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E. L. T[illegible], M. D.

A TREATISE OF ALGEBRA, IN THREE PARTS.

CONTAINING

I. The fundamental RULES and OPERATIONS.

II. The COMPOSITION and RESOLUTION of EQUATIONS of all DEGREES; and the different AFFECTIONS of their ROOTS.

III. The APPLICATION of ALGEBRA and GEOMETRY to each other.

To which is added, An

APPENDIX,

Concerning the general PROPERTIES of GEOMETRICAL LINES.

By COLIN MACLAURIN, M. A.

Late Professor of MATHEMATICS in the University of EDINBURGH, and Fellow of the ROYAL SOCIETY.

The FOURTH EDITION.

LONDON,

Printed for J. NOURSE, W. STRAHAN, J. F. and C. RIVINGTONS, W. JOHNSTON, T. LONGMAN, G. ROBINSON, and T. CADELL. 1779.

MU

TO

HIS GRACE

THOMAS,

LORD ARCHBISHOP

OF

CANTERBURY,

Primate of all ENGLAND:

As a testimony of GRATITUDE for the friendship and generous protection, with which HIS GRACE was pleased to honour my deceased husband,

THIS

TREATISE

IS INSCRIBED

BY HIS GRACE'S

MOST OBLIGED

AND

OBEDIENT HUMBLE SERVANT,

ANNE MACLAURIN.

TO THE

READER.

THAT Mr. MACLAURIN *had, many years ago, intended to publish a Treatise on this subject, appears from a letter, dated April* 19, 1729, *to his honoured friend* MARTIN FOLKES, *Esq. now President of the Royal Society* *. *And we find, in one of his manuscripts, the plan of such a Work, agreeing, almost in every article, with the contents of this Volume.*

Had the celebrated Author lived to publish his own Work, his name would, alone, have been sufficient to recommend it to the notice of the Publick: but that Task having, by his lamented premature death, devolved to the gentlemen whom he left entrusted with his Papers, the Reader may reasonably expect some account of the materials of which it consists, and of the care that has been taken in collecting and disposing them, so as best to answer the Author's intention, and fill up the Plan he had designed.

* Phil. Trans. N° 408.

He seems, in composing this Treatise, to have had these three Objects in view.

1. *To give the general* Principles *and* Rules *of the Science, in the shortest, and, at the same time, the most clear and comprehensive manner that was possible. Agreeable to this, though every Rule is properly exemplified, yet he does not launch out into what we may call, a* Tautology *of examples. He rejects some applications of* Algebra, *that are commonly to be met with in other writers; because the number of such applications is endless: and, however useful they may be in Practice, they cannot, by the rules of good* method, *have place in an* elementary *Treatise. He has likewise omitted the Algebraical solution of* particular *Geometrical problems, as requiring the knowledge of the Elements of* Geometry; *from which those of* Algebra *ought to be kept, as they really are, entirely distinct; reserving to himself to treat of the mutual relation of the two Sciences in his* Third Part, *and, more generally still, in the* Appendix. *He might think too, that such an application was the less necessary, that* Sir ISAAC NEWTON's *excellent Collection of Examples is in every body's hands, and that there are few Mathematical writers,*

who

who do not furnish numbers of the same kind.

2. *Sir* ISAAC NEWTON'*s Rules, in his* Arithmetica Universalis, *concerning the Resolutions of the higher equations, and the Affections of their roots, being, for the most part, delivered without any demonstration, Mr.* MACLAURIN *had designed, that his Treatise should serve as a Commentary on that Work. For we here find all those difficult passages in Sir* ISAAC'*s Book, which have so long perplexed the Students of Algebra, clearly explained and demonstrated. How much such a Commentary was wanted, we may learn from the words of a late eminent Author*.* "The ablest Mathematicians of the last age " *(says he)* did not disdain to write Notes on " the Geometry of DES CARTES; and surely Sir ISAAC NEWTON's Arithmetick no " less deserves that honour. To excite some " one of the many skilful Hands that our " times afford to undertake this Work, and " to shew the necessity of it, I give this " *Specimen*, in an explication of two pas-

* *s'Gravesande*, in Præfat. ad Specimen Comment. in Arith. Univers.

"sages * of the *Arithmetica Universalis*; "which, however, are not the most difficult in that Book."

What this learned Professor so earnestly wished for, we at last see executed; not separately, nor in the loose disagreeable form which such Commentaries generally take, but in a manner equally natural and convenient; every Demonstration being aptly inserted into the Body of the Work, as a necessary and inseparable Member; an Advantage which, with some others, obvious enough to an attentive Reader, will, it is hoped, distinguish this Performance from every other, of the kind, that has hitherto appeared.

3. *After having fully explained the Nature of* Equations, *and the Methods of finding their Roots, either infinite expressions, when it can be done, or in infinite* converging series; *it remained only to consider the Relation of* Equations *involving two variable quantities, and of Geometrical* Lines *to each other; the Doctrine of the* Loci; *and the* Construction *of Equations. These make the Subject of the* Third Part.

* *Viz.* The finding of *Divisors*, and the evolution of *Binomial Surds*. See § 59—72. Part II. § 127. Part I.

Upon

Upon this Plan Mr. MACLAURIN *composed a system of* Algebra, *soon after his being chosen Professor of Mathematicks in the University of* Edinburgh; *which he, thenceforth, made use of in his ordinary Course of Lectures, and was occasionally improving to the Perfection he intended it should have, before he committed it to the Press. And the best Copies of his Manuscript having been transmitted to the Publisher, it was easy, by comparing them, to establish a correct and genuine Text. There were, besides, several detached Papers, some of which were quite finished, and wanted only to be inserted in their proper places. In a few others, the Demonstrations were so concisely expressed, and couched in Algebraical characters, that it was necessary to write them out at more length, to make them of a piece with the rest. And this is the only liberty the Publisher has allowed himself to take; excepting a few inconsiderable additions, that seemed necessary to render the Book more complete within itself, and to save the trouble of consulting others who have written on the same Subject.*

The Rules concerning the Impossible roots *of Equations, our Author had very fully considered, as appears from his Manuscript papers: but as he had no-where reduced any thing on that Subject to a better form, than what was*

long

long ago published in the Philosophical Transactions, N° 394, *and* 408. *we thought it best to take the substance of* Chap. 11. Part II. *from thence; especially as the latter of these Papers furnishes a demonstration of the original Rule, which pre-supposes only what the Reader has been taught in a preceding Chapter.*

The Paper that is subjoined, on the Sums of the Powers of the Roots of an Equation, *is taken from a Letter of the Author* (8 Jul. 1743) *to the Right Honourable the Earl* STANHOPE; *communicated to the Publisher, with some things added by his Lordship, which were wanting to finish the Demonstration.*

Of these Materials, carefully collected and put in order, the following Elementary Treatise *is composed; which we have chosen rather to give in a Volume that is within the reach of every Student, than in one more pompous, which might be less generally useful. And we hope, from the pains it has cost us, its blemishes are not many, nor such as a candid intelligent Reader may not forgive.*

The Latin Appendix *is a proper Sequel, and a high Improvement, of what had been demonstrated in* Part III. *concerning the Relation of* Curve lines *and* Equations; *a Subject which*

* A translation of which is now given to this edition, by the Rev. Mr. Lawson.

our

our Author had been early and intimately acquainted with; witness his Geometria Organica, *printed in* 1719, *when he was not full twenty-one years of age, and which, though so juvenile a work, gained him, at once, that distinguished Rank among Mathematicians, which he thenceforth held with great lustre. Yet he frankly owns, he was led to many of the Propositions in this* Appendix, *from a Theorem of Mr.* COTES, *communicated to him, without any demonstration, by the Reverend and Learned Dr.* SMITH, *Master of* Trinity-College, Cambridge. *How he has profited of that Hint, the Learned will judge: Thus much we can venture to say, that he himself set some value upon this Performance; having, we are told, employed some of the latest hours he could give to such Studies, in revising it for the Press; to bequeath it as his last Legacy to the Sciences and to the Publick.*

The gentlemen to whom Mr. MACLAURIN *left the care of his Papers, are* MARTIN FOLKES, *Esq. President of the Royal Society;* ANDREW MITCHEL, *Esq. and the Reverend Mr.* HILL, *Chaplain to his Grace the Archbishop of* Canterbury; *with whom he had lived in a most intimate friendship. And by their direction this Treatise is published.*

CON-

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PART II.

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APPENDIX.

A

TREATISE

OF

ALGEBRA.

PART I.

CHAP. I.

Definitions and Illustrations.

§ 1. ALGEBRA is a general method of computation by certain signs and symbols which have been contrived for this purpose, and found convenient. It is called an UNIVERSAL ARITHMETICK, and proceeds by operations and rules similar to those in common arithmetick, founded upon the same principles. This, however, is no argument against its usefulness or evidence; since arithmetick is not to be the less valued

valued that it is common, and is allowed to be one of the most clear and evident of the sciences. But as a number of symbols are admitted into this science, being necessary for giving it that extent and generality which is its greatest excellence; the import of those symbols is to be clearly stated, that no obscurity or error may arise from the frequent use and complication of them.

§ 2. In Geometry, lines are represented by a line, triangles by a triangle, and other figures by a figure of the same kind; but, in Algebra, quantities are represented by the same letters of the alphabet; and various signs have been imagined for representing their affections, relations, and dependencies. In Geometry the representations are more natural, in Algebra more arbitrary: the former are like the first attempts towards the expression of objects, which was by drawing their resemblances; the latter correspond more to the present use of languages and writing. Thus the evidence of Geometry is sometimes more simple and obvious; but the use of Algebra more extensive, and often more ready: especially since the mathematical sciences have acquired so vast an extent, and have been applied to so many enquiries.

§ 3. In those sciences, it is not barely magnitude that is the object of contemplation: but there

there are many affections and properties of quantities, and operations to be performed upon them, that are neceſſarily to be conſidered. In eſtimating the ratio or proportion of quantities, magnitude only is conſidered (*Elem.* 5. *Def.* 3.) But the nature and properties of figures depend on the poſition of the lines that bound them; as well as on their magnitude. In treating of motion, the direction of motion as well as its velocity; and the direction of powers that generate or deſtroy motion, as well as their forces, muſt be regarded. In OPTICS, the poſition, brightneſs, and diſtinctneſs of images, are of no leſs importance than their bigneſs; and the like is to be ſaid of other ſciences. It is neceſſary therefore that other ſymbols be admitted into Algebra beſide the letters and numbers which repreſent the magnitude of quantities.

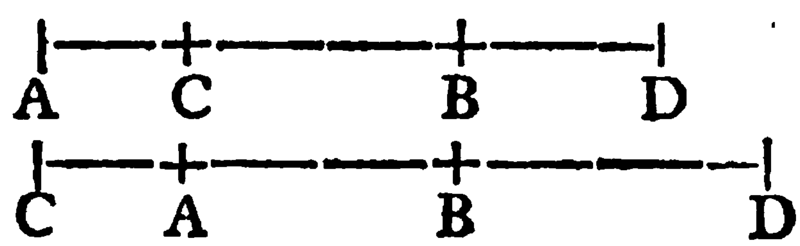
§ 4. The relation of equality is expreſſed by the ſign $=$; thus to expreſs that the quantity repreſented by a is equal to that which is repreſented by b, we write $a = b$. But if we would expreſs that a is greater than b, we write $a \sqsubset b$; and if we would expreſs algebraically that a is leſs than b, we write $a \sqsupset b$.

§ 5. QUANTITY is what is made up of parts, or is capable of being greater or leſs. It is increaſed by *Addition*, and diminiſhed by *Subtraction*; which are therefore the two primary operations that relate to quantity. Hence it is, that any quantity may be ſuppoſed to enter into algebraic computations two different ways which have contrary effects; either as an *increment* or as a *decrement*; that is, as a quantity to be added or as a quantity to be ſubtracted. The ſign $+$ (*plus*) is the mark of *Addition*, and the ſign $-$ (*minus*) of *Subtraction*. Thus the quantity being repreſented by a, $+a$ imports that a is to be added, or repreſents an increment; but $-a$ imports that a is to be ſubtracted, and repreſents a decrement. When ſeveral ſuch quantities are joined, the ſigns ſerve to ſhew which are to be added, and which are to be ſubtracted. Thus $+a+b$ denotes the quantity that ariſes when a and b are both conſidered as increments, and therefore expreſſes the ſum of a and b. But $+a-b$ denotes the quantity that ariſes when from the quantity a the quantity b is ſubtracted; and expreſſes the exceſs of a above b. When a is greater than b, then $a-b$ is itſelf an increment; when $a=b$, then $a-b=0$; and when a is leſs than b, then $a-b$ is itſelf a decrement.

§ 6. As addition and ſubtraction are oppoſite, or an increment is oppoſite to a decrement, there is an analogous oppoſition between the affections of quantities that are conſidered in the mathematical ſciences. As between exceſs and defect; between the value of effects

or

or money due to a man, and money due by him; a line drawn towards the right, and a line drawn to the left; gravity and levity; elevation above the horizon, and depression below it. When two quantities equal in respect of magnitude, but of those opposite kinds, are joined together, and conceived to take place in the same subject, they destroy each other's effect, and their amount is *nothing*. Thus 100 *l.* due to a man and 100 *l.* due by him balance each other, and in estimating his stock may be both neglected. Power is sustained by an equal power acting on the same body with a contrary direction, and neither have effect. When two unequal quantities of those opposite qualities are joined in the same subject, the greater prevails by their difference. And when a greater quantity is taken from a lesser of the same kind, the remainder becomes of the opposite kind. Thus if we add the lines AB and BD together, their sum is AD; but if we are to subtract BD from AB,



then BC = BD is to be taken the contrary way towards A, and the remainder is AC; which, when BD, or BC exceeds AB, becomes a line on the other side of A. When two powers or forces are to be added together, their sum acts upon

upon the body: but when we are to ſubtract one of them from the other, we conceive that which is to be ſubtracted to be a power with an oppoſite direction; and if it be greater than the other, it will prevail by the difference. This change of quality however only takes place where the quantity is of ſuch a nature as to admit of ſuch a contrariety or oppoſition. We know nothing analogous to it in quantity abſtractly conſidered; and cannot ſubtract a greater quantity of matter from a leſſer, or a greater quantity of light from a leſſer. And the application of this doctrine to any art or ſcience is to be derived from the known principles of the ſcience.

§ 7. A quantity that is to be added is likewiſe called a *poſitive* quantity; and a quantity to be ſubtracted is ſaid to be *negative:* they are equally real, but oppoſite to each other, ſo as to take away each other's effect, in any operation, when they are equal as to quantity. Thus $3-3=0$, and $a-a=0$. But though $+a$ and $-a$ are equal as to quantity, we do not ſuppoſe in Algebra that $+a=-a$; becauſe to infer equality in this ſcience, they muſt not only be equal as to quantity, but of the ſame quality, that in every operation the one may have the ſame effect as the other. A decrement may be equal to an increment, but it has in all operations a contrary effect; a motion down-

wards

wards may be equal to a motion upwards, and the depreſſion of a ſtar below the horizon may be equal to the elevation of a ſtar above it; but thoſe poſitions are oppoſite, and the diſtance of the ſtars is greater than if one of them was at the horizon ſo as to have no elevation above it, or depreſſion below it. It is on account of this contrariety that a negative quantity is ſaid to be leſs than nothing, becauſe it is oppoſite to the poſitive, and diminiſhes it when joined to it. whereas the addition of o has no effect. But a negative is to be conſidered no leſs as a real quantity than the poſitive. Quantities that have no ſign prefixed to them are underſtood to be poſitive.

§ 8. The number prefixed to a letter is called the numeral *coefficient*, and ſhews how often the quantity repreſented by the letter is to be taken. Thus $2a$ imports that the quantity repreſented by a is to be taken twice; $3a$ that it is to be taken thrice; and ſo on. When no number is prefixed, *unit* is underſtood to be the coefficient. Thus 1 is the coefficient of a or of b.

Quantities are ſaid to be *like* or *ſimilar*, that are repreſented by the ſame letter or letters equally repeated. Thus $+3a$ and $-5a$ are like; but a and b, or a and aa are unlike.

A quantity is ſaid to conſiſt of as many *terms* as there are parts joined by the ſigns + or or $-$; thus $a+b$ confifts of two terms, and is called a *binomial*; $a+b+c$ confifts of three terms, and is called a *trinomial*. Thefe are called *compound* quantities: a *fimple* quantity, confifts of one term only, as $+a$, or $+ab$, or $+abc$.

The other fymbols and definitions neceffary in *Algebra* fhall be explained in their proper places.

CHAP. II.

Of ADDITION.

§ 9. CASE I. To add quantities that are like and have like figns.

Rule. *Add together the coefficients, to their fum prefix the common fign, and fubjoin the common letter or letters.*

EXAMPLES.

To	$+5a$	to	$-6b$	to	$a+b$
Add	$+4a$	add	$-2b$	add	$3a+5b$
Sum	$+9a$	Sum	$-8b$	Sum	$4a+6b$

To

To	$3a - 4x$
Add	$5a - 8x$
Sum	$8a - 12x$

Cafe II. To add quantities that are like but have unlike figns.

Rule. *Subtract the leffer coefficient from the greater, prefix the fign of the greater to the remainder, and fubjoin the common letter or letters.*

EXAMPLES.

To	$-4a$	$+5b-6c$
Add	$+7a$	$-3b+8c$
Sum	$+3a$	$2b+2c$

To	$a+6x-5y+8$	$2a-2b$
Add	$-5a-4x+4y-3$	$-2a+2b$
Sum	$-4a+2x-y+5$	$0 \,.\,.\, 0$

This rule is eafily deduced from the nature of pofitive and negative quantities.

If there are more than two quantities to be added together, firft add the pofitive together into one fum, and then the negative (by *Cafe* I.) Then add thefe two fums together (by *Cafe* II.)

EXAMPLE.

$$\begin{array}{lr} & +\ 8a \\ & -\ 7a \\ & +10a \\ & -12a \\ \hline \text{Sum of the positive} \ldots & +18a \\ \text{Sum of the negative} \ldots & -19a \\ \hline \text{Sum of all} \ldots\ldots & -\ a \end{array}$$

Case III. To add quantities that are unlike.

Rule. *Set them all down one after another, with their signs and coefficients prefixed.*

EXAMPLES.

$$\begin{array}{ll|l} \text{To} & +2a & +3a \\ \text{Add} & +3b & -4x \\ \hline \text{Sum} & 2a+3b & 3a-4x \end{array}$$

$$\begin{array}{ll} \text{To} & 4a+4b+3c \\ \text{Add} & -4x-4y+3z \\ \hline \text{Sum} & 4a+4b+3c-4x-4y+3z \end{array}$$

CHAP.

CHAP. III.

Of SUBTRACTION.

§ 10. GEneral rule. *Change the signs of the quantity to be subtracted into their contrary signs, and then add it so changed to the quantity from which it was to be subtracted (by the rules of the last chapter:) the sum arising by this addition is the remainder.* For, to subtract any quantity, either positive or negative, is the same as to add the opposite kind.

EXAMPLES.

From	$+5a$	$8a-7b$
Subtract	$+3a$	$3a+4b$
Remaind.	$5a-3a$, or $2a$	$5a-11b$

From	$2a-3x+5y-6$
Subtract	$6a+4x+5y+4$
Remaind.	$-4a-7x \cdot 0-10$

It is evident, that to subtract or take away a decrement is the same as adding an equal increment. If we take away $-b$ from $a-b$, there remains a; and if we add $+b$ to $a-b$, the sum is likewise a. In general, the subtraction of a negative quantity is equivalent to adding its positive value.

CHAP.

CHAP. IV.

Of Multiplication.

§ 11. IN Multiplication the General rule for the sign is, That *when the signs of the factors are like* (i. e. both $+$, or both $-$,) *the sign of the product is* $+$; but *when the signs of the factors are unlike, the sign of the product is* $-$.

Case I. When any positive quantity, $+a$, is multiplied by any positive number, $+n$, the meaning is, That $+a$ is to be taken as many times as there are units in n; and the product is evidently na.

Case II. When $-a$ is multiplied by n, then $-a$ is to be taken as often as there are units in n, and the product must be $-na$.

Case III. Multiplication by a positive number implies a repeated addition: but multiplication by a negative implies a repeated subtraction. And when $+a$ is to be multiplied by $-n$, the meaning is, That $+a$ is to be subtracted as often as there are units in n: therefore

fore the product is negative, being $-na$.

Case IV. When $-a$ is to be multiplied by $-n$, then $-a$ is to be subtracted as often as there are unites in n; but (by § 10.) to subtract $-a$ is equivalent to adding $+a$, consequently the product is $+na$.

The IId and IVth cases may be illustrated in the following manner.

By the definitions, $+a-a=0$; therefore, if we multiply $+a-a$ by n, the product must vanish or be 0, because the factor $a-a$ is 0. The first term of the product is $+na$ (by Case I.) Therefore the second term of the product must be $-na$, which destroys $+na$; so that the whole product must be $+na-na=0$. Therefore $-a$ multiplied by $+n$ gives $-na$.

In like manner, if we multiply $+a-a$ by $-n$, the first term of the product being $-na$, the latter term of the product must be $+na$, because the two together must destroy each other, or their amount be 0, since one of the factors (*viz* $a-a$) is 0. Therefore $-a$ multiplied by $-n$ must give $+na$.

In this general doctrine the multiplicator is always considered as a number. A quantity of any kind may be multiplied by a number: but a pound a pound is not to be multiplied by a pound, or a debt by a debt, or a line by a line. We shall afterwards consider the analogy there is betwixt rectangles in Geometry and a product of two factors.

§ 12. If the quantities to be multiplied are *simple* quantities, *find the sign of the product by the last rule; after it place the product of the coefficients, and then set down all the letters after one another as in one word.*

EXAMPLES.

Mult.	$+a$	$-2a$	$6x$
By	$+b$	$+4b$	$-5a$
Prod.	$+ab$	$-8ab$	$-30ax$

Mult.	$-8x$	$+3ab$
By	$-4a$	$-5ac$
Prod.	$+32ax$	$-15aabc$

§ 13. To multiply *compound* quantities, you must *multiply every part of the multiplicand by all the parts of the multiplier taken one after another, and then collect all the products into one sum: that sum shall be the product required.*

EX-

EXAMPLES.

$$\begin{array}{ll}
\text{Mult.} & a+b \\
\text{By} & a+b \\
\hline
\text{Prod.} & \left\{\begin{array}{l} aa+ab \\ \quad\;\, +ab+bb \end{array}\right. \\
\hline
\text{Sum} & aa+2ab+bb
\end{array}
\qquad\Bigg|\qquad
\begin{array}{l}
2a-3b \\
4a+5b \\
\hline
\left\{\begin{array}{l} 8aa-12ab \\ \qquad\;\, +10ab-15bb \end{array}\right. \\
\hline
8aa-2ab-15bb
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & 2a-4b \\
\text{By} & 2a+4b \\
\hline
\text{Prod.} & \left\{\begin{array}{l} 4aa-8ab \\ \qquad\;\, +8ab-16bb \end{array}\right. \\
\hline
\text{Sum} & 4aa\ldots 0-16bb
\end{array}
\qquad\Bigg|\qquad
\begin{array}{l}
xx-ax \\
x+a \\
\hline
\left\{\begin{array}{l} xxx-axx \\ \qquad\;\, +axx-aax \end{array}\right. \\
\hline
xxx\ldots 0-aax
\end{array}$$

$$\begin{array}{ll}
\text{Mult.} & aa+ab+bb \\
\text{By} & a-b \\
\hline
\text{Prod.} & \left\{\begin{array}{l} aaa+aab+abb \\ \qquad\;\, -aab-abb-bbb \end{array}\right. \\
\hline
\text{Sum} & aaa\ldots 0\ldots 0-bbb
\end{array}$$

§ 14. Products that arise from the multiplication of two, three, or more quantities, as *abc*, are said to be of two, three, or more *dimensions*; and those quantities are called *factors* or *roots*.

If

If all the factors are equal, then theſe products are called *powers*; as aa or aaa are powers of a. Powers are expreſſed ſometimes by placing above the root to the right-hand a figure expreſſing the number of factors that produce them. Thus,

$$\left.\begin{array}{l} a\\ aa\\ aaa\\ aaaa\\ aaaaa \end{array}\right\}\text{is called the}\left\{\begin{array}{l}1\text{ſt}\\ 2\text{d}\\ 3\text{d}\\ 4\text{th}\\ 5\text{th}\end{array}\right\}\begin{array}{l}\text{Power of the}\\ \text{root, } a \text{ and}\\ \text{is ſhortly}\\ \text{expreſſed}\\ \text{thus,}\end{array}\left\{\begin{array}{l}a\\ a^2\\ a^3\\ a^4\\ a^5\end{array}\right.$$

§ 15. Theſe figures which expreſs the number of factors that produce powers are called their *indices* or *exponents*; thus 2 is the index of a^2. And *powers of the ſame root are multiplied by adding their exponents.* Thus $a^2 \times a^3 = a^5$, $a^4 \times a^3 = a^7$, $a^3 \times a = a^4$.

§ 16. Sometimes it is uſeful not actually to multiply compound quantities, but to ſet them down with the ſign of multiplication ($\times$) between them, drawing a line over each of the compound factors. Thus $\overline{a+b} \times \overline{a-b}$ expreſſed the product of $a+b$ multiplied by $a-b$.

CHAP.

CHAP. V.

Of Division.

§ 17. THE same rule for the signs is to be observed in Division as in Multiplication; that is, *If the signs of the dividend and divisor are like, the sign of the quotient must be +; if they are unlike, the sign of the quotient must be −.* This will be easily deduced from the rule in Multiplication, if you consider that the quotient must be such a quantity as multiplied by the divisor shall give the dividend.

§ 18. The General rule in Division is, *to place the dividend above a small line, and the divisor under it, expunging any letters that may be found in all the quantities of the dividend and divisor, and dividing the coefficients of all the terms by any common measure.* Thus when you divide $10ab+15ac$ by $20ad$, expunging a out of all the terms, and dividing all the coefficients by 5, the quotient is $\frac{2b+3c}{4d}$; and

$$2b)\, ab+bb \left(\frac{a+b}{2}\right.$$

$$12ab)\, 30ax-54ay \left(\frac{5x-9y}{2b}\right.$$

$4aa)\,8ab+6ac\,(\frac{4b+3c}{2a}.$

And $2bc)\,5abc\,(\frac{5a}{2}.$

§ 19. *Powers of the same root are divided by subtracting their exponents as they are multiplied by adding them.* Thus if you divide a^5 by a^2, the quotient is a^{5-2} or a^3. And b^6 divided by b^4 gives b^{6-4} or b^2; and a^7b^5 divided by a^2b^3 gives a^5b^2 for the quotient.

§ 20. *If the quantity to be divided is compound, then you must range its parts according to the dimensions of some one of its letters, as in the following example.* In the dividend $a^2+2ab+b^2$, they are ranged according to the dimensions of a, the quantity a^2 where a is of two dimensions being placed first, $2ab$ where it is of one dimension next, and b^2, where a is not at all, being placed last. *The divisor $a+b$, must be ranged according to the dimensions of the same letters; then you are to divide the first term of the dividend by the first term of the divisor, and to set down the quotient, which, in this example, is a; then multiply this quotient by the whole divisor, and subtract the product from the dividend, and the remainder shall give a new dividend, which in this example is $ab+b^2$.*

$a+b$

$$\begin{array}{r|l} a+b) & a^2+2ab+b^2\,(a+b \\ & \underline{a^2+\ ab} \\ & \quad\ ab+b^2 \\ & \quad\ \underline{ab+b^2} \\ & \quad\ \ 0\ \ \ 0 \end{array}$$

Divide the first term of this new dividend by the first term of the divisor and set down the quotient (which in this example is b) with its proper sign. Then multiply the whole divisor by this part of the quotient, and subtract the product from the new dividend; and if there is no remainder, the division is finished: If there is a remainder, you are to proceed after the same manner till no remainder is left; or till it appear that there will be always some remainder.

Some Examples will illustrate this operation.

EXAMPLE I.

$$\begin{array}{r|l} a+b) & a^2-b^2\,(a-b \\ & \underline{a^2+ab} \\ & \quad -ab-b^2 \\ & \quad \underline{-ab-b^2} \\ & \quad\ \ \ 0\ \ \ \ 0 \end{array}$$

EXAMPLE II.

$$
\begin{array}{l}
a-b)\,aaa-3aab+3abb-bbb\,(aa-2ab+bb \\
\quad aaa-aab \\
\hline
\quad\quad -2aab+3abb-bbb \\
\quad\quad -2aab+2abb \\
\hline
\quad\quad\quad\quad\quad abb-bbb \\
\quad\quad\quad\quad\quad abb-bbb \\
\hline
\quad\quad\quad\quad\quad\ 0 \quad\quad 0
\end{array}
$$

EXAMPLE III.

$$
\begin{array}{l}
a-b)\,aaa-bbb\,(aa+ab+bb \\
\quad\quad aaa-aab \\
\hline
\quad\quad\quad\quad aab-bbb \\
\quad\quad\quad\quad aab-abb \\
\hline
\quad\quad\quad\quad\quad\quad abb-bbb \\
\quad\quad\quad\quad\quad\quad abb-bbb \\
\hline
\quad\quad\quad\quad\quad\quad\ 0 \quad\quad 0
\end{array}
$$

EX.

EXAMPLE IV.

$$
\begin{array}{rl}
3a-6)\ 6aaaa-96\ (2aaa+4aa+8a+16 & \\
\underline{6aaaa-12aaa} & \\
12aaa-96 & \\
\underline{12aaa-24aa} & \\
24aa-96 & \\
\underline{24aa-48a} & \\
48a-96 & \\
\underline{48a-96} & \\
0 \quad 0 &
\end{array}
$$

§ 21. It often happens that the operation may be continued without end, and then you have an *infinite Series* for the quotient; and *by comparing the first three or four terms you may find what law the terms observe; by which means, without any more division, you may continue the quotient as far as you please.* Thus, in dividing 1 by $1-a$, you find the quotient to be $1+a+aa+aaa+aaaa+$ &c. which Series can be continued as far as you please by adding the powers of a.

The operation is thus:

$$
\begin{array}{l}
1-a)\,1\,(1+a+aa+aaa\ \text{\&c.} \\
\underline{1-a} \\
\quad +a \\
\quad \underline{+a-aa} \\
\qquad\quad +aa \\
\qquad\quad \underline{+aa-aaa} \\
\qquad\qquad\quad +aaa \\
\qquad\qquad\quad \underline{+aaa-aaaa} \\
\qquad\qquad\qquad\quad +aaaa\ \text{\&c.}
\end{array}
$$

Another Example.

$$
\begin{array}{l}
a+x)\,aa+xx\,(a-x+\frac{2xx}{a}-\frac{2x^3}{a^2}+\frac{2x^4}{a^3}-\text{\&c.} \\
\quad \underline{aa+ax} \\
\qquad -ax+xx \\
\qquad \underline{-ax-xx} \\
\qquad\qquad +2xx \\
\qquad\qquad \underline{+2xx-\frac{2x^3}{a}} \\
\qquad\qquad\qquad -\frac{2x^3}{a} \\
\qquad\qquad\qquad \underline{-\frac{2x^3}{a}-\frac{2x^4}{a^2}} \\
\qquad\qquad\qquad\qquad +\frac{2x^4}{a^2} \\
\qquad\qquad\qquad\qquad \text{\&c.}
\end{array}
$$

In this last example the signs are alternately $+$ and $-$, the coefficient is constantly 2, after the first two terms, and the letters are the powers of x and a; so that the quotient may be continued as far as you please without any more division.

But in Division, after you come to a remainder of one term, as $2xx$ in the last example, it is commonly set down with the divisor under it, after the other terms, and these together give the quotient. Thus, the quotient in the last example is found to be $a - x + \frac{2x^2}{a+x}$. And $bb + ab$ divided by $b - a$ gives for the quotient $b + \frac{2ab}{b-a}$.

Note, The sign $\div$ placed between any two quantities, expresses the quotient of the former divided by the latter. Thus $\overline{a+b} \div \overline{a-x}$ is the quotient of $a + b$ divided by $a - x$.

CHAP. VI.

Of FRACTIONS.

§ 22. IN the last Chapter it was said that the quotient of any quantity a divided by b is expressed by placing a above a small line and b under it, thus, $\frac{a}{b}$. These quotients are also called *Fractions*; and the dividend or quantity placed above the line is called the *Numerator* of the fraction, and the divisor or quantity placed under the line is called the *Denominator*. Thus $\frac{2}{3}$ expresses the quotient of 2 divided by 3; and 2 is the numerator and 3 the denominator of the fraction.

§ 23. *If the numerator of a fraction is equal to the denominator, then the fraction is equal to unity. Thus* $\frac{a}{a}$ *and* $\frac{b}{b}$ *are equal to unit. If the numerator is greater than the denominator, then the fraction is greater than unit.* In both these cases, the fraction is called *improper*. But *if the numerator is less than the denominator, then the fraction is less than unit*, and is called *proper*. Thus $\frac{5}{3}$ is an improper fraction; but $\frac{3}{4}$ and $\frac{2}{3}$ are proper fractions. A *mixt* quantity is that

whereof

whereof one part is an *integer* and the other a *fraction*. As $3\frac{4}{5}$ and $5\frac{2}{3}$, and $a+\frac{a^2}{b}$.

PROBLEM I.

§ 24. *To reduce a* MIXT *quantity to an* IMPROPER FRACTION.

Rule. *Multiply the part that is an integer by the denominator of the fractional part; and to the product add the numerator; under their sum place the former denominator.*

Thus $2\frac{3}{5}$ reduced to an improper fraction, gives $\frac{13}{5}$; $a+\frac{a^2}{b}=\frac{ab+a^2}{b}$; and $a-x+\frac{a^2-ax}{x}=\frac{a^2-x^2}{x}$.

PROBLEM II.

§ 25. *To reduce an* IMPROPER *fraction to a* MIXT QUANTITY.

Rule. *Divide the numerator of the fraction by the denominator, and the quotient shall give the integral part; the remainder set over the denominator shall be the fractional part.*

Thus $\frac{12}{5}=2\frac{2}{5}$; $\frac{ab+a^2}{b}=a+\frac{a^2}{b}$; $\frac{ax+2xx}{a+x}=x+\frac{x^2}{a+x}$; $\frac{aa+xx}{a-x}=a+x+\frac{2xx}{a-x}$.

PROBLEM III.

§ 26. *To reduce fractions of different denominations to the fractions of equal value that shall have the same denominator.*

Rule. *Multiply each numerator, separately taken, into all the denominators but its own, and the products shall give the new numerators. Then multiply all the denominators into one another, and the product shall give the common denominator.*

Thus the fractions $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, are respectively equal to these fractions $\frac{acd}{bcd}$, $\frac{bbd}{bcd}$, $\frac{ccb}{bcd}$, which have the same denominator bcd. And the fractions $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, are respectively equal to these $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$.

PROBLEM IV.

§ 27. *To* ADD *and* SUBTRACT *Fractions.*

Rule. *Reduce them to a common denominator, and add or subtract the numerators, the sum or difference set over the common denominator, is the sum or remainder required.*

Thus

Thus $\frac{a}{b}+\frac{c}{d}+\frac{d}{e}=\frac{ade+bce+d^2b}{bde}$;

$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$; $\frac{2}{3}+\frac{3}{4}=\frac{8+9}{12}=\frac{17}{12}$

$=1\frac{5}{12}$; $\frac{3}{4}-\frac{2}{3}=\frac{9-8}{12}=\frac{1}{12}$;

$\frac{4}{5}-\frac{3}{4}=\frac{16-15}{20}=\frac{1}{20}$;

$\frac{x}{2}-\frac{x}{3}=\frac{3x-2x}{6}=\frac{x}{6}$.

PROBLEM V.

§ 28. *To* MULTIPLY *Fractions*.

Rule. *Multiply their numerators one into another to obtain the numerator of the product; and their denominators multiplied into one another shall give the denominator of the product.*

Thus $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$; $\frac{2}{3}\times\frac{4}{5}=\frac{8}{15}$; and $\frac{a+b}{c}\times\frac{a-b}{d}=\frac{a^2-b^2}{cd}$.

If a mixt quantity is to be multiplied, first reduce it to the form of a fraction (by *Prob.* I.) And if an integer is to be multiplied by a fraction, you may reduce it to the form of a fraction by placing unit under it.

EXAMPLES.

$5\frac{2}{3}\times\frac{3}{4}=\frac{17}{3}\times\frac{3}{4}=\frac{51}{12}$; $9\times\frac{2}{3}=\frac{9}{1}\times\frac{2}{3}$

=

$= \frac{18}{3} = 6$; $\overline{b + \frac{bx}{a}} \times \frac{a}{x} = \frac{ba + bx}{a} \times \frac{a}{x} = \frac{a^2b + abx}{ax} = \frac{ab + bx}{x}$.

PROBLEM VI.

§ 29. *To* DIVIDE *Fractions.*

Rule. *Multiply the numerator of the dividend by the denominator of the divisor, their product shall give the numerator of the quotient. Then multiply the denominator of the dividend by the numerator of the divisor, and their product shall give the denominator.*

Thus $\frac{4}{5}\Big)\frac{2}{3}\Big(\frac{10}{12}$; $\frac{3}{7}\Big)\frac{5}{8}\Big(\frac{35}{24}$; $\frac{c}{d}\Big)\frac{a}{b}\Big(\frac{ad}{cb}$; $\frac{a+b}{a-b}\Big)\frac{a-b}{a}\Big(\frac{a^2 - 2ab + b^2}{a^2 + ab}$.

§ 30. These last four Rules are easily demonstrated from the definition of a fraction.

1. It is obvious that the fractions $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, are respectively equal to $\frac{adf}{bdf}$, $\frac{cbf}{dbf}$, $\frac{ebd}{fbd}$, since if you divide adf by bdf, the quotient will be the same as of a divided by b; and cbf divided by dbf gives the same quotient as c divided by d; and ebd divided by fbd the same quotient as e divided by f.

2. Fractions reduced to the same denomination are added by adding their numerators and sub-

subscribing the common denominator. I say $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. For call $\frac{a}{b} = m$, and $\frac{c}{b} = n$, and it will be $a = mb$, $c = nb$, and $mb + nb = a + c$, and $m + n = \frac{a+c}{b}$; that is, $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. After the same manner, $\frac{a}{b} - \frac{c}{b} = m - n = \frac{a-c}{b}$.

3. I say $\frac{a}{b} \times \frac{c}{d} (= m \times n) = \frac{ac}{bd}$; for $bm = a$, $dn = c$; and $bdmn = ac$, and $mn = \frac{ac}{bd}$; that is, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

4. I say $\frac{a}{b}$ divided by $\frac{c}{d}$, or $\frac{m}{n}$, gives $\frac{ad}{cb}$; for $mb = a$, and $mbd = ad$; $nd = c$, and $nbd = cb$; therefore $\frac{mbd}{nbd} = \frac{ad}{cb}$; that is, $\frac{m}{n} = \frac{ad}{cb}$.

PROBLEM VII.

§ 31. *To find the greatest common Measure of two numbers*; that is, the greatest number that can divide them both without a remainder.

Rule. *First divide the greater number by the lesser, and if there is no remainder the lesser number is the greatest common divisor required.*

If

If there is a remainder, divide your last divisor by it; and thus proceed continually dividing the last divisor by its remainder, till there is no remainder left, and then the last divisor is the greatest common Measure required.

Thus the greatest common measure of 45 and 63 is 9; and the greatest common measure of 256 and 48 is 16.

```
45) 63 (1                    48) 256 (5
    45                           240
    --                           ---
    18) 45 (2                     16) 48 (3
        36                            48
        --                            --
         9) 18 (2                      0
            18
            --
             0
```

§ 32. Much after the same manner the greatest common measure of algebraic quantities is discovered; only *the remainders that arise in the operation are to be divided by their simple divisors,* and *the quantities are always to be ranged according to the dimensions of the same letter.*

Thus to find the greatest common measure of $a^2 - b^2$ and $a^2 - 2ab + b^2$;

$$a^2 - b^2)\ a^2 - 2ab + b^2\ (1$$
$$\underline{a^2 - b^2}$$
$$-2ab + 2b^2 \text{ Remainder,}$$

which

which divided by $-2b$ is reduced to

$$a-b)\ a^2-b^2\ (a+b$$
$$\underline{a^2-b^2}$$
$$0\quad 0$$

Therefore $a-b$ is the greatest common measure required.

The ground of this operation is, That any quantity that measures the divisor and the remainder (if there is any) must also measure the dividend; because the dividend is equal to the sum of the divisor multiplied into the quotient, and of the remainder added together *. Thus in the last example, $a-b$ measures the divisor a^2-b^2, and the remainder $-2ab+2b^2$; it must therefore likewise measure their sum $a^2-2ab+b^2$. You must observe in this operation to make that the dividend which has the highest powers of the letter, according to which the quantities are ranged.

PROBLEM VIII.

§ 33. *To reduce any Fraction to its lowest terms.*

Rule. *Find the greatest common measure of the numerator and denominator; divide them by that common measure and place the quotients in their room, and you shall have a fraction equivalent* to

* See Chap. XIV.

CHAP. VII.

Of the INVOLUTION of Quantities.

§ 36. THE products arising from the continual multiplication of the same quantity were called (in *Chap.* IV.) the *powers* of that quantity. Thus a, a^2, a^3, a^4, &c. are the powers of a; and ab, a^2b^2, a^3b^3, a^4b^4, &c. are the powers of ab. In the same Chapter, the rule for the multiplication of powers of the same quantity is to "Add the exponents and make their sum the exponent of the product." Thus $a^4 \times a^5 = a^9$; and $a^3b^3 \times a^6b^2 = a^9b^5$. In *Chap.* V. you have the rule for dividing powers of the same quantity, which is, "To subtract the exponents and make the difference the exponent of the quotient."

Thus $\frac{a^6}{a^4} = a^{6-4} = a^2$; and $\frac{a^5b^3}{a^4b} = a^{5-4}b^{3-1} = ab^2$.

§ 37. *If you divide a lesser power by a greater, the exponent of the quotient must,* by this Rule, *be negative.* Thus $\frac{a^4}{a^6} = a^{4-6} = a^{-2}$. But $\frac{a^4}{a^6} = \frac{1}{a^2}$; and hence $\frac{1}{a^2}$ is expressed also by a^{-2} with a negative exponent.

It

It is alſo obvious that $\frac{a}{a} = a^{1-1} = a^{0}$; but $\frac{a}{a} = 1$, and therefore $a^{0} = 1$. After the ſame manner $\frac{1}{a} = \frac{a^{0}}{a} = a^{0-1} = a^{-1}$; $\frac{1}{aa} = \frac{a^{0}}{a^{2}} = a^{0-2} = a^{-2}$; $\frac{1}{aaa} = a^{0-3} = a^{-3}$; ſo that the quantities a, 1, $\frac{1}{a}$, $\frac{1}{a^{2}}$, $\frac{1}{a^{3}}$, $\frac{1}{a^{4}}$, &c. may be expreſſed thus, a^{1}, a^{0}, a^{-1}, a^{-2}, a^{-3}, a^{-4}, &c. Thoſe are called the *negative* powers of a which have negative exponents; but they are at the ſame time *poſitive* powers of $\frac{1}{a}$ or a^{-1}.

§ 38. *Negative powers (as well as poſitive) are multiplied by adding, and divided by ſubtracting their exponents.* Thus the product of a^{-2} (or $\frac{1}{a^{2}}$) multiplied by a^{-3} (or $\frac{1}{a^{3}}$) is $a^{-2-3} = a^{-5}$ (or $\frac{1}{a^{5}}$); alſo $a^{-6} \times a^{4} = a^{-6+4} = a^{-2}$ (or $\frac{1}{a^{2}}$); and $a^{-3} \times a^{3} = a^{0} = 1$. And, in general, *any poſitive power of a multiplied by a negative power of a of an equal exponent gives* UNIT *for the product*; for the poſitive and negative deſtroy each other, and the product gives a^{0}, which is equal to unit.

D Like-

Likewise $\frac{a^{-5}}{a^{-2}} = a^{-5+2} = a^{-3} = \frac{1}{a^3}$; and $\frac{a^{-2}}{a^{-5}} = a^{-2+5} = a^3$. But also, $\frac{a^{-2}}{a^{-5}} = \frac{a^{-2}}{a^{-2} \times a^{-3}}$ $= \frac{1}{a^{-3}}$; therefore $\frac{1}{a^{-3}} = a^3$: And, in general, "any quantity placed in the denominator of a fraction may be transposed to the numerator, if the sign of its exponent be changed." Thus $\frac{1}{a^3} = a^{-3}$, and $\frac{1}{a^{-3}} = a^3$.

§ 39. The quantity a^m expresses any power of a in general; the exponent (m) being undetermined; and a^{-m} expresses $\frac{1}{a^m}$, or a negative power of a of an equal exponent: and $a^m \times a^{-m}$ $= a^{m-m} = a^0 = 1$ is their product. a^n expresses any other power of a; $a^m \times a^n = a^{m+n}$ is the product of the powers a^m and a^n, and a^{m-n} is their quotient.

§ 40. To raise any simple quantity to its second, third, or fourth power, is to add its exponent twice, thrice, or four times to itself; therefore the second power of any quantity is had by doubling its exponent, and the third by trebling its exponent; and, in general, *the power expressed by m of any quantity is had by multiplying the exponent by m*, as is obvious from the multiplication of powers. Thus the second power or square of a is $a^{2 \times 1} = a^2$; its third power

power or cube is $a^{3\times 1}=a^3$; and the *m*th power of a is $a^{m\times 1}=a^m$. Alſo, the ſquare of a^4 is $a^{2\times 4}=a^8$; the cube of a^4 is $a^{3\times 4}=a^{12}$; and the *m*th power of a^4 is $a^{4\times m}$. The ſquare of abc is $a^2b^2c^2$, the cube is $a^3b^3c^3$, the *m*th power $a^mb^mc^m$.

§ 41. The raiſing of quantities to any power is called *Involution*; and *any ſimple quantity is involved by multiplying the exponent by that of the power required*, as in the preceding Examples.

The coefficient muſt alſo be raiſed to the ſame power by continual multiplication of itſelf by itſelf, as often as unit is contained in the exponent of the power required. Thus the cube of $3ab$ is $3\times 3\times 3\times a^3b^3=27a^3b^3$.

As to the Signs, *When the quantity to be involved is poſitive, it is obvious that all its powers muſt be poſitive.* And *when the quantity to be involved is negative, yet all its powers whoſe exponents are even numbers muſt be poſitive*, for any number of multiplications of a negative, if the number is even, gives a poſitive; ſince $-\times-=+$, therefore $-\times-\times-\times-=+\times+=+$; and $-\times-\times-\times-\times-\times-=+\times+\times+=+$.

The power then only can be negative when its exponent is an odd number, though the quantity to be involved be negative. The powers of $-a$ are $-a$, $+a^2$, $-a^3$, $+a^4$, $-a^5$, &c.

&c. Those whose exponents are 2, 4, 6, *&c.* are positive; but those whose exponents are 1, 3, 5, *&c.* are negative.

§ 42. The involution of *compound* quantities is a more difficult operation. The powers of any *binomial* $a + b$ are found by continual multiplication of it by itself as follows.

$$
\begin{array}{l}
a + b = \text{Root.} \\
\times\, a + b \\
\hline
a^2 + ab \\
\quad + ab + b^2 \\
\hline
a^2 + 2ab + b^2 = \text{the Square or 2d Power.} \\
\times\, a + b \\
\hline
a^3 + 2a^2b + ab^2 \\
\quad + a^2b + 2ab^2 + b^3 \\
\hline
a^3 + 3a^2b + 3ab^2 + b^3 = \text{Cube or 3d Power.} \\
\times\, a + b \\
\hline
a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
\quad + a^3b + 3a^2b^2 + 3ab^3 + b^4 \\
\hline
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 = \text{Biquadrate or 4th Power.} \\
\times\, a + b \\
\hline
a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \\
\quad + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5 \\
\hline
a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = \text{5th Power.} \\
\times\, a + b \\
\hline
a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab^5 \\
\quad + a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab^5 + b^6 \\
\hline
a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 = \text{6th Power, \&c.}
\end{array}
$$

§ 43.

§ 43. If the powers of $a - b$ are required, they will be found the same as the preceding, only the terms in which the exponent of b is an odd number will be found negative; "because an odd number of multiplications of a negative produces a negative." Thus the cube of $a - b$ will be found to be $a^3 - 3a^2b + 3ab^2 - b^3$: where the 2d and 4th terms are negative, the exponent of b being an odd number in these terms. In general, "The terms of any power of $a - b$ are positive and negative by turns."

§ 44. It is to be observed, That "in the first term of any power of $a \mp b$, the quantity a has the exponent of the power required, that in the following terms, the exponent of a decrease gradually by the same difference (*viz.* unit) and that in the last terms it is never found. The powers of b are in the contrary order; it is not found in the first term, but its exponent in the second term is unit, in the third term its exponent is 2; and thus its exponent increases, till in the last term it becomes equal to the exponent of the power required."

As the exponents of a thus decrease, and at the same time those of b increase, "the sum of their exponents is always the same, and is equal to the exponent of the power required." Thus in the 6th power of $a + b$, viz. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, the exponents of a decrease in this order, 6, 5, 4, 3, 2, 1, 0; and those of b increase in the contrary order, 0, 1, 2, 3, 4, 5, 6. And the sum of their exponents in any term is always 6.

§ 45. To find the coefficient of any term, the coefficient of the precedent term being known; you are to "divide the coefficient of the preceding term by the exponent of b in the given term, and to multiply the quotient by the exponent of a in the same term, increased by unit. Thus to find the coefficients of the terms of the 6th power of $a+b$, you find the terms are

$$a^6,\ a^5b,\ a^4b^2,\ a^3b^3,\ a^2b^4,\ ab^5,\ b^6;$$

and you know the coefficient of the first term is unit; therefore, according to the rule, the coefficient of the 2d term will be $\frac{1}{1} \times \overline{5+1} = 6$; that of the 3d term will be $\frac{6}{2} \times \overline{4+1} = 3 \times 5 = 15$; that of the 4th term will be $\frac{15}{3} \times \overline{3+1} = 5 \times 4 = 20$; and those of the following will be 15, 6, 1, agreeable to the preceding Table.

§ 46. In general, if $a+b$ is to be raised to any power m, the terms, without their coefficients, will be, a^m, $a^{m-1}b$, $a^{m-2}b^2$, $a^{m-3}b^3$, $a^{m-4}b^4$, $a^{m-5}b^5$, &c. continued till the exponent of b becomes equal to m.

The

The coefficients of the respective terms, according to the last rule, will be

$1,\ m,\ m \times \frac{m-1}{2},\ m \times \frac{m-1}{2} \times \frac{m-2}{3},\ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4},\ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5},$ &c. continued until you have one coefficient more than there are units in m.

It follows therefore by these last rules, that

$$\overline{a+b}^m = a^m + ma^{m-1}b + m \times \frac{m-1}{2} \times a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4}b^4 + \&c.$$

which is the general *Theorem* for raising a quantity consisting of two terms to any power m.

§ 47. If a quantity consisting of three, or more terms is to be involved, "you may distinguish it into two parts, considering it as a binomial, and raise it to any power by the preceding rules; and then by the same rules you may substitute instead of the powers of these compound parts their values."

Thus $\overline{a+b+c}^2 = \overline{\overline{a+b}+c}^2 = \overline{a+b}^2 + 2c \times \overline{a+b} + c^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

And $\overline{a+b+c}^3 = \overline{a+b}^3 + 3c \times \overline{a+b}^2 + 3c^2 \times \overline{a+b} + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

In these examples, $a+b+c$ is considered as composed of the compound part $a+b$ and the simple part c; and then the powers of $a+b$ are formed by the preceding rules, and substituted for $\overline{a+b}^3$ and $\overline{a+b}^2$.

CHAP. VIII.

Of EVOLUTION.

§ 48. THE reverse of Involution, or the resolving of powers into their roots is called *Evolution*. The roots of single quantities are easily extracted *by dividing their exponents by the number that denominates the root required*. Thus the square root of a^8 is $a^{\frac{8}{2}}=a^4$; and the square root of $a^4b^8c^2$ is a^2b^4c. The cube root of a^6b^3 is $a^{\frac{6}{3}}b^{\frac{3}{3}}=a^2b$; and the cube root of $x^9y^6z^{12}$ is $x^3y^2z^4$. The ground of this rule is obvious from the rule for Involution. The powers of any root are found by multiplying its exponent by the index that denominates the power; and therefore, when any power is given, the root must be found by dividing the exponent of the given power by the number that denominates the kind of root that is required.

§ 49.

§ 49. It appears from what was ſaid of Involution, that "*any power that has a poſitive ſign may have either a poſitive or negative root, if the root is denominated by any even number.*" Thus the ſquare root of $+ a^2$ may be $+ a$ or $- a$, becauſe $+ a \times + a$ or $- a \times - a$ gives $+ a^2$ for the product.

But if a power have a negative ſign, "*no root of it denominated by an even number can be aſſigned,*" ſince there is no quantity that multiplied into itſelf an even number of times can give a negative product. Thus the ſquare root of $- a^2$ cannot be aſſigned, and is what we call an "*impoſſible* or *imaginary* quantity."

But if the root to be extracted is denominated by an *odd* number, *then ſhall the ſign of the root be the ſame as the ſign of the given number whoſe root is required.* Thus the cube root of $- a^3$ is $- a$, and the cube root of $a^6 b^3$ is $- a^2 b$.

§. 50. If the number that denominates the root required is a diviſor of the exponent of the given power, then ſhall the root be only a "*lower power of the ſame quantity.*" As the cube root of a^{12} is a^4, the number 3 that denominates the cube root being a diviſor of 12.

But if the number that denominates what ſort of root is required is not a diviſor of the exponent of the given power, "*then the root required ſhall have a fraction for its exponent.*"

Thus

Thus the ſquare root of a^3 is $a^{\frac{3}{2}}$; the cube root of a^5 is $a^{\frac{5}{3}}$, and the ſquare root of a itſelf is $a^{\frac{1}{2}}$

These powers that have fractional exponents are called "*Imperfect powers* or *ſurds*;" and are otherwiſe expreſſed by placing the given power within the radical ſign $\sqrt{}$, and placing above the radical ſign the number that denominates what kind of root is required. Thus $a^{\frac{3}{2}} = \sqrt[2]{a^3}$; $a^{\frac{5}{3}} = \sqrt[3]{a^5}$; and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. In numbers the ſquare root of 2 is expreſſed by $\sqrt[2]{2}$, and the cube root of 4 by $\sqrt[3]{4}$.

§ 51. Theſe imperfect powers or ſurds are "*multiplied and divided*, as other powers, *by adding and ſubtracting their exponents.*" Thus $a^{\frac{1}{2}} \times a^{\frac{5}{2}} = a^{\frac{6}{2}} = a^3$; $a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{2}{3}+\frac{3}{4}} = a^{\frac{17}{12}} = \sqrt[12]{a^{17}}$; and $\frac{a^{\frac{7}{2}}}{a^{\frac{3}{2}}} = a^{\frac{7-3}{2}} = a^{\frac{4}{2}} = a^2$.

They are *involved* likewiſe and *evolved* after the ſame manner as perfect powers. Thus the ſquare of $a^{\frac{3}{2}}$ is $a^{2 \times \frac{3}{2}} = a^3$; the cube of $a^{\frac{2}{3}}$ is $a^{3 \times \frac{2}{3}} = a^{\frac{6}{3}}$. The ſquare root of $a^{\frac{2}{3}}$ is $a^{\frac{2}{3 \times 2}} = a^{\frac{1}{3}}$, the cube root of $a^{\frac{1}{4}}$ is $a^{\frac{1}{12}}$. But we ſhall have occaſion to treat more fully of Surds hereafter.

§ 52. The ſquare root of any compound quantity, as $a^2 + 2ab + b^2$ is diſcovered after this

this manner. "*First, take care to dispose the terms according to the dimensions of the alphabet, as in division; then find the square root of the first term* aa, *which gives* a *for the first member of the root. Then subtract its square from the proposed quantity, and divide the first term of the remainder* ($2ab + b^2$) *by the double of that member*, viz. $2a$, *and the quotient* b *is the second member of the root. Add this second member to the double of the first, and multiply their sum* ($2a + b$) *by the second member* b, *and subtract the product* ($2ab + b^2$) *from the foresaid remainder* ($2ab + b^2$) *and if nothing remains, then the square root is obtained*;" and in this example it is found to be $a + b$.

The manner of operation is thus,

$$\begin{array}{rl}
 & a^2 + 2ab + b^2\ (a + b \\
 & a^2 \\
\hline
2a + b) & 2ab + b^2 \\
\times\ b) & 2ab + b^2 \\
\hline
 & \quad 0 \qquad 0
\end{array}$$

But if there had been a remainder, you must have divided it by the double of the sum of the two parts already found, and the quotient would have given the third member of the root.

Thus if the quantity proposed had been $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$, after proceeding as above you would have found the remainder

$2ac$

$2ac + 2bc + c^2$, which divided by $2a + 2b$ gives c to be annexed to $a + b$ as the 3d member of the root. Then adding c to $2a + 2b$ and multiplying their sum $2a + 2b + c$ by c, subtract the product $2ac + 2bc + c^2$ from the foresaid remainder; and since nothing now remains, you conclude that $a + b + c$ is the square root required.

The operation is thus;

$$
\begin{array}{r|l}
 & a^2 + 2ab + 2ac + b^2 + 2bc + c^2 \; (a + b + c \\
 & \underline{a^2} \\
2a + b & 2ab + 2ac + b^2 + 2bc + c^2 \\
\times b & \underline{2ab \qquad\quad + b^2} \\
2a + 2b + c & 2ac + 2bc + c^2 \\
\times c & \underline{2ac + 2bc + c^2} \\
 & 0 \quad 0 \quad 0
\end{array}
$$

Another Example.

$$
\begin{array}{r|l}
 & xx - ax + \tfrac{1}{4}aa \; (x - \tfrac{1}{2}a \\
 & \underline{xx} \\
2x - \tfrac{1}{2}a & -ax + \tfrac{1}{4}aa \\
\times -\tfrac{1}{2}a & \underline{-ax + \tfrac{1}{4}aa} \\
 & 0 \quad 0
\end{array}
$$

The square root of any number is found out after the same manner. If it is a number under 100, its nearest square root is found by the following Table; by which also its cube root is found

found if it be under 1000, and its biquadrate if it be under 10000.

The Root	1	2	3	4	5	6	7	8	9
Square	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729
Biquad.	1	16	81	256	625	1296	2401	4096	6561

But if it is a number above 100, then its ſquare root will conſiſt of two or more figures, which muſt be found by different operations by the following

RULE.

§ 53. *Place a point above the number that is in the place of units, paſs the place of tens, and place again a point over that of hundreds, and go on towards the left hand, placing a point over every 2d figure; and by theſe points the number will be diſtinguiſhed into as many parts as there are figures in the root. Then find the ſquare root of the firſt part, and it will give the firſt figure of the root; ſubtract its ſquare from that part, and annex the ſecond part of the given number to the remainder. Then divide this new number (neglecting its laſt figure) by the double of the firſt figure of the root, annex the quotient to that double, and multiply the number thence ariſing by the ſaid quotient, and if the product is leſs than your dividend, or equal to it, that quotient ſhall be the ſecond figure of the root. But if the product is greater than the dividend, you muſt take a leſs number for the ſecond figure*

figure of the root than that quotient. Much after the same manner may the other figures of the quotient be found, if there are more points than two places over the given number.

To find the square root of 99856, I first point it thus $\dot{9}9\dot{8}5\dot{6}$, then I find the square root of 9 to be 3, which therefore is the first figure of the root; I subtract 9, the square of 3, from 9, and to the remainder I annex the second part 98, and divide (neglecting the last figure 8) by the double of 3, or 6, and I place the quotient after 6, and then multiply 61 by 1, and subtract the product 61 from 98. Then to the remainder (37) I annex the last part of the proposed number (56) and dividing 3756 (neglecting the last figure 6) by the double of 31, that is by 62, I place the quotient after, and multiplying 626 by the quotient 6, I find the product to be 3756, which subtracted from the dividend and leaving no remainder, the exact root must be 316.

EXAMPLES.

$\dot{9}9\dot{8}5\dot{6}$ (316
9
61) 98
×1) 61
626) 3756
×6) 3756
0

$\dot{2}7\dot{3}9\dot{4}7\dot{5}6\ (5234$	$\dot{5}\dot{2}9\ (23$
25	4
$102)\ 239$	$43)\ 129$
$\times 2)\ 204$	$\times 3)\ 129$
$1043)\ 3547$	0
$\times 3)\ 3129$	
$10464)\ 41856$	
$\times 4)\ 41856$	
0	

§ 54. In general, to extract any root out of any given quantity, "*First range that quantity according to the dimensions of its letters, and extract the said root out of the first term, and that shall be the first member of the root required. Then raise this root to a dimension lower by unit than the number that denominates the root required, and multiply the power that arises by that number itself; divide the second term of the given quantity by the product, and the quotient shall give the second member of the root required.*"

Thus to extract the root of the 5th power out of $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$, I find that the root of the 5th power out of a^5 gives a, which I raise to the 4th power, and multiplying by 5, the product is $5a^4$; then dividing the second term of the given quantity $5a^4b$ by $5a^4$, I find b to be the second member; and

and raising $a+b$ to the 5th power and subtracting it, there being no remainder, I conclude that $a+b$ is the root required. If the root has three members, the third is found after the same manner from the first two considered as one member, as the second member was found from the first; which may be easily understood from what was said of extracting the square root.

§ 55. In extracting roots it will often happen that the exact root cannot be found in finite terms; thus the square root of $a^2 + x^2$ is found to be

$$a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \&c.$$

The operation is thus;

$$a^2 + x^2 \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} -, \&c. \right.$$

$$a^2$$

$$\left. 2a + \frac{x^2}{2a} \right) * + x^2$$

$$\left. \times \frac{x^2}{2a} \right) = x^2 + \frac{x^4}{4a^2}$$

$$\left. 2a + \frac{x^2}{a} - \frac{x^4}{8a^3} \right) - \frac{x^4}{4a^2}$$

$$\left. \times - \frac{x^4}{8a^3} \right) = - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$$

$$+ \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

&c.

After

After the ſame manner, the cube root of $a^3 + x^3$ will be found to be

$$a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}} + \text{\&c.}$$

§ 56. "The general Theorem which we gave for the Involution of binomials will ſerve alſo for their Evolution;" becauſe to extract any root of a given quantity is the ſame thing as to raiſe that quantity to a power whoſe exponent is a fraction that has unity for its numerator, and the number that expreſſes what kind of root is to be extracted for its denominator. Thus, to extract the ſquare root of $a + b$, is to raiſe $a + b$ to a power whoſe exponent is $\frac{1}{2}$: Now ſince $\overline{a+b}^m = a^m + m \times a^{m-1}b + m \times \frac{m-1}{2} a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}b^3$ &c.

ſuppoſing $m = \frac{1}{2}$, you will find

$\overline{a+b}|^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{1}{2} \times a^{-\frac{1}{2}}b + \frac{1}{2} \times -\frac{1}{4} \times a^{-\frac{3}{2}}b^2 + \frac{1}{2} \times -\frac{1}{4} \times -\frac{1}{2}a^{-\frac{5}{2}}b^3$ &c. $= a^{\frac{1}{2}} + \frac{b}{2a^{\frac{1}{2}}} - \frac{b^2}{8a^{\frac{3}{2}}} + \frac{b^3}{16a^{\frac{5}{2}}} -$ &c. And after this manner you will find that

$\overline{a^2+x^2}|^{\frac{1}{2}} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} -$ &c. as before.

§ 57. The roots of numbers are to be extracted as thoſe of algebraic quantities. "*Place a point over the units, and then place points over* *every third, fourth or fifth figure towards the left hand, according as it is the root of the cube, of the 4th or 5th power that is required; and if there be any decimals annexed to the number, point them after the same manner, proceeding from the place of units towards the right hand. By this means the number will be divided into so many periods as there are figures in the root required. Then enquire which is the greatest cube, biquadrate, or 5th power in the first period, and the root of that power will give the first figure of the root required. Subtract the greatest cube, biquadrate, or 5th power from the first period, and to the remainder annex the first figure of your second period, which shall give your dividend.*

Raise the first figure already found to a power less by unit than the power whose root is sought, that is, to the 2d, 3d, or 4th power, according as it is the cube root, the root of the 4th, or the root of the 5th power that is required, and multiply that power by the index of the cube, 4th, or 5th power, and divide the dividend by this product, so shall the quotient be the second figure of the root required.

Raise the part already found of the root, to the power whose root is required, and if that power be found less than the two first periods of the given number, the second figure of the root is right. But if it be found greater, you must diminish the second figure of the root till that power *be found equal to or less than those periods of the given number. Subtract it, and to the remainder annex the next period; and proceed till you have gone through the whole given number, finding the 3d figure by means of the two first, as you found the second by the first; and afterwards finding the 4th figure (if there be a 4th period) after the same manner from the three first."*

Thus to find the cube root of 13824; point it 13824; find the greatest cube in 13, *viz.* 8, whose cube root 2 is the first figure of the root required. Subtract 8 from 13, and to the remainder 5 annex 8 the first figure of the second period; divide 58 by triple the square of 2, *viz.* 12, and the quotient is 4, which is the second figure of the root required, since the cube of 24 gives 13824 the number proposed. After the same manner the cube root of 13312053 is found to be 237.

OPERATION.

13824 (24

Subtr. $8 = 2 \times 2 \times 2$

$3 \times 4 = 12)\ 58\ (4$

Subtract $13824 = 24 \times 24 \times 24$

Rem. 0

$$
\begin{array}{r}
\dot{1}3\dot{3}12\dot{0}5\dot{3}\ (237 \\
8 = 2\times2\times2 \\
\hline
12)\ 53\ (4\ \text{or})\ 3 \\
\text{Subtract}\ \underline{12167} = 23\times23\times23 \\
3\times23\times23 = 1587)\ 11450\ (7 \\
\text{Subtract}\ \underline{13312053} = 237\times237\times237 \\
\text{Remain.}\quad 0
\end{array}
$$

In extracting of roots, after you have gone through the number proposed, if there is a remainder, you may continue the operation by adding periods of cyphers to that remainder, and find the true root in decimals to any degree of exactness.

CHAP. IX.

Of Proportion.

§ 58. WHEN quantities of the same kind are compared, it may be considered either how much the one is greater than the other, and what is their *difference*; or, it may be considered how many times the one is contained in the other; or, more generally, what

what is their *quotient*. The first relation of quantities is expressed by their *Arithmetical ratio*; the second by their *Geometrical ratio*. That term whose ratio is enquired into is called the *antecedent*, and that with which it is compared is called the *consequent*.

§ 59. When of four quantities the difference betwixt the first and second is equal to the difference betwixt the third and fourth, those quantities are call *Arithmetical proportionals*; as the numbers 3, 7, 12, 16. And the quantities, a, $a + b$, e, $e + b$. But quantities form a *series* in arithmetical proportion, when they "*increase or decrease by the same constant difference.*" As these, a, $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, &c. x, $x - b$, $x - 2b$, &c. or the numbers, 1, 2, 3, 4, 5, &c. and 10, 7, 4, 1, -2, -5, -8, &c.

§ 60. In four quantities *arithmetically proportional*, "*the sum of the extremes is equal to the sum of the mean terms.*" Thus a, $a + b$, e, $e + b$, are arithmetical proportionals, and the sum of the extremes $(a + e + b)$ is equal to the sum of the mean terms $(a + b + e)$. Hence, to find the fourth quantity arithmetically proportional to any three given quantities; "Add the second and third, and from their sum subtract the first term, the remainder shall give the fourth arithmetical proportional required."

§ 61. In a *series* of arithmetical proportionals "*the sum of the first and last term is equal to the sum of any two terms equally distant from the extremes.*" If the first terms are a, $a+b$, $a+2b$, &c. and the last term x, the last term but one will be $x-b$, the last but two $x-2b$ the last but three $x-3b$, &c. So that the first half of the terms, having those that are equally distant from the last term set under them, will stand thus;

$$\begin{array}{l} a,\ a+b,\ a+2b,\ a+3b,\ a+4b, \\ x,\ x-b,\ x-2b,\ x-3b,\ x-4b, \\ \hline a+x,\ a+x,\ a+x,\ a+x,\ a+x,\ \&c. \end{array} \&c.$$

And it is plain that if each term be added to the term above it, the sum will be $a+x$ equal to the sum of the first term a and the last term x. From which it is plain, that "*the sum of all the terms of an arithmetical progression is equal to the sum of the first and last taken half as often as there are terms,*" that is, the sum of an arithmetical progression is equal to the sum of the first and last terms multiplied by half the number of terms. Thus in the preceding series, if n be the number of terms, the sum of all the terms will be $\overline{a+x} \times \frac{n}{2}$.

§ 62. The common difference of the terms being b, and b not being found in the first term, it is plain that "its coefficient in any term

term will be equal to the number of terms that precede that term." Therefore in the last term x you must have $\overline{n-1} \times b$, so that x must be equal to $a + \overline{n-1} \times b$. And the sum of all the terms being $\overline{a+x} \times \frac{n}{2}$, *it will also be equal to* $\frac{2an + n^2b - nb}{2}$, *or to* $a + \frac{nb-b}{2} \times n$. Thus for example, the series $1+2+3+4+5$ &c. continued to a hundred, must be equal to $\frac{2 \times 100 + 10000 - 100}{2} = 5050$.

§ 63. If a series have (o) nothing for its first term, then "*its sum shall be equal to half the product of the last term multiplied by the number of terms.*" For then, a being $= 0$, the sum of the terms, which is in general $\overline{a+x} \times \frac{n}{2}$, will in this case be $\frac{nx}{2}$. From which it is evident, that "the sum of any number of arithmetical proportionals beginning from nothing, is equal to half the sum of as many terms equal to the greatest term.

Thus $0+1+2+3+4+5+6+7+8+9 = \frac{9+9+9+9+9+9+9+9+9+9}{2} = \frac{10 \times 9}{2} = 45$.

§ 64. If of four quantities the quotient of the *first* and *second* be equal to the quotient of the *third* and *fourth*, then those quantities are said to be in *Geometrical proportion.*" Such are the numbers 2, 6, 4, 12; and the quantities a, ar, b, br; which are expressed after this manner;

$$2 : 6 :: 4 : 12.$$
$$a : ar :: b : br.$$

And you read them by saying, As 2 is to 6, so is 4 to 12; or as a is to ar, so is b to br.

In four quantities geometrically proportional, "*the product of the extremes is equal to the product of the middle terms.*" Thus $a \times br = ar \times b$. And, if it is required to find a fourth proportional to any three given quantities, "*multiply the second by the third, and divide the product by the first, the quotient shall give the fourth proportional required.*" Thus, to find a fourth proportional to a, ar, and b, I multiply ar by b, and divide the product arb by the first term a, the quotient br is the fourth proportional required.

§ 65. In calculations it sometimes requires a little care to place the terms in due order; for which you may observe the following Rule.

"*First set down the quantity that is of the same kind with the quantity sought, then consider, from the nature of the question, whether that which is given is greater or less than that which is sought; if it is greater, then place the greatest of the other two quantities on the left hand; but if it is less, place the least of the other two quantities on the left hand, and the other on*

the

the right." Then ſhall the terms be in due order; and you are to proceed according to the rule, multiplying the ſecond by the third, and dividing their product by the firſt.

EXAMPLE.

If 30 *men do any piece of work in* 12 *days, how many men ſhall do it in* 18 *days?*

Becauſe it is a number of men that is ſought, firſt ſet down 30, the number of men that is given: I eaſily ſee that the number that is given is greater than the number that is ſought, therefore I place 18 on the left hand, and 12 on the right; and find a fourth proportional to 18, 30, 12, *viz.* $\frac{30 \times 12}{18} = 20$.

§ 66. When a ſeries of quantities increaſe by one common multiplicator, or decreaſe by one common diviſor, they are ſaid to be in "*Geometrical proportion continued.*"

As $a, ar, ar^2, ar^3, ar^4, ar^5$, &c. or,

$$a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}, \frac{a}{r^5}, \text{\&c.}$$

The common multiplier or diviſor is called their "*common ratio.*"

In ſuch a ſeries, "*the product of the firſt and laſt is always equal to the product of the ſecond and laſt but one,* or *to the product of any two terms equally remote from the extremes.* In the ſeries a, ar, ar^2, ar^3, &c. if y be the laſt term,

then

then ſhall the four laſt terms of the ſeries be $y, \frac{y}{r}, \frac{y}{r^2}, \frac{y}{r^3}$; now it is plain that $a \times y = ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2} = ar^3 \times \frac{y}{r^3}$ &c.

§ 67. "*The ſum of a ſeries of geometrical proportionals wanting the firſt term, is equal to the ſum of all but the laſt term multiplied by the common ratio.*

For $ar + ar^2 + ar^3$ &c. $+ \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y$ $= r \times \overline{a + ar + ar^2 \text{ \&c.} + \frac{y}{r^4} + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r}}$. Therefore if s be the ſum of the ſeries, $s - a$ will be equal to $\overline{s - y} \times r$; that is $s - a = sr - yr$, or $sr - s = yr - a$, and $s = \frac{yr - a}{r - 1}$ *.

§ 68. Since the exponent of r is always increaſing from the ſecond term, if the number of terms be n, in the laſt term its exponent will be $n - 1$. Therefore $y = ar^{n-1}$; and $yr = ar^{n-1+1} = ar^n$; and $s = \left(\frac{yr - a}{r - 1}\right) = \frac{ar^n - a}{r - 1}$. So that having the firſt term of the ſeries, the number of the terms, and the common ratio, you may eaſily find the ſum of all the terms.

If it is a decreaſing ſeries whoſe ſum is to be found, as of $y + \frac{y}{r} + \frac{y}{r^2} + \frac{y}{r^3}$ &c. $+ ar^3 + ar^2 + ar + a$, and the number of the terms be

* See the Rules in the following Chapter.

ſup-

supposed infinite, then shall a, the last term, be equal to nothing. For, because n, and consequently r^{n-1} is infinite, $a = \frac{y}{r^{n-1}} = 0$. The sum of such a series $s = \frac{yr}{r-1}$; which is a finite sum, though the number of the terms be infinite.

Thus $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +$ &c. $= \frac{1 \times 2}{2-1} = 2$.
and $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} +$ &c. $= \frac{1 \times 3}{3-1} = \frac{3}{2}$.

CHAP. X.

Of EQUATIONS that involve only one unknown Quantity.

§ 69. AN *equation* is "*a proposition asserting the equality of two quantities.*" It is expressed most commonly by setting down the quantities, and placing the sign ($=$) between them.

An equation gives the value of a quantity, when that quantity is alone on one side of the equation: and that value is known, if all those that are on the other side are known. Thus if I find that $x = \frac{4 \times 6}{3} = 8$, I have a known value of x. These are the last conclusions we are to to seek in questions to be resolved; and if there be only one unknown quantity in a given equation, and only one dimension of it, such a value may always be found by the following Rules.

RULE I.

§ 70. "*Any quantity may be transposed from one side of the equation to the other, if you change its sign.*"

For to take away a quantity from one side, and to place it with a contrary sign on the other side, is to subtract it from both sides; and it is certain, that "when from equal quantities you subtract the same quantity, the remainders must be equal."

By this Rule, when the known and unknown quantities are mixed in an equation, you may separate them by bringing all the unknown to one side, and the known to the other side of the equation; as in the following Examples.

Suppose $5x + 50 = 4x + 56$,
By transposit. $5x - 4x = 56 - 50$, or, $x = 6$.
And if $2x + a = x + b$,
$2x - x = b - a$, or, $x = b - a$.

RULE II.

§ 71. *Any quantity by which the unknown quantity is multiplied may be taken away, if you divide all the other quantities on both sides of the equation by it.*"

For

For that is to divide both sides of the equation by the same quantity; and when you divide equal quantities by the same quantity, the quotients must be equal. Thus,

$$\text{If } ax = b,$$
$$\text{then } x = \frac{b}{a}.$$

$$\text{And if } 3x + 12 = 27,$$
$$\text{by Rule 1. } 3x = 27 - 12 = 15,$$
$$\text{and by Rule 2. } x = \frac{15}{3} = 5.$$

$$\text{Also if } ax + 2ba = 3cc,$$
$$\text{by Rule 1. } ax = 3cc - 2ba,$$
$$\text{and by Rule 2. } x = \frac{3cc}{a} - 2b.$$

RULE III.

§ 72. *If the unknown quantity is divided by any quantity, that quantity may be taken away if you multiply all the other members of the equation by it.*" Thus,

$$\text{If } \frac{x}{b} = b + 5,$$
$$\text{then shall } x = bb + 5b.$$

$$\text{If } \frac{x}{5} + 4 = 10,$$
$$\text{then } x + 20 = 50,$$
$$\text{and by Rule 1. } x = 50 - 20 = 30.$$

If

$$\text{If } \frac{4x}{3} + 24 = 2x + 6,$$
$$\text{then } 4x + 72 = 6x + 18,$$
$$\text{by Rule 1. } 72 - 18 = 6x - 4x, \text{ or } 54 = 2x.$$
$$\text{and by Rule 2. } x = \frac{54}{2} = 27.$$

By this Rule an equation, whereof any part is a fraction, may be reduced to an equation that ſhall be expreſſed by integers. If there are more fractions than one in the given equation, you may, by reducing them to a common denominator, and then multiplying all the other terms by that denominator, abridge the calculation thus;

$$\text{If } \frac{x}{5} + \frac{x}{3} = x - 7,$$
$$\text{then } \frac{3x + 5x}{15} = x - 7,$$
$$\text{and by this Rule } 3x + 5x = 15x - 105,$$
$$\text{and by Rule 1 and 2. } x = \frac{105}{7} = 15.$$

RULE IV.

§ 73. "*If that member of the equation that involves the unknown quantity be a* ſurd *root, then the equation is to be reduced to another that ſhall be free from any ſurd, by bringing that member firſt to ſtand alone upon one ſide of the equation, and then taking away the radical ſign from it, and raiſing the other ſide of the equation to the power denominated by the ſurd.*"

Thus

Thus if $\sqrt{4x+16}=12$,
then $4x+16=144$,
and $4x=144-16=128$,
and $x=\frac{128}{4}=32$.

If $\sqrt{ax+b^2}-c=d$,
then $\sqrt{ax+b^2}=d+c$;
$ax+b^2=d^2+2dc+c^2$.
and $x=\frac{d^2+2dc+c^2-b^2}{a}$.

If $\sqrt[3]{a^2x-b^2x}=a$,
then $a^2x-b^2x=a^3$,
and $x=\frac{a^3}{a^2-b^2}$.

RULE V.

§ 74. "*If that side of the equation that contains the unknown quantity be a complete square, cube, or other power; then extract the square root, cube root, or the root of that power, from both sides of the equation, and thus the equation shall be reduced to one of a lower degree.*

If $x^2+6x+9=20$,
then $x+3=\pm\sqrt{20}$,
and $x=\pm\sqrt{20}-3$.

If

$$\text{If } x^2 + ax + \frac{a^2}{4} = b^2,$$
$$\text{then } x + \frac{a}{2} = \pm b$$
$$\text{and } x = \pm b - \frac{a}{2}.$$

$$\text{If } x^2 + 14x + 49 = 121,$$
$$\text{then } x + 7 = \pm 11,$$
$$\text{and } x = \pm 11 - 7 = 4, \text{ or } -18.$$

RULE VI.

§ 75. "*A proportion may be converted into an equation, asserting the product of the extreme terms equal to the product of the mean terms; or any one of the extremes equal to the product of the means divided by the other extreme.*"

$$\text{If } 12 - x : \frac{x}{2} :: 4 : 1,$$
$$\text{then } 12 - x = 2x \ldots\ 3x = 12 \ldots \text{ and } x = 4.$$
$$\text{Or if } 20 - x : x :: 7 : 3,$$
$$\text{then } 60 - 3x = 7x \ldots 10x = 60 \ldots \text{ and } x = 6.$$

RULE VII.

§ 76. *If any quantity be found on both sides of the equation with the same sign prefixt, it may be taken away from both:" "Also, if all the quantities in the equation are multiplied or divided*

divided by the same quantity, it may be struck out of them all." Thus,

If $3x + b = a + b \ldots 3x = a \ldots$ and $x = \frac{a}{3}$.

If $3ax + 5ab = 8ac \ldots 3x + 5b = 8c \ldots$ and $x = \frac{8c - 5b}{3}$.

If $\frac{2x}{3} + \frac{8}{3} = \frac{16}{3} \ldots 2x + 8 = 16 \ldots$ and $x = 4$.

RULE VIII.

§ 77. "*Instead of any quantity in an equation you may substitute another equal to it.*"

Thus, if $3x + y = 24$,
and $y = 9$;
then $3x + 9 = 24 \ldots x = \frac{24 - 9}{3} = 5$.

If $3y + 5x = 120$,
and $y = 5x$;
then $15x + 5x \; (= 20x) = 120$,
and $x = \frac{120}{20} = 6$.

The further improvement of this Rule shall be taught in the following chapter.

CHAP. XI.

Of the Solutions of questions that produce simple equations.

SIMPLE equations are those "wherein the unknown quantity is only of one dimension:" In the solution of which we are to observe the following directions.

DIRECTION I.

§ 78. *After forming a distinct idea of the question proposed, the unknown quantities are to be expressed by letters, and the particulars to be translated from the common language into the algebraic manner of expressing them, that is, into such equations as shall express the relations or properties that are given of such quantities.*

Thus, if the sum of two quantities must be 60, that condition is expressed thus, $x + y = 60$.

If their difference must be 24, that condition gives . . . . $x - y = 24$.

If their product must be 1640, then $xy = 1640$.

If their quotient must be 6, then . . . $\frac{x}{y} = 6$.

If their proportion is as 3 to 2, then $x : y :: 3 : 2$, or $2x = 3y$; because the product of the extremes

tremes is equal to the product of the mean terms.

DIRECTION II.

§ 79. "*After an equation is formed, if you have one unknown quantity only, then, by the Rules of the preceding Chapter, bring it to stand alone on one side, so as to have only known quantities on the other side:*" *thus you shall discover its value.*

EXAMPLE.

A person being asked what was his age, answered that $\frac{3}{4}$ *of his age multiplied by* $\frac{1}{12}$ *of his age gives a product equal to his age.* Qu. *what was his age?*

It appears from the question, that if you call his age x, then shall .... $\frac{3x}{4} \times \frac{x}{12} = x$,

that is ... $\frac{3x^2}{48} = x$;

and by Rule 3. ... $3x^2 = 48x$.

and by Rule 7. ... $3x = 48$

whence by Rule 2. ... $x = 16$.

DIRECTION III.

§ 80. "*If there are two unknown quantities, then there must be two equations arising from the conditions of the question: Suppose the quantities x and y; find a value of x or y, from* *each of the equations, and then by putting these two values equal to each other, there will arise a new equation involving one unknown quantity; which must be reduced by the Rules of the former Chapter.*

EXAMPLE I.

Let the sum of two quantities be s, and their difference d. Let s and d be given, and let it be required to find the quantities themselves. Suppose them to be x and y, then, by the supposition,

$$x+y=s$$
$$x-y=d$$

$$\text{whence}\begin{cases}x=s-y\\x=d+y\end{cases}$$

$$\text{and } d+y=s-y$$
$$2y=s-d$$
$$y=\frac{s-d}{2}$$
$$\text{and } x=\frac{s+d}{2}.$$

EXAMPLE II.

Let it be required to find two numbers whose sum is s, and their proportion as a to b. Let the numbers be x and y, then shall

Suppose

$$\text{Suppos.}\ \begin{cases} x + y = s \\ x : y :: a : b \end{cases}$$

$$bx = ay$$

$$x = \frac{ay}{b}$$

$$x = s - y$$

$$\frac{ay}{b} = s - y$$

$$\frac{ay}{b} + y = s$$

$$ay + by = bs$$

$$\overline{a + b} \times y = bs$$

$$y = \frac{bs}{a + b}$$

$$x = \frac{ay}{b} = \frac{as}{a + b}.$$

EXAMPLE III.

A privateer running at the rate of 10 *miles an hour, discovers a ship* 18 *miles off making way at the rate of* 8 *miles an hour: It is demanded how many miles the ship can run before she be overtaken?*

Let the number of miles the ship can run before she be overtaken be called x; and the number of miles the privateer must run before she come up with the ship, be y; then shall (by *Supp.*) .... $y = x + 18$ .... and $x : y :: 8 : 10$, whence $10x = 8y$ .... $x = \frac{4y}{5}$ .... and $x = y - 18$.

 Whence

Whence $y - 18 = \frac{4y}{5}$, and $y = 90 \ldots x = y - 18 = 72$.

To find the time, say, if 8 miles give 1 hour; 72 miles will give 9 hours. — Thus, 8 : 1 :: 72 : 9.

EXAMPLE IV.

Suppose the distance between London *and* Edinburgh *to be* 360 *miles, and that a courier sets out from* Edinburgh *running at the rate of* 10 *miles an hour; another sets out at the same time from* London, *and runs* 8 *miles an hour. It is required to know where they will meet?* Suppose the courier that sets out from *Edinburgh* runs x miles, and the other y miles before they meet; then shall

$$\text{by suppos.} \begin{cases} x + y = 360 \\ x : y :: 5 : 4 \end{cases}$$

$$x = \frac{5y}{4}$$

$$x = 360 - y$$

$$\frac{5y}{4} = 360 - y$$

$$\frac{5y}{4} + y = 360$$

$$9y = 1440$$

$$y = \frac{1440}{9} = 160$$

$$x = 360 - y = 200.$$

EXAMPLE V.

Two persons discoursing of their revenues, says A, *if* B *would yield him a post he has of* 25l. *a year,*

year, their revenues would be equal: Says B, *if* A *would give him a place he holds of* 22l. per annum, *the revenue of* B *would be double that of* A. Qu. *their revenues?*

Let the revenue of *A* be called x, that of *B*, y; then,

$$\text{by supp.} \begin{cases} x + 25 = y - 25 \\ y + 22 = 2x - 44 \end{cases}$$

$$\begin{aligned} y &= x + 25 + 25 = x + 50 \\ y &= 2x - 44 - 22 = 2x - 66 \\ 2x - 66 &= x + 50 \\ x &= 66 + 50 = 116. \\ y &= x + 50 = 166. \end{aligned}$$

EXAMPLE VI.

A gentleman distributing money among some poor people, found he wanted 10s. *to be able to give* 5s. *to each; therefore he gives each* 4s. *only, and finds that he has* 5s. *left.* Qu. *the number of shillings and poor people?*

Call the number of the poor x, and the number of shillings y; then,

$$\text{by supp.} \begin{cases} 5x = y + 10 \\ 4x = y - 5 \end{cases}$$

$$\begin{aligned} y &= 5x - 10 \\ y &= 4x + 5 \\ 5x - 10 &= 4x + 5 \\ 5x - 4x &= 15 \\ x &= 15 \\ y &= 4x + 5 = 65 \end{aligned}$$

EXAMPLE VII.

Two merchants were copartners; the sum of their stock was 300l. *One of their stocks continued in company* 11 *months; but the other drew out his stock in* 9 *months; when they made up their accounts they divided the gain equally.* Qu. *What was each man's stock?* Suppose the stock of the first to be x, and the stock of the other to be y; then,

$$\text{by supp.}\begin{cases} x + y = 300 \\ 11x = 9y \end{cases}$$

$$x = \frac{9y}{11} = 300 - y$$

$$11y + 9y = 3300$$

$$20y = 3300$$

$$y = \frac{3300}{20} = 165 \,..\, x = 300 - y = 135.$$

EXAMPLE VIII.

There are two numbers whose sum is the 6th part of their product, and the greater is to the lesser as 3 *to* 2. Qu. *What are these numbers?* Call them x and y; then,

$$\text{supp.}\begin{cases} x + y = \frac{xy}{6} \\ x : y :: 3 : 2 \end{cases}$$

$$yx = 6x + 6y$$

$$yx - 6x = 6y$$

$$\overline{y - 6} \times x = 6y$$

$$x = \frac{6y}{y - 6}$$

$$x = \frac{3y}{2}, \text{ whence*}$$

$$*\frac{6y}{y - 6} = \frac{3y}{2}$$

$$12y = 3yy - 18y$$

$$30y = 3yy$$

$$30 = 3y$$

$$\therefore y = \frac{30}{3} = 10$$

$$x = \frac{3 \times 10}{2} = 15.$$

DI-

DIRECTION IV.

§ 81. "*When in one of the given equations, the unknown quantity is of one dimension, and in the other of a higher dimension; you must find a value of the unknown quantity from that equation where it is of one dimension, and then raise that value to the power of the unknown quantity in the other equation; and by comparing it, so involved, with the value you deduce from that other equation, you shall obtain an equation that will have only one unknown quantity, and its powers.*"

That is, when you have two equations of different dimensions, if you cannot reduce the higher to the same dimension with the lower, you must raise the lower to the same dimension with the higher.

EXAMPLE IX.

The sum of two quantities, and the difference of their squares, being given, to find the quantities. Suppose them to be x and y, their sum s, and difference of their squares d. Then,

$$\begin{cases} x+y=s \\ x^2-y^2=d \end{cases}$$

$$x=s-y$$
$$x^2=s^2-2sy+y^2$$
$$x^2=d+y^2$$
$$d+y^2=s^2-2sy+y^2$$
$d=s^2-2sy$, whence *

$$* \; 2sy=s^2-d$$
$$y=\frac{s^2-d}{2s}.$$
and $x=\dfrac{s^2+d}{2s}$

EX-

EXAMPLE X.

Let the proportion of two numbers and the sum of their squares be given, and let it be required to find the numbers themselves. Suppose their proportion to be the same as that of a to b, and let the sum of their squares be c; that is, let

$$\begin{cases} x : y :: a : b \\ x^2 + y^2 = c \end{cases}$$

then $x = \frac{ay}{b}$,

and $x^2 = \frac{a^2y^2}{b^2}$;

but $x^2 = c - y^2$,

whence $c - y^2 = \frac{a^2y^2}{b^2}$

$$b^2y^2 + a^2y^2 = cb^2$$

$$\overline{a^2 + b^2} \times y^2 = cb^2$$

$$y^2 = \frac{cb^2}{a^2 + b^2}$$

$$y = \sqrt{\frac{cb^2}{a^2 + b^2}}, \text{ and } x = \sqrt{\frac{ca^2}{a^2 + b^2}}.$$

EXAMPLE XI.

Let the proportion of two numbers be that of a to b, and the difference of their cubes be d. Qu. *What are the numbers?* Then,

$x : y$

$$\begin{cases} x : y :: a : b \\ x^3 - y^3 = d \end{cases}$$

$$x = \frac{ay}{b}, \text{ and } x^3 = \frac{a^3y^3}{b^3};$$

$$\text{but } x^3 = d + y^3$$

$$\text{whence} \qquad d + y^3 = \frac{a^3y^3}{b^3},$$

$$\text{and } a^3y^3 - b^3y^3 = db^3$$

$$y^3 = \frac{db^3}{a^3 - b^3}$$

$$y = \sqrt[3]{\frac{db^3}{a^3 - b^3}}$$

$$\text{and } x = \sqrt[3]{\frac{da^3}{a^3 - b^3}}.$$

DIRECTION V.

§. 82. *If there are three unknown quantities, there must be three equations in order to determine them, by comparing which you may, in all cases, find two equations involving only two unknown quantities; and then, by* Direction 3, *from these two you may deduce an equation involving only one unknown quantity; which may be resolved by the Rules of the last Chapter.*"

From three equations involving any three unknown quantities, x, y, and z, to deduce two equa-

equations involving only two unknown quantities, the following Rule will always serve.

RULE.

"*Find three values of x from the three given equations; then, by comparing the first and second value, you will find an equation involving only y and z; again, by comparing the first and third, you will find another equation involving only y and z;" and lastly, those equations are to be resolved by* Direction 3.

EXAMPLE XII.

Suppose

$$\left.\begin{array}{l} x+y+z=12 \\ x+2y+3z=20 \\ \frac{x}{3}+\frac{y}{2}+z=6 \end{array}\right\} \text{then, } x= \left\{\begin{array}{l} 12-y-z \\ 20-2y-3z \\ 18-\frac{3y}{2}-3z \end{array}\right\} \begin{array}{l} \text{1st} \\ \text{2d} \\ \text{3d} \end{array} \Big\} \text{value.}$$

$$12-y-z=20-2y-3z$$

$$12-y-z=18-\frac{3y}{2}-3z$$

These two last equations involve only y and z, and are to be resolved, by *Direction* 3, as follows.

$$\left\{\begin{array}{l} 2y - y + 3z - z = 20 - 12 = 8 \\ \underline{y + 2z = 8} \end{array}\right.$$

$$36 - 3y - 6z = 24 - 2y - 2z$$
$$12 = y + 4z$$

whence $y = \left\{\begin{array}{l} 8 - 2z \ldots \text{1ſt value.} \\ \underline{12 - 4z} \ldots \text{2d value.} \end{array}\right.$

$$8 - 2z = 12 - 4z$$
$$2z = 12 - 8 = 4$$
$$\therefore \text{ and } z = 2$$
$$y (= 8 - 2z) = 4$$
$$x (= 12 - y - z) = 6.$$

§ 83. This method is general, and will extend to all equations that involve three unknown quantities: but there are often eaſier and ſhorter methods to deduce an equation involving one unknown quantity only; which will be beſt learned by practice.

EXAMPLE XIII.

Suppoſing $\left\{\begin{array}{l} x + y + z = 26 \\ x - y = 4 \\ \underline{x - z = 6} \end{array}\right.$

by addition $3x = 36$

$$\left\{\begin{array}{l} x = \frac{36}{3} = 12 \\ y = x - 4 = 8 \\ z = x - 6 = 6 \end{array}\right.$$

EX-

EXAMPLE XIV.

$$\text{Supposing}\begin{cases} x+y=a \\ x+z=b \\ y+z=c \end{cases}$$

$$x=a-y$$

$$a-y+z=b$$

$$y+z=c$$

$$a+0+2z=b+c$$

$$2z=b+c-a$$

$$\begin{cases} z=\dfrac{b+c-a}{2} \\ y\,(=c-z)=\dfrac{c+a-b}{2} \\ x\,(=a-y)=\dfrac{a+b-c}{2}. \end{cases}$$

§ 84. It is obvious from the 3d and 5th Directions, in what manner you are to work if there are four, or more, unknown quantities, and four, or more, equations given. By comparing the given equations, you may always at length discover an equation involving only one unknown quantity; which, if it is a simple equation, may always be resolved by the Rules of the last Chapter. We may conclude then, that " When there are as many simple equations given as quantities required, these quantities may be discovered by the application of the preceding Rules."

§ 85.

§ 85. "If indeed there are more quantities required than equations given, then the question is not limited to determinate quantities; But is capable of an infinite number of solutions." And, "If there are more equations given than there are quantities required, it may be impossible to find the quantities that will answer the conditions of the question;" because some of these conditions may be inconsistent with others.

CHAP. XII.

Containing some General Theorems for the exterminating unknown Quantities in given Equations.

IN the following *Theorems*, we call those coefficients of the "*same order*" that are prefixt to the same unknown quantities in the different equations. Thus, in *Theor.* 2. a, d, g, are of the same order, being the coefficients of x: also b, e, h, are of the same order, being the coefficients of y: and those are of the same order that affect no unknown quantity.

But those are called "*opposite*" coefficients that are taken each from a different equation, and

and from a different order of coefficients: As, a, e, and d, b, in the first Theorem; and a, e, k, in the second; also a, b, f; and d, b, k, &c.

THEOREM I.

§ 86. Suppose that two equations are given, involving two unknown quantities, as

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

then shall $y = \frac{af - dc}{ae - db}$.

Where the numerator is the difference of the products of the opposite coefficients in the orders in which y is not found, and the denominator is the difference of the products of the opposite coefficients taken from the orders that involve the two unknown quantities:

For from the first equation, it is plain that

$$ax = c - by \ldots \text{and } x = \frac{c - by}{a},$$

from the 2d, $dx = f - ey \ldots$ and $x = \frac{f - ey}{d}$,

therefore $\frac{c - by}{a} = \frac{f - ey}{d}$, and $cd - dby = af - aey$;

whence $aey - dby = af - cd$,

and $y = \frac{af - cd}{ae - db}$;

after the same manner, $x = \frac{ce - bf}{ae - db}$.

EX-

EXAMPLE I.

$$\text{Supp.}\begin{cases} 5x + 7y = 100 \\ 3x + 8y = \ \ 80 \end{cases}$$

$$\text{then } y = \frac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \frac{100}{19} = 5\tfrac{5}{19}$$

$$\text{and } x = \frac{240}{19} = 12\tfrac{12}{19}.$$

EXAMPLE II.

$$\begin{cases} 4x + 8y = \ \ 90 \\ 3x - 2y = 160 \end{cases}$$

$$y = \frac{4 \times 160 - 3 \times 90}{4 \times -2 - 3 \times 8} = \frac{640 - 270}{-8 - 24} = \frac{370}{-32} = -11\tfrac{9}{16}$$

THEOREM II.

§ 87. Suppose now that there are three unknown quantities and three equations, then call the unknown quantities x, y, and z.

$$\text{Thus}\begin{cases} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{cases}$$

$$\text{Then shall } z = \frac{aep - ahn + dhm - dbp + gbn - gem}{aek - ahf + dhc - dbk + gbf - gec}.$$

Where the numerator consists of all the different products that can be made of three opposite coefficients taken from the orders in which z is not found; and the denominator consists of all the products that can be made of the three

G opposite

opposite coefficients taken from the orders that involve the three unknown quantities. For, from the last it appears, that

$$y = \frac{an - afz - dm + dcz}{ae - db}, \text{ and that}$$

$$y = \frac{ap - akz - gm + gcz}{ab - gb}; \text{ therefore}$$

$$\frac{an - afz - dm + dcz}{ae - db} = \frac{ap - akz - gm + gcz}{ab - gb}, \text{ and}$$

$$\overline{an - afz - dm + dcz} \times ab - gb \times \overline{an - afz} + gbdm - gbdcz = \overline{ap - gm - akz + gcz} \times ae - db \times \overline{ap - akz} + gbdm - gbdcz.$$

Take $gbdm - gbdcz$ from both sides, and divide by a, so shall

$$\overline{an - dm - afz + dcz} \times b - gbn + gbfz = \overline{ap - gm - akz + gcz} \times e - dbp + dbkz.$$

Transpose and divide, so shall you find

$$z = \frac{aep - abn + dbm - dbp + gen - gem}{aek - abf + dbc - dbk + gbf - gec}.$$ The values of x and y are found after the same manner, and have the same denominator. *Ex. gr.*

$$y = \frac{afp - akn + dkm - dep + gen - gfm}{aek - abf + dbc - dbk + gbf - gec}.$$

If any term is wanting in any of the three given equations, the values of z and y will be found more simple. Suppose, for example, that f and k are equal to nothing, then the term fz will vanish in the second equation, and kz in the third, and $z = \frac{aep - anb + dbm - dbp + gnb - gem}{dbc - gec}$;

$$y = \frac{gcn - dep}{dbc - gec}.$$

If

If four equations are given, involving four unknown quantities, their values may be found much after the ſame manner, by taking all the products that can be made of four oppoſite coefficients, and always prefixing contrary ſigns to thoſe that involve the products of two oppoſite coefficients.

