A TREATISE OF FLUXIONS: OR, AN INTRODUCTION TO Mathematical Philosophy.

CONTAINING

A full Explication of that METHOD by which the Most Celebrated Geometers of the present Age have made such vast Advances in *MECHANICAL PHILOSOPHY*.

A WORK *very Useful for those that would know how to apply Mathematicks to Nature.*

By CHARLES HAYES, *Gent.*

LONDON,

Printed by *Edw. Midwinter*, for *Dan. Midwinter* and *Tho. Leigh*, at the *Rose and Crown* in St. *Paul's Church-Yard*. 1704.

TO
Sir Dalby Thomas; Kt.

General *and* Chief Director *for the* Royal African Company *of* England, *on the* Coast *of* GUINEA *in* AFRICA.

SIR,

SINCE the ensuing Treatise was Written with a design to propagate Mathematical Learning, and Your many Kindnesses so exceedingly influenc'd the Undertaking; The Author thinks himself in Gratitude oblig'd to acknowledge your Favours: And if the Performance can any ways Merit Your Acceptance, He is very sensible how much in Justice he is bound to Inscribe it to Your Name.

Pardon then, Sir, this Address, since it is not the effect of Flattery, nor with a Design to Court (as having no reason to doubt of) Your Favours; But proceeds from a sincere Desire he has to take the first Opportunity to express his Thankfulness for those he stands indebted for already; and to offer something as a mark of the Respect and Esteem he has for his best Friend.

And

And indeed, To whom can he Dedicate any thing of his, if not to Him, of whose Generosity and Natural Inclinations to do Good to every one, he is so fully Convinc'd, That He has ventur'd to Encounter many Dangers for the Honour and Happiness he Enjoys in his Company; and now when far from all other Friends, Daily receives fresh Assurances of Real Friendship.

The Design of the following Sheets is certainly Noble, being to instil the most Extensive Principles of Humane Knowledge into the Minds of Men; and to lead them step by step through the Labyrinths that hitherto have so much perplext all Natural Sciences. If the Work then be not a sufficient Monument of Gratitude, yet let the Greatness of the Design cover the Imperfections that a stricter scrutiny might observe: And when both Conspire to the best Advantage, let it be deem'd only a faint Resemblance of what it ought to Come up to.

That you may long live to promote the Good of the Royal *African Company* of *England*, in whose Interest You have so heartily Engag'd Your Self; and that You may ever Succeed in Your own Private Concerns, And live to Enjoy All the Blessings accompanying a Prudent and Wise Management of Affairs, is the Earnest Prayer of *Honoured SIR,*

Your most Oblig'd and

Obedient Servant

C. H.

THE PREFACE TO THE READER.

A Preface is expected from every Author, and when tending to inform the Readers of the motives that induced him to write: And of the means they must use to understand what is wrote, is very proper; the Former to justifie himself from the Imputation of Vanity, and the Latter to quicken and forward their Industry.

As to the first, 'Tis manifest, since the World has been convinced of the mischief of Dogmatizing, either in Philosophy or Mathematicks, that by allowing themselves a freedom of Thought, and boldly venturing forwards; the Advances in each are equally wonderful, that it is a difficult matter to resolve, whether an easie Acquiescence in all the Ancient Discoveries more obstructed, or the more generous Essays of modern Hero's improved our Knowledge; that since Men resumed their Native priviledge, and allowed themselves the Liberty of enquiring freely into things, they have extended their Dominions over all the Earth, and their Knowledge far above the Clouds; that these being the undeniable results of Mathematical Studies; it is farther plain, that they create in us more awful Thoughts and juster Notions of the works of God; *that the admirable Harmony which only they discover, both in things in Heaven, and in things in Earth, undeniably prove* one *and the* same great Author; *in a word, that by them we encrease our Riches, enlarge our Power, and improve our Reason; these then being some of the great Advantages Mankind receive from Mathematical Learning, Who would not incessantly aspire after such useful Knowledge? Who can be blam'd for using these endeavours to propagate the same?*

As to the ensuing Treatise, the Author has been well assur'd that there are in England *as many Lovers of the Mathematicks as in any part of the World; that multitudes of excellent Judgments and natural Parts, merely for want of a competent Knowledge in other Languages, have hitherto been deprived of the Opportunities of improving them, to the great disadvantage of the most* Flourishing Island *in the* World; *that in other Nations the best pieces of*

Learning are written in their own mother Tongues, for the good of their Country which we seem purposely to slight, seeking a little empty applause by writing in a Language not easily attain'd, as if the Knowledge of things *and* words *had a necessary dependance on each other; and in a word, that such a Treatise was wanting in the* English *Tongue, as should contain a full and plain account of the best Methods, the most celebrated Geometers of our Age have made use of in their wonderfull Discoveries; and which would put it in the Power of every industrious Person to make use of those parts which* God *and Nature has bestow'd upon him to the best purposes: These, he says, were the principal motives that induced him to this difficult undertaking, and he hopes the sincerity of his design will at least merit a favourable Censure from the World. He knows there are Persons better qualified for such an undertaking, but none appearing, hopes his forwardness to serve the Publick will be no objection against him. He professes all along, he has endeavoured to express things in as plain Terms as the nature of the subject will allow; and he assures his Readers, that he is not conscious of omitting any thing that might really conduce to their Instruction.*

The Author is sufficiently aware that this Science is of a vast extent, and therefore does not pretend to treat of every thing it has or may be applied to. What incredible advances natural Philosophy has received since it began to flourish is apparent to those that are but a little conversant in the profound Writings of the Great Mr. Newton, *whose* Immortal Genius *will be a* lasting Ornament *to the* English Nation; *and the application of* Fluxions *in* Astronomy, Mechanicks, Dioptricks, Catoptricks, Gunnery, Navigation *&c. are numberless: The Reader therefore is not to expect a Treatise comprehending all the uses of Fluxions, that being a Task not to be performed till this Science arrives to it's utmost Perfection: All that can be expected is to teach the Principles of the Science, and to shew how to apply them thro' most of the general parts of that* Geometry, *which is best able to assist us in our After-Enquiries: And the Author hopes when he has done this, his Performance may modestly deserve the Title of* An Introduction to Mathematical Philosophy.

And since he has not made such frequent mention of several excellent Persons as some People may expect, and by that means incur the Censure of a Plagiary; he thinks himself bound by all the tyes of Justice, Honour and Gratitude, freely to acknowledge how much this Treatise is indebted to those worthy and generous Persons, who have already infinitely oblig'd the World with writings of this Nature. Dr. Wallis, Dr. Barrow, Mr. Newton, Mr. Libnitz, the Marquess De l'Hospital, Mess^rs. Bernoulli; Mr. Craig, Dr. Cheyne, Dr. Gregory, Mr. Tchirnhaus, Mr. De

Moivre;

Moivre, Mr. Fatio, Mr. Varignion, Mr. Newintiit, Mr. Carre, *are Persons of such Merit, and have furnish'd the World with such extraordinary Inventions of this nature, as will transmit their names with the greatest Respect to all the succeeding Generations. And tho' in the ensuing Treatise, the Author has made no scruple to borrow from any of those excellent Persons as occasion requires, yet he acknowledges himself more particularly indebted to* Mr. Newton, Mr. Libnitz, the Mess^rs^ Bernouilli, the Marquess De l'Hospital *and* Mr. Craig, *Persons who have given surprizing and innumerable proofs of their profound Penetration into this Science: And where he has not particularly mention'd the Authors themselves in the body of the Book, he declares that either Brevity, or want of due Information who they are, were the only motives that induced him to silence.*

And on this occasion also he thinks himself oblig'd to acknowledge the obligations of thanks he lies under to that Industrious and Learned Mathematician Mr. John Harris; *who notwithstanding the many laudable Designs (particularly that of his* Lexicon Technicum Magnum,*) he is daily engaged in for encouraging and promoting the Mathematical Learning; took the trouble upon him to revise the greatest part of the Sheets as they came from the Press, that the Errors there might not discourage the Reader, nor stop him in his Progress.*

Secondly, As to the means; Arithmetick, Geometry *and* Specious Algebra, *will be indispensably necessary to prepare the Readers for the following Treatise; and since there is enough already in the* English *Tongue on these subjects, the Author has suppos'd the Reader to be acquainted with them. And because* Conic-Sections *are also of great use in* Mathematical Philosophy, *and frequent mention will be made of them in several places, there is prefixt to the beginning of the Book a short discourse of* Conic-Sections, *extracted in a great measure from a late learned Treatise of* Conic-Sections, *published at* Oxford *by* D. Milnes: *These things the Reader ought to be well acquainted with, and then let him read in order, seeing it has been the Author's aim in composing the ensuing Treatise, to dispose the several parts in such order, that beginning and reading the Book as it lies, there might be nothing wanting to a full understanding of what he Reads, but a perfect Knowledge of what preceeds.*

This is not so strictly meant, that the Reader must understand every single Proposition or Article before he can proceed farther; for there are several Propositions relating to mixt Mathematicks, which Beginners will hardly comprehend at first reading; and therefore may omit them, without mistrusting their own Abilities to go on, or being discouraged in their Studies. The

great

great Business at first is to be perfectly Master of the design, and principles on which the several parts are grounded; and what applications are scatter'd up and down, will be more or less understood, according as the Reader is acquainted with the principles of the Science they belong to.

Lastly, He desires his Readers Impartially to peruse his Book; and then he hopes, when they duly consider how easie it is in such variety of matter to be mistaken, they will not rigorously censure every little Fault, which had he had more time and leasure to spare, and could his other Business have permitted him to have seen the whole compleatly Printed, might possibly have come forth much more full and correct: In a word, he humbly craves leave to assure his Readers, that he will be willing to own and retract his Errors on better Information; but withal desires them throughly to weigh and examine both what he has written, and what themselves have to object, which possibly may prevent several useless disputes.

THE

THE
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ERRATA.

Page	Line	Read
6	8	$\frac{1}{4}\dot{x}\dot{z}$.
9	12	$x^{\frac{1}{4}}$.
13	6	z^n.
15	1	$\dot{q}$.
26	35	P m D = (*Art.* 47.) S m D.
27	*penult.*	(m β.)
Ibid	*ult.*	$\frac{1}{2}$ P m + $\frac{1}{2}$ m P.
35	36	$\frac{x}{y} +$.
38	13	FP = x.
39	25	dele to x.
47	*antepe.*	$\dot{z} = -\frac{x\dot{y}}{p}$.
54	26	$= b, \frac{b\,x}{r}, \frac{b\,x}{r^2}, \frac{b\,x}{r^3}$, &c.
55	26	$p = 2a - 3x$.
56	19	$y\dot{x} + \frac{\dot{y}\dot{x}}{2}$.
59	7	$m \pm 1$.
60	23	$\frac{x^{\overline{1 \mp m}}}{\overline{1 \mp m}}$.
62	*antep.*	$\frac{1}{2} \times \overline{2rx - xx}\vert^{-\frac{1}{2}}$.
64	27	$+ m a^{m-1} x$.
68	5	$\overline{e + f x^n}\vert^{m+1}$.
Ibid	6	$\frac{\overline{n + r - 1} \times e}{\overline{mn + r - n + 1} \times f}$.
Ibid	7	$\frac{\overline{2n - r - 1} \times e}{\overline{mn + r - 2n + 1} \times f}$.
78	8	Diameter.
81	20	$\sqrt{\frac{41}{64} a a}$.
83	21	$b\dot{x}$.
Ibid	22	$- z\dot{x}$.
84	4	$d = 2c \mp p$.
Ibid	11	$r\dot{y}$.
Ibid	12	$\frac{r x \dot{x}}{y} = 2c \mp p$.
86	5	Sections.
Ibid	10	For D read F.
Ibid	*ult*	$\frac{m x y y}{\overline{2m \pm 4n} \times r}$.

Page

ERRATA.

Page	Line	Read
87	21	$y = \frac{r x^{\frac{n}{m}}}{c^{\frac{n}{m}}}$.
89	21	A θ.
97	3	$+ c a x + e a^2$.
Ibid	4	$\overline{b x^2 + c a x + e a^2}$.
Ibid	13	$e a^2 x$.
99	16	in V.
111	antep.	$\sqrt[m]{x + a}$.
103	7	$\frac{-2mm}{\overline{2m+1} \times \overline{m+1}}$.
Ibid	8	$-2mmaa$.
204	4	$\overline{n+1} \times m b x^{n+1}$.
106	1.2	$x^2 \times \overline{x+a}\vert^{-1} = y^2$.
111	7.8	$\frac{nfy}{mr + nr} = b$.
Ibid	9	$n^n f^n y^n = \overline{m+n}\vert^n r^n b^n$.
Ibid	16	$\frac{m}{2m + 4n} \times rc$.
112	17	x RQ.
113	36	$p\gamma$.
117	12	$\dot{z} \dot{x} \sqrt{\dot{x}^2 + \dot{y}^2}$.
Ibid	25	$\ddot{x} z \sqrt{\dot{x}^2 + \dot{y}^2}$.
118	22	$\sqrt{\dot{x}^2 + \dot{y}^2}$.
119	ult	$\ddot{x}$.
124	20	$2ax + ab = 2ac \pm bx$.
128	36	$-\frac{2rx\dot{x} - 2rb\dot{x}}{2\sqrt{x}}$.
144	20	tCG $+ t$GD.
150	3	$+ \frac{3x^3 \dot{z}}{4a^3}$.
151	1	$\frac{y\dot{x}}{\dot{z}^4}$.
155	7	B m.
156	14	$\frac{\dot{x}\dot{z}}{y}$.
157		$2a^3 \dot{x}^2 \times \overline{xx + aa}\vert^{-2}$.
158	1	$\dot{y} = \frac{3}{5} \overline{x - a}\vert^{-\frac{2}{5}} \times \dot{x}$.
Ibid	2	$-\frac{6}{25} \times \overline{x-a}\vert^{-\frac{7}{5}} \dot{x}^2 = \frac{-6\dot{x}^2}{25\sqrt[5]{\overline{x-a}\vert^7}} = o$.
Ibid	4	$\frac{-6\dot{x}^2}{25\sqrt[5]{\overline{x-a}\vert^7}}$.
Ibid	19	and $\dot{u}$.
159	32	$-2r\dot{y}^2 + 2y^2\ddot{y} - 2ry\ddot{y} = o$.
Ibid	33. 35	$y\ddot{y} = \frac{r\dot{y}^2 - 2y\dot{y}^2}{y - r}$.
164	22	E$m = y$.
Ibid	24	$mn = y$.
169	24	$\frac{cyy\dot{x}}{r}$.
170	18	$y = r$.
171	8	$-cx^{\frac{1}{m}-1}\dot{x}$
		$= \dot{x}$.
Ibid	20	$y = \frac{br}{x}$.
172	14	OB $= 2b + d$.
173	6	$2rbcx\dot{x}$.
Ibid	11	a.
184	12	KL to MN.
186	15	
Ibid	26	$y + x : x + 3y$.
187	27	$y + \dot{y}$.
188	penult.	$3y^2\dot{y}$.
189	penult.	$\frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$.
190	8	dele $=$.
Ibid	9	$\dot{x} = t\dot{y}$.
Ibid	24	$r b x^{r-1} y^s \dot{x}$.
Ibid	penult.	$\frac{p\dot{q} - q\dot{p}}{pp}$.
192	27	$\dot{x}^2$.
198	2	$\frac{y\dot{x}^2 + y\dot{y}}{-2\dot{y}^2}$.
198	22	$\frac{\dot{x}\ddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}\vert^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}$.
Ibid	23	$= 3\dot{x}\dot{y}\ddot{y}^2$.
206	13	$\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2}$.
207	10	$my^m\dot{y}$
208	12	$\dot{x} \times \sqrt{\frac{2a - x}{x}}$.
211	34	$\frac{b \pm 2a}{b}$.
216	2	dele MG m.
Ibid	15	$r = s + q$.
Ibid	22	KM $= c$.
218	24	PN (u)
219	24	$\dot{z}$.
Ibid	ult.	$\sqrt{2r - x}$.
222	4	$r c^{\frac{n}{m}} \dot{x}$.
234	24	CM.
238	ult.	$\frac{\dot{y}^2}{\dot{x}^2}$.
250	penult.	$\frac{mmy}{bn + ny}$.
258	14	BG.
259	4	$+\frac{4}{9}xx$.
Ibid	5	$-\frac{1}{9}xx$.
265	21	BV $= v$.
Ibid	24	$2vr = a^2 + v^2$.
Ibid	antepe,	$+\frac{2}{3}a^3$.
Ibid	ult.	cr^2.
266	3	$\frac{1}{2}cr^2$.
267	21. 23	$\frac{8r^2}{3C}$.
284	5	$-x^{\frac{1}{m}+1}\dot{x}$.
294	14	:: S$b \times bdq$.
Ibid	15	bn : BN :: $\frac{bdq}{Sb} : \frac{BDq}{SB}$.
300	28	BCq.
302	3	$\frac{1}{4}$.
221		Upon the Suppoſition that an infinitely ſmall Arch deſcrib'd with the Radius AM, coincides with an infinitely ſmall part of the Spiral Curve, the expreſſion of Mm comes to $\frac{y\dot{x}}{r}$; and this is true in the Spurious Spiral whoſe generation in this manner Dr. *Wallis* deſcribes. But otherwiſe Mm is really $\overline{\frac{y^2\dot{x}^2}{r^2} + \dot{y}^2}\vert^{\frac{1}{2}}$, the Fluent of which may be found by the foregoing methods.

