inferred to obtain universally through the material system.

183. Cor. In the impact of bodies therefore the quantity of motion, estimated in the same direction, is the same before and after impact. If *A*, moving with a velocity equal to *a*, impinge upon *B*, moving in the same, or an opposite, direction, with a velocity equal to *b*, the sum of their moments after impact, is equal to $Aa + Bb$, or $Aa - Bb$, according as they move in the same, or opposite directions; for, if they move in the same direction, the en-

creafe

creafe of moment communicated to *B* is deftroyed in *A*, and their fum continues the fame; and, if they move in oppofite directions, the leaft moment, and a part of the other body's moment equal to it, are deftroyed, fo that their fum after impact continues to be equal to *Aa* — *Bb*, the fame as before impact.

COMPOSITION AND RESOLUTION OF FORCES.

184. * The fame motion may be communicated to a body *B* by a fingle force *Z*, and any number of confpiring forces, into which it is faid to be refolvable, and they are faid to be compounded into

PLATE III. FIG. XXVII. *Z*; thus the body *B* may defcribe the line *BP*, in the fame time, by a force *Z* acting in that direction, or by two, three, &c. forces *X*, *Y*, &c. inclined to it. For it is evident, that *B*, acted upon at the fame time, by two forces *X*, *Y*, in directions making an angle, which ceafe to act after the body has left the point *B*, will move in fome intermediate rectilineal direction, and this, by changing the inclination and magnitude of *X* and *Y*, which are variable without limit, may be any intermediate line whatever *BP*. The force *Z* is faid to be refolvable into the two, *X* and *Y*, producing the fame effect with it; and, vice versâ, *X* and *Y* are faid to be compounded into one, *Z*, which produces the fame effect with them.

185. PROP. *A body* B *urged at the fame time, by two forces* x, y, *whofe action ceafes when the body has left the point* B, *and whofe mag-*

FIG. XXVIII. *nitudes and directions are as two right lines* BM, BN, *making any angle, will move as if it were impelled by one force* z, *whofe magnitude and direction are as* BP, *the diagonal of a parallelogram, whofe fides are* BM *and* BN.†

DEM.

* Newt. Princip. p. 14. Mufchen. Ch. X. Helfham, Lect. IV. Maclaurin's Newt. Ch. II. Gravef. Lect. I. Ch. XIII. Hermannus, Lect. I. Ch. II. Varignon. Tom. I. Sect. I.

† The magnitudes of forces, whether animal exertions as percuffions, protrufions, &c. or philofophic powers, as gravity, magnetifm, &c. can only be compared by meafuring their effects, which are fuppofed to be produced uniformly in the fame time. And, in this fuppofition, it is evident, that the conclufions will be the fame, whether the forces *X*, *Y* be impulfes, or the fame number of impulfes always equal to themfelves.

DEM. The accefs of B to, or recefs from, MP, is not affected by the action of y; nor its accefs to, or recefs from, NP, by the action of x (fecond law of motion): therefore they will carry B to MP and NP refpectively, in the fame time, whether they act together or feparately. But if they act together, B, being carried to both MP and NP, muft be found at P their interfection, and defcribe BP uniformly (firft law of motion); which is the magnitude and direction of z (180). Q. E. D.

Otherwife,

Let BM, moving parallel to NP uniformly, arrive at NP in the fame time that B defcribes BM uniformly, which confequently at the end of that time will arrive at P. And if npm, be any new pofition of BM, and the body at p; $Bn : np$ :: velocity of BM : velocity of the body :: $BN : BM$ (106); therefore p is always in the diagonal BP; Bp encreafes uniformly becaufe Bn does; and BP is the magnitude and direction of z (180). Q. E. D.*

186. Cor. 1. A force z, producing the fame effect with two x and y, which are as BM, BN, is as twice Bs, fuppofing s to be the bifection of BP.

187. Cor. 2. If any two forces act upon a body, in directions which make an angle, it cannot be quiefcent.

188. Cor. 3. A force reprefented by a fide of a triangle BP, may be fubftituted for two reprefented by the remaining fides BM, MP; and it is faid to be equivalent to them, becaufe they produce

* This propofition, which is deduced from the fecond law of motion, may be demonftrated not inelegantly from the third. The forces x and y, urging the body B obliquely, partly oppofe each other, and fince, from the third law of motion, the oppofite parts are equal, the path of B will be equally diftant from any cotemporary pofitions of it, M and N, when they act feparately; which therefore is Bs, s being the bifection of BP. And becaufe each of the forces x, y carry B through Bs, they will both carry it through $2Bs$ or BP, the diagonal of a parallelogram, whofe fides are BM and BN.

duce the same effect, or make a body describe BP in the same time. And these three forces BM, BP, BN are in the same plane (EUC. B. XI. P. II.)

FIG. XXVIII. 189. Cor. 4. A force BM is equivalent to BA and AM; and BN to BC and CN; and therefore these forces BA, AM, BC, CN, are equivalent to, and may be substituted for, the force BP. And any number of forces, whose directions are in the same plane, may be reduced to one in that plane producing the same effect with them.

FIG. XXIX. 190. Cor. 5. A force AH is equivalent to two represented by AB and AG; and AD to AH and AF; therefore any number of forces AG, AB, AF, &c. in different planes, may be reduced to one AD producing the same effect with them.

FIG. XXVIII. XXIX. 191. Cor. 6. The converse of this is true, and one force BP, or AD, may be resolved into any number, either in the same, or different planes, producing the same effect with it.

FIG. XXX. 192. Cor. 7, The velocity generated by z : velocity generated by $x+y :: BP : BM + BN$; and the quantity of motion is therefore diminished by composition, and encreased by resolution; but, when estimated in any given direction BL, it remains invariable. For let BP, BM, BN, be each resolved into two, one in the direction BL, and the other perpendicular to it; and from similar triangles

$$Py : Dy :: yL : yn,$$

and comp. $PD : nL :: Dy : yn :: BE : Bm$,

therefore $BL = Bm + Bn$.

193. Cor. 8. The same force z, and consequently the same velocity, may be generated by an infinite number of pairs of forces, because

becaufe the fame right line may be the diagonal of an infinite number of parallelograms.

194. Cor. 9. The parts, of the two forces, directly oppofed to each other, are to the parts, acting in the diagonal, as $Mm + Nn$ to $Bm \mp Bn$, according as the angle MBN is not greater, or greater than a right angle. If x and y are given, and the angle MBN be fuppofed variable, and become equal to two right angles, then N will come to N', P to P', s to s' (s always bifecting BP and MN) and $2Bs'$, or the confpiring parts of x and y, $= BP' = BM - BN$; if this angle vanifh, then N will come to N'', P to P'', s to s'', and $2Bs''$, or the confpiring parts of x and y, $= BP'' = BM + BN$. FIG. XXXI. FIG. XXXII.

195. Cor. 10. If BP and the angle BMP be given, the curve line paffing through M, in different pofitions of M, is a circular arc fubtended by BP; if BP and $BM + MP$ be given, this curve line is an ellipfe, whofe focal diftance is BP; and if BP and the difference of BM and MP be given, the curve is an hyperbola and BP its focal diftance. PLATE IV. FIG. XXXIII.

196. PROP. *Any three forces* x, y, z, *whofe directions are* Bx, By, BN *in the fame plane, acting upon a body* B *without producing motion, are to each other as the three fides of a triangle refpectively parallel to their directions.* FIG. XXXIV.

DEM. Let BN be the magnitude of z, and be refolved into two, in the directions yB, xB; viz. BM, BP; which muft be refpectively equal to x and y, becaufe in equilibrio with them, and acting in the fame directions; therefore x, y, z, are to each other as BM, BP (MN), BN refpectively. Q. E. D.

197. Cor. 1. x, y, z are to each other as the three fides of a triangle refpectively perpendicular, or equally inclined to their directions; for this is fimilar to the former.

198. Cor. 2. Any two of these forces are to each other inversely as the sines of the angles, which their directions make with the direction of the third. For,

$$z : y :: BN : BP :: \text{sin.} \angle BPN \text{ or } PBM : \text{sin.} \angle BNP \text{ or } NBM,$$
$$z : x :: BN : BM :: \text{sin.} \angle BMN \text{ or } MBP : \text{sin.} \angle MNB \text{ or } NBP,$$
$$\text{and } x : y :: BM : MN :: \text{sin.} \angle BNM \text{ or } NBP : \text{sin.} \angle MBN.$$

199. Cor. 3. The angle xBy, and magnitude of x and y, being given, the magnitude and direction of z may be found; for BM, MN and the angle BMN, the supplement to the angle xBy, being known, BN, and the angles MBN and MNB or NBP, may be found by trigonometry.

FIG. XXXV. 200. Cor. 4. Any number of forces in the same plane, acting upon a body B which remains quiescent, may be reduced to two equal and opposite: if BM, BN, BD, act upon B, they are equivalent to BP and BC in opposite directions, which must be equal, because B is quiescent.

201. Cor. 5. Any number of forces in different planes, urging a body without producing motion, may be reduced to two in the same plane, equal and opposite. If BD and DC be elevated above the plane BNM, and PBC be the common intersection of the planes BMN and BDC; BP, which is equivalent to BM and BN, must be equal to BC, which is equivalent to BD and DC, because they are opposite and destroy each other.

202. Cor. 6. If more than three forces BM, BN, and BD, each of which is combined from two, act in different planes, upon a body B without producing motion, and are estimated in any direction BL by letting fall parallel lines upon it, the forces resulting will produce no motion in that direction; for the parts of BM, BN, resulting from this resolution, which are in the direction BL, are equal and opposite to the part of BD reduced to the same direction, and the sums of the opposite parts, on each side of BL, must also be equal, because B is supposed to be quiescent.

203. Cor.

203. Cor. 7. If the forces *BM*, *BN*, *BC*, which act upon a body without producing motion, be reduced to any plane *FH*, by resolving each into two, one perpendicular to it, and the other parallel to the lines joining the points, where the perpendiculars meet it, the forces resulting will produce no motion in that plane. For *BP*, being taken equal and opposite to *BC*, will be equivalent to *BN* and *BM*, and *BC*, *BP* will be projected into equal straight lines *bc*, *bp*; but *BN*, *MP*, being equal and parallel, are projected into equal and parallel lines *bn*, *mp*, and the figure *bmpn*, is a parallelogram. And *bm*, *bn*, being equivalent to *bp*, will be equivalent to its equal and opposite *bc*. FIG. XXXVI.

204. PROP. *If a body be acted upon by two similar variable forces, in directions parallel to* BP, BQ, *making any angle, which act when the body hath left the point* B, *it will describe a right line.* FIG. XXXVII.

DEM. Let the forces act by impulses at the beginning of equal particles of time, and let *BD*, *DE*, *EF*, and *BG*, *GH*, *HI* be the relative magnitudes of corresponding impulses. By the action of the two first impulses *BD*, *BG*, the body will move in the direction *BK* (185); by the two next *DE*, *GH*, it will describe, in the same direction, *KL*, &c.; and, because the forces are similar, *BD* : *DE* :: *BG* : *GH*, and componendo *BD* : *BG* :: *BE* : *BH*, and consequently *BKL* is a right line. Q. E. D.

205. Cor. 1. Since this demonstration does not depend upon the magnitude of the particles of time, into which the whole time is supposed to be divided, it will obtain when these particles are evanescent and the forces act incessantly.

206. Cor. 2. If the forces be constant, the velocity in *BM* is uniformly accelerated; for, in this supposition, *BD*, *DE*, &c. are equal, and *BD* : *DE* :: *BK* : *KL*; but *BK*, *KL* which are therefore equal, &c. measure the increments of velocity communicated at *B* and *K* (106).

 207. Cor.

207. Cor. 3. Every body moving in a curve line, muſt be acted upon by, at leaſt, two diſſimilar forces.

208. Cor. 4. The whole force in the direction *BP* is the ſum of the impulſes *BD*, *DE*, *EF*, &c. and the force in the direction *BM*, is the ſum of the impulſes *BK*, *KL*, *LM*, &c. and theſe forces are ſimilar; for *BD*, *DE*, &c. meaſure the intenſity of the force in the direction *BP* at the points *B* and *D*; and *BK*, *KL*, &c. meaſure the intenſity of the force in the direction *BM*, at *B* and *K*, and *BD* : *DE* :: *BK* : *KL*.

FIG. XXXVII. 209. PROP. 86. *If a body be urged at the ſame time by two conſtant, or ſimilar variable, forces* x, y, *whoſe magnitudes and directions are the two ſides of a parallelogram* BF, BI, *it will move in the ſame manner as if it were urged by a conſtant or ſimilar variable force* z, *repreſented by the diagonal.*

DEM. The impulſe *BK* is equivalent to *BD* and *BG*; *KL* to *DE* and *GH*; *LM* to *EF* and *HI*, &c. conſequently the ſum of theſe impulſes *BM* is equivalent to *BF* and *BI*. Q. E. D.

210. Cor. 1. A force repreſented by one ſide of a triangle *BM*, may be ſubſtituted for two ſimilar forces repreſented by the remaining ſides *BF*, *FM*.

211. Cor. 2. A force repreſented by *BM* may be reduced into any number of forces *BR*, *RS*, *SM*, ſimilar to it, and vice verſâ.

212. Cor. 3. Three ſimilar variable forces, urging a body without producing motion, are to one another as the three ſides of a triangle, parallel or perpendicular to their directions; and they are to each other inverſely as the ſines of the angles, which their directions

directions make with the direction of the third (198); and any number of similar forces in the same or different planes, in equilibrio, may be reduced to two, in the same plane, equal and opposite.

SCHOLIUM.

213. Prop. (204) may be demonstrated per saltum, though perhaps less intelligibly, by the following process. Let a body B be urged, at the same time, by any forces x and y, in the directions BM, BN, which carry it respectively through Bm, Bn and BM, BN, in equal times; and it is evident that, at the end of those times, B will be in d and D. If x and y act only at the point B, or if $Bm : BM :: Bn : BN$, the path of B is the diagonal; if they act at B, m, n, M, N, &c. and Bm be to BM as the times of description, and Bn be to BN as the squares, cubes, &c. of those times; $Bn : BN :: Bm^2 : BM^2$ or $Bm^3 : BM^3$, &c.; and the path of B is a parabola, &c. FIG. XXXVIII.

214. PROP. 87. *A body* B *impelled by two forces tending to two equidistant points,* Q, R, *which at equal distances are equal, and at unequal distances are variable according to any law whatever, will describe a right line.* FIG. XXXIX.

DEM. Let these forces act by impulses at the beginning of equal particles of time, and BE, BF being the magnitude of the two first, will make B describe BD, which produced bisects the base QR; and DL, DM, the two next, will make it describe DN, bisecting the base QR: therefore BN is a right line; and the demonstration obtains when the particles of time are evanescent, and the action of the forces incessant. Q. E. D.

215. Cor. The direction of the combined attraction of all the particles, composing a sphere, passes through the center: for let it be divided into thin laminæ parallel and contiguous to each other, and

and let *Q*, *R*, be any corpuſcules in a lamina, equidiſtant from *PB* paſſing through the center, and what has been proved of theſe two may be proved of every two equidiſtant from *P*, which compoſe the whole lamina; and the ſame may be proved of every lamina, and conſequently of the whole ſphere.

SCHOLIUM.

216. Three forces, acting upon a body without producing motion, are to each other as the three ſides of a triangle parallel to their direction, which conſequently meet in the ſame point of the body, and are in the ſame plane (EUC. B. XI. P. II.). This may be evinced otherwiſe; for ſince a body, acted upon by any number of forces, cannot be at reſt, unleſs they oppoſe each other, and the oppoſite parts be equal, and in the ſame direction; it is clear that
PLATE V. FIG. XL. a plane *AB* may paſs through *P*, acted upon by x, y, z in different planes, ſo that they ſhall be on the ſame ſide of *AB*, and therefore, as they partly conſpire, they muſt communicate motion to the point *P*, and to the whole body. And if any two of theſe forces, x, y,
FIG. XLI. act againſt the ſame point p of the body *AB*, and x acts againſt q; x and y will communicate motion to p (187) and z to q, and *AB* cannot therefore be quieſcent. If the directions of x and y meet
FIG. XLII. in a point p without the body, and z act at the point q, *AB* cannot be at reſt; becauſe x and y have the ſame effect with, and are equivalent to, ſome force in an intermediate direction pd, which is not oppoſite to the direction of z, and cannot therefore deſtroy it. It is ſuppoſed, that the directions of theſe forces are inclined to each other; for, if they be parallel, *BA* may be quieſcent, when they act upon different points.

FIG. XLIII. 217. PROP. *If a body* B *be acted upon by any number of forces, at the ſame time, whoſe magnitudes and directions are* BC, BA, BF, BH, *&c. and from the biſection,* m, *of the diagonal* BD *of a parallelogram, whoſe ſides are* BC, BA, *a right line be drawn to the extremity* F, *of* BF, *cutting the diagonal* BE, *of a parallelogram whoſe ſides are* BD, BF, *in* n, *and from* n *a right line be drawn to the extremity,* H, *of* BH, *cutting*

cutting the diagonal BG *of a parallelogram, whose sides are* BE, BH, *in* p, Bp : BG :: *unity* : *number of forces.*

DEM. The triangles Bpn and GpH, Bnm and nEF, are similar, and consequently $Bp : pG :: Bn : GH(Bn + nE) :: Bm : Bm + EF(Bm + BD$ or $3Bm)$; therefore $Bp : BG :: Bn : 4Bm :: 1 : 4$. The process is similar when there are more forces. Q. E. D.

218. Cor. 1. It is evident, from the construction of the figure, and similar triangles, that $CA : Cm :: 2 : 1$,
$mF : mn :: 3 : 1$,
$nH : np :: 4 : 1$; and if the number of forces were equal to any number, m, nH would be to $np :: m : 1$.

219. Cor. 2. The force BG, resulting from the action of any number of forces BC, BA, BF, BH, &c. either in the same, or a different, plane, is to their sum as $BG : BC + BA + BF + BH$, &c. and if the magnitudes and directions of these forces be given, the magnitude and direction of BG may be found; for CB, BA, and the $\angle CBA$ being given, BD, and the $\angle$s CBD, DBA, are known; and DB, BF, and the $\angle DBF$ (which may be found) being given, BE is also given, and in the same manner any other diagonals may be investigated.

220. Cor. 3. If a body therefore be acted upon by any number of forces BA, BC, BD, BE, BF, BG, the directions of the forces resulting from the action of two, three, four, &c. are found by joining AC, bisecting it in m, and taking $mD : mn :: 3 : 1$, $nE : nP :: 4 : 1$, $pF : pq :: 5 : 1$, $qG : qr :: 6 : 1$, and the directions are Bm, Bn, Bp, Bq, Br. Their magnitudes are $2Bm$, $3Bn$, $4Bp$, $5Bq$, $6Br$, &c. FIG. XLIV.

221. Cor.

FIG. XLV. 221. Cor. 4. The same force BP may be generated by the action of 1, 2, 3, &c. forces, either in the same, or different, planes, whose directions and quantities are variable in infinitum: for let it result from the combined action of four forces, and taking $BP : Bq :: 4 : 1$, through q draw any line whatever qA, and take $qA : qP :: 3 : 1$; through p draw any line pC, and let pC be to $pn :: 2 : 1$; and through n draw any line nD, and taking nE equal to it; the forces BA, BC, BD, BE, are equivalent to BP. The lines qA, pC, nD, &c. may be drawn in any directions, in any different planes, and of any different lengths.

FIG. XLVI. 222. PROP. *The directions of three forces* x, y, z, *acting upon a body* B *without producing motion, will meet in the center of gravity of a triangle, whose distances from the angular points, are as the magnitudes of the forces.*

DEM. Produce AB, and taking $BE = AB = z$, complete the parallelogram $BDEC$ by drawing from E lines parallel to BC, BD; BE is equivalent to BC and BD, and equal to z, which is equivalent to x and y, which must therefore be equal to BC, and BD, respectively; but $BG = \frac{BE}{2} = \frac{AB}{2}$, and B is therefore the center of gravity of the triangle. Q. E. D.

Otherwise:

Let BA, BC, BD, be the magnitudes of z, x, y, respectively, and joining DC and bisecting it in G, and joining GA, $AB = 2BG$ (217). Q. E. D.

FIG. XLVII. 223. PROP. *If a body* B, *be acted upon by four forces, in different planes,* x, y, z, w, *or* BA, BC, BE, BF, *and remain at rest, they are to each other as the three sides and diagonal of a parallelopiped parallel to their directions respectively.*

DEM.

DEM. The forces BA, BC are equivalent to BI, and BI, BE to BD, which muſt be equal and oppoſite to w or BF, becauſe B is quieſcent. Q. E. D.

224. Cor. An infinite number of parallelopipeds may be deſcribed, whoſe ſides and diagonal are the ſame: for let x, y, z, w, be invariable, and taking BC, BA, equal reſpectively to y and x, making any angle whatever ABC, and BD, ID in a different plane, reſpectively equal to w and z, draw BE parallel to ID, and DE to BI. Biſect BI in H, and join EH, interſecting DB in G; and becauſe the triangles BGH, EGD are ſimilar, $EG:GH::ED$ $(2BH):BH::2:1$; therefore theſe two forces combined with BE produce a force paſſing through G (217); and becauſe $DG:GB::EG:GH::2:1$ and $DB:GB::3:1$, the force w is equal to DB. But the angle CBA is infinitely variable, and BD, FD may be drawn in any planes whatever. FIG. XLVII.

225. PROP. *If four forces,* x, y, z, w, *which are as* BA, BD, BE, BF *reſpectively, act in different planes, upon the body* B *without producing motion,* B *is the center of gravity of a triangular pyramid whoſe baſe is* EAD *and vertex* F. FIG. XLVIII.

DEM. Join AD, and biſect it in m, and, taking mn to $nE::1:2$, n will be the center of gravity of the triangle EAD, and nB is the direction of the force reſulting from x, y, z combined, and its magnitude is equal to $3\times Bn$, which muſt be equal and oppoſite to w or BF, becauſe B is quieſcent: BF is therefore equal to $3Bn$, and B conſequently the center of gravity of a pyramid, whoſe baſe is EAD and vertex F. Q. E. D.

226. Cor. 1. It is evident that $BF=3Bn$, becauſe the force reſulting from $x+y+z$ is to thoſe forces as $3Bn:BA+BD+BE$ (220) and $w:x+y+z::BF:BA+BD+BE$ (hypoth.) therefore $w=BF=3Bn$.

M Cor.

227. Cor. 2. The force resulting from the combined action of *BA*, *BD*, *BE* passes through the center of gravity of a triangle, which is formed by joining *A*, *E*, *D*; and the direction of a force, equivalent to them, passes through the same point.

228. Cor. 3. An indefinite number of pyramids may be described, having the distance of their angular points from the center of FIG. XLIX. gravity always the same; for let *BA*, *BD*, *BE*, *BF*, be invariable, and making the angle *ABD* of any magnitude whatever, draw *CD*, *BD*, in any plane whatever, equal respectively to *BE*, *BF*, and complete the parallelogram *BCDE*, and the process is the same as that in (224).

CHAP.

CHAP. V.

OF ATTRACTION.

229. DEF. *THAT power or principle in nature, by whofe influence bodies, or the conftituent parts of bodies, accede, or have a tendency to accede, to each other, without any fenfible material impulfe, is called attraction.*

230. DEF. *That natural power, by whofe influence different bodies, or different parts of the fame body, recede, or have a tendency to recede, from each other, without any fenfible material effect, is repulfion.*

When the changes of motion in the body *A* are uniformly obferved to depend upon the fituation and diftance of another body *B*, a connexion is underftood to obtain between them, expreffing fome quality or mechanical affection of matter, fuch as gravity, cohefion, elafticity, magnetifm, electricity, which appears to refide in matter, and by whofe agency this change of motion is conceived to be produced. If the direction of *A*'s motion, or tendency to motion, be towards *B*, it is faid to be attracted by *B*; and if it be from *B*, it is faid to be repelled by *B*.

231. That there are in nature motions and tendencies to motion, conatus accedendi & recedendi, both in aggregate bodies, and in their minute conftituent particles, without any fenfible caufe, and indeed inexplicable by any known properties of matter, is unqueftionably certain. Thefe demonftrate a fource of motion diftinct from, and repugnant to, any material impulfe with which

we are acquainted, which is denominated attraction, a term indifferently applied to the mechanical affection and its effects, as all inexplicable powers are only measurable by their effects. These effects being found by observation to be different at different distances, the intensity of the power producing them is inferred to be variable, and the law of variation is discovered by an actual mensuration of these effects at different known distances. By this process, these material affections, though intelligible only in their effects, are considered as quantities capable of mathematical comparison, and their ratios are compared with those of lines and numbers. Attraction is usually divided into two kinds: *first*, that which operates at sensible distances, as gravity, electricity, magnetism; and, *secondly*, that, whose effects are limited to almost insensible distances, which is called the attraction of cohesion.

ATTRACTION OF GRAVITY.

232. The existence of this species of attraction is evidenced by uniform experience. Matter when supported is heavy, and, the support being removed, accedes towards the earth; when projected obliquely, it deviates from the line of projection, and vibrates when suspended by a string inclined to the earth's surface. And the direction of this motion of matter, and tendency to motion, is invariably the same, towards the center of the earth nearly. A mechanical affection is justly inferred from these phenomena (230), which is conceived to produce this motion and tendency to motion, and is promiscuously called gravity, gravitation, or the attraction of gravity. But more correctly the tendency of a body towards the earth, which is measured by the velocity acquired in a given time, is called the accelerating force of gravity; the weight or moment of a body, which is measured by the product of the quantity of matter and the accelerating force, is called its vis motrix; and the power conceived to be in the earth, which produces this weight and tendency towards it, is called the absolute force or attraction of gravity; and a body influenced by it, is said to be attracted by, or gravitate towards, the earth. All hard and soft bodies

bodies are ſubject to this attractive power, and have weight; air, water, and other fluids, and the vapours ariſing from them, and all odorous ſubſtances, and exhalations from all terreſtrial bodies, gravitate towards the earth, as they may be collected in a balance and actually weighed; and the weight of theſe laſt ſubſtances is alſo rightly inferred from the decrement of weight ſuſtained by the evaporable matter. Even fire and light, and the moſt volatile vapours, ſeem to be under the controul of this univerſal principle, and no matter in the vicinity of the earth, acceſſible to experiments, hath yet been diſcovered to be uninfluenced by it, and conſequently all matter may be conſidered as gravitating towards the earth, till future ſatisfactory experiments induce a different opinion. This attractive power is not confined to bodies in the vicinity of the earth, but extends alſo to the moon; for the moon deſcribes equal areas in equal times about the earth's center, and is therefore urged by a force directed to that center (NEWT. Sect. II. P. II.) and coincident in direction with gravity: and it appears, from aſtronomical obſervations, that it is equal in quantity to the force of gravity at that diſtance, and they are conſequently the ſame power. And becauſe the revolutions of their ſatellites round jupiter and ſaturn, and of the primary planets round the ſun, are phenomena ſimilar to that of the moon round the earth, both being acted upon by forces directed to their reſpective centers, which vary according to the ſame law; the ſatellites therefore gravitate towards their primaries, and theſe towards the ſun (7). The primary planets gravitate towards each other; for jupiter and ſaturn, in conjunction or at their leaſt diſtance, are diſcovered to produce very ſenſible irregularities in each other's motions; and the motions of their ſatellites are ſaid alſo to be ſubject to irregularities, which are ſenſible at their leaſt diſtance where their action is greateſt. All the great bodies, compoſing the ſolar ſyſtem, are therefore impreſſed with this principle of attraction; which, being attached to the whole of any body, muſt pervade every conſtituent portion; and the minuteſt particles of matter gravitate, though perhaps inſenſibly, towards each other. The operation of this principle between two ſmall bodies, at the ſurface of the earth, is only rendered inſenſible from the predominant influence of the earth;

earth; for a pendulous body was obſerved to be conſiderably deflected from its vertical ſituation by the attraction of the mountain Chimborazo in Peru; and, by ſome ingenious experiments made with great preciſion, Dr. Maſkelyne aſcertained the quantity of attraction of the mountain Schehallien in Scotland.* Every mountain therefore, and every leſs portion of the earth, poſſeſſes this quality; which might have been inferred from (215). For if a particle of matter be attracted by every part of a ſphere of matter, equally at equal diſtances, the direction of the combined attraction will paſs through its center and *v.v*; but it appears, from experience, that all bodies deſcend in directions perpendicular to the earth's ſurface, and, becauſe the earth is nearly ſpherical, the whole force, producing this deſcent, is directed nearly to the center, and is conſequently combined of the force of every particle.

233. PROP. *The accelerating force of gravity at equal diſtances from the earth's center, is the ſame in all bodies, whether quieſcent or moving, and whatever be their magnitude, figure, denſity.*

DEM. This propoſition is only demonſtrable from experiments. Two equal wooden boxes, ſuſpended by threads of eleven feet in length, one of which was filled with wood, and an equal weight of gold fixed in the center of oſcillation of the other, were diſcovered by Sir I. Newton, to perform all equal vibrations in the ſame time; and numberleſs experiments demonſtrate that all bodies,

* If a mountain do poſſeſs an attractive power, a body ſuſpended near it will be deflected from its vertical poſition, and point to a falſe zenith, and the arc, meaſuring the diſtance of this from the true, is the meaſure of the mountain's attraction. In an obſervation on the north ſide of Schehallien, the plumb line, being deflected towards the mountain, pointed too much to the ſouth, and gave the diſtance of a ſtar from the zenith too much to the north; and from an obſervation made on the north ſide, the diſtance was too much to the ſouth. The difference of the latitude of the two ſtations, collected by theſe obſervations, muſt be greater than it really is, by the ſum of the arcs meaſuring the deflections from the two zeniths. From obſervations of ten ſtars near the zenith, Dr. Maſkelyne found the difference of the latitude of the two ſtations to be 54″6; and, from a menſuration of their diſtance, it was only 43″: the difference of theſe is 11″6, and the half of it 5″8, meaſures the attraction of the mountain. This experiment is moſt convincing, and deciſive in ſupport of the univerſality of gravitation.

dies, however different their magnitude, figure, and denfity, defcend in an exhaufted receiver, whatever be its height, exactly in the fame time; and confequently the tendency of bodies towards the earth, or their accelerating force, is the fame whether quiefcent or moving, &c.* Q. E. D.

234. PROP. *The weight of bodies, at the furface of the earth, are proportional to their quantities of matter.*

DEM. The weight of a body is evidently as the number of equal particles or quantity of matter contained in it, multiplied into the tendency towards the earth, or accelerating force of each, and this being given (233) varies as the quantity of matter. Q. E. D.

Otherwife:

The weight of any two bodies A and B, or the forces producing them, are clearly proportional to their vis inertiæ; for if the vis inertiæ of A were to that of B, as 1, 2, 3, &c. : 1, it is evident that the weights or forces, producing the fame velocities in A and B in the fame time, will be as 1, 2, 3, &c, : 1; and confequently the weight, being as the vis inertiæ, will be as the quantity of matter (166). Q. E. D.

235. Cor. If the accelerating force of gravity were encreafed in any ratio, the weight of a given body would be encreafed in the fame ratio. Subftituting therefore W, Q, F, for the weight, quantity of matter, and force of gravity, refpectively, and fuppofing them to be variable; W will be as $Q \times F$; F as $\frac{W}{Q}$; Q as $\frac{W}{F}$, and, if W be given, F and Q are inverfely as each other.

SCHO-

* All bodies, whether great or fmall, denfe or rare, acquire a velocity in falling 1″, which would carry them uniformly through 32.2 feet in 1″, and an encreafe of velocity, equal to this, is found to be added in every fucceeding fecond of time. e

SCHOLIUM.*

236. In the ſame place, and at the ſame diſtance from the æquator, bodies are found to deſcend through the ſame ſpace in the ſame time always; but, the force of gravity being diminiſhed unequally by the diurnal rotation of the earth, they do not, and ought not to, deſcend through equal ſpaces, or perform equal vibrations, in the ſame time, at different diſtances from the æquator. However, this unequal diminution of gravity, ariſing from the different centrifugal forces, does not afford a ſolution for the different ſpaces fallen through; and there muſt be ſome other cauſes of inequality, which are probably the difference of diſtance from the center, and different ſpecies of matter near the places of obſervation.

237. PROP. *The force of gravity varies as the ſquare of the diſtance from the center of the earth inverſely.*

PLATE VI. FIG. LI.

DEM. Gravity acts in right lines, and, whatever be the cauſe, is equally diffuſed over every point of the ſurface $ABDC$, equidiſtant from the center of the earth S; and that ſame influence is alſo equally diffuſed over the ſurface $abdc$, ſimilar to the former and ſimilarly ſituated. Taking $almn$ equal and ſimilar to $ABDC$, the influence of gravity upon $almn$: influence upon $abdc$ or $ABDC :: almn\,(ABDC) : abdc :: SA^2 : Sa^2$. Q. E. D.

Another

PLATE VI. FIG. L.

* PROP. *Suppoſing the earth to be ſpherical, the diminution of gravity, ariſing from the centrifugal force, varies as the ſquare of the coſine of latitude.*

DEM. Let Pp be the earth's axis, and $ÆQ$ its æquator, and the centrifugal force at Q: centrifugal force at $A :: QC : AD$; the centrifugal force at A: that part oppoſite to gravity : $AN : AL$ (ſuppoſing NL to be perpendicular to AC) :: $AC : AD$; and conſequently the centrifugal force, or diminution of gravity at Q: the diminution of gravity at $A :: QC^2 : AD^2$. This diminution of gravity varies therefore as AD^2, or as the ſquare of the coſine of the place's diſtance from Q. Q. E. D.

Another demonſtration:

The magnitude of the earth's attractive force, at different diſtances, varies as the ſpace through which it impels a body in equal times; but a body at the earth's ſurface falls through 16.1 feet in 1″, and the moon, at her mean diſtance, nearly 60 ſemidiameters of the earth, falls through $\frac{16.1}{60 \times 60}$ feet in 1″ (from obſervation): preſuming therefore that the moon is retained in her orbit by the earth's attractive power, the force of gravity at the earth's ſurface, is to the force at the moon, as $16.1 : \frac{16.1}{60 \times 60} :: 60 \times 60 : 1$. Q. E. D.

238. Cor. 1. The firſt of theſe demonſtrations is applicable to all forces diffuſed by rectilineal effluvia of matter from a center, as heat, ſmells, &c.

239. Cor. 2. Putting d for the ſemidiameter of the earth, and W for the weight of a body at the earth's ſurface, its weight at any other diſtance nd (n being a number) will be equal to $\frac{W}{n^2}$. A weight, of 3600 lb. at the earth's ſurface, is equal to $\frac{3600}{60 \times 60}$ or 1 lb. at the diſtance of the moon.

240. Cor. 3. If BC and bc be homogeneous and ſimilar, their weights at unequal diſtances are equal; for they are to each other as $\frac{BC}{SA^2} : \frac{bc}{Sa^2}$ (237) as 1 : 1. If they be not homogeneous, their weights will be as their quantities of matter, that is, as their weights at equal diſtances, divided by the ſquares of their diſtances.

SCHOLIUM I.

241. The variation of the force of gravity was collected from obſervation in Peru by Meſſ. Condamine and Bouguer. Condamine found that the ſame pendulum upon mount Pichinca, at Quito, and upon the banks of the river of the Amazons, performed reſpectively 98720, 98740 and 98770 vibrations, in twenty-four hours. And Bouguer obſerved upon the ſummit of Pichinca, and upon the ſea-ſhore, being $\frac{1}{1349}$th part nearer to the center of the earth, that the lengths of iſocronous pendulums were to each other as 438.71 to 439.7. Theſe obſervations demonſtrate a diminution of the force of gravity in receding from the earth's ſurface; but the law of variation cannot be collected from them, or any other ſimilar experiments, with preciſion; becauſe the different ſtates of the atmoſphere, of heat and cold, and different denſities of the earth contiguous to the different places of obſervation, would proportionably affect the experiments, and theſe cannot be aſcertained with ſufficient accuracy. From the obſervations of Bouguer, the force of gravity upon the mountain and upon the ſea-ſhore, is as 99575 : 99802; and, ſuppoſing it to vary inverſely as the ſquare of the diſtance, it is as 99575 : 99722; which differs leſs from the true law than could be expected.

SCHOLIUM II.

242. An inveſtigation of the cauſe of attractions and their mode of operation, and particularly of this mechanical affection of matter, its gravity, ſo univerſally prevalent, hath long been the object of the philoſopher's reſearches, but without effect; the ſubject is ſtill involved in impenetrable darkneſs, and his wiſhes are unſatisfied. Dr. Halley refers it to the immediate agency of the Creator; Mr. Cotes deems it eſſential to matter, like extenſion and mobility, &c. and Sir I. Newton ſeems to entertain a different opinion, and attributes it to ſome undiſcovered and inviſible mechanical affection of matter. Electrical and magnetic experiments prove the exiſt-

ence

ence of a subtle fluid pervading the pores of the densest bodies, and the gravity of bodies may arise from the action of a fluid, whose particles are so small as to pass through all bodies, and does not therefore resist their motion; and, if endued with a strong repulsive power, it will by expanding itself press upon gross bodies. This æther, Newton supposes, is, from its repulsive force, much rarer within the dense bodies of the sun and planets, than in empty space, and in receding from them becomes perpetually denser and more elastic, and therefore occasions their gravity, every body being impelled from the denser towards the rarer parts of the fluid. This however is conjectural; and its truth or falsehood can only be established by future experiments. All that is certainly known about this mysterious cause, is, that it cannot arise from the pressure of a fluid similar to those with which we are acquainted, whether soft or elastic, quiet or agitated, for the following reasons. *First*, Because a fluid, subtle enough to penetrate the inmost recesses of the densest bodies, and so rare as not to impede their motion, cannot be conceived capable of communicating motion to them. *Secondly*, Because the attraction of gravity penetrates the inmost recesses of bodies, and their weights are proportional to their quantities of matter, not magnitude of surface; but the pressure of every fluid with which we are acquainted, varies as the surface opposed to it. *Thirdly*, Because the attraction of gravity acts with equal intensity upon bodies, whether quiescent or moving; but all known fluids act upon quiescent and moving bodies with different forces. But, however unsearchable the efficient cause of gravity may be, its final cause is most conspicuous, being the preservation of the earth and other planets, and of their periodical revolutions, which are only continued by an uninterrupted exertion of this mechanical affection of matter.

encreafe, or decreafe, of force refulting from it, would be finite, and confequently infinitely lefs than that refulting from contact. But if the furface of P were enlarged, and in contact with new matter, attracting according to this law, it's cohefive force would be proportionably encreafed.

246. Cor. 3. If this attraction, at a finite diftance, had a finite ratio to that of gravity, at the point of contact it would be infinitely fuperior to that of gravity; and, if at the point of contact, the ratio between this attraction and gravity were finite, at any affignable diftance it would be evanefcent.

247. Cor. 4. The attraction of cohefion, being great in contact and infenfible at a fmall diftance, varies in a higher than the inverfe duplicate ratio of the diftance.

FIG. LII. LIV. 248. Cor. 5. Since the cohefive force of any particle P, in contact with the body PAC, varies as the quantity of furface, it will be greateft when the furfaces of contact are plane. Two fpecies of matter conftructed of diffimilar particles, $a, b, c, d, e, f, g,$ &c. $l, m, n, o, p, q,$ &c. have different degrees of hardnefs; for the furfaces of contact, and cohefive forces of $a, b, c, d,$ &c. are greater than thofe of $l, m, n,$ &c.

249. Cor. 6. Of unequal fimilar particles, the fmalleft are capable of the firmeft cohefion, becaufe they have the greateft furfaces of contact compared with their magnitudes, the furfaces being diminifhed only as the fquares, and the magnitudes as the cubes, of any lines fimilarly placed in them.

250. Cor. 7. Hard bodies therefore may be conceived to be conftructed of very fmall particles, or fuch as are terminated by plane furfaces; for collections of thefe particles cemented by this principle

principle of cohesion, with which they are proved to be impressed, would constitute hard bodies. And soft bodies may be conceived to be combined from large spherical particles, or such as have many angular points, and do not allow of great surfaces of contact.

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251. The existence of this natural power is collected, and a general idea of it formed, from the following familiar observations and experiments. *First, An attraction subsists between the component parts of hard and soft bodies.* The force required to separate the contiguous parts of hard bodies, which is equal to the force of cohesion, is much superior to their gravity, though they be full of pores and consist of portions collected into one mass, and touching only a few points; and it must be much greater between the smallest particles, whose surfaces are in contact, without any great intermediate vacuities. This attraction is very strong in all tenacious and viscid bodies, as pitch, rosin, &c. which adhere to every thing in contact with them; and if a body very soft and viscid be extended, it will again contract its dimensions by the tendency of the parts to each other. Large ships, floating near each other in a calm sea, have a great tendency to each other, and are with difficulty kept from coming together. If the surfaces of the segments of two leaden bullets, whose diameters are not greater than $\frac{1}{4}$th of an inch, be polished, and compressed together by a gentle twist, it is said that a weight of 100 lb. is frequently required to separate them. And if two polished plates of glass, marble, brass, or any metalline substances, be compressed together and suspended in an exhausted receiver, the weight of the lower plate will not dissolve their cohesion; and if any fine thread be wrapped round one of them several times, at considerable intervals, after compression, their cohesion is still sensible, though they be removed from actual contact by the thickness of the thread. The cohesive forces of these substances is augmented by moistening their surfaces with water, oil, grease, &c. which expels the air from their pores, or, if very tenacious, keeps it confined, and prevents the action

action of its repulsive power. The cohesive force of two polished plane surfaces of metal, whose diameters were two inches, heated in boiling water, and besmeared with grease, oil, &c. was overcome by the following weights:

		Cold grease.	Hot grease.
Planes of	glass	by 130 lb.	by 300 lb.
	brass	150	800
	copper	200	850
	marble	225	600
	silver	150	250
	iron	300	950.

When these surfaces were moistened with water, oil, &c. and compressed together, their cohesion was overcome by the weights in the following table:

With water	by	12 oz.
oil		18
Venice terpentine		24
rosin		850 lb.
tallow-candle		800
pitch		1400.

Secondly, *An attraction subsists between the particles of any portions of water, oil, mercury, and all other fluids, except air, fire, and light.* Small portions of these fluids form themselves into globular drops both in the open air and in vacuo; and if a drop of mercury be poured upon clean paper, or a drop of water upon the leaf of a plant, their spherical figure is not changed, the gravity of their parts being unable to dissolve their cohesion; and two drops of any fluid, not much attracted by the surface on which they are placed, will coalesce into one when in contact, as is often observable in drops of water lodged upon the leaves of plants. If mercury, well purged of air, be poured into a clean glass tube 70 inches long, so that its parts be contiguous to each other and to the glass, after inversion the whole column will remain suspended; but the pressure of the atmosphere sustains only 29 or 30 inches, and the remainder must be supported by some other agent, which is chiefly the mutual adhesion of the parts; for if they be discontinued

tinued by a bubble of air intervening, or by ſhaking the glaſs, the column ſubſides to the height of 29 or 30 inches.

Thirdly, This ſpecies of attraction ſubſiſts between the particles of different fluids, and between them and other ſubſtances. If a piece of fir wood, whoſe ſurface is one ſquare inch, be ſoaked in water and float upon its ſurface, a weight of 50 grains, beſides an equivalent to its own weight, is required to detach it from the fluid; and when its ſurface is enlarged, a proportionably greater weight is required. Water riſes near the ſides of the veſſel containing it, round a glaſs bubble floating upon its ſurface, up capillary tubes of glaſs, plants, &c. A remarkable inſtance of this attraction occurs in the new invented water-pump. If two wheels, or pullies, *B* and *D* in the ſame plane, be made to revolve with great FIG. LV. velocity, and *D* be immerſed in water, a column of water will aſcend with the rope; and, if there be two or more grooves in *B* and *D*, and as many ropes paſs over them, at the diſtance of about an inch, the columns raiſed by each rope will cohere, and the quantity of water be much augmented.* Air is incorporated with moſt hard bodies, poſſeſſes their interſtices, and probably ſerves as a bond of union to their conſtituent parts. It appears from many experiments, that the quantity of air, detached from ſome hard bodies in which it was conſolidated, by the action of fire, or ſome particular fermentation, is very conſiderable; and Dr. Hale diſcovered that $\frac{2}{3}$ of a calculus humanus were air. It is attracted by water, and perhaps all fluids, into whoſe pores it inſinuates itſelf, is intimately mixed

* There are two of theſe machines, at Windſor, of different dimenſions. The depth of the well at the round tower is 178 feet: *D* is the wheel in the water, its diameter = 12 inches, the thickneſs of the rope *DB* = $\frac{1}{2}$ inch nearly; diameter of the upper wheel *B* = 13 inches; diameter of *C* = 11 inches; diameter of *E* = 4 feet 6 inches; a power, applied at *F* turns the wheel *E* round, and that, by means of the ſtring *EC*, communicates motion to *G*, which has the ſame axis with *B*. FIG. LV.

In the other machine the depth of the well is 95 feet, and, in this, the quantity of water, raiſed by the utmoſt efforts of a man, was at the rate of nine gallons in a minute. At Paris 500 pounds of water were elevated through an altitude equal to 240 feet, in ten minutes, when the diameter of the rope, ſurrounding the pullies, did not exceed ſix French lines. In the beginning of the motion, the column, adhering to the rope, is always leſs than when it has been worked for ſome time, and continues to encreaſe, till the ſurrounding air partake of its motion.

mixed with them, and coheres too ſtrongly to be ſeparated without the agency of fire, or ſome more attractive ſubſtance. If ſal ammoniac, or corroſive mercury ſublimate, be diſſolved in water, bubbles of air will diſengage themſelves from the water, and, adhering very tenaciouſly to the thin particles of the ſalt, aſcend with them, and burſt upon the ſurface. When water, beer, &c. are poured into a glaſs, or other veſſel, many bubbles of air are detached from the fluid by the ſtronger attraction of the veſſel, and adhere to its ſides and bottom; and, if theſe be rough, or have more points of contact, the number of bubbles will be encreaſed. Light is attracted by glaſs, water, and all tranſparent mediums, as appears by its refraction and reflection; and alſo by all other ſubſtances, as is obſervable in an experiment of Sir I. Newton.* A beam of light admitted through an aperture into a dark chamber, and paſſing by the edge of any ſubſtance, will be deflected from a rectilineal courſe, and by repeated attractions and repulſions deſcribe an undulating line in its paſſage.

OF HARDNESS, SOFTNESS, AND ELASTICITY.

252. DEF. *A hard body is that whoſe parts are not eaſily moved from their places; as wood, metal, ſtones, &c.*

Bodies conſtituted in ſuch a manner that their component parts do not give way to compreſſion, and whoſe coheſive force is inſuperable, may be called perfectly hard. The hardeſt bodies with which we are acquainted, as adamant, flint, gems, tempered ſteel, &c. are full of vacuities, and contiguous portions of them, not being in contact all round, are penetrable, and ſeparable by the action of a ſufficient force. Fire inſinuates itſelf amongſt the vacuities of the denſeſt and hardeſt bodies, and produces fuſion or expanſion, which cannot be effected without an intire ſeparation, and change of place, of their component parts. Perfect hardneſs therefore ſeems to be confined to the elementary particles, or leaſt parts into which bodies can be divided, whoſe figure and dimenſions cannot be ſeparated by any known power, and are impenetrable.

253. DEF.

* Newton's Optics, p. 317.

253. DEF. *A soft body is that whose parts change their position by the action of a small force, and retain it when the force is removed; as butter, snow, wax, &c.*

Bodies constructed in such a manner that their vacuities are not replete with any fluid, and whose parts do not repel each other, and are only retained in their places by their inertia, may be called perfectly soft; but no species of matter is perfectly soft, for the particles of all bodies, not possessed of a repelling power, require a much greater force to separate them, than what is equivalent to their inertia. The degrees of hardness and softness are infinitely variable, and the limit where hardness ends, and softness begins, cannot be defined. In the congress of hard and soft bodies, there is no cause of separation after they come into contact, and they will therefore either move on together, or be quiescent, after impact; and, the effects of collision being ascertained when there is no repulsive power, allowance must afterwards be made for their different degrees of repulsion.

254. DEF. *An elastic body is that which changes its figure, or the position of its parts, by the action of a force, and recovers, or has a tendency to recover, its figure.*

Elasticity is said to be perfect, when the parts of the body return to their first situation with a force equal to the force displacing them, and imperfect, when they do not.

255. PROP. *If a perfectly hard body* A *impinge upon a perfectly elastic immoveable body* B, *it will be reflected with a velocity equal to that of impact.*

DEM. The particles, composing the surface of B, are continually removed from their places by the compression of A, till it and their resistance become equal; and then, returning to their first situation with a force equal to that of compression, and in a direction opposite to *A*'s motion, *A* must evidently be repelled with a velocity equal to that of impact. Q. E. D.

256. Cor. 1. If *A* and *B* be both perfectly elastic, the particles, composing their surfaces, recede equally, though the recess is less than when *A* was perfectly hard; but the effect is the same, the whole force of restitution being equal to that of compression, and conspiring to repel *A*.

257. Cor. 2. If *B* be moveable, the velocity lost by *A* is double of that lost by impact only; for the parts of *B*'s surface, contiguous to the point of impact, restoring themselves with a force equal to that which displaced them and acting against *A*, it will be equally retarded by the forces of impact and restitution; and, for the same reason, the velocity communicated to *B* is double of what it would be, were both bodies perfectly hard.

258. Cor. 3. The relative velocity of *A* and *B* is the same before and after impact, or the velocity with which they accede to each other before impact, is equal to the velocity with which they recede from each other after impact; for the forces of impact and restitution, being equal and opposite, produce the same effects in opposite directions, and the relative velocity, which is destroyed by impact, must be restored by the force of elasticity.