CHAP. XIII.

Of Quadratic EQUATIONS.

§ 88. IN the ſolution of any queſtion where you have got an equation that involves one unknown quantity, but involves at the ſame time the ſquare of that quantity, and the product of it multiplied by ſome known quantity, then you have what is called a *Quadratic equation*; which may be reſolved by the following

RULE.

1. "*Tranſport all the terms that involve the unknown quantity to one ſide, and the known terms to the other ſide of the equation.*
2. *If the ſquare of the unknown quantity is multiplied by any coefficient, you are to divide all the terms by that coefficient, that the coefficient of*

of the square of the unknown quantity may be unit.

3. *Add to both sides the square of half the coefficient prefixed to the unknown quantity itself, and the side of the equation that involves the unknown quantity will then be a complete square.*

4. *Extract the square root from both sides of the equation; which you will find, on one side, always to be the unknown quantity with half the foresaid coefficient joined to it; so that by transposing this half you may obtain the value of the unknown quantity expressed in known terms.*" Thus,

Suppose $y^2 + ay = b$,

Add the square of $\frac{a}{2}$ to both sides $\Big\}$ $y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}$,

Extract the root, $y + \frac{a}{2} = \pm\sqrt{b + \frac{a^2}{4}}$,

Transpose $\frac{a}{2}$, $y = \pm\sqrt{b + \frac{a^2}{4}} - \frac{a}{2}$.

§ 89. The square root of any quantity, as $+aa$, may be $+a$, or $-a$; and hence, "All quadratic equations admit of two solutions." In the last example, after finding that $y^2 + ay + \frac{a^2}{4} = b + \frac{a^2}{4}$, it may be inferred that $y + \frac{a}{2} = +\sqrt{b + \frac{a^2}{4}}$, or to $-\sqrt{b + \frac{a^2}{4}}$; since

—

$-\sqrt{b+\frac{a^2}{4}} \times -\sqrt{b+\frac{a^2}{4}}$ gives $b+\frac{a^2}{4}$, as well as $+\sqrt{b+\frac{a^2}{4}} \times +\sqrt{b+\frac{a^2}{4}}$. There are therefore two values of y; the one gives $y = +\sqrt{b+\frac{a^2}{4}} - \frac{a}{2}$, the other $y = -\sqrt{b+\frac{a^2}{4}} - \frac{a}{2}$.

§ 90. Since the ſquares of all quantities are poſitive, it is plain that "The ſquare root of a negative quantity is imaginary, and cannot be aſſigned." Therefore there are ſome quadratic equations that cannot have any ſolution. For example,

Suppoſe $y^2 - ay + 3a^2 = 0$,
then $y^2 - ay = -3a^2$;
add $\frac{a^2}{4}$ to both, $y^2 - ay + \frac{a^2}{4} = -3a^2 + \frac{a^2}{4} = -\frac{11a^2}{4}$,
extract the root, $y - \frac{a}{2} = \pm\sqrt{-\frac{11a^2}{4}}$,
and $y = \frac{a}{2} \pm \sqrt{-\frac{11a^2}{4}}$;

whence the two values of y muſt be imaginary or impoſſible, becauſe the root of $-\frac{11a^2}{4}$ cannot poſſibly be aſſigned.

But of this we ſhall treat more fully in the Second Part.

Suppofe that the quadratic equation propofed to be refolved is $y^2 - ay = b$;

$$\text{then } y^2 - ay + \frac{a^2}{4} = b + \frac{a^2}{4},$$

$$y - \frac{a}{2} = \pm\sqrt{b + \frac{a^2}{4}},$$

$$y = \frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}.$$

If the fquare root of $b + \frac{a^2}{4}$ cannot be extracted exactly, you muft, in order to determine the value of y, nearly approximate to the value of $\sqrt{b + \frac{a^2}{4}}$, by the Rules in *Chap.* 8. The following examples will illuftrate the Rule for quadratic equations.

EXAMPLE I.

To find that number, which if you multiply by 8, the product fhall be equal to the fquare of the fame number, having 12 *added to it.*

Call the number y; then

$$y^2 + 12 = 8y,$$

$$\text{tranfp. } y^2 - 8y = -12,$$

$$\text{Add the fq. of } 4,\ y^2 - 8y + 16 = -12 + 16 = 4,$$

$$\text{extract the root } y - 4 = \pm 2,$$

$$\text{tranfp. } y = 4 \pm 2 = 6, \text{ or } 2.$$

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EXAMPLE II.

To find a number such that if you subtract it from 10, *and multiply the remainder by the number itself, the product shall give* 21.

Call it y; then

$$\overline{10-y}\times y=21.$$

that is, $10y-yy=21$;

transp. $y^2-10y=-21$,

add the sq. of 5, $y^2-10y+25=-21+25=4$,

extr. the sq. root $y-5=\pm\sqrt{4}=\pm 2$,

and $y=5\pm 2=7$, or 3.

EXAMPLE III.

The sum of two quantities is a, their product b. Qu. *What are the quantities?*

Suppose $\begin{cases} x+y=a\ldots\text{then } x=a-y, \\ xy=b\ldots\ldots\text{then } x=\dfrac{b}{y}, \end{cases}$

therefore $a-y=\dfrac{b}{y}$,

and $ay-y^2=b$;

transp. $y^2-ay=-b$,

add $\dfrac{a^2}{4}\ldots y^2-ay+\dfrac{a^2}{4}=-b+\dfrac{a^2}{4}$,

extract $\sqrt{}$, $y-\dfrac{a}{2}=\pm\sqrt{-b+\dfrac{a^2}{4}}$,

and $y=\dfrac{a}{2}\pm\sqrt{-b+\dfrac{a^2}{4}}$,

$x\,(=a-y)=\dfrac{a}{2}\pm\sqrt{-b+\dfrac{a^2}{4}}$.

EXAMPLE IV.

The sum of two quantities is a, and the sum of their squares b. Qu. *the quantities?*

Suppose $\begin{cases} x + y = a \ldots \text{then } x = a - y, \\ x^2 + y^2 = b \ldots\ldots x^2 = b - y^2, \end{cases}$

invol. $x^2 = a^2 - 2ay + y^2$

whence $a^2 - 2ay + y^2 = b - y^2$;

transp. and divide $\begin{cases} 2y^2 - 2ay = b - a^2, \\ y^2 - ay = \dfrac{b - a^2}{2}, \end{cases}$

add $\dfrac{a^2}{4}$, $y^2 - ay + \dfrac{a^2}{4} = \dfrac{b - a^2}{2} + \dfrac{a^2}{4} = \dfrac{2b - a^2}{4}$,

extr. $\surd$, $y - \dfrac{a}{2} = \pm\sqrt{\dfrac{2b - a^2}{4}}$; and $y = \dfrac{a}{2}$
$\pm\sqrt{\dfrac{2b - a^2}{4}}$; $x (= a - y) = \dfrac{a}{2} \mp \sqrt{\dfrac{2b - a^2}{4}}$.

Or thus, $y = \dfrac{a \pm \sqrt{2b - a^2}}{2}$, and $x = \dfrac{a \mp \sqrt{2b - a^2}}{2}$.

EXAMPLE V.

A company dining together in an inn, find their bill amounts to 175 *shillings; two of them were not allowed to pay, and the rest found that their shares amounted to* 10s. *a man more than if all had paid.* Qu. *How many were in company?*

Suppose their number x; then if all had paid each man's share would have been $\frac{175}{x}$, seeing $x - 2$ is the number of those that pay. It is therefore, by the question,

175

$$\frac{175}{x-2} - \frac{175}{x} = 10,$$

and $175x - 175x + 350 = 10x^2 - 20x$;
that is, $10x^2 - 20x = 350$,
and $x^2 - 2x = 35$;
add $1 \ldots x^2 - 2x + 1 = 35 + 1 = 36$.
extr. $\surd \ldots x - 1 = \pm 6$,
$x = 1 \pm 6 = 7$, or -5.

It is obvious that the positive value 7 gives the solution of the question; the negative value -5 being, in the present case, useless.

EXAMPLE VI.

There are three numbers in continual geometrical proportion; the sum of the first and second is 10, *and the difference of the second and* 3*d is* 24. Qu. *the numbers?*

Let the first be x, and the second will be $10-x$, and the third, $34-x$; therefore,

$$x : 10 - x :: 10 - x : 34 - x,$$

and $34x - x^2 = 100 - 20x + x^2$;

transp. and divid. $\begin{cases} 54x = 100 + 2x^2, \\ x^2 - 27x = -50, \end{cases}$

add $\frac{27}{2} \times \frac{27}{2} \ldots x^2 - 27x + \frac{729}{4} = \frac{729}{4} - 50 = \frac{529}{4}$,

extract $\surd \ldots x - \frac{27}{2} = \pm \sqrt{\frac{529}{4}} = \pm \frac{23}{2}$,

and $x = \frac{27+23}{2}$, or $= \frac{27-23}{2} = 25$, or 2.

So

So the three continued proportionals are

$$2 : 8 : 32, \text{ or}$$
$$25 : -15 : 9$$

§91. Any equation of this form $y^{2m} + ay^m = b$, where the greatest index of the unknown quantity y is double to the index of y in the other term, may be reduced to a quadratic $z^2 + az = b$, by putting $y^m = z$, and consequently $y^{2m} = z^2$. And this quadratic resolved as above, gives $z = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}$.

And seeing $y^m = z = -\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}$, $y = \sqrt[m]{-\frac{a}{2} \pm \sqrt{b + \frac{a^2}{4}}}$.

EXAMPLE I.

The product of two quantities is a, and the sum of their squares b. Qu. the quantities?

$$\text{Supp.} \begin{cases} xy = a \ldots\ldots \text{or, } x = \frac{a}{y},\ x^2 = \frac{a^2}{y^2}, \\ x^2 + y^2 = b \ldots x^2 = b - y^2, \end{cases}$$

whence $b - y^2 = \frac{a^2}{y^2}$;

mult. by y^2.. $by^2 - y^4 = a^2$,

transp. $y^4 - by^2 = -a^2$.

Put now $y^2 = z$... and consequently $y^4 = z^2$, and it is

$z^2 -$

$$z^2 - bz = -a^2;$$

add $\frac{b^2}{4}$, $z^2 - bz + \frac{b^2}{4} = \frac{b^2}{4} - a^2$,

ext. $\surd$, $z - \frac{b}{2} = \pm\sqrt{\frac{b^2}{4} - a^2}$,

and $z = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}$; and, seeing $y = \sqrt{z}$,

$$y = \pm\sqrt{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}}.$$

EXAMPLE II.

To find a number from the cube of which if you subtract 19, *and multiply the remainder by that cube, the product shall be* 216.

Call the number required x; and then, by the question,

$$\overline{x^3 - 19} \times x^3 = 216,$$
$$x^6 - 19x^3 = 216.$$

Put $x^3 = z \ldots\ldots x^6 = z^2$, and it will be

$$z^2 - 19z + \frac{361}{4} = 216 + \frac{361}{4} = \frac{1225}{4},$$

and $\surd \ldots z - \frac{19}{2} = \pm \frac{35}{2}$;

whence $z = \frac{19 \pm 35}{2} = 27$, or $= -8$.

But $x = \sqrt[3]{z}$; wherefore $x = +3$, or -2.

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EXAMPLE III.

To find the value of x, supposing that $x^3 - 7x^{\frac{3}{2}} = 8$.

$$\text{Put } x^{\frac{3}{2}} = z, \text{ and } x^3 = z^2;$$
$$\text{then } z^2 - 7z = 8,$$
$$z^2 - 7z + \frac{49}{4} = \frac{81}{4},$$
$$z - \frac{7}{2} = \pm \frac{9}{2},$$
$$z = 8.$$

But $x^3 = z^2$, and $x = \sqrt[3]{z^2} = \sqrt[3]{64} = 4$.

CHAP. XIV.

Of Surds.

§ 92. If the lesser quantity measures a greater so as to leave no remainder, as $2a$ measures $10a$, being found in it five times, it is said to be an *aliquot* part of it, and the greater is said to be a *multiple* of the lesser. The lesser quantity in this case is the *greatest common measure* of the two quantities; for as it measures the greater, so it also measures itself, and no quantity can measure it that is greater than itself.

When a third quantity measures any two proposed quantities, as $2a$ measures $6a$ and $10a$, it is is ſaid to be a *common meaſure* of theſe quantities; and if no greater quantity meaſure them both, it is called their greateſt *common meaſure.*

Thoſe quantities are ſaid to be *commenſurable* which have any common meaſure; but if there can be no quantity found that meaſures them both, they are ſaid to be *incommenſurable*; and if any one quantity be called *rational*, all others that have any common meaſure with it, are alſo called rational: But thoſe that have no common meaſure with it, are called *irrational* quantities.

§ 93. If any two quantities a and b have any common meaſure x, this quantity x ſhall alſo meaſure their ſum and difference $a \mp b$. Let x be found in a as many times as unit is found in m, ſo that $a = mx$; and in b, as many times as unit is found in n, ſo that $b = nx$; then ſhall $a \mp b = mx \mp nx = \overline{m \mp n} \times x$; ſo that x ſhall be found in $a \mp b$, as often as unit is found in $m \mp n$: Now ſince m and n are integer numbers, $m \mp n$ muſt be an integer number or unit, and therefore x muſt meaſure $a \mp b$.

§ 94. It is alſo evident, that if x meaſure any number as a, it muſt meaſure any multiple of that number. If it be found in a as many times as unit is found in m, ſo that $a = mx$, then it will be found in any multiple of a, as na, as many times as unit is found in mn; for $na = mnx$.

§ 95.

§ 95. If two quantities a and b are proposed, and b measure a by the units that are in m (that is, be found in a as many times as unit is found in m) and there be a remainder c; and if x be supposed to be a common measure of a and b, it shall be also a measure of c. For by the supposition $a = mb + c$, since it contains b as many times as there are units in m, and there is c besides of remainder; therefore $a - mb = c$. Now x is supposed to measure a and b, and therefore it measures mb (*Art.* 94.) and consequently $a - mb$ (*Art.* 93.) which is equal to c.

If c measures b by the units in n, and there be a remainder d, so that $b = nc + d$, and $b - nc = d$, then shall x also measure d; because it is supposed to measure b, and it has been proved that it measures c, and consequently nc, and $b - nc$ (by *Art.* 94.) which is equal to d. Whence, as after subtracting b as often as possible from a, the remainder c is measured by x; and after subtracting c as often as possible from b, the remainder d is also measured by x; so, for the same reason, if you subtract d as often as possible from c, the remainder (if there be any) must still be measured by x: and if you proceed, still subtracting every remainder from the preceding remainder, till you find some remainder which subtracted from the preceding leaves no further remainder, but exactly measures it, this

last

last remainder will still be measured by x, any common measure of a and b.

§ 96. The last of these remainders, *viz*, that which exactly measures the preceding remainder, must be a common measure of a and b: suppose that d was this last remainder, and that it measured c by the units in r, then shall $c = rd$, and we shall have these equations,

$$a = mb + c,$$
$$b = nc + d,$$
$$c = rd.$$

Now it is plain that since d measures c, it must also measure nc, and therefore must measure $nc + d$, or b. And since it measures b and c, it must measure $mb + c$, or a; so that it must be a common measure of a and b. But further, it must be their *greatest* common measure; for every common measure of a and b must measure d, by the last article; and the greatest number that measures d, is itself, which therefore is the greatest common measure of a and b.

§ 97. But if, by continually subtracting every remainder from the preceding remainder, you can never find one that measures that which precedes it, exactly, no quantity can be found that will measure both a and b; and therefore they will be *incommensurable* to each other.

For if there was any common measure of these quantities, as x, it would necessarily measure all

all the remainders c, d, &c. For it would measure $a-mb$, or c, and consequently $b-nc$, or d; and so on. Now these remainders decrease in such a manner, that they will necessarily become at length less than x, or any assignable quantity: for c must be less than $\frac{1}{2}a$; because c is less than b, and therefore less than mb, and consequently less than $\frac{1}{2}c+\frac{1}{2}mb$, or $\frac{1}{2}a$. In like manner d must be less than $\frac{1}{2}b$, for d is less than c, and consequently less than $\frac{1}{2}d+\frac{1}{2}nc$, or $\frac{1}{2}b$. The third remainder, in the same manner, must be less than $\frac{1}{2}c$, which is itself less than $\frac{1}{4}a$: thus these remainders decrease so, that every one is less than the half of that which preceded it next but one. Now if from any quantity you take away more than its half, and from the remainder more than its half, and proceed in this manner, you will come at a remainder less than any assignable quantity. It appears therefore that if the remainders c, d, &c. never end, they will become less than any assignable quantity, as x, which therefore cannot possibly measure them, and therefore cannot be a common measure of a and b.

§ 98. In the same way, the greatest common measure of two numbers is discovered. Unit is a common measure of all integer numbers, and two numbers are said to be *prime* to each other, when they have no greater common measure than unit; such as 9 and 25. Such always

are

are the leaſt numbers that can be aſſumed in any given proportion; for if theſe had any common meaſure, then the quotients that would ariſe by dividing them by that common meaſure would be in the ſame proportion, and being leſs then the numbers themſelves, theſe numbers would not be the leaſt in the ſame proportion; againſt the ſuppoſition.

§ 99. The leaſt numbers in any proportion always meaſure any other numbers that are in the ſame proportion. Suppoſe a and b to be the leaſt of all integer numbers in the ſame proportion, and that c and d are other numbers in that proportion, then will a meaſure c, and b meaſure d.

For if a and b are not aliquot parts of c and d, then they muſt contain the ſame number of the ſame kind of parts of c and d, and therefore dividing a into parts of c, and b into an equal number of like parts of d, and calling one of the firſt m, and one of the latter n; then as m is to n, ſo will the ſum of all the ms be to the ſum of all the ns; that is, $m : n :: a : b$; therefore a and b will not be the leaſt in the ſame proportion; againſt the ſuppoſition. Therefore a and b muſt be aliquot parts of c and d. Hence we ſee that numbers which are prime to each other are the leaſt in the ſame proportion; for if there were others in the ſame proportion leſs than them, theſe would meaſure them by the ſame number, which therefore would be their common meaſure againſt the ſuppoſition, for we ſuppoſed them to be prime to each other.

§ 100. If two numbers a and b are prime to one another, and a third number c meaſures one of them a, it will be prime to the other b. For if c and b were not prime to each other, they would have a common meaſure, which becauſe it would meaſure c, would alſo meaſure a, which is meaſured by c, therefore a and b would have a common meaſure, againſt the ſuppoſition.

§ 101. If two numbers a and b are prime to c, then ſhall their product ab be alſo prime to c: For if you ſuppoſe them to have any common meaſure as d, and ſuppoſe that d meaſures ab by the units in e, ſo that $de = ab$, then ſhall $d : a :: b : e$. But ſince d meaſures c, and c is ſuppoſed to be prime to a, it follows (by *Art.* 100.) that d and a are prime to each other; and therefore (by *Art.* 99.) d muſt meaſure b; and yet ſince d is ſuppoſed to meaſure c, which is prime to b, it follows that d is alſo prime to b: that is, d is prime to a number which it meaſures, which is abſurd.

§ 102. It follows from the laſt article, that if a and c are prime to each other, then a^2 will be prime to c: For by ſuppoſing that a is equal to b, then ab will be equal to a^2; and conſequently a^2 will be prime to c. In the ſame manner c^2 will be ptime to a.

§ 103.

§ 103. If two numbers a and b, are both prime to other two c, d, then shall the product ab be prime to the product cd; for (by *Art.* 101.) ab will be prime to c and also to d, and therefore, by the same article, cd will be prime to ab.

§ 104. From this it follows, that if a and c are prime to each other, then shall a^2 be prime to c^2, by supposing, in the last, that $a = b$, and $c = d$. It is also evident that a^3 will be prime to c^3, and in general any power of a to any power of c whatsoever.

§ 105. Any two numbers, a and b, being given, to find the least numbers that are in the same proportion with them, *divide them by their greatest common measure* x, and *the quotients* c *and* d *shall be the least numbers in the same proportion with* a *and* b.

For if there could be any other numbers in that proportion less than c and d, suppose them to be e and f, and these being in the same proportion as a and b would measure them: And the number by which they would measure them, would be greater than x, because e and f are supposed less than c and d, so that x would not be the greatest common measure of a and b; against the supposition.

§ 106. Let it be required to find the least number that any two given numbers as a and b can measure. First, *if they are prime to each*

 other,

other, then their product ab is the least number which they can both measure.

For if they could measure a less number than ab as c, suppose that c is equal to ma, and to nb; and since c is less than ab, therefore ma will be less than ab, and m less than b; and nb being less than ab, it follows that n must be less than a; but since $ma = nb$, and consequently $a : b :: n : m$, and a and b are prime to each other, it would follow that a would measure n, and b measure m; that is, a greater number would measure a less, which is absurd.

But if the numbers a and b are not prime to each other, and their greatest common measure is x, which measures a by the units in m, and measures b by the units in n, so that $a = mx$, and $b = nx$; then shall an (which is equal to bm, because $a : b :: mx : nx : m : n$, and therefore $an = bm$) be the least number that a and b can both measure. For if they could measure any number c less than na, so that $c = la = kb$, then $a : b :: m : n :: k : l$; and because x is supposed to be the greatest common measure of a and b, it follows that m and n are the least of all numbers in the same proportion, and therefore m measures k, and n measures l. But as c is supposed to be less than na, that is, la less than na, therefore l is less than n, so that a greater would measure a lesser, which is absurd. Therefore a and b cannot measure any number less than

na;

na; which they both meaſure, becauſe $na = mb$.

It follows from this reaſoning, that if a and b meaſure any quantity c, the leaſt quantity na, which is meaſured by a and b, will alſo meaſure c. For if you ſuppoſe as before that $c = la$, you will find that n muſt meaſure l, and na muſt meaſure la or c.

§ 107. Let a expreſs any integer number, and $\frac{m}{n}$ any fraction reduced to its loweſt terms, ſo that m and n may be prime to each other, and conſequently $an + m$ alſo prime to n, it will follow that $\overline{an + m}^2$ will be prime to n^2, and conſequently $\frac{\overline{an + m}^2}{n^2}$ will be a fraction in its leaſt terms, and can never be equal to an integer number. Therefore the ſquare of the mixt number $a + \frac{m}{n}$ is ſtill a mixt number, and never an integer. In the ſame manner the cube, biquadrate, or any power of a mixt number, is ſtill a mixt number, and never an integer. It follows from this, that *the ſquare root of an integer muſt be an integer or an incommenſurable.* Suppoſe that the integer propoſed is B, and that the ſquare root of it is leſs than $a + 1$, but greater than a, than it muſt be an incommenſurable; for if it is a commenſurable, let it be $a + \frac{m}{n}$ where $\frac{m}{n}$ repreſents any fraction reduced to its leaſt terms; it would follow that $a + \frac{m}{n}$ ſquared would give an integer number B, the contrary of which we have demonſtrated.

§ 108. It follows from the laſt article, that *the ſquare roots of all numbers but of* 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, &c. (which are the ſquares of the integer numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, *&c.*) *are incommenſurables*; after the ſame manner, *the cube roots of all numbers but of the cubes of* 1, 2, 3, 4, 5, 6, 7, 8, 9, *&c.* are *incommenſurables:* and quantities that are to one another in the proportion of ſuch numbers muſt alſo have their ſquare roots or cube roots incommenſurable.

§ 109. The roots of ſuch numbers being incommenſurable are expreſſed therefore by placing the proper radical ſign over them; thus, $\sqrt[2]{2}$, $\sqrt[2]{3}$, $\sqrt[2]{5}$, $\sqrt[2]{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, *&c.* expreſs numbers incommenſurable with unit. Theſe numbers, though they are incommenſurable themſelves with unit, are *commenſurable in power* with it, becauſe their powers are integers, that is, multiples of unit. They may alſo be commenſurable ſometimes with one another, as the $\sqrt[2]{8}$, and the $\sqrt[2]{2}$, becauſe they are to one another as 2 to 1: And when they have a common meaſure, as $\sqrt[2]{2}$ is the common meaſure of of both, then their ratio is reduced to an expression in the least terms, as that of commensurable quantities, by dividing them by their greatest common measure. This common measure is found as in commensurable quantities, only the root of the common measure is to be made their common divisor. Thus $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{4} = 2$, and $\frac{\sqrt{18a}}{\sqrt{2}} = 3\sqrt{a}$.

§ 110. A rational quantity may be reduced to the form of any given surd, by raising the quantity to the power that is denominated by the name of the surd, and then setting the radical sign over it thus, $a = \sqrt[2]{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4} = \sqrt[5]{a^5} = \sqrt[n]{a^n}$, and $4 = \sqrt{16} = \sqrt[3]{64} = \sqrt[4]{256} = \sqrt[5]{1024} = \sqrt[n]{4^n}$.

§ 111. As surds may be considered as powers with fractional exponents, *they are reduced to others of the same value that shall have the same radical sign, by reducing those fractional exponents to fractions having the same value and a common denominator.* Thus $\sqrt[n]{a} = a^{\frac{1}{n}}$, and $\sqrt[m]{a} = a^{\frac{1}{m}}$, and $\frac{1}{n} = \frac{m}{nm}$, $\frac{1}{m} = \frac{n}{nm}$, and therefore $\sqrt[n]{a}$ and $\sqrt[m]{a}$, reduced to the same radical

 sign,

ſign, become $\sqrt[nm]{a^m}$ and $\sqrt[nm]{a^n}$. If you are to reduce $\sqrt[2]{3}$ and $\sqrt[3]{2}$ to the ſame denominator, conſider $\sqrt[2]{3}$ as equal to $3^{\frac{1}{2}}$, the $\sqrt[3]{2}$ as equal to $2^{\frac{1}{3}}$, whoſe indices reduced to a common denominator, you have $3^{\frac{1}{2}} = 3^{\frac{3}{6}}$ and $2^{\frac{1}{3}} = 2^{\frac{2}{6}}$, and conſequently $\sqrt[2]{3} = \sqrt[6]{3^3} = \sqrt[6]{27}$, and $\sqrt[3]{2} = \sqrt[6]{2^2} = \sqrt[6]{4}$; ſo that the propoſed ſurds $\sqrt[2]{3}$ and $\sqrt[3]{2}$ are reduced to other equal ſurds $\sqrt[6]{27}$ and $\sqrt[6]{4}$, having a common radical ſign.

§ 112. *Surds of the ſame rational quantity are multiplied by adding their exponents, and divided by ſubtracting them.*

Thus $\sqrt[2]{a} \times \sqrt[3]{a} = a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{3+2}{6}} = a^{\frac{5}{6}} = \sqrt[6]{a^5}$; and $\dfrac{\sqrt[3]{a}}{\sqrt[5]{a}} = \dfrac{a^{\frac{1}{3}}}{a^{\frac{1}{5}}} = a^{\frac{1}{3}-\frac{1}{5}} = a^{\frac{5-3}{15}} = a^{\frac{2}{15}} = \sqrt[15]{a^2}$; $\sqrt[m]{a} \times \sqrt[n]{a} = a^{\frac{m+n}{mn}}$; $\dfrac{\sqrt[m]{a}}{\sqrt[n]{a}} = a^{\frac{n-m}{mn}}$; $\sqrt[2]{2} \times \sqrt[3]{2} = \sqrt[6]{2^5} = \sqrt[6]{32}$; $\dfrac{\sqrt[2]{2}}{\sqrt[3]{2}} = \sqrt[6]{2}$.

§ 113. If the ſurds are of different rational quantities, as $\sqrt[n]{a^2}$ and $\sqrt[n]{b^5}$, and have the ſame ſign, *multiply theſe rational quantities into one another*

another, or divide them by one another, and set the common radical sign over their product or quotient. Thus $\sqrt[n]{a^2}\times\sqrt[n]{b^3}=\sqrt[n]{a^2b^3}$; $\sqrt[2]{2}\times\sqrt[2]{5}=\sqrt[2]{10}$; $\dfrac{\sqrt[m]{a^4}}{\sqrt[m]{b^3a}}=\sqrt[m]{\dfrac{a^4}{b^3a}}=\sqrt[m]{\dfrac{a^3}{b^3}}$; $\dfrac{\sqrt[3]{9}}{\sqrt[3]{24}}=\sqrt[3]{\dfrac{9}{24}}=\sqrt[3]{\dfrac{3}{8}}=\frac{1}{2}\sqrt[3]{3}$.

If the surds have not the same radical sign, *reduce them by the 111th* Art. *to such as shall have the same radical sign, and proceed as before.* Thus

$\sqrt[m]{a}\times\sqrt[n]{b}=\sqrt[nm]{a^nb^m}$; $\dfrac{\sqrt[m]{a}}{\sqrt[n]{x}}=\sqrt[mn]{\dfrac{a^n}{x^m}}$; $\sqrt[2]{2}\times\sqrt[3]{4}=$

$2^{\frac{1}{2}}\times4^{\frac{1}{3}}=2^{\frac{3}{6}}\times4^{\frac{2}{6}}=\sqrt[6]{2^3\times4^2}=\sqrt[6]{8\times16}$

$=\sqrt[6]{128}$; $\dfrac{\sqrt[3]{4}}{\sqrt[2]{2}}=\dfrac{4^{\frac{1}{3}}}{2^{\frac{1}{2}}}=\dfrac{4^{\frac{2}{6}}}{2^{\frac{3}{6}}}=\sqrt[6]{\dfrac{4^2}{2^3}}=\sqrt[6]{\dfrac{16}{8}}$

$=\sqrt[6]{2}$. If the surds have any rational coefficients, their product or quotient must be prefixed. Thus $2\sqrt[2]{3}\times5\sqrt[2]{6}=10\sqrt[2]{18}$.

§ 114. The powers of surds are found as the powers of other quantities, *by multiplying their exponents by the index of the power required.* Thus the square of $\sqrt[3]{2}$ is $2^{\frac{1}{3}\times2}=2^{\frac{2}{3}}=\sqrt[3]{4}$; the cube

cube of $\sqrt[2]{5} = 5^{\frac{1}{2}\times 3} = 5^{\frac{3}{2}} = \sqrt[2]{125}$. Or you need only, in involving ſurds, *raiſe the quantity under the radical ſign to the power required, continuing the ſame radical ſign*; *unleſs the index of that power is equal to the name of the ſurd, or a multiple of it, and in that caſe the power of the ſurd becomes rational.* Evolution is performed *by dividing the fraction which is the exponent of the ſurd by the name of the root required.*

Thus the ſquare root of $\sqrt[3]{a^4}$ is $\sqrt[3]{a^2}$, or $\sqrt[6]{a^4}$.

§ 115. The ſurd $\sqrt[m]{a^m x} = a\sqrt[m]{x}$; and the like manner, if a power of any quantity of the ſame name with the ſurd divides the quantity under the radical ſign without a remainder, as here a^m divides $a^m x$, and 25 the ſquare of 5 divides 75, the quantity under the ſign in $\sqrt[2]{75}$, without a remainder, then place the root of that power rationally before the ſign, and the quotient under the ſign, and thus the ſurd will be reduced to a more ſimple expreſſion. Thus $\sqrt[2]{75} = 5\sqrt{3}$; $\sqrt[2]{48} = \sqrt[2]{3 \times 16} = 4\sqrt[2]{3}$; $\sqrt[3]{81} = \sqrt[3]{27 \times 3} = 3\sqrt[3]{3}$.

§ 116. When ſurds by the laſt article are rereduced to their leaſt expreſſions, if they have the ſame irrational part, they are added or ſubtracted.

tracted, *by adding or subtracting their rational coefficients, and prefixing the sum or difference to the common irrational part.*

Thus $\sqrt[2]{75}+\sqrt[2]{48}=5\sqrt{3}+4\sqrt{3}=9\sqrt{3}$; $\sqrt[3]{81}+\sqrt[3]{24}=3\sqrt[3]{3}+2\sqrt[3]{3}=5\sqrt[3]{3}$; $\sqrt[2]{150}-\sqrt[2]{54}=5\sqrt{6}-3\sqrt{6}=2\sqrt{6}$; $\sqrt{a^2x}+\sqrt{b^2x}=a\sqrt{x}+b\sqrt{x}=\overline{a+b}\times\sqrt{x}$.

§ 117. *Compound surds* are such as consist of two or more joined together. The simple surds are commensurable in power, and by being multiplied into themselves give at length rational quantities; yet compound surds multiplied into themselves commonly give still irrational products. But when any compound surd is proposed, *there is another compound surd which multiplied into it gives a rational product.* Thus $\sqrt{a}+\sqrt{b}$ multiplied by $\sqrt{a}-\sqrt{b}$ gives $a-b$, and *the investigation of that surd which multiplied into the proposed surd will give a rational product*, is made easy by the following Theorems.

THEOREM I.

§ 118. Generally, if you multiply a^m-b^m by $a^{n-m}+a^{n-2m}b^m+a^{n-3m}b^{2m}+a^{n-4m}b^{3m}$, &c. continued till the terms be in number equal to $\frac{n}{m}$, the product shall be a^n-b^n: for

a^{n-m}

$a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m} + a^{n-4m}b^{3m}$, &c. . . b^{n-m}
$\times \overline{a^m - b^m}$

$a^n + a^{n-m}b^m + a^{n-2m}b^{2m} + a^{n-3m}b^{3m}$, &c.
$\quad - a^{n-m}b^m - a^{n-2m}b^{2m} - a^{n-3m}b^{3m}$, &c. $- b^n$

$a^n \quad * \quad * \quad * \quad - b^n$

THEOREM II.

$a^{n-m} - a^{n-2m}b^m + a^{n-3m}b^{2m} - a^{n-4m}b^{3m}$, &c. multiplied by $\overline{a^m + b^m}$, gives $a^n \mp b^n$, which is demonstrated as the other. Here the sign of b^n is positive, when $\frac{n}{m}$ is an odd number.

§ 119. When any binomial surd is proposed, *suppose the index of each number equal to m, and let n be the least integer number that is measured by m, then shall* $a^{n-m} \pm a^{n-2m}b^m + a^{n-3m}b^{2m}$, &c. *give a compound surd, which multiplied into the proposed surd* $a^m \mp b^m$ *will give a rational product.* Thus to find the surd which multiplied by $\sqrt[3]{a} - \sqrt[3]{b}$, will give a rational quantity. Here $m = \frac{1}{3}$, and the least number which is measure by $\frac{1}{3}$ is unit; let $n = 1$, then shall $a^{n-m} + a^{n-2m}b^m + a^{n-3m}b^{2m}$, &c. $= a^{1-\frac{1}{3}} + a^{1-\frac{2}{3}}b^{\frac{1}{3}} + a^0 b^{\frac{2}{3}} = a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}} = \sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$, which multiplied by $\overline{\sqrt[3]{a} - \sqrt[3]{b}}$, gives $a - b$.

To

To find the surd which multiplied by $\sqrt[4]{a^3} + \sqrt[4]{b^3} = a^{\frac{3}{4}} + b^{\frac{3}{4}}$, gives a rational product. Here $m = \frac{3}{4}$ and $n = 3$, and $a^{n-m} - a^{n-2m}b^m + a^{n-3m}b^{2m}$, &c. $= a^{3-\frac{3}{4}} - a^{3-\frac{6}{4}}b^{\frac{3}{4}} + a^{3-\frac{9}{4}}b^{\frac{3}{2}} - a^{3-3}b^{\frac{9}{4}} = a^{\frac{9}{4}} - a^{\frac{6}{4}}b^{\frac{3}{4}} + a^{\frac{3}{4}}b^{\frac{6}{4}} - b^{\frac{9}{4}} = \sqrt[4]{a^9} - \sqrt[4]{a^6b^3} + \sqrt[4]{a^3b^6} - \sqrt[4]{b^9}$.

THEOREM III.

§ 120. Let $a^m \pm b^l$ be multiplied by $a^{n-m} \mp a^{n-2m}b^l + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l} +$ *&c.* and the product shall give $a^n \pm b^{\frac{n}{m}l}$: *therefore n must be taken the least integer that shall give $\frac{nl}{m}$ also an integer.*

Dem. $a^{n-m} \mp a^{n-2m}b^l + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l} \ldots\ldots$

$$
\begin{array}{llllll}
\times\, \overline{a^m \pm b^l} & & & & & [\ldots\text{&c. } a^0 b^{\overline{\frac{n}{m}-1}\times l} \\
\hline
a^n & \mp a^{n-m}b^l & + a^{n-2m}b^{2l}, \text{ &c.} & & & \\
 & \mp a^{n-m}b^l & - a^{n-2m}b^{2l}, \text{ &c.} & & & \pm b^{\frac{n}{m}l} \\
\hline
a^n & * & * & * & & \pm b^{\frac{n}{m}l}
\end{array}
$$

The sign of $b^{\frac{nl}{m}}$ is positive only when $\frac{n}{m}$ is an odd number, and the binomial proposed is $a^m + b^l$.

§ 121.

§ 121. If any binomial surd is proposed whose two numbers have different indices, let these be m and l, and take n equal to the least integer number that is measured by m and by $\frac{m}{l}$; and $a^{n-m} \mp a^{n-2m}b^{l} + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l}$, &c. shall give a compound surd, which multiplied by the proposed $a^{m} \pm b^{l}$ shall give a rational product. Thus $\sqrt[2]{a} - \sqrt[3]{b}$ being given, suppose $m = \frac{1}{2}$, $l = \frac{1}{3}$, and $\frac{m}{l} = \frac{3}{2}$, therefore you have $n = 3$, and $a^{n-m} + a^{n-2m}b^{l} + a^{n-3m}b^{2l} + a^{n-4m}b^{3l}$ + &c. $= a^{3-\frac{1}{2}} + a^{3-1}b^{\frac{1}{3}} + a^{3-\frac{3}{2}}b^{\frac{2}{3}} + a^{3-2}b + a^{3-\frac{5}{2}}b^{\frac{4}{3}} + a^{0}b^{\frac{5}{3}} = a^{\frac{5}{2}} + a^{2}b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} + ab + a^{\frac{1}{2}}b^{\frac{4}{3}} + b^{\frac{5}{3}} = \sqrt{a^{5}} + a^{2} \times \sqrt[3]{b} + \sqrt[2]{a^{3}} \times \sqrt[3]{b^{2}} + ab + \sqrt[2]{a} \times \sqrt[3]{b^{4}} + \sqrt[3]{b^{5}} = a^{2}\sqrt{a} + a^{2} \times \sqrt[3]{b} + a\sqrt{a} \times \sqrt[3]{b^{2}} + ab + b\sqrt{a} \times \sqrt[3]{b} + b \times \sqrt[3]{b^{2}}$, which multiplied by the $\sqrt[2]{a} - \sqrt[3]{b}$, gives $a^{n} - b^{\frac{nl}{m}} = a^{3} - b^{2}$.

§ 122. By these Theorems any binomial surd whatsoever being given, you may find a surd, which multiplied by it shall give a rational product.

Suppose that a binomial surd was to be divided by another, as $\sqrt[2]{20} + \sqrt[2]{12}$, by $\sqrt[2]{5} - \sqrt[2]{3}$, the

the quotient may be expreſſed by $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$. But it may be expreſſed in a more ſimple form by *multiplying both numerator and denominator by that ſurd which multiplied into the denominator gives a rational product.* Thus $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}} =$

$$\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} = \frac{\sqrt{100+2\sqrt{60}+6}}{5-3}$$
$$= \frac{16+2\sqrt{60}}{2} = 8+2\sqrt{15}.$$

§ 123. In general, when any quantity is divided by a binomial ſurd, as $a^m \pm b^l$, where m and l repreſent any fractions whatſoever, *take n the leaſt integtr number that is meaſured by m and $\frac{m}{l}$, multiply both numerator and denominator by $a^{n-m} + a^{n-2m}b^l + a^{n-3m}b^{2l}$,* &c. *and the denominator of the product will become rational, and equal to $a^n - b^{\frac{nl}{m}}$; then divide all the members of the numerator by this rational quantity, and the quote ariſing will be that of the propoſed quantity divided by the binomial ſurd, expreſſed in its leaſt terms.*

Thus $\frac{3}{\sqrt{5}-\sqrt{2}} = \frac{3\sqrt{5}+3\sqrt{2}}{3} = \sqrt{5}+\sqrt{2}$;

$$\frac{\sqrt{6}}{\sqrt{7}-\sqrt{3}} = \frac{\sqrt{42}+\sqrt{18}}{4}; \quad \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} = \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}$$
$$\times \frac{\sqrt[3]{16}+2+\sqrt[3]{4}}{\sqrt[3]{16}+2+\sqrt[3]{4}} = \frac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} \times \frac{2\sqrt[3]{2}+2+\sqrt[3]{4}}{2\sqrt[3]{2}+2+\sqrt[3]{4}}$$
$$=$$

$$=\frac{2\sqrt[3]{40}+2\sqrt[3]{20}+\sqrt[3]{80}}{2}=2\sqrt[3]{5}+\sqrt[3]{20}+\sqrt[3]{10}:$$

Also $\frac{\sqrt[2]{10}}{\sqrt[2]{2}-\sqrt[3]{3}}=$ (because $m=\frac{1}{2}$, $l=\frac{1}{3}$, $n=3$, and $a^n-b^{\frac{nl}{m}}=8-9=-1)=$

$$\frac{4\sqrt{20}+4\sqrt{10}\times\sqrt[3]{3}+2\sqrt{20}\times\sqrt[3]{9}+6\sqrt{10}+3\sqrt{20}\times\sqrt[3]{3}+3\sqrt{10}\times\sqrt[3]{9}}{-1}$$

$$=-8\sqrt{5}-4\sqrt{10}\times\sqrt[3]{3}-8\sqrt{5}\times\sqrt[3]{9}-6\sqrt{10}-6\times\sqrt{5}\times\sqrt[3]{3}-3\sqrt{10}\times\sqrt[3]{9}.$$

§ 124. When the ſquare root of a ſurd is required, it may be found nearly *by extracting the root of a rational quantity that approximates to its value.* Thus to find the ſquare root of $3+2\sqrt{2}$, we firſt calculate $\sqrt{2}=1,41421$, and therefore $3+2\sqrt{2}=5,82842$, whoſe root is found to be nearly $2,41421$: ſo that $\sqrt[2]{3+2\sqrt{2}}$ is nearly $2,41421$. But ſometimes we may be able to expreſs the roots of ſurds exactly by other ſurds; as in this example the ſquare root of $3+2\sqrt{2}$ is $\overline{1+\sqrt{2}}$, for $\overline{1+\sqrt{2}}\times\overline{1+\sqrt{2}}=1+2\sqrt{2}+2=3+2\sqrt{2}$.

In order to know when and how this may be found, let us ſuppoſe that $x+y$ is a binomial ſurd, whoſe ſquare will be x^2+y^2+2xy: If x and y are quadratic ſurds, then x^2+y^2 will be rational, and $2xy$ irrational; ſo that $2xy$ ſhall

ſhall always be leſs than $x^2 + y^2$, becauſe the difference is $x^2 + y^2 - 2xy = \overline{x-y}^2$, which is always poſitive. Suppoſe that a propoſed ſurd conſiſting of a rational part A, and an irrational part B, coincides with this, then $x^2 + y^2 = A$ and $xy = \frac{1}{2}B$: Therefore by what was ſaid of Equations, *Chap.* 13th,

$$y^2 = A - x^2 = \frac{B^2}{4x^2}, \text{ and therefore}$$

$$Ax^2 - x^4 = \frac{B^2}{4}, \text{ and } x^4 - Ax^2 + \frac{B^2}{4} = 0;$$

from whence we have $x^2 = \frac{A + \sqrt{A^2 - B^2}}{2}$ and $y^2 = \frac{A - \sqrt{A^2 - B^2}}{2}$. Therefore when a quantity partly rational and partly irrational is propoſed to have its root extracted, *call the rational part* A, *the irrational* B, *and the ſquare of the greateſt member of the root ſhall be* $\frac{A + \sqrt{A^2 - B^2}}{2}$, *and the ſquare of the leſſer part ſhall be* $\frac{A - \sqrt{A^2 - B^2}}{2}$.

And as often as the ſquare root of $A^2 - B^2$ can be extracted, the ſquare root of the propoſed binomial ſurd may be expreſſed itſelf as a binomial ſurd. For example, if $3 + 2\sqrt{2}$ is propoſed, then $A = 3$, $B = 2\sqrt{2}$, and $A^2 - B^2 = 9 - 8 = 1$. Therefore $x^2 = \frac{A + \sqrt{A^2 - B^2}}{2} = 2$, and $y^2 = \frac{A - \sqrt{A^2 - B^2}}{2} = 1$. Therefore $x + y = 1 + \sqrt{2}$.

To find the square root of $-1\ \sqrt{-8}$, suppose $A=-1$, $B=\sqrt{-8}$, so that $A^2-B^2=9$, and $\frac{A+\sqrt{A^2-B^2}}{2}=\frac{-1+3}{2}=1$, and $\frac{A-\sqrt{A^2-B^2}}{2}=\frac{-1-3}{2}=-2$, therefore the root required is $1+\sqrt{-2}$.

§ 125. But though x and y are not quadratic surds or roots of integers, if they are the roots of like surds, as if they are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{z}}$, where m and n are integers, then $A=\overline{m+n}\times\sqrt{z}$ and $\frac{1}{2}B=\sqrt{mnz}$; $A^2-B^2=\overline{m-n}|^2\times z$, and $x^2=\frac{A+\sqrt{A^2-B^2}}{2}=\frac{\overline{m+n}\sqrt{z}+\overline{m-n}\sqrt{z}}{2}=m\sqrt{z}$, $y^2=\frac{A-\sqrt{A^2-B^2}}{2}=n\sqrt{z}$, and $x+y=\sqrt{m\sqrt{z}}+\sqrt{n\sqrt{z}}$. The part A here easily distinguishes itself from B by its being greater.

§ 126. If x and y are equal to $\sqrt{m\sqrt{z}}$ and $\sqrt{n\sqrt{t}}$, then $x^2+2xy+y^2=m\sqrt{z}+n\sqrt{t}+2\sqrt{mn\sqrt{zt}}$. So that if z or t be not multiples one of the other, or of some number that measures them both by a square number, then will A itself be a binomial.

§ 127. Let $x+y+z$ express any trinomial surd, its square $x^2+y^2+z^2+2xy+2xz+2yz$ may be supposed equal to $A+B$ as before. But

rather

rather multiply any two radicals as $2xy$ by $2xz$, and divide by the third $2yz$, which gives the quotient $2x^2$ rational, and double the square of the surd x required. The same rule serves when there are four quantities, $\overline{x^2+y^2+z^2+s^2+2xy+2xs+2xz+2yz+2ys+2zs}$, multiply $2xy$ by $2xs$, and the product $4x^2sy$ divided by $2sy$ gives $2x^2$ a rational quotient, half the square of $2x$. In like manner $2xy \times 2yz = 4y^2xz$, which divided by $2xz$ another member gives $2y^2$, a rational quote, the half of the square of $2y$. In the same manner z and s may be found; and their sum $x+y+z+s$, the square root of the septinomial $\overline{x^2+y^2+z^2+s^2+2xy+2xs+2xz+2yz+2ys+2xs}$, discovered.

For example, to find the square root of $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$; I try $\frac{\sqrt{24}\times\sqrt{40}}{\sqrt{60}}$, which I find to be $\sqrt{16}=4$, the half of the square root of the double of which, *viz.* $\frac{1}{2}\times\sqrt{8}=\sqrt{2}$, is one member of the square root required; next $\frac{\sqrt{24}\times\sqrt{60}}{\sqrt{40}}=6$, the half of the square root of the double of which is $\sqrt{3}$, another member of the root required; lastly, $\frac{\sqrt{40}\times\sqrt{60}}{\sqrt{24}}=10$, which gives $\sqrt{5}$ for the third member of the root required: From which we conclude that the square root of $10+\sqrt{24}+\sqrt{40}+\sqrt{60}$ is $\sqrt{2}+\sqrt{3}+\sqrt{5}$; and trying you find it succeeds, since

 multi-

multiplied by itself it gives the proposed quadrinomial.

§ 128. For extracting the higher roots of a binomial, whose two members being squared are commensurable members, there is the following

RULE.

* "*Let the quantity be* A ± B, *whereof* A *is the greater part, and c the exponent of the root required. Seek the least number n whose power n^c is divisible by* AA − BB, *the quotient being* Q. *Compute* $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$ *in the nearest integer number, which suppose to be r. Divide* A $\sqrt{Q}$ *by its greatest rational divisor, and let the quotient be s, and let* $\dfrac{r + \frac{n}{r}}{2s}$, *in the nearest integer number, be t, so shall the root required be* $\dfrac{ts \pm \sqrt{t^2 s^2 - n}}{\sqrt[2c]{Q}}$, *if the c root of* A ± B *can be extracted.*

EXAMPLE I.

Thus to find the cube root of $\sqrt{968} + 25$, we have $A^2 - B^2 = 343$, whose divisors are 7, 7, 7, whence $n = 7$, and $Q = 1$. Further, $\overline{A+B} \times \sqrt{Q}$, that is, $\sqrt{968} + 25$ is a little more

* Arithm. Universal. p. 59.

than

than 56, whose nearest cube root is 4. Wherefore $r = 4$. Again, dividing $\sqrt{968}$ by its greatest rational divisor, we have $A\sqrt{Q} = 22\sqrt{2}$, and the radical part $\sqrt{2} = s$, and $\frac{r + \frac{n}{r}}{2s}$, or $\frac{5}{2\sqrt{2}}$, in the nearest integers, is $2 = t$. And lastly, $ts = 2\sqrt{2}$, $\sqrt{t^2s^2 - n} = 1$, and $\sqrt[2c]{Q} = \sqrt[6]{1} = 1$. Whence $2\sqrt{2} + 1$ is the root, whose cube, upon trial, I find to be $\sqrt{968} + 25$.

EXAMPLE II.

To find the cube root of $68 - \sqrt{4374}$; we have $A^2 - B^2 = 250$, whose divisors are 5, 5, 5, 2. Thence $n = 5 \times 2 = 10$, and $Q = 4$, and $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$, or $\sqrt[3]{\overline{68 + \sqrt{4374}} \times 2}$ is nearly $7 = r$; again, $A\sqrt{Q}$, or $68 \times \sqrt{4} = 136 \times \sqrt{1}$, that is, $s = 1$, and $\frac{r + \frac{n}{r}}{2s}$, or $\frac{7 + \frac{10}{7}}{2}$, is nearly $= 4 = t$. Therefore $ts = 4$, $\sqrt{t^2s^2 - n} = \sqrt{6}$, and $\sqrt[2c]{Q} = \sqrt[6]{4} = \sqrt[3]{2}$, whence the root to be tried is $\frac{4 - \sqrt{6}}{\sqrt[3]{2}}$.

EXAMPLE III.

Suppose the fifth root of $29\sqrt{6} + 41\sqrt{3}$ is demanded, $A^2 - B^2 = 3$, and $n = 3$; $Q = 81$,

 $r = 5$,

$r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{t^2s^2-n}=\sqrt{3}$, and $\sqrt[2c]{Q}=\sqrt[10]{81}=\sqrt[5]{9}$. And therefore trial is to be made with $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

In these operations, if the quantity is a fraction, or if its parts have a common divisor, you are to extract the root of the *numerator* and *denominator*, or of the *factors* separately. Thus, to extract the cube root of $\sqrt{242}-12\frac{1}{2}$, this reduced to a common denominator is $\frac{\sqrt{968}-25}{2}$. And the roots of the numerator and denominator, separately found, give the root $\frac{2\sqrt{2}-1}{\sqrt[3]{2}}$. And if you seek any root of $\sqrt[3]{3993}+\sqrt[6]{17578125}$, divide its parts by the common divisor $\sqrt[3]{3}$, and the quotient being $11+\sqrt{125}$, the root of the quantity proposed will be found by taking the roots of $\sqrt[3]{3}$ and of $11+\sqrt{125}$, and multiplying them into each other.

§ 129. The ground of this Rule may be explained from the following

THEOREM

Let the sum or difference of two quantities x and y be raised to a power whose exponent is c, *and*

and let the 1st, 3d, 5th, 7th, &c. terms of that power, collected into one sum, be called A, *and the rest of the terms, in the even places, call* B; *the difference of the squares of* A *and* B *shall be equal to the difference of the squares of* x *and* y *raised to the same power* c.

For the terms in the c power of $x+y$ (writing for their coefficients, respectively, 1, c, d, e, &c. are
$x^c + cx^{c-1}y + dx^{c-2}y^2 + ex^{c-3}y^3 +$ &c. $=$ A $+$ B, and the same power of $x-y$ (changing the signs in the even places) is
$x^c - cx^{c-1}y + dx^{c-2}y^2 - ex^{c-3}y^3 +$ &c. $=$ A $-$ B, and therefore $\overline{x+y}|^c \times \overline{x-y}|^c = \overline{A+B} \times \overline{A-B}$ $= A^2 - B^2$ $(= \overline{\overline{x+y} \times \overline{x-y}}|^c) = \overline{x^2-y^2}|^c$. Q. E. D.

Let one, or both, of the quantities x, y, be a quadratic *surd*, that is, let $x+y$, the c root of the proposed binomial A $+$ B belong to one of these forms, $p + l\sqrt{q}$, $k\sqrt{p} + q$, or $k\sqrt{p} + l\sqrt{q}$. And it follows,

1. If $x+y = p + l\sqrt{q}$, that, c being any whole number, A, the sum of the *odd* terms, will be a rational number; and B, the sum of the terms in the *even* places, each of which involves an *odd* power of y will be a rational number multiplied into the quadratic surd $\sqrt{q}$.

2. Let c, the exponent of the root sought, be an odd number, as we may always suppose

 it,

it, because if it is even, it may be halved by the extraction of the square root, till it becomes odd; and let $x + y = k\sqrt{p} + q$. Then A will involve the surd $\sqrt{p}$, and B will be rational.

3. But if both members of the root are irrational ($x + y = k\sqrt{p} + l\sqrt{q}$) A and B are both irrational, the one involving $\sqrt{p}$, and the other the surd $\sqrt{q}$.

And in all these cases, it is easily seen that when x is greater than y, A will be greater than B.

§ 130. From this composition of the binomial A + B, we are led to its resolution, as in the foregoing rule, by these steps.

I.

When A is *rational*, and $A^2 - B^2$ is a perfect c power.

1. By the *Theorem*, $A^2 - B^2 = \overline{x^2 - y^2}|^c$ *accurately*; and therefore extracting the c root of $A^2 - B^2$ it will be $x^2 - y^2$. Call this root n.

2. Extract in the nearest integer, the c root of A + B, it will be (*nearly*) $x + y$. Which put $= r$.

3. Divide $x^2 - y^2$ ($= n$) by $x + y$ ($= r$) the quotient is (*nearly*) $x - y$; and the sum of the divisor and quotient is (*more nearly*) $2x$; that is, if an integer value of x it to be found, it will be the nearest to $\dfrac{r + \frac{n}{r}}{2}$.

4.

4. $x^2 - \overline{x^2 - y^2} = y^2$; or, $\overline{\frac{r + \frac{n}{r}}{2}}\Big|^2 - n = y^2$:

whence $y = \sqrt{\overline{\frac{r + \frac{n}{r}}{2}}\Big|^2 - n}$, and therefore, putting $t = \frac{r + \frac{n}{r}}{2}$, the root fought $x + y = t + \sqrt{t^2 - n}$; the fame expreffion as in the rule, when $Q = 1$, $s = 1$, that is, when $A^2 - B^2$ is a perfect c power, and the greater member A is rational.

II.

When A is *irrational*, and $Q = 1$.

By the fame procefs, $x = \frac{r + \frac{n}{r}}{2}$ $(= T)$ and $y = \sqrt{T^2 - n}$. But feeing A is fuppofed irrational, and c an odd number, x will be irrational likewife; and they will both involve the fame irreducible furd $\sqrt{p}$, or s, which is found by dividing A by its greateft rational divifor. Write therefore for x or T, its value $t \times s$, and $x + y = ts + \sqrt{t^2 s^2 - n}$.

III.

If the c root of $A^2 - B^2$ cannot be taken, multiply $A^2 - B^2$ by a number Q, fuch as that the product may be the (leaft) perfect c power n^c $(= A^2 Q - B^2 Q.)$ And now (inftead of

A +

$A+B$) extract the c root of $\overline{A+B} \times \sqrt{Q}$, which, found as above, will be $ts+\sqrt{t^2s^2-n}$; and consequently the c root of $A+B$ will be $ts+\sqrt{t^2s^2-n}$, divided by the c root of $\sqrt{Q}$; that is, $\frac{ts+\sqrt{t^2s^2-n}}{\sqrt[2c]{Q}}$.

It is required in the rule that *a perfect c power* (n^c) *be found which shall be a multiple of* A^2-B^2 *by the whole number* Q. To find this power, let the given number A^2-B^2 be represented by the product $a^m b^p df$; whose single divisors let be $a, a, a, \ldots . b, b, b, \ldots . d, f$; and the product of these divisors raised to the power c, which is $a^c b^c d^c f^c$, divided by $a^m b^p df$ will give the quotient $a^{c-m} b^{c-p} d^{c-1} f^{c-1} = Q$ a whole number, provided some index, as m or p, be not greater than c. If it is, take, instead of the single divisor a or b, a^2 or b^2, a^3 or b^3, &c. till there be no negative index in the quotient; that is, till Q be a whole number.

§ 131. We may add the following remarks

1. If the residual $A-B$ is given, it is evident from its genesis by involution, that the same rule gives its root $x-y$.

2. The extracting the c root of $A+B$, or of $\overline{A+B} \times \sqrt{Q}$, in the nearest integer, neglecting the fractional part, will always give $x+y$ such, that the value of x which results in the operation shall not differ from its true value by unity;

unity; that is, it shall be the true integer value sought.

For f being some proper fraction, let $x+y \pm f$ be the accurate value of $\sqrt[c]{A+B\times\sqrt{Q}}$, and let the quotient of x^2-y^2 divided by it be $x-y\mp g$, then the sum of the divisor and quotient being $2x\pm f\mp g$, if our reckoning the fractional part could make a difference of unity in the value of x, it would follow that $f-g$ or $g-f=2$. Which is absurd, g, as well as f, being a proper fraction.

3. If both A and B are irrational; or, if the lesser of the two members is rational, no root denominated by an even number can be found.

4. When the greater member is rational, and the exponent c is an even number, it is ambiguous whether the greater member of the root is rational or surd. And though a root in the form of $p+l\sqrt{q}$ is not found, yet a root in the form of $k\sqrt{p}+q$, or, that failing, in the form $k\sqrt{p}+l\sqrt{q}$, may be obtained.

If we look for a root $k\sqrt{p}+q$, we are now to subtract $x-y$ from $x+y$, and half the remainder will give y (or q) the rational part And to x^2-y^2 ($=n$) adding y^2, the sum will be x^2.

So that $y=\dfrac{r-\frac{n}{r}}{2}$, and $x=\sqrt{\left|\dfrac{r-\frac{n}{r}}{2}\right|^2+n}$; the expressions being the same as when c is odd, with

with the sign of n changed. If this does not succeed, and a prime number stands under the radical sign, no farther trial need be made.

But if a composite number stands under the radical sign, the root may possibly belong to the form $k\sqrt{p} + l\sqrt{q}$; and that composite number being $p \times q$, since $k^2 p - l^2 q = n$, and $k\sqrt{p} = x$, the numbers k, l, may be sought for in the nearest integers, and trial made with $k\sqrt{p} + l\sqrt{q}$; as in this

EXAMPLE.

To find the fourth root of $49849 - 2895\sqrt{224}$.

The 4th root of $A^2 - B^2$ is $157 = x^2 - y^2 = n$, and the 4th root of $A - B$, that is, $x - y = r = 9$ *nearly*: and $\frac{n}{r} = \frac{157}{9} = 17$ *nearly*. Whence $x = \frac{9 + 17}{2} = 13$. But now the least radical factor in B being $\sqrt{14} = \sqrt{7 \times 2}$, I put $13 (= x) = k\sqrt{7}$, and k in the nearest integer $= 5$. Again $k^2 p - l^2 q = n = 175 - l^2 \times 2 = 157$; that is, $l^2 \times 2 = 18$, and $l = 3$; which gives the root $5\sqrt{7} - 3\sqrt{2}$.

In this manner the even roots may be sought immediately. But to avoid ambiguity and needless trouble, it is better first to depress them by extracting the square root, as in § 124.

A SUP-

A SUPPLEMENT

TO THIS

CHAPTER.

§ 132. THERE occur sometimes, especially in the resolution of *cubic* equations by *Cardan*'s Rule (*Part* II. § 79.) binomials of this form $A \pm B\sqrt{-q}$, whose cube roots must be found. To these the foregoing rule cannot be applied throughout, because the imaginary factor $\sqrt{-q}$. Yet if the root is expressible in rational numbers, the first step of that rule will often lead us to it in a short way, not merely tentative, the trials being confined to known limits.

For it being, universally, $\sqrt[c]{A^2 - B^2} = x^2 - y^2$ and, in the present case, $\sqrt[3]{A^2 + B^2q}$ ($= x^2 - y^2$ $= p^2 + l^2 \times q$); if we divide the part under the radical sign by its greatest rational divisor, the quote is the imaginary surd $\sqrt{-q}$, and from $\sqrt[3]{A^2 + B^2q}$, subtracting p^2 the square of some divisor of A, the remainder is $l^2 \times q$, a known multiple of the square of l a divisor of B.

That

That p and l are divisors of A and B respectively is evident; for cubing $p + l\sqrt{-q}$, you find $A = p \times \overline{p^2 - 3l^2q}$, $B = l \times \overline{3p^2 - l^2q}$. And the signs of p and l must be such as will give the products of $p \times \overline{p^2 - 3l^2q}$, $l \times \overline{3p^2 - l^2q}$ of the same signs as A and B respectively.

EXAMPLE

To find the cube root of $81 + \sqrt{-2700} = 81 + 30\sqrt{-3}$.

Here $A = 81$, $B = 30$, $q = 3$; $\sqrt[3]{81 \times 81 + 2700} = 21 = p^2 + l^2q$. Subtracting therefore from 21, the square of $(p) \pm 3$, which is a divisor of A, there remains $(l^2 \times q =)\ 2 \times 2 \times 3$. And $(l =)\ 2$ is a divisor of 30. Lastly, A $(= p \times \overline{p^2 - 3l^2q})$ being positive, and the factor $p^2 - 3l^2q$ negative, p must have the negative sign; and for the like reason $l = +2$. So that the root is $-3 + 2\sqrt{-3}$.

It will be shewn in the second Part of this Treatise that "every cube or other power has as many roots, real and imaginary, as there are units in the exponent of the power;" particularly, that *unity* itself has the cube roots 1, $\frac{-1 + \sqrt{-3}}{2}$, and $\frac{-1 - \sqrt{-3}}{2}$. If therefore we would find the other two cube roots, in this example, seeing $z^3 = z^3 \times 1$, and $\sqrt[3]{z^3} \times \sqrt[3]{1} = z$

(z^3

(z^3 repreſenting any cube whatever, and z any of its roots) we are to multiply $-3+2\sqrt{-3}$, the root already found, by $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, and by $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, and the products $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$, and $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$ will be the roots required.

Or, becauſe the denominator of the imaginary roots of unity is 2, taking $p=\frac{3}{2}$, one half of a diviſor of A, we have $21-\frac{9}{4}=\frac{75}{4}=\frac{25}{4}\times 3=l^2q$, that is $l=\frac{5}{2}$; and p^2-3l^2q as well as $3p^2-l^2q$ being negative, both p and l muſt be negative, and the root is $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$. Again take $p=\frac{9}{2}$, and you ſhall find $l=+\frac{1}{2}$; ſo the remaining root is $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, as before.

We may here obſerve that the operation ought to be abridged, where it can be done, by dividing the given binomial by the greateſt cube that it contains; and finding the root of the quotient; which multiplied by the root of the cube by which you divided, will give the root required. Thus, in the foregoing Example, $81+\sqrt{-2700}=27\times 3+\sqrt{-\frac{100}{27}}$, and the roots of $3+\sqrt{-\frac{100}{27}}$ being now, more eaſily, found to be $-1+2\sqrt{-\frac{1}{3}}$; $-\frac{1}{2}-\frac{5}{2}\sqrt{-\frac{1}{3}}$, and $\frac{3}{2}+\frac{1}{2}\sqrt{-\frac{1}{3}}$, theſe multiplied by 3, the cube root of 27, gives the roots required the ſame as above.

"If the coefficient of the imaginary member of the binomial has a contrary ſign, the roots will be the ſame, with the ſigns of the imaginary

imaginary parts changed." Thus the cube roots of $81 - \sqrt{-2700}$, or $81 - 30\sqrt{-3}$, will be $-3 - 2\sqrt{-3}$, $-\frac{3}{2} + \frac{5}{2}\sqrt{-3}$, and $\frac{9}{2} - \frac{1}{2}\sqrt{-3}$. And therefore $\sqrt[3]{81 + \sqrt{-2700}} + \sqrt[3]{81 - \sqrt{-2700}} = -3 \times 2 = -6$, or $= -\frac{3}{2} \times 2 = -3$, or $= \frac{9}{2} \times 2 = 9$, the imaginary parts vaniſhing by the contrariety of their ſigns.

We may obſerve likewiſe, that ſuch roots, whether expreſſible in rational numbers, or not, may be found by evolving the binomial $A + B\sqrt{-q}$ by the *Theorem* in *pag.* 41, and ſumming the alternate terms. As, in the foregoing example, $\overline{81 + 30\sqrt{-3}}|^{\frac{1}{3}}$, or rather $81^{\frac{1}{3}} \times \overline{1 + \frac{10}{27}\sqrt{-3}}|^{\frac{1}{3}}$, being expanded into a ſeries, the ſum of the odd terms will continually approach to $4.5 = \frac{9}{2}$, and the ſum of the coefficients of the even terms to $\frac{1}{2}$, which is the coefficient of the imaginary part. But for a general and elegant ſolution, recourſe muſt be had to Mr. *de Moivre*'s Appendix to Dr. *Saunderſon*'s Algebra, and the continuation of it in *Philoſ. Tranſ.* N°. 451. What has been explained above may ſerve, for the preſent, to give the Learned ſome notion of the compoſition and reſolution of thoſe cubes; that he need not hereafter be ſurpriſed to meet with expreſſions of real quantities which involve imaginary roots.

End of the FIRST PART.

A

TREATISE

OF

ALGEBRA.

PART II.

Of the Genesis and Resolution of EQUATIONS of all Degrees; and of the different Affections of the ROOTS.

CHAP. I.

Of the Genesis and Resolution of Equations in general; and the number of roots an equation of any degree may have.

§ 1. AFTER the same manner as the higher powers are produced by the multiplication of the lower powers of the same root; equations of superior orders are generated by the multiplication of equations of inferior orders involving the same unknown quantity. And *an equation of*

any

any dimension may be considered as produced by the multiplication of as many simple equations as it has dimensions; or *of any other equations whatsoever, if the sum of their dimensions is equal to the dimension of that equation.* Thus any *cubic* equation may be conceived as generated by the multiplication of *three* simple equations, or of *one quadratic* and *one simple* equation. A *biquadratic* as generated by the multiplication of *four simple* equations, or of *two quadratic* equations; or lastly, of *one cubic* and *one simple* equation.

§ 2. If the equations which you suppose multiplied by one another are *the same*, then the equation generated will be nothing else but some power of those equations, and the operation is merely *involution*; of which we have treated already: and, when any such equation is given, the simple equation by whose multiplication it is produced is found by *evolution*, or the extraction of a root.

But when the equations that are supposed to be multiplied by each other are *different*, then other equations than powers are generated; which to resolve into the simple equations whence they are generated, is a different operation from involution, and is what is called, *the resolution of equations.*

But as evolution is performed by observing and tracing back the steps of involution; so to discover

discover the rules for the resolution of equations, we must carefully observe their *generation.*

§ 3. Suppose the unknown quantity to be x and its values in any simple equations to be $a, b, c, d,$ &c. then those simple equations, by bringing all the terms to one side, become $x-a=0, x-b=0, x-c=0,$ &c. And, the product of any two of these, as $\overline{x-a}\times\overline{x-b}=0$ will give a *quadratic* equation, or an equation of two dimensions. The product of any three of them, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$ will give a *cubic* equation, or one of three dimensions. The product of any four of them will give a *biquadratic* equation, or one of four dimensions, as $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$. And, *in general,* "*In the equation produced, the highest dimension of the unknown quantity will be equal to the number of simple equations that are multiplied by each other.*

§ 4. When any equation equivalent to this biquadratic $\overline{x-a}\times\overline{x-b}\times\overline{x-c}\times\overline{x-d}=0$ is proposed to be resolved, the whole difficulty consists in finding the simple equations $\overline{x-a}=0$, $\overline{x-b}=0, \overline{x-c}=0, \overline{x-d}=0$, by whose multiplication it is produced; for each of these simple equations gives one of the values of x, and one solution of the proposed equation. For, if any of the values of x deduced from thoſe ſimple equations be ſubſtituted in the propoſed equation, in place of x, then all the terms of that equation will vaniſh, and the whole be found equal to nothing. Becauſe when it is ſuppoſed thas $x = a$, or $x = b$, or $x = c$, or $x = d$, then the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$ does vaniſh, becauſe one of the factors is equal to nothing. There are therefore four ſuppoſitions that give $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$ according to the propoſed equation; that is, there are four roots of the propoſed equation. And after the ſame manner, "*Any other* equation admits of as many ſolutions as there are ſimple equations multiplied by one another that produce it," or "as many as there are units in the higheſt dimenſion of the unknown quantity in the propoſed equation.

§ 5. But as there are no other quantities whatſoever beſides theſe four (a, b, c, d,) that ſubſtituted in the product $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$, in the place of x, will make the product vaniſh; therefore, the equation $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} = 0$, cannot poſſibly have more than theſe four roots, and cannot admit of more ſolutions than four. If you ſubſtitute in that product a quantity neither equal to a, nor b, nor c, nor d, which ſuppoſe e, then ſince neither $e-a$, $e-b$, $e-c$, nor $e-d$ is equal to nothing; their

product

product $\overline{e-a} \times \overline{e-b} \times \overline{e-c} \times \overline{e-d}$ cannot be equal to nothing, but muſt be ſome real product; and therefore there is no ſuppoſition beſide one of the foreſaid four, that gives a juſt value of x according to the propoſed equation. So that it can have no more than theſe four roots. And after the ſame manner it appears, that "*No equation can have more roots than it contains dimenſions of the unknown quantity.*"

§ 6. To make all this ſtill plainer by an example, in numbers; ſuppoſe the equation to be reſolved to be $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, and that you diſcover that this equation is the ſame with the product of $\overline{x-1} \times \overline{x-2} \times \overline{x-3} \times \overline{x-4}$, then you certainly infer that the four values of x are 1, 2, 3, 4; ſeeing any of theſe numbers placed for x makes that product, and conſequently $x^4 - 10x^3 + 35x^2 - 50x + 24$, equal to nothing, according to the propoſed equation. And it is certain that there can be no other values of x beſides theſe four: ſince when you ſubſtitute any other number for x in thoſe factors $x-1$, $x-2$, $x-3$, $x-4$, none of the factors vaniſh, and therefore their product cannot be equal to nothing according to the equation.

§ 7. It may be uſeful ſometimes to conſider equations as generated from others of an inferior

rior ſort beſides ſimple ones. Thus a *cubic* equation may be conceived as generated from the *quadratic* $x^2 - px + q = 0$, and the *ſimple* equation $x - a = 0$, multiplied by each other; whoſe product

$$\left.\begin{array}{l} x^3 - px^2 + qx - aq \\ \quad - ax^2 + apx \end{array}\right\} = 0$$ may expreſs any cubic equation whoſe roots are the quantity (a) the value of x in the ſimple equation, and the two roots of the quadratic equation, *viz.*

$\frac{p + \sqrt{p^2 - 4q}}{2}$ and $\frac{p - \sqrt{p^2 - 4q}}{2}$; as appears from *Chap.* 13. *Part* I. And, according as theſe roots are *real* or *impoſſible*, two of the roots of the cubic equation are *real* or *impoſſible*.

§ 8. In the doctrine of involution we ſhewed that "the ſquare of any quantity poſitive or negative, is always poſitive," and therefore, "the ſquare root of a negative is impoſſible or imaginary." For example, the $\sqrt{a^2}$ is either $+a$ or $-a$, but $\sqrt{-a^2}$ can neither be $+a$ nor $-a$, but muſt be *imaginary*. Hence is underſtood that "a quadratic equation may have no impoſſible expreſſion in its coefficients, and yet when it is reſolved into the ſimple equations that produce it, they may involve impoſſible expreſſions." Thus the quadratic equation $x^2 + a^2 = 0$ has no impoſſible coefficient, but the ſimple equations from which it is produced, *viz.* $x + \sqrt{-a^2} = 0$, and $x - \sqrt{-a^2} = 0$, both

both involve an imaginary quantity; as the square $-a^2$ is a real quantity, but its square root is imaginary. After the same manner a biquadratic equation, when resolved, may give four simple equations, each of which may give an impossible value for the root: and the same may be said of any equation that can be produced from quadratic equations only; that is, whose dimensions are of the even numbers.

§ 9. But "a cubic equation (which cannot be generated from quadratic equations only, but requires one simple equation besides to produce it) if none of its coefficients are impossible, will have, at least, one real root," the same with the root of the simple equation whence it is produced. The square of an impossible quantity may be real, as the square of $\sqrt{-a^2}$ is $-a^2$; but "the cube of an impossible quantity is still impossible," as it still involves the square root of a negative: as, $\sqrt[2]{-a^2} \times \sqrt[2]{-a^2} \times \sqrt[2]{-a^2} = \sqrt[2]{-a^6} = a^3\sqrt{-1}$, is plainly imaginary. From which it appears, that though two simple equations involving impossible expressions, multiplied by one another, may give a product where no impossible expression may appear; yet, "if three such simple equations be multiplied by each other, the impossible expression will not disappear in their product." And hence it is plain, that though a quadratic equation whose coefficients are all real may have its two roots impossible, yet "a cubic equation whose coefficients are real cannot have all its three roots impossible."

§ 10. In general, it appears that the impossible expressions cannot disappear in the equation produced, but when their number is *even*; that there are never in any equations, whose coefficients are real quantities, single impossible roots, or an odd number of impossible roots, but "that the roots become impossible in pairs;" and that "an equation of an odd number of dimensions has always one real root."

§ 11. The roots of equations are either *positive* or *negative* according as the roots of the simple equations whence they are produced are positive or negative." If you suppose $x = -a$, $x = -b$, $x = -c$, $x = -d$, &c. then shall $x + a = 0$, $x + b = 0$, $x + c = 0$, $x + d = 0$; and the equation $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} = 0$ will have its roots, $-a$, $-b$, $-c$, $-d$, &c. negative.

But to know when the roots of equations are positive and when negative, and how many there are of each kind, shall be explained in the next chapter.

CHAP.

CHAP. II.

Of the Signs and Coefficients of EQUATIONS.

§ 12. WHEN any number of ſimple equations are multiplied by each other, it is obvious that the higheſt dimenſion of the unknown quantity in their product is equal to the number of thoſe ſimple equations; and, the term involving the higheſt dimenſion is called the *firſt* term of the equation generated by this multiplication. The term involving the next dimenſion of the unknown quantity, leſs than the greateſt by unit, is called the *ſecond* term of the equation; the term involving the next dimenſion of the unknown quantity, which is leſs than the greateſt by two, the *third* term of the equation, *&c.* And that term which involves no dimenſion of the unknown quantity, but is ſome known quantity, is called the *laſt* term of the equation.

"*The number of terms is always greater than the higheſt dimenſion of the unknown quantity by unit.*" And when any term is wanting, an *aſteriſk* is marked in its place. The *ſigns* and *coefficients* of equations will be underſtood by conſidering the following TABLE, where the ſimple

ſimple equations $x-a$, $x-b$, &c. are multiplied by one another, and produce ſucceſſively the higher equations.

$$\begin{array}{l} x - a = 0 \\ \times\, x - b = 0 \\ \hline = x^2 \left.\begin{array}{l} -ax \\ -bx + ab \end{array}\right\} = 0, \text{ a Quadratic.} \\ \times\, x - c = 0 \\ \hline = x^3 \left.\begin{array}{r} -a \\ -b \\ -c \end{array}\right\} \times x^2 \left.\begin{array}{r} +ab \\ +ac \\ +bc \end{array}\right\} \times x - abc = 0, \text{ a Cubic.} \\ \times\, x - d = 0 \\ \hline = x^4 \left.\begin{array}{r} -a \\ -b \\ -c \\ -d \end{array}\right\} \times x^3 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +bc \\ +bd \\ +cd \end{array}\right\} \times x^2 \left.\begin{array}{r} -abc \\ -abd \\ -acd \\ -bcd \end{array}\right\} \times x + abcd = 0, \text{ a Biquadratic.} \\ \times\, x - e = 0 \\ \hline = x^5 \left.\begin{array}{r} -a \\ -b \\ -c \\ -d \\ -e \end{array}\right\} \times x^4 \left.\begin{array}{r} +ab \\ +ac \\ +ad \\ +ae \\ +bc \\ +bd \\ +be \\ +cd \\ +ce \\ +de \end{array}\right\} \times x^3 \left.\begin{array}{r} -abc \\ -abd \\ -abe \\ -acd \\ -ade \\ -ace \\ -bcd \\ -bce \\ -bde \\ -cde \end{array}\right\} \times x^2 \left.\begin{array}{r} +abcd \\ +abce \\ +abde \\ +acde \\ +bcde \end{array}\right\} \times x - abcde = 0, \text{ a Surſolid.} \end{array}$$

&c.

§ 13.

§ 13. From the inspection of these equations it is plain, that the coefficient of the first term is *unit*.

The coefficient of the second term is *the sum of all the roots* (*a*, *b*, *c*, *d*, *e*) *having their signs changed*.

The coefficient of the third term is *the sum of all the products that can be made by multiplying any two of the roots* (*a*, *b*, *c*, *d*, *e*) *by one another*.

The coefficient of the fourth term is *the sum of all the products that can be made by multiplying into one another any three of the roots, with their signs changed*. And after the same manner all the other coefficients are formed.

The last term is always *the product of all the roots having their signs changed*, multiplied by one another.

§ 14. Although in the Table such simple equations only are multiplied by one another as have positive roots, it is easy to see, that "the coefficients will be formed according to the same rule when any of the simple equations have negative roots." And, *in general*, if $x^3 - px^2 + qx - r = 0$ represent any cubic equation, then shall p be the sum of the roots; q the sum of the products made by multiplying any two of them; r the product of all the three: and, if $-p$, $+q$, $-r$, $+s$, $-t$, $+u$, &c. be the coefficients of the 2d, 3d, 4th, 5th, 6th, 7th, *&c.*

&c. terms of any equation, then shall p be the sum of all the roots, q the sum of the products of any two, r the sum of the products of any three, s the sum of the products of any four, t the sum of the products of any five, u the sum of the products of any six, *&c.*

§ 15. When therefore any equation is proposed to be resolved, it is easy to find the sum of the roots, (for it is equal to the coefficient of the second term having its sign changed:) or, to find the sum of the products that can be made by multiplying any determinate number of them.

But it is also easy "to find the sum of the squares, or of any powers, of the roots."

The sum of the squares is always $p^2 - 2q$. For calling the sum of the squares B, since the sum of the roots is p; and "the square of the sum of any quantities is always equal to the sum of their squares added to double the products that can be made by multiplying any two of them," therefore $p^2 = B + 2q$, and consequently $B = p^2 - 2q$. For example, $\overline{a+b+c}|^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; that is, $p^2 = B + 2q$. And $\overline{a+b+c+d}|^2 = a^2 + b^2 + c^2 + d^2 + 2 \times \overline{ab+ac+ad+bc+bd+cd}$, that is again $p^2 = B + 2q$, or $B = p^2 - 2q$. And so for any other number of quantities. In general therefore, "B the sum of the squares of the roots

roots may always be found by subtracting $2q$ from p^2;" the quantities p and q being always known, since they are the coefficients in the proposed equation.

§ 16. "The sum of the cubes of the roots of any equation is equal to $p^3 - 3pq + 3r$, or to $Bp - pq + 3r$." For $\overline{B - q} \times p$ gives always the excess of the sum of the cubes of any quantities above the triple sum of the products that can be made by multiplying any three of them. Thus $\overline{a^2 + b^2 + c^2 - ab - ac - bc} \times \overline{a + b + c}$ ($= \overline{B - q} \times p$) $= a^3 + b^3 + c^3 - 3abc$. Therefore if the sum of the cubes is called C, then shall $\overline{B - q} \times p = C - 3r$, and $C = Bp - qp + 3r$ (because $B = p^2 - 2q$) $= p^3 - 3pq + 3r$.

After the same manner, if D be the sum of the 4th powers of the roots, you will find that $D = pC - qB + pr - 4s$: and if E be the sum of the 5th powers then shall $E = pD - qC + rB - ps + 5t$. And after the same manner the sum of any powers of the roots may be found; the progression of these expressions of the sum of the powers being obvious.

§ 17. As for the signs of the terms of the equation produced, it appears from inspection that the signs of all the terms in any equation in the table are alternately $+$ and $-$: these equations are generated by multiplying continually $x - a$, $x - b$, $x - c$, $x - d$, &c. by one another.

another. The first term is always some pure power of x, and is positive; the second is a power of x multiplied by the quantities $-a$, $-b$, $-c$, &c. And since these are all negative, that term must therefore be negative. The third term has the products of any two of these equations ($-a$, $-b$, $-c$, &c.) for its coefficient: which products are all positive, because $- \times -$ gives $+$. For the like reason, the next coefficient, consisting of all the products made by multiplying any three of these quantities, must be negative; and the next positive. So that the coefficients, in this case, will be positive and negative by turns. But, "in this case the roots are all positive;" since $x = a$, $x = b$, $x = c$, $x = d$, $x = e$, &c. are the assumed simple equations. It is plain then, that "*when all the roots are positive, the signs are alternately + and —.*"

§ 18. But if the roots are all negative, then $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d}$, &c. $=0$, will express the equation to be produced; all whose terms will plainly be positive; so that "*when all the roots of an equation are negative, it is plain there will be no changes in the signs of the terms of that equation.*"

§ 19. In general, "there are as many positive roots in any equation as there are changes in the signs of the terms from + to —, or from — to +; and the remaining roots are negative."

negative." The Rule is general, if the impossible roots be allowed to be either positive or negative.

§ 20. In quadratic equations, the two roots are either both positive, as in this

$$(\overline{x-a}\times\overline{x-b}=)\ \begin{matrix} x^2-ax+ab=0, \\ -bx \end{matrix}$$

where there are two changes of the signs: Or they are both negative, as in this

$$(\overline{x+a}\times\overline{x+b}=)\ x^2\left.\begin{matrix}+a\\+b\end{matrix}\right\}x+ab=0.$$

where there is not any change of the signs. Or there is one positive and one negative, as in

$$(\overline{x-a}\times\overline{x+b}=)\ x^2\left.\begin{matrix}-a\\+b\end{matrix}\right\}x-ab=0,$$

where there is necessarily one change of the signs; because the first term is positive, and the last negative, and there can be but one change whether the second term be $+$ or $-$.

Therefore the rule given in the 19th section extends to all quadratic equations.

§ 21. In cubic equations, the roots may be,

1°. All positive as in this, $\overline{x-a}\times\overline{x-b}\times\overline{x-c}=0$, in which the signs are alternately $+$ and $-$, as appears from the Table; and there are three changes of the signs.

2°. The roots may be all negative, as in the equation $\overline{x+a}\times\overline{x+b}\times\overline{x+c}=0$, where there can be no change of the signs. Or,

3°.

3°. There may be two positive roots and one negative, as in the equation $\overline{x-a} \times \overline{x-b} \times \overline{x+c} = 0$, which gives

$$\left.\begin{matrix} x^3 - a \\ -b \\ +c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x + abc = 0.$$

Here there must be two changes of the signs: because if $a + b$ is greater than c, the second term must be negative, its coefficient being $-a, -b + c$.

And if $a + b$ is less than c, then the third term must be negative, its coefficient $+ab - ac - bc$ $(ab - c \times \overline{a + b})$ * being in that case negative. And there cannot possibly be three changes of the signs, the first and last terms having the same sign.

4°. There may be one positive root and two negative, as in the equation $\overline{x+a} \times \overline{x+b} \times \overline{x-c} = 0$, which gives

$$\left.\begin{matrix} x^3 + a \\ +b \\ -c \end{matrix}\right\} x^2 \left.\begin{matrix} +ab \\ -ac \\ -bc \end{matrix}\right\} x - abc = 0.$$

Where there must be always one change of the signs, since the first term is positive and the last negative. And there can be but one change of the signs, since if the second term is negative, or $a + b$ less than c, the third must be negative

* Because the rectangle $a \times b$ is less than the square $\overline{a+b} \times \overline{a+b}$, and therefore much less than $\overline{a+b} \times c$.

negative alſo, ſo that there will be but one change of the ſigns. Or, if the ſecond term is affirmative, whatever the third term is, there will be but one change of the ſigns. It appears therefore, in general, that in cubic equations, there are as many affirmative roots as there are changes of the ſigns of the terms of the equation.

The ſame way of reaſoning may be extended to equations of higher dimenſions, and the rule delivered in § 19, extended to all kinds of equations.

§ 22. There are ſeveral conſectaries of what has been already demonſtrated, that are of uſe in diſcovering the roots of equations. But before we proceed to that, it will be convenient to explain ſome transformations of equations, by which they may often be rendered more ſimple, and the inveſtigation of their roots more eaſy.

L CHAP.

CHAP. III.

Of the Transformation of Equations; and exterminating their intermediate terms.

§ 23. We now proceed to explain the transformations of equations that are most useful: and first, "*The affirmative roots of an equation are changed into negative roots of the same value, and the negative roots into affirmative, by only changing the signs of the terms alternately, beginning with the second.*" Thus the roots of the equation $x^4 - x^3 - 19x^2 + 49x - 30 = 0$ are $+1$, $+2$, $+3$, -5; whereas the roots of the same equation having only the signs of the second and fourth terms changed, viz. $x^4 + x^3 - 19x^2 - 49x - 30 = 0$, are -1, -2, -3, $+5$.

To understand the reason of this rule, let us assume an equation, as $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times \overline{x-e}$ &c. $= 0$, whose roots are $+a$, $+b$, $+c$, $+d$, $+e$, &c. and another having its roots of the same value, but affected with contrary signs, as $\overline{x+a} \times \overline{x+b} \times \overline{x+c} \times \overline{x+d} \times \overline{x+e}$ &c. $= 0$. It is plain, that the terms taken alternately, beginning from the first, are the same in in both equations, and have the same sign, "being products of an even number of the roots;" the product of any two roots having the same sign as their product when both their signs are changed; as $+a \times -b = -a \times +b$.

But the second terms and all taken alternately from them, because their coefficients involve always the products of an odd number of the roots, will have contrary signs in the two equations. For example, the product of four, *viz.* $abcd$, having the same sign in both, and one equation in the fifth term having $abcd \times +e$, and the other $abcd \times -e$, it follows that their product $abcd$ must have contrary signs in the two equations: these two equations therefore that have the same roots, but with contrary signs, have nothing different but the signs of the alternate terms, beginning with the second. From which it follows, "that if any equation is given, and you change the signs of the alternate terms, beginning with the second, the new equation will have roots of the same value, but with contrary signs.

§ 24. It is often very useful "*to transform an equation into another that shall have its roots greater or less than the roots of the proposed equation by some given difference.*"

Let the equation proposed be the cubic $x^3 - px^2 + qx - r = 0$. And let it be required to transform it into another equation whose roots

 shall

shall be less than the roots of this equation by some given difference (e), that is, suppose $y = x - e$, and consequently $x = y + e$; then instead of x and its powers, substitute $y + e$ and its powers, and there will arise this new equation.

$$(A) \left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array}\right\} = 0.$$

whose roots are less than the roots of the preceding equation by the difference (e).

If it had been required to find an equation whose roots should be greater than those of the proposed equation by the quantity (e), then we must have supposed $y = x + e$, and consequently $x = y - e$, and then the other equation would have had this form;

$$(B) \left.\begin{array}{r} y^3 - 3ey^2 + 3e^2y - e^3 \\ - py^2 + 2pey - pe^2 \\ + qy - qe \\ - r \end{array}\right\} = 0.$$

If the proposed equation be in this form, $x^3 + px^2 + qx + r = 0$, then by supposing $x + e = y$ there will arise an equation agreeing in all respects with the equation (A), but that the second and fourth terms will have contrary signs.

And

And by supposing $x - e = y$, there will arise an equation agreeing with (B) in all respects, but that the second and fourth terms will have contrary signs to what they have in (B).

The first of these suppositions gives this equation,

$$(C)\left.\begin{array}{l} y^3 - 3ey^2 + 3e^2y - e^3 \\ \quad + py^2 - 2pey + pe^2 \\ \quad\quad + qy \quad - qe \\ \quad\quad\quad\quad + r \end{array}\right\} = 0.$$

The second supposition gives the equation

$$(D)\left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad + py^2 + 2epy + pe^2 \\ \quad\quad + qy \quad + qe \\ \quad\quad\quad\quad + r \end{array}\right\} = 0.$$

§ 25. The first use of this transformation of equations is to shew, "*how the second (or other intermediate) term may be taken away out of an equation.*"

It is plain that in the equation (A) whose second term is $\overline{3e - p} \times y^2$, if you suppose $e = \frac{1}{3}p$, and consequently $3e - p = 0$, then the second term will vanish.

In the equation (C) whose second term is $\overline{-3e + p} \times y^2$, supposing $e = \frac{1}{3}p$, the second term also vanishes.

Now the equation (A) was deduced from $x^3 - px^2 + qx - r = 0$, by supposing $y = x - e$: and

and the equation (*C*) was deduced from $x^3 + px^2 + qx + r = 0$; by supposing $y = x + a$. From which this Rule may easily be deduced for exterminating the second term out of any cubic equation.

RULE.

"*Add to the unknown quantity of the given equation the third part of the coefficient of the second term with its proper sign*, viz. $\mp \frac{1}{3}p$, *and suppose this aggregate equal to a new unknown quantity* (y). *From this value of* y *find a value of* x *by transposition, and substitute this value of* x *and its powers in the given equation, and there will arise a new equation that shall want the second term.*"

EXAMPLE.

Let it be required to exterminate the second term out of this equation, $x^3 - 9x^2 + 26x - 34 = 0$, suppose $x - 3 = y$, or $y + 3 = x$; and substituting according to the Rule, you will find

$$\left.\begin{array}{r} y^3 + 9y^2 + 27y + 27 \\ - 9y^2 - 54y - 81 \\ + 26y + 78 \\ - 34 \end{array}\right\} = 0.$$

$$y^3 \quad * \quad - y \quad - 10 = 0.$$

In

In which there is no term where y is of two dimensions, and an asterisk is placed in the room of the second term, to shew it is wanting.

§ 26. Let the equation proposed be of any number of dimensions represented by (n); and let the coefficient of the second term with its sign prefixed be $-p$, then supposing $x - \frac{p}{n} = y$, and consequently $x = y + \frac{p}{n}$, and substituting this value for x in the given equation, there will arise a new equation that shall want the second term.

It is plain from what was demonstrated in *Chap.* 2. that the sum of the roots of the proposed equation is $+p$; and since we suppose $y = x - \frac{p}{n}$, it follows, that in the new equation, each value of y will be less than the respective value of x by $\frac{p}{n}$; and, since the number of the roots is n, it follows that the sum of the values of y will be less than $+p$, the sum of the values of x, by $n \times \frac{p}{n}$ the difference of any two roots, that is, by $+p$: therefore the sum of the values of y will be $+p-p=0$.

But the coefficient of the second term of the equation of y is the sum of the values of y, viz. $+p-p$, and therefore that coefficient is equal to nothing; and consequently, in the equation of y, the second term vanishes. It follows then, that the second term may be exterminated out of any given equation by the following

RULE.

"*Divide the coefficient of the second term of the proposed equation by the number of dimensions of the equation; and assuming a new unknown quantity y, add to it the quotient having its sign changed. Then suppose this aggregate equal to x the unknown quantity in the proposed equation; and for x and its powers, substitute the aggregate and its powers, so shall the new equation that arises want its second term.*"

§ 27. If the proposed equation is a quadratic, as $x^2 - px + q = 0$, then, according to the rule, suppose $y + \frac{1}{2}p = x$, and substituting this value for x, you will find,

$$\left.\begin{array}{r} y^2 + py + \frac{1}{4}p^2 \\ - py - \frac{1}{2}p^2 \\ + q \end{array}\right\} = 0.$$

$$y^2 \;*\; - \tfrac{1}{4}p^2 + q = 0.$$

And from this example the use of exterminating the second term appears; for commonly the solution of the equation that wants the second term is more easy. And, if you can find the value

value of y from this new equation, it is easy to find the value of x by means of the equation $y + \frac{1}{2}p = x$. For example,

Since $y^2 + q - \frac{1}{4}p^2 = 0$, it follows that

$y^2 = \frac{1}{4}p^2 - q$, and $y = \pm\sqrt{\frac{1}{4}p^2 - q}$, so that $x = y + \frac{1}{2}p = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, which agrees with what we demonstrated, *Chapter 13. Part* I.

If the proposed equation is a biquadratic, as $x^4 - px^3 + qx^2 - rx + s = 0$, then by supposing $x - \frac{1}{4}p = y$, or $x = y + \frac{1}{4}p$, an equation shall arise having no second term. And if the proposed is of five dimensions, then you must suppose $x = y \pm \frac{1}{5}p$. And so on.

§ 28. When the second term in any equation is wanting, it follows, that, "the equation has both affirmative and negative roots," and that the sum of the affirmative roots is equal to the sum of the negative roots: by which means the coefficient of the second term, which is the sum of all the roots of both sorts, vanishes, and makes the second term vanish.

In general, "the coefficient of the second term is the difference between the sum of the affirmative roots and the sum of the negative roots:" and the operations we have given serve only to diminish all the roots when the sum of the affirmative is greatest, or increase the roots when the sum of the negative is greatest, so

as

as to balance them, and reduce them to an equality.

It is obvious, that in a quadratic equation that wants the second term, there must be one root affirmative and one negative; and these must be equal to one another.

In a cubic equation that wants the second term, there must be either, two affirmative roots equal, taken together, to a third root that must be negative; or, two negative equal to a third that must be positive.

"Let an equation $x^3 - px^2 + qx - r = 0$ be proposed, and let it be now required to exterminate the third term."

By supposing $y = x - e$, the coefficient of the third term in the equation of y is found (see equation *A*) to be $3e^2 - 2pe + q$. Suppose that coefficient equal to nothing, and by resolving the quadratic equation $3e^2 - 2pe + q = 0$, you will find the value of e, which substituted for it in the equation $y = x - e$, will shew how to transform the proposed equation into one that shall want the third term.

The quadratic $3e^2 - 2pe + q = 0$ gives $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$. So that the proposed cubic will be transformed into an equation wanting the third term by supposing $y = x - \frac{p - \sqrt{p^2 - 3q}}{3}$, or $y = x - \frac{p + \sqrt{p^2 - 3q}}{3}$.

If

If the proposed equation is of n dimensions, the value of e, by which the third term may be taken away, is had by resolving the quadratic equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times n-1} = 0$, supposing $-p$ and $+q$ to be the coefficients of the second and third terms of the proposed equation.

The fourth term of any equation may be taken away by solving a cubic equation, which is the coefficient of the fourth term in the equation when transformed, as in the second article of this chapter. The fifth term may be taken away by solving a biquadratic; and after the same manner the other terms can be exterminated if there are any.

§ 29. There are other transmutations of equations, that on some occasions are useful.

An equation, as $x^3 - px^2 + qx - r = 0$, *may be transformed into another that shall have its roots equal to the roots of this equation multiplied by a given quantity*, as f, by supposing $y = fx$, and consequently $x = \frac{y}{f}$, and substituting this value for x in the proposed equation, there will arise $\frac{y^3}{f^3} - \frac{py^2}{f^2} + \frac{qy}{f} - r = 0$, and multiplying all by $f^3 \ldots y^3 - fpy^2 + f^2qy - f^3r = 0$, where the coefficient of the second term of the proposed equation multiplied into f makes the coefficient of the

the second term of the transformed equation; and the following coefficients are produced by the following coefficients of the proposed equation (as q, r, &c.) multiplied into the powers of f (f^2, f^3, &c.)

Therefore "to transform any equation into another whose roots shall be equal to the roots of the proposed equation multiplied by a given quantity" (f), you need only multiply the terms of the proposed equation, beginning at the second term, by f, f^2 f^3, f^4, &c. and putting y instead of x there will arise an equation having its roots equal to the roots of the proposed equation multiplied by (f) as required.

§ 30. The transformation mentioned in the last article is of use when the highest term of the equation has a coefficient different from unity; for, by it, the equation may be transformed into one that shall have the coefficient of the highest term unit.

If the equation proposed is $ax^3 - px^2 + qx - r = 0$, then transform the equation into one whose roots are equal to the roots of the proposed equation multiplied by (a). That is, suppose $y = ax$, or $x = \frac{y}{a}$, and there will arise $\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$; so that $y^3 - py^2 + qay - ra^2 = 0$.

From

From which we eafily draw this

RULE

"*Change the unknown quantity x into another y, prefix no coefficient to the highest term, pafs the fecond, multiply the following terms, beginning with the third, by a, a^2, a^3, a^4, &c. the powers of the coefficient of the higheft term of the propofed equation, refpectively.*"

Thus the equation $3x^3 - 13x^2 + 14x + 16 = 0$, is transformed into the equation

$y^3 - 13y^2 + 14 \times 3 \times x + 16 \times 9 = 0$, or

$y^3 - 13y^2 + 42x + 144 = 0$.