A TREA-

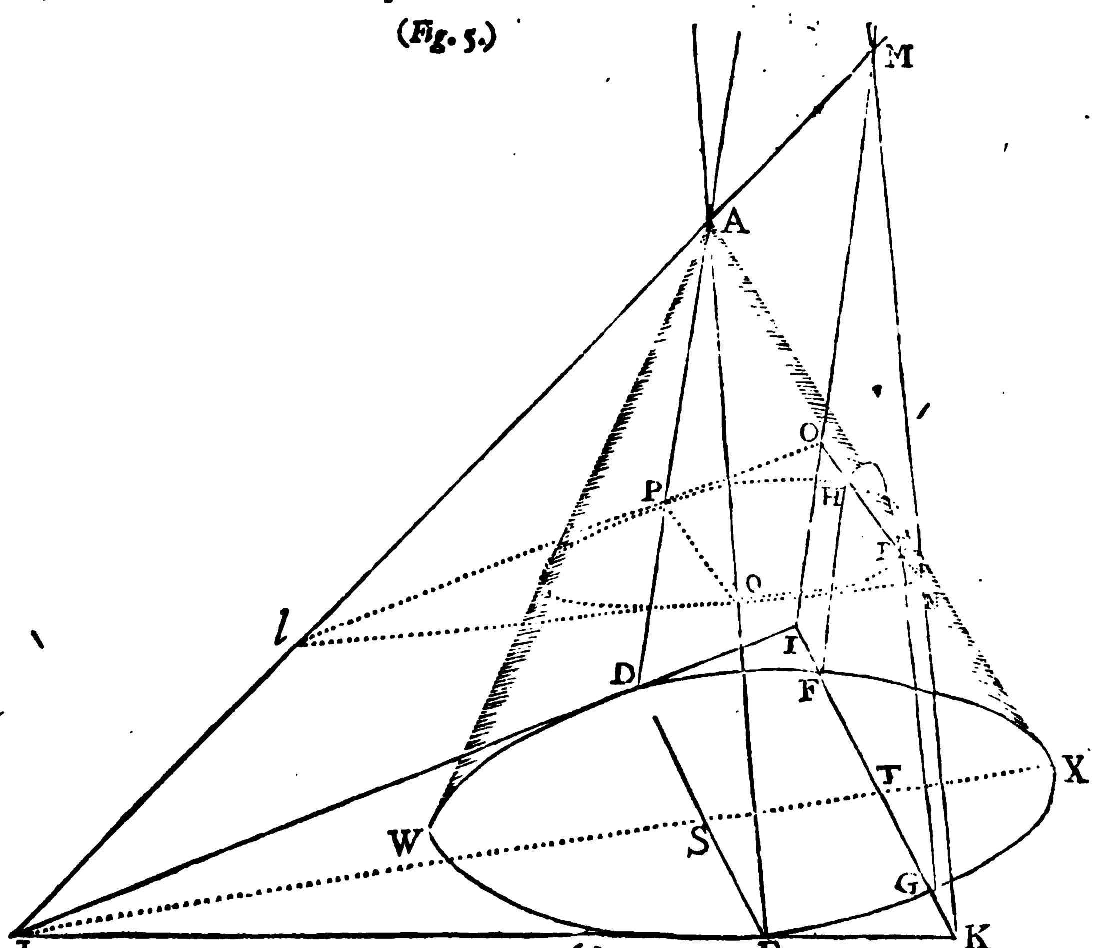
A SHORT
DISCOURSE
CONCERNING
Conick Sections.

TO know the Fundamental Properties of the three principal Sections of the Cone, *viz.* the Parabola, Hyperbola and Ellipsis, is absolutely necessary for all those that have a desire to make any reasonable Advances in real Philosophy. And the Design of this Treatise being to explain the first Principles, upon which our most Scientific Knowledge is built, and to treat of that inexhaustable Source, to which the most amazing Discoveries of latter Years owe their rise; I have thought fit, the better to accomplish my Design, to premise a brief Account of the Origin and Properties of the said Figures, which I shall comprise in the following Definitions and Lemmas.

DEFINITION I.

If the Cone AWX be cut by any Plane ADE, passing through the Vertex A, and also by another Plane IMK, parallel to the Plain ADE; then the common Section of that Plane with the Surface of the Cone, *v. g.* GHF, is call'd an *Hyperbola*, and the Plane IMK is call'd the *Plain of the Section.*

(*Fig.* 5.)



(a)

DEFINITION II.

If two Plains ADI, AEK, touching the Surface of the Cone in the right Lines DA*d*, EA*e*, cut the foresaid parallel Plane in the Lines IM*i*, KM*k*; these right Lines being infinitely produc'd, will never touch the Cone, and being in the Plain of the Section, they are call'd the Asymptotes of the Hyperbola GHF: For it is evident that the Plains ADI and AEK can never touch the Curve GHF, because they touch the Cone already in the Lines AD and AE, and yet notwithstanding they approach nearer and nearer to GHF; because as the Cone is produc'd, the Circle of the Base increases, and at the same time its Convexity decreases, and that *in Infinitum*.

LEMMA I.

If IFGK be the common Section of the Plane of the Section and the Plane of the Base, (or any other Plane parallel to the Base,) and if it intersect the Asymptotes in I, K, and the Hyperbola in F, G; and if through any point H, in either of (for if the Plane of the Section IMK be produc'd until it intersect the opposite Cone, that Section is also an Hyperbola) the opposite Section, the right Line OSH be drawn parallel to IK, and intersecting the Asymptotes in O and N, and the Hyperbola again in R; I say that IF × FK = KG × GI = OH × HN = NR × RO.

DEMONSTRATION.

If Planes be drawn through IK and ON parallel to the Base of the Cone, their common Sections with the Cone DFGE, PHRQ will be Circles, and the common Sections of those Planes with the Planes forming the Asymptotes, *viz.* DI, EK, PO, QN, will touch the Circles in the points D, E, P, Q, and because the Plains are parallel, DI will be = PO, and EK = QN, and if the Tangents meet in L, *l*, then is LE = LD, and *l*Q = *l*P, and drawing the Diameter LST, it will bisect the Parallels DE and FG in S and T; and because DE is bisected in S, therefore IK is (by similar Triangles) bisected in T, whence IF = GK and IG = FK.

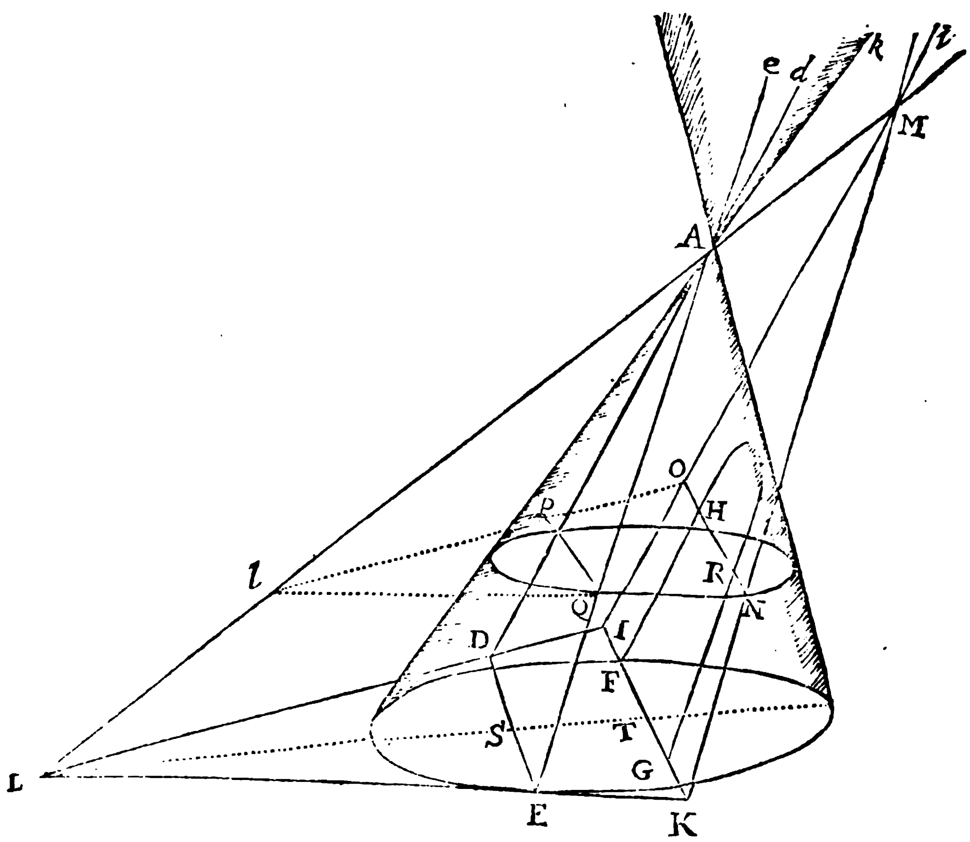
In like manner OH = RN and OR = HN, and therefore DI*q* = (*Prop.* 36. *El.* 3.) IF × IG = IF × FK = KG × GI = (because DI = PO) PO*q* = OH × OR = OH × HN = NR × RO.

LEMMA

LEMMA II.

If any two parallel right Lines, either both in one Section, or one in each Section, or both in the opposite Sections, terminating in B,C,F,G, or touching the opposite Sect-

(*Fig.* 2.)



Paste this at the Bottom of Page 25 in Conick-Sections.

DEMONSTRATION.

If the Right Lines BC, FG be parallel to the common Section of the Plane of the Section and the Plane of the Base, then this is the same with the preceding Lemma; But if not, then through any two of the four points B, C, G, F, draw the right Lines ICK, LGQ parallel to the common Section of the Plane of the Base with the Plane of the Section, until they cut the Asymptotes in I, K, Q, L, then the Triangles DCK, HGQ are similar; as also ICA, LGE, therefore

IC : CA :: LG : GE,

And CK : CD :: GQ : GH;

And by multiplication IC × CK : CA × CD :: LG × GQ : GE × GH.

But IC × CK = (*Lemma* I.) LG × GQ.

Therefore CA × CD = GE × GH.

In

In like manner (drawing Lines through the points B and F parallel to ICK and LGQ.) It may be demonstrated that $AB \times BD = CD \times CA = EF \times FH$.

COROLLARY I.

Hence $CD = BA$.

For $AB \times BD = CD \times CA$.

And $AB \times BC + AB \times CD = CD \times BC + CD \times AB$.

And taking away that which is common to both, we have $AB \times BC = CD \times BC$, therefore $AB = CD$, and for the like reason $EF = GH$.

COROLLARY II.

Hence $AB \times AC = AB \times BD = EF \times FH = EF \times EG$.

COROLLARY III.

The right Line ACD touching the Section in C, and terminating in the Asymptotes in A and D, is bisected in C the point of Contact; for AB is every where equal to CD, and in this case the points B and C coincide; and in like manner, if the Line BC be in both the opposite Sections, and pass through M the Center of the Asymptotes, it will be bisected in the said Center: for CD is always = AB, and in this Case the points A and D coincide.

COROLLARY IV.

If the Line ACD touch the Section in C, and be parallel to any other Line, as FG, then $EF \times FH = ACq = CDq$, but if BC pass through the Center of the Asymptotes, then is $EF \times FH = DBq = CAq$. because the points D and A coincide in M.

COROLLARY V.

If the right Line ACD terminating in the Asymptotes in A and D, and meeting the Curve in C, be therein bisected, it will touch the Section in the point C, for if it be said to meet the Curve again, *v. g.* in B, then is $AC = BD = CD$, that is the point C will be the same with B; in like manner, if GC be bisected by an Assymptote, the point of Intersection is the Centers of the Asymptotes.

COROLLARY VI.

The two right Lines HGE, ACD parallel and touching the opposite Sections, and terminating in the Asymptotes, are equal; for $HG \times GE = CA \times CD$, and $HG = GE$, and $AC = CD$, therefore $HE = AD$.

COROLLARY VII.

The right Line connecting the points of Contact G and C, passes through M the Center of the Asymptotes; for the Triangles MHE, MDA are similar, and $AC = \frac{1}{2} AD = CD = HG = GE$, and $MG = MC$; *ergo* M is the Center of the Asymp otes.

DIFNITION IV.

If the Plain (*Fig. in Pag.* I.) ADE touch the Surface of the Cone in the Line AW and the Plain of the Section be parallel to the same, then the Section GHF is called a Parabola.

DEFINITION IV.

And if the Plain A D E passing through the Vertex A be altogether without the Cone, and the Plain of the Section be parallel to the same, then the Section is called an Ellipsis; and hence it appears that of all the three, the Ellipsis only includes a Space, the other two being infinite.

LEMMA. III.

In every Conick Section, and in the Opposite Sections, if any two right Lines (*Fig.* 4. 5. 6.) BI, CI, touching the same or the opposite Sections in B and C, and meeting in I (or perhaps in the Ellipsis and opposite Sections being parallel) be connected in the points of Contact by the right Line B C, and if any other Line E F be drawn on either side parallel to B C, Intersecting the Curve in F and E, and the Tangents in G and D; I say that D E = F G.

Fig. 3.

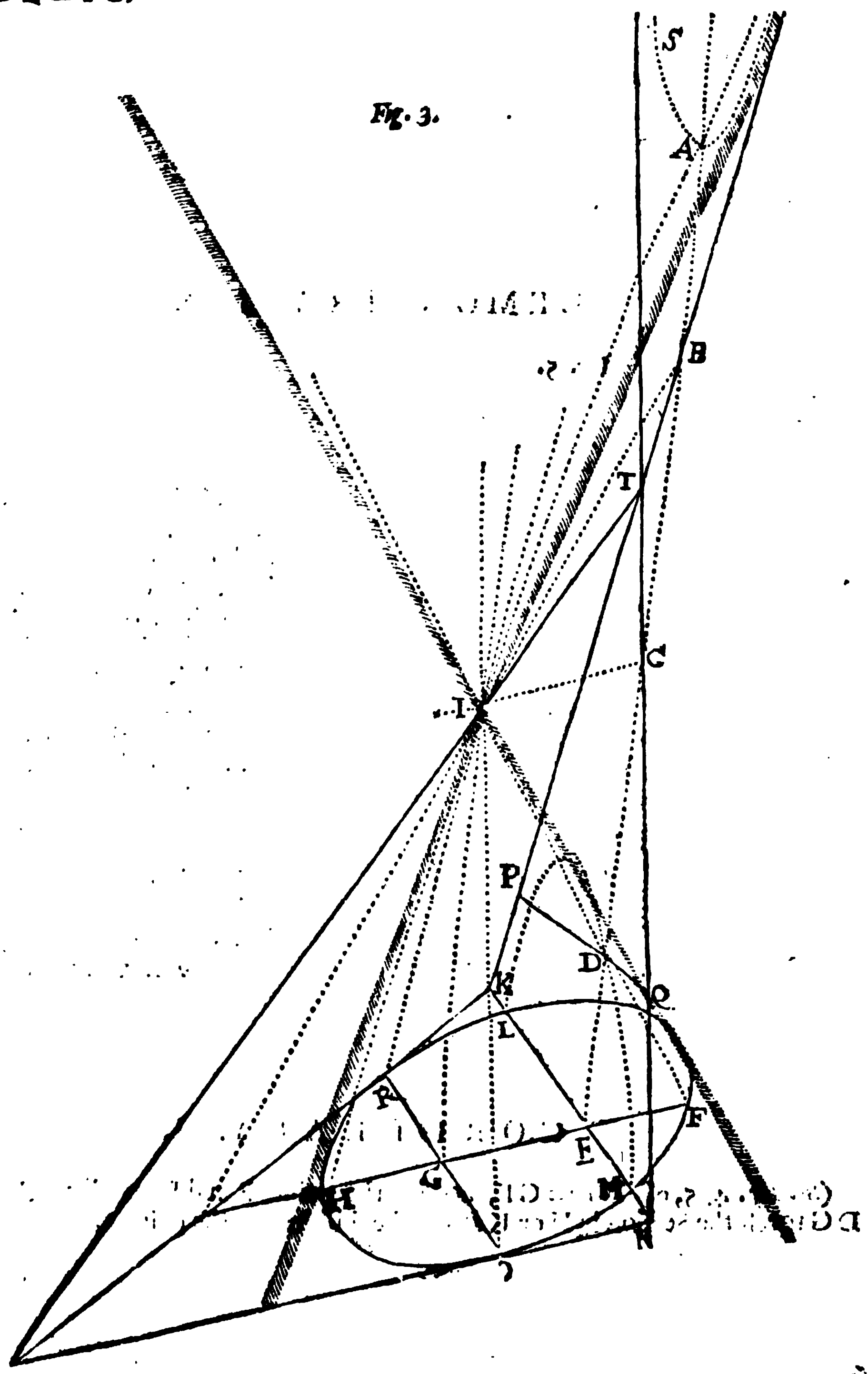


Fig. 4.

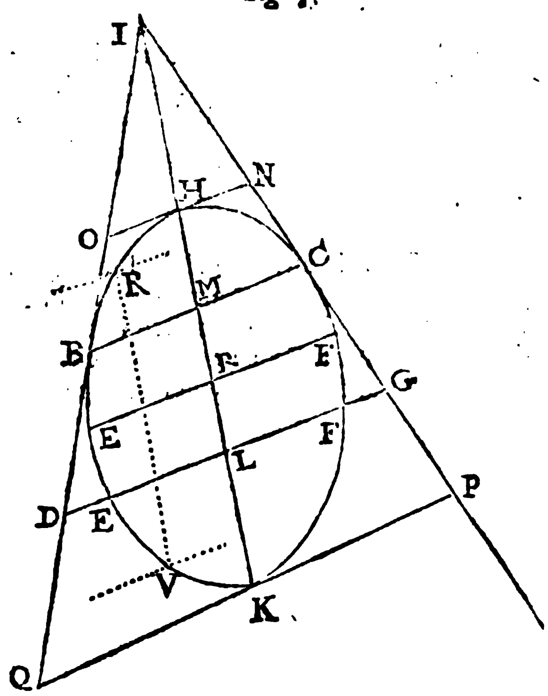
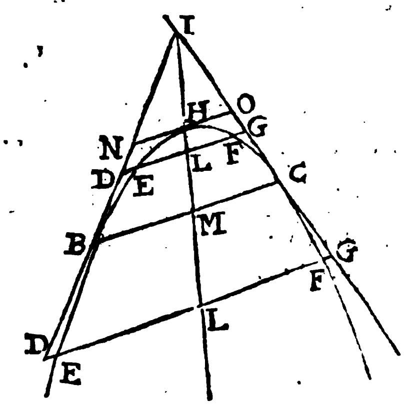
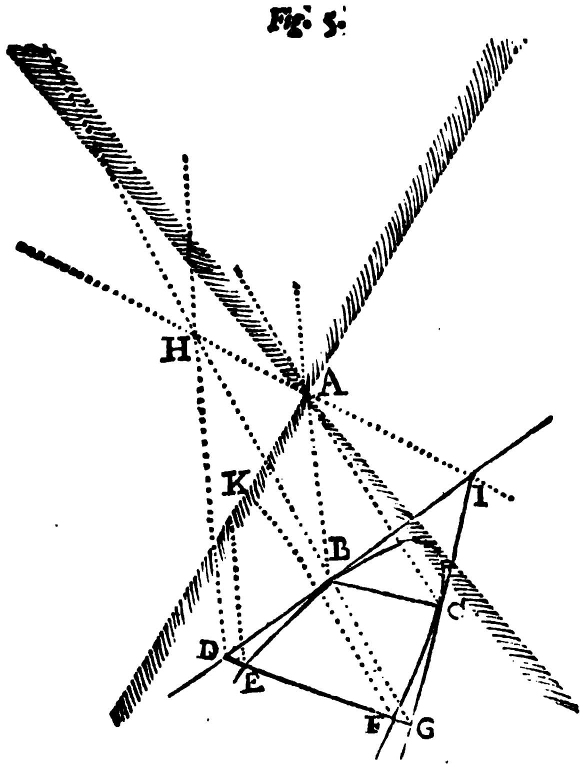


Fig. 6.