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259. The whole time of contact of *A* and *B*, may be divided into two equal periods, the former between their first contact and the cessation of *A*'s compression, and the latter between this point and their last contact, when they are separated by the restoration of their relative velocity. And, if the time of *A*'s compression be divided into very small equal parts, *A*'s decrements of velocity produced in them will perpetually encrease, from the first contact

contact to the end of compression, where its force of protrusion vanishes, and it is stationary for a moment, if *B* be fixed, or, if *B* be moveable, its velocity is equal, for a moment, to that of *B*. The particles then returning, by their elasticity, to their first situation, will communicate, in equal times, perpetually decreasing increments of velocity to *A*, equal to the corresponding decrements during its compression, which vanish when the surface has regained its natural state, when *A* leaves *B*, and is reflected with a velocity equal to that lost by impact. When *A* and *B* are both perfectly elastic, the particles, contiguous to the point of impact, give way equally; and when *A* is perfectly hard, and *B* imperfectly elastic, or when they are both elastic in different degrees, the particles yield to impact unequally, and a pit or cavity must be formed in one of them.

260. Cor. 4. If the times of compression and restitution be divided into the same number of equal moments, and the decrements and increments of velocity, in corresponding moments of compression and dilatation, be as $n : 1$, the velocity lost by impact is to that communicated in an opposite direction, in the same ratio of $n : 1$ (Euc. B.V. P.XII.). And, if this ratio between these velocities always obtain, their corresponding increments and decrements will always be to each other in that ratio.

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261. *The existence of elasticity is demonstrated by numberless experiments. When two bodies impinge, they must coalesce if the surfaces of contact be immoveable, or, after receding, remain immoveable; and their separation is a proof of elasticity. Metals, semimetals, stones, gems, fossils, cartilages, most fluids, as air and even water, exert an influence opposite to the direction of the force compressing them, and discover a tendency to return to their natural state, which is, in all of them, imperfect and less than the force impressed; but most perfect in glass, ivory, hardened steel and cartilages.

* Muschenbroek, Ch. XVI. Desaguliers, Lect. VI.

lages. Elasticity is encreased by encreasing the density of a body; for metals are rendered more elastic by being beaten with a hammer, and their elasticity, which was not perceptible before, becomes after this very sensible. Steel is more elastic when tempered, and its density is then encreased in the ratio of 7809 : 7738. It is also sometimes encreased by cold, as the range of a cannon ball is greater when the cannon is cold, than when heated, and the string of a violin, or a steel lamina, is inflected, and recovers its situation with less force in hot than in cold weather. The sphere of action of the component parts of an elastic body, and the moments of time in which they lose, and recover, their situation, are too small to allow of decisive experiments for ascertaining their intensity, and laws of operation, at different distances from their natural state; and the result of experiments is only the relative magnitude of the velocity lost by impact, or the whole effect of compression, and that communicated by the whole aggregate force of restitution. Metal fibres, and thin steel laminæ, exhibit no elasticity unless stretched to a certain degree, and inflected by a certain force, as appears from lax chords, which, if a little stretched and removed from their natural state, discover no tendency to return to it; and, when the inflection of a fibre is very great, the influence of elasticity seems to be annihilated, as appears from fibres of wood, which, if inflected beyond a certain limit, remain quiescent and have no tendency to recover their situation. This is also observable in elastic bodies, for their elasticity is only discovered by impact, and the force of impact may be so small as to excite no sensible motion of the constituent parts, or so great as to destroy their elasticity; but the limits where it begins and terminates are unknown. A general idea of elasticity may be formed by considering the most simple cases of the vibrations of fibres, or thin steel laminæ, and conceiving elastic bodies to be composed of them.

FIG. LVI. EXP. I. If any fibre, metalline chord, or thin lamina of steel, whose length is *AB*, be stretched and fixed to two immoveable points *A* and *B*, and inflected into the position *ACB* by a power which ceases to act at *C*, it will return by its elasticity to its natural state

AB,

AB, and, proceeding with the velocity acquired, continue to perform nearly equal vibrations on each side of AB till its motion be destroyed by friction, and the resistance of the air.

EXP. II. If the distance of this fibre from a table, to which it is parallel, be equal to D, and a spherical ball P, whose diameter is $2D$, be rolled against it, the fibre will be inflected, and this inflection will encrease till P be quiescent, and, then returning to its first situation, P will be repelled and detached from it, when arrived at AB, with a velocity nearly equal to that of impact.

EXP. III. Very small inflections PD, PC, of the same fire AB, are found to be nearly as the inflecting forces; but when the inflections are considerable, they vary as some power of the force, whose exponent is less than unity or fractional. If AB be a small brass wire tended by a weight W of 3 lb. and inflected at P by weights equal to $\frac{1}{2}$ oz. and 1 oz. successively, $PD : PC :: \frac{2}{5} : \frac{4}{5} :: \frac{1}{2} : 1$; or the inflections are as the inflecting forces.

EXP. IV. If the lengths of the wire be multiplied by 2, 3, &c. and the small inflections PD, PC be always to each other as $\frac{1}{2} : 1$; the inflecting forces are found to be as $1 : 2$ and $1 : 3$, &c. or inversely as the lengths of the wire.

EXP. V. If the lengths of the wire be the same as in Exp. III, W be equal to 3 lb. and 6 lb. and $PD : PC :: 1 : 2$, the same as before; the inflecting forces are as $1 : 2$, or they vary directly as W.

262. Cor. 1. If F represent the inflecting force, L the length of the wire, I a small inflection PD or PC; I is as F (Exp. III.); F is as $\frac{1}{L}$ (Exp. IV.); and F is as W (Exp. V.). And consequently

if

if I, F, W be supposed to vary, the inflecting force F will vary as $\frac{I \times W}{L}$.

263. Cor. 2. Fibres of unequal thicknesses may be conceived to be composed of a greater or less number of finer fibres of the same thickness, and if W, L, I, be given, it is evident that F will be as the number of smaller fibres, or as the area of a section of the fibre composed of them, or as the square of its diameter (D^2); and consequently F will vary as $\frac{I \times W \times D^2}{L}$.

264. Cor. 3. It is collected from these and other similar experiments, that the elasticity of a stretched fibre appears as soon as it is inflected by the impact of the ball, and continues to encrease to the limit of inflection, where, the moment of the ball and resistance of the fibre becoming equal and opposite, the protrusion ceases; and, because the velocities of impact and resilition are always nearly equal whatever be the velocity of impact, the velocities of P are equal, at equal distances from the limit of inflection, both in its progress and regress.

FIG. LVII. 265. Cor. 4. Elastic bodies may be conceived to be formed of elastic fibres or strata, such as AB; for let the sphere DBE be imagined to be composed of such strata, and stricken at D by a body perfectly hard, and the parts nearest to D, receding by the force of impact, will communicate motion to the contiguous parts, and these to the next, till the different strata be inflected as is represented by the dotted lines in the figure. When the motion of the impinging body is extinguished, and the particles, composing the several strata, are no longer protruded, they will return, by their elasticity, to their first situation; and, if their velocity be perfect, with the same force in acceding to, and receding from, D; and consequently the impinging body will be reflected with a force equal

equal to the force of impact. And that this is not merely hypothetical, but that motion is diffused from the point of impact to the remote parts of elastic bodies, is presumed from the following experiments.

Exp. VI. If a spherical ball of ivory *A* be pressed against another *B*, whose surface is fresh painted with any colour, it will receive a small point of that colour upon its surface; but if *A* impinge upon *B* with any velocity, the breadth of the spot will be magnified, and become still greater as the velocity of impact is encreased. And, because the ball retains its spherical figure after impact, the parts of its surface must have lost, and recovered, their first situation.

Exp. VII. If two glass balls impinge with a proper degree of velocity, the interior parts of the balls will be broken, though the exterior, contiguous to the point of impact, be unbroken.

Exp. VIII. If two ivory balls *A* and *B* be suspended from the same point by two strings of the same length, and the less ball *A* impinge upon *B* at rest with a given velocity, *A* will be reflected always to the same height, and *B* will be impelled to the same height, upon the graduated periphery of a circle whose radius is the length of the string. But if either *A* or *B* be hollowed and lead inserted in the center, or nearer to the posterior surface, neither ball, though their weight be the same, will ascend as high as before the insertion of lead.

266. Cor. 1. Motion is therefore communicated from the point of impact to the contiguous parts (Exp. vi.), and diffused from thence to the remote parts of every elastic body (Exp. vii. & viii.)

267. Cor. 2. It is evident, from Exp. VIII., that the progressive motion of the parts, from the point of impact, is stopped by the insertion of lead, and consequently that the force of restitution, and change of figure, is less, than before it was inserted.

Exp. IX. A stroke or friction upon the edge of a glass, filled with water, communicates a tremulous motion to the parts of the glass, which is visibly communicated to the water. A reed or stick, placed across the bottom of a large glass bell, will fall when the bell is struck, the stroke producing a change of figure; and, if a piece of metal be fixed near the brim or lip of the bell without touching it, and the bell be stricken by any hard body, it is seen to touch the metal, and a successive of sounds, perpetually decaying, may be heard. If the edge of the bell be pinched, and the fingers suddenly withdrawn, the same sound is heard without producing any sensible motion towards the metal, or displacing the reed across it.

268. Cor. The motion diffused from the point of impact, to the remote parts of an elastic body, is continued for some time, and diminished gradually till it vanishes. And there seem to be two kinds of vibrations of the parts of an elastic body, one of which is quick, and called a tremor of its minute parts, and the other slower and longer, by which its figure is changed, and an impinging body repelled.

CHAP.

CHAP. VII.

MECHANICAL POWERS.

*THE exiftence, and intenfity of operation, of the mechanical affections, gravity, cohefion and elafticity, and the nature of the other qualities of matter, being afcertained experimentally, they are affumed as eftablifhed principles, and their efficacy in the production of preffure, motion, and other phenomena, is the next object of mechanical philofophy. There are fix fimple machines, commonly ftiled mechanical powers, from the effects produced, with their intervention, by the action of gravity and animal exertions, viz. the lever, wheel and axis, pulley, wedge, inclined plane and fcrew. They are all calculated to communicate motion to bodies, and fuftain their preffure, for which, the power unaffifted by them, is incompetent; and the artifice in all confifts in diftributing the weight amongft fuch a number of agents, that the part fuftained by the power may bear a fmall ratio to the whole. Thus, a power incapable of communicating motion to, or fupporting the preffure of, a body, without mechanical affiftance, may effect its purpofe by transferring a part of the weight upon a fulcrum, diftributing it amongft a number of pulleys, or placing it upon an inclined plane or fcrew; and, by this artifice, a power *P* may keep a weight fufpended which exceeds it in any affigned ratio, though without any acquifition of moment in a given direction; for motion is only communicable according to the eftablifhed natural relations fubfifting between matter and motion, and the magnitudes of two powers, in equilibrio, are always inverfely as their velocities.

* Keil's Phyfics, Lect. X. Helfham, Lect. VI. Emerfon, Prop. 18, &c. Graves, L. I. C. X. Mufchenbroek, Ch. VIII. Varignon, pag. 305. Maclaurin's Newton, B. II. Ch. III. Hamilton's Effay on the Principles of Mechanics. Morgan's Notes to Rohault.

LEVER.

269. DEF. *A lever is a bar of wood or metal, and is usually represented by an inflexible line, without gravity, revolving about a fixed point, called the fulcrum, by the action of a power upon its arms.*

PLATE VII. FIG. LVIII. The points *W*, *P*, where the weight and power act, are the points of suspension, and the immoveable point *F*, about which every point of the lever revolves, is called promiscuously the fulcrum, hypo-mochlion, and center of motion. There are three kinds of levers: 1. the fulcrum is between the power and weight, as in the common balance, sciffars, snuffers, &c. 2. The weight is between the power and centre of motion, as the oars and rudder of a boat, cutting knives fixed at one end, doors, &c. 3. The power is between the weight and fulcrum, as a ladder raised against a wall, a weight raised by the arm, where the center of motion is at the shoulder, &c.

FIG. LIX. 270. Cor. 1. Forces whose magnitudes are to each other as *PA*, *PB*, *PC*, &c. acting at *P* and terminated by a line *AC*, parallel to *PF*, have the same effect: for each may be resolved into two forces, one perpendicular, and the other parallel, to *PF*; of which, the perpendicular parts are equal and entirely employed in producing a rotation of the lever round *F*, or in supporting a body *W* placed on the other side of *F*, and acting perpendicularly to *FW*, and the parallel parts only produce a motion in the direction *PF*, and do not produce any rotation, or contribute to the support of *W*.

FIG. LX. 271. Cor. 2. If a given power, represented by *PB*, act at the same point *P* in any direction *PD*, its efficacy to turn the lever round *F*, or support any body *W*, is as the chord *PC* of the circle whose diameter is *PB*; for *PD* being taken equal to *PB* and resolved

resolved into two forces, one DE perpendicular, and the other PE parallel, to PF, it appears from similar triangles, that DE, the only effective part of the force, is equal to PC.

272. PROP. *Two powers* W, P, *acting upon a lever* PW, *whose center of motion is* F, *at the points* P *and* W *in the directions* WM, PL *in the same plane, and in equilibrio, are to each other inversely as the perpendiculars let fall from* F *upon their directions.* PLATE VII. FIG. LXI.

DEM. From F as a center, with the longer perpendicular FL, describe a circular arc cutting the direction of W in D, and, because the efficacy of these forces is the same to whatever points of their directions they are applied, let them be applied at L and D. Let DE represent the magnitude of W and be resolved into two forces, one DG in the direction FD, and the other EG perpendicular to it; and, because DG has no effect in making DF revolve, nor consequently the lever which makes an invariable angle with it, and EG and P act at equal distances from F, in directions perpendicular to those distances, and are in equilibrio, P is equal to EG, and $W : P :: DE : EG :: DF\ (FL) : FM$, from similar triangles. Q. E. D.

273. Cor. 1. In any lever FP or FHK, those parts of P and W, which are opposite to each other, are inversely as their rectilineal distances from F; for let WA and PD, be the respective magnitudes of W and P, which act at W and P, and resolving each into two forces, WB, PE, coincident with the lever, or arm FP which makes an invariable angle with it, and AB, DE, parallel and opposite, and drawing the perpendiculars AC, DG, to FP; and FM, FL, to the directions; it appears, from similar triangles, that $DE : AB :: DG : AC :: \frac{DP \times FL}{FP} : \frac{WA \times FM}{FW}$ ($:: \frac{1}{FP} : \frac{1}{FW}$, because $DP \times FL = WA \times FM$ from this proposition. FIG. LXII.

274. Cor.

274. Cor. 2. If the directions of P and W, acting upon the arms of a straight lever, and in equilibrio, be parallel, they are therefore inversely as their distances, or inversely as the portions of any line drawn through F and terminated by their directions (273). This follows also from the proposition, because the distances, or the segments of any right line on each side of F and terminated by their directions, are, in this supposition, as the perpendiculars let fall from F upon their directions.

276. Cor. 3. The velocity of any point either in a straight, or curved lever, varies as its rectilineal distance from the center of motion F; for all points describe similar circular arcs, having their centers in F. If P and W act upon the same right line, and their directions be perpendicular, or inclined in the same angle, to their rectilineal distances, their velocities will be as those distances, being measured by the bases of similar triangles which are described in the same time: their velocities are therefore to each other inversely as the opposite parts of P and W (273).

FIG. LXIII. 277. Cor. 4. If a lever be moveable about an axis AB, or fixed to an axis which is moveable about two centers A and B, and perpendiculars PF, WF, be drawn to this axis from P and W, to which their directions are perpendicular and in the same plane, an equilibrium obtains when $P : W$ inversely as their perpendicular distances from the axis.

FIG. LXII. 278. Cor. 5. There will be an equilibrium upon the lever when $P \times FP \times \text{sin.} \angle FPL$ (or the angle which P's direction makes with FP) is equal to $W \times FW \times \text{sin.} \angle FWM$ (or the angle which W's direction makes with FW; for $P : W :: FM : FL :: \frac{FW \times \text{sin.} \angle FWM}{\text{rad.}} : \frac{FP \times \text{sin.} \angle FPL}{\text{rad.}}$; and $F \times FP \times \text{sin.} \angle FPL = W \times FW \times \text{sin.} \angle FWM$, supposing the radius to be given.

279. Cor.

279. Cor. 6. The intensities or moments of any powers A, B, C, &c. whose directions are parallel, vary as their magnitudes multiplied into their distances from the center of motion; for let Q, R, S, &c. acting at the same point E, in directions parallel to those of A, B, C, &c. be in equilibrio with them respectively, or (Cor. 2.) let $A\times AF=Q\times EF$, $B\times BF=R\times EF$, $C\times FC=S\times EF$, &c.; and the efficacy of A, B, C, &c. to make the lever revolve, or support any power acting against them at E, is evidently as $Q+R+S$, &c. or as $\overline{Q+R+S}\times EF$, or, substituting their equals, as $A\times AF+B\times BF+C\times CF$, &c. FIG. LXIV.

280. Cor. 7. The intensities or moments of A, B, C, &c. the sines of whose directions with their rectilineal distances to the same radius are a, b, c, &c. respectively, will be as $A\times a\times AF+B\times BF\times b+C\times CF\times c$, &c.; for if Q, R, S act at the same point E, in the same direction, the sine of whose inclination to $EF=z$, and be in equilibrio with them, $\overline{Q+R+S}\times EF\times z=A\times AF\times a+B\times BF\times b+C\times CF\times c$, &c. (278).

281. Cor. 8. If more than two powers act upon a lever, there will be an equilibrium when the sum of the products arising from multiplying each into the perpendicular distance of its direction from the center of motion; or, if their directions be parallel and the lever straight, into its distance from F, on one side, is equal to the sum of the products on the other side. Whatever be the form of the lever, the value of the perpendicular may be substistituted for it, and an equilibrium obtains when the sums of the products on each side of F are equal.

282. Cor. 9. Because the efficacy of P and W is the same to whatever part of their direction they are applied, a bended or curved lever may be reduced to a straight one, making an invariable angle with it, and P and W may therefore be always supposed to act in the same right line.

283. PROP.

PLATE VIII. FIG. LXV. 283. PROP. *When any number of levers* WC, CE, EP, *&c. are combined together in the same direction, the ratio of* P *to* W, *acting in the same plane at their extremities, in parallel directions, and in equilibrio, is that of* $WB \times CD \times EF : BC \times DE \times FP$.

DEM. Let the forces Q, R, P, acting at the points C, E, P, in directions parallel to those of W and P, be respectively in equilibrio with W, Q, R, and consequently (274) $W : Q :: BC : WB$

$$Q : R :: DE : CD$$
$$R : P :: FP : EF;$$

and, by composition of ratios, $W : P :: BC \times DE \times FP : WB \times CD \times EF$. Q.E.D.

284. Cor. 1. If P and W act in different directions, and Q and R act also in any other different directions, and Q, R, P be respectively in equilibrio with W, Q, R, perpendiculars, let fall from the centers of motion, B, D, F, upon their directions, are to be substituted for the distances.

285. Cor. 2. If any of the forces, in this and the preceding proposition and corollaries, act in different planes, they are to be reduced to the same plane by resolving each into two forces, one in that plane and the other perpendicular to it, of which the former only are effective, and are to be used in the several analogies.

FIG. LXVI. 286. PROP. *If lines be drawn from* F *parallel to the directions of* P *and* W *which meet in* A; P, W, *and pressure upon* F (Pr) *are to each other as the sides and diagonal* AB, AC, AF, *of the parallelogram* CABF, *respectively.*

DEM. From F let fall the perpendiculars FL, FM, upon the directions of P and W, and the triangles FCM, FLB (having a right angle in each, and the angles FCM, FBL, either equal to or the

the supplements of CAB) are similar; therefore $P : W :: FM : FL :: FC : FB$; and a force, whose quantity and direction are AF, is equivalent to P and W (185). Q. E. D.

287. Cor. 1. A power therefore acting at F, whose magnitude is to $P + W$ as AF to $AB + AC$, and whose directions are FA, AB and AC, respectively, will prevent all motion.

288. Cor. 2. The magnitude of P, W or Pr, is as the sine of the angle formed by the directions of the other two; for

$Pr : P :: FA : AB ::$ sin. of $\angle FBA$ or BAC : sin. of $\angle AFB$ or FAC;

$Pr : W :: FA : AC ::$ sin. of $\angle FCA$ or $\angle CAB$: sin. of $\angle CFA$ or FAB;

and $W : P :: AC : FC ::$ sin. of $\angle CFA$ or FAB : sin. of $\angle CAF$.

289. Cor. 3. Any two of these forces, pressure upon F, P, and W, are to each other inversely as the perpendiculars let fall upon their directions from any point F in the direction of the third; for $Pr : P ::$ sin. of $\angle FCA$ or WAH : sin. of $\angle CAF :: WH : WI$ (supposing WH and WI to be perpendicular to the directions of P and Pr, and AW to be the radius); and $Pr : W ::$ sin. of $\angle FBA$ or PAK : sin. of $\angle BAF$ or $BAN :: PK : PN$, supposing PK and PN to be perpendicular, respectively, to the directions of W and Pr, and AP to be the radius, &c. And if the lever be straight and the directions of P and W parallel, the magnitudes of P, W and Pr, are as the distances of the other two, the perpendiculars then becoming the distances.

290. Cor. 4. If the extremities of the perpendiculars FL, FM, be joined by a right line LM, Pr, P and W are to each other respectively as LM, FM and FL, for this triangle is similar to FAB, as easily appears by describing a circle upon FA as a diameter.

FIG. LXVII. 291. PROP. *The distance from* F, *quantity and direction, of any two forces* Q *and* R *acting upon a straight lever in the same plane, being given, to find the magnitude, direction and distance of a force equivalent to them.*

Let the directions of *Q* and *R* meet in *A*, and taking *AB* : *AD* :: *Q* : *R*, and completing the parallelogram, the diagonal *AE* is the magnitude and direction of a force equivalent to them (185). But *QR* and the angles *AQR*, *ARQ* being given, *AQ*, *AR*, and the angle *QAR* may be found; and *AB*, *AD*, and the angle *BAD* being known, *AE*, the magnitude of the combined force, and the angle *BAE* may be found; and *QA*, and the angles *AQN*, *QAN* being found, the angle *QNA*, or inclination of *AN*, to *FN* and *QN*, are known. Q. E. I.

FIG. LXVIII. 292. Cor. 1. If any number of forces *Q*, *R*, *S* in the same plane, whose magnitudes and directions are *AB*, *AC*, *ER*, act upon the lever *FW*, and be in equilibrio with any other forces *T*, *V*, *W*, whose quantities and directions are *HG*, *HI*, *LN*, they may be reduced to two which are in equilibrio; for *AB* and *AC* are equivalent to *AD*, and taking *EF* equal to *AD*, *EF* and *ER* are equivalent to *EQ*; and in the same manner *HG*, *HI*, *LN* are equivalent to *LP*, which is in equilibrio with *EQ*, because the lever is at rest. And the directions, and distances from *F*, of *LP* and *EQ*, are found as in the proposition.

FIG. LXIX. 293. Cor. 2. Any number of forces in the same plane, whose magnitudes and directions are *AB*, *AC*, *EX*, *HG*, *HI*, *LN*, may be reduced to one, which is equal to the pressure upon the fulcrum; for, supposing the forces *EQ*, and *LP*, resulting from the other forces combined, to act at their intersection *S*, a third force equivalent to them, and consequently to the pressure upon *F*, must pass through *S* (185), and, taking *SU*, *SV*, respectively equal to *EQ*, *LP*, and completing the parallelogram, *SY* will be its magnitude and direction.

294. Cor.

294. Cor. 3. If any number of forces, whose quantities and directions are given, act upon a lever, the position of a fulcrum, about which they will be in equilibrio, and the quantity and direction of pressure upon it, may be found; for, let the directions of Q and R meet in A, and taking $AB : AC :: Q : R$, and, completing the parallelogram, its diagonal AD produced, will cut the lever in a point, which being supported, Q and R will be in equilibrio (286); and combining AD with S, and the force, resulting from these, with another, &c. the diagonal will always intersect the lever in a point F, about which they will be in equilibrio. FIG. LXVIII.

295. LEMMA. *If right lines be drawn from any point* P *to the extremities of the diagonal, and sides, of the parallelogram* ABCD, *the triangle* PAC, *having the diagonal for its base, is equal to the difference, or sum of the triangles* PAB, PAD, *having the sides for their bases, according as* P *is situated between the lines forming the angle* BAD, *or those which form its supplement to two right angles.* FIG. LXX.

DEM. CASE I. Let P be situated between the lines forming the angle BAD; and, drawing Pnm perpendicular to AB or CD, the triangle $APC = ADPC - ADP$; but $ADPC = ADC + DPC = AB \times \frac{mn}{2} + DC \times \frac{Pn}{2} = AB \times \frac{Pm}{2} = APB$; therefore $APC = APB - APD$.

CASE II. Let P be placed between the lines forming the supplemental angle to BAD, and the triangles $ADC \pm DPC = \frac{AB}{2} \times \overline{mn \pm Pn}\,(Pm) = ABP$; and, adding APD to both, $APC = ABP + ADP$. Q.E.D. FIG. LXXI.

296. Cor.

PLATE IX. FIG. LXXII. 296. Cor. 1. If P be in one of the fides containing the angle BAD, the triangle $APC = ADC + DPC = \frac{AB}{2} \times \overline{mn \mp Pn}$ $(Pm) = PAB$.

FIG. LXXIII. 297. Cor. 2. If P be in the diagonal, the triangle $PAB = ABC \mp BPC = BC \times \frac{mn}{2} \mp BC \times \frac{Pn}{2} = BC \times \frac{Pm}{2} = PAD$. The upper fign is to be ufed when P is without the figure, the lower when it is within.

298. Cor. 3. If perpendiculars be drawn from P to the diagonal and fides, Pd, Pm, Pq; $AC \times Pd = AB \times Pm \mp AD \times Pq$, and when P is in the diagonal AC, $AB \times Pq = AD \times Pm$.

FIG. LXXIV. 299. Prop. *If there be any number of forces* Q, R, S, T; *in the fame plane, which are combined as before, the fums of the products, arifing from multiplying each into the perpendicular diftance of its direction from any point* F, *in the diagonal* CL, *are equal on each fide of it.*

Dem. Let CD, CG, CH, CK be the relative magnitudes of Q, R; S; T, refpectively, and compounding them, and drawing perpendiculars, from any point F in the laft diagonal, upon their directions, viz. Fa, Fb, Fc, Fd, Fg, Fh; from the preceding lemma, $CK \times Fh = CI \times Fg = CE \times Fc - CH \times Fd = CD \times Fa + CG \times Fb - CH \times Fd$, and confequently CK or $T \times Fh + CH$ or $S \times Fd = CD$ or $Q \times Fa + CG$ or $R \times Fb$. Q. E. D.

300. Cor. A lever therefore paffing through any point F in the diagonal LC, produced in any direction, and in the fame plane with the forces which act upon it, will be in equilibrio.

301. Lemma.

301. LEMMA. *If a right line* PW *revolve round a fixed point* F, *and lines are drawn from its extremities to another fixed point* Q, *to which perpendiculars* FL, FM, *and* Fl, Fm *are drawn from* F, *and* W *is supposed to ascend,* FL *has to* FM *a greater ratio than* Fl : Fm. FIG. LXXV.

DEM. Take $Pr = py = FW$, and, drawing the perpendiculars rv, yz; $FL : rv ::$ sin. $\angle FWL$: sin. $\angle rPv :: PQ : WQ$; $rv : FM :: Pr (FW) : FP$, and comp. $FL : FM :: PQ \times FW : WQ \times FP$. By a similar process, $Fm : Fl :: wQ \times FP : pQ \times FW$; and adding these analogies together, $FL \times Fm : FM \times Fl :: PQ \times FW \times wQ \times FP : WQ \times FP \times pQ \times FW :: PQ \times wQ : WQ \times pQ$. But PQ is greater than pQ and wQ than WQ, and consequently $PQ \times wQ$ is greater than $WQ \times pQ$, and $FL \times Fm$ than $FM \times Fl$; and the ratio of $FL : FM$ is greater than that of $Fl : Fm$. Q. E. D.

302. PROP. *If any two powers* P *and* W, *whose directions always meet in the same point* Q, *be in equilibrio upon the lever* PW *in any one position of* PW, *they cannot be in equilibrio when it is in any other position* pw.

DEM. Let the lever revolve, and be in any other situation pw, and because $P : W :: FL : FM$, or in a greater ratio than $Fl : Fm$ (301) P will preponderate. Q. E. D.

303. Cor. 1. It is evident that the lever cannot rest till it pass through the points F, Q.

304. Cor. 2. If the directions of P and W be parallel to each other, they will be in equilibrio in any position of the lever, because the perpendiculars drawn from F to their directions are always as the distances, and consequently in a given ratio to each other. And, for the same reason, if there be ever so many forces, acting in parallel directions upon the arms of a straight lever,

O and

and in equilibrio in any one situation of the lever, they will be in equilibrio in every situation of it.

FIG. LXXVI. 305. PROP. *If the center of motion* F *be placed above the straight lever* PW, *and* P *and* W, *acting always in parallel directions, equilibrate in any position* PW, *they do not equilibrate in any other*, pw.

DEM. From F draw FM and Fm perpendicular to the lever, and $P : W :: WM : PM :: wm : pm$, or in a less ratio than that of $Lw : Lp$ or $LV : LR$, and consequently much less than that of the perpendiculars from the center of motion F upon the directions, or $MV : MR$. Q. E. D.

306. Cor. 1. Because $P : W$ in a less ratio than that of $MV : MR$, $P \times MR$ is less than $W \times MV$, and consequently W will descend.

PLATE X. FIG. LXXVII. 307. Cor. 2. If F be placed on the other side of the lever, the descending body will preponderate; for $P : W :: WM : PM :: mw : mp$, and consequently $P : W$ in a greater ratio than that of $Lw : Lp$ or $LV : LR$, and therefore much greater than that of the perpendiculars upon the directions or $MV : MR$; and $P \times MR$ is greater than that of $W \times MV$.

FIG. LXXVIII. 308. PROP. *If* P *and* W *act always at the same distance from the lever, in directions parallel to* FM, *and in equilibrio about* F *in the position* PFW, *and the lever be moved, the descending body will continue to preponderate.*

DEM. Let FN, ps, wr, be drawn parallel to the directions in which the bodies act, pQw be the lever, and Fn be the position of FN; and from similar Δs $Qr : Qs :: Qw : Qp$, and therefore $Qr : Qs$ in a greater ratio than $nw : np$ or $NW : NP$ or $WM : PL$ or

PL or P to W; but $Nr : Ns$ in a greater ratio than $Qr : Qs$, and consequently $P : W$ in a less ratio than $Nr : Ns$, and the body W will continue to preponderate.

309. Cor. When the bodies are placed under the lever, the descending body will continue to descend, because $P : W$ in a less ratio than $Nr : Ns$, as easily appears by turning the figure.

SCHOLIUM.

310. In all communications of motion by impact, the quantities of motion lost and gained being equal and opposite, the quantity of motion estimated in the same direction is invariable, and the quantities of matter vary inversely as the velocities lost and gained; and if two bodies A and B act upon a lever, or any other machine, they are so connected that A cannot descend without making B ascend with the same quantity of motion, and their quantities of matter are therefore inversely as each other. These cases, having such marks of coincidence, are inferred to be similar in every respect, and the cause of an equilibrium in the mechanical powers, is often immediately assigned from this equality of momenta; but they are not exactly similar, because when A impinges upon B, some part of its motion is transferred to B, and A's motion necessarily precedes this communication of motion; but, when they act upon any machine, the descending body A cannot be said to communicate any part of its motion to B ascending; because, from their connection, their motion must necessarily commence and be extinguished together; and besides, the power of the lever ought to be considered. The ratio of the power and weight may however be assigned, in every machine, from their incipient momenta; for P and W, acting upon the arms of a lever, exert a pressure and have a tendency to move; and if a and b be the velocities with which A and B strike the lever, estimated in such directions that their pressures are solely employed in resisting each other's efforts to produce motion, no part being lost by obliquity of direction, $A \times a \times$ into its velocity or distance from the centre of motion.

motion is equal to $B \times b \times$ into its velocity or diftance, when there is an equilibrium; and by fubftituting preffures or P and W, for $A \times a$ and $B \times b$, the ratio of $P:W$ is found to be the fame as was before collected from the refolution of motion. As the velocity of any point of a lever varies as its rectilineal diftance from the center of motion, all points defcribing fimilar circular arcs round it in the fame time, if the directions of P and W be perpendicular to their rectilineal diftances, their velocities will be the fame as thofe of the points where they act and wholly efficient, and when there is an equilibrium, P will be to W inverfely as their rectilineal diftances. But if the direction of P or W be enclined to their rectilineal diftance, their efficient velocity will not be equal to that of the point where they act; if P's direction be the line PD, it is evident that P will have two motions whilft the lever revolves, one acceding to, or receding from, F, according as the angle DPF is lefs or greater than a right angle, and the other producing the rotation of the lever; for, defcribing a circular arc with F as a center, and FD as radius, the power will act at every intermediate point in PE whilft the lever defcribes the angle PFD. The motion in the direction of the lever is inefficient, and if PD reprefent the direction and quantity of A's velocity, and be refolved into two, PC in the direction of PF, and DC perpendicular to it, this laft only is efficient; and if the angle at F be very fmall, $A \times DC \times$ its perpendicular velocity or into FD, or $\frac{A \times FP \times PD \times FL}{FP}$, fuppofing FL to be perpendicular to the direction, is A's efficacy to turn the lever. The effective part of B's velocity being found in the fame manner, and thefe values of a and b being fubftituted for them in the fuppofition of an equality of moments; whatever be their directions, the ratio of P to W is the fame as that difcovered by other principles. Thus in an equilibrium $A \times PD \times FL$ is given, and confequently $A \times PD$ or the preffure of P in the line PD is as $\frac{1}{FL}$. The demonftration of this fundamental propofition, afcribed to Archimedes, depends upon this principle, that if a number of weights be fufpended upon the arms of a lever, at points equidiftant from each other, whether on the fame fide of the fulcrum or not, their efficacy

FIG. LXXIX.

efficacy to make the lever revolve is the ſame as if they were united in a point biſecting the diſtance of the points of ſuſpenſion. Mr. Huygens ſays, that many fruitleſs attempts have been made to remedy the defects of this demonſtration, and propoſes another founded on this principle, that when two equal bodies are placed upon the arms of a lever, that which is moſt remote from the fulcrum will preponderate. But this principle is not more evident than that of Archidemes, and beſides, his proceſs is prolix and tireſome. Mr. Maclaurin hath given a demonſtration of this propoſition when the arms of the lever are commenſurate, and his method might eaſily be extended to caſes in which they are incommenſurate, and be made general; but this would add to the length of a proceſs already very long. The only principle in Sir I. Newton's demonſtration that hath been controverted is this, which is taken for granted, "that the ſame power will have the ſame effect to whatever point of the direction, in which it acts, it be applied;" yet no doubts are entertained of the truth of it, though, perhaps, from its ſimplicity and intuitive evidence, it cannot be demonſtrated by more ſimple principles. If the line *PC* and the angle *PFC* be invariable, the radii *PF*, *CF* being fixed to *PC* at the points *P* and *C*, it is evident, that two equal forces *P* and *Q*, acting upon the points *P* and *C* in the directions *PC* and *CP*, will deſtroy each other, and the line *PC*, and conſequently *PF* and *CF*, will be quieſcent. If therefore *P* make a body deſcribe the arc *Ww* in any ſmall time, *Q* will make it deſcribe *wW* in the ſame time, or they would not deſtroy each other's effects. Equal forces therefore *P* and *Q*, acting at different points of the ſame direction, in the ſame time make the radii *PF*, *CF* deſcribe angles at *F*, *PFp* and *CFc* equal to *WFw*, and therefore to each other. Though this demonſtration of Newton be general, conciſe, and perfectly ſatisfactory, the great utility of the propoſition may poſſibly render other demonſtrations of it not undeſerving of attention. PLATE VII. FIG. LIX.

DEMONSTRATION OF ARCHIMEDES.

FIG. LXXX. Let AB be a homogeneous cylinder, and C the bisection of its axis, and it is evident, that if a fulcrum, or power equal and opposite to the pressure upon it, be applied at C, the parts AC and CB will be in equilibrio. Let any point D be taken, and, bisecting AD in E, and DB in F, it is clear that two powers, respectively equal to the weights of AD and DB, applied at E and F, will support them, and have the same effect with the fulcrum, or power, applied at C, and be in equilibrio about C: but $CE = CA - AE = \frac{AB - AD}{2} = \frac{DB}{2}$; and $CF = CB - BF = \frac{AB - DB}{2} = \frac{AD}{2}$; and consequently $CE : CF :: DB : AD ::$ force applied at F : force applied at E. Q. E. D.

FIG. LXXXI. 311. Lemma. *If from any point* P, *in the diagonal of a parallelogram,* ABCD, *two lines* Pm *and* Pq, *be drawn perpendicular to the sides, the perpendiculars and sides are inversely as each other.*

Dem. Draw CE, CF, perpendicular to the sides, and, from similar triangles, $Pq : CF :: AP : AC :: Pm : CE$; therefore $Pq : Pm :: CF : CE :: CD\ (AB) : CB\ (AD)$ (sim. triangles). Or, this follows from (298), where it is proved, that $AB \times Pm = AD \times Pq$, and consequently, $Pm : Pq :: AD : AB$. Q. E. D.

FIG. LXXXII. 312. Prop. *If any three forces* W, P, Z, *whose magnitudes and directions are* AB, AC, AD, *act upon the lever* WP, *which is at rest,* W *is to* P *inversely as the perpendiculars let fall from* F *upon their directions.*

Dem.

DEM. The directions of these forces must be in the same plane, and meet in the same point, (216), and are respectively as the sides and diagonal of a parallelogram parallel to their directions; and W: $P :: AB : AC$ (196) $:: FL : FM$ (311). The effect will evidently be the same when a fulcrum is applied at F instead of the force Z. Q.E.D.

*Dr. Hamilton's demonstration of this proposition depends upon the same principle with the above. Let the forces W, P, Z, act upon the inflexible line WP at the points W, P, F, and let the directions of P and W meet in C, and the direction of a force Z, equivalent to them, must pass through C. Therefore P, W, Z, are to each other as the sides and diagonal of a parallelogram respectively parallel to their directions, FA, FB, FC, and $W : P :: AC : BC ::$ sin. $\angle AFC$ (FCB) $::$ sin. $\angle ACF :: FM : FL$. FIG. LXXXIII.

$W : Z :: AC : FC ::$ sin. $\angle FCB$: sin. $\angle FBC$ or $BCA :: PQ : PN$, supposing PQ and PN to be perpendicular to the directions of Z and W.

$P : Z :: FA : FC ::$ sin. $\angle FCA$: sin. $\angle FAC$ or $ACB :: WG : WR$, supposing WG and WR to be perpendicular to the directions of Z and P.

The parts of W and P, which act in directions exactly opposite to those of Z, are found by resolving AC and QA (BC) into two forces, one parallel to the right line joining W, P, as Am and Bn, and the other parallel to FC as Qm, Qn. The opposite parts Am, Bn, are equal and destroy each other, and the conspiring parts $Qm + Qn$ must be equal to Z; and consequently when two forces W and P are in equilibrio with a third force Z, and their directions are all parallel, FIG. LXXXIV.

$W : P :: Qn : Cn :: PB : BC :: PQ : QW$;
$W : Z :: Cm : QC :: CA : CW :: QP : PW$; and
$P : Z :: Cn : CQ :: CB : CP :: WQ : WP$.

†BALANCE.

313. DEF. *The ancient balance, commonly called the statera romana, or steelyard, is a lever of the first kind supported at the point* F, *placed near* PLATE XI. FIG. LXXXV.

* Essay on the Principles of Mechanics. † Helsham, Lect. VI. Muschenb. Ch. VIII. § CCCLXXXIII. Desaguliers, pag. 95.

near one extremity, about which the brachia FA, FN *equiponderate. On one side of* F *at the extremity* A, *an unknown weight*, W, *is suspended, and on the other side a known weight*, P, *is moveable upon the arm* FN, *which is divided into parts equal to* FA, *each of which is also divided into* 10, 100, *&c. equal parts.*

314. Cor. If P be at x, the fourth division from I, which is the n^{th} division from F, when an equilibrium obtains between it and W, and IK be divided into m equal parts, the weight of W will be equal to $P \times \overline{n + \frac{4}{m}}$; for $W : P :: Fx : FA$ (274) $:: FA \times \overline{n + \frac{4}{m}} : FA :: \overline{n + \frac{4}{m}} : 1$, and consequently $W = P \times \overline{n + \frac{4}{m}}$.

FIG. LXXXVI. 315. Def. *The common, or modern, balance, or a pair of scales, is a lever of the first kind, as* AB, *supported at its bisection* F: *to the extremities* A *and* B *are suspended basins or scales, and the brachia and basins, on each side of* F, *are supposed to equiponderate.*

316. Cor. 1. If any known weight, P, placed in one scale equiponderate with one unknown, W, placed in the other, the weight of W is known, being equal to that of P; for $P : W :: AF : BF$ (274) $:: 1 : 1$, and consequently $P = W$.

317. Cor. 2. A balance, whose arms are unequal in length, is fallacious; for if AF and BF be unequal, P and W, when in equilibrio, cannot be equal, for $W = \frac{P \times BF}{AF}$. But the relation of $BF : AF$ being known, W is also known.

318. Cor. 3. If a man, whose weight is equal to W, standing in one scale and in equilibrio with P placed in the other, press the

the beam upwards in D with a force equal to Q, the diminution of W's moment is equal to $Q \times FD$; and because the reaction at the scale is equal to Q, the encrease of W's moment is equal to $Q \times FA$, and consequently W will descend with a force equal to $Q \times AD$. If the pressure be upwards at E, W will descend with a force, resulting from this pressure, equal to $Q \times EF$, and, from the reaction, with a force equal to $Q \times FA$; and therefore the whole force of descent is equal to $Q \times EA$. When the pressure is downwards at D, the encrease of W's moment is equal to $Q \times FD$, and the diminution of its moment $= Q \times FA$; and, consequently, W will ascend with a force equal to $W \times DA$. If the pressure be downwards at E, the diminution of W's moment, or encrease of P's moment, is equal to $Q \times EF$, and a part, Q, of W's weight being transferred to E, the diminution of its moment, on that account, is equal to $Q \times FA$; and consequently the whole diminution of W's moment, or force of P's ascent is equal to $Q \times EA$.

319. Cor. 4. If the center of motion C be in the right line joining the centers of suspension, the equilibrium of equal weights P and W will obtain in every position; the perpendiculars let fall from C upon the directions being always equal to each other. But when C is above or below WP, an equilibrium of equal weights does not obtain, unless WP coincide with the horizontal line AB. When WP coincides with AB, the perpendiculars let fall from C upon the directions of W and P are equal to GB and GA, CG being perpendicular to AB; but when the balance is in any other position WP, the perpendicular CI is greater than CH, because gL, which is less than CI, is equal to gM, which is greater than CH. W will descend and continue to vibrate till its motion be destroyed by friction. This corollary is also deducible from (305). FIG. LXXXVII.

320. Cor. 5. If P and W be unequal, and C, be in the right line WP, the heavier of them will descend till WP be perpendicular to the horizon, or, if the center of motion be not in WP, till $P \times CH = W \times CI$.

321. Cor.

FIG. LXXXVI. 321. Cor. 6. In a balance whose arms are unequal, the weight of W may be still ascertained; for let W, suspended at A, be in equilibrio with a known weight Q, suspended at B, and $W \times FA = Q \times FB$, and, suspended at B, let it be in equilibrio with a known weight R suspended at A, and $W \times FB = R \times FA$; consequently, by multiplying these equations together, $W^2 \times FA \times FB = Q \times R \times FA \times FB$, and $W = \sqrt[2]{Q \times R}$.

FIG. LXXXVIII. 322. *Cor. 7. If the beam of the balance be supposed to have weight and be similar and homogeneous in every part, its center of gravity is in the bisection F; but if it be not homogeneous, or the center of motion be not in the bisection, let G be its center of gravity, and an equilibrium will obtain when $P \times PF + B$ (weight of the balance) $\times GF = W \times FW$.

323. Cor. 8. If from F, a style FD, perpendicular to WP, be raised, the equilibrium of the balance will be affected by it, except it be in an horizontal situation, the moment of the style being measured by its weight multiplied into the distance of its center of gravity from the line FH, perpendicular to the horizon. But the equilibrium is restored by continuing the style to the other side of F, so that the moments on each side may be equal and opposite.

324. Cor. 9. If the center of gravity of the balance, scales, and weights, be in the center of motion, F, an equilibrium obtains in every position of the balance; but if this center be above or below F, the balance cannot be quiescent till the right line joining F and this center be perpendicular to the horizon. The best position therefore of the center of gravity is below F, and as little below it as possible, that the arcs described by it, during its vibrations, may be small and soon described. The points of suspension should

* These corollaries may be omitted till the chapter upon the center of gravity be read.

ſhould be in the ſame right line with the center of motion, which ought accurately to biſect their diſtance (317).

325. Cor. 10. The arms of the balance ſhould be as long as can be uſed conveniently; becauſe the moment of a given body varies as its diſtance from the fulcrum, and, therefore, the greater the diſtance, the more diſtinguiſhable will be the moment ariſing from any ſmall difference between *P* and *W*. And to diſtinguiſh very minute differences of weight, the friction upon the axis, in the motion of the beam and ſcales, ought to be as little as poſſible.

CHAP.

CHAP. VIII.

*WHEEL AND AXIS.

326. DEF. *A WHEEL and axis or axis in peritrochio, is a machine composed of a circular wheel, in whose center*
FIG. LXXXIX. *a cylindrical axis is inserted and fixed; and the wheel revolving by the action of a power* P, *the axis, whose extremities are supported, revolves with it, and the rope, to which a body* W *is appended, is tied to the axis and wrapped round it during its motion.*

327. Cor. It is evident that every point of the axis *B*, *G*, &c. describes a circle round its corresponding center *y*, *z*, &c. in the time of a revolution of the wheel; and that any points of the wheel and axis, *A* and *B* describe similar arcs of circles in the same time, and, consequently, their velocities are as the peripheries or radii of the circle described by them.

328. PROP. *If the directions of* P *and* W *be perpendicular to the radii of the wheel and axis respectively, they are in equilibrio when* P : W :: *radius of the axis* : *radius of the wheel.*

DEM. The same power is required to support *W* to whatever point of the axis it be applied, because the distance from the corresponding center of motion is the same, and the wheel and axis may be reduced to a bent lever, and consequently there will be an equilibrium when *P* : *W* :: *W*'s distance from the center of motion : *P*'s distance :: radius of the axis : radius of the wheel (277). Q. E. D.

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* Keil's Physics, Lect. X. Helsham, Lect. VII. Muschenb. Ch. VIII. Sect. CCCCXLIII. Emerson, Prop. XXIV. Varignon, Tom. I. Sect. IV.

Otherwise:

This proposition is usually proved by the following process: since the directions of P and W are perpendicular to their respective distances from their centers of motion, they are wholly efficient, and P's velocity is to W's velocity as the periphery of the wheel to the periphery of the axis, and consequently, when there is an equilibrium, $P : W ::$ periphery of the axis : periphery of the wheel :: radius of the axis : radius of the wheel (272). Q. E. D.

329. Cor. 1. If the thickness of the rope, to which W is appended, be not inconsiderable, it ought not to be neglected; for, when one or more spires of the ropes are folded about the axis, the distance of W's direction from the center of motion is encreased, being equal to the semidiameters of the axis and ropes; and there is an equilibrium when $P : W ::$ the distance of W's direction from the center of motion : semidiameter of the wheel.

330. Cor. 2. If P and W act in the same plane, and in the directions PD, and WD, meeting in D, and be in equilibrio, they are equivalent to a third force, or pressure upon the axis at A, whose direction meets PD, and WD in D (216); and, producing PD, WD, these three forces are to each other, as the sides DF, DE, and diagonal DG, of the parallelogram EF; therefore $P : W :: DF : DE$, or drawing AN, AM, perpendicular to WD and FD respectively, $P : W :: AN : AM$ (311). FIG. XC.

331. Cor. 3. The pressure upon the axis at $A (Pr) : P :: DG : DF ::$ sin. $\angle DFG$ or PDW : sin. $\angle FGD$ or ADW; $Pr : W :: DG : DE ::$ sin. $\angle DEG$ or PDW : sin. $\angle DGE$ or ADP; and $P : W ::$ sin. $\angle ADW$: sin. $\angle ADP$. When the angle PDW is infinitely small, or PD and WD are parallel, the perpendiculars AN, AM, are to each other as $AW : PA$.

FIG. XCI. 332. PROP. *In a combination of wheels* GH, FK, EL, *whose axes are* QM, RN, BD, *an equilibrium obtains when* P : W :: QD × RC × AB : DG × CF × AE.

DEM. Let Q, R, W, be respectively in equilibrio with P, Q, R; and $P : Q :: QD : DG$, (328)
$Q : R :: RC : FC$,
$R : W :: AB : AE$, and, componendo, $P : W :: QD \times RC \times AB : DG \times FC \times AE$. Q. E. D.

333. Cor. 1. If the ratios of $QD : DG$, $CR : CF$, $AB : AE$ be the same, and the number of wheels be n, $P : W :: QD^n : DG^n$; and if this given ratio be that of $1 : r$, $P : W :: 1 : r^n$ and $P = \frac{W}{r^n}$.

334. Cor. 2. If the peripheries QM, RN, were to touch the peripheries FK, EL, and operate upon them by means of teeth made in each, this analogy would still obtain, and P would be to W as the product of the semidiameters of the axes, to the product of the semidiameters of the wheels. As if XY be a combination of
FIG. XCII. wheels, and W be appended upon the axis YF, and the power P act at P; $P : W$ :: products of the semidiameters of the pinions B, D, F : product of the semidiameters of the wheels A, C, E.

335. Cor. 3. Because the number of teeth in the wheels and pinions are to each other as their peripheries, or radii, $P : W$ :: semidiameter of the axis to which W is appended multiplied into the number of teeth in the pinions : the length of the lever where P acts multiplied into the number of teeth in the several wheels.