Then finding the roots of this equation, it will eafily be difcovered what are the roots of the propofed equation: fince $3x = y$, or $x = \frac{1}{3}y$. And therefore fince one of the values of y is -2, it follows that one of the values of x is $-\frac{2}{3}$.

§ 31. By the laft Rule "*an equation is eafily cleared of fractions.*" Suppofe the equation propofed is $x^3 - \frac{p}{m}x^2 + \frac{q}{n}x - \frac{r}{e} = 0$. Multiply all the terms by the product of the denominators, you find

$mne \times x^3 - nep \times x^2 \times meq \times x - mnr = 0$.

Then (by laft fection) transforming the equation into one that fhall have unit for the coefficient of the higheft term, you find

$y^3 - nep \times y^2 + m^2e^2nq \times y - m^3n^3e^2r = 0$.

Or,

Or, neglecting the denominator of the last term $\frac{r}{e}$, you need only multiply all the equation by mn, which will give

$mn \times x^3 - np \times x^2 + mq \times x - \frac{mnr}{e} = 0$. And then $y^3 - np \times y^2 + m^2nq \times y - \frac{m^2n^2r}{e} = 0$.

Now after the values of y are found, it will be easy to discover the values of x; since, in the first case, $x = \frac{y}{mne}$; in the second, $x = \frac{y}{mn}$.

For example, the equation

$x^3 * - \frac{4}{3}x - \frac{146}{27} = 0$; is first reduced to this form $3x^3 * - 4x - \frac{146}{9} = 0$, and then transformed into $y^3 * - 12y - 146 = 0$.

Sometimes, by these transformations, "*Surds are taken away.*" As for example,
The equation $x^3 - p\sqrt{a} \times x^2 + qx - r\sqrt{a} = 0$, by putting $y = \sqrt{a} \times x$, or $x = \frac{y}{\sqrt{a}}$, is transformed into this equation.

$\frac{y^3}{a\sqrt{a}} - p\sqrt{a} \times \frac{y^2}{a} + q \times \frac{y}{\sqrt{a}} - r\sqrt{a} = 0$.

Which by multiplying all the terms by $a\sqrt{a}$, becomes $y^3 - pay^2 + qay - ra^2 = 0$, an equation free of surds. But in order to make this succeed, the surd ($\sqrt{a}$) must enter the alternate terms beginning with the second.

§ 32.

§ 32. An equation, as $x^3 - px^2 + qx - r = 0$, *may be transformed into one whose roots shall be the quantities reciprocal of* x; by supposing $y = \frac{1}{x}$, and $y = \frac{z}{r}$, or, (by one supposition) $x = \frac{r}{z}$, becomes $z^3 - qz^2 + prz - r^2 = 0$.

In the equation of y, it is manifest that the order of the coefficients is inverted; so that if the second term had been wanting in the proposed equation, the last but one should have been wanting in the equations of y and z. If the third had been wanting in the equation proposed, the last but two had been wanting in the equations of y and z.

Another use of this transformation is, that "*the greatest root in the one is transformed into the least root in the other.*" For since $x = \frac{1}{y}$, and $y = \frac{1}{x}$, it is plain that when the value of x is greatest, the value of y is least, and conversely.

How an equation is transformed so as to have all its roots affirmative, shall be explained in the following chapter.

CHAP.

CHAP. IV.

Of finding the Roots of Equations when two or more of the Roots are equal to each other.

§ 33. BEFORE we proceed to explain how to resolve equations of all sorts, we shall first demonstrate "*how an equation that has two or more roots equal, is depressed to a lower dimension*;" and its resolution made, consequently, more easy. And shall endeavour to explain the grounds of this and many other rules we shall give in the remaining part of this Treatise, in a more simple and concise manner than has hitherto be done.

In order to this, we must look back to § 24. where we find that if any aquation, as $x^3 - px^2 + qx - r = 0$, is proposed, and you are to transform it into another that shall have its roots less than the value of x by any given difference, as e, you are to assume $y = x - e$, and substituting for x its value $y + e$, you find the transformed equation,

$$\left.\begin{array}{rrr} y^3 + 3ey^2 & + 3e^2y & + e^3 \\ - py^2 & - 2pey & - pe^2 \\ & + qy & + qe \\ & & - r \end{array}\right\} = 0.$$

Where

Where we are to obſerve,

1° That the laſt term ($e^3 - pe^2 + qe - r$) is the very equation that was propoſed, having e in place of x.

2° The coefficient of the laſt term but one is $3e^2 - 2pe + q$, which is the quantity that ariſes by multiplying every term of the laſt coefficient $e^3 - pe^2 + qe - r$ by the index of e in each term, and dividing the product $3e^3 - 2pe^2 + qe$ by the quantity e that is common to all the terms.

3° The coefficient of the laſt term but two is $3e - p$, which is the quantity that ariſes by multiplying every term of the coefficient laſt found ($3e^2 - 2pe + q$) by the index of e in each term, and dividing the whole by $2e$.

§ 34. Theſe ſame obſervations extend to equations of all dimenſions. If it is the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ that is propoſed, then by ſuppoſing $y = x - e$, it will be transformed into this other,

$$\left.\begin{array}{rrrrr} y^4 + 4ey^3 & + 6e^2y^2 & + 4e^3y & + e^4 \\ - py^3 & - 3pey^2 & - 3pe^2y & - pe^3 \\ & + qy^2 & + 2qey & + qe^2 \\ & & - ry & - re \\ & & & + s \end{array}\right\} = 0.$$

Where again it is obvious that the laſt term is the equation that was propoſed, having e in place of x. That the laſt term but one has

M for

for its coefficient the quantity that arises by multiplying the terms of the last quantity by the indices of e in each term, and dividing the product by e. That the coefficient of the last term but two (*viz.* $6e^2 - 3pe + q$) is deduced in the same manner from the term immediately following; that is, by multiplying every term of $4e^3 - 3pe^2 + 2qe - r$ by the index of e in that term, and dividing the whole by e multiplied into the index of y in the term sought, that is, by $e \times 2$. And the next term is $4e - p = \frac{6e^2 \times 2 - 3pe \times 1}{3e}$.

The demonstration of this may easily be made general by the Theorem of finding the powers of a binomial, since the transformed equation consists of the powers of the binomial $y + e$ that are marked by the indices of e in the last term, multiplied each by their coefficients 1, $-p$, $+q$, $-r$, $+s$, &c. respectively.

§ 35. From the last two articles we can easily find the terms of the transformed equation without any involution. The last term is had by substituting e instead of x in the proposed equation; the next term, by multiplying every part of that last term by the index of e in each part, and dividing the whole by e; and the following terms in the manner described in the foregoing article; the respective divisors being

being the quantity e multiplied by the index of y in each term.

The demonſtration for finding when two or more roots are equal will be eaſy, if we add to this, that, "*when the unknown quantity enters all the terms of any equation, then one of its values is equal to nothing.*" As in the equation $x^3 - px^2 + qx = 0$, where $x - 0 = 0$ being one of the ſimple equations that produce $x^3 - px^2 + qx = 0$, it follows that one of the values of x is 0. In like manner two of the values of x are equal to nothing in this equation $x^3 - px^2 = 0$; and three of them vaniſh in the equation $x^4 - px^3 = 0$.

It is alſo obvious *(converſely)* that "if x does not enter all the terms of the equation, *i. e.* if the laſt term be not wanting, then none of the values of x can be equal to nothing;" for if every term be not multiplied by x, then $x - 0$ cannot be a diviſor of the whole equation, and conſequently 0 cannot be one of the values of x, If x^2 does not enter into all the terms of the equation, then two of the values of x, cannot be equal to nothing. If x^3 does not enter into all the terms of the equation, then three of the values of x cannot be equal to nothing, &c.

§ 36. Suppoſe now that two values of x are equal to one another, and to e; then it is plain that two values of y in the transformed equation will be equal to nothing: ſince $y = x - e$. And conſequently, by the laſt article, the two laſt terms of the transformed equation muſt vaniſh.

Suppoſe it is the cubic equation of § 33, that is propoſed, *viz.* $x^3 - px^2 + qx - r = 0$; and becauſe we ſuppoſe $x = e$, therefore the laſt term of the transformed equation, *viz.* $e^3 - pe^2 + qe - r$ will vaniſh. And ſince two values of y vaniſh, the laſt term but one *viz.* $3e^2y - 2pey + qy$ will vaniſh at the ſame time. So that $3e^2 - 2pe + q = 0$. But, by ſuppoſition, $e = x$; therefore, when two values of x, in the equation $x^3 - px^2 + qx - r = 0$, are equal, it follows, that $3x^2 - 2px + q = 0$. And thus "the propoſed cubic is depreſſed to a quadratic that has one of its roots equal to one of the roots of that cubic."

If it is the biquadratic that is propoſed, *viz.* $x^4 - px^3 + qx^2 - rx + s = 0$, and two of its roots be equal; then ſuppoſing $e = x$, two of the values of y muſt vaniſh, and the equation of § 34 will be reduced to this form,

$$\left.\begin{array}{rr} y^4 + 4ey^3 & + 6e^2y^2 \\ - py^3 & - 3pey^2 \\ & + qy^2 \end{array}\right\} ** = 0. \quad \text{So that}$$

$4e^3\ 3pe^2 + 2qe - r = 0$; or, becauſe $x = e$, $4x^3\ 3px^2 + 2qx - r = 0$.

In

In general, when two values of x are equal to each other, and to e, the two laſt terms of the transformed equation vaniſh: and conſequently, "if you multiply the terms of the propoſed equation by the indices of x in each term, the quantity that will ariſe will be $= 0$, and will give an equation of a lower dimenſion than the propoſed, that ſhall have one of its roots equal to one of the roots of the propoſed equation."

That the laſt two terms of the equation vaniſh when the values of x are ſuppoſed equal to each other, and to e, will alſo appear by conſidering, that ſince two values of y then become equal to nothing, the product of the values of y muſt vaniſh, which is equal to the laſt term of the equation; and becauſe two of the four values of y are equal to nothing, it follows alſo that one of any three that can be taken out of theſe four muſt be $= 0$; and therefore, the products made by multiplying any three muſt vaniſh; and conſequently the coefficient of the laſt term but one, which is equal to the ſum of theſe products, muſt vaniſh.

§ 37. After the ſame manner, if there are three equal roots in the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$, and if e be equal to one of them; three values of y $(= x - e)$ will vaniſh, and conſequently y^3 will enter all the terms of the transformed equation; which will have this form,

$$\left.\begin{matrix} y^4 + 4ey^3 \\ - py^3 \end{matrix}\right\} * * * = 0.$$ So that here

$6e^2 - 3pe + q = 0$; or, since $e = x$, therefore $6x^2 - 3px + q = 0$: and one of the roots of this quadratic will be equal to one of the roots of the proposed biquadratic.

In this case, two of the roots of the cubic equation $4x^3 - 3px^2 + 2qx - r = 0$ are roots of the proposed biquadratic, because the quantity $6x^2 - 3px + q$ is deduced from $4x^3 - 3px^2 + 2qx - r$, by multiplying the terms by the indexes of x in each term.

In general, "whatever is the number of equal roots in the proposed equation, they will all remain but one in the equation that is deduced from it by multiplying all the terms by the indexes of x in them; and they will all remain but two in the equation deduced in the same manner from that;" and so of the rest.

§ 38. What we observed of the coefficients of equations transformed by supposing $y = x - e$, leads to this easy demonstration of this Rule; and will be applied in the next chapter to demonstrate the Rules for finding the limits of equations.

It is obvious however, that though we make use of equations whose signs change alternately, the

the ſame reaſoning extends to all other equations.

It is a conſequence alſo of what has been demonſtrated, that "if two roots of any equation, as $x^3 - px^2 + qx - r = 0$, are equal, then multiplying the terms by any arithmetical ſeries, as $a + 3b$, $a + 2b$, $a + b$, a, the product will be $= 0$.

For ſince $ax^3 - apx^2 + aqx - ar = 0$; and $\overline{3x^2 - 2px + q} \times bx = 0$, it follows that
$ax^3 + 3bx^3 - apx^2 - 2bpx^2 + aqx + bqx - ar = 0$.
Which is the product that ariſes by multiplying the terms of the propoſed equation by the terms of the ſeries, $a + 3b$, $a + 2b$, $a + b$, a; which may repreſent any arithmetical progreſſion.

CHAP. V.

Of the Limits of Equations.

§ 39. WE now proceed to ſhew how to diſcover the limits of the roots of equations, by which their ſolution is much facilitated.

Let any equation, as $x^3 - px^2 + qx - r = 0$, be propoſed; and transform it, as above, into the equation.

$$\left.\begin{array}{l} y^3 + 3ey^2 + 3e^2y + e^3 \\ \quad - py^2 - 2pey - pe^2 \\ \qquad\quad + qy + qe \\ \qquad\qquad\quad - r \end{array}\right\} = 0.$$

Where the values of y are leſs than the reſpective values of x by the difference e. If you ſuppoſe e to be taken ſuch as to make all the coefficients, of the equation of y, poſitive, *viz.* $e^3 - pe^2 + qe - r$, $3e^2 - 2pe + q$, $3e - p$; then there being no variation of the ſigns in the equation, all the values of y muſt be negative; and conſequently, the quantity e, by which the values of x are diminiſhed, muſt be greater than the greateſt poſitive value of x: and conſequently muſt be the limit of the roots of the equation $x^3 - px^2 + qx - r = 0$.

It

It is sufficient therefore, in order to find the limit, to "enquire what quantity substituted for x in each of these expressions $x^3 - px^2 + qx - r$, $3x^2 - 2px + q$, $3x - p$, will give them all positive;" for that quantity will be the limit required.

How these expressions are formed from one another, was explained in the beginning of the last chapter.

EXAMPLE.

§ 40. If the equation $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ is proposed; and it is required to determine the limit that is greater than any of the roots; you are to enquire what integer number substituted for x in the proposed equation, and following equations deduced from it by § 35; will give, in each, a positive quantity.

$$5x^4 - 8x^3 - 30x^2 + 60x + 63$$
$$5x^3 - 6x^2 - 15x + 15$$
$$5x^2 - 4x - 5$$
$$5x - 2.$$

The least integer number which gives each of these positive, is 2; which therefore is the limit of the roots of the proposed equation; or a number that exceeds the greatest positive root.

If

If the limit of the *negative* roots is required, you may, by § 23, change the negative into positive roots, and then proceed as before to find their limits. Thus, in the example, you will find that -3 is the limit of the negative roots. So that the five roots of the proposed equation are betwixt -3 and $+2$.

§ 41. Having found the limit that surpasses the greatest positive root, call it m. And if you assume $y=m-x$, and for x substitute $m-y$, the equation that will arise will have all its roots positive; because m is supposed to surpass all the values of x, and consequently $m-x$ $(=y)$ must always be affirmative. And by this means, *any equation may be changed into one that shall have all its roots affirmative.*

Or if $-n$ represent the limit of the negative roots, then by assuming $y=x+n$, the proposed equation shall be transformed into one that shall have all its roots affirmative; for $+n$ being greater than any negative value of x, it follows that $y=x+n$ must be always positive.

§ 42. " *The greatest negative coefficient of any equation increased by unit, always exceeds the greatest root of the equation.*

To demonstrate this, let the cubic $x^3-px^2-qx-r=0$ be proposed; where all the terms are negative except the first. Assuming $y=x-e$ it will be transformed into the following equation.

(A)

$$(A)\left.\begin{array}{r}y^3+3ey^2+3e^2y+e^3\\ -py^2-2pey-pe^2\\ -qy\;\;-qe\\ -r\end{array}\right\}=0.$$

1°. Let us suppose that the coefficients p, q, r, are equal to each other; and if you also suppose $e=p+1$, then the last equation becomes

$$(B)\left.\begin{array}{l}y^3+2py^2+p^2y+1\\ \quad\;\;+3y^2\;\;+3py\\ \qquad\qquad\;\;+3y\end{array}\right\}=0.$$

Where all the terms being positive, it follows that the values of y are all negative, and that consequently e, or $p+1$, is greater than the greatest value of x in the proposed equation.

2°. If q and r be not $=p$, but less than it, and for e you still substitute $p+1$ (since the negative part $\left(\begin{array}{c}-qy-qe\\ -r\end{array}\right)$ becomes less, the positive remaining undiminished) *a fortiori*, all the coefficients of the equation *(A)* become positive. And the same is obvious if q and r have positive signs, and not negative signs, as we supposed. It appears therefore, "that, if, in any cubic equation, p be the greatest negative coefficient, then $p+1$ must surpass the greatest value of x."

§ 43. 3°. By the same reasoning it appears, that if q be the greatest negative coefficient of the

the equation, and $e = q + 1$, then there will be no variation of the figns in the equation of y: for it appears from the laft article, that if all the three (p, q, r) were equal to one another, and e equal to any one of them increafed by unit, as to $q + 1$, then all the terms of the equation (A) would be pofitive. Now if e be fuppofed ftill equal to $q + 1$, and p and r to be lefs than q, then, *a fortiori*, all thefe terms will be pofitive, the negative part, which involves p and r, being diminifhed, while the pofitive part and the negative involving q remain as before.

4° After the fame manner it is demonftrated, that if r is the greateft negative coefficient in the equation, and e is fuppofed $= r + 1$, then all the terms of the equation (A) of y will be pofitive; and confequently $r + 1$ will be greater than any of the values of x.

What we have faid of the *cubic* equation $x^3 - px^2 + qx - r = 0$, is eafily applicable to others.

In general, we conclude that "the greateft negative coefficient in any equation increafed by unit, is always a limit that exceeds all the roots of that equation."

But it is to be obferved at the fame time, that the greateft negative coefficient increafed by unit, is very feldom the *neareft* limit: that is beft difcovered by the Rule in the 39th article.

§ 44.

§ 44. Having ſhewn in § 41. how to change any propoſed equation into one that ſhall have all its roots affirmative; we ſhall only treat of ſuch as have all their roots poſitive, in what remains relating to the limits of equations.

Any ſuch equation may be repreſented by $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d}$ &c. $= 0$, whoſe roots are a, b, c, d, &c.

And of all ſuch equations two limits are eaſily diſcovered from what precedes, *viz.* 0, which is leſs than the leaſt, and e, found according to § 39. which ſurpaſſes the greateſt root of the equation.

But beſides theſe, we ſhall now ſhew how "*to find other limits betwixt the roots themſelves.*" And, for this purpoſe, will ſuppoſe a to be the leaſt root, b the ſecond root, c the third, and ſo on; it being arbitrary.

§ 45. If you ſubſtitute 0 in place of the unknown quantity, putting $x = 0$, the quantity that will ariſe from that ſuppoſition is the laſt term of the equation, all the others, that involve x, vaniſhing.

If you ſubſtitute for x a quantity leſs than the leaſt root a, the quantity reſulting will have the ſame ſign as the laſt term; that is, will be poſitive or negative according as the equation is of an even or odd number of dimenſions. For all the factors $x - a$, $x - b$, $x - c$, &c. will be negative, and their product will be *poſitive*

or

or *negative* according as their number is *even* or *odd*.

If you substitute for x a quantity greater than the least root a, but less than all the other roots, then the sign of the quantity resulting will be contrary to what it was before; because one factor $(x - a)$ becomes now positive, all the others remaining negative as before.

If you substitute for x a quantity greater than the two least roots, but less than all the rest, both the factors $x - a$, $x - b$, become positive, and the rest remain as they were. So that the whole product will have the same sign as the last term of the equation. Thus successively placing instead of x quantities that are limits betwixt the roots of the equation, the quantities that result will have alternately the signs $+$ and $-$. And, *conversely*, "if you find quantities which substituted in place of x in the proposed equation, do give alternately positive and negative results, those quantities are the limits of that equation."

It is useful to observe, that, in general, "when, by substituting any two numbers for x in any equation, the results have contrary signs, one or more of the roots of the equation must be betwixt those numbers." Thus, in the equation $x^3 - 2x^2 - 5 = 0$, if you substitute 2 and 3 for x, the results are -5, $+4$; whence it follows that the roots are betwixt 2 and

and 3: for when these results have different signs, one or other of the factors which produce the equations must have changed its sign; suppose it is $x = e$, then it is plain that e must be betwixt the numbers supposed equal to x.

§ 46. Let the cubic equation $x^3 - px^2 + qx - r = 0$ be proposed, and let it be transformed, by assuming $y = x - e$ into the equation

$$\left.\begin{array}{r} y^3 + 3ey^2 + 3e^2y + e^3 \\ - py^2 - 2pey - pe^2 \\ + qy + qe \\ - r \end{array}\right\} = 0.$$

Let us suppose e equal successively to the three values of x, beginning with the least value; and because the last term $e^3 - pe^2 + qe - r$ will vanish in all these suppositions, the equation will have this form,

$$\left.\begin{array}{r} y^2 + 3ey + 3e^2 \\ - py - 2pe \\ + q \end{array}\right\} = 0;$$

where the last term $3e^2 - 2pe + q$ is, from the nature of equations, produced of the remaining values of y, or of the excesses of two other values of x above what is supposed equal to e, since always $y = x - e$. Now,

1°. If e be equal to the least value of x, then those two excesses being both positive, they will give a positive product, and consequently $3e^2 - 2pe + q$ will be, in this case, positive.

2.

2°. If e be equal to the second value of x, then, of those two excesses one being negative and one positive, their product $3e^2 - 2pe + q$, will be negative.

3°. If e be equal to the third and greatest value of x, then the two excesses being both negative, their product $3e^2 - 2pe + q$ is positive. Whence,

If in the equation $3e^2 - 2pe + q = 0$, you substitute successively in the place of e, the three roots of the equation $e^3 - pe^2 + qe - r = 0$, the quantities resulting will successively have the signs $+$, $-$, $+$; and consequently the three roots of the cubic equation are the limits of the roots of the equation $3e^2 - 2pe + q = 0$ (by § 45.) That is, the least of the roots of the cubic is less than the least of the roots of the other; the second root of the cubic is a limit between the two roots of the other; and the greatest root of the cubic is the limit that exceeds both the roots of the other.

§ 47. We have demonstrated that the roots of the cubic equation $e^3 - pe^2 + qe - r = 0$ are limits of the quadratic $3e^2 - 2pe + q$; whence it follows (*conversely*) that the roots of the quadratic $3e^2 - 2pe + q = 0$ are the limits between the first and second, and between the second and third roots of the cubic $e^3 - pe^2 + qe - r = 0$. So that if you find the limit that exceeds the greatest

greateſt root of the cubic, by § 39. you will have (with o, which is the limit leſs than any of the roots) four limits for the three roots of the propoſed cubic.

It was demonſtrated in § 35. how the quadratic $3e^2 - 2pe + q$ is deduced from the propoſed cubic $e^3 - pe^2 + qe - r = 0$, *viz.* by multiplying each term by the index of e in it, and then dividing the whole by e; and what we have demonſtrated of cubic equations is eaſily extended to all others; ſo that we conclude, "that the laſt term but one of the transformed equation is the equation for determining the limits of the propoſed equation." Or, that the equation ariſing by multiplying each term by the index of the unknown quantity in it, is the equation whoſe roots give the limits of the propoſed equation; if you add to them the two mentioned in § 44.

§ 48. For the ſame reaſon, it is plain that the root of the ſimple equation $3e - p = 0$, (*i. e.* $\frac{1}{3}p$) is the limit between the two roots of the quadratic $3e^2 - 2pe + q = 0$. And, as $4e^3 - 3pe^2 + 2qe - r = 0$ gives three limits of the equation $e^4 - pe^3 + qe^2 - re + s = 0$, ſo the quadratic $6e^2 - 3pe + q = 0$ gives two limits that are betwixt the roots of the cubic $4e^3 - 3pe^2 + 2qe - r = 0$; and $4e - p = 0$ gives one limit that is betwixt the two roots of the quadratic $6e^2 - 3pe + q = 0$. So that we have a complete series of these equations arising from a simple equation to the proposed, each of which determines the limits of the following equation.

§ 49. If two roots in the proposed equation are equal, then "the limit that ought to be betwixt them must, in this case, become equal to one of the equal roots themselves." Which perfectly agrees with what was demonstrated in the last chapter, concerning the Rule for finding the equal roots of equations.

And, the same equation that gives the limits, giving also one of the equal roots, when two or more are equal, it appears, that "if you substitute a limit in place of the unknown quantity in an equation, and, instead of a positive or negative result, it be found $= 0$, then you may conclude, that not only the limit itself is a root of the equation, but that there are two roots in that equation equal to it and to one another.

§ 50. It having been demonstrated that the roots of the equation $x^3 - px^2 + qx - r = 0$ are the limits of the roots of the equation $3x^2 - 2px + q = 0$, the three roots of the cubic equation, which suppose to be a, b, c, substituted for x in the quadratic $3x^2 - 2px + q$, must give the results positive and negative alternately.

nately. Suppose these three results to be $+N$, $-M$, $+L$; that is, $3a^2-2pa+q=N$, $3b^2-2pb+q=-M$, $3c^2-2pc+q=L$; and since $a^3-pa^2+qa-r=0$, and $3a^3-2pa^2+qa=N\times a$, subtracting the former multiplied into 3 from the latter, the remainder is $pa^2-2qa+3r=N\times a$. In the same manner $pb^2-2qb+3r=-M\times b$, and $pc^2-2qc+3r=+L\times c$. Therefore $px^2-2qx+3r$ is such a quantity that if, for x, you substitute in it successively a, b, c, the results will be $+N\times a$, $-M\times b$, $+L\times c$. Whence a, b, c, are limits of the equation $px^2-2qx+3r=0$ (by § 45.) and, *conversely*, the roots of the equation $px^2-2qx+3r=0$ are limits between the first and second, and between the second and third roots of the cubic $x^3-px^2+qx-r=0$. Now the equation $px^2-2qx+3r=0$ arises from the proposed cubic by multiplying the terms of this latter by the arithmetical progression $0, -1, -2, -3$. And in the same manner it may be shewn that the roots of the equation $px^3-2qx^2+3rx-4s=0$ are limits of the equation $x^4-px^3+qx^2-rx+s=0$.

Or multiply the terms of the equation

$$x^3-px^2+qx-r=0$$

by $a+3b, a+2b, a+b, a$

$$ax^3-apx^2+aqx-ar\ (=0)$$

$$+3bx^3-2bpx^2+bqx\ (=\overline{3x^2-2px+q}\times bx).$$

 Any

Any arithmetical series where a is the least term and b the common difference, and the products (if you substitute for x, successively, a, b, c, the three roots of the proposed cubic) shall be $+N\times bx$, $-M\times bx$, $+L\times bx$. For the first part of the product $a\times\overline{x^3-px^2+qx-r}=0$; and a, b, c, being limits in the equation $3x^2-2px+q=0$, their substitution must give results, N, M, L, alternately positive and negative.

In general, the roots of the equation $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+$ &c. $=0$ are limits of the roots of the equation $nx^{n-1}-\overline{n-1}\times px^{n-2}+\overline{n-2}\times qx^{n-3}-\overline{n-3}\times rx^{n-4}+$ &c. $=0$; or of any equation that is deduced from it by multiplying its terms by any arithmetical progression $a\mp b$, $a\mp 2b$, $a\mp 3b$, $a\mp 4b$, &c. *And conversely*, the roots of this new equation will be limits of the proposed equation.

$$x^n-px^{n-1}+qx^{n-2}-\text{\&c.}=0.$$

" *If any roots of the equation of the limits are impossible, then must there be some roots of the proposed equation impossible.*" For as (in § 46.) the quantity $3e^2-2pe+q$ was demonstrated to be equal to the product of the excesses of two values of x above the third supposed equal to e; if any impossible expression be found in those excesses, then there will of consequence be found impossible expressions in these two values of x.

And

And "from this obſervation rules may be deduced for diſcovering when there are impoſſible roots in equations." Of which we ſhall treat afterwards.

§ 51. Beſides the method already explained, there are others by which limits may be determined, which the root of an equation cannot exceed.

Since the ſquares of all real quantities are affirmative, it follows, that "*the ſum of the ſquares of the roots of any equation muſt be greater than the ſquare of the greateſt root.*" And the ſquare root of that ſum will therefore be a limit that muſt exceed the greateſt root of the equation.

If the equation propoſed is $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, then the ſum of the ſquares, of the roots (by § 15.) will be $p^2 - 2q$. So that $\sqrt[2]{p^2 - 2q}$ will exceed the greateſt root of that equation.

Or if you find, by § 16. the ſum of the 4th powers of the roots of the equation, and extract the biquadratic root of that ſum, it will alſo exceed the greateſt root of the equation.

§ 52. If you find a mean proportional between the ſum of the ſquares of any two roots, a, b, and the ſum of their biquadrates $(a^4 + b^4)$, this mean proportional will be $\sqrt[2]{a^6 + a^2b^4 + a^4b^2 + b^6}$. And the ſum of the cubes is $a^3 + b^3$. Now since $a^2 - 2ab + b^2$ is the square of $a - b$, it must be always positive; and if you multiply it by a^2b^2, the product $a^4b^2 - 2a^3b^3 + a^2b^4$ will also be positive; and consequently $a^4b^2 + a^2b^4$ will be always greater than $2a^3b^3$. Add $a^6 + b^6$, and we have $a^6 + a^4b^2 + a^2b^4 + b^6$ greater than $a^6 + 2a^3b^3 + b^6$; and extracting the root $\sqrt[2]{a^6 + a^4b^2 + a^2b^4 + b^6}$ greater than $a^3 + b^3$. And the same may be demonstrated of any number of roots whatever.

Now if you add the sum of all the cubes taken *affirmatively* to their sum with their *proper* signs, they will give double the sum of the cubes of the affirmative roots. And if you subtract the second sum from the first, there will remain double the sum of the cubes of the negative roots. Whence it follows, that "half the sum of the mean proportional betwixt the sum of the squares and the sum of the biquadrates, and of the sum of the cubes of the roots with their proper signs, exceeds the sum of the cubes of the affirmative roots;" and "half their difference exceeds the sum of the cubes of the negative roots." And by extracting the cube root of that sum and difference, you will obtain limits that shall exceed the sums of the affirmative and of the negative roots. And since it is easy, from what has been

been already explained, to diminish the roots of an equation so that they all may become negative but one, it appears how by this means you may approximate very near to that root. But this does not serve when there are impossible roots.

Several other Rules like these might be given for limiting the roots of equations. We shall give one not mentioned by other Authors.

In a cubic $x^3 - px^2 + qx - r = 0$ find $q^2 - 2pr$, and call it e^4; then shall the greatest root of the equation always be greater than $\frac{e}{\sqrt{3}}$, or $\sqrt[4]{\frac{e^4}{3}}$. And,

In any equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ find $\frac{q^2 - 2pr + 2s}{n}$, and extracting the root of the fourth power out of that quantity, it shall always be less than the greatest root of the equation.

CHAP. VI.

Of the Resolution of Equations, all whose Roots are commensurate.

§ 53. IT was demonstrated, in *Chap.* 2, that the last term of any equation is the product of its roots: from which it follows, that the roots of an equation, when commensurable quantities, will be found among the divisors of the last term. And hence we have for the resolution of equations this

RULE.

Bring all the terms to one side of the equation, find all the divisors of the last term, and substitute them successively for the unknown quantity in the equation. So shall that divisor which, substituted in this manner, gives the result $= 0$, *be the root of the proposed equation.*

For example, suppose this equation is to be resolved,

$$\left.\begin{matrix} x^3 - 3ax^2 + 2a^2x - 2a^2b \\ - bx^2 + 3abx \end{matrix}\right\} = 0,$$

where the last term is $2a^2b$, whose simple literal

ral divisors are a, b, $2a$, $2b$, each of which may be taken either positively or negatively: but as here we find there are variations of signs in the equation, we need only take them positively. Suppose $x = a$ the first of the divisors, and substituting a for x, the equation becomes

$$\left.\begin{matrix} a^3-3a^3+2a^3-2a^2b \\ -a^2b+3a^2b \end{matrix}\right\} \text{or, } 3a^3-3a^3+3a^2b-3a^2b=0.$$

So that, the whole vanishing, it follows that a is one of the roots of the equation.

After the same manner, if you substitute b in place of x, the equation is

$$\left.\begin{matrix} b^3-3ab^2+2a^2b-2a^2b \\ -b^3+3ab^2 \end{matrix}\right\} = 0,$$

which vanishing shews b to be another root of the equation.

Again, if you substitute $2a$ for x, you will find all the terms destroy one another so as to make the sum $= 0$. For it will then be.

$$\left.\begin{matrix} 8a^3-12a^3+4a^3-2a^2b \\ -4a^2b+6a^2b \end{matrix}\right\} = 0.$$

Whence we find that $2a$ is the third root of the equation. Which, after the first two ($+a$, $+b$) had been found, might have been collected from this, that the last term being the product of the three roots, $+a$, $+b$ being known, the

the third must necessarily be equal to the last term divided by the product ab, that is, $= \frac{2a^2b}{ab} = 2a$.

Let the roots of the cubic equation

$x^3 - 2x^2 - 33x + 90 = 0$ be required.

And first the divisors of 90 are found to be 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90. If you substitute 1 for x, you will find $x^3 - 2x^2 - 33x + 90 = 56$; so that 1 is not a root of the equation. If you substitute 2 for x, the result will be 24: but putting $x = 3$, you have

$x^3 - 2x^2 - 33x + 90 = 27 - 18 - 99 + 90 = 117 - 117 = 0$.

So that 3 is one of the roots of the proposed equation. The other affirmative root is $+5$; and after you find it, as it is manifest from the equation, that the other root is negative, you are not to try any more divisors taken positively, but to substitute them, negatively taken, for x: and thus you find that -6 is the third root. For putting $x = -6$, you have

$x^3 - 2x^2 - 33x + 90 = -216 - 72 + 198 + 90 = 0$.

This last root might have been found by dividing the last term 90, having its sign changed, by 15, the product of the two roots already found.

§ 55. When one of the roots of an equation is found, in order to find the rest with less trouble, divide the proposed equation by the simple equation which you are to deduce from the root already

already found, and the quotient shall give an equation of a degree lower than the proposed; whose roots will give the remaining roots required.

As for example, the root $+3$, first found, gave $x=3$ or $x-3=0$, whence dividing thus,

$$
\begin{array}{l}
x-3)\ x^3-2x^2-33x+90\ (x^2+x-30 \\
\quad\ \underline{x^3-3x^2} \\
\qquad\qquad x^2-33x+90 \\
\qquad\qquad \underline{x^2-\ \ 3x} \\
\qquad\qquad\quad -30x+90 \\
\qquad\qquad\quad \underline{-30x+90} \\
\qquad\qquad\qquad 0 \quad\ \ 0
\end{array}
$$

The quotient shall give a quadratic equation $x^2+x-30=0$, which must be the product of the other two simple equations from which the cubic is generated, and whose roots therefore must be two of the roots of that cubic.

Now the roots of that quadratic equation are easily found by *Chap.* 13. *Part* I. to be $+5$ and -6. For,

$$x^2+x=30;$$

add $\frac{1}{4}$ . . . $x^2+x+\frac{1}{4}=30+\frac{1}{4}=\frac{121}{4}$,

$\surd$ . . . . . . $x+\frac{1}{2}=\pm\sqrt{\frac{121}{4}}=\pm\frac{11}{2}$,

and . . . . . $x=\pm\frac{11}{2}-\frac{1}{2}=+5$, or -6.

§ 56.

§ 56. After the ſame manner, if the biquadratic $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$ is to be resolved; by ſubſtituting the diviſors of 120 for x, you will find that $+3$, one of thoſe diviſors, is one of the roots; the ſubſtitution of 3 for x giving $81 - 54 - 225 + 78 + 120 = 279 - 279 = 0$. And therefore dividing the propoſed equation by $x - 3$, you muſt enquire for the roots of the cubic $x^3 + x^2 - 22x - 40 = 0$, and finding that $+5$, one of the diviſors of 40, is one of the roots, you divide that cubic by $x - 5$, and the quotient gives the quadratic $x^2 + 6x + 8 = 0$, whoſe two roots are -2, -4. So that the four roots of the biquadratic are $+3$, $+5$, -2, -4.

§ 57. This Rule ſuppoſes that you can find all the diviſors of the laſt term; which you may always do thus.

" *If it is a ſimple quantity, divide it by its leaſt diviſor that exceeds unit, and the quotient again by its leaſt diviſor, proceeding thus till you have a quotient that is not diviſible by any number greater than unit.* This quotient, with theſe diviſors, are the firſt or *ſimple diviſors* of the quantity. And the products of the multiplication of any 2, 3, 4, &c. of them are the *compound diviſors*.

As, to find the diviſors of 60; firſt I divide by 2, and the quotient 30 again by 2, then the next

next quotient 15 by 3, and the quotient of this division 5 is not farther divisible by any integer above unit; so that

The simple divisors are . . . . . . 2, 2, 3, 5.
The products of two, ... 4, 6, 10, 15.
The products of three, ... 12, 20, 30.
The products of all four, ... 60.

The divisors of 90 are found after the same manner.

Simple divisors, ... 2, 3, 3, 5.
The products of two, ... 6, 9, 10, 15.
The products of three, ... 18, 30, 45.
The product of all four, ... 90.

The divisors of $21abb$.

The simple divisors, ... $3, 7, a, b, b$.
The products of two, $21, 3a, 3b, 7a, 7b, ab, bb$.
The products of three, $21a, 21b, 3ab, 3bb, 7ab, 7bb, abb$.
The products of four, $21ab, 21bb, 3abb, 7abb$.
The product of the five, ... $21abb$.

§ 58. But as the last term may have very many divisors, and the labour may be very great to substitute them all for the unknown quantity, we shall now show how it may be abridged, by limiting to a small number the divisors you are to try. And first it is plain, from § 42. that " any divisor that exceeds the greatest

greatest negative coefficient by unity is to be neglected."[a] Thus in resolving the equation $x^4 - 2x^3 - 25x^2 + 26x + 120 = 0$, as 25 is the greatest negative coefficient, we conclude that the divisors of 120 that exceed 26 may be neglected.

But the labour may be still abridged, if we make use of the Rule in § 39; that is, if we find the number, which substituted in these following expressions,

$$x^4 - 2x^3 - 25x^2 + 26x + 120$$
$$2x^3 - 3x^2 - 25x + 13,$$
$$6x^2 - 6x - 25,$$
$$2x - 1,$$

will give in them all a positive result; for that number will be greater than the greatest root, and all the divisors of 120 that exceed it may be neglected.

That this investigation may be easier, we ought to begin always with that expression, where the negative roots seem to prevail most; as here in the quadratic expression $6x^2 - 6x - 25$; where finding that 6 substituted for x gives that expression positive, and gives all the other expressions at the same time positive, I conclude that 6 is greater than any of the roots, and that all the divisors of 120 that exceed 6 may be neglected.

If

If the equation $x^3 + 11x^2 + 10x - 72 = 0$ is proposed, the Rule of § 42 does not help to abridge the operation; the last term itself being the greatest negative term. But, by § 39. we enquire what number substituted for x will give all these expressions positive:

$$x^3 + 11x^2 + 10x - 72,$$
$$3x^2 + 22x + 10,$$
$$3x + 11.$$

Where the labour is very short, since we need only attend to the first expression; and we see immediately that 4 substituted for x gives a positive result, whence all the divisors of 72 that exceed 4 are to be rejected; and thus by a few trials we find that $+ 2$ is the positive root of the equation. Then dividing the equation by $x - 2$, and resolving the quadratic equation that is the quotient of the division, you find the other two roots to be $- 9$, and $- 4$.

§ 59. But there is another method that reduces the divisors of the last term, that can be useful, still to more narrow limits.

Suppose the cubic equation $x^3 - px^2 + qx - r = 0$ is proposed to be resolved. Transform it to an equation whose root shall be less than the values of x by unity, assuming $y = x - 1$. And the last term of the transformed equation will be $1 - p + q - r$; which is found by substituting unit, the difference of x and y, for x, in in the proposed equation; as will easily appear from § 24, where, when $y = x - e$, the last term of the transformed equation was $e^3 - pe^2 + qe - r$.

Transform again the equation $x^3 - px^2 + qx - r = 0$, by assuming $y = x + 1$, into an equation whose roots shall exceed the values of x by unit, and the last term of the transformed equation will be $-1 - p - q - r$, the same that arises by substituting -1, the difference betwixt x and y, for x, in the proposed equation.

Now the values of x are some of the divisors of r, which is the term left when you suppose $x = 0$; and the values of the y's are some of the divisors of $+1 - p + q - r$, and of $-1 - p - q - r$, respectively. And these values are in arithmetical progression increasing by the common difference unit; because $x - 1$, x, $x + 1$, are in that progression. And it is obvious the same reasoning may be extended to any equation of whatever degree. So that this gives a general method for the resolution of equations whose roots are commensurable.

RULE.

" *Substitute in place of the unknown quantity successively the terms of the progression* 1, 0, -1, &c. *and find all the divisors of the sums that result; then take out all the arithmetical progressions you can find among these divisors, whose*

whose common difference is unit, and the values of x will be among the divisors arising from the substitutions of $x = 0$ that belong to these progressions." The values of x will be affirmative when the arithmetical progression increases, but negative when it decreases.

EXAMPLE.

§ 60. Let it be required to find one of the roots of the equation $x^3 - x^2 - 10x + 6 = 0$. The operation is thus:

Suppofit.	*Refult.*		*Divifors.*	*Arith. prog: decr.*
$x = 1$		-4	1, 2, 4	4
$x = 0$	$x^3 - x^2 - 10x + 6 =$	$+6$	1, 2, 3, 6	3 gives $x = -3$
$x = -1$		$+14$	1, 2, 7, 14	2

Where the suppositions of $x = 1$, $x = 0$, $x = -1$ give the quantity $x^3 - x^2 - 10x + 6$ equal to -4, 6, 14; among whose divisors we find only one arithmetical progression, 4, 3, 2; the term of which opposite to the supposition of $x = 0$, being 3, and the series decreasing, we try if -3 substituted for x makes the equation vanish; which succeeding one of its roots must be -3. Then dividing the equation by $x + 3$, we find the roots of the (quadratic) quotient

$$x^2 - 4x + 2 = 0 \text{ are } 2 \pm \sqrt{2}.$$

§ 61. If it is required to find the roots of the equation $x^3 + 3x^2 - 46x - 72 = 0$, the operation will be thus:

Suppos.	Results	Divisors.	Progressions.			
$x = 1$ —	120	1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.	8	3	4	5
$x = 0$ —	72	1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.	9	2	3	4
$x = -1$ —	30	1, 2, 3, 5, 6, 10, 15, 30.	10	1	2	3

Of these four arithmetical progressions having their common difference equal to unit, the first

first gives $x=9$, the others give $x=-2$, $x=-3$, $x=-4$; all which succeed except $x=-3$: so that the three values of x are $+9$, -2, -4.

CHAP. VII.

Of the Resolutions of Equations by finding the equations of a lower degree that are their divisors.

§ 62. TO find the roots of an equation is the same thing as to find the *simple* equations, by the multiplication of which into one another it is produced, or to find the simple equations that divide it without a remainder.

If such simple equations cannot be found, yet if we can find the *quadratic* equations from which the proposed equation is produced, we may discover its roots afterwards by the resolution of these quadratic equations. Or, if neither these simple equations nor these quadratic equations can be found, yet, by finding a *cubic* or *biquadratic* that is a divisor of the proposed equation, we may depress it lower, and make the solution more easy.

Now, in order to find the Rules by which these divisors may be discovered, we shall suppose that

$$\left.\begin{array}{l} mx - n \\ mx^2 - nx + r \\ mx^3 - nx^2 + rx - s \end{array}\right\} \text{are the} \left\{\begin{array}{l} \text{simple} \\ \text{quadratic} \\ \text{cubic} \end{array}\right.$$

divisors of the proposed equation; and if E represent the quotient arising by dividing the proposed equation by that divisor, then

$$E \times \overline{mx - n},$$

$$E \times \overline{mx^2 - nx + r},$$

or, $E \times \overline{mx^3 - nx^2 + rx - s}$, will represent the proposed equation itself. Where it is plain, that "since m is the coefficient of the highest term of the divisors, it must be a divisor of the coefficient of the highest term of the proposed equation."

§ 63. Next we are to observe, that, supposing the equation has a simple divisor $mx - n$, if we substitute in the equation $E \times \overline{mx - n}$, in place of x, any quantity, as a, then the quantity that will result from this substitution will necessarily have $ma - n$ for one of its divisors; since, in this substitution, $mx - n$ becomes $ma - n$.

If we substitute successively for x any arithmetical progression, a, $a - e$, $a - 2e$, &c. the quantities that will result from these substitutions, will have among their divisors

$ma - n,$

$ma - me - n,$

$ma - 2me - n,$ which are also in arithmetical progression, having their common difference equal to me.

If, for example, we substitute for x the terms of this progression, 1, 0, $-$ 1, the quantities that result have among their divisors the arithmetical progression $m - n$, $- n$, $- m - n$; or, changing the signs, $n - m$, n, $n + m$. Where the difference of the terms is m, and the term belonging to the supposition of $x = 0$ is n.

§ 64. It is manifest therefore, that when an equation has any simple divisor, if you substitute for x the progression 1, 0, $-$ 1, there will be found amongst the divisors of the sums that result from these substitutions, one arithmetical progression at least, whose common difference will be unit or a divisor m of the coefficient of the highest term, and which will be the coefficient of x in the simple divisor required: and whose term, arising from the supposition of $x = 0$, will be n the other member of the simple divisor $mx - n$.

From which this Rule is deduced for discovering such a simple divisor, when there is any.

RULE.

"*Substitute for x in the proposed equation successively the numbers* 1, 0, $-$ 1. *Find all the divisors of the sums that result from this substitution, and take out all the arithmetical progressions you can find amongst them, whose difference is unit, or some divisor of the coefficient of the highest term of the equation. Then suppose n equal to that term of any one progression that arises from the supposition of $x = 0$, and $m =$ the foresaid divisor of the coefficient of the highest term of the equation, which m is also the difference of the terms of this progression; so shall you have $mx \pm n$ for the divisor required.*"

You may find arithmetical progressions giving divisors that will not succeed; but if there is any divisor, it will be found thus by means of these arithmetical progressions.

§ 65. If the equation proposed has the coefficient of its highest term $= 1$, then it will be $m = 1$, and the divisor will be $x - n$, and the rule will coincide with that given in the end of the last chapter, which we demonstrated after a different manner; for the divisor being $x - n$, the value of x will be $+ n$, the term of the progression that is a divisor of the sum that arises from supposing $x = 0$. Of this case we gave

gave examples in the last chapter; and though it is easy to reduce an equation whose highest term has a coefficient different from unit, to one where that coefficient shall be unit, by § 30; yet, without that reduction, the equation may be resolved by this rule, as in the following

EXAMPLE.

§ 66. Suppose $8x^3 - 26x^2 + 11x + 10 = 0$, and that it is required to find the values of x; the operation is thus:

Suppos.	Results.		Divisors.	Progr.	
$x = 1$		$+5$	1, 5.	3	3
$x = 0$	$8x^3 - 26x^2 + 11x + 10 =$	$+10$	1, 2, 5, 10.	2	5
$x = -1$		-35	1, 5, 7, 35.	1	7

The difference of the terms of the last arithmetical progression is 2, a divisor of 8, the coefficient of the highest term x^3 of the equation, therefore supposing $m = 2$, $n = 5$, we try the divisor $2x - 5$; which succeeding, it follows that $2x - 5 = 0$, or $x = 2\frac{1}{2}$.

The quotient is the quadratic $4x^2 - 3x - 2 = 0$, whose roots are $\frac{3 + \sqrt{41}}{8}$, and $\frac{3 - \sqrt{41}}{8}$, so that the three roots of the proposed equation are $2\frac{1}{2}$, $\frac{3 + \sqrt{41}}{8}$, $\frac{3 - \sqrt{41}}{8}$. The other arithmetical progression gives $x+2$; for a divisor. But it does not succeed.

§ 67. If the proposed equation has no simple divisor, then we are to enquire if it has not some quadratic divisor (if itself is an equation of more than three dimensions).

An equation having the divisor $mx^2 - nx + r$ may be expressed as in the first article of this chapter by $E \times \overline{mx^2 - nx + r}$, and if we substitute for x any known quantity a, the sum that will result will have $ma^2 - na + r$ for one of its divisors; and, if we substitute successively for x the progression a, $a - e$, $a - 2e$, $a - 3e$, &c. the sums that arise from this substitution will have

$$ma^2 - na + r,$$

$$m \times \overline{a - e}^2 - n \times \overline{a - e} + r,$$

$$m \times \overline{a - 2e}^2 - n \times \overline{a - 2e} + r,$$

$$m \times \overline{a - 3e}^2 - n \times \overline{a - 3e} + r,$$

&c.

amongst their divisors, respectively.

These terms are not now, as in the last case, in arithmetical progression; but if you subtract them from the squares of the terms a, $a - e$, $a - 2e$, $a - 3e$, &c. multiplied by m a divisor of the highest term of the proposed equation, that is from

ma^2,

ma^2,

$m \times \overline{a-e}|^2$,

$m \times \overline{a-2e}|^2$,

$m \times \overline{a-3e}|^2$, &c. the remainders,

$na - r$,

$n \times \overline{a-e} - r$,

$n \times \overline{a-2e} - r$,

$n \times \overline{a-3e} - r$, &c. shall be in arithmetical progression, having their common difference equal to $n \times e$.

If, for example, we suppose the assumed progression a, $a-e$, $a-2e$, $a-3e$, &c. to be 2, 1, 0, -1, the divisors will be

$$\left.\begin{array}{r} 4m - 2n + r, \\ m - n + r, \\ + r, \\ m + n + r, \end{array}\right\} \begin{array}{c}\text{which} \\ \text{subtracted} \\ \text{from}\end{array} \left\{\begin{array}{c} 4m \\ m \\ 0 \\ m \end{array}\right\} \text{respectively,}$$

$$\begin{array}{rr} \text{leave} & 2n - r, \\ & n - r, \\ & - r, \\ & - n - r, \end{array}$$

an arithmetical progression, whose difference is $+ n$; and whose term arising from the substitution of 0 for x is $- r$.

From which it follows, that by this operation, if the proposed equation has a quadratic divisor, you will find an arithmetical progression

sion that will determine to you n and r, the coefficient m being supposed known; since it is unit, or a divisor of the coefficient of the highest term of the equation. Only you are to observe, that if the first term mx^2 of the quadratic divisor is negative, then, in order to obtain an arithmetical progression, you are not to subtract, but add the divisors $-4m-2n+r$, $-m-n+r$, $+r$, $-m+n+r$, to the terms $4m$, m, 0, m.

§ 68. The general Rule therefore, deduced from what we have said, is,

"*Substitute, in the proposed equation for x the terms* 2, 1, 0, − 1, &c. *successively. Find all the divisors of the sums that result, adding and subtracting them from the squares of these numbers*, 2, 1, 0, − 1, &c. *multiplied by a numerical divisor of the highest term of the proposed equation, and take out all the arithmetical progressions that can be found amongst these sums and differences. Let r be that term in any progression that arises from the substitution of $x = 0$, and let $\mp n$ be the difference arising from subtracting that term from the preceding term in the progression; lastly, let m be the aforesaid divisor of the highest term; then shall $\overline{mx^2 \pm nx - r}$ be the divisor that ought to be tried.*" And one or other of the

the divisors found in this manner will succeed, if the proposed equation has a quadratic divisor.

§ 69. Suppose, for example, the biquadratic $x^4 - 5x^3 + 7x^2 - 5x - 6 = 0$ is proposed, which has no simple divisor; then to discover if it has any quadratic divisor, the operation is thus:

Suppose

Suppos.	Results.	Divisors.	Squares.	Sums and differences of the divisors and squares.	Arithmetical Progressions.			
$x = 2$	-12	1, 2, 3, 4, 6, 12.	4	$-8, -2, 0, 1, 2, 3, 5, 6, 7, 8, 10, 16.$	8	1	-8	3
$x = 1$	-8	1, 2, 4, 8.	1	$-7, -3, -1, 0, 2, 3, 5, 9.$	5	-1	-7	2
$x = 0$	-6	1, 2, 3, 6.	0	$-6, -3, -2, -1, 1, 2, 3, 6.$	2	-3	-6	1
$x = -1$	$+12$	1, 2, 3, 4, 6, 12.	1	$-11, -5, -3, -2, -1, 0, 2, 3, 4, 5, 7, 13.$	-1	-5	-5	0

The

The first arithmetical progression gives the divisor $x^2 - 3x - 2$; the second gives $x^2 - 2x + 3$: both which succeed, so that the roots of the two equations $x^2 - 3x - 2 = 0$, and $x^2 - 2x + 3 = 0$, *viz.* $\frac{3 \pm \sqrt{17}}{2}$ and $1 \pm \sqrt{-2}$, are the four roots of the proposed equation, the two last of which are impossible. The divisors which the other arithmetical progressions give, do not succeed.

§ 70. After the same manner a Rule may be discovered for finding the cubic divisors, or those of higher dimensions, of any proposed equation.

Suppose the cubic divisor to be $mx^3 - nx^2 + rx - s$, and by supposing x equal to the terms of the arithmetical progression, it will be as follows:

Suppos.	*Results.*	*Cubes of terms of progr. × m*	*1st Differ.*	*2d Diff.*	*3d Diff.*
$x = 3$	$27m - 9n + 3r - s$	$27m$	$9n - 3r + s$	$5n - r$	$2n$
$x = 2$	$8m - 4n + 2r - s$	$8m$	$4n - 2r + s$	$3n - r$	$2n$
$x = 1$	$m - n + r - s$	m	$n - r + s$	$n - r$	$2n$
$x = 0$	$-s$	0	$+s$	$-n - r$	
$x = -1$	$-m - n - r - s$	$-m$	$n + r + s$		

Where the first differences are not themselves in arithmetical progression, as in the last case, but the differences of its terms, or the second differences, are in arithmetical progression, the

common

common difference being $2n$; whence n is known. The quantity r is found in the column of the second difference, and s is always to be assumed some divisor of the last term of the proposed equation, as m is of the coefficient of the first term. Whence all the coefficients of a divisor $mx^3 - nx^2 + rx - s$, with which trial is to be made, may be determined.

If it is a divisor of four dimensions that is required, by proceeding in like manner, you will obtain a series of differences whose second differences are in arithmetical progression. If it is a divisor of five dimensions that is required, you will obtain, in the same manner, a progression whose third differences will be in arithmetical progression; and by observing these progressions, you may discover rules for determining the coefficients of the divisor required.

The foundation of these Rules being, that, if an arithmetical progression a, $a + e$, $a + 2e$, $a + 3e$, &c. is assumed, the first differences of their squares will be in arithmetical progression; these differences being $2ae + e^2$, $2ae + 3e^2$, $2ae + 5e^2$, &c. whose common difference is $2e^2$. And the second differences of their cubes, and the third differences of their fourth powers are likewise in arithmetical progression, as is easily demonstrated.

§ 71.

§ 71. Hitherto we have only shewn how to find the divisors of equations that involve but *one letter*. But the same rules serve for discovering the divisors when there are two letters, if all the terms have the same dimensions; for, "*by supposing one of the letters equal to unit, find the divisor by the preceding Rules, and then by completing the dimensions of the divisor, substituting the letter again for unit, you will have the divisor required.*"

Suppose, for example, you are to find the divisor of $8x^3 - 26ax^2 + 11a^2x + 10a^3 = 0$, by putting $a = 1$, that quantity becomes $8x^3 - 26x^2 + 11x + 10 = 0$; whose divisor was found, § 66. to be $2x - 5$; now multiply the term -5 by $+a$, to bring it to the same dimensions as the other, and the divisor required is $2x - 5a$.

§ 72. Besides the method hitherto explained for finding the divisors of lower dimensions that may divide the proposed equation, there are others that deserve to be considered. The following is applicable to equations of all sorts, though we give it only for those of four dimensions.

Let the biquadratic $x^4 - px^3 + qx^2 - rx + s = 0$ be the equation proposed; and let us suppose it is the product of these two quadratic equations.

x^2

$$\begin{array}{l} x^2 - mx + n = 0 \\ \times\, x^2 - kx + l = 0 \\ \hline \end{array}$$

$$\left.\begin{array}{l} x^4 - k \\ \quad\; - m \end{array}\right\} x^3 \left.\begin{array}{l} + l \\ + n \\ + mk \end{array}\right\} x^2 \left.\begin{array}{l} - ml \\ - nk \end{array}\right\} x + ln = 0,$$

the terms of which will be equal, respectively, to the terms of the proposed equation.

In this equation, l and n being divisors of the last term s, we may consider one of them (*viz.* l) as known; and in order to find m or k, we need only compare the terms of this equation with the terms of the proposed equation respectively, which gives,

1°. $k + m = p$.

2°. $mk + l + n = q$.

3°. $ml + nk = r$.

4°. $nl = s$.

Now in order to find an equation that shall involve only k, and known terms, take the two values of m that arise from the first and third equations, and you will find,

$m = p - k = \frac{r - nk}{l}$ (because $n = \frac{s}{l}$, by equation the fourth) $= \frac{r - \frac{ks}{l}}{l} = \frac{rl - ks}{l^2}$; whence $pl^2 - kl^2 = rl - ks$, and $k = \frac{pl^2 - rl}{l^2 - s}$; and the quadratic $x^2 - kx + l = 0$ becomes

$$x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0.$$

To

To apply this to practice, you must substitute successively for l all the divisors of s, the last term of the proposed equation, till you find one of them such, that $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l$ can divide the proposed equation without a remainder.

EXAMPLE.

§ 73. If the equation $x^4 - 6x^3 + 20x^2 - 34x + 35 = 0$ is proposed. The divisors of 35 are 1, 5, 7, 35; if you put $l = 1$, the quadratic that arises will not succeed. But if you suppose $l = 5$, then the equation $x^2 - kx + l$, that is $x^2 - \frac{pl^2 - rl}{l^2 - s} \times x + l = 0$ becomes

$$x^2 - \frac{6 \times 25 - 34 \times 5}{25 - 35} x + 5 = 0 = x^2 - 2x + 5,$$

which divides the proposed equation without a remainder, and gives the quotient $x^2 - 4x + 7 = 0$.

"In this operation it is unnecessary to try any divisor l, that exceeds the square root of s, the last term of the proposed equation." And, if the proposed equation is literal, "you need only try those divisors of the last term that are of two dimensions."

If, in any supposition of l, the value of k, viz. $\frac{pl^2 - rl}{l^2 - s}$, becomes a fraction, then that supposition is to be rejected, and another value of l to be tried.

§ 74. By comparing the second and fourth equations of the last article, you may obtain another value of k. For $n = q - l - mk = \frac{s}{l}$; so that ($m$ being equal to $p - k$) $\frac{s}{l} = q - l - pk + k^2$, and $k^2 - pk + q - l - \frac{s}{l} = 0$. Which gives $k = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q + l + \frac{s}{l}}$. So that the quadratic divisor required becomes

$$x^2 - \frac{1}{2}p \mp \sqrt{\frac{1}{4}p^2 - q + l + \frac{s}{l}} \times x + l = 0.$$

This divisor must be tried when $l = \frac{s}{l}$, and at the same time $l = \frac{r}{p}$, the former expression not serving in that case.

By this formula, divisors may be found whose second terms may be *irrational*.

How the divisors of higher equations may be found, when they have any, may be understood from whas has been said of those of four dimensions.

SUP-

Supplement to Chap. VII.

Of the Reduction of Equations by Surd divisors.

AN equation of four, six, or more dimensions, although it may admit of no rational divisor, may have one that is irrational. As the biquadratic $x^4 + px^3 + qx^2 + rx + s = 0$, which we suppose to be irreducible by any rational divisor, may yet, by adding a square $k^2x^2 + 2klx + l^2$ multiplied into some quantity n, be completed into a square $\overline{x^2 + \frac{1}{2}px + Q}|^2$. In which case we shall have $x^2 + \frac{1}{2}p + Q = \sqrt{n} \times \overline{kx + l}$, and n is found by the resolution of an affected quadratic equation.

To reduce a biquadratic equation in this manner, we have the following

RULE.

I.* If the biquadratic is $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, s, represent the given coefficients under their proper signs, *put* $q - \frac{1}{4}p^2 = \alpha$, $r - \frac{1}{2}\alpha p = \beta$, $s - \frac{1}{4}\alpha^2 = \zeta$. And for n *take some integer common divisor of* β *and* 2ζ, *that is not a square number, and which, if either* p *or* r *is an odd number, must be odd, and, di-*

* *Arith. Univers.* pag. 264.

vided by 4, leave the remainder unity. Write likewise for k some divisor of $\frac{\beta}{n}$ *if p is an even number, or the half of an odd divisor if p is odd, or* 0 *if* $\beta = 0$. *Subtract* $\frac{\beta}{nk}$ *from* $\frac{1}{2}pk$, *and let the remainder be l. For* Q *put* $\frac{\alpha + nk^2}{2}$, *and try if, dividing* $Q^2 - s$ *by n, the root of the quotient is rational and equal to l; if it is, add* $nk^2x^2 + 2nklx + nl^2$ *to both sides of the equation, and extracting the root you shall have* $x^2 + \frac{1}{2}px + Q = n^{\frac{1}{2}} \times \overline{kx + l}$.

EXAMPLE I.

Let the equation proposed be $x^4 + 12x - 17 = 0$, and because $p = 0$, $q = 0$, $r = 12$, $s = -17$, we shall have $\alpha = 0$, $\beta = 12$, $\zeta = -17$. And β and 2ζ, that is 12 and -34, having only 2 for a common divisor, it must be $n = 2$. Again, $\frac{\beta}{n} = 6$, whose divisors 1, 2, 3, 6, are to be successively put for k, and -3, $-\frac{3}{2}$, -1, $-\frac{1}{2}$ for l respectively.

But $\frac{\alpha + nk^2}{2}$, that is k^2, is equal to Q; and $\sqrt{\frac{Q^2 - s}{n}} = l$. And when the even divisors 2 and 6 are substituted for k, Q becomes 4 and 36, and

and $\mathcal{Q}^2 - s$ being an odd number, is not divisible by n $(= 2)$. Wherefore 2 and 6 are to be set aside. But when 1 and 3 are written for k, $\mathcal{Q}$ is 1 or 9, and $\mathcal{Q}^2 - s$ is 18 or 98 respectively; which numbers can be divided by 2, and the roots of the quotients extracted, being ± 3 and ± 7; but only one of them, *viz.* -3, coincides with l. I put therefore $k = 1$, $l = -3$, $\mathcal{Q} = 1$, and adding to both sides of the equation $nk^2x^2 + 2nklx + nl^2$, that is, $2x^2 - 12x + 18$, there results $x^4 + 2x^2 + 1 = 2x^2 - 12x + 18$, and extracting the root of each, $x^2 + 1 = \pm\sqrt{2} \times \overline{x - 3}$. And again, extracting the root of this last, the four values of x, according to the varieties in the signs, are

$$-\tfrac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \tfrac{1}{2}},\quad -\tfrac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \tfrac{1}{2}};$$

$$\tfrac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \tfrac{1}{2}},\quad \tfrac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \tfrac{1}{2}},$$

being the roots of $x^4 + 12x - 17 = 0$, the equation at first proposed.

EXAMPLE II.

Let the equation be $x^4 - 6x^3 - 58x^2 - 114x - 11 = 0$, and writing -6, -58, -114, -11 for p, q, r, s, respectively, we have $-67 = \alpha$, $-315 = \beta$, and $-1133\tfrac{1}{4} = \zeta$. The numbers β and 2ζ, that is -315 and $-\frac{4533}{2}$, have but one common divisor 3, that is $n = 3$. And the divisors

divisors of $-105 = \frac{\beta}{n}$ are 3, 5, 7, 15, 21, 35, and 105. Wherefore I first make trial with $3 = k$, and dividing $\frac{\beta}{n}$ or -105, by it get the quotient -35, and this subtracted from $\frac{1}{2}pk = -3 \times 3$, leaves 26, whose half, 13, ought be equal to l. But $\frac{\alpha + nk^2}{2}$, or $\frac{-67 + 27}{2}$, that is, -20 is equal to Q; and $Q^2 - s = 411$, which is indeed divisible by $n = 3$; but the root of the quotient 137 cannot be extracted. Therefore I reject the divisor 3, and try with $5 = k$; by which dividing $\frac{\beta}{n} = -105$, the quotient is -21, and this taken from $\frac{1}{2}pk = -3 \times 5$, leaves $6 = 2l$. At the same time, $Q \left(= \frac{\alpha + nk^2}{2}\right) = \frac{75 - 67}{2} = 4$. And $Q^2 - s$, or $16 + 11$, is divisible by n, and the root of the quotient 9, that is, 3, coincides with l. Whence I conclude that putting $l = 3$, $k = 5$, $Q = 4$, $n = 3$, adding to both sides of the equation the quantity $nk^2x^2 + 2nklx + nl^2$, that is, $75x^2 + 90x + 27$, and extracting the roots, it will be

$$x^2 + \tfrac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}, \text{ or}$$
$$x^2 + 3x + 4 = \pm\sqrt{3} \times \overline{5x + 3}.$$

EXAMPLE III.

In like manner in the equation $x^4 - 9x^3 + 15x^2 - 27x + 9 = 0$ writing -9, $+15$, -27, $+9$, for

for p, q, r, s, there refult $\alpha = -5\frac{1}{4}$, $-50\frac{5}{8} = \beta$, $2\frac{7}{64} = \zeta$. The common divifors of β and 2ζ, that is, of $\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, 135; but 9 is a fquare, and 3, 15, 27, 135 divided by 4 do not leave unity for a remainder, as is required when p is an odd number. Setting thefe afide there remain only 5 and 45 to be tried for n. Firft let $n = 5$, and the halves of the odd divifors of $\frac{\beta}{n} = -\frac{81}{8}$, that is, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be tried for k. If $k = \frac{1}{2}$, the quotient $-\frac{81}{4}$ of $\frac{\beta}{n}$ divided by k, taken from $\frac{1}{2}pk$ or $-\frac{9}{4}$, leaves $18 = 2l$: and $\mathcal{Q}\left(= \frac{\alpha + nk^2}{2}\right) = -2$, $\mathcal{Q}^2 - s = -5$, which is divifible by 5, but the root of the quotient -1, which fhould be $l = 9$, is imaginary. Put next $k = \frac{3}{2}$, and the quotient of $\frac{\beta}{n}$ divided by k, or of $-\frac{81}{8}$ by $\frac{3}{2}$, is $-\frac{27}{4}$. This fubtracted from $\frac{1}{2}pk = -\frac{27}{4}$, leaves nothing, that is $l = 0$. Again,
$\mathcal{Q}\left(= \frac{\alpha + nk^2}{2}\right) = 3$, and $\mathcal{Q}^2 - s = 0$, and $l\left(= \sqrt{\frac{\mathcal{Q}^2 - s}{n}}\right) = 0$. From which coincidence I infer that $n = 5$, $k = \frac{3}{2}$, $l = 0$, and adding $nk^2x^2 + 2nlkx + nl^2$, that is, $\frac{45}{4}x^2$ to both fides of the equation, I find $x^2 - 4\frac{1}{2}x + 3 = \sqrt{5} \times \frac{3}{2}x$.

Literal equations may be treated much in the same way. And, if you put $n = 1$, the same Rule will give you the *rational* divisor of a biquadratic equation, if it admits of one. Thus for the equation $x^4 - x^3 + 5x^2 + 12x - 6 = 0$, putting $n = 1$ I find $k = \frac{5}{2}$, $l = -\frac{5}{2}$, and the equation is reduced to the two quadratics $x^2 - 3x + 3 = 0$ and $x^2 + 2x - 2 = 0$.

When the divisors of $\frac{\beta}{n}$ are so many that it would be troublesome to make trial with them all for k, their number may be reduced by finding all the divisors of $\alpha s - \frac{1}{4}r^2$. For to one of these, or to its half when odd, the number Q must be equal.

The ground of this Rule is as follows.

If a biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$, in which p, q, r, s, are the given coefficients with their signs, and the equation is supposed clear of fractions and surds; if this equation can be completed into a square, in the manner already described, we shall have $x^4 + px^3 + qx^2 + rx + s + nk^2x^2 + 2nklx + nl^2 = \overline{x^2 + \frac{1}{2}px + Q}^2$, that is, $x^4 + px^3 + \overline{q + nk^2} \times x^2 + \overline{r + 2nkl} \times x + s + nl^2 = x^4 + px^3 + \overline{2Q + \frac{1}{4}p^2} \times x^2 + pQ \times x + Q^2$. And comparing the terms, we get these *three* equations,

$$1. \; q + nk^2 = 2Q + \frac{1}{4}p^2,$$

2. r

$$2.\ r + 2nkl = p\mathfrak{Q},$$
$$3.\ s + nl^2 = \mathfrak{Q}^2;$$

in which there being four unknown quantities, they can be found only by trial.

The values of $\mathfrak{Q}$, taken from the first and second equations and made equal to each other, give $n = \frac{\frac{1}{2}pq - \frac{1}{8}p^3 - r}{2kl - \frac{1}{2}pk^2}$ (writing, as in the Rule, $q - \frac{1}{4}p^2 = \alpha$, and $r - \frac{1}{2}\alpha p = \beta) = \frac{\beta}{k \times \overline{\frac{1}{2}pk - 2l}}$. Whence, if the quantities n, k, l, $\mathfrak{Q}$, are to be found, it follows, (1°.) That n being a divisor of β, giving the quote $k \times \overline{\frac{1}{2}pk - 2l}$, k will be a divisor of $\frac{\beta}{n}$, giving the quote $\frac{1}{2}pk - 2l$; and that subtracting this quote from $\frac{1}{2}pk$, l will be equal to half the remainder. (2°.) In the first equation we had $\mathfrak{Q} = \frac{\alpha + nk^2}{2}$, and, from the third, $l^2 = \frac{\mathfrak{Q}^2 - s}{n}$. (3°.) Because $\mathfrak{Q} = \frac{1}{2}\alpha + \frac{1}{2}nk^2$ and $nl^2 = \mathfrak{Q}^2 - s$, $n = \frac{\frac{1}{4}\alpha^2 - s}{l^2 - \frac{1}{2}\alpha k^2 - \frac{1}{4}nk^4} = \frac{2s - \frac{1}{2}\alpha^2}{k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - 2l^2}$ (if $\zeta = s - \frac{1}{4}\alpha^2) = \frac{2\zeta}{k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - 2l^2}$, that is, n divides 2ζ by $k^2 \times \overline{\alpha + \frac{1}{2}nk^2} - l^2$. And if the several values of the quantities n, k, l, $\mathfrak{Q}$, answer to those conditions, or coincide, it is a proof that they have been rightly assumed; and that adding to the given equation the quantity $n \times \overline{kx + l}^2$, it will be completed into the square $\overline{x^2 + \frac{1}{2}p + \mathfrak{Q}}^2$.

It

It was ſaid that Q will always be ſome diviſor of $\alpha s - \frac{1}{4}r^2$. For $\alpha s = \alpha Q^2 - \alpha n l^2$, and taking from both $\frac{1}{4}r^2 = \frac{1}{4}p^2 Q^2 - pQnkl + n^2 k^2 l^2$, ſeeing the remainder $\alpha Q^2 - \overline{\alpha + nk^2} \times nl^2 - \frac{1}{4}p^2 Q^2 + pQnkl = \alpha Q^2 - 2Q \times nl^2 - \frac{1}{4}p^2 Q^2 + pQnkl$, has Q in every term; the thing is manifeſt.

It is needleſs to be particular as to the ſeveral limitations in the Rule, ſeeing they follow eaſily from the algebraical expreſſions of the quantities. You are not, for inſtance, if you ſeek a ſurd diviſor, to take n a ſquare number, for if n is a ſquare number, $\sqrt{n} \times \overline{kx + l}$ would be rational. Or if n is a multiple of a ſquare, as $\nu \times m^2$, then, at leaſt, $m \times \overline{kx + l}$ would be rational, and n would be depreſſed to ν.

Let us examine one caſe, *when p is even and r odd*; and by the Rule n muſt be an odd number, a multiple of 4 more unity.

1. Seeing $\beta = r - \frac{1}{2}\alpha p$, or $\beta + \frac{1}{2}\alpha p = r$, of the numbers β and $\frac{1}{2}\alpha p$ one muſt be even and the other odd, that their ſum r may be odd. If β is odd, its diviſor n muſt be odd likewiſe. Suppoſe β to be even, then $\frac{1}{2}\alpha p$, and conſequently $\frac{1}{2}p$ and α are both odd. But if α is odd, $2\zeta = 2s - \frac{1}{2}\alpha^2$ will be half an odd number, and n its diviſor is odd.

In this caſe, Q is half an odd number. For let it be an integer, pQ will be an even number. But if Q is an integer, ſo muſt l, becauſe

$s + nl^2 = Q^2$; and $2nkl$ must be even. And $r + 2nkl$ (an odd number) $= pQ$ an even number, *which is absurd.*

2. Let N represent any number in general, I an odd number; then I say, "every odd number is a multiple of *four*, more or less *unity*," that is, $I = 4N \pm 1$. "The square of an odd number is $4N + 1$," (that is some multiple of 4, more unity;) and "if from such a square there be taken any multiple of 4, the remainder, if greater than unity, will be $4N + 1$."

Hence it follows that $n = 4N + 1$. For seeing $nl^2 = Q^2 - s$; because l and Q are the halves of odd numbers, we have, according to the present notation, $\frac{n \times I^2}{4} = \frac{I^2 - 4s}{4}$, or without the common denominator $n \times I^2 = I^2 - 4s$, that is, $n \times \overline{4N + 1} = 4N + 1$, and consequently, $n = 4N + 1$. For it is not $4N - 1$ but $4N + 1$ that can give the product $4N + 1$. *

In

* In the former Editions, there were here inserted two Rules for the Case of $\beta = 0$: which, though true, Mr. *Thomas Simpson* has, in his *Miscellaneous Tracts*, published 1757, shewn to be unnecessary. In *this*, therefore, they are omitted.

It is only to be regretted that Mr. *Simpson* should, through inattention, have placed this inaccuracy, not to the account of the Editor, as he ought to have done, but to that of Mr. MACLAURIN. The whole explanation of Sir *Isaac*'s Method of Reducing Equations by means of *Surd Divisors*, is (pag. 213.) professedly a *Supplement*; as is likewise the Addition to *Chap.* 14. *Part.* I. And the Edi-

In like manner the other limitations may be determined: and what has been ſaid may lead to the invention or demonſtration of ſimilar Rules for the higher equations of even dimenſions, if any one pleaſes to take the trouble.

CHAP. VIII.

Of the Reſolution of Equations by *Cardan*'s Rule, and others of that kind.

§ 75. WE now proceed to ſhew how an expreſſion of the root of an equation can be obtained that ſhall involve only known quantities. In *Chap.* 11. *Part.* I. we ſhewed how to reſolve *ſimple* equations; and in *Chap.* 13. we ſhewed how to reſolve any *quadratic* equation, by adding to the ſide of the equation that involves the unknown quantity, what was neceſſary to make it a complete ſquare, and then extracting the ſquare root on both ſides. In § 27 of this *Part*, we gave another method

Editor thought he had, in his Preface, ſufficiently intimated that a few ſuch inſertions had been made, and the reaſon why: though he cannot recollect any others worth mentioning; if it is not §§ 123, 124, of *Part* II.

of

of reſolving *quadratic* equations, by taking away the ſecond term: where it appeared that if $x^2 - px + q = 0$, $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$.

§ 76. The ſecond term can be taken away out of any *cubic* equation, by § 25; ſo that they all may be reduced to this form, $x^3 * + qx + r = 0$.

Let us ſuppoſe that $x = a + b$; and $x^3 + qx + r = a^3 + 3a^2b + 3ab^2 + b^3 + qx + r = a^3 + 3ab \times \overline{a+b} + b^3 + qx + r = a^3 + 3abx + b^3 + qx + r$ (by ſuppoſing $3ab = -q$) $= a^3 + b^3 + r = 0$.

But $b = -\frac{q}{3a}$, and $b^3 = -\frac{q^3}{27a^3}$, and conſequently, $a^3 - \frac{q^3}{27a^3} + r = 0$; or, $a^6 + ra^3 + \frac{q^3}{27}$.

Suppoſe $a^3 = z$, and you have $z^2 + rz = \frac{q^3}{27}$; which is a quadratic whoſe reſolution gives

$$z = -\tfrac{1}{2}r \pm \sqrt{\tfrac{1}{4}r^2 + \frac{q^3}{27}} = a^3,$$

and $a = \sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and

$$x = a + b = a - \frac{q}{3a} = \sqrt[3]{-\tfrac{1}{2}r \pm \sqrt[2]{\tfrac{1}{4}r^2 + \frac{q^3}{27}}} - \frac{q}{3 \times \sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}}$$

: in which expreſſions there are only known quantities.

§ 77.

§ 77. The values of x may be found a little differently, thus:

Since $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, it follows,

that $a^3 + r = +\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, and

$b^3 (= -a^3 - r) = -\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; so that

$b = \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and $x\ (= a + b) =$

$\sqrt[3]{-\frac{1}{2}r \pm \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r \mp \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}}$;

which gives but one value of x, because when, in the value of a the surd $\sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ is positive, it is negative in the value of b, and there is only the difference of this sign in their values. So that we may conclude

$$x = \sqrt[3]{-\frac{1}{2}r + \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt[2]{\frac{1}{4}r^2 + \frac{q^3}{27}}},$$

§ 79. * The values of x may be discovered without exterminating the second term.

Any cubic equation may be reduced to this form,

$$\left.\begin{array}{l} x^3 - 3px^2 - 3px - 2r \\ \qquad\qquad + 3p^2x - p^3 \\ \qquad\qquad\qquad + 3pq \end{array}\right\} = 0,$$

* Vid. Phil. Transf. 309.

which

which, by ſuppoſing $x = z + p$, will be reduced to $z^3 * - 3qz - 2r = 0$, in which the ſecond term is wanting. But by the laſt article, ſince $z^3 * - 3qz - 2r = 0$, it follows that

$z = \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$ (if you ſuppoſe that the cubic root of the binomial $r + \sqrt{r^2 - q^3}$ is $m + \sqrt{n}$) $= m + \sqrt{n} + m - \sqrt{n} = 2m$. And ſince $x = z + p$, it follows that $x = p + 2m$.

§ 79. But as the ſquare root of any quantity is *twofold*, "the cube root is *threefold*," and can be expreſſed three different ways.

Suppoſe the cube root of unit is required, and let $y^3 = 1$, or $y^3 - 1 = 0$, then ſince unit itſelf is a cube root of 1, one of the values of y is 1, ſo that the equation $y - 1 = 0$ ſhall divide the firſt equation $y^3 - 1 = 0$, and the quotient $y^2 + y + 1 = 0$ reſolved, gives $y = \frac{-1 \mp \sqrt{-3}}{2}$, ſo that the three expreſſions of $\sqrt[3]{1}$ are $1 \frac{-1 + \sqrt{-3}}{2}$ and $\frac{-1 - \sqrt{-3}}{2}$. And, in general, the cube root of any quantity A^3 may be A, or $\frac{-1 + \sqrt{-3}}{2} \times A$, or $\frac{-1 - \sqrt{-3}}{2} \times A$; ſo that the cube root of the binomial $r + \sqrt{r^2 - q^3}$ may be $m + \sqrt[2]{n}$, as we ſuppoſed above, or $\frac{-1 + \sqrt{-3}}{2}$

$\times m$

$\times\overline{m+\sqrt[2]{n}}$, or $\frac{-1-\sqrt{-3}}{2}\times\overline{m+\sqrt[2]{n}}$. And hence we have three expressions for x, *viz.*

$$1^{\circ}.\ x=p+2m,$$
$$2^{\circ}.\ x=p-\ m+\sqrt{-3n},$$
$$3^{\circ}.\ x=p-\ m-\sqrt{-3n};$$

and these give the three roots of the proposed cubic equation.

EXAMPLE I.

§ 80. Let it be required to find the roots of the equation $x^3-12x^2+41x-42=0$.

Comparing the coefficients of this equation with those of the general equation

$$x^3-3px^2\left.\begin{array}{l}-3q\\+3p^2\end{array}\right\}x\left.\begin{array}{l}-2r\\-\ p^3\\+3pq\end{array}\right\}=0, \text{ you find,}$$

$1^{\circ}.\ 3p=12$, so that . . . . $p=4$,

$2^{\circ}.\ 3p^2-3q\,(=48-3q)=41$ . . . . $q=\frac{7}{3}$,

$3^{\circ}.\ 3pq-p^3-2r\,(=-36-2r)=-42$ . . $r=3$;

and consequently, $r^2-q^3=9-\frac{343}{27}=-\frac{100}{27}$,

and $r+\sqrt{r^2-q^3}=3+\sqrt{-\frac{100}{27}}$. Now the cube root of this binomial is found to be

$-1+\sqrt[2]{-\frac{4}{3}}\,(=m+\sqrt{n})$*. Whence,

$1^{\circ}.\ x=p+2m=4-2=2.$

$2^{\circ}.\ x=p-m-\sqrt{-3n}=4+1-\sqrt{4}=5-2=3.$

$3^{\circ}.\ x=p-m+\sqrt{-3n}=5+2=7.$

* Section 131. Part I.

So

So that the three roots of the propofed equation are 2, 3, 7.

You may find other two expreffions of the cube root of $3 + \sqrt{-\frac{100}{27}}$, befides $-1 + \sqrt{-\frac{4}{3}}$, *viz.* $\frac{3}{2} + \sqrt{-\frac{1}{12}}$, and $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$; but thefe fubftituted for $m + \sqrt{n}$ give the fame values for x, as are already found.

EXAMPLE II.

In the equation $x^3 + 15x^2 + 84x - 100 = 0$, you find $p = -5$, $q = -3$, $r = 135$, and $r + \sqrt{r^2 - q^3} = 135 + \sqrt{18252}$, whofe cube root is $3 + \sqrt{12}$; fo that $x\ (= p + 2m) = -5 + 6 = 1$. The other two values of x, viz. $-8 + \sqrt{-36}$, $-8 - \sqrt{-36}$, are impoffible.

After the fame manner, you will find that the roots of the equation $x^3 + x^2 - 166x + 660 = 0$, are -15, $7 \pm \sqrt{5}$. The Rule by which we may difcover if any of the roots of an equation are *impoffible*, fhall be demonftrated afterwards.

§ 82. The roots of biquadratic equations may be found by reducing them to cubes, thus.

Let the fecond term be taken away by the Rule given in *Chap.* 3. And let the equation that refults be

$$x^4 * + qx^2 + rx + s = 0.$$

And let us fuppofe this biquadratic to be the product of thefe two quadratic equations,

$$\begin{array}{l} x^2 + ex + f = 0 \\ x^2 - ex + g = 0 \\ \hline x^4 * \left.\begin{array}{l} +f \\ +g \\ -e^2 \end{array}\right\} \times x^2 \left.\begin{array}{l} +eg \\ -ef \end{array}\right\} \times x + fg = 0. \end{array}$$

Where e is the coefficient of x in both equations but affected with contrary signs; because when the second term is wanting in an equation, the sum of the affirmative roots must be equal to the sum of the negative.

Compare now the proposed equation with the above product, and the respective terms put equal to each other will give $f + g - e^2 = q$, $eg - ef = r$, $fg = s$. Whence it follows, that $f + g = q + e^2$, and $g - f = \frac{r}{e}$; and consequently $f + g + g - f (= 2g) = q + e^2 + \frac{r}{e}$, and $g = \frac{q + e^2 + \frac{r}{e}}{2}$, the same way, you will find, by subtraction, &c. $f = \frac{q + e^2 - \frac{r}{e}}{2}$, and $f \times g\ (= s) = \frac{1}{4} \times \overline{q^2 + 2qe^2 + e^4 - \frac{r^2}{e^2}}$; and multiplying by $4e^2$, and ranging the terms, you have this equation,

$$e^6 + 2qe^4 + \overline{q^2 - 4s} \times e^2 - r^2 = 0.$$

Suppose $e^2 = y$, and it becomes $y^3 + 2qy^2 +$

q^2

$\overline{q^2-4s}\times y - r^2 = 0$, a cubic equation whose roots are to be discovered by the preceding articles. Then the values of y being found, their square root will give e (since $y = e^2$); and having e, you will find f and g from the equations

$$f = \frac{q + e^2 - \frac{r}{e}}{2},\quad g = \frac{q + e^2 + \frac{r}{e}}{2}.$$

Lastly, extracting the roots of the equations $x^2 + ex + f = 0$, $x^2 - ex + g = 0$, you will find the four roots of the biquadratic $x^4 * qx^2 + rx + s = 0$; for either $x = -\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - f}$, or, $x = +\frac{1}{2}e \pm \sqrt{\frac{1}{4}e^2 - g}$.

§ 83. Or if you want to find the roots of the biquadratic without taking away the second term; suppose it to be of this form,

$$x^4 - 4px^3 \left.\begin{matrix} -2q \\ +4p^2 \end{matrix}\right\} x^2 \left.\begin{matrix} -8r \\ +4pq \end{matrix}\right\} x \left.\begin{matrix} -4s \\ +q^2 \end{matrix}\right\} = 0,$$

and the values of x will be

$$\left.\begin{matrix} x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}} \\ \text{and } x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}} \end{matrix}\right\},\ \text{where}$$

a^2 is equal to the root of the cubic,

$$y^3 \left.\begin{matrix} -p^2 \\ -q \end{matrix}\right\} y^2 \left.\begin{matrix} +2pr \\ +s \end{matrix}\right\} y - r^2 = 0.$$

The demonstration is reduced from the last article, as the 78th is from the preceding.

CHAP. IX.

Of the Methods by which you may approximate to the roots of *numeral* Equations by their limits.

§ 84. WHEN any equation is propofed to be refolved, firft find the limits of the roots (by *Chap.* 5.) as for example, if the roots of the equation $x^2 - 16x + 55 = 0$ are required, you find the limits are 0, 8, and 17, by § 48; that is, the leaft root is between 0 and 8, and the greateft between 8 and 17.

In order to find the firft of the roots, I confider that if I fubftitute 0 for x in $x^2 - 16x + 55$, the refult is pofitive, *viz.* + 55, and confequently any number betwixt 0 and 8 that gives a pofitive refult, muft be lefs than the leaft root, and any number that gives a negative refult, muft be greater. Since 0 and 8 are the limits, I try 4, that is, the mean betwixt them, and fuppofing $x = 4$, $x^2 - 16x + 55 = 16 - 64 + 55 = 7$, from which I conclude that the root is greater than 4. So that now we have the root limited between 4 and 8. Therefore I next try 6, and fubftituting it for x we find $x^2 - 16x + 55 = 36 - 96 + 55 = -5$; which refult being negative, I conclude that 6 is greater than the root required, which therefore is limited now between 4 and

and 6. And ſubſtituting 5, the mean between them in place of x, I find $x^2 - 16x + 55 = 25 - 80 + 55 = 0$; and conſequently 5 is the leaſt root of the equation. After the ſame manner you will diſcover 11 to be the greateſt root of that equation.

§ 85. Thus by diminiſhing the greater, or increaſing the leſſer limit, you may diſcover the true root when it is a commenſurable quantity. But by proceeding after this manner, when you have two limits, the one greater than the root, the other leſſer, that differ from one another but by unit, then you may conclude the root is *incommenſurable*.

We may however, by continuing the operation in fractions, approximate to it. As if the equation propoſed is $x^2 - 6x + 7 = 0$, if we ſuppoſe $x = 2$, the reſult is $4 - 12 + 7 = -1$, which being negative, and the ſuppoſition of $x = 0$ giving a poſitive reſult, it follows that the root is betwixt 0 and 2. Next we ſuppoſe $x = 1$; whence $x^2 - 6x + 7 = 1 - 6 + 7 = +2$, which being poſitive, we infer the root is betwixt 1 and 2, and conſequently incommenſurable. In order to approximate to it, we ſuppoſe $x = 1\frac{1}{2}$, and find $x^2 - 6x + 7 = 2\frac{1}{4} - 9 + 7 = \frac{1}{4}$; and this reſult being poſitive, we infer the root muſt be betwixt 2 and $1\frac{1}{2}$. And therefore we try $1\frac{3}{4}$, and find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7$ $= 3\frac{1}{16} - 10\frac{8}{16} + 7 = -\frac{7}{16}$, which is negative; ſo that we conclude the root to be betwixt $1\frac{1}{2}$ and $1\frac{3}{4}$. And therefore we try next $1\frac{5}{8}$, which giving alſo a negative reſult, we conclude the root is betwixt $1\frac{1}{2}$ (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try therefore $1\frac{9}{16}$, and the reſult being poſitive, we conclude that the root muſt be betwixt $1\frac{9}{16}$ and $1\frac{10}{16}$, and therefore is nearly $1\frac{19}{32}$.

§ 86. Or you may approximate more eaſily by transforming the equation propoſed into another whoſe roots ſhall be equal to 10, 100, or 1000 times the roots of the former (by § 29.) and taking the limits greater in the ſame proportion. This transformation is eaſy; for you are only to multiply the ſecond term by 10, 100, or 1000, the third term by their ſquares, the fourth by their cubes, *&c.* The equation of the laſt example is thus transformed into $x^2 - 600x + 70000 = 0$, whoſe roots are 100 times the roots of the propoſed equation, and whoſe limits are 100 and 200. Proceeding as before, we try 150, and find $x^2 - 600x + 70000 = 22500 - 90000 + 70000 = 2500$, ſo that 150 is leſs than the root. You next try 175, which giving a negative reſult muſt be greater than the root: and thus proceeding you find the root to be betwixt 158 and 159: from which you

you infer that the leaſt root of the propoſed equation $x^2 - 6x + 7 = 0$ is betwixt 1.58 and 1.59, being the hundreth part of the root of $x^2 - 600x + 70000 = 0$.

§ 87. If the cubic equation $x^3 - 15x^2 + 63x - 50 = 0$ is propoſed to be reſolved, the equation of the limits will be (by § 48) $3x^2 - 30x + 63 = 0$, or $x^2 - 10x + 21 = 0$, whoſe roots are 3, 7; and by ſubſtituting 0 for x the value of $x^3 - 15x^2 + 63x - 50$ is negative, and by ſubſtituting 3 for x, that quantity becomes poſitive, $x = 1$ gives it negative, and $x = 2$ gives it poſitive, ſo that the root is between 1 and 2, and therefore incommenſurable. You may proceed as in the foregoing examples to approximate to the root. But there are other methods by which you may do that more eaſily and readily; which we proceed to explain.

§ 88. When you have diſcovered the value of the root to leſs than an unit (as in this example, you know it is a little above 1) ſuppoſe the difference betwixt its real value and the number that you have found nearly equal to it, to be repreſented by f; as in this example. Let $x = 1 + f$. Subſtitute this value for x in this equation, thus,

$$
\begin{array}{rcl}
x^3 & = & 1 + 3f + 3f^2 + f^3 \\
-15x^2 & = & -15 - 30f - 15f^2 \\
+63x & = & 63 + 63f \\
-50 & = & -50 \\
\hline
x^3 - 15x^2 + 63x - 50 & = & -1 + 36f - 12f^2 + f^3 = 0.
\end{array}
$$

 Now

Now becauſe f is ſuppoſed leſs than unit, its powers f^2, f^3, may be neglected in this approximation; ſo that aſſuming only the two firſt terms we have $-1+36f=0$, or, $f=\frac{1}{36}=.027$; ſo that x will be nearly 1.027.

You may have a nearer value of x by conſidering, that ſeeing $-1+36f-12f^2+f^3=0$, it follows that

$$f=\frac{1}{36-12f+f^2} \text{ (by ſubſtituting } \frac{1}{36} \text{ for } f)$$

$$\text{nearly}=\frac{1}{36-12\times\frac{1}{36}+\frac{1}{36}\times\frac{1}{36}}=\frac{1296}{46225}=.02803.$$

§ 89. But the value of f may be corrected and determined more accurately by ſuppoſing g to be the difference betwixt its real value, and that which we laſt found nearly equal to it. So that $f=.02803+g$. Then by ſubſtituting this value for f in the equation $f^3-12f^2+36f-1=0$, it will ſtand as follows,

$$\left.\begin{array}{rll} f^3 = & 0.0000220226 & +0.002357g+0.0849g^2+g^3 \\ -12f^2 = - & .00942816 & -0.67272g-12g^2 \\ +36f = & 1.00908 & +36g \\ -1 = & -1. & \end{array}\right\}=0$$

$$=-0.0003261374+35.329637g-11.9195g^2+g^3=0.$$

Of which the firſt two terms, neglecting the reſt, give $35.329637\times g=0.0003261374$, and $g=\frac{.0003261374}{35.329637}=0.0000092312$7. So that $f=0.0280392312$7; and $x=1+f=1.0280392312$7; which is very near the true root of the equation that was propoſed.

If

If ſtill a greater degree of exactneſs is required, ſuppoſe h equal to the difference betwixt the true value of g, and that we have already found, and proceeding as above you may correct the value of g.

§ 90. For another example; let the equation to be reſolved be $x^3 - 2x - 5 = 0$, and by ſome of the preceding methods you diſcover one of the roots to be between 2 and 3. Therefore you ſuppoſe $x = 2 + f$, and ſubſtituting this value for it, you find

$$\left.\begin{array}{rcl} x^3 &=& 8 + 12f + 6f^2 + f^3 \\ -2x &=& -4 - 2f \\ -5 &=& -5 \end{array}\right\} = 0$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that $10f = 1$ nearly, or $f = 0.1$. Then to correct this value, we ſuppoſe $f = 0.1 + g$, and find

$$\left.\begin{array}{rcl} f^3 &=& 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 &=& 0.06 + 1.2\,g + 6.\,g^2 \\ 10f &=& 1. + 10.\,g \\ -1 &=& -1. \end{array}\right\} = 0$$

$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

ſo that $g = \dfrac{-0.061}{11.23} = -0.0054$

Then by ſuppoſing $g = -.0054 + h$, you may correct its value, and you will find that the root required is nearly 2.09455147.

§ 91.

§ 91. It is not only one root of an equation that can be obtained by this method, but, by making use of the other limits, you may discover the other roots in the same manner. The equation of § 87, $x^3 - 15x^2 + 63x - 50 = 0$, has for its limits 0, 3, 7, 50. We have already found the least root to be nearly 1.028039. If it is required to find the middle root, you proceed in the same manner to determine its nearest limits to be 6 and 7; for 6 substituted for x gives a positive, and 7 a negative result. Therefore you may suppose $x = 6 + f$, and by substituting this value for x in that equation, you find $f^3 + 3f^2 - 9f + 4 = 0$, so that $f = \frac{4}{9}$ nearly. Or since $f = \frac{4}{9 - 3f - f^2}$, it is (by substituting $\frac{4}{9}$ for f) $f = \frac{4}{9 - \frac{4}{3} - \frac{16}{81}} = \frac{324}{605}$, whence $x = 6 + \frac{324}{605}$ nearly. Which value may still be corrected as in the preceding articles. After the same manner you may approximate to the value of the highest root of the equation.

§ 92. "In all these operations, you will approximate sooner to the value of the root, if you take the three last terms of the equation, and extract the root of the quadratic equation consisting of these three terms."

Thus, in § 88, instead of the two last terms of the equation $f^3 - 12f^2 + 36f - 1 = 0$, if you

you take the three laſt and extract the root of the quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true value than what you diſcover by ſuppoſing $36f - 1 = 0$.

It is obvious that this method extends to all equations.

§ 93. "By aſſuming equations affected with general coefficients, you may, by this method, deduce General Rules or *Theorems* for approximating to the roots of porpoſed equations of whatever degree."

Let $f^3 - pf^2 + qf - r = 0$ repreſent the equation by which the fraction f is to be determined, which is to be added to the limit, or ſubtracted from it, in order to have the near value of x. Then $qf - r = 0$ will give $f = \frac{r}{q}$. But ſince $f = \frac{r}{f^2 - pf + q}$, by ſubſtituting $\frac{r}{q}$ for f, we have this Theorem for finding f nearly, *viz.*

$$f = \frac{r}{\frac{r^2}{q^2} - \frac{pr}{q} + q} = \frac{q^2 \times r}{q^3 - pqr + r^2}.$$

After the ſame manner, if it is a biquadratic, by which f is to be determined, as $f^4 - pf^3 + qf^2 - rf + s = 0$, then f being very little, we ſhall have $f = \frac{s}{r}$; which value is corrected by conſidering

ſidering that $f = \frac{s}{r - qf + pf^2 - f^3}$ (by ſubſtituting $\frac{s}{r}$ for f) $= \frac{s}{r - \frac{qs}{r} + \frac{ps^2}{r^2} - \frac{s^3}{r^3}}$, whence we have this Theorem for all biquadratic equations,

$$f = \frac{r^3 \times s}{-s^3 + ps^2r - qsr^2 + r^4}.$$

§ 94. Other Theorems may be deduced by aſſuming the three terms of the equation, and extracting the root of the quadratic which they form.

Thus, to find the value of f in the equation $f^3 - pf^2 + qf - r = 0$ where f is ſuppoſed to be very little, we neglect the firſt term f^3, and extract the root of the quadratic $pf^2 - qf + r = 0$, or of $f^2 - \frac{q}{p} \times f + \frac{r}{p} = 0$; and we find $f = \frac{q}{2p} \pm \sqrt{-\frac{r}{p} + \frac{q^2}{4p^2}} = \frac{q \pm \sqrt{q^2 - 4pr}}{2p}$ nearly.

But this value of f may be corrected by ſuppoſing it equal to m, and ſubſtituting m^3 for f^3 in the equation $f^3 - pf^2 + qf - r = 0$, which will give $m^3 - pf^2 + qf - r = 0$, and $pf^2 - qf + r - m^3 = 0$; the reſolution of which quadratic equation gives $f = \frac{q \pm \sqrt{q^2 - 4pr + 4pm^3}}{2p}$, very near the true value of f.

After the ſame manner you may find like Theorems for the roots of biquadratic equations, or of equations of any dimenſion whatever.

§ 95.