DEMONSTRATION.

Fig. 5.



Describe any of the Conick Sections on the Surface of a Cone, and drawing the Lines as in the Figure; through A the Vertex of the Cone, draw the Lines AB, AC, and then the Plains passing through AB, BI, and AC, CI, will touch the Surface of the Cone.

Through DEFG draw a Plain parallel to the Plains ABC, making the Section or opposite Sections EKF, and cutting the Contingent Plains in DH, HG; then EKF will be an (*Defin.* 1.) Hyperbola, and the Lines HD, HG will be its (*Defin.* 2.) Asymptotes, *Ergo* (*Lem.* 2. *Cor.* 1.) DE = FG.

COROLLARY. I.

(*See Fig.* 4, 5, 6.) DF = GE, and if (the points E and F meeting) the right Line DG touch the Section in H or K it will be bisected in the point of Contact.

COR-

COROLLARY II.

(*See Fig.* 4, 5, 6.) If through the point I where the Tangents concur, and M the middle point between B and C be drawn the right Line I M, (or the Tangents being parallel, if I M be drawn parallel to either of them) it will bisect all the Lines E F parallel to C B; for because the Triangles I B C, I D G are similar, and M B = M C, therefore L D = L G, and (*Lem.* III.) D E = F G, therefore L E = L F.

COROLLARY III.

(*See Fig.* 5. 6.) In the Hyperbola and Parabola the Line I M infinitely produced intersects the Section only in one point, and in the Hyperbola it intersects also the opposite Section, and in the (*See Fig.* 3,) Ellipsis it intersects the Curve in two points.

COROLLARY IV.

And because all those Lines which intersect the Parabola in one point only are parallel to the Principal Axis, it follows that I M is so too, and all the Lines I L are parallel to one another.

DEFINITION V.

The right Lines (*See Fig.* 4, 5, 6.) I M generated in this manner, and infinitely produced are called Diameters.

DEFINITION VI.

And the right Lines M C, and M B, and all their parallels L G, L E are called *Ordinates* applied to the Diameter I M.

DEFINITION VII.

In the Hyperbola or the opposite Sections, and in the Ellipsis, the portion of the Diameter H K is called the Transverse Diameter, and in every Section the point H or K is called the Vertex.

DEFINITION VIII.

In every Section the Portions of the Diameter H M, H L, intercepted between the Vertex and the Ordinates, are called the Intercepted Diameters, or Abscissæ.

DEFINITION IX.

In every Section, if the Diameter I M intersect the right Line B C, and its Parallels E F, &c. at right Angles, then the Diameter I M is called the principal Diameter or Axis of the Section.

COROLLARY V.

(*See Fig.* 4, 5, 6.) In every Section H N or K P drawn through the Vertex H or K parallel to the Ordinates touches the Section, & *conversim*; for if any part of the same be within the Section, it will be (*Cor.* 2.) bisected by the Diameter, which yet it is supposed not to meet but in the point H or K. 2. If it touch the Section in the Vertex, it must be parallel to the Ordinates, else two right Lines might touch the Section in the same point.

CON-

COROLLARY VI.

In the Ellipsis and the opposite Sections, the right Line K H joyning the points of Contact of the parallel Tangents, is the Diameter of all the right Lines B C, E F, *&c.* Parallel to those Tangents; for if any other right Line as V R be their Diameter, then (*Cor.* 4.) right Lines drawn through R and V parallel to B C, E F, *&c.* must touch the Section, but this cannot be untill R and K, V and H Coincide, *ergo* K H is the Diameter.

COROLLARY VII.

A right Line bisecting any two parallel Lines within the Section is their Diameter, and the Diameter of all others parallel to them.

COROLLARY VIII.

The Diameter of a Parabola bisects no Line terminating in the Section, but its own Ordinates; for if any Line be drawn, then its own Diameter must bisect it, *ergo* (*Cor.* 3.) (this Diameter being parallel to that) that cannot bisect it.

COROLLARY IX.

(*See Fig.* 4, 5, 6.) In every Conick Section and in the opposite Sections, it is evident from the Genesis of Diameters, that Tangents drawn to touch the Section in C, B the extremities of the Ordinate (B C) to the Diameter I M, will meet in some point as I, in the same Diameter produced.

LEMMA IV.

In the Hyperbola and in the opposite Sections, if any two right Lines K N, S T, either both in the same, or one in each, or *one in one, and the other in both Sections* be produced (if need be) untill they intersect each other in E, and both the Asymptotes in R, V, M, L. I say.

$$\left.\begin{matrix} KM \times MN \\ LN \times MN \end{matrix}\right\} : RT \times TV :: KE \times EN : SE \times ET.$$

DEMONSTRATION.

Draw the right Line Y N X through the point N (where either of the right Lines intersects the Curve) parallel to the (other) S T and intersecting the Asymptotes in X and Y, then the Triangles L N Y, L E R; M N X, M E V are similar.

Whence $LE : LN :: ER : NY$

And $ME : MN :: EV : NX$.

Therefore $LE \times ME : LN \times MN :: ER \times EV : NY \times NX$.

And because $RS = TV$, and $KL = MN$.

Therefore $RE \times EV = RS \times ET + RS \times TV + SE \times ET + SE \times TV$

And $RT \times TV = RS \times TV + SE \times TV + ET \times \left\{\begin{matrix} TV \\ RS \end{matrix}\right.$

Whence $RE \times EV - RT \times TV = SE \times ET$, and $ER \times EV = SE \times ET + RT \times TV$

And in like manner $LE \times ME = KE \times EN + KM \times MN$.

And by substitution, the Analogy last found, will be

$$KE \times EN + KM \times MN : \left\{\begin{matrix} LN \times MN \\ KM \times MN \end{matrix}\right\} :: SE \times ET + RT \times TV : \left\{\begin{matrix} NY \times NX \\ RT \times TV \end{matrix}\right.$$

And by Division, $KE \times EN : KM \times MN :: SE \times ET : RT \times TV$.

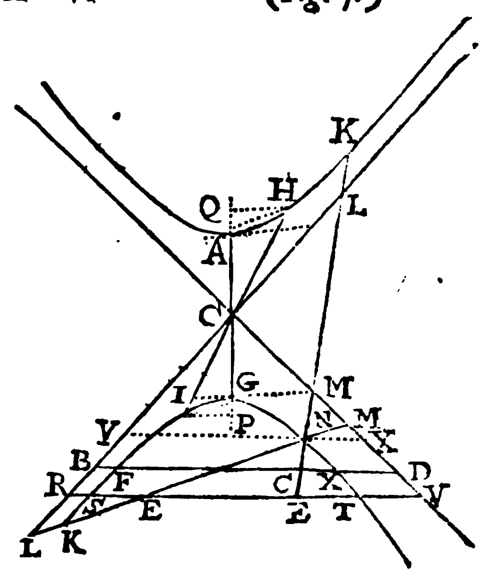
COROLLARY.

If K M pass through C the Center of the Assymptotes, then it will be $KE \times EN : KM q$ or $MN q :: SE \times ET : RT \times TV$.

LEMMA V. (*Fig.* 7.)

In the Hyperbola and in the opposite Sections; if the right Line (*Fig.* 7.) KN terminating in the same or *in the opposite Sections*, and produced (if need be) intersect any other two parallel Lines F X S T, terminating (either) in the same (or in the opposite Sections) in the points C, E, then, I say,

$KC \times CN : XC \times CF :: KE \times EN : ET \times ES.$



DEMONSTRATION.

Let the right Lines be produced until they intersect the Asymptotes in B, D, R, V, then by Lem. 4

$$KC \times CN : XC \times CF :: KM \times MN : BX \times XD.$$

$$\text{And } KE \times EN : SE \times ET :: KM \times MN : \begin{cases} RT \times TV \\ BX \times XD. \end{cases}$$

$$\text{Therefore } KC \times CN : XC \times CF :: KE \times EN : ET \times ES.$$

COROLLARY.

If K N bisect the parallel Lines F X, S T, then K O N (infinitely produced) is their (*Cor.* 7. *Lem.* 3.) Diameter, and then $KC \times CN : KE \times EN :: CX q : ET q$.

LEMMA. VI.

In the Parabola and Ellipsis, if the right Line F H terminating in the Section, intersect any other two parallel Lines L M, R O in the points E G, then I say.

$$GO \times GR : LE \times EM :: HG \times GF : HE \times EF.$$

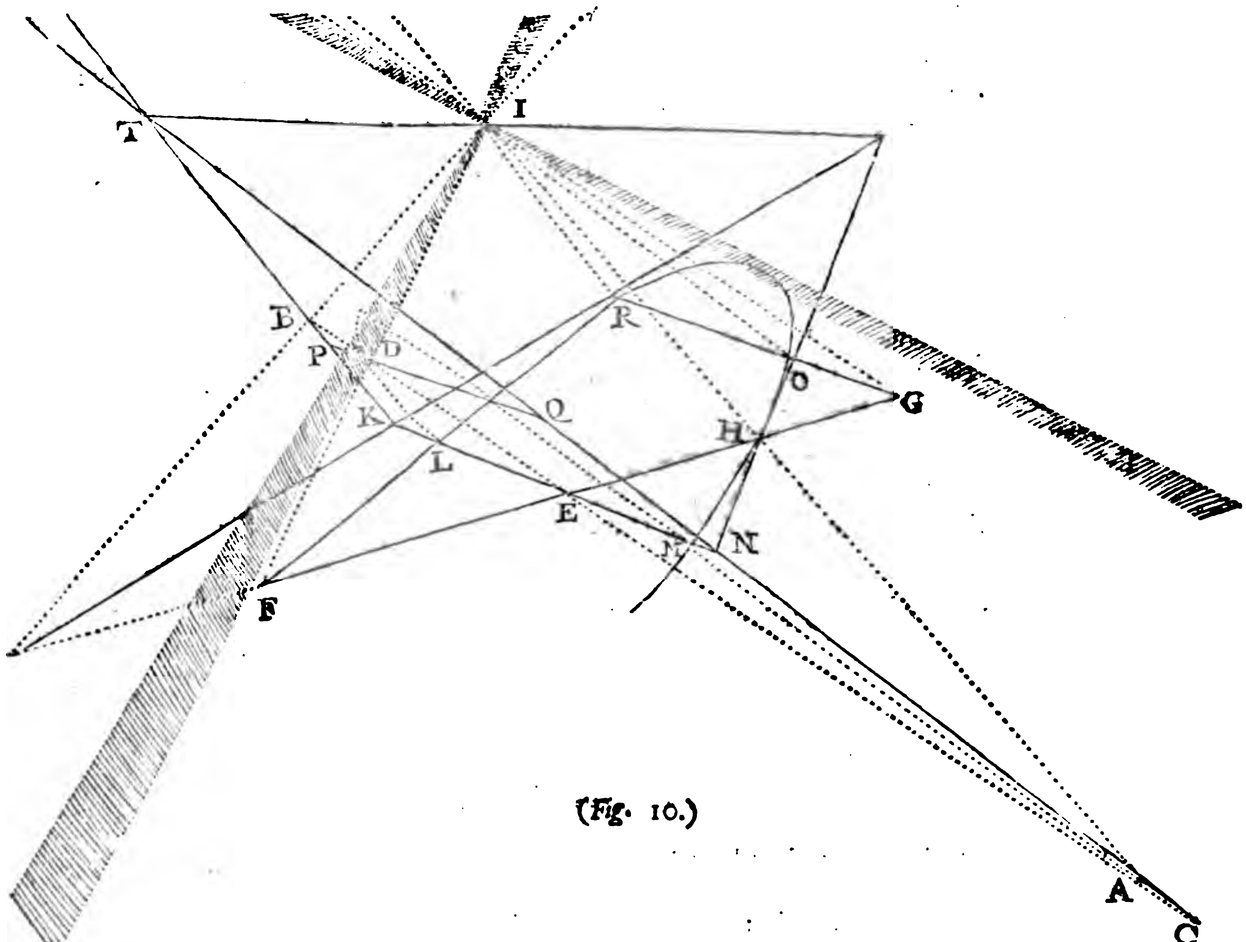
DEMONSTRATION.

Let the Ellipsis or Parabola be describ'd on the Surface of a (*Fig.* 10.) Cone, with the same Lines as in the present Figures.

Through either of the right Lines R O or L M, as R O, and I the Vertex of the Cone; suppose the Plain R I O to pass, cutting the Surface of the Cone in the right Lines I R, I O, in which let the Plains I R K, I O N touch the Cone, and their mutual Intersection I T, and suppose their common Sections with the Plain of the Section to be R K, O N.

Through L M draw a Plain parallel to the Plain R I O, and making the opposite Sections L D M A S, and cutting the Plains which touch the Cone, in the Lines T K, T N, which are therefore Asymptotes.

Through H F and I the Vertex of the Cone draw a Plain cutting the Surface of the Cone in I H A, I D F, the Plain R I O in I G, the Plain of the opposite Sections in A B C D E, and the Plain forming the Asymptotes in I B, I C.



(Fig. 10.)

The right Line AD is terminated either in one or in both the opposite Sections, and intersects the Asymptotes in B and C.

Through the point D in the Plain of the opposite Sections draw PDQ parallel to KLEMN, then because the Plains are parallel, and the right Lines PDQ, KLEMN parallel, the Triangles RGI and PDB, GIO and DCQ, HGI and HEA, FDE and FIG are similar, therefore.

IG : DE :: GF : FE, and IG : AE :: HG : HE.

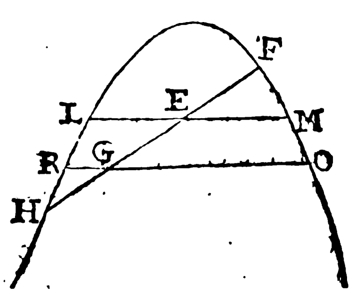
Therefore IGq : DE × AE :: GF × HG : FE × HE.

Again IG : GO :: CD : DQ, and IG : GR :: BD : DP.

Therefore IGq : GO × GR :: CD × BD : $\left\{\begin{matrix} DQ \times DP \\ KM \times MN \end{matrix}\right\}$ :: (*Lem.* 4.) AE × ED : LE × EM.

And by Alternation } IGq : AE × ED :: GO × GR : LE × EM :: GF × HG : FE × HE.

(*Fig.* 9.) COROLLARY. I.



In the Parabola, if the right Line FH intersect the Curve, only in F, and be produced infinitely towards H, that is if FH, be a Diameter, (*Cor.* 4. *Lem.* 3.) then GH and EH are equal, both being infinite, therefore FE : FG :: LE × EM : RG × GO.

And because the right Line (*See Fig.* 6.) HM bisects all the Ordinates EF, BC in L and M, therefore HL : HM :: LFq : MCq.

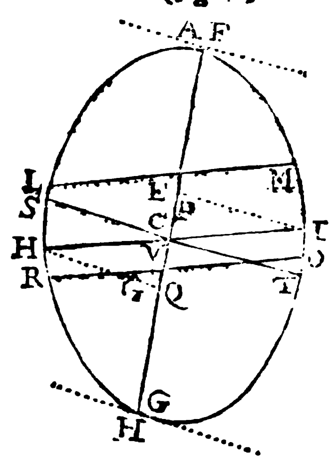
COROLLARY II.

And in the Ellipsis, if the Diameter (*See Fig.* 4.) HK bisect the Ordinates BC, EF, then it is HM × MK : HL × LK :: MCq : LFq. LEM-

LEMMA. VII.

(*See Fig.* 7 8.) In the Ellipsis and in the opposite Sections, any Line ICH passing through C the middle point of the Diameter AG, and terminating in the points I, H, is bisected in C.

(*Fig.* 8.)



DEMONSTRATION.

If ICH be an Ordinate to the Diameter AG, then 'tis evident it is bisected in C; but if not, through the points I, H, draw the Ordinates IP, HQ, to the Diameter AG, which being parallel, the Triangles CHQ, CIP will be similar; therefore by *Cor.* 2. *Lem.* 6. and *Cor. Lem.* 5. It will be.

$AP \times PG : AQ \times QG :: PIq : QHq :: PCq : QCq.$

Whence by Alternation and Composition in the Ellipsis and Division in the opposite Sections

$$\left.\begin{array}{c} AP \times PG + PCq \\ PCq - AP \times PG \\ CGq \end{array}\right\} : PCq :: \left\{\begin{array}{c} AQ \times QG + QCq \\ QCq - AQ \times QG \\ CGq \end{array}\right\} : QCq.$$

Whence $PC = QC$, and consequently $CI = CH$.

LEMMA VIII.

In the Hyperbola and in the opposite Sections, all the Diameters meet in the point C, where the Asymptotes meet; and all the determinate Diameters mutually bisect one another in the said point; and in the Ellipsis, all the Diameters (*See Fig.* 7. 8.) meet and mutually bisect one another in a certain common point C within the Section.

Because the determinate Diameters in the Hyperbola or the opposite Sections, connect the points of Contact of the parallel Tangents (by *Cor.* 6. *Lem.* 3.) therefore they all pass through the point C where the Asymptotes meet (by *Cor.* 7. *Lem.* 2.) and are therein bisected (by *Cor.* 3. *Lem.* 2.)

And in the Ellipsis if through C and V the middle points of the two Diameters AG, HI be drawn SCVT terminating either way in the Section, it will be bisected both in C and V (by *Lem.* 7.) therefore the points C and H coincide or they are the same.

DEFINITION. X.

The point C wherein all the Diameters meet is called the Center of the Ellipsis or the Center of the opposite Sections.

General CONSECTARIES.