336. Cor. 4. The number of revolutions of a pinion or wheel being inversely as the time of one revolution, or inversely as the periphery or number of teeth in it, the number of revolutions of

the wheel where P acts, is to the number of revolutions, in the same time, of the axis to which W is appended, as the product of the number of teeth in the wheels to the product of the number of teeth in the pinions.

337. PROP. *If a wheel with teeth* C D A *revolve about* C, *and impel the wheel* S B A *round with it, by the action of teeth* B D, b d, *&c. upon* B *and* b, *whose curvature is such that* B, b, *describe the lines* B D, b d, *whilst they describe* B A *and* b A *respectively, the moments of these wheels are equal.* FIG. XCIII.

DEM. Because the points B and D come into contact at A, at the same time, every part of BA is applied to AD, and is therefore equal to it, and consequently B and D move towards it with equal velocities; and therefore if equal weights act at B and D perpendicular to SB, CD, they must be in equilibrio. Q. E. D.

338. Cor. 1. If $C D n B$ be a bent lever whose fulcrum is at C, equal forces, acting at D and B in directions perpendicular to CD, SB, would keep the levers $C D n B$ and $S B$ in equilibrio.

339. Cor. 2. Because the point B moves over BD whilst the arc BA rolls over DA, the figure of the tooth $B n D$ is an epicycloidal arc; and the effect is the same whether B describe the concave or convex side of $B n D$, and consequently whether the wheel CDA impel $S B A$ by the action of the convex side of the tooth upon B; or $S B A$ impel CDA round with it by the action of B upon the concave side of the tooth $B n D$.

340. Cor. 3. If any number of epicycloidal teeth DB, db and Aa, be inserted in the periphery AD, at equal distances from each other, and teeth B, b, A, be inserted in the other wheel AB, at equal distances, and $Ab = Ad$, and $AB = AD$, the teeth

S 2 B, b, A,

$B, b, A,$ will all act together with equal moments; for because $Ab = Bb$, the velocities and consequently moments of B and b are equal, and they act together, because b is always found in bd and B in BnD.

341. Cor. 4. The effect is the same when the epicycloidal teeth are upon the periphery of the wheel SBA; for, whilst B and D move to A, B describes the epicycloid BnD, and D describes another epicycloid DmB, whose base is BA.

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342. The teeth should not act upon each other before they arrive at SC, joining their centers, and the machine is more or less complete according to the number of teeth acting together. The action of any tooth should not cease before that of the succeeding tooth begins.

PULLEY*.

FIG. XCV. 343. Def. *A pulley is a small circular wheel, as* mEn, *revolving about an axis passing through its center, by the action of a power which is applied to a rope passing over the pulley.*

344. Prop. *In a single fixed pulley, an equilibrium obtains when the power* P, *is equal to the weight* W.

Dem. When a rope is stretched and quiescent, it is evident that the tension of every part is the same, otherwise motion would ensue; therefore the tension of Pm is equal to that of Wn and $P = W$. Q. E. D.

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* Muschenbroek, Ch. VIII. Sect. CCCCXCIV. Varignon, Tom. I. Sect. III. Keil's Physics, Lect. X. Helsham, Lect. VII. Hamilton's Principles of Mechanics, pag. 162. Emerson, Prop. XXVII. Desaguliers, pag. 99.

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345. This propoſition is ſometimes proved by conſidering the pulley as a lever; for the moments of P and W being the ſame to whatever parts of their directions they are applied, but, if applied at m and n, mEn is a ſtraight lever whoſe center of motion, E, is in its biſection, and conſequently when there is an equilibrium $P = W$ (274). And the concluſion thus deduced is certainly ſatisfactory without any other demonſtration.

346. Cor. 1. If the ſame rope paſs over any number of fixed pulleys, P is equal to W when there is an equilibrium, becauſe the tenſion of the rope is, in every part, the ſame.

347. PROP. *If the weight*, W, *be ſuſtained by a power* P, *applied to a rope paſſing over a moveable pulley* E, P : W :: 1 : 2. FIG. XCVI.

DEM. Let the tenſion of the rope $PA = m$, and that of BC, and DF, is each equal to m, and conſequently the tenſion of EW is equal to $2m$; therefore $P : W :: m : 2m :: 1 : 2$. Q. E. D.

348. Cor. 1. If W be ſupported by P, applied to a rope paſſing over any number of moveable pulleys (n), $P : W :: 1 : 2n$; for the number of ropes ſupporting the weight is equal to $2n$, each ſupports an equal part of it, and the tenſion of the rope, to which P is applied, is equal to that of one of them.

349. Cor. 2. If W be ſupported by P, applied to a rope paſſing over each pulley in two blocks, to the lower of which W is appended; $P : W ::$ 1 : number of ropes at the lower block. For all the ropes A, B, C, D, E, ſupport W, and each ſupports an equal part, becauſe their tenſion is the ſame, and the tenſion of each is equal to that of F to which P is applied. FIG. XCVII.

350. Cor.

350. Cor 3. If a moveable pulley, L, be annexed to this ſyſtem, P will ſupport a weight equal to $2 \times W$; for the tenſions of the ropes G and H are equal and they both ſuſtain the weight, which is therefore equal to $2 \times W$.

351. Cor. 4. It is evident that P acquires no moment or quantity of motion by this diſtribution of the weight amongſt a number of pulleys; for the velocity of P is to that of W as the number of ropes, ſupporting the weight, to unity. If W be elevated through any ſpace equal to s, any point in each of the ropes ſupporting it, muſt move through a ſpace equal to s, and P conſequently through a ſpace equal to s multiplied into the number of ſtrings.

FIG. XCVIII. 352. PROP. *If the body* W *be ſupported by the power* P, *in a ſyſtem compoſed of one fixed, and any number of moveable pulleys, each having a ſeparate ſtring,* P : W :: *unity* : *that power of* 2 *whoſe index is the number of moveable pulleys.*

DEM. Let the tenſion of the ſtring, to which P is applied, be equal to m, and the tenſion of A and B is each equal to m; that of C and D is each equal to $2m$; that of E and F is each equal to $4m$; and that of GH to $8m$; therefore $P : W :: m : 8m :: 1 : 8$; and, if the number of moveable pulleys be equal to n, it is evident that the tenſion of the ſtring ſupporting W is equal to the n^{th} term of the geometric ſeries 2, 4, 8, &c. or to 2^n and $P : W :: 1 : 2^n$. Q. E. D.

353. Cor. 1. $P \times 2^n = W$ and $P = \frac{W}{2^n}$; and if the number of moveable pulleys and P or W be given, the number of moveable pulleys, or n, may be found.

354. Cor.

354. Cor. 2. The parts of W, sustained by the several moveable pulleys, &c. are to each other as 2, 4, 8, 16, &c.

355. Prop. *If the body,* W, *be sustained by the power,* P, *in a system of pulleys where the rope passing over each pulley is immediately fixed to* W, P : W :: *unity : that power of two, diminished by unity, whose exponent is the number of ropes, or number of moveable pulleys encreased by unity.* FIG. XCIX.

Dem. Let the tension of the rope, to which P is applied, be equal to m, and it is evident that the tensions of the ropes G, B, D, F, are equal to m, $2m$, $4m$, $8m$, respectively; but these ropes entirely support W, and consequently $P : W :: m : \overline{1+2+4+8} \times m :: 1 : 15$. If the number of ropes fixed to W be equal to n, it is evident that $P : W :: 1 : 1+2+4+8$, &c. continued to n terms $:: 1 : 2^n - 1$. Q. E. D.

356. Cor. 1. $P \times \overline{2^n - 1} = W$ and $P = \frac{W}{2^n - 1}$; and if any two of these magnitudes P, W, or n, be given, the other may be found.

357. Cor. 2. m, $2m$, $4m$, $8m$, &c. express the ratio of the parts of W respectively, supported by G, B, D, F, &c.

358. Cor. 3. If the rope, to which P is applied, instead of being fixed to W pass over a pulley to which W is appended, $P : W :: m : 4m :: 1 : 4$; for W is supported by the ropes E, G, F, whose tensions, compared with that to which P is applied, are respectively equal to m, $2m$, m, and the sum of their tensions $= 4m$. FIG. C.

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FIG. CI. 359. When the directions of the ropes PA, QB, to which two powers Q and P are applied, and the direction in which W acts, are parallel to each other, as is supposed in the preceding propositions, it is evident that W is exactly equal to $P+Q$, because they just support it, and their force is all effective, no part being lost by obliquity of direction. This also appears by considering BCA to be a lever, for (287) $W : P+Q :: CB+CA : AB$, and consequently $P+Q=W$ and $P=Q=\frac{W}{2}$. But if the direction in which P acts be changed to ZA, touching the pulley in D, motion will ensue, parallel to the horizon, as P acts partly in that direction, and the quantity and direction of Q must be changed to restore the equilibrium. It is evident that the pulley cannot be at rest, till the horizontal parts of the forces P and Q be equal to each other, and the parts contributing to the support of W be each equal to $\frac{W}{2}$. This is also obvious from (216); for the pulley BCA is acted upon by three forces, whose directions are not parallel, and is quiescent; and these forces are therefore in the same plane, their directions meet in the same point D, and are consequently equally inclined to the direction of W, or to the horizontal line BA.

FIG. CII. 360. PROP. *If* W *be supported by two powers* P *and* Q, *whose directions touch the pulley in* A *and* B, W : P *or* Q :: *sin.* $\angle$ *contained between* PA *and* QB : *sin. of half that angle.*

DEM. Because the pulley is acted upon by three forces P, Q, W, and kept at rest, and their directions are perpendicular to the sides of the triangle CBA, $W : P :: BA : CA$, and $W : Q :: BA : CB$; therefore $W : P$ or $Q ::$ sin. $\angle BCA$ or its supplement to two right angles BDA : sin. $\angle BDC$ (equal to the $\angle CBA$ or CAB). Q. E. D.

361. Cor.

361. Cor. 1. Becaufe $P : W :: CA : BA$ and $Q : W :: CB : BA$; therefore $P + Q : W :: CA + CB : BA$.

362. Cor. 2. If AB be an arc of 60 degrees, or the angle ADB be equal to 120 degrees, AB is equal to AC or CB, and confequently $W = P$ or Q.

363. Cor. 3. Becaufe $P : W :: AC : AB$, $P = \frac{W \times AC}{AB}$. If therefore the arc AB vanifh, or the angle ADB be equal to 180°, $P = \frac{W \times AC}{0}$, or P is infinitely great compared with W. As the arc AB encreafes to a femicircle, P decreafes and becomes the leaft poffible when it is a femicircle, becaufe the chord AB is then the greateft poffible. In this cafe PA and QB are parallel to each other and to CD, and $W = P + Q$, becaufe $AB = AC + CB$, and confequently $P = \frac{W \times AC}{AB} = \frac{W}{2}$. As the arc BA encreafes beyond the femicircle, P encreafes and becomes infinite when it is equal to the periphery.

364. Cor. 4. If P be finite, W is either finite or evanefcent; for $W = \frac{P \times AB}{AC} = 0$, when AB vanifhes, and is finite when AB is finite.

365. Prop. *If* W *be fuftained by* P *in a fyftem of moveable pulleys* v, y, x, *each of which has a feparate rope, and the angles contained by the directions of the ropes be* FSE, DTC, BRA; W : P :: EF × DC × BA : vE × yC × xA. FIG. CIII.

DEM. Let W, T, R, P represent the tensions of the strings vW, yT, xR, PG respectively; and (360) $W : T :: EF : vE$
$T : R :: CD : yC$
$R : P :: AB : xA$; and consequently
$W : P :: EF \times CD \times AB : vE \times yC \times xA$.
Q. E. D.

366. Cor. 1. If the pulleys and the angles FSE, DTC, BRA be equal, $W : P :: EF^3 : vE^3$; and, if the number of moveable pulleys be equal to n, $W : P :: EF^n : vE^n$.

367. Cor. 2. If the directions of the ropes become parallel, $W : P :: 2Ev \times 2yC \times 2xA : Ev \times Cy \times Ax$, and, when the pulleys are equal, $W : P :: 2 \times 2 \times 2 : 1$; or as that power of two, whose exponent is the number of moveable pulleys, to unity, which coincides with (352).

368. Cor. 3. If the tension of the rope PG, to which P is applied, be equal to m, $R = \frac{m \times AB}{Ax}$, $T = \frac{m \times AB \times CD}{Ax \times Cy}$, $W = \frac{m \times AB \times CD \times EF}{Ax \times Cy \times Ev}$.

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369. The conclusions, derived from considering the tensions of the several ropes in any system of pulleys, may also be investigated by supposing the moments of P and W to be equal, the velocities involved in these moments being reduced to opposite directions. 1. In fig. 99. the ropes being all parallel to the direction in which W acts, the velocities of P and W are in opposite directions and entirely efficient; and, when an equilibrium obtains, the product of P and its velocity is equal to the product of W and its velocity: for let $Ll = Mm = Nn = Oo = v$, and if the point L be

be elevated to l, through a ſpace equal to v, the pulley EF will deſcend through a ſpace equal to v, and any point of the rope E through a ſpace equal to $2v$, and, when the point M is elevated to m, through $3v$; therefore the pulley BA will deſcend through a ſpace equal to $3v$, and when the ropes A and B are ſtretched, A muſt deſcend through a ſpace equal to $6v$, and when the point N is elevated to n, through $7v$, &c.; conſequently the velocities of W and the ſeveral ſtrings E, C, A, P, are reſpectively as v, $3v$, $7v$, $15v$, and when an equilibrium obtains $P:W::1:15$. 2. In fig. 98. let W aſcend through a ſpace equal to v, and the ropes F,E,A will evidently be elevated through ſpaces equal to $2v$, $4v$, $8v$, reſpectively, and if the number of moveable pulleys be equal to n, it is clear that the velocity of P : velocity of $W :: 2^n \times v : v :: 2^n : 1$, and when there is an equilibrium P and W are inverſely as their velocities, or as $1 : 2^n$. 3. If the ropes, ſuſtaining the pulleys be not parallel to each other; let EFV be a pulley ſuſtained by two powers E and F, whoſe directions meet in S; and if W be raiſed through a very ſmall ſpace equal to SV, E and F, acting parallel to ES and FS, will deſcribe the ſpaces SE and SF reſpectively, and the velocities of W, E, F, are to each other as SV, SE, SF. Reſolve SF and SE, each into two, Sm, mF, and Sn, mE, of which mF and mE being equal and oppoſite deſtroy each other, and the remainders are wholly efficient; therefore, in an equilibrium, $\overline{E+F} \times Sm = W \times SV$; but $\overline{E+F} \times SF : \overline{E+F} \times Sm$ (or $W \times SV$) $:: SF : Sm : FV : Fm$, and $\frac{\overline{E+F} \times SF}{2}$ $(E \times SE)$: $W \times SV :: \frac{FV}{2} : Fm :: FV : FE$, or the power or tenſion at E is to the moment of W as $EV : FE$, which coincides with what is proved in (360). FIG. CIV.

I 2 INCLINED

INCLINED PLANE.

PLATE XIII. FIG. CV. 370. DEF. *A plane surface inclined, in any angle, to an horizontal plane, is called an inclined plane. If* BA *be drawn upon the inclined plane, and* BC *upon the horizontal plane, from the same point* B *of their common intersection, perpendicular to it, the angle* ABC *is the angle of elevation or inclination; and if* PW *be the direction of the power, the angle* PWA *is called the angle of traction.*

371. PROP. *If a body, whose weight is* W, *be just supported upon the inclined plane* AB *by a power* P, *acting in the direction* PW; P : W :: PW : PC *(supposing* PC *to be perpendicular to the horizontal line* BC) :: *sin.* ∠ *inclination* ABC : *cos.* ∠ AWP.

DEM. The weight *W*, acting in the direction *WL* or *PC* perpendicular to the horizon, is supported by the power *P*, and reaction of the plane, acting respectively in the directions *PW*, and *WC* perpendicular to the plane; and consequently *P*, *W*, and pressure upon the plane, are to each other respectively as the sides *PW*, *PC* and *WC* of the triangles *PWC* (196), and $P : W :: PW : PC ::$ sin. $\angle PCW$ or ABC : sin. $\angle PWC$ or cos. of $\angle PWA$. Q. E. D.

Another demonstration:

FIG. XVI. Let *WZ*, representing the weight of *W*, or its tendency to descend in a direction perpendicular to the horizon, be resolved into two, one *SZ* perpendicular to the plane, and the other *WS* parallel to it. *WS* represents the tendency of *W* to descend upon the plane, and, taking $Wx = WS$, and drawing xE perpendicular to the plane, any force *WD*, *WE*, &c. drawn from *W* and terminated by xE, will be in equilibrio with *W*, because the only effi-

cient

cient part of WE, resulting from resolution, is equal and opposite to WS; but $P : W :: WE : WZ :: \frac{WX \times \text{rad.}}{\text{cos.} \angle EWX} : \frac{WS \times \text{rad.}}{\text{sin.} \angle SZW} ::$ sin. $\angle SZW$ or ACB : cos. $\angle EWA$.

372. Cor. 1. The same force, acting in parallel directions, is required to support the same weight upon every part of the plane; because the angle of traction is the same, and $P = \frac{W \times \text{sin.} \angle \text{inclin.}}{\text{cos.} \angle \text{of traction}}$, which is a given quantity.

373. Cor. 2. Because $P = \frac{W \times \text{sin. of} \angle \text{inclin.}}{\text{cos.} \angle \text{traction}}$, if W and the plane be given, P is the least possible when the cosine of the angle AWP is the greatest possible, or when PW is parallel to the plane: as the angle PWA encreases from hence, its cosine decreases, and consequently P encreases, and becomes equal to W, when PW is perpendicular to the horizon, and infinitely greater than W when PW coincides with WG or WC. FIG. CV.

374. Cor. 3. When PW is parallel to the plane, $P : W ::$ sin. $\angle$ of inclin. : radius $:: AC : AB$; when it is parallel to the base, $P : W ::$ sin. $\angle$ of inclin. : sin. $\angle BAC$ (this angle being the complement of the angle of traction) $:: AC : BC$.

375. Cor. 4. If AB be perpendicular to the horizon, and the angle $PWA = 0$, $P = W$; for the cos. of $\angle PWA =$ sin. of 90° = sin. of $\angle$ of inclination. If, in this supposition, PW be parallel to the horizon, P is infinite; for cos. of $\angle$ of traction = 0.

376. Cor. 5. Let Pr represent the pressure upon the plane, and $Pr : W ::$ sin. $\angle WPC$: sin. $\angle PWC$ or cos. $\angle$ of traction, and $Pr =$

$W \times$

$\frac{W \times \text{sin.} \angle WPC}{\text{cos.} \angle PWA} = 0$, when PW is perpendicular to the horizon, and infinite when the $\angle PWA$ is a right angle.

377. Cor. 6. Let CI be perpendicular to P's direction, and the sides of the triangle CIB, CI, CB, and IB, are respectively perpendicular to the directions of P, W, and pressure upon the plane, and consequently P, W, and Pr, are respectively as CI, CB and IB.

378. Cor. 7. If WCP were an angular balance revolving upon the fulcrum C, and acted upon by two forces in the directions PW and WL, an equilibrium would obtain between them when $P : W$ inversely as the perpendiculars let fall from C upon their directions, or $P : W :: CL : CK$ (272); but $CL : CK ::$ sin. $\angle CWL$ or WCP : sin. $\angle CWK$ or $CWP :: PW : PC$.

FIG. CVII. 379. Cor. 8. If P and W be in equilibrio, their perpendicular velocities are to each other inversely as their magnitudes; for let W descend through an infinitely small space WA, and its perpendicular descent is WY, and P's perpendicular ascent is equal to the difference between PA and PW, or Am, if Em be perpendicular to PA. Draw FDn perpendicular to PA, and DZ to FA, and $Am : WY ::$ P's velocity : W's velocity $:: An : DZ :: AF : DF$ (sim. triangles) $:: W : P$ (377).

FIG. CVIII. 380. Prop. *If two bodies* W *and* V, *placed upon the inclined planes* AB, AD, *support each other by means of a rope passing over the pulley* P, V : W :: *cos.* $\angle$ AVP × *sin.* $\angle$ ABC : *cos.* $\angle$ PWA × *sin.* $\angle$ ADC.

Dem. The tension of the rope VPW is every where the same, or the same power, P, is required to support V and W; but

P :

$P : W :: \text{fin.} \angle ABC : \text{cof.} \angle PWA$; and
$V : P :: \text{cof.} \angle AVP : \text{fin.} \angle ADC$; therefore
$V : W :: \text{fin.} \angle ABC \times \text{cof.} \angle AVP : \text{cof.} \angle PWA \times \text{fin.} \angle ADC$.
Q.E.D.

381. Cor. 1. If PV and PW be parallel to AD and AB refpectively, $P : W :: AC : AB$,
$V : P :: AD : AC$; therefore
$V : W :: AD : AB$.
This conclufion is alfo derived from the general expreffion, the cofines of the angles PVA and PWA becoming AD and AB.

382. Cor. 2. If PV and PW be parallel to the bafes of the planes, $P : W :: AC : BC$,
$V : P :: DC : AC$, and
$V : W :: DC : BC$.
This alfo follows from the general expreffion, the complements of the $\angle$s AVP and AWP, being, in this fuppofition, refpectively equal to DAC and BAC.

WEDGE.

383. Def. *A wedge is a hard body, generally of a triangular prifmatic figure as* AF, *which is generated by the motion of the triangle* AED *upon the right line* EF *always perpendicular to its plane.* FIG. CIX.

If AED be an ifofceles triangle, it is called an ifofceles wedge, if fcalene, a fcalene wedge. $ABCD$ is the back of the wedge, upon which a force is impreffed chiefly by percuffion; and $ABFE$, $DEFC$, are the fides of the wedge, upon which the refiftances of wood, &c. act and counterpoife the force of percuffion. The angle AED is the vertical angle of the wedge.

384. Prop. *If two equal refiftances, acting in equal angles upon the fides of an ifofceles wedge* ABV, *be in equilibrio with a power,* P, *acting*

FIG. CX. *ing in a direction perpendicular to the back; P is to the resistances, as the rectangle of the sines of half the vertical angle of the wedge and the inclination of the resistance to the sides, to the square of the radius.*

DEM. Let the directions and quantities of the resistances be CD and Cd; resolve each into two, DE and de, in the directions of the sides, and CE, Ce, perpendicular to them, and DE, de, are lost by their obliquity. Resolve EC, eC into two forces, EF, eF, parallel to the base, and FC, FC perpendicular to it; EF, ef, being equal and opposite, destroy each other, and the forces $2FC$, being exactly opposite to, and in equilibrio with P, are equal to it. But $FC : EC ::$ sin. $\angle FEC$ or $\angle BVC$: rad.

$EC : DC ::$ sin. $\angle EDC$: rad.; and

ex æquo $FC : DC ::$ sin. $\angle BCV \times$ sin. $\angle EDC : \overline{\text{rad.}}|^2 :: 2FC$ or $P : 2DC$ or resistances. Q. E. D.

385. Cor. 1. If the resistances be perpendicular to the sides, the sine of the angle ADC becomes the radius; and consequently P : resistances :: sin. $\angle CVA$: radius :: $AC : AV :: AB : AV + BV$.

386. Cor. 2. If the resistances be perpendicular to the axis, P : resist. :: $BC \times CV : BV^2$; and, when the angle AVB is equal to two right angles, P : resist. :: $BC \times 0 : BC^2 :: 0 : BC$, or P is infinitely less than the resistances.

387. Cor. 3. When the resistances are perpendicular to the back AB, the angle $EDC = BVC$, and P : resist. :: sin.^{2} $\angle BVC$: rad.^{2} :: $BC^2 : BV^2$. If the angle BVA be equal to two right angles $BC = BV$, and $P =$ resistances; and if the angle BVA be diminished without limit, the resistances are encreased without limit.

388. Cor.

388. Cor. 4. If P be the same in the two suppositions of cor. 1. and 2; resist. in cor. 1 : resist. in cor. 2 :: $CV : BV$.

389. Cor. 5. If the resistances be given in cor. 2, P is the greatest possible when the angle BVA is a right angle.

390. Cor. 6. If the quantities of the resistances be given, and their directions be variable; P is the greatest possible in the common supposition of cor. 1.; for it varies as the sine of the angle EDC, which is the greatest when that angle is right.

SCHOLIUM.

391. The relation, subsisting between the power and resistances in the wedge, has been traced through various processes and modes of reasoning, by different philosophical writers*; and different, and frequently contradictory, conclusions have been deduced from different demonstrations, some of which must consequently be erroneous. As in all equilibria of forces, they must either obliquely, or directly, oppose each other, and their intensities, estimated in opposite directions, must be equal; one source of error hath resulted from a misapplication of this equality of moments, which is universally true and will never lead to false conclusions, if the velocities be reduced to opposite directions, and the efficient parts only of the moments be supposed equal. 1. Let two equal resistances R and r act perpendicularly upon the sides of an isosceles wedge ABD, and be in equilibrio with a power P acting perpendicularly upon the base AB; and taking Dd, the continuation of PD, very small, and drawing dE, de parallel, and DE, De perpendicular, to the sides, the velocities of P, R, r, are to each other as Dd, DE, De respectively, these being described by them in the same time. Resolve DE, De into two, EL and eL parallel FIG. CXI.

* Rohault Not. ad a, 9. Maclaurin's Newt. Chap. III. Art. 21. Graves, Lect. I. C. X. Desaguliers, p. 107. Emerson, Prop. 30. Muschenbroek, Sect. CCCCLXIII. and CCCCCXCI.

rallel to the base, which being equal and opposite destroy each other, and DL, dl perpendicular to the base, which being opposite to P's direction, are wholly efficient; and because there is an equilibrium $\overline{R+r} \times DL = P \times Dd$, but $\overline{R+r} \times DL$ or $P \times Dd : \overline{R+r} \times DE :: DL : DE :: AP : AD :: AB : AB + AD$, the same as (385).

When R and r act in any other direction, the relation of their moments to that of P is discoverable by a similar process. If they be parallel to the base, draw Mdm parallel to their direction, and the velocities of P, R and r, are respectively as Dd, DM and dm; and resolving DM, dm into two DE, de perpendicular, and ME, me parallel, to the sides, the last not being opposed to the sides are quite inefficient; and repeating the process above $\overline{R+r} \times DL = P \times Dd$; and $P \times Dd : \overline{R+r} \times DE :: AP : AD$

$$\overline{R+r} \times DE : \overline{R+r} \times DM :: DE : DM :: PD : AD$$

and $P \times Dd : \overline{R+r} \times DM :: AP \times PD : AD^2$; which is the same as (386).

A different analogy is investigated by Rowning, from the same principle, viz. the power : the resistances :: $AP : PD$.

2. Another source of error is an improper use of this principle, "that if three forces act upon a body, which remains at rest, they are to each other as the three sides of a triangle, respectively parallel or perpendicular to their directions;" which certainly is untrue, if any parts of them be inefficient and lost by obliquity of direction; for the force FG will have exactly the same effect upon the sides of the wedge AD with FH, FI, or the perpendicular FL, supposing GL to be parallel to AD, and of the parts, resulting from a resolution of each into two forces, one perpendicular to, and the other coincident with, the sides, these last, Fx, &c. are lost. When the wedge does not fill the cleft, and the resistances act perpendicularly upon its sides, Maclaurin and Varignon have applied the above principle and deduced different conclusions, both of which cannot be true. Dr. Hamilton, Gravesands, Desaguliers, Emerson, Muschenbroek assert, that the power is to the resistances as half the back is to the height, and Keil, Whiston and

and Nicholson, assign this ratio to be that of the whole back to the height. The first analogy coincides with this general theorem, "that the power is to the resistances, when in equilibrio, as a line drawn from the bisection of the base to one side, parallel to the resistance upon that side, to the height of the wedge; and this theorem is presumed to be erroneous for the following reasons.

1. If the directions of the resistances be parallel to the axis, their sum is equal to the power, according to the theorem, which cannot be true; because the power acting perpendicularly is wholly efficient, but part of the resistance is lost by obliquity of direction.

2. If the vertical angle of the wedge be diminished without limit, a line drawn, from the bisection of the base to a side, parallel to the resistance, accedes to equality with the height when the direction of the resistance is parallel to the height, and the power and resistance become equal, according to the theorem; but, the power continuing the same, the directions of the resistances accede to a parallelism with the sides, and they become wholly inefficient.

3. When the resistances act perpendicularly to the axis, and the vertical angle of the wedge accedes to two right angles; according to the theorem, the ratio of the power to the resistances encreases without limit, which cannot be true, because the inefficiency of the resistances evidently encreases without limit, and consequently the ratio of the resistances to the power encreases without limit, which is directly the reverse of the theorem.

4. If the power be the same in the two cases where the directions of the resistances are perpendicular to the sides and axis, of the wedge, the resistances are, according to the theorem, as the side and height of the wedge respectively; or greater resistances are required to sustain a given power, when their directions are perpendicular to the sides, and therefore entirely efficient, than when they are oblique and consequently not wholly efficient, which certainly is untrue.

It is presumed, for these reasons, that the analogy, derived from the theorem, and consequently the theorem itself, cannot be generally true; and were these reasons less decisive, it would not be difficult to point out the several oversights in the demonstration which induced a false conclusion*.

FIG. CXIII. 392. PROP. *If the power and resistances* P, R, *and* r, *act perpendicularly upon the back and sides of a scalene wedge* ABD, *and be in equilibrio*, P : R + r :: *back of the wedge* : *sum of its sides*.

DEM.

*THEOREM. If a power P, acting perpendicularly upon the back of an isosceles wedge ABC, be in equilibrio with the equal resistances E and F, acting perpendicularly upon its sides, $P : E + F$ as a line PE drawn from the bisection of the base P to one side, parallel to the resistance upon it, to the height of the wedge PB.

FIG. CXII.

DEM. Let the directions of E and F meet in the bisection of the base P, and $PE = PF$, and, completing the parallelogram, $P : E + F :: PN : PE + PF$ (196) $:: PH : PE :: PE : PB$.

Cor. If PN be the force of P, PE and PF will represent the force with which P protrudes the resistances in directions perpendicular to the sides.

THEOREM II. If the directions of the resistances be any other lines PD, PO, equally inclined to the sides, the power is to the resistances $(E + F)$ as PD to PB.

DEM. Resolve PE and PF into two, PG and PK in the directions of PD and PO, and GE, FK perpendicular to them; and these last forces, acting perpendicularly to the resistances, are lost. If PN be the force of P, PE and PF are its force perpendicular to the sides, and $PG + PK$ the force in equilibrio with the resistances; therefore $P : E + F :: PN : PG + PK :: PH : PG :: PD : PB$, because $PE^2 = PG \times PD = PH \times PB$, and consequently $PH : PG :: PD : PB$. Q. E. D.

In this demonstration, EG and FK, being inclined to the sides of the wedge, are not inefficient, and, not being opposite, they do not destroy each other, and consequently ought not to be neglected. And, besides, if the power estimated in the directions of PD and PO be equal to $PG + PK$, these will be equal to the power estimated in the direction PN, or equal to PN. Resolve PG and PK into two, GL, OL, perpendicular to the axis, which being equal and opposite, destroy each other, and PL and PL are the only remaining parts of the force, and these are never equal to PN, unless PD and PE coincide. *Hamilton on the Principles of Mechanics.*

DEM. Becaufe the power and refiftances act perpendicularly upon the back and fides, they are wholly efficient; their directions are in the fame plane, meet in the fame point (216), and their magnitudes are to each other as the three fides of a triangle parallel, or perpendicular, to their directions; therefore $P:R+r :: AB:AD+DB$. Q.E.D.

393. Cor. If PM, RE, re, be the relative magnitudes of P, R, r, in equilibrio, and lines be drawn through M, E, and e, parallel to the fides of the wedge refpectively, any forces, whofe magnitudes are PL, RF, rf, drawn from P, R and r, terminated by the lines drawn parallel to the fides, will be in equilibrio; for their perpendicular and only efficient parts PM, RE and re are in equilibrio.

SCREW.

*394. DEF. *If a right line, divided into equal parts* AC, CE, EG, *&c. reprefenting equal inclined planes, be fo wrapped round the convex and concave furfaces of two cylinders with equal bafes, that* AB, CD, EF, *&c. the horizontal bafes of the planes, may be bent into the peripheries of circles parallel and equal to the bafes of the cylinders; equal fpirals will be formed upon their furfaces, whofe lengths are* AC, CE, EG, *&c. and diftances* CB, ED, FG, *&c. the perpendicular heights of the planes. The convex cylinder is inferted in the concave, and fo adapted to it that the fpirals protuberant upon one cylinder may exactly fill the excavated fpiral or groove upon the other: the firft is called the external, and the fecond the internal, fcrew.* PLATE XIV. FIG. CXIV.

Another definition:

If a cylinder move uniformly about its axis, whilft a point moves uniformly upon its furface in a right line parallel to the axis, the line, defcribed by this componnd motion, is a fpiral, which, being raifed upon the external furface of the cylinder, forms the external fcrew; and a fimilar fpiral groove being cut upon the internal furface of a hollow cylinder,

* Maclaurin, Chap. III. XXII. Keil's Phyfics, Lect. X. Rohault, Not. ad 9. Emerfon's Mechanics, Prop. 29.

linder, of the same diameter, to receive the protuberant spiral, forms the internal screw. Either of these screws is fixed, and the other is moveable by a lever passing through the center, and in the plane, of its base.

FIG. CXV. 395. PROP. *There is an equilibrium upon the screw, when the power,* P, *is to the weight,* W, *or resistances acting parallel to the axis, as the distance between two contiguous spirals, to the periphery of the circle described by the power.*

DEM. It is evident that W, acting upon either of the screws, in a direction parallel to its axis, will equally press every point of the spirals of the other in contact with it, in directions perpendicular to their bases. If ABD be the base of one spiral, p, the magnitude of a power acting at B perpendicular to BC, and in equilibrio with the pressure pr, upon one point of the spiral, P' a power acting at P perpendicularly to PC, and in equilibrio with p or pr; (374) $p : pr$ :: distance between two contiguous spirals $(d) : ABD$; $P' : p :: BC : PC :: ABD$: periphery of the circle described by the power, therefore $P' : pr :: d$: periphery of this circle, and the sum of all the P''s, or P is to the sum of all the pr's, or W in the same ratio, that is, $P : W :: d$: periphery of the circle described by P. Q. E. D.

Another demonstration from def. 2.

Whilst B or P makes one revolution, W is elevated through a space equal to d, therefore the velocity of P is to the velocity of W, as the periphery of the circle described by P to the distance between two contiguous spirals, and consequently when there is an equilibrium, $P : W$ :: dist. between two spirals : to the periphery described by P. Q. E. D.

396. Cor. 1. If the direction of W be inclined to the axis, and P do not act in the plane of the base, but in any other direction, the ratio of P to W may be found by art. 371.

397. Cor.

397. Cor. 2. The pressure sustained by any point of the spirals is to the part of W incumbent upon it, as the length of a spiral to its base ABD.

398. Cor. 3. In an endless or perpetual screw, acting upon a wheel, the distance of whose teeth is equal to AB, the distance of two contiguous spirals, and in equilibrio with a body W, suspended from the axis EF; $P : W :: AB \times EF$: diameter of the wheel $\times$ periphery of a circle whose radius is PG; for if R be the resistance of a tooth to a spiral of the screw, $P : R :: AB$: per. of the circle described by P, $R : W :: EF$: diameter of the wheel; therefore $P : W :: AB \times EF$: per. of the circle described by $P \times$ diameter of the wheel. PLATE XII. FIG. XCIV.

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399. The second demonstration of this proposition is generally deemed unsatisfactory, because the moments of P and W are not reduced to opposite directions, but the ratio of P to W may be investigated by finding the opposite and efficient parts of a power sustaining a body upon an inclined plane, and combining it with the power of the lever. Let WC be an inclined plane, or spiral of the screw, W a weight acting perpendicularly to the horizon WE, and supported by a power P, acting parallel to WB; and if W be elevated through a very small space Ww, or perpendicular height Hw, it is evident that P, acting parallel to WE, will descend through a space equal to $WE - we$ or WH; and consequently the velocities of W and P, estimated in opposite directions, are as Hw to HW, or as the height of the plane to its base, and when an equilibrium obtains, $P : W ::$ height of the plane : its base or ABD. But, from the nature of the lever, a power, acting at the distance PC, must be diminished in the ratio of $BC : PC$, or of the circumference ABD : circumference described by the power acting at P. FIG. CXVI. FIG. CXV.

CHAP.

CHAP. IX.

CENTER OF GRAVITY.

400. Def. *The center of gravity of a body or system of bodies is a point about which the parts of the body or system are in equilibrio.*

401. Prop. *To find the center of gravity of a body.*

PLATE XIV. FIG. CXVII. Let A, B, C, D, &c. be particles of the body, and finding the centers of equilibrium p and q, of A and B, C and D respectively (274); let $A+B$ be placed in p, and $C+D$ in q, and their center of equilibrium, G, will be the center of gravity of the particles A, B, C, D, &c. Because the force of gravity acts upon the particles in parallel directions, the efficacy of A to communicate motion to G is $A \times AG$, and that of B is $B \times BG$ (279) or $A \times \overline{Ap+pG}$ and $B \times \overline{Bp+pG}$, which are equivalent to them (181), or $\overline{A+B} \times pG$, since $A \times Ap$ and $B \times Bp$ are equal and opposite, and consequently destroy each other. The sum of the moments of C and D is found, by a similar process, to be the same as if they were placed in q; and consequently G, which is the center of gravity of $A+B$ and $C+D$, placed in p and q respectively, is the center of gravity of A, B, C, D, placed at the points A, B, C, D, &c. Q. E. D.

402. Cor. 1. The particles of the body cannot be in equilibrio about any other point except G; for, if possible, let X be such a point, and it is proved, as before, that the efforts of A and B to move $X = \overline{A+B} \times pX$, and of C and $D = \overline{C+D} \times qX$; therefore the point X is kept in equilibrio by two forces, $\overline{A+B} \times pX$ and

and $\overline{C+D} \times qX$, not acting in opposite directions, which is impossible (187).

403. Cor. 2. In every situation of the body composed of the particles A, B, C, D, &c. if the point G be supported, the body will be at rest; for the force of gravity acting always in parallel directions upon the particles, their moments, or efforts to move G, will always be as $A \times AG$, $B \times BG$, &c. which by the process used in this proposition, will always be reduced to two forces that are equal and opposite.

404. Cor. 3. If $A+B+C+D$, &c. be equal to Q, and the pressure of each in parallel directions be equal to q, a force as $Q \times q$, acting at the point G, in a direction opposite to that in which the particles press, will remove their pressure. Or if A, B, C, &c. be destitute of gravity, and only resist the action of a force by their inertia, a force P acting at G will communicate equal velocities to every particle; because their resistances, being exerted in directions opposite to that of P (3d law of motion) and therefore parallel to each other, vary as their distance from G, and consequently the sums of the resistances on each side of G are equal. And $v.v$, if Q be moving and without gravity, a force applied, at G (the center of inertia) equal to the moment of Q, will destroy all motion.

SCHOLIUM.

405. The particles which compose a body, being connected together by the force of cohesion, every line of particles may be considered as a lever, impressed in different points by the action of gravity, or any force which acts in parallel directions. The particle A therefore is connected with G by the cohesion of the intermediate particles, and, an infinite number of levers terminating in G will therefore be formed, upon which the particles, in any one line, act with intensities varying as their magnitudes multiplied into their distances.

FIG. CXVIII. 406. PROP. *The sum, or difference, of the products, which results from multiplying each particle* A, B, C, D *into its perpendicular distance from any plane* LN, *according as they are on the same or different sides of the plane, is equal to the product of all the particles multiplied into the distance of their center of gravity,* G, *from that plane.*

DEM. Let P and Q be the centers of gravity of A and B, C and D, and drawing right lines through P, Q, G, parallel to the plane, which intersect the perpendiculars drawn from those points respectively; and $A:B::BP:AB::Bn$ or $Bb-Pp:Am$ or $Pp-Aa$,

$$\text{and } A\times\overline{Pp-Aa}=B\times\overline{Bb-Pp}, \text{ or}$$

$$A\times Aa+B\times Bb=\overline{A+b}\times Pp.$$

By a similar process it appears, that $C\times Cc+D\times Dd=\overline{C+D}\times Qq$. But $A+B:C+D::QG:PG::Gv$ or $Gg\mp Qq:Px$ or $Pp-Gg$, and $\overline{A+B}\times\overline{Pp-Gg}=\overline{C+D}\times\overline{Gg\pm Qq}$; or, by transposition and a substitution of equals, $A\times Aa+B\times Bb\pm C\times Cc\pm D\times Dd=\overline{A+B+C+D}\times Gg$, where the higher or lower signs are to be used, according as the bodies are on the same, or a different, side of the plane. Q. E. D.

FIG. CXIX. 407. Cor. 1. If the particles be placed upon the same right line, or Aa, Bb, Cc, Dd, Gg, become Ag, Bg, Cg, Dg, Gg, respectively, it is evident, that $A\times Ag+B\times Bg\pm C\times Cg\pm D\times Dg=\overline{A+B+C+D}\times Gg$; or the sum, or difference, of the products resulting from the multiplication of each particle into its distance from any point g, according as they are on the same, or a different, side of that point, is equal to the product of their sum multiplied into the distance of their center of gravity from that point.

408. Cor. 2. The whole moment of a body, acting upon a lever, being equal to that of every particle, or to the sum of the products which results from the multiplication of each particle into its distance from the center of motion (279), is equal therefore to the

the product of the whole body into the diſtance of the center of gravity from the center of motion, and is conſequently the ſame as if it were collected in the center of gravity. The demonſtration of this propoſition obtains therefore when A, B, C, D, are collections of particles or bodies, whoſe centers of gravity are the points A, B, C, D. And to find the center of gravity of a ſyſtem of bodies, it is evident, that, in the propoſition (401), bodies, whoſe centers of gravity are A, B, C, D, &c. may be ſubſtituted for particles.

409. Cor. 3. If A, B, C, D, be bodies acting upon any plane LN, in parallel directions, the ſum of their efforts to move it is the ſame as if they were collected in their center of gravity; for, if A, B, C, D, be the reſpective centers of gravity of each body, this ſum is equal to $A \times Aa \times Bb \pm C \times Cc \pm D \times Dd = \overline{A+B+C+D} \times Gg$; or, if they be placed upon a lever, the ſum of their efforts to make it revolve is the ſame, as if they were placed at G. When the center of gravity therefore is in the plane, or at the fulcrum of a lever, the plane and lever are quieſcent. And if any point Z be taken in NL, $A \times aZ + B \times bZ + C \times cZ + D \times dZ = \overline{A+B+C+D} \times gZ$; for if a plane paſs through Z, the proof is the ſame as that of this propoſition. FIG. CXVIII.

410. Cor. 4. The diſtance of any plane from the common center of gravity of A, B, C, D, &c. or Gg is equal to $\dfrac{A \times Aa + B \times Bb \pm C \times Cc \pm D \times Dd}{A+B+C+D, \&c.}$; and its diſtance from a plane paſſing through any point Z is equal to $\dfrac{A \times Za + B \times Zb \pm C \times Zc \pm D \times Zd}{A+B+C+D}$, where the lower ſigns are to be uſed for thoſe bodies not on the ſame ſide of Z with A and B.

FIG. CXX. 411. Cor. 5. A right line drawn from A through the center of gravity G of any number of bodies A, B, C, D, &c. will paſs through the center of gravity of the remainder; for $B \times Bb + D \times Dd = C \times Cc$, and conſequently the center of gravity of B, C, D is in the plane paſſing through AG, and if this plane revolve, their center of gravity is always in the plane paſſing through AG, and conſequently it muſt be in the line AG produced, which is the common interſection of the planes. If r be this center, $\overline{B+C+D} \times Gr = A \times AG$, and if the bodies be equal and n their number, $AG = \overline{n-1} \times Gr$.

FIG. CXXI. 412. Cor. 6. If a circle or ſphere be deſcribed about the center of gravity G, of any number of bodies A, B, C, &c. and any point P be taken in the periphery of the circle, or ſurface of the ſphere, $PA^2 \times A + PB^2 \times B + PC^2 \times C$, &c. is a given quantity; for, drawing GP and the perpendiculars to it Aa, Bb, Cc, $A \times Ga = B \times Gb + C \times Gc$ (409), or, by ſubſtitution of equals, $A \times \frac{\overline{GA^2 - PA^2 + GP^2}}{2GP} = B \times \frac{\overline{PB^2 - BG^2 - GP^2}}{2GP} + C \times \frac{\overline{PC^2 - GP^2 - GC^2}}{2GP}$, or, $A \times PA^2 + B \times PB^2 + C \times PC^2 = A \times \overline{GA^2 + GP^2} + B \times \overline{GB^2 + GP^2} + C \times \overline{GC^2 + GP^2}$, and this ſide of the equation is invariable in whatever point of the periphery or ſurface P be placed.

FIG. CXXII. 413. PROP. *If* A, B, C, D, *&c. be particles of a body urged by forces in parallel directions, whoſe magnitudes are* Aa, Bb, Cc, *&c. the ſum of their weights is equal to the weight of* A + B + C, *&c. acted upon by a force whoſe magnitude is* Gg.

DEM. The weights of A, B, C, &c. are $A \times Aa$, $B \times Bb$, $C \times Cc$, &c. (235), and conſequently the ſum of their weights is equal to $\overline{A+B+C}$, &c. $\times Gg$; but this product is the weight of $\overline{A+B+C}$, &c. acted upon by the force Gg. Q. E. D.

414. Cor.

414. Cor. If the forces Aa, Bb, Cc, &c. be equal to each other, Gg is equal to one of them, or if the particles A, B, C, &c. be acted upon by the same force, their weight is the same as if they were collected in their center of gravity and acted upon by that force. The tendency therefore of a body to descend is the same as if it were collected in its center of gravity, and, consequently, if a line drawn from that center perpendicular to the horizon, fall within the base of the body, it cannot fall, and, if without the base, it cannot stand.

415. PROP. *If any number of bodies* A, B, C, *&c. move in parallel directions, with any velocities, the center of gravity will describe a right line parallel to them.* FIG. CXXIII.

DEM. Let A and B, a and b, be cotemporary positions of the bodies A and B, and G, g, their centers of gravity, and through g draw a line xy parallel, and consequently equal, to AB. From the nature of the center of gravity, $A : B :: BG : AG :: bg : ag :: yg : xg$ (sim. triangles); and the point g divides the parallel and equal lines AB, xy, in the same ratio, and Gg is a right line parallel to Aa or Bb. If H be the center of gravity of A, B, C, it is proved in the same manner, that it cuts the parallel and equal lines GC, vz, in the same ratio, and Hh is consequently a right line parallel to Gg. Q. E. D.

416. Cor. 1. If any number of bodies A, B, C, &c. ascend, or descend in parallel right lines, the sum of the products resulting from the multiplication of each body into the space described by it, is equal to the product of their sum and the space described by their center of gravity G; for, let a, b, c, g, be cotemporary positions of A, B, C, G, and, drawing any plane NL, $A \times Am + B \times Bn + C \times Cq = \overline{A + B + C} \times Gh$ (406), and $A \times am + B \times bv + C \times cq = \overline{A + B + C} \times gh$; and consequently by addition $A \times Aa + B \times Bb + C \times Cc = \overline{A + B + C} \times Gg$. FIG. CXXII.

417. Cor.

FIG. CXXIII. CXXIV. 417. Cor. 2. If any number of bodies therefore move in parallel directions with any unequal velocities, or they be placed upon the lever XY, and receive unequal impulses from any force at the same time in parallel directions, the center of gravity will, in the beginning of its motion, move uniformly in a right line parallel to them, and its velocity is equal to the products of each body into its velocity, divided by the sum of the bodies; for the spaces Aa, Bb, Cc, Gg are described in the same time, and vary as the velocities, and Gg (or velocity of G) $= \frac{A \times Aa + B \times Bb + C \times Cc}{A + B + C}$.

FIG. CXXV. 418. PROP. *The surface, or solid, described by the line or surface* AB, *moving round an axis passing through* C, *is equal to the surface or solid formed by the multiplication of* AB *into the line described by the center of gravity*, G, *of* AB.

DEM. For if AB be covered with physical points of the same thickness, AD, DE, EF, &c. $\overline{AD \times CA + DE \times CD + EF \times CE, \text{\&c.}} = \overline{AD + DE + EF, \text{\&c.}}$ or $AB \times CG$ (407), or because the arcs Aa, Dd, Ee, &c. are similar, $AD \times Aa + DE \times Dd + EF \times Ee$, &c. (= the sum of all the superficial, or solid, annuli Ad, De, &c.) $= AB \times Gg$.

FIG. CXXVI. 419. PROP. *The content of the solid generated by the revolution of a plane figure* ABC *round an axis* AC, *is equal to that of the solid whose base is* ABC, *and perpendicular height the periphery of the circle described by the center of gravity*, G.

DEM. For the moment of ABC or $ABC \times GD$, is equal to the sum of the moments of the elementary parts of ABC, or to the sum of all the small rectangles En multiplied into the distance of their center of gravity from AC, or to the sum of all the $y \times mn \times \frac{y}{2}$ or $\frac{y^2}{2} \times mn$; therefore $ABC \times GD$ is equal to the sum of all

all the $\frac{y^2}{2} \times mn$, and $p \times ABC \times GD$ to the ſum of all the $\frac{py^2}{2} \times mn$; but if p be the periphery of a circle whoſe radius is unity, $p \times GD$ is the periphery of a circle whoſe radius is GD; $\frac{p \times y^2}{2}$ is the area of a circle whoſe radius is y, and the ſum of all the $\frac{py^2}{2} \times mn$, is therefore equal to the ſolid deſcribed by ABC.

420. DEF. *The center of gravity of a line or ſurface, is the center of gravity of equal phyſical points, which are ſuppoſed to compoſe that line or ſurface.*

421. Cor. 1. The center of gravity of a right line is evidently in its biſection, and the center of gravity of a parallelogram, priſm, cylinder, are in the biſection of the diameter or axis, becauſe the number of phyſical lines in a parallelogram, or of phyſical laminæ in the priſm and cylinder, on each ſide of this point, are equal.