§ 95. In general, let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$ &c. $+ A = 0$ represent an equation of any dimensions n, were A is supposed to represent the absolute known term of the equation. Let k represent the limit next less than any of the roots, and supposing $x = k + f$, substitute the powers of $k + f$ instead of the powers of x, and there will arise $\overline{k+f}^n + p \times \overline{k+f}^{n-1} + q \times \overline{k+f}^{n-2} + r \times \overline{k+f}^{n-3}$, &c. $+ A = 0$, or by involution, disposing the terms according to the dimensions of f . . . .

$$\left.\begin{array}{l}
k^n + nk^{n-1} \times f + n \times \frac{\overline{n-1}}{2} k^{n-2} f^2 + \text{\&c.} \\
pk^{n-1} + p \times \overline{n-1} k^{n-2} \times f + p \times \overline{n-1} \times \frac{\overline{n-2}}{2} k^{n-3} f^2 + \text{\&c.} \\
qk^{n-2} + q \times \overline{n-2} k^{n-3} \times f + q \times \overline{n-2} \times \frac{\overline{n-3}}{2} k^{n-4} f^2 + \text{\&c.} \\
rk^{n-3} + r \times \overline{n-3} k^{n-4} \times f + r \times \overline{n-3} \times \frac{\overline{n-4}}{2} k^{n-5} f^2 + \text{\&c.} \\
\text{\&c.} \qquad + A
\end{array}\right\} = 0,$$

where

where neglecting all the powers of f after the first two terms, you find

$$f = \frac{-A - k^n - pk^{n-1} - qk^{n-2} - rk^{n-3} \text{ \&c.}}{nk^{n-1} + p \times \overline{n-1}k^{n-2} + q \times \overline{n-2}k^{n-3} + r \times \overline{n-3}k^{n-4} \text{ \&c.}}, \text{ and}$$

$$x (= k + f) = \frac{-A + \overline{n-1}k^n + p \times \overline{n-2}k^{n-1} + q \times \overline{n-3}k^{n-2} + r \times \overline{n-4}k^{n-3} \text{ \&c.}}{nk^{n-1} + p \times \overline{n-1}k^{n-2} + q \times \overline{n-2}k^{n-3} + r \times \overline{n-3}k^{n-4} \text{ \&c.}}$$

whence particular Theorems for extracting the roots of equations may be deduced.

§ 96. "By this method you may discover Theorems for approximating to the roots of

pure

pure powers;" as to find the n root of any number A; suppose k to be the nearest less root in integers, and that $k+f$ is the true root then shall $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2$ &c. $= A$; and assuming only the two first terms,

$f = \frac{A-k^n}{nk^{n-1}}$: or, more nearly, taking the three first terms,

$f = \frac{A-k^n}{nk^{n-1} + n \times \frac{n-1}{2}k^{n-2}f}$, and (taking $\frac{A-k^n}{nk^{n-1}} = f$)

$$f = \frac{A-k^n}{nk^{n-1} + \frac{n^2-n}{2}k^{n-2} \times \frac{A-k^n}{nk^{n-1}}} = \frac{A-k^n}{nk^{n-1} + \frac{\overline{n-1}}{2k} \times \overline{A-k^n}}$$

(putting $m = A-k^n$) $= \frac{km}{nk^n + \frac{n-1}{2} \times m}$; which is a *rational* Theorem for approximating to f.

You may find an *irrational* Theorem for it, by assuming the three first terms of the power of $k+f$, viz. $k^n + nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A$.

For $nk^{n-1}f + n \times \frac{n-1}{2}k^{n-2}f^2 = A - k^n = m$; and resolving this quadratic equation, you find

$$f = -\frac{k}{n-1} \pm \sqrt{\frac{2m}{n \times \overline{n-1} \times k^{n-2}} + \frac{k^2}{\overline{n-1}|^2}} =$$

$$-\frac{k}{n-1} \pm \sqrt{\frac{2mn - 2m + nk^n}{n \times \overline{n-1}|^2 k^{n-2}}}.$$

In

In the application of these Theorems, when a near value of f is obtained, then adding it to k, substitute the aggregate in place of k in the formula, and you will by a new operation, obtain a more correct value of the root required; and, by thus proceeding, you may arrive at any degree of exactness.

Thus to obtain the cube root of 2, suppose $k = 1$, and $f\ (= \frac{km}{nk^n + \frac{n-1}{2}m}) = \frac{1}{4} = 0.25$. In the second place, suppose $k = 1.25$, and f will be found, by a new operation, equal to 0.009921, and consequently, $\sqrt[3]{2} = 1.259921$ nearly. By the irrational Theorem, the same value is discovered for $\sqrt[3]{2}$.

CHAP.

CHAP. X.

Of the Method of *Series* by which you may approximate to the roots of *literal* equations.

§ 97. IF there be only two letters, x and a, in the proposed equation, suppose a equal to unit, and find the root of the numeral equation that arises from the substitution, by the rules of the last chapter. Multiply these roots by a, and the products will give the roots of the proposed equation.

Thus the roots of the equation $x^2 - 16x + 55 = 0$ are found, in § 84, to be 5 and 11. And therefore the roots of the equation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. The roots of the equation $x^3 + a^2x - 2a^3 = 0$ are found by enquiring what are the roots of the numeral equation $x^3 + x - 2 = 0$, and since one of these is 1, it follows that one of the roots of the proposed equation is a; the other two are *imaginary*.

§ 98. If the equation to be resolved involves more than two letters, as $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$, then the value of x may be exhibited in a series having its terms composed of the powers of a and y with their respective coefficients; which will "*converge the sooner the less y is in respect of a, if the terms are continually multiplied by the powers of y, and divided by those of a.*" Or, "will *converge the sooner the greater y is in respect of a, if the terms be continually multiplied by the powers of a, and divided by those of y.*" Since when y is very little in respect of a, the terms y, $\frac{y^2}{a}$, $\frac{y^3}{a^2}$, $\frac{y^4}{a^3}$, $\frac{y^5}{a^4}$, &c. decrease very quickly. If y vanish in respect of a, the second term will vanish in respect of the first, since $\frac{y^2}{a} : y :: y : a$. And after the same manner $\frac{y^3}{a^2}$ vanishes in respect of the term immediate preceding it.

But when y is vastly great in respect of a, then a is vastly great in respect of $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in respect of $\frac{a^3}{y^2}$; so that the terms a, $\frac{a^2}{y}$, $\frac{a^3}{y^2}$, $\frac{a^4}{y^3}$, $\frac{a^5}{y^4}$, &c. in this case *decrease* very swiftly. In either case, the series converge swiftly that consist of such terms; and a few of the first terms will give a near value of the root required.

§ 99. If a series for x is required from the proposed equation that shall converge the sooner, the less y is in respect of a; to find the first term of this series, we shall suppose y to vanish; and extracting the root of the equation

x^3

$x^3 + a^2x - 2a^3 = 0$, consisting of the remaining parts of the equation that do not vanish with y, we find, by § 97, that $x = a$; which is the true value of x when y vanishes; but is only near its value when y does not vanish, but only is very little. To get a value still nearer the true value of x, suppose the difference of a from the true value to be p; or that $x = a + p$. And substituting $a + p$ in the given equation for x, you will find,

$$\left.\begin{array}{r} x^3 = a^3 + 3a^2p + 3ap^2 + p^3 \\ + a^2x = a^3 + a^2p \\ - 2a^3 = -2a^3 \\ + ayx = a^2y + apy \\ - y^3 = -y^3 \end{array}\right\} = 0$$

$$\left.\begin{array}{r} = 4a^2p + 3ap^2 + p^3 \\ a^2y + apy - y^3 \end{array}\right\} = 0.$$

But since, by supposition, y and p are very little in respect of a, it follows that the terms $4a^2p$, a^2y, where y and p are separately of the *least* dimensions, are vastly great in respect of the rest; so that, in determining a near value of p, the rest may be neglected: and from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer value of p, and consequently of x, suppose $p = -\frac{1}{4}y + q$, and substituting this value for it in the last equation, you will find,

$$\left.\begin{array}{rl} p^3 = & -\frac{1}{64}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\ 3ap^2 = & \quad \frac{3}{16}ay^2 - \frac{3}{2}ayq + 3aq^2 \\ 4a^2p = & - \quad a^2y + 4a^2q \\ ayp = & -\frac{1}{4}ay^2 + ayq \\ a^2y = & \quad a^2y \\ -y^3 = & - \quad y^3 \end{array}\right\} = 0$$

$$\left.\begin{array}{l} = -\frac{65}{64}y^3 + \frac{3}{16}y^2q - \frac{3}{4}yq^2 + q^3 \\ \quad -\frac{1}{16}ay^2 - \frac{1}{2}ayq + 3aq^2 \\ \quad + 4a^2q \end{array}\right\} = 0.$$

And since, by the supposition, q is very little in respect of p, which is nearly $= -\frac{1}{4}y$, therefore q will be very little in respect of y; and consequently all the terms of the last equation will be very little in respect of these two, *viz.* $-\frac{1}{16}ay^2$, $+4a^2q$, where y and q are of least dimensions separately: particularly the term $-\frac{1}{2}ayq$ is little in respect of $4a^2q$, because y is very little in respect of a; and it is little in respect of $-\frac{1}{16}ay^2$, because q is little in respect of y.

Neglect therefore the other terms and supposing $-\frac{1}{16}ay^2 + 4a^2q = 0$, you will have $q = \frac{1}{64} \times \frac{y^2}{a}$; so that $x = a - \frac{1}{4}y + \frac{1}{64} \times \frac{y^2}{a}$. And by proceeding in the same manner you will find $x = a - \frac{y}{4} + \frac{y^2}{64a} - \frac{131y^3}{512a^2} + \frac{509y^4}{16384a^3} -$ &c.

§ 100. When it is required to find a series for x that shall converge sooner, the *greater* y is in in respect of any quantity a; you need only suppose a to be very little in respect of y, and proceed by the same reasoning as in the last example on the supposition of y being very little.

Thus, to find a value for x in the equation $x^3 - a^2x + ayx - y^3 = 0$ that shall converge the sooner the greater y is in respect of a, suppose a to vanish, and the remaining terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vastly great, it appears that $x = y$ nearly.

But to have the value of x more accurately, put $x = y + p$, then

$$\left.\begin{array}{rl} x^3 = & y^3 + 3y^2p + 3yp^2 + p^3 \\ -a^2x = & -a^2y - a^2p \\ +ayx = & ay^2 + ayp \\ -y^3 = & -y^3 \end{array}\right\} = 0$$

$$= +3y^2p + 3yp^2 + p^3 - a^2y - a^2p + ay^2 + ayp:$$

where the terms $3y^2p + ay^2$ become vastly greater than the rest, y being vastly greater than a or p; and consequently $p = -\frac{1}{3}a$ nearly.

Again, by supposing $p = -\frac{1}{3}a + q$, you will transform the last equation into

$$\left.\begin{array}{l} -\frac{8}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ -a^2y - ayq - aq^2 \\ \quad -\frac{2}{3}a^2q \end{array}\right\} = 0;$$

where the two terms $3qy^2 - a^2y$ must be vastly greater than any of the rest, a being vastly less

 than

than y, and q vaſtly leſs than a, by the ſuppoſition; ſo that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this manner, you may correct the value of y, and find that

$$x = y - \tfrac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3} \ \&c.$$

which ſeries converges the ſooner the greater y is ſuppoſed to be taken in reſpect of a.

§ 101. In the ſolution of the firſt *Example* thoſe terms were always compared in order to determine p, q, r, &c. in which y and thoſe quantities p, q, r, &c. were *ſeparately* of feweſt dimenſions. But in the ſecond *Example*, thoſe terms were compared in which a and the quantities p, q, r, &c. were of leaſt dimenſions ſeparately. And theſe always are the proper terms to be compared together, becauſe they become vaſtly greater than the reſt, in the reſpective hypotheſes.

In general; to determine the firſt, or any, term in the ſeries, *ſuch terms of the equation are to be aſſumed together only, as will be found to become vaſtly greater than the other terms*; that is, which give a value of x, which ſubſtituted for it in all the terms of the equation ſhall raiſe the dimenſions of the other terms all above, or all below, the dimenſions of the aſſumed terms, according as y is ſuppoſed to be vaſtly little, or vaſtly great in reſpect of a.

Thus

Thus to determine the first term of a converging series expressing the value of x in the last equation $x^3 - a^2x + ayx - y^3 = 0$, the terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which substituted for x, the equation becomes

$$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0,$$

where the first term is of more dimensions than the assumed terms ayx, $-y^3$; and the second of fewer: so that the two first terms cannot be neglected in respect of the two last, neither when y is very great nor very little, compared with a. Nor are the terms x^3, ayx, fit to be compared together in order to obtain the first term of a series for x, for the like reason.

But x^3 may be compared with $-a^2x$, as also $-a^2x$ with $-y^3$ for that end. These two give the first term of a series that converges the sooner the less y is; as $x^3 = y^3$ gives the first term of a series that converges the sooner the greater y is. The last series was given in the preceding article. The comparing x^3 with $-a^2x$ gives these two series,

$$x = a - \tfrac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3} \text{ \&c.}$$

$$x = -a + \tfrac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3} \text{ \&c.}$$

The comparing $-a^2x$ with $-y^3$ gives

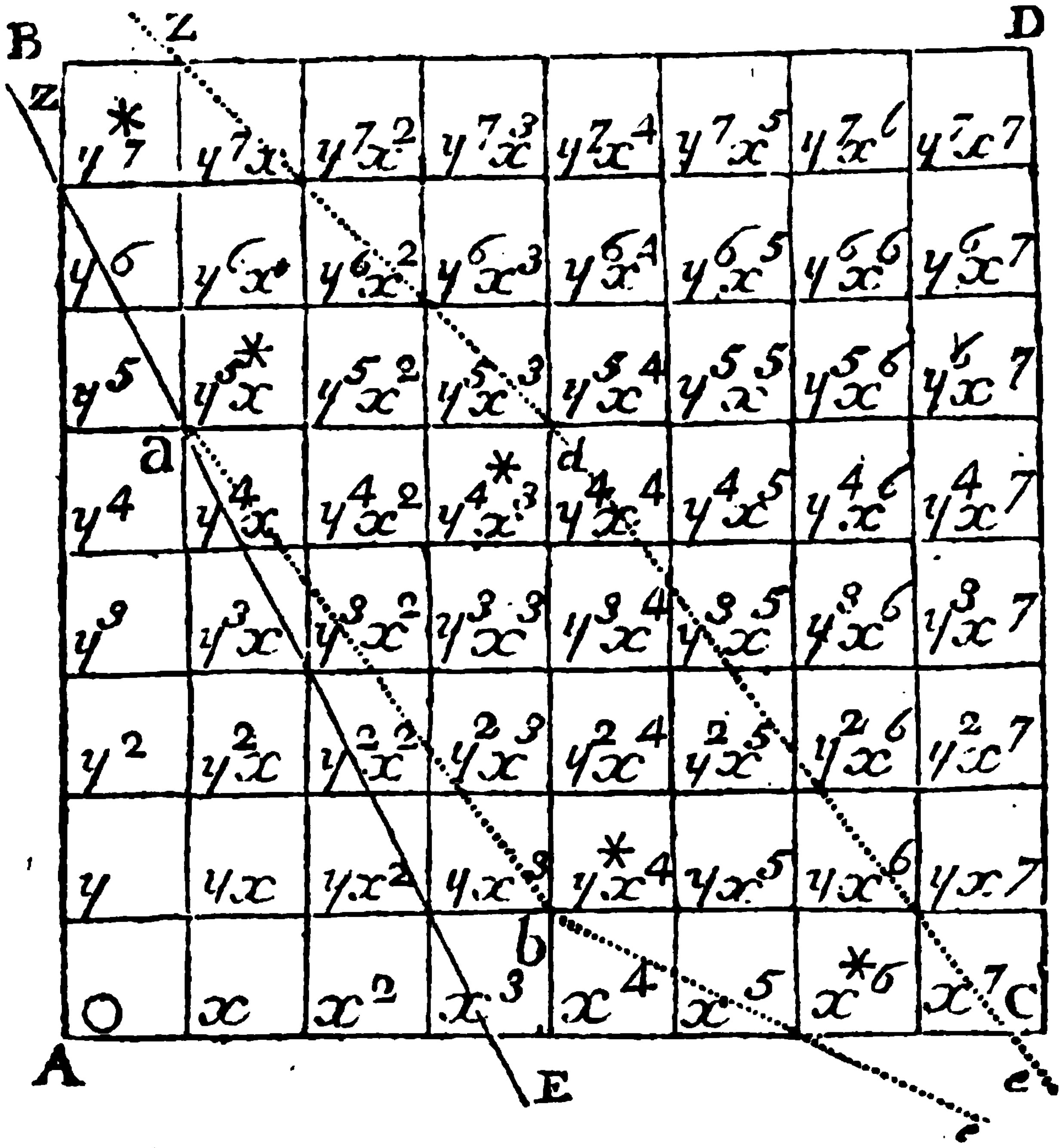
$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} - \text{\&c.}$$

And these series give three values of x when y is very little; the last of which is itself also very little in that case, as it appears indeed from the equation, that when y vanishes, the three values of x become $+a$, $-a$, and o, because when y vanishes, the equation becomes $x^3 - a^2x = o$, whose roots are a, $-a$, o.

§ 102. It appears sufficiently from what we have said, that when an equation is proposed involving x and y, and the value of x is required in a converging series, the difficulty of finding the first term of the series is reduced to this; "to find what terms assumed in order to determine a value of x expressed in some dimensions of y and a will give such a value of it, as substituted for it in the other terms will make them all of more dimensions of y, or all of less dimensions of y than those assumed terms."

To determine this, draw BA and AC at right angles to each other, complete the parallelogram ABCD and divide it into equal squares, as in the figure. In these squares place the powers of x from A towards C, and the powers of y from A towards B, and in any other square place that power of x that is directly below it in the line AC, and that power of y that is in a parallel with it in the line AB; so that the index of x in any square may express its distance from the line AB, and the index of y in any square

ſquare may expreſs its diſtance from the line AC. Of this ſquare we are to obſerve,



1. That the terms are not only in geometrical progreſſion in the vertical column AB, or the horizontal AC, and their parallels; but alſo in the terms taken in any oblique ſtraight line whatever; for in any ſuch term it is manifeſt that the *indices* of *y* and *x* will be in arithmetical progreſſion. The indices of *y*, becauſe thoſe terms will remove equally from the line AC, or approach equally to it, and the indices of

of y in any such terms are as their distances from that line AC. The indices of x will also be in arithmetical progression, because these terms equally remove from, or approach to the line AB. Thus for example, in the terms y^7, $y^5 x$, $y^3 x^2$, $y x^3$, the indices of y decreasing by the common difference 2, while the indices of x increase in the progression of the natural numbers, the common ratio of the terms is $\frac{x}{y^2}$. It follows,

2. From the last observation, that "if any two terms be supposed equal, then all the terms in the same straight line with these terms, will be equal;" because by supposing these two terms equal, the common ratio is supposed to be a ratio of equality; and from this it follows, that "if you substitute every where for x the value that arises for it by supposing any two terms equal, expressed in the powers of y, the dimensions of y in all the terms that are found in the same straight line will be equal;" but "the dimensions of y in the terms *above* that line will be *greater* than in those in that line;" and "the dimensions of y in the terms *below* the said line will be *less* than its dimensions in that line." Thus, by supposing $y^7 = y x^3$, we find $x^3 = y^6$, or $x = y^2$; and substituting this value for x in all the squares, the dimensions of y in the terms y^7, $y^5 x$, $y^3 x^2$, $y x^3$, which are all found in the

the ſame ſtraighf line, will be 7, but the dimenſions in all the terms above that line will be more than 7, and in all the terms below that line will be leſs than 7.

§ 103. From theſe two obſervations we may eaſily find a method for diſcovering what terms ought to be aſſumed from an equation in order to give a value for x which ſhall make the other terms all of *higher*, or all of *lower* dimenſions of y than the aſſumed terms: *viz.* "after all the terms of the equation are ranged in their proper ſquares (by the laſt article) ſuch terms are to be aſſumed as lie in a ſtraight line, ſo that the other terms either lie all above the ſtraight line, or fall all below it."

For example, ſuppoſe the equation propoſed is $y^7 - ay^5x + y^4x^3 + a^2yx^4 - ax^6 = 0$, then marking with an aſteriſk the ſquares in the laſt article which contain the ſame dimenſions of x and y as the terms in the equation, imagine a ruler ZE to revolve about the firſt ſquare marked at y^7, and as it moves from A towards C, it will firſt meet the term ay^5x, and while the ruler joins theſe two terms, all the other terms lie above it: from which you infer, that by ſuppoſing theſe terms equal, you ſhall obtain a value of x, which ſubſtituted for it, will give all the other terms of higher dimenſions of y, than thoſe terms: and hence we conclude that the value of x deduced from ſuppoſing theſe terms, equal,

equal, *viz.* $\frac{y^2}{a}$, is the firſt term of a ſeries that will converge the ſooner the leſs y is in reſpect of a.

If the ruler be made to revolve about the ſame ſquare the contrary way from D towards C, it will firſt meet the term y^4x^3, and by ſuppoſing $y^7 + y^4x^3 = 0$, we find $y = x$, which gives the firſt term of a ſeries for x, that converges the ſooner the greater that y is. And this is the celebrated Rule invented by Sir *Iſaac Newton* for this purpoſe.

§ 104. This Rule may be extended to equations having terms that involve powers of x and y with *fractional* or *ſurd* indices; "by taking diſtances from A in the lines AC and AB proportional to theſe fractions and ſurds," and thence determining the ſituation of the terms of the propoſed equation in the parallelogram ABCD.

It is to be obſerved alſo, that when the line joining any two terms has all the other terms on one ſide of it, by them you may find the firſt term of a converging ſeries for x, and thus "various ſuch ſeries can be deduced from the ſame equation." As, in the laſt Example, the line joining y^5x and yx^4 has all the terms above it; and therefore ſuppoſing $-ay^5x + a^2yx^4 = 0$, we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which is the firſt

term

term of another converging ſeries for x. Again, the ſtraight line joining yx^4 and x^6 has all the other terms above it, and therefore, ſuppoſing $a^2yx^4 - ax^6 = 0$, we find $ay = x^2$, and $x = a^{\frac{1}{2}}y^{\frac{1}{2}}$, the firſt term of another ſeries for x, converging alſo the ſooner the leſs y is. There are two ſeries converging the ſooner the greater y is, to be deduced from ſuppoſing $y^7 = -y^4x^3$, or $y^4x^3 = ax^6$. And, to find all theſe ſeries, "deſcribe a *polygon Zabed*, having a term of the equation in each of its angles, and including all the other terms within it, then a ſeries may be found for x, by ſuppoſing any two terms equal that are placed in any two adjacent angles of the polygon."

§ 105. If the ruler ZE be made to move parallel to itſelf, all the terms which it will touch at once will be of the ſame dimenſions of y: for they will bear the ſame proportion to one another as the terms in the line ZE themſelves. The terms which the Ruler will touch firſt will have fewer dimenſions of y, than thoſe it touches afterwards in the progreſs of its motion, if it moves towards D; but more dimenſions than they, if it moves towards A. The terms in the ſtraight line ZE, ſerve to determine the firſt term of the converging ſeries required. Theſe with the terms it touches afterwards ſerve to determine the ſucceeding terms of the converg-

ing

ing series; all the rest vanishing compared with these, when y is very little and the ruler moves from A towards D, or when y is vastly great and the ruler moves from D towards A.

§ 106. The same Author gives another method for discovering the first term of a series that shall converge the sooner the less y is. "Suppose the term where y is separately of fewest dimensions to be Dy^l; compare it successively with the other terms, as with $Ey^m x^s$, and observe where $\frac{l-m}{s}$ is found *greatest*; and putting $\frac{l-m}{s} = n$, Ay^n will be the first term of a series that shall converge the sooner the less y is:" for in that case Dy^l and $Ey^m x^s$ will be infinitely greater than any other terms of the proposed equation. Suppose $Fy^e x^k$ is any other term of the equation, and, by the supposition, $\frac{l-m}{s} (=n)$ is greater than $\frac{l-e}{k}$, and consequently, multiplying by k, you find nk greater than $l-e$, and $nk+e$ greater than l; now if for x you substitute Ay^n, then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vanish compared with Dy^l (since $nk+e$ is greater than l) when y is infinitely little. Thus therefore all the terms will vanish compared with Dy^l and $Ey^m x^s$ which are supposed equal; and consequently they will give the first term of a series that will converge the sooner the less y is.

§ 107.

§ 107. If you obſerve "when $\frac{l-m}{s}$ is found *leaſt* of all, and ſuppoſe it equal to n, then will Ay^n be the firſt term of a ſeries that will converge the ſooner the *greater* y is." For in that caſe Dy^l and $Ey^m x^s$ will be infinitely greater than $Fy^e x^k$, becauſe $\frac{l-m}{s}$ $(=n)$ being leſs than $\frac{l-e}{k}$, it follows that nk is leſs than $l-e$, and $nk+e$ leſs than l; and conſequently $Fy^e x^k$ $(=FA^k y^{nk+e})$ vaſtly leſs than Dy^l, when y is very great.

After the ſame manner, if you compare any term $Dy^l x^b$, where both x and y are found, with all the other terms, and obſerve where $\frac{l-m}{s-b}$ is found *greateſt* or *leaſt*, and ſuppoſe $\frac{l-m}{s-b}=n$, then may Ay^n be the firſt term of a converging ſeries. For ſuppoſing that $Fy^e x^k$ is any other term of the equation, if $\frac{l-m}{s-b}$ $(=n)$ is greater than $\frac{l-e}{k-b}$, then ſhall $nk-nb$ be greater than $l-e$, and $nk+e$ greater than $l+nb$. But $nk+e$ are the dimenſions of y in $Fy^e x^k$ when $x=Ay^n$, and $l+nb$ are the dimenſions of y in $Ey^m x^s$, therefore $Fy^e x^k$ is of more dimenſions of y than $Ey^m x^s$, and therefore vaniſhes compared to it when y is ſuppoſed infinitely little. In the ſame manner, if $\frac{l-m}{s-b}$ is leſs than $\frac{l-e}{k-b}$, then will

will $Ey^m x^l$ be infinitely greater than $Fy^e x^k$, when y is infinite.

§ 108. When the firſt term (Ay^n) of the ſeries is found by the preceding method, then by ſuppoſing $x = Ay^n + p$, and ſubſtituting this binomial and its powers for x and its powers, there will ariſe an equation for determining p the ſecond term of the ſeries. This new equation may be treated in the ſame manner as the equation of x, and by the Rule of § 103, the terms that are to be compared in order to obtain a near value of p, may be diſcovered; by means of which terms p may be found: which ſuppoſe equal to By^{n+r}, then by ſuppoſing $p = By^{n+r} + q$, the equation may be transformed into one for determining q the third term of the ſeries, and by proceeding in the ſame manner you may determine as many terms of the ſeries as you pleaſe; finding $x = Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$ &c. where the dimenſions of y aſcend or deſcend according as r is poſitive or negative; and always "in arithmetical progreſſion, that this value of x being ſubſtituted for it in the propoſed equation, the terms involving y and its powers may fall in with one another, ſo that more than one may always involve the ſame dimenſion of y, which may mutually deſtroy each other and make the whole equation vaniſh, as it ought to do.

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It is obvious that as the dimensions of y in $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. are in an arithmetical progression whose difference is r, the square, cube, or any power s of $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. will consist of terms wherein the dimensions of y will constitute an arithmetical progression having the same common difference r; for these dimensions will be sn, $sn + r$, $sn + 2r$, $sn + 3r$, &c. Therefore, if in any term $Ey^m x^s$ you substitute for x the series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$ &c. the terms of the series expressing $Ey^m x^s$ will consist of these dimensions of y, *viz.* $m + sn$, $m + sn + r$, $m + sn + 2r$, $m + sn + 3r$, &c. and by a like substitution in any other term as $Fy^e x^k$, the dimensions of y will be $e + nk$, $e + nk + r$, $e + nk + 2r$, $e + nk + 3r$, &c. The former series of indices must coincide with the latter series, that the terms in which they are found may be compared together, and be found equal with opposite signs so as to destroy one another, and make the whole equation vanish.

The first series consists of terms arising by adding some multiple of r to $m + sn$, the latter by adding some multiple of r to $e + nk$; and that these may coincide, some multiple of r added to $m + sn$ must be equal to some other multiple of r added to $e + nk$. From which it appears that the difference of $m + sn$ and $e + nk$ is always

 a mula multiple of r; and consequently that r is a divisor of the difference of dimensions of y in the terms $Ey^m x^e$ and $Fy^e x^k$, supposing $x = Ay^n$. It follows therefore "that r is a common divisor of the differences of the dimensions of y in the terms of the equation, when you have substituted Ay^n for x in all the terms." And if r be assumed equal to the *greatest* common divisor (excepting some cases afterward to be mentioned) you will have the true form of a series for x. And now the dimensions y^n, y^{n+r}, y^{n+2r}, y^{n+3r} &c. being known, there remains only, by calculation, to determine the general coefficients A, B, C, D, &c. in order to find the series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. $= x$.

§ 109. This leads us to Sir *Isaac Newton's second general method* of series; which consists in assuming a series with undetermined coefficients expressing x, as $Ay^n + By^{n+r} + Cy^{n+2r} +$ &c. where A, B, C, &c. are supposed as yet unknown, but n and r are discovered by what we have already demonstrated; and substituting this every where for x, you must suppose, in the new equation that arises, the sum of all the terms that involve the same dimensions of y to vanish, by which means you will obtain particular equations, the *first* of which will give A, the *second* B, the *third* C, &c. and these values being

being substituted in the assumed series for A, B, C, &c. the series for x will be obtained as far as you please.

Let us apply, for example, this method to the equation (of § 98) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppose it is required to find a series converging the sooner the less y is: its first term (by § 99 or 102) is found to be a, so that $n = 0$. Substitute a for x in the equation, and the terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the differences of the indices are 0, 1, 2, 3; whose greatest common measure is 1, so that $r = 1$. Assume therefore $x = A + By + Cy^2 + Dy^3$, &c. and substitute this series for x in the equation. Then

$$\begin{array}{llllll}
x^3 = & A^3 + 3A^2By + & 3AB^2y^2 + & B^3y^3 & + \&c. \\
 & & + 3A^2Cy^2 + & 3A^2Dy^3 & + \&c. \\
 & & & + 6ABCy^3 & + \&c. \\
+ a^2x = & a^2A + a^2By + & a^2Cy^2 + & a^2Dy^3 & + \&c. \\
+ ayx = & \quad aAy + & aBy^2 + & aDy^3 & + \&c. \\
- 2a^3 = & - 2a^3 & & & \\
- y^3 = & \ldots\ldots\ldots & & - 1 \times y^3. &
\end{array}$$

Now since $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the sum of these series involving y must vanish. But that cannot be if the coefficient of every particular term does not vanish. For every term where y is infinitely little, is infinitely greater than the following terms, so that

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if every term does not vanish of itself, the addition or subtraction of the following terms which are infinitely less than it, or of the preceding terms which are infinitely greater, cannot destroy it; and therefore the whole cannot vanish. It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an equation for determining A, and gives $A = a$.

In order to determine B, you must suppose the sum of the coefficients affecting y to vanish, *viz.* $\overline{3A^2B + a^2B + aA} \times y = 0$, or, since $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To determine C, in the same manner suppose $3AB^2y^2 + 3A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or, substituting for A and B their values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$; and consequently $C = \frac{1}{64a}$. And, by proceding in the same manner, $D = \frac{131}{512a^2}$, so that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2 + \frac{131}{512a^2}y^3$ &c. as we found before in § 99.

§ 110. By this method you may transfer series from one undetermined quantity to another, and obtain Theorems for the *reversion* of series.

Suppose that $x = ay + by^2 + cy^3 + dy^4 + \&c.$ and it is required to express y by a series consisting of the powers of x. It is obvious that when

when x is very little, y is alſo very little, and that in order to determine the firſt term of the ſeries, you need only aſſume $x = ay$. And therefore $y = \frac{x}{a}$; ſo that $n = 1$. By ſubſtituting $\frac{x}{a}$ for y, you find the dimenſions of x in the terms will be 1, 2, 3, 4, &c. ſo that $r = 1$ alſo. You may therefore aſſume $y = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ And by the ſubſtitution of this value of y you will find

$$\begin{array}{llll} ay = & aAx + aBx^2 + & aCx^3 & + \&c. \\ by^2 = & & bA^2x^2 + 2bABx^3 & + \&c. \\ cy^3 = & & cA^3x^3 & + \&c. \\ \&c. & & \&c. & \end{array}$$

But the firſt term being already found to be $\frac{x}{a}$, you have $A = \frac{1}{a}$; and ſince $aB + bA^2 = 0$, it follows that $B = -\frac{b}{a^3}$. After the ſame manner you will find $C = \frac{2b^2 - ac}{a^4}$. Whence $y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^4}x^3 + \&c.$

§ 111. Suppoſe again you have $ax + bx^2 + cx^3 + dx^4 + \&c. = gy + hy^2 + iy^3 + ky^4$ &c. to find x in terms of y. You will eaſily ſee, by § 103, that the firſt term of the ſeries for x is $\frac{gy}{a}$, that $n = 1$, $r = 1$. Therefore aſſume $x = Ay + By^2 + Cy^3$ &c. and by ſubſtituting

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this value for x and bringing all the terms to one side, you will have

$$\begin{array}{rcl}
ax & = & aAy + aBy^2 + aCy^3 + \&c. \\
bx^2 & = & bA^2y^2 + 2bABy^3 + \&c. \\
cx^3 & = & cAy^3 + \&c. \\
\&c. & & \&c. \\
-gy & = & -gy \\
-hy^2 & = & \ldots -hy^2 \\
-iy^3 & = & \ldots\ldots -iy^3 \\
\&c. & & \&c.
\end{array}$$

From whence we see, first, that $aA = g$, and $A = \frac{g}{a}$. 2°. That $aB + bA^2 - h = 0$, and $B = \frac{h}{a} - \frac{bg^2}{a^3}$. 3°. That $aC + 2bAB + cA^3 - i = 0$, and therefore $C = \frac{i - 2bAB - cA^3}{a}$. And thus the three first terms of the series $Ay + By^2 + Cy^3$ &c. are known*.

§ 112. Before we conclude it remains to clear a difficulty in this method that has embarrassed some late ingenious writers, concerning "the value of r to be assumed when *two* or *more* of the values of the first term of a series for expressing x are found equal;" a correction of the preceding Rule being necessary in that case. And the author of that correction having only collected it from experience, and given it us with

* See Mr. *De Moivre*, in *Phil. Transf.* 240.

without proof, it is the more neceſſary to demonſtrate it here.

It is to be obſerved then, that in order that the ſeries $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. may expreſs x, it is not only neceſſary that when it is ſubſtituted for x in the propoſed equation $Dy^l + Ey^m x^s + Fy^e x^k = 0$, the indices $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. ſhould fall in with the indices $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. in order that the terms may be compared together to determine the coefficients A, B, C, &c. but it is alſo neceſſary, that in the particular equations for determining any of thoſe coefficients, as B for example, thoſe terms that involve B ſhould not deſtroy each other. Thus the equation $3A^2B - 3A^2B - aA = 0$ can never determine B, becauſe $3A^2B - 3A^2B = 0$, and thus B exterminates itſelf out of the equation; beſides the contradiction ariſing from $-A = 0$, when A perhaps has been determined already to be equal to ſome real quantity.

In order to know how to evite this abſurdity, let us ſuppoſe that the firſt order of terms in the propoſed equation are, as before, Dy^l, $Ey^m x^s$, &c. and if Ay^n is found to be the firſt term of a ſeries for x, then the dimenſions of y in the firſt order of terms, ariſing by ſubſtituting in them Ay^n for x, will be $m + ns$, and the dimenſions of y ariſing by ſubſtituting

$Ay^n + By^{n+r} + Cy^{n+2r}$ &c. for x will be $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. Suppose that $Fy^e x^k$ is the next order of terms and, by the same substitution, the dimensions of y arising from it will be

(because $Fy^e x^k = Fy^e \times \overline{Ay^n + By^{n+r} + Cy^{n+2r} + \&c.}|^k = FA^k y^{e+nk} + kFBA^{k-1}y^{e+nk+r}$ &c.) $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. Now it is plain, that $e + nk$ must coincide with some of the dimensions $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. that the terms involving them may be compared together. And therefore, as we observed in § 108, r must be the difference of $e + nk$ and $m + ns$, or some divisor of that difference. *In general*, r must be assumed such a divisor of that difference as may allow not only $e + nk$ to coincide with some one of the series $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. but as may make all the indices of the other orders besides $e + nk$ likewise to coincide with one of that series: that is, if $Gy^f x^b$ is another term in the equation, r must be so assumed that the series $f + nb$, $f + nb + r$, $f + nb + 2r$, &c. arising by substituting in it $Ay^n + By^{n+r} + Cy^{n+2r}$ &c. for x, may coincide somewhere with the first series $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. And therefore we said, in § 108, "that r must be assumed so as to be equal to some common divisor of the differences of the indices $m + ns$, $e + nk$, $f + nb$, which arise in the proposed equation by by substituting in it for x the first term already known Ay^n." For by assuming r equal to a common divisor of these differences, the three series

$$m+ns, m+ns+r, m+ns+2r, m+ns+3r, \&c.$$
$$e+nk, e+nk+r, e+nk+2r, e+nk+3r, \&c.$$
$$f+nb, f+nb+r, f+nb+2r, f+nb+3r, \&c.$$

will coincide with one another, since some multiples of r added to $m+ns$ will give $e+nk$ and all that follow it in the *second* series, and some multiples of r added to $m+ns$ will also give $f+nb$ and all that follow it in the *third* series. It is also obvious, that, if no particular reason hinder it, r ought to be assumed equal to the *greatest* common measure of these differences. For example, if the indices $m+ns$, $e+nk$, $f+nb$, happen to be in arithmetical progression, then r ought to be assumed equal to the common difference of the terms, and the first of the second series will coincide with the second of the first, and the first of the third series, will coincide with the second of the second series, and with the third of the first, and so on.

§ 113. These things being well understood, we are next to observe that after you have substituted $Ay^n + By^{n+r} + Cy^{n+2r}$&c. for x in the first order of terms in the equation, the terms that involve $m+ns$ dimensions of y will destroy one another; for $x - Ay^n$ must be a divisor of the

the aggregate of these terms, since they give Ay^n as one value of x: let $\overline{x - Ay^n} \times P$ represent that aggregate, and, substituting for x its value $Ay^n + By^{n+r} + Cy^{n+2r}$ &c. that aggregate becomes $\overline{Ay^n + By^{n+r} + Cy^{n+2r} \&c. - Ay^n} \times P$ $= \overline{By^{n+r} + Cy^{n+2r} \&c.} \times P$. Now the lowest dimension in $\overline{x - Ay^n} \times P$ was supposed to be $m + ns$, whence the dimension of P, in the same terms, will be $m + ns - n$, and the lowest dimension in $\overline{By^{n+r} + Cy^{n+2r} + \&c.} \times P$ will be $n + r + m + ns - n = m + ns + r$. Suppose again that *two* values of x, determined from the first order of terms, are equal, and then $\overline{x - Ay^n}|^2$ will be a divisor of that aggregate of the first order of terms. Suppose that aggregate now $\overline{x - Ay^n}|^2 \times P$, which by substitution of $Ay^n + By^{n+r} + Cy^{n+2r}$ &c. for x will become $\overline{By^{n+r} + Cy^{n+2r} + \&c.}|^2 \times P$, in which the lowest term will now be of $m + ns$ dimensions, since in $\overline{x - Ay^n}|^2 \times P$ the lowest term is supposed of $m + ns$ dimensions; and consequently, in these terms, the dimension of P itself is $m + ns - 2n$.

In general, if the number of values of x supposed equal to Ay^n be p, then must $\overline{x - Ay^n}|^p$ be a divisor of the aggregate of the terms of the first order. And that aggregate being expressed by $\overline{x - Ay^n}|^p \times P$, in the lowest terms, the dimensions

sions of y in P will be $m + ns - pn$, that in $\overline{x - Ay^n}|^p$ they may be $m + ns$, as we always suppose. Substitute in $\overline{x - Ay^n}|^p \times P$ for $x - Ay^n$ its value $By^{n+r} + Cy^{n+2r} + \&c.$ and in the result $\overline{By^{n+r} + Cy^{n+2r} + \&c.}|^p \times P$ the lowest dimensions of y will be $pn + pr + m + ns - pn = m + ns + pr$.

§ 114. From what has been said we conclude that when you have substituted for x in the first order of terms of the equation proposed the series $Ay^n + By^{n+r} + Cy^{n+2r} + \&c$, the first term of which Ay^n is known, and the values of x whose number is p are found equal, then the terms arising that involve $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. till you come to $m + ns + pr$, will destroy each other and vanish; so that the first term with which the terms of the second order $e + nk$ can be compared must be that which involves $m + ns + pr$; and therefore supposing $e + nk = m + ns + pr$, or $r = \frac{e + nk - m - ns}{p}$, "the highest value you can give r must be the difference of $e + nk$ and $m + ns$ divided by p the number of equal values of the first term of the series." If this value of r is a common measure of all the differences of the indices, then is it a just value of r; but if it is not, such a value of r must be assumed, as may measure this and all the differences; that is, " such a value as may be be the greatest common measure of the least difference divided by p (*viz.* $\frac{e+nk-m-ns}{p}$) and of the common measure of all the differences." For thus the indices $m+ns$, $m+ns+r$, $m+ns+2r$, &c. will coincide with $e+nk$, $e+nk+r$, $e+nk+2r$, &c. and with $f+nb$, $f+nb+r$, $f+nb+2r$, &c. and you shall always have terms to be compared together sufficient to determine B, C, D, &c. the general coefficients of the series assumed for x.

§ 115. To all this it may be added, that if $x-Ay^n$ be a divisor of the aggregate of the terms of the *second* order Fy^ex^k, &c. then, by substituting for x the series $Ay^n+By^{n+r}+Cy^{n+2r}$ + *&c.* there vanish not only as many terms of the series involving $m+ns$, $m+ns+r$, $m+ns+2r$ &c. as there are equal values of the first term Ay^n; but the terms involving $e+nk$ dimensions of y vanish also; and therefore it is then only necessary that $e+nk+r$ coincide with $m+ns+pr$, so that, in that case, you need only take $r=\frac{e+nk-m-ns}{p-1}$. And if $\overline{x-Ay^n}|^{p-1}$ be a divisor of the aggregate of the second order of terms, then the terms after substituting for x the series $Ay^n+By^{n+r}+Cy^{n+2r}$ &c.) which involve $e+nk$, $e+nk+r$, $e+nk+2r$, &c. will vanish to the term $e+nk+\overline{p-1}\times r$; so that, sup-

supposing $e + nk + \overline{p-1} \times r = m + ns + pr$, you have $r = e + nk - m - ns$, that is, to the *least* difference of the indices $m + ns$, $e + nk$, $f + nb$, &c. provided that difference be a measure of the other differences; although there may be as many values of the first term of the series equal as there are units in p. Or, if that does not happen, r must be taken, as formerly, equal to the greatest common measure of the differences.

§ 116. Suppose that the orders of terms of the equation can be expressed the *first* by $\overline{x - Ay^n}|^p \times P$, the *second* by $\overline{x - Ay^n}|^q \times Q$, the *third* by $\overline{x - Ay^n}|^l \times L$, &c. and suppose that $Ey^m x^r$ is one of the first, $Fy^e x^k$ one of the second, $Gy^f x^h$ one of the third, and so on; then it is plain that, substituting for x the series $Ay^n + By^{n+r} + Cy^{n+2r}$ &c. the lowest term that will remain in the first will be $m + ns + pr$ dimensions of y, the lowest term that will remain in the second will be of $e + nk + qr$, and the lowest term remaining in the third of $f + nb + lr$ dimensions of y. For by the same reasoning as we used, in § 113, to demonstrate that, in the first order of terms $\overline{x - Ay^n}|^p \times P$, the lowest dimensions of y are $m + ns + pr$, we shall find that, in the subsequent orders, the lowest dimensions of y in the terms $\overline{x - Aq^n}|^q \times Q = \overline{By^{n+r} + Cy^{n+2r} \text{ &c.}}|^q \times Q$ must

must be $e+nk-qn+qn+qr=e+nk+qr$, and so of the other terms $\overline{x-Ay^n}|^l \times L$ the lowest dimensions must be $f+nb+lr$. The indices therefore of the terms that do not vanish being

$$\ast\ \ast\ m+ns+pr,$$
$$\ast\ \ast\ \ast\ \ast\ e+nk+qr,$$
$$\ast\ \ast\ \ast\ \ast\ \ast\ \ast\ f+nb+lr,$$

if r be taken equal to $\frac{e+nk-m-ns}{p-q}$, then will $m+ns+pr$ and $e+nk+qr$ coincide: and if at the same time r be a divisor of $f+nb-m-ns$, and be found in it a number of times greater than $p-l$, or if r be less than $\frac{f+nb-m-ns}{p-l}$, then r will be rightly assumed. *In general*, "take all the quotients $\frac{e+nk-m-ns}{p-q}$, $\frac{f+nb-m-ns}{p-l}$, and either the *least* of these, or a number whose denominator, exceeding $p-q$ by an integer, measures it and all the differences $f+nb-m-ns$, gives r;" supposing p, q, and l integers. But if p, q, and l are fractions, you are to "take r so that it be equal to $\frac{e+nk-m-ns}{p-q+K} = \frac{f+nb-m-ns}{p-l+M}$, and so that K and M may be integers." Suppose, for example, $m+ns=\frac{7}{3}$,

$p=$

$p = \frac{5}{2}$; $e + nk = \frac{10}{3}$, $q = \frac{3}{2}$; $f + nb = \frac{9}{2}$, and $l = \frac{1}{2}$: then putting — — — — — ($r =$) $\frac{e+nk-m-ns}{p-q+K} = \frac{1}{1+K} = \frac{f+nb-m-ns}{p-l+M} = \frac{\frac{13}{6}}{2+M}$, $M = \frac{1}{6} + \frac{13}{6}K$; whence it is easily seen that 5 and 11 are the least integers that can be assumed for K and M. And that $r = \frac{1}{1+K} = \frac{1}{6}$; and therefore $m + ns + pr = \frac{33}{12}$, $e + nk + qr = \frac{43}{12}$, and $f + nb + lr = \frac{55}{12}$. That is, the terms of the first series whose dimensions are $m + ns + \overline{p+K} \times r$, $m + ns + \overline{p+M} \times r$ fall in with the first terms of the second and third series respectively *.

* See on this subject, *Colson.* Epist. in Animadv. D. *Moivrei.* *Taylor* Meth. Incr. *Stirling* Lin. iii. Ord. *s'Gravesande* Append. Elem. Algebrae. *Stewart* on the Quadrature of Curves.

CHAP.

CHAP. XI.

Of the Rules for finding the number of impossible Roots in an equation.

§ 117. THE number of *impossible* roots in an equation may, for most part, be found by this

RULE.

"*Write down a series of fractions whose denominators are the numbers in this progression* 1, 2, 3, 4, 5, &c. *continued to the number which expresses the dimension of the equation. Divide every fraction in the series by that which precedes it, and place the quotients in order over the middle terms of the equation. And if the square of any term multiplied into the fraction that stands over it gives a product greater than the rectangle of the two adjacent terms, write under the term the sign* +, *but if that product is not greater than the rectangle, write* −; *and the signs under the extreme terms being* +, *there will be as many* imaginary *roots as there are changes of the signs from* + *to* −, *and from* − *to* +.

Thus

Thus, the given equation being $x^3 + px^2 + 3p^2x - q = 0$, I divide the second fraction of the series $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$, by the first, and the third by the second, and place the quotients $\frac{1}{3}$ and $\frac{1}{3}$ over the middle terms in this manner;

$$\begin{array}{ccccccccc} x^3 & + & \overset{\frac{1}{3}}{px^2} & + & \overset{\frac{1}{3}}{3p^2x} & - & q & = & 0. \\ + & & - & & + & & + & & \end{array}$$

Then because the square of the second term multiplied into the fraction that stands over it, that is, $\frac{1}{3} \times p^2x^4$ is less than $3p^2x^4$ the rectangle under the first and third terms, I place under the second term the sign $-$: but as $\frac{1}{3} \times 9p^4x^2$ $(= 3p^4x^2)$ the square of the third term multiplied into its fraction is greater than *nothing*, and consequently much greater than $-pqx^2$ the negative product of the adjoining terms, I write under the third term the sign $+$. I write $+$ likewise under x^3 and $-q$ the first and last terms, and finding in the signs thus marked two changes, one from $+$ to $-$, and another from $-$ to $+$, I conclude the equation has two impossible roots.

In like manner the equation $x^3 - 4x^2 + 4x - 6 = 0$ has two impossible roots;

$$\begin{array}{ccccccccc} x^3 & - & \overset{\frac{1}{3}}{4x^2} & + & \overset{\frac{1}{3}}{4x} & - & 6 & = & 0; \\ + & & + & & - & & + & & \end{array}$$

and the equation $x^4 * - 6x^2 - 3x - 2 = 0$ the same number

$$\begin{array}{ccccc} \frac{3}{8} & \frac{4}{9} & \frac{3}{8} & & \\ x^4 * & - 6x^2 & - 3x & - 2 & = 0. \\ + + & + & - & + & \end{array}$$

For the series of fractions $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$ yields, by dividing them as the Rule directs, the fractions $\frac{3}{8}$, $\frac{4}{9}$, $\frac{3}{8}$ to be placed over the terms. Then the square of the second term, which is *nothing*, multiplied by the fraction over it being still *nothing*, and yet greater than $-6x^6$ the negative product of the adjacent terms, I write under (*) the term that is wanting, the sign +, and proceeding as in the former examples, I conclude, from the two changes that happen in the series + + + − +, that the equation has two of its roots impossible.

The same way we discover two impossible roots in the equation

$$\begin{array}{cccccc} & \frac{2}{5} & \frac{1}{2} & \frac{1}{2} & \frac{2}{5} & \\ x^5 & - 4x^4 & + 4x^3 & - 2x^2 & - 5x & - 4 = 0. \\ + & + & - & + & + & + \end{array}$$

When two or more terms are wanting in the equation, under the first of such terms place the sign −, under the second +, under the third −, and so on alternately; only when the two terms to the right and left of the deficient terms have

have contrary signs, you are always to write the sign + under the last deficient term.

As in the equations

$$\begin{array}{cccccccc} x^5 & +ax^4 & * & * & * & +a^5 & =0 \\ + & + & - & + & - & + & \end{array}$$

$$\begin{array}{cccccccc} x^5 & +ax^4 & * & * & * & -a^5 & =0 \\ + & + & - & + & + & + & \end{array}$$

the first of which has *four* impossible roots, and the other *two*. Thus likewise the equation

$$\begin{array}{cccccccccc} & \frac{3}{7} & \frac{5}{9} & \frac{3}{5} & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & & \\ x^7 & -2x^6 & +3x^5 & -2x^4 & +x^3 & * & * & -3 & =0 \\ + & - & + & - & + & - & + & + & \end{array}$$

has *six* impossible roots.

Hence too we may discover if the imaginary roots lie hid among the affirmative, or among the negative roots. For the signs of the terms which stand over the signs below that change from + to − and − to +, shew, by the number of their variations, how many of the impossible roots are to be reckoned affirmative; and that there are as many negative imaginary roots as there are repetitions of the same sign. As in the equation

$$\begin{array}{cccccc} x^5 & -4x^4 & +4x^3 & -2x^2 & -5x & -4=0 \\ + & + & - & + & + & + \end{array}$$

the signs (− + −) of the terms $-4x^4+4x^3$ $-2x^2$ which stand over the signs + − + point-ing out two affirmative roots*, we infer that two impoſſible roots lie among the affirmative; and the three changes of the ſigns in the equation (+ − + − − −) giving three affirmative roots and two negative, the five roots will be one real affirmative, two negative, and two imaginary affirmatives. If the equation had been

$$\begin{array}{cccccc} x^5 - & 4x^4 - & 4x^3 - & 2x^2 - & 5x - & 4 = 0, \\ + & + & - & - & + & + \end{array}$$

the terms $-4x^4 - 4x^3$ that ſtand over the firſt variation + −, ſhew, by the repetition of the ſign −, that one imaginary root is to be reckoned negative, and the terms $-2x^2 - 5x$ that ſtand over the laſt variation − +, give, for the ſame reaſon, another negative impoſſible root; ſo that the ſigns of the equation (+ − − − − −) giving one affirmative root, we conclude that of the four negative roots, two are imaginary.

This always holds good, unleſs, which ſometimes may happen, there are more impoſſible roots in the equation than are diſcoverable by the Rule."

This Rule hath been inveſtigated by ſeveral eminent Mathematicians in various ways; and others, ſimilar to it, invented and publiſhed†. *But the*

* See § 19.

† See *Stirling's* Linea iij. Ord. *Newton.* p. 59. *Phill. Tranſ.* N° 394, 404, 408.

original

original Rule being, on account of its ſimplicity and eaſy application, if not preferable to all others, at leaſt the fitteſt for this place, it is ſufficient to direct the Reader where he may find the ſubject more fully treated; and to add the demonſtration our Author has given of it towards the end of his Letter to Mr. Folkes, Phil. Tranſ. N° 408, *as it depends only on what has been demonſtrated in* Chap. 5. *concerning the limits of the roots of equations.*

§ 118. Let $ax^2 \pm px \pm q = 0$ be any adfected quadratic equation; and, by § 88, *Part* I. its roots will be $\frac{1}{2a} \times \overline{\mp p \pm \sqrt{p^2 \mp 4aq}}$: whence it is plain that, the ſign of q in the given equation being $+$, the roots will be impoſſible as oft as $4aq$ is greater than p^2, or $\frac{1}{4}p^2$ leſs than $a \times q$.

§ 119. It was ſhewn, *in general* (§ 45 .... 50) *that the roots of the equation* $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, *are the* limits *of the roots of the equation* $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3}$ &c. $= 0$, *or of any equation that is deduced from it by multiplying its terms by any arithmetical progreſſion* $l \mp d$, $l \mp 2d$, $l \mp 3d$, &c. *and converſely the roots of this new equation will be the* limits *of the roots, of the propoſed equation* $x^n - Ax^{n-1} + Bx^{n-2}$ &c. $= 0$.

And that *if any roots of the equation of the limits are impoſſible, there muſt be ſome roots of the propoſed equation impoſſible.*

§ 120. Let $x^3 - Ax^2 + Bx - C = 0$ be a cubic equation, and the *equation of limits* $3x^2 - 2Ax + B = 0$. If the two roots of this last are imaginary, there are two imaginary roots of the given equation $x^3 - Ax^2 + Bx - C = 0$, by the last *Art.* But, by the preceding *Art.* this happens as oft as $\frac{1}{3}A^2$ is less than B; and, in that case, the given equation has two imaginary roots.

Again, multiplying the terms of the equation by the terms of the progression, 0, − 1, − 2, − 3, we get another *equation of the limits* $Ax^2 - 2Bx + 3C = 0$; whose two roots, and consequently two roots of the given equation, are imaginary when $\frac{1}{3}B^2$ is less than $A \times C$.

Hence likewise the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary roots, if two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two roots of the equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two roots of the equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ must be imaginary, when two roots of the quadratic $6x^2 - 3Ax + B = 0$, or of the quadratic $3Ax^2 - 4Bx + 3C = 0$, are imaginary, because the roots of these quadratic equations are the *limits* of the roots of that cubic, and for the same reason two roots of the cubic equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ must be imaginary, when the roots of the quadratic $3Ax^2 - 4Bx + 3C = 0$, or or of the quadratic $Bx^2 - 3Cx + 6D = 0$ are impossible. Therefore two roots of the biquadratic $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ must be imaginary when the roots of any one of these three quadratic equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is less than B, $\frac{4}{9}B^2$ less than AC, or $\frac{3}{8}C^2$ less than BD.

§ 121. By proceeding in the same manner, you may deduce from any equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3}$ &c. $= 0$, as many quadratic equations as there are terms excepting the first and last, whose roots must be all real quantities, if the proposed equation has no imaginary roots. The quadratic deduced from the three first terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifestly have this form, $n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times x^2 - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times Ax + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times B = 0$, continuing the factors in each till you have as many as there are units in $n - 2$. Then dividing the equation by all the factors $n - 2$, $n - 3$, $n - 4$, &c. which are found in each coefficient, the equation will become $n \times \overline{n-1} \times x^2 - \overline{n-1} \times 2Ax + 2 \times 1 \times B = 0$, whose roots will be imaginary, by § 118, when $n \times \overline{n-1} \times 2 \times 4B$ exceeds $\overline{n-1}|^2 \times 4A^2$, or when B exceeds $\frac{n-1}{2n}A^2$: so that the proposed equation must have some imaginary roots when B exceeds $\frac{n-1}{2n}A^2$. The quadratic equation deduced in the same manner from the three first terms of the equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3}$ &c. $= 0$, will have this form, $\overline{n-1} \times \overline{n-2} \times \overline{n-3}$ &c. $\times Ax^2 - \overline{n-2} \times \overline{n-3} \times \overline{n-4}$ &c. $\times 2Bx + \overline{n-3} \times \overline{n-4} \times \overline{n-5}$ &c. $\times 3C = 0$, which dividing by the factors common to all the terms, is reduced to $\overline{n-1} \times \overline{n-2} \times Ax^2 - \overline{n-2} \times 4Bx + 6C = 0$, whose roots must be imaginary when $\frac{2}{3} \times \frac{n-2}{n-1} \times B^2$ is less than AC; and therefore in that case some roots of the proposed equation must be imaginary.

§ 122. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three terms of the equation, $x^n - A^{n-1} + Bx^{n-2}$ &c. $= 0$, that immediately follow one another; multiply the terms of this equation first by the progression $n, n-1, n-2$, &c. then by the progression $n-1, n-2, n-3$, &c. then by $n-2, n-3, n-4$, &c. till you have multiplied by as many progressions as there are units in $n-r-1$: then multiply the terms of the equation that arises, as often by the progression 0, 1, 2, 3, *&c.* as there are units in $r-1$, and you will at length arrive at a quadratic of this form;

$\overline{n-r+1} \times \overline{n-r} \times \overline{n-r-1} \times \overline{n-r-2}$ &c. $\times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4}$ &c. $\times Dx^2$

$-$

$-\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}$ &c.
$\times r\times\overline{r-1}\times\overline{r-2}\times\overline{r-3}$ &c. $\times Ex$
$+\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\times\overline{n-r-4}$
&c. $\times\overline{r+1}\times r\times\overline{r-1}\times\overline{r-2}$ &c. $\times F=0$:
and dividing by the factors $n-r-1$, $n-r-2$, *&c.* and $r-1$, $r-2$, *&c.* which are found in each coefficient, this equation will be reduced to $\overline{n-r+1}\times\overline{n-r}\times 2\times 1\times Dx^2-\overline{n-r}\times 2\times r\times 2Ex+2\times 1\times\overline{r+1}\times rF=0$, whose roots must be imaginary, by § 118, when $\frac{n-r}{n-r+1}\times\frac{r}{r+1}\times E^2$ is less than DF. From which it is manifest, that if you divide each term of this series of fractions $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. $\frac{n-r+1}{r}$, $\frac{n-r}{r+1}$, by that which precedes it, and place the quotients above the terms of the equation $x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}$ &c. $=0$, beginning with the second: then if the square of any term multiplied by the fraction over it be found less than the product of the adjacent terms, some of the roots of that equation must be imaginary quantities.

§ 123. An equation may have impossible roots although none are discovered by the Rule: because, " though *real* roots in the *given equation* always give *real* roots in the *equation of limits*; yet it does not follow, *conversely*, that when the roots of the *equation of limits* are real, those of of the equation from which it is produced must be such likewise. Thus the *cubic*

$$\left.\begin{matrix} x^3 - 2m \\ -\ q \end{matrix}\right\} \times x^2 \left.\begin{matrix} + m^2 \\ + 2qm \\ + n \end{matrix}\right\} \times x - q \times \overline{m^2 + n} = 0,$$

has two of its roots imaginary, $m + \sqrt{-n}$, $m - \sqrt{-n}$, the third being $+q$: and yet in the *equation of limits* $3x^2 - \overline{4m + 2q} \times x^2 + m^2 + 2qm + m = 0$, if $\overline{m-q}|^2$ exceeds $3n$, the roots of the equation of limits will be real. Or if the other equation of limits $\overline{2m + q} \times x^2 - 2 \times \overline{m^2 + 2qm + n} \times x + 3q \times \overline{m^2 + n} = 0$ is found by multiplying by the progression 0, -1, -2, -3; it will have its roots real as oft as $\overline{m^2 + 2qm + n}|^2$ exceeds $\overline{2m + q} \times 3q \times \overline{m^2 + n}$. And the like may be shewn of higher equations.

§ 124. The reason why this Rule, and perhaps every other that depends on the comparison of the square of a term with the rectangles of the terms on either side of it, must sometimes fail to discover the impossible roots, may appear likewise from this consideration: that the number of such comparisons being always less by *unit* than the number of the quantities q, m, n, &c. in the general equation; they cannot include and fix the relations of these quantities,

on

on which the ratio of *greater* or *lesser* inequality of the squares and rectangles depends; no more than equations fewer in number than the quantities sought can furnish a determinate solution of a problem.

CHAP. XII.

Containing a general demonstration of Sir *Isaac Newton*'s Rule for finding the sums of the powers of the roots of an equation*.

LET the equation be $\overline{x-a} \times \overline{x-b} \times \overline{x-c} \times \overline{x-d} \times$ &c. $= 0$, or,

$$\left.\begin{array}{l} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \ldots \\ \ldots - Ix^3 + Kx^2 - Lx + M \end{array}\right\} = 0.$$

It is known that $A = a + b + c + d +$ &c. $B = ab + ac + ad + bc + bd + cd +$ &c. $C = abc + abd + bcd +$ &c. $D = abcd +$ &c. the *parts* or *terms* of the coefficients A, B, C, D, &c. being of 1, 2, 3, 4, *&c. dimensions*; that is, containing as many *roots* or *factors* as there are terms of the equation preceding them, respectively.

* See *Arith Univers.* pag. 157. And *Chap.* II. § 15—17, of this *Part*.

CASE I.

Let r be an *index equal* to n, or *greater* than n, then, multiplying the equation by x^{r-n}, and substituting successively a, b, c, d, &c. for x, you obtain

$$\left.\begin{array}{l} a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \ldots \\ \ldots - La^{r-n+1} + Ma^{r-n} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \ldots \\ \ldots - Lb^{r-n+1} + Mb^{r-n} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \ldots \\ \ldots - Lc^{r-n+1} + Mc^{r-n} \end{array}\right\} = 0,$$

&c.

Whence, by transposition and addition, this *Theorem* results, that, in this case, "the sum of the powers of the roots, of the exponent r, is equal to the sum of their powers of the exponent $r-1$ multiplied by A, *minus* the sum of their powers of the exponent $r-2$ multiplied by B, + the sum of those of the exponent $r-3$ multiplied by C, and so on."

It remains to find the sums of the powers of the roots, when the exponents are *less* than n the exponent of the equation.

CASE II.

If r is less than n, and H be the coefficient in the equation, of the *dimensions* r; that is, if H be taken so that the number of terms preceding it in the equation be equal to r, or the number

ber of *factors* in its parts $abcdefgh$, $abcdefgi$, &c. equal to r, then the Theorem may be expressed in the following manner.

$$a^r + b^r + c^r + d^r + \&c.$$

$$= \left\{\begin{matrix} +a^{r-1} \\ +b^{r-1} \\ +c^{r-1} \\ +d^{r-1} \\ +\&c. \end{matrix}\right\} \times A - \left.\begin{matrix} -a^{r-2} \\ -b^{r-2} \\ -c^{r-2} \\ -d^{r-2} \\ -\&c. \end{matrix}\right\} \times B + \left.\begin{matrix} +a^{r-3} \\ +b^{r-3} \\ +c^{r-3} \\ +d^{r-3} \\ +\&c. \end{matrix}\right\} \times C \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots - r \times H.$$

The case when $r = n - 1$ is easily demonstrated; for, dividing the equation by x, we have

$$x^{n-1} - Ax^{n-2} + Bx^{n-3} \ldots - L + \frac{M}{x} = 0.$$

Whence

$$a^{n-1} - Aa^{n-2} + Ba^{n-3} \ldots - L + \frac{M}{a} = 0,$$

$$b^{n-1} - Ab^{n-2} + Bb^{n-3} \ldots - L + \frac{M}{b} = 0,$$

$$c^{n-1} - Ac^{n-2} + Bc^{n-3} \ldots - L + \frac{M}{c} = 0,$$

&c. &c.

and (because $L = \frac{M}{a} + \frac{M}{b} + \frac{M}{c} + \frac{M}{d} + \&c.$) we shall have $a^{n-1} + b^{n-1} + c^{n-1} + \&c.$

$$= \left\{\begin{matrix} +a^{n-2} \\ +b^{n-2} \\ +c^{n-2} \\ +\&c. \end{matrix}\right\} \times A - \left.\begin{matrix} -a^{n-3} \\ -b^{n-3} \\ -c^{n-3} \\ -\&c. \end{matrix}\right\} \times B + \left.\begin{matrix} +a^{n-4} \\ +b^{n-4} \\ +c^{n-4} \\ +\&c. \end{matrix}\right\} \times C \ldots$$

$$\ldots\ldots\ldots\ldots\ldots + \overline{n-1} \times L.$$

When

When $r = n - 2$, the demonstration is derived from hence, that $\overline{a^2 + b^2 + c^2 + d^2 + \&c.} = A^2 - 2B$ (*pag.* 142:) as follows.

By § 32. transform the given equation, *viz.*

$$\left.\begin{array}{l} x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \ldots \\ \ldots - Ix^3 + Kx^2 - Lx + M \end{array}\right\} = 0 \text{ into}$$

the equation

$$\left.\begin{array}{l} z^n - \frac{L}{M}z^{n-1} + \frac{K}{M}z^{n-2} - \frac{I}{M}z^{n-3} \ldots \\ \ldots - \frac{C}{M}z^3 + \frac{B}{M}z^2 - \frac{A}{M}z + \frac{1}{M} \end{array}\right\} = 0;$$

the roots α, β, γ, δ, &c. of which new equation shall be respectively equal to the reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, &c. of the roots of the original equation.

Divide now the original equation by x^2, and in the quotient substitute for x the roots a, b, c, d, &c. successively, so shall you have

$$\left.\begin{array}{l} a^{n-2} - Aa^{n-3} + Ba^{n-4} - Ca^{n-5} \ldots \\ \ldots - Ia + K - \frac{L}{a} + \frac{M}{a^2} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} b^{n-2} - Ab^{n-3} + Bb^{n-4} - Cb^{n-5} \ldots \\ \ldots - Ib + K - \frac{L}{b} + \frac{M}{b^2} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} c^{n-2} - Ac^{n-3} + Bc^{n-4} - Cc^{n-5} \ldots \\ \ldots - Ic + K - \frac{L}{c} + \frac{M}{c^2} \end{array}\right\} = 0,$$

&c.

Add all these equations together, and for $n - 2$ substitute its value r, and it will be

$$\left.\begin{matrix} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{matrix}\right\} \times A \left.\begin{matrix} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{matrix}\right\} \times B \ldots$$

$$\ldots \left\{\begin{matrix} -a \\ -b \\ -c \\ -\&c. \end{matrix}\right\} \times I + nK - \frac{L}{M} \times L + M \times \left\{\begin{matrix} +\alpha^2 \\ +\beta^2 \\ +\gamma^2 \\ +\&c. \end{matrix}\right\} = 0.$$

But by the principle adduced from *pag.* 142, $\alpha^2 + \beta^2 + \gamma^2 + \&c. = \frac{L^2}{M^2} - \frac{2K}{M}$: wherefore, by multiplication and transposition, it will follow that

$$2K - \frac{L}{M} \times L + M \times \left\{\begin{matrix} + \alpha^2 \\ + \beta^2 \\ + \gamma^2 \\ + \&c. \end{matrix}\right\} = 0.$$

Which equation being subtracted from the preceding, there remains

$$\left.\begin{matrix} a^r - a^{r-1} \\ + b^r - b^{r-1} \\ + c^r - c^{r-1} \\ + \&c. - \&c. \end{matrix}\right\} \times A \left.\begin{matrix} + a^{r-2} \\ + b^{r-2} \\ + c^{r-2} \\ + \&c. \end{matrix}\right\} \times B \ldots$$

$$\ldots \left\{\begin{matrix} - a \\ - b \\ - c \\ - \&c. \end{matrix}\right\} \times I + r \times K = 0.$$ Which was to be proved.

But to shew it *universally*, we may use the following LEMMA:

"That

"That if A is the coefficient of one dimension, or the coefficient of the second term, in an equation; G any other coefficient, H the coefficient next after it; the difference of the dimensions of G and A being $r - 2$: if likewise $A' \times G'$ represent the sum of all those terms of the product $A \times G$ in which the square of any root, as a^2, or b^2, or c^2, &c. is found; then will $A' \times G' = AG - rH$."

This is a particular case of *Prop.* VI. concerning the *impossible roots* in *Phil. Trans.* N° 408; which, by continuing the Table of Equations in *pag.* 140, and observing how the coefficients are formed, may be thus demonstrated.

Let the coefficient of a term of the equation, as D ($= abcd + abce + abcf$ &c. $+ bcde + bcdf$ &c.) be multiplied by A ($= a + b + c + d +$ &c.) and, in the product $A \times D$, setting aside all the terms, $A' \times D'$, in which a^2, b^2, c^2, &c. are found, any one of the remaining terms will arise as often as there are factors in the terms of the following coefficient E. Thus the term $abcde$ will arise *five* times: because it is made up of any one of the five roots (or terms of A) a, b, c, d, e, multiplied into the other four that make a term of D: the like is true of every other term, as $abcdf$, $bcdef$, &c. each of which will arise *five* times in the product $A \times D$. And the sum of these terms $abcde + abcdf +$ &c. making up the coefficient E, it follows that

$A \times$

$A \times D - A' \times D' = 5E$, or $A' \times D' = AD - 5E$, And the same holds of any two coefficients G, H, whose dimensions are $r - 1$ and r respectively.

To apply this to the present purpose, it is to be observed, that, in each of the coefficients A, B, C, D, &c. except the last M, which is the product of all the roots a, b, c, d, &c. we may distinguish two several portions or members, in one of which any particular root, as a, is contained, but in the whole remaining portion of the same coefficient, that particular root (a) is wholly absent. Now if, for brevity's sake, we denote that portion of any coefficient wherein any root, as a, is contained, by annexing the symbol of the said root with the sign $+$ in an *uncus* to the symbol, as G, of the coefficient (thus $G^{(+a}$;) and if we denote the remaining portion of the same coefficient, from which the same root a is totally absent by annexing the symbol of the said root with the sign $-$ in an *uncus* to the symbol G of the same coefficient (thus $G^{(-a}$) it will appear that (if G be any coefficient and H the following coefficient)

$$G = G^{(+a} + G^{(-a} \text{ and } H^{(+a} = aG^{(-a},$$
$$G = G^{(+b} + G^{(-b} \text{ and } H^{(+b} = bG^{(-b},$$
&c. &c.

Divide now the equation proposed by x^{n-r}, and it will become

$$\left.\begin{array}{l} x^r - Ax^{r-1} + Bx^{r-2} - Cx^{r-3} \ldots\ldots \\ \ldots + Gx - H + \frac{I}{x} - \frac{K}{x^2} + \frac{L}{x^3} - \frac{M}{x^{n-r}} \end{array}\right\} = 0,$$

in which substituting a, b, c, &c. successively for x, we obtain

$$\left.\begin{array}{l} a^r - Aa^{r-1} + Ba^{r-2} - Ca^{r-3} \ldots\ldots \\ \ldots + Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} b^r - Ab^{r-1} + Bb^{r-2} - Cb^{r-3} \ldots\ldots \\ \ldots + Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} \end{array}\right\} = 0,$$

$$\left.\begin{array}{l} c^r - Ac^{r-1} + Bc^{r-2} - Cc^{r-3} \ldots\ldots \\ \ldots + Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} \end{array}\right\} = 0,$$

But, by the notation here used, and explained as above.

$$\begin{array}{lllll} Ga = & aG^{(+a} + aG^{(-a} & & & \\ -H = & & -aG^{(-a} - H^{(-a} & & \\ +\frac{I}{a} = & & & +H^{(-a} + \frac{I}{a}^{(-a} & \\ -\frac{K}{a^2} = & -\frac{I}{a}^{(-a} - \frac{K}{a^2}^{(-a} & & & \\ +\frac{L}{a^3} = & & +\frac{K}{a^2}^{(-a} + \frac{L}{a^3}^{(-a} & & \\ -\frac{M}{a^{n-3}} = & & -\frac{L}{a^3}^{(-a} & \&c. & \end{array}$$

Whence

Whence

$$Ga - H + \frac{I}{a} - \frac{K}{a^2} + \frac{L}{a^3} - \frac{M}{a^{n-r}} = aG^{(+a}$$

$$Gb - H + \frac{I}{b} - \frac{K}{b^2} + \frac{L}{b^3} - \frac{M}{b^{n-r}} = bG^{(+b}$$

$$Gc - H + \frac{I}{c} - \frac{K}{c^2} + \frac{L}{c^3} - \frac{M}{c^{n-r}} = cG^{(+c}$$

&c.

And the sum of these $= aG^{(+a} + bG^{(+b} + cG^{(+c} +$ &c. $=$ (by this notation) $A \times G =$ by the lemma)

$$\left.\begin{array}{l} +a \\ +b \\ +c \\ +\&c. \end{array}\right\} \times G - rH.$$

Compare this last conclusion with that which followed from dividing the proposed equation by x^{n-r}, and substituting for x the roots a, b, c, &c. and you will have

$$\left.\begin{array}{l} \left.\begin{array}{ll} a^r & -a^{r-1} \\ +b^r & -b^{r-1} \\ +c^r & -c^{r-1} \\ +\&c. & -\&c. \end{array}\right\} \times A \left.\begin{array}{l} +a^{r-2} \\ +b^{r-2} \\ +c^{r-2} \\ +\&c. \end{array}\right\} \times C \ldots\ldots \\ \ldots\ldots\ldots\ldots \left\{\begin{array}{l} +a \\ +b \\ +c \\ +\&c. \end{array}\right\} \times G - rH \end{array}\right\} = 0,$$

which was to be demonstrated.

From these two Theorems Sir *Isaac Newton*'s Rule manifestly follows.

But, to illustrate the reasoning here used by some examples: suppose $r = 3$, then we are to take C for H, because three terms only precede C in the equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \&c. = 0$; and we are to prove that

$$\left.\begin{array}{l} a^3 \\ +b^3 \\ +c^3 \\ +d^3 \\ +\,\&c. \end{array}\right\} = \left.\begin{array}{l} a^2 \\ +b^2 \\ +c^2 \\ +d^2 \\ +\,\&c. \end{array}\right\} \times A - \left.\begin{array}{l} -a \\ -b \\ -c \\ -d \\ -\,\&c. \end{array}\right\} \times B + 3C.$$

That this may appear, observe that

$a^3+b^3+c^3+d^3+\&c. = \overline{a^2+b^2+c^2+d^2+\&c.} \times \overline{a+b+c+d+\&c.} - a^2 \times \overline{b+c+d+\&c.} - b^2 \times \overline{a+c+d+\&c.} - c^2 \times \overline{a+b+d+\&c.} - d^2 \times \overline{a+b+c+\&c.} - \&c. =$ (because $A'B' = a \times \overline{ab+ac+ad+\&c.} + b \times \overline{ab+bc+bd+\&c.} + c \times \overline{ac+bc+dc+\&c.} + d \times \overline{ad+bd+cd+\&c.} + \&c.$) $= \overline{a^2+b^2+c^2+d^2+\&c.} \times A - A'B'$ (by the Lemma) $= \overline{a^2+b^2+c^2+d^2+\&c.} \times A - AB + 3C.$

In like manner, $\overline{a^4+b^4+c^4+d^4}+\&c. = \overline{a^3+b^3+c^3+d^3+\&c.} \times \overline{a+b+c+d+\&c.} - \overline{a^2+b^2+c^2+d^2+\&c.} \times \overline{ab+ac+ad+bc+bd+cd+\&c.} + a^2 \times \overline{bc+bd+cd+\&c.} + b^2 \times \overline{ac+ad+cd+\&c.}$

$$+\&c. + c^2 \times \overline{ab+ad+bd+\&c.} + d^2 \times \overline{ab+ac+bc+\&c.} + \&c. = \overline{a^3+b^3+c^3+d^3+\&c.} \times A - \overline{a^2+b^2+c^2+d^2+\&c.} \times B + AC = \overline{a^3+b^3+c^3+d^3+\&c.} \times A - \overline{a^2+b^2+c^2+d^2+\&c.} \times B + \overline{a+b+c+d+\&c.} \times C - 4D.$$

End of the SECOND PART.

A TREATISE OF ALGEBRA.

PART III.

Of the Application of *Algebra* and *Geometry* to each other.

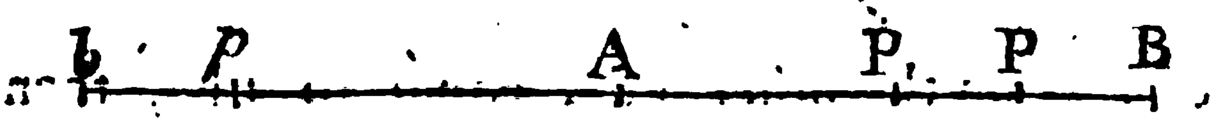
CHAP. I.

Of the Relation between the equations of Curve Lines and the figure of those Curves, in general.

§ 1. IN the two first parts we considered Algebra as independent of Geometry; and demonstrated its operations from its own principles. It remains that we now explain the use of Algebra in the resolution of geometrical problems; or or reaſoning about geometrical figures; and the uſe of geometrical lines and figures in the reſolution of equations. The mutual intercourſe of theſe ſciences has produced many extenſive and beautiful Theories, the chief of which we ſhall endeavour to explain, beginning with the relation betwixt curve lines and their equations.

§ 2. We are now to conſider quantities as repreſented by *lines*; a *known* quantity by a *given* line, and an *unknown* by an *undetermined* line.

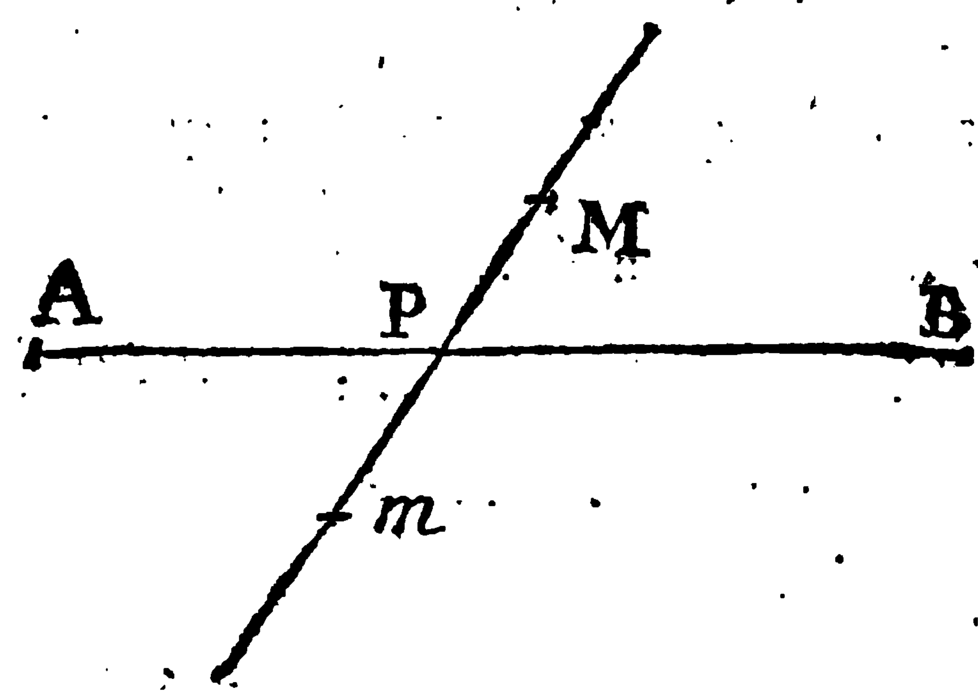
But as it is ſufficient that it be indetermined on one ſide, we may ſuppoſe one extremity to be known.



Thus the line AB, whoſe extremities A and B are both determined, may repreſent a given quantity: while AP, whoſe extremity P is undetermined, may repreſent an undetermined quantity. A leſſer undetermined quantity may be repreſented by AP, taking P nearer to A; and, if you ſuppoſe P to move towards A, then will AP, ſucceſſively, repreſent all quantities leſs than the firſt AP; and after P has coincided with A, if it proceed in the ſame direction to the place *p*, then will A*p* repreſent a negative quantity, if AP was ſuppoſed poſitive.

If AP represent x, and $Ap = AP$, then will Ap represent $-x$; and for the same reason, if AP represent $(+a,)$ then will Ab ($=AB$) represent $(-a)$.

§ 3. After the same manner, if PM represent $+y$, and you take Pm, the continuation



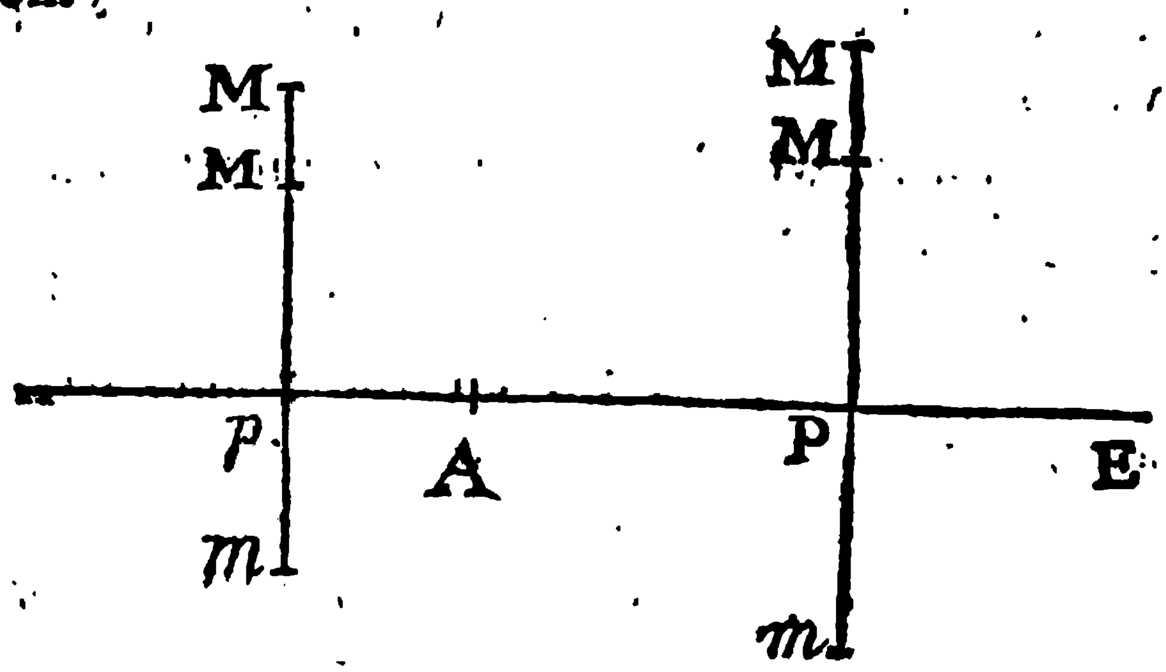
of PM on the other side, equal to PM, then will Pm represent $-y$: for, by supposing M to move towards P, the line PM decreases; when M comes to P, then PM vanishes; and after M has passed P, towards m, it becomes negative.

§ 4. In Algebra, the root of an equation, when it is an impossibe quantity, has its expression; but in Geometry, it has none. In Algebra you obtain a general resolution, and there is an expression, in all cases, of the thing required; only, within certain bounds, that expression represents an *imaginary* quantity, or rather, "*is the symbol of an operation which, in that case, cannot be performed*;" and serves only

only to shew the *genesis* of the quantity, and the *limits* within which it is possible.

In the geometrical resolution of a question, the thing required is exhibited only in those cases when the question admits of a *real* solution; and, beyond those limits, no solution appears. So in finding the intersections of a given circle and a straight line, if you determine them by an equation, you will find two general expressions for the distances of the points of intersection from the perpendicular drawn from the center on the given line. But, geometrically, those intersections will be exhibited only when the distance of the straight line from the center is less than the radius of the given circle.

§ 5. "When in any equation there are two undetermined quantities, x and y, then for each particular value of x, there may be as many values of y as it has dimensions in that equation."



So

So that, if AP (a part of the indefinite line AE) represent x, and the perpendiculars PM represent the corresponding values of y, then there will be as many points (M,) the extremities of these perpendiculars or *ordinates*, as there are dimensions of y in the equation. And the values of PM will be the roots of the equation arising by substituting for x its particular value AP in any case.

From which it appears, how, when an equation is given, you may determine as many of the points M as you please, and draw the line that shall pass through all these points; "which is called the *locus* of the equation."

§ 6. When any equation involving two unknown quatities (x and y) is proposed, then substituting for x any particular value AP, if the equation that arises has all its roots positive, the points M will lie on one side of AE: but if any of them are found negative, then these are to be set off on the other side of AE towards m.

If, for x, which is supposed undetermined, you substitute a negative quantity, as Ap, then you will find the points M, m, as before: and the *locus* is not complete till all the points M, m, are taken in, that it may shew all the values of y corresponding to all the possible values of x.

"If, in any case, one of the values of y vanish,

vanish, then the point M coincides with P, and the *locus* meets with AE in that point."

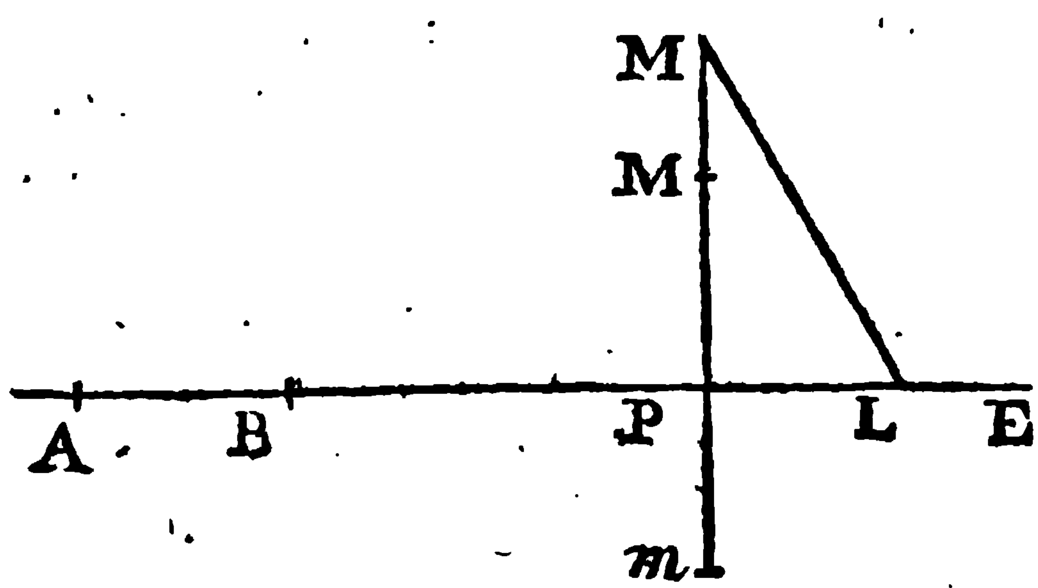
"If one of the values of y becomes infinite, then it shews that the curve has an *infinite arc*: and, in that case, the line PM becomes an *asymptote* to the curve, or touches it at an infinite distance," if AP is itself finite.

"If, when x is supposed infinitely great, a value of y vanish, then the curve approaches to AE produced as an asymptote."

"If any values of y become *impossible*, then so many points M vanish."

§ 7. From what has been said it appears, that when an equation is proposed involving two undetermined quantities (x and y) "there may be as many intersections of the curve that is the *locus* of the equation, and of the line PM as there are dimensions of y in the equation; and as many intersections of the curve and the line AE as there are dimensions of x in the equation."

If you draw any other line LM meeting the



same curve in M, and the line AE in the given

given angle ALM. Suppose LM $= u$, and AL $= z$; "then the equation involving u and z, shall not rise to more dimensions than y and x had in the proposed equation, or, than the sum of their dimensions in any of its terms."

For, since the angles PLM, MPL, PML, are given, it follows that, the sines of these angles being supposed to one another as l, m, n, PM : ML ($y : u$) :: $l : m$; and consequently $y = \frac{lu}{m}$: and that PL : ML :: $n : m$; so that PL $= \frac{nu}{m}$, and $x =$ AP ($=$ AL $-$ PL) $= z - \frac{nu}{m}$. Substitute, for y and x, in the proposed equation these values $\frac{lu}{m}$ and $z - \frac{nu}{m}$, and it is obvious (since u and z are of one dimension only in the values of y and x) that in the equation which will arise, z and u will not have more dimensions than the highest dimension of x and y in the proposed equation, or the highest sum of their dimensions taken together in the terms where they are both found: and consequently, "LM drawn any where in the plane of the curve will not meet it in more points than there are units in the highest dimension of x or y, or in the highest sum of their dimensions, in the terms where both are found." Now the dimension of the equation or curve being denominated from the highest dimension of x or y in it, or from the sum of their dimensions where

where they are moſt; we conclude, that "the number of points in which the curve can meet with any ſtraight line, is equal to the number that expreſſes the dimenſion of the curve.

It appears alſo from this article, how, when an equation of a curve is given expreſſing the relation of the ordinate PM and abſciſſe AP, you may transform it, ſo as to expreſs the relation between any other ordinate ML and the abſciſſe AL, by ſubſtituting for y its value $\frac{lu}{m}$, and for x its value $z-\frac{nu}{m}$.

Or, if you would have the abſciſſe begin at any other point B, ſuppoſing $AB=e$, ſubſtitute for x not $z-\frac{nu}{m}$, but $z-\frac{nu}{m}+e$.

§ 8. Thoſe curve lines that can be deſcribed by the reſolution of equations, the relation of whoſe ordinates PM and abſciſſes AP can be expreſſed by an equation involving nothing but determined quantities beſides theſe ordinates and abſciſſes, are called "*geometrical* or *algebraic* curves."

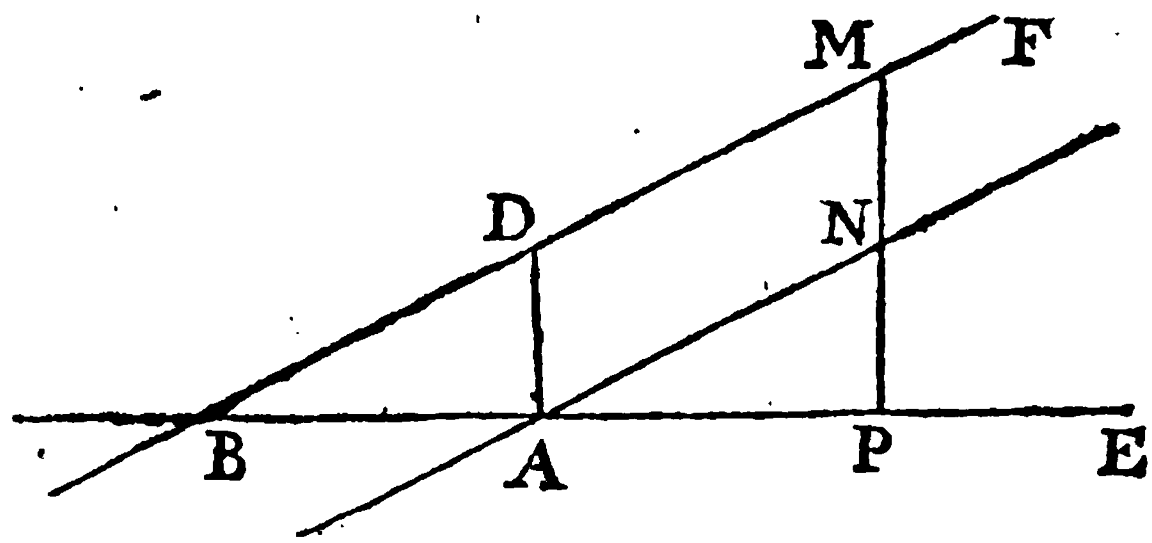
They are divided into *orders* according to the dimenſions of their equations, or number of points in which they can interſect a ſtraight line.

The *ſtraight lines* themſelves conſtitute the *firſt* order of lines; and when the equation expreſſing the relation of x and y is of one dimenſion

dimension only, the points M must be all found in a straight line constituting a given angle with AE.

Suppose, for example, that the equation given is $ay - bx - cd = 0$, and that the *locus* is required.

Since $y = \frac{bx + cd}{a}$, it follows, that, APM being a right angle, if you draw AN making the



angle NAP such that its cosine be to its sine as a to b; and drawing AD parallel to the ordinates PM, and equal to $\frac{cd}{a}$, through D you draw DF parallel to AN, DF will be the locus required. Where you are to take AD on the same side of the line AE, with PN, if bx and cd have the same sign, but on the contrary side of AE if they have contrary signs.

§ 9. Those curves whose equations are of *two* dimensions constitue the *second order* of lines, and the *first kind* of *curves*. Their intersections

tersections with a straight line can never exceed two, by § 7.

The curves whose equations are of *three* dimensions form the *third order* of lines, or *second kind* of *curves*: and their intersections with a straight line can never exceed Three. And, after the same manner, the curves are determined that belong to the *higher* orders, to infinity.

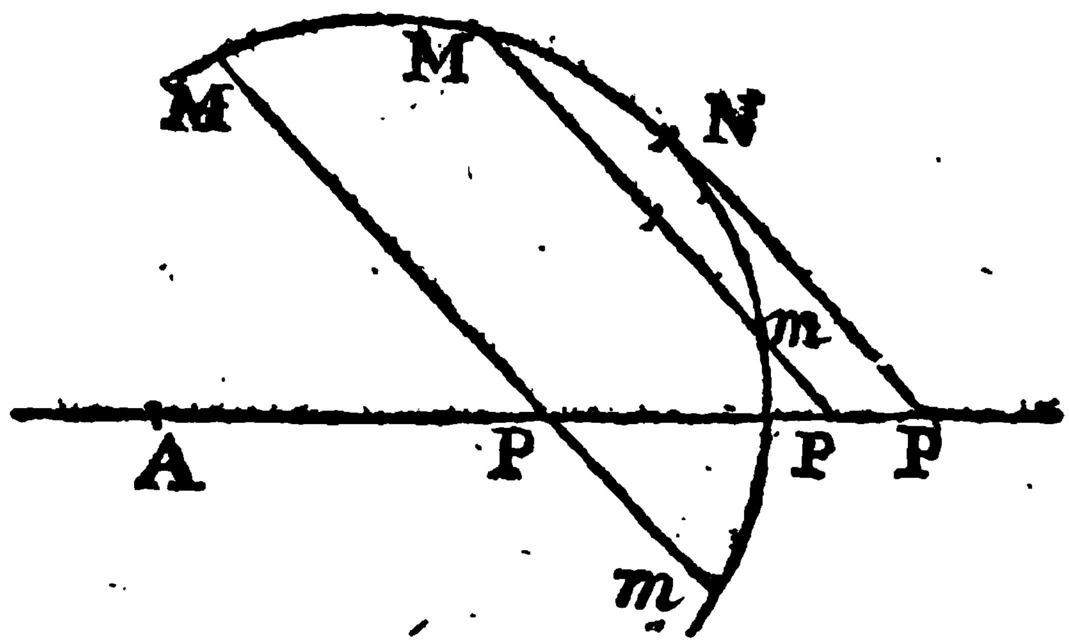
Some curves, if they were completely described, could cut a straight line in an infinite number of points; but these belong to none of the orders we have mentioned; they are not geometrical or algebraic curves, for the relation betwixt their ordinates and abscisses cannot be expressed by a finite equation involving only ordinates and abscisses with determined quantities.

§ 10. As "the roots of an equation become impossible always in pairs, so the intersections of the curve and its ordinate PM must vanish in pairs," if any of them vanish.

Let PM cut the curve in the points M and *m*, and by moving parallel to itself come to touch it in the point N; then the two points of intersection, M and *m*, go into one point of contact N. If PM still move on parallel to itself, the points of intersection will, beyond

N,

N, become imaginary; as the two roots of



an equation first become equal and then imaginary.

§ 11. The curves of the 3d, 5th, 7th orders, and all whose dimensions are odd numbers, must have, at least, two infinite arcs; since equations whose dimensions are odd numbers have always one real root *at least*; and consequently, for every value of x, the equation by which y is determined must, at least, have one real root: so that as x (or AP may be increased *in infinitum* on both sides, it follows that M must go off *in infinitum* on both sides, without limit.

Whereas, in the curves whose dimensions are even numbers, as the roots of their equations may become all impossible, it follows that the figure of the curve may be like a *circle* or *oval* that

that is limited within certain bounds, beyond which it cannot extend.

§ 12. When two roots of the equation by which y is determined become equal, either "the ordinate PM touches the curve," two points of intersection, in that case, going into a point of contact; or, "the point M is a *punctum duplex* in the curve;" two of its arcs intersecting each other there: or, "some oval that belongs to that kind of curve becoming infinitely little in M, it vanishes into what is called a *punctum conjugatum*."

If, in the equation, y be supposed $= 0$, then "the roots of the equation by which x is determined, will give the distances of the points where the curve meets AE from A." And, if two of those roots be found equal, then either "the curve *touches* the line AE;" or, "AE passes through a *punctum duplex* in the curve." When y is supposed $= 0$, if one of the values of x vanish, the curve, in that case, passes through A." If two vanish, then either "AE touches the curve in A;" or, "A is a *punctum duplex*."

"As a *punctum duplex* is determined from the equality of two roots, so is a *punctum triplex* determined from the equality of *three* roots.