I. Any right Line (*See Fig.* 7. 8.) AG passing through any point A in the Curve and the Center C is the Diameter of all the right Lines drawn in the Section parallel to the Tangent in A.

II. A Diameter bisects not any Line terminating in the Section (except in the Center) but its own Ordinates, for such a right Line would also be bisected by its own Diameter; now a Line cannot be bisected by two other right Lines, but in the point where these two Lines meet, which in this Case is the Center.

III. If one of the Ordinates as EF (*Fig.* 4.) pass through the Center R, then it will be $HRq : RFq :: HM \times MK : MCq$.

DEFINITION XI.

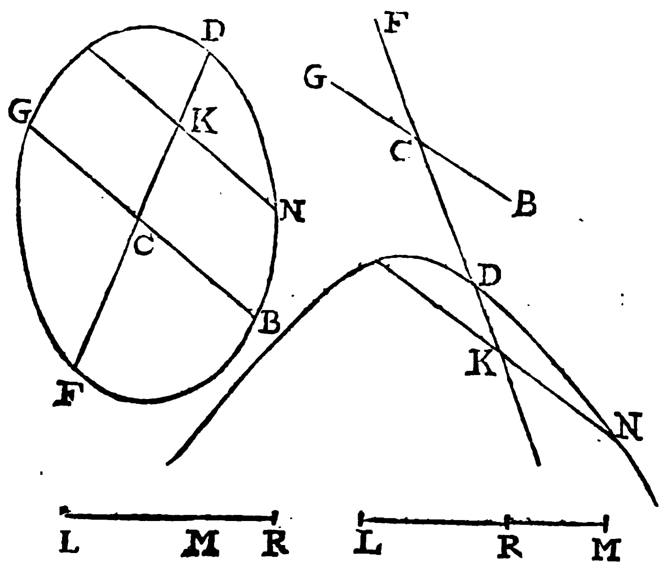
The Diameter ERL being parallel to the Tangents in H and K is called the conjugate Diameter to the Diameter HK, and because this Diameter in the Hyperbola is infinite, a Line drawn parallel to the same and touching the Curve in that point where

where the Transverse Diameter intersects the same, and bounded by the Asymptotes or a portion of that equal to this, is called the conjugate Diameter.

LEMMA IX.

In the Hyperbola and Ellipsis, let D F be any (Determinate) Diameter, its Vertex D and its opposite Vertex F; G B the conjugate Diameter, and K N any Ordinate applied to the Diameter D F; then let L R be a third proportional to F D the Transverse and G B the conjugate Diameter, and therein (produced in the Hyperbola) take the point M so that FD : DK :: LR : MR. I say KNq = DK × LM.

(*Fig.* 11. 12.) DEMONSTRATION.



Because FD : DK :: LR : MR, it will be by composition in the Hyperbola and Division in the Ellipsis.

$$\left.\begin{matrix}FD \pm DK \\ FK\end{matrix}\right\} DK :: \left\{\begin{matrix}LR \pm MR \\ LM\end{matrix}\right\} : MR$$

Whence FK × MR = DK × LM, and again, because FD : GB :: GB : LR it will be FD : LR :: FDq : GBq :: (their Subquadruples) CDq : CBq :: (*Cor.* 3. *After Definition* 10.) DK × FK : KNq :: (by Hypoth.) DK : MR :: DK × FK : MR × FK, ergo KNq = MR × FK = DK × LM.

DEFINITION XII.

The right Line L R is called the *Latus Rectum* or Parameter.

COROLLARY I.

In every Ellipsis, the Transverse and conjugate Diameters are two mean proportionals between their respective Parameters. For DF : GB :: GB : L (=to the parameter of DF) and by inversion GB : DF :: L : GB; again GB : DF :: DF : *l*, therefore L : GB :: DF : *l*.

COROLLARY II.

GBq = DF × Param. = to the Figure of the Diameter, and consequently CBq = $\frac{1}{4}$ the Figure of the Diameter.

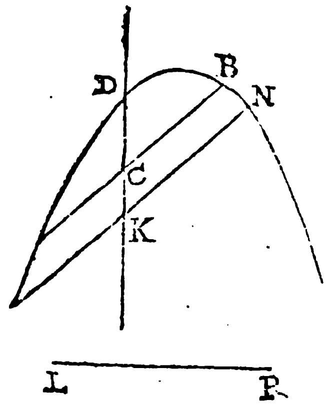
COROLLARY III.

DF : LR :: DK × FK : KNq, this is one of the Steps.

COROLLARY IV.

Because it is (by Supposition) DF : GB :: GB : LR therefore DF : LR :: DFq : GBq, and consequently DFq : GBq :: DK × FK : KNq.

LEMMA. X



In the Parabola, let D C be any Diameter, and B C an Ordinate to the same; then take L R a third proportional to the intercepted Diameter D C, and B C an Ordinate to the same; I say the Square of any Ordinate K N is equal to the Rectangle contained under L R, and D K the respective Abscissa, *viz.* KNq = LR × DK.

Forbecause DC : CB :: CB : LR.

Therefore DC × LR = CBq.

$$\left.\begin{matrix}(\textit{Cor. 1. Lem. 6.})\ CBq \\ \text{And}\quad DC \times LR\end{matrix}\right\} : KNq :: DC : DK :: DC \times LR : DK \times LR$$

Therefore DK × LR = KNq; universally.

FINIS.

A TREATISE OF FLUXIONS, OR AN INTRODUCTION TO Mathematical Philosophy.

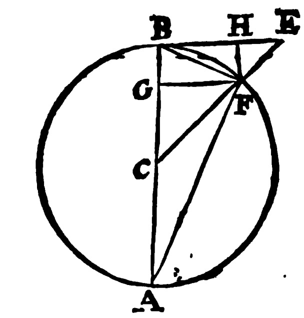
SECT. I.

The Nature of Fluxions.

1. MAGNITUDE is divisible in *infinitum*, and the Parts after this infinite Division, being infinitely little, are what Analysts call *Moments* or *Differences*; And if we consider Magnitude as Indeterminate and perpetually Increasing or Decreasing, then the infinitely little Increment or Decrement is call'd the *Fluxion* of that Magnitude or Quantity: And whether they be call'd *Moments*, *Differences* or *Fluxions*, they are still suppos'd to have the same Proportion to their Whole's; as a Finite Number has to an Infinite; or as a finite Space has to an infinite Space. Now those infinitely little Parts being extended, are again infinitely Divisible; and these infinitely little Parts of an infinitely little Part of a given Quantity, are by Geometers call'd *Infinitesimæ Infinitesimarum* or *Fluxions* of *Fluxions*. Again, one of those infinitely little Parts may be conceiv'd to be Divided into an infinite Number of Parts which are call'd *Third Fluxions*, &c.

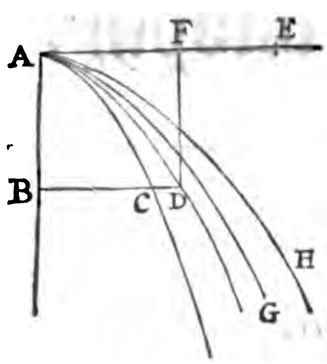
2. Because this Doctrine may seem hard to most Readers at first, I shall endeavour to prove that there are Quantities infinitely less than a given Quantity, which are also infinitely greater than another Quantity; and consequently, that if there be Quantities infinitely little, there are others infinitely less than they: And in a Word, that Quantity is not only divisible *in infinitum*, but that there is also an endless Progression of such infinite Divisions.

That there are Quantities infinitely less than an infinitely little Quantity may be prov'd thus. In the Circle ABF draw the Diameter AB, and let BF be a part of the Periphery infinitely little, then its Chord BF will also be infinitely little; that is, the Chord BF will have the same Proportion to AB as a finite Number has to an infinite. From F let fall the Perpendicular FG, and draw the Line AF; I say BG will be infinitely less than BF; for the Angle AFB in the Semicircle is a (*Prop* 31. *Elem.* 3.) Right-angle, and FG is Perpendicular to the Base AB, therefore the Triangles ABF, FBG are (*Prop.* 8. *Elem.* 6.) similar; and Consequently, (*Prop.* 4. *Elem.* 6.) AB : BF :: BF : BG. But BF is infinitely less than BA, therefore BG is infinitely less than BF; that is, a Quantity may be infinitely less than another Quantity infinitely little; or a Quantity infinitely less than one may be infinitely greater than another. Q. E. D.



It is evident from the nature of the Circle, that the Tangent of an Arch is greater than the Sine of that Arch; and that the Tangent of an Arch is greater, and its Sine less than the Arch it self: This being suppos'd, in the Circle ABF, whose Center C and Diameter AB, take the Arch BF infinitely little, BE the Tangent, FG the Sine, and BG the vers'd Sine thereof. Draw FH Parallel to AB, then HE will be the difference between the Right-sine of the Arch BF and its Tangent. Now the Chord BF is infinitely little in Comparison of BC, and BG is infinitely little in respect of BF; and again, HE will be infinitely less than BG, because the Triangles CGF, FHE are Similar, and CG : GF :: GB = FH : HE. Hence 'tis evident that if the Quantities CG, BF, BG, HE be given, then CG is infinitely greater than BF, and *infinito*-infinitely greater than BG, and *infinito-infinito*-infinitely greater than HE. And thus, from the bare Consideration of the Circle we have arriv'd to *Third Fluxions*.

And for a clearer Illustration of this Doctrine, take the following Example from the Incomparable Mr. *Newton*, which I find Demonstrated by a late Ingenious Author thus: Let AC be a common Parabola, AB its Axis, and AE a line touching the same in the principle Vertex A. Then it is evident from the nature of the Curve, that the Angle of Contact FAC is less than any rectilineal Angle. To the same Axis AB and Vertex A, describe a Parabola of another kind, *v. g.* a cubical Parabola AD, whose Ordinates encrease in a subtriplicate Proportion of the intercepted Diameters; I say the Angle of Contact FAD will be infinitely less than the Angle of Contact FAC. Or which is the same thing, it is impossible so to diminish the Angle of Contact (of the Apollonian Parabola AC,) FAC, that it shall be equal to or less than the Angle of Contact FAD, let the Parameter of AC be never so great. Which is thus Demonstrated.



Let the Parameter of AC be $= a$; and the Parameter of the Cubical Parabola AD $= b$. Take the Point E in the Tangent line: So that, $a : b :: b :$ AE, and then $a \times$ AE $= b^2$. Through F the middle Point between A and E draw FD Parallel to the Axis, and intersecting the Curve AD in D, draw DCB Parallel to the Tangent-line AE, then suppose BD $= z$, BC $= y$, and AB $= x$; then is $ax = y^2$, and $b^2 x = z^3$, and $\frac{y^2}{a} = x = \frac{z^3}{b^2}$. Therefore $b^2 y^2 = a z^3$; and reducing this Equation to an Analogy, it will be $b^2 : az :: z^2 : y^2$. That is, $a \times$ AE : $a \times$ BD ($= a \times$ AF) :: BDq : BCq; but $a \times$ AE is greater than $a \times$ AF (by supposition) therefore BDq is greater than BCq, and BD is greater than BC : Therefore the Point C in the Apollonian Parabola falls within the Cubical Parabola AD: What we have thus Demonstrated of BC holds true in all the Ordinates of the Parabola AC, so long as they are less than AE; and therefore the Portion of the Parabola AC at the Vertex A falls within the Parabola AD. Therefore the Angle of Contact DAF is infinitely less than the Angle of Contact CAF, because this Angle being infinitely diminish'd, is still greater than that.

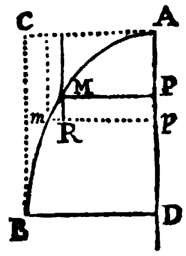
In like manner if the Curve AG be describ'd, whose Ordinates increase in a subquadruplicate Proportion of the Intercepted Diameters, the Angle FAG, might be Demon-

demonstrated to be infinitely less than the Angle F A D, which is infinitely less than the Angle F A C, which is infinitely less than any rectilineal Angle. Again, if the Curve A H be describ'd, whose Ordinates Increase in a subquintuplicate Proportion of the Intercepted Diameters, then the Angle of Contact F A H might be prov'd to be infinitely less than the Angle of Contact F A G, *&c.*

And thus we have a series of Angles of Contact, which might be continu'd in *infinitum*, and every one of them is infinitely greater than that which immediately follows.

And thus I think I have briefly Demonstrated that there are degrees of Infinity, and that the same Quantity may be Consider'd as infinitely little and infinitely great in different Respects. And having thus premis'd a general Account of *Fluxions*, I shall now come to more particular Instances.

3. By the Doctrine of *Fluxions*, we understand the *Arithmetick* of infinitely small Increments or Decrements of Indeterminate or variable Quantities, and by such Quantities we understand those which in the Generation of (*v. g.*) a Curve by local Motion perpetually increase or decrease, which are therefore more properly call'd (by the Incomparable Mr. *Newton*) *Flowing Quantities.* For Instance, let A M B represent the Curve of a Parabola, Hyperbola, Circle, Ellipsis, or any other Geometrical Figure. The Parameter of any Figure is a determinate and invariable Quantity, and so is the Transverse Diameter of an Hyperbola, Ellipse or Circle: And consequently in the Arithmetick of Fluxions they remain always the same (*i. e.* They have one and the same determinate Value) throughout all the Work.



But if we suppose a Line as P M, to move with one Extremity upon the Diameter A D (and with the other to touch the Curve) from A to D, always Parallel to it self, 'tis evident that according as it descends or recedes from the Vertex A, and comes nearer and nearer to D, it increases in length, as does also the portion of the Diameter intercepted between the same and the Vertex A; Thus *pm* is greater than P M and the Intercepted Diameter A *p* is greater than A P, and the portion of the Curve A M *m* is greater than A M. Now if the Line M R be drawn Parallel to the Axis A P, 'tis manifest, that A P being the Abscissa, P M is the Ordinate; and again, if A P + P *p* be the Abscissa, then the Corresponding Ordinate is P M + R *m* = *pm.* So that if the Increment of the Abscissa be suppos'd = P *p*, then the Increment of the Ordinate is = R *m*; and the Increment of the Curve is = M *m*. And if P *p* be suppos'd to be infinitely little, then P *p*, R *m*, M *m* are call'd the Moments, Differentials, infinitely little Increments, or (more properly, as I have intimated before) Fluxions of the intercepted Diameter A P, Ordinate P M, and the Curve A M respectively.

4. All Surfaces may be consider'd, as Compos'd of an infinite Number of Parallel-lines, Streight or Crooked, and those Lines (being streight) may be suppos'd Parallelograms of an infinitely little height, and may be call'd the *Elementa* of the Surface.

For instance, in the Parabolic Space A M B D; Imagine the Axis or height A D to be divided into an infinite Number of Equal Parts, and suppose the Ordinates P M, *p m* to be drawn through every Point of the Axis, then 'tis evident that they will Occupy the whole Parabolic Space A M B D. And if we multiply every one of the Ordinates P M by an infinitely little Part of the Axis P *p*, there will be produc'd the infinitely little Surfaces or Parallelograms M *p* (because the Ordinates M P, *m p* being infinitely near each other, the Triangle M R *m* is infinitely little in respect of the Parallelogram M *p*, and consequently may be rejected.) Now as P *p*, R *m*, M *m*, are the Fluxions of the Abscissa, Ordinate and Curve respectively: So the infinitely little Parallelogram M *p* is the Moment, infinitely little Increment or Fluxion of the Area or Parabolic Space A M P, and the Summ of all those Parallelograms is equal to the said Parabolic Space A M B D.

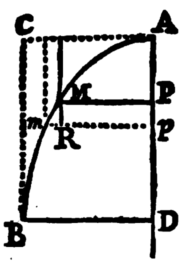
5. And if we would Contemplate a Solid, we may consider it as Compos'd of an infinite Number of Parallel Plains or Surfaces comprehended by streight or Curve-lines. And those Surfaces or Plains may be taken for Solids whose heights are equal and infinitely little. So that Plains or infinitely thin Solids may very properly be call'd the *Elementa* of Bodies.

Thus

Thus in the Parabolical Conoid, which is a Solid form'd by the Revolution of the Semi-parabola AMBD about the Axis AD; 'tis evident that every Ordinate PM, *pm*, describes a Circular Surface in one whole Revolution, which being multiplied by P*p* an infinitely little part of the Axis, there will be generated an infinitely thin Solid, which is the Fluxion or Element of the given Conoid, and the Summ of all the said infinitely thin Solids, (the Solids generated by the Revolution of the Triangles MR*m* being infinitely little, and consequently vanishing when compar'd with the others) is equal to, or Constitutes the Parabolical Conoid.

6. And thus we consider Quantities as indeterminate and variable, and perpetually increasing or decreasing by local Motion. But we must take great heed, not to consider the Fluxions, or Increments, or Decrements as finite Quantities: For being once Finite they are no longer Moments or Fluxions, it being in a manner repugnant to their perpetual Increment or Decrement. They are the very first Principles (*Principia jamjam nascentia*) of finite Magnitudes. *Nor is it necessary that we should so much consider the Magnitude of those* Moments, *as the Proportions between them as they begin to be.* And therefore it is the same thing for our purpose, if instead of the Moments themselves, we consider the Velocities of the Increments or Decrements, or even finite Quantities proportional to the said Velocities.

Thus in the Parabola AMB, we draw any two Ordinates PM, *pm* infinitely near, or at least suppose them to be so; and having drawn MR Parallel to the Axis AD, we call MR the infinitely little Increment of the Abscissa AP, and R*m* that of the Ordinate, and M*m* that of the Curve, which Lines more properly denote the Proportion between the respective Increments of the Abscissa, Ordinate, *&c.* or the Proportion between their respective Velocities, which they have when they begin to Contribute to the Augmentation of the said Abscissa, Ordinate or Curve respectively.



7. And as the Lines P*p*, R*m*, M*m* are call'd *Fluxions*, so the finite Quantities AP, PM, AM are call'd *Flowing Quantities* (which are the same with (*Art.* 3.) indeterminate or variable Quantities,) and I chuse to use these Names in the ensuing Treatise, because the Generation of Figures and Quantity by continu'd Motion is more Natural and more easily conceiv'd, and the Schemes in this Method are more simple than in that of Parts. But when the (*Art.* 6.) Proportion of Fluxions is to be investigated, or any way conduce to the Solution of a Problem, then I call the indefinite little Lines P*p*, R*m*, M*m* the Fluxions of the respective flowing Quantities AP, PM, AM, tho' being but Finite Quantities only, they do but represent the Proportions between the respective Fluxions of those flowing Quantities.