422. Cor. 2. The center of gravity of any number of lines AB, CD, EF, &c. is found by joining the centers of gravity m, n, of any two AB, CD, and taking a point p, ſo that $pm : pn :: CD : AB$, and joining p, and q the center of gravity of EF, and taking a point r ſo that $pr : rq :: EF : AB + CD$, &c. (401). The center of gravity of the ſides of a triangle ABC may be found as above, or it may be found by biſecting the ſides in m, n, p, joining the points of biſection, and biſecting the angles mpn, and nmp, by the lines pr and mq, whoſe interſection, G, is the center of gravity; for $pq : qn :: pm : mn :: AC : BC$, or the center of gravity is in qm; and $mr : rn :: mp : pn :: AC : AB$, or this center is in pr, and therefore muſt be in their interſection G. FIG. CXXVII. FIG. CXXVIII.

423. PROP.

FIG. CXXIX. 423. PROP. *If a regular polygon* ABCDE *be inscribed in the segment of a circle whose center is* F, *half the sum of its sides is to the sine of half the arc :: perpendicular let fall from* F *upon a side : the distance of* G *from* F.

DEM. For let m, n, p, q, be respectively the centers of gravity of the sides, and joining vs, the centers of AB, BC, and CD, DE, it is clear that the center of gravity of all the sides will be in vs, and it will be also in FC, and consequently at G the intersection of FC and vs. But from the similar triangles sFq, EID, $ED : EI (:: ED + DC : EC) :: Fq : Fs$, and from the similar triangles GsF, CEH, $CE : HE :: Fs : FG$, and, by adding these analogies together, $ED + DC : EH :: Fq : FG$. Q. E. D.

424. Cor. 1. If a regular polygon be inscribed in a circle, its center of gravity is the center of the circle; for $FG = \frac{\text{sine of half the periphery} \times Fq}{\frac{1}{2}\text{ the sum of the sides}} = \frac{0 \times Fq}{\frac{1}{2}\text{ the sum of the sides}}$.

425. Cor. 2. As the arc EC : sine of that arc :: FE : distance of the center of gravity of twice EC from the center F; for, let the sides of the polygon inscribed, be diminished without limit, so as to coincide with the circular arc and be equal to it, and Fq is then equal to the radius. If the arc be a semicircle, the quadrant $CK : FC :: FC : FG$; and the center of gravity of the whole periphery is in F, because $FG = \frac{0 \times FC}{KCL} = 0$.

426. PROP. *The distance of the center of gravity of a triangle from any of its angles, is equal to two third parts of the line drawn from that angle to the bisection of the side opposite to it.*

DEM.

DEM. Let Ap bisect the opposite side BC, and drawing ab, cd, ef, &c. infinitely near and parallel to BC, each will be bisected, and the center of gravity of each, and consequently of all, which make up the triangular surface, will be in Ap; and, for the same reason, it is in Bq bisecting AC, and therefore must be in their intersection G; but, from similar triangles, $AG : Gp :: AB : pq :: BC : pC :: 2 : 1$. Q. E. D. FIG. CXXX.

427. Cor. 1. The center of gravity of any right lined figure is found by dividing it into triangles, and considering them as bodies: let m be the center of gravity of ABC, and n of BDC and joining mn, and taking a point G, so that $mG : nG :: DBC : BAC$, it will be the center of gravity of the trapezium $ABDC$ (401). The process is similar when the figure has more sides. FIG. CXXXI.

428. Cor. 2. The center of gravity of a regular polygon, inscribed in a circle, is the center of that circle; for drawing a right line BH, from any of the angles through the center, the figures on each side are similar and equal, and the center of gravity is in BH; and, for the same reason, it is in DI, and therefore must be at C, and consequently the center of gravity of a circle is its center. FIG. CXXXII.

429. PROP. *The distance of the center of gravity of a regular polygon, inscribed in a sector of a circle* FEAC, *from its center, is equal to two thirds of distance of the center of gravity of the periphery* EDABC. FIG. CXXXIII.

DEM. Find the center of gravity, H, of the periphery (423), and, taking $IE = \frac{2}{3} FE$, describe a polygon similar to $EDABC$. The centers of gravity of the triangles FED, FDA, FAB, FBC, are at L, S, T, V, the bisections of IK, KN, NP, PQ; and the centers of gravity of $FEDA$ and $FABC$, are at R, and X, the bisections of SL, VT, and the center of the whole figure is at G, the bisection of RX; but $FE : EI :: FM : LM$, therefore $LM = \frac{1}{3} FM$, and, for the same reason, $RO = \frac{1}{3} FO$ and $GH = \frac{1}{3} FH$. Q. E. D.

430. Cor. 1. The diſtance of the center of gravity of a ſector of a circle from the center, is therefore equal to two thirds of the diſtance of the center of gravity of the circular arc upon which it ſtands.

431. Cor. 2. As the arc AE : ſine of AE :: radius : diſtance of the center of gravity of twice AE from F, or FH (425); but $FG = \frac{2}{3} \times FH$; therefore AE : ſine of AE (:: $EABC$: its ſubtenſe EC) :: $\frac{2}{3}$ radius : FG.

SCHOLIUM.

432. The diſtance, from the vertex, of the center of gravity of a parabolic ſpace is equal to three fifths, and of a paraboloid to two thirds, of the axis; but theſe, and the moſt difficult propoſitions upon the center of gravity, are beſt inveſtigated by a fluxional proceſs.

FIG. CXXXIV. 433. PROP. *If one or more of the bodies* A, B, C, *&c. move uniformly in the ſame right line, with velocities equal to* a, b, c, *&c. their common center of gravity will move uniformly.*

DEM. Let A and B move uniformly in the ſame, or an oppoſite, direction, P be their center of gravity, and D their diſtance: and becauſe the motions of A and B are uniform, D either continues the ſame, or encreaſes and decreaſes uniformly; but $AP = \frac{D \times B}{A+B}$, and conſequently varies as D, and P moves uniformly. If another body C move uniformly in the ſame right line, and R be the center of gravity of A, B, C; the diſtance CP either continues the ſame, or encreaſes and decreaſes uniformly, becauſe C and P move uniformly; but $PR = \frac{CP \times C}{A+B+C}$, and conſequently varies as CP, or R moves uniformly. Q. E. D.

434. Cor.

434. Cor. 1. The velocity of the center of gravity is equal to $\frac{Aa \pm Bb \pm Cc}{A+B+C}$; for, let p, r, a, b, c, be cotemporary positions of P, R, and the bodies, and (407) $A \times Ap$ or $\times \overline{Aa+ap} - B \times Bp$ or $\times \overline{bp \pm Bb} = \overline{A+B} \times Pp$, and $Pp = \frac{A \times Aa \pm B \times Bb}{A+B}$, because $A \times ap - B \times bp = 0$. And, placing $A+B$ in P and repeating the above process, it appears that $Rr =$ the velocity of R (106) $= \frac{A \times Aa \pm B \times Bb \pm C \times Cc}{A+B+C}$. It is from hence again collected, that the velocity of R is uniform; because Aa, Bb, Cc are constant, and consequently their sum, or difference, multiplied into the same given quantities, or the velocity of R, is always the same.

435. Cor. 2. Because $\overline{A+B+C} \times Rr = A \times Aa \pm B \times Bb \pm C \times Cc$, the velocity of the center of gravity is such as would be communicated to the sum of the bodies acted upon by a force equal to $A \times Aa \pm B \times Bb \pm G \times Cc$.

436. *Prop. *If one or more bodies* A, B, C, *&c. move uniformly in right lines, either in the same or different planes, their common center of gravity* S *will move uniformly in a right line.* FIG. CXXXV.

DEM.

* This proposition is proved very neatly, though, perhaps, less intelligibly by the following process.

CASE I. Let two bodies move, in the same plane, in the directions DE, AB; and let D and A, E and B be cotemporary positions, and H, K, the centers of gravity in those positions, respectively; and taking $BP = AD$, joining EP and drawing DL parallel to HK, $DE : AB$ in the given ratio of the motions of the bodies; and, because the angle EDP is given, all the angles of the triangle EDP are given, and DP is to PE in a given ratio: but $PE : PL :: BE : BK$, which is a given ratio, therefore $DP : PL$ in a given ratio; and, because all the angles of the triangle DPL are given, the angle PDL is given, and L is always in DL. By the nature of the center of gravity; $DA : DH :: EB : EK :: PB$ or $DA : LK$; therefore $DH = LK$, and $DHKL$ is a parallelogram, HK is parallel to DL, and the angle BHK is given, and the center of gravity K is always in the right line HK given in position: And, because all the angles of the triangles DPL and DLE are given, the lines DP, DE, FIG. CXXXVI.

 DL,

DEM. 1. Let B describe Bb uniformly in the time T, and P, Q, be the centers of gravity of A and B; and $A+B : B :: AB : AP :: Ab : AQ :: Bb : PQ$ (EUC. B. 6. P. 5.), and PQ is parallel to Bb, and equal to $\frac{Bb \times B}{A+B}$, and varies therefore as Bb or uniformly. Let A describe Aa uniformly in the time T, either in the same plane with Bb, or not, and R be the center of gravity of A, and B placed at b; and QR, the path of the center of gravity, will appear, by the same process with the above, to be parallel to Aa, and equal to $\frac{Aa \times A}{A+B}$, and consequently it varies as Aa, or encreases uniformly. When both bodies move at the same time, the point P will have two motions PQ and QR, and will consequently describe the diagonal PR uniformly in the time T (185).

2. Let a third body C be added, and the common center of gravity be S, and CS produced will pass through the center of gravity of A and B (411). From the nature of the center of gravity, $A+B+C : A+B :: CP : CS :: CQ : CT :: QP : ST$, and $ST =$

$A+B$

DL, that is, AB, DE, HK, are in a given ratio, and consequently the point K moves uniformly in HK. The demonstration is the same if one of the bodies moves from B towards A.

CASE II. Let the paths of the bodies AB and DE be in different planes; and through AB draw a plane Bde parallel to DE, and through DE draw the plane $DdeE$ perpendicular to Bde; produce BA to d, and let Dd, Ee, be perpendicular to de, and the planes DdA, EeB, will be perpendicular to the plane edB. Let A and D, B and E be cotemporary positions of the bodies. If the body at D were to move in de, the center of gravity would move uniformly in some line HK (case 1.); through HK erect the plane $HKkh$ perpendicular to HBK. From similar triangles, and the nature of the center of gravity, $Ab : bD :: AH : Hd :: BK : Ke :: Bk : kE$; therefore hk is the path of the center of gravity of the bodies moving in AB, DE. And, because $Dd : Hb :: Ad : AH :: Be : BK :: Ee$ or $Dd : Kk$, $Hb = Kk$ and kb is equal and parallel to HK; therefore the center of gravity of the bodies, moving uniformly in AB, DE, moves uniformly in hk.

CASE III. The common center of gravity of two bodies and a third body, is either at rest, or moves uniformly in a right line; for two may be placed in their common center of gravity, which was proved to move uniformly, and the center of gravity of the three or more bodies is proved, by the same process as before, to move uniformly.

$\frac{\overline{A+B} \times QP}{A+B+C}$, and varies as QP or uniformly; and for the same reason TV, the motion of T arising from A's motion, is equal to $\frac{QR \times \overline{A+B}}{A+B+C}$, and therefore varies as QR or uniformly (case 1.). When A and B move together, the motions ST, TV, will be combined into one, SV; and if C describe Cc uniformly in the time T, the common center of gravity will describe VY, and this new motion, combined with SV, will make it describe SY uniformly in the time T. Q. E. D.

437. Cor. 1. It is evident, that the paths of the center of gravity, arising from the motion of any one body, is always parallel to that of the moving body: PQ and ST are parallel to Bb; QR and TV are parallel to Aa and VY to Cc.

438. Cor. 2. The centers of gravity of two, three, &c. bodies will describe polygons or curves similar to that of the moving body to which their motion is owing; and if the velocity of the body be variable, the velocity of each center will be variable according to the same law.

439. Cor. 3. The velocity of the center of gravity of two, three, &c. bodies is the same as if they were placed in it, and acted upon by forces, equal to the moments of the moving bodies, in their respective planes and directions; for $B \times Bb = \overline{A+B} \times PQ$ and $A \times Aa = \overline{A+B} \times QR$; and if $A+B$ were placed at P, and acted upon by forces equal to $B \times Bb$ and $A \times Aa$ in the planes and directions of Bb and Aa, they would describe the diagonal PR (185).

440. Prop. *The common center of gravity of two or more bodies is not affected by any action of the bodies upon each other.*

Dem.

FIG. CXXXVIII. DEM. Let A and B be two bodies in a syſtem, acting upon each other, G their common center of gravity, and Aa, Bb, the velocities loſt by A and gained by B reſpectively in oppoſite directions, and $A \times Aa = B \times Bb$ (3d law of motion), or $A : B :: Bb : Aa :: BG : AG :: bg : aG$, or $A : B ::$ B's diſtance from the center of gravity : A's diſtance from it; and conſequently the ſame point G is ſtill the center of gravity of A and B, or it has been immoveable. What is proved of theſe two, is true of every two bodies, and therefore of all. Q. E. D.

441. Cor. If two parts of a ſyſtem A and B, attract or repel each other, or moving with unequal rectilineal motions, diſturb each others motion by the force of their inertia, the center of gravity will not be affected by their mutual action.

CHAP.

CHAP. X.

COMMUNICATION OF MOTION BY DIRECT IMPACT.

442. DEF. *TWO bodies are said to impinge directly, when the right line, in which their centers of gravity move, passes through the point of contact.*

443. DEF. *The center of gravity of a body, or system of bodies, supposed to be without gravity, is the same point with that center when its influence obtains.*

LEMMA. If two bodies X and Y, consisting of any number of particles without gravity A, B, C, and D, E, &c., be immoveably connected to the right line SR, passing through their centers of gravity G and H, which is perpendicularly impressed at F by a force F, the moments communicated to these bodies are inversely as the distances of their centers of gravity from the point where F acts. FIG. CXXXIX

DEM. The sum of the resistances of D and E to the action of F is $D \times Fd + E \times Fe$ (279) $= \overline{D+E} \times FH$ (407), and consequently if d, e, h, be the respective velocities of D, E, and H, and $D + E = Y$ be placed in H, $D \times d + E \times e = Y \times h$; and for the same reason $A \times a + B \times b + C \times c$, &c. $= X \times g$, supposing a, b, c, g, to be the respective velocities of A, B, C, and of X collected at G. Let a fulcrum be placed at G, and because F is in equilibrio with

with $D \times d + E \times e$ or $Y \times b$, $F : Y \times b :: GH : FG$, and if a fulcrum be placed at H, $X \times g : F :: FH : GH$, and ex æquo $X \times g : Y \times b :: FH : FG$. Q. E. D.

444. Cor. Becauſe $g : b :: FH \times Y : FG \times X :: D \times Fd + E \times Fe : A \times Fa + B \times Fb + C \times Fc$, the velocities of G and H, and of the points A, B, C, &c. are the ſame, whether the particles be placed in thoſe points, or collected in their centers of gravity G and H; and the effect is evidently the ſame, whether G and H be in the right line SR, or in PQ, making an invariable angle with SR, and at the ſame perpendicular diſtances from F's direction.

445. Prop. *If a material ſurface* SUR, *compoſed of the particles* A, B, C, D, *&c. be impreſſed by a force equal to* F, *acting in the plane of the ſurface at the center of gravity* F, *and perpendicularly to a right line* SFR *paſſing through it, the particles will move with equal velocities in directions parallel to that of* F.

Dem. Let the particles, on one ſide of F's direction, $A + B + C$, &c. $= X$ be placed at their center of gravity G, and $D + E$, &c. on the other ſide at their center of gravity H, and GFH is a right line (411). But $g : b :: Y \times FH : X \times FG :: 1 : 1$ and $g = b$; and the incipient motions of X and Y are parallel to F's direction (2d law of motion). Since X and Y begin to move with equal velocities, in parallel directions, they are relatively at reſt, and conſequently will not diſturb each others motions; and becauſe the relative ſituations of the points A, B, C, &c. are always the ſame, they will move with velocities equal to thoſe of G and H, and in parallel directions. The effects are the ſame whether A, B, C, &c. be in their real places, or collected in their centers of gravity (444); and the initial motions of G and H are not diſturbed by the mutual actions of the particles (441). Q. E. D.

446. Cor.

446. Cor. 1. If any particles B, C, &c. be not in the ſuperficies SR, but on different ſides of it, their center of gravity will be in q (411), and their efforts, reſulting from their inertia, to diſturb the incipient motion of the particles, are meaſured by the products of each particle into its perpendicular diſtance from the plane SR(279), and theſe being equal and oppoſite, and conſequently deſtroying each other, the particles will continue to move with their initial velocities. If a body therefore be impelled by any power F, in a direction paſſing through its center of gravity, its conſtituent particles will move with equal velocities, in directions parallel to that of F; and if the particles be equal, and their number n, the velocity of each, and of the center of gravity, is the ſame whether that center be impelled by the force F, or each particle by a force equal to $\frac{F}{n}$, in the direction of F. FIG. CXXXIX.

447. Cor. 2. In the impact of one body A upon another body B, either moving or quieſcent, when the right line joining their centers of gravity, and the lines in which they move, are in the ſame right line, the forces of impact and reſiſtance produce a change of velocity, in each body, only in this line, or their impact is direct.

448. Prop.* *Let a body* A, *moving with a velocity equal to* a, *impinge directly upon* B, *moving in the ſame, or an oppoſite, direction, with a velocity equal to* b, *to find their common velocity.*

If A and B be inelaſtic, (3d law of motion) $Aa \pm Bb$ is the ſame before and after impact (where the higher ſign is to be uſed when the bodies move in the ſame direction, and the lower when they move in oppoſite directions,) and, becauſe they move together, if v be their common velocity, $Aa \pm Bb = \overline{A+B} \times v$ and $v = \frac{Aa \pm Bb}{A+B}$. Q. E. I.

449. Cor.

* Newt. Cor. 3. ad leges motus. Maclaurin's Newt. Chap. IV. Rohault. Not. Part I. Ch. II. Art. 6. Helſham. Lect. V. and Appendix. Nov. Com. Petropol. Tom. XVII. pag. 315.

449. Cor. 1. The velocity, which A loses by impact, is equal to the difference between its velocities before and after impact, or to $a - v$; and the velocity which B gains by impact, in the direction of A's motion, is equal to the difference between its velocities after and before impact, or to $v \mp b$. Substituting the value of v, $a - v = a - \frac{Aa \mp Bb}{A+B} = \frac{Aa + Ba - Aa \mp Bb}{A+B} = \frac{B \times \overline{a \mp b}}{A+B} = l$, and, resolving this equation into a proportion, $A + B : B :: a \mp b : l$. The velocity gained by $B = g = v \mp b = \frac{Aa \pm Bb}{A+B} \mp b = \frac{Aa \pm Bb \mp Ab \mp Bb}{A+B} = \frac{A \times \overline{a \mp b}}{A+B}$, and $A + B : A :: a \mp b : g$. From hence we have the following rules.

1.* The sum of the bodies is to the struck body as the difference or sum (according as they move in the same, or an opposite, direction) of the velocities, before impact, to the velocity lost by the striking body.

2. The sum of the bodies is to the striking body as the difference or sum of the velocities, before impact, to the velocity gained by the struck body.

450. Cor. 2. If A and B move in opposite directions with velocities that are always inversely as the bodies, their common center of gravity will not be affected by their motion; but, l and g, representing the velocities lost by A and gained by B, are made in opposite

* These rules are investigated differently by the following process:

1. $A \times l = B \times g$ (3d law of motion) and $g = \frac{A \times l}{B}$; but $a - l$ (A's velocity after impact) $= g \pm b$ (B's velocity after impact) $= \frac{A \times l}{B} \pm b$ and $B \times a - B \times l = A \times l \pm B \times b$ and $l \times \overline{A+B} = B \times \overline{a \mp b}$, and $\overline{A+B} : B :: \overline{a \mp b} : l$.

2. $l = \frac{B \times g}{A}$ and $a - l = a - \frac{B \times g}{A} = g \pm b$, (because the bodies move on together) and $Aa - B \times g = Ag \pm A \times b$ and $\overline{A+B} \times g = A \times \overline{a \mp b}$, and $A + B : A :: a \mp b : g$.

opposite directions, and $A \times l = B \times g$ or $A : B :: g : l$, or g and l are inversely as the bodies, and consequently the velocity of the center of gravity of A and B is not affected by impact, and is, therefore, both before and after impact, equal to their common velocity, v.

451. PROP. *If* A, *moving with a velocity equal to* a, *impinge upon* B, *moving in the same, or an opposite, direction with a velocity equal to* b, *and* A *and* B *be perfectly elastic, or one of them be perfectly hard, and the other perfectly elastic, to find the velocity lost by* A (L), *and that gained by* B (G).

When A and B are inelastic, $A + B : B :: a \mp b : l$, and
$A + B : A :: a \mp b : g$; but $L = 2l$, and $G = 2g$ (257); therefore $A + B : 2B :: a \mp b : L$, and $A + B : 2A :: a \mp b : G$; and we have these rules.

1. The sum of the bodies is to twice the struck body as the difference or sum of the velocities, before impact, to the velocity lost by the striking body.

2. The sum of the bodies is to twice the striking body as the difference or sum of the velocities, before impact, to the velocity gained by the struck body. Q. E. I.

452. Cor. 1. $L : G :: 2l : 2g :: B : A$ and $A \times L = B \times G$; therefore in all perfectly elastic bodies $L = \frac{B \times G}{A}$ and $G = \frac{A \times L}{B}$.

453. Cor. 2. A's velocity after impact is equal to the difference between its velocity before impact and the velocity lost, or to $a - \frac{2B \times \overline{a \mp b}}{A + B} = \frac{Aa + Ba - 2Ba \pm 2Bb}{A + B} = \frac{\overline{A - B} \times a \pm 2Bb}{A + B}$.
B's velocity after impact is equal to the sum of its velocity before,

and

and the velocity gained, or to $\pm b + \frac{2A \times \overline{a \mp b}}{A+B} = \frac{2Aa \pm Bb \mp Ab}{A+B}$.
If $A = B$, $\frac{\overline{A-B} \times a \pm 2Bb}{A+B} = \pm b$, and $\frac{2Aa \pm Bb \mp Ab}{A+B} = a$, or the bodies move with interchanged velocities.

454. Cor. 3. If A be greater or lefs than B, L is lefs or greater than $a \mp b$, and G greater or lefs than $a \mp b$. If A be to B, as m to n, $L = \overline{a \mp b} \times \frac{2n}{m+n}$, and $G = \overline{a \mp b} \times \frac{2m}{m+n}$.

455. Cor. 4. If there be any number of bodies A, B, C, D, &c. in geometric progreffion, encreafing or decreafing, and A, moving with a velocity equal to a, communicate motion to B at reft, and B, moving with the velocity gained, communicate motion to C at reft, &c. the velocities communicated to B, C, D, &c. will be in geometric progreffion, decreafing or encreafing. For (451)
$A+B : 2A :: a$: velocity of B
$B+C : 2B ::$ vel. of B : vel. of C
$C+D : 2C ::$ vel. of C : vel. of D
$D+E : 2D ::$ vel. of D : vel. of E. And, becaufe A, B, C, &c. are in geometric progreffion, $A+B : 2A :: B+C : 2B :: C+D : 2C :: D+E : 2D$; and confequently a : vel. of B :: vel of B : vel of C :: vel of C : vel. of D, &c. And if the bodies encreafe or decreafe in magnitude, a is greater or lefs than the velocity of B, or the velocities communicated decreafe or encreafe.

456. Cor. 5. If the number of bodies in geometric progreffion be equal to n, a : velocity of the laft body :: $\overline{A+B}|^{n-1} : \overline{2A}|^{n-1}$; for a : vel. of the laft body in a ratio compounded of the ratios of a : vel. of B, vel. of B : vel. of C, vel. of C : vel. of D, &c. each of which is equal to the ratio of $A+B : 2A$, and the number of ratios is equal to $n-1$. The velocity of A, or a, is alfo to the velocity of the laft body :: $\overline{A+B} \times \overline{B+C} \times \overline{C+D}$, &c. : $A \times B \times C$, &c. (the product of all except the laft) $\times 2^{n-1}$.

457. Cor.

457. Cor. 6. The velocities lost by A, B, C, &c. are in geometric progression encreasing or decreasing, according as the bodies decrease or encrease; for,

$A+B:2B::a:$ vel. lost by A

$B+C:2C::$ vel. of $B:$ vel. lost by B

$C+D:2D::$ vel. of $C:$ vel. lost by C, &c.

But A, B, C, &c. being in geometric progression, $A+B:2B::B+C:2C::C+D:2D$, &c. and consequently $a:$ vel. lost by A :: vel. of $B:$ vel. lost by B :: vel. of $C:$ lost by C, &c.; therefore the ratios of the velocities lost and gained are the same, and the last being in geometric progression (456), the first are so too.

458. Prop. *If there be three perfectly elastic bodies*, A, X, Q, *and* A, *moving with a velocity equal to* a, *impinge upon* X *at rest, and* X *impinge upon* Q *at rest,* Q's *velocity is the greatest possible when* X *is a mean proportional between* A *and* Q.

Dem. Let q be the velocity of Q, and $a:q::\overline{A+X}\times\overline{X+Q}$ $4AX$ (456) $::A+X+Y+Q:4A$ (supposing A to be to X as Y to Q, and consequently $A+X:X::Y+Q:Q$, and for $\overline{A+X}\times Q$, substituting its equal $\overline{Y+Q}\times X$); and $q=\dfrac{4Aa}{A+X+Y+Q}$, and, $4Aa$ being given, q varies $\dfrac{1}{A+X+Y+Q}$, and is greatest when their sum is least, or when $X=Y$; for $A\times Q$ being given, $X\times Y$ is given, and the sum of two factors containing a given product is least * when they are equal. Q. E. D.

459. Cor. 1. The velocity, communicated to Q, will be encreased, by encreasing the number of bodies in geometric progression, between it and A, and arrive at its limit when that number

* Let the given product be represented by the given rectangle AB, contained by two unequal lines AL, LB. Make the square LB equal to the rectangle AB and $AL+LB=AD=2CM$ is always greater than $2LM$ or $LM+LF$. FIG. CXL.

number is infinitely great, and then $a : q :: \sqrt{Q} : \sqrt{A}$. Let the succeſſive differences of the bodies be x, y, z, &c. or $B = A + x$, $C = B + y$, $D = C + z$, &c. and let their reſpective velocities be a, b, c, c, d, &c.; becauſe $A + B : 2A :: a : b$ and $B = A + x$, by ſubſtitution, $A + A + x : 2A$ or $A + \frac{x}{2} : A :: a : b :: \sqrt{B} : \sqrt{A}$ (31); $B + C : 2B$ or $2B + y : 2B$ or $B + \frac{y}{2} : B :: b : c :: \sqrt{C} : \sqrt{B}$; and, for the ſame reaſon, $C + \frac{z}{2} : C :: c : d :: \sqrt{D} : \sqrt{C}$, &c. but $a : d$ in a ratio compounded of the ratios of $a : b$, $b : c$, $c : d$, &c. or their equals $\sqrt{B} : \sqrt{A}$, $\sqrt{C} : \sqrt{B}$, $\sqrt{D} : \sqrt{C}$, &c. or of the ſquare root of the laſt body to the ſquare root of the firſt (41); and conſequently $a : q :: \sqrt{Q} : \sqrt{A}$.

460. Cor. 2. Becauſe $A \times L = B \times G$, $A : B :: G : L$; but G and L are made in oppoſite directions, and are to each other as the ſpaces deſcribed by the bodies in the ſame time, and being inverſely as the diſtances from the center of gravity, this center will evidently not be affected by impact, but will continue to move with a velocity equal to v, their common velocity after impact when inelaſtic. This alſo appears from article 440.

461. Prop. *If two perfectly elaſtic bodies* A *and* B *impinge directly, the ſums of the products, reſulting from multiplying each body into the ſquare of its velocity, is the ſame before and after impact.*

Dem. Retaining the ſame notation, $a - v = l$, and $2a - 2v = L$, and $2v \mp 2b = G$; and conſequently A's velocity after impact is equal to $a - 2a + 2v = 2v - a$, and B's velocity $= \pm b + G = 2v \mp b$; but $A \times \overline{2v - a}|^2 + B \times \overline{2v \mp b}|^2 = A \times 4v \times \overline{v - a} + a^2 + B \times 4v \times \overline{v \mp b} + b^2 = A \times a^2 + B \times b^2$, becauſe $A \times a + B \times \pm b = \overline{A + B} \times v$, and conſequently $A \times \overline{v - a} +$

B ×

$B \times \overline{v \mp b} = 0$, or, multiplying this equation by $4v$, $A \times 4v \times \overline{v - a} + B\,4v \times \overline{v \mp b} = 0$*. Q. E. D.

462. Cor. 1. Whatever therefore be the number of elastic bodies A, B, C, &c. which impinge successively, beginning with A, the sums of the products, resulting from the multiplication of each body into the square of its velocity, are the same before and after impact.

463. Cor. 2. The relative velocities of A and B are the same before and after impact; for, before impact, the relative velocity is equal to the velocity of A diminished or encreased by that of B, according as they move in the same, or an opposite, direction, or equal to $a \mp b$; and, after impact, it is equal to B's velocity diminished or encreased by that of A, or equal to $2v \mp b - 2v + a = a \mp b$, the same as before impact.

464. Cor. 3. In this proposition the force of elasticity is equal to that of compression, and this obtains very nearly in some kinds of bodies, as glass, ivory, tempered steel, &c. and, in these, the conservatio vis vivæ is preserved, or the products of the bodies into the

* This proposition may be proved a little differently by the following process: let p and q be equal to the velocities of A and B respectively, after impact, and $A \times a \pm B \times b = B \times q \pm Ap$ (3d law of motion) and $Aa \mp Ap = Bq \mp Bb$. But the relative velocities of A and B are the same before and after impact, or $a \mp b = q \mp p$, and $a \pm p = q \pm b$, and, multiplying $Aa \mp Ap$ into $a \pm p$, and $Bq \mp Bb$ into $q \pm b$, the results are

$$A \times a^2 \mp Apa \pm Apa - Ap^2 = B \times q^2 \mp Bqb \pm Bqb - Bb^2, \text{ or}$$

$$A \times a^2 + B \times b^2 = B \times q^2 + A \times p^2.$$

EXAMP. Let the ratios of $A : B$, and $a : b$, be respectively as $1 : 9$, and $6 : 1$, and, before impact, $A \times a^2 + B \times b^2 = 1 \times 36 + 9 \times 1 = 45$, and, after impact, $A \times p^2 = A \times \overline{a - \frac{2B \times \overline{a - b}}{A + B}}\Big|^2$, and $B \times q^2 = B \times \overline{b + \frac{2A \times \overline{a - b}}{A + B}}\Big|^2$; the first $= \overline{6 - \frac{18 \times 5}{10}}\Big|^2 = \overline{\frac{60 - 90}{10}}\Big|^2 = \overline{-3}\big|^2$ or 9, and the second $= 9 \times \overline{1 + \frac{2 \times 5}{10}}\Big|^2 = 9 \times 4 = 36$, and $A \times p^2 + B \times q^2 = 45$.

the ſquare of their velocities, are the ſame before and after impact. But in all bodies, endued with imperfect degrees of elaſticity, the equality of theſe products is not preſerved.

465. PROP. *When either* A *or* B *is perfectly hard, and the other imperfectly elaſtic, or when they both are imperfectly elaſtic, and the whole force of reſtitution is to that of compreſſion, as any number* n : 1, *to find the velocities of* A *and* B *after impact.*

DEM. When A and B are inelaſtic, let the velocities loſt by A and gained by B, reſpectively, by the force of impact, be l and g; and $1 : n :: l : nl =$ the velocity which A loſes by the force of reſtitution, and $nl + l = A$'s whole loſs of velocity. Let $m = n + 1$ and $a - ml = A$'s velocity after impact, $=$ (becauſe $l = \frac{B \times g}{A}$) $\frac{Aa - mgB}{A}$. B's velocity after impact $= \pm b + g + ng = \pm b + mg$. Q.E.I.

466. Cor. 1. Becauſe $l = \frac{B \times \overline{a \mp b}}{A + B}$, $\overline{n + 1} \times l$, or A's whole loſs of velocity, $= \overline{n + 1} \times \frac{B \times \overline{a \mp b}}{A + B}$, and conſequently $A + B : \overline{n + 1} \times B :: a \mp b :$ velocity loſt by A. And becauſe $\overline{n + 1} \times g$, or B's whole gain of velocity, $= \overline{n + 1} \times \frac{A \times \overline{a \mp b}}{A + B}$, $A + B : \overline{n + 1} \times A :: a \mp b :$ velocity gained by B.

467. Cor. 2. It is proved in the ſame manner, as when the bodies were perfectly elaſtic, that the velocities loſt and gained, by a ſeries of imperfectly elaſtic bodies in geometric progreſſion, which are all at reſt, except the firſt, are in geometric progreſſion; for let the

the velocities of A, B, C, D, &c. be a, b, c, &c. and

$$A + B : \overline{n+1} \times A :: a : b$$
$$B + C : \overline{n+1} \times B :: b : c$$
$$C + D : \overline{n+1} \times C :: c : d,$$

and because $A : B :: B : C :: C : D$, &c.
$\overline{A + B} : \overline{n+1} \times A :: B + C : \overline{n+1} \times B :: C + D : \overline{n+1} \times C$, &c.;
or $a : b :: b : c :: c : d$, &c.

468. Cor. 3. If elasticity be equal to the resistance made to compression, $n = 1$, and, substituting p and q for the velocities of A and B after impact, $p = a - \frac{mBg}{A}$, and $q = mg \mp b$. Let elasticity be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. of the resistance, or $n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.; and $p = a - \frac{\frac{3}{2}Bg}{A}$, $a - \frac{\frac{4}{3}Bg}{A}$, $a - \frac{\frac{5}{4}Bg}{A}$, &c. and $q = \frac{3}{2}g \mp b$, $\frac{4}{3}g \pm b$, $\frac{5}{4}g \pm b$, &c.: if therefore A, B, g and n be given, the ratio of the relative velocities, or of $a \mp b : q \mp p$ may be found; and $v : v$, if the ratio of the relative velocities, and A, B, a, g be known, the degree of elasticity, or n, may be found: for $p = a - \frac{mBg}{A}$ and $q = \pm b + mg$, and the ratio of $q - p$ $(\pm b - a + \overline{g + \frac{Bg}{A}} \times m)$ $: a - b$ is a given ratio, and consequently $m\ (n+1)$ is known.

469. Cor. 4. The converse of prop. (461) is true, and if the products of A and B into the squares of their velocities, be the same before and after impact, the force of restitution is equal to that of compression; and generally if the products of A and B into that power of the velocity, whose exponent is any number r, be supposed equal before and after impact, the relation between the forces of elasticity and resistance may be found: for $a - \frac{mBg}{A} = p$ and $\pm b + mg = q$, and by solving the equation $A \times a^r +$

$B \times b^r = A \times \overline{a - \frac{mBg}{A}}\,^r + B \times \overline{\pm b + mg}\,^r$, m, and consequently n may be found.

FIG. CXLI.

470. LEMMA. *If two sperical bodies* A *and* B *move from* A *and* B *at the same time, with velocities which are to each other as* a : b, *or* AC : BD, *to find a plane that shall touch both bodies at the point of impact.*

Join AB, and describe the parallogram AH, whose sides are AC and AB; join DH, and from C, as a center, describe a circular arc, with a radius equal to the semidiaters of the bodies, cutting DH in L; draw LE parallel to AC, and EF parallel to the line joining C and L, and the bodies will at the same time be in E and F, and a plane perpendicular to EF will touch both bodies at the point of impact; for $DE : EL$ or $FC :: DB : BH$ or AC, and

$DB : DB \pm DE$ or $BE :: AC : AC \pm FC$ or AF,

and $DB : AC :: BE : AF :: b : a$, and BE, AF are therefore described in the same time by B and A, and because $EF = CL =$ the sum of the semidiameters of the bodies, they will be in contact at P, with a plane perpendicular to EF. Q. E. I.

FIG. CXLII.

471. LEMMA. *If two sperical bodies* A *and* B *inpinge obliquely, to determine the velocities lost by* A *and gained by* B, *and the velocity of each after impact.*

Case 1. Suppose A and B to be inelastic, and to meet in D, and Ll to be a plane in contact with both the bodies, and the velocities to be FD and GD. Resolve each velocity into two, FL, and GM, perpendicular to the plane, and DL, DM, parallel to it: DL and DM are not affected by impact, and FL, GM, are directly opposite to each other. Taking therefore $DH = \frac{A \times FL - B \times GM}{A + B}$. (448), (448), and Dm, Dl respectively equal to DM, DL, A and B will describe the lines DQ, DR respectively.

Case 2. If A and B be either perfectly or imperfectly elastic, they will be reflected, and the velocities DE, DH be equal respectively to $FL - \frac{2B \times \overline{FL + GM}}{A+B}$ and $-GM + \frac{2A \times \overline{FL + GM}}{A+B}$ (453): taking therefore $Dm = DM$ and $Dl = DL$, A will be reflected along DK, and B along DP; for the velocities DM, DL remain after impact, and these, compounded with DH and DE respectively, will make A and B describe the diagonals of parallelograms, whose sides are DE, Dl, and DH, Dm. Q. E. I. FIG. CXLIII.

472. *Lemma. *To find the spontaneous center of conversion of the particles* P *and* Q, *without gravity, or a point about which the right line* SR, *adhering to those particles, begins to revolve, when impressed perpendicularly by a force* F, *acting at a point* F *which is not the center of gravity* G. FIG. CXLIV.

Let pqS be a new position of SR, infinitely near to the former, and the point S is evidently stationary whilst P and Q describe the small spaces Pp and Qq: but $Q \times FQ : P \times FP :: Pp : Qq$ (443) :: SP or $SF + FP : SQ$ or $SF - FQ$, and $Q \times FQ \times SF - Q \times FQ^2 = P \times FP^2 + P \times FP \times SF$, and $SF = \frac{P \times FP^2 + Q \times FQ^2}{Q \times FQ - P \times FP}$ $= \frac{P \times FP^2 + Q \times FQ^2}{P + Q \times FG}$. Q. E. I.

473. Cor. 1. Whilst F acts at the same point, the point S is always the same whatever be the magnitude of F, because $\frac{P \times FP^2 + Q \times FQ^2}{P + Q \times F}$ is a given quantity.

474. Cor.

* Philos. Trans. for 1780, pag. 550, where this subject is treated with great perspicuity by Mr. Vince.

474. Cor. 2. To whatever point of QP the same force F be applied, the incipient motions of Q and P or $Q \times Qq + P \times Pp$, are invariably the same (2d law of motion); and consequently the velocity of the center of gravity, or Gg, is always the same as if both particles were placed at G and acted upon by the same force F; for $Gg = \frac{P \times Pp + Q \times Qq}{P + Q}$ (406).

FIG. CXLV. 475. Cor. 3. If SUR be a material surface composed of the particles A, B, C, &c. and a force F act in the line PF placed in the surface, and perpendicular to a line SG drawn through the center of gravity G, each particle will receive an impulse in a direction parallel to PF (2d law of motion); and because G is not affected by the action of the particles upon each other, it will move in a right line (440), and the center of spontaneous conversion will be in some point of the line SG, perpendicular to PF. Let S be that center, and the whole plane will begin to revolve about S; but if a force Q equal to F were to act in an opposite direction, all motion would evidently be destroyed, and consequently the combined force of every particle in the surface (revolving round S, which results from the action of F), to make it revolve about F and destroy the action of Q would be equal to nothing. But the force of any particle $A = A \times SA$ multiplied into the perpendicular distance of its direction from $F = A \times SA \times Fm$, or, because (letting fall Sa perpendicular to SF) $Fm : Fn :: Sa : SA$, and $Fm = \frac{Fn \times Sa}{SA}$ and $A \times SA \times Fm = A \times Fn \times Sa = A \times Sa \times \overline{SF - Sn}$ $= A \times Sa \times SF - A \times SA^2$: and by a similar process $B \times Sb \times SF - B \times SB^2 = B$'s force: consequently $SF = \frac{A \times SA^2 + B \times SB^2 \&c.}{A \times Sa + B \times Sb \&c.}$ $= \frac{A \times SA^2 + B \times SB^2 \&c.}{A + B \&c. \times SG}$. If S be the center of suspension, F will therefore be the center of oscillation (art. 512).

476. Cor.

476. Cor. 4. If pqS be the next position of SR, and xz be drawn parallel to it, P has been progressive through xp, and Q regressive through zq, subtending an angle at g equal to that at S. The initial motion of the center of gravity is not affected by the unequal motions of P and Q, because their actions upon it, or $P \times pg$ and $Q \times qg$ are equal and opposite, and the motion of G is consequently rectilineal and uniform (1st law of motion); and because the incipient angular motions of P and Q about g continue to be uniform, it is evident that when the angle GSg would become equal to four right angles, or P and Q have made one revolution round g, Gg, the line described by the center of gravity, would be equal to the periphery of a circle whose radius is SG. FIG. CXLIV.

477. PROP. *Let a body* B, *be impelled by a force* F, *acting in the direction* FD *not passing through its center of gravity* G, *to define its motion.* FIG. CXLVI.

Through G draw a right line SR perpendicular to FD produced, and, if S be the spontaneous center of conversion of the plane surface SUR, every particle will receive an impulse, and begin to move in a direction parallel to this surface (2d law of motion), and consequently to revolve round an axis passing through S perpendicular to it. And if the parts of every section of the body, perpendicular to DF, on each side of the plane SUR, be similar and similarly situated, the plane of conversion SUR will not be affected by either the progressive or rotatory motion of the particles: for their effects upon SUR, arising from their progressive motion, are as the sums of the products resulting from the multiplication of each particle into its distance from this plane, which are equal and opposite; and if p and q be two equal particles, similarly situated in any plane perpendicular to DF, their effects, arising from their rotation, being measured by the product of each particle into the distance from the axis, will be in that plane, and equal and opposite to each other. Consequently the incipient plane of rotation remains unaltered, and if a right line be drawn through G, perpendicular to the plane SUR, every particle

ticle will move round this line as an axis, whilst G moves uniformly in a right line.

478. Cor. 1. In a body, B, revolving by the action of a force impelling it in a direction not passing through the center of gravity, the distance of the spontaneous center of conversion from the intersection of F's direction with a right line passing through the center of gravity, that is, SD is equal to the sum of the particles multiplied into the square of the distance of each, divided by B multiplied into SG.

FIG. CXLV. 479. Cor. 2. $A \times SA^2 = A \times \overline{SG^2 + GA^2 + 2SG \times Ga}$ (Euc. B. 2. P. 12).

$$B \times SB^2 = B \times \overline{SG^2 + GB^2 + 2SG \times Gb}$$
$$C \times SC^2 = C \times \overline{SG^2 + GC^2 + 2SG \times Gc}$$
$$D \times SD^2 = D \times \overline{SG^2 + GD^2 - 2SG \times Gd}, \text{ \&c.}$$

and, because $A \times 2SG \times Ga + B \times 2SG \times Gb + C \times 2SG \times Gc - D \times 2SG \times GD$, &c. $= 0$, $A \times SA^2 + B \times SB^2 + C \times SC^2 + D \times SD^2 = A \times \overline{SG^2 \times GA^2} + B \times \overline{SG^2 + GB^2} + C \times \overline{SG^2 + GC^2}$, &c.; therefore $SF = SG + GF = \dfrac{A \times \overline{SG^2 + GA^2} + B \times \overline{SG^2 + GB^2}}{A + B + C, \text{ \&c.} \times SG}$, &c. &c.; and $FG = \dfrac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C, \text{ \&c.} \times SG}$, &c., $SG \times GF$ is equal to a given quantity, or SG and GF vary inversely as each other.

FIG. CXLVI. 480. Cor. 3. Because the velocity of G is the same as if the particles were concentrated in G, and acted upon by the force F, the time of describing the angle GSg or qgz is the same to whatever point F be applied: but, Gg being given, the angle GSg varies inversely as SG, or directly as GF (last cor.). If different forces act at the same point, Gg, or the angle of rotation, will evidently be as the magnitude of the force; and, consequently, the angular velocity will vary generally as $F \times GF$, or as the magnitude of the force multiplied into the perpendicular distance of the center of gravity from its direction.

481. PROP.

481. PROP. *Let a body, whose quantity of matter is Q, moving with a velocity equal to V, impinge upon the body B, in the direction FC passing through the center of gravity of Q, to find the velocity of the center of gravity, G, of the body B.* FIG. CXLVI.

Let g be the velocity of G, and S the spontaneous center of conversion of B; and $SG : SD :: g$: velocity of D, which is therefore equal to $\frac{SD \times g}{SG}$. $V - \frac{SD \times g}{SG}$ is therefore equal to the velocity lost by Q in the direction FC, and (3d law of motiou) $Q \times \overline{V - \frac{SD \times g}{SG}} = B \times g$, and $\frac{Q \times \overline{V \times SG - SD \times g}}{SG} = B \times g$, and $g = \frac{Q \times V \times SG}{B \times SG + Q \times SD}$. Q.E.I.

482. Cor. 1. If the bodies be perfectly elastic, the velocity of G, after impact, is equal to $\frac{2Q \times V \times SG}{B \times SG + Q \times SD}$.

483. Cor. 2. Because the velocity of Q, after impact, is equal to that of D, or to $\frac{SD \times g}{SG}$, or, substituting the velocity of g, equal to $\frac{Q \times V \times SD}{B \times SG + Q \times SD}$; $V - \frac{Q \times V \times SD}{B \times SG + Q \times SD}$ is Q's loss of velocity, when the bodies are inelastic, which is equal to $\frac{B \times V \times SG}{B \times SG + Q \times SD}$ $= Q$'s loss of velocity, and consequently Q's velocity after impact, when the bodies are perfectly elastic, is equal to $V - \frac{2B \times V \times SG}{B \times SG + Q \times SD}$ $= \frac{V \times \overline{Q \times SD - B \times SG}}{B \times SG + Q \times SD}$.

484. Cor.

484. Cor. 3. If FC pass through the center of gravity G, of the body B, or the impact become direct, $SG = SD$, and G's velocity $= \frac{2Q \times V \times SG}{B \times SG + Q \times SG} = \frac{2Q \times V}{B + Q}$. Q's velocity $= \frac{V \times \overline{Q \times SD - B \times SD}}{B \times SG + Q \times SD}$ $= V \times \frac{\overline{Q - B}}{B + Q}$; and rules are deduced from these expressions, which are the same as in direct impact before investigated.

485. Cor. 4. Because $Q \times V = \overline{Q + B} \times g$ (3d law of motion) $g = \frac{Q \times V}{Q + B}$, or the progressive motion of B is the same upon whatever point of B the body Q impinges.

486. Cor. 5. If Y be the magnitude of a body placed at D, which receives the same velocity from the impact of Q, that was communicated to the point D, and, consequently, the velocity of Y, after impact, will be equal to $\frac{Q \times V}{Q + Y} = \frac{Q \times V \times SD}{B \times SG + Q \times SD}$, and $Y = \frac{B \times SG}{SD}$. Hence if the direction does not pass through the center of gravity of Q, find the distance of the spontaneous center of conversion of Q, s, and its center of gravity I, and a body whose magnitude is equal to $Q \times \frac{sI}{sd}$ impinging directly, would have the same effect upon B, whose velocity consequently may be estimated as above.

CHAP.

CHAP. XI.

*CENTERS OF PERCUSSION, GYRATION AND OSCILLATION.

487. DEF. *THE center of percussion of a body, or system of bodies, is a point, which being stopped by an immoveable obstacle, the body or system is perfectly quiescent.*

If *A*, *B*, *C*, &c. be particles of a body, or bodies, whose centers of gravity are the points *A*, *B*, *C*, &c. connected by inflexible lines, and moving with equal velocities, in directions parallel to any right line *DN*, an immoveable obstacle opposed to the center of gravity *G* in the direction *ND* will destroy all motion. For, when *G* is stopped, each particle will endeavour, by its inertia, to proceed in the line in which it was moving, with a moment equal to the product of its quantity of matter and velocity; and, if any plane be drawn through *DGN*, the effort of each particle to communicate motion to it varies as its moment multiplied into its perpendicular distance (279). But, the velocities of *A*, *B*, *C*, &c. being equal, the sums of these efforts to move the plane, on each side of it, are equal (409), and being opposite they consequently destroy each other, and *A*, *B*, *C*, &c. are perfectly quiescent. If different bodies, or particles of the same body *A*, *B*, *C*, &c., whose relative situation is unchangeable, revolve about an axis passing through any point *S*, and a plane be drawn through their center of gravity *G* perpendicular to the axis, each particle will describe a plane parallel to this plane, their angular velocities will be equal, and their lineal velocities will be as their perpendicular distances from the axis of suspension. FIG. CXLVII. FIG. CXLVIII.

* Simpson's Fluxions, pag. 210. Lyons's Fluxions, pag. 244. Emerson's Mechanics, Sect. VI.

sion. And if, as before, a plane be drawn through G, parallel to the axis, the sums of the moments of A, B, C, &c. multiplied into their perpendicular distances, on each side of this plane, are not equal, because the velocities are not equal; and consequently if G be stopped, A, B, C, &c. will not be quiescent. But the lateral efforts to move the plane passing through SG perpendicularly to the axis, being supposed equal, it is evident that the center of percussion will be in SG.