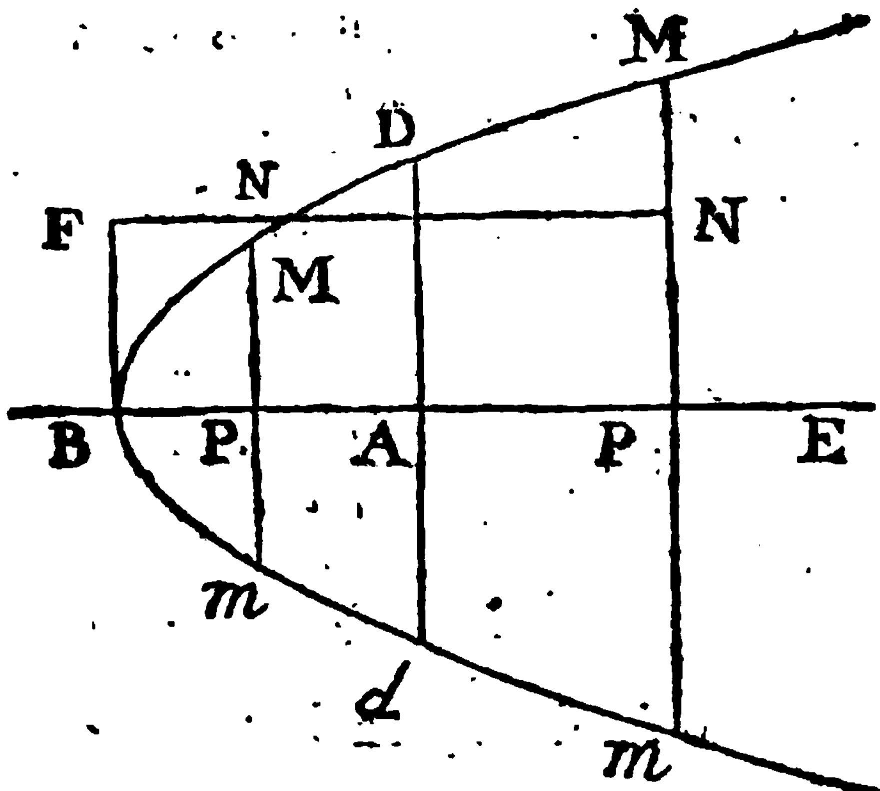
§ 13. A few examples will make these observations very plain. Suppose it is required to describe the line that is the *locus* of this equation, $y^2 = ax + ab$, or $y^2 - ax - ab = 0$. Since $y = \pm\sqrt{ax + ab}$, and since a and b, are given in variable quantities, if you assume AP ($=x$) of a known value, it will be easy to find $\sqrt{ax + ab}$; and setting off PM on one side equal to $\sqrt{ax + ab}$, and Pm on the other equal to PM, the points M and m will belong to the locus required. And for every positive value of AP you will thus obtain a point of the locus on each side. The greater AP ($= x$) is taken, the greater does the $\sqrt{ax + ab}$ become, and consequently PM and Pm become the greater.

If AP be supposed infinitely great, PM and Pm will also become infinitely great; and consequently the locus has two infinite arcs that go off to an infinite distance from AE and from AD. If you suppose x to vanish, $y = \pm\sqrt{ab}$; so that y does not vanish in that case but passes through D and d, taking AD and Ad $= \sqrt{ab}$ a mean proportional betwixt a and b.

If you now suppose that the point P moves to the other side of A, then you must, in the equation, suppose x to become negative, and $y = \pm\sqrt{ab - ax}$; so that y will have two values as before, while x is less than b. But if AB $= b$, and you suppose the point P to come to B, then $ab = ax$, and $y = \pm\sqrt{ab - ax} = 0$. That is, PM and P$m$ vanish; and the curve there meets the line AE. If you suppose P to move from

A

A beyond B, than x becomes greater than b, and ax greater than ab, so that $ab - ax$ being



negative, $\sqrt{ab - ax}$ becomes imaginary, and the two values of y become imaginary; that is, beyond B there are no ordinates that meet the curve, and consequently, on that side, the curve is limited in B.

All this agrees very well with what is known by other methods, that the curve whose equation is $y^2 = ax + ab$, is a parabola whose vertex is B, axis BE, and parameter equal to a. For since BP $= b \pm x$, and PM $= y$, if BF be equal to a; then the rectangle BN ($= ab \pm ax$) will be equal to PMq ($= y^2$;) which is the known property of the *parabola*. And it is obvious, that the figure of the parabola is such as we

 have

have determined this locus to be from the consideration of its equation.

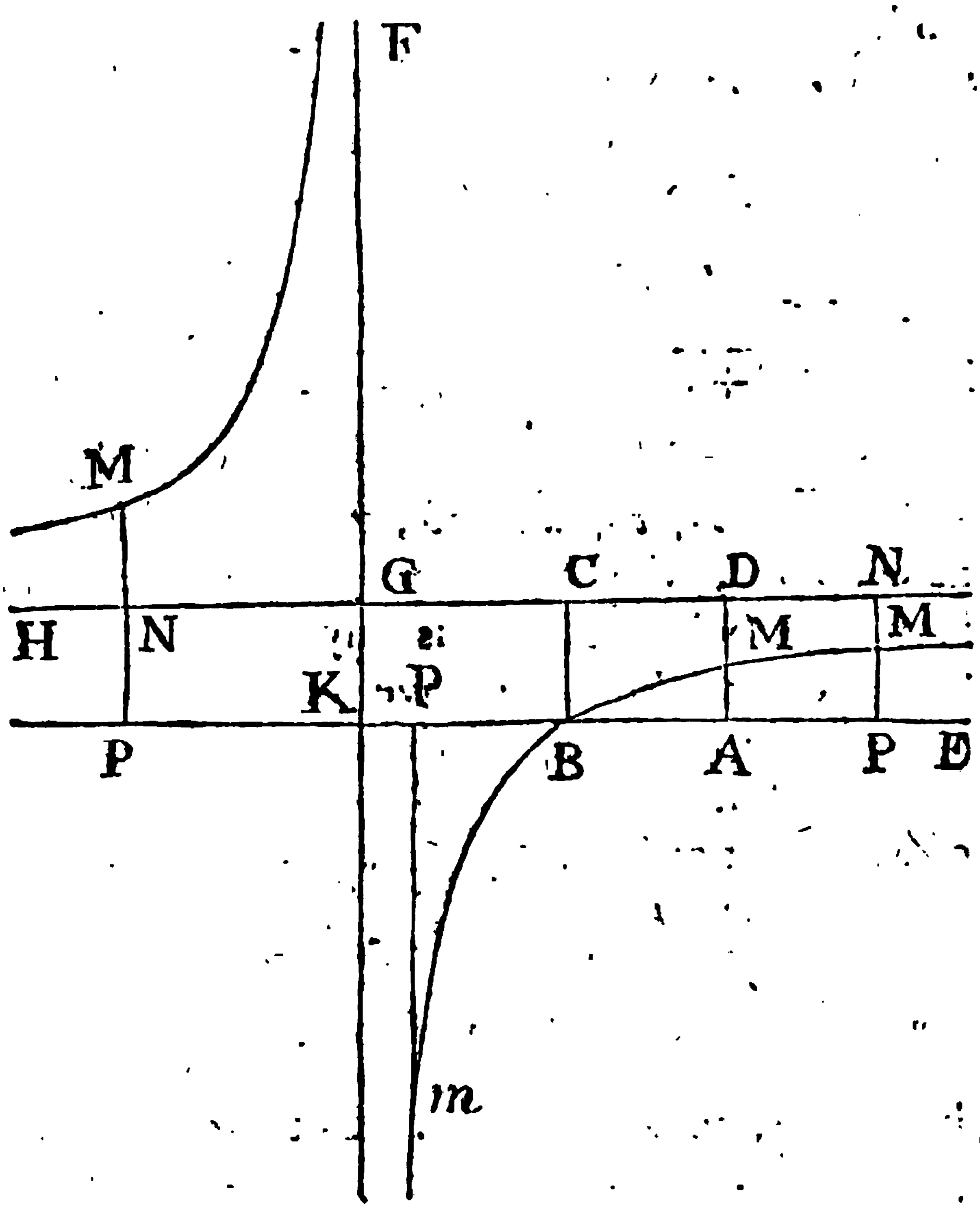
§ 14. Let it be required to describe the line that is the *locus* of this equation, $xy + ay + cy = bc + bx$, or $y = \frac{bc + bx}{a + c + x}$.

Here, it is plain, the ordinate PM can meet the curve in one point only, there being but one value of y corresponding to each value of x. When $x = 0$, then $y = \frac{bc}{a + c}$, so that the curve does not pass through A. If x be supposed to increase, then y will increase, but will never become equal to b, since $y = b \times \frac{c + x}{a + c + x}$, and $a + c + x$ is always greater than $c + x$. If x be supposed infinite, then the terms a and c vanish compared with x, and consequently $y = b \times \frac{x}{x} = b$; from which it appears, that taking AD $= b$, and drawing GD parallel to AE, it will be an *asymptote*, and touch the curve at an infinite distance.

If x be now supposed negative, and AP be taken on the other side of A, then shall $y = b \times \frac{c - x}{a + c - x}$; and if x be taken, on that side, $= c$, then shall $y = b \times \frac{c - c}{a} = 0$; so that the curve must pass through B, if AB $= c$.

If x be supposed greater than c, then will $c - x$ become negative, and the ordinate will become negative

negative and lie on the other side of AE, till x becomes equal to $a+c$, and then $y=b\times\frac{-a}{0}$,



or *infinite*; so that if AK be taken $=a+c$, the ordinate KL will be an *asymptote* to the curve.

If x be taken greater than $a+c$, or AP greater than AK, then both $c-x$ and $a+c-x$ become negative; and consequently $y\ (=b\times\frac{x-c}{x-a-c})$ becomes positive; and since $x-c$ is always greater than $x - a - c$, it follows that y will be always greater than b or KG, and consequently the rest of the curve lies in the angle FGH. And, as x increases, since the ratio of $x - c$ to $x - a - c$ approaches still nearer to a ratio of equality, it follows that PM approaches to an equality with PN, and the curve to its asymptote GH on that side also.

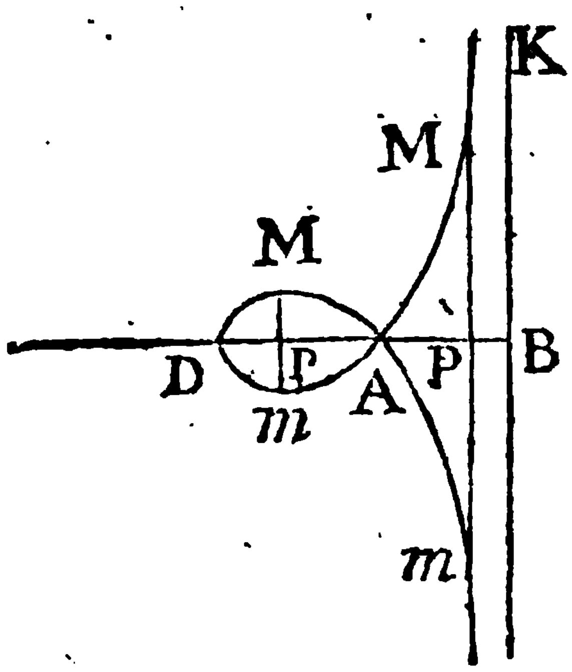
This curve is the common *hyperbola*; for since $b \times \overline{c + x} = y \times \overline{a + c + x}$, by adding ab to both sides $b \times \overline{a + c + x} = y \times \overline{a + c + x} + ab$; and $\overline{b - y} \times \overline{a + c + x} = ab$; that is, NM × GN = GC × BC, which is the property of the common hyperbola. And it is easy to see how the figure of the *locus* we have been considering agrees with the figure of the hyperbola.

§ 15. Let it be required to describe the *locus* of the equation $cy^2 - xy^2 = x^3 + bx^2$. Where since $y^2 = \frac{x^3 + bx^2}{c - x}$ and $y = \pm\sqrt{\frac{x^3 + bx^2}{c - x}}$, it follows that PM and Pm must be taken equal, on both sides, to $\sqrt{\frac{x^3 + bx^2}{c - x}}$. But that when x is taken equal to c, if AB $= c$, and BK be perpendicular to AB, then BK must be an *asymptote* to the curve. If x be supposed greater than c, or AP greater than AB, then $c - x$ being negative, the fraction $\frac{x^3 + bx^2}{c - x}$ will become negative, and its square root impossible. So that no part of of the *locus* can be found beyond B. If x be supposed negative, or P taken on the other side of A, then $y = \pm\sqrt{\frac{-x^3 + bx^2}{c + x}}$, the sign of x^3 and x being changed, but not the sign of bx^2; because the square of a negative is the same as the square of a positive, but its cube is negative: while x is less than b, the values of y will be real and equal; but if $x = b$, then the values of y vanish, because, in that case,

$$y = \pm\sqrt{\frac{-x^3 + bx^2}{c - x}} = \sqrt{\frac{-b^3 + b^3}{c - x}} = 0;$$ and consequently, if AD be taken $= b$, the curve will pass through D, and there touch the ordinate.

If x be taken greater than b then $\pm\sqrt{\frac{-x^3 + bx^2}{c + x}}$ will become *imaginary*, so that no part of the curve is found beyond D.

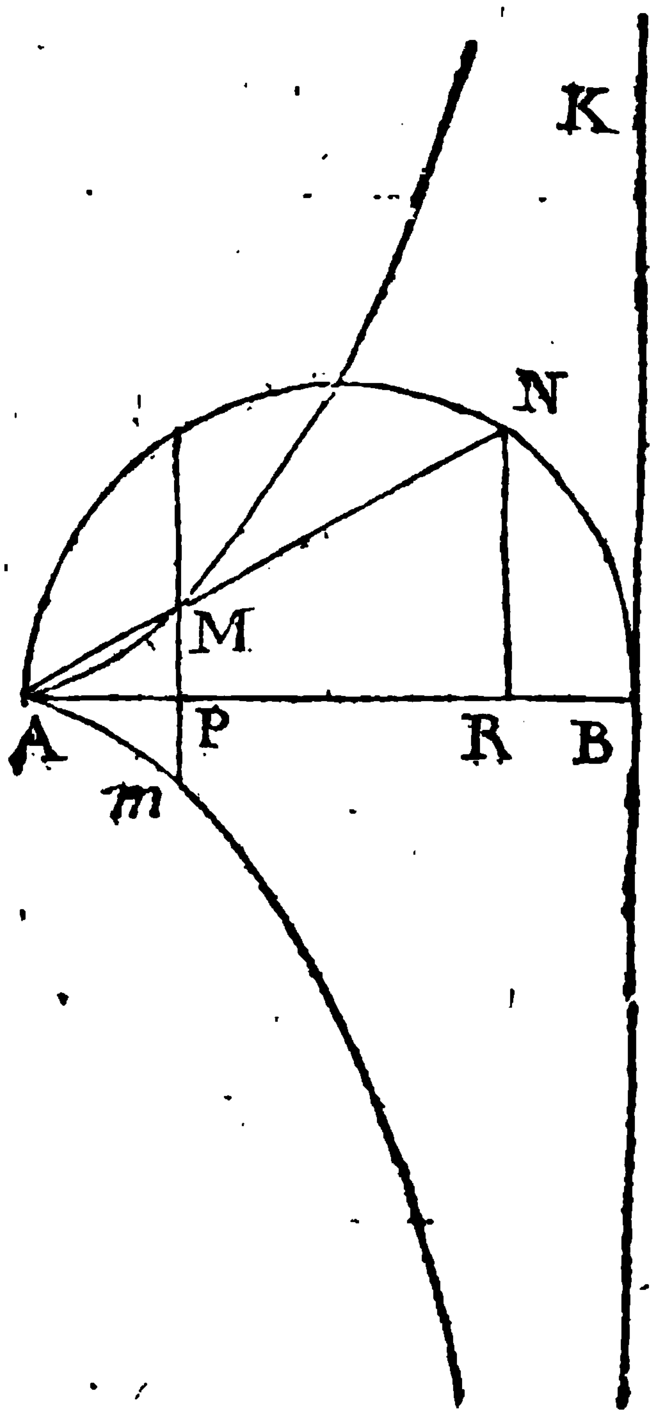
If you suppose $y = 0$, then will $x^3 + bx^2 = 0$



 be

be an equation whose roots are $-b$, 0, 0, from which it appears that the curve passes twice through the point A, and has, in A, a *punctum duplex*. This *locus* is a line of the third order, BK is its asymptote, and it has a *nodus* betwixt A and D.

If you suppose b to vanish in the equation, so that $cy^2 - xy^2 = x^3$, then will A and D coin-

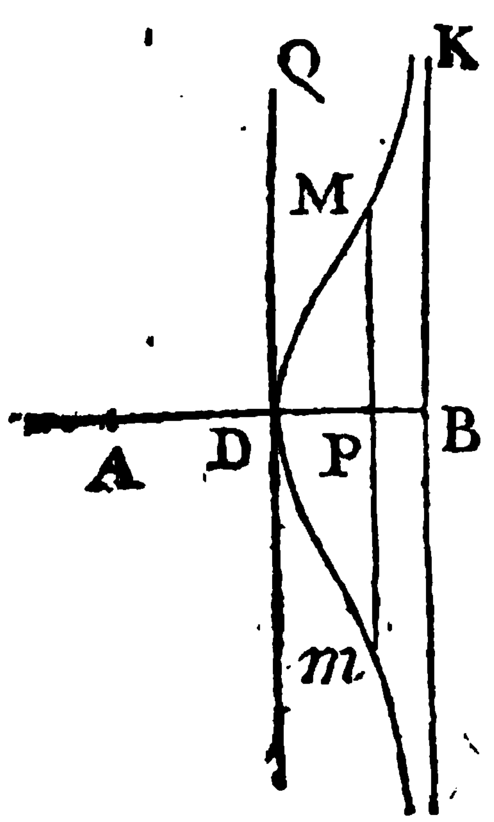


cide, and the *nodus* vanish, and the curve will have in the point A a *cuspis*, the two arcs AM and

and Am touching one another in that point. And this is the same curve which by the ancients was called the *cissoid* of *Diocles*, the line AB being the diameter of the generating circle, and BK the asymptote.

For, if BR be equal to AP, and the ordinate RN be raised meeting the circle in N, and AN be drawn, it will cut the perpendicular PM in M a point of the cissoid. So that if M be a point in the cissoid, AP : PM :: AR : RN :: $\sqrt{AR} : \sqrt{BR} :: \sqrt{BP} : \sqrt{AP}$, and consequently $BP \times PMq = AP$ cub. that is, $\overline{c - x} \times y^2 = x^3$: which is the equation the *locus* of which was required.

If, instead of supposing b positive, or equal to nothing, we now suppose it negative, the equation will be $cy^2 - xy^2 = x^3 - bx^2$, the curve will pass through D, as before, and taking



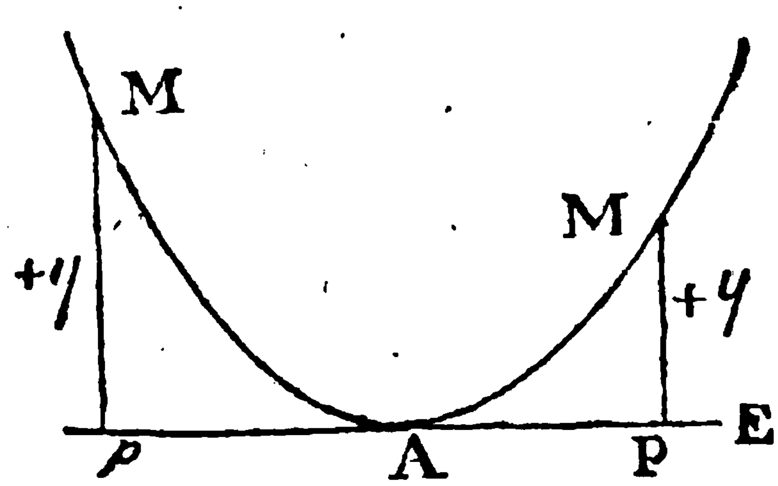
$AB = c$, BK will be its asymptote: it will have a *punctum conjugatum* in A, because when y vanishes, two values of x vanish, and the third becomes equal to b or AD. The whole curve, besides this *point* A, lies between DQ and BK. These are demonstrated after the same manner as in the first case.

§ 16.

§ 16. If an equation is propoſed, as $y = ax^n + bx^{n-1} + cx^{n-2}$, &c. and n is an even number, then will the *locus* of the equation have two infinite arcs lying on the ſame ſide of AE. For, if x become infinite, whether poſitive or negative, x^n will be poſitive, and ax^n have the ſame ſign in either caſe; and as ax^n becomes infinitely greater than the other terms bx^{n-1}, cx^{n-2}, &c. it follows that the infinite values of y will have the ſame ſign in theſe caſes; and conſequently, the two infinite arcs of the curve will lie on the ſame ſide of AE.

But if n be an odd number, then when x is negative, x^n will be negative, and ax^n will have the contrary ſign to what it has when x is poſitive; and therefore the two infinite arcs, in this caſe, will lie on different ſides of AE, and tend towards parts directly oppoſite.

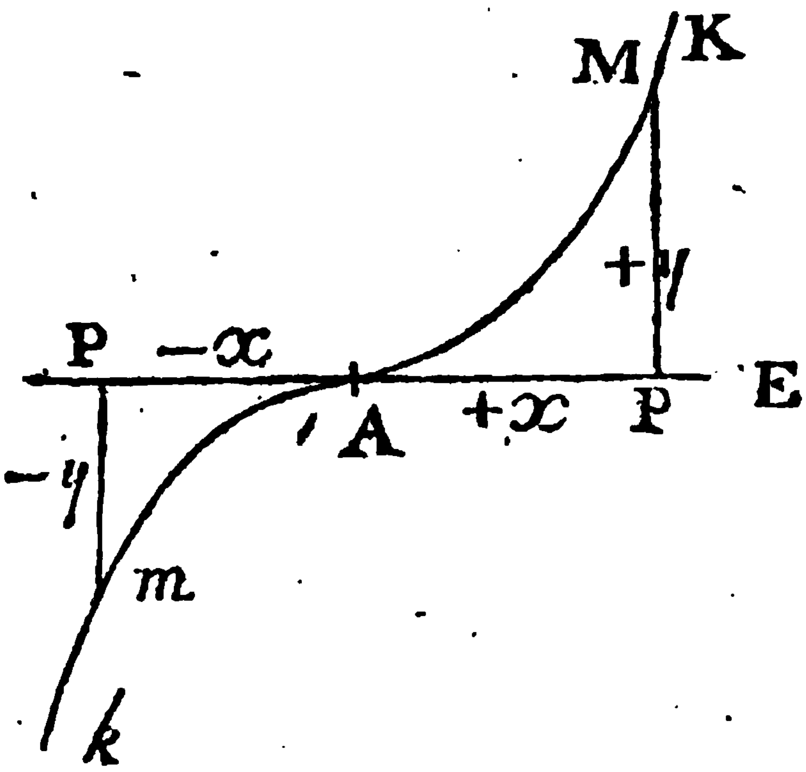
Thus the locus of the equation $ay = x^2$ is the parabola. A is the vertex, AE is the tan-



gent at the vertex; and the two infinite arcs lie manifeſtly on the ſame ſide of AE.

But

But the *locus* of the equation $a^2y = x^3$, where the index of x is an odd number, has its two

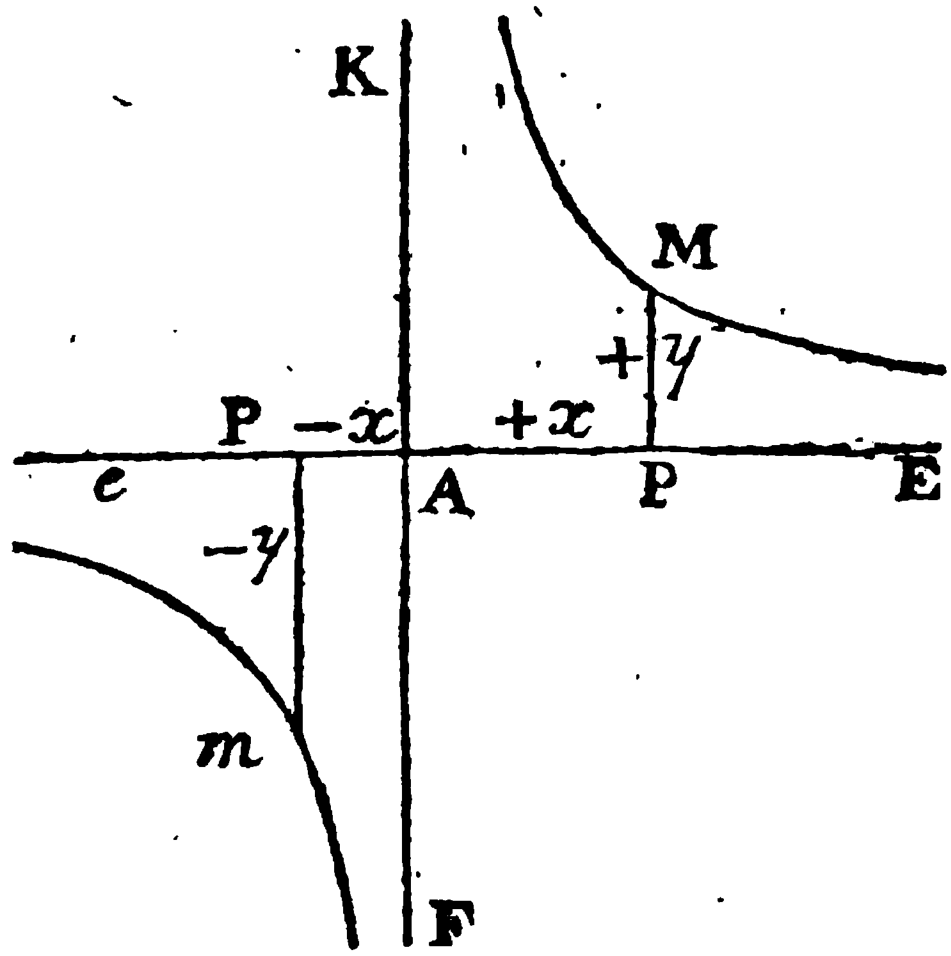


arcs on different sides of AE, tending towards opposite parts, as AMK, and A*mk*. This curve is called the *cubical parabola*, and is a line of the third order.

The *locus* of the equation $a^3y = x^4$ is of a figure like the common parabola; and "all those loci, in whose equations y is of one dimension, x of an even number of dimensions: But those *loci* are like the cubical parabola, in whose equations y is of one dimension only, and x of an *odd* number of dimensions." And this Rule is even true of the locus of the equation $y = x$, which is a straight line cutting AE in an angle of 45°; which manifestly goes off as the cubical parabola does to infinity, towards opposite parts, and on different sides of AE.

§ 17.

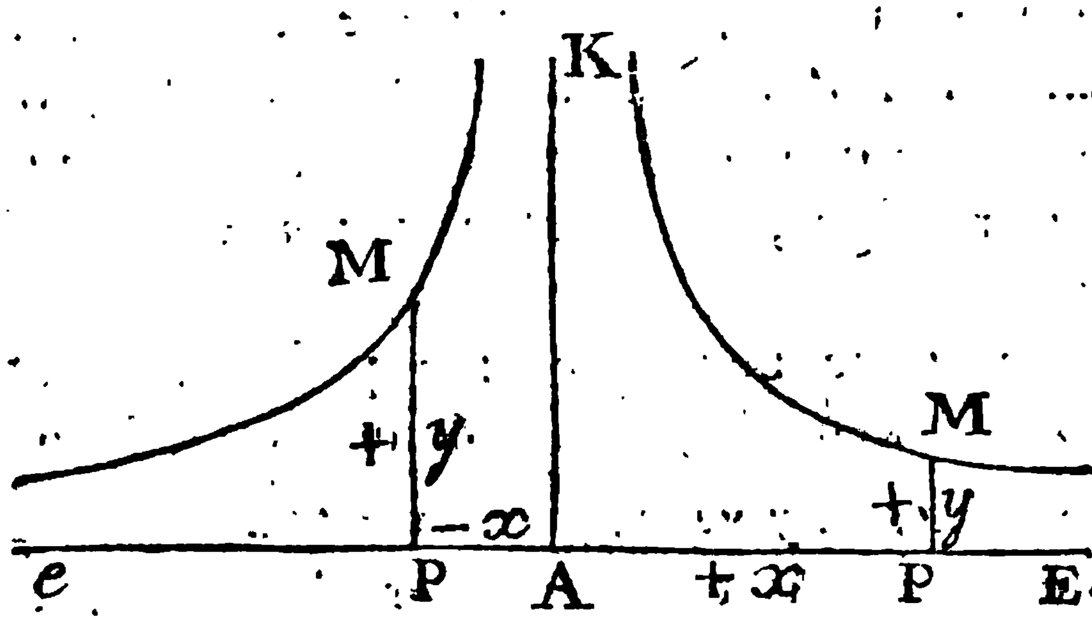
§ 17. If the *locus* of the equation $yx^n = a^{n+1}$ is required.



If n is an *odd* number, then when x is positive, $y = \frac{a^{n+1}}{x^n}$; but when x is negative, then $y = -\frac{a^{n+1}}{x^n}$; so that this curve must all lie in the vertically *opposite* angles KAE, FA*e*, (as the common *hyperbola*:) FK, E*e*, being asymptotes.

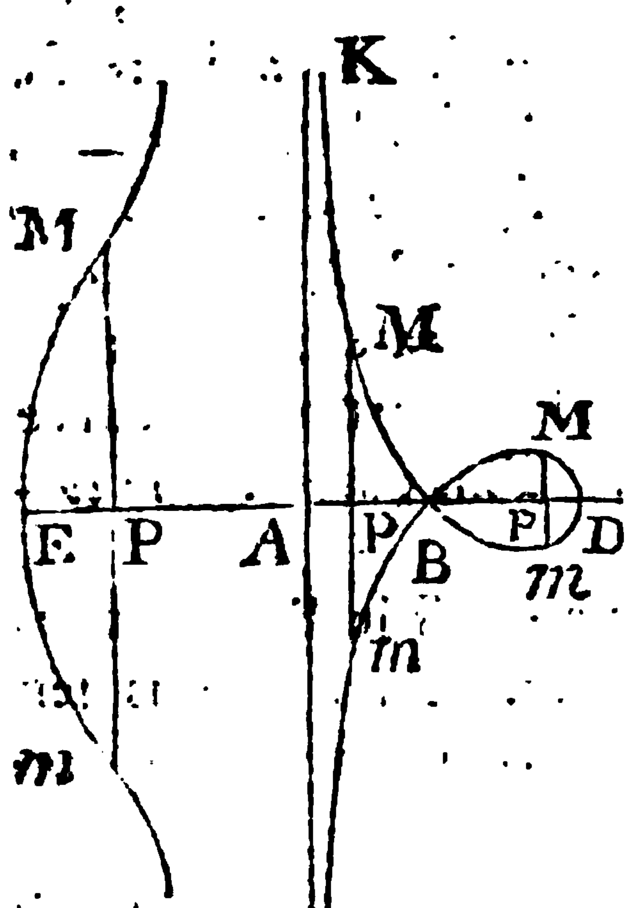
But if n is an *even* number, then y is always positive, whether x be positive or negative, because x^n, in this case, is always positive; and there-

therefore the curve must all lie in the two *ad-*



jacent angles KAE and KA*e*, and have AK and AE for its two asymptotes.

§ 18. Let the equation given be $\overline{a^2 - x^2} \times \overline{x-b}^2 = x^2y^2$; so that $y = \pm\sqrt{a^2 - x^2} \times \frac{x-b}{x}$.



If $x = 0$, then y becomes infinite, and therefore the ordinate at A is an *asymptote* to the curve. If AB $= b$, and P be taken betwixt A and B, then shall PM and P*m* be equal, and lie on different sides of the absciffe AP. If $x = b$, then the two values of y vanish, because

because $x - b = 0$; and consequently, the curve passes through B, and has there a *punctum duplex*. If AP be taken greater than AB, then shall there be two values of y, as before, having contrary signs, that value which was positive before being now become negative, and the negative value being become positive. But if AD be taken $= a$, and P comes to D, then the two values of y vanish, because $\sqrt{a^2 - x^2} = 0$. And if AP is taken greater than AD, then $a^2 - x^2$ becomes negative, and the value of y *impossible*: and therefore, the curve does not go beyond D.

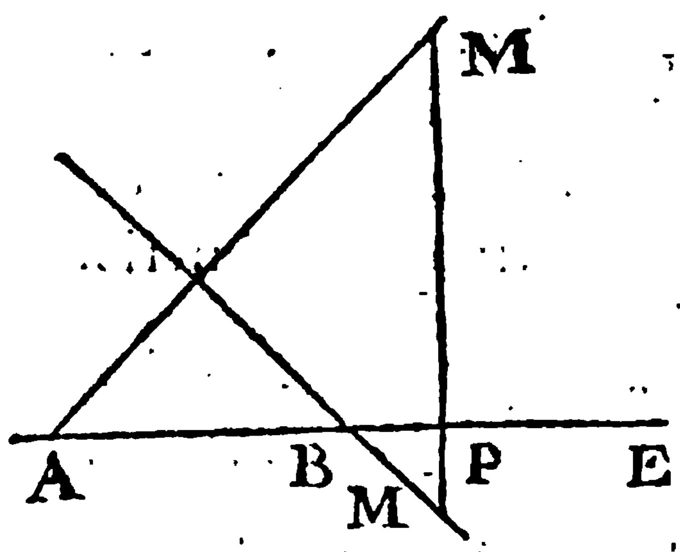
If x now be supposed negative, we shall find $y = \pm\sqrt{a^2 - x^2} \times \overline{b + x} \div x$. If x vanish, both these values of y become infinite, and consequently, the curve has two infinite arcs, on each side of the *asymptote* AK. If x increase, it is plain y diminishes, and if x becomes $= a$, y vanishes, and consequently the curve passes through E, if AE be taken $=$ AD, on the opposite side. If x be supposed greater than a, then y becomes *impossible*; and no part of the curve can be found beyond E. This curve is the *conchoid* of the ancients.

If $a = b$, it will have a *cuspis* in B, the nodus betwixt B and D vanishing. And if a is less than b, the point B will become a *punctum conjugatum*.

From

From what has been ſaid an error may be corrected of an Author in the *Memoirs de l'Acad. Royale des Sciences*, who gives this curve no infinite arcs, but only a *double nodus*. Some other errors of the ſame kind may be corrected in that Treatiſe, from what we have ſaid.

§ 19. If the propoſed equation can be reſolved into two equations of lower dimenſions, without affecting either y or x with any radical ſign, then the *locus* ſhall conſiſt of the two *loci* of thoſe inferior equations. Thus the locus of the equation $y^2 - 2xy + by + x^2 - bx = 0$ is found to be two ſtraight lines cutting the ab-



ſciſſe AE in angles of 45°, in the points A and B, whoſe diſtance AB $= b$, becauſe that equation is reſolved into theſe two $y - x = 0$, and $y - x + b = 0$.

After the ſame manner, ſome *cubic* equations can be reſolved into three ſimple equations, and then the *locus* is *three* ſtraight lines; or may be reſolved into a *quadratic* and *ſimple* equation,

equation, and then the *locus* is a *conic section* and a a *straight line*.

In general, "the curves of the superior orders include all the curves of the inferior orders; and whatever is demonstrated generally of any one order, is also true of the inferior orders." So, for example, any general property of the conic sections hold true of two straight lines as well as of a conic section: Particularly that "the rectangles of the segments of parallels bounded by them, will be always to one another in a given ratio." The general properties of the lines of the third *order* are true of three straight lines, or of any one straight line and a conic section. And, as the general properties of the higher orders of lines descend also to those of the inferior orders, so there is scarce any property of the inferior orders, but has an analogy to some property of the higher orders; of which it is but a particular case or instance. And hence, the properties of the inferior orders lead to the discovery of those of the superior orders*.

§ 20. We have shewed how to judge of the figure of a *locus* from the consideration of its equation. And when a *locus* is to be described exactly, for every value of x you must, by the resolution of equations, according to the Rules

* See the APPENDIX.

in

in *Part* II. find the correſponding values of y, and determine from theſe values the points of the *locus*.

But there are geometrical *conſtructions* by which the roots of equations can be determined more commodiouſly for this purpoſe. And, as by theſe conſtructions we deſcribe the *loci* of the equations, ſo reciprocally when *loci* are deſcribed, they are uſeful in determining the roots of equations; both which ſhall be explained in the following *Chapter*. Then we ſhall give an account of the moſt general and ſimple methods of deſcribing theſe *loci* by the mechanical motion of angles and lines, whoſe interſections trace the curve; or of conſtructing them by finding geometrically any number of their points.

CHAP. II.

Of the Construction of Quadratic Equations; and of the Properties of the Lines of the second order.

§ 21. THE general equation expressing the nature of the lines of the second order, having all its terms and coefficients, will be of this form;

$$\left.\begin{array}{l} y^2 + axy + cx^2 \\ \quad + by + dx \\ \qquad\quad + e \end{array}\right\} = 0.$$

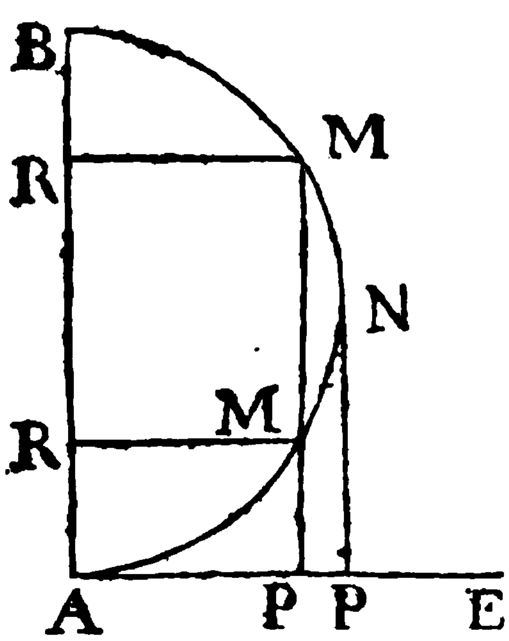
Where a, b, c, d, e, represent any given quantities with their proper signs prefixed to them.

If a quadratic equation is given, as $y^2 + py + q = 0$, and, by comparing it with the preceding, if you take the quantities a, b, c, d, e, and x such that $ax^2 + b = p$, and $cx + dx + e = q$, then will the values of y in the first equation be equal to the values of it in the second; and if the *locus* be described belonging to the first equation, the two values of the ordinate when $ax + b = p$ and $cx^2 + dx + e = q$, will be the two roots of the equation $y^2 + py + q = 0$.

And as *four* of the given quantities a, b, c, d, e, may be taken at pleasure, and the *fifth*, with

with the abſciſſe x, determined, ſo that $ax+b$ may be ſtill equal to p, and $cx^2+dx+e=q$; hence there are innumerable ways of conſtructing the ſame equation. But thoſe *loci* are to be preferred which are deſcribed moſt eaſily; and therefore, the *circle*, of all conic ſections, is to be preferred for the reſolution of quadratic equations.

§ 22. Let AB be perpendicular to AE, and upon AB deſcribe the ſemicircle BMMA. If AP be ſuppoſed equal to x, AB $=a$, and PM $=y$, then making MR, MR, perpendiculars to the diameter AB, ſince AR $\times$ RB $=$ RMq, and AR $=y$, RB $=a-y$, RM $=x$, it follows that $\overline{a-y}\times y=x^2$, and $y^2-ay+x^2=0$. And, if an equation $y^2-py+q=0$, be propoſed to be reſolved, its roots will be the ordinate to the circle, PM and PM, to its tangent AE, if $a=p$, and $x^2=q$: becauſe then the equation of the circle $y^2-ay+x^2=0$, will be changed into the propoſed equation $y^2-py+q=0$.



We have therefore this conſtruction for finding the roots of the quadratic equation $y^2-py+q=0$; take AB $=p$, and on AB deſcribe a

 ſemi-

ſemicircle; then raiſe AE perpendicular to AB, and on it take $AP = \sqrt{q}$, that is, a mean proportional between 1 and q (*by* 13 *El.* 6.) then draw PM parallel to AB, meeting the ſemicircle in M, M, and the lines PM, PM ſhall be the roots of the propoſed equation.

It appears from the conſtruction that if $q = \frac{p^2}{4}$, or $\sqrt{q} = \frac{1}{2}p$, then $AP = \frac{1}{2}AB$, and the ordinate PN touches the curve in N, the two roots PM, PM, in that caſe, becoming equal to one another and PN.

If AP be taken greater than $\frac{1}{2}AB$, that is, when $\sqrt{q}$ is greater than $\frac{1}{2}p$, or q greater than $\frac{1}{4}p^2$, the ordinates do not meet the circle, and the roots of the equation become *imaginary*: as we demonſtrated, in another manner, in *Part* II.

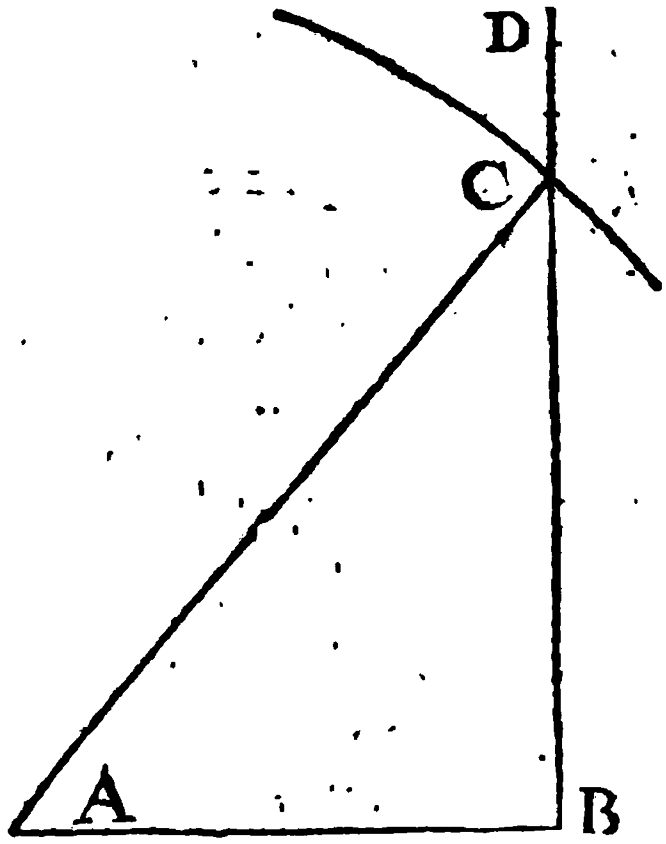
§ 23. The roots of the ſame equation may be otherwiſe thus determined.

Take $AB = \sqrt{q}$, and raiſe BD perpendicular to AB; from A as a center with radius equal to $\frac{1}{2}p$, deſcribe a circle meeting BD in C, then the two roots of the equation $y^2 - py + q = 0$, ſhall be $AC + CB$, and $AC - CB$.

For theſe roots are $\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, and $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$; and $AC = \frac{1}{2}p$, $CB = \sqrt{AC^2 - CB^2} = \sqrt{\frac{1}{4}p^2 - q}$, and conſequently theſe roots are $AC \pm CB$.

The

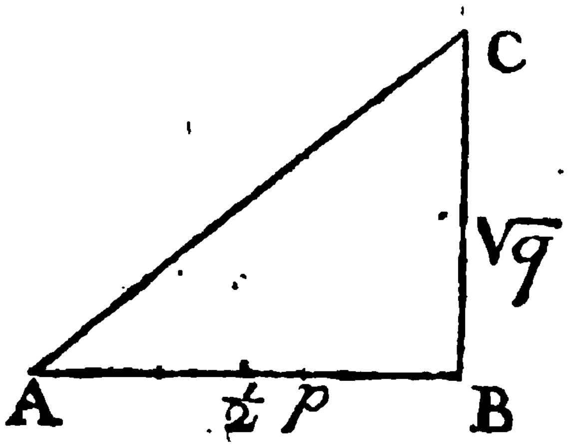
The roots of the equation $y^2 + py + q = 0$ are



$- AC \pm CB$; as is demonſtrated in the ſame manner.

§ 24. The roots of the equation $y^2 - py - q = 0$ are determined by this conſtruction.

Take $AB = \frac{1}{2}p$, $BC = \sqrt{q}$, draw AC; and the two roots ſhall be $AB \pm AC$. If the ſecond



cond term is poſitive, then the roots ſhall be $- AB \pm AC$.

And all quadratic equations being reducible to these four forms,

$$y^2 - py + q = 0,$$
$$y^2 + py - q = 0,$$
$$y^2 - py - q = 0,$$
$$y^2 + py + q = 0,$$

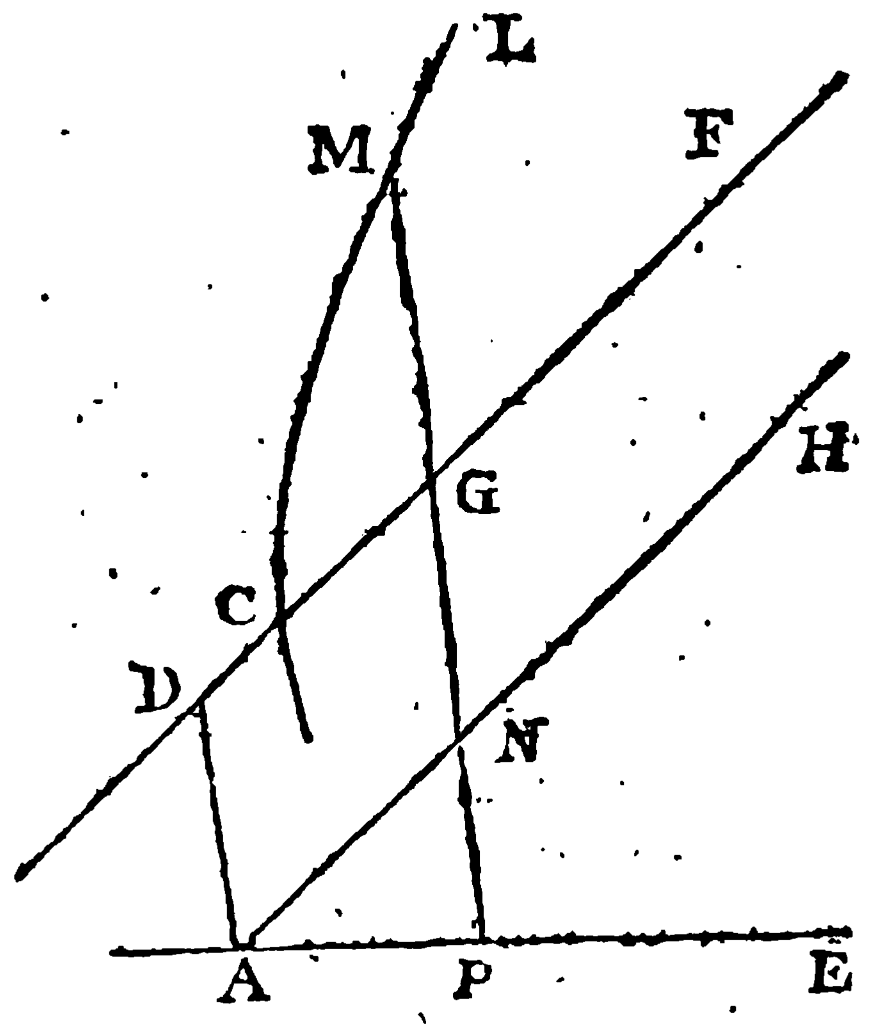
it follows, that they may be all constructed by this and the last two articles.

§ 25. By these geometrical constructions, the locus of any equation of two dimensions may be described; since, by their means, the values of y that correspond to any given value of x may be determined. But if we demonstrate that these *loci* are always *conic sections*, then they may more easily be described by the methods that are already known for describing these curves.

In order to prove this, we shall enquire what equations belong to the different *conic sections*; and, as it will appear that there is no equation of two dimensions but must belong to one or other of them, it will follow that they are *loci* of all equations of two dimensions.

§ 26. Let CML be a *parabola*; AE any line drawn in the same plane; and let it be required to find the equation expressing the relation betwixt the ordinate PM forming any given angle with AE, and the absciss AP begin-

beginning at A any given point in the line AE.



Let CF be the diameter of the parabola whose ordinates are parallel to PM. Draw AH parallel to CF meeting PM in N; and AD parallel to PM meeting CF in D. Because the angles HAE, APN, ANP, are given the lines AP, PN, AN, will be in a given ratio to each other: suppose them to be always as a, b, c; let AD $= d$, DC $= e$; and seeing AP $(= x)$: PN :: $a : b$, PN $= \frac{b}{a}x$; likewise AP : AN :: $a : c$, or AN $= \frac{c}{a}x$. And GM $=$ PM $-$ PN $-$ NG $= y - \frac{b}{a}x - d$. But CG $=$ DG $-$

 DC

$DC = AN - DC = \frac{c}{a}x - e$. If now the *parameter* of the diameter CF be called p, then, from the nature of the parabola, $p \times CG = GMq$: and consequently, $p \times \overline{\frac{c}{a}x - e} = \overline{y - \frac{b}{a}x - d}|^2$, from which this equation follows,

$$y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \left.\begin{matrix} \frac{2bd}{a} \\ -\frac{pc}{a} \end{matrix}\right\} x \left.\begin{matrix} + d^2 \\ + pe \end{matrix}\right\} = 0.$$

Whence, if any equation is proposed, and such values of a, b, c, d, e, p can be assumed as to make *that* equation and *this* coincide, then the *locus* of that equation will be a *parabola*. The construction of which may be deduced from this article.

§ 27. In this general equation for the parabola, the coefficient of x^2 is the square of half the coefficient of xy; and, "when any equation is proposed that has this property, the *locus* of it is a *parabola*." For, whatever coefficients affect the three last terms, they may be made to agree with the coefficients of the last terms of the general equation, by assuming proper values of p, c, and e.

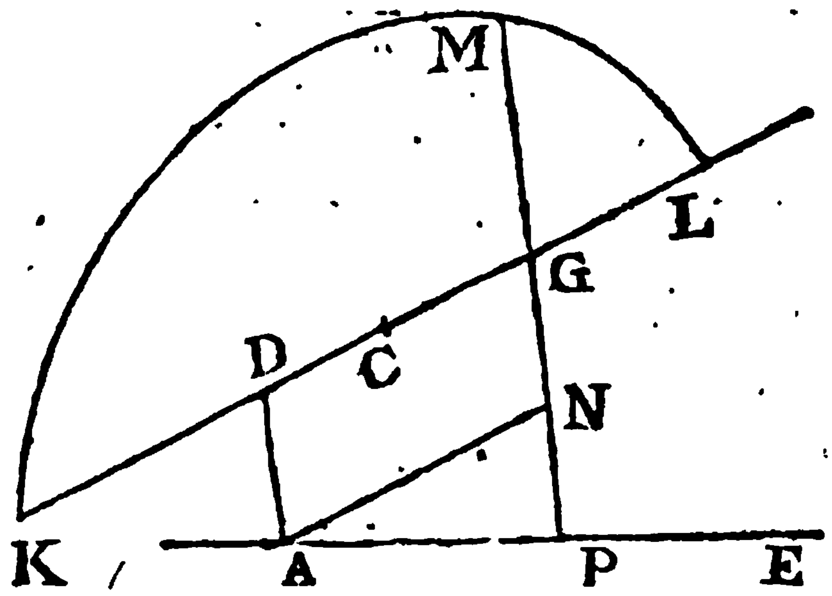
It appears also, that "if the locus be a parabola, and the term xy be wanting, the term x^2 must also be wanting." And, "if any equation of two dimensions be proposed that

wants

wants both the terms, xy and x^2, it may be always accommodated to a *parabola*.

§ 28. The general equation for the *ellipse* is deduced from the property of the ordinates of any diameter, in the same manner; the construction of the figure being the same as in § 26. Only, in place of the *parabola*,

Let KML be an *ellipse* whose diameter is KL, having its ordinates parallel to PM, and



let C be the center of the *ellipse*. Suppose CL $= t$, and the *parameter* of that diameter $= p$, then $GMq : CLq - CGq :: p : 2t$. But, as in § 26, $GM = y - \frac{b}{a}x - d$, and $CG = \frac{c}{a}x - e$; therefore, $\overline{y - \frac{b}{a}x - d}\,\big|^2 \times \frac{2t}{p} = t^2 - \frac{c^2}{a^2}x^2 + \frac{2ce}{a}x - e^2$: whence this equation:

$$y^2 - \frac{2b}{a}xy + \left.\begin{matrix}\frac{b^2}{a^2}\\ +\frac{pc^2}{2ta^2}\end{matrix}\right\} x^2 - 2dy + \left.\begin{matrix}\frac{2bd}{a}\\ -\frac{pce}{at}\end{matrix}\right\} x \left.\begin{matrix}+d^2\\ -\frac{pt}{2}\\ +\frac{pe^2}{2t}\end{matrix}\right\} = 0.$$

And

And if any equation is proposed that can be made to agree with this general equation, by assuming proper values of a, b, c, d, p and e; then the *locus* of that equation will be an *ellipse*.

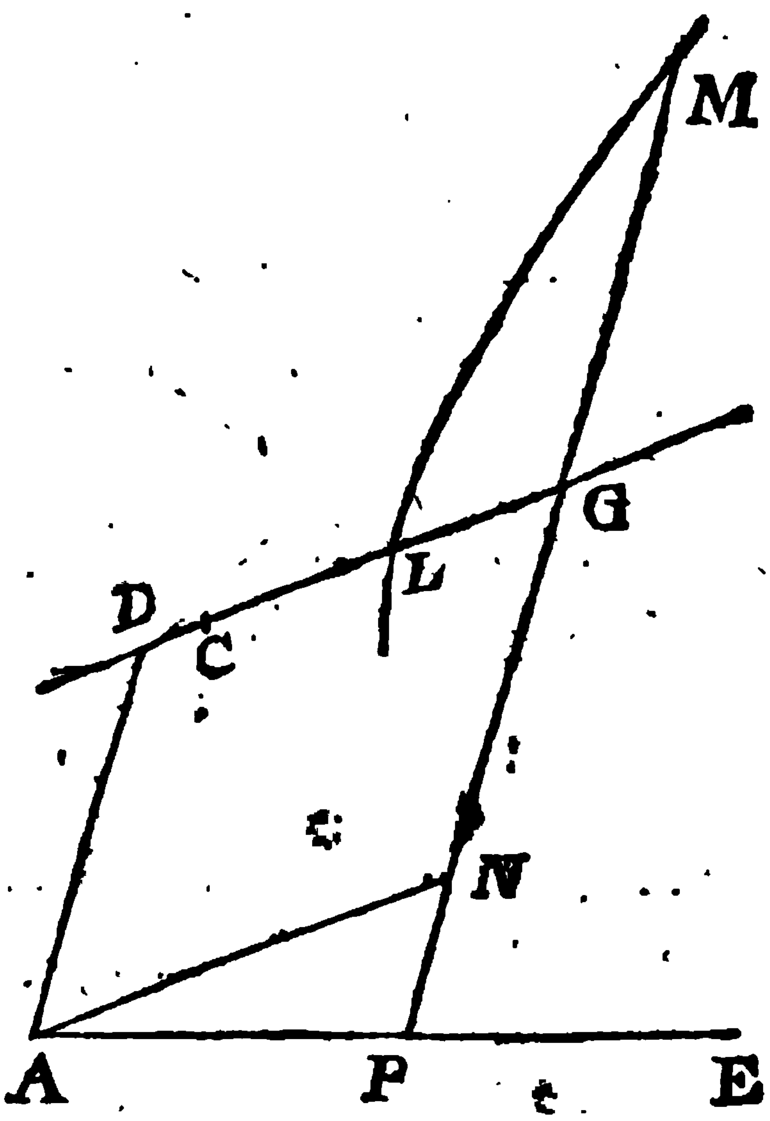
§ 29. "In the general equation for the *ellipse*, the terms x^2 and y^2 have the same sign: and the coefficient of x^2 is always greater than the square of half the coefficient of xy, because $\frac{b^2}{a^2} + \frac{pc^2}{2ta^2}$ is greater than $\frac{b^2}{a^2}$. And although the term xy be wanting, yet the term x^2 must remain, its coefficient, in that case, being $\frac{p}{2t}$, which must be always real and positive. On the other hand, if an equation is proposed in which the coefficient of x^2 exceeds the square of half the coefficient of xy; or, an equation that wants xy, but has x^2 and y^2, of the same sign, its *locus* must be an *ellipse*."

§ 30. In the hyperbola, as $GMq : CGq - CLq :: p : 2t$; when t is a *first* diameter, the equation that arises will differ from the equation of the *ellipse* only in the signs of the values of CGq and CLq, and consequently will have this form,

$$\left.\begin{array}{l} y^2 - \frac{2b}{a}xy + \frac{b^2}{a^2}x^2 - 2dy + \frac{2bd}{a}x + d^2 \\ \qquad\qquad - \frac{pc^2}{2ta^2}x^2 \qquad\quad + \frac{pce}{at}x + \frac{pt}{2} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad - \frac{pe^2}{2t} \end{array}\right\} = 0.$$

If

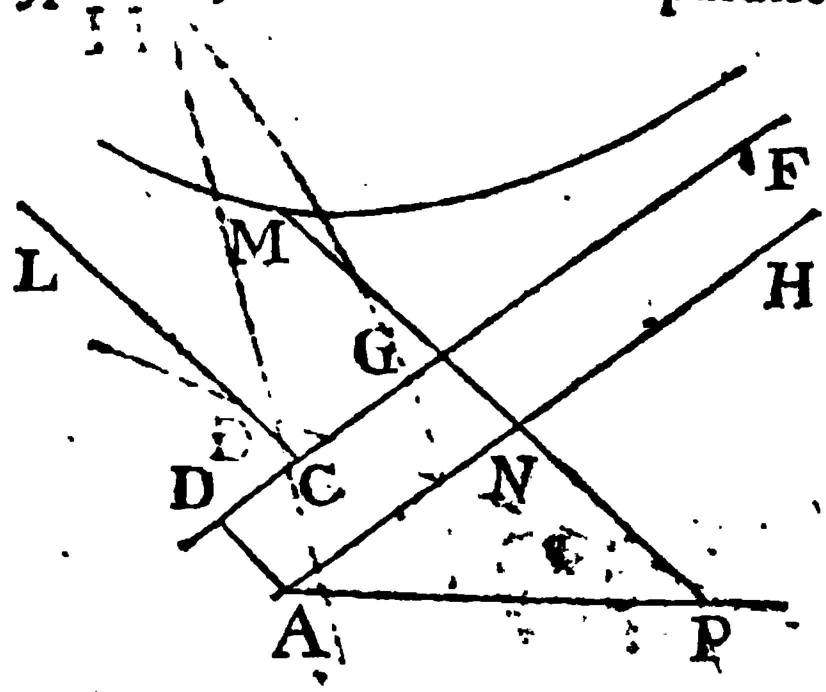
If t be a *second* diameter, then $\frac{pt}{2}$ will be negative.



In this equation, it is manifest that the coefficient of the term x^2 is less than the square of half the coefficient of xy; and, that when the term xy is wanting, the term x^2 must be negative. And, reciprocally, "if an equation is proposed where the coefficient of x^2 is less than the square of half the coefficient of xy; or where xy is wanting and y^2 and x^2 have contrary signs, the locus of that equation must be an *hyperbola*."

§ 31.

§ 31. The equation of the *hyperbola* when its ordinates PM are parallel to an asymptote does not come under the general equation of the last article. Let CF and CL be the *asymptotes* of the *hyperbola*, and let PM be parallel to CL.



Then CG × GM will be equal to a given rectangle (which suppose ga). Then, $CG = DG - DC = \frac{c}{a}x - e$, $GM = y - \frac{b}{a}x - d$, and consequently $\overline{y - \frac{b}{a}x - d} \times \overline{\frac{c}{a}x - e} = g \times a$: whence this equation,

$$\left.\begin{array}{l} xy - \frac{b}{a}x^2 + \frac{eb}{c} \\ \qquad\qquad\quad - d \end{array}\right\} x \left.\begin{array}{r} - \frac{ea}{c}y + \frac{eda}{c} \\ - \frac{ga^2}{c} \end{array}\right\} = 0.$$

Where only one of the terms y^2, x^2, can be found with xy; and where xy will be found without either of these terms, if AE and AH coincide, that is, if AE is parallel to the asymptote DF.

It appears from this, that "if an equation is proposed that either has xy the only term of two dimensions; or, has xy and either x^2 or y^2 besides, but not both of them, the locus of the equation shall be an *hyperbola*, one of whose *asymptotes* shall be parallel to y or x according as it is y^2 or x^2 that is wanting in the equation."

§ 32. From all these compared together, it follows, that "*the* locus *of any equation of two dimensions is a* conic section."

For if the term xy is wanting in the equation, and but one of the terms y^2, x^2 is found in it, the *locus* shall be a *parabola*; by § 27.

If xy is wanting, and x^2, y^2, have the same sign, then the *locus* is an *ellipse*. § 29.—But, when they have different signs, it is an *hyperbola*. § 30.

If xy is found in the equation, and x^2, y^2, are both wanting, or either of them, the *locus* is an *hyperbola*. § 31.

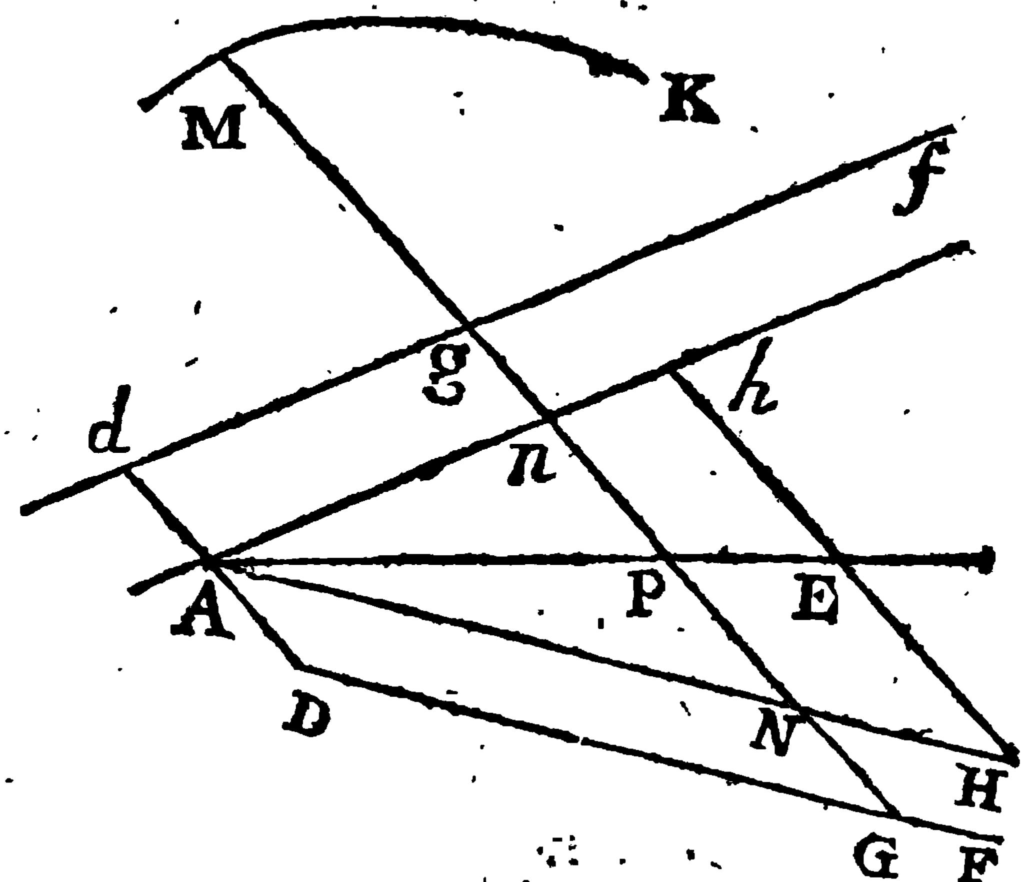
If both x^2 and y^2 are found in it, having contrary signs, the *locus* is still an *hyperbola*.

If y^2 and x^2 have the same signs, then, according as the coefficient of x^2 is *greater*, *equal*, or *less* than the square of half the coefficient of xy, the locus shall be an *ellipse*, *parabola*, or *hyperbola*. § 27, 29, 30.

In any case therefore the *locus* of the equation is some *conic section*.

§ 33.

§ 33. These may all be demonstrated more directly from the consideration of the general equation of the lines of the *second order* in § 21. For it is obvious that, by § 25. *Part* II.



the second term of that general equation may be exterminated by assuming $z = y + \frac{ax + b}{2}$, and it will be transformed into

$$\left.\begin{array}{l} z^2 * + \frac{4c - a^2}{4} \times x^2 \\ + \overline{d - \frac{ab}{2}} \times x \\ \qquad\qquad + e - \frac{b^2}{4} \end{array}\right\} = 0.$$

which, by transposing the last term, is,

$$z^2 = \overline{\frac{a^2}{4} - c} \times x^2 + \overline{\frac{ab}{2} - d} \times x \left.\begin{array}{l} + \frac{b^2}{4} \\ - e \end{array}\right\}.$$

Let

Let MK be the locus of the equation: and if AH be drawn so that HE be to AE as $\frac{1}{2}a$ to unit, and AD, parallel to PM, be $= \frac{1}{2}b$, and through D the line DF be drawn parallel to AH, meeting PM in G, then shall GM ($=$ PM $+$ PN $+$ NG $= y + \frac{1}{2}ax + \frac{1}{2}b$) $= z$. And if AH $= f$, then DG $=$ AN $= fx$.

Suppose DG $= u$, and $x = \frac{u}{f}$. Instead of x substitute $\frac{u}{f}$, and the equation that results will express the relation of GM and DG, of this form,

$$z^2 = \frac{a^2 - 4c}{4f^2} \times u^2 + \frac{ab - 2d}{2f} \times u + \frac{1}{4}b^2 - e = 0.$$

Which will be an *hyperbola*, *parabola*, or *ellipsis*, according as the term $\frac{a^2 - 4c}{4f^2}$ is *positive*, *nothing*, or *negative*. That is, according as $\frac{a^2}{4}$ is *greater*, *equal* to, or *less* than c. But a was the coefficient of xy; from which it appears, that "the locus is an *ellipse*, *parabola*, or *hyperbola*, according as the coefficient of x^2 is *greater*, *equal* to, or *less* than the square of half the coefficient of xy."

It appears also, that "if the term xy be wanting, or $a = 0$, then the locus will be an *ellipse*, *parabola*, or *hyperbola*, according as the term cx^2 is *positive*, *nothing*, or *negative*."

Hence

Hence likewise, if the term x^2 be wanting, and the term xy not wanting, then the term $\frac{a^2-4c}{4f^2}u^2$ being positive (because $\frac{a^2}{4f^2}$ is always positive, whatever a or f be) "*the locus must be an* hyperbola."

Note, That part of the figure, on the other side of AE, which is marked with small letters, answers to the case when the coefficient of y, in the general equation, *viz.* $ax+b$, is negative.

§ 34. The lines of the *second order* have some general properties which may be demonstrated from the consideration of the general equation representing them.

The general equation of § 21. by exterminating the second term can be transformed into the equation,

$$z^2 = \frac{a^2-4c}{4f^2} \times u^2 + \frac{ab-2d}{2f} \times u + \frac{b^2}{4} - e.$$

From which we have

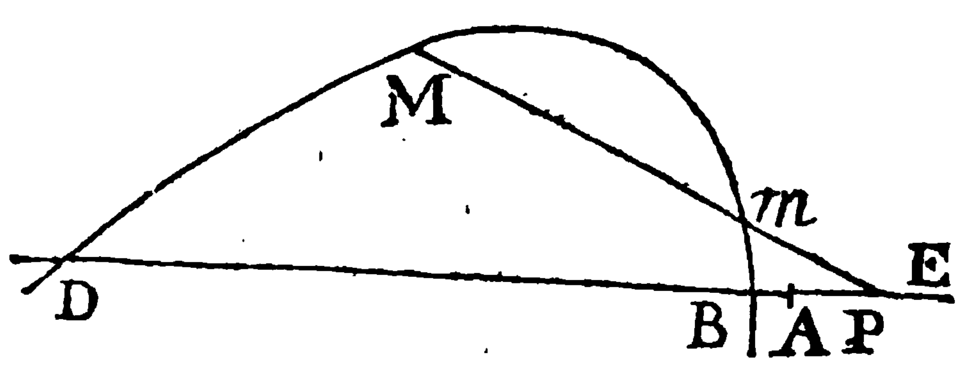
$$z = \pm\sqrt{\frac{a^2-4c}{4f^2} \times u^2 + \frac{ab-2d}{2f} + \frac{b^2}{4} - e.}$$

Where the two values of z are always equal, and have contrary signs, so that the line DF, on which the abscisses are taken, must bisect the ordinates, and consequently, is a *diameter* of the conic section. And, as this has been demonstrated generally, in any situation of the lines PM, it follows that if any parallels, as Mm, Mm, Mm, be drawn meeting a conic ſection*, there is a line DF which can biſect all theſe parallels. And conſequently if any two parallels, Mm, Mm, are biſected in G and g, the line Gg that biſects theſe two, will biſect all the other lines parallel to them, terminated by the curve. "Which is a general property of all the conic ſections."

There is one caſe which muſt be excepted, when PM is parallel to an aſymptote, becauſe in that caſe it meets with the conic ſection only in one point.

§ 35. In the general equation of § 21, if you ſuppoſe $y=0$, there will remain $cx^2+dx+e=0$, by which the points are determined where the curve meets the abſciſſe AE.

Suppoſe it meets it in B and D, and that AB = A, and AD = B. Then ſhall − A and − B be the two roots of the equation $x^2+\frac{d}{c}x$ $+\frac{e}{c}=0$; and therefore $\overline{x+A}\times\overline{x+B}=x^2$



$+\frac{d}{c}x+\frac{e}{c}$: but $x+A=BP$, and $x+B=DP$;

* Supply the figure.

Z therefore

therefore $BP \times DP = x^2 + \frac{d}{c}x + \frac{e}{c}$. Now, it is manifest from the nature of equations, that if PM meet the curve in M and *m*, the rectangle of the roots PM, and P*m* shall be equal to $cx^2 + dx + e$ the last term of the equation

$$\left.\begin{array}{l} y^2 + axy + cx^2 \\ \quad + by + dx \\ \qquad\quad + e \end{array}\right\} = 0.$$

We have therefore $PM \times Pm = cx^2 + dx + e$; and $BP \times DP = x^2 + \frac{d}{c}x + \frac{e}{c}$; so that $PM \times Pm : BP \times DP :: cx^2 + dx + e : x^2 + \frac{d}{c}x + \frac{e}{c} :: c : 1$. That is, "the rectangle of the ordinates PM, P*m* is to the rectangle of the segments of the abscisses, as, in a given ratio, *c* is to 1." Which is another general property of the lines of the *second order*.

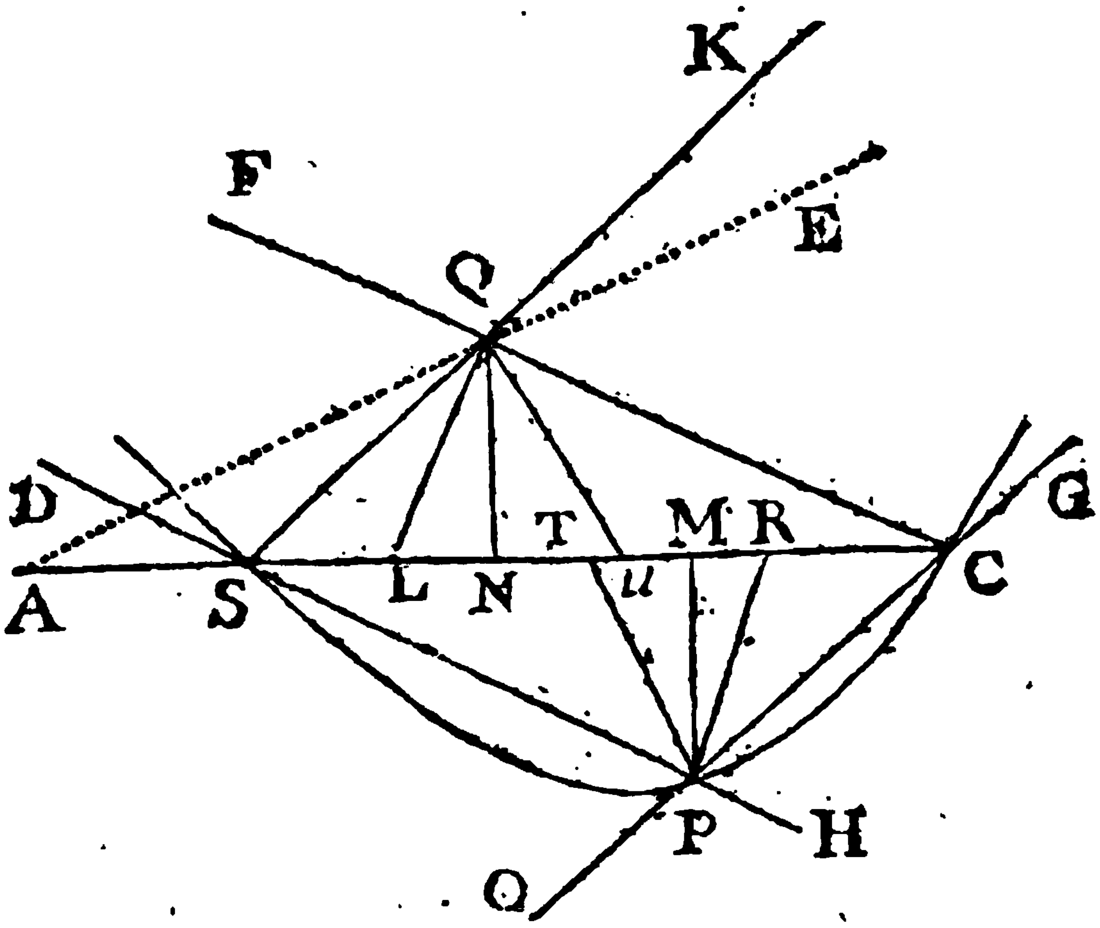
In a similar manner the analogous properties of the lines of the higher orders are demonstrated*.

§ 36. There are many different ways of describing the lines of the *second order*, by motion. The following is Sir *Isaac Newton*'s.

† Let the two points C and S be given, and the straight line AE in the same plane. Let the given

* See the APPENDIX.
† See *Geometria Organica*, Prop. I.

given angles FCO, KSH, revolve about the points C and S as poles, and let the interfection of the fides CF, SK, be carried along the



ſtraight line AE, and the interſection of the ſides CO, SH, will deſcribe a line of the *ſecond order*.

Let the ſides CF, SK interſect each other in Q, and the ſides CO, SH, in P; let PM and QN be perpendicular on CS. Then draw PR, QU; PT, QL; ſo that CUQ = CRP = FCG; and SLQ = STP = KSD.

The angle RCP = CQU, ſince RCP makes two right ones with RCQ and QUC. So that the triangles CUQ and CRP will be ſimilar. And after the ſame manner you may demon-

 ſtrate

ſtrate that the triangles SLQ, STP are ſimilar; whence,

$$CR : PR :: QU : CU,$$
$$\text{and } ST : PT :: QL : SL.$$

Suppoſe $CS = a$, $CA = b$, the ſine of the angle FCO to its coſine as d to a; ſin. angle CAE to coſin. as c to a, and ſin. KSH to coſin. as e to a. Put alſo $PM = y$, $CM = x$, $QN = z$.

Then $RM : PM :: a : d$, $PR : PM :: \sqrt{a^2+d^2} : d$, $AN : QN :: a : c$. So that $RM = \frac{ay}{d}$, $CR\ (= CM - RM) = x - \frac{ay}{d}$, $PR = y\frac{\sqrt{a^2+d^2}}{d}$.

Likewiſe $QU = \frac{z\sqrt{a^2+d^2}}{d}$, and $CU\ (= CA - AN - NU) = b - \frac{a}{c}z - \frac{a}{d}z$. And it being $CR : PR :: QU : CU$, it follows that $dx - ay : y\sqrt{a^2+d^2} :: z\sqrt{a^2+d^2} : b - \overline{\frac{a}{c} + \frac{a}{d}}\,z$ d. So that $z = \frac{bc \times \overline{dx - ay}}{\overline{dc - a^2} \times y + \overline{d + c} \times ax}$.

In like manner you will find $ST = a - x - \frac{a}{e}y$, $PT = \frac{y\sqrt{a^2+e^2}}{e}$, $QL = \frac{z\sqrt{a^2+e^2}}{e}$, and $SL\ (= AN - AS - NL) = a - b + \frac{\overline{e - c} \times az}{ee}$. But

is

it was ST : PT :: QL : SL, that is, $a - x - \frac{a}{e}y : \frac{y\sqrt{a^2+e^2}}{e} :: \frac{z\sqrt{a^2+e^2}}{e} : a - b + \frac{\overline{e-c}\times az}{ec}$.

Whence $QN = z = \frac{\overline{a-b}\times c\times\overline{ae-ex-ay}}{\overline{ec+a^2}\times y+\overline{ax}\times\overline{e-c}+a^2\times\overline{c-e}}$.

And from the equation of these two values of z this equation results;

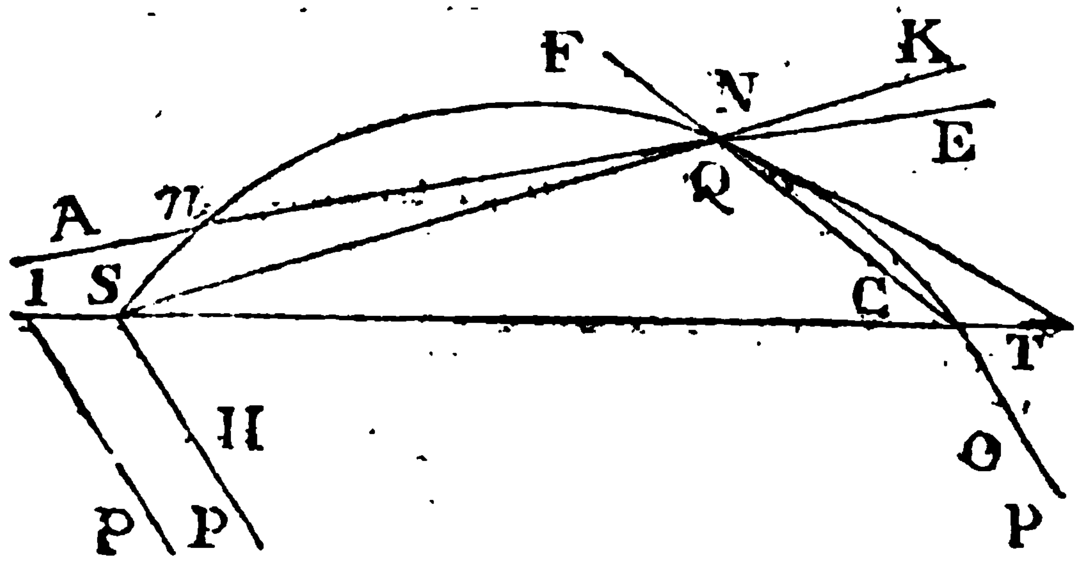
$$\left.\begin{matrix}\overline{a-b}\times ce\\ +\overline{ae-bc}\times d\end{matrix}\right\}x^2 \left.\begin{matrix}+a^2\times\overline{d+c-e}\\ +dce\end{matrix}\right\}xy \left.\begin{matrix}+a\times\overline{-a^2+cd}\\ -bc\times\overline{e+d}\end{matrix}\right\}y^2$$

$$\left.\begin{matrix}+abc\times\overline{d+e}\\ -a^2e\times\overline{d+e}\end{matrix}\right\}x \left.\begin{matrix}-bc\times\overline{a^2-ed}\\ -ae\times\overline{dc-a^2}\end{matrix}\right\}y=0;$$

where since x and y are only of two dimensions, it appears that the curve described must be a line of the second *order*, or a conic section, according to what has been already demonstrated.

§ 37. As the angles FCO, KSH revolve about the poles C and S, if the angle CQS becomes equal to the supplement of these given angles to four right ones, then the angle CPS must vanish, that is, the lines CO and SH must become parallel: and the intersection P must go off to an infinite distance. And the lines CO and SH become, in that case, parallel to one of the *asymptotes*.

In order to determine if this may be, describe on CS an arc of a circle that can have inscribed in it an angle equal to the supplement of the angles FCO, KSH, to four right angles: If this arc meet the line AE in two points N, *n*, then when Q the intersection of the sides CF, SK comes to either of these points, as it is car-



ried along the line AE, the point P will go off to infinity, and the lines SH, CO, become parallel to each other and to an *asymptote* of the curve.

If that arc only touch the line AE, the point P will go off to infinity but once. If the arc neither cut the line AE nor touch it, the point P cannot go off to infinity. In the first case the conic section is an *hyperbola*, in the second a *parabola*, in the third an *ellipse*.

The *asymptotes*, when the curve has any, are determined by the following construction.

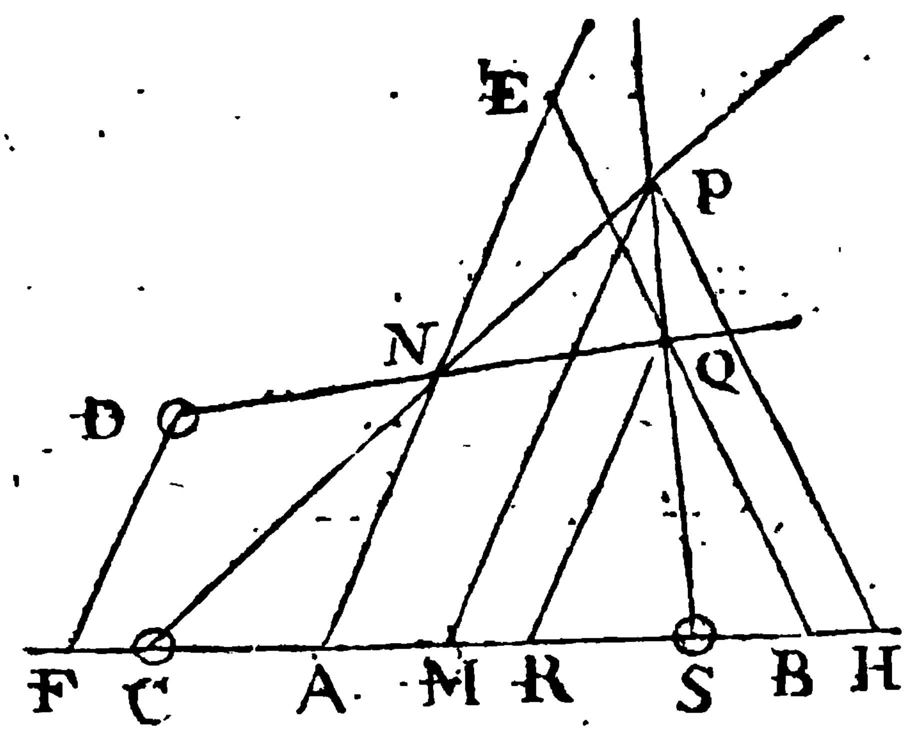
Draw NT constituting the angle CNT = SNA, meeting SC in T; then take SI = CT, and always towards opposite parts, and through I draw IP parallel to SH or CO, and IP will be one asymptote of the curve. The other is determined in like manner, by bringing Q to *n*.

And

And the two asymptotes meet in the center, constituting there an angle = NS*n*.

From this construction it is obvious, that when the circular arc CN*n*S touches the line AE, the angle SNA being then = SCN, the line NT will become parallel to CS; and therefore CT and SI become infinite, that is, the *asymptote* IP going off to infinity, the curve becomes a *parabola*.

§ 38. There is another general method of describing the lines of the *second order*, that deserves our consideration.



Instead of angles we now use three rulers DQ, CN, SP, which we suppose to revolve about the poles D, C, S, and cut one another always in three points N, Q and P; and carrying any two of these intersections, as N and Q, along the given straight lines AE, BE, the third intersection P will describe a conic section.

Through the points D, P, Q, draw DF, PM, QR, parallel to AE, meeting CS in F, M, R; alfo through P draw PH parallel to BE meeting CS in H.

Then putting $PM = y$, $CM = x$, $CS = a$, $CA = b$, $SB = c$, $DF = k$, $AF = l$, $AE = d$, $BE = e$, $AB\ (= a - b + c) = f$; fince the triangles PMH, AEB are fimilar, therefore $PH = \frac{ey}{d}$, $MH = \frac{fy}{d}$, $SH = \frac{dx + fy - ad}{d}$. And fince $CA : AN :: CM : PM$; ... $AN = \frac{by}{x}$; and fince $SB : BQ :: SH : PH$, ... $BQ = \frac{cey}{dx + fy - ad}$.

But,

$BQ : QR :: BE : AE$ ... $QR = \frac{cdy}{dx + fy - ad}$,

and ... $BR = \frac{cfy}{dx + fy - ad}$.

Now $AN - DF : RQ - AN :: AF : AR$; this is,

$$\frac{by}{x} - k : \frac{cdy}{dx + fy - ad} - \frac{by}{x} :: l : f - \frac{cfy}{dx + fy - ad}.$$

And multiplying the extremes and means, and ordering the terms, it is,

$$\left.\begin{array}{l} bf \times \overline{c - l - f} \times y^2 + \overline{c \times \overline{ld - kf} - bd \times \overline{l + f} + kff} \times xy \\ + bad \times \overline{l + f} \times y - adfk \times x + dfk \times x^2 \end{array}\right\} = 0.$$

In which equation, the fign of fome terms may vary by varying the fituation of the poles and

and lines; but x and y not rising to more than two dimensions, it appears that the point P always describes a *conic section*. Only in some particular cases the conic section becomes a straight line. As for example, when D is found in the straight line CS; for then DF vanishing the terms $dfkx^2 - adfkx$ vanish, and the remaining terms being divisible by y, the equation becomes,

$$bf \times \overline{c-l-f} \times y + \overline{cld - bd \times \overline{l+f}} \times x + bad \times \overline{l+f} = 0.$$

Which is a *locus* of the *first order*, and shews, that, in this case, P must describe a *straight line*.

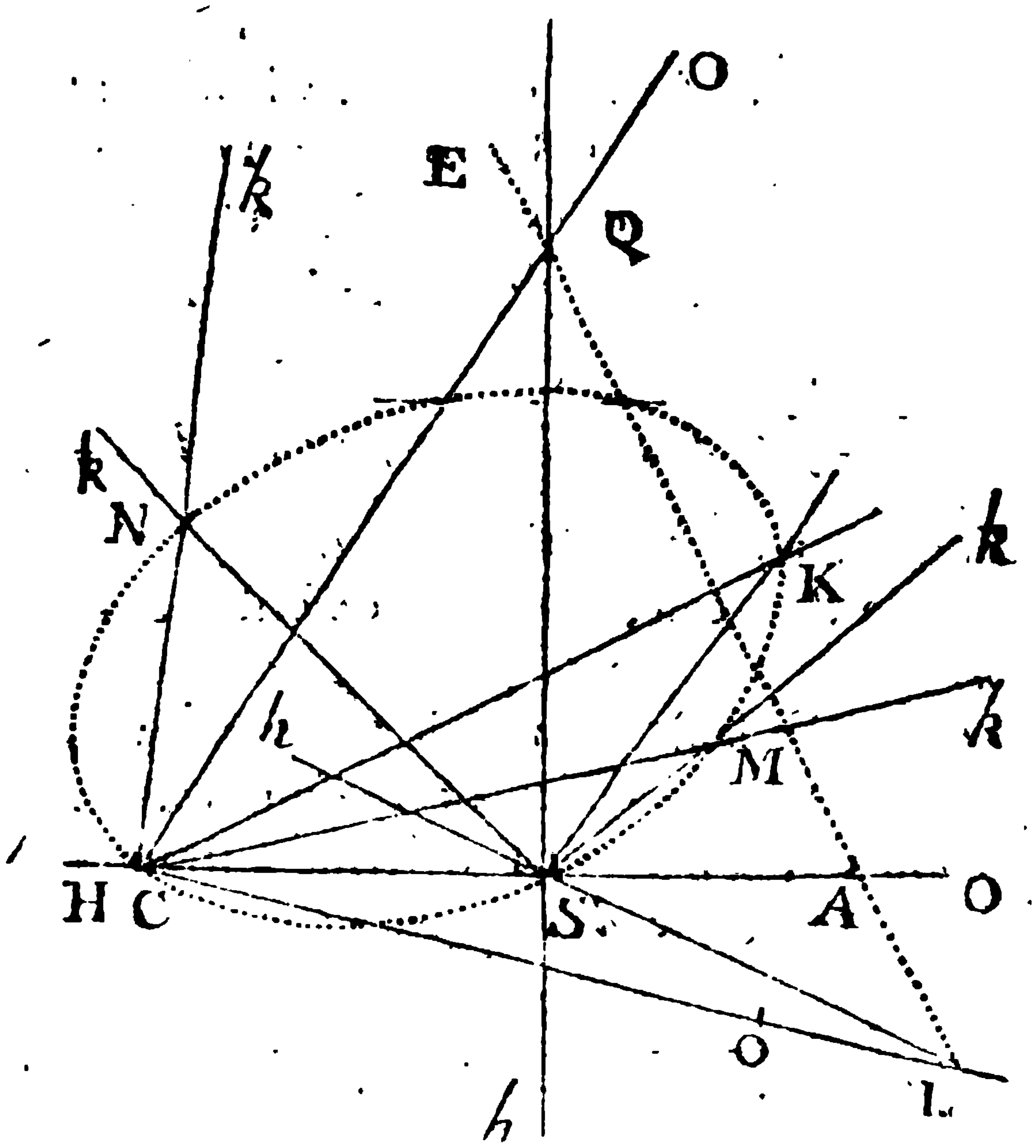
After the same manner it appears that if the point E the intersection of the lines AE, BE, falls in CS, then will P describe a straight line. For in that case d vanishes, and the equation becomes,

$$b \times \overline{c-l-f} \times y - f \times \overline{c-k} \times x = 0.$$

§ 39. These two descriptions furnish, each, a general method of "describing a line of the second order through any five given points whereof three are not in the same straight line."

Suppose the five given points are C, S, M, K, N; join any three of them, as C, S, K, and let angles revolve about C and S equal to the angles KCS, KSC. Apply the intersection of the legs CK, SK first to the point N, and let the inter-

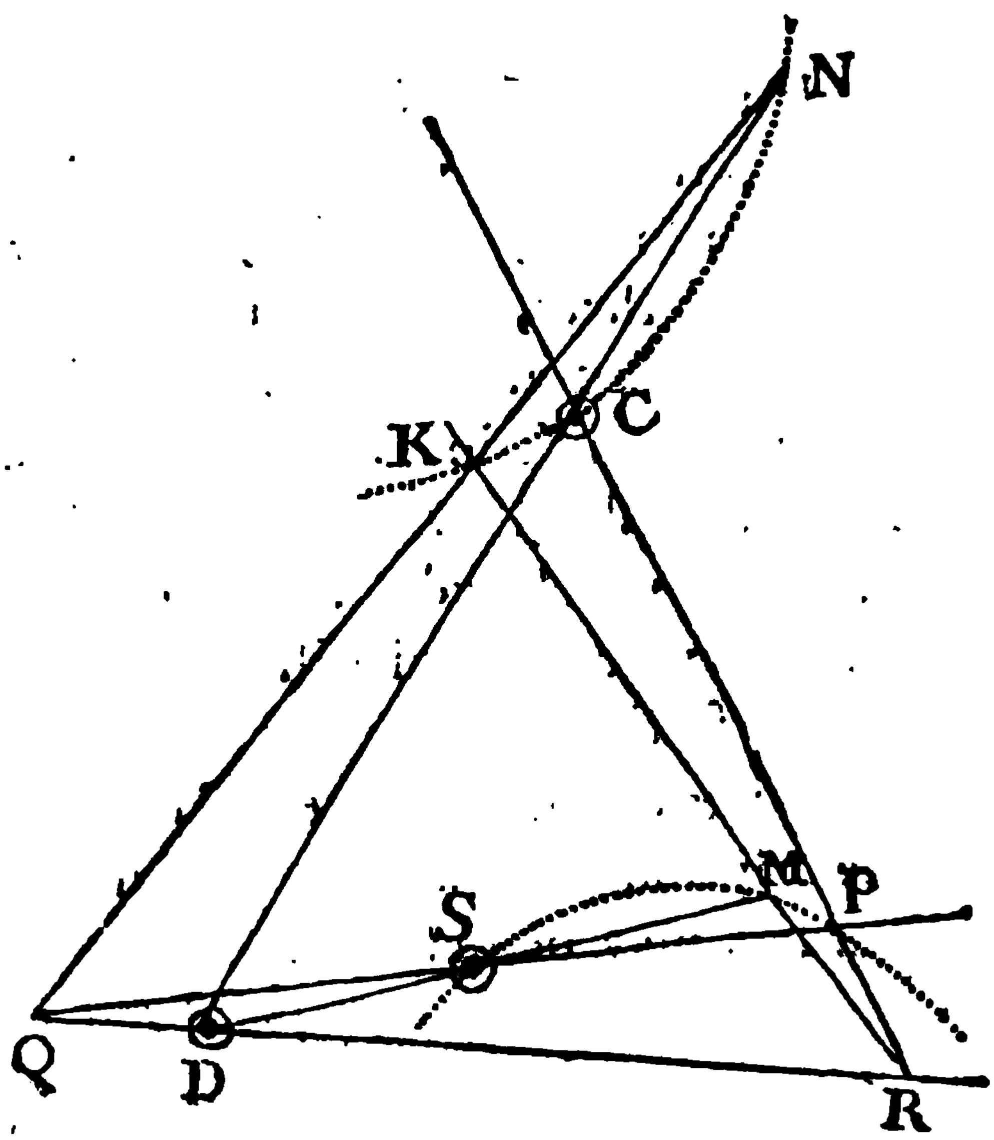
intersection of the legs CO and SH be Q; secondly apply the intersection of the same legs CK, SK, to the remaining point M, and let the



intersection of the legs CO, SH be L. Draw a line joining Q and L, and it will be the line AE along which if you carry the intersection of the legs CO, SH, the intersection of the other legs will describe a conic section passing through the five given points C, S, M, K, N.

It must pass through C and S from the construction: when the intersection of CO, SH comes to A, the curve will pass through K. And

And when it becomes to Q and L, it passes through N, M.



§ 40. From the second description we have this solution of the same problem.

Let C, S, M, K, N be the five given points: draw lines joining them; produce two of the lines NC, MS, till they meet in D. Let three rulers revolve about the three poles C, S, D, *viz.* CP, SQ, DR. Let the intersection of the rulers CP, DR, be carried over the given line MK, and the intersection of the rulers SQ, DR be carried through the line NK; and the point P, the intersection of the rulers that

that revolve about C and S, will describe a conic section that passes through the five points C, S, M, K, N.

§ 41. It is a remarkable property of the conic sections, that "if you assume any number of poles whatsoever, and make rulers revolve about each of them, and all the intersections but one, be carried along given right lines, that one shall never describe a line above a conic section;" if, instead of rulers you substitute given angles which you move on the same poles, the curve described will still be no more then a conic section.

By carrying one of the intersections necessary in the description over a *conic* section, lines of higher orders may be described.

CHAP.

CHAP. III.

Of the Conſtruction of *cubic* and *biquadratic* Equations.

§ 42. "THE roots of any equation may be determined by the interſections of a ſtraight line with a curve of the ſame dimenſions as the equation:" or, "by the interſections of any two curves whoſe indices multiplied by each other give a product equal to the index of the propoſed equation."

Thus the roots of a *biquadratic* equation may be determined by the interſections of two conic ſections; for the equation by which the ordinates from the four points in which theſe conic ſections may cut one another can be determined will ariſe to four dimenſions: and the conic ſections may be aſſumed in ſuch a manner, as to make this equation coincide with any propoſed biquadratic: ſo that the ordinates from theſe four interſections will be equal to the roots of the propoſed biquadratic.

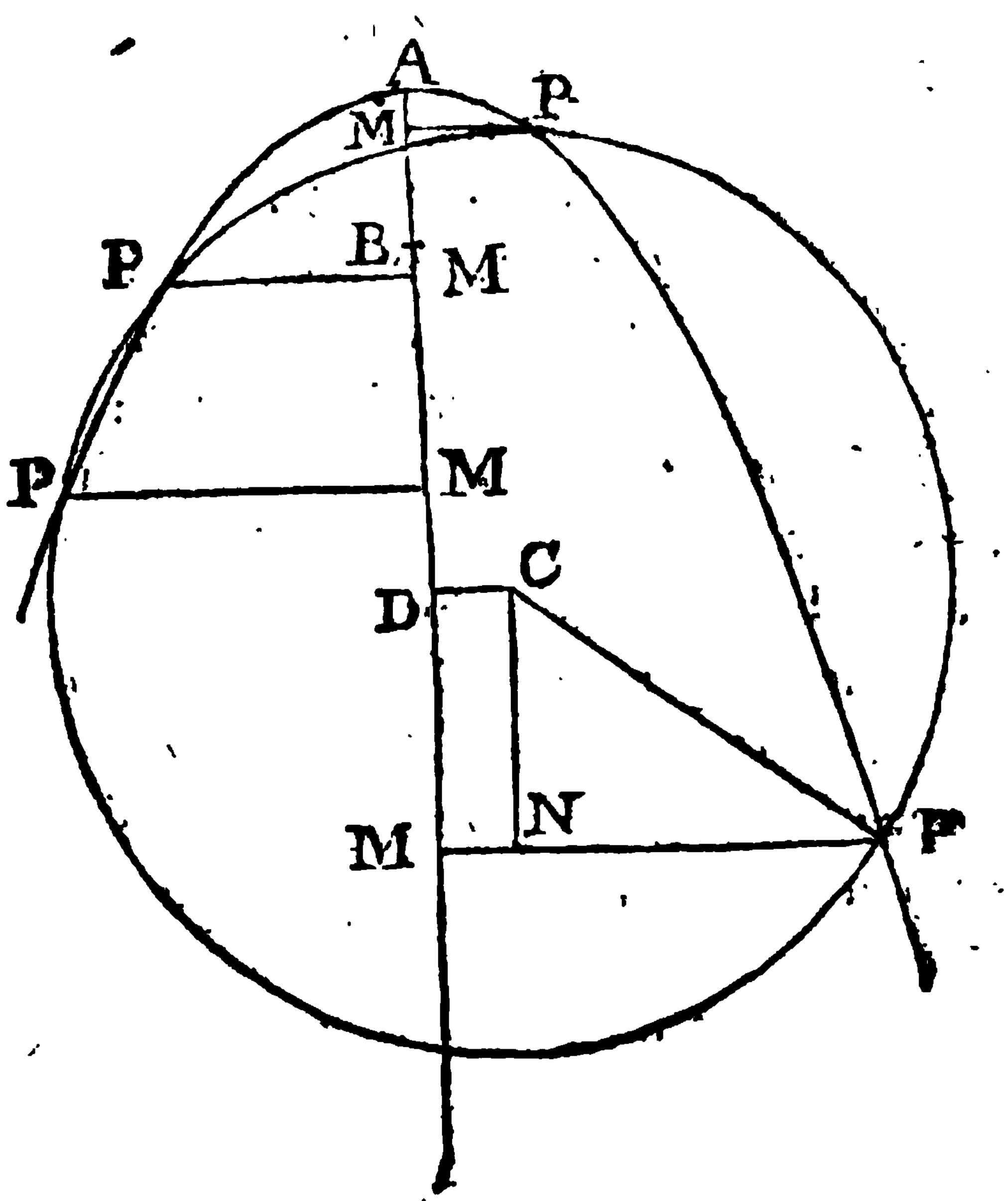
If one of the interſections of the conic ſection falls upon the *axis*, then "one of the ordinates vaniſhes, and the equation by which theſe ordinates are determined will then be of three dimenſions only; or a *cubic*," to which any propoſed cubic equation may be accommodated.