8. And 'tis manifest that in this Method we consider all Curve-lines, as Compos'd of an infinite Number of infinitely little Streight-lines, or as Polygons of an infinite Number of Sides. Thus the Particle of the Curve M*m*, being suppos'd infinitely little is consider'd as a Straight-line. And then by considering the Fluxions of finite Quantities, and their mutal Relations in infinitely little Streight-lines, we come to discover the Relations and Proportions between the given Quantities themselves. For all Curves being Polygons of an infinite Number of Sides, 'tis evident that one differs from another in nothing else but in the Angles Comprehended between those infinitely little Sides; and consequently to find the Curvature of any Line, is the same thing as to Determine the Position of the said Sides. But this will appear more plain afterwards, when we come to shew how to draw Tangents to all sorts of Curves, *&c.*

9. And as in Specious Algebra, all sorts of Quantities are denoted by Letters, so here to avoid Confusion, and to ease the Memory as much as possible, we always denote the Abscissa or intercepted Diameter of any Curve, as AP by the Letter x, the Ordinate PM by the Letter y, and the Curve AM by the Letter z, then the Quantities x, y, z, are call'd (*Art.* 3. and 8.) flowing Quantities, and the Fluxions P*p*, R*m*, M*m* are represented by the Letters representing the respective flowing Quantities, with Pricks over them, in this manner, P*p* $= \dot{x}$, R*m* $= \dot{y}$, M*m* $= \dot{z}$.

And

And if the flowing Quantity be expreſs'd Fraction-wiſe *v.g.* $\frac{yy}{b-x}$, then the Fluxion thereof is denoted thus $\dot{\overline{\frac{yy}{b-x}}}$, or if the flowing Quantity be a Surd as $\sqrt{aa-xx}$, Then the Fluxion thereof is written thus $\dot{\overline{\sqrt{aa-xx}}}$.

Note, that generally (unleſs it be otherwiſe expreſs'd) the laſt Letters of the Alphabet denote flowing Quantities, as x, y, z, and the firſt Letters, as a, b, c, d, always denote invariable Quantities.

SECT. II.

The Algorithm or Arithmetick of Fluxions.

PROPOSITION I.

To find the Fluxion of one or more ſimple Quantities Connected with the Signs $+$ *or* $-$.

10. LET it be requir'd to find the Fluxion of x. Suppoſe x to repreſent a Line, as AP, (*Fig. in Art.* 7.) then 'tis evident that the Fluxion of x is $Pp = \dot{x}$.

Again, let it be requir'd to find the Fluxion of $a + x + y - z$. If we ſuppoſe x to be augmented by an infinitely little part $\dot{x}$, that is if x become $= x + \dot{x}$, then y will become $= y + \dot{y}$, and $z = z + \dot{z}$, and becauſe a is an (*Art.* 9) invariable Quantity, it remains always the ſame, therefore the Quantity propos'd $a + x + y - z$, will become $a + x + \dot{x} + y + \dot{y} - z - \dot{z}$, and the Fluxion of that given Quantity, or the exceſs of this above that, is $\dot{x} + \dot{y} - \dot{z}$, and hence ariſes this,

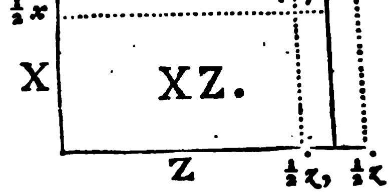
RULE I.

11. For ſimple Quantities Connected with the Signs $+$ or $-$. Take the Fluxion of every one of the Quantities propos'd, and Connect them with the Signs of their reſpective flowing Quantities, and the Summ will be the Fluxion requir'd.

PROP. II.

If X *be multiplied into* Z, *and if the Product be* XZ. *I deſire to know the Fluxion of the Rectangle* XZ; *That is, ſuppoſing the Sides* X *and* Z *to be augmented or diminiſh'd each by an infinitely little Quantity, I would know how much the new Rectangle exceeds or is exceeded by the given Rectangle* XZ.

12. If X be multiplied by Z the Rectangle is XZ: Now ſuppoſe half the infinitely little increment of X to be $\frac{1}{2}\dot{x}$, and half the Fluxion or infinitely little Increment of Z to be $\frac{1}{2}\dot{z}$; 'tis evident that the Fluxion or Increment of the Rectangle is $= X \times \frac{1}{2}\dot{z} + Z \times \frac{1}{2}\dot{x} + r$. Again, ſuppoſe half the infinitely little Decrement of X and Z to be $\frac{1}{2}\dot{x}$ and $\frac{1}{2}\dot{z}$ reſpectively, then the Fluxion or Decrement of the Rectangle XZ is $= X \times \frac{1}{2}\dot{z} + Z \times \frac{1}{2}\dot{x} - r$, and adding the Increment and Decrement into one Summ, we have $X \times \dot{z} + Z \times \dot{x}$ for the Fluxion of Rectangle XZ. Q. E. I.



The Incomparable Mr. *Newton*, shews how to find the Fluxion of any Rectangle in a manner nothing differing from this, only express'd another way. For instance, to find the Fluxion of the Rectangle $X \times Z$: He supposes the Fluxion of X and Z to be $\dot{x}$ and $\dot{z}$. Then,

$$\begin{array}{ll} X+\frac{1}{2}\dot{x} & X-\frac{1}{2}\dot{x} \\ Z+\frac{1}{2}\dot{z} & Z-\frac{1}{2}\dot{z} \\ \hline XZ+\frac{1}{2}\dot{x}Z & XZ-\frac{1}{2}\dot{x}Z \\ \quad +\frac{1}{2}\dot{z}X+\frac{1}{4}\dot{x}\dot{z} & \quad -\frac{1}{2}\dot{z}X+\frac{1}{4}\dot{x}\dot{z} \end{array}$$

The $\square$ $XZ+\frac{1}{2}$ the Moment $\}$ $= XZ+\frac{1}{2}\dot{x}Z+\frac{1}{2}\dot{z}X+\frac{1}{4}\dot{x}\dot{z}$ $\quad XZ-\frac{1}{2}\dot{x}Z-\frac{1}{2}\dot{z}X+\frac{1}{4}\dot{x}\dot{z}$

The $\square$ $XZ-\frac{1}{2}$ the Moment $\}$ $= XZ-\frac{1}{2}\dot{x}Z-\frac{1}{2}\dot{z}X+\frac{1}{4}\dot{x}\dot{z}$

And the difference $\dot{x}Z+\dot{z}X$ is = to the Fluxion of the Rectang. XZ.

There is yet another way to find the Fluxion of any Rectangle XZ; which is thus, the Fluxions of the Sides X and Z are (*Art.* 10.) $\dot{x}$ and $\dot{z}$, and therefore the Sides of the Rectangle become $X+\dot{x}$ and $Z+\dot{z}$, and the Rectangle it self is $XZ+\dot{x}Z+\dot{z}X+\dot{z}\dot{x}$, from which subtracting the given Rectangle XZ, the remainder $\dot{x}Z+\dot{z}X$ (the term $\dot{z}\dot{x}$ being infinitely little in comparison of either of these) is Fluxion the of the Rectangle XZ. Q. E. I.

PROP. III.

To find the Fluxion of the Product of any Number of flowing Quantities multiplied into one another.

13. Let it be requir'd to find the Fluxion of xyz; this may be done several ways, as 1°. Suppose xy to be one Quantity, and z another, then xyz may be consider'd as a Rectangle. Now the Fluxion of xy is (*Art.* 12.) $\dot{x}y+\dot{y}x$, which being multiplied by the other side z, the product is $\dot{x}zy+\dot{y}zx$, and the Fluxion of the side z is (*Art.* 10.) $\dot{z}$, by which multiplying the other side xy, the product is $\dot{z}xy$, and adding both Products together we have $\dot{x}zy+\dot{y}zx+\dot{z}xy$, which is the Fluxion of the given Product or Quantity xyz. Q. E. I.

Or 2°. The Fluxion of xyz may be found thus; for y put $y+\dot{y}$, for x put $x+\dot{x}$ and for z put $z+\dot{z}$, then $\overline{y+\dot{y}}\times\overline{x+\dot{x}}\times\overline{z+\dot{z}}$ will be $= xyz+\dot{x}zy+\dot{y}zx+\dot{z}xy+\dot{x}\dot{z}y+\dot{x}\dot{y}z+\dot{z}\dot{y}x+\dot{x}\dot{z}\dot{y}$, from which subtracting the given Product or Quantity xyz, the Remainder $\dot{x}zy+\dot{y}zx+\dot{z}xy$ (rejecting all those Terms that follow as being incomparably less than any of these) is the Fluxion or instantaneous Increment of the given Quantity xyz. Q. E. I.

And if it be requir'd to find the Fluxion of $xyzu$, I take xyz for one Quantity, and taking the (*Art.* 13.) Fluxion thereof, *viz.* $\dot{x}yz+\dot{y}xz+\dot{z}xy$. I multiply the same by the other Term u, and the Product is $\dot{x}yzu+\dot{y}xzu+\dot{z}xyu$: Then I multiply the Fluxion of u, *viz.* $\dot{u}$ by the other Term xzy and the Product is $\dot{u}xyz$; lastly, I add both Products together, and then the Summ $\dot{x}yzu+\dot{y}xzu+\dot{z}xyu+\dot{u}xyz$ is the Fluxion of the given Quantity $xyzu$. And hence arises,

RULE

RULE II.

To find the Fluxions of any Number of flowing Quantities multiplied into one another.

Multiply the Fluxion of every particular Quantity by the Product of all the others, then the Summ of all those Products is the Fluxion requir'd.

Thus the Fluxion of bx is $\dot{b}x + b\dot{x} =$ (*Art.* 3. and 9.) $b\dot{x}$. And the Fluxion of $\overline{b+z} \times \overline{c-x}$ is $= c\dot{z} - x\dot{z} - b\dot{x} - z\dot{x}$. And the Fluxion of $x \times \overline{a+y}$ is $= a\dot{x} + y\dot{x} + x\dot{y}$.

And to find the Fluxion of $xy + zu$, The Fluxion of the Term xy is (*Art.* 12.) $\dot{x}y + \dot{y}x$, and the Fluxion of the other Term zu is $\dot{z}u + \dot{u}z$, and consequently the Fluxion of $xy + zu$ is $= \dot{x}y + \dot{y}x + \dot{z}u + \dot{u}z$.

And the Fluxion of $ax + bxy + czu$ is $= a\dot{x} + by\dot{x} + bx\dot{y} + cu\dot{z} + cz\dot{u}$.

PROP. IV.

To find the Fluxion of any Fraction.

14. Let it be requir'd to find the Fluxion of $\frac{x}{y}$. Suppose $\frac{x}{y} = z$, then is $x = yz$. Now it is evident that as these variable Quantities are always equal between themselves, whither they be suppos'd to increase or decrease, their Fluxions must be so too, and therefore $\dot{x} = y\dot{z} + z\dot{y}$, and $\dot{x} - z\dot{y} = y\dot{z}$, and dividing by y, $\frac{\dot{x} - z\dot{y}}{y} = \dot{z} =$ (by putting $\frac{x}{y} = z$) $\frac{y\dot{x} - x\dot{y}}{yy}$. But z being $= \frac{x}{y}$, therefore $\dot{z} =$ to the Fluxion of $\frac{x}{y}$, and consequently, $\frac{y\dot{x} - x\dot{y}}{yy}$ $(= \dot{z})$ is equal to the Fluxion of the Fraction $\frac{x}{y}$.

Again, Let it be propos'd to find the Fluxion of this Fraction $\frac{x}{a+x}$. Suppose $\frac{x}{a+x} = z$, then is $x = az + xz$, and $\dot{x} = a\dot{z} + z\dot{x} + x\dot{z}$, and by Transposition $\dot{x} - z\dot{x} = a\dot{z} + x\dot{z}$, and by Division $\frac{\dot{x} - z\dot{x}}{a+x} = \dot{z}$, and Substituting $\frac{x}{a+x}$ for z in the Equation, we have $\frac{a\dot{x} + x\dot{x} - x\dot{x}}{aa + 2ax + xx} = \dot{z}$, that is $\frac{a\dot{x}}{aa + 2ax + xx}$ $(= \dot{z}) =$ to the Fluxion of the given Fraction $\frac{x}{a+x}$.

And if it be requir'd to find the Fluxion of $\frac{b}{y}$ put $\frac{b}{y} = z$, then is $b = yz$, and (the Fluxion of b being (*Art.* 9.) $= 0$) $0 = z\dot{y} + y\dot{z}$, and by Transposition $-z\dot{y} = y\dot{z}$, and $\frac{-z\dot{y}}{y} = \dot{z}$, and by restitution, putting $\frac{b}{y}$ for z, $\frac{-b\dot{y}}{yy}$ $(= \dot{z}) =$ to the Fluxion of the given Fraction $\frac{b}{y}$. And hence we have,

RULE III.

To find the Fluxion of any Fraction.

Multiply the Fluxion of the Numerator by the Denominator, and after it place (with the Sign $-$) the Fluxion of the Denominator multiplied into the Numerator: And divide the whole by the Square of the Denominator. So shall you have the Fluxion of the given Fraction.

Thus

Thus the Fluxion of $\frac{ax}{b+x}$ is $= \frac{ab\dot{x} + ax\dot{x} - ax\dot{x}}{bb + 2bx + xx} = \frac{ab\dot{x}}{bb + 2bx + xx}$

And the Fluxion of $\frac{a}{xy}$ is $= \frac{-ay\dot{x} - ax\dot{y}}{xxyy}$ and that of $\frac{a}{x+y}$ is $\frac{-a\dot{x} - a\dot{y}}{xx + 2xy + yy}$

Before we can proceed farther to find the Fluxions of Powers, it will be necessary to explain the Analogy between Powers and their Exponents, which I shall do in the following

LEMMA I.

If a Rank of Number be in a Geometrical Progression, and if the first Term be Unity and the second any Quantity as x, and if under every Term its own Exponent be plac'd, 'tis evident that those Exponents will form an Arithmetical Progression.

15. For instance, Geom. Progression $1,\ x,\ xx,\ xxx,\ xxxx,\ x^5,\ x^6,\ x^7,$ &c.
Arithmetical Progression $0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,$ &c.

And if the Terms of the Geometrical Progression be continu'd downwards from Unity, and those of the Arithmetical downwards from Nothing; The Terms of this Progression will be the Exponents of the respective Terms of that. Thus the Exponent of $\frac{1}{x}$ will be -1, and that of $\frac{1}{xx}$ will be -2, &c. as is evident in these *Series*.

Geometrical Progression $x,\ 1,\ \frac{1}{x},\ \frac{1}{xx},\ \frac{1}{x^3},\ \frac{1}{x^4},\ \frac{1}{x^5},$ &c.

Arithmetical Progression $1,\ 0,\ -1,\ -2,\ -3,\ -4,\ -5,$ &c.

Or if we suppose the first Term of a Geometrical Series to be (*v. g.*) $\frac{1}{xxxx}$ and that every Term is produc'd by multiplying the preceeding Term by x, then the Series will be,

$$\frac{1}{xxxx},\ \frac{1}{xxx},\ \frac{1}{xx},\ \frac{1}{x},\ 1,\ x,\ xx,\ xxx,\ xxxx.$$

And the Corresponding Arithmetical Series (the common difference by which the Terms rise being 1) will be.

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

And if the Geometrical Series be express'd by help of these Exponents, it will stand thus.

$$x^{-4},\ x^{-3},\ x^{-2},\ x^{-1},\ x^{0},\ x^{1},\ x^{2},\ x^{3},\ x^{4}.$$

Whence it is evident that the Exponents of perfect Powers Ascending, are positive Numbers, and those of perfect Powers Descending, are Negative Numbers.

But if it so happen that the Exponent of the Power is not a whole Number but a Broken; that is, if the Power be any intermediate between the Root and the Square or the Square and the Cube, &c. then to find the Exponent thereof, we must take the Corresponding Number in the Arithmetical Series.

Thus the Exponent of $\sqrt[2]{x}$ is $\frac{1}{2}$, because as $\sqrt[2]{x}$ is a mean Proportional between 1 and x in the Geometrical Series: So $\frac{1}{2}$ is an Arithmetical Mean between their Exponents 0 and 1.

And the Exponent of $\sqrt[3]{x}$ is $\frac{1}{3}$: Because as $\sqrt[3]{x}$ is the first of two mean Proportionals between 1 and x; So $\frac{1}{3}$ is the first of two Arithmetical Means between their Exponents 0 and 1, and for the like reason, the Exponent of $\sqrt[5]{x^4}$ is $\frac{4}{5}$; and that of $\sqrt[2]{x^3}$ is $\frac{3}{2}$.

If

If there be three mean Proportionals between 1 and x, Then the Geometrical Series will stand thus,

$$1,\ \sqrt[4]{x},\ \sqrt[4]{xx},\ \sqrt[4]{xxx},\ x,\ \sqrt[4]{xxxxx},\ \sqrt[4]{xxxxxx},\ \sqrt[4]{xxxxxxx},\ xx,\ \&c.$$

And the Corresponding Arithmetical Series will be,

$$0,\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ 1,\ \frac{5}{4},\ \frac{6}{4},\ \frac{7}{4},\ 2,\ \&c.$$

And contracting the Terms of the Geometrical Series by help of the Exponents of the Powers to which the Letter x is actually rais'd, it will then stand in this Form,

$$1,\ \sqrt[4]{x^1},\ \sqrt[4]{x^2},\ \sqrt[4]{x^3},\ x^1,\ \sqrt[4]{x^5},\ \sqrt[4]{x^6},\ \sqrt[4]{x^7},\ x^2,\ \&c.$$

Or (transferring 4 out of $\sqrt{}$, and making the same a common Denominator to the Exponents of x) thus,

$$1,\ x^{\frac{1}{4}},\ x^{\frac{2}{4}},\ x^{\frac{3}{4}},\ x^1,\ x^{\frac{5}{4}},\ x^{\frac{6}{4}},\ x^{\frac{7}{4}},\ x^2,\ \&c.$$

Whence it is manifest that, *v. g.* $x^{\frac{1}{4}}$ is the first of three mean Proportionals between 1 and x, and that the Exponent thereof $(\frac{1}{4})$ is the same with the respective Term $(\frac{1}{4},)$ in the Arithmetical Series; which is also, the first of three Arithmetical mean Proportionals, between 0 and 1.