FIG. CXLVIII. 488. PROP. *Let* A, B, C, *&c. be particles of a body, revolving about an axis passing through the point* S, *it is required to find the center of percussion.*

Let A, B, C, &c. be the places of the particles reduced to the plane passing through SG, perpendicular to the axis, by letting fall perpendiculars from them upon it; and let A's moment or $A \times SA$, because its velocity is as SA, be represented in quantity and direction by Ap perpendicular to SA, and resolved into two forces Aq perpendicular, and pq parallel to SG. The whole moment of A : its force perpendicular to $SG :: Ap\ (A \times SA) : Aq :: SA : Sa$ (sim. triangles), and $Aq = \frac{A \times SA \times Sa}{SA} = A \times Sa$; and, supposing P to be the center of percussion, the efficacy of A to communicate motion to P is equal to its perpendicular moment multiplied into the distance at which it acts $=$ $A \times Sa \times Pm = A \times Sa \times \overline{SP - Sm} = A \times Sa \times \overline{SP - \frac{SA^2}{Sa}} =$ $A \times Sa \times SP - A \times SA^2$. By a similar process it appears, that the efficacy of B, C, &c. to communicate motion to P, is $B \times Sb \times SP - B \times SB^2$, $C \times SC^2 - C \times Sc \times SP$, &c.; and, because P is quiescent, these forces are equal on each side of it, or $A \times Sa \times SP - A \times SA^2 + B \times Sb \times SP - B \times SB^2 = C \times SC^2 - C \times Sc \times SP$, and consequently $SP = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{A \times Sa + B \times Sb + C \times Sc}$, &c. Q. E. I.

489. Cor.

489. Cor. 1. Becaufe $A \times Sa + B \times Sb + C \times Sc$, &c. $= \overline{A+B+C}$, &c. $\times SG$ (410), $SP = \dfrac{A \times SA^2 + B \times SB^2 + C \times SC^2}{\overline{A+B+C}, \text{\&c.} \times SG}$, &c.

490. Cor. 2. The diftance of the center of percuffion from the center of gravity, or GP, is equal to $\dfrac{A \times GA^2 + B \times GB^2 + C \times GC^2}{\overline{A+B+C}, \text{\&c.} \times SG}$, &c. (479).

491. Cor. 3. In the fame body, or fyftem of bodies, A, B, C, &c. $\overline{A+B+C}$, &c. $\times SG \times GP = A \times GA^2 + B \times GB^2 + C \times GC^2$, &c. $SG \times GP$ is alfo a given quantity, wherever the point of fufpenfion be placed, becaufe $\dfrac{A \times GA^2 + B \times GB^2}{A+B, \text{\&c.}}$, &c. is given; and if SG be infinitely great, or the velocities of A, B, C, &c. be equal to each other, GP is evanefcent.

492. Cor. 4. If a circle be defcribed from G as a center, with a radius equal to SG, and, the plane of motion remaining as before, the points of fufpenfion be in different points of the periphery of this circle, the diftance between the centers of gravity and percuffion will be invariable; for SG being given, GP is given (laft cor.)

493. Cor. 5. If particles, whofe magnitudes are $\dfrac{A \times SA^2}{SP^2}$, $\dfrac{B \times SB^2}{SP^2}$, $\dfrac{C \times SC^2}{SP^2}$, &c. be concentrated in P, the fame angular velocity will be generated, in thefe particles, by a force F acting for a given time, and in A, B, C, &c. at their refpective diftances SA, SB, SC, &c. For, let X, Y, Z, &c. be refpectively in equilibrio with A, B, C, &c. and their moments are inverfely as their perpendicular diftances (272), or $A \times SA : X \times SP :: SP : SA$, and $A \times SA^2 = X \times SP^2$,

$\times SP^2$, and $X = \frac{A \times SA^2}{SP^2}$. By a ſimilar proceſs it is proved that $Y = \frac{B \times SB^2}{SP^2}$, and $Z = \frac{C \times SC^2}{SP^2}$.

494. Cor. 6. A pendulous body moving with a given angular velocity, will have the greateſt effect upon another body againſt which it impinges, when the point of impact is the center of percuſſion; for in this caſe all its motion is communicated. But when the direction of impact does not paſs through this center, it will have a lateral motion, or endeavour to continue its rotation.

495. Cor. 7. If the points A, B, C, &c. and G be reduced to any plane perpendicular to the axis, and the center of percuſſion be ſuppoſed to be in this plane, its diſtance from the axis of ſuſpenſion is found, by ſubſtituting the diſtances of their places in this plane from the axis, inſtead of SA, SB, &c. in the expreſſion $\frac{A \times SA^2 + B \times SB^2}{A + B, \&c. \times GS}$, &c.

496. LEMMA. *The velocity* V, *generated in a body whoſe quantity of matter is* Q, *by the action of a conſtant force* F, *for a given time, will vary as the force directly and quantity of matter inverſely.*

DEM. Let V, Q, F, be variable, and it is evident that V will be encreaſed and diminiſhed in the ſame ratio with F directly, and in the ſame ratio with which the inertia or (pag. 63.) Q is diminiſhed and encreaſed; or V varies as $\frac{F}{Q}$. Q. E. D.

497. Cor. Becauſe the velocity is equal to the moment of a body divided by its quantity of matter, if m repreſent the quantity of motion generated in a given time by F, or the numeral product of

of the velocity and quantity of matter in a given body, V will always be equal to $\frac{m}{Q}$.

498. PROP. *If any constant force* F *act, for a given time, perpendicularly upon the line* SV, *at the point* V, *and* SV *and the particles* A, B, C, *&c. whose center of gravity is* G, *are connected to an axis passing through a fixed point* S, *the velocity communicated to the point* V *will be as* $\frac{F \times SV^2}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, *&c.* FIG. CXLVIII.

DEM. Suppose particles, whose magnitudes are $\frac{A \times SA^2}{SV^2}$, $\frac{B \times SB^2}{SV^2}$, $\frac{C \times SC^2}{SV^2}$, &c. to be concentrated in V; and they would be in equilibrio with A, B, C, &c. respectively (493), and consequently equally resist the action of F. But the velocity generated in these particles, collected in V, is as $\frac{F}{Q}$ (496), or as $\frac{F}{\frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{SV^2}}$, &c. or as $\frac{F \times SV^2}{A \times SA^2 + B \times SB^2 + C \times SC^2}$. Q. E. D.

499. Cor. 1. If m, as before (497), represent the absolute quantity of motion generated by F in a given time; then the velocity of $V = \frac{m \times SV^2}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c.

500. Cor. 2. Because all points describe equal angles at the axis, in equal times, the velocity of V is to the velocity of $A :: SV : SA$, and the velocity of $A =$ velocity of $V \times SA = \frac{m \times SV \times SA}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c.; and by a similar process, the

velocity

velocity of $B = \frac{m \times SV \times SB}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c.; and the velocity of $C = \frac{m \times SV}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c.

501. Cor. 3. The angular velocity of a body, being as its lineal velocity directly and distance inversely, the angular velocity of A, or B, &c. $= \frac{m \times SV}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c. And the moment of a body being measured by the quantity of matter and velocity, the sum of the moments communicated to A, B, C, &c. will be equal to $\frac{\overline{A \times SA + B \times SB + C \times SC, \text{&c.}} \times m \times SV}{A \times SA^2 + B \times SB^2 + C \times SC^2}$ (last cor.), &c.

FIG. CXLIX. 502. Cor. 4. If A, B, C, &c. be collected in any point L, the quantity of motion generated in them, in a given time, by the action of the constant force F at V is equal to $\frac{\overline{A+B+C, \text{&c.}} \times SL \times m \times SV}{\overline{A+B+C, \text{&c.}} \times SL^2}$ $(500) = \frac{\overline{A \times SA + B \times SB + C \times SC, \text{&c.}} \times m \times SV}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c. and consequently $SL = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2, \text{&c.}}{A \times SA + B \times SB + C + SC, \text{&c.}}$: but $A \times SA + B \times SB + C \times SC = \overline{A + B + C}, \text{&c.} \times SG$, and $SL = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{\overline{A + B + C}, \text{&c.} \times SG}$, &c., and L is the center of percussion. If therefore any system of particles be concentrated in the center of percussion, the quantities of motion generated in them, in a given time, placed in that point and at their respective distances, SA, SB, &c., by the action of F at any point V, are the same.

SCHOLIUM.

503. In the preceding proposition and corollaries A, B, C, &c. are supposed to be unaffected by gravity, and motion to be communicated

municated by ſome conſtant external force F, which is only reſiſted by their inertia; but the rules inveſtigated are applicable when F is the force of gravity, or that force acts in conjunction with F: for during an infinitely ſmall time the force of gravity may be conſidered as conſtant. If a ſyſtem of bodies A, B, C, &c. whoſe center of gravity is G, be ſuſpended from an horizontal axis paſſing through S, the ſum of their efforts to deſcend is the ſame as if they were collected in G, and they will therefore deſcend till SG be perpendicular to the horizon. In any other poſition of SG, let GL repreſent the force of deſcent in a direction perpendicular to the horizon, and be reſolved into two forces, GE touching the arc, and EL in the direction of SG. EL is the tendency from S, and GE is that part of the whole force of gravity which produces a rotation about S, and may be deemed conſtant whilſt G deſcribes an infinitely ſmall arc, and is to be added to, or ſubtracted from, F, according as they conſpire with, or oppoſe, each other. FIG. CL.

CENTER OF GYRATION.

504. DEF. *The center of gyration of a body, or ſyſtem of bodies,* A, B, C, *&c. is a point* Y, *in which, if* A + B + C, *&c. be collected, the ſame angular velocity will be generated, in a given time, by any conſtant force* F, *acting at any point* V *in a given direction, whether* A + B + C, *&c. be collected in* Y, *or be at their original diſtances* SA, SB, SC, *&c.* FIG. CLI.

505. PROP. *To find the center of gyration of the particles* A, B, C, *&c. which are connected to an axis paſſing through the point* S, *and preſerve their relative ſituation.*

Let SA, SB, SC, &c. be the perpendicular diſtances of A, B, C, &c. from the axis; and, ſuppoſing the quantity of motion generated by F, in a given time, to be m, their angular velocity is equal

to

to $\frac{m \times SV}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c. (501); and, when $A + B + C$, &c. are concentrated in Y, their angular velocity $= \frac{m \times SV}{\overline{A + B + C, \text{\&c.}} \times SY^2}$. But, from the definition of the center of gyration, $\frac{m \times SV}{A \times SA^2 + B \times SB^2, \text{\&c.}} = \frac{m \times SV}{\overline{A + B + C} \times SY^2}$, and $SY^2 = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2, \text{\&c.}}{A + B + C, \text{\&c.}}$, and $SY =$ the distance of the center of gyration from $S = \sqrt{\frac{A \times SA^2 + B \times SB^2 + C \times SC^2}{A + B + C, \text{\&c.}}}$, &c.

506. Cor. 1. If therefore $p =$ a particle of a material surface or body, B, $d =$ its perpendicular distance from the axis of suspension passing through S, the distance of the center of gyration from the axis is equal to the sum of all the $\frac{d^2 \times p}{B}$.

507. Cor. 2. If any part of a system of particles A, B, &c. be collected in their center of gyration, the center of gyration of the whole system will continue the same; for the same force will communicate an equal angular velocity to A and B and $A + B$ placed in their center of gyration. A, B, C, &c. may consequently be considered as bodies of any magnitude, whose centers of gyration are the points A, B, C, &c. respectively.

508. Cor. 3. Because $\overline{A + B + C}$, &c. $\times SY^2 = A \times SA^2 + B \times SB^2 + C \times SC^2$, &c. $= \overline{A + B + C} \times SG \times SP$ (489), therefore $SY^2 = SG \times SP$, or the distance of the center of gyration from the axis of suspension is a mean proportional between the distances of the center of gravity and the center of percussion from that axis.

509. Cor.

509. Cor. 4. If a syſtem of bodies A, B, C, &c. whoſe ſum $= Q$, connected as in this propoſition, be ſtruck by a given moment at the point V, in a direction perpendicular to a line SV, drawn from V through the center of gravity and perpendicular to an axis paſſing through S, their angular velocity may be found: for, let Y be the center of gyration, G the center of gravity, and the angular velocity of A, B, C, &c. is the ſame, whether Q be collected at Y, or a body equal to $\frac{Q \times SY^2}{SV^2}$ (489), or $\frac{Q \times SG \times SP}{SV^2}$ be directly oppoſed to the impinging body at the point V. But $A+B+C$, &c. or Q being given, and alſo the quantity of matter and velocity of the ſtriking body, the velocity of Q may be found by the common rules in the direct impact of bodies (449). And the arc deſcribed in a given time by Q and the diſtance SV, being known, the angle which it ſubtends may be found. The converſe of this, or the velocity of the impinging body, may be found, if its quantity of matter, and Q and its velocity, be given. FIG. CLL.

510. Cor. 5. In a given ſyſtem of bodies A, B, C, &c. whoſe center of gravity is G, if a circle be deſcribed from G as a center with any radius, and the center of ſuſpenſion S be in its periphery, the diſtance of the center of gyration from S is always the ſame; for $SY^2 = \frac{A \times SA^2 + B \times SB^2 + C \times SC^2, \text{\&c.}}{A+B+C, \text{\&c.}}$, which is a given quantity (412).

511. Cor. 6. If the periphery of a circle, compoſed of the particles A, B, C, &c. were to revolve about an axis perpendicular to its plane and paſſing through its center, the center of gyration will be in the periphery; for $SY = \sqrt{\frac{A+B+C, \text{\&c.} \times \text{rad.}^2}{A+B+C}} =$ radius. The angular velocity is therefore the ſame as if all the matter were collected in any one point of the periphery; and if all the matter in a gyrating body is to be placed in the center of gyration Y, it may be placed in any point of the periphery of a

circle

circle whofe radius is SY, or collected into two equal portions, and placed in two points diametrically oppofite to each other, and whofe diftances from $S = SY$; for then the center of gravity will be in S, and there will be no lateral motion.

CENTER OF OSCILLATION.

FIG. CLII. 512. DEF. *The center of ofcillation of a body, or fyftem of bodies,* A, B, C, *&c. connected to an axis paffing through* S, *and moving by the action of gravity, is a point, in which, if the body or fyftem be collected, it, and* A, B, C, *&c. placed at their original diftances* SA, SB, SC, *&c. will defcribe equal angles at the axis in the fame times.*

513. PROP. *Suppofe* A, B, C, *&c. to be particles of a body connected to an horizontal axis paffing through* S, *and acted upon by the force of gravity in a direction parallel to the vertical line* SV, *to find their center of ofcillation.*

Let SGO be drawn in the plane defcribed by the center of gravity G, and O be the center of ofcillation, and the lines SO, SA, SB, &c. will defcribe equal angles at the axis in equal times. If the weight of A, which is as A (234), acted perpendicularly to SA, its moment would be $A \times SA$: let this be reprefented by Ap, perpendicular to the horizon, aud refolved into two, Aq perpendicular to SA, and qp parallel to it, and this laft being loft, Aq is the only efficient part of A's moment. But Ap $(A \times SA) : Aq :: SA : Aa$ (fim. triangles) and $Aq = \frac{A \times SA \times Aa}{SA} = A \times Aa$*; and the angular velocity of A generated in a given time is (501) $\frac{A \times Aa}{A \times SA^2 + B \times SB^2 + C \times SC^2}$, &c. By a fimilar procefs the angular velocities of B, C, &c. generated in the fame time, are

$= B \times b$

* This follows immediately from (279), for Aa is equal to the perpendicular let fall from the center of motion upon the direction of A's weight.

$\frac{-B\times Bb}{A\times SA^2+B\times SB^2+C\times SC^2}$, &c., $\frac{-C\times Cc}{A\times SA^2+B\times SB^2+C\times SC^2}$, &c.
When $A+B+C$, &c. are concentrated in O, the angular velocity of O generated in the same time, would be as $\frac{\overline{A+B+C,\text{\&c.}}\times Oo}{\overline{A+B+C,\text{\&c.}}\times SO^2}$ $=\frac{Oo}{SO^2}=\frac{Gg}{SG\times SO}$ (sim. trian.): therefore $\frac{A\times Aa-B\times Bb-C\times Cc}{A\times SA^2+B\times SB^2+C\times SC^2}$ $=\frac{Gg}{SG\times SO}$; but $A\times Aa-B\times Bb-C\times Cc=\overline{A+B+C,\text{\&c.}}\times Gg$, and consequently by substitution, $SO=\frac{A\times SA^2+B\times SB^2+C\times SC^2}{\overline{A+B+C,\text{\&c.}}\times SG}$.

514. Cor. 1. The distances of the centers of oscillation and percussion from the axis are equal, each being equal to $\frac{A\times SA^2+B\times SB^2+C\times SC^2}{\overline{A+B+C}\times SG}$, &c., and consequently $SG\times GO$ equal is to $\frac{A\times GA^2+B\times GB^2}{A+B+C,\text{\&c.}}$, &c. (490), and SG and GO are inversely as each other. If SG be infinitely great, or A, B, C, &c. move with equal velocities in parallel lines, GO is evanescent, or the centers of gravity and oscillation coincide.

515. Cor. 2. If any number of particles be connected to an axis and revolve round it, retaining their relative situation, by the action of a force whose direction is in the plane of motion and perpendicular to the axis, the time of describing any given angle is the same as if they were concentrated in O and urged in the same direction by the same force; and the points where the forces act are transferred from A, B, C, &c. to O. But in the center of gyration Y, the same force is supposed to act at the same point, when A, B, C, &c. are placed at their respective distances SA, SB, SC, &c. and concentrated in Y.

 516. Cor.

516. Cor. 3. If A, B, C, &c. be bodies composed of any number of particles, let l, m, n be the respective distances of their centers of gravity, and p, q, r, the distances of their centers of oscillation from the axis, and SO will be equal to $\frac{l \times p \times A + m \times q \times B + n \times r \times C}{A + B + C, \&c. \times SG}$: for let a, b, c, be particles of A, B, C, and x, y, z, their respective distances from the axis, and, by this proposition, $SO = \frac{\text{sum of all the } a \times x^2 + \text{sum of } b \times y^2 + \text{sum of } c \times z^2}{SG \times \overline{A + B + C}}$; but the sums of all the $\frac{a \times x^2}{l \times A}, \frac{b \times y^2}{m \times B}, \frac{c \times z^2}{n \times C}$ are respectively equal to p, q, r, and consequently, the sums of $a \times x^2 + b \times y^2 + c \times z^2 = p \times l \times A + q \times m \times B + r \times n \times C$, and $SO = \frac{l \times p \times A + q \times m \times B + r \times n \times C}{A + B + C, \&c.}$.

FIG. CXLVIII. 517. Cor. 4. If A, B, C, consisting of a number of particles, were urged by a constant force F at the point V, as in (498): let q, r, s, &c. be particles in A, t, u, v, &c. particles in B, and x, y, z, &c. particles in C, and $A = q + r + s$, &c., $B = t + u + v$, &c., $C = x + y + z$, &c.; and the velocity communicated to V in a given time is as $\frac{F \times SV^2}{q \times Sq^2 + r \times Sr^2 + s \times Ss^2, \&c. + t \times St^2 + u \times Su^2, \&c. + x \times Sx^2 + y \times Sy^2, \&c.}$; but if P, p, π be the centers of percussion, and G, g, γ the centers of gravity of A, B, C, &c. respectively, then (489)

$$q \times Sq^2 + r \times Sr^2 + s \times Ss^2, \&c. = SP \times SG \times A$$
$$t \times St^2 + u \times Su^2 + v \times Sv^2, \&c. = Sp \times Sg \times B$$
$$x \times Sx^2 + y \times Sy^2 + z \times Sz^2, \&c. = S\pi \times S\gamma \times C;$$

and the velocity of V is as $\frac{F \times SV^2}{SP \times SG \times A + Sp \times Sg + B + S\pi \times S\gamma \times C}$.

FIG. CLIII. 518. Prop. *If O be the center of oscillation when the axis of suspension passes through S, the point S will be the center of oscillation when the axis of suspension passes through O, the plane of motion being supposed to be unaltered.*

Dem.

DEM. When S is the point of suspension, the distance of the center of oscillation from the center of gravity G, or $GO =$ $\dfrac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C, \&c. \times SG}$, &c. (514); and when the axis passes through O, the distance of the center of oscillation from $G = D = \dfrac{A \times AG^2 + B \times BG^2}{A + B + C \times OG}$, &c.; and consequently $SG \times GO$ $= D \times GO$ and $SG = D$, and the distance of the center of oscillation from $O = SO$. Q. E. D.

519. Cor. 1. If two circles be described from G as a center with the radii GS and GO, the distances of the centers of oscillation from the points of suspension would be the same in whatever parts of the peripheries they were placed; for SG, or GO being given, the other is given (491), and consequently the times of describing equal angles at the axis will be equal, if the points of suspension be any where in these peripheries.

520. Cor. 2. If the axis of rotation passed through the center of gravity G, and Y were the center of gyration, GY^2 would be equal to $\dfrac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C}$ (505) $= SG \times GO$, and consequently GO is a third proportional to SG and GY.

521. Cor. 3. Because $SG \times GO$ is a given quantity, $SG + GO$ is the least possible when the peripheries of the circle, whose radii are SG and GO, meet in Y, because in that supposition $SG = GO$ (note to 458). And because in this case SO is the least possible, the time of describing a given angle at the axis is the least possible. But because GO encreases as GS decreases, the time of describing a given

a given angle at the axis decreases and arrives at its limit when V and S coincide in the center of gyration Y, and no point of suspension can be assumed, where the time would not be greater than when the axis passes through Y. As GS encreases from GY to infinity, GO is diminished without limit, and the time of describing a given angle, whether the axis passes through S or O, is perpetually encreased.

CHAP.

CHAP. XII.

RECTILINEAL MOTION OF BODIES.

522. LEMMA. *IF a side* AB *of a right-angled triangle* ABC *be divided into very small equal parts,* A b, b c, c d, d e, *&c. and rectangular parallelograms be described upon them about the triangle, the sum of the rectangles is equal to the area of the triangle, when* A b, b c, *&c. are diminished without limit.* FIG. CLIV.

For the difference between the triangle and the sum of the rectangles, is the sum of the triangles $Alm+mnp+pqr$,&c. which is equal to $\frac{Ab \times BC}{2} = 0$ when Ab vanishes. Q. E. D.

523. PROP.* *If a quiescent body be acted upon by a constant force in the same right line, and any right line* AB *represent the time, and* LM, *perpendicular to it, the velocity communicated in the time* AL, *the space described in the time* AL *will vary as the triangle* ALM.

DEM. Let AB be divided into very small equal parts Ab, bc, cd, &c. and let the forces act only at the beginning of each part, and the velocity consequently during each will be uniform. If the velocity communicated at A be represented by bm, the increments communicated at b, c, d, &c. will be each equal to bm, and consequently LM encreases in the same ratio with AL, and AM is a right line. But the spaces described in the particles of time Ab,

* Keil's Physics, Lect. XI. Graves, Lect. I. Ch. XIV. Muschenbroek, Ch. VI. CLVIII. Cotes de Descensu Gravium. Morgan's Notes to Rohault. Maclaurin's Phil. Disc. B. II. Ch. V. Emerson, p. 5, &c.

Ab, bc, cd, ef, &c. are as the rectangles Am, bp, cr, ds, &c. (107), and the whole space described in the time AL, is as the sum of these rectangles, which, when Ab, bc, cd, &c. vanish and the force acts incessantly, are equal to the triangular area (ALM). Q. E. D.

524. Cor. 1. If the velocity and time be represented by V and T, and be divided into very small increments V' and T', whose number is n, $V = n \times V'$ and $T = n \times T'$.

525. Cor. 2. If V, T and v, t, represent corresponding velocities and times, $V : v :: T : t$ and $\frac{V}{v} = \frac{T}{t}$; and, if for v a number of feet uniformly described in t seconds be substituted, $V = \frac{v \times T''}{t''}$ will be the number of feet described in t'' by the velocity V.

526. Cor. 3. Whatever be the magnitude of the constant force, the space described, S, will be represented by a triangular area, and consequently S, or the spaces described by the action of different constant forces, are generally as $V \times T$, and, when numbers are substituted for V and T, S is equal to the product of $\frac{V \times T}{2}$*. In the same constant force, the triangular areas ALM and ABC are always similar, and V and T are directly as each other, and consequently S is as V^2 or T^2. If therefore the time be divided into equal parts, the spaces described in these times are as the odd numbers 1, 3, 5, 7, &c.; for the spaces described in 1, 2, 3, 4, &c. parts of time, are as 1, 4, 9, 16, &c. and consequently the spaces described in the 1st, 2d, 3d, &c. alone are as 1, 3, 5, 7, &c.

527. Cor.

* This equation may be deduced from finding the sum of an arithmetic progression: let V' and T' be increments of V and T, and $S = \overline{1+2+3+4 \ldots n} \times V'T' = \frac{n^2}{2} \times V'T'$ (because n is infinitely great, and consequently vanishes compared with n^2). $= \frac{nV' \times nT'}{2} = \frac{V \times T}{2}$.

527. Cor. 4. The ſpace which is deſcribed in any time T from a ſtate of reſt, is half of that deſcribed, in the ſame time, with the laſt velocity, V, continued uniform: for, in the firſt caſe, the ſpace $= \frac{V \times T}{2}$ (526), and, in the ſecond, the ſpace $= V \times T$ (107).

528. Cor. 5. If a body, projected with any velocity, be acted upon by a conſtant force, in a direction oppoſite to its motion, it will be uniformly retarded: for the force will evidently deſtroy equal parts of velocity in equal times. A body therefore projected with the laſt acquired velocity, will aſcend to the point from whence it fell, and the velocities in the aſcent and deſcent are the ſame at the ſame point. And, if bodies be projected with different velocities and reſiſted by the ſame conſtant force, the time that elapſes, T, till the velocity be deſtroyed, will vary as the velocity of projection V; the ſpace will vary as the ſquare of the time T^2, or ſquare of velocity V^2; the ſpaces deſcribed in equal portions of time, will be as the odd numbers in a retrograde order 7, 5, 3, 1; and, whether theſe retarding forces be the ſame or different, the ſpace will vary as $V \times T$, and be equal to $\frac{V \times T}{2}$.

SCHOLIUM.

529. At the ſurface of the earth the force of gravity is conſtant, for a body deſcends through $16\frac{1}{12}$ feet nearly in the firſt ſecond, and conſequently acquires a velocity that would make it deſcribe $32\frac{1}{12}$ feet uniformly in $1''$; and this is found to be the velocity communicated in every ſucceeding ſecond. Any of theſe quantities, ſpace deſcribed S, velocity acquired, or time T, being known, the others may be diſcovered.

1. The proper meaſure of velocity is the ſpace uniformly deſcribed by it in any given time: if therefore a body fall by the force of gravity for t'', and acquire a velocity which would make

 it

it describe V feet uniformly in 1″, V and 32 are the proper measures of the velocities communicated in t'' and 1″ (106), and $V : 32 :: t'' : 1''$ (525), and $V = 32 \times t''$. If $t = 2, 3, 4$, &c. seconds, $V = 2 \times 32,\ 3 \times 32,\ 4 \times 32$, &c. feet respectively.

2. If V, or the number of feet uniformly described in 1″, by the velocity acquired in falling for an unknown time, be given, this time may be found; for it is equal $\frac{V}{32}$ seconds. If $V = 192$ feet, the time of falling $= \frac{192''}{32} = 6''$.

3. If Y be the space described in falling t'', $16 : Y :: 1^2 : t''^2$ (526), and $Y = 16 \times t''^2$ feet. If $t = 2, 3, 4$, &c. seconds, $Y = 16 \times 2^2$, 16×3^2, 16×4^2, &c. feet respectively. Or Y may be found by knowing the velocity (V) at the end of the descent; for $Y : 16 :: V^2 : 32^2$, and $Y = \frac{16 \times V^2}{\overline{32}|^2} = \frac{16 \times V^2}{4 \times \overline{16}|^2} = \frac{V^2}{64}$ feet. If $V = 192$ feet, $Y = \frac{\overline{192}|^2}{64} = 576$ feet.

4. If Y, or the number of feet described in falling, be known, the time of descent, and velocity acquired, may be found; for $Y = 16 \times t''^2$, and consequently $t'' = \frac{\sqrt{Y}}{4}$. If $Y = 576$ feet, $t = \frac{\sqrt{576}}{4} = \frac{24}{4} = 6$ seconds. And because $Y = \frac{V^2}{64}$, $V = \sqrt{Y} \times 8$. If $Y = 576$ feet, $V = \sqrt{576} \times 8 = 24 \times 8 = 192$ feet.

530. PROP. *If bodies be acted upon by different constant forces, the velocities communicated will vary in a ratio compounded of the forces and times.*

Let

Let S, V, T, reprefent force, velocity and time, and be fuppofed variable, and it is evident that the velocity will be encreafed and diminifhed in the fame ratio with both the force and time, and, thefe being independent of each other, V will be as $F \times T$. Q.E.D.

531. Cor. 1. V therefore is as $F \times T$, and if F be compared with the force of gravity f, or any other known force, capable of generating a velocity equal to v in the time t, then $V : v :: F \times T : f \times t$, and $\frac{V}{v} = \frac{F \times T}{f \times t}$.

532. Cor. 2. Becaufe, in all conftant forces, the fpace varies generally in a ratio compounded of the velocity and time, and the velocity varies generally as the force and time; the following analogies are general:

S is as $V \times T$, or as $F \times T^2$, or as $\frac{V^2}{F}$;

T varies as $\frac{S}{V}$, or as $\sqrt{\frac{S}{F}}$;

V varies as $\frac{S}{T}$, or as $\sqrt{S \times F}$; and

F varies as $\frac{V}{T}$, or as $\frac{S}{T^2}$, or as $\frac{V^2}{S}$.

If numbers be fubftituted for V, F, and T; or the number of feet defcribed in a given time 1, by the action of any conftant force F, be fubftituted for the force; the number of feet which would be uniformly defcribed in this given time 1, by the velocity acquired in the time T, $= V$; and $T =$ the number of parts of time each equal to 1, contained in the whole time in which the velocity is communicated: then (106) V and $2F$ are proper meafures of the velocity acquired in the times T and 1, and $V : 2F :: T : 1$ (525) and $V = 2F \times T$; and fubftituting this value of V in the general equation ($S = \frac{V \times T}{2}$), the following general equations are deduced:

$$S = \frac{V \times T}{2} = F \times T^2 = \frac{V^2}{4F} \text{ feet;}$$

$$T = \frac{2S}{V} = \sqrt{\frac{S}{F}}, \text{equal portions of time, each of which is equal to 1; and}$$

$$V = \frac{2S}{T} = \sqrt{S \times 4F} \text{ feet.}$$

533. Cor. 3. Theſe equations and analogies are applicable to all accelerating and retarding forces, ſuch as gravity, and to all reſiſtances that are conſtant and produce equal decrements of velocity in equal times. If the motion of a ball ſhot into a bank of earth or piece of wood, be uniformly reſiſted, and the magnitude of this reſiſtance compared with gravity, and the velocity of impact, be known; the depth, or ſpace through which it moves before all motion is deſtroyed, may be diſcovered; and vice verſâ *.

FIG. CLV. 534. PROP. *If a body deſcend down an inclined plane* AB *by the action of a conſtant force* F, *whoſe direction is perpendicular to any right line* BC; F *will be to that part of* F *which makes the body deſcend, as the radius to the ſine of the plane's inclination to* BC.

DEM. Let any given line LM, perpendicular to BC, repreſent the conſtant force F, in quantity and direction, and be reſolved into two forces, LN parallel, and MN perpendicular, to the plane; MN is deſtroyed by the reſiſtance of the plane, and LN is the only part of F that communicates motion to the body; therefore F:

* EXAMPLE. Suppoſe a reſiſtance were equal to the reſiſtance of gravity F, when a body is projected perpendicularly upwards, multiplied into any number n, and all motion were deſtroyed in t'', then S (being equal to the force multiplied into the ſquare of the time) $= n \times 16 \times t^2$. If the velocity of projection were ſuch as would make a body deſcribe f feet in a ſecond uniformly, then S (being equal to $\frac{\text{vel.}^2}{4 \times \text{force}}$) $= \frac{f^2}{4 \times n \times 16} = \frac{f^2}{n \times 64}$. If the ſpace deſcribed and force be known, then the time is equal to $\sqrt{\frac{S}{n \times 16}}''$, and the velocity $= \sqrt{S \times n \times 64}$ feet.

F : that part of F which makes the body move (A) :: $LM : LN$ :: $AB : AC$ (fim. triangles) :: rad. : fin. of $\angle ABC$. Q. E. D.

535. Cor. 1. F is to that part of F which is deftroyed by the reaction of the plane as $LM : MN :: AB : BC$ (fim. triangles) :: radius : cof. $\angle ABC$; and confequently the part of F, deftroyed by the refiftance of the plane, $= \frac{F \times \text{cof.} \angle ABC}{\text{radius}}$, which is conftant upon the fame, or parallel, planes, and upon different planes, not parallel, varies as the $\frac{\text{bafe}}{\text{length}}$, if F be given.

536. Cor. 2. Becaufe the accelerating force $LN = A = \frac{F \times AC}{AB}$, and F is given, A is conftant upon the fame, or parallel planes, and, upon different planes, inclined in unequal angles to BC, it varies as the height divided by the length, or as $\frac{H}{L}$, calling H the perpendicular height and L the length.

537. Cor. 3. Becaufe the force upon the fame plane is conftant, the analogies and equations in (526) are applicable to the motion of a body upon the fame plane; and the analogies and equations in (532) obtain when bodies move upon different planes. If BC be horizontal, and F be the force of gravity, then the fpace defcribed upon the plane in $T'' = S = A \times T''^2 = F \times T^2 \times \frac{H}{L}$ (536) $= \frac{16 T^2 \times H}{L}$; $V = \frac{2F \times T'' \times H}{L} = \frac{32 T'' \times H}{L} = 8 \times \sqrt{H}$ feet; and $T'' = \frac{V \times L}{32 \times H} = \frac{L}{4 \times \sqrt{H}}$.

538. Becaufe the fpace defcribed is generally as the force multiplied into the fquare of the time (532), the fpaces that are to each

each other as the forces muſt be deſcribed in the ſame time, or when S is as F, T is given. If therefore two bodies deſcend at the ſame inſtant, one down the plane AB, and the other in AC perpendicular to BC, and CP be drawn perpendicular to AB, P and C are cotemporary poſitions of the bodies; for $AP : AC :: LN : LM :: A : F$, that is, the ſpace is as the force, and T^2 and T are given. If the diameter AD of a circle be perpendicular to the horizon, a body will deſcend through it, and any chord AP drawn from its extremity A, in the ſame time, becauſe the angle APD is a right one; and the times of deſcent through all the chords drawn from A are conſequently equal to each other.

FIG. CLVI.

539. Cor. 5. The velocities acquired in falling down different inclined planes, are as the ſquare roots of their perpendicular heights; for V^2 is always as the ſpace multiplied into the force, or as $L \times A$, or as $\frac{L \times H}{L}$ (536), or as H, and V is as $\sqrt{H}$; this alſo follows from art. 537. The velocities therefore acquired in falling down the perpendicular AC, and any planes AB, AE, AG, &c. drawn from A and terminated by the baſe CB, are equal to each other. And becauſe V is always equal to $2A \times T$, and in this cor. V is given, the times of falling are as the forces inverſely, or as the lengths of the planes. Or becauſe S is as $V \times T$, and V is given, T is as S, that is, as the length.

FIG. CLV.

540. Cor. 6. The times of deſcending through different planes are as the lengths directly and ſquare roots of the heights inverſely; for the time is generally as the ſpace deſcribed directly and laſt velocity inverſely (532), or T is as $\frac{L}{V}$, or as $\frac{L}{\sqrt{H}}$ (laſt cor.); this alſo follows from art. 537. The times therefore of deſcribing different planes, equally inclined to the direction of F, are as the ſquare roots of their lengths; for H is as L (ſim. triangles), and conſequently $\frac{L}{\sqrt{H}}$ is as $\frac{L}{\sqrt{L}}$, or as $\sqrt{L}$ or T.

541. PROP.

541. PROP. *The velocity acquired in falling down any number of contiguous planes, supposing none to be lost in passing from one plane to another, is equal to the velocity acquired in falling down the same perpendicular altitude.* FIG. CLVII.

DEM. The velocities acquired in falling down AB and EB, $AB + BC$ and EC or PC, $AB + BC + CD$ and PD, or down the perpendicular PR, are equal (539). Q. E. D.

542. Cor. The velocities, acquired in falling down any systems of planes, are therefore as the square roots of their perpendicular heights; and if the body be projected from D with the velocity acquired, it will ascend through this or any other system of planes to the same perpendicular altitude.

543. PROP. *The velocity lost in passing from any plane to the next, is to the whole velocity as the sine of the angle of their inclination to the radius.* FIG. CLVIII.

DEM. Let CB represent the velocity acquired in falling down AC, and be resolved into two, CP coincident with the plane CD; and BP perpendicular to it; and this last is evidently destroyed by the resistance of the plane: but $BP : CB ::$ sin. of $\angle PCB$ or $\angle PCA$: radius. Q. E. D.

544. Cor. 1. $CB : CP$ (velocity upon the plane CD) :: radius : cosine of $\angle PCB$ or PCA. If the angle ACD become equal to two right angles, or ACD be any circular arc, CP and CB coincide, or the velocity lost is equal to nothing; and consequently a body moving in any circular arc sustains no loss of velocity by changing the direction of its motion; and a body, projected with the velocity acquired, up any curvilineal arc, will rise to the same height from whence it fell.

545. Cor.

545. Cor. 2. If ACD be any circular arc, the velocity acquired in falling from A to D is consequently equal to that acquired in falling through the same perpendicular height; and the velocities acquired or destroyed in descending or ascending through two curve lines are as the square roots of their perpendicular altitudes: for every curve may be considered as consisting of an indefinite number of inclined planes.

FIG. CLVII. 546. PROP. *The times of describing two systems of inclined planes* ABCD *and* abcd, *whose number, inclinations, and ratios of their lengths are the same, are to each other as the square roots of the lengths of the planes.*

DEM. Because the planes are equally inclined to the direction of the force, the time of falling down AB is as $\sqrt{AB}$ (540), or as $\sqrt{EB}$, or $\sqrt{EC}$ (hypoth.), or as the time down EB, or EC; and, dividendo, the time of falling down BC, after having fallen down AB or EB, is as $\sqrt{AB}$; consequently the time of falling down $AB + BC$ is as $\sqrt{AB}$, or as $\sqrt{AB + BC}$, &c.: for from the supposition $\sqrt{AB} : \sqrt{ab} :: \sqrt{AB + BC} : \sqrt{ab + bc}$, &c. Q. E. D.

547. Cor. 1. The times of describing similar parts of similar curves, equally inclined, in similar parts, to the direction of the force, are as the square roots of their lengths.

548. Cor. 2. If two bodies vibrate in similar circular arcs, the times of performing these vibrations are as the square roots of the lengths of these arcs, or as the square roots of their radii.

SCHOLIUM.

549. In the preceding propositions and corollaries, inclined planes and bodies, descending or ascending upon them, are supposed to be without any asperities upon their surfaces, and the bodies are supposed to move by the action of a force in a given direction,

direction, without any retardation from friction, or rotation about an axis. In this supposition, the tendency of each particle of a body to descend, when impelled by the force of gravity, is equal to the particle multiplied into $\frac{H}{L}$, and the tendency of the body W is equal to $W \times \frac{H}{L}$, and is the same as if it were collected in its center of gravity. But if the parts of W and the inclined plane adhere together, the force of this adhesion must be estimated and allowed for in practice. If W be a spherical body, and the parts of its surface in contact with the plane adhere by the pressure of W upon it, their tendency to descend will be diminished by this adhesion, and consequently they will be regressive with regard to the center of gravity G, and revolve round it. The absolute velocities of different particles in W, and consequently their accelerating forces, are different; and the tendency to descend of the center of gravity is diminished by this motion of rotation round it. The angular velocity of every particle round G is the same, and the same as if all the particles were concentrated in their center of gyration Y, or a body whose magnitude is $\frac{W \times GY^2}{GA^2}$ were collected in A (498).

550. PROP. *Suppose the same spherical body to slide and roll down the same inclined plane, to find the ratio of the forces acting upon it.* FIG. CLIX.

From the nature of the circle, the initial regressive velocity of the point A is equal to the progressive velocity of the center of the sphere, or center of gravity G; but the regressive velocity of a body whose magnitude is $\frac{W \times GY^2}{GA^2}$ placed at A (last scholium), would be destroyed by the action of a force in a contrary direction equal to $\frac{H}{L}$; and in this case G would not be impeded by the regressive motion of A, but G and A would slide down with equal velocities; consequently

the

the force acting upon W when it slides : the force when it rolls down the plane :: $\overline{W + \frac{W \times GY^2}{GA^2}} \times \frac{H}{L} : W \times \frac{H}{L}$. Q. E. I.

551. Cor. If the center of gyration be found, and GY, and GA be expreffed in numbers, the ratio of thefe forces will be expreffed in terms of the weight.

SCHOLIUM.

552. In the communication of motion by the external applica-cation of forces fuch as impact, protrufion, &c. their magnitude, or capacity to communicate motion, is not always meafurable by their cotemporary effects; but in this chapter every particle of matter is fuppofed to be impelled by a force, fimilar to that of gravity, with the fame intenfity in parallel directions, which will confequently communicate the fame velocity, or change of velocity, to bodies of different magnitudes, whether quiefcent or moving. The magnitude of thefe forces at any inftant, or the magnitude of their accumulated action, is meafurable by their cotemporary effects, or by the velocity generated or deftroyed in equal times; for the increments or decrements of velocity, produced in equal times, by the fame conftant force, being equal, the whole velocity is encreafed or diminifhed uniformly; and when the forces are unequal, the changes of velocity being as the forces, it is evident that the velocities are proper meafures of the intenfity of forces, and will vary as the forces multiplied into the times in which they are generated or deftroyed. And the magnitudes of variable forces, whofe intenfities and cotemporary effects are perpetually encreafed or diminifhed, are meafured at any inftant of time by their conftant action at that inftant, or by the velocities which they would produce in the fame time were their intenfity conftant during that time; and the magnitude of their accumulated actions for any finite time is meafured by the addition of thefe cotemporary effects.

553. PROP.

453. *PROP. *If* F, V, T, *represent any finite variable force, velocity and time, respectively, and* $\dot{V}$, $\dot{T}$, *be any small changes of* V *and* T, $\dot{V}$ *will vary as* F × $\dot{T}$.

DEM. The force F being always finite, and being ſuppoſed to encreaſe or decreaſe according to the ſame law, an encreaſe or diminution of it in any finite time muſt be finite, and conſequently, in an infinitely ſmall time, evaneſcent compared with the whole force F, which therefore may be conſidered as conſtant during the time $\dot{T}$ in which $\dot{V}$ is generated or deſtroyed; therefore (530) $\dot{V}$ varies as $F \times \dot{T}$. Q. E. D.

554. Cor. 1. If $\dot{V}$ be the fluxion of the velocity, or the change of velocity generated by F at any inſtant, ſuppoſed to act conſtantly for any time $\dot{T}$, $\dot{V}$ will therefore vary as $F \times \dot{T}$.

555. Cor. 2. $\dot{T}$ is therefore as $\frac{\dot{V}}{F}$ and F as $\frac{\dot{V}}{\dot{T}}$, and the relation of any two being known, the other may be found; and becauſe $\dot{T}$ is as $\frac{\dot{S}}{V}$, $V\dot{V}$ is as $F \times \dot{S}$.

556. PROP. *If any right line* AS *repreſent the time, and* OR *be as the force at any point* O, *the change of velocity at* O *will be as the area* AR. FIG. CLX.

DEM. The increment of velocity $\dot{V}$ is as $F \times \dot{T}$, or as $AB \times Ap$ or as Aq; and the next increment of velocity is as ps, and the ſum of the increments, or the whole change of velocity, is as the ſum of theſe areas; conſequently the velocity communicated in the time AO is as AR. Q. E. D.

557. Cor. 1. If the time in acquiring any given velocity be known, and the areas AR can be ſquared, the velocities communicated in any other time AO may be found; and vice verſâ.

558. Cor.

* Newt. Prin. Tom. I. Sect. VII. Euler's Mechan. Ch. III.

558. Cor. 2. If the force F be finite, and the time Ap infinitely ſmall, the increment of velocity, $\dot{V}$, is evaneſcent; becauſe $AB \times Ap =$ a finite quantity multiplied into one that is evaneſcent $= 0$; and conſequently the increment of velocity, or $\dot{V}$, is infinitely ſmall, and no finite change of velocity can be generated by F in an inſtant.

559. Cor. 3. If the body aſcend or deſcend, and the velocity be as the time in which it is acquired or deſtroyed, the force is invariable; for, from this propoſition, the change of velocity is as AR, or as AO (hypoth.), and conſequently AR is a rectangular parallelogram, or OR is conſtant.

FIG. CLXI. 560. Prop. *If* PO *repreſent the ſpace deſcribed by the action of a force tending to* S, *and the ordinate* OR *be the relative magnitude of the force at the point* O, *the velocity*, V, *acquired at* O *will vary in a ſubduplicate ratio of the correſponding area* PR.

Dem. Let V, F, S, be the velocity, force, and ſpace deſcribed, reſpectively, and becauſe $\dot{V}$ is as $F \times \dot{T}$ (554), and $\dot{T}$ is as $\frac{\dot{S}}{V}$; $V\dot{V}$ is as $F \times \dot{S}$, or as $PB \times Pp$, and the fluent of $V\dot{V}$, or $\frac{V^2}{2}$ is as the fluent of $PB \times Pp$ or PR, and V is as $\sqrt{PR}$. Q. E. D.

561. Cor. 1. If x be the diſtance of the moving body from the center of force, S, and the force be as any power of the diſtance, whoſe exponent is $n-1$; $V\dot{V}$ is as $F \times -\dot{x}$, or as $-x^{n-1}\dot{x}$, and $\frac{V^2}{2}$ is as $\frac{-x^n}{n}$ + correction: but when $V = 0$ at P, $x = SP = p$, and $\frac{x^n}{n} = \frac{p^n}{n}$; therefore the correction $= \frac{p^n}{n}$, and V is as $\sqrt{p^n - x^n}$.

562. Cor.

562. Cor. 2. If the force be constant, or $n=1$, then V being as $\sqrt{p^n-x^n}$ is as $\sqrt{p-x}$, or in a subduplicate ratio of the space described, either in acceding to S from a state of rest, or receding from S till the velocity be destroyed. The fluxion of the time $\dot{T}$ is as the fluxion of the space directly and velocity inversely; or as $\frac{-\dot{x}}{\sqrt{p-x}}=-\dot{x}\times\overline{p-x}|^{-\frac{1}{2}}$, and T is as $\frac{\overline{p-x}|^{\frac{1}{2}}}{\frac{1}{2}}$, and when $T=0$, $x=p$, and $\frac{\overline{p-x}|^{\frac{1}{2}}}{\frac{1}{2}}=0$; therefore T is as the square root of the space described from rest, which coincides with art. 526.

563. Cor. 3. If the force vary directly as the distance, or $n=2$, then V varies as $\sqrt{p^2-x^2}$, or as the right sine of a circular arc, described from S as a center, and radius equal to SP or p, whose versed sine is the space described. And describing this circle, the fluxion of the time $\dot{T}$ is as $\frac{\dot{S}}{V}$, or as $\frac{Oo}{OR}$, or as $\frac{Rn}{Sn}$ (sim. triang.), and the fluent T is as the fraction $\frac{PRD}{SD}$, which is a constant quantity. If the force therefore vary directly as the distance, the times of descent to S from a state of rest are equal wherever P be taken. FIG. CLXII.

564. Cor. 4. If the force vary inversely as any power of the distance, or n be negative, or less than 1, let this power be expressed by the number $-m-1$; and V is as $\sqrt{\frac{1}{m\times x^m}-\frac{1}{m\times p^m}}$, or as $\sqrt{\frac{p^m-x^m}{m\times p^m x^m}}$. If the force be inversely as the distance, or $m=0$, this expression does not shew the variation of velocity. If the force be inversely as the square of the distance, or $m=1$, V is as $\frac{\sqrt{p-x}}{\sqrt{m\times px}}$, or the velocity is as the square root of the space described directly, and inversely as the square root of the distance from S.

565. Prop.

FIG. CLXIII. 565. PROP. *If the ordinates* PB, AL, OR, *be always proportional to the forces at those points, and the force at* A, *continued constant through the space* AM, *communicate a velocity*, V, *equal to that acquired by the body at* O, *descending by the variable force, the area* PR *will be equal to the rectangle* AN, *and the body must have fallen from* P.

DEM. Let $\dot{v}$ and $\dot{V}$ be the fluxions of the velocities communicated, by the variable and constant forces respectively, in any periods of their descent; and $\dot{v} : \dot{V} :: \frac{PB \times P\dot{O}}{v} : \frac{AL \times A\dot{M}}{V}$ (555),

and $v\dot{v} : V\dot{V} :: PB \times P\dot{O} : AL \times A\dot{M}$, and, taking the fluents, $v^2 : V^2 :: PR : AN$. If therefore $v = V$, the areas PR and AN are equal, and the body must have fallen from P. Q. E. D.

566. Cor. 1. If the areas PR and AN be equal, the velocities, acquired by the action of the constant and variable forces, whilst the bodies describe, from rest, the spaces AM and PO, will also be equal.

567. Cor. 2. Because the areas AN and PR are always equal, when the velocities at M and O are equal, their increments Or and Mn must be equal; and if F be the given force at A, and $y =$ the space described AM, $F \times \dot{y} = Or = OR \times Oo =$ the force at O multiplied into the fluxion of the space.

568. PROP. *If a body be attracted towards the point* S, *by forces which always vary as that power of the distance whose exponent is* n—1, *and begin to move at any given distance* S P, *it is required to assign the velocity*, V, *acquired in describing any space* PO, *supposing the magnitude of the force at any given distance* SA *to be known.*

Let

Let the magnitude of the force at A be to the force of gravity as $F : 1$, and let

$SA = a$,

$SP = p$,

$SO = x$; and, from the supposition, F : force at any point $O :: a^{n-1} : x^{n-1}$; and the force at $O = \frac{F \times x^{n-1}}{a^{n-1}}$. But, if y be the space through which a body must fall, when acted upon by a constant force equal to F, to acquire a velocity equal to that at O, from (567) $F \times \dot{y} = \frac{F \times \overline{x^{n-1} - \dot{x}}}{a^{n-1}}$, and the fluents are equal, or $F \times y = \frac{-Fx^{n}}{n \times a^{n-1}} +$ correction; but when $F \times y$ vanishes, $x = p$, and consequently the fluent corrected $= \frac{F \times \overline{p^{n} - x^{n}}}{n \times a^{n-1}}$. But if s be the space described by the force of gravity in 1'', or any other given time 1, and the force of gravity be expressed by 1, $V =$ *$\sqrt{4sFy} = \sqrt{\frac{4sF \times \overline{p^{n} - x^{n}}}{n \times a^{n-1}}}$ feet in 1''. Q. E. I.