So

So that the three remaining ordinates will be the three roots of that proposed cubic.

§ 43. Those conic sections ought to be preferred for this purpose that are most easily described. They must not however be both *circles*; for their intersections are only two, and can serve only for the resolution of *quadratic* equations.

Yet the circle ought to be one, as being most easily described; and the *parabola* is commonly assumed for the other. Their intersections are determined in the following manner.



Let

Let APE be the common Apollonian parabola. Take on its axis the line $AB =$ half of its parameter. Let C be any point in the plane of the parabola, and from it as a center describe, with any radius CP, a circle meeting the parabola in P. Let PM, CD, be perpendiculars on the axis in M and D, and let CN, parallel to the axis, meet PM in N.

Then will always $CPq = CNq + NPq$ (47 *E.* 1.)

Put $CP = a$, the parameter of the parabola $= b$, $AD = c$, $DC = d$, $AM = x$, $PM = y$.

Then $CNq = \overline{x \overset{\backsim}{+} c}|^2$, $NPq = \overline{y \overset{\backsim}{+} d}|^2$; and $\overline{x \overset{\backsim}{+} c}|^2 + \overline{y \overset{\backsim}{+} d}|^2 = a^2$*. That is,

$$x^2 \pm 2cx + c^2 + y^2 \pm 2dy + d^2 = a^2.$$

But, from the nature of the parabola, $y^2 = bx$, and $x^2 = \frac{y^4}{b^2}$; substituting therefore these values for x^2 and x, it will be,

$$\frac{y^4}{b^2} \pm \frac{2cy^2}{b} + y^2 \pm 2dy + c^2 + d^2 - a^2 = 0.$$

Or, multiplying by b^2,

$$y^4 \pm \overline{2bc + b^2} \times y^2 \pm 2db^2 \times y + \overline{c^2 + d^2 - a^2} \times b^2 = 0.$$

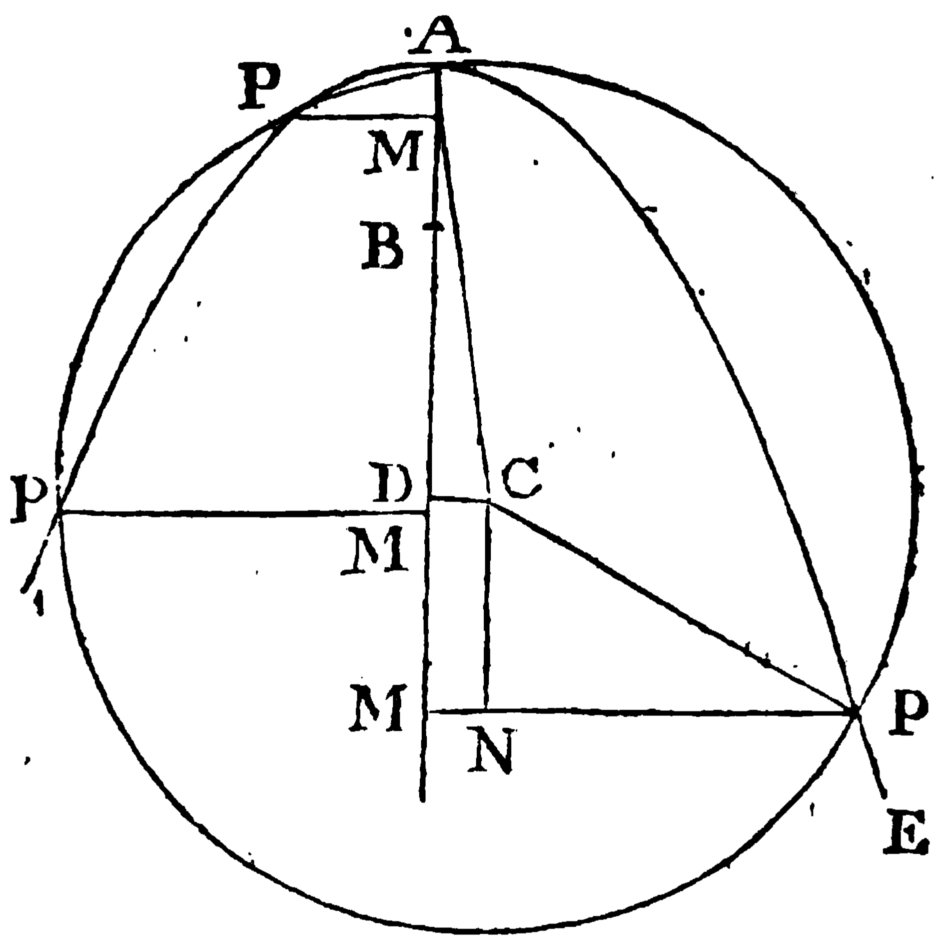
Which may represent any biquadratic equation that wants the second term; since such values may be found for a, b, c, and d, by comparing this with any proposed biquadratic, as to make them coincide. And then the ordinates from the points P, P, P, P, on the axis will be equal to the roots of that proposed biquadratic. And

* $x \backsim c$ is the difference of x and c indefinitely, whichever of the two is greatest.

this

this may be done, though the parameter of the parabola (*viz.* b) be given: that is, if you have a parabola already made or given, by it alone you may resolve all biquadratic equations, and you will only need to vary the center of your circle and its radius.

§ 44. If the circle described from the center C pass through the vertex A, then $CPq = CAq = CDq + ADq$, that is, $a^2 = d^2 + c^2$; and the



last term of the biquadratic ($c^2 + d^2 - a^2$) will vanish; therefore, dividing the rest by y, there arises the cubic,

$$y^3 * \pm \overline{2bc + b^2} \times y \pm 2db^2 = 0.$$

Let the cubic equation proposed to be resolved be $y^3 * \pm py \pm r = 0$. Compare the terms of these

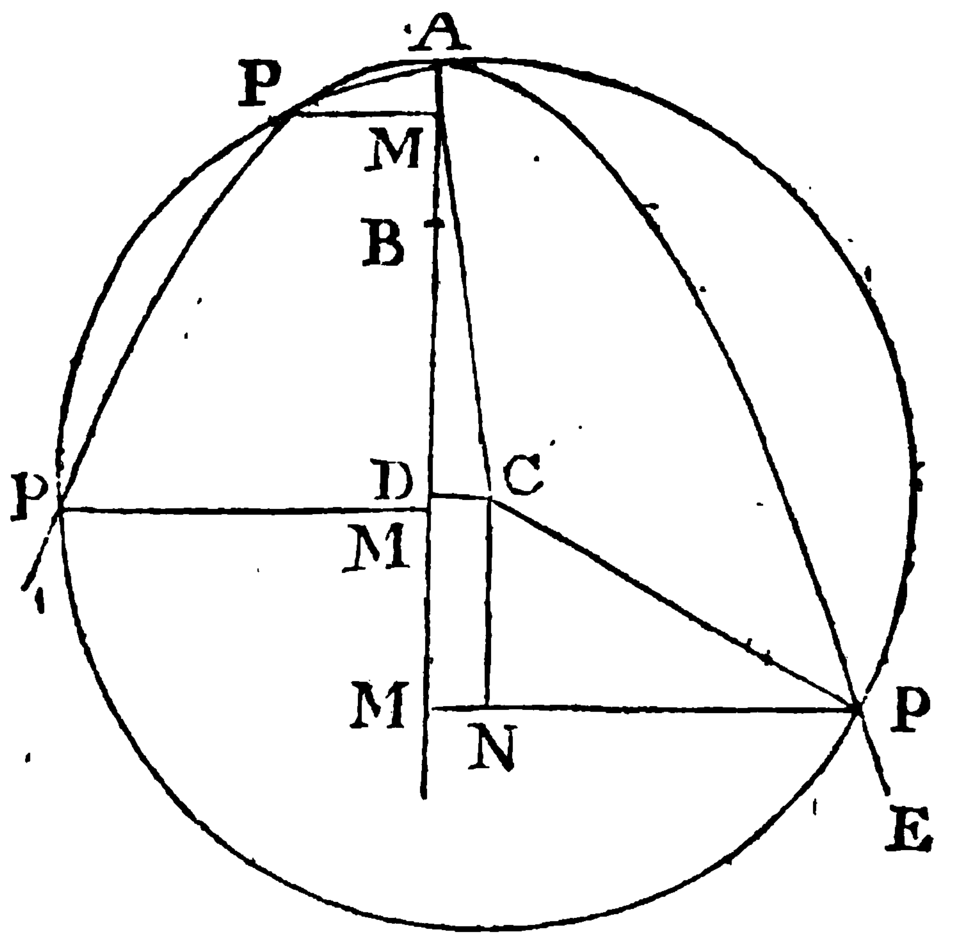
these two equations, and you will have $\pm 2bc + b^2 = \pm p$, and $\pm 2db^2 = \pm r$, or, $\mp c = \frac{b}{2} \mp \frac{p}{2b}$, and $d = \pm \frac{r}{2b^2}$. From which you have this construction of the cubic $y^3 * \pm py \pm r = 0$, by means of any given parabola APE.

"*From the point B take in the axis (forward if the equation has* $-p$, *but backwards if* p *is positive) the line* $BD = \frac{p}{2b}$; *then raise the perpendicular* $DC = \frac{r}{2b^2}$, *and from C, describe a circle passing through the vertex A, meeting the parabola in P, so shall the ordinate PM be one of the roots of the cubic* $y^3 * \pm py \pm r = 0$."

The ordinates that stand on the same side of the axis with the center C are negative or affirmative, according as the last term r is *negative* or *affirmative*; and those ordinates have always *contrary* signs that stand on *different* sides of the *axis*. The roots are found of the same value, only they have contrary signs, when r is positive as when it is negative; the second term of the equation being wanting; which agrees with what has been demonstrated elsewhere.

§ 45. In resolving *numerical* equations, you may suppose the parameter b to be unit; then this may be done, though the parameter of the parabola (*viz.* b) be given: that is, if you have a parabola already made or given, by it alone you may resolve all biquadratic equations, and you will only need to vary the center of your circle and its radius.

§ 44. If the circle described from the center C pass through the vertex A, then $CPq = CAq = CDq + ADq$, that is, $a^2 = d^2 + c^2$; and the



last term of the biquadratic $(c^2 + d^2 - a^2)$ will vanish; therefore, dividing the rest by y, there arises the cubic,

$$y^3 * \pm \overline{2bc + b^2} \times y \pm 2db^2 = 0.$$

Let the cubic equation proposed to be resolved be $y^3 * \pm py \pm r = 0$. Compare the terms of these

these two equations, and you will have $\pm 2bc + b^2 = \pm p$, and $\pm 2db^2 = \pm r$, or, $\mp c = \frac{b}{2} \mp \frac{p}{2b}$, and $d = \pm \frac{r}{2b^2}$. From which you have this construction of the cubic $y^3 * \pm py \pm r = 0$, by means of any given parabola APE.

"*From the point B take in the axis (forward if the equation has $-p$, but backwards if p is positive) the line* $BD = \frac{p}{2b}$; *then raise the perpendicular* $DC = \frac{r}{2b^2}$, *and from C, describe a circle passing through the vertex A, meeting the parabola in P, so shall the ordinate PM be one of the roots of the cubic* $y^3 * \pm py \pm r = 0$."

The ordinates that stand on the same side of the axis with the center C are negative or affirmative, according as the last term r is *negative* or *affirmative*; and those ordinates have always *contrary* signs that stand on *different* sides of the *axis*. The roots are found of the same value, only they have contrary signs, when r is positive as when it is negative; the second term of the equation being wanting; which agrees with what has been demonstrated elsewhere.

§ 45. In resolving *numerical* equations, you may suppose the parameter b to be unit; then

$AD = \frac{1}{2} \mp \frac{1}{2}p$, and $DC = \frac{1}{2}r$; and the ordinate PM muſt then be meaſured on a ſcale where the *parameter*, or 2 AB is unit. Or, if it be more convenient, the *parameter* may be ſuppoſed to expreſs 10, 100, &c. or any other number, and PM will be found by meaſuring it on a ſcale where the parameter is 10, 100, &c. or that other number.

§ 46. "When the circle meets the *parabola* in one point only beſides the vertex, the equation has only one real root, and the other two imaginary."

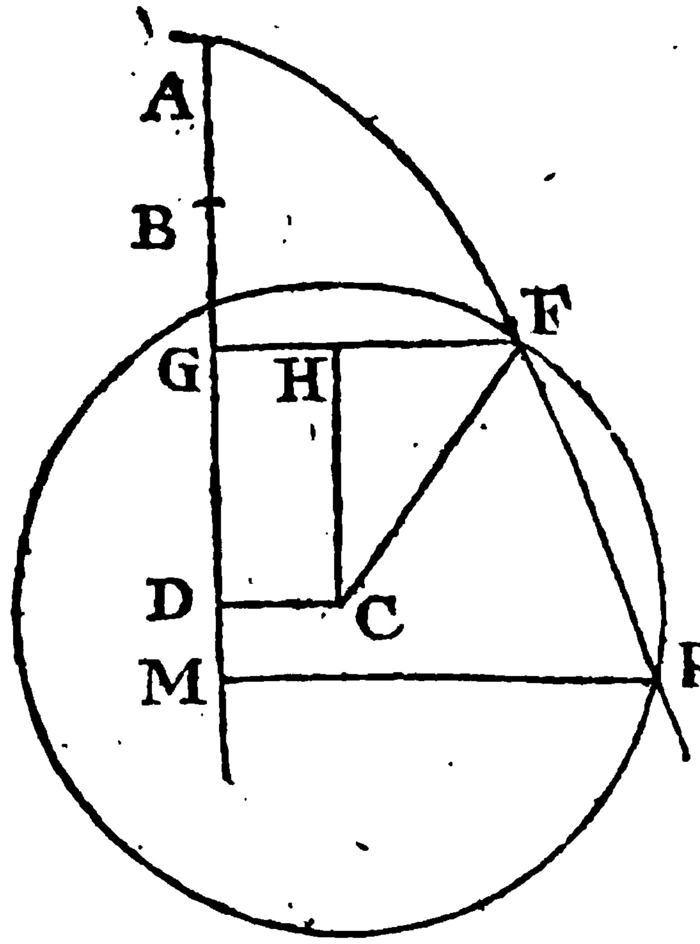
Thus, if the equation has $+p$, or if D falls on the ſame ſide of B as A does, the circle can meet the parabola in two points only, whereof A is one; and therefore the equation muſt have two *imaginary* roots; as we demonſtrated elſewhere. If the circle *touch* the *parabola*, then two roots of the equation are equal.

It is alſo obvious, that the equation muſt neceſſarily have *one real* root; becauſe, ſince the circle meets the *parabola* in the vertex A, it muſt meet it in one other point, at leaſt, beſides A.

§ 47. Inſtead of making the circle paſs through the *vertex* A, you may ſuppoſe it to paſs through ſome other given point in the parabola, and that interſection being given, the *biquadratic* found for determining the interſections, in § 43, may be reduced to a *cubic*.

Let

Let the ordinate belonging to that given



intersection be g, then one of the values of y being g, it follows that the biquadratic

$$y^4 * \left.\begin{array}{r} \pm 2bc \\ + b^2 \end{array}\right\} y^2 \pm 2db^2 y + \overline{d^2 + c^2 - a^2} \times b^2 = 0$$

will be divisible by $y - g$, which will reduce it to a *cubic* that shall have the second term. And thus we have a construction for *cubic* equations that have all their terms.

For example, let us suppose that the *parameter* is AG, and the ordinate at G is GF meeting the curve in F. Suppose now that the circle is always to pass through F; then shall $CFq\,(=a^2) = CHq + HFq = \overline{c \pm b}^2 + \overline{b \pm d}^2 = c^2 + d^2 \pm 2cb \pm 2db + 2b^2$, and substituting

 in

in the equation of § 43 this value of a^2, it becomes

$$y^4 * \left.\begin{matrix}\pm 2cb \\ + b^2\end{matrix}\right\} y^2 \pm 2db^2 \times y \left.\begin{matrix}- 2b^4 \\ \mp 2cb^3 \\ \mp 2db^3\end{matrix}\right\} = 0.$$

Where c in the last term has a contrary sign to what is has in the third, and d a contrary sign to what it has in the fourth.

This biquadratic has FG, or b, for one of its roots; and being divided by $y - b$, there arises this cubic,

$$y^3 + by^2 \left.\begin{matrix}\pm 2cb \\ + 2b^2\end{matrix}\right\} y \left.\begin{matrix}\pm 2db^2 \\ \pm 2cb^2 \\ + 2b^3\end{matrix}\right\} = 0,$$

having all its terms complete. If C had been taken on the other side of the axis, the second term by^2 had been negative.

Let now any cubic equation be proposed to be resolved, as $y^3 + py^2 + qy - r = 0$. And by comparing it with the preceding, you will find

$$\left.\begin{matrix}p = b \\ q = 2b^2 \pm 2bc \\ -r = 2b^3 \pm 2cb^2 \pm 2db^2\end{matrix}\right\} \text{whence} \left\{\begin{matrix}b = p, \\ \mp c = p - \frac{q}{2p}, \\ d = \frac{q}{2p} + \frac{r}{2p^2}.\end{matrix}\right.$$

Therefore, to construct the proposed cubic *equation* $y^3 + py^2 + qy - r = 0$, *let the parameter of your* parabola *be equal to* p; *take, on the axis from the*

the vertex A, the line $AD = p - \frac{q}{2p^2}$, *and raise the perpendicular* $DC = \frac{q}{2p} + \frac{r}{2p^2}$, *and from C describe a circle through F, meeting the parabola in P, so shall the ordinate PM be a root of the equation.*

If the equation proposed is a *literal* equation of this form $y^3 + py^2 + pqy - p^2r = 0$, having all the terms of three dimensions, then this construction will only require $AD = 1 - \frac{1}{2}q$, and $DC = \frac{1}{2}q + \frac{1}{2}r$.

§ 48. If you suppose the parabola to pass through any point F taken any where in the parabola (*vid. Fig. preced.*) and call the ordinate $FG = e$, then $\overline{c - \frac{e^2}{b}}\Big|^2 + \overline{e - d}\Big|^2 = a^2$, and the general biquadratic may have this form,

$$y^4 * \left.\begin{matrix} +2cb \\ +b^2 \end{matrix}\right\} y^2 - 2db^2y \left.\begin{matrix} +2ce^2b \\ +2deb^2 \\ -e^4 \\ -e^2b^2 \end{matrix}\right\} = 0.$$

But since $FG = e$ is one of the values of y, the equation will be divisible by $y - e$, and the quotient is found to be this *cubic*,

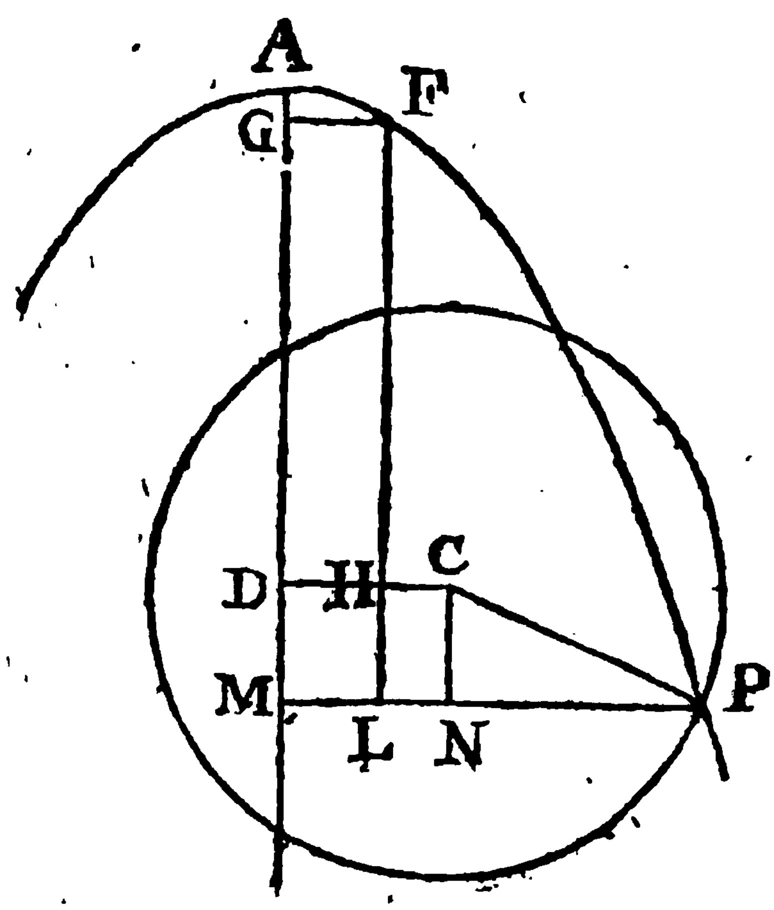
$$y^3 + ey^2 \left.\begin{matrix} -2bc \\ +b^2 \\ +e^2 \end{matrix}\right\} y \left.\begin{matrix} -2db^2 \\ -2ceb \\ +eb^2 \\ +e^3 \end{matrix}\right\} = 0.$$

 Which

Which compared with $y^4 + py^3 + pqy - p^2r = 0$, gives FG (or e) $= p$, AD $(= c) = \frac{p^2 + b^2 - pq}{2b}$, and DC $= d = \frac{p^2 \times \overline{q + r}}{2b^2}$. And by this construction the roots of a complete cubic equation may be found by any parabola whatsoever.

§ 49. It is easy to see from § 43. how to construct the roots of a biquadratic by any parabola, after the second term is taken away. But "the roots of a biquadratic may be determined by any parabola:" only they cannot be the ordinate on the axis, but "may be equal to the perpendiculars on a line parallel to the axis, meeting the parabola in F, CD in H, and PM in L."

Let FG be an ordinate to the axis in G; and



the rest remaining as before, let FL $= x$, PL $= y$, the

the parameter $= b$, CP $= a$, FH $= c$, CH $= d$, FG $= e$.

And since PMq $(= \overline{PL + HD}|^q) =$ AM $\times b$, therefore $y^2 + 2ey + e^2 = \overline{AG + FL} \times b = \overline{\frac{e^2}{b} + x} \times b$; and consequently $y^2 + 2ey = bx$.

But CNq + NPq = CPq; that is, $\overline{x - c}|^2 + \overline{y - d}|^2 = a^2$. And substituting for x^2 and x their values $\overline{\frac{y^2 + 2ey}{b^2}}|^2$ and $\frac{y^2 + 2ey}{b}$; you will find

$$y^4 + 4ey^3 + \left.\begin{matrix} 4e^2 \\ -2cb \\ +b^2 \end{matrix}\right\} y^2 - \left.\begin{matrix} 4ecb \\ 2db^2 \end{matrix}\right\} y + \left.\begin{matrix} c^2b^2 \\ + d^2b^2 \\ + a^2b^2 \end{matrix}\right\} = 0,$$

which is a complete biquadratic equation. And by comparing with it the equation $y^4 + py^3 + bqy^2 - b^2ry - b^3s = 0$, you will find FG $(= e) = \frac{1}{4}p$, FH $(= c) = \frac{\frac{1}{4}p^2 + b^2 - bq}{2b}$, HC $(= d) = \frac{b^2r - pbc}{2b^2}$, and CP $(= a) = \sqrt{b^3s + c^2 + d^2}$: which gives a general construction for any such biquadratic equation by any parabola whatsoever. If the signs of p, q, r, or s, are different, it is easy to make the necessary alterations in the construction. *Ex. gr.* If p is negative, then FG must be taken on the other side of the axis.

If you suppose the circle to pass through F, the equation will become a *cubic* having all its

 terms;

terms: the laſt term $\overline{c^2 + d^2 - a^2} \times b^2$ vaniſhing, becauſe then $c^2 + d^2 = a^2$. It will have this form,

$$y^3 + 4ey^2 \left.\begin{array}{l} + 4e^2 \\ - 2cb \\ + b^2 \end{array}\right\} y \left.\begin{array}{l} - 4ecb \\ - 2db^2 \end{array}\right\} = 0;$$

and then "the conſtruction will give the roots of a complete cubic equation."

§ 50. We have ſufficiently ſhewed, how the roots of cubic and biquadratic equations may be conſtructed by the parabola and circle; we ſhall now ſhew how *other* conic ſections may be determined by whoſe interſections the ſame roots may be diſcovered.

Let the equation propoſed be $y^4 * + bpy^2 + b^2qy - b^3r = 0$; and let us ſuppoſe, that,

1°. $bx = y^2$; then ſhall we have by ſubſtitution of b^2x^2 for y^4, and dividing by bp,

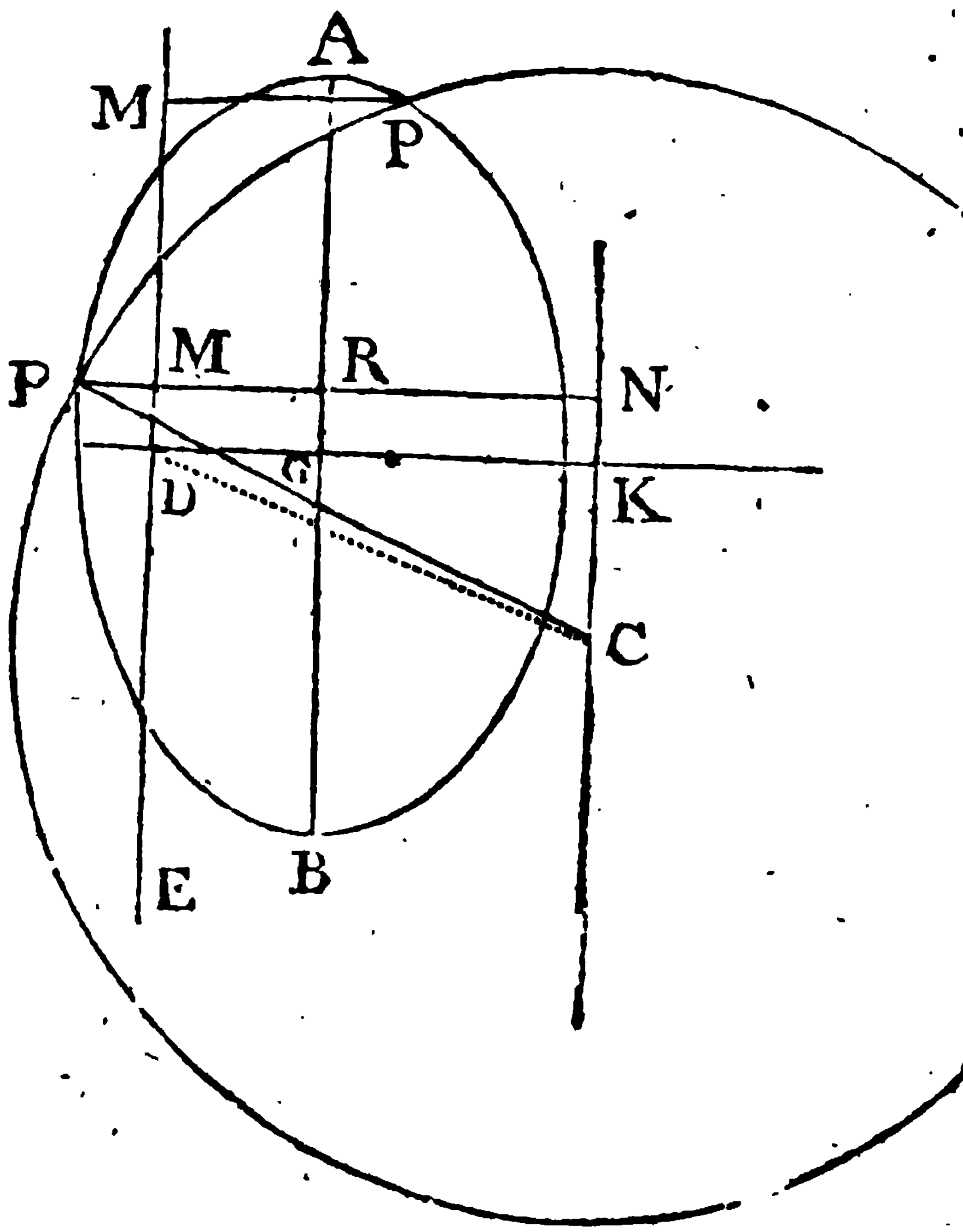
2°. $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, which has its *locus* an *ellipſe*. Then by ſubſtituting (in this laſt) bx for y^2, and multiplying all the terms by $\frac{p}{b}$, you find,

3°. $x^2 + px + qy - br = 0$, an equation to a *parabola* Then, adding to this equation $y^2 - bx = 0$, you will have,

4°. $x^2 + y^2 \left.\begin{array}{l} + p \\ - b \end{array}\right\} x + qy - br = 0$, an equation to a *circle*.

The

The roots of the equation $y^4 * + bpy^2 + b^2qy - b^3r = 0$ may be determiued by the interfection of any *two* of thefe *loci*; as for example,



by the interfections of the *ellipfe* that is the *locus* of the equation $y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, and of the *circle* which is the *locus* of

$x^2 + y^2 \left.\begin{matrix}+p\\-b\end{matrix}\right\} x + qy - br = 0$, from which we deduce this conftruction.

Let

Let AB be the axis of an ellipsis, equal to $\sqrt{br + \frac{bq^2}{4p}}$, *let G be the center of the ellipse, and the axis to the parameter as p to b. At G, raise a perpendicular to the axis, and on it take* $GD = \frac{bq}{2p}$, *and on the other side in the perpendicular continued take* $GK = \frac{1}{2}q \times \frac{p-b}{p}$. *Let DE and KC be parallel to the axis: take* $KC = \frac{1}{2}b - \frac{1}{2}p$, *and from C as a center, with the radius* $\sqrt{DCq + br}$ *describe a circle meeting the ellipse in P, and the ordinate PM, on the line DE, shall be one of the roots of the proposed equation.*

Let PM ($=y$) produced meet AB in R, and KC in N; and calling DM $= x$, then CPq = NPq + NCq, that is, $\frac{1}{4}q^2 + \frac{1}{4}b^2 - \frac{1}{2}pb + \frac{1}{4}p^2 + br = \overline{\frac{1}{2}b - \frac{1}{2}p - x}\,\big|^2 + \overline{y + \frac{1}{2}q}\,\big|^2$; and therefore,

$$1°.\ y^2 + x^2 + qy \left.\begin{matrix} -b \\ +p \end{matrix}\right\} x - br = 0,$$ the equation to the circle, which was to be constructed.

And since PRq : GBq — GRq :: $b : p$, therefore $\overline{y + \frac{bq}{2p}}\,\big|^2 : br + \frac{bq^2}{4p} - x^2 :: b : p$; and consequently,

$$2°.\ y^2 + \frac{b}{p}x^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0;$$ which is the equation that was to be constructed.

Now that their intersections will give the roots required, appears thus.

For

For x^2 in the first equation substitute the value you deduce for it from the second, *viz.* $br - \frac{p}{b}y^2 - qy$, and there will arise

$$\left.\begin{matrix} y^2 - \frac{p}{b}y^2 - b \\ + p \end{matrix}\right\} x = 0, \text{ or } \frac{b-p}{b} \times y^2 = \overline{b-p} \times x;$$

that is, $\frac{y^2}{b} = x$, and $x^2 = \frac{y^4}{b^2}$; which substituted for x^2 and x in the first equation, gives

$\frac{y^4}{b^2} + y^2 + \overline{p-b} \times \frac{y^2}{b} + qy - br = 0$; that is, $y^4 * + bpy^2 + b^2qy - b^3r = 0$.

And if you substitute them in the second equation, there will arise

$\frac{b}{pb^3}y^4 + y^2 + \frac{bq}{p}y - \frac{b^2r}{p} = 0$, that is, $y^4 * + bpy^2 + b^2qy - b^3r = 0$, the very same as before; and thus it appears that the roots of the equation $y^4 * - bpy^2 + b^2qy - b^3r = 0$ are the ordinates that are common to the *circle* and *ellipse*, or that are drawn from their intersection.

End of the THIRD PART.

APPEN-

APPENDIX:

DE

Linearum Geometricarum Proprietatibus generalibus TRACTATUS.

DE

Linearum Geometricarum

Proprietatibus generalibus.

De lineis secundi ordinis, sive sectionibus conicis, scripserunt uberrime geometræ veteres & recentiores; de figuris quæ ad superiores linearum ordines referuntur pauca & exilia tantum ante Newtonum tradiderunt. Vir illustrissimus, in Tractatu de *Enumeratione Linearum tertii Ordinis*, doctrinam hanc, cum diu jacuisset, excitavit, dignamque esse in qua elaborarent geometris ostendit. Expositis enim harum linearum proprietatibus generalibus, quæ vulgatis sectionum conicarum affectionibus sunt adeo affines ut velut ad eandem normam compositæ videantur, alios suo exemplo impulit ut analogiam hanc sive similitudinem quæ tam diversis intercedit figurarum generibus bene cognitam & satis firme animo conceptam atque comprehensam habere studerent. In qua illustranda & ulterius indaganda curam operamque merito posuerunt; cum nihil sit omnium quæ in disciplinis purè mathematicis tractantur quod pulchrius dicatur aut ad animum veri investigandi cupidem oblectandum aptius, quam rerum tam diversarum consensus sive harmonia, ipsiusque doctrinæ compositio & nexus admirabilis, quo posterius priori convenit, quod sequitur superiori respondet,

quæque

quæque ſimpliciora ſunt ad magis ardua viam conſtanter aperiunt.

Linearum tertii ordinis proprietates generales a *Newtono* traditæ parallelarum ſegmentá & aſymptotos pleræque ſpectant. Alias harum affectiones quaſdam diverſi generis breviter indicavimus in tractatu de fluxionibus nuper edito, Art. 324, & 401. Celeberrimus *Coteſius* pulcherrimam olim detexit linearum geometricarum proprietatem, hucuſque ineditam, quam abſque demonſtratione nobis communicavit vir Reverendus D. *Robertus Smith*, Collegii S. S. Trinitatis apud Cantabrigienſes præfectus, doctrina operibuſque ſuis pariter ac fide & ſtudio in amicos clarus. De his meditantibus nobis alia quoque ſe obtulerunt theoremata generalia; quæ cum ad arduam hanc geometriæ partem augendam & illuſtrandam conducere viderentur, ipſa quaſi in faſciculum congerenda & una ſerie breviter exponenda & demonſtranda putavimus.

SECTIO I.

De Lineis Geometricis in genere.

§ 1. LIneæ ſecundi ordinis ſectione ſolidi geometrici, coni ſcilicet, definiuntur, unde earum proprietates per vulgarem geometriam optime derivantur. Verum diverſa eſt ratio figurarum quæ ad ſuperiores linearum ordines referuntur. Ad has definiendas, earumque proprietates eruendas, adhibendæ ſunt æquationes generales co-ordinatarum relationem exprimentes. Repræ-
Fig. 1. ſentet x abſciſſam AP, y ordinatam PM figuræ FMH, denotentque a, b, c, d, e, &c. coefficientes quaſcunque inva-

invariabiles; & dato angulo APM si relatio co-ordinatarum x & y definiatur æquatione quæ, præter ipsas co-ordinatas, solas involvat coefficientes invariabiles, linea FMH geometrica appellatur; quæ quidem auctoribus quibusdam linea algebraica, aliis linea rationalis dicitur. Ordo autem lineæ pendet ab indice altissimo ipsius x vel y in terminis æquationis a fractionibus & surdis liberatæ; vel a summa indicis utriusque in termino ubi hæc summa prodit maxima. Termini enim x^2, xy, y^2 ad secundum ordinem pariter referuntur; termini x^3, x^2y, xy^2, y^3 ad tertium. Itaque æquatio $y = ax + b$, sive $y - ax - b = 0$, est primi ordinis & designat lineam sive locum primi ordinis, quæ quidem semper recta est. Sumatur enim in ordinata PM recta PN ita ut PN sit ad AP ut $+a$ ad unitatem; constituatur AD parallela ordinatæ PM æqualis ipsi $+b$, & ducta DM parallela rectæ AN erit locus cui æquatio proposita respondebit. Nam PM $=$ PN $+$ NM $=$ ($a \times$ AP $+$ AD) $ax + b$. Quod si æquatio sit formæ $y = ax - b$ vel $y = - ax + b$, recta AD, vel PN, sumenda est ad alteram partem abscissæ AP; contrarius enim rectarum situs contrariis coefficientium signis respondet. Si valores affirmativi ipsius x designent rectas ad dextram ductas a principio abscissæ A, valores negativi denotabunt rectas ab eodem principio ad sinistram ductas; & similiter si valores affirmativi ipsius y ordinatas repræsentent supra abscissam constitutas, negativi designabunt ordinatas infra abscissam ad oppositas partes ductas. Fig. 1.

Æquatio generalis ad lineam secundi ordinis est hujus formæ

$$\left.\begin{matrix} y^2 - axy + cx^2 \\ - by - dx \\ + e \end{matrix}\right\} = 0,$$

& æquatio generalis ad lineas tertii ordinis est $y^3 - \overline{ax + b} \times y^2 + \overline{cxx - dx + e} \times y - fx^3 + gx^2 - hx + k = 0$. Et similibus æquationibus definiuntur lineæ geometricæ superiorum ordinum.

§ 2. Linea geometrica occurrere potest rectæ in tot punctis quot sunt unitates in numero qui æquationis vel lineæ ordinem designat, & nunquam in pluribus. Occursus curvæ & abscissæ AP definiuntur ponendo $y = 0$, quo in casu restat tantum ultimus æquationis terminus quem y non ingreditur. Linea tertii ordinis *ex. gr.* occurrit abscissæ AP cum $fx^3 - gx^2 + hx - k = 0$, cujus æquationis si tres radices sint reales abscissa secabit curvam in tribus punctis. Similiter in æquatione generali cujuscunque ordinis index altissimus abscissæ x equalis est numero qui lineæ ordinem designat, sed nunquam major, adeoque is est numerus maximus occursuum curvæ cum abscissa vel alia quavis rectâ. Cum autem æquationis cubicæ unica saltem radix sit semper realis, idemque constet de æquatione quavis quinti aut imparis cujusvis ordinis (quoniam radix quævis imaginaria aliam necessario semper habet comitem), sequitur lineam tertii aut imparis cujuscunque ordinis rectam quamvis asymptoto non parallelam in eodem plano ductam in uno saltem puncto necessario secare. Si vero recta sit asymptoto parallela, in hoc casu vulgo dicitur curvæ occurrere ad distantiam infinitam. Linea igitur imparis cujuscunque ordinis duo saltem habet crura in infinitum progredientia. Æquationis autem quadraticæ vel paris cujusvis ordinis radices omnes nonnunquam fiunt imaginariæ, adeoque fieri potest ut recta in plano lineæ paris ordinis ducta eidem nullibi occurrat.

§ 3. Æquatio secundi aut superioris cujuscunque ordinis quandoque componitur ex tot simplicibus, a surdis & fractis liberatis, in se mutuo ductis quot sunt ipsius æquationis propositæ dimensiones; quo in casu figura FMH non est curvilinea sed conflatur ex totidem rectis, quæ per simplices has æquationes definiuntur ut in *Art.* 1. Similiter si æquatio cubica componatur ex æquationibus duabus in se mutuo ductis, quarum altera sit quadratica altera simplex, locus non erit linea tertii ordinis proprie sic dicta, sed sectio conica cum rectâ adjunctâ. Proprietatis autem quæ de lineis geometricis superiorum ordinem generaliter demonstrantur, affirmandæ sunt quoque de lineis inferiorum ordinum, modo numeri harum ordines designantes simul sumpti numerum compleant qui ordinem dictæ superioris lineæ denotat. Quæ de lineis tertii ordinis (*ex. gr.*) generaliter demonstrantur affirmanda quoque sunt de tribus rectis in eodem plano ductis, vel de sectione conica cum unica quavis recta simul in eodem plano descriptis. Ex altera parte, vix ulla assignari potest proprietas lineæ ordinis inferioris satis generalis cui non respondeat affectio aliqua linearum ordinum superiorum. Has autem ex illis derivare non est cujusvis diligentiæ. Pendet hæc doctrina magna ex parte a proprietatibus æquationum generalium, quas hic memorare tantum convenit.

§ 4. In æquatione quacunque coefficiens secundi termini æqualis est excessui quo summa radicum affirmativarum superat summam negativarum; & si desit hic terminus, indicio est summas radicum affirmativarum & negativarum, vel summas ordinatarum ad diversas partes abscissæ constitutarum, æquales esse. Sit equatio generalis ad lineam ordinis n,

$y^n - \overline{ax+b} \times y^{n-1} + \overline{cxx - dx + e} \times y^{n-2} - \&c. = 0$, supponatur $u = y - \frac{ax+b}{n}$, pro y substituatur ipsius valor $u + \frac{ax+b}{n}$; & in equatione transformata deerit secundus terminus u^{n-1}; ut ex calculo, vel ex doctrinâ æquationum passim traditâ facile patet: & hinc quoque constat, quod per hypothesim valor quisque ipsius u minor sit valore correspondente ipsius y differentia $\frac{ax+b}{n}$; unde sequitur summam valorum ipsius u (quorum numerus est n) deficere a summa valorum ipsius y (quæ summa est $ax+b$) differentia $\frac{ax+b}{n} \times n = ax + b$, adeoque priorem summam evanescere & secundum terminum deesse in æquatione qua u definitur, vel affirmativos & negativos valores ipsius u æquales summas conficere. Si itaque sumatur $PQ = \frac{ax+b}{n}$, ut sit $QM = u$, rectæ ex utraque parte puncti Q ad curvam terminatæ eandem conficient summam. Locus autem puncti Q est recta BD quæ abscissam ultra principium A

Fig. 3. productam secat in B ita ut $AB = \frac{b}{a}$, & ordinatam AD ipsi PM parallelam in D ita ut sit $AD = \frac{1}{n} \times b$; si enim hæc recta ordinatæ PM occurrat in puncto Q, erit PQ ad PB (seu $\frac{b}{a} + x$) ut AD ad AB vel a ad n, adeoque $PQ = \frac{ax+b}{n}$, ut oportebat. Atque hinc constat rectam semper duci posse quæ parallelas quasvis lineæ geometricæ occurrentes in tot punctis quot sunt figuræ dimensiones ita secabit ut summa segmentorum cujusvis paral-

parallelæ ex una secantis parte ad curvam terminatorum semper æqualis fit summæ segmentorum ejusdem ex altera secantis parte. Manifestum autem est rectam quæ duas quasvis parallelas hac ratione secat ipsam necessario esse quæ similiter alias omnes parallelas secabit. Atque hinc patet veritas theorematis *Newtoniani*, quo continetur proprietas linearum geometricarum generalis, notissimæ sectionum conicarum proprietati analoga. In his enim recta quæ duas quasvis parallelas ad sectionem terminatas bisecat diameter est, & bisecat alias omnes hisce parallelas ad sectionem terminatas. Et similiter recta quæ duas quasvis parallelas lineæ geometricæ occurrentes in tot punctis quot ipsa est dimensionum ita secat ut summa partium ex uno secantis latere consistentium & ad curvam terminatarum æqualis sit summæ partium ejusdem parallelæ ex altero secantis latere consistentium ad curvam terminatarum, eodem modo secabit alias quasvis rectas his parallelas.

§ 5. In æquatione quavis terminus ultimus, sive is quem radix y non ingreditur, æqualis est facto ex radicibus omnibus in se mutuo ductis; unde ad aliam ducimur non minus generalem linearum geometricarum proprietatem. Occurrat recta PM lineæ tertii ordinis in M, m & μ, eritque $PM \times Pm \times P\mu = fx^3 - gx^2 + bx - k$. Secet abscissa AP curvam in tribus punctis I, K, L; & AI, AK, AL, erunt valores abscissæ x, posita ordinata $y = 0$, quo in casu æquatio generalis dat $fx^3 - gx^2 + bx - k = 0$ pro his valoribus determinandis, ut in Art. 2. exposuimus. Æquationis igitur $x^3 - \frac{gx^2}{f} + \frac{bx}{f} - \frac{k}{f} = 0$ tres radices sunt AI, AK, AL; adeoque hæc æquatio componitur ex tribus $x - AI$, $x - AK$, $x - AL$ in se Fig. 1.

ſe mutuo ductis; eſtque $x^3 - \frac{gx^2}{f} + \frac{hx}{f} - \frac{k}{f} = \overline{x - AI} \times \overline{x - AK} \times \overline{x - AL} = \overline{AP - AI} \times \overline{AP - AK} \times \overline{AP - AL} = IP \times KP \times LP = \frac{1}{f} \times PM \times Pm \times P\mu$. Factum igitur ex ordinatis PM, Pm, Pμ ad punctum P & curvam terminatis eſt ad factum ex ſegmentis IP, KP, LP, rectæ AP, eodem puncto & curva terminatis in ratione invariabili coefficientis f ad unitatem. Simili ratione demonſtratur, dato angulo APM, ſi rectæ AP, PM, lineam geometricam cujuſvis ordinis ſecent in tot punctis quot ipſa eſt dimenſionum, fore ſemper factum ex ſegmentis prioris ad punctum P & curvam terminatis ad factum ex ſegmentis poſterioris eodem puncto & curva terminatis in ratione invariabili.

§ 6. In articulo præcedente ſuppoſuimus, cum *Newtono*, rectam AP lineam tertii ordinis ſecare in tribus punctis I, K, L; verum ut theorema egregium reddatur generalius, ſupponamus abſciſſam AP in unico tantum puncto curvam ſecare; ſitque id punctum A.

Fig. 4. Quoniam igitur evaneſcente y evaneſcat quoque x, ultimus æquationis terminus, in hoc caſu, erit $fx^3 - gx^2 + hx = fx \times \overline{xx - \frac{gx}{f} + \frac{h}{f}} = fx \times \overline{x - \frac{g}{2f}}^2 + \frac{h}{f} - \frac{gg}{4ff}$ (ſi ſumatur Aa verſus P æqualis $\frac{g}{2f}$, & ad punctum a erigatur perpendicularis $ab = \left(\frac{\sqrt{4fh - gg}}{2f}\right) = f \times AP \times \overline{aP^2 + ab^2} = f \times AP \times bP^2$; unde cum PM $\times$ Pm $\times$ Pμ, ſit æqualis ultimo termino $fx^3 - gx^2 + hx$,

ut

ut in articulo præcedente; erit PM $\times$ Pm $\times$ Pμ ad AP $\times$ bP^2 in ratione conſtante coefficientis f ad unitatem. Valor autem rectæ perpendicularis ab eſt ſemper realis quoties recta AP curvam in unico puncto ſecat; in hoc enim caſu radices æquationis quadraticæ $fx^2 + gx + h$ ſunt neceſſario imaginariæ, adeoque $4fh$ major quam gg, & quantitas $\sqrt{4fh - gg}$ realis. Cum igitur recta quævis in unico puncto A ſecat lineam tertii ordinis, eſt ſolidum ſub ordinatis PM, Pm, Pμ ad ſolidum ſub abſciſſa AP & quadrato diſtantiæ puncti P a puncto dato b in ratione conſtanti. Juncta Ab eſt ad Aa, ſive radius ad coſinum anguli bAP, ut $\sqrt{4fh}$ ad g, & A$b = \sqrt{\frac{h}{f}}$. Idem vero punctum b ſemper convenit eidem rectæ AP, qualiſcunque ſit angulus qui abſciſſâ & ordinatâ continetur.

§ 7. Sit figura ſectio conica, cujus æquatio generalis ſit $yy - \overline{ax - b} \times y + cxx - dx + e = 0$ ut ſupra; & Fig. 5.
ſi æquationis $cxx - dx + e = 0$ radices ſint imaginariæ, recta AP ſectioni non occurret. In hoc autem caſu quantitas $4ec$ ſemper ſuperat ipſam dd; unde cum ſit $cxx - dx + e = c \times \overline{x - \frac{d}{2c}}^2 + e - \frac{dd}{4c}$ (ſi ſumatur A$a = \frac{d}{2c}$ & erigatur ab perpendicularis abſciſſæ in a ita ut $ab = \frac{\sqrt{4ec - dd}}{2c}$) $= c \times \overline{a\text{P}^2 + ab^2} = c \times b\text{P}^2$, ſitque PM $\times$ P$m = cxx - dx + e$, erit PM $\times$ Pm ad bP^2 ut c ad unitatem. Itaque in ſectione quavis conica ſi recta AP ſectioni non occurrat, erit, dato angulo APM, rectangulum contentum ſub rectis ad punctum P conſiſtentibus & ad curvam terminatis ad quadratum

tum distantiæ puncti P a puncto dato b in ratione constanti, quæ in circulo est ratio æqualitatis. Manifestum autem est eandem methodum adhiberi posse lineæ quarti ordinis quam abscissa secat in duobus tantum punctis, vel lineæ ordinis cujuscunque quam abscissa secat in punctis binario paucioribus numero qui figuræ ordinem designat.

§ 8. Hisce præmissis, progredimur ad linearum geometricarum proprietates minus obvias exponendas eodem fere ordine quo se nobis obtulerunt. Utebamur autem lemmate sequenti ex fluxionum doctrina petito, quodque in tractatus de hisce nuper editi Art. 717, demonstravimus; harum tamen aliquas per algebram vulgarem demonstrari posse postea observavimus.

Lemma. Si quantitatibus x, y, z, u, &c. simul fluentibus, ut & quantitatibus X, Y, Z, V, &c. sit factum ex prioribus ad factum ex posterioribus in ratione constanti quacunque, erit $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{v}}{v} + \&c. = \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} + \frac{\dot{V}}{V} + \&c.$ Porro, brevitatis gratia, quantitates appellamus sibi mutuo *reciprocas*, quarum in se mutuo ductarum factum est unitas, sic $\frac{1}{x}$ dicimus *reciprocam* esse ipsius x, & $\frac{1}{y}$ ipsius y.

§ 9. Theor. I. *Occurrat recta quavis per punctum datum ducta lineæ geometricæ cujuscunque ordinis in tot punctis quot ipsa est dimensionum; rectæ figuram in his punctis contingentes abscindant ab alia rectâ positione data per idem punctum datum ductâ segmenta totidem hoc puncto terminata; & horum segmentorum* reciproca *eandem*

dem semper conficient summam, modo segmenta ad contrarias partes puncti dati sita contrariis signis afficiantur.

Sit P punctum datum, PA & Pa rectæ quævis duæ ex P ductæ quarum utraque curvam secat in tot punctis A, B, C, &c. a, b, c, &c. quot ipsa est dimensionum. Abscindant tangentes AK, BL, CM, &c. et ak, bl, cm, &c. a recta EP per punctum datum P ducta segmenta PK, PL, PM, &c. et Pk, Pl, Pm, &c. dico fore $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c. = \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} + \&c.$ atque hanc summam manere semper eandem manente puncto P & recta PE positione data. Fig. 6.

Supponamus enim rectas ABC, abc motibus sibi parallelis deferri, ita ut earum occursus P progrediatur in recta PE positione data; cumque sit semper AP × BP × CP × &c. ad aP × bP × cP in ratione constanti per Art. 5. repræsentet $\dot{A}$P fluxionem ipsius AP, $\dot{B}$P fluxionem rectæ BP, & $\dot{C}$P, $\dot{E}$P, &c. fluxiones rectarum CP, EP, &c. respectivas, ut vitetur inutilis symbolorum multiplicatio, eritque (per Art. 8.) $\frac{\dot{A}P}{AP} + \frac{\dot{B}P}{BP} + \frac{\dot{C}P}{CP} + \&c. = \frac{\dot{a}P}{aP} + \frac{\dot{b}P}{bP} + \frac{\dot{c}P}{cP} + \&c.$ Verum cum recta AP motu sibi semper parallelo deferatur, notissimum est $\dot{A}$P fluxionem rectæ AP esse ad $\dot{E}$P fluxionem rectæ EP ut AP ad subtangentem PK, adeoque $\frac{\dot{A}P}{AP} = \frac{\dot{E}P}{PK}$. Similiter $\frac{\dot{B}P}{BP} = \frac{\dot{E}P}{PL}$, $\frac{\dot{C}P}{CP} = \frac{\dot{E}P}{PM}$, $\frac{\dot{a}P}{aP} = \frac{\dot{E}P}{Pk}$, $\frac{\dot{b}P}{bP} = \frac{\dot{E}P}{Pl}$, & $\frac{\dot{c}P}{cP} = \frac{\dot{E}P}{Pm}$, unde $\frac{\dot{E}P}{PK} + \frac{\dot{E}P}{PL}$ +

$$+\frac{EP}{PM}+\&c.=\frac{EP}{Pk}+\frac{EP}{Pl}+\frac{EP}{Pm}+\&c.\ \text{et}\ \frac{1}{PK}$$
$$+\frac{1}{PL}+\frac{1}{PM}+\&c.=\frac{1}{Pk}+\frac{1}{Pl}+\frac{1}{Pm}+\&c.$$

Hæc ita se habent quoties puncta K, L, M, &c. et *k*, *l*, *m*, &c. sunt omnia ad easdem partes puncti P, adeoque fluxiones rectarum AP, BP, CP, &c. *a*P, *b*P, *c*P, &c. omnes ejusdem signi. Si vero, cæteris manentibus, puncta quævis M et *m* cadant ad contrarias partes puncti P, tum crescentibus reliquis ordinatis AP, BP, &c. necessario minuuntur ordinatæ CP & *c*P, earumque fluxiones pro subdititiis seu negativis habendæ sunt; adeoque in hoc casu $\frac{1}{PK}+\frac{1}{PL}-\frac{1}{PM}$ &c. $=\frac{1}{Pk}+\frac{1}{Pl}-\frac{1}{Pm}$, &c. & generaliter in summis hisce colligendis, termini iisdem vel contrariis signis afficiendi sunt, prout segmenta cadunt ad easdem vel ad contrarias partes puncti dati P.

Fig. 7.

§ 10. Si recta PE occurrat curvæ in tot punctis D, E, I, &c. quot ipsa est dimensionum, summa $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}+$ &c. quam constantem seu invariatam manere ostendimus, æqualis erit summæ seu aggregato $\frac{1}{PD}+\frac{1}{PE}+\frac{1}{PI}+$ &c. i. e. summæ reciprocarum segmentis rectæ PE positione datæ puncto dato P & curva terminatis: in qua, si segmentum quodvis sit ad alteras partes puncti P, hujus reciproca subducenda est.

§ 11.

§ 11. Si figura ſit ſectio conica, cui recta PE nullibi occurrat, inveniatur punctum b ut in Art. 7. jungatur Pb, huic ducatur ad rectos angulos bd rectam PE ſecans in d, eritque $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$. Eſt enim PA × PB ad bP^2 in ratione conſtanti, adeoque (per Art. 8.) $\frac{\dot{A}P}{AP} + \frac{\dot{B}P}{BP} = \frac{2b\dot{P}}{bP}$, unde (quoniam $\dot{A}P$ eſt ad $\dot{E}P$ ut AP ad PK, $\dot{B}P$ ad $\dot{E}P$ ut BP ad PL, & $b\dot{P}$ ad $\dot{E}P$ ut bP ad dP) $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$. Fig. 8.

§ 12. Similiter ſi recta EP occurrat lineæ tertii ordinis in unico puncto D, inveniatur punctum b ut in Art. 6. recta bd perpendicularis in junctam bP occurrat rectæ EP in d, & quoniam AP × BP × CP eſt ad DP × bP^2 in ratione conſtanti (*ibid.*) erit $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} = \frac{1}{PD} + \frac{2}{Pd}$. Si autem Pb perpendicularis ſit in rectam EP, evaneſcet $\frac{2}{Pd}$. Fig. 9.

§ 13. Aſymptoti linearum geometricarum ex data plaga crurum infinitorum per hanc propoſitionem determinantur; eæ enim conſiderari poſſunt tanquam tangentes cruris in infinitum producti. Recta PA aſymptoto parallela curvæ occurrat in punctis A, B, &c. recta autem PE curvam ſecet in D, E, I, &c. ſumatur in hac recta PM ita ut $\frac{1}{PM}$ ſit æqualis exceſſui quo ſumma $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} +$ &c. ſuperat ſummam $\frac{1}{PK} +$ $\frac{1}{PL} +$ &c. & aſymptotos tranſibit per M, ſi vero Fig. 10.

æquales

æquales ſint hæ ſummæ, crus curvæ parabolicum erit, aſymptoto abeunte in infinitum.

§ 14. Ad *curvaturam* linearum geometricarum unico
Fig. 11. theoremate generali definiendam, ſit CDR circulus cui occurrant recta PR in D & R, & recta PC in C & N; ſecet tangens CM rectam PD in M, atque manente recta DR, ſupponamus rectam PCN deferri motu ſibi ſemper parallelo donec coincidant puncta P, D, C, & quæratur ultimus valor differentiæ $\frac{1}{PM} - \frac{1}{PD}$. In recta PN ſumatur punctum quodvis q, occurrat qv parallela tangenti CM rectæ DR in v; ducatur DQ parallela ipſi PN, & QV (parallela rectæ circulum contingenti in D) ſecet DR in V. Erit itaque $\frac{1}{PM} - \frac{1}{PD} = \frac{DM}{PM \times PD}$ (quoniam $DM \times MR = CM^2$) $= \frac{CM^2 \times PM}{PM^2 \times PD \times MR}$ $= \frac{qv^2 \times PM}{Pv^2 \times MR \times PM + Pv^2 \times MR \times MD}$ (cum $MR \times MD$, ſeu CM^2, ſit ad PM^2 ut qv^2 ad Pv^2) $=$ $\frac{qv^2 \times PM}{Pv^2 \times MR \times PM + qv^2 \times PM^2} = \frac{qv^2}{Pv^2 \times MR + qv^2 \times PM}$, cujus ultimus valor, evaneſcente PM & coincidentibus qv & Pv cum QV & DV, eſt $\frac{QV^2}{DV^2 \times DR}$. Atque idem eſt valor ultimus differentiæ $\frac{1}{PM} - \frac{1}{PD}$ ſi D & C ſint in arcu lineæ cujuſvis ejuſdem curvaturæ cum circulo CDR.

Fig. 12. § 15. Theor. II. *Ex puncto quovis* D *lineæ geometricæ ducantur duæ quævis rectæ* DE, DA, *quarum utraque eam ſecet in tot punctis* D, I, E, *&c.* & D, A, B, *&c. quot ipſa eſt dimenſionum; abſcindant tangentes* AK,

AK, BL, *&c. a recta* DE *segmenta* DK, DL, *&c. Occurrat recta quævis*, QV *tangenti* DT *parallela ipsis* DA *&* DE *in* Q *&* V, *sitque* QV^2 *ad* DV^2 *ut m ad* 1; *sumatur in* DE *recta* DR *ita ut* $\frac{m}{DR}$ *æqualis sit excessui summæ* $\frac{1}{DE} + \frac{1}{DI} +$ *&c. supra summam* $\frac{1}{DK} + \frac{1}{DL} +$ *&c. & circulus supra chordam* DR *descriptus rectam* DT *contingens erit circulus osculatorius, sive ejusdem curvaturæ cum linea geometrica proposita, ad punctum* D.

Ostendimus enim in Art. 10. (Fig. 6.) generaliter summam $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. $= \frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} +$ &c. & in Art. præcedente invenimus valorem ultimum differentiæ $\frac{1}{PM} - \frac{1}{PD}$, coincidentibus punctis P, D & C, esse $\frac{QV^2}{DV^2 \times DR} = \frac{m}{DR}$ si circulus ejusdem curvaturæ cum linea geometrica ad punctum D rectæ DE occurrat in R. Unde sequitur fore $\frac{m}{DR} = \frac{1}{DE} + \frac{1}{DI} +$ &c. $- \frac{1}{DK} - \frac{1}{DL} -$ &c. sive reciprocam ipsi $\frac{1}{m} \times DR$ esse æqualem excessui quo summa reciprocarum segmentis puncto D & curva terminatis superat summam reciprocarum segmentis eodem puncto & tangentibus AK, BL, &c. terminatis. Quoties autem excessus hic evadit negativus, chorda DR sumenda est ad alteras partes puncti D, semperque adhibenda est regula superius descripta pro signis terminorum dignoscendis. Si recta DA bisecet angulum EDT recta

recta DE & tangente DT contentum, theorema fit paulo simplicius. Hoc enim in casu $QV=DV$, $m=1$, & $\frac{1}{DR}$ æqualis excessui quo $\frac{1}{DE}+\frac{1}{DI}+$ &c. superat $\frac{1}{DK}+\frac{1}{DL}+$ &c.

§ 16. Ex eodem principio consequitur theorema generale quo determinatur variatio curvaturæ vel mensura anguli contactus curva & circulo osculatorio contenti, in linea quavis geometrica; præmittenda tamen est explicatio brevis variationis curvaturæ, cum hæc non satis dilucide apud auctores descripta sit. Linea quævis curva a tangente flectitur per curvaturam suam, cujus eadem est mensura ac anguli contactus curva & tangente contenti; & similiter curva a circulo osculatorio inflectitur per variationem curvaturæ suæ, cujus variationis eadem est mensura ac anguli contactus curva & circulo osculatorio comprehensi. Occurrat recta TE tangenti DT perpendicularis curvæ in E & circulo osculatorio in r, & variatio curvaturæ erit ultimo ut Er subtensa anguli contactus EDr si detur DT; cumque dato angulo contactus EDr sit Er ultimo ut DT^3, ut ex Art. 369. tractatus de fluxionibus colligitur, generaliter curvaturæ variatio erit ultimo ut $\frac{Er}{DT^3}$. Utimur circulo ad curvaturam aliarum figurarum definiendam; verum ad variationem curvaturæ mensurandam, quæ in circulo nulla est, adhibenda est parabola vel sectio aliqua conica. Quemadmodum autem ex circulis numero indefinitis qui curvam datam in puncto dato contingere possunt, unicus dicitur osculatorius qui curvam adeo intime tangit ut nullus alius circulus inter hunc & curvam duci possit; similiter omnium parabolarum quæ eandem habent curvaturam cum linea proposita ad punctum datum

Fig. 13.

(sunt

(sunt autem hæ quoque numero infinitæ) ea eandem simul habet curvaturæ variationem, quæ, non solùm arcum curvæ tangit & osculat, sed adeo premit ut nullus alius arcus parabolicus duci possit inter eas, reliquis omnibus arcubus parabolicis transeuntibus vel extra vel intra utrasque. Qua vero ratione hæc parabola determinari possit, ex iis quæ alibi fusius explicavimus facile intelligitur.

Sit DE arcus curvæ, DT tangens, TEK recta tangenti perpendicularis, sitque rectangulum ET × TK semper æquale quadrato tangentis DT, & curva SKF locus puncti K, qui rectæ DS curvæ normali occurrat in S, quemque tangat in S recta SV tangentem TD secans in V. Recta DS erit diameter circuli osculatorii, & bisecta DS in *s*, erit *s* centrum curvaturæ; juncta autem V*s*, si angulus SDN constituatur æqualis angulo *s*VD ex altera parte rectæ DS, & recta DN circulo osculatorio occurrat in N; tum parabola diametro & parametro DN descripta, quæque rectam DT contingit in D, ipsa erit cujus contactus cum linea proposita in D intimus erit atque maxime perfectus seu proximus. Omnes autem parabolæ aliâ quavis chorda circuli osculatorii tanquam diametro & parametro descripta, & rectam DT contingentes in D, eandem habent curvaturam cùm linea proposita in puncto D. Qualitas curvaturæ a *Newtono* in opere posthumo nuper edito explicata est potius variatio radii curvaturæ; est enim ut fluxio radii curvaturæ applicata ad fluxionem curvæ, vel (si R denotet radium circuli osculatorii & S arcum curvæ ut $\frac{\dot{R}}{\dot{S}}$. Ipsa autem curvatura est inverse ut radius R, & variatio curvaturæ ut $\frac{-\dot{R}}{RR\dot{S}}$, quæ est mensura anguli

guli contactus curvâ & circulo osculatorio contenti. Harum autem una ex alterâ datâ facile derivatur. Variatio radii curvaturæ in curvâ quavis DE est ut tangens anguli DVS vel DV*s*, & in parabolâ quavis est semper ut tangens anguli contenti diametro per punctum contactus transeunte & rectâ ad curvam perpendiculari. Hæc ex theoremate sequenti generali deduci possunt.

Fig. 14. § 17. Theor. III. *Sit* D *punctum in linea quavis geometrica datum, occurrat* DS *diameter circuli osculatorii per* D *ducta curvæ in tot punctis* D, A, B, *&c. quot ipsa est dimensionum; ducatur* DT *curvam contingens in* D, *quæ curvam secet in punctis* I, *&c. binario paucioribus, & occurrat tangentibus* AK, BL, *&c. in* K, L, *&c. eritque variatio curvaturæ, sive mensura anguli contactus curva & circulo osculatoria comprehensi, directe ut excessus quo summa reciprocarum segmentis tangentis* DT *puncto contactus* D *& tangentibus* AK, BL, *&c. terminatis superat summam reciprocarum segmentis eodem puncto & curva terminatis, & inverse ut radius curvaturæ, i. e. ut* $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} + \&c. - \frac{1}{DI} - \&c.}$

Ducatur enim recta D*k* curvam secans in *e*, *i*, &c. circulum osculatorium in R; sitque angulus *k*DT quam minimus; hujus supplementum ad duos rectos bisecetur recta D*ab*, quæ lineæ geometricæ propositæ occurrat in punctis D, *a*, *b*, &c. & ductæ tangentes *ak*, *bl*, &c. secent rectam D*k* in punctis *k*, *l*, &c. eritque per propositionem præcedentem $\frac{1}{DR} = \frac{1}{De} + \frac{1}{Di} - \frac{1}{Dk} - \frac{1}{Dl}$, &c. Unde $\frac{1}{DR} - \frac{1}{De}$ (sive $\frac{Re}{DR \times De}$) $= \frac{1}{Di} -$

$-\frac{1}{Dk}-\frac{1}{Dl}-$ &c. Proinde coincidentibus rectis Dk & DK, seu evanescente angulo kDK, erit ultimo $\frac{Re}{DR \times De}$ æqualis $\frac{1}{DI}-\frac{1}{DK}-\frac{1}{DL}-$ &c. Sit erT perpendicularis tangenti in T, atque occurrat circulo osculatorio in r; cumque sit re ultimo ad Re ut eT ad De, erit ultimo $\frac{Re}{DR \times De}=\frac{re}{DR \times eT}=\frac{re \times DS}{DR \times DT^2}$ sive $\frac{re \times DS}{DT^3}$. Mensura autem anguli contactus rDe curva & circulo osculatorio contenti, sive variatio curvaturæ, est ut $\frac{re}{DT^3}$ adeoque ut $\frac{1}{DS} \times \frac{1}{DI}-\frac{1}{DK}-\frac{1}{DL}$, &c.

§ 18. Variatio autem radii curvaturæ, sive hujus qualitas a *Newtono* descripta, ex priori facillime colligitur. Junctis enim SI, SK, SL, &c. erit hæc variatio radii osculatorii ut excessus quo summa tangentium angulorum DKS, DLS, &c. superat summam tangentium angulorum DIS, &c. Crescit autem curvatura a puncto D versus e, & minuitur radius osculatorius, quoties arcus De tangit circulum osculatorium DR interne, vel cum $\frac{1}{DK}+\frac{1}{DL}+$ &c. superat $\frac{1}{DI}+$ &c. at contrà minuitur curvatura a D versus e, & augetur radius circuli osculatorii, quoties arcus curvæ De tangit arcum circularem externe vel transit intra circulum & tangentem adeoque cum DR sit ultimo minor quam De vel cum $\frac{1}{DI}+$ &c. superat $\frac{1}{DK}+\frac{1}{DL}+$ &c.

§ 19. Sumatur igitur in tangente DT recta DV ita ut $\frac{1}{DV}=\frac{1}{DK}+\frac{1}{DL}+$ &c. $-\frac{1}{DI}-$ &c: jungatur ſV, conſtituatur angulus SDN æqualis DVſ, atque occurrat rectâ DN circulo oſculatorio in N; & parabola diametro DN deſcripta, cujus parameter eſt DN, quæque rectam DT contingit in D, eandem habebit variationem curvaturæ cum linea geometrica propoſita in puncto D. Ex iiſdem principiis alia quoque theoremata deducuntur, quibus variatio curvaturæ in lineis geometricis generaliter definitur.

§ 20. Ut hæc theoremata ad formam magis geometricam reducantur, lemmata quædam ſunt præmittenda, quibus doctrina de diviſione rectarum harmonicâ amplior & generalior reddatur. In recta quavis DI, ſumptis æqualibus ſegmentis DF & FG, ducantur a puncto quovis V quod non eſt in rectâ DI tres rectæ VD, VF, VG, & quarta VL ipſi DI parallela, atque hæ quatuor rectæ, a Cl. D. *De la Hire*, Harmonicales dicuntur. Recta vero quævis, quæ quatuor harmonicalibus occurrit ab iiſdem harmonice ſecatur. Occurrat recta DC harmonicalibus VD, VF, VG, & VL in punctis D, A, B, C; eritque DA ad DC ut AB ad BC. Ducatur enim per punctum A recta MAN ipſi DI parallela, quæ occurrat rectis VD & VG in M & N; & ob æquales DF & FG, æquales erunt MA & AN. Eſt autem DA ad DC ut AM (ſive AN) ad VC, adeoque ut AB ad BC. Manifeſtum eſt rectam, quæ uni harmonicalium parallela eſt, dividi in æqualia ſegmenta a tribus reliquis. Occurrat recta BH parallela ipſi VF reliquis VG, VC, VD in B, K, & H; eritque VK ad KB ut FG (vel DF) ad VF adeoque ut VK ad KH, & proinde BK = KH.

Fig. 15.

§ 21. Hinc ſequitur, ſi recta quævis a quatuor rectis ab eodem puncto ductis ſecetur harmonice, aliam quamvis

vis rectam quæ his quatuor rectis occurrit harmonice secari ab iisdem; eam vero quæ parallela est uni quatuor rectarum in segmenta æqualia dividi a tribus reliquis, Sit DA ad DC ut AB ad BC, jungantur VA, VB, VC, & VD; occurrant rectæ MAN, DFG ipsi VC parallelæ rectis VD, VA, & VB in M, A, N & D, F, G; eritque MA ad VC ut DA ad DC, vel AB ad BC, adeoque ut AN ad VC; MA = AN, & DF = FG, &, per præcedentem, recta quævis quæ ipsis VD, VA, VB, VC occurrit harmonice secabitur ab iisdem.

§ 22. Ex puncto D ducantur duæ rectæ DAC, D*ac* rectas VA & VC secantes in punctis A, C atque *a*, *c*; junctæ A*c* & *a*C sibi mutuo occurrant in Q, & ducta VQ harmonice secabit rectam DAC vel aliam quamvis rectam ex puncto D ad easdem rectas ductam. Secet enim VQ rectam AC in B, & per punctum Q ducatur recta MQN parallela ipsi DC, quæ occurrat rectis D*a*, VA & VC in punctis M, R, & N; cumque sit MR ad MQ ut DA ad DC, & MQ ad MN in eadem ratione, erit quoque RQ ad QN ut DA ad DC. Sed RQ est ad QN ut AB ad BC. Quare DA est ad DC ut AB ad BC. Hæc est Prop. 20ma, Lib. I. sectionum conicarum Cl. *De la Hire*. Fig. 16. n. 1.

§ 23. Sit DA ad DC ut AB ad BC, eritque $\frac{2}{DB}$ æqualis summæ vel differentiæ ipsarum $\frac{1}{DA}$ & $\frac{1}{DC}$ prout puncta A & C sunt ad easdem vel contrarias partes puncti D. Sint imprimis puncta A & C ad easdem partes puncti D, cumque sit $DA \times BC = DC \times AB$, i. e. $DA \times \overline{DC - DB} = DC \times \overline{DB - DA}$, vel $DA \times \overline{DB - DC} = DC \times \overline{DA - DB}$ erit $2DA \times DC = DA \times DB +$

 DC

$DC \times DB$, adeoque $\frac{2}{DB} = \frac{1}{DA} + \frac{1}{DC}$. Sint nunc
n. 2 & 3. puncta A & C ad contrarias partes puncti D, eritque vel $DA \times \overline{DB - DC} = DC \times \overline{DB + DA}$, vel $DA \times \overline{DB + DC} = DC \times \overline{DB - DA}$, adeoque $\frac{2}{DB} = \frac{1}{DC} - \frac{1}{DA}$ cum puncta B, & C sunt ad easdem partes puncti D, vel $\frac{2}{DB} = \frac{1}{DA} - \frac{1}{DC}$ quoties puncta A & B sunt ad easdem partes puncti D. Si igitur, datis puncto D & rectis VF & VC positione, ducatur ex puncto D recta quævis illis occurrens in punctis A & C, & in eadem recta sumatur semper DB ita ut $\frac{2}{DB} = \mp \frac{1}{DA} \mp \frac{1}{DC}$; ubi supponitur terminos $\frac{1}{DA}$ & $\frac{1}{DC}$ iisdem vel contrariis signis afficiendos esse prout puncta A & C sunt ad easdem vel contrarias partes puncti D, erit locus puncti B ipsa harmonicalis VG quæ rectam DFG rectæ VC parallelam secat in G ita $FG = DF$; quæque transit per punctum Q ubi (ductâ Dac quæ iisdem rectis VF & VC occurrat in a et c) junctæ Ac et aC se mutuo decussant.

Fig. 17. § 24. Si in recta DA sumatur semper Db ita ut $\frac{1}{Db} = \frac{1}{DA} \mp \frac{1}{DC}$; ducatur DF parallela rectæ VC quæ rectæ VF occurrat in F, & DH parallela rectæ VF quæ rectæ VC occurrat in H, & ducta diagonalis HF erit locus puncti b; nam ex hypothesi $\frac{1}{Db} = \frac{2}{DB}$, & $DB = 2Db$; adeoque cum VG sit locus puncti B erit punctum b ad rectam HF, si puncta A & C sint ad

easdem

easdem partes puncti D. Si autem supponatur $\frac{1}{Db} = \frac{1}{DA} - \frac{1}{DC}$, eadem constructio inserviet pro determinando puncto b, si substituatur loco rectæ VC alia vc rectæ VC parallela ad æqualem distantiam a puncto D sed ad contrarias partes.

§ 25. Ex puncto dato D ducatur recta quævis DM quæ tribus rectis positione datis occurrat in punctis A, C, E; & sumatur semper DM ita ut $\frac{1}{DM} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$ (ubi termini sunt contrariis signis afficiendi quoties rectæ DA, DC vel DE sunt ad contrarias partes puncti D); supponatur $\frac{1}{DA} + \frac{1}{DC} = \frac{1}{DL}$, eritque L ad rectam positione datam per præcedentem; adeoque, cum sit $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE}$, erit punctum M ad positione datam, per eandem. Compositio autem problematis facile ex dictis perficitur. Sint VA, VC & vE tres rectæ positione datæ, & compleatur parallelogrammum DFVH, ducendo DF & DH rectis VC & VF respective parallelas, & occurrat recta vE diagonali in v; deinde compleatur parallelogrammum Dfvb ducendo rectas Df & Db rectis vE & HF parallelas quæ rectis HE & vE occurrant in punctis f & b; & diagonalis bf erit locus puncti M. Occurrat enim recta DA rectis HF & bf in L & M; eritque, ex præcedentibus, $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$. Alia constructio ex Art. 22. deducitur.

§ 26. Recta quævis ex puncto dato D ducta occurrat rectis positione datis in punctis A, B, C, E, &c. et in hac recta sumatur semper $\frac{1}{DM} = \frac{1}{DA} \mp \frac{1}{DB} \mp \frac{1}{DC} \mp \frac{1}{DE}$, &c. eritque locus puncti M semper ad rectam positione datam. Demonstratur ad modum præcedentis.

Fig. 18. § 27. Theor. IV. *Circa datum punctum* P *revolvatur recta* PD *quæ occurrat lineæ geometricæ cujuscunque ordinis in tot punctis* D, E, I, *&c. quot ipsa est dimensionum, & si in eadem recta sumatur semper* PM *ita ut* $\frac{1}{DM} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ *&c. (ubi signa terminorum regulam sæpius descriptam observare supponimus) erit locus puncti* M *linea recta.*

Ducatur enim ex polo P recta quævis positione data PA, quæ curvæ occurrat in tot punctis A, B, C, &c. quot ipsa est dimensionum. Ducantur rectæ AK, BL, CN curvam in his punctis contingentes, quæ occurrant rectæ PD in totidem punctis K, L, N, &c. et per Art. 10. $\frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. $= \frac{1}{PK} \mp \frac{1}{PL} \mp \frac{1}{PN} \mp$ &c. Unde $\frac{1}{PM}$ æqualis est huic summæ, cumque positione detur recta PA, & maneant rectæ AK, BL, CN, &c. dum recta PD circa polum P revolvitur, erit punctum M ad lineam rectam, per articulum præcedentem; quæ per superius ostensa ex datis tangentibus AK, BL, &c. determinari potest.

§ 28. Sicut recta Pm medium est harmonicum inter duas rectas PD & PE, cum $\frac{2}{Pm} = \frac{1}{PD} + \frac{1}{PE}$; similiter

militer Pm dicatur *medium* harmonicum inter rectas quaslibet PD, PE, PI, &c. quarum numerus est n, & cum $\frac{n}{Pm} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. Et si ex puncto dato P recta quævis ducta lineam geometricam secet in tot punctis quot ipsa est dimensionum, in qua sumatur semper Pm medium harmonicum inter segmenta omnia ductæ ad punctum datum P & curvam terminata, erit punctum m ad rectam lineam. Erit enim $\frac{1}{PM} = \frac{n}{Pm}$ adeoque Pm ad PM ut n ad unitatem; cumque punctum M sit ad rectam lineam, per præcedentem, erit m quoque ad rectam lineam. Atque hoc est theorema *Cotesii*, vel eidem affine.

§ 29. Sint a, b, c, d, &c. radices æquationis ordinis n, V ultimus ejus terminus quem ordinata seu radix y non ingreditur, P coefficiens termini penultimi, M medium harmonicum inter omnes radices, seu $\frac{n}{M} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} +$ &c. Cum igitur sit V factum ex radicibus omnibus a, b, c, &c. in se mutuo ductis, sitque P summa factorum cum radices omnes unâ dempta in se mutuo ducuntur, erit $P = \frac{V}{a} + \frac{V}{b} + \frac{V}{c} + \frac{V}{d}$ + &c. $= \frac{nV}{M}$, adeoque $M = \frac{nV}{P}$. Sic, si æquatio sit quadratica, cujus radices duæ sint a et b, erit $M = \frac{2ab}{a+b}$ (assumptâ æquatione generali sectionum conicarum Art. 1. propositâ) $= \frac{2cxx - 2dx + 2e}{ax - b}$. In æquatione cubica cujus tres radices sunt a, b, c, erit $M = \frac{3abc}{ab + ac + bc}$ (si assumatur æquatio generalis linearum tertii ordinis ibidem propositi) $= \frac{3fx^3 - 3gx^2 + 3hx - 3k}{cxx - dx + e}$.

Fig. 19. § 30. Occurrant rectæ quævis duæ Pm & $P\mu$, ex puncto P ductæ, lineæ geometricæ in punctis D, E, I, &c. et d, e, i, &c. sitque Pm medium harmonicum inter segmenta prioris ad punctum P & curvam terminata, & $P\mu$ medium harmonicum inter segmenta similia posterioris rectæ; juncta μm occurrat abscissæ AP in H, eritque $PH = \frac{nV\dot{x}}{\dot{V}}$ vel PH ad Pm ut P ad $\frac{\dot{V}}{\dot{x}}$. Secet enim abscissa curvam in tot punctis B, C, F, &c. quot ipsa est dimensionum; cumque ultimus terminus æquationis (i. e. V) sit ad $BP \times CP \times FP \times$ &c. in ratione constanti, ut supra (Art. 5.) ostendimus, erit (per Art. 8.) $\frac{\dot{V}}{V} = \frac{\dot{x}}{BP} \mp \frac{\dot{x}}{CP} \mp \frac{\dot{x}}{FP} \mp$ &c. adeoque $\frac{n}{PH} = \frac{1}{BP} \mp \frac{1}{CP} \mp \frac{1}{FP} \mp$ &c. $= \frac{\dot{V}}{V\dot{x}}$, & $PH = \frac{nV\dot{x}}{\dot{V}}$ (quoniam recta $PM = \frac{nV}{P}$) $= Pm \times \frac{P\dot{x}}{\dot{V}}$. In sectionibus conicis est PH ad Pm ut $ax - b$ ad $2cx - d$; & in lineis tertii ordinis ut $cxx - dx + e$ ad $3fxx - 2gx + h$.

§ 31. Si desideretur propositionis præcedentis demonstratio ex principiis pure algebraicis petita, ea ope sequentis *Lemmatis* perfici poterit. Sit abscissa $AP = x$, ordinata $PD = y$, ultimus terminus æquationis lineam geometricam definientis $V = Ax^n + Bx^{n-1} + Cx^{n-2}$ + &c. penultimi coefficiens $P = ax^{n-1} + bx^{n-2} + cx^{n-3}$ + &c. et sit Q quantitas quæ formatur ducendo terminum quemque quantitatis V in indicem ipsius

ſius x in hoc termino & dividendo per x, i. e. ſit $Q = nAx^{n-1} + \overline{n-1} \times Bx^{n-2} + \overline{n-2} \times Cx^{n-3} + \&c.$ (quæ ipſa eſt quantitas quam $\frac{\dot{V}}{\dot{x}}$ dicimus.) Ducatur ordinata Dp quæ angulum quemvis datum ApD cum abſciſſa conſtituat, ſintque rectæ PD, pD et Pp ut datæ l, r et k; dicatur $p\mathrm{D} = u$, $\mathrm{A}p = z$, & tranſmutetur æquatio propoſita ad ordinatam u & abſciſſam z; & æquationis novæ, cum ſit $z = \mathrm{AP}$, terminus ultimus v erit æqualis ipſi V, penultimi autem coefficiens p erit æqualis $\pm \frac{Qk + Pl}{r}$.

Cum enim ſit PD ($= y$) ad pD ($= u$) ut l ad r, erit $y = \frac{lu}{r}$; ſit autem Pp ad pD ($= u$) ut k ad r, erit $\mathrm{P}p = \frac{ku}{r}$, & $\mathrm{AP} = x = \mathrm{A}p \pm \mathrm{P}p = z \pm \frac{ku}{r}$. His autem valoribus pro y et x ſubſtitutis in æquatione propoſita lineæ geometricæ, prodibit æquatio relationem co-ordinatarum z et u definiens. Ad hujus ultimum terminum v & penultimum pu determinandum, ſufficit hos valores ſubſtituere in ultimo V, & penultimo Py, æquationis propoſitæ, atque terminos reſultantes colligere in quibus ordinata u vel non reperitur, vel unius tantum dimenſionis; horum enim ſumma dat pu, illorum v. Subſtituatur pro x ipſius valor $z \pm \frac{ku}{r}$ in quantitate V vel $Ax^n + Bx^{n-1} + Cx^{n-2} + \&c.$ et termini reſultantes $Az^n \pm \frac{nAz^{n-1}ku}{r} + Bz^{n-1} \pm \overline{n-1} \times \frac{Bz^{n-2}ku}{r} + Cz^{n-2} \pm \overline{n-2} \times \frac{Cz^{n-3}ku}{r} +$ &c. ſoli ad rem faciunt de qua nunc agitur. Subſtituatur deinde pro x idem valor, & pro y ipſius valor

valor $\frac{lu}{r}$, in quantitate $Py = \overline{ax^{n-1} + bx^{n-2} + cx^{n-3}}$ $\overline{+ \&c.} \times y$; & termini resultantes soli $\overline{az^{n-1} + bz^{n-2} + cz^{n-3} + \&c.} \times \frac{lu}{r}$ sunt nobis retinendi. Supponatur nunc $z = x$, & summa priorum sit æqualis $V \pm \frac{Qku}{r}$, & posteriorum summa $= \frac{Plu}{r}$. Unde manifestum est ultimum terminum æquationis novæ $v = V$, & penultimum $pu = \frac{Pl \pm Qk}{r} \times u$.

§ 32. Sit nunc Pm medium harmonicum inter segmenta PD, PE, PI, &c. et $P\mu$ medium harmonicum inter segmenta Pd, Pe, Pi, &c. ut in Art. 30. juncta μm secet abscissam in H; & supponamus $P\mu$ ordinatæ pD parallelam esse. Ducatur μs abscissæ parallela, quæ rectæ Pm occurrat in s; eritque Ps ad $P\mu$ ut PD ad pD vel ut l ad r, et μs ad $P\mu$ ut k ad r. Cumque sit $P\mu = \frac{nv}{p}$ (per Articulum præcedentem) $\frac{nVr}{Pl \pm Qk}$, erit $ms = Pm \pm Ps = \frac{nV}{P} \pm \frac{nvl}{pr} = \frac{nV}{P} \pm \frac{nVl}{Pl + Qk} = \frac{nVQk}{P \times \overline{Pl \pm Qk}}$. Est autem ms ad $s\mu$ ut Pm ad PH, i. e. $\frac{nVQk}{P \times \overline{Pl \pm Qk}}$ ad $\frac{nVk}{Pl \pm Qk}$ ut Pm ad PH; adeoque Q ad P ut Pm ad PH, vel $PH = Pm \times \frac{P}{Q}$ vel $\frac{nV}{Q}$. Cum igitur valor rectæ PH non pendeat a quantitatibus, l, k et r; sed, his mutatis, sit semper idem, erit punctum μ ad rectam positione datam, ut in Theor. 4. aliter ostendimus. Quin & valor rectæ PH is est quem in Art. 29. alia methodo definivimus; & recta Hm omnes

nes rectas ex P ductas secat harmonice, secundum definitionem sectionis harmonicæ in Art. 28. generaliter propositam.

SECTIO II.

De Lineis secundi ordinis, sive sectionibus coinicis.

§ 33. EX iis quæ generaliter de lineis geometricis in sectione primâ demonstrata sunt, sponte fluunt proprietates linearum secundi, tertii, & superiorum ordinum. Quæ ad sectiones conicas spectant optime derivantur ex proprietatibus circuli, quæ figura basis est coni. Verum ut usus theorematum præcedentium clarius pateat, & figurarum analogia illustretur, operæ pretium erit harum quoque affectiones ex premissis deducere. Doctrina autem conica de diametris, earumque ordinatis (quibus parallelæ sunt rectæ sectionem contingentes ad vertices diametri) & de parallelarum segmentis quæ rectis quibuscunque occurrunt, & asymptotis, tota facillime fluit ex iis quæ Art. 4. et 5. ostensa sunt.

§ 34. Rectæ AB & FG sectioni conicæ inscriptæ occurrant sibi mutuo in puncto P; ductæ AK, BL, FM, GN sectionem contingentes occurrant rectæ PE, per P ductæ in punctis K, L, M, N; eritque semper Fig. 20.
$\frac{1}{PK} \pm \frac{1}{PL} = \frac{1}{PM} \mp \frac{1}{PN}$ (si recta PE curvæ occurrat in punctis D & E) $= \frac{1}{PD} \mp \frac{1}{PE}$. Segmentis autem

tem quæ sunt ad easdem partes puncti P eadem præponuntur signa; iisque quæ sunt ad oppositas partes
Fig. 21. puncti P signa præponuntur contraria. Hinc si bisecetur DE in P, & ex puncto P ducatur recta quævis sectionem secans in punctis A et B, unde ducantur rectæ AK et BL curvam contingentes quæ rectam DE
Fig. 22. secent in K et L; erit semper PK = PL. Quod si DE sectioni non occurrat, sitque P Punctum ubi diameter quæ bisecat rectas ipsi DE parallelas eidem occurrit; erit in hoc quoque casu PK = PL.

Fig. 23. § 35. Concurrant rectæ AB et FG sectioni conicæ inscriptæ in puncto P; ducantur rectæ sectionem contingentes in punctis A et F quæ sibi mutuo occurrant in K, & juncta PK transibit per occursum rectarum quæ sectionem contingunt in punctis B et G. Si enim recta PK non transeat per occursum rectarum sectionem tangentium in B et G, huic occurrat in N illi in L; cumque $\frac{1}{PK} \mp \frac{1}{PL} = \frac{1}{PK} \mp \frac{1}{PN}$ per præcedentem, erit PL = PN; & coincidunt puncta L et N contra hypothesin.

§ 36. Eadem ratione patet rectas AG et BF sibi mutuo occurrere in π puncto rectæ LK; adeoque puncta P, K, π, L esse in eadem recta linea. Hinc datis tribus punctis contactus A, B, et F, cum duabus tangentibus AK et FK, sectio conica facile describitur. Revolvatur enim recta KπP circa tangentium occursum K ut polum, quæ occurrat rectis AB et FB in punctis P et π; & junctæ Aπ, FP occursu suo G describent sectionem conicam quæ transibit per tria puncta data A, B, F & continget rectas AK et FK in A et F.

§ 37.

§ 37. Cæteris manentibus, occurrant rectæ AF et BG sibi mutuo in puncto p, tangentes AK et BL in R, atque tangentes FK et GL in Q; & puncta R, π, Q et p erunt in eadem recta linea; similiter occurrant tangentes AK et GQ in m; tangentes BR et FK in n; & puncta P; m, n, p erunt in eadem recta linea. Demonstratur ad modum Art. 35. Fig. 24.

§ 38. Hinc datis quatuor punctis contactus, A, B, F, G cum unica tangente AK, occursus rectarum AB et FG, AF et BG, atque AG et BF, dabunt puncta P, p, et π; junctæ autem Pp, Pπ, et $p\pi$ secabunt tangentem datam AK in tribus punctis m, K et R unde ductæ mG, FK, RB sectionem conicam contingent in punctis datis G, F et B.

§ 39. Datis quatuor tangentibus RK, KQ, QL, LR et unico puncto contactus A, occursus tangentium RK et LQ, LR et QK dabunt puncta m et n. Jungantur LK et nm; & occursus rectarum LK et RQ, LK et nm, RQ et nm, dabunt puncta π, P et p; junctæ vero PA, πA et pA secabunt tangentes RL, QK et QL in punctis contactus B, G et F.

§ 40. Datis quinque punctis contactus A, B, F, G, et f, junctæ GF et Gf rectæ AB occurrant in punctis P et X; junctæ AF et Af occurrant rectæ BG in p et x; & junctæ Pp, Xx occursu suo dabunt punctum m; unde ductæ mA et mG sectionem conicam tangent in A et G; & similiter determinantur rectæ quæ curvam contingent in punctis reliquis B, F et f.

§ 41. Dentur quinque rectæ sectionem conicam contingentes, VK, KQ, QL, Lu, et uV; occursus tangentium VK et LQ dabit punctum m; occursus tangentium

KQ

KQ et L*u* dabit punctum *n*; jungantur *mn*, LK, VL et *mu*; recta LK secabit rectam *mn* in P; & recta LV secabit ipsam *mu* in X; juncta autem PX secabit tangentes VK et *u*L in punctis contactus A et B. Similiter reliqua puncta contactus determinantur.

Fig. 25. § 42. Datis tribus tangentibus AK, BK, et RL, cum duobus punctis contactus A et B, facillime determinatur tertium, per Art. 35. Occurrat enim tangens RL reliquis tangentibus in R et L, atque junctæ AL et BR, se mutuo decussent in π, juncta Kπ secabit tangentem RL in tertio puncto contactus F; & sectio conica describi potest ut in Art. 36.

Fig. 26. § 43. Dentur quatuor tangentes KQ, QL, LR, et RK cum unico puncto D sectionis conicæ quod non sit in aliquâ quatuor tangentium. Inveniantur puncta P, p et π ut in Art. 39. Jungantur PD, pD, et πD; & ducta PZ rectæ pD parallela occurrat rectæ RQ in Z; & bifariam secetur PZ in S; & ducta pS secabit rectam PD in E puncto curvæ; vel occurrat PD rectæ RQ in *z*, et (per Art. 23.) secetur PD harmonice in *z* et E. Ducta autem Dπ secabit junctam pE in *e*, et Eπ secabit ipsam pD in *d*, ita ut hæc quoque puncta *d*, *e* sint ad curvam.

Fig. 27. n. 1. § 44. Ex puncto K ducantur duæ tangentes ad sectionem conicam in A et B; ex puncto A ducantur rectæ duæ AF et AG sectioni occurrentes in F et G; juncta BG secet AF in P, et juncta BF secet rectam AG in π; eruntque puncta P, K, π in eadem recta linea, per Art. 36.

n. 2. Verum propositio hæc generalior est. Si enim a puncto quovis K ducantur duæ rectæ KA*a*, KB*b* sectione ctionem secantes in punctis A, a et B, b; et ex punctis A et a ducantur rectæ ad sectionem AF et aG; juncta autem BF secet aG in P, & ducta bG secet AF in π, erunt puncta P, K, π in eadem recta linea; quod variis modis aliàs demonstravimus, unde expeditam methodum olim deduximus sectionem conicam describendi per data quævis quinque puncta. Sint A, a, B, b, et F puncta quinque data; concurrant rectæ Aa et Bb in K; jungantur AF et BF; revolvatur recta PKπ circa polum K, quæ occurrat his rectis in π et P; et ductæ aP, $b\pi$ concursu suo G sectionem describent.

§ 45. Sit P punctum datum extra sectionem conicam, unde ducta quævis sectioni occurrat in D et E; et si $\frac{2}{PM} = \frac{1}{PD} \mp \frac{1}{PE}$ erit M ad lineam rectam AB quæ sectioni occurrit in punctis A et B, ita ut ductæ PA et PB, erunt contingentes sectionis. Si vero punctum p sit in medio puncto rectæ AB intra sectionem, sitque $\frac{2}{pm} = \frac{1}{pd} \mp \frac{1}{pe}$, locus puncti m erit recta ab per P ducta ipsi AB parallela. Tangentes ad puncta D et E semper concurrunt in recta AB, et tangentes ad puncta d et e in recta ab. Fig. 28.

§ 46. Contingat recta DT sectionem in D, unde ducantur duæ quævis rectæ DE et DA, quæ sectioni occurrant in E et A. Occurrat DE rectæ AK sectionem contingenti, in K; et ductæ EN, KM tangenti DT parallelæ secent DA in N et M, sumatur in recta DE, DR ad EN ut KM ad KE, & circulus ejusdem curvaturæ cum sectione in D transibit per R. Nam per Art. 15. est $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} - \frac{1}{DK} = \frac{KE}{DE \times DK}$ et Fig. 29. n. 1.

DR

$DR = \frac{DE \times DK}{KE} \times \frac{QV^2}{DV^2}$ (quoniam $QV : DV ::$

n. 2. $KM : DK :: EN : DE) = \frac{KM \times EN}{KE}$. Quod si fuerit tangens AK parallela rectæ DE, (i. e. si DE sit ordinata diametri per A transeuntis) erit $DR = \frac{EN^2}{DE}$, vel DR ad DE ut EN^2 ad DE^2; ut alibi demonstravimus Art. 373. tractatus de fluxionibus. Si in hoc casu DE sit diameter, erit $\frac{EN^2}{DE}$, adeoque DR, æqualis parametro diametri DE; ut satis notum est.

Fig. 30. n. 1. § 47. Ducantur rectæ DT, DE, quarum prior sectionem conicam contingat in D, posterior eidem occurrat in E. Ducatur DA quæ bisecet angulum EDT et sectioni occurrat in A; jungatur AE, cui occurrat in V recta DV parallela rectæ quæ curvam contingit in A; et ductâ VR parallelâ rectæ DA, hæc secabit DE in R ubi circulus osculatorius occurrit rectæ DE; eritque DR diameter curvaturæ si angulus EDT sit rectus. Erit enim VR ad AD ut ER ad DE, et ut DR ad DK; unde DR ad DK ut DE ad EK, adeoque $\frac{1}{DR} = \frac{1}{DE} -$

n. 2. $\frac{1}{DK}$, ut oportebat, per Art. 15. Si autem sit tangens AK parallela rectæ DE (quo in casu tangentes AK et DT æquales constituunt angulos cum recta DA quæ proinde perpendicularis est axi figuræ) coincident puncta R et E, & circulus osculatorius transibit per punctum E. Sequitur quoque ex dictis rectas EK, DE, et ER esse in progressione geometrica.

Fig. 31. § 48. Occurrat recta quævis DE sectioni conicæ in D et E, concurrant rectæ curvam contingentes ad D et E in

in puncto V. Sit DOA diameter per D curvæ, & si constituatur angulus $DVr = EDO$, erit DR $(= 2Dr)$ chorda circuli osculatorii. Ducatur enim AK sectionem contingens quæ rectæ DE occurrat in K, et tangenti EV in Z; ducatur EN parallela tangenti DT rectam DA secans in N; cumque sit DR ad KA ut EN ad EK; sitque KZ $(= \frac{1}{2} AK)$ ad EK ut VD ad DE, erit VD ad DE ut $\frac{1}{2}$DR ad EN; adeoque triangula DVr et EDN similia et angulus DVr æqualis angulo EDO. Hanc methodum determinandi circulum osculatorium demonstravimus in tractatu de fluxionibus, Art. 375. sed non adeo breviter.