And in like manner in the Negative Series, the Exponent of $\frac{1}{\sqrt[3]{x^8}}$ is $-\frac{8}{3}$; and the Exponent of $\frac{1}{\sqrt[3]{x^5}}$ is $-\frac{5}{3}$, and that of $\frac{1}{\sqrt[3]{x^7}}$ is $-\frac{7}{3}$.

Thus if all the Terms of the Geometrical Series be less than unity, as in the first part of the third Series, and if between every two Terms there be two mean Proportionals, then we shall have this Series,

$$\frac{1}{xxxx},\ \frac{1}{\sqrt[3]{x^{11}}},\ \frac{1}{\sqrt[3]{x^{10}}},\ \frac{1}{xxx},\ \frac{1}{\sqrt[3]{x^8}},\ \frac{1}{\sqrt[3]{x^7}},\ \frac{1}{xx},\ \frac{1}{\sqrt[3]{x^5}},\ \frac{1}{\sqrt[3]{x^4}},\ \frac{1}{x}.\ \&c.$$

And the Corresponding Arithmetical Series will be,

$$-4,\ -\tfrac{11}{3},\ -\tfrac{10}{3},\ -3,\ -\tfrac{8}{3},\ -\tfrac{7}{3},\ -2,\ -\tfrac{5}{3},\ -\tfrac{4}{3},\ -1.\ \&c.$$

And consequently the Geometrical Series may be written thus,

$$x^{-4},\ x^{-\frac{11}{3}},\ x^{-\frac{10}{3}},\ x^{-3},\ x^{-\frac{8}{3}},\ x^{-\frac{7}{3}},\ x^{-2},\ x^{-\frac{5}{3}},\ x^{-\frac{4}{3}},\ x^{-1}.\ \&c.$$

The reason of this will farther appear, if we consider these or such like Series's.

Geometrical Progression $1,\ \sqrt{x},\ x.\ 1,\ \sqrt[3]{x},\ \sqrt[3]{xx},\ x.\ 1,\ \sqrt[4]{x^1},\ \sqrt[4]{x^2},\ \sqrt[4]{x^3},\ x.$

Arithmetical Progression $0,\ \frac{1}{2},\ 1.\ 0,\ \frac{1}{3},\ \frac{2}{3},\ 1.\ 0,\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ 1.$

Geometrical Progression, $\frac{1}{x},\ \frac{1}{\sqrt{x^3}},\ \frac{1}{xx}.\ \frac{1}{x},\ \frac{1}{\sqrt[3]{x^4}},\ \frac{1}{\sqrt[3]{x^5}},\ \frac{1}{xx}.$

Arithmetical Progression, $-1,\ -\frac{3}{2},\ -2.\ -1,\ -\frac{4}{3},\ -\frac{5}{3},\ -2.$

Whence it is evident that a Geometrical Series *v. g.*

$$\frac{1}{xxxx},\ \frac{1}{\sqrt[2]{xxxxxxx}},\ \frac{1}{xxx},\ \frac{1}{\sqrt[2]{xxxxx}},\ \frac{1}{xx},\ \frac{1}{\sqrt[2]{xxx}},\ \frac{1}{x},\ \frac{1}{\sqrt[2]{x}},\ 1,\ \sqrt[2]{x},\ x,\ \&c.$$

May also be express'd thus,

$$\frac{1}{x^4},\ \frac{1}{\sqrt[2]{x^7}},\ \frac{1}{x^3},\ \frac{1}{\sqrt[2]{x^5}},\ \frac{1}{x^2},\ \frac{1}{\sqrt[2]{x^3}},\ \frac{1}{x^1},\ \frac{1}{\sqrt[2]{x^1}},\ 1,\ \sqrt[2]{x^1},\ x^1,\ \&c.$$

Or thus,

$$\frac{1}{x^4},\ \frac{1}{x^{\frac{7}{2}}},\ \frac{1}{x^3},\ \frac{1}{x^{\frac{5}{2}}},\ \frac{1}{x^2},\ \frac{1}{x^{\frac{3}{2}}},\ \frac{1}{x^1},\ \frac{1}{x^{\frac{1}{2}}},\ 1,\ x^{\frac{1}{2}},\ x^1,\ \&c.$$

Or thus,

$$x^{-4},\ x^{-\frac{7}{2}},\ x^{-3},\ x^{-\frac{5}{2}},\ x^{-2},\ x^{-\frac{3}{2}},\ x^{-1},\ x^{-\frac{1}{2}},\ x^0,\ x^{\frac{1}{2}},\ x^1,\ \&c.$$

And which way soever these Terms be written, they must be read as they are Express'd in the first or second Rank. Thus, $x^{-\frac{7}{2}}$ is one divided by the Square Root of the seventh Power of x, and $x^{-\frac{5}{2}}$ is one divided by the Square Root of the fifth Power of x, and $x^{-\frac{11}{3}}$ is one divided by the Cube Root of the eleventh Power of x, and so of the rest.

And because the Exponents of Powers above Unity are positive Numbers; as the Exponents of those that are below (or less than) unity are negative Numbers: Therefore that may be call'd the positive or ascending, and this the negative or descending Series in respect of 1, the first Term of each Series.

But if we suppose the first Term of a Series to be less than Unity, and the following Terms to be produced by a successive Multiplication by x, or by any Power of x, then the Exponent of every Term between that first Term and Unity, will be Negative; that of Unity nothing, and those of the following Terms Positive: And even the Terms of the Series, whose Exponents are Negative Numbers, as well as those whose Exponents are positive Numbers, are ascending in respect of the first Term of the Progression.

And in General, the whole Contrivance lies in adapting Numbers in an Arithmetical Progression to those in a Geometrical, without altering or disturbing the Indices in the two first Ranks in this Lemma.

CONSECTARIES.

16. Hence it is evident that, $\sqrt{x}$, $\sqrt[3]{x}$, $\sqrt[3]{x^4}$, $\frac{1}{x}$, $\frac{1}{xx}$, $\frac{1}{\sqrt{x^3}}$, &c. may be express'd thus, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{4}{3}}$, x^{-1}, x^{-2}, $x^{-\frac{3}{2}}$, &c. Respectively, and both ways represent the same thing.

17. The Sum of the Exponents of any two Terms in a Geometrical Progression is the Exponent of that Term in the Series, which is produc'd by the Multiplication of the two given Terms. Thus x^{2+3} or x^5 is the Product of x^2 multiplied by x^3. And $x^{\frac{1}{2}+\frac{1}{3}}$ or $x^{\frac{5}{6}}$ is the Product of $x^{\frac{1}{2}}$ multiplied by $x^{\frac{1}{3}}$. And $x^{-\frac{1}{2}+\frac{1}{4}}$ or $x^{-\frac{1}{4}}$, is the Product of $x^{-\frac{1}{2}}$ or $\frac{1}{x^{\frac{1}{2}}}$ multiplied by $x^{\frac{1}{4}}$.

More Examples of Multiplication.

To find the Product of $\frac{1}{x} \times \frac{1}{\sqrt[3]{x^5}}$.

$$\frac{1}{x} \times \frac{1}{\sqrt[3]{x^5}} = x^{-1} \times x^{-\frac{5}{3}} = x^{-\frac{3}{3}} \times x^{-\frac{5}{3}} = x^{-\frac{8}{3}} = \frac{1}{x^{\frac{8}{3}}} = \frac{1}{\sqrt[3]{x^8}}.$$

To

To find the Product of $\frac{1}{x^{\frac{4}{3}}} \times \frac{1}{x^{\frac{7}{5}}}$.

$$\frac{1}{x^{\frac{4}{3}}} \times \frac{1}{x^{\frac{7}{5}}} = x^{-\frac{4}{3}} \times x^{-\frac{7}{5}} = x^{-\frac{20}{15}} \times x^{-\frac{21}{15}} = x^{-\frac{41}{15}} = \frac{1}{x^{\frac{41}{15}}} = \frac{1}{\sqrt[15]{x^{41}}}.$$

To find the Product of $\frac{1}{\sqrt[2]{ax^5}} \times \frac{1}{\sqrt[3]{x^7}}$.

$$\frac{1}{\sqrt[2]{ax^5}} \times \frac{1}{\sqrt[3]{x^7}} = a^{-\frac{1}{2}} \times x^{-\frac{5}{2}} \times x^{-\frac{7}{3}} = a^{-\frac{1}{2}} \times x^{-\frac{15}{6}} \times x^{-\frac{14}{6}} = a^{-\frac{1}{2}} \times x^{-\frac{29}{6}}$$

$$= \frac{1}{a^{\frac{1}{2}} \times x^{\frac{29}{6}}} = \frac{1}{\sqrt[2]{a^5}\,\sqrt[6]{x^{29}}}.$$

To find the Product of $\frac{1}{x^{\frac{3}{2}}} \times x^{\frac{7}{2}}$.

$$\frac{1}{x^{\frac{3}{2}}} \times x^{\frac{7}{2}} = x^{-\frac{3}{2}} \times x^{\frac{7}{2}} = x^{\frac{4}{2}} = xx.$$

To find the Product of $\frac{1}{\sqrt[3]{x^8}} \times x^{\frac{3}{2}}$.

$$\frac{1}{\sqrt[3]{x^8}} \times x^{\frac{3}{2}} = x^{-\frac{8}{3}} \times x^{\frac{3}{2}} = x^{-\frac{16}{6}} \times x^{\frac{9}{6}} = x^{-\frac{7}{6}} = \frac{1}{\sqrt[6]{x^7}}.$$

Again, the Square of $x^{\frac{1}{4}}$ or $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ is $= x^{\frac{2}{4}} = x^{\frac{1}{2}}$, and the Cube of x^2 is $x^{2+2+2} = x^6$; if the Indices be Negative. The Square of $\frac{1}{x^{\frac{1}{2}}}$ or of $x^{-\frac{1}{2}}$ is $x^{-\frac{1}{2}-\frac{1}{2}} = x^{-1}$ or $\frac{1}{x}$. The Cube of $x^{-\frac{1}{3}}$ is $x^{-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}}$ or $x^{-\frac{3}{3}} = x^{-1}$. Whence 'tis evident, that Double, Triple, Quadruple, *&c.* the Exponent of any Term in a Geometrical Series, is the Index or Exponent of the Square, Cube, Biquadrate, *&c.* of the said Term.

18. The difference between the Exponents of any two Terms in a Geometrical Progression, is the Exponent of the Quotient, one Term being divided by the other. Thus $x^{\frac{2}{3}-\frac{1}{2}} = x^{\frac{4}{6}-\frac{3}{6}} = x^{\frac{1}{6}}$ is the Quotient of $x^{\frac{2}{3}}$ divided by $x^{\frac{1}{2}}$. And if $x^{\frac{1}{3}}$ be to be divided by $x^{\frac{1}{5}}$ the Quotient will be $x^{\frac{1}{3}-\frac{1}{5}} = x^{\frac{2}{15}}$, and if $x^{-\frac{1}{4}}$ be to be divided by $x^{\frac{1}{3}}$ the Quotient will be $x^{-\frac{1}{4}-\frac{1}{3}} = x^{-\frac{7}{12}}$.

More Examples of Division.

To Divide $\frac{1}{\sqrt[3]{x^5}}$ by $\frac{1}{x}$.

$$\left.\frac{1}{x}\right) \frac{1}{\sqrt[3]{x^5}} \left(= x^{-1}\right) x^{-\frac{5}{3}} \left(= x^{-\frac{5}{3}}\right) x^{-\frac{2}{3}} \left(= x^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{xx}}\right.$$

To Divide $x^{\frac{7}{2}}$ by $\frac{1}{\sqrt[2]{x^3}}$.

$$\left.\frac{1}{\sqrt[2]{x^3}}\right) x^{\frac{7}{2}} \left(= x^{-\frac{3}{2}}\right) x^{\frac{7}{2}} \left(= x^{\frac{10}{2}} = x^5\right.$$

To Divide $\frac{1}{\sqrt[3]{ax}}$ by $x^{\frac{1}{3}}$.

$$\left.x^{\frac{1}{3}}\right) \frac{1}{\sqrt[3]{ax}} \left(= x^{\frac{1}{3}}\right) a^{-\frac{1}{3}} \times x^{-\frac{1}{3}} \left(= a^{-\frac{1}{3}} \times x^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{axx}}\right.$$

PROP.

PROP. V.

To find the Fluxions of Powers, when the Exponents are whole Numbers.

19. First, If the Exponent be a positive Number: Let it be requir'd to find the Fluxions of xx or x^2, 'tis manifest that xx is $= x \times x$. But the Fluxion of $x \times x$ is $=$ (*Art.* 12.) $\dot{x}x + x\dot{x} = 2x\dot{x}$.

In like manner the Fluxion of the Cube of x or of x^3, or of $x \times x \times x$ is $=$ (*Art.* 13.) $\dot{x}xx + x\dot{x}x + xx\dot{x} = 3xx\dot{x}$.

And if the Fluxion of x^4 be requir'd, it will be found to be $= \dot{x}x^3 + \dot{x}x^3 + \dot{x}x^3 + \dot{x}x^3 = 4x^3\dot{x}$, and the same Method is to be us'd in finding the Fluxions of all other Powers (whose Exponents are positive and whole Numbers) in *infinitum*.

Hence if m represent any positive whole Number at pleasure, then the Fluxion of the Power x^m is $= mx^{m-1}\dot{x}$.

Secondly, If the Exponent of the Power be Negative: Let it be requir'd to find the Fluxion of x^{-1}. The flowing Quantity x^{-1} is $=$ (*Art.* 15. 16.) $\frac{1}{x}$, and the Fluxion thereof is $=$ (*Art.* 14.) $\frac{-\dot{x}}{xx} = -x^{-2}\dot{x}$.

If it be requir'd to find the Fluxion of x^{-2} or $\frac{1}{xx}$, it will be found to be $\frac{-\dot{x}x - x\dot{x}}{x^4} = \frac{-\dot{x} - \dot{x}}{x^3} = -2x^{-2-1}\dot{x} = -2x^{-3}\dot{x}$.

In like manner, the Fluxion of x^{-3} or $\frac{1}{x^3}$ is $= -3x^{-3-1}\dot{x}$.

The Fluxions of such negative Powers may be investigated thus: To find the Fluxion of x^{-3} or $\frac{1}{x^3}$. Suppose $\frac{1}{x^3} = z$, then is $1 = zx^3$, and the Fluxions of both Sides of the Equation are, (*Art.* 9. 13.) $0 = 3zx^2\dot{x} + x^3\dot{z}$, and by Transposition $-3zx^2\dot{x} = x^3\dot{z}$; and by Division $\frac{-3x^2z\dot{x}}{x^3}$ or $\frac{-3z\dot{x}}{x}$ is $= \dot{z}$, and substituting $\frac{1}{x^3}$ for z, we have $\frac{-3\dot{x}}{x^4}$ or $-3x^{-3-1}\dot{x}$ $(= \dot{z})$ $=$ to the Fluxion of the negative Power x^{-3}.

Hence Universally if m represent any negative whole Number, then the Fluxion of the negative Power x^{-m} is $= -mx^{-m-1}\dot{x}$.

PROP. VI.

To find the Fluxions of Powers, when the Exponents are broken Numbers.

This Proposition differs but little from the former; And

20. First, If the Exponent be a Fraction and positive: Let it be requir'd to find the Fluxion of $x^{\frac{1}{2}}$. Suppose $x^{\frac{1}{2}} = z$, then is $x = z^2$, and the Fluxions of both sides of the Equation (*Art.* 10. 19.) are $\dot{x} = 2z\dot{z}$, and by Division $\frac{\dot{x}}{2z} = \dot{z}$, and Substituting $x^{\frac{1}{2}}$ for z, we have $\frac{\dot{x}}{2x^{\frac{1}{2}}} = \dot{z}$, and Consequently $\frac{\dot{x}}{2x^{\frac{1}{2}}}$ or $\frac{1}{2}x^{\frac{1}{2}-1}\dot{x}$ is the Fluxion of the given Power $x^{\frac{1}{2}}$.

And

And if it be requir'd to find the Fluxion of $x^{\frac{1}{3}}$. Suppose $x^{\frac{1}{3}} = z$, then $x = z^3$ and $\dot{x} = 3z^2\dot{z}$, and $\frac{\dot{x}}{3z^2} = \dot{z} =$ (by Substitution) $\frac{\dot{x}}{3x^{\frac{2}{3}}} =$ (*Art.* 15. and 16.) $\frac{1}{3}x^{\frac{1}{3}-1}\dot{x}$.

And Universally, to find the Fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$. Suppose $x^{\frac{m}{n}} = z$, then is $x^m = z^n$, and their Fluxions are equal, *viz.* (*Art.* 19.) $mx^{m-1}\dot{x} = nz^{n-1}\dot{z}$, and by Division $\frac{mx^{m-1}\dot{x}}{nz^{n-1}} = \dot{z} =$ (by substitution, because $x^{\frac{m}{n}} = z$, and $x^m = z^n$, and consequently $x^{m-\frac{m}{n}} =$ (*Art.* 18.) x^m divided by $x^{\frac{m}{n}} = z^{n-1}$, and multiplying by n, $nx^{m-\frac{m}{n}} = nz^{n-1}$) $\frac{mx^{m-1}\dot{x}}{nx^{m-\frac{m}{n}}} =$ (*Art.* 18.) $\frac{m}{n}x^{\frac{m}{n}-1}\dot{x} =$ to the Fluxion of $\sqrt[n]{x^m}$.

Secondly, If the Exponent be a Fraction and Negative: Let it be requir'd to find the Fluxion of $x^{-\frac{1}{3}}$ or (*Art.* 15.16.) $\frac{1}{x^{\frac{1}{3}}}$. Suppose $\frac{1}{x^{\frac{1}{3}}} = z$, then is $1 = x^{\frac{1}{3}}z$, and $\frac{1}{z}$ is $= x^{\frac{1}{3}}$. Therefore the Fluxions of both sides of the Equation are (*Art.* 14. 20.) $\frac{-\dot{z}}{zz} = \frac{1}{3}x^{-\frac{2}{3}}\dot{x}$ and by Multiplication and then changing all the Signs of the Equation, we have $\dot{z} = -\frac{1}{3}zzx^{-\frac{2}{3}}\dot{x} =$ (because $z = \frac{1}{x^{\frac{1}{3}}}$) $-\frac{1}{3}x^{-\frac{4}{3}}\dot{x} = -\frac{1}{3}x^{-\frac{1}{3}-1}\dot{x}$; which is the Fluxion of the Power $x^{-\frac{1}{3}}$.