569. Cor. 1. If the force be as some negative power of the distance, or $n - 1 = -m$, then F : the force at $O :: a^{-m} : x^{-m}$, and the force at $O = \frac{F \times x^{-m}\dot{x}}{a^{-m}}$; the fluxion of $F \times y = \frac{F \times x^{-m}\dot{x}}{a^{-m}}$ and its fluent $= \frac{F \times x^{1-m}}{1 - m \times a^{-m}} +$ cor. $= \frac{F \times a^{m}}{1 - m \times x^{m-1}} - \frac{F a^{m}}{1 - m \times p^{m-1}}$

$$= \frac{F \times a^{m} \times \overline{p^{m-1} - x^{m-1}}}{1 - m \times p^{m-1} x^{m-1}}, \text{ and } V = \sqrt{\frac{4sF \times a^{m} \times \overline{p^{m-1} - x^{m-1}}}{1 - m \times p^{m-1} x^{m-1}}}.$$

570. Cor.

* Let $f =$ the force of gravity,
$s =$ the space described by the constant action of f in 1'' $= 16$ feet nearly, and
$v =$ the velocity acquired in falling 1''; and
$V^2 : v^2 :: F \times y : f \times s$ and $V^2 = \frac{v^2 \times F \times y}{f \times s}$; but if $f = 1$, $v = 2s$ and $V^2 = 4sFy$ and $V = \sqrt{4sFy}$.

570. Cor. 2. Becauſe $4Fs$ and a^{n-1} are given quantities, V varies as $\sqrt{p^n - x^n}$.

571. Cor. 3. If the body deſcend to the center, or $x = 0$, and the force vary according to any direct law of the diſtance, or inverſe law leſs than the ſimple, that is, if n be any affirmative, whole number, or fraction leſs than unity, $V = \sqrt{\frac{4sF \times p^n}{n \times a^{n-1}}}$ is a real finite quantity. Let the force be as that power of x, whoſe exponent is $-\frac{1}{2}$, 1, 2, 3, &c. or $n = \frac{1}{2}$, 2, 3, 4, &c. and $V = \sqrt{\frac{4sF \times p^{\frac{1}{2}}}{\frac{1}{2} \times a^{-\frac{1}{2}}}}$, $\sqrt{\frac{4sF \times p^2}{2 \times a}}$, $\sqrt{\frac{4sF \times p^3}{3 \times a^2}}$, $\sqrt{\frac{4sF \times p^4}{4 \times a^3}}$, &c.

572. Cor. 4. If the force be inverſely as the diſtance, or $n = 0$, and the body deſcend to the center, as before, $V = \sqrt{\frac{4sF \times p^0}{0 \times a^{-1}}}$ is infinitely great. If the force be inverſely as the ſquare of the diſtance, or $n = -1$, $F \times \dot{y} = -\frac{Fx^{-2}\dot{x}}{a^{-2}}$ and $Fy = \frac{Fa^2}{x} +$ cor. $= \frac{Fa^2}{x} - \frac{Fa^2}{p} = \frac{Fa^2 \times \overline{p - x}}{px}$, and $V = \sqrt{4sFa^2 \times \frac{p-x}{px}}$. From hence it appears, as before (564), that the velocity varies as $\sqrt{\frac{PO}{OS}}$.

573. Prop. *Suppoſe a body to be impelled by a force varying as that power of the diſtance from the center of force* S, *whoſe exponent is* n − 1, *it is required to aſſign the time of deſcribing any ſpace* PO, *ſuppoſing it to fall from a ſtate of reſt.*

The fluxion of the ſpace $\dot{S} = V \times \dot{T}$, and conſequently $\dot{T}$ is equal to the fluxion of the ſpace divided by the velocity $= -\frac{\dot{x}}{V} =$

$-\frac{\dot{x}\times\sqrt{na^{n-1}}}{\sqrt{4sF\times\overline{p^n-x^n}}}$; and the time is equal to the fluent of $\frac{\dot{x}\times\sqrt{na^{n-1}}}{\sqrt{4sF\times\overline{p^n-x^n}}}$. Or, if $n-1$ be negative and $=-m$, then $\dot{T}=-\dot{x}\times\sqrt{\frac{1-m\times p^{m-1}x^{m-1}}{4sFa^m\times\overline{p^{m-1}-x^{m-1}}}}$, and $T=$ its fluent. Q. E. I.

574. Cor. 1. Let the force vary as the diſtance from the center of force S, or $n=2$, and $T=\sqrt{\frac{2a}{4sF}}\times$ into the fluent of $\frac{-\dot{x}}{\sqrt{p^2-x^2}}=$ cor. — length of a circular arc, whoſe radius is unity and right ſine $\frac{x}{p}$. If the time vaniſh, or $x=p$, the cor. $=$ length of a circular arc whoſe radius is 1 and right ſine is $\frac{p}{p}$, or 1, and therefore $=$ a quadrantal arc; conſequently the time of deſcribing any ſpace PO is equal to $\sqrt{\frac{2a}{4sF}}\times$ into a quadrant $-\sqrt{\frac{2a}{4sF}}\times$ into the arc whoſe right ſine is $\frac{x}{p}$, the radius being unity. When the body deſcends to the center, or $\frac{x}{p}=0$, the time is equal to $\sqrt{\frac{2a}{4sF}}\times$ into a quadrant. If the radius of the circle $=r$, and a quadrantal arc of that circle $=Q$, the quadrant of a circle whoſe radius is unity $=\frac{Q}{r}$; and the time of deſcent to $S=\sqrt{\frac{2a}{4sF}}\times\frac{Q}{r}=$ a given quantity.

575. Cor. 2. If the force vary inverſely as the ſquare of the diſtance from S, then $V=\sqrt{\frac{4sFa^2\times\overline{p-x}}{px}}$ and $\dot{T}=\dot{x}\times$

FIG. CLXVI. $\sqrt{\frac{p}{4sFa^2}} \times \sqrt{\frac{x}{p-x}}$, and $T = \sqrt{\frac{p}{4sFa^2}} \times$ into the fluent of $\dot{x} \times \sqrt{\frac{x}{p-x}}$. With SP or p as a diameter, and from the center C describe a circle, and the fluent of $\dot{x} \times \sqrt{\frac{x}{p-x}}$ is as the area PRS; for this area $=$ sector $PCR +$ the triangle $SRC = \overline{\text{arc } PR + \text{sine } OR} \times \frac{SP}{4}$, and the fluxion of the area $= \overline{Rr + mr}\ (P\dot{R} + O\dot{R}) \times \frac{SP}{4}$ $= \frac{\dot{x} \times CR}{OR} + \dot{x} \times \frac{CO}{OR}$ (sim. triang.), (or $\frac{\dot{x} \times x}{OR}$, or $\frac{x \times \dot{x}}{\sqrt{x \times p - x}}$, or $\dot{x} \times \sqrt{\frac{x}{p-x}}$) $\times \frac{SP}{4}$, and consequently the fluent of $\sqrt{\frac{p}{4sFa^2}} \times \dot{x} \sqrt{\frac{x}{p-x}} = \sqrt{\frac{p}{4sFa^2}} \times \overline{\text{arc } PR + \text{sine } OR}$. And because $\frac{p}{4sFa^2}$ is given, the time of describing PQ varies as $PR + OR$, or as $\overline{PR + OR} \times \frac{SP}{4}$, or as the area PRS.

FIG. CLXIII. 576. PROP. *If a body begin to fall from* A, *it is required to determine the space* AO *through which it must descend, when impelled by a force varying as that power of the distance whose exponent is* n — 1, *to acquire a velocity,* V, *equal to that communicated, whilst the body describes a space equal to* $\frac{AS}{2}$, *by the constant action of a force equal to that at* A.

Let the force at $A = 1$

$SA = r$

$SO = x$. And the force at A, or 1 : the force at $Q :: r^{n-1} : x^{n-1}$; and the force at $Q = \frac{x^{n-1}}{r^{n-1}}$, and $\frac{r}{2} \times$ the force at A =

$= \frac{v^2}{2} =$ the fluent of $- \frac{x^{n-1}\dot{x}}{r^{n-1}}$ (567) $= - \frac{x^n}{n \times r^{n-1}} +$ cor. $=$ $\frac{r^n - x^n}{n \times r^{n-1}}$, and $r^n - x^n = \frac{n r^n}{2}$, and $x = \frac{\overline{2-n}|^{\frac{1}{n}} \times r}{2|^{\frac{1}{n}}}$. Q. E. I.

577. Cor. 1. If the force vary directly as the distance from the center S, or $n = 2$, $x = \frac{\overline{2-2}|^{\frac{1}{2}} \times r}{2^{\frac{1}{2}}} = 0$, and the body must fall to the center to acquire a velocity equal to that acquired in descending through $\frac{SA}{2}$, when acted upon by a constant force equal to that at A.

578. Cor. 2. If the force vary inversely as the distance, or $n = 0$, $x = \frac{\overline{2-0}|^{\frac{1}{0}} \times r}{2|^{\frac{1}{0}}}$, which expression does not shew the value of x: but it may be found by the following process. The force at A or 1 : force at O :: x : r, and the force at $O = \frac{r}{x}$, and $\frac{v^2}{2} = r \times$ flu. $- \frac{\dot{x}}{x} = r \times -$ hyp. log. of $x +$ cor. $= r \times -$ log. of x $+ r \times$ log. of $r = r \times$ log. of $\frac{r}{x}$.

579. Cor. 3. If the force vary inversely as the square of the distance from S, or $n = -1$; $x = \frac{\overline{2+1}|^{-1} \times r}{2^{-1}} = \frac{2}{3} \times r$, and the body must fall through a third of the distance SA, or to m. If the force vary according to any law therefore between the direct simple and inverse duplicate ratio of the distance from S, the body must fall to some intermediate space between S and m. If the force vary as that power of the distance inversely, whose exponent

ponent is 3, 4, &c. or $n = -2, -3,$ &c.; then $x = \frac{\overline{2+2}|^{-\frac{1}{2}} \times r}{2^{-\frac{1}{2}}}$; $\frac{\overline{2+3}|^{-\frac{1}{3}} \times r}{2^{-\frac{1}{3}}}$, &c. $= \frac{2^{\frac{1}{2}} \times r}{2|^{\frac{1}{2}}}$, $\frac{2^{\frac{1}{3}} \times r}{5^{\frac{1}{3}}}$, &c.; and the ſpace fallen through $= r - \frac{2^{\frac{1}{2}} \times r}{4^{\frac{1}{2}}}$, $r - \frac{2^{\frac{1}{3}} \times r}{5^{\frac{1}{3}}}$, &c.

FIG. CLXIV. 580. PROP. *Suppoſe the force at* A *to be* 1, *and in other places to vary as that power of the diſtance whoſe exponent is* n − 1, *it is required to determine the height to which a body will aſcend when acted upon by this variable force, and projected from* A *in the direction* SA *with a velocity equal to that acquired in falling through* $\frac{SA}{2}$, *as in the laſt propoſition.*

Let B be the place to which the body aſcends, and the force at A or 1 : force at any other point $O :: r^{n-1} : x^{n-1}$, and the force at $O = \frac{x^{n-1}}{r^{n-1}}$. But if the body were impelled by a conſtant force equal to that at A, it would aſcend through a ſpace equal to $\frac{r}{2}$, and (567) $\frac{r \times 1}{2}$ = the fluent of $\frac{x^{n-1}\dot{x}}{r^{n-1}} = \frac{x^n}{n \times r^{n-1}} +$ cor. $= \frac{x^n - r^n}{n \times r^{n-1}}$, and $\frac{n \times r^n}{2} = x^n - r^n$, and $x = \frac{\overline{n+2}|^{\frac{1}{n}} \times r}{2|^{\frac{1}{n}}}$. Q. E. I.

581. Cor. 1. Suppoſe the force to vary directly as the diſtance from S, or $n = 2$; then $\frac{\overline{n+2}|^{\frac{1}{n}} \times r}{2|^{\frac{1}{n}}} = \frac{4^{\frac{1}{2}} \times r}{2^{\frac{1}{2}}}$, and x $(= SB) : r :: 4^{\frac{1}{2}} : 2^{\frac{1}{2}} :: 2^{\frac{1}{2}} : 1$.

582. Cor. 2. If the force vary inverſely as the diſtance from S, or $n = 0$, $x = \frac{\overline{0+2}|^{\frac{1}{0}} \times r}{2^{\frac{1}{0}}}$, which expreſſion does not ſhew the

mag-

magnitude of x; but it may be determined by the procefs in (578).

583. Cor. 3. If the force vary inverfely as the fquare of the diftance from S, or $n = -1$, then $x = \dfrac{\overline{2-1}|^{\frac{1}{-1}} \times r}{\overline{2}|^{\frac{1}{-1}}} = 2r = SD$; If the force therefore vary according to any law between the direct fimple and inverfe duplicate ratio of the diftance from S, the body will rife to fome intermediate altitude between B and D. If the force vary inverfely as the cube of the diftance, or $n = -2$, then $x = \dfrac{\overline{2-2}|^{\frac{1}{-2}} \times r}{2^{-\frac{1}{2}}} = \dfrac{2^{\frac{1}{2}} \times r}{0}$, and is infinitely great. If therefore the force vary according to a law between the inverfe duplicate and triplicate, the body will afcend to fome intermediate fpace between D and infinity.

CHAP.

CHAP. XIII.

PENDULOUS MOTION*.

FIG. CLXV. 584. Def. *A BODY, or number of bodies, connected to a right line* SP, *and moving about a point* S *to which it is suspended, by the force of gravity, is called a pendulum.*

585. Def. *The motion of a pendulum in the same direction, from a state of rest, till it begins to return in an opposite direction, is one vibration or oscillation.*

586. Cor. The velocity acquired in descending from P to V, supposing SV to be perpendicular to the horizon, will make the body P describe an arc Vp, whose perpendicular altitude is equal to that of VP (542), and the motion through VP is equal to one half of a vibration. And because a pendulum, composed of any number of bodies, will perform a given vibration, or equal parts of a vibration, in the same time as if they were collected in their center of oscillation (512); a pendulum composed of any number of bodies may be reduced to one, where one body is connected to a right line SP, and the length of a pendulum is always understood to mean the distance between the centers of oscillation and suspension.

587. Prop. *If a body vibrate in a circular arc* PVp, *that part of the force of gravity, which accelerates and retards it, varies as the right sine of the arc intercepted between the body and the lowest point.*

Dem.

* Keil's Phys. Lect. XV. Helsham, Lect. X. Muschenb. Ch. XII. CCXCVII. Maclaurin's Newt. Book II. Ch. V. Emerson's Mechan. Prop. XI.. Hugen. Horol. Oscil. Part II. Prop. XVI. Rohault's Notes, Part II. Ch. XXVIII.

DEM. Let the force of gravity be represented, in quantity and direction, by a given line LP perpendicular to the horizon, and be resolved into two forces, LM parallel to the string SP, and PM touching the circle in P, of which LM is the tension of the string and has no effect upon the body's motion, and it is accelerated and retarded by PM only; therefore the accelerating force (A) : force of gravity (G) :: $PM : PL :: PN : SP$ (sim. triang.), and $A = \frac{G \times PN}{SP}$, and consequently A varies PN. Q. E. D.

588. Cor. 1. The velocity at any point Q is equal to that acquired in falling through the perpendicular altitude ND (541), and varies as $\sqrt{ND}$, or $\sqrt{NV - VD}$, or $\sqrt{NV \times 2VS - VD \times 2VS}$, or $\sqrt{PV^2 - QV^2}$ (PV and QV being chords of the arcs PV and QV), or as the right sine of a circular arc, whose radius is the chord PV, and versed sine the difference between the chords PV and QV.

589. Cor. 2. If the arc mV be supposed equal to two inclined planes mn and nV, touching it at m and V, these planes are equal, and the velocity in the horizontal plane nV is uniform, and the time of describing it is equal to half the time of falling down mn (527): but the time of falling down mn (t) : time (T) of falling down mV $(= 2\,mn)$, or down the diameter $2SV :: mn : mV :: 1 : 2$, and $t = \frac{T}{2}$; and consequently the times of describing mn and nV, or the time of half a vibration $= \frac{3T}{4}$, and the time of a whole vibration : $T :: 3 : 2$.

SCHOLIUM.

590. If the body P be acted upon by a force F, perpendicular to the horizon, which is to the force of gravity as the arc PV to its sine PN, all vibrations are isochronal; for, let SP represent the constant force of gravity, and $PL = F$; and by a resolution of

PL into two, LM parallel to SP, and PM coincident with the tangent, this laſt is the only part of F that accelerates P. But $PL : PS :: PM : PN$ (ſim. triang.), and

$PL : PS :: PV : PN$ (hypoth.); and conſequently $PM = PV$, or the accelerating force is as the diſtance from the loweſt point V, and P will always arrive at V in the ſame time (563). The iſochronal force in a circle, F, is therefore equal to $G \times \frac{PV}{PN}$; and, when a pendulum is urged by the force of gravity, the time of a vibration will be encreaſed with the arc of vibration, becauſe the exceſs of F above G is encreaſed with that arc.

FIG. CLXVII. 591. DEF. *If a circle* FPE *revolve upon the right line* BA, *the curve line* BVA *deſcribed by any point* P *of the periphery in one revolution, is called a cycloid:* BA *is the baſe,* VD *biſecting* BA *at right angles is the axis,* V *the vertex,* PM *parallel to the baſe is an ordinate, and* FPE *is the generating circle, of the cycloid.*

592. Cor. Becauſe every point of the periphery of the generating circle has been applied to the baſe BA, it is evident that BA is equal to the periphery, and BD to the ſemiperiphery of the generating circle.

593. LEMMA. *If a circle be deſcribed upon the axis as a diameter, and an ordinate* PM *be drawn from any point* P, *the part of this ordinate* PL, *contained between the point from whence it is drawn and the periphery of the circle, is equal to the circular arc* VL, *contained between the vertex and the interſection of the ordinate and periphery.*

DEM. Let FPE be any poſition of the generating circle, and, becauſe every point of the arc PE has been applied to BE, the right line $BE =$ the arc $PE =$ the arc LD, and the arc $LV = ED = MN = PL$. Q.E.D.

594. Cor.

594. Cor. Because PL is always equal to the arc LV or PF, their cotemporary increments or decrements are equal, that is, the initial motions of the point P, which traces out the cycloidal arc, one parallel to the base BA, and the other in the direction of the tangent to the circle at P, are equal to each other.

595. LEMMA. *An ordinate being drawn from any point* P *of the cycloidal arc, cutting the periphery of the generating circle, whose diameter is the axis* VD, *in* L, *the chord* VL *of the circular arc is parallel to the line touching the cycloid at* P.

DEM. Let GP be a tangent to the circle at P, and producing EP to C, the $\angle CPq = \angle GPE = \angle GEP$ (the tangents DP and DE being equal, EUC. B. III. p. 36.) $= \angle EPN$: but the initial motions of P, in the directions GP and PL, being equal (594), the path of P or Pb will bisect the angle qPp (composition of motion), and consequently the $\angle CPb = \angle EPb$ or Pb is at right angles to EP, and FP, which is parallel to LV, is also perpendicular to EP, and therefore is a tangent to the curve at P. Q. E. D.

596. Cor. A tangent to the cycloid at the vertex V is therefore perpendicular to the axis VD and parallel to the base BA.

597. LEMMA. *The generating circle being described upon the axis as a diameter, and an ordinate being drawn from any point* O, *cutting the periphery of the circle in* R, *the cycloidal arc* VO *is equal to twice the corresponding chord of the circular arc* VR.

DEM. Draw an ordinate ro infinitely near to RO, and rs perpendicular to VR produced, and $Vr = Vs$, the angle at V being evanescent; and Oo ($= Rv$) and Rr are cotemporary increments of the arc VO and chord VR: drawing tangents to the circle at V and R, the triangles VRT and Rrv, having the angles at R ver-

 tical,

tical, and the angles TVR and Rvr alternate, are similar, and consequently $Rr = vr$ and Rv or $Oo = 2Rs$. The increment of the arc VO is therefore equal to twice the corresponding increment of the chord VR, and, because they are nascent and evanescent at the same time, $VO = 2VR$. Q. E. D.

FIG. CLXVIII.

598. PROP. *To make a body oscillate in a given cycloid* AVB.

Produce the axis VD making $SD = DV$, and through S draw a line KSH parallel to BA, and let two circles, each equal to DLV, generate two semicycloids SA, SB, each equal to BV or AV; and if one extremity of a string SCX, whose length is equal to SV or SCA, be fixed at S, a body collected in the other extremity X, will always be found in the cycloidal arc XVP; for, because SCX is a tangent to the cycloid at C, CX is parallel to EA (595); and $CG = EA$; but $CX = 2AE$ (597) $= 2CG$ and $CG = GX$, and the ordinates XL and GE are equidistant from AD, or $AF = DN$, and the arc AE = the arc LD, and the chord LD is parallel to AE or XG: but $AG = CE$ = the arc AE, and consequently the remaining arc $HE (= LV) = GD = XL$, and X is in the cycloidal arc AXV (593).

599. Cor. Because SCX touches the curve at C, it is evident that whilst X describes a very small arc, C may be deemed quiescent, or CX is always perpendicular to the cycloidal arc at X; and very near the vertex V, where SV is perpendicular to the curve, a circular arc, whose radius is SV, will coincide for a small distance with the cycloidal arc.

600. PROP. *If a pendulum begin to vibrate from any point* P, *and a circle be described, whose radius is equal to the cycloidal arc* VP, *the velocity in different points will vary as the right sine of an arc of this circle, whose versed sine is the space described by* P.

DEM.

DEM. The velocity at any point Q is equal to that acquired in falling through the same perpendicular altitude NR (545), which is as $\sqrt{NR}$, or $\sqrt{NV - RV}$, or $\sqrt{NV \times VD - VR \times VD}$, or $\sqrt{VF^2 - VE^2}$, or $\sqrt{VP^2 - VQ^2}$, or $\sqrt{Vn^2 - Vq^2}$, or qn. Q.E.D.

601. Cor. 1. The velocity therefore encreases from P to V, where it is the greatest, being as Vv, and then it decreases in such a manner that the velocities at all equal distances from V are equal; and consequently the cotemporary changes, or increments as the body accedes to V, and decrements as it recedes from it, are equal at equal distances from V.

602. Cor. 2. If s be the number of inches through which a body descends from rest by the force of gravity in 1″, and V be the velocity acquired in vibrating through PQ, or in falling down the perpendicular NR, then $NR : s$ as the squares of the velocities acquired in falling through NR and s, or as $V^2 : 4s^2$, and $V = \sqrt{NR \times 4s}$ $= \sqrt{4s \times NV - VR} = \sqrt{\frac{4s}{DV}} \times \sqrt{FV^2 - EV^2} = \sqrt{\frac{4s}{DV}} \times$ into the right sine $\frac{qn}{2}$; and at the vertex the velocity is equal to $\sqrt{\frac{4s}{DV}} \times \frac{Vv}{2}$ inches in 1″.

603. Cor. 3. If a pendulum begin to oscillate at different distances PV and QV from the vertex V, the velocities acquired at V in these different vibrations, are as $\sqrt{NV} : \sqrt{RV}$, or as $VF : VE$, or as $VP : VQ$; or the velocity acquired in vibrating from rest is as the distance from the vertex at which it begins to vibrate.

604. Cor. 4. If s be the space described from rest, as in cor. 2. the velocities acquired at V, in vibrating from different points P

and Q are equal to $\sqrt{\frac{4s}{DV}} \times \frac{VP}{2}$, and $\sqrt{\frac{4s}{VD}} \times \frac{qn}{2}$, or such as would carry the body over these numbers of inches in 1″.

605. Prop. *The time in which the pendulum vibrates through any arc* QH, *is equal to the time in which a body would describe the corresponding circular arc* n o, *with the greatest velocity* V v, *continued uniformly.*

Let M be infinitely near to Q, so that QM may be described uniformly, and taking Vp, Vq, Vm, Vb, respectively equal to VP, VQ, VM, VH, and drawing the right line nk parallel to Vp; nk (qm or QM) : nl :: qn : Vv, or QM and nl are to each other as the velocities with which they are described, and are therefore described in the same time. The same may be proved of the other corresponding parts of QH and no, which are, consequently, described in the same time. Q. E. D.

606. Cor. 1. The times of describing any arcs PQ, QV are to each other as the corresponding circular arcs pn, nv, because they are described with the same uniform velocity; and the time of a whole vibration is as the semiperiphery pvx.

607. Cor. 2. The velocity, with which QM is described, is actually equal to $\sqrt{\frac{4s}{DV}} \times \frac{qn}{2}$, and the velocity with which nl is described, is equal to $\sqrt{\frac{4s}{DV}} \times \frac{Vp}{2}$. The time of a vibration is equal to the time in which a body would describe the semiperiphery pvx with an uniform velocity equal to $\sqrt{\frac{4s}{DV}} \times \frac{Vv}{2}$ inches in a second.

608. Cor.

608. Cor. 3. If a body begin to oscillate from different points P and Q, the times of their vibrations are equal to the times of describing the semiperipheries of circles, whose radii are VP and VQ, with uniform velocities, which are respectively equal to $\sqrt{\frac{4s}{DV}} \times \frac{VP}{2}$, and $\sqrt{\frac{4s}{DV}} \times \frac{VQ}{2}$; but these times are as the spaces or semiperipheries divided by the velocities, or divided by their radii VP and VQ, and are consequently equal: the times therefore of all vibrations, however different, are equal.

609. Prop. *Supposing a pendulum to begin to oscillate from any point* P, *the time in which it performs one vibration, is to the time of descent down the axis as the periphery of a circle to its diameter.*

Dem. The times of descent down DV and FV are equal (538); and in this time a body would describe a space equal to $2FV$ or PV or pV, with the velocity acquired in falling down FV or PV continued uniform; but this is to the time of describing pvx with the same velocity, or to the time of one vibration (607) as the spaces described, or as $pV : pvx$, or as the diameter of a circle to its periphery. Q. E. D.

610. Cor. 1. The periphery of a circle, being to its diameter in a given ratio, it appears again that the times of all vibrations are as the times of descent down the axis, and consequently are given.

611. Cor. 2. The velocity with which the semiperiphery pvx is described, when the time of describing it is equal to a vibration, is equal to $\sqrt{\frac{4s}{DV}} \times \frac{PV}{2}$ (607), and the time is equal to $\frac{pvx}{\text{vel.}} = \frac{2pvx}{PV} \times \sqrt{\frac{DV}{4s}}$; and the time of falling down $DV = \sqrt{\frac{DV}{s}}$;

con-

consequently the time of a vibration is to the time of falling down $DV :: \frac{2 \times pvx}{PV} \times \sqrt{\frac{DV}{4s}} : \sqrt{\frac{DV}{s}} :: \frac{2 \times pvx}{PV} \times \frac{1}{\sqrt{4}} : 1 :: 2pvx : 2PV$, the ſame analogy as that in this propoſition.

612. Cor. 3. The times of vibrations in different cycloids, being equal to the times of deſcent down their axes multiplied into the ſame given quantity, will vary as the times of deſcent down the axes, or lengths of the pendulums; or ſuppoſing L to be the length of a pendulum, and T the time of a vibration, and to be variable; T will be as $\sqrt{L}$, and L as T^2.

613. Cor. 4. If the vibrations be very ſmall, the ſtring is not ſenſibly affected by the cycloidal arcs SA, SB, and the pendulum SP will deſcribe a circular arc, and hence it appears that very ſmall vibrations in circular arcs are performed in equal times. The times of deſcent down a circular arc and its chord are therefore unequal, the former being to the time of deſcent down the axis as half the periphery of a circle to its diameter; and the latter is equal to the time of deſcent down the diameter, or four times the axis, and is conſequently to the time of deſcent down the axis as 2 : 1.

614. Cor. 5. If the pendulum begin to oſcillate from B, half the time of a vibration is to the time of deſcent down the inclined plane BV as $BD : BV$; for half the time of a vibration is to the time of deſcent down the axis, as the ſemiperiphery of a circle to its diameter, or as $DFEV(BD)$ is to DV, and the time down DV : time down the inclined plane $BV :: DV : BV$, and, ex æquo, half the time of a vibration : time of deſcent down the inclined plane $BV :: BD : BV$.

615. Cor. 6. The ſpace deſcribed by a falling body in 1″, may be diſcovered, if the length of a pendulum performing one vibration

tion in a fecond be known: let the length of this pendulum be 39.2 inches, and (609) 1″ : time of defcent through $\frac{39.2}{2}$ (t) :: periphery of a circle to its diameter :: 3.14159, &c. (p) : 1 and $t = \frac{1}{p}$. But $\frac{39.2}{2}$: fpace fallen through in 1″ (x) :: $\frac{1}{p^2}$: 1^2 and $x = \frac{39.2}{2} \times p^2 = 193.1$ inches nearly $= 16.1$ feet nearly.

616. Cor. 7. The length of the pendulum performing one vibration in 1″, may be difcovered, if the fpace through which a body defcends in 1″ be known; for let this fpace = 193.1 inches, and (laft cor.) the length of the pendulum $= \frac{2 \times 193.1}{\overline{3.14159}|^2}$ inches $= 39.2$ inches.

617. Cor. 8. The time of one vibration may be difcovered, if the length of the pendulum be known; for let L be the given length of the pendulum in inches, and T be the time of one vibration, and T : time of defcent through $\frac{L}{2}$ :: 3.14159, &c. : 1 :: p : 1; but the time of defcent through $\frac{L}{2} = \sqrt{\frac{L}{2s}}$, and confequently $T = p \times \sqrt{\frac{L}{2s}}$ feconds, or parts of feconds. If $L = 39.2$ inches, and $s = 16.1 \times 12$ inches, $T = 3.14159 \times \sqrt{\frac{39.2}{2 \times 16.1 \times 12}}$ $= 1''$ nearly.

618. Cor. 9. The number of vibrations in a given time, being inverfely as the time of one vibration, will be as $\frac{1}{\sqrt{L}}$. If N be the

the number of vibrations in a given time, T; and t be the time of one vibration, then $N = \frac{T}{t}$; but, the length of the pendulum being L, $t = p \times \sqrt{\frac{L}{2s}}$, and consequently $N = \frac{T}{p} \times \sqrt{\frac{2s}{L}}$. If $T = 60 \times 60''$, then $N = \frac{60 \times 60}{3.14159} \times \sqrt{\frac{193 \times 2}{39.2}} = 60 \times 60$ very nearly.

619. PROP. *Let a pendulum, whose length is L inches, lose or gain any number of vibrations expressed by n, in any number of hours, h, to find the length of a pendulum that shall vibrate once in 1″.*

Let Y = the length of the pendulum required, and let the number of vibrations performed by Y in h hours, or $h \times 60 \times 60'' = m$, and L performs $m \pm n$, vibrations in h hours; but $L : Y :: m^2 : \overline{m \pm n}|^2$ (618) and $Y = L \times \frac{\overline{m \pm n}|^2}{m^2}$.

620. PROP. *If the length of a pendulum and the force of gravity be variable, the time of an oscillation varies as the square root of the length of the pendulum L, divided by the square root of the force of gravity, G.*

DEM. The time of one oscillation varies as the time of descent down the axis, and that is as the square root of the axis directly and force inversely, or as $\sqrt{\frac{DV}{G}}$ (532), or as $\sqrt{\frac{L}{G}}$. Q. E. D.

621. COR. 1. Therefore L is as $T^2 \times G$, and T being given, L is as G: if therefore the lengths of two pendulums, performing a vibration, or the same number of vibrations, in the same time be known,

known, the ratio of the forces of gravity, being the same with the ratio of these lengths, will consequently be known.

622. Cor. 2. Or, if the number of vibrations performed in the same time, in different places, by two pendulums whose lengths are L and l, be equal to m and $m+n$ respectively, the forces of gravity G and g, in the places of observation, may be found: for, the times of one vibration of the pendulums L and l are as $\sqrt{\frac{L}{G}}$: $\sqrt{\frac{l}{g}}$, and $m+n : m$ inversely as the times of one vibration, or as $\sqrt{\frac{L}{G}} : \sqrt{\frac{l}{g}}$ (620), and consequently $G : g :: m^2 \times L : \overline{m+n}^2 \times l$.

623. Prop. *That part of the force of gravity, which accelerates or retards a pendulum vibrating in a cycloid, varies directly as its distance from the vertex.*

Dem. Let the force of gravity be represented, in quantity and direction, by the line DV, and be resolved into two forces, DF parallel to the string, and VF parallel to the tangent at P, or direction in which P moves; and VF ($= 2VP$), which only accelerates P, varies as VP. Q. E. D.

Otherwise: The velocities at Q and M are as qn, ml; the time of describing QM is as the included circular arc nl; and, supposing the time nl to be very small and given, the accelerating force at Q is as the change of velocity kl (453), or as $\frac{Vq \times nl}{Vn}$ (sim. triangles), or, because nl and Vn are given, as Vq or VQ. Q. E. D.

624. Cor. 1. The force accelerating or retarding the pendulum (A) : whole force of gravity (G) :: $VF : VD :: VP : VB$ and $A =$

 $G \times VP$

$\frac{G \times VP}{VB}$; and the tension of the string : G :: $DF : DV$, and this tension $= \frac{G \times DF}{DV}$. At the vertex where $DF = DV$ the tension of the string $= G$, and $A = o$; and, at B, $VP = VB$ and $A = G$, and the tension $=$ o.

625. Cor. 2. The tension of the string : force accelerating or retarding P :: $DF : VF$ :: $\sqrt{DN} : \sqrt{VN}$ (sim. triangles).

CHAP.

CHAP. XIV.

PROJECTILE MOTION.

626. DEF. *THE range or random of a projectile, is the rectilineal distance between the point of projection and impulse upon any obstacle; the horizontal range is called the amplitude; and the angle of elevation is the angle contained between the horizon and the direction of projection.*

627. PROP. *A body projected in any direction* DN, *and acted upon by a constant force,* G, *in a direction parallel to a right line* DO *inclined in any angle to* DN, *will describe a parabola.* FIG. CLXIX.

DEM. Let DO be described from rest, by the action of G, in the same time, T, in which DE is described by the velocity of projection, and, completing the parallelogram, the body will, at the end of the time T, evidently be at R; but DO varies as T^2 (526), or (because the velocity in DN is uniform from the first law of motion, and T varies as DE) as DE^2 or OR^2, which is a property peculiar to the common parabola. Q. E. D.

628. Cor. 1. The parameter belonging to any diameter DO is equal to $\frac{OR^2}{DO}$ or $\frac{DE^2}{ER}$ (conic sect.); and if the parameter belonging to any point D be equal to $\frac{DE^2}{ER}$, the parabola will pass through the point R.

 629. Cor.

629. Cor. 2. If the velocity of projection be the same, the parameter belonging to DO will be the same, whatever be the angle of projection; for the projectile velocity and G being given, DE, ER, and consequently $\frac{DE^2}{ER}$, or the parameter, are given.

630. Cor. 3. Let Y be the space through which a body falls from rest, when acted upon by the constant force G, to acquire a velocity, V, equal to that in any point D of the curve, and let v be the velocity acquired in falling from E to R, by the action of same force G, and $V : v :: DE : 2ER$ (106): but $Y : ER :: V^2 : v^2$ (526) $:: DE^2 : 4ER^2$, and $Y = \frac{DE^2}{4ER} = \frac{1}{4}$th of the parameter belonging to DO.

631. Cor. 4. If G be the force of gravity, the axis and all diameters of the parabola, are perpendicular to the horizon; the velocities are equal at equal distances from the principal vertex; the time of arriving at the vertex of the axis, or at the greatest altitude, is equal to half the time of the flight; the parabola cuts the horizon, and all lines parallel to the horizon, in equal angles; and DE, which is parallel to the ordinate OR, is a tangent to the parabola at D.

FIG. CLXX. 632. Cor. 5. The velocity in different points varies in a subduplicate ratio of the parameter belonging to those points, or, if IX be the directrix and DI perpendicular to it, in a subduplicate ratio of DI or DS, S being the focus, or, in the same parabola, as the perpendicular SY upon the tangent DY. The horizontal velocity DH, determined by drawing perpendiculars from the body upon the horizon, varies as DE, and is consequently uniform; and the velocity, therefore, in any point D, being as DE, is as the secant of the angle of elevation; and at the principal vertex, the velocity in the curve and the horizontal velocity, are equal.

633. Prop.

633. **Prop.** *To find the directions in which a body, being projected with any velocity,* V, *from a given point* D, *will pass through any point* P. FIG. CLXXI.

Because V, or the number of feet uniformly described in 1″, with the velocity of projection, is given, the space DE, described from rest to acquire this velocity, is known, being equal to $\frac{V^2}{64}$ (art. 529). Draw DA, equal to $4DE$, perpendicular to the horizon, BC perpendicular to DA through its bisection G, and DC perpendicular to DP; and from C as a center, at the distance CD, describe a circle, and, if R, r, be the intersections of its periphery with a right line, passing through P perpendicular to the horizon, DR, and Dr will be the directions required; for (sim. triang.) $DA : DR :: DR : RP$, and $DA : Dr :: Dr : rP$, and DA (= the parameter) $= \frac{DR^2}{RP}$ or $\frac{Dr^2}{rP}$; and the parabola described by a body projected in either of the directions DR or Dr with the velocity V, will pass through P (628). Q. E. I.

634. Cor. 1. Because the arcs BR and Br are equal, the line DB makes equal angles with the corresponding directions DR and Dr. Drawing BQ a tangent to the circle at B, a body projected in the direction DB, will pass through Q, and DQ is the greatest distance upon that line to which the body can be projected with the velocity V.

635. Cor. 2. If DP be horizontal, the range upon the horizon $=$ $DF = RN$, the sine of twice the angle of elevation FDR; for, by the property of the circle $\angle FDR = \angle RAD = 2 \angle RCD$. The horizontal range is therefore the greatest, when the angle of elevation is $45°$, the sine of twice that angle, or of $90°$, being equal to $\frac{1}{2} DA$, or $\frac{1}{2}$ of the parameter at D; or it is equal to the latus rectum, for that is always equal to $\frac{DM^2}{MV}$, which, in this case, is equal to $\frac{DC^2}{4 \times \frac{1}{4} DC} = DC$. FIG. CLXXII.

Because

Becauſe the ſine of any angle varies as the radius or diameter, if the circle be deſcribed upon any other diameter DE or $\frac{1}{4}$th of DA, the horizontal range will be equal to the ſine of twice the angle of elevation of this circle multiplied into $\frac{DA}{DE}$, or into 4.

636. Cor. 3. If the axis MV, produced, be interſected by the tangent in T, the greateſt altitude $MV = \frac{1}{4}MT$ (conic ſect.) $= \frac{1}{4}FR = \frac{1}{4}$th of the verſed ſine of twice the angle of elevation; and when this angle is right, $\frac{1}{4}$th of the verſed ſine $= \frac{2}{4}DA$. Becauſe the verſed ſines of any angles are as the diameters of the circles deſcribed, if the circle be deſcribed with a diameter equal to DE, the altitude is equal to the verſed ſine of twice the angle of elevation of this circle.

637. Cor. 4. The time of the flight is equal to the time of deſcribing DR with the projectile velocity, and varies as DR, or as $\frac{1}{2}DR$, which is the ſine of the angle of elevation; and conſequently the time is the greateſt when the angle $FDR = 90°$, and it is then equal to the time of deſcent from reſt through the parameter AD. The time is alſo equal to the time of falling from reſt through RF, and therefore varies as $\sqrt{RF}$, or as the ſquare root of the verſed ſine of twice the angle of elevation, or as the ſquare root of $\frac{1}{4}RF$, or the greateſt altitude.

638. Prop. *The velocity of projection*, V, *and the angle of elevation being given, to deſcribe the path of the projectile.*

FIG. CLXXIII. Let V be the number of feet uniformly deſcribed in 1″, and $\frac{V^2}{64} = DE = \frac{1}{4}$th of the parameter, is the ſame whatever be the angle of elevation, and the periphery of a circle, whoſe radius is DE, and center D, will paſs through the foci of all parabolas deſcribed by a body projected from D with the velocity V. Let DE be perpendicular to the horizon, and DR the direction of pro-

projection, and making the angle $SDR = \angle RDE$; S will be the focus: EX drawn perpendicular to DE is the directrix; V the bisection of SX drawn parallel to DE, is the principal vertex, and SV the axis; DM perpendicular to VM is an ordinate; SE bisects DT in R and is perpendicular to it, and RV is a tangent to the parabola at V (conic sect.).

639. Cor. Again, it appears that the amplitude DF is (as in art. 635) equal to four times the sine of twice the angle of elevation, for $\angle MDR = \angle RDC = 2\angle RCN$, and $DF = 2DM = 2NV = 4RN$.

640. Prop. *To find the least velocity with which a body projected, from a given point* D, *will hit a given point* P. FIG. CLXXIV.

Draw BPH perpendicular to the horizon, and, making PB equal to PD joined, bisect the angle DPB by the right line PC, meeting DC perpendicular to DP, in C; and, because the triangles CBP and CDP are equal and similar, $CB = CD$, the $\angle$s CBP and CDP are right angles, and BP is a tangent to the circle described from C, as a center, at the distance CB. Bisect DG, drawn perpendicular to the horizon, in E, and a body projected in the direction DB, with a velocity equal to that acquired in falling through ED, will pass through P (634), and, because DP is the greatest range (634), the velocity must be the least possible. Q. E. I.

641. Cor. If P be in the horizon, CD and GD coincide, and the angle of elevation BDH is equal to 45°.

642. Prop. *The range upon any plane* DP, *is equal to the product of the parameter and sines of the angles formed by the plane and direction of projection, and the plane and a perpendicular to the horizon, divided by the square of the cosine of the plane's elevation.* FIG. CLXXI.

Dem.

DEM. From trigon. $DP : DR :: \text{sin.}\angle DRP \text{ or } \angle ADR : \text{sin.}\angle DPF$,
$DR : DA :: \text{sin.}\angle DAR \text{ or } \angle PDR : \text{sin.} ARD \text{ or } \angle RPD \text{ or } DPF$;
and consequently $DP : DA :: \text{sin.}\angle ADR \times \text{sin.}\angle PDR : \text{sin.}^2\angle DPF$ or $\text{cos.}^2 \angle PDF$, and $DP = \dfrac{DA \times \text{sin.} \angle ADR \times \text{sin.} \angle PDR}{\text{cos.}^2 \angle PDF}$.
Q. E. D.

FIG. CLXXV. 643. Cor. 1. Let $S =$ the sine of the angle contained between the plane and the direction; $Q =$ the sine of the angle contained between the direction and the parameter DA perpendicular to the horizon; and $C =$ the cosine of the angle contained between the plane and the horizon; and the range $DP = \dfrac{DA \times S \times Q}{C^2}$, and, because DA varies as the square of the velocity, or V^2, DP is as $\dfrac{V^2 \times S \times Q}{C^2}$. This corollary may be demonstrated differently by the following process: let V be the velocity of projection, or number of feet described uniformly in $1''$, and $V : DR :: 1'' : \dfrac{DR''}{V} =$ the time of describing DR or RP, and $RP = 16 \times \dfrac{DR^2}{V^2}$ (529) $= DP \times \dfrac{S}{Q}$; but $DR = DP \times \dfrac{C}{Q}$, therefore $DP \times \dfrac{S}{Q} = 16 \times \dfrac{DP^2 \times C^2}{V^2 \times Q^2}$, and consequently $DP = \dfrac{S \times Q \times V^2}{16 \times C^2}$.

644. Cor. 2. If DP be bisected in M and MV be drawn perpendicular to the horizon, the greatest altitude $MV = \frac{1}{2} MT = \frac{1}{4} PR$; but $PR : DP :: S : Q$ and $PR = \dfrac{DP \times S}{Q} =$ (substituting the value of DP) $\dfrac{V^2 \times S^2}{C^2}$; and consequently MV, or the greatest altitude varies as $\dfrac{V^2 \times S^2}{C^2}$;

645. Cor.

645. Cor. 3. The time of the flight is equal to the time of describing DR, and is as $\frac{DR}{\text{vel.}}$; or, if $V =$ the number of feet described uniformly in 1″ with the velocity of projection, the time is actually equal to $\frac{DR''}{V} =$ (fig. 171.) $\frac{DA \times S}{C \times V}$; and because DA is as V^2, the time varies as $\frac{V \times S}{C}$. The time is also equal to the time of descent from rest through RP, and consequently varies as $\sqrt{PR}$, or as $\sqrt{MV}$. FIG. CLXXV.

646. Cor. 4. Supposing V to be given, the range DP is the greatest when $S = Q$, or when the direction bisects the angle contained between the plane and the vertical DA; and if the plane DP coincide with the horizon, the amplitude $DP = DF$ $\left(= \frac{DA \times S \times Q}{C^2}\right) = \frac{DA \times FR \times DF}{DR^2}$ (making DR the radius) $=$ DF or RN the sine of twice the angle of elevation, DA being equal to $\frac{DR^2}{FR}$, and $\frac{DA \times FR}{DR^2}$ being given. The greatest altitude $= \frac{DA \times S^2}{4C^2} = \frac{DA \times FR^2}{4DR^2} = \frac{1}{4}FR$.

647. Prop. *If both the velocity of projection,* V, *and the angle of elevation, vary, the horizontal range or amplitude will vary as the square of the velocity and sine of twice the angle of elevation.*

Dem. The range, being generally as $\frac{V^2 \times S \times Q}{C^2}$ (643), will, when DP coincides with the horizon, be as $\frac{V^2 \times S \times Q}{\text{rad.}^2}$; and, if the radius be unity, $S \times Q = \frac{1}{2}$ of the sine of twice the angle of elevation (trigonom.), and the amplitude consequently is as $V^2 \times$ sine of twice that angle. Q. E. D. FIG. CLXXI.

 Another

Another demonstration:

FIG. CLXXVII. Let $V =$ the number of feet described uniformly in 1″, by the velocity of projection, and $V : DL :: 1'' : \frac{DL''}{V}$, or the time of describing DL with the velocity of projection, which is equal to the time of describing LF from rest, and $LF = \frac{s \times DL^{2''}}{V^2}$, supposing s to be the descent from rest in the first second (529); but, describing a circle with any diameter DA, $LF = \frac{DL \times DR}{DA}$ (sim. triang.), and consequently $\frac{s \times DL^2}{V^2} = \frac{DL \times DR}{DA}$, and $DL = \frac{DR \times V^2}{s \times DA}$; and $DF \left(= \frac{DL \times RN}{DR}\right.$ from sim. triang.$\left.\right) = \frac{RN \times V^2}{s \times DA}$, and varies as $V^2 \times RN$, because s and DA are given. Q. E. D.

648. Cor. The greatest altitude is generally as $\frac{V^2 \times S^2}{C^2}$ (644), or when DP is horizontal, as $\frac{V^2 \times S^2}{\text{rad.}^2}$, or, the radius being unity, as $V^2 \times$ versed sine of twice the angle of elevation; for $S^2 = \frac{1}{2}$ of this versed sine.* This corollary is also deducible from the second demonstration; for the greatest altitude $= \frac{LF}{4} = \frac{DL \times DR}{4DA} =$ (substituting the value of DL) $\frac{DR^2 \times V^2}{4s \times DA^2} = \frac{V^2 \times DN}{4s \times DA}$, and consequently varies as $V^2 \times DN$, s and DA being given.

649. Prop.

FIG. CLXXVI. * Let the arcs DF, EF, and the angles DCF and ECF, be equal, and, FCD being the angle of elevation, EL drawn perpendicular to the radius CD, is the sine, and DL the versed sine, of twice that angle; and, joining ED and drawing GH from G, the intersection of CF and ED, parallel to DL, $EH = \frac{1}{2}$ the sine, and $GH = \frac{1}{2}$ the versed sine. But the triangles EGH and ECG are similar, the angles at H and G being right, and the $\angle EGH = \angle EDC = \angle CED$; therefore EG $(= FN = S) : EH :: CE\ (= 1) : CG\ (= CN = Q)$, and $EH = S \times Q$. And EG or $S : GH\ (= \frac{1}{2} LD) :: CE : EG$, and consequently $GH = S^2$.

649. PROP. *The velocity of projection,* V, *or number of feet described uniformly in* 1″ *with that velocity, and the angle of elevation,* E, *being given, to find the amplitude, altitude, and time of flight.*

1. The space described from rest to acquire the velocity of projection is equal to $\frac{V^2}{4s}$, and the amplitude, when the angle of elevation $= 45°$, is equal to $\frac{V^2}{2s}$ (635); but the amplitudes are as the times of twice the angles of elevation, and consequently sin. of 90° : sin. of $2E :: \frac{V^2}{2s}$, or ½ parameter : amplitude, which is therefore known.

2. When the angle of elevation $= 45°$, the altitude $= \frac{1}{4}$ of the parameter $= \frac{V^2}{8s}$, and, the altitude being always as the versed sine of twice the angle of elevation, the versed sine of 90° : versed sine of $2E :: \frac{V^2}{8s}$: altitude, which is therefore known.

3. When the angle of elevation is 90°, the time of the flight is equal to the time of falling, from rest, down the parameter, $\frac{V^2}{s}$, and $= \sqrt{\frac{V^{2''}}{s^2}} = \frac{V''}{s}$, and the times of flight being as the sines of angles of elevation (637), the sine of 90° : sine of $E :: \frac{V''}{s}$: time of the flight, which is therefore known. Q. E. I.

650. PROP. *The amplitude and the angle of elevation,* E, *being given, to find the velocity of projection, and the altitude.*

1. Let

1. Let the parameter $= P$, and, when the angle of elevation is $45°$, the amplitude $= \frac{P}{2}$; therefore, the sin. of $2E$: sin. of $90°$ :: given amplitude : $\frac{P}{2}$, which is therefore known; and the velocity of projection, being equal to that acquired in falling from rest through $\frac{P}{4}$, is also known, being equal to $\sqrt{4s \times \frac{P}{4}}$ feet in $1''$ (529).