§ 49. Variatio curvaturæ, sive tangens anguli contactus sectione conica & circulo osculatorio comprehensi, est directe ut tangens anguli contenti diametro quæ per contactum ducitur & normali ad curvam, & inverse ut quadratum radii curvaturæ. Sit enim DR diameter curvaturæ, & hæc variatio ad punctum D erit ut $\frac{1}{DR \times DV}$, per Art. 17. adeoque, cum sit DV ad Dr ut DE ad EN, ut $\frac{EN}{DE \times DR^2}$. Variatio autem radii curvaturæ est ut tangens anguli EDO. Quod si recta DO circulo osculatorio occurrat in n, parabola diametro & parametro Dn descripta, quæque contingit rectam DT in D, ea erit cujus contactus cum sectione est intimus, per Art. 19. Fig. 32.

§ 50. Cæteris manentibus, ex puncto V ducatur recta VH circulum osculatorium contingens in H; jungatur HD, cumque sit angulus RDH complementum anguli DrV ad rectum erit $RDH = DVr = EDO$; adeoque variatio radii curvaturæ erit ut tangens anguli Fig. 32.

D d RDH;

RDH; & coincidentibus rectis DR et DH variatio evanescit.

[✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱✱]

SECTIO III.

De Lineis tertii Ordinis.

§ 51. DE lineis tertii ordinis sive curvis secundi generis, uberius nobis agendum est. Doctrinam conicam, variis modis usque ad fastidium fere, tractarunt permulti. Hanc autem geometriæ universalis partem, pauci adtigerunt; eam tamen nec sterilem esse nec injucundam ex sequentibus, ut spero, patebit, cum præter proprietates harum figurarum a *Newtono* olim traditas, aliæ sunt plures geometrarum attentione non indignæ. Ostendimus supra, rectam secare posse lineam tertii ordinis in tribus punctis, quoniam æquationis cubicæ tres sunt radices, quæ omnes reales esse possunt. Recta autem quæ lineam tertii ordinis in duobus punctis secat, eidem in tertio aliquo puncto necessario occurrit, vel parallela est asymptoto curvæ, quo in casu dicitur ei occurrere ad distantiam infinitam: æquationis enim cubicæ si duæ radices sint realis, tertia necessario realis erit. Hinc recta quæ lineam tertii ordinis contingit, eam in aliquo puncto semper secat; cum contactus pro duabus intersectionibus coincidentibus habendus sit. Recta autem quæ curvam in puncto flexus contrarii contingit, simul pro secante habenda est. Ubi duo arcus curvæ sibi mutuo occurrent, punctum *duplex* formatur, & recta quæ alterum arcum ibi contingit in eodem puncto alterum

secat.

secat. Recta autem alia quævis ex puncto duplice ducta in uno alio puncto curvam secat, sed non in pluribus.

§ 52. PROP. I. Sint duæ parallelæ, quarum utraque secet lineam tertii ordinis in tribus punctis; recta quæ utramque parallelam ita secat ut summa duarum partium parallelæ ex uno secantis latere ad curvam terminatarum æqualis sit tertiæ parti ejusdem ex altero secantis latere ad curvam terminatæ similiter secabit omnes rectas his parallelas quæ curvæ in tribus punctis occurrunt; per Art. 4.

§ 53. PROP. II. Occurrat recta positione data lineæ tertii ordinis in tribus punctis; ducantur duæ quævis parallelæ quarum utraque curvam secet in totidem punctis; & solida contenta sub segmentis parallelarum ad curvam & rectam positione datam terminatis erunt in eadem ratione ac solida sub segmentis hujus rectæ ad curvam & parallelas terminatis, per Art. 5.

Hæ duæ proprietates a *Newtono* olim expositæ fuerunt.

§ 54. PROP. III. Cæteris manentibus ut in propositione præcedente, occurrat recta positione data lineæ tertii ordinis in unico puncto A, & solidum sub segmentis PM, Pm, Pμ unius parallelæ contentum erit semper ad solidum sub segmentis pN, pn, $p\nu$, alterius parallelæ ut solidum AP $\times$ bP² contentum sub segmento AP & qua- Fig. 33.

drato distantiæ bP puncti P a puncto quodam b ad solidum $Ap \times bp^2$ contentum sub segmento Ap et quadrato distantiæ puncti p ab eodem puncto b, per Art. 6.

Fig. 34. n. 1. § 55. Prop. IV. Ex dato quovis puncto P ducatur recta PD quæ lineæ tertii ordinis occurrat in tribus punctis D, E, F, & alia quævis recta PA quæ eandem secet in tribus punctis A, B, C. Ducantur tangentes AK, BL, CM, quæ rectæ PD occurrant in K, L, et M; et medium harmonicum inter tres rectas PK, PL, PM, coincidet cum medio harmonico inter tres rectas
n. 2. PD, PE, et PF, per Art. 10. & 28. Si autem recta PD curvæ occurrat in unico puncto D, inveniatur punctum d ut in Art. 6 & medium harmonicum inter tres rectas PK, PL, PM, erit ad medium harmonicum inter duas rectas PD et $\frac{1}{2}Pd$ in ratione 3 ad 2, per Art. 12.

§ 56. Prop. V. Revolvatur recta PD circa polum P, sumatur semper PM in recta PD æqualis medio harmonico inter tres rectas PD, PE, et PF, eritque locus puncti M linea recta, per Art. 28.

Atque hæc est proprietas harum linearum a *Cotesio* inventa.

Fig. 35. § 57. Prop. VI. Sint tria puncta lineæ tertii ordinis in eadem recta linea; ducantur rectæ curvam in his punctis contingentes, quæ eandem

secent

secent in aliis tribus punctis; atque hæc tria puncta erunt etiam in recta linea.

Occurrat recta FGH lineæ tertii ordinis in tribus punctis F, G, et H. Rectæ FA, GB, HC, curvam in his punctis contingentes eandem secent in punctis A, B, C; & hæc puncta erunt in recta linea. Jungatur enim AB, & hæc transibit per C; si enim fieri potest, occurrat curvæ in alio puncto M, tangenti HC in N et rectæ FGH in P; cumque sit $\frac{1}{PA} + \frac{1}{PB} + \frac{1}{PM} = \frac{1}{PA} + \frac{1}{PB} + \frac{1}{PN}$, per Prop. IV. erit $PN = PM$; quod fieri nequit nisi coincidant puncta N, M, et C. Recta igitur AB transit per C.

§ 58. *Corol.* Hinc si A, B, C, sint tria puncta lineæ tertii ordinis in eadem recta linea, ductæ autem AF et BG curvam contingant in F et G, & juncta FG curvam denuo secet in H, juncta CH curvam continget in H. Si enim recta curvam contingeret in H quæ eandem secaret non in C sed in alio quovis puncto, foret hoc punctum cum tribus aliis A, B, C, in eadem recta quæ igitur secaret lineam tertii ordinis in quatuor punctis. Hoc autem fieri non potest. Incidi autem primo in hanc propositionem via diversa sed minus expedita, eandem deducendo ex Prop. II. Similiter si recta Af curvam quoque contingat in f, & ducta Gf curvæ occurrat in h, juncta Ch erit tangens ad punctum h. Et si a punctis A, B, C, lineæ tertii ordinis in eadem recta sitis, ducantur tot rectæ curvam contingentes quot duci possunt, erunt semper tres contactus in eadem rectâ.

§ 59. Prop. VII. Ex puncto quovis lineæ tertii ordinis ducantur duæ rectæ curvam con- Fig. 36.

 tingentes,

tingentes, & recta contactus conjungens denuo secet curvam in alio puncto, rectæ curvam in hoc puncto & in primo puncto contingentes se mutuo secabunt in puncto aliquo curvæ.

Ex puncto A ducantur rectæ curvam contingentes in F et G, juncta FG curvam secet in H, eandemque contingat in H recta HC quæ curvæ occurrat in C, & ducta AC erit curvæ tangens ad punctum A. Sequitur ex *Corollario* præcedente, coincidentibus enim A et B recta CA fit tangens ad punctum A.

§ 60. *Corol.* 1. Si ex puncto curvæ C ducantur duæ rectæ eandem contingentes in A et H, & ex puncto alterutro A contingentes AF et AG ad curvam, recta per contactus F et G ducta transibit per alterum punctum H.

Fig. 37. § 61. *Corol.* 2. Contingat recta AC curvam in A, eamque secet in C, ductæ autem AF et CH curvam contingant in F et H, recta per contactus ducta eam denuo secet in G, & juncta AG curvam continget in G. Quod si alia recta Cb ex puncto C ducatur ad curvam eam contingens in b; & junctæ bF, bG, curvæ occurrant in f et g, ductæ Af et Ag erunt tangentes ad puncta f et g.

Fig. 38. § 62. *Corol.* 3. Sit A punctum flexus contrarii unde ductæ AF et AG curvam contingant in F et G juncta FG secet curvam in H, & ducta AH curvam continget in H. Si enim tangens ad punctum H curvæ in alio quovis puncto ab A diverso occurreret, recta ex hoc occursu ad punctum flexus contrarii A ducta curvam in A contingeret, quot fieri nequit. Manifestum autem est tres tantum duci posse rectas ex puncto flexus contrarii

curvam

curvam contingentes præter eam quæ in hoc ipso puncto simul tangit & secat, atque tres contactus cadere in eandem rectam lineam. Ex solo puncto flexus contrarii tres rectæ ductæ curvam ita contingunt ut tres contactus sint in eadem rectâ. Sint enim F, G, H, in eadem recta, unde tangentes ductæ conveniant in eodem puncto curvæ *a*, quod non sit punctum flexus contrarii; ducatur *ae* curvam contingens in *a*, quæque ei occurrat in *e*, & juncta *e*H curvam tanget in H, per hanc propositionem; adeoque rectæ *e*H et *a*H curvam contingerent in eodem puncto H. *Q. E. A.*

§ 63. Prop. VIII. Ex puncto quovis lineæ tertii ordinis ducantur tres rectæ curvam contingentes in tribus punctis; recta duos quosvis contactus conjungens occurrat denuo curvæ, & ex occursu ad tertium contactum ducta curvam denuo secabit in puncto ubi recta ad primum punctum ducta curvam continget.

Ex puncto A, lineæ tertii ordinis ducantur tres rectæ AF, AG, et A*f*, curvam contingentes in tribus punctis F, G, et *f*; recta G*f* quæ horum duo quævis conjungit secet curvam denuo in N, et recta ex hoc puncto ad tertium contactum F ducta curvam secet in *g*, tum juncta A*g* curvam continget in *g*. Ducatur enim recta AC curvam contingens in A quæ eandem secet in C; cumque puncta G, N, et *f*, sint in eadem recta, & tangentes ad puncta G et *f* transeant per A, sequitur (per Prop. VII.) tangentem ad punctum N transire per C. Cumque puncta F, N, *g*, sint in eadem rectâ, tangentes autem FA et NC curvæ occurrant in A et C, sitque AC tangens ad punctum A, tangens ad punctum *g* transibit per A. Fig. 37.

§ 64. *Corol.* Hinc si curva describatur, ex datis tribus punctis contactus ubi tres rectæ ex eodem puncto curvæ ductæ eam contingunt, invenitur quartum punctum contactus ubi recta ex eodem puncto curvæ ducta eam contingit. Atque hinc colligitur ex eodem curvæ puncto quatuor tantum rectas duci posse lineam tertii ordinis contingentes præter rectam quæ in hoc ipso puncto curvam contingit. Si enim rectæ ex eodem curvæ puncto duci possent eam in quinque punctis contingentes, plures rectæ numero indefinitæ curvam contingentes ex eodem puncto duci possent; ut ex præmissis facile colligitur. Hoc autem Corollarium postea facilius demonstrabitur. Vide infra, Art. 77.

Fig. 38. § 65. PROP. IX. Ex puncto flexus contrarii ducantur tres tangentes ad curvam, & recta contactus conjungens harmonice secabit rectam quamvis ex puncto flexus contrarii ductam & ad curvam terminatam.

Sit A punctum flexus contrarii, AF, AG, et AH, rectæ curvam contingentes in punctis F, G, et H. Ex puncto A ducatur recta quævis curvam secans in B et C, & rectam FH in P; eritque PB ad PC ut BA ad AC. Cum enim tres tangentes ad puncta F, G, et H, in eodem puncto A conveniant, erit per Prop. IV. $\frac{1}{BP} + \frac{1}{PA} - \frac{1}{PC} = \frac{3}{PA}$, adeoque $\frac{1}{PB} - \frac{1}{PC} = \frac{2}{PA}$, i. e. PA est medium harmonicum inter duas rectas PB et PC ad curvam terminatas. Quæ linearum tertii ordinis proprietas est simplicitatis insignis.

§ 66.

§ 66. *Corol.* 1. Recta quæ duas quasvis rectas ex puncto flexus contrarii ductas ad curvam secat harmonice, secabit quoque alias quasvis rectas ex eodem puncto eductas & ad curvam terminatas.

§ 67. *Corol.* 2. Si recta asymptoto parallela per punctum flexus contrarii ducta occurrat rectæ FH in R & curvæ in O, erit $\frac{1}{RO} = \frac{2}{RA}$, adeoque $RA = 2RO$.

§ 68. Prop. X. Recta duo puncta flexus contrarii conjungens vel transit per 3^um punctum flexus contrarii vel dirigitur in eandem plagam cum crure infinito curvæ. Fig. 39.

Sint A et *a* puncta flexus contrarii, juncta A*a* curvæ occurrat in *a*, eritque a quoque punctum flexus contrarii. Si enim tangens figuræ in puncto a curvæ occurreret in alio quovis puncto *e*, forent A, *a*, *e*, in eadem recta. Verum ex hypothesi sunt A, *a*, et a in eadem rectæ, quæ igitur lineæ tertii ordinis occurreret in punctis quatuor. Sit A punctum flexus contrarii, & recta AO asymptoto parallela curvæ occurrat in O ducatur OQ curvam contingens in O, & secans in Q, juncta AQ, transibit per D ubi curva asymptoton secat.

§ 69. Prop. XI. Ductis ex puncto flexus contrarii A tangentibus ad curvam AF, AG, AH; & duabus secantibus quibuscunque ABC, A*bc*, junctæ B*b* et C*c* vel B*c* et *b*C se mutuo secabunt in recta FH quæ contactus conjungit. Fig. 38.

Occurrat enim recta B*b* ipsi FH in Q, et BC eidem in P; jungantur QA et QC; cumque sit AB ad AC ut PB ad PC, per Prop. IX. erunt QA, QB, QP et QC, harmo-

harmonicales, adeoque A*b* secabit rectam QC in *c* et ipsam FH in *p*, ita ut A*b* sit ad A*c* ut *pb* ad *pc*; & proinde erit *c* punctum curvæ, per Prop. IX. unde sequitur converse rectas B*b* et C*c* convenire in puncto Q rectæ FH; & similiter ostenditur rectas B*c* et *b*C sibi mutuo occurrere in puncto *q* ejusdem rectæ.

§ 70. *Corol.* 1. Ex puncto quovis Q rectæ FH ducantur ad curvam rectæ QB, QC, eam secantes in punctis B, *b*, M et C, *c*, N; tum junctæ CB, *cb*, MN, convenient in puncto flexus contrarii A; junctæ B*c* et *b*C, M*c* et N*b*, B*b* et C*c*, NB et MC, convenient in recta FH.

§ 71. *Corol.* 2. Tangentes ad puncta B et C conveniunt in puncto aliquo T rectæ FH; & si a puncto quovis T in recta FH sito ducantur tangentes ad curvam, rectæ contactus conjungentes vel transibunt per punctum flexus contrarii, vel convenient in recta FH.

§ 72. *Corol.* 3. Dato puncto flexus contrarii A, & punctis B, C, *b*, *c*, ubi duæ rectæ ex eo ductæ curvam secant, datur recta FH positione; junctæ enim B*b* et C*c* occursu suo dabunt punctum Q, & junctarum B*c* et *b*C occursus dabit *q*, ductaque Q*q* ea est quæ contactus F, G, et H, conjungit. His autem quinque punctis datis cum aliis duobus M et *m*, determinatur linea tertii ordinis quæ per hæc septem puncta A, B, C, *b*, *c*, M, *m*, transit & in puncto A habet flexum contrarium. Ex punctis enim M et *m* dantur N et *n*, ubi ductæ AM et A*m* curvam secant, & his novem conditionibus linea determinatur. Si autem dentur tria puncta M, *m*, et S; hæc dabunt tria alia N, *n*, et *s*; unde darentur undecim conditiones ad figuram determinandam, quæ nimiæ sunt,

ſunt. Similiter dato puncto flexus contrarii A cum punctis F, G, (adeoque tangentibus AF et AG) et punctis M et *m* quibuſcunque, datur recta FG, adeoque puncta N et *n*, et determinatur curva.

§ 73. *Corol.* 4. Contingant rectæ HB, HC, curvam in B et C, et juncta CB tranſibit per A, junctæ CG et FB concurrent in puncto curvæ V, et ducta VH curvam continget in V. Tangens autem ad punctum flexus contrarii A determinatur ducendo AV cui occurrat in L, recta PL ipſi AH parallela & biſecanda PL in X; juncta enim AX erit tangens ad punctum A. Occurrat enim tangens ad A rectæ FH in S; eritque $\frac{1}{PS} + \frac{2}{PH} = \frac{1}{PH} + \frac{1}{PG} - \frac{1}{PF}$, adeoque $\frac{1}{PS} + \frac{1}{PH} = \frac{1}{PG} - \frac{1}{PF}$ (quoniam AC ſecatur harmonice in P et B, adeoque VA, VF, VP, et VG, harmonicales) $= \frac{2}{PK}$. Eſt igitur PK medium harmonicum inter PS et PH; unde ſi PL parallela rectæ AH occurrat rectis AV et AS in X et L, erit PX = XL. Fig. 40.

§ 74. Prop. XII. Ex puncto lineæ tertii ordinis A ducantur duæ rectæ curvam contingentes in F & G, juncta FG curvæ occurrat in H, & tangens ad punctum A ſecet curvam in M; jungatur HM, cui occurrat FLK ipſi AH parallela in L, & ſumatur FK = 2FL; tum juncta HK, recta quævis AB ex A ducta harmonice ſecabitur a rectis HK et HF in N, P, et a curva in B, C; ita ut NB erit ad NC ut BP ad PC. Fig. 41.

Occurrat

Occurrat enim recta AB tangenti HM in T, eritque $\frac{1}{PB} + \frac{1}{PA} - \frac{1}{PC} = \frac{2}{PA} + \frac{1}{PT}$, adeoque $\frac{1}{PB} - \frac{1}{PC}$ $= \frac{1}{PA} + \frac{1}{PT}$ (per constructionem, & *harmonice*) $= \frac{2}{PN}$. Unde sequitur rectam NC secari harmonice in punctis B et P, vel NB esse ad NC ut BP ad PC.

§ 75. *Corol.* 1. Hinc si duæ quævis rectæ ex A ductæ secentur in N harmonice ita ut PC sit ad PB ut CN ad BN, omnes rectæ ex A eductæ a rectis HF et HK similiter harmonice secabuntur.

§ 76. *Corol.* 2. Si curva punctum duplex non habeat, eamque secet recta HK in duobus punctis *f* et *g*, ductæ A*f* et A*g* erunt rectæ curvam contingentes in his punctis. Coincidat enim punctum B cum puncto N, quando N pervenit ad *f* occursum rectæ HK cum curva; adeoque cum $\frac{1}{PB} \mp \frac{1}{PC} = \frac{1}{PN}$, erit $\frac{1}{PC} = \frac{1}{PN}$, et coincidit C cum B, & recta ex A ducta curvam tunc contingit. Ex altera parte, si recta A*f* curvam contingat transibit recta HK per *f*; ob æquales enim PB, PC, in hoc casu, coincidunt puncta B et C cum N.

§ 77. *Corol.* 3. Si recta HK in solo puncto H curvæ occurrat, duæ tantum tangentes duci poterunt a puncto A ad curvam, viz. AF et AG. Quatuor tantum ad summum tangentes duci possunt a puncto quovis lineæ tertii ordinis ad curvam ut AF, AG, A*f*, A*g*. Si enim alia quævis tangens duci posset a puncto A ad curvam ut Aφ, recta HK transiret per punctum φ, et quatuor

puncta

puncta lineæ tertii ordinis forent in eadem rectâ, viz: H, *f*, *g*, φ. *Q*, *E*. *A*.

§ 78. Prop. XIII. Si ex puncto lineæ tertii ordinis duci possunt quatuor rectæ curvam contingentes, rectæ contactus conjungentes convenient semper in puncto aliquo curvæ, & recta quævis a primo puncto ducta harmonice secabitur a curva & rectis binos contactus conjungentibus:

Sit a punctum curvæ, AF, AG, A*f*, et A*g*, rectæ curvam contingentes in punctis F, G, *f*, et *g*. Jungantur FG et *fg*, quibus occurrat recta quævis ABC (ex A ducta curvamque secans in B et C) in P et N; & recta NC harmonice secabitur in B et P, ita ut semper sit NC ad NB ut CP ad PB: sequitur ex Corol. 2. præcedentis. Rectæ autem FG et *fg* concurrunt in puncto curvæ H; & similiter rectæ F*f* et G*g* conveniunt in E, atque F*g* et G*f* in R; et ER erunt puncta curvæ, per idem corollarium. Atque hæc est posterior duarum proprietatum linearum tertii ordinis quas descripsimus in tractatu de fluxionibus, Art. 402. Quod si recta AM curvam contingat in A, et secet in M, junctæ ME, MR, MH, curvam tangent in punctis E, R, H; & rectarum AE et HR, AR et HE, AH et RE occursus erunt quoque in curva*.

§ 79. *Corol.* Cum igitur sint rectæ HK, HB, HP, et HC, harmonicales; si rectæ HB et HC curvæ occurrant in *b* et *c*; erunt puncta A, *b*, et *c*, in eadem recta linea. Occurrat enim juncta A*b* curvæ in *b* et *c* at-

* Supple quæ desunt in Schemate.

que

que ipsi HF in p, et HK in n; cumque sit nc ad nb ut pc ad pb patet c esse in recta HC; & reciproce, si c sit in recta HC et b in recta HB, erunt A, b, c, in eadem rectâ.

Fig. 42. § 80. Prop. XIV. Habeat linea tertii ordinis punctum duplex O. Ex puncto quovis curvæ A ducantur duæ rectæ AF et AG curvam contingentes in F et G; ducta FG curvam secet in H; jungatur OH. Recta quævis AB ex A ducta curvæ occurrat in punctis B et C, rectæ FG in P, & rectæ OH in N; & recta NP *harmonice* secabitur in punctis B et C, ita ut PB sit ad PC ut BN ad NC.

Jungatur enim AO quæ rectæ FG occurrat in p et tangenti HL in t; cumque sit O punctum duplex, erit $\frac{2}{pO} + \frac{1}{pA} = \frac{1}{pA} + \frac{1}{pt}$, adeoque $\frac{1}{pA} + \frac{1}{pt} = \frac{2}{pO}$. Secatur igitur pA harmonice in t et O, ita ut pt sit ad pA ut tO ad OA, et harmonicales sunt Hp, Ht, HO, et HA. Occurrat recta PA tangenti LH in T, cumque sit $\frac{1}{PC} + \frac{1}{PB} + \frac{1}{PA} = \frac{2}{PA} + \frac{1}{PT}$, erit $\frac{1}{PC} + \frac{1}{PB} = \frac{1}{PA} + \frac{1}{PT} = \frac{2}{PN}$; consequenter PC est ad NC ut PB ad BN.

§ 81. *Corol.* Si tangens HL occurrat rectæ GZ ipsi AH parallelæ in Z, & sumatur GV = 2GZ, ductæ HV transibit per punctum duplex O, si modo curva tale punctum habeat. Vel si recta Gra occurrat rectis AH et HR in a et r, junctæ rA et Ra, se decussent in m, juncta Hm transibit per punctum duplex O.

§ 82.

§ 82. Prop. XV. Ex puncto lineæ tertii ordinis ducantur duæ tangentes, & ex alio quovis ejusdem puncto ducantur rectæ ad contactus curvam in duobus aliis punctis secantes; tangentes ad hæc duo nova puncta in eodem puncto curvæ convenient.

Ex puncto A ducantur rectæ AF et AG curvam contingentes in F et G. Sumatur punctum quodvis curvæ P, jungantur PF et PG curvam secantes in punctis K et L; atque tangentes ad puncta K et L concurrent in puncto aliquo curvæ B. Determinatur autem punctum B, ducendo rectam PC quæ curvam contingit in P, et secat in C; si enim jungatur AC occurret denuo curvæ in puncto B. Fig. 43.

Cum enim puncta F, K, P, sint in eadem rectâ, & tangentes ad puncta F et P curvam secent in A et C; sequitur tangentem ad punctum K transituram per B. Et ob rectam LGP, tangens ad punctum L transibit quoque per B.

§ 83. *Corol.* Sint igitur A et B duo quævis puncta in linea tertii ordinis; ex utroque ducantur quatuor rectæ curvam in aliis quatuor punctis contingentes, viz. AF, AG, A*f*, A*g*; et BK, BL, B*k*, B*l*. Junctæ FK et GL, FL et GK, F*l* et G*k*, G*l* et F*k*; sibi mutuo occurrent in quatuor punctis curvæ, P, Q, *q*, *p*; & si ducantur tangentes ad hæc quatuor puncta, hæ occurrent curvæ & sibi mutuo in puncto C ubi recta AB curvam secat. Unde si sint tria puncta lineæ tertii ordinis in eadem rectâ, & ex singulis ducantur quatuor rectæ curvam contingentes in quatuor aliis punctis, recta per duo quævis puncta contactus ducta curvam semper secabit in alio Fig. 44.

aliquo

aliquo puncto contactus; & quatuor hujusmodi rectæ per idem punctum contactus semper transibunt.

Fig. 43. § 84. Prop. XVI. Sint F et G puncta duo lineæ tertii ordinis, ita sumpta ut rectæ FA et GA curvam in his punctis contingentes conveniant in puncto aliquo curvæ A. Sumatur in curva aliud quodvis punctum P, unde ducantur ad puncta F et G rectæ PF et PG quæ curvæ occurrunt in K et L; jungantur FL et GK, atque harum occursus Q erit in curva. Tangentes autem ad puncta K et L sibi mutuo & curvæ occurrent in puncto aliquo curvæ B, atque tangentes ad puncta P et Q convenient in puncto curvæ C, ita ut tria puncta A, B, C, sint in eadem recta.

Ducatur enim tangens ad punctum P quæ curvæ occurrat in C, & ducta AC secet eandem in B; & ductæ BK, BL, erunt tangentes ad puncta K et L, per præcedentem. Occurrat recta LF curvæ in Q; & si recta GK non transeat per Q, occurrat curvæ in *q*. Quoniam igitur tria puncta L, F, Q, sunt in eadem rectâ, tangentes vero ad L et F curvam secent in B et A, sequitur (per Prop. VII.) tangentum ad punctum Q transire per punctum C. Similiter, cum sint puncta G, K, et *q*, in eadem rectâ, tangentes autem ad puncta G & K transeant per A et B, tangens ad punctum *q* transibit quoque per punctum C. Utraque igitur recta GQ, C*q* curvam contingit prior in Q, posterior in *q*. Coincidunt igitur puncta Q et *q*, si enim diversa esse ponamus, sequitur per Prop. VIII. plures quam quatuor tangentes duci posse ad curvam ex eodem puncto C. Sint enim

As

Af et Ag rectæ quæ curvam contingant in f & g, & ductæ Lf, Lg, curvam secent in m & n; & rectæ Cm, Cn, erunt tangentes ad puncta m et n. Quare haberemus quinque tangentes ex C ad curvam ductas, CP, CQ, Cm, Cn, & Cq; quod repugnat Corol. 3. Prop. XII.

§ 85. *Corol.* 1. Dato puncto P, ubicunque sumantur puncta F & G, modo tangentes ad hæc puncta in curva conveniant, datur punctum Q, ubi junctæ FL & GK occurrunt sibi mutuo & curvæ. Et si a puncto P ducatur recta quævis PNM quæ curvæ occurrat in N et M, & junctæ QM, QN, eam secent in m & n; erunt puncta P, n, & m, in eadem recta linea. Ostendimus enim tangentes ad puncta P & Q, se mutuo decussare in puncto curvæ*.

§ 86. *Corol.* 2. Si sumantur quatuor puncta F, G, K, L, in linea tertii ordinis, ita ut tangentes ad puncta F & G, conveniant in aliquo puncto curvæ, & tangentes, ad puncta K et L, conveniant quoque in aliquo puncto curvæ, ductæ FK & GL concurrent in puncto curvæ, & ductæ FL & GK sibi mutuo occurrent in puncto curvæ. Fig. 43.

§ 87. Prop. XVII. Sint F et G duo quævis puncta lineæ tertii ordinis, ubi si rectæ ducantur curvam contingentes, hæ se mutuo secabunt in puncto aliquo curvæ. Sumantur alia quatuor puncta curvæ L, K, f, g, ita ut ductæ LF et GK conveniant in curva, atque rectæ Ff et Gg, in eâ quoque conveniant; tunc ductæ Lf et gK, se mutuo secabunt in curva, ut & ductæ Lg et Kf.

* Supple quod deest in Schemate.

Tangentes enim ad puncta f et g se mutuo decussant in curva, per Prop. XIV. ut & tangentes ad puncta K et L, per eandem. Adeoque per Corol. 2. praecedentis, junctae fL et Kg conveniunt in curva, ut et fK et gL.

Fig. 45. n. 1. § 88. *Lemma.* Dentur tres rectae IC, IH, et CH, positione; & tria puncta F, G, S, quae sint in eadem recta linea. Sumatur punctum quodvis Q in recta IC, juncta QF occurrat rectae IH in L, & juncta QG rectae HC in P; jungatur FP, ducta SL occurrat rectis FP et QP in k et N; atque puncta k et N erunt ad rectas positione datas. Jungatur enim IN, quae occurrat rectae GS in m, & ducatur per N parallela rectae FS quae occurrat rectis IC, IH, et LQ, in punctis x, u, et r; occurrat recta FG rectis IC, IH, et HC, in a, b, et h. Quoniam Nx est ad Nr ut Ga ad GF, et Nr ad Nu ut SF ad Sb, erit Nx ad Nu (adeoque ma ad mb) ut Ga × SF ad GF × Sb, i. e. in data ratione. Datur igitur punctum m, adeoque recta INm positione; & similiter est punctum k ad positione datam.

n. 2. § 89. *Corol.* Coincidentibus punctis S et G, coincidit quoque punctum m cum puncto G. Jungatur igitur IG quae rectae HC occurrat in D, & ducta CF occurrat rectae HI in E, tum juncta DE erit locus puncti K ubi ductae GL et FP se mutuo decussant.

Fig. 46. § 90. Prop. XVIII. Sit PGLFQK quadrilaterum inscriptum figurae, cujus sex anguli tangant lineam tertii ordinis ut in Prop. XVI. Ducantur rectae curvam contingentes IC, CH, HI, in tribus punctis Q, P, L, quae non sint in eadem rectâ; jungatur IG quae tangenti CH

occurrat

occurrat in D, et HF quæ tangenti CI occurrat in E; erunt puncta D, K, E, in eadem recta linea, quæ quidem curvam in puncto K contingit.

Supponamus enim rectas QFL et FKP moveri circa polum F, & rectas LGP et QKG circa polum G, puncta autem Q, L, et P, deferri in tangentibus QI, LI et PC; tum punctum K movebitur in recta DE, per Corol. præcedens. Unde si puncta Q, L, P, ferantur in curva quæ has rectas QI, LI, et PC, in his punctis contingit, movebitur quoque in curva quam recta DE contingit: Sed per Prop. XV. si puncta Q, L, P, ferantur in lineâ tertii ordinis proposita, punctum K movebitur in eâdem, quam igitur recta DE contingit in K.

§ 91. *Corol.* 1. Similiter si rectæ AF et AG (quæ curvam contingunt in F et G) occurrant rectæ IH (quæ curvam contingit in L) in punctis M et N; juncta MP secet tangentem AG in *d*, & juncta QN tangentem AF in *e*; recta *de* transibit per K, & curvam in hoc puncto continget; atque quatuor puncta D, *d*, *e*, E, erunt in eadem rectâ linea.

§ 92. *Corol.* 2. Ex duobus punctis curvæ quibuscunque C et B ducantur ad curvam quatuor contingentes binæ ex singulis, CQ et CP ex puncto C, BL et BK ex puncto B, sintque harum tangentium occursus I, H, E, et D; tum ductæ LQ et EH se mutuo secabunt in puncto curvæ F; atque junctarum LP et ID occursus erit in puncto curvæ G; tangentes autem ad puncta F et G se mutuo secabunt in puncto curvæ A quod est in eadem recta cum punctis C et B.

§ 93. *Corol.* 3. Datis tribus punctis lineæ tertii ordinis quæ sint in eadem rectâ, & duabus tangentibus ex horum singulis ductis ad curvam positione datis, sex puncta contactus determinantur per hanc propositionem. Sint A, B, C, tria curvæ puncta data in eadem recta, AM et AN tangentes ex A, BMI, et BDE, tangentes ex B quæ prioribus occurrant in M, N, e, et d; sintque CD et CE tangentes ex tertio puncto C ductæ; atque occurrat CD ipsis BM, BD, AM, et AN, in H, D, b, et c, & CE iisdem in I, E, n et m. His positis, juncta Ne secabit tangentem CI in puncto contactus Q, Md secabit tangentem CD in puncto contactus P, ID secabit tangentem AN in puncto contactus G, EH tangentem AM in contactu F, mb secabit tangentem BH in L, & denique nc tangentem BE in K. Quamvis autem problema in hoc casu determinatum sit, solutiones tamen plures admittit. Diversæ enim lineæ tertii ordinis, sed numero definitæ, per tria puncta A, B, et C, duci possunt contingentes sex rectas positione datas AM, AN, BM, BD, CD, et CE. Occurrat enim Ne tangenti CD in p, recta Md tangenti CE in q, ID tangenti AM in f, EH tangenti AN in g, nc tangenti BM in l, et mb tangenti BD in k; atque linea tertii ordinis quæ conditionibus propositis satisfacit continget rectas CD et CE vel in P et Q, vel in p et q. Ea continget rectas AM et AN vel in punctis F et G vel in f et g; rectas autem BM et BD vel in L et K, vel in l et k. Constat igitur plures lineas tertii ordinis problematis conditionibus satisfacere posse, sed numero determinatas, adeoque problema esse determinatum *.

§ 94. *Corol.* 4. Datis duobus punctis lineæ tertii ordinis A et B, tangentibus quoque AM, AN, BM, BD positione datis cum tribus punctis contactus F, G, et L, datur punctum K ubi recta BD curvam contingit.

* Supple quæ desunt in Schemate.

Si

Si enim ducantur rectæ Ne et LF, harum occursu dabitur punctum Q, & ducta QG secabit contingentem BD in puncto contactus K. Datur quoque punctum P occursus rectarum LG et Md, vel rectarum Md et FK; tres enim rectæ LG, Md, et FK, necessario conveniunt in puncto P. Sit $MedN$ quadrilaterum quodvis, sumatur punctum quodvis Q in diagonali Ne et P in diagonali Md, recta quævis QFL ex Q ducta secet latera Me et MN in F et L, ducta PL secet latus Nd in G; jungatur QG quæ latus de secet in K; atque puncta F, K, P, erunt semper in eadem recta linea, per superius ostensa. Unde constat problema non ideo fieri impossibile, quod oporteat tres rectas LG, Md, et FK, in eodem puncto convenire.

§ 95. Prop. XIX. Sint D, E, F, puncta lineæ tertii ordinis in eadem recta, sintque tres rectæ curvam in his punctis contingentes sibi mutuo parallelæ. In recta DF sumatur punctum P ita ut 2PF sit medium harmonicum inter PD et PE; & si alia quævis recta per P ducta curvæ occurrat in f, d, et e, erit semper $2Pf$ medium harmonicum inter Pd et Pe. Supponimus autem puncta d et e esse ad easdem partes puncti P, punctum autem f esse ad contrarias. Fig. 47.

Occurrant enim tangentes DK, EL, FM, rectæ df in punctis K, L, et M; eritque per Art. 9. $\frac{1}{Pf} - \frac{1}{Pd} - \frac{1}{Pe} = \frac{1}{PM} - \frac{1}{PK} - \frac{1}{PL}$ (si recta Qq tangentibus parallela harmonice secet rectam PD ita ut PE sit ad EQ ut PD ad DQ, & Qq occurrat rectæ fd in in $q) = \frac{1}{PM} - \frac{2}{Pq}$ (quoniam Pq est ad PM ut PQ ad PF, & ex hypothesi $2PF = PQ$, adeoque $2PM = Pq$) $= \frac{1}{PM} - \frac{1}{PM} = 0$; unde $\frac{1}{Pf} = \frac{1}{Pd} + \frac{1}{Pe}$, adeoque $2Pf$ est medium harmonicum inter Pd et Pe.

§ 96. *Corol.* 1. Jungantur Dd et Ee quæ conveniant in puncto V, junctæ VQ et Ff erunt parallelæ; & productâ VQ donec occurrat rectæ fd in r, erit $Pf = \frac{1}{2}Pr$. Recta enim PD secatur harmonice in E et Q, ex hypothesi, adeoque etiam recta Pd secatur harmonice in e et r, per Art. 21. unde $Pf = \frac{1}{2}Pr$; cumque sit $PF = \frac{1}{2}PQ$; sequitur rectam Ff parallelam esse harmonicali VQr.

§ 97. *Corol.* 2. Similiter si sumatur in recta DF punctum p ita ut $2pD$ sit æqualis medio harmonico inter pE et pF, & recta quævis ex p ducta curvæ occurrat in tribus punctis, erit segmentum hujus rectæ ex una parte puncti p ad curvam terminatum æquale dimidio medii harmonici inter duo segmenta eodem puncto p et curvâ ad alteras partes terminata.

Fig. 48. § 98. *Lemma.* Ex centro gravitatis trianguli ducatur recta quævis quæ tribus lateribus trianguli occurrat, & segmentum hujus rectæ centro gravitatis & uno trianguli latere terminatum erit dimidium medii harmonici inter segmenta ejusdem rectæ centro gravitatis & duobus aliis trianguli lateribus terminata. Sit P centrum gravitatis trianguli VTZ, occurrat recta FDE per P ducta lateribus in F, D, E; sintque puncta D et E ad easdem partes puncti P; eritque $\frac{1}{PF} = \frac{1}{PD} + \frac{1}{PE}$. Ducatur enim per punctum P, recta MPL lateri VZ parallela, quæ

quæ lateribus VT, ZT, occurrat in L et M et rectæ VN parallelæ lateri ZT in N; cumque sit $MP = PL$, et $TL = 2VL$, ob similia triangula TLM, VLN, erit $LM = 2LN$, unde $LN = LP$, et $PN = 2PM$, proinde si PD occurrat rectæ VN in K, erit (per Art. 21. & 23.) $\frac{1}{PD} + \frac{1}{PE} = \frac{2}{PK} = \frac{1}{PF}$.

§ 99. Prop. XX. Contingant tres rectæ VT, VZ, TZ, lineam tertii ordinis transeatque eadem recta linea per tres contactus & per P centrum gravitatis trianguli VTZ; recta quævis per hoc centrum ducta curvæ occurrat in puncto c ex una parte & in punctis a et b ex altera ejusdem centri gravitatis parte, eritque $2Pc$ medium harmonicum inter segmenta Pa et Pb. Fig. 49.

Occurrat enim recta Pc lateribus trianguli VTZ in f, d, et e; & rectæ VN lateri TZ parallelæ in k; eritque $2Pf = Pk$, adeoque $\frac{1}{Pf} = \frac{2}{Pk} = \frac{1}{Pd} + \frac{1}{Pe} = \frac{1}{Pa} + \frac{1}{Pb} - \frac{1}{Pc} + \frac{1}{Pf}$, adeoque $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$, unde Pc est dimidium medii harmonici inter rectas Pa et Pb.

§ 100. Prop. XXI. Sit V punctum duplex in linea tertii ordinis, VT et VZ rectæ curvam in hoc puncto contingentes, quibus in T et Z occurrat recta TZ curvam contingens in F ita ut $FT = FZ$: jungatur FV, in qua sumatur $FP = \frac{1}{3}FV$; & si recta quævis per P ductæ curvæ occurrat in tribus punctis a, b, c, quorum a et b sint ad easdem partes puncti P, c ad partes con- Fig. 50.

 trarias,

trarias, erit semper $2Pc$ medium harmonicum inter segmenta Pa et Pb, seu $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$.

Cum enim bisecetur TZ in F, sitque $FP = \frac{1}{3}FV$, manifestum est punctum P esse centrum gravitatis trianguli VTZ; cumque sit punctum P in rectâ FV quæ per contactus transit, sequitur propositio ex præcedente.

§ 101. *Corol.* 1. Si jungatur rectæ Va, Vb, et Fc, erit P quoque centrum gravitatis trianguli hisce rectis contenti, ut et trianguli tribus rectis curvam in a, b, c, contingentibus comprehensi; & si ductæ Va et Vb occurrant rectæ Fc in m et n, erit semper Fm æqualis Fn.

§ 102. *Corol.* 2. Recta per punctum duplex ducta parallela rectæ Fc harmonice secabit ipsam Pa in k ita ut Pa erit ad ak ut Pb ad Pk; quæ vero ducitur a puncto k ad x occursum rectarum curvam in a et b contingentium parallela est recta cy figuram contingenti in c.

§ 103. *Corol.* 3. Datis duobus punctis a et c ubi recta quævis ex P ducta curvæ occurrit, datur tertium b; junguntur enim Va et Fc quæ sibi mutuo occurrant in m; sumatur Fn ex altera parte puncti F æqualis ipsi Fm; et juncta Vn secabit rectam Pa in b.

Fig. 51. § 104. Prop. XXII. Ducatur per punctum quodvis P recta quæ dirigatur in plagam crurum infinitorum & occurrat curvæ in punctis a et c; ducatur per idem punctum recta quævis curvam secans in punctis D, E, F, quæque rectis curvam in a et c contigentibus occurrat in k et m, atque

asymptoto

asymptoto cruris infiniti in l; & si puncta D, E, k, l, m, sint ad easdem partes puncti P, punctum vero F ad contrarias, erit $\frac{1}{Pl} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF} - \frac{1}{Pk} - \frac{1}{Pm}$, ubi termini cujusvis signum est mutandum quoties segmentum ad oppositas partes puncti P protenditur.

Sequitur ex Theor. I. Art. 9. est enim per hoc theorema $\frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF}$.

§ 105. *Corol.* 1. Si recta PD ducatur per concursum tangentium ak et cm; & sumatur PM æqualis medio harmonico inter rectas PD, PE, PF, secundum Art. 28, erit $\frac{1}{Pl} = \frac{3}{PM} - \frac{2}{Pk}$, adeoque $\frac{1}{3}$PM erit medium harmonicum inter Pl et $\frac{1}{2}$Pk. Quod si tangentes ak et cm concurrant in ipso puncto M, asymptotos quoque per M transibit.

§ 106. *Corol.* 2. In casu Prop. XIX. ubi tres contactus sunt in eadem recta linea & tres tangentes parallelæ, sumatur punctum P ut in Propositione XIX. sitque aPc asymptoto parallela, occurrant ak et cm tangentes rectæ PD in k et m, eritque $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$, sive P$l$ æqualis dimidio medii harmonici inter Pk et Pm. Quod si tangentes ak et cm concurrant in eodem puncto rectæ PD, erit Pl = $\frac{1}{2}$Pk; quoniam vero in Prop. XIX. $\frac{1}{Pf} = \frac{1}{Pa} + \frac{1}{Pc}$, erit P$a$ = Pc. Fig. 47.

§ 107.

Fig. 49. § 107. *Corol.* 3. Idem dicendum est de casu Prop. XX. ubi tres contactus D, E, F, sunt in eadem rectâ quæ transit per P centrum gravitatis trianguli VTZ tangentibus contenti. Si autem altera rectarum curvam in *a* vel *c* contingentium (posita *aPc* asymptoto parallelâ) sit rectæ DP parallela, abibit asymptotos in infinitum, eritque crus parabolicum.

§ 108. *Corol.* 4. Iisdem positis ac in Prop. XXI. Sit *cPa* asymptoto parallela, occurrant tangentes *ak*, *cm*, rectæ VF in *k* et *m*, eritque $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$. Unde si curva diametrum habet, cum hæc necessario transeat

Fig. 52. per punctum duplex V, & per punctum curvæ F ubi bisecatur tangens TFZ, sumatur ab F versus V, $FP = \frac{1}{3}FV$, ducatur *cPa* asymptoto parallela, & tangens *ak* quæ diametro occurrat in *k*, & ex altera parte puncti P sumatur, super rectam PV, $Pl = \frac{1}{2}Pk$, & recta per *l* ducta ordinatim applicatis parallela erit asymptotos curvæ. Si vero tangens *ak* sit diametro parallela, erit crus curvæ generis parabolici. Propositio *Newtoni* de segmentis rectæ cujusvis tribus asymptotis & curva terminatis facile sequitur ex Art. 4. ut ab aliis olim ostensum est.

Fig. 53. § 109. Prop. XXIII. Ex puncto quovis D lineæ tertii ordinis ducantur duæ quævis rectæ DEI, DAB, quæ curvæ occurrant in punctis, E, I, et A, B; ducantur tangentes AK, BL, quæ rectæ DE occurrant in K et L. Sit DG medium harmonicum inter segmenta DE, DI, ad curvam terminata, atque DH medium harmonicum inter segmenta DK, DL, ejusdem rectæ tangentibus abscissa. Sit DV medium geometricum inter

DG

DG et DH, ducatur VQ parallela tangenti DT, quæ occurrat rectæ DA in Q; & si circulus ejusdem curvaturæ cum linea tertii ordinus proposita in puncto D occurrat rectæ DE in R, erunt HG, QV et 2DR continue proportionales.

Nam per Theor. II. (Art. 15.) est $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL} = \frac{2}{DG} - \frac{2}{DH} = \frac{2DH - 2DG}{DG \times DH} = \frac{2HG}{DV^2}$ (quoniam $DV^2 = DG \times DH$;) unde $QV^2 = 2HG \times DR$, adeoque HG ad QV ut QV ad 2DR.

§ 110. *Corol.* 1. Sumatur igitur Dr in recta DE tertia proportionalis rectis HG et $\frac{1}{2}$QV, & perpendicularis rectæ DE ad punctum r secabit normalem tangenti DT ad punctum D in centro circuli osculatorii sive circuli ejusdem curvaturæ cum linea proposita, in puncto O. Si puncta E, I, K, L, sint ad easdem partes ejusdem puncti prout DH major est vel minor quam DG, i. e. prout medium harmonicum inter segmenta DK, DL tangentibus abscissa majus est vel minus medio harmonico inter segmenta DE, DI, ad curvam terminata.

§ 111. *Corol.* 2. Si angulus EDT bisecetur recta DA, erit $QV = DV$, et $2HG \times DR = DV^2 = DG \times DH$, adeoque HG ad DG ut DH ad 2DR.

§ 112. *Corol.* 3. Revolvatur recta DA circa polum D, manente rectâ DE, et HG, differentia mediorum harmonicorum DH et DG, augebitur vel minuetur in duplicata ratione rectæ VQ. Quippe ob datam chordam circuli osculatorii DR, manet quantitas $\frac{QV^2}{HG}$ quæ æqualis est 2DR.

Fig. 54. § 113. *Corol.* 4. Si tangentium AK et BL altera, ut BL, sit rectæ DE parallela, ducantur GX et KZ parallelæ rectæ DT curvam in D contingenti, quæ ipsi AB occurrant in X, Z; eritque $\frac{GX \times KZ}{DG \times DK \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} = \frac{2}{DG} - \frac{1}{DK} = \frac{2DK - DG}{DG \times DK}$, adeoque $\frac{GX \times KZ}{DR} = 2DK - DG$, & proinde erit ut $2DK - DG$ ad KZ ita GX ad DR. Si tangens AK evadat quoque parallela rectæ DE (quod in his figuris contingere potest) erit DG ad GX ut GX ad 2DR; nam in hoc casu $\frac{GX^2}{DG^2 \times DR} = \frac{2}{DG}$, adeoque $GX^2 = DG \times 2DR$.

§ 114. *Corol.* 5. Si recta DE sit asymptoto parallela, adeoque curvæ occurrat in uno puncto E præter ipsum D, sitque simul tangens BL asymptoto parallela, ducatur EY parallela tangenti DT quæ occurrat rectæ DA in Y, eritque KE ad KZ ut EY ad DH *.

§ 115. *Corol.* 6. Si sit D punctum flexus contrarii, coincidet punctum H cum G, evanescente lineâ HG, adeoque evadit DR infinite magna, i. e. curvaturâ minor est ad punctum flexus contrarii quam in circulo quantumvis magno; ut alibi quoque ostendimus, tractatus de fluxionibus, Art. 378.

Fig. 55. § 116. *Corol.* 7. Sit V punctum duplex, DA asymptoto parallela, & occurrant rectæ VQ, KZ, tangenti DT parallelæ rectæ DA in Q et Z, atque occurrat DV asymptoto in L, sitque DH medium harmonicum inter DK et DL, eritque $2DH - DG$ ad KZ ut DL ad DK,

* Supple figuram.

atque

atque VH : HN : : VQ : DR. Si recta DA bisecet angulum TDV, erit DR : DV : : DH : 2VH.

§ 117. PROP. XXIV. Sit D punctum quodvis lineæ tertii ordinis, occurrat tangens ad D curvæ in I, sitque DS diameter circuli osculatorii, quæ curvæ occurrat in A et B; unde rectæ ductæ curvam contingentes secent DI in K et L; sit DH medium harmonicum inter DK et DL, & sumatur DV ad DI ut DH ad differentiam rectarum 2DI et DH; eritque variatio curvaturæ inverse et rectangulum SD × DV; & junctâ VS, variatio radii curvaturæ ut tangens anguli DVS. Fig. 56.

Nam per Theor. III. (Art. 17.) variatio curvaturæ est ut $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} - \frac{1}{DI}} = \frac{1}{DS} \times \overline{\frac{2}{DH} - \frac{1}{DI}} = \frac{1}{DS} \times \frac{2DI - DH}{DH \times DI} = \frac{1}{DS \times DV}$. Variatio autem radii osculatorii est ut $\frac{DS}{DV}$, adeoque ut tangens anguli DVS, per Art. 18. parabola autem quæ eandem habebit curvaturam & eandem variationem curvaturæ cum linea proposita, determinatur ut in Art. 19.

§ 118. *Corol.* Si tangens BL sit tangenti ad D parallela, erit DV ad DI ut DK ad IK; & si utraque tangentium AK, BL, fiat parallela ipsi DT, erit DV, $=$DI, adeoque variatio curvaturæ inverse ut DS × DI. Quod si in hoc casu sit DT parallela asymptoto curvæ, evanescet variatio curvaturæ. Quemadmodum igitur evanescit variatio curvaturæ in verticibus axium sectionum conicarum; ea similiter evanescit in verticibus diame- Fig. 57. Fig. 58.

diametrorum linearum tertii ordinis quæ ad rectos angulos ordinatim applicatas bisecant.

Fig. 59. *Schol.* Sunt autem alia plurima theoremata de tangentibus & curvatura linearum tertii ordinis. Sint, *ex. gr.* F et G duo puncta lineæ tertii ordinis unde tangentes ductæ concurrunt in curva in A. Producatur FG donec curvæ occurrat in H. Sit TAC tangens ad punctum A, & constituatur angulus FAN = GAT ad contrarias partes rectarum FA, GA, secetque AN rectam FG in N. Et si circuli osculatorii occurrunt rectæ FG in B et *b*, erit GB ad F*b* ut rectangulum NFH ad NGH. Sit enim puncta *a* ipsi A quamproximum, & puncta *f*, *g*, *h*, ipsis F, G, H, quamproxima, eritque AF*a* : FG*f* : : GF : FB. FG*f* (=HG*h*) : HF*h* : : FH : GH. HF*h* (= GF*g*) : AG*a* : : *h*G : GF; unde AF*a* : AG*a* : : FH× *h*G : FB × GH : : GN : FN; unde FB : G*b* : : NFH : NGH. *Sed de his satis.*

FINIS.

APPENDIX:

BEING A

TREATISE

CONCERNING THE

GENERAL PROPERTIES

OF

GEOMETRICAL LINES.

Translated from the LATIN

BY JOHN LAWSON, B. D.

CONCERNING THE

GENERAL PROPERTIES

OF

GEOMETRICAL LINES.

CONCERNING the lines of the second order, or the conic sections, the ancient and modern geometers have written very fully; concerning the figures which are referred to the superior orders of lines, little has been delivered before NEWTON. That most illustrious man, in his tract concerning the *Enumeration of Lines of the Third Order*, has revived this subject, which had long lain neglected, and has shewn it to be worthy of the geometer's notice. For the general properties of these lines, which he has laid down, are so consonant to the known properties of the conic sections, that they seem to be conformable to the same law, and from his example many others have been since induced to make this subject their study, and have clearly comprehended and explained the analogy which there is between figures of such very different kinds. The pains which they have been at in the illustration and further investigation of these matters, have deservedly met with applause, since there is nothing in pure mathematics which can be called more beautiful, or that is more apt to delight a mind desirous of investigating truth, than the agreement and harmony of different things, and the admirable connection of the succeeding with the preceding, where the more simple always open the way to those which are more difficult.

Most of the general properties of lines of the third order, delivered by *Newton*, relate to segments of parallels and asymptotes. Some other of their affections, of a different kind, I have briefly pointed out in my Treatise of Fluxions, lately published, Art. 324, and 401. The famous *Cotes* formerly discovered a most beautiful property of geometrical lines, hitherto unpublished, which has been communicated to me by the Rev. Dr. *Robert Smith*, master of Trinity College, Cambridge, a gentleman not less remarkable for his learning and works, than for his fidelity and regard for his friends. Whilst I had these under consideration, some other general theorems offered themselves; which, as they seem to conduce to the augmentation and illustration of this difficult part of geometry, I have thought fit to throw together, and briefly to expound in order, and demonstrate.

SECTION I.

Of Geometrical Lines in general.

§ 1. LINES of the second order are defined by the section of a geoemtrical solid, viz. a cone, whence their properties are best derived by common geometry. But the nature of the figures which are referred

referred to the superior orders of lines is different. To define and draw out their properties, general equations must be applied, expressing the relation of the co-ordinates. Let x represent the abcissa AP, y the ordinate PM of the figure PMH, and let a, b, c, d, e, &c. denote any invariable coefficients; and having the angle APM given, if the relation of the co-ordinates x and y be defined by an equation which, besides the co-ordinates themselves, involves only invariable coefficients, the line FMH is called a geometrical one; which indeed by some authors is called an algebraical line, by others a rational line. But the order of the line depends upon the highest index of x or y in the terms of the equation freed from fractions and surds, or upon the sum of the indices of both in a term where that sum is the greatest. For the terms x^2, xy, y^2 are equally referred to the second order; the terms x^3, x^2y, xy^2, y^3 to the third. Therefore the equation $y = ax + b$, or $y - ax - b = 0$, is of the first order and denotes a line or the locus of the first order, which indeed is always a right line. For let there be taken in the ordinate PM the right line PN, so that PN be to AP as $+ a$ to unity; let AD, parallel to PM, be made equal to $+ b$, and DM, drawn parallel to AN, will be the locus to which the proposed equation will answer. For PM = PN + NM = $(a \times \text{AP} + \text{AD}) ax + b$. But if the equation be of the form $y = ax - b$, or $y = - ax + b$, the right line AD, or PN, is to be taken on the other side of the abscissa AP; for the contrary situation of right lines answers to the contrary signs of the coefficients. If the affirmative values of x denote right lines drawn from A, the beginning of the abscissa, to the right hand, the negative values will denote right lines drawn from the same beginning to the left; and in like manner if the

Fig. 1.

Fig. 2.

the affirmative values of y represent the ordinates constituted above the abscissa, the negative ones will denote the ordinates below the abscissa, drawn the opposite way.

The general equation for a line of the second order is of this form,

$$\begin{array}{l} yy - axy + cx^2 = 0 \\ \quad\; - by \;\; - dx \\ \qquad\qquad\;\; + e \end{array}$$

and the general equation for lines of the third order is $y^3 - \overline{ax + b} \times y^2 + \overline{cxx - dx + e} \times y - fx^3 + gx^2 - hx + k = 0$. And by similar equations geometrical lines of superior orders are defined.

§ 2. A geometrical line may meet a right line in as many points as there are units in the number which denotes the order of the equation or line, and never in more. The number of times that any curve will meet its abscissa AP is determined by putting $y = 0$, in which case there remains only the last term of the equation into which y does not enter. For example, a line of the third order meets the abscissa AP when $fx^3 - gx^2 + hx - k = 0$, of which equation if there be three real roots, in three points. In like manner in the general equation of any order the highest index of the abscissa x is equal to the number which denotes the order of the line, but never greater, and of course expresses the number of times that the curve will meet the abscissa or any other right line. But since one root of a cubic equation is always real, and that the same is true of an equation of the fifth or any odd order (because every imaginary root has necessarily its fellow), it follows that a line of the third or any other odd order cuts any right line, not parallel to the asymptote

asymptote drawn in the same plane, in one point at least. But if the right line be parallel to the asymptote, in this case it is commonly said to meet the curve at an infinite distance. A line therefore of any odd order has necessarily two branches which may be produced *in infinitum*. But of a quadratic, or any other equation of an even number of roots, all the number of roots may be sometimes imaginary, therefore it may be that a right line drawn in the plane of a curve of an even order may never meet it.

§. 3. An equation of the second, or of any higher order is sometimes compounded of so many simple ones, freed from surds and fractions, multiplied into each other as often the proposed dimensions of that equation express; in which case the figure FMH is not curvilinear, but is made up of so many right lines as are described by the simple equations thus determined, as in § 1. In like manner if a cubic equation be compounded of two equations multiplied into each other, one of which is a quadratic and the other a simple one, the the locus will not be a line of the third order, properly so called, but a conic section joined with a right line. Now the properties which are generally demonstrated of geometrical lines of higher orders are to be affirmed also of lines of inferior orders, if the numbers denoting their orders, taken together, make up the number which denotes the order of the said superior line. Those which, for example, are generally demonstrated of lines of the third order, are also to be affirmed of three right lines drawn in the same plane, or of a conic section together with one right line described in the same plane. On the other hand, there can scarce any property of a line of an inferior order be assigned sufficiently general to which some affection of lines of superior orders does not correspond. But to derive these from those, it is not every one that can take the pains. This doctrine in a great measure depends upon the properties of general equations, which it is here only proper to mention.

§ 4. In every equation the coefficient of the second term is equal to the excess of the sum of the affirmative roots above the sum of the negative ones; and if that term be wanting, it is an indication that the sums of the affirmative and negative roots, or the sums of the ordinates constituted on different sides of the abscissa, are equal. Let the general equation be for a line of the order n, $y^n - \overline{ax + b} \times y^{n-1} + \overline{cxx - dx + e} \times y^{n-2} - \&c. = 0$, suppose $u = y - \frac{ax + b}{n}$, for y let be substituted its value $u + \frac{ax + b}{n}$; and in the transformed equation the second term u^{n-1} will be wanting; as appears from the calculation, or from the doctrine of equations, every where delivered: and from hence it also appears, that by hypothesis every value of u is less than the corresponding value of y by $\frac{ax + b}{n}$; from whence it follows that the sum of the values of u (whose number is n) falls short of the sum of the values of y (whose sum is $ax + b$) by the difference $\frac{ax + b}{n} \times n = ax + b$, so that the first sum vanishes, and the second term is wanting in the equation by which u is determined, or that the affirmative and negative values of u make equal sums. If therefore PQ be taken

=

$=\frac{ax+b}{n}$, ſo that QM may $=x$, right lines on both ſides the point Q, terminated at the curve, will make the ſame ſum. Now the locus of the point Q is the right line BD which cuts the abſciſſa, produced beyond its beginning A, in B, ſo that $AB=\frac{b}{a}$, and the ordinate AD, parallel to PM, in D, ſo that $AD=\frac{1}{n}\times b$; for if this right line meets the ordinate PM in the point Q, PQ will be to PB (or $\frac{b}{a}+x$) as AD to AB, or a to n; ſo that $PQ=\frac{ax+b}{n}$, as it ought to do. And from hence it appears, that a right line may always be drawn which ſhall ſo cut any number of parallels, meeting a geometrical line in as many points as the dimenſions of the figure expreſs, that the ſum of the ſegments of every parallel, terminated at the curve on one ſide of the cutting line, may always be equal to the ſum of the ſegments of the ſame on the other ſide the cutting line. Now it is manifeſt that a right line which cuts any two parallels in this manner is neceſſarily that which will cut all other parallels in the ſame manner. And from hence appears the truth of the *Newtonian* theorem, in which is contained the general property of geometrical lines, analogous to that well-known property of the conic ſections. For in theſe a right line which biſects any two parallels, terminated at the ſection, is a diameter, and biſects all others parallel to theſe, and terminated at the ſection. And, in like manner a right line, which cuts any two parallels, meeting a geometrical line in as many points as it has dimenſions, ſo that the ſum of the parts ſtanding on

Fig. 3.

one

one side of the cutting line and terminated at the curve may be equal to the sum of the parts of the same parallel standing on the other side of the cutting line terminated at the curve, will in the same manner cut all other right lines parallel to these.

§ 5. In every equation the last term, or that into which the root y does not enter, is equal to the product of all the roots multiplied into each other; from whence we are led to another property of geometrical lines, not less general than that above. Let the right Fig. 1. line PM meet a line of the third order in M, m and μ, and it will be $PM \times Pm \times P\mu = fx^3 - gx^2 + hx - k$. Let the abscissa AP cut the curve in the three points I, K, L; and AI, AK, AL will be the values of the abscissa x, the ordinate being put $= 0$, in which case the general equation gives $fx^3 - gx^2 + hx - k = 0$ for determining these values, as we explained in Art. 2. Therefore of the equation $x^3 - \frac{gx^2}{f} + \frac{hx}{f} - \frac{k}{f} = 0$ the three roots are AI, AK, AL; and so this equation is compounded of the three $x - AI$, $x - AK$, $x - AL$ multiplied into each other; and $x^3 - \frac{gx^2}{f} + \frac{hx}{f} - \frac{k}{f} = \overline{x - AI} \times \overline{x - AK} \times \overline{x - AL} = \overline{AP - AI} \times \overline{AP - AK} \times \overline{AP - AL} = IP \times KP \times LP = \frac{1}{f} \times PM \times Pm \times P\mu$. Therefore the product of the ordinates PM, Pm, Pμ, terminated by the point P and the curve, is to the product of the segments IP, KP, LP, of the right line AP, terminated by the same point and the curve, in the invariable ratio of the coefficient

efficient f to unity. In like manner it is demonstrated, that having given the angle APM, if the right lines AP, PM, cut a geometrical line of any order in as many points as it has dimensions, that the product of the segments of the first, terminated by P and the curve, will always be to the product of the segments of the latter, terminated by the same point and the curve, in an invariable ratio.

§ 6. In the preceding article we have supposed, with *Newton*, that the right line AP cuts a line of the third order in three points I, K, L; but that this famous theorem may be rendered more general, let us suppose that the abscissa AP cuts the curve in only one point; and let that be A. Therefore because y vanishes let x vanish also, the last term of the equation, in this case, will be $fx^3 - gx^2 + hx = fx \times \overline{xx - \frac{gx}{f} + \frac{h}{f}} = fx \times \overline{x - \frac{g}{2f}}^2 + \frac{h}{f} - \frac{gg}{4ff}$ (if Aa be taken towards P equal $\frac{g}{2f}$, and at the point a be erected a perpendicular $ab = \frac{\sqrt{4fh - gg}}{2f}$) $= f \times AP \times \overline{aP^2 + ab^2} = f \times AP \times bP^2$; from whence, when $PM \times Pm \times P\mu$ is equal to the last term $fx^3 - gx^2 + hx$, as in the preceding article, $PM \times Pm \times P\mu$ will be to $AP \times bP^2$ in the constant ratio of the coefficient f to unity. Now the value of the right line perpendicular to ab is always real, as often as the right line AP cuts the curve in one point only; for in this case the roots of the quadratic equation $fx^2 - gx + h$ are necessarily imaginary, so that $4fh$ is greater than gg, and the quantity $\sqrt{4fh - gg}$ real. When therefore Fig. 4.

fore any right line cuts a line of the third order in one point A only, the solid under the ordinates PM, Pm, Pμ will be to the solid under the abscissa AP and the square of the distance of the point P from a given point b in a constant ratio. Ab, being joined is to Aa, as radius to the cosine of the angle bAP, as $\sqrt{4fb}$ to g, and A$b = \sqrt{\frac{b}{f}}$. But the same point b always agrees to the same right line AP, whatever be the angle which is contained by the abscissa and ordinate.

§ 7. Let the figure be a conic section, whose general equation is $yy - \overline{ax - b} \times y + cxx - dx + e = 0$ as above; and if the roots of the equation $cxx - dx + e = 0$ be imaginary, the right line AP will not meet the section. Now, in this case the quantity $4ec$ always exceeds dd; whence, when $cxx - dx + e = c \times \overline{x - \frac{d}{2c}}^2 + e - \frac{dd}{4c}$ (if Aa be taken $= \frac{d}{2c}$, and ab be erected perpendicular to the abscissa at a, so that $ab = \frac{\sqrt{4ec - dd}}{2c}$) $= c \times \overline{aP^2 \times ab^2} = c \times bP^2$, and PM $\times$ Pm $= cxx - dx + e$, then PM $\times$ Pm is to bP^2 as c to unity. Therefore in any conic section, if the right line AP does not meet the section, the angle APM being given, the rectangle contained under right lines standing at the point b and terminated at the curve is to the square of the distance of the point P from the given point b in a constant ratio, which in a circle is that of equality. Now it is manifest that the same method may be applied to a line of the fourth order which the abscissa cuts in two points only, or to a line of any order which

Fig. 5.

which the abſciſſa cuts in points leſs by two than the number which denotes the order of the figure.

§ 8. This being premiſed, I proceed to explain the leſs obvious properties of geometrical lines almoſt in the ſame order in which they occurred to me. Now I uſed the following lemma, derived from the doctrine of fluxions, and which I have demonſtrated in my treatiſe on that ſubject, lately publiſhed Art. 717. yet I have ſince obſerved that ſome of them may be demonſtrated by common algebra.

Lemma. If the quantities x, y, z, u, &c. flowing together, and alſo the quantities X, Y, Z, V, &c. the product of the former be to the product of the latter in any conſtant ratio, then $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} + \frac{\dot{v}}{v} +$ &c. $= \frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z} + \frac{\dot{V}}{V} +$ &c. Moreover, for brevity's ſake I call thoſe quantities mutually *reciprocal*, when, being multiplied into each other, the product is unity, ſo $\frac{1}{x}$ I call the *reciprocal* of x, and $\frac{1}{y}$ of y.

§ 9. Theor. I. *Let any right line, drawn through a given point, meet a geometrical line of any order in as many points as it has dimenſions; and let right lines, touching the figure in theſe points, cut off from another right line given in poſition and drawn through the ſame given point, as many ſegments terminated by this point; the* reciprocals *of theſe ſegments will always make the ſame ſum, if the ſegments lying on the contrary ſide of the given point be affected with the contrary ſigns.*

Let

Fig. 6. Let P be the given point, PA and Pa any two right lines drawn from P, of which both meet the curve in as many points A, B, C, and a, b, c, &c. as it has dimensions. Let the tangents AK, BL, CM, &c. and ak, bl, cm &c. cut off from the right line EP, drawn through the point P, the segments PK, PL, PM, &c. and Pk, Pl, Pm + &c. I say that $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. $= \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} +$ &c. and that this sum always remains the same, the point P remaining, and the right line PE being given in position.

For let us suppose the right lines ABC, abc to be carried by motions parallel to themselves, so that their concourse P proceeds in the right line PE given in position; since AP × PB × CP × &c. is always to aP × bP × cP in a constant ratio by Art. 5. let $\dot{A}$P represent the fluxion of AP, $\dot{B}$P the fluxion of BP, and $\dot{C}$P, $\dot{E}$P &c. the fluxions of the right lines CP, EP, &c. respectively, that an useless multiplication of symbols may be avoided, then (by Art. 8.) $\frac{\dot{A}P}{AP} + \frac{\dot{B}P}{BP} + \frac{\dot{C}P}{CP} +$ &c. $= \frac{\dot{a}P}{aP} + \frac{\dot{b}P}{bP} + \frac{\dot{c}P}{cP} +$ &c. But when the right line AP is carried by a motion parallel to itself, it is well-known that $\dot{A}$P, the fluxion of the right line AP, is to $\dot{E}$P, the fluxion of the right line EP, as AP to the subtangent PK, and so $\frac{\dot{A}P}{AP} = \frac{\dot{E}P}{PK}$. In like manner $\frac{\dot{B}P}{BP} = \frac{\dot{E}P}{PL}$, $\frac{\dot{C}P}{CP} = \frac{\dot{E}P}{PM}$, $\frac{\dot{a}P}{aP} = \frac{\dot{E}P}{Pk}$, $\frac{\dot{b}P}{bP} = \frac{\dot{E}P}{Pl}$ and

$\dot{c}$P

$\frac{c\dot{P}}{cP} = \frac{E\dot{P}}{Pm}$, whence $\frac{E\dot{P}}{PK} + \frac{E\dot{P}}{PL} + \frac{E\dot{P}}{PM} + \&c. = \frac{E\dot{P}}{Pk} + \frac{E\dot{P}}{Pl} + \frac{E\dot{P}}{Pm} + \&c.$ and $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c. = \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} + \&c.$

Things are so whenever the points K, L, M, &c. and k, l, m, &c. are all on the same side of the point P, and so the fluxions of the right lines AP, BP, CP, &c. aP, bP, cP, &c. have all the same sign. But if, other things remaining the same, some points M and m fall on the contrary side of P, then while the rest of the ordinates AP, BP, &c. increase, the ordinates CP and cP are necessarily diminished, and their fluxions are to be accounted subtractive, or negative; and so in this case $\frac{1}{PK} + \frac{1}{PL} - \frac{1}{PM}$, &c. $= \frac{1}{Pk} + \frac{1}{Pl} - \frac{1}{Pm}$, &c. and in general, in collecting these sums, the terms are to be affected with the same or contrary signs, as the segments fall on the same or contrary side of the given point P. Fig. 7.

§ 10. If a right line PE meets a curve in as many points D, E, I, &c. as its dimensions express the sum $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} + \&c.$ which we have shewn to be constant or invariable, will be equal to the sum or aggregate $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} + \&c.$ i. e. to the sum of the reciprocals to the segments of the right line PE, given in the position, and determined by the given point P and the curve; in which, if any segment be on the other side of the point P, its reciprocal is to be subtracted.

§ 11.

Fig. 8. § 11. If the figure be a conic section, which the right line PE no where meets, let the point b be found as in Art. 7. and Pb joined, and at right angles to this let bd be drawn, cutting the right line PE in d, then will $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$. For PA × PB is to bP^2 in a constant ratio, and so (by Art. 8.) $\frac{AP}{AP} + \frac{BP}{BP} = \frac{2bP}{bP}$, whence (because $\dot{A}P$ is to $\dot{E}P$ as AP to PK, $\dot{B}P$ to $\dot{E}P$ as BP to PL, and $b\dot{P}$ to $\dot{E}P$ as bP to dP) $\frac{1}{PK} + \frac{1}{PL} = \frac{2}{Pd}$.

§ 12. In like manner if the right line EP meets a line of the third order in only one point D, let the point b be found as in Art. 6. and let the right line bd, perpendicnlar to bP, meet the right line EP in d, and because AP × BP × CP is to DP × bP^2 in a constant ratio (*ibid.*) $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} = \frac{1}{PD} + \frac{2}{Pd}$. But if Pb be perpendicular to the right line EP, $\frac{2}{Pd}$ will vanish.

Fig. 10. § 13. The asymptotes of geometrical lines are determined from the given direction of their infinite branches or legs by this proposition; for they may be considered as tangents to the legs produced *in infinitum*. Let the right line PA, parallel to the asymptote, meet the curve in the points A, B, &c. but the right line PE cut the curve in D, E, I, &c. Let PM be taken in this so that $\frac{1}{PM}$ may be equal to the excess by

which

Appen^x. pl. 448.

Fig. 1

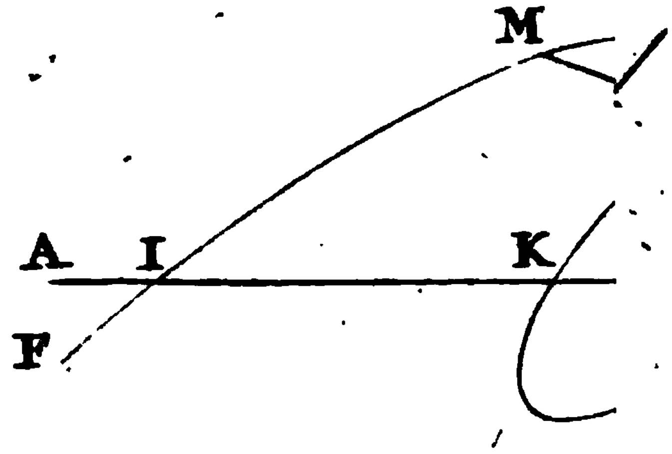


Fig. 4.

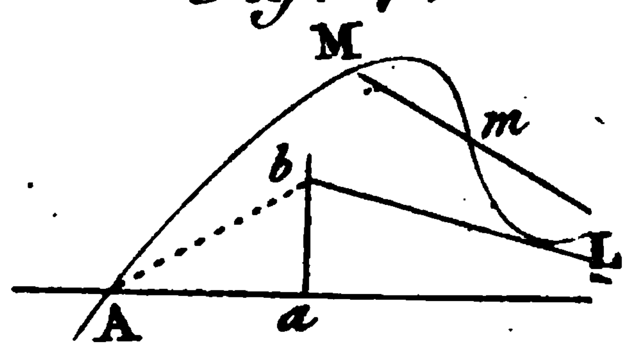
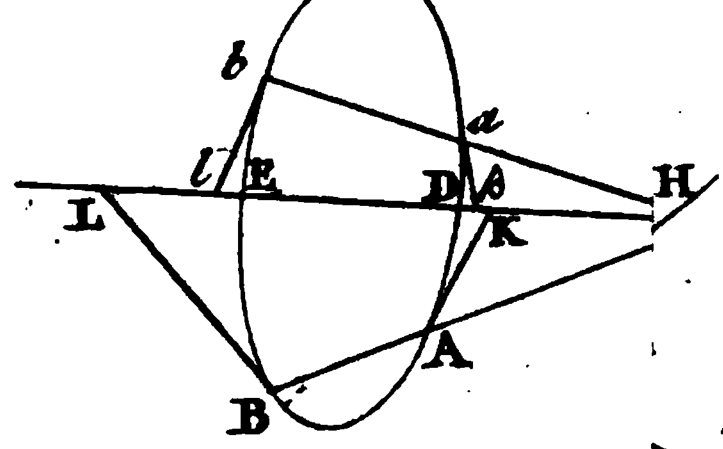


Fig.



P.

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which the ſum $\frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} +$ &c, exceeds the ſum $\frac{1}{PK} + \frac{1}{PL} +$ &c. and the aſymptote will paſs through M; but if theſe ſums be equal, the curve will be a parabola, the aſymptote going off *in infinitum*.

§ 14. To determine the *curvature* of geometrical lines by one general theorem, let CDR be a circle which the right line PR meets in D and R, and the right line PC in C and N; let the tangent CM cut the right line PD in M, and the right line DR remaining fixed, let us ſuppoſe the right line PCN to be carried by a motion always parallel to itſelf till the points P, D, C, coincide, and let the laſt value of the difference $\frac{1}{PM} - \frac{1}{PD}$ be required. In the right line PN take any point q, let qv, parallel to the tangent CM, meet the right line DR in v; let DQ be drawn parallel to PN, and let QV (parallel to a line touching the circle in D) cut DR in V. Therefore $\frac{1}{PM} - \frac{1}{PD} = \frac{DM}{PM \times PD}$ (becauſe $DM \times MR = CM^2$) $= \frac{CM^2 \times PM}{PM^2 \times PD \times MR}$ $= \frac{qv^2 \times PM}{Pv^2 \times MR \times PM + Pv^2 \times MR \times MD}$ (ſince $MR \times MD$, or CM^2, is to PM^2 as qv^2 to Pv^2) $= \frac{qv^2 \times PM}{Pv^2 \times MR \times PM + qv^2 \times PM^2} = \frac{qv^2}{Pv^2 \times MR + qv^2 \times PM}$, whoſe laſt value, PM vaniſhing, and qv and Pv coinciding with QV and DV, is $\frac{QV^2}{DV^2 \times DR}$. And this is alſo the laſt value of the difference $\frac{1}{PM} - \frac{1}{PD}$ if D

Fig. 11.

 and

and C are in the arc of any line of the same curvature with the circle CDR.

Fig. 12. §. 15. Theor. II. *From any point* D *of a geometrical line let there be drawn any two right lines* DE, DA, *and both of them cut it in as many points* D, I, E, *&c. and* D, A, B, *&c. as it has dimensions; let the tangents* AK, BL, *&c. cut off from the right line* DE *the segments* DK, DL, *&c. let any right line* QV, *parallel to the tangent* DT, *meet* DA *and* DE *in* Q *and* V, *and let* QV^2 *be to* DV^2 *as* m *to* 1; *moreover let there be taken in* DE *the right line* DR *such that* $\frac{m}{DR}$ *may be equal to the excess of the sum* $\frac{1}{DE} + \frac{1}{DI} +$ &c. *above the sum* $\frac{1}{DK} + \frac{1}{DL} +$ *&c. and a circle described upon the chord* DR, *touching the right line* DT *will be the osculatory circle, or of the same curvature with the geometrical line proposed, at the point* D.

For we have shewn in general, Art. 10. (Fig. 6.) that the sum $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} +$ &c. $= \frac{1}{PD} + \frac{1}{PE} + \frac{1}{PI} +$ &c. and in the preceding Art. we have found the last value of the difference $\frac{1}{PM} - \frac{1}{PD}$, when the points P, D and C coincide, to be $\frac{QV^2}{DV^2 \times DR} = \frac{m}{DR}$ if a circle of the same curvature with the geometrical line at the point D meets the right line DE in R. From whence it follows that $\frac{m}{DR}$ will be $= \frac{1}{DE} + \frac{1}{DI}$ + &c. $- \frac{1}{DK} - \frac{1}{DL} -$ &c. or that the reciprocal of of $\frac{1}{m} \times DR$ is equal to the excefs by which the fum of the reciprocals of the fegments terminated by the point D and the curve, furpaffes the fum of the reciprocals of the fegments terminated by the fame point and the tangents AK, BL, &c. But as often as this excefs comes out negative, the chord DR is to be taken on the other fide of the point D, and the rule above defcribed is always to be applied for diftinguifhing the figns of the terms. If the right line DA bifects the angle EDT, made by the right line DE and the tangent DT, the theorem becomes a little more fimple. For in this cafe $QV = DV$, $m = 1$, and $\frac{1}{DR} =$ the excefs by which $\frac{1}{DE} + \frac{1}{DI} +$ &c. exceeds $\frac{1}{DK} + \frac{1}{DL}$ + &c.

§ 16. From the fame principle follows a general theorem by which the variation of curvature is determined, or the meafure of the angle of contact contained by the curve and the ofculatory circle, in any geometrical line; yet a brief explication of the variation of curvature muft be premifed, fince this is not clearly defcribed by authors. Every curve is bent from its tangent by its curvature, of which the meafure is the fame as of the angle of contact contained by the curve and tangent; and in like manner a curve is bent from its ofculatory circle by the variation of its curvature, of which variation the meafure is the fame as of the angle of contact contained by the curve and ofculatory circle. Let the right line TE perpendicular to the tangent DT meet the curve in E and the ofculatory circle in r, and the variation of curvature will be Fig. 13.

 ultimately

ultimately as Er the subtense of the angle of contact EDr, if DT be given; and since, when the angle of contact EDr is given, Er is ultimately as DT^3, as may be collected from Art. 369. of the Treatise of Fluxions; in general the variation of curvature will be ultimately as $\frac{Er}{DT^3}$. We use a circle for determining the curvature of other figures; but to measure the variation of curvature, which is nothing in a circle, a parabola or some conic section is to be applied. Now as of the circles indefinite in number which may touch a given curve in a given point, one only is called osculatory, which so closely touches the curve that no other can be drawn between this and the curve; in like manner of all parabolas which have the same curvature with the line proposed at a given point (for these are also infinite in number) that only has the same variation of curvature, which not only touches the arc of the curve and kisses it, but presses so close that no other parabolic arc can be drawn between them, all other parabolic arcs passing either without or within both. By what method this parabola is to be determined may be easily understood from what I have elsewhere more fully explained.

Let DE be the arc of a curve, DT a tangent, TEK a right line perpendicular to the tangent, and let the rectangle ET × TK be always equal to the square of the tangent DT, and the curve SKF the locus of the point K, which meets the line DS perpendicular to the curve in S, and which touches the right line SV in S cutting the tangent TD in V. The right line DS will be the diameter of the osculatory circle, and DS being bisected in f, f will be the center of curvature; now

now V*ſ* being joined, if the angle SDN be made equal to the angle *ſ*VD on the other ſide of the right line DS, and the right line DN meet the oſculatory circle in N; then the parabola deſcribed with the diameter and parameter DN, and which touches the right line DT in D, will be that whoſe contact with the line propoſed in D will be the cloſeſt and moſt perfect or neareſt that can be deſcribed. But all other parabolas, deſcribed with any other chord of the oſculatory circle although deſcribed with the diameter and parameter, and touching the right line in D, have the ſame curvature in D with the line propoſed. The quality of curvature explained by *Newton* in a poſthumous work lately publiſhed, is rather a variation of the radius of curvature; for it is as the fluxion of the radius of curvature divided by the fluxion of the curve, or (if R denotes the radius of the oſculatory circle and S the arc of the curve) as $\frac{\dot{R}}{\dot{S}}$. Now the curvature is inverſely as the radius R, and the variation of curvature as $\frac{-\dot{R}}{RR\dot{S}}$, which is the meaſure of the angle of contact contained between the curve and the oſculatory circle. Now of theſe the one is eaſily derived from the other. The variation of the radius of curvature in any curve DE is as the tangent of the angle DVS or DV*ſ*, and in any parabola it is always as the tangent of the angle contained by a diameter paſſing through the point of contact and a right line perpendicular to the curve. Theſe things may be deduced from the following general theorem.

Fig. 14. § 17. Theor. III *Let there be a point* D *given in any geometrical line, and let* DS, *the diameter of the osculatory circle, drawn through* D, *meet the curve in as many points* D, A, B, *&c. as it has dimensions; let* DT *be drawn touching the curve in* D, *and let it cut the curve in the points* I, *&c, fewer by two, and meet the tangents* AK, BL, *&c. in* K, L, *&c. and the variation of curvature, or the measure of the angle of contact made by the curve and the osculatory circle, will be directly as the excess by which the sum of the reciprocals to the segments of the tangent* DT, *terminated by the point of contact* D *and the tangents* AK, BL, *&c. exceeds the sum of the reciprocals to the segments terminated by the same point and the curve, and inversely as the radius of curvature,* i. e. *as* $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} + \&c. - \frac{1}{DI} - \&c.}$

For let there be drawn Dk cutting the curve in e, i, &c. and the osculatory circle in R; and let the angle kDT be very small, let the supplement of this to two right angles be bisected by the right line Dab, which let meet the proposed geometrical line in the points D, a, b, &c. and let the tangents ak, bl, &c. when drawn, cut the right line Dk in the points k, l, &c. then by the preceding proposition $\frac{1}{DR} = \frac{1}{De} + \frac{1}{Di} - \frac{1}{Dk} - \frac{1}{Dl}$, &c. From whence $\frac{1}{DR} - \frac{1}{De}$ $\left(\text{or } \frac{Re}{DR \times De}\right) = \frac{1}{Di} - \frac{1}{Dk} - \frac{1}{Dl} -$ &c. Therefore the right lines Dk and DK coinciding, or the angle kDK vanishing, $\frac{R}{LR \times De}$ will be ultimately equal to $\frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL}$ —

— &c. Let erT be perpendicular to the tangent at T, and meet the osculatory circle in r; and since re is ultimately to Re as eT to De, ultimately $\frac{Re}{DR \times De}$ $= \frac{re}{DR \times eT} = \frac{re \times DS}{DR \times DT^2}$ or $\frac{re \times DS}{DT^3}$. Now the measure of the angle of contact rDe contained by the curve and osculatory circle, or the variation of curvature, is as $\frac{re}{DT^3}$, and therefore as $\overline{\frac{1}{DS}} \times \overline{\frac{1}{BI} - \frac{1}{DK} - \frac{1}{DL}}$, &c.

§ 18. Now the variation of the radius of curvature, or the quality of it described by *Newton* is most easily collected from the former. For SI, SK, SL, &c. being joined, this variation of the osculatory radius will be as the excess by which the sum of the tangents of the angles DKS, DLS, &c. exceeds the sum of the tangents of the angles DIS, &c. Now the curvature increases from the point D towards E, and the osculatory radius is diminished, as often as the arc DE touches the osculatory circle DR internally, or when $\frac{1}{DK} + \frac{1}{DL}$ + &c. exceeds $\frac{1}{DI}$ + &c. and on the contrary the curvature from D towards e is diminished, and the radius of the osculatory circle is increased, as often as the arc De of the curve touches the circular arc externally or passes between the circle and tangent, therefore when DR is ultimately less than De, or when $\frac{1}{DI}$ + &c. exceeds $\frac{1}{DK} + \frac{1}{DL}$ + &c.

§ 19. Let therefore the line DV be taken in the tangent DT so that $\frac{1}{DV} = \frac{1}{DK} + \frac{1}{DL} + \&c. - \frac{1}{DI} - \&c.$ let ſV be joined, let the angle SDN be made equal DVſ, and the line DN meet the osculatory circle in N; and a parabola described with the diameter DN whose parameter is DN, and which touches the right line DT in D, will have the same variation of curvature with the proposed geometrical line in the point D. From the same principles other theorems are also deduced, by which the variation of curvature in geometrical lines is in general determined.

§ 20. That these theorems may be reduced into a more geometrical form, some lemmas are to be premised, by which the doctrine of the harmonical division of right lines is made more full and general. In
Fig. 15. any right line DI having taken equal segments DF and FG, let there be drawn from any point V, which is not in the right line DI, three right lines VD, VF, VG, and a fourth VL parallel to DI, and these four right lines are, by *De la Hire*, called Harmonicals. But any right line which meets four harmonicals is cut by the same harmonically. Let the right line DC meet the harmonials VD, VF, VG, and VL in the points D, A, B, C; and it will be DA to DC as AB to BC. For through the point A let there be drawn the line MAN parallel to DI, which meets the lines VD and VG in M and N; and because of the equals DF and FG, MA and AN will be equal. Now DA is to DC as AM (or AN) to VC, and therefore as AB to BC. It is manifest that a right line, which is parallel to one of the harmonicals, is divided into equal segments

ments by the remaining three. Let the line BH parallel to VF meet the remaining lines VG, VC, VD in B, K, and H; and it will be as VK to KB, so FG (or DF) to VF, and therefore as VK to KH, and consequently BK = KH.

§ 21. Hence it follows, if any right line be cut harmonically by four right lines drawn from the same point, that any right line which meets these four lines will also be cut harmonically by the same; but that that which is parallel to one of the four is divided into equal segments by the remaining three. Let DA be to DC as AB to BC, let VA, VB, VC, and VD be joined; let the right lines MAN, DFG parallel to VC meet the lines VD, VA, and VB in M, A, N, and D, F, G; and it will be MA to VC as DA to DC or AB to BC, and therefore as AN to VC; hence MA = AN, and DF = FG; and, by the preceding, any right line which meets VD, VA, VB, VC will be harmonically cut by the same.

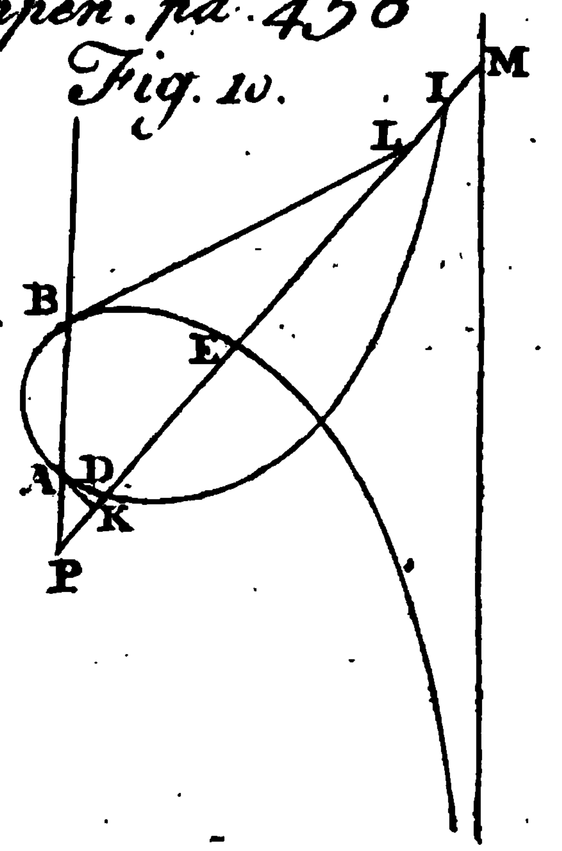
§ 22. From the point D let there be drawn two right lines DAC, D*ac* cutting the lines VA and VC in the points A, C and *a*, *c*; let A*c* and *a*C joined meet each in Q, and VQ drawn will cut the line DAC harmonically, or any other right line drawn from the point D to the same right lines. For let VQ cut the line AC in B, and through the point Q let there be drawn the line MQN parallel to DC, which meets the lines D*a*, VA and VC in the points M, R, and N; and since MR is to MQ as DA to DC, and MQ to MN in the same ratio, RQ will be to QN as DA to DC. But RQ is to QN as AB to BC. Wherefore DA is to Fig. 16. n. 1.
DC

DC as AB to BC. This is the 20th Prop. of *De la Hire's* first Book of conic sections.

§ 23. Let DA be to DC as AB to BC, and $\frac{2}{DB}$ will be the sum or the difference of $\frac{1}{DA}$ and $\frac{1}{DC}$, according as the points A and C are on the same or contrary sides of the point D. First let the points A and C be on the same side of the point D, and since $DA \times BC = DC \times AB$, i. e. $DA \times \overline{DC - DB} = DC \times \overline{DB - DA}$, or $DA \times \overline{DB - DC} = DC \times \overline{DA - DB}$, it will be $2DA \times DC = DA \times DB + DC \times DB$, and there-

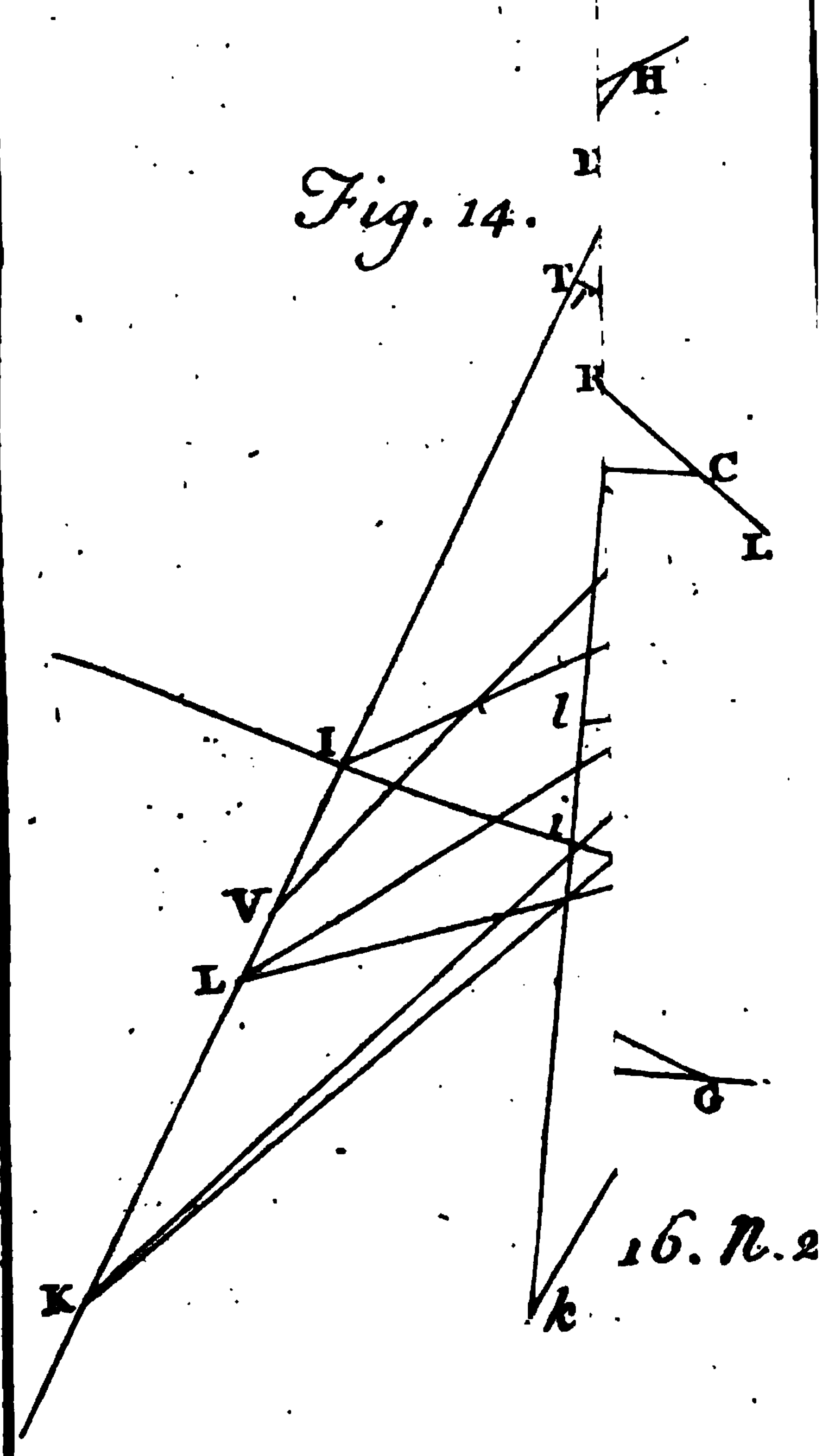
n. 2 & 3. fore $\frac{2}{DB} = \frac{1}{DA} + \frac{1}{DC}$. Let now the points A and C be on the contrary sides of the point D, and it will be either $DA \times \overline{DB - DC} = DC \times \overline{DB + DA}$, or $DA \times \overline{DB + DC} = DC \times \overline{DB - DA}$, and therefore $\frac{2}{DB} = \frac{1}{DC} - \frac{1}{DA}$ when the points B and C are on the same side of D, or $\frac{2}{DB} = \frac{1}{DA} - \frac{1}{DC}$ when the points A and B are on the same side of the point D. If therefore, having given the point D and the right lines VF and VC in position, any right line be drawn through the point D meeting them in the points A and C, and in the same right line DB be always taken so that $\frac{2}{DB} = \mp \frac{1}{DA} \mp \frac{1}{DC}$, where the terms $\frac{1}{DA}$ and $\frac{1}{DC}$ are supposed to be affected with the same or contrary signs as the points A and C are on the same or contrary sides of the point D, the locus of the point B will be the harmonical VG which cuts the line DFG parallel appen. pa. 458

Fig. 10.



.13.

Fig. 14.



16. N.2

parallel to VC in G so that $FG = DF$; and which passes through the point Q where (Dac being drawn which meets the same right lines VF and VC in a and c) Ac and aC being joined, cross each other.

§ 24. If in a right line DA Db be always taken so that $\frac{1}{Db} = \frac{1}{DA} \mp \frac{1}{DC}$; let DF be drawn parallel to the line VC which meets VF in F, and DH parallel to the line VF which meets the line VC in H, and the diagonal HF being drawn will be the locus of the point b; for by hypothesis $\frac{1}{Db} = \frac{2}{DB}$, and $DB = 2Db$; therefore since VG is the locus of the point B, the point b will be in the right line HF, if the points A and C are on the same side of the point D. But if it be supposed that $\frac{1}{Db} = \frac{1}{DA} + \frac{1}{DC}$, the same construction will serve for determining the point b, if instead of the right line VC be substituted another vc parallel to VC at an equal distance from the point D, but on the contrary side. Fig. 17.

§ 25. If from a given point D be drawn any right line DM which meets three lines given in position in the points A, C, E; and DM be always taken so that $\frac{1}{DM} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$ (where the terms are to be affected with the contrary signs as often as the lines DA, DC, or DE are on the contrary side of the point D); let it be supposed that $\frac{1}{DA} + \frac{1}{DC} = \frac{1}{DL}$, and L will be in a line given in position by the preceding; and therefore when $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE}$, the point will be be in a line given in position, by the same. Now the composition of the problem is easily performed from what has been said. Let VA, VC and vE be three lines given in position, and let the parallelogram DFVH be completed, by drawing DF and DH respectively parallel to VC and VF, and let vE meet the diagonal in v; then let the parallelogram $Dfvb$ be completed by drawing Df and Db parallel to the lines vE and HF, which meet the lines HE and vE in the points f and b; and the diagonal bf will be the locus of the point M; and it will be, from what goes before, $\frac{1}{DM} = \frac{1}{DL} + \frac{1}{DE} = \frac{1}{DA} + \frac{1}{DC} + \frac{1}{DE}$. Another construction is deduced from Art. 22.

§ 26. Let any right line drawn from the given point D meet right lines given in position in the points A, B, C, E, &c. and in this right line let there be taken $\frac{1}{DM}$ always $= \frac{1}{DA} \mp \frac{1}{DB} \mp \frac{1}{DC} \mp \frac{1}{DE}$, &c. the locus of the point M will always be in a right line given in position. It is demonstrated in the same manner as the preceding.