Again, Let it be requir'd to find the Fluxion of $x^{-\frac{1}{2}}$ or $\frac{1}{x^{\frac{1}{2}}}$. Put $\frac{1}{x^{\frac{1}{2}}} = z$, then is $\frac{1}{z} = x^{\frac{1}{2}}$, and $\frac{-\dot{z}}{zz} = \frac{1}{2}x^{\frac{1}{2}-1}\dot{x}$, and by Multiplication and Changing all the Signs of Equation, there will arise $\dot{z} = -\frac{1}{2}zzx^{\frac{1}{2}}\dot{x} =$ (because $z = x^{-\frac{1}{2}}$) $-\frac{1}{2}x^{-\frac{3}{2}}\dot{x} =$ to the Fluxion of the given Power $x^{-\frac{1}{2}}$.

And Universally, To find the Fluxion of $x^{-\frac{m}{n}}$ or $\frac{1}{x^{\frac{m}{n}}}$. Suppose $x^{-\frac{m}{n}} = z$, then is $\frac{1}{z} = x^{\frac{m}{n}}$, and $\dot{z} = -\frac{m}{n}zzx^{\frac{m}{n}-1}\dot{x} = -\frac{m}{n}x^{-\frac{m}{n}-1}\dot{x} =$ to the Fluxion of $x^{-\frac{m}{n}}$. Q. E. I.

And from the two preceeding Propositions may be deduced this General

RULE IV.

To find the Fluxions of all sorts of Powers.

Multiply the given Power by its Exponent, and multiply that Product by the Fluxion of the Root; And lastly, from the Index of the Power, Subtract one or Unity, and then this last Quantity is the Fluxion of the given Power.

Thus if m represent any Number whole or broken, positive or negative; and if x be the flowing Quantity, then the Fluxion of x^m is $= mx^{m-1}\dot{x}$.

PROP. VII.

To find the Fluxions of Surd Quantities.

21. Let it be requir'd to find the Fluxion of $\sqrt{2rx - xx}$, or $\overline{2rx - xx}|^{\frac{1}{2}}$. Suppose $\overline{2rx - xx}|^{\frac{1}{2}} = z$, then is $2rx - xx = zz$, and Consequently $r\dot{x} - x\dot{x} = z\dot{z}$, and by Division $\frac{r\dot{x} - x\dot{x}}{z} = \dot{z} =$ (by Substitution) $\frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$ = to the Fluxion of $\sqrt{2rx - xx}$.

Let it be requir'd to find the Fluxion of $\overline{ay - xx}|^3$; for $\overline{ay - xx}|^3$ put z, and then $ay - xx = z^{\frac{1}{3}}$, and $a\dot{y} - 2x\dot{x} = \frac{1}{3}z^{-\frac{2}{3}}\dot{z}$, And Multiplying by 3, $3a\dot{y} - 6x\dot{x} = z^{-\frac{2}{3}}\dot{z}$, and consequently $3az^{\frac{2}{3}}\dot{y} - 6z^{\frac{2}{3}}x\dot{x} = \dot{z} =$ (Substituting $\overline{ay - xx}|^2 = z^{\frac{2}{3}}$) $3a^3y^2\dot{y} - 6a^2x^2y\dot{y} + 3ax^4\dot{y} - 6a^2y^2x\dot{x} + 12ayx^3\dot{x} - 6x^5\dot{x}$ = to the Fluxion of $\overline{ay - xx}|^3$.

The Fluxions of imperfect Powers may be also investigated by (*Art.* 20.) the general Rule, and express'd otherwise and more briefly thus:

The Fluxion of $\overline{2rx - xx}|^{\frac{1}{2}}$ is $= \frac{1}{2} \times \overline{2rx - xx}|^{-\frac{1}{2}} \times 2r\dot{x} - 2x\dot{x} =$ (*Art.* 15. 16.) $\frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$.

The Fluxion of $\overline{ay - xx}|^3$ is $= 3 \times \overline{ay - xx}|^2 \times \overline{a\dot{y} - 2x\dot{x}}$, which being Reduced will be found equal to the Fluxion thereof formerly found.

The Fluxion of $\sqrt{xy + yy}$ is $= \frac{1}{2} \times \overline{xy + yy}|^{-\frac{1}{2}} \times y\dot{x} + x\dot{y} + 2y\dot{y} =$ (*Art.* 15.) $\frac{y\dot{x} + x\dot{y} + 2y\dot{y}}{2\sqrt{xy + yy}}$. The Fluxion of $\sqrt{a^4 + axyy}$ is $= \frac{1}{2} \times \overline{a^4 + axyy}|^{-\frac{1}{2}} \times \overline{ay^2\dot{x} + 2axy\dot{y}} = \frac{ay^2\dot{x} + 2axy\dot{y}}{2\sqrt{a^4 + axyy}}$.

The Fluxion of $\sqrt{ax + xx + \sqrt{a^4 + axyy}}$ is = by the (*Art.* 20.) Rule and The preceeding Example;

$$\frac{1}{2} \times \overline{ax + xx + \sqrt{a^4 + axyy}}|^{-\frac{1}{2}} \times \overline{a\dot{x} + 2x\dot{x} + \frac{ay^2\dot{x} + 2axy\dot{y}}{2\sqrt{a^4 + axyy}}}$$

$$= \frac{a\dot{x} + 2x\dot{x}}{2\sqrt{ax + xx + \sqrt{a^4 + axyy}}} + \frac{ay^2\dot{x} + 2axy\dot{y}}{2\sqrt{ax + xx + \sqrt{a^4 + axyy}} \times 2\sqrt{a^4 + axyy}}.$$

The Fluxion of $\frac{\sqrt[3]{ax + xx}}{\sqrt{xy + yy}}$ is = (*Art.* 14. 20.) (finding the Fluxions of the Numerator and Denom.)

$$\frac{\frac{a\dot{x} + 2x\dot{x}}{3\sqrt[3]{\overline{ax + xx}|^2}} \times \sqrt{xy + yy} - \frac{y\dot{x} + x\dot{y} + 2y\dot{y}}{2\sqrt{xy + yy}} \times \sqrt[3]{ax + xx}}{xx + yy}$$

To find the Fluxions of Quantities Compounded of Rational and Surd Quantities: Let it be requir'd to find the Fluxion of $\overline{bx^2 + cax + ea^2} \times \sqrt{xx + aa} = z$. Put $\overline{bx^2 + cax + ea^2} = p$, and $\sqrt{xx + aa} = q$. Then the given Quantity is

$pq = z$; and the Fluxion thereof is (*Art.* 12.) $p\dot{q} + q\dot{p} = \dot{z}$. But $\dot{z}$ is $= \frac{x\dot{x}}{\sqrt{xx+aa}}$, and $\dot{p}$ is $= 2bx\dot{x} + ca\dot{x}$. Therefore in the Equation $p\dot{q} + q\dot{p} = \dot{z}$, if in place of $p, q, \dot{p}, \dot{q}$, we restore the Quantities they represent, we shall have

$$\frac{\overline{bx^3 + cax^2 + ea^2x} \times \dot{x}}{\sqrt{xx+aa}} + 2bx \times \sqrt{xx+aa} \times \dot{x} + ca \times \sqrt{xx+aa} \times \dot{x} = \dot{z}.$$

Which being Reduc'd to one Denomination, gives

$$\frac{\overline{3bx^3 + 2acx^2 + ea^2x + 2ba^2x + ca^3} \times \dot{x}}{\sqrt{xx+aa}} = \dot{z} =$$ to the Fluxion of the given Quantity.

I might now shew how to find the Fluxions of Powers when the Exponents themselves are also flowing or variable Quantities: But this being a Business too intricate for Beginners, I shall refer it to a more proper place, and conclude this Section with one observation, which ought carefully to be remember'd.

22. In all the preceeding Propositions, we have suppos'd (in taking the Fluxions of flowing Quantities) that when the variable Quantity x Increases, the others z, y, *&c.* increase also; that is, that when x becomes $= x + \dot{x}$, we have suppos'd that y and z become equal to $y + \dot{y}$ and $z + \dot{z}$ respectively. But if it so happen that while one Increases, all or any of the others Decrease, we must Consider the Fluxions of those that Decrease as Negative Quantities in comparison of the Fluxions of the others which increase at the same time: And consequently we must change the Signs of those Terms wherein the negative Fluxions are found. Thus if x Increases, while y and z Decrease, that is, if x become $x + \dot{x}$, and y become $y - \dot{y}$, and z become $z - \dot{z}$, and if I would find the Fluxion of the Product xyz, If all the Quantities be suppos'd to Increase, then the Fluxion of xyz is (*Art.* 13.) $= \dot{x}yz + \dot{y}xz + \dot{z}yx$. But if y and z Decrease, while x increases, then I must change the Signs of those Terms, wherein the Fluxions of z and y are found, and then the Fluxion of xyz is $= \dot{x}yz - \dot{y}xz - \dot{z}yx$.

SECT.

SECT. III.

The Use of Fluxions

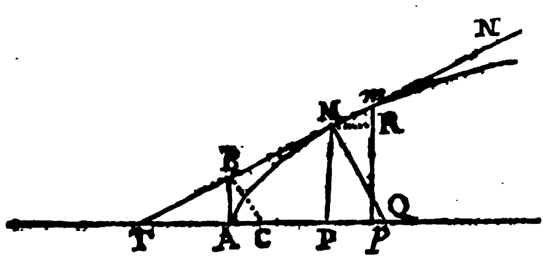
In drawing Tangents to all sorts of Curve Lines, Algebraic and Transcendent.

DEFINITION I.

ALL Curves are Polygons of an (*Art.* 8.) Infinite Number of Sides. And If any of the infinitely little Sides as M*m* be produced both ways, then that Line will touch the Curve in M or *m*, and is therefore call'd a *Tangent* to the Curve in either of the said Points.

DEFINITION II.

All Curves Respect either a certain determinate *Point* or an *Axis*. Thus the Curve AM*m* is describ'd to the Axis AP. And if AP be the *Axis*, then the Curve is said to begin in the same, in the *Vertex* A. And Lines drawn from any Point M in the Curve, as MP, Perpendicular to the Axis, are call'd *Ordinates*. And the Portion of the Axis intercepted between the Vertex A and the Ordinate MP is call'd the *intercepted Diameter* to that Ordinate.



DEFINITION III.

If the Nature of the Curve-line AM*m* be express'd by an Equation, in which the two indeterminate Quantities denote straight Lines only, then the said Curve is call'd an *Algebraic* or *Geometrical Curve*.

Thus if the Nature of the Curve AM*m* be such, that the intercepted Diameter AP retain always the same Proportion to its Corresponding Ordinate PM: For instance, If the Product of AP multiplied into a determinate Quantity, be always equal to the Square of PM, then the Equation Expressing the Nature of the Curve AM*m* will be (supposing $AP = x$; $PM = y$; and the determinate Quantity or *latus rectum* of the Figure $= a$) $ax = yy$. And because the two indeterminate or flowing Quantities x and y, denote streight Lines, Therefore the Curve AM*m* is call'd an *Algebraic* or *Geometrical Curve*.

And it is manifest that the Number of such Curves is infinite: Because all the *possible variety of Relations* between the Ordinate and intercepted Diameter is *endless*.

DEFINITION IV.

And if the Nature of any Curve be express'd by an Equation, wherein one of the flowing Quantities represents a Curve Line, then that Curve is call'd a *Transcendent Curve*; And if the Curve which enters the Equation be *Geometrical*, or a Curve of the *first kind or degree*, then the *Transcendent Curve* is call'd a Curve of the *second kind or degree*; And if the said indeterminate Quantity represent a Curve of the *second kind*, then the *Transcendent Curve* is call'd a Curve of the *third kind*. And so on Infinitely.

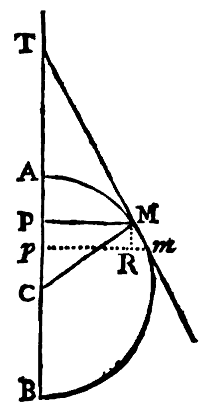
I know that all sorts of Curves might be more accurately reduced under proper Heads, from the Consideration of their *Foci*: But what I have advanc'd already will be sufficient for my purpose.

PROP.

PROP. I.

To draw a Tangent to a Circle.

23. In the Circle AMB: Let the Diameter AB be $= 2a$, and let the infinitely little part of the Curve Mm be prolong'd until it cut the Diameter produc'd in T, and draw the Line MC Perpendicular to TM, Intersecting the Diameter in C. Draw the Ordinate $PM = y$, and another Ordinate pm infinitely near to PM, and draw MR Parallel to the Diameter AB. Now If the Intercepted Diameter AP be $= x$, then (*Art.* 3. 10.) the Fluxion thereof is $= \dot{x}$, and for the like reason, the Fluxion of the Ordinate Rm is $= \dot{y}$. Now the Position of PM being given, and MC being suppos'd Perpendicular to the Tangent MT, it remains only to find the length of PC, which determines the Perpendicular CM, and consequently the Tangent MT.



The Triangles RMm, PMC are similar, (for the Angles at P and R are right Angles, and the Angles PMR, CMm being right Angles, if the Angle CMR which is common be subtracted from both the remaining Angles PMC, RMm will be equal, and consequently PCM is $= RmM$.) Therefore MR $(= Pp = \dot{x})$: Rm $(\dot{y})$ :: PM (y) : $PC = \frac{y\dot{y}}{\dot{x}}$. But the property of the Circle is that $AP \times PB = PMq$. Therefore the Equation expressing the Nature of the Curve is $2ax - xx = yy$, and finding the Fluxions of both sides of the Equation, we have (*Art.* 9. 12. 19.) $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$ or $a\dot{x} - x\dot{x} = y\dot{y}$, and by Division $\dot{x} = \frac{y\dot{y}}{a-x}$. But $PC = \frac{y\dot{y}}{\dot{x}} =$ (substituting $\frac{y\dot{y}}{a-x}$ for $\dot{x}$) $\frac{ay\dot{y} - xy\dot{y}}{y\dot{y}} = a - x$. Whence it is evident that the Point C falls in the Center of the Circle, and consequently 'tis manifest that if a Line be drawn from C the Center of the Circle, to any Point in the Circumference as M, and if MT be drawn Perpendicular to CM, it will touch the Circle in M.

And if it be requir'd to find the Length of PT which determines the Intersection of the Tangent MT in the Diameter BA produc'd; It may be done thus: The Triangles mRM, MPT are similar, therefore mR $(\dot{y})$: RM $(\dot{x})$ :: MP (y) : PT $= \frac{y\dot{x}}{\dot{y}} =$ (because $\dot{x} = \frac{y\dot{y}}{a-x}$) $\frac{yy}{a-x} = \frac{2ax - xx}{a-x}$. And $PT - AP = \frac{2ax - xx}{a-x} - x = \frac{ax}{a-x} = AT$.

Another way.

24. Retaining the same Symbols as (*Art.* 23.) before; suppose $PT = t$, then because the Triangles TPM, MRm are similar, it is TP (t) : PM (y) :: MR $(\dot{x})$: $Rm = \frac{y\dot{x}}{t}$. And $Ap = x + \dot{x}$; $Bp = 2a - x - \dot{x}$; and $pm = y + \frac{y\dot{x}}{t}$. Then $2ax - xx + 2a\dot{x} - 2x\dot{x} - \dot{x}\dot{x} =$ (because $Ap \times Bp = pmq$, by the property of the Curve) $yy + \frac{2yy\dot{x}}{t} + \frac{y^2\dot{x}\dot{x}}{tt}$: From which subtracting the Equation of the Curve $2ax - xx = yy$, and rejecting the Terms $\dot{x}\dot{x}$ and $\frac{y^2\dot{x}\dot{x}}{tt}$ as being incomparably little in respect of any of the others, we have

$2a\dot{x} - 2x\dot{x} = \frac{2yy\dot{x}}{t}$, and dividing both sides of the Equation by $2\dot{x}$ there will arise $a - x = \frac{yy}{t}$, and consequently $\frac{yy}{a-x} = t = PT$. Which was requir'd.

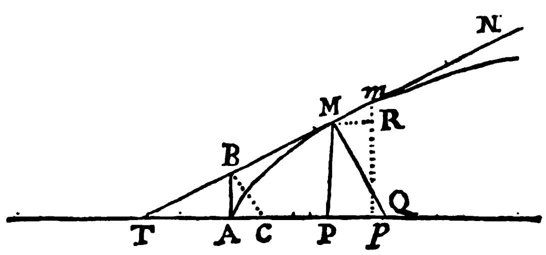
DEFINITION V.

The Line PT, which determines the Intersection of the Tangent MT in the Axis is call'd the *Sub-tangent*, and the Line PC which determines the Intersection of the Perpendicular (MC) to the Tangent in the Point of Contact M, in the Axis AB, is call'd the *Subnormal.*

PROP. II.

To draw Tangents to all sorts of Paraboloides.

25. Let the Curve AM*m* be drawn. Then if the Ordinates PM be in a subduplicate Proportion of the intercepted Diameters AP, the Curve AM*m* is a Parabola; But if the Ordinates MP be in a Subtriplicate, Subquadruplicate, *&c.* Proportion of the intercepted Diameters AP, then the Curve AM*m* is call'd a Paraboloide.



Suppose the intercepted Diameter $AP = x$, the Ordinate $PM = y$, and the Parameter of the Figure $= a$: Then the Equation expressing the Nature of the Parabola AM*m* is, $ax = yy$. let it be requir'd to draw the Line MT to touch the Curve in M. Suppose the thing done, and that MT is the Tangent requir'd, intersecting the Axis produced in T; it is requir'd to find the Sub-tangent PT. Draw the Ordinate P*m* infinitely near PM, and draw MR Parallel to AP, then $Pp = \dot{x} = MR$, and $Rm = \dot{y}$; and because the Triangles *m*RM, MPT, are similar, it is, *m*R $(\dot{y})$: RM $(\dot{x})$ :: MP (y) : $PT = \frac{y\dot{x}}{\dot{y}}$. Now the Equation of the Curve is $ax = yy$, therefore $a\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{2y\dot{y}}{a}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}}$ is $= \frac{2yy}{a}$ = (by substituting ax for yy) $\frac{2ax}{a} = 2x = 2AP$.