2. The altitude is found when the parameter is known, by the last proposition.

FIG. CLXXVII. Or, The altitude is easily found by the following process: Bisect the amplitude DF in M, and, drawing MT perpendicular to DF, the angles in the triangle DMT, and the side DM, are known, and MT may be found from trigonometry; and consequently $MV = \frac{1}{2} MT$ is known. Q. E. I.

SCHOLIUM.

651. By a similar process the converse of these propositions are easily solved, that is, 1. the amplitude and altitude being given, to find the velocity of projection and angle of elevation; 2. the velocity of projection and altitude being given, to find the amplitude and angle of elevation; and, 3. and the angle of elevation and altitude being given, to find the amplitude and velocity.

SCHOLIUM II.

652. In the theory of projectiles, the medium is supposed to be void of resistance, which suppositiou does not obtain in practice, the resistance of the air being variable according to the different velocities and magnitudes of the projectiles, and always considerable.

able. 1. If a musket ball, of $\frac{3}{4}$th of an inch diameter, be fired, from a piece forty-five inches long, with half its weight of powder, its velocity is nearly 1700 feet in 1″, and its horizontal range, when the angle of elevation is 45°, ought to be about seventeen miles; but from practical writers it appears, that the range is less than half a mile. An iron ball of 24 lb. weight, discharged with a full charge of powder, has a velocity of 1650 feet in 1″, and its horizontal range at 45° would be about sixteen miles, were the path described a parabola; but, from experiments, it appears not to be three miles, and not $\frac{1}{5}$th part of the space investigated by the theory. When the velocity of the shot is about 400 feet in 1″, the resistance of the air is still considerable, and neither the amplitude, height or time of flight, correspond with the theory. This resistance is confirmed by the constant observation of all conversant in the projection of bombs; for the ranges at elevations equally distant from 45°, ought, according to the theory, to be equal; but the ranges of a shell projected at an elevation of 15° or 20°, are always found to be greater than those projected at elevations equal to 60° or 65°, though equally distant from 45°.

Projectiles, whose motion is sensible to the eye, are seen to descend in a curve obviously shorter, and inclined in a greater angle to the horizon, than that in which they ascended; and whoever views, in a proper situation, the flight of stones, arrows, shells, &c. projected to any considerable distance, may see evidently that the vertex of the curve or greatest altitude, divides the path described into two unequal parts, and is more remote from the point of projection, than where the projectile falls to the ground.

SCHOLIUM III.

653. A body, projected with a considerable velocity, is often not only affected by the resistance of the air, and deflected from a parabolic path in a direction perpendicular to the horizon, but is made to deviate laterally and change the plane of motion. The path of a tennis-ball, struck with great force, is plainly observed to be incurvated sideways as well as downwards, and to move in a different plane from that arising from the combined

bined action of gravity and force of projection. Bullets are not only depressed beneath the line of projection, but deflected to the right or left of that direction by the resistance of the air, or action of some other force. Mr. Robins fixed a barrel, carrying a ball of $\frac{3}{4}$th of an inch diameter, and, firing at a mark, one foot and $\frac{5}{7}$th square, at the distance of 180 feet, missed it only once in sixteen successive trials; but when the same barrel was fired with a smaller quantity of powder, he found the ball to be deflected 100 yards to the right or left of the line aimed at and placed at the distance of 760 yards; and its direction in the perpendicular was equally uncertain, its range differing sometimes 200 yards. Because the force of gravity acts always in a direction perpendicular to the horizon, a body projected in any direction would, if unresisted, be always in the same vertical plane, which, as is observed, is not true in fact; and, therefore, besides the resistance of the air to the progressive motion of the body, there is another lateral force producing a deviation from the plane of projection, which is probably the inequality of resistance upon its surface: for, if the surface of a projectile, protruding the particles of a fluid, be equally resisted in every point, it is obvious that the plane in which it first moves is not altered. But supposing the resistances upon different sides of a body to be unequal, it will be impelled, by the greater resistance, towards those parts where the resistance is least; and thus either diverge laterally from the first plane of motion, or ascend, or descend in that plane, beyond the altitude investigated by the theory. A motion of rotation round an axis will produce this inequality of resistance upon the different sides of a projectile, and consequently its irregular motions; for the parts of a revolving body are exposed to the air, which is protruded, in different angles of obliquity and with different forces, and must therefore be differently resisted. If the axis of rotation be perpendicular to the direction, the body will be deflected naturally from the vertical plane; and because the velocity of the body, and consequently resistance of the air, are variable, this deflection will be different in different points, or it will be a curve line. That this lateral deviation is produced by a motion of rotation seems to be confirmed by experiments. Let a sphere of wood, suspended freely by a cord in a current of air, be made to revolve round a ſtring as an axis, and the parts of its ſurface, on oppoſite ſides, oppoſing or conſpiring with the motion of the particles of the fluid, are unequally impelled by it, and the ſphere is always deflected towards that ſide whoſe reſiſtance is leaſt. Or, let a wooden ball, loaded with lead, be ſuſpended freely in a ſtream of water by a twiſted cord; and, as the cord returns to its natural ſtate, the ball revolves round it, and moves gradually towards the ſide where the reſiſtance is leaſt, or where the parts conſpire with the motion of the water. When the ball arrives at its utmoſt extent, it is quieſcent for a moment, and returns gradually to its firſt ſituation, and is again quieſcent till the motion of the ball twiſt the cord the contrary way, and it then moves alſo towards the other ſide; and in this manner the ball continues to vibrate till its motion be deſtroyed by friction, and it remain quieſcent in its firſt ſituation. The magnitude and direction of reſiſtance of the air, and conſequent deviations from theory, can only be aſcertained by a ſeries of experiments; which, reſulting from the operation of cauſes apparently fluctuating and uncertain, produce an inconſiſtency in the ſame experiments, and render the ſubject at leaſt very complicated and difficult *.

* See Robins's Tracts of Gunnery, p. 183, &c. and Euler's True Principles of Gunnery, by Mr. Brown.

ERRATA ET ADDENDA.

Pag. 2. lin. 7. *for* of that, *read* that.
10. art. 15. *for* the existence of ratios, *read* the existence of the measures of ratios.
24. art. 48. ex. 1. for *LA*, read *Lm*, and for *La*, read *LA*.
29. ex. 2. *dele* or closeness.
37. art. 78. *for* all infinitely great, &c. *read* all infinitely great or small magnitudes compared with each other, admit of the same inequality of ratio with finite magnitudes.
42. art. 85. *dele* the remoteness of the elementary particles.
67. lin. 8. *after* velocity, *read* when in motion.
fig. 26. the letters *L*, *M*, wanting.
81. art. 194. *for MBN*, *read PBN*.
82. art. 200. *after BD*, *put BE*.
83. art. 204. *dele* in the same direction.
fig. 48. the letter *D* wanting.
111. ex. 3. *for* fire, *read* fibre.
115. line the last. *after* velocities, *put* in opposite directions.
148. *for* fig. 16. *read* fig. 106.
152. l. . by the direction of the resistances is understood the right line in which the resisting forces act, and by *the quantities of the resistances* are understood, not the quantities really exerted upon the wedge, but which would be exerted if no part were lost.
186. the following corollary wanting to art. 470. If the circle does not cut or touch the line *DH*, the bodies will never meet.
197. art. 500. *for* ± velocity of $V \times SA$, *read* $= \frac{\text{velocity of } V \times SA}{SV}$.
198. art. 500. l. 2. for $m \times SV$, read $m \times SV \times SC$.
231. art. 589. *for* arc, *read* chord, in which corollary, the body is supposed to vibrate in the chords.

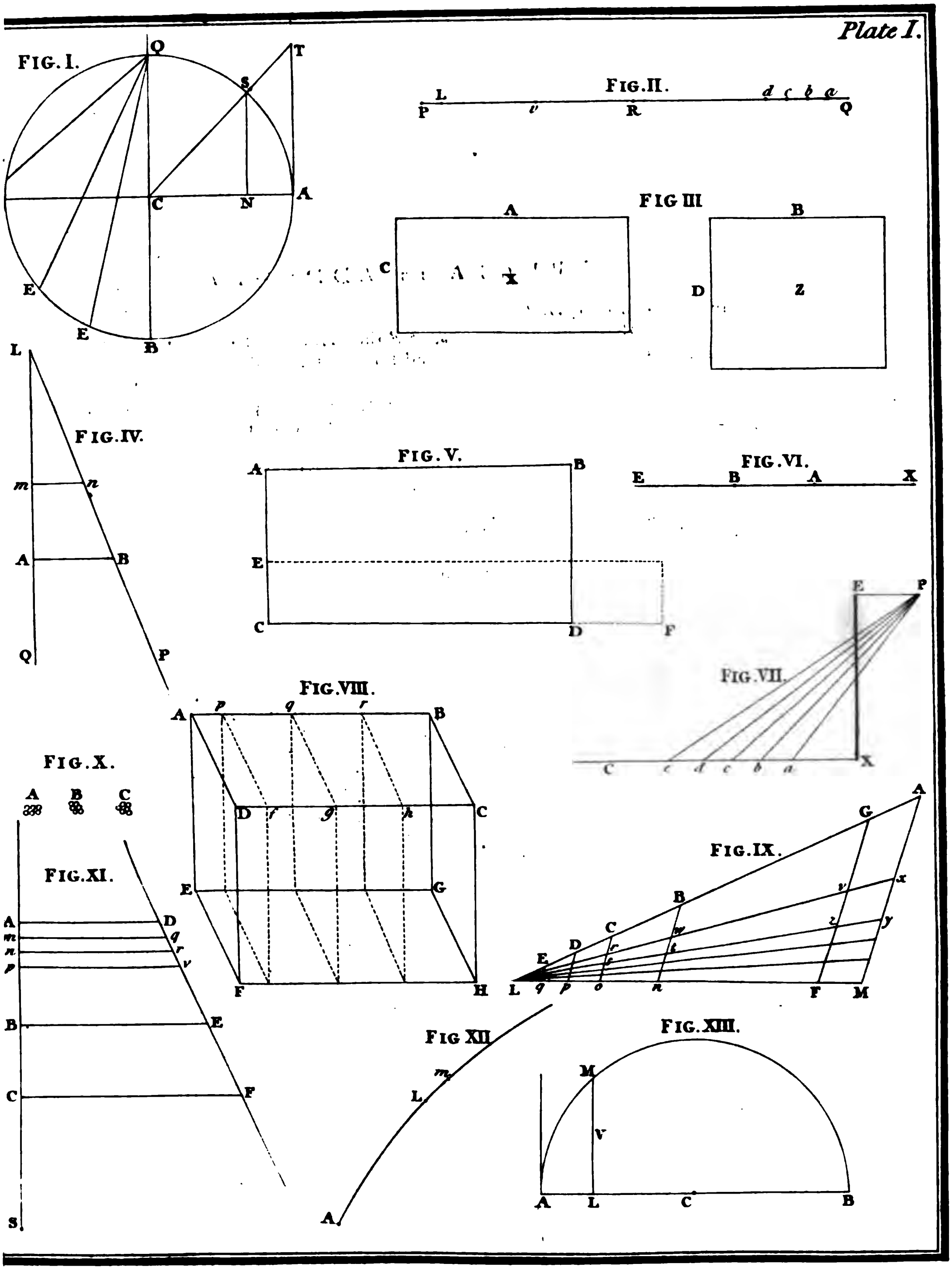


Plate I.
FIG. I.
FIG. II.
FIG III
FIG. IV.
FIG. V.
FIG. VI.
FIG. VII.
FIG. VIII.
FIG. IX.
FIG. X.
FIG. XI.
FIG XII
FIG. XIII.

Plate II.

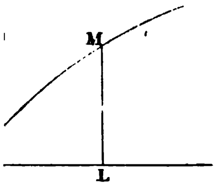


FIG. XIV.

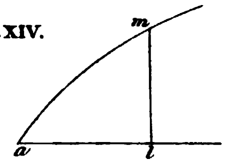


FIG. XV.

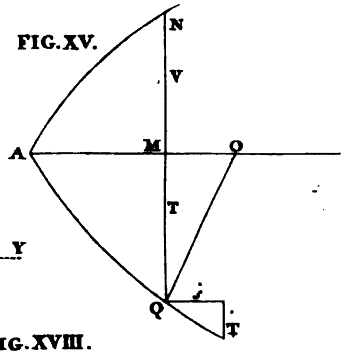


FIG. XVI.

FIG. XVII.

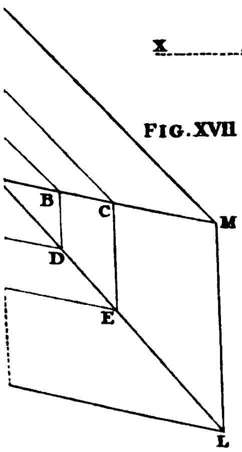


FIG. XVIII.

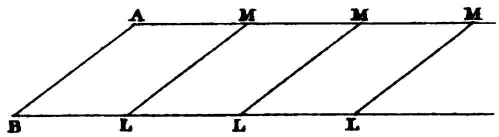


FIG. XIX.

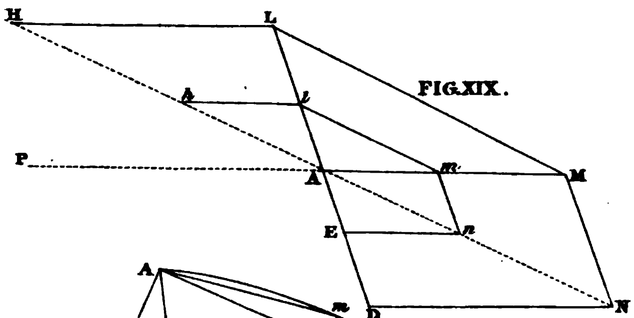


FIG. XX.

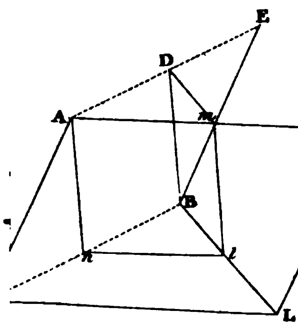


FIG. XXI.

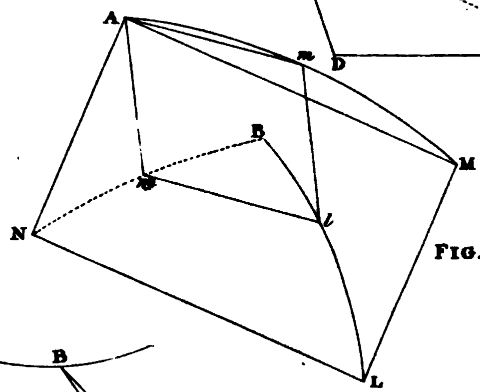


FIG. XXII.

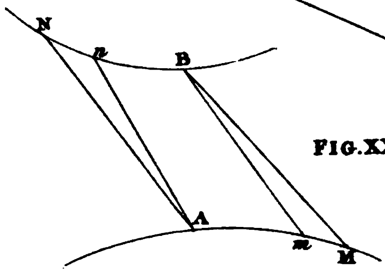


Plate III

FIG. XXIII.

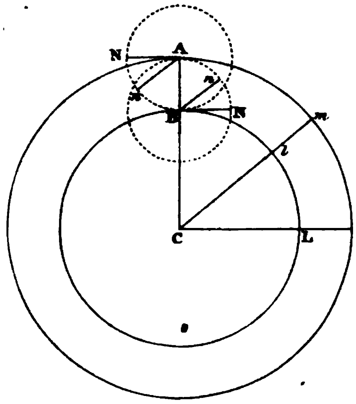


FIG. XXIIII.

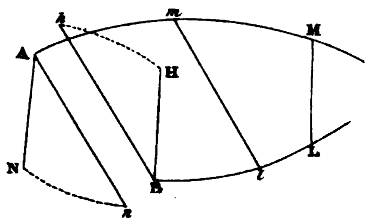


FIG. XXV.

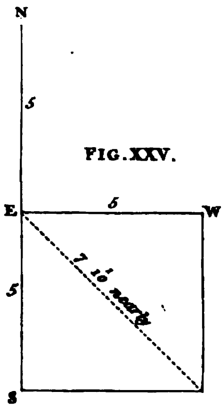


FIG. XXVI.

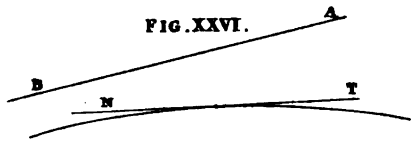


FIG. XXVII.

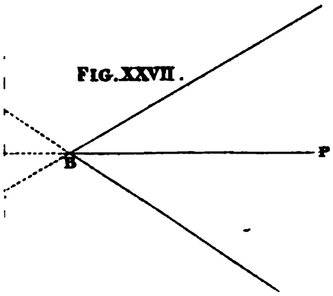


FIG. XXVIII.

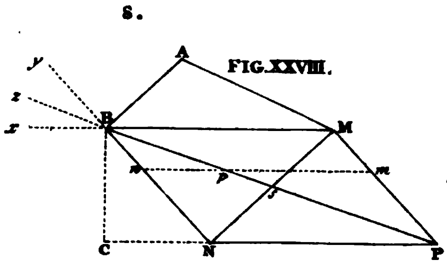


FIG. XXIX.

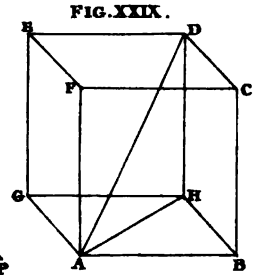


FIG. XXX.

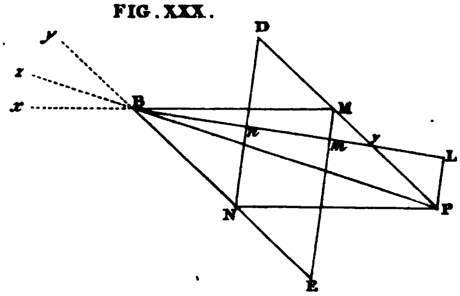


FIG. XXXI.

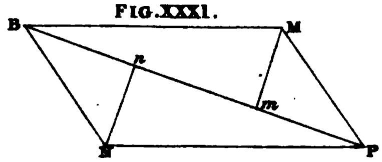


FIG. XXXII.

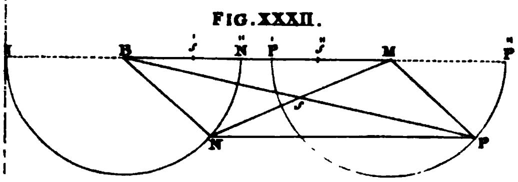
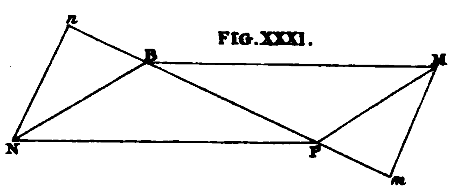


FIG. XXXI.



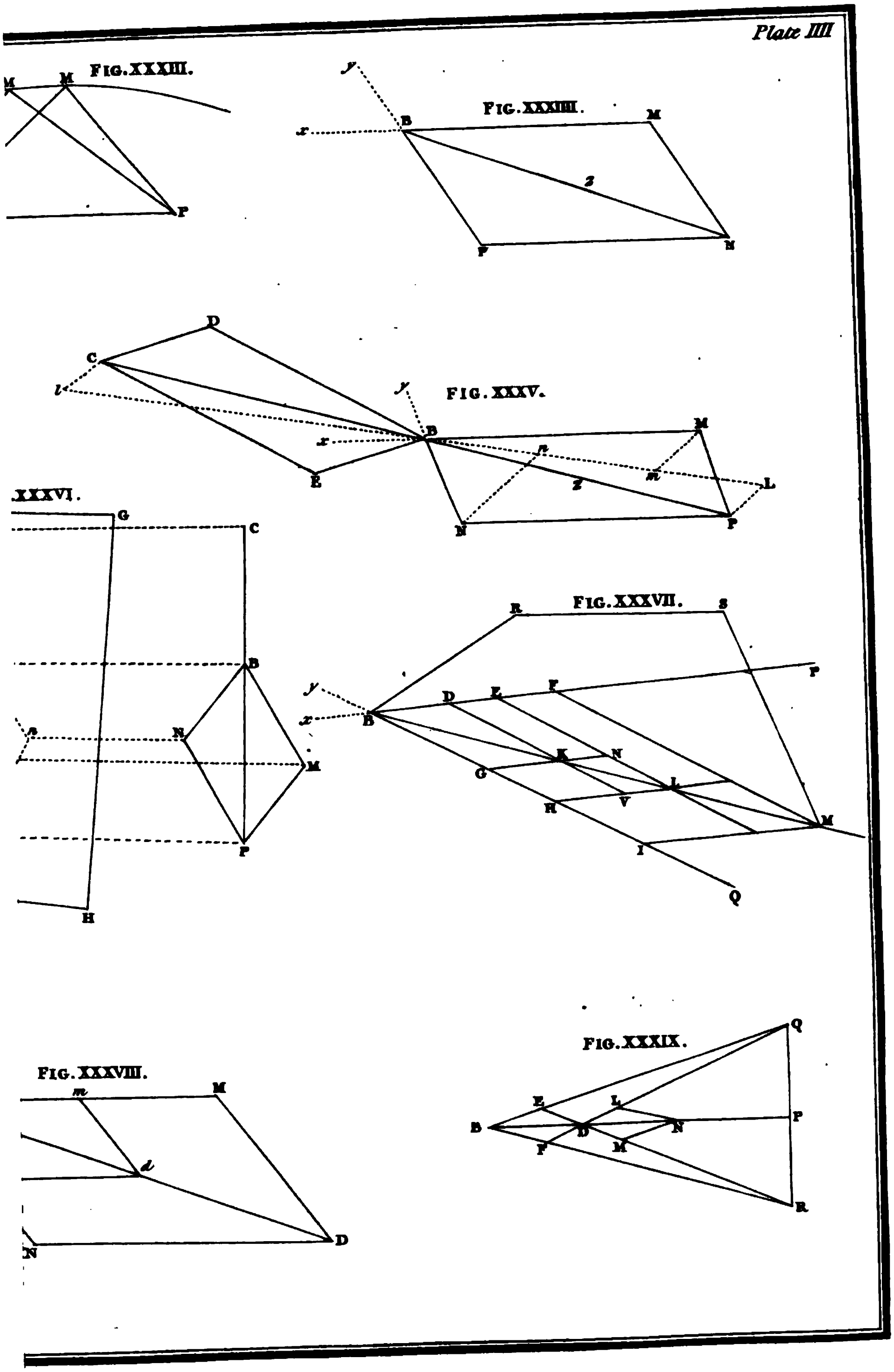


Plate IIII
FIG. XXXIII.
FIG. XXXIIII.
FIG. XXXV.
XXXVI.
FIG. XXXVII.
FIG. XXXVIII.
FIG. XXXIX.

Plate V.

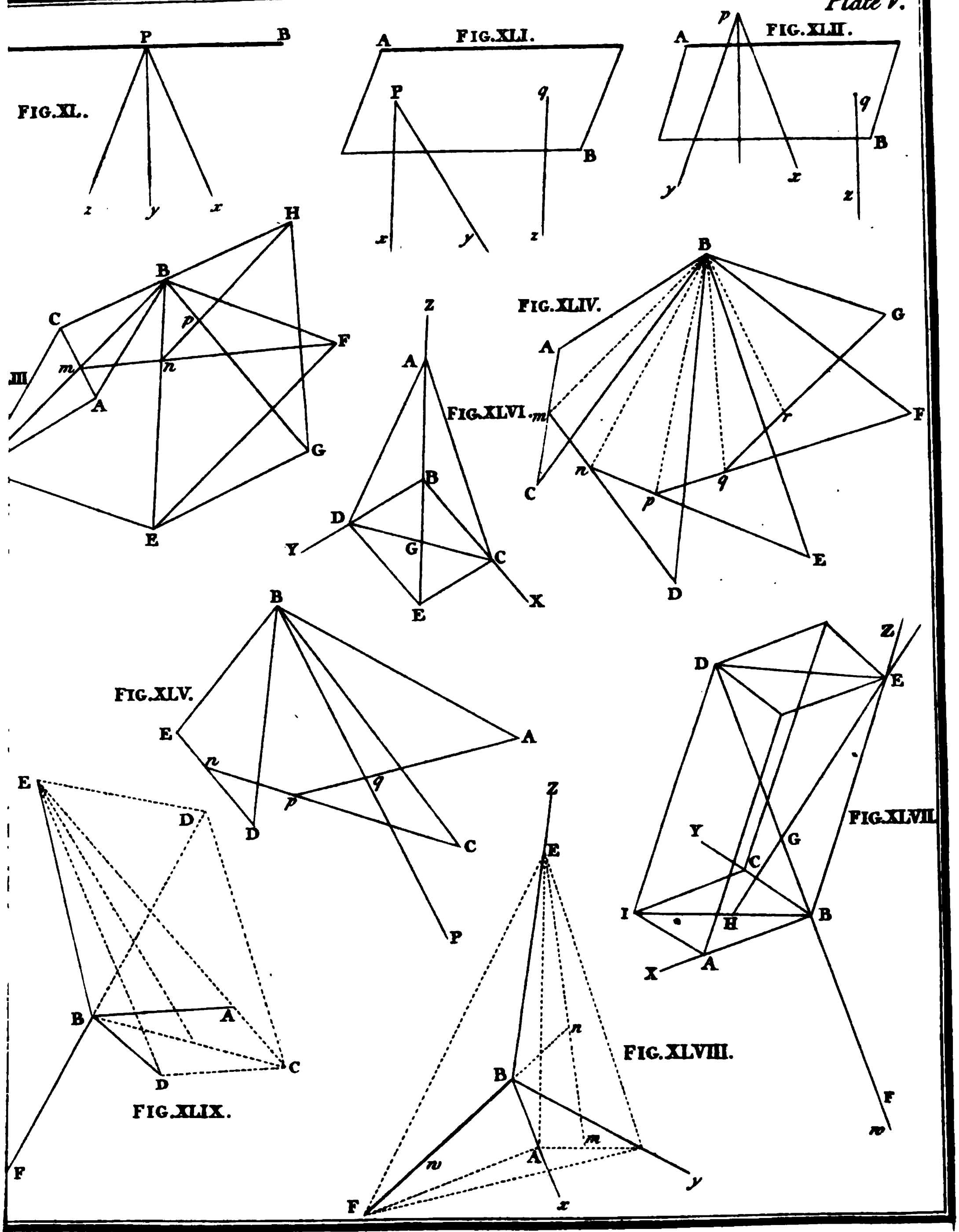


FIG. XL.
FIG. XLI.
FIG. XLII.
FIG. XLIV.
FIG. XLVI.
FIG. XLV.
FIG. XLVII.
FIG. XLVIII.
FIG. XLIX.

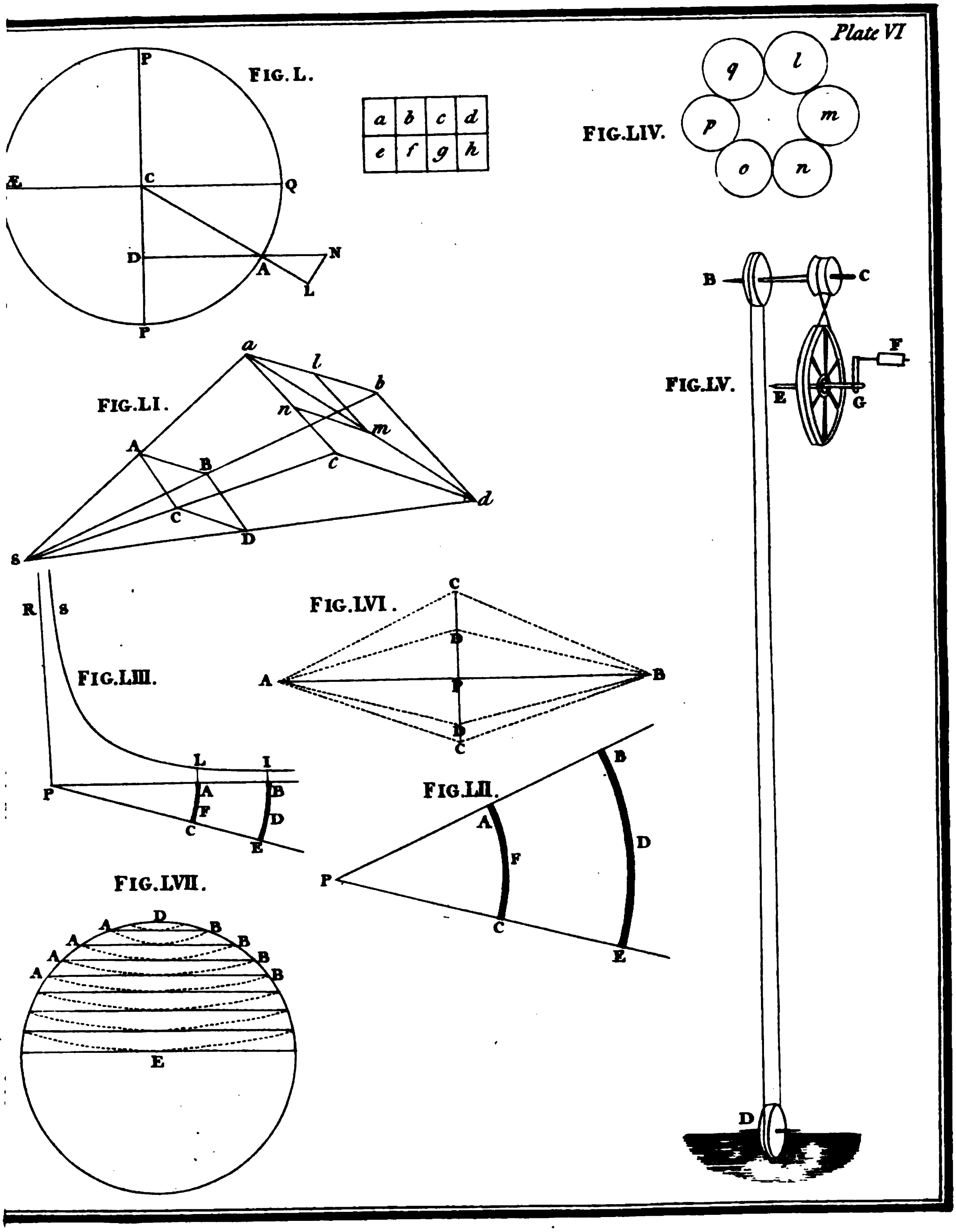


Plate VI
FIG. L.
FIG. LI.
FIG. LII.
FIG. LIII.
FIG. LIV.
FIG. LV.
FIG. LVI.
FIG. LVII.

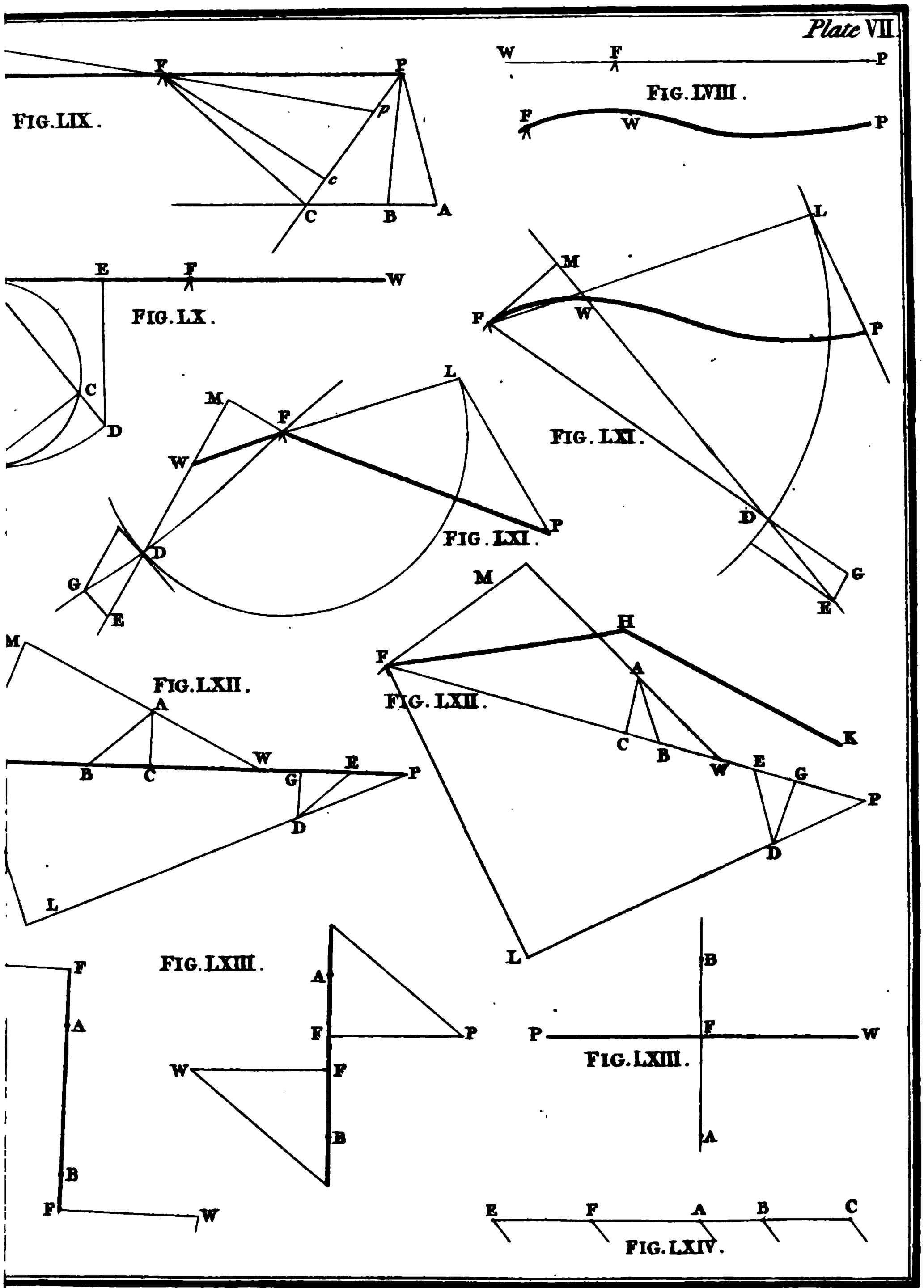


Plate VII.
FIG. LVIII.
FIG. LIX.
FIG. LX.
FIG. LXI.
FIG. LXI.
FIG. LXII.
FIG. LXII.
FIG. LXIII.
FIG. LXIII.
FIG. LXIV.

Plate VIII.

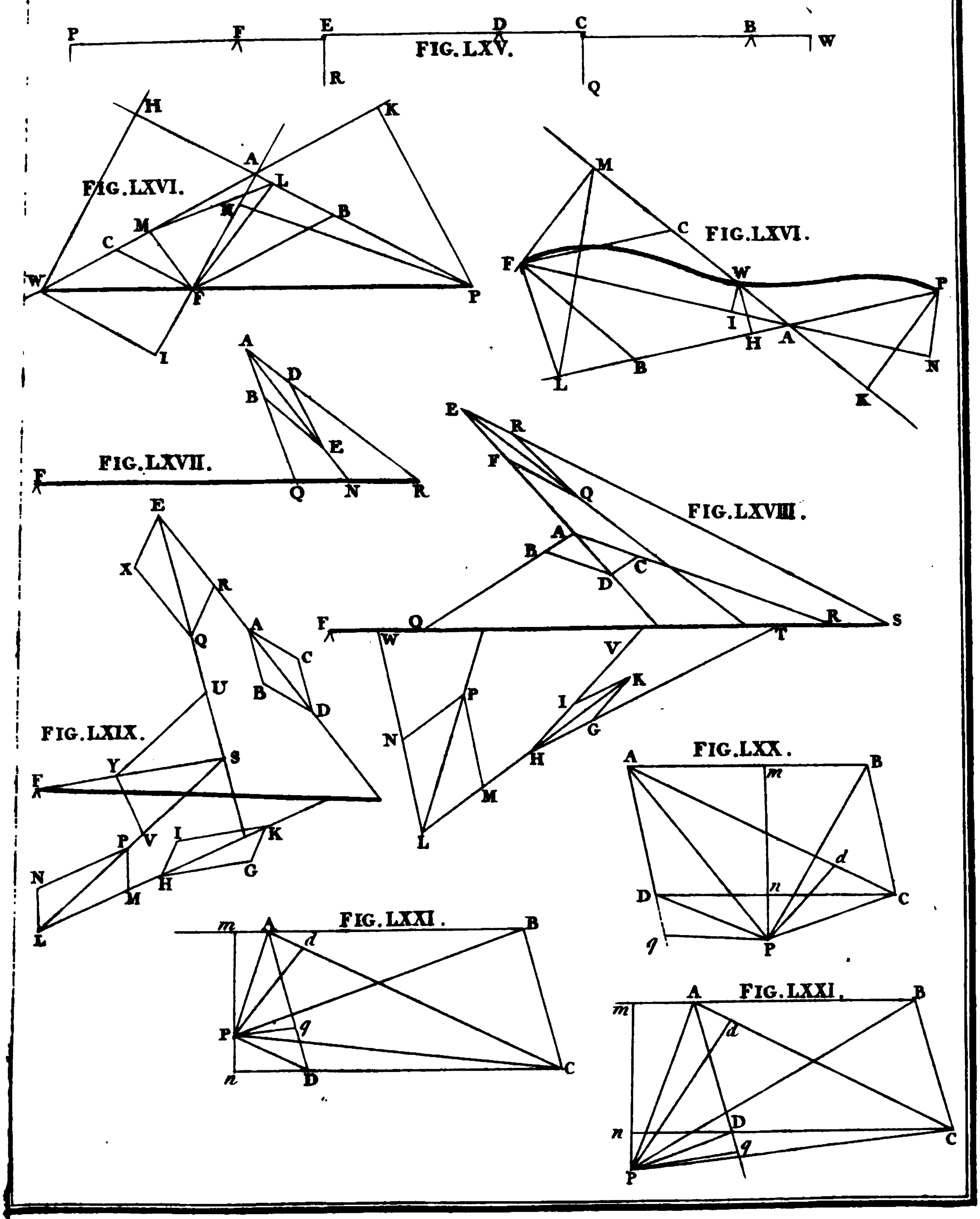


FIG. LXV.
FIG. LXVI.
FIG. LXVI.
FIG. LXVII.
FIG. LXVIII.
FIG. LXIX.
FIG. LXX.
FIG. LXXI.
FIG. LXXI.

Plate IX.

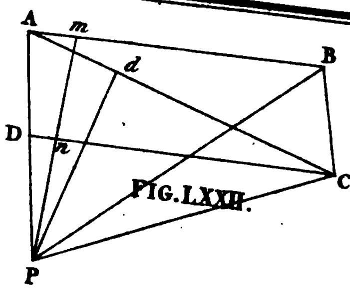


FIG. LXXII.

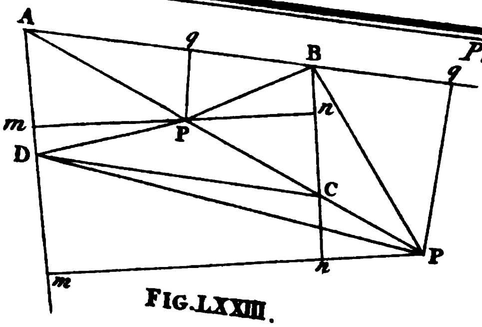
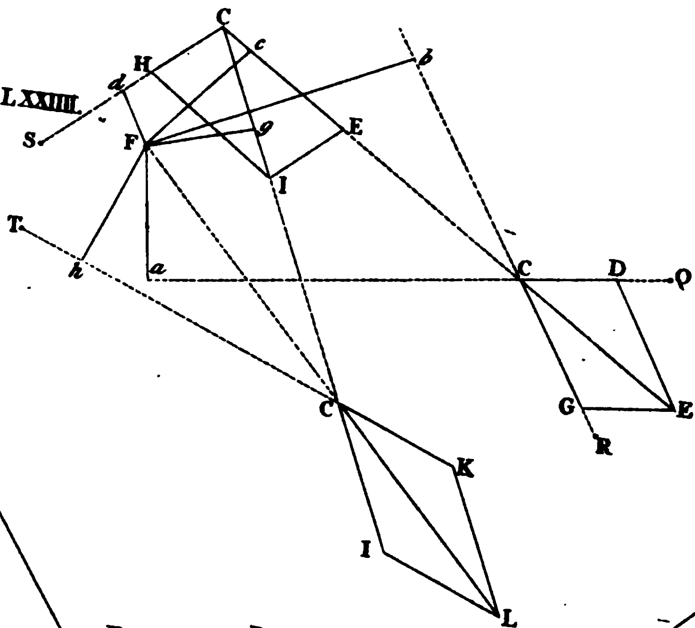


FIG. LXXIII.

FIG. LXXIIII.



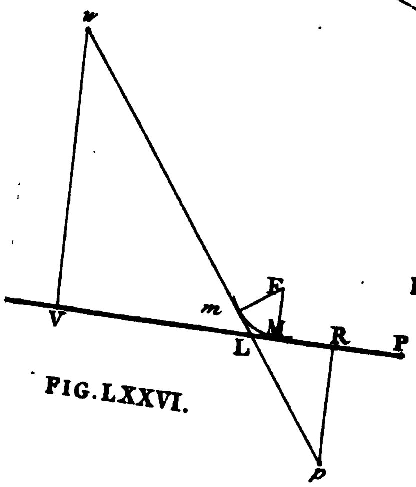


FIG. LXXVI.

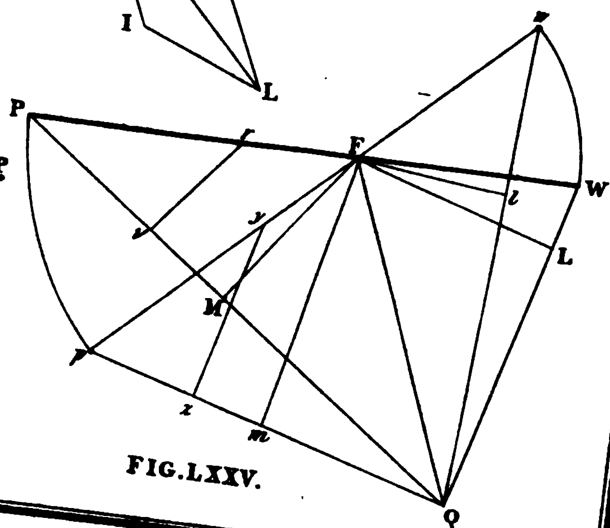


FIG. LXXV.

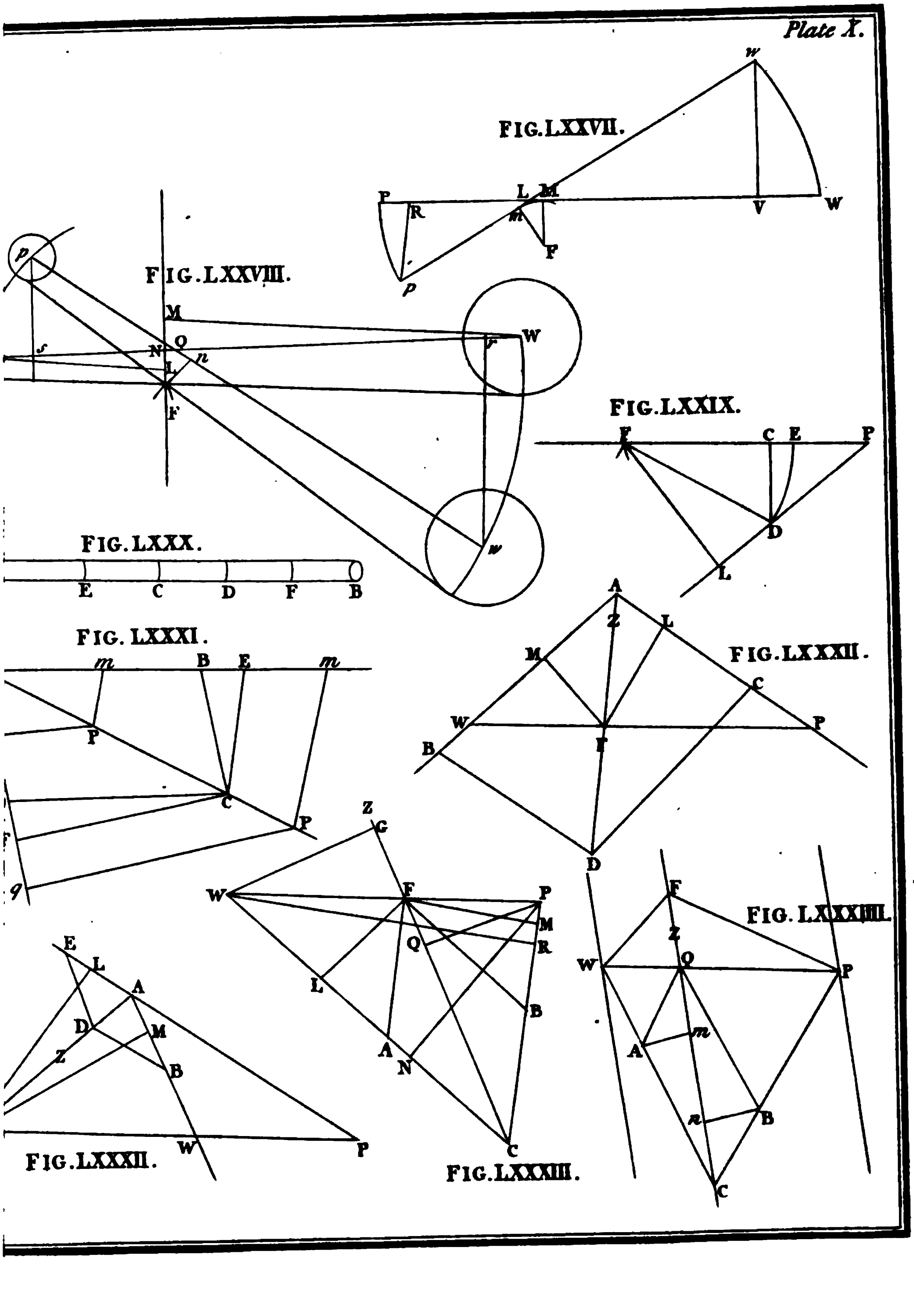


Plate X.

FIG. LXXVII.

FIG. LXXVIII.

FIG. LXXIX.

FIG. LXXX.

FIG. LXXXI.

FIG. LXXXII.

FIG. LXXXII.

FIG. LXXXIII.

FIG. LXXXIIII.

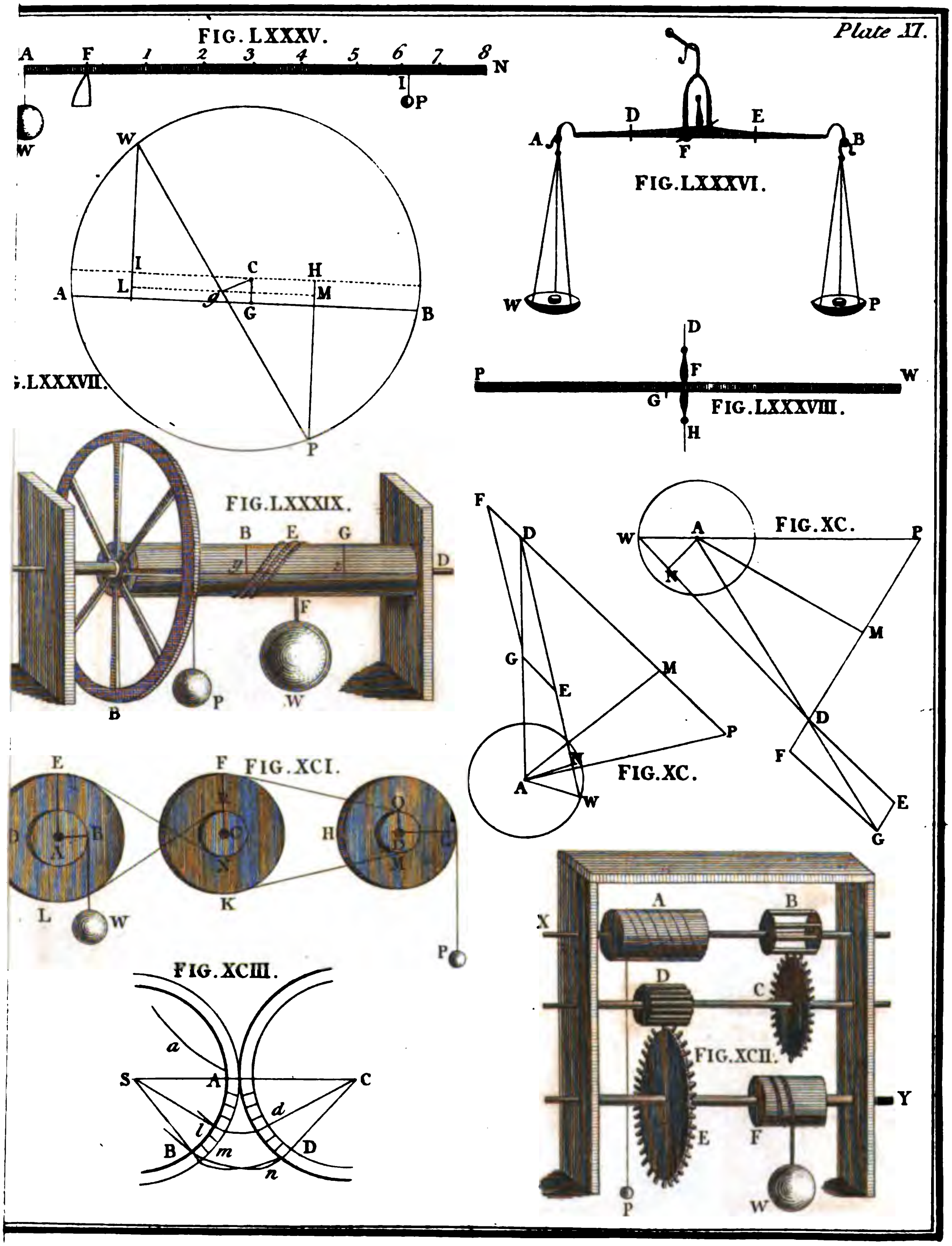


Plate XI.
FIG. LXXXV.
FIG. LXXXVI.
.LXXXVII.
FIG. LXXXVIII.
FIG. LXXXIX.
FIG. XC.
FIG. XC.
FIG. XCI.
FIG. XCII.
FIG. XCIII.