Fig. 18. § 27. Theor. IV. *About the given point* P *let the right line* PD *revolve which meets a geometrical line of any order in as many points* D, E, I, *&c. as it has dimensions, and if in the same right line be always taken* PM *so that* $\frac{1}{PM} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ *&c. (where we suppose the signs of the terms to keep the rule repeatedly given) the locus of the point* M *will be a right line.*

For

For let there be drawn from the pole P any right line given in position PA, which let meet the curve in as many points A, B, C, &c. as it has dimensions. Let there be also drawn the right lines AK, BL, CN touching the curve in these points, which let meet PD in as many points K, L, N, &c. and by Art. 10. $\frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. $= \frac{1}{PK} \mp \frac{1}{PL} \mp \frac{1}{PN} \mp$ &c. Whence $\frac{1}{PM}$ is equal to this sum, and when the line PA is given in position, and the right lines AK, BL, CN, &c. remain fixed, whilst the right line PD revolves about the pole P, the point M will be in a right line, by the preceding Article; which may be determined by what has been shewn above from the given tangents AK, BL, &c.

§ 28. As the right line Pm is a mean harmonical between the two lines PD and PE, when $\frac{2}{Pm} = \frac{1}{PD} + \frac{1}{PE}$; in like manner Pm may be called a *mean harmonical* between any right lines PD, PE, PI, &c. whose number is n, when $\frac{n}{Pm} = \frac{1}{PD} \mp \frac{1}{PE} \mp \frac{1}{PI} \mp$ &c. And if any right line drawn from a given point P cut a geometrical line in as many points as it has dimensions, in which let Pm be always taken an harmonical mean between all the segments of the drawn line terminated by the point and the curve, the point m will be in a right line. For $\frac{1}{PM}$ will $= \frac{n}{Pm}$, and therefore Pm is to PM as n to unity; and since the point M is in a right line, by the preceding, the point m will

m will also be in a right line. And this is *Cotes's* theorem, or nearly related to it.

§ 29. Let a, b, c, d, &c. be the roots of an equation of the order n, V its last term into which the ordinate or root y does not enter, P the coefficient of the last term but one, M the harmonical mean between all the roots, or $\frac{n}{M} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} +$ &c. Therefore since V is the product of all the roots a, b, c, &c. multiplied into each other, and P is the sum of the products when all the roots, one excepted, are multiplied into each other, P will $= \frac{V}{a} + \frac{V}{b} + \frac{V}{c} + \frac{V}{d} +$ &c. $= \frac{nV}{M}$, and therefore $M = \frac{nV}{P}$. So, if the equation be a quadratic, whose two roots are a and b, M will $= \frac{2ab}{a+b}$ (having assumed the general equation for conic sections given in Art. 1.) $= \frac{2exx - 2dx + 2e}{ax - b}$. In a cubic equation, whose three roots are a, b, c, M will $= \frac{3abc}{ab + ac + bc}$ (if there be assumed the general equation for lines of the third order there given) $= \frac{3fx^3 - 3gx^2 + 3bx - 3k}{cxx - dx + e}$.

Fig. 19. §. 30. Let any two lines Pm and Pμ, drawn from the point P, meet a geometrical line in the points D, E, I, &c. and d, e, i, &c. and let Pm be an harmonical mean between the segments of the former terminated by the point P and the curve, and Pμ an harmonical mean between the like segments of the latter line;

line; let μm, being joined, meet the abscissa AP in H, then will $PH = \frac{nV\dot{x}}{\dot{V}}$ or PH is to Pm as P to $\frac{\dot{V}}{\dot{x}}$. For let the abscissa cut the curve in as many points B, C, F, &c. as it has dimensions; and since the last term of the equation (i. e. V) is to BP × CP × FP × &c. in a constant ratio, as we have shewn above (Art. 5.), it will be (by Art. 8.) $\frac{\dot{V}}{V} = \frac{\dot{x}}{BP} \mp \frac{\dot{x}}{CP} \mp \frac{\dot{x}}{FP} \mp$ &c. and therefore $\frac{n}{PH} = \frac{1}{BP} \mp \frac{1}{CP} \mp \frac{1}{FP} \mp \&c. = \frac{\dot{V}}{V\dot{x}}$, and $PH = \frac{nV\dot{x}}{\dot{V}}$ (because the line $PM = \frac{nV}{P}$) $= Pm \times \frac{P\dot{x}}{\dot{V}}$. In conic sections it is PH to Pm as $ax - b$ to $2cx - d$; and in lines of the third order as $cxx - dx + e$ to $3fxx - 2gx + h$.

§ 31. If a demonstration of the preceding proposition be desired from principles purely algebraical, it may be had by help of the following *Lemma*. Let the abscissa $AP = x$, the ordinate $PD = y$, V the last term of the equation defining the geometrical line $= Ax^n + Bx^{n-1} + Cx^{n-2} +$ &c. P the coefficient of the last term but one $= ax^{n-1} + bx^{n-2} + cx^{n-3} +$ &c. and let Q be the quantity which arises from multiplying every term of the quantity V into the index of x in this term, and dividing by x, i. e. let $Q = nAx^{n-1} + \overline{n-1} \times Bx^{n-2} + \overline{n-2} \times Cx^{n-3} +$ &c. (which is the quantity which we call $\frac{\dot{V}}{\dot{x}}$). Let there be drawn the

the ordinate Dp which makes any given angle ApD with the abſciſſa, and let the right lines PD, pD, and Pp be as the given ones l, r and k; let $p\mathrm{D} = u$, $\mathrm{A}p = z$; and let the propoſed equation be transformed into another expreſſing the relation between the ordinate u and abſciſſa z; and ſince $z = \mathrm{AP}$, the laſt term v of the new equation will be equal V, but p the coefficient of the laſt term but one, will be equal to $\mp \frac{\mathrm{Q}k + \mathrm{P}l}{r}$.

For ſince PD ($= y$) is to pD ($= u$) as l to r, $y = \frac{lu}{r}$; but let Pp be to pD ($= u$) as k to r, then Pp $= \frac{ku}{r}$, and $\mathrm{AP} = x = \mathrm{A}p \pm \mathrm{P}p = z \pm \frac{ku}{r}$. Now theſe values being ſubſtituted for y and x in the propoſed equation of the geometrical line, there will come out an equation determining the relation of the coordinates z and u. To determine the laſt term of this v and the laſt but one pu, it is ſufficient to ſubſtitute theſe values in the laſt V, and in the laſt but one Py, of the propoſed equation, and to collect the reſulting terms in which the ordinate u is either not found, or of one dimenſion only; for the ſum of theſe gives pu, and of thoſe v. Let for x be ſubſtituted its value $z \pm \frac{ku}{r}$ in the quantity V or $\mathrm{A}x^n + \mathrm{B}x^{n-1} + \mathrm{C}x^{n-2}$ + &c. and the reſulting terms $\mathrm{A}z^n \pm \mathrm{B}z\frac{n\mathrm{A}z^{n-1}ku}{r}$ $+ \mathrm{B}z^{n-1} \pm \overline{n-1} \times \frac{\mathrm{B}z^{n-2}ku}{r} + \mathrm{C}z^{n-2} \pm \overline{n-2}$ $\times \frac{\mathrm{C}z^{n-3}ku}{r}$ + &c. will alone ſerve for the purpoſe we we are about. Then let be substituted for x the same value, and for y its value $\frac{lu}{r}$, in the quantity $Py = \overline{ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c.} \times y$; and the resulting terms alone $\overline{az^{n-1} + bz^{n-2} + cz^{n-3} + \&c.} \times \frac{lu}{r}$ are to be retained. Let it be supposed now that $z = x$, and the sum of the first be equal $V \pm \frac{Qku}{r}$, and of the latter the sum $= \frac{Plu}{r}$. From whence it is manifest that the last term of the new equation $v = V$, and the last but one $pu = \frac{Pl \pm Qk}{r} \times u$.

§ 32. Let now Pm be an harmonical mean between the segments PD, PE, PI, &c. and Pμ an harmonical mean between the segments Pd, Pe, Pi, &c. as in Art. 30. let μm, being joined, cut the abscissa in H; and let us suppose Pμ to be parallel to the ordinate pD. Let μs be drawn parallel to the abscissa, which let meet the right line Pm in s; and Ps will be to Pμ as PD to pD or as l to r, and μs to Pμ as k to r. And since $P\mu = \frac{nv}{p}$ (by the preceding Article) $= \frac{nVr}{\overline{Pl \pm Qk}}$, ms will $= Pm \pm Ps = \frac{nV}{P} \pm \frac{nvl}{pr} = \frac{nV}{P} \pm \frac{nVl}{Pl + Qk} = \frac{nVQk}{P \times \overline{Pl \pm Qk}}$. Now ms is to $s\mu$ as Pm to PH, i. e. $\frac{nVQk}{P \times \overline{Pl \pm Qk}}$ to $\frac{nVk}{Pl \pm Qk}$ as Pm to PH; and so Q is to P as Pm to PH, or $PH = Pm \times \frac{P}{Q}$ or $\frac{nV}{Q}$. Since

 there-

therefore the value of the right line PH does not depend upon the quantities l, k and r; but, these being changed, is always the same, the point μ will be at a right line given in position, as we have otherwise shewn in Theor. 4. Moreover also the value of the line PH is that which in Art. 29. we have determined by another method; and the right line Hm cuts all right lines drawn through P harmonically, according to the definition of harmonical section given in general in Art. 28.

SECTION II.

Of Lines of the second Order, or the Conic Sections.

§ 33. FROM what has been demonstrated in general concerning geometrical lines in the first section, the properties of lines of second, third, and superior orders naturally flow. What relate to the conic sections are best derived from the properties of the circle, which figure is the base of the cone. But that the use of the preceding theorems may more clearly appear, and the analogy of the figures be illustrated, it will be worth while to deduce the properties of these also from what has been premised. Now the whole conic doctrine about diameters, and their ordinates (to which right lines touching the section at the vertices of the diameters are parallel) and about the segments of parallels which meet any right lines, and about asymp-

asymptotes, flows very easily from what has been shewn in Art. 4. and 5.

§ 34. Let the right lines AB and FG inscribed in a conic section meet each other in the point P; let AK, BL, FM, GN drawn touching the section meet PE, drawn through P in the points K, L, M, N; and it will always be $\frac{1}{PK} \pm \frac{1}{PL} = \frac{1}{PM} \mp \frac{1}{PN}$ (if the right line PE meets the curve in points D and E) $= \frac{1}{PD} \mp \frac{1}{PE}$. But to the segments which are on the same side of the point P the same signs are to be prefixed, and to those which are on opposite sides of P contrary signs are to be prefixed. Hence if DE be bisected in P, and from the point P be drawn any right line cutting the section in the points A and B, from whence let be drawn the right lines AK and BL touching the curve which cut DE in K and L; PK will always = PL. But if DE does not meet the section, and P be the point where the diameter which bisects right lines parallel to DE meets the same; in this case also PK will = PL. Fig. 20. Fig. 21. Fig. 22.

§ 35. Let the right lines AB and FG inscribed in a conic section meet in the point P; let right lines touching the section in the point A and F being drawn meet each other in K, and PK being joined will pass through the concourse of right lines which touch the section in the points B and G. For if the line PK does not pass through the concourse of the lines touching the section in B and G, let it meet one of them in N, and the other in L; and since $\frac{1}{PK} \mp$ Fig. 23.

$\frac{1}{PL} = \frac{1}{PK} \mp \frac{1}{PN}$ by the preceding, PL will $=$ PN; and the points L and N coincide contrary to the hypothesis.

§ 36. For the same reason it appears that the right lines AG and BF meet each other in the point π of the right line LK; and therefore the points P, K, π, L are in the same right line. Hence having three points of contact A, B, and F given, with two tangents AK and FK, the conic section is easily described. For let the right line KπP revolve about the concourse of the tangents K as a pole, which let meet the right lines AB and FB in the points P and π; and Aπ, FP being joined will, by their concourse G, describe the conic section which will pass through the three given points A, B, F, and touch the right lines AK and FK in the points A and F.

§ 37. The same things remaining, let the right lines AF and BG meet each other in the point p, the tangents AK and BL in R, and tangents FK and GL in Q; and the points R, π, Q, and p will be in the same right line; in like manner let the tangents AK and GQ meet in m; the tangents BR and FK in n; and the points P, m, n, p will be in the same right line. This is demonstrated in the same manner as in Art. 35.

Fig. 24.

§ 38. Hence having four points of contact A, B, F, G given with one tangent AK, the concourse of the right lines AB and FG, AF and BG, and of AG and BF will give the points P, p, and π; and Pp, Pπ, $p\pi$ being joined will cut the given tangent AK in three points m, K, and R, from whence mG, KF, RB being drawn,

TAB. III.

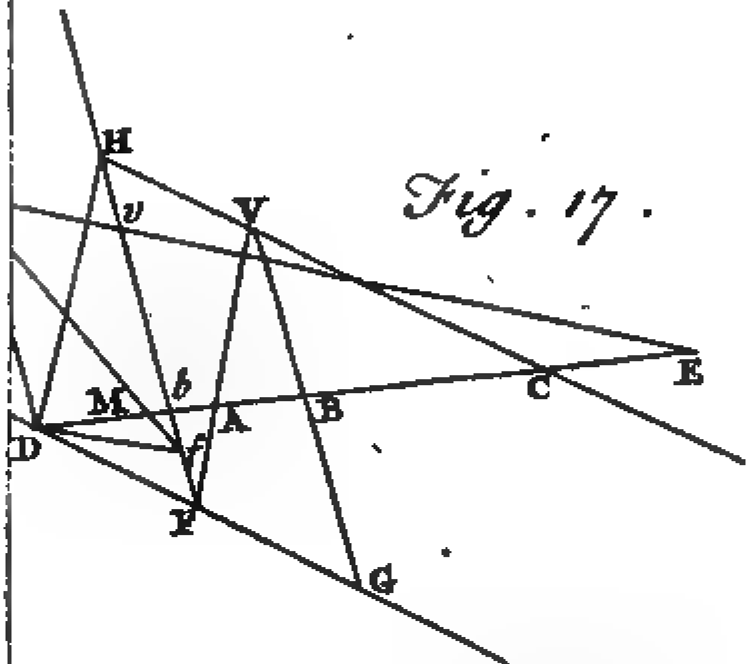
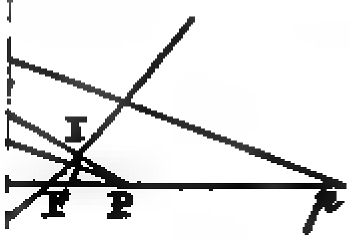


Fig. 17.

19.



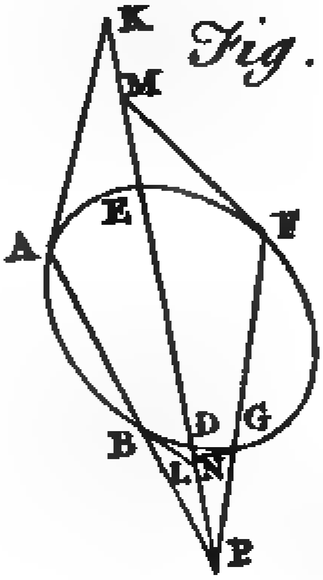


Fig. 20.

Fig. 22.

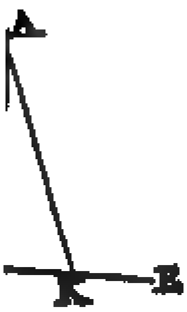
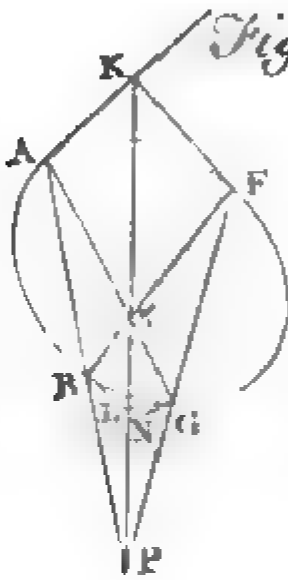


Fig. 23.



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drawn, will touch the conic section in the given points G, F and B.

§ 39. Having four tangents RK, KQ, QL, LR given and one point of contact A, the concourse of the tangents RK and LQ, LR and QK will give the points *m* and *n*. Let LK and *mn* be joined, and the concourse of the right lines LK and RQ, LK and *mn*, RQ and *mn*, will give the points π, P, *p*; but PA, πA and *p*A being joined will cut the tangents RL, QK and QL in the points of contact B, G, and F.

§ 40. Having given five points of contact A, B, F, G and *f*, let GF and G*f* being joined meet the line AB in the points P and X; let AF and A*f* being joined meet the line BG in *p* and *x*; and P*p*, X*x* being joined will by their concourse give the point *m*; from whence *m*A and *m*G being drawn will touch the conic section in A and G; and in like manner are determined the lines which will touch the curve in the remaining points B, F, and *f*.

§ 41. Let there be five lines given touching a conic section, VK, KQ, QL, L*u*, and *u*V; the concourse of the tangents VK and LQ will give the point *m*; the concourse of the tangents KQ and L*u* will give the point *n*; let be joined *mn*, LK, VL and *mu*; the line LK will cut the line *mn* in P; and the line LV will cut *mu* in X; now PX being joined will cut the tangents VK and *u*L in the points of contact A and B. And in like manner the remaining points of contact are determined.

§ 42. Having three tangents AK, BK, and RL given, and two points of contact A and B, the third Fig. 25.

 is

is easily determined, by Art. 35. For let the tangent RL meet the others in R and L, and let AL and BR being joined cross each other in π, Kπ being joined will cut the tangent RL in the third point of contact F; and the conic section may be described as in Art. 36.

Fig. 26. § 43. Let there be given four tangents KQ, QL, LR, and RK with one point of contact D of the conic section which is not in any of the four tangents. Let be found the points P, p, and π, as in Art. 39. Let there be joined PD, pD, and πD; and let PZ, being drawn parallel to pD, meet the line RQ in Z; and let PZ be bisected in S; and pS being drawn will cut the line PD in E a point of the curve; or let PD meet the line RQ in z, and (by Art. 23.) let PD be cut harmonically in z and E. Now Dπ being drawn will cut pE being joined in e, and Eπ cut pD in d, so that also these points d, e may be also in the curve.

Fig. 27. n. 1. § 44. If from the point K be drawn two lines touching a conic section in A and B; from the point A let be drawn also two lines AF, AG meeting the section in F and G; let BG being joined cut AF in P, and BF being joined cut AG in π; then will the points P, K, π be in the same right line, by Art. 36.

But this proposition is more general. For if from n. 2. any point K be drawn two lines KAa, KBb cutting the section in the points A, a, and B, b; and from the points A and a be drawn to the section the lines AF and aG; now let BF being joined cut aG in P, and bG being drawn cut AF in π; the points P, K, π will be in the same right line: which we have in various

rious ways otherwise demonstrated, from whence I formerly deduced an expeditious method of describing a conic section through any five given points. Let A, a, B, b and F be the five given points, let the lines Aa and Bb meet in K; let AF and BF be joined; let the line PKα revolve about the pole K, and let it meet these lines in π and P; and aP, $b\pi$ being drawn will, by their concourse G, describe the section.

§ 45. If P be a given point out of a conic section from whence any right line drawn to the section meets it in D and E; and if $\frac{2}{PM} = \frac{1}{PD} \mp \frac{1}{PE}$, M will be at a right line which meets the section in the points A and B, so that PA and PB being drawn, they will be tangents to the circle. But if the point p be the middle point of AB within the section, and it be also $\frac{2}{pm} = \frac{1}{pd} \mp \frac{1}{pe}$, the locus of the point m will be a right line ab, drawn through P parallel to AB. Tangents at the points D and E always meet in the right line AB, and tangents at the points d and e in the right line ab. Fig. 28.

§ 46. Let a right line DT touch a section in D, from whence let be drawn any two right lines DE and DA, which meet the section in E and A. Let DE meet in K the line AK which touches the section; and let EN, KM drawn parallel to the tangent DT cut DA in N and M, let be taken in the line DE, DR to EN as KM to KE, and a circle of the same curvature with the section in D will pass through R. For by Art. 15. it is $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} - \frac{1}{DK} =$ Fig. 29. n. 1.

$\frac{KE}{DE \times DK}$, and $DR = \frac{DE \times DK}{KE} \times \frac{QV^2}{DV^2}$ (becauſe QV
n. 2. : DV : : KM : DK : : EN : DE) $= \frac{KM \times EN}{KE}$. But if the tangent AK was parallel to the line DE (i. e. if DE was an ordinate to the diameter paſſing through the point A) then DR would $= \frac{EN^2}{DE}$, or DR would be to DE as EN^2 to DE^2; as I have elſewhere demonſtrated in Art. 373. of the Treatiſe of Fluxions. If in this caſe DE be a diameter, $\frac{EN^2}{DE}$, and ſo DR, will be equal to the parameter of the diameter DE; as is well known.

Fig. 30. n. 1. § 47. Let be drawn the right lines DT, DE, of which let the firſt touch a conic ſection in D, and the latter meet it in E. Let DA be drawn which biſects the angle EDT and meets the ſection in A; let AE be joined, which let meet in V the line DV parallel to the line which touches the curve in A; and VR being drawn parallel to DA, it will cut DE in R where the oſculatory circle meets the line DE; and DR will be the diameter of curvature, if the angle EDT be a right one. For VR will be to AD as ER to DE, and as DR to DK; whence DR is to DK as DE to EK, and ſo $\frac{1}{DR} = \frac{1}{DE} - \frac{1}{DK}$, as it ought, by Art. 15.
n. 2. Now if the tangent AK be parallel to DE (in which caſe the tangents AK and DT make equal angles with the line DA which is therefore perpendicular to the axis of the figure) the points R and E will coincide, and the oſculatory circle will paſs through the point E. It follows alſo from what has been ſaid that

the

TAB. IV.

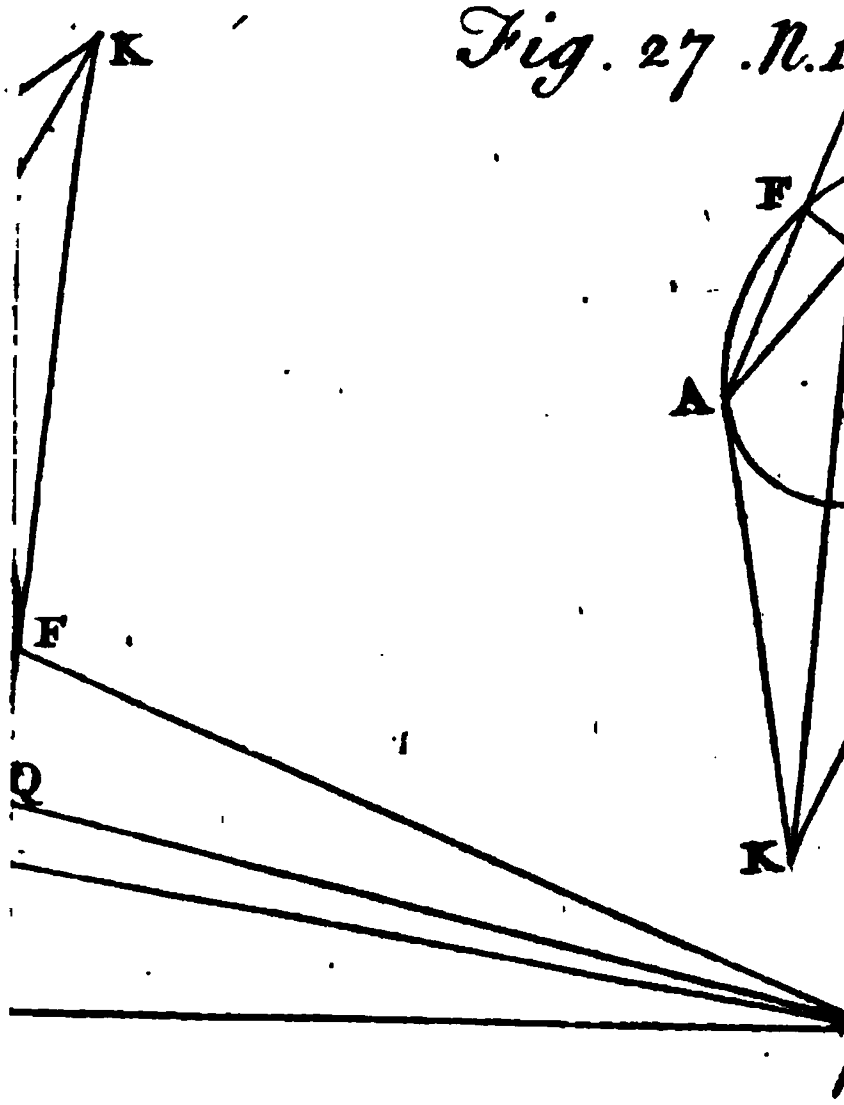


Fig. 27. N. 1.

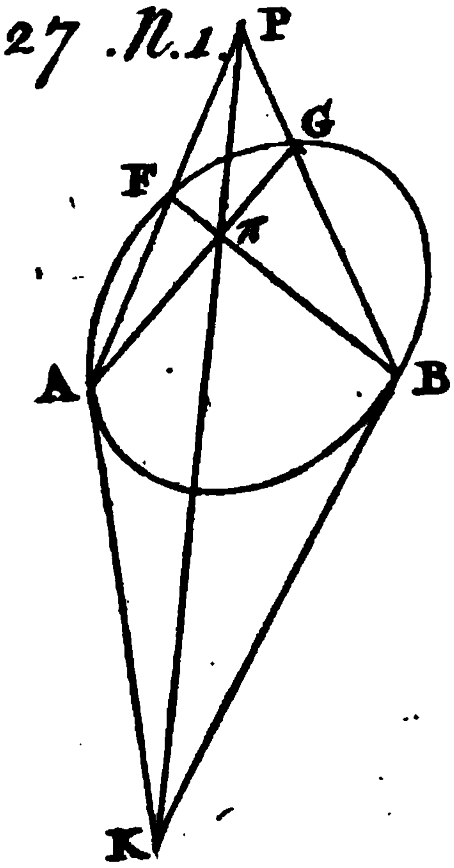
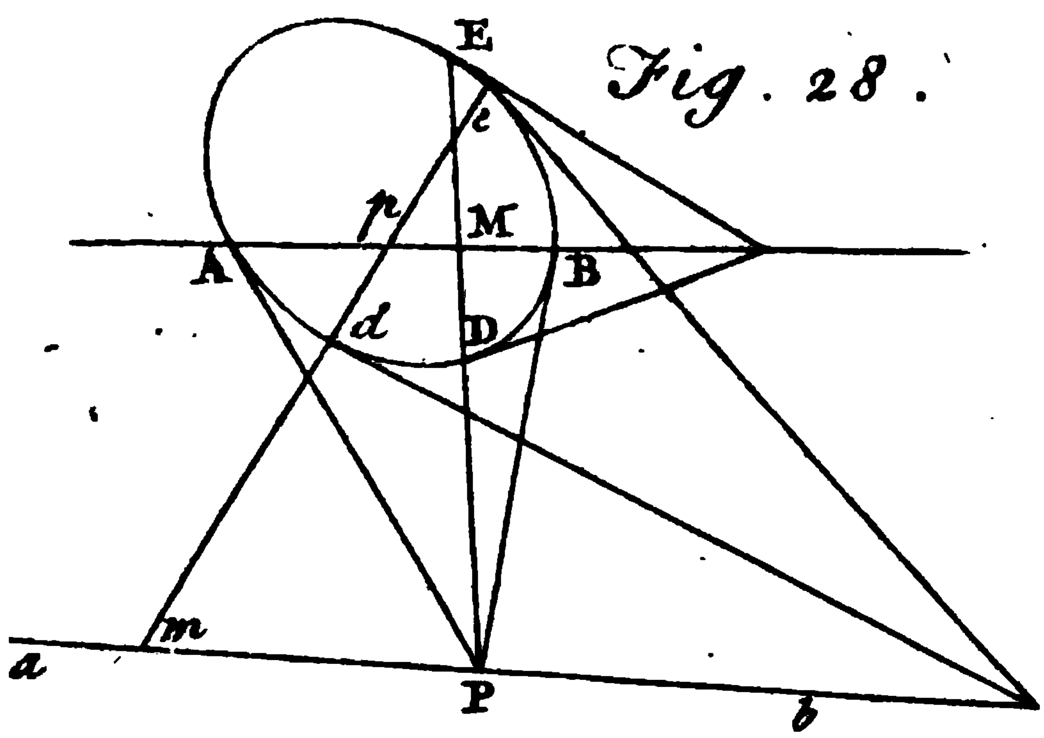
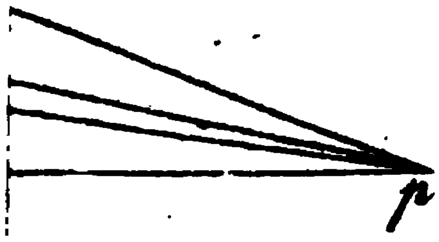
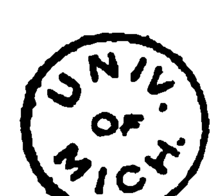


Fig. 28.



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the lines EK, DE, and ER are in geometrical progression.

§ 48. Let any right line DE meet a conic section in D and E, let tangents to the curve at D and E meet in the point V. Let DOA be a diameter of the curve through D, and if the angle DVr be constituted $=$ EDO, DR ($= 2Dr$) will be a chord of the osculatory circle. For let AK be drawn touching the section which meets DE in K, and the tangent EV in Z; let EN be drawn parallel to the tangent DT cutting DA in N; and since DR is to KA as EN to EK; and KZ ($= \frac{1}{2}$AK) to EK as VD to DE, it will be as VD to DE so $\frac{1}{2}$DR to EN; and so the triangles DVr and EDN will be similar, and the angle DVr equal to the angle EDO. This method of determining the osculatory circ[le] [we] have demonstrated in the Treatise of Fluxions, Art. 375. but not so shortly. Fig. 31.

§ 49. The variation of curvature, or the tangent of the angle of contact made by a conic section and the osculatory circle, is directly as the tangent of the angle contained by the diameter which is drawn through the point of contact and perpendicular to the curve, and inversely as the square of the radius of curvature. For let DR be the diameter of curvature, and this variation at the point D will be as $\frac{1}{DR \times DV}$, by Art. 17. so that, since DV is to Dr as DE to EN, it will be as $\frac{EN}{DE \times DR^2}$. But the variation of the radius of curvature is as the tangent of the angle EDO. But if the line DO meets the osculatory circle in n, a parabola described with the diameter and parameter Dn, Fig. 32.

and

and which touches the line DT in D, will be that whose contact with the section is the closest, by Art. 19.

Fig. 32. § 50. The rest remaining, let from the point V be drawn VH touching the osculatory circle in H; let HD be joined, and since the angle RDH is the complement of the angle DrV to a right one, RDH will $= DVr = EDO$; and so the variation of the radius of curvature will be as the tangent of the angle RDH; and the right lines DR and DH coinciding, that variation vanishes.

SECTION III.

Concerning Lines of the third Order.

§ 51. WE must treat more fully about lines of the third order or curves of the second kind. Very many have handled the doctrine of conics, and in such various methods as almost to cloy. But few have touched upon this part of universal geometry; yet it will appear from what follows, as I hope, to be neither barren nor unpleasant, since besides the properties of these figures formerly delivered by *Newton*, there are many others not unworthy the attention of geometers. I have shewn above, that a right line may cut a line of the third order in three points, because there are three roots of a cubic equation, which may all be real. Now a right line which cuts a line of the third order in two points, necessarily meets the same in some third point, or is parallel to the asymptote of the

the curve, in which caſe it is ſaid to meet it at an infinite diſtance: for if two roots of a cubic equation are real, the third will neceſſarily be real. Hence a right line which touches a line of the third order, always cuts it in ſome point, ſince the contact is to be looked upon as two coincident interſections. But a right line which touches the curve in the point of contrary flexure, is at the ſame time to be eſteemed a ſecant. When two arcs of the curve meet each other, there is a *double* point formed, and a right line which touches either arc there, in the ſame point cuts the other. But any other right line drawn from the double point cuts the curve in one other point but not in more.

§ 52. PROP. I. Let there be two parallels, each of which let cut a line of the third order in three points; a right line which ſo cuts both of the parallels, that the ſum of the two parts of the parallel terminated at the curve on one ſide of the cutting line may be equal to the third part of the ſame terminated at the curve on the other ſide of the cutting line, will in like manner cut all other right lines parallel to theſe which meet the curve in three points; by Art. 4.

§ 53. PROP. II. Let a right line given in poſition meet a line of the third order in three points; let any two parallels be drawn, both of which let cut the curve in as many points; and the ſolids contained under the ſegments of the

the parallels terminated by the curve and the line given in position, will be in the same ratio as the solids under the segments of this right line terminated by the parallels, by Art. 5.

These two properties were formerly exhibited by *Newton.*

Fig. 33. § 54. PROP. III. The rest remaining as in the preceding position, let the right line given in position meet a line of the third order in one only point A, and the solid contained under the segments PM, Pm, Pμ of one parallel will always be to the solid under the segments pN, pn, $p\nu$, of the other parallel, as the solid AP $\times$ bP^2 contained under the segment AP and the square of the distance bP of the point P from a certain point b, to the solid Ap $\times$ bp^2 contained under the segment Ap and the square of the distance of the point p from the same point b, by Art. 6.

Fig. 34. n. 1. § 55. PROP. IV. From any point P let be drawn a right line PD which may meet a line of the third order in three points D, E, F, and any other right line PA which may cut the same in three points A, B, C. Let be drawn the tangents AK, BL, CM, which let meet PD in K, L, and M; and the harmonical mean between the three lines PK, PL, PM, coincides with the harmonical mean between the

the three lines PD, PE, PF, by Art. 10. and 28. But if the right line PD meets the curve in one only point D, let be found the point d, as in Art. 6. and the harmonical mean between the three lines PK, PL, PM, will be to the harmonical mean between the two right lines PD and $\frac{1}{2}$Pd in the ratio of 3 to 2, by Art. 12. n. 2.

§ 56. PROP. V. Let the right line PD revolve about the pole P, let PM be always taken in PD equal to the harmonical mean between the three lines PD, PE, and PF, and the locus of the point M will be a right line, by Art. 28.

And this is a property of these lines invented by *Cotes*.

§ 57. PROP. VI. Let there be three points of a line of the third order in the same right line; let right lines touching the curve in these points be drawn, which may cut the same in three other points; these three points will also be in a right line. Fig. 35.

Let the right line FGH meet a line of the third order in the points F, G, and H. Let the lines FA, GB, HC touching the curve in these points cut the same in the points A, B, C; and these points will be in a right line. For let AB be joined, and this will pass through C; for if it be possible, let it meet the curve

curve in any other point M, the tangent HC in N, and the right line FGH in P; and since $\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PM}=\frac{1}{PA}+\frac{1}{PB}+\frac{1}{PN}$, by Prop. IV. PN will = PM; which cannot be, unless the points N, M, and C coincide. Therefore the right line AB passes through C.

§ 58. *Corol.* Hence if A, B, C be three points of a line of the third order in the same right line, and AF and BG being drawn touch the curve in F and G, and FG, being joined, cut the curve again in H, CH being joined, will touch the curve in H. For if a right line should touch the curve in H, which should not cut it in C but in some other point, this point would be with the three others A, B, C, in the same right line which would therefore cut a line of the third order in four points. But this cannot be. I first hit upon this proposition in a different way, but less expeditious, by deducing the same from Prop. II. In like manner if the right line A*f* also touches the curve in *f*, and G*f* being drawn, meets the curve in *b*, C*b*, being joined, will be a tangent at the point *b*. And if from the points A, B, C, of a line of the third order situated in the same right line, be drawn as many right lines touching the curve as can be drawn, there will always be three points of contact in the same right line.

Fig. 36. § 59. Prop. VII. From any point of a line of the third order let be drawn two lines touching the curve, and let the line joining the points of contact cut the curve in another point, the tangents to the curve at this other point and

at

Appen. pa. 478.

Fig. 29.
N. 1.

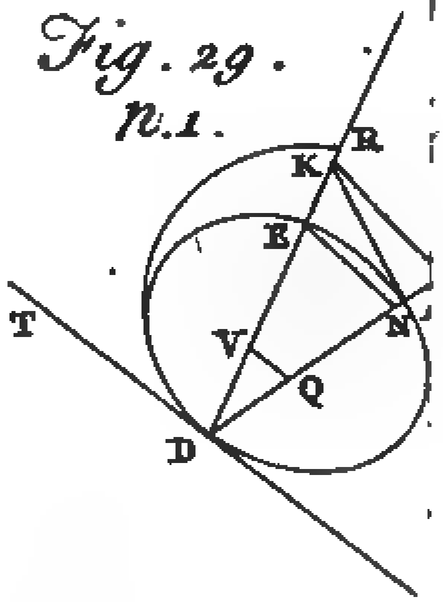


Fig. 30.
N. 1.

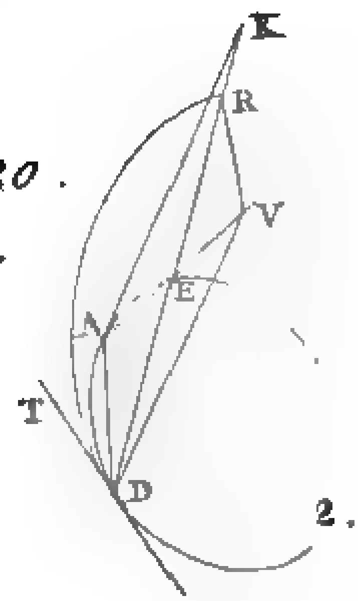
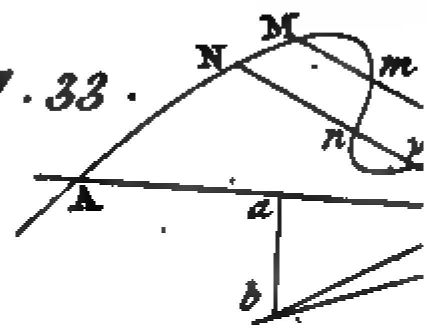


Fig. 33.



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at the first point will cut each in some point of the curve.

From the point A let be drawn lines touching the curve in F and G, let FG, being joined, cut the curve in H, and let these touch the same in H, the line HC which meets the curve in C, and AC being drawn will be a tangent to the curve at A. It follows from the preceding *Corollary*, for A and B coinciding the line CA is a tangent at the point A.

§ 60. *Corol.* 1. If from the point C of the curve be drawn two lines touching the same in A and H, and from either point A tangents to the curve AF and AG be drawn, the line drawn through F and G the points of contact will pass through the other point H.

§ 61. *Corol.* 2. Let the line AC touch the curve in A, and cut it in C, and let AF and CH touch the curve in F and H, and the line drawn through the points of contact cut it again in G, AG, being joined, will touch the curve in G. But if there be any other line drawn from C as C*b* touching the curve in *b*; and *b*F, *b*G, being joined, meet the curve in *f* and *g*, A*f* and A*g* being drawn will be tangents at the points *f* and *g*. Fig. 37.

§ 62. *Corol.* 3. Let A be a point of contrary flexure, from whence let AF and AG being drawn touch the curve in F and G, and let FG, being joined, cut the curve in H, and AH being drawn will touch the curve in H. For if the tangent at the point H should meet the curve in any other point different from A, the right line drawn from this point of meeting to the point of contrary Fig. 38.

contrary flexure A would touch the curve in A, which cannot be. Now it is manifest that only three lines can be drawn from the point of contrary flexure touching the curve besides that which both touches and cuts it in that same point, and that the three points of contact fall in the same right line. From the point of contrary flexure alone, three right lines can be drawn so touch the curve that the three points of contact can be in the same right line. For let F, G, H, be in the same right line, from which tangents being drawn, may meet in the same point of the curve *a*, which is not the point of contrary flexure; let *ae* be drawn touching the curve in *a*, and which meets it in *e*, and *e*H, being joined, will touch the curve in H, by this proposition; and so the lines *e*H and *a*H would touch the curve in the same point H, which is absurd.

§ 63. Prop. VIII. From any point of a line of the third order let be drawn three lines touching the curve in three points; let a right line joining two of the points of contact meet the curve again, and a right line drawn from that point of meeting to the third point of contact will again cut the curve in a point where a right line to the first point will touch the curve.

Fig. 37. From the point A of a line of the third order let be drawn three lines AF, AG, and A*f*, touching the curve in the three points F, G, and *f*; let the line G*f*, which joins two of them, meet the curve again in N, and a line drawn from this point to the third point of contact F cut the curve in *g*, then A*g*, being joined,

joined, will touch the curve in g. For let there be drawn AC touching the curve in A and cutting it in C; and since the points G, N, and f are in the same right line, and the tangents at the points G and f pass through A, it follows (by Prop. VII.) that the tangent at the point N passes through C. And since the points F, N, g are in a right line, but the tangents FA and NC meet the curve in A and C, and AC is a tangent at the point A, the tangent at the point g will pass through A.

§ 64. *Corol.* Hence if a curve be described, from three given points of contact where three lines drawn from the same point of the curve touch it, a fourth point of contact is found where a line drawn from the same point of the curve touches it. And from hence it is collected that only four lines can be drawn from the same point of the curve touching a line of the third order besides that which touches it in that same point. For if lines might be drawn from the same point of the curve touching in five points, more lines indefinite in number might be drawn from the same point touching the curve; as is easily gathered from what goes before. Now this Corollary we shall afterwards demonstrate more easily. See below, Art. 77.

§ 65. PROP. IX. Let three tangents to the curve be drawn from a point of contrary flexure, and a right line joining the points of contact will cut harmonically any right line drawn from the point of contrary flexure and terminated by the curve. Fig. 38.

Let A be the point of contrary flexure, AF, AG, AH tangents to the curve at the points F, G, and H.

 From

From the point A let be drawn any right line cutting the curve in B and C, and the right line FH in P; and it will be as PB to PC as BA to AC. For since three tangents at the points F, G, and H meet in the same point A, by Prop. IV. $\frac{1}{PB} + \frac{1}{PA} - \frac{1}{PC} = \frac{3}{PA}$, therefore $\frac{1}{PB} - \frac{1}{PC} = \frac{2}{PA}$, i. e. PA is an harmonical mean between the two lines PB and PC terminated at the curve. Which is a property of lines of the third order of admirable simplicity.

§ 66. *Corol.* 1. A line which cuts any two right lines drawn from a point of contrary flexure to the curve harmonically, will also cut any other two lines drawn from the point and terminated by the curve harmonically.

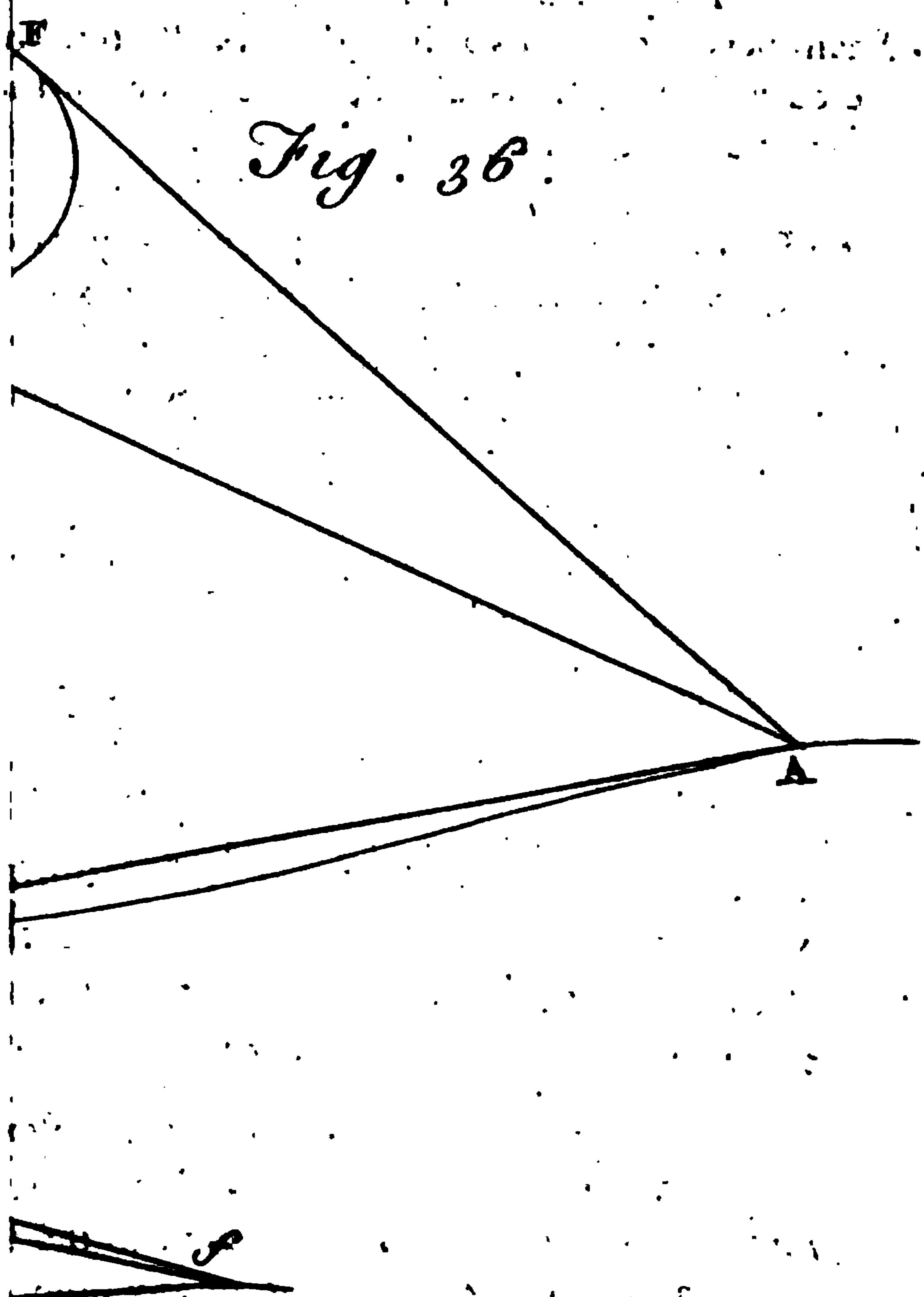
§ 67. *Corol.* 2. If a right line parallel to the asymptote drawn through a point of contrary flexure meets the line FH in R and the curve in O, then $\frac{1}{RO} = \frac{2}{RA}$, and so RA = 2RO.

Fig. 39. § 68. Prop. X. A right line joining two points of contrary flexure either passes through a 3d point of contrary flexure or is in the same direction with an infinite leg of the curve.

Let A and *a* be two points of contrary flexure, let A*a* joined meet the curve in a, a will also be a point of contrary flexure. For if a tangent to the figure in the point a should meet the curve in any other point *e*, A, *a*, *e*, would be in the same right line. But by hypo-

TAB. VI.

Fig. 36.



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hypothesis A, *a*, and a are in the same right line, which therefore would meet a line of the third order in four points. Let A be a point of contrary flexure, and let the line AO parallel to the asymptote meet the curve in O, let OQ be drawn touching the curve in O, and cutting it in Q, AQ, being joined, will pass through D where the curve cuts the asymptote.

§ 69. PROP. XI. Having drawn from a point of contrary flexure A the tangents to the curve AF, AG, AH; and any two cutting it ABC, A*bc*, then B*b* and C*c* or B*c* and *bc* will mutually cut each other in the right line FH which joins the points of contact. Fig. 38.

For let the line B*b* meet FH in Q, and B the same in P; let be joined QA and QC; and since it is as AB to AC so PB to PC, by Prop. IX. QA, QB, QP and QC will be harmonicals, and so A*b* will cut the line QC in *c* and FH in *p*, so that A*b* is to A*c* as *pb* to *pc*; and therefore *c* will be a point of the curve, by Prop. IX. from which it follows conversely that the lines B*b* and C*c* meet in the point Q of the line FH; and in like manner it is shewn that B*c* and *b*C meet each in a point *q* of the same line.

§ 70. *Corol.* 1. From any point Q of the line FH let be drawn to the curve the lines QB, QC cutting it in the points B, *b*, M and C, *c*, N; then CB, *cb*, MN, will meet in the point of contrary flexure A; B*c* and *b*C, M*c* and N*b*, B*b* and C*c*, NB and MC will meet in the line FH.

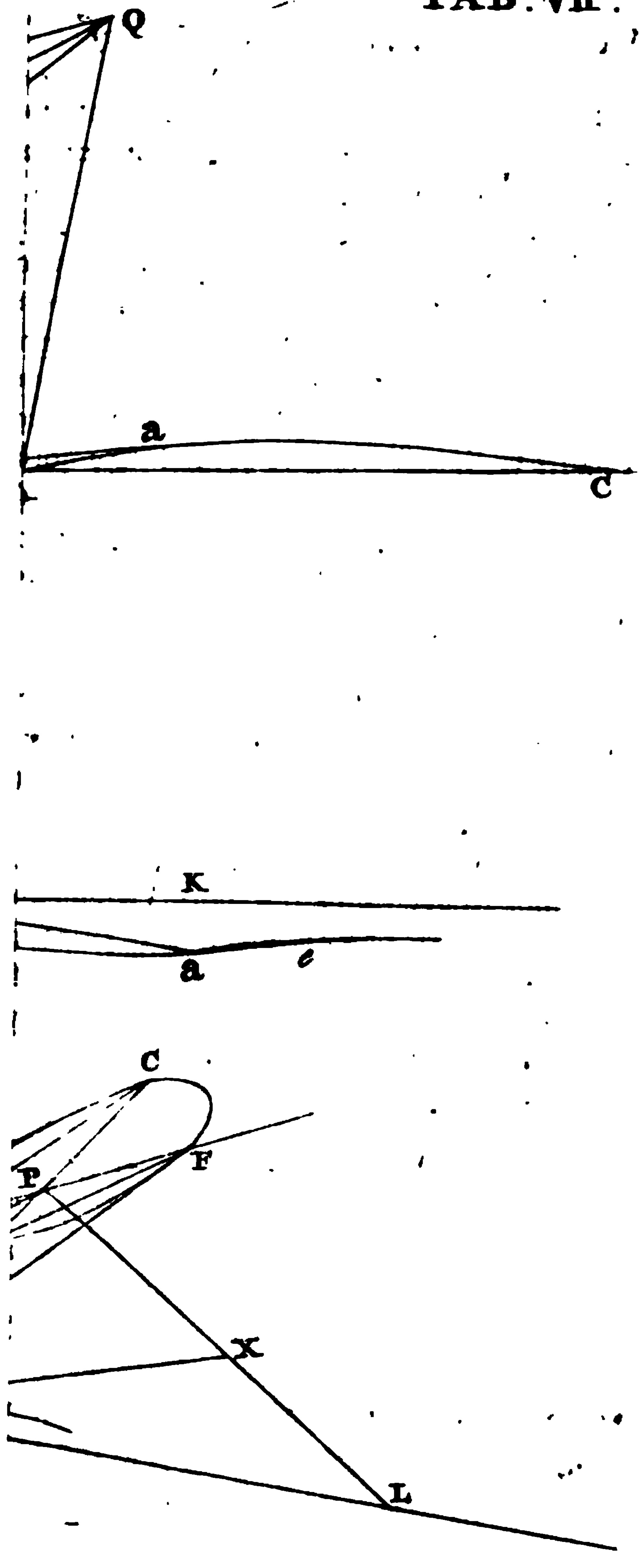
§ 71. *Corol.* 2. Tangents at the points B and C meet in some point T of the line FH; and if from any point T situated in FH be drawn tangents to the curve they will pass through the point of contrary flexure, or meet in the line FH.

§ 72. *Coral.* 3. Having given the point of contrary flexure A, and the points B, C, *b*, *c*, where two right lines drawn from it cut the curve, the right line FH is given in position; for B*b* and C*c* joined will by their concourse give the point Q, and the concourse of B*c* and *b*C joined will give *q*, and Q*q* joined is that which joins the points of contact F, G, and H. Now these five points being given with other two M and *m*, a line of the third order is determined which passes through these seven points A, B, C, *b*, *c*, M, *m*, and has its contrary flexure in A. For from the points M and *m* are given the points N and *n*, where AM and A*m* being drawn cut the curve, and from these nine conditions the line is determined. Now if three points M, *m*, and S were given; these would give three others N, *n*, and *s*; whence would be given eleven conditions to determine the figure, which are too many. In like manner having given the point of contrary flexure A with the points F, G, (and so the tangents AF and AG) and the points M and *m* any whatever, the right line FG is given, and so the points N and *n*, and the curve is determined.

Fig. 40. § 73. *Corol.* 4. Let the lines HB, HC touch the curve in the points B and C, and CB joined will pass through A, CG and FB will meet in a point of the curve V, and VH drawn will touch the curve in V. Now the tangent at the point of contrary flexure A is determined by drawing AN, which let PL parallel to AH meet in L, and by bisecting PL in X; for AX, being joined, will be the tangent at the point A. For

let

TAB. VII.



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let the tangent at A meet the line FH in S; and it will be $\frac{1}{PS} + \frac{2}{PH} = \frac{1}{PH} + \frac{1}{PG} - \frac{1}{PF}$, and so $\frac{1}{PS} + \frac{1}{PH} = \frac{1}{PG} - \frac{1}{PF}$ (because AC is harmonically cut in P and B, and therefore VA, VF, VP, and VG are harmonicals) $= \frac{2}{PK}$. Therefore PK is an harmonical mean between PS and PH; whence if PL parallel to AH meets the lines AV and AS in X and L, PX will = XL.

§ 74. PROP. XII. From a point of a line of the third order A let be drawn two lines touching the curve in F and G, and let FG joined meet the curve in H, and let a tangent at the point A cut the curve in M; let HM be joined, which let meet in L the line FLK parallel to AH, and let FK be taken = 2FL; then HK being joined, any right line AB drawn from A will be cut harmonically by the lines HK and HF in N, P, and by the curve in B, C; so that NB will be to NC as BP to PC. Fig. 41.

For let AB meet the tangent HM in T, and it will be $\frac{1}{PB} + \frac{1}{PA} - \frac{1}{PC} = \frac{2}{PA} + \frac{1}{PT}$, and so $\frac{1}{PB} - \frac{1}{PC} = \frac{1}{PA} + \frac{1}{PT}$ (by construction, and *harmonically*) $= \frac{2}{PN}$. Whence it follows that the line NC is cut harmonically in the points B and P, or that NB is to NC as BP to PC.

§ 75. *Corol.* 1. Hence if any two lines drawn from A are cut in N harmonically so that PC be to PB as CN to BN; all lines drawn from A will in like manner be cut harmonically by the lines HF and HK.

§ 76. *Corol.* 2. If the curve has not a double point, and the right line HK cuts it in two points, f and g, Af and Ag being drawn will be tangents to the curve in these points. For let the point B coincide with N when N comes to f the concourse of HK with the curve; and therefore when $\frac{1}{PB} \mp \frac{1}{PC} = \frac{1}{PN}$, it will be $\frac{1}{PC} = \frac{1}{PN}$, and C coincides with B, and the line drawn from A then touches the curve. On the other side, if the line Af touches the curve, the line HK will pass through f; for because of PB, PC being equal in this case, the points B and C will coincide with N.

§ 77. *Corol.* 3. If the line HK meets the curve in only one point H, only two tangents can be drawn from the point A to the curve, viz. AF and AG. Four tangents at most can be drawn from any point of a line of the third order to the curve as AF, AG, Af, Ag. For if any other tangent could be drawn from A as $A\varphi$, the line HK would pass through the point φ, and four points of a line of the third order would be in the same right line, viz. H, f, g, φ, which is absurd.

§ 78. Prop. XIII. If from a point of a line of the third order four tangents to the curve may be drawn, the lines joining the points of contact will always meet in some point of the curve,

curve, and any right line drawn from the first point will be cut harmonically by the curve and the lines joining two points of contact.

Let A be a point of the curve, AF, AG, Af, Ag tangents in the points F, G, f, g. Let be joined FG and fg, which let meet any right line ABC (drawn from A and cutting the curve in B and C) in P and N; and the line NC will be harmonically cut in B and P, so that always NC is to NB as CP is PB: this follows from Corol. 2. of the preceding. Now the lines FG and fg meet in the point of the curve H; and in like manner the lines Ff and Gg meet in E, and Fg and Gf in R; and E and R will be points of the curve, by the same corollary. And this is the latter of the two properties of lines of the third order which I described in the Treatise of Fluxions, Art. 402. But if the line AM touches the curve in A, and cuts it in M, ME, MR, MH, being joined, will touch the curve in the points E, R, H; and the concourse of the lines AE and HR, AR and HE, AH and RE will be also in the curve*.

§ 79. *Corol.* Therefore since the lines HK, HB, HP, and HC are harmonicals; if the lines HB and HC meet the curve in *b* and *c*; the points A, *b*, *c*, will be in the same right line. For let AB joined meet the curve in *b* and *c*, and HP in *p*, and NK in *n*; and since *nc* is to *nb* as *pc* to *pb*, it appears that *c* is in the line HC; and reciprocally if *c* be in the line NC and *b* in the line HB, A, *b*, *c*, will be in the same right line.

§ 80. PROP. XIV. Let a line of the third order have a double point O. From any point Fig. 42.

* Supply what is wanting in the scheme.

A of the curve let be drawn two lines AF, and AG touching the curve in F and G; let FG joined cut the curve in H; let OH be joined. Let any right line AB, drawn from A, meet the curve in the points B and C, the line FG in P, and the line OH in N; and the line NP will be cut harmonically in the points B and C, so that PB is to PC as BN to NC.

For let AO joined meet FG in p and the tangent HL in t; and since O is a double point, it will be $\frac{2}{pO} + \frac{1}{pA} = \frac{2}{pA} + \frac{1}{pt}$, and so $\frac{1}{pA} + \frac{1}{pt} = \frac{2}{pO}$. Therefore pA is cut harmonically in t and O, so that pt is to pA as tO to OA, and Hp, Ht, HO, and HA are harmonicals. Let the line PA meet the tangent LH in T, and since $\frac{1}{PC} + \frac{1}{PB} + \frac{1}{PA} = \frac{2}{PA} + \frac{1}{PT}$, it will be $\frac{1}{PC} + \frac{1}{PB} = \frac{1}{PA} + \frac{1}{PT} = \frac{2}{PN}$; consequently PC is to NC as PB to BN.

§ 81. *Corol.* If the tangent HL meets the line GZ parallel to AH in Z, and GV be taken = 2GZ, HV drawn will pass through the double point O, if the curve has such a point. Or if the line Gra meets the lines AH and HR in a and r, let rA and Ra cross each other in m, then Hm joined will pass through the double point O.

§ 82. Prop. XV. From a point of a line of the third order let be drawn two tangents, and from any other point of the same let be drawn lines to the points of contact cutting the

curve

curve in two other points; tangents to these new points will meet the curve in the same point.

From the point A let be drawn AF and FG, touching the curve in F and G. Let any point of the curve P be taken, let PF and PG be joined cutting the curve in the points K and L; and the tangents at the points K and L will meet in some point of the curve B. Now the point B is determined, by drawing the line PC, which touches the curve in P, and cuts it in C; for if AC be joined, it will meet again the curve in the point B. Fig. 43.

For since the points F, K, P, are in the same right line, and tangents at the points F and P cut the curve in A and C; it follows that the tangent at the point K will pass through B. And because LGP is a right line, the tangent at L will pass also through B.

§ 83. *Corol.* Therefore let A and B be any two points in a line of the third order; from both of them let be drawn four lines touching the curve in four other points, viz. AF, AG, A*f*, A*g*; and BK, BL, B*k*, B*l*. Let FK and GL, FL and GK, F*l* and G*k*, G*l* and F*k*, meet each other in four points of the curve P, Q, *q*, *p*; and if tangents be drawn to these four points, these will meet the curve and each other in the point C where the line AB cuts the curve. Whence if there be three points of a line of the third order in the same right line, and from each of them be drawn four lines touching the curve in four other points, a right line drawn through any two points of contact will always cut the curve in some other point of contact; and four of these lines will always pass through the same point of contact. Fig. 44.

§ 84.

Fig. 43. § 84. PROP. XVI. Let F and G be two points of a line of the third order so taken that FA and GA touching the curve in these points may meet in any point A of the curve. Let be taken in the curve any other point P, from which let be drawn to the points F and G the lines PF and PG, which may meet the curve in K and L; let FL and GK be joined, and their concourse Q will be in the curve. Now the tangents at the points K and L will meet each other and the curve in some point of the curve B, and the tangents at the points P and Q will meet in a point of the curve C, so that the three points A, B, C, may be in the same right line.

For let the tangent at the point P be drawn, which let meet the curve in C, and let AC cut the same in B; and BK, BL being drawn will be tangents at the points K and L, by the preceding. Let the line LF meet the curve in Q; and if the line GK does not pass through Q, let it meet the curve in *q*. Therefore, because the three points L, F, Q, are in the same right line, but the tangents at L and F cut in B and A, it follows (by Prop. VII.) that the tangent at the point *q* will pass also through the point C. Therefore both lines CQ, C*q* touch the curve, the first in Q, the latter in *q*. Therefore the points Q and *q* coincide, for if we put them different, it follows from Prop. VIII. that more than four tangents might be drawn to the curve from the same point C. For let A*f* and A*g* be tangents at *f* and *g*, and let L*f*,

L*g*

Lg drawn cut the curve in m and n; and Cm, Cn will be tangents at m and n. Wherefore we should have five tangents drawn from C to the curve CP, CQ, Cm, Cn, and Cq; which is contrary to Corol. 3. Prop. XII.

§ 85. *Corol.* 1. The point P being given, and the points F and G taken any where, so that tangents at these points meet in the curve, the point Q is given, where FL and GK joined meet each other and the curve. And if from the point P any right line PMN be drawn to meet the curve in N and M, and QM, QN joined cut it in m and n; the points P, n, and m, will be in the same right line. For we have shewn that tangents at the points P and Q cross each other in a point of the curve *.

§ 86. *Corol.* 2. If four points F, G, K, L, be taken in a line of the third order, so that tangents at the points F and G meet in some point of the curve, and tangents at the points K and L meet also in some point of the curve, FK and GL drawn will meet in a point of the curve, and FL and GK drawn will meet each other in a point of the curve. Fig. 43.

§ 87. PROP. XVII. Let F and G be two points of a line of the third order, where, if lines be drawn touching the curve, these shall cut each other in some point of the curve. Let be taken four other points of the curve L, K, f, g, so that LF and GK drawn may meet in the curve, and Ff and Gg may meet in it

* Supply what is wanting in the Scheme.

also;

also; then Fl and Gk drawn will cut each other in the curve, as also Lg and Kf when drawn.

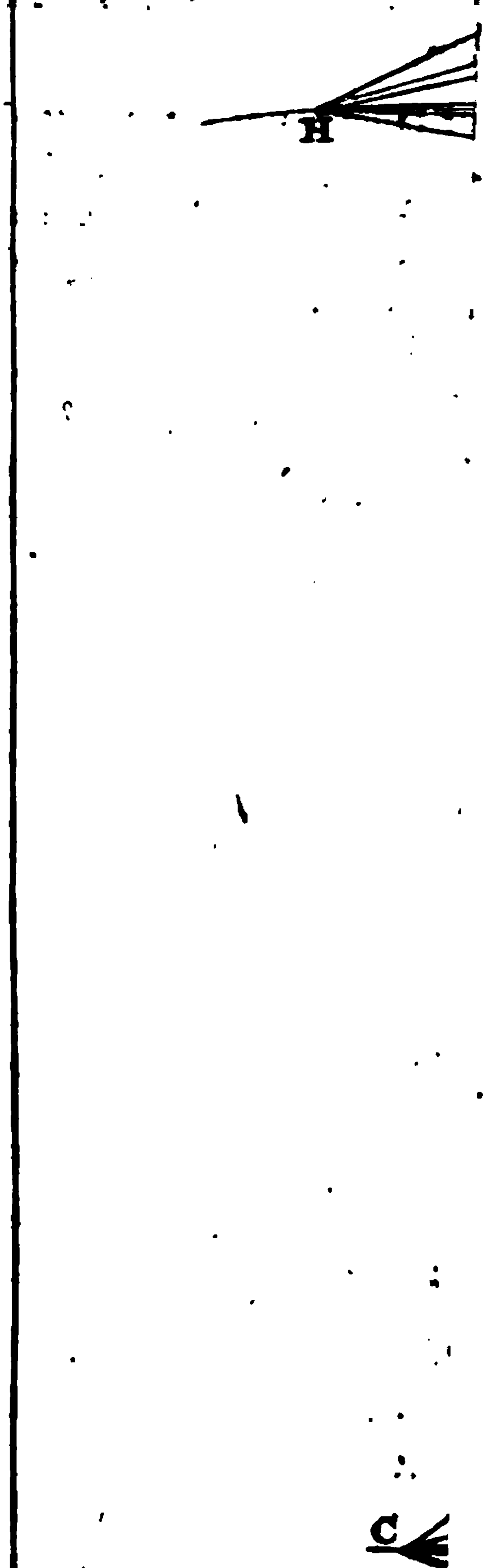
For tangents at the points f and g cross each other in the curve, by Prop. XIV. as also tangents at the points K and L, by the same. And therefore, by Corol. 2. of the preceding, fL and Kg meet in the curve, as also fK and gL.

Fig. 45. n. 1. § 88. *Lemma.* Let there be three right lines given in position IC, IH, and CH; and three points F, G, S, which are in the same right line. Let any point Q be taken in the line IC, and let QF joined meet the line IH in L, and QG joined meet HC in P; let FP be joined, let SL drawn meet FP and QP in k and N; and k and N will be at right lines given in position. For let IN be joined, which let meet GS in m, and through N let be drawn a parallel to FS, which let meet the lines IC, IH, and LQ in the points x, u, and r; let the line FG meet the lines IC, IH, and HC in the points a, b, and b. Because Nx is to Nr as Ga to GF, and Nr to Nu as SF to Sb, it will be as Nx to Nu (and therefore ma to mb) as $Ga \times SF$ to $GF \times Sb$, i. e. in a given ratio. Therefore the point m is given, and so the right line INm is given in position; and in like manner the point k is at a line given in position.

n. 2. § 89. *Corol.* The points G and S coinciding, the point m will also coincide with the point G. Therefore let IG be joined and meet HC in D, and CF drawn and meet the line HI in E, the line DE joined will be the locus of the point K, where GL and FP cross each other.

§ 90.

Appen. pa. 492.



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§ 90. PROP. XVIII. Let PGLFQK be a quadrilateral inscribed in a figure, whose six angles touch a line of the third order, as in Prop. XVI. Let lines IC, CH, HI be drawn touching the curve in three points Q, P, L, which are not in the same right line; let IG joined meet the tangent CH in D, and HF meet the tangent CI in E; the points D, K, E, will be in the same right line, which will touch the curve in the point K. Fig. 46.

For let us suppose the lines QFL and FKP to be moved about the pole F, and the lines LGP and QGK about the pole G, but the points Q, L, and P, to be carried along in the tangents QI, LI, and PC; then the point K will be carried in the line DE, by the preceding Corol. Whence, if the points Q, L, P, be carried in a curve which touches these lines QI, LI, and PC, in these points, KI will also be carried in a curve which the line DE touches. But by Prop. XV. if the points Q, L, P, be carried in the proposed line of the third order, the point K will be carried in the same, which therefore the line DE touches in K.

§ 91. *Corol.* 1. In like manner if the lines AF and AG (which touch the curve in F and G) meet the line IH (which touches the curve in L) in the points M and N; let MP joined cut the tangent AG in *d*, and QN joined the tangent AF in *e*, *de* will pass through K, and touch the curve in that point; and the four points D, *d*, *e*, E, will be in the same right line.

§ 92. *Corol*, 2. Let be drawn from any two points of the curve C and B four tangents two from each,

CQ

CQ and CP from the point C, BL and BK from the point B, and let the intersections of these tangents be I, H, E, and D; then LQ and EH drawn will cut each other in a point of the curve F; and the concourse of LP and ID joined will be in a point of the curve G; but the tangents at the points F and G will cut each other in a point of the curve A, which will be in the same right line with the points C and B.

§ 93. *Corol.* 3. Having given three points of a line of the third order which are in the same right line, and two tangents drawn from each of these to the curve being given in position, the six points of contact are determined by this proposition. Let A, B, C be the three given points of the curve in one right line, AM and AN tangents from A, BMI and BDE tangents from B, which meet the former in M, N, *e*, and *d*; and let CD and CE be tangents from the third point C; and let CD meet BM, BD, AM, and AN, in H, D, *b*, and *c*, and CE the same in I, E, *n*, and *m*. These things supposed, N*e* joined will cut the tangent CI in the point of contact Q, M*d* will cut the tangent CD in the point of contact P, ID will cut the tangent AN in the point of contact G, EH the tangent AM in the point of contact F, *mb* will cut the tangent BH in L, and lastly *nc* the tangent BE in K. Now though the problem in this case is determinate, yet it admits of several solutions. For different lines of the third order, but definite in number, may be drawn through the three points A, B, C, touching the six right lines given in position AM, AN, BM, BD, CD, and CE. For let N*e* meet the tangent CD in *p*, M*d* the tangent CE in *q*, ID the tangent AM in *f*, EH the tangent AN in *g*, *nc* the tangent BM in *l*, and *mb* the tangent BD in *k*; and a line of the third order which satisfies the proposed conditions will touch the lines CD and

CE

CE either in P and Q, or in p and q. That will touch the lines AM and AN either in the points F and G, or in f and g; but the lines BM and BD either in L and K, or in l and k. It appears therefore that several lines of the third order may satisfy the conditions of the problem, but determinate in number, and therefore the problem is determinate*.

§ 94. *Corol.* 4. Having given two points A and B of a line of the third order, also the tangents AM, AN, BM, BD given in position, with three points of contact F, G, and L, the point K is given where the line BD touches the curve. For to it let be drawn the lines Ne and LF, by their concourse the point Q will be given, and QG drawn will cut the tangent BD in the point of contact K. P the point of concourse of the lines LG and Md, or the lines Md and FK, is also given; for the three lines LG, Md, and FK necessarily meet in the point P. Let MedN be any quadrilateral, let any point Q be taken in the diagonal Ne and P in the diagonal Md, let any right line drawn from Q cut the sides Me and MN in F and L, let PL cut the side Nd in G, let QG be joined, which let cut the side de in K; and the points F, K, P, will be always in the same right line, by what is shewn above. Whence it appears that the problem does not become impossible, because it is necessary that three right lines LG, Md, and FK must meet in the same point.

§ 95. Prop. XIX. Let D, E, F, be points in a line of the third order in the same right line, and let there be three lines touching the curve in these points parallel to each other. In Fig. 47.

* Supply what is wanting in the Scheme.

the

the line DF let be taken the point P so that 2PF may be an harmonical mean between PD and PE; and if any other right line drawn through P meets the curve in f, d, and e, $2Pf$ will always be an harmonical mean between Pd and Pe. But we suppose that the points d and e are on the same side the point P, but the point f on the contrary.

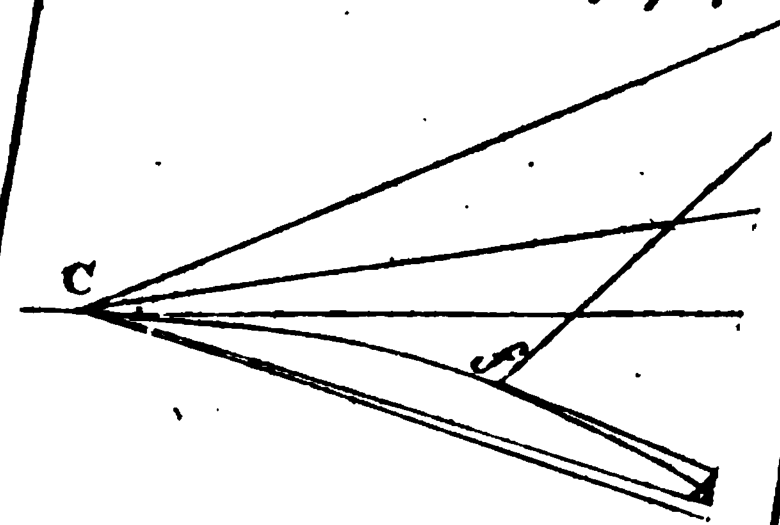
For let the tangents DK, EL, FM, meet the line df in the points K, L, and M; and it will be by Art. 9. $\frac{1}{Pf} - \frac{1}{Pd} - \frac{1}{Pe} = \frac{1}{PM} - \frac{1}{PK} - \frac{1}{PL}$ (if the line Qq parallel to the tangents harmonically cut PD so that PE be to EQ as PD to DQ, and Qq meet the line fd in q) $= \frac{1}{PM} - \frac{2}{Pq}$ (because Pq is to PM as PQ to PF, and by hypothesis $2PF = PQ$, and so $2PM = Pq$) $= \frac{1}{PM} - \frac{1}{PM} = 0$; whence $\frac{1}{Pf} = \frac{1}{Pd} + \frac{1}{Pe}$, and therefore $2Pf$ is an harmonical mean between Pd and Pe.

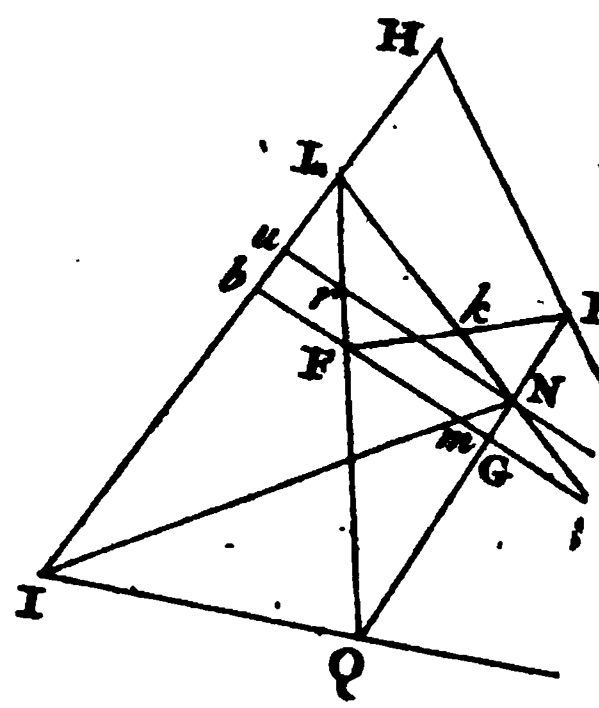
§ 96. *Corol.* 1. Let Dd and Ee joined meet in the point V, VQ and Ff joined will be parallel; and VQ being produced to meet the line fd in r, Pf will $= \frac{1}{2}Pr$. For PD is cut harmonically in E and Q, by hypothesis, and therefore the line Pd is also cut harmonically in e and r, by Art. 21. whence $Pf = \frac{1}{2}Pr$; and since $PE = \frac{1}{2}PQ$; it follows that the line Ff is parallel to the harmonical VQr.

§ 97. *Corol.* 2. In like manner if be taken the point p in the line DF so that $2pD$ be equal to the harmonical mean between pE and pF, and any line drawn from p meet the curve in three points, the segment of this

Appen. pa. 496.

Fig. 44.





this line on one side the point p terminated at the curve will be equal to half the harmonical mean between the two segments on the other side the same point p terminated by it and the curve.

§ 98. *Lemma.* From the center of gravity of a triangle let be drawn any right line which meets the three sides of the triangle, and the segment of this line terminated by the center of gravity and one side of the triangle will be half an harmonical mean between the segments of the same line terminated by the center of gravity and the two other sides of the triangle. Let P be the center of gravity of the triangle VTZ, let the line FDE drawn through P meet the sides in F, D, E; and let the points D and E be on the same side of the point P; it will be $\frac{1}{PF} = \frac{1}{PD} + \frac{1}{PE}$. For let be drawn through the point P the line MPL parallel to the side VZ which may meet the sides VT, ZT, in L and M and the line VN parallel to ZT in N; and since MP = PL, and TL = 2VL, because of the similar triangles TLM, VLN, LM will = 2LN, whence LN = LP, and PN = 2PM, therefore if PD meet the line VN in K, it will be (by Art. 21. and 23.) $\frac{1}{PD} + \frac{1}{PE} = \frac{2}{PK} = \frac{1}{PF}$. Fig. 48.

§ 99. PROP. XX. Let three lines VT, VZ, TZ, touch a line of the third order, and let the same right line pass through the three points of contact and the center of gravity of the triangle VTZ; let any right line drawn through this center meet the curve in c on one side and in a and b on the other of the center of gravity, and 2Pc will be an harmonical mean between the segments Pa and Pb. Fig. 49.

For let Pc meet the sides of the triangle in f, d, and e, and the line VN parallel to TZ in k; and $2Pf$ will $= PK$, and so $\frac{1}{Pf} = \frac{2}{Pk} = \frac{1}{Pd} + \frac{1}{Pe} = \frac{1}{Pa} + \frac{1}{Pb} - \frac{1}{Pc} + \frac{1}{Pf}$, and therefore $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$, whence Pc is half the harmonical mean between Pa and Pb.

Fig. 50. § 100. PROP. XXI. **Let V be a double point in a line of the third order, VT and VZ lines touching the curve in that point, to which let the line TZ touching the curve in F so meet in T and Z that FT = FZ; let FV be joined, in which let be taken $FP = \frac{1}{3}FV$; and if any right line drawn through P meet the curve in three points a, b, c, of which a and b are on the same side of the point P, c on the contrary, $2Pc$ will always be an harmonical mean between the segments Pa and Pb, $\frac{1}{Pc} = \frac{1}{Pa} + \frac{1}{Pb}$.**

For since TZ is bisected in F, and $FP = \frac{1}{3}FV$, it is manifest that the point P is the center of gravity of the triangle VTZ; and since the point P is in the line FV which passes through the points of contact, the proposition follows from the preceding.

§ 101. *Corol.* 1. If the lines Va, Vb, and Fc be joined, P will also be the center of gravity of the triangle contained by them; as also of the triangle contained by three lines touching the curve in a, b, c; and if Va and Vb meet the line Fc in m and n, Fm will be always equal to Fn.

§ 102. *Corol.* 2. A line drawn through the double point parallel to Fc will cut Pa harmonically in k, so that Pa will be to ak as Pb to Pk; but the line which is drawn from k to x the concourse of the tangents at a and b is parallel to the line cy touching the figure in c.

§ 103.

§ 103. *Corol.* 3. Two points a and c being given where any line drawn from P meets the curve, the third b is also given; for let be joined Va and Fc which meet each other in m; let be taken on the other side F the line Fn equal to Fm, and Vn joined will cut the line Pa in b.

§ 104. PROP. XXII. Let be drawn through any point P in the direction of the infinite legs a line to meet the curve in a and c; and through the same point any line cutting the curve in the points D, E, F, and which may meet the lines touching the curving in a and c, in k and m, and the asymptote of the infinite leg in l, and if the points D, E, k, m, l, are on the same side of P, but the point F on the contrary, it will be $\frac{1}{Pl} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF} - \frac{1}{Pk} - \frac{1}{Pm}$, where the sign of every term is to be changed as often as the segment goes off to the opposite side of P. Fig. 51.

This follows from Theor. I. Art. 9. for by that theorem $\frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm} = \frac{1}{PD} + \frac{1}{PE} - \frac{1}{PF}$.

§ 105. *Corol.* 1. If the line PD be drawn through the concourse of the tangents ak and cm; and PM be taken equal to an harmonical mean between the lines PD, PE, PF, according to Art. 28. it will be $\frac{1}{Pl} = \frac{3}{PM} - \frac{2}{Pk}$, and so $\frac{2}{3}$PM will be the harmonical mean between Pl and $\frac{1}{2}$Pk. But if the tangents ak and cm meet in the point M itself, the asymptote will also pass through M.

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Fig. 47. § 106. *Corol.* 2. In the case of Prop. XIX. where three points of contact are in the same right line, and the three tangents parallel, let be taken the point P as in Prop. XIX. and let aPc be parallel to the asymptote, let the tangents ak and cm meet the line PD in k and m, and it will be $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$, or Pl equal to half the harmonical mean between Pk and Pm. But if the tangents ak and cm meet in the same point of the line PD, Pl will $= \frac{1}{2}Pk$; but because in Prop. XIX. $\frac{1}{Pf} = \frac{1}{Pa} + \frac{1}{Pc}$, Pa will $= Pc$.

Fig. 49. § 107. *Corol.* 3. The same is to be said of the case in Prop. XX. where three points of contact D, E, F, are in the same right line which passes through P the center of gravity of the triangle VTZ contained by the tangents. But if one of the lines touching the curve in a or c (supposing aPc parallel to the asymptote) be parallel to the line DP, the asymptote will go off *in infinitum*, and the leg will be parabolical.

Fig. 52. § 108. *Corol.* 4. The same things supposed as in Prop. XXI. let cPa be parallel to the asymptote, let the tangents ak, cm meet the line VF in k and m, and it will be $\frac{1}{Pl} = \frac{1}{Pk} + \frac{1}{Pm}$. Whence if the curve has a diameter, since this necessarily passes through the double point V, from the point of the curve F where the tangent TFZ is bisected let be taken from F towards V, $FP = \frac{1}{3}FV$, let cPa be drawn parallel to the asymptote, and the tangent ak which may meet the diameter in k, and on the other side the point P let be taken upon the line PV, $Pl = \frac{1}{2}Pk$, and a line drawn through l parallel to the ordinates will be the asymptote to the curve. But if the tangent ak be parallel to the diameter, the leg of the

TAB. X.

Fig. 48.

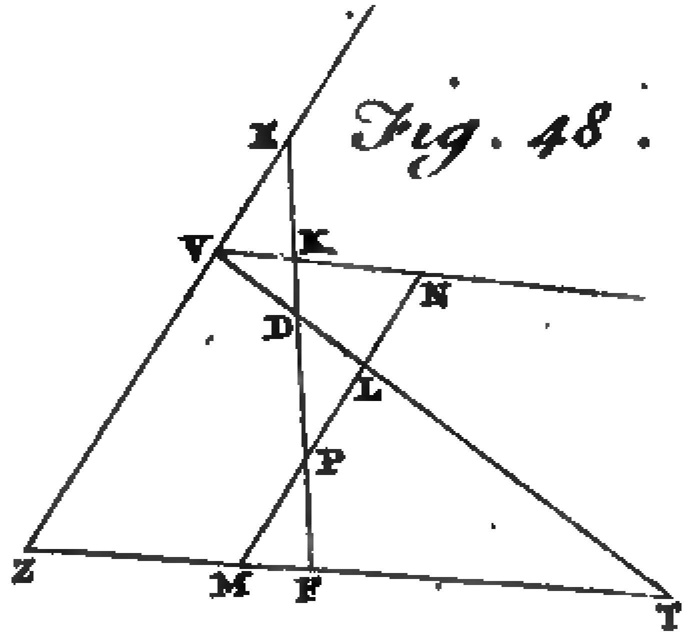
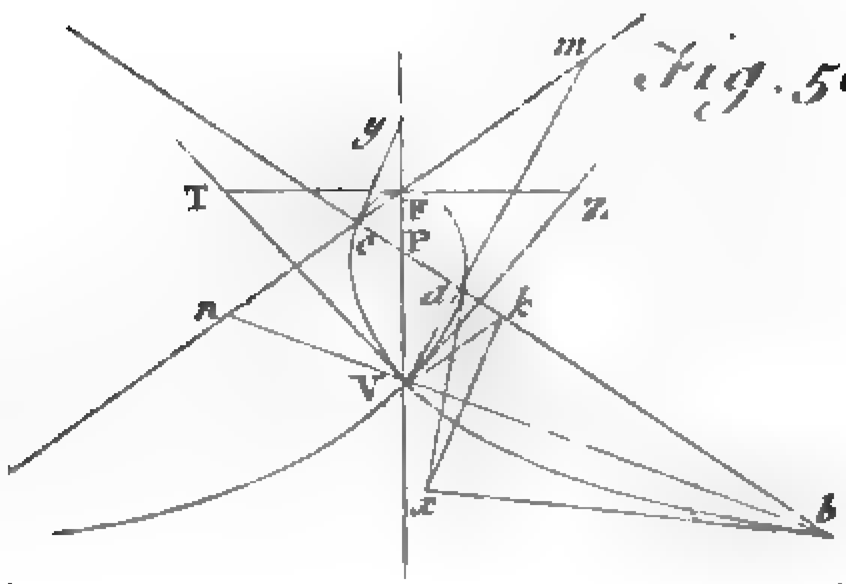


Fig. 50.



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the curve will be of the parabolic kind. *Newton's* proposition about the segments of any right line terminated by three asymptotes and the curve easily follows from Art. 4. as has before been shewn by others.

§ 109. PROP. XXIII. From any point D of a line of the third order let be drawn any two lines DEI, DAB, which let meet the curve in E, I, and A, B; let the tangents AK, BL be drawn, which let meet the line DE in K and L. Let DG be an harmonical mean between the segments DE, DI, terminated at the curve, and DH an harmonical mean between the segments DK, DL, of the same line cut off by the tangents. Let DV be a geometrical mean between DG and DH, let be drawn VQ parallel to the tangent DT, which let meet the line DA in Q; and if a circle of the same curvature with the proposed line of the third order in D meets the line DE in R, HG, QV, and 2DR will be continual proportionals. Fig. 53.

For by Theor. II. (Art. 15.) $\frac{QV^2}{DV^2 \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} - \frac{1}{DL} = \frac{2}{DG} - \frac{2}{DH} = \frac{2DH - 2DG}{DG \times DH} = \frac{2HG}{DV^2}$ (because $DV^2 = DG \times DH$;) whence $QV^2 = 2HG \times DR$, and so HG is to QV as QV to 2DR.

§ 110. *Corol.* 1. Let therefore be taken D*r* in the line DE a third proportional to the lines HG and $\frac{1}{2}$QV, and a perpendicular to the line DE at the point *r* will cut a perpendicular to the tangent DT at the point D in the center of the osculatory circle or the circle of the same curvature with the proposed line, in the point O. If the points E, I, K, L be on the same side of the point D, D, the point r is to be taken on the ſame or contrary ſide of the ſame point as DH is greater or leſs than DG, *i. e.* as an harmonical mean between the ſegments DK, DL cut off by the tangents is greater or leſs than an harmonical mean between the ſegments DE, DI, terminated by the curve.

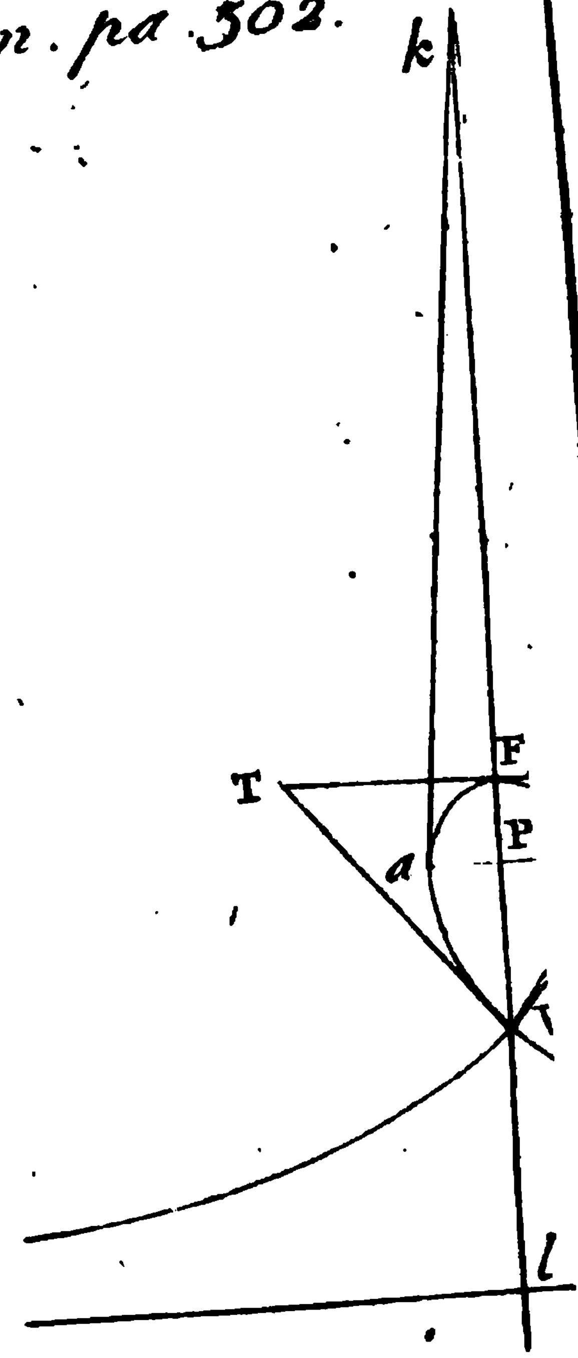
§ 111. *Corol.* 2. If the angle EDT be biſected by the line DA, QV will $=$ DV, and $2HG \times DR = DV^2 = DG \times DH$, and ſo HG is to DG as DH to 2DR.

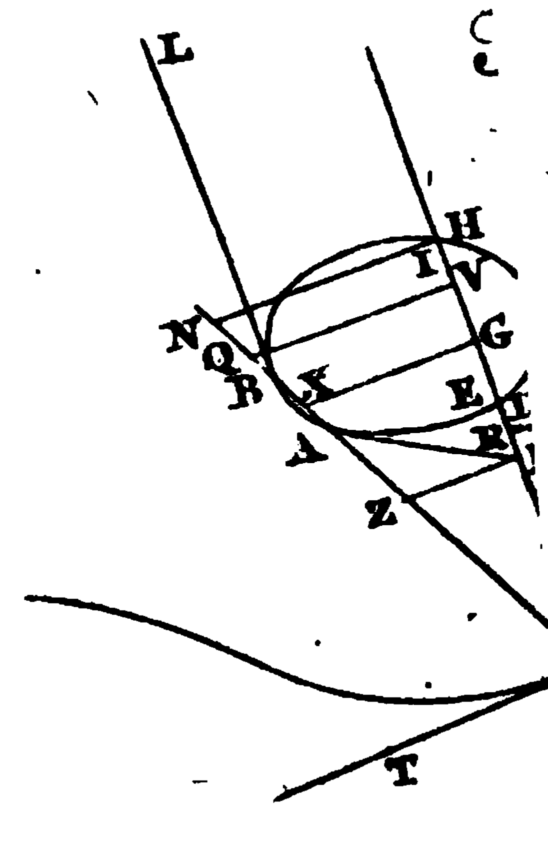
§ 112. *Corol.* 3. Let the line DA revolve about the pole D, the line DE remaining, and HG, the difference of the harmonical means DH and DG, will be increaſed or diminiſhed in the duplicate ratio of the line VQ. For as much as, becauſe DR the chord of the oſculatory circle being given, there remains the quantity $\frac{QV^2}{HG}$ which is equal to 2DR.

Fig. 54. § 113. *Corol.* 4. If of the tangents AK, BL one of them as BL be parallel to the line DE, let be drawn GX and KZ parallel to DT touching the curve in D, which let meet AB in X and Z; and it will be $\frac{GX \times KZ}{DG \times DK \times DR} = \frac{1}{DE} + \frac{1}{DI} - \frac{1}{DK} = \frac{2}{DG} - \frac{1}{DK} = \frac{2DK - DG}{DG \times DK}$, and ſo $\frac{GX \times KZ}{DR} = 2DK - DG$, and therefore it will be as $2DK - DG$ to KZ ſo GX to DR. If the tangent AK alſo comes out parallel to the line DE (which in theſe figures may happen) it will be as DG to GX as GX to 2DR: for in this caſe $\frac{GX^2}{DG^2 \times DR} = \frac{2}{DG}$, and ſo $GX^2 = DG \times 2DR$.

§ 114. *Corol.* 5. If the line DE be parallel to the aſymptote, and ſo meets the curve in one point E beſides D, and at the ſame time the tangent BL be parallel to the aſymptote, let EY be drawn parallel to the tangent

Appen. pa 302.





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tangent DT, which let meet the line DA in Y, and it will be as KE to KZ so EY to DR*.

§ 115. *Corol.* 6. If D be a point of contrary flexure, the point H will coincide with G, and the line HG vanishing, and so DR comes out infinitely great, i.e. the curvature at a point of contrary flexure is less than in any circle however great; as I have elsewhere shewn, in the Treatise of Fluxions, Art. 378.

§ 116. *Corol.* 7. Let V be a double point, DA parallel to the asymptote, and let the lines VQ, KZ, parallel to the tangent DT meet the line DA in Q and Z, and let DV meet the asymptote in L, and let DH be an harmonical mean between DK and DL, and it will be as 2DH − DG to KZ so DL to DK, and VH : HN : : VQ : DR. If the line DA bisects the angle TDV, it will be DR : DV : : DH : 2VH. Fig. 55.

§ 117. PROP. XXIV. Let D be any point of the third order, let the tangent at D meet the curve in I, and let DS be the diameter of the osculatory circle, which let meet the curve in A and B; from whence let lines drawn touching the curve cut DI in K and L; let DH be an harmonical mean between DK and DL, and let be taken DV to DI as DH to the difference of the lines 2DI and DH; the variation of curvature will be inversely as the rectangle SD × DV; and VS being joined, the variation of the radius of curvature as the tangent of the angle DVS. Fig. 56.

For by Theor. III. (Art. 17.) the variation of curvature is as $\frac{1}{DS} \times \overline{\frac{1}{DK} + \frac{1}{DL} - \frac{1}{DI}} = \frac{1}{DS} \times \overline{\frac{2}{DH} - \frac{1}{DI}}$

* Supply the figure.

=

$= \frac{1}{DS} \times \frac{2DI - DH}{DH \times DI} = \frac{1}{DS \times DV}$. But the variation of the osculatory radius is as $\frac{DS}{DV}$, and so as the tangent of the angle DVS, by Art. 18. Now the parabola which will have the same curvature and the same variation of curvature with the line proposed, is determined as in Art. 19.

Fig. 57. § 118. *Corol.* If the tangent BL be parallel to the tangent at D, it will be as DV to DI so DK to IK; and if both the tangents AK, BL be parallel to DT, DV will $=$ DI, and the variation of curvature inversely as $DS \times DI$. But if in this case DT be parallel to the asymptote of the curve, the variation of curvature will vanish. Therefore as the variation of curvature will vanish in the vertices of the axes of conic sections; so it will in like manner vanish in the vertices of the diameters of lines of the third order which bisect their ordinates at right angles.

Fig. 58.

Fig. 59. *Schol.* Now there are many other theorems about the tangents and curvature of lines of the third order. Let, for example, F and G be two points of a line of the third order, from whence tangents drawn meet the curve in A. Let FG be produced till it meet the curve in H. Let TAC be a tangent at A, and let be constituted the angle FAN $=$ GAT on the contrary side of FA, GA, and let AN cut FG in N. And if the osculatory circles meet the lines FG in B and b, GB will be to Fb as the rectangle NFH to NGH. For let the point a be very near to A, and the points f, g, h, very near to F, G, H, and it will be AFa : FGf : : GF : FB. FGf ($=$ HGb) : HFb : : FH : GH. HFb ($=$ GFg) : AGa : : bG : GF; whence AFa : AGa : : $FH \times bG$: $FB \times GH$: : GN : FN; whence FB : Gb : : NFH : NGH. *But enough on this subject.*

FINIS.

Appen. pa 504.

Fig. 57.

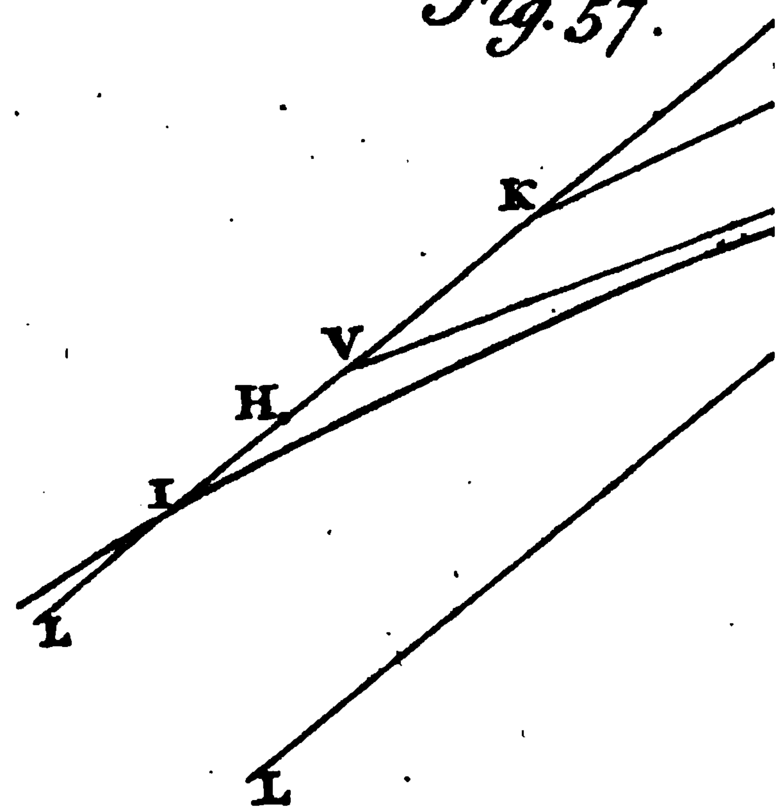
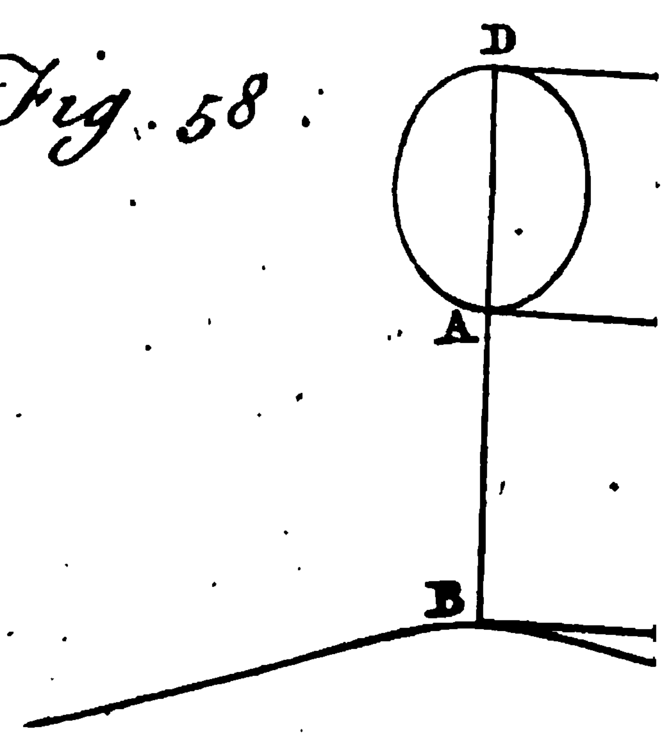


Fig. 58.



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