Plate XII.

FIG. XCIV.

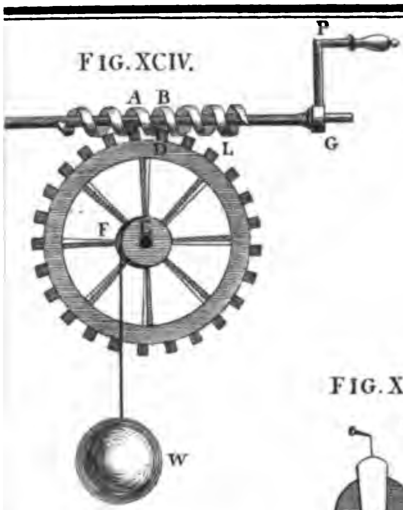


FIG. XCV.

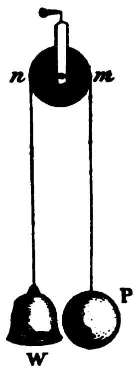


FIG. XCVI.

FIG. XCVII.

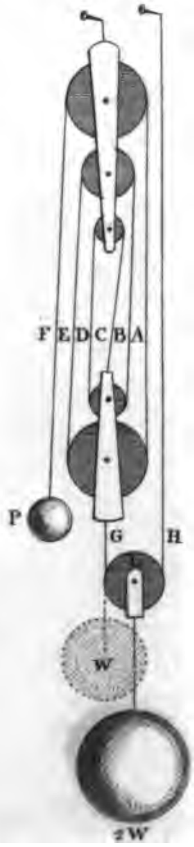
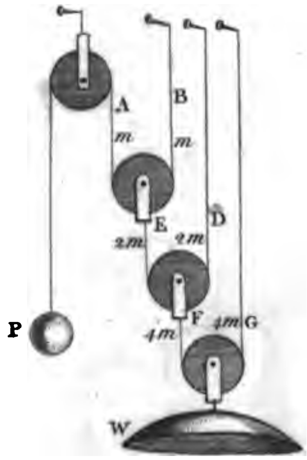


FIG. XCVIII.



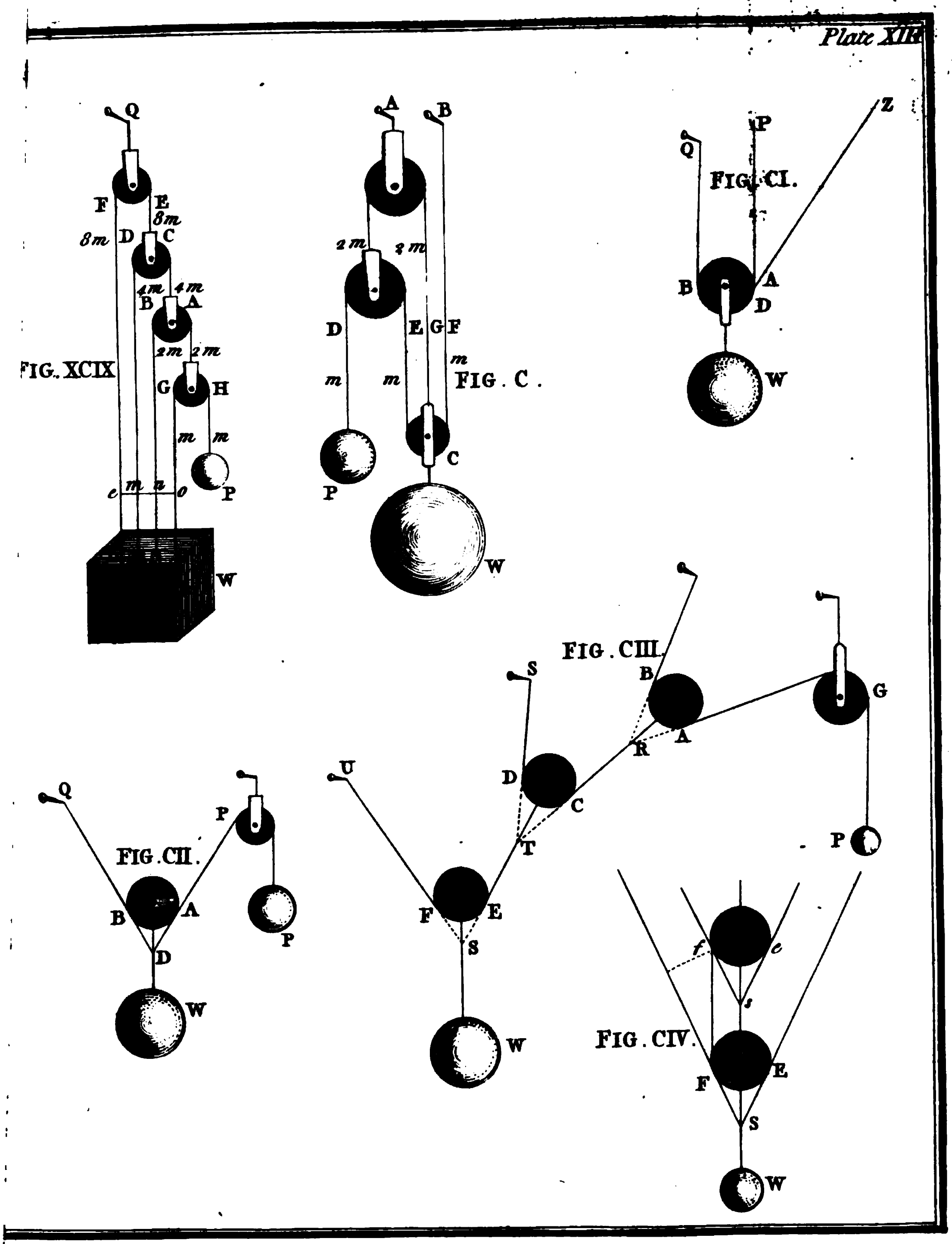


Plate XIII
FIG. XCIX
FIG. C.
FIG. CI.
FIG. CII.
FIG. CIII.
FIG. CIV.

Plate XIIII

FIG. CV.

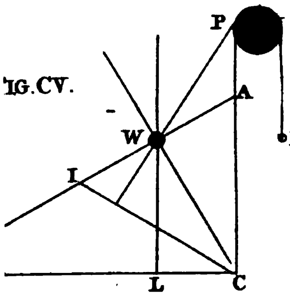


FIG. CVI.

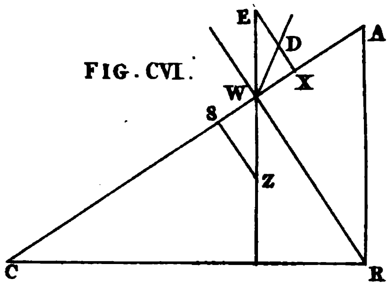


FIG. CVII.

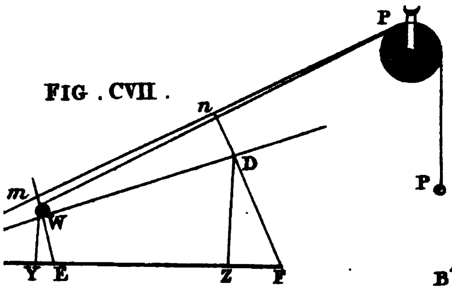


FIG. CVIII.

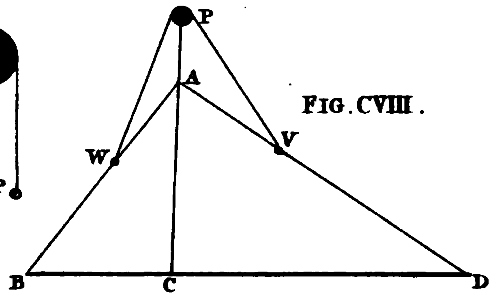


FIG. CIX.

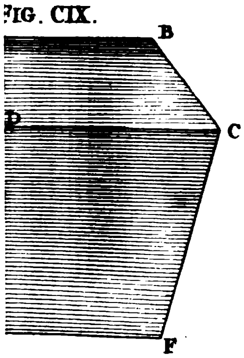


FIG. CX.

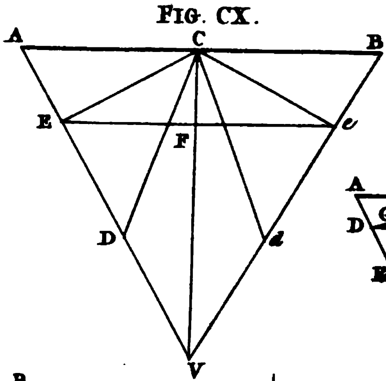


FIG. CXII.

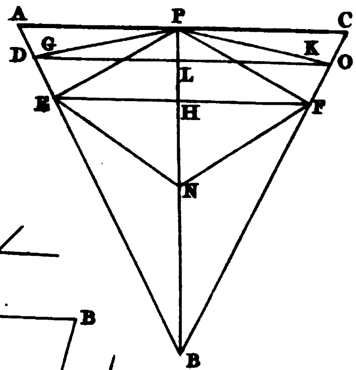
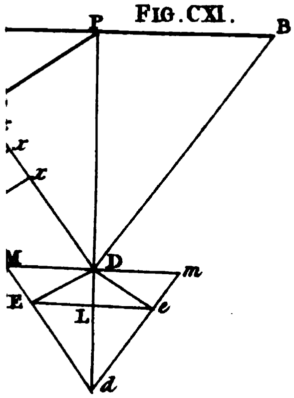


FIG. CXI.



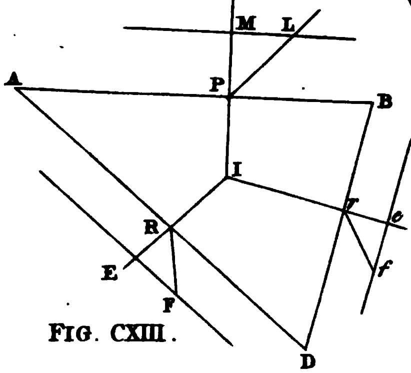


FIG. CXIII.

FIG. CXIV.

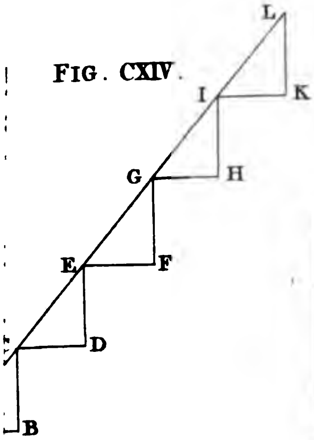


FIG. CXV.

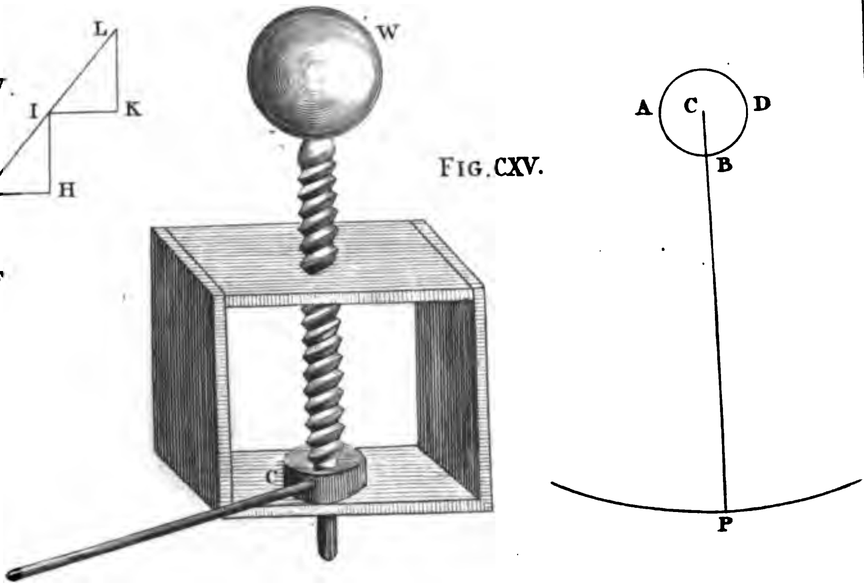


FIG. CXVI.

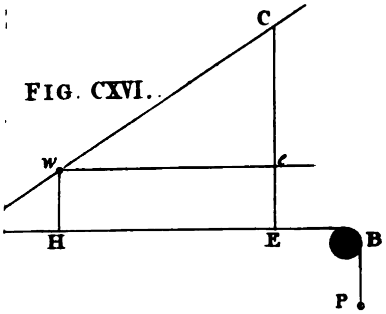


FIG. CXVII.

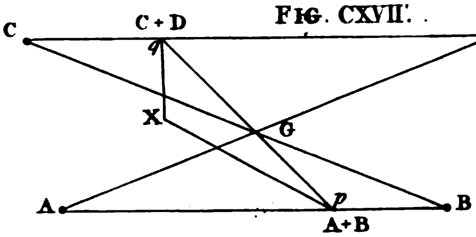


FIG. CXVIII.

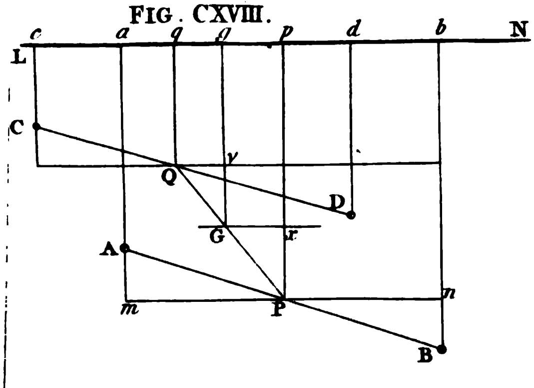
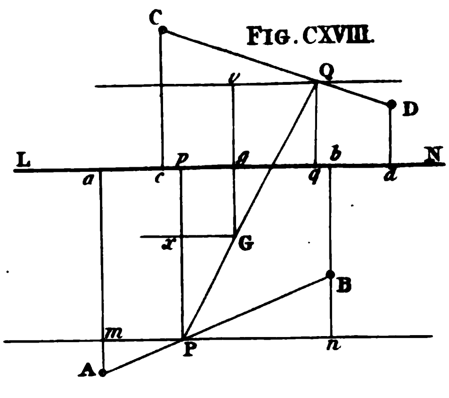


FIG. CXVIII.



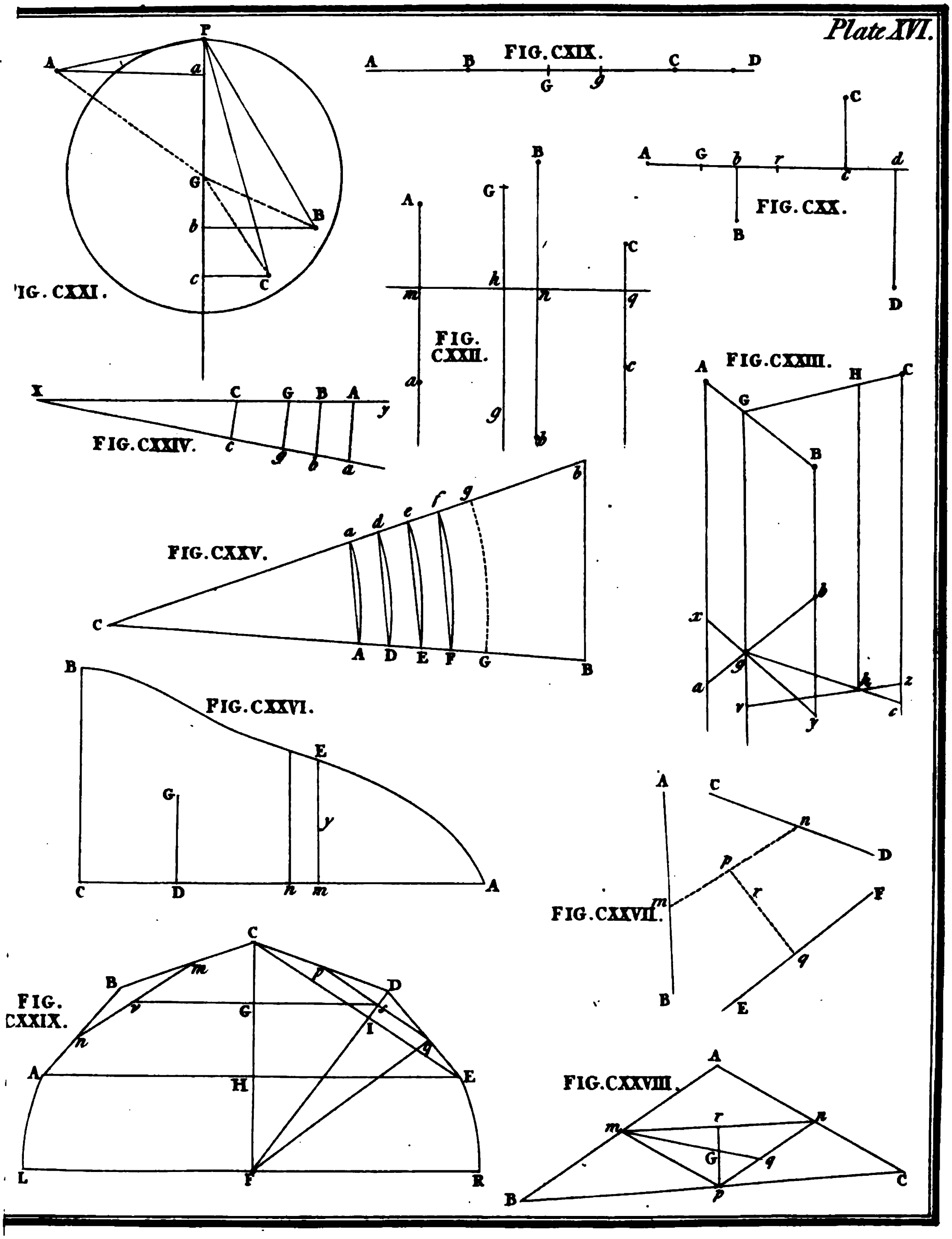


Plate XVI.
FIG. CXIX.
FIG. CXX.
FIG. CXXI.
FIG. CXXII.
FIG. CXXIII.
FIG. CXXIV.
FIG. CXXV.
FIG. CXXVI.
FIG. CXXVII.
FIG. CXXVIII.
FIG. CXXIX.

Plate XVII.

FIG. CXXX.

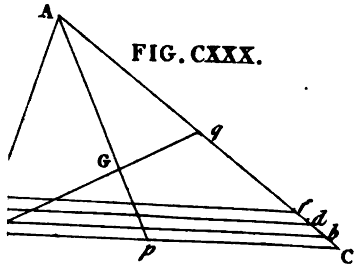
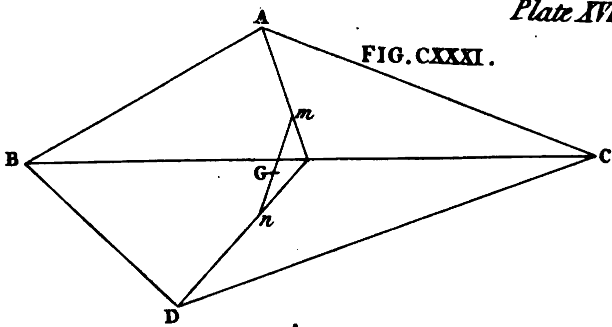
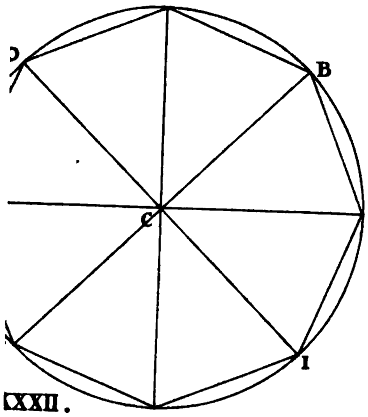


FIG. CXXXI.





XXXII.

FIG. CXXXIII

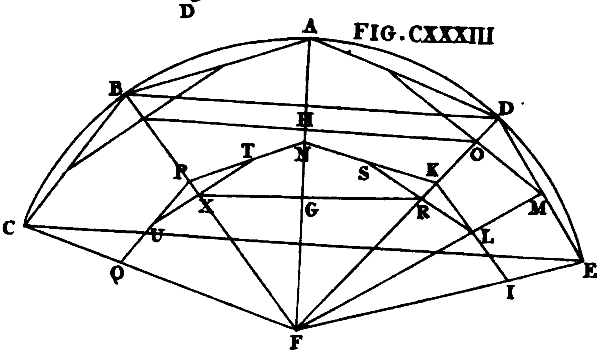


FIG. CXXXIV.

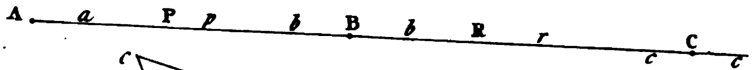
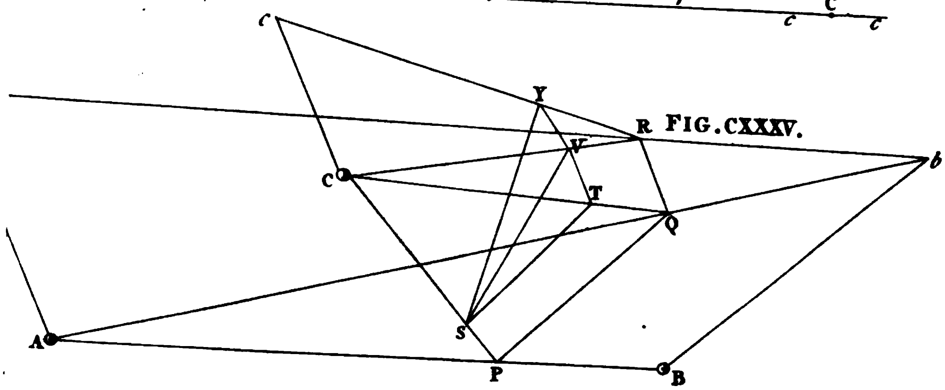
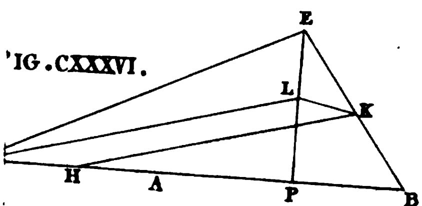
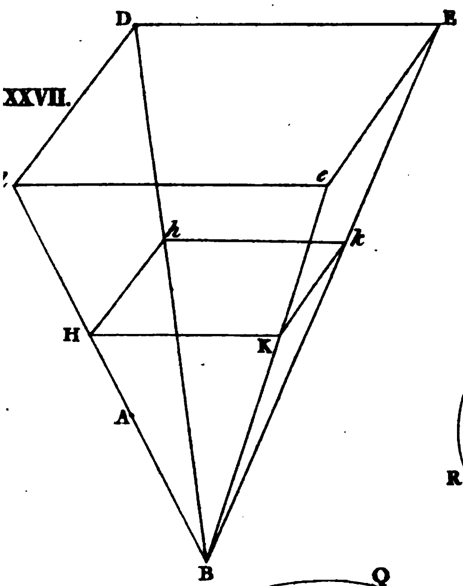


FIG. CXXXV.



IG. CXXXVI.





XXVII.

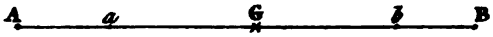


FIG. CXXXVIII.

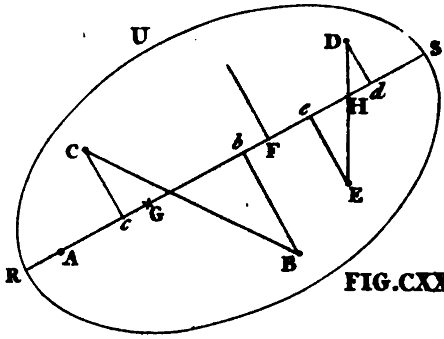
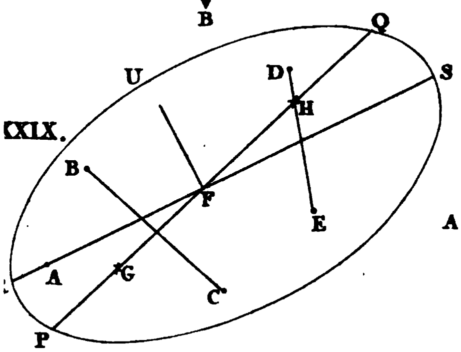


FIG. CXXXIX.



XIX.

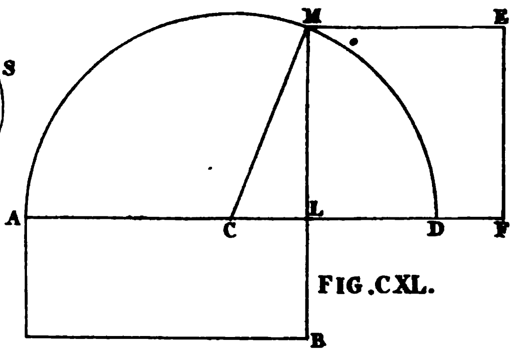
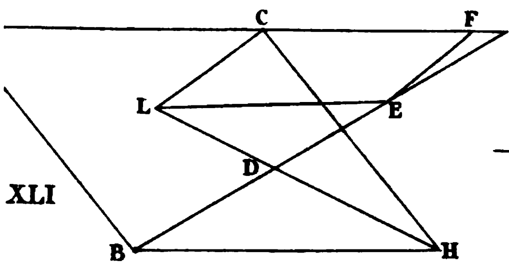


FIG. CXL.



XLI

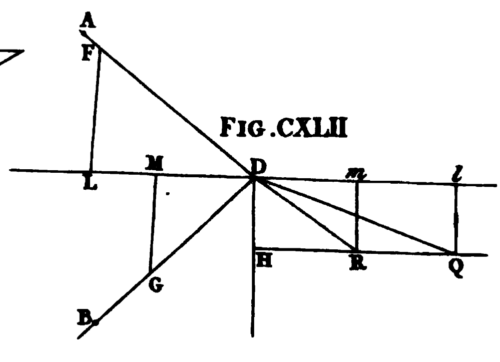


FIG. CXLII

FIG. CXLIII.

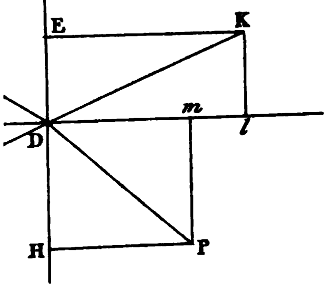
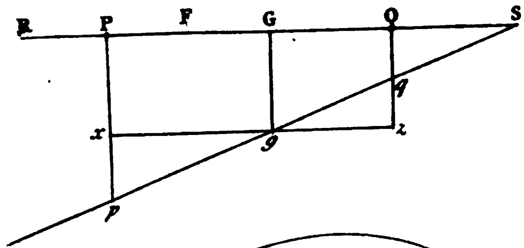
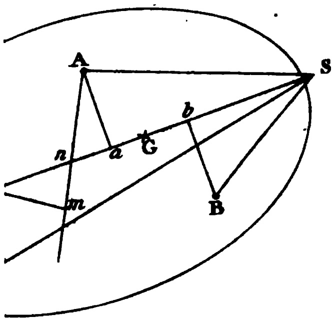


FIG. CXLIV.





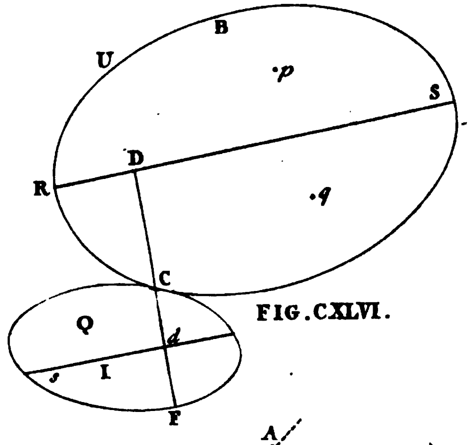
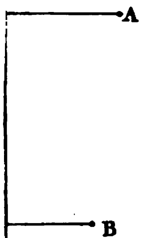


FIG. CXLVI.



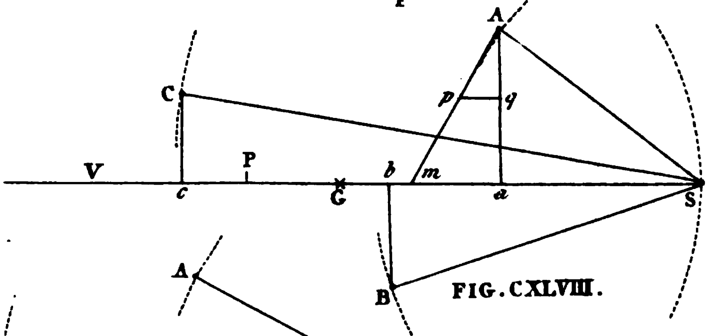


FIG. CXLVIII.

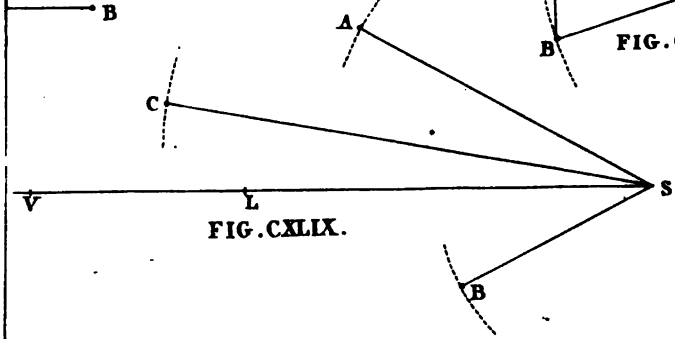


FIG. CXLIX.

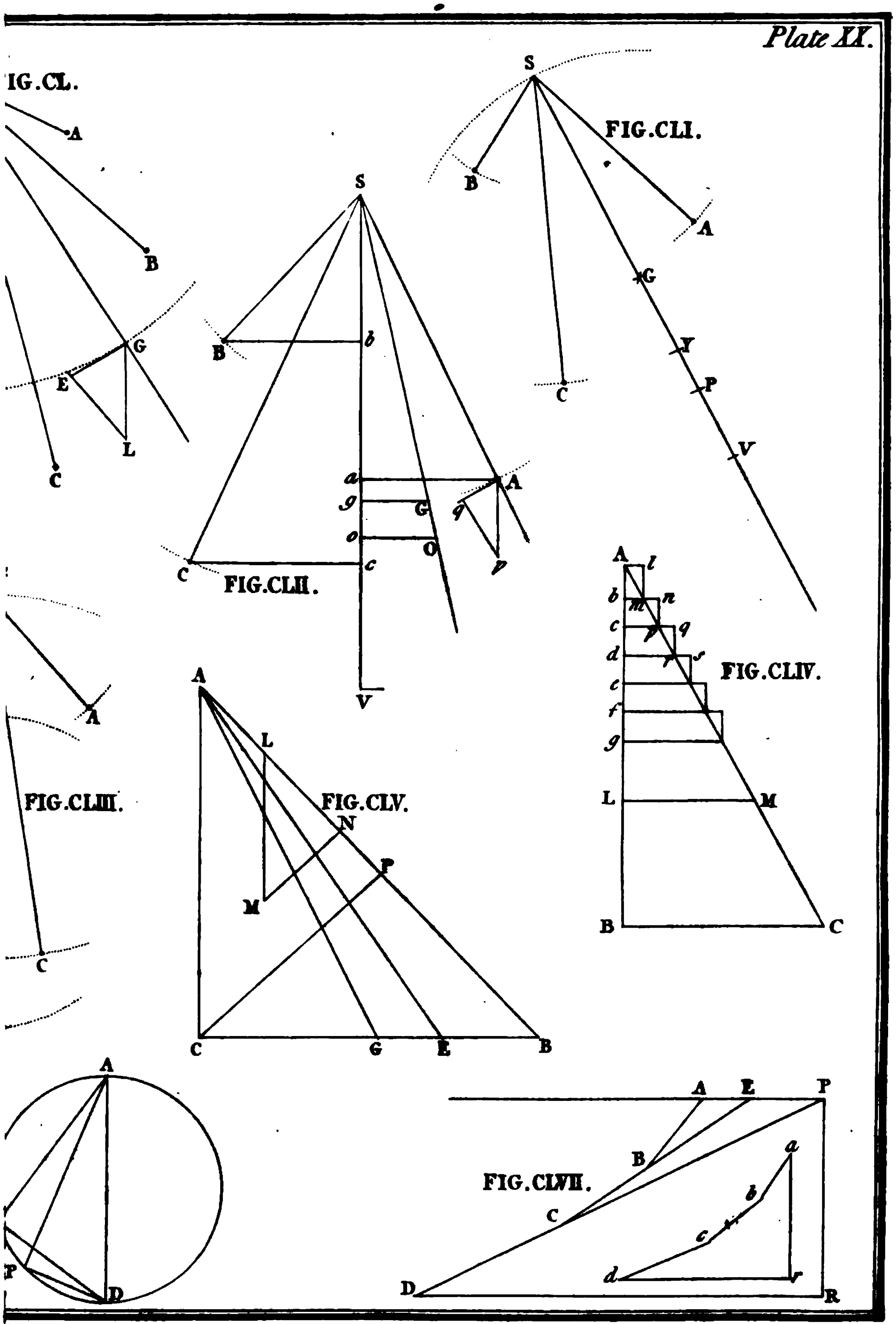


Plate XX.
IG.CL.
FIG.CLI.
FIG.CLII.
FIG.CLIII.
FIG.CLIV.
FIG.CLV.
FIG.CLVII.

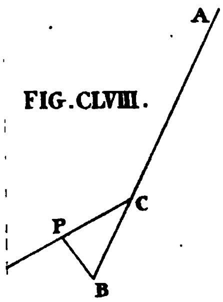


FIG. CLVIII.

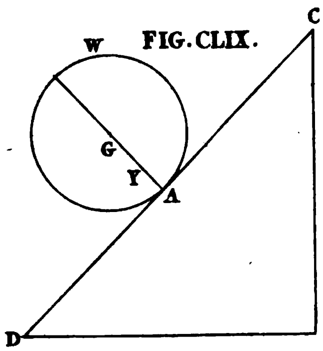


FIG. CLIX.

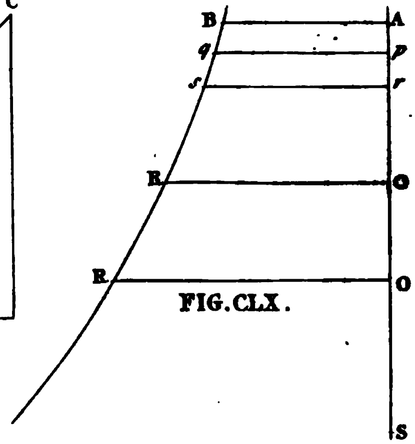


FIG. CLX.

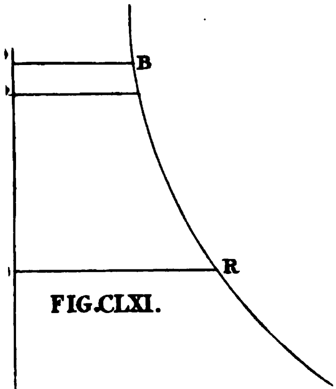


FIG. CLXI.

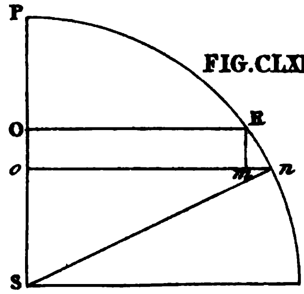


FIG. CLXII.

FIG. CLXIV.

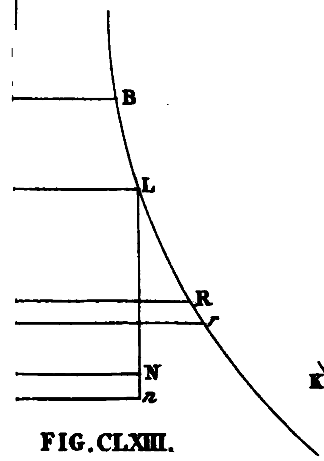


FIG. CLXIII.

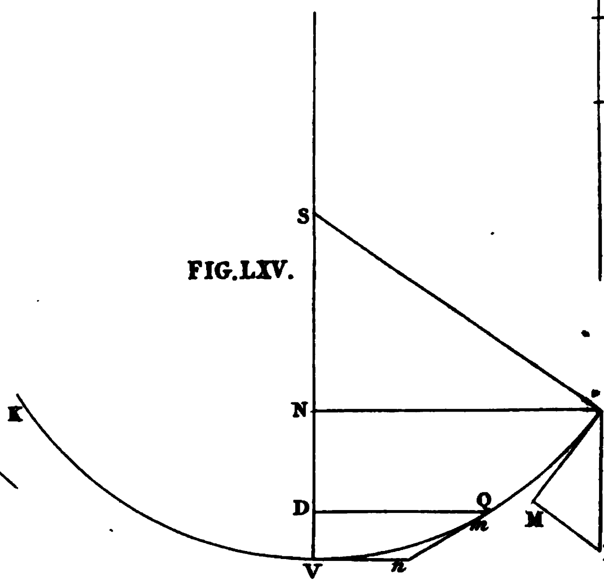


FIG. LXV.

Plate XXII.

FIG. CLXVI.

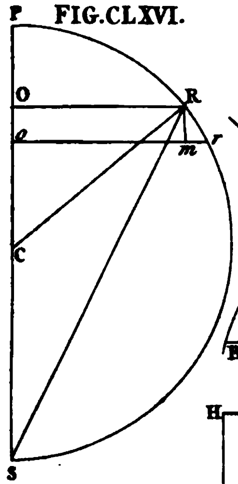


FIG. CLXVII.

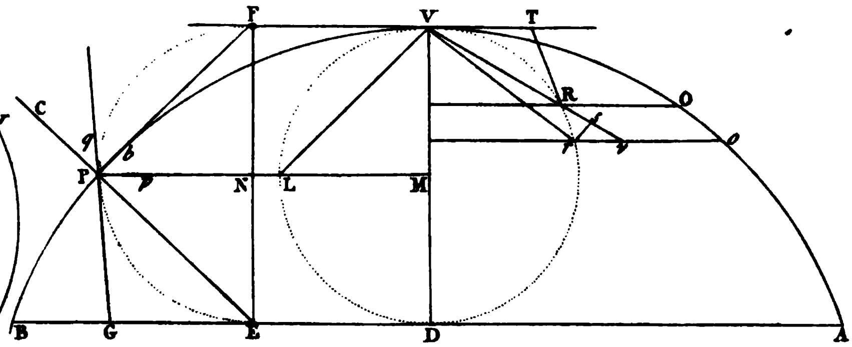


FIG. CLXVIII.

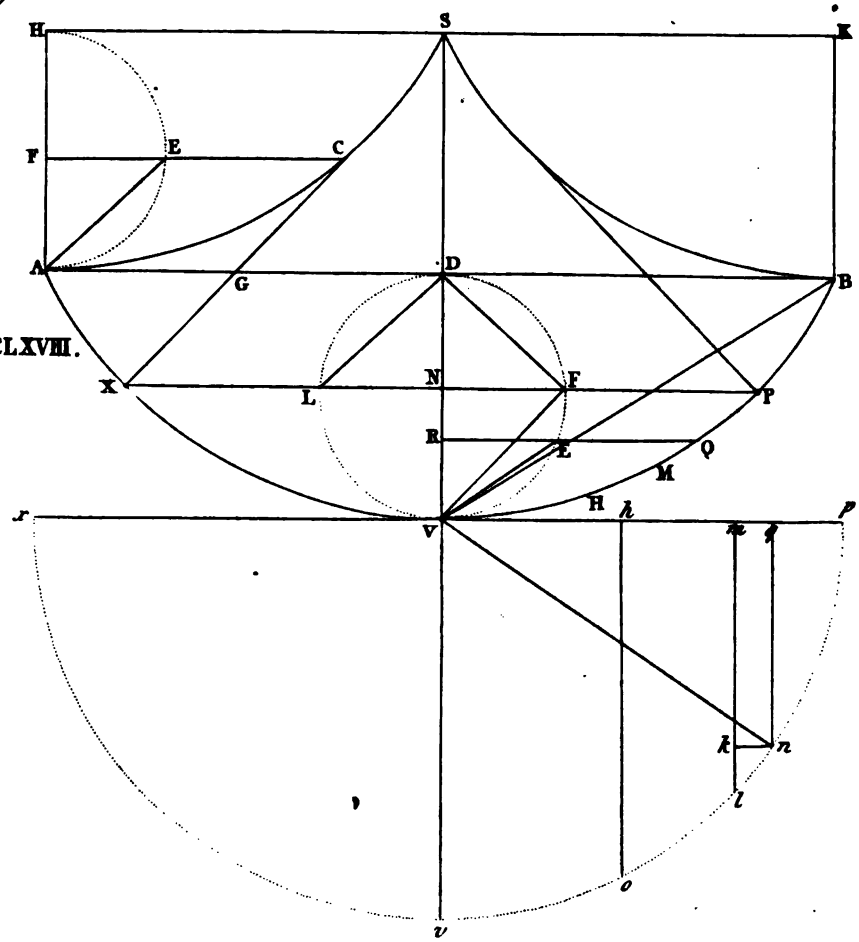


Plate XXIII.

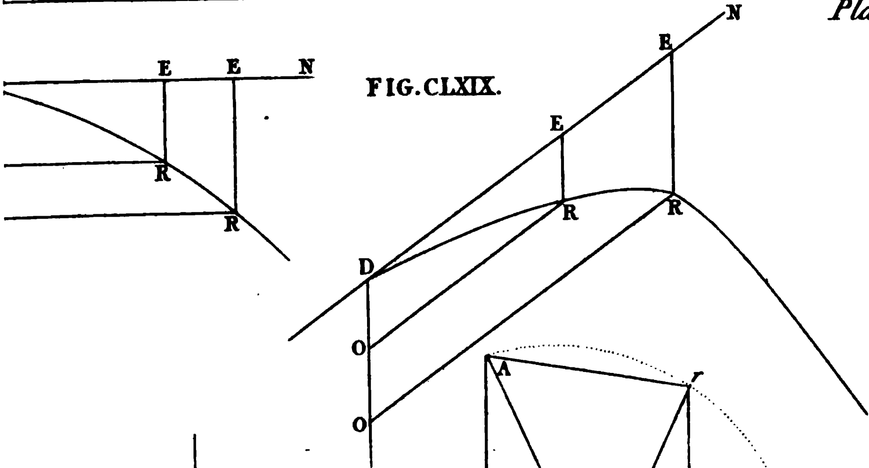
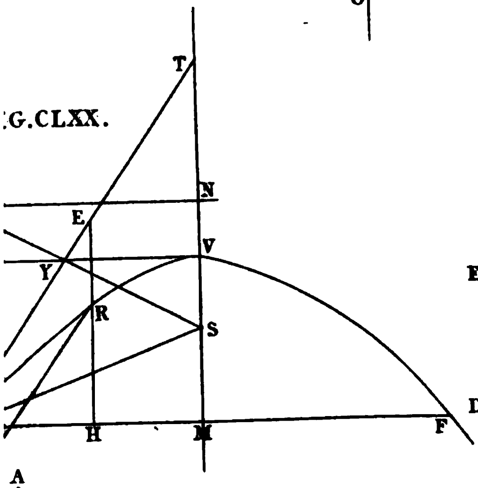


FIG.CLXIX.



:G.CLXX.

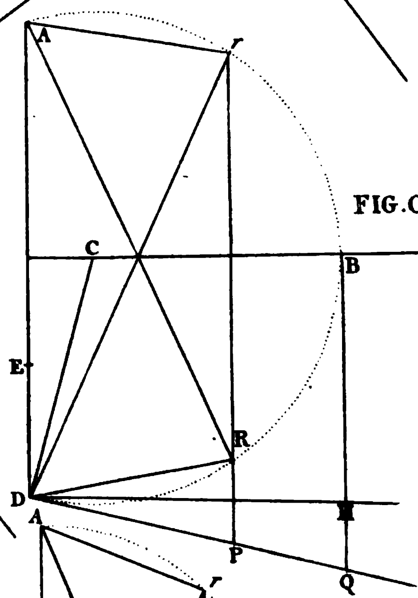


FIG.CLXXI.

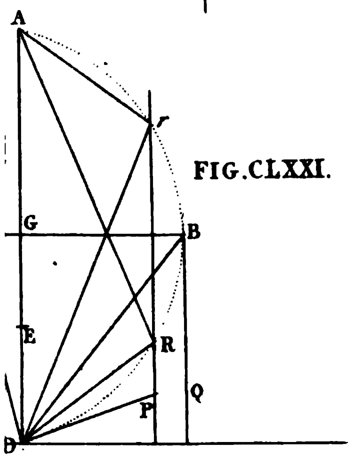


FIG.CLXXI.

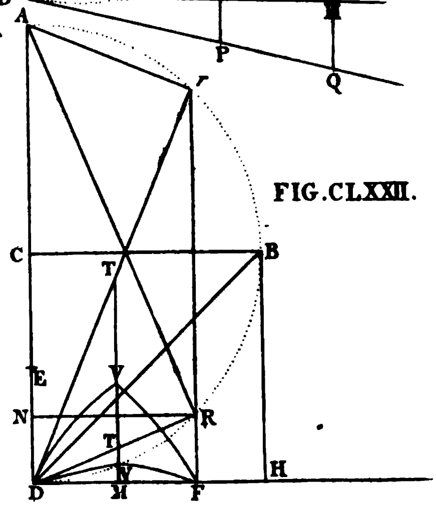


FIG.CLXXII.

Plate XXIV.

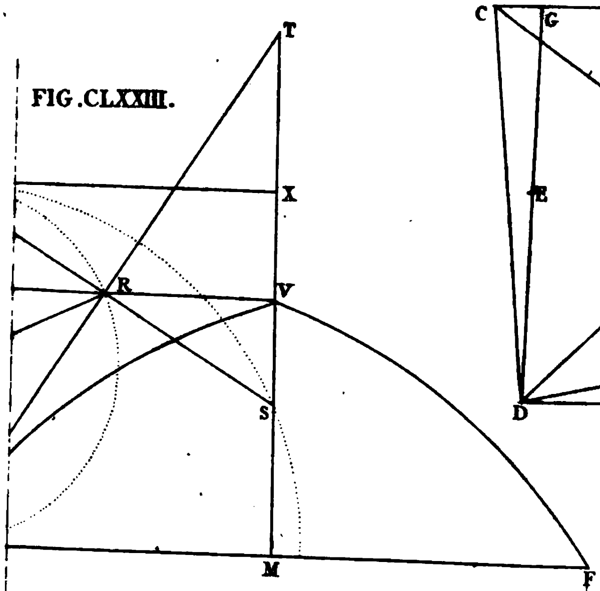


FIG. CLXXIII.

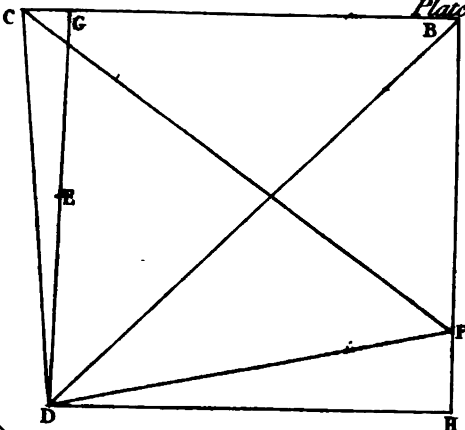


FIG. CLXXIV.

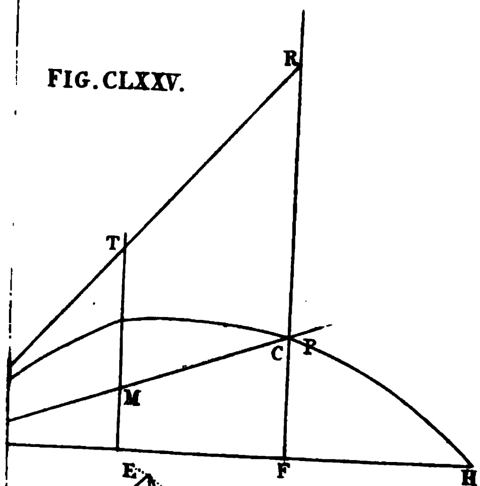


FIG. CLXXV.

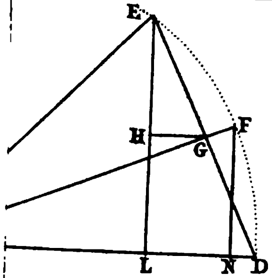


FIG. CLXXVI.

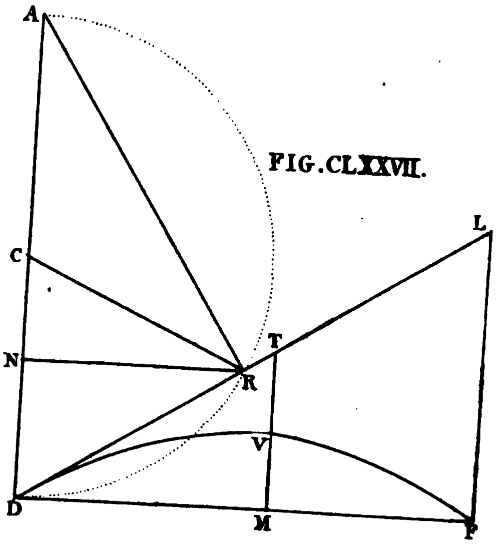


FIG. CLXXVII.