Hence it is manifest, that in the Parabola, the Sub-tangent PT is equal to twice the intercepted Diameter AP.

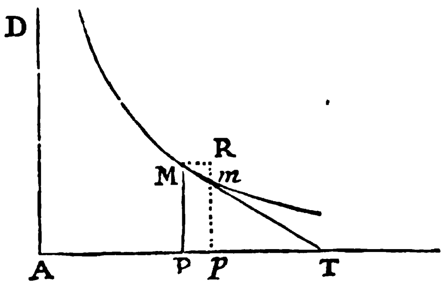
And if the Equation of the Curve be $aax = y^3$, then the Curve AM*m* is a Cubical Paraboloide, and the Sub-tangent PT is = (*Art.* 25.) $\frac{y\dot{x}}{\dot{y}}$ = (because $aax = y^3$, and $\dot{x} = \frac{3y^2\dot{y}}{aa}$) $\frac{3y^3}{aa}$ = (because $aax = y^3$) $\frac{3aax}{aa} = 3x = 3AP$; whence 'tis evident that in this Curve the Sub-tangent PT is equal to three times the intercepted Diameter AP.

PROP.

PROP. III.

To draw Tangents to all sorts of Hyperboloides.

26. Let AD, and AT represent the Asymptotes of the Equilateral Hyperbola Mm, then the Point A is the Center of the Opposite Sections. Draw the Ordinate PM, and another Ordinate pm infinitely near the same; then suppose the Parameter of the Figure $= a$, the intercepted Diameter $AP = x$, $Pp = \dot{x}$; the Ordinate $PM = y$, and $Rm = \dot{y}$. And let it be requir'd to draw the Line MT to touch the Curve in M. Suppose the thing done; then the Triangles mRM, MPT are



similar, therefore $mR\ (\dot{y}) : RM\ (\dot{x}) :: MP\ (y) : PT = \frac{y\dot{x}}{\dot{y}}$. But the Equation of the Curve is $aa = xy$, therefore reducing the Equation to Fluxions, we have (*Art.* 9. 12.) $o = y\dot{x} + x\dot{y}$, and consequently $-\dot{y} = \frac{y\dot{x}}{x}$. Therefore $PT = \frac{y\dot{x}}{\dot{y}}$ which is $=$ (*Art.* 22.) $\frac{y\dot{x}}{-\dot{y}}$ is $=$ (substituting $\frac{y\dot{x}}{-x}$ for $-\dot{y}$) $-x = AP$.

COROLLARY. I.

27. If the value of the Sub-tangent PT come out positive, then it is a sign that the point T falls on the same side of the Ordinate PM with the point A the beginning of x, as in the Parabola: But if the value of the Sub-tangent come out Negative, then the point T falls on the contrary side of the Ordinate PM in respect of A the beginning of x, as in the Hyperbola.

COROL. II.

28. In the Parabola and Hyperbola, if the Parameter be suppos'd $= 1$, then $y^m = x$ expresses the nature of all sorts of Parabola's, when m is a positive whole or broken Number, and the same Equation expresses the Nature of all sorts of Hyperboliform Figures when m is a negative Number. And *Universally* in either, the length of the Sub-tangent $PT =$ (*Art.* 25. 26.) $\frac{y\dot{x}}{\dot{y}}$ is $=$ (because the general Equation for both is $y^m = x$, and consequently $my^{m-1}\dot{y} = \dot{x}$) $= my^m =$ (because $y^m = x$) $= mx$.

Hence if m be $= \frac{1}{2}$, the Equation of the Curve is $y^{\frac{1}{2}} = x$ or (putting the Parameter $1 = a$) $axx = y^3$, which expresses the Nature of one of the Cubical Parabola's, and the length of the Sub-tangent PT is $= \frac{1}{2}$ AP.

If $m = -3$, then the general Equation is $y^{-3} = x$, or (*Art.* 15.) $1 = y^3 x$, that is (supposing the Parameter $1 = a$) $a^4 = y^3 x$, which expresses the Nature of a Hyperboliform Figure, and the Sub-tangent $PT = -3x = -3$ AP.

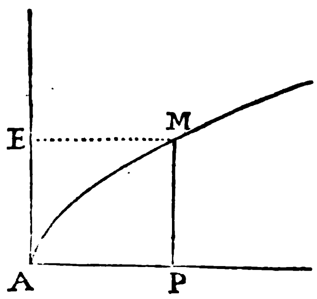
And *Universally*, in all Paraboliform and Hyperboliform Figures, the *Sub-tangent is equal to the Exponent of the Power of the Ordinate multiplied into the Abscissa.*

COROL. III.

29. In the Parabola, the Sub-tangent $PT = 2x$, and by the property of the Curve, PMq $(= yy) = ax$, therefore (by similar Triangles) the Sub-normal $PQ = \frac{ax}{2x} = \frac{1}{2}a =$ to $\frac{1}{2}$ the Parameter of the Figure, and consequently it is an inva-

invariable Quantity. Which is a Remarkable property of the Apollonian Parabola; and thus the Sub-normals of all sorts of Paraboliform Figures may be Investigated.

COROL. IV.



30. If m be a Fraction and positive, *v.g.* $= \frac{1}{3}$, then the Equation of the Curve is $y^{\frac{1}{3}} = x$, that is $y = x^3$, and consequently the line AE which touches the Curve in the Vertex A becomes an Axis to the same, that is the Convexity of the Curve is towards the Axis. For in this Case, the Ordinate PM becomes the intercepted Diameter AE, and the intercepted Diameter AP becomes the Ordinate EM.

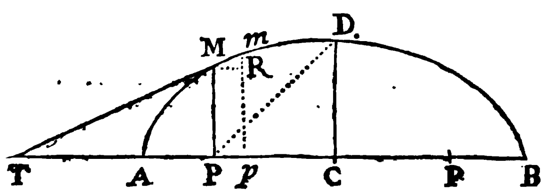
COROL. V.

31. And to draw a Tangent to the Vertex of any Parabola, being the beginning of x, we must investigate the Proportion between $\dot{x}$ and $\dot{y}$ in that point, which is done, thus: $m y^{m-1} \dot{y} =$ (*Art.* 28.) $\dot{x}$, and reducing the same to an Analogy $\dot{x} : \dot{y} :: m y^{m-1} : 1$. Now 'tis evident, that in (*Fig. Art.* 25.) A, the Ordinate y vanishes or is equal to nothing, therefore the fourth Term of the Analogy is infinite in respect of the third, and consequently $\dot{y}$ is infinite in respect of $\dot{x}$, that is, the Tangent in the Vertex A is Parallel to the Ordinates. But if m be less than Unity, then y becomes (*Art.* 30.) the intercepted Diameter; and the Tangent to the Vertex A Co-incides with AE the Axis of the Curve.

PROP. IV.

To draw Tangents to all sorts of Ellipses.

32. Let A M m be an Ellipsis, and 'tis requir'd to draw MT which shall touch the same in the point M. Suppose the Transverse Axis AB $= a$, the Parameter $= b$. Draw the Ordinates MP, mp infinitely near each other, and MR Parallel to AP, and suppose AP $= x$; Pp $= \dot{x}$; PM $= y$; and Rm $= \dot{y}$. Then by the property of the Curve AP × PB ($x \times \overline{a-x}$) : PMq (yy) :: AB (a) : Param. b. and the Equation expressing the Nature



of the Curve is $\frac{ayy}{b} = ax - xx$ Therefore (*Art.* 12. 19.) $\frac{2ay\dot{y}}{b} = a\dot{x} - 2x\dot{x}$ and $\dot{y} = \frac{ab\dot{x} - 2b x\dot{x}}{2ay}$ and consequently PT $= \frac{y\dot{x}}{\dot{y}}$ is $= \frac{2ayy}{ab - 2bx} =$ (substituting $ax - xx$ for $\frac{ayy}{b}$) $\frac{2ax - 2xx}{a - 2x}$. And PT − AP $= \frac{2ax - 2xx}{a - 2x} - x$, $= \frac{ax}{a - 2x} =$ AT.

And *Universally*, If m be the Exponent of the Power of AP, and n that of PB (where note that Exponent of PM $= y$, is = to the Sum of the Exponents of AP $= x$, and PB $= \overline{a-x}$) then the Equation expressing the nature of all sorts of Ellipses will appear in this form, is $\frac{ay^{m+n}}{b} = x^m \times \overline{a-x}|^n$, and Consequently

quently (*Art.* 19. 21.) $\frac{\overline{m+n}\,a y^{m+n-1}\dot{y}}{b} = m x^{m-1}\dot{x}\times\overline{a-x}^{n} - n\overline{a-x}^{n-1}\dot{x}\times x^{m}$ and then (putting $b=1$) by division $\frac{\overline{m+n}\,a y^{m+n-1}\dot{y}}{m x^{m-1}\times\overline{a-x}^{n} - n\overline{a-x}^{n-1}\times x^{m}} = \dot{x}$.

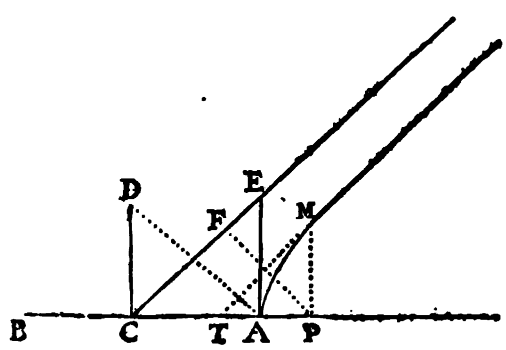
But PT $= \frac{y\dot{x}}{\dot{y}} =$ (by substitution) $\frac{\overline{m+n}\,a y^{m+n}}{m x^{m-1}\times\overline{a-x}^{n} - n\overline{a-x}^{n-1}\times x^{m}} =$ (by substituting $x^{m}\times\overline{a-x}^{n}$ for $\frac{a y^{m+n}}{b}$ or $a y^{m+n}$) $\frac{\overline{m+n}\times x^{m}\times\overline{a-x}^{n}}{m x^{m-1}\times\overline{a-x}^{n} - n\overline{a-x}^{n-1}\times x^{m}}$

$=$ (dividing by x^{m-1}) $\frac{\overline{m+n}\times x\times\overline{a-x}^{n}}{m\times\overline{a-x}^{n} - n\overline{a-x}^{n-1}\times x} =$ (dividing by $\overline{a-x}^{n-1}$)

$\frac{\overline{m+n}\times x\times\overline{a-x}}{m\times\overline{a-x} - n\times x} = \frac{\overline{m+n}\times\overline{ax-xx}}{m\overline{a-x} - n\times x}$. Therefore PT$-$AP $= \frac{\overline{m+n}\times\overline{ax-xx}}{m\overline{a-x}-nx} - x = \frac{nax}{m\overline{a-x}-nx} =$ AT.

COROL. I.

33. The same things being suppos'd as (*Art.* 32.) before, only if the point B fall on the other side of the point A in respect of the Ordinate PM, then we have this Equation $\frac{ay^{m+n}}{b} = x^{m}\times\overline{a+x}^{n}$, which expresses the Nature of all sorts of Hyperbola's in respect of their Diameters. Whence it will be found, that the Sub-tangent PT is $= \frac{\overline{m+n}\,\overline{ax+xx}}{m\overline{a+x}+nx}$ and AT $= \frac{nax}{ma+\overline{m+n}\,x}$.



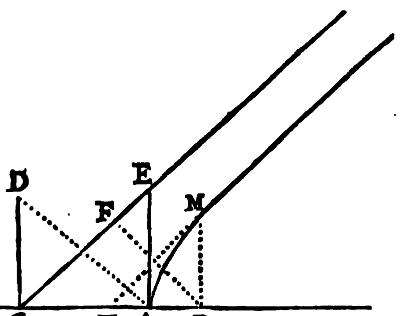
COROL. II.

34. If we suppose the intercepted Diameter A P to be infinitely produc'd, then x will be infinite, and the Tangent MT will touch the Curve at an infinite Distance, that is, it will become the *Asymptote* CE, and in that Case AT $=$ (*Art.* 33.) $\frac{nax}{ma+\overline{m+n}\,x}$ will be $=$ (because the Term ma is infinitely little in respect of $\overline{m+n}\,x$, and so may be rejected) $\frac{nax}{\overline{m+n}\,x} = \frac{n}{m+n}\times a =$ AC.

And if $m=1$, and $n=1$, then the Curve is a common Hyperbola; and AC $= \frac{1}{2}a = \frac{1}{2}$AB an invariable Quantity; that is, *the Asymptotes of the common Hyperbola intersect each other in the Center of the opposite Sections.*

COROL. III.

35. And because the Equation expressing the Nature of all sorts of Hyperbola's in Finite Cases is (*Art.* 33.) $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x^n}$, or $ay^{m+n} = bx^m \times \overline{a+x^n}$ therefore in this Case, when x is infinite we have $ay^{m+n} =$ (rejecting the Term a as being incomparably little in respect of x) $= bx^m \times x^n = bx^{m+n}$. And putting $r = m+n$, we have $ay^r = bx^r$. And extracting the Root r of each side of the Equation, there will arise $y\sqrt[r]{a} = x\sqrt[r]{b}$, and consequently (*Art.* 21.) $\dot{y}\sqrt[r]{a} = \dot{x}\sqrt[r]{b}$. Now if we suppose the Asymptote to be infinitely produc'd, until it touch the Curve, and if we conceive the Fluxionary Triangle to be formed in that point, and if AE be drawn Parallel to the Ordinates, then 'tis evident that that Triangle and the Triangle CAE will be similar, Therefore $\dot{x} : \dot{y} :: (\sqrt[r]{a} : \sqrt[r]{b} ::)$ AC $\left(\frac{n}{m+n}a = \frac{n}{r}a\right)$: AE $= \frac{n}{r}\sqrt[r]{ba^{r-1}}$.



B C T A P

Which determines the position of the Asymptote CE.

For instance, if m be $= 1$, and $n = 1$, then is $r = 2$, and the Curve is the Apollonian Hyperbola, and AE $= \frac{n}{r}\sqrt[r]{ba^{r-1}} = \frac{1}{2}\sqrt[2]{ab}$. That is, AE is equal to $\frac{1}{2}$ the Conjugate Diameter of the Hyperbola; for the Conjugate Diameter is a mean Proportional between the Parameter b and the Transverse Axis a.

And if a be $= b$, then AE $= \frac{1}{2}\sqrt[2]{ab} = \frac{1}{2}\sqrt[2]{aa} = \frac{1}{2}a =$ (*Art.* 34.) AC; and the Angle ECA $= \frac{1}{2}$ a Right-angle, and consequently the Asymptotes are Perpendicular to each other in C, and the Curve AM is call'd *an Equilateral Hyperbola.*

COROL. IV.

36. In the Ellipsis (*Fig. Art.* 32.) AMB, the Equation expressing the Nature thereof is (*Art.* 32.) $\frac{ayy}{b} = ax - xx$, that is, when the point P falls in C the middle of the Axis AB, the relation between the Ordinate and intercepted Diameter will be express'd by $\frac{ayy}{b} = \frac{1}{4}aa$, and consequently $yy = \frac{1}{4}ab$ and by equal Extraction $y = \frac{1}{2}\sqrt{ba}$; that is, the Conjugate Diameter in the Ellipsis is a mean Proportional between the Parameter b, and the Transverse Axis a.

And in that point the Fluxion $\dot{y}$ is infinitely little in respect of $\dot{x}$, therefore the Subtangent is infinitely great in respect of the Ordinate y, that is, the Tangent in that point will be Parallel to the Axis.

COROL. V.

37. If the Ordinate PM $= y$ be suppos'd $= \frac{1}{2}$ the Parameter of the Figure $= \frac{1}{2}b$, then the Equation of the Curves (*Art.* 32, 33.) $\frac{ayy}{b} = ax \pm xx$, will become $\frac{1}{4}ab = ax \pm xx = \frac{1}{2}$ the Parameter $\times \frac{1}{2}$ the Transverse Axis. And the point P is call'd *the Focus of the Figure*, or *the Umbilick Point*, or *Punctum ex Comparatione.*

COROL. VI.

38. If CD (*Fig. Art.* 32.) be $= \frac{1}{2}$ the Conjugate Diameter, and P the Focus of the Figure, then PD is $=$ AC $= \frac{1}{2}$ the Transverse Axis. For PD$q =$ CD$q +$ PC$q =$ (*Art.* 36.) $\frac{1}{4}ab +$ PC$q =$ (*Art.* 37.) $ax - xx +$ PC$q =$ AP $\times$ PB $+$ PC$q =$ ACq, and consequently PD $=$ AC. And

And in the Hyperbola, If C be the Center of the Asymptotes, and $CD = \frac{1}{2}$ the Conjugate Diameter, then is $CD =$ (*Art.* 35.) AE. I say, also $AD = CE = CP =$ to the Distance of the Focus P from the Center C. For $ADq = CEq = AEq + ACq =$ (*Art.* 35.) $\frac{1}{4}ab + ACq =$ (*Art.* 37.) $ax + xx + ACq = AP \times PB + ACq = CPq$. Therefore $CEq = CPq$ and $CE = CP$.

If the Axes of the Ellipsis (*Fig. Art.* 32.) are equal, then the Curve AMB is a Circle; and both the Foci P, P, unite in the Center. For $AP \times PB =$ (*Art.* 36. 37.) $CDq =$ (by supposition) ACq. Ergo the Points P, C, P, Co-incide.

COROL. VII.

39. If the Curve AM be an Equilateral Hyperbola, Then I say, BP : CA : AP are continually Proportional. For $AP \times BP =$ (*Art.* 37.) $\frac{1}{4}ab =$ (*Art.* 35.) $\frac{1}{4}aa = ACq$, Therefore BP : AC : AP ∺.

If in either of the Asymptotes of an Equilateral Hyperbola, as CE, you take $CF = CA$, and draw the Perpendicular FP, it will intersect the Axis in the Focus P. Because $FCP = \frac{1}{2}$ Right-angle, therefore $FP = FC = AC =$ (*Art.* 35.) DC. Therefore $CP = AD$, and (*Art.* 38.) P is the Focus.

COROL. VIII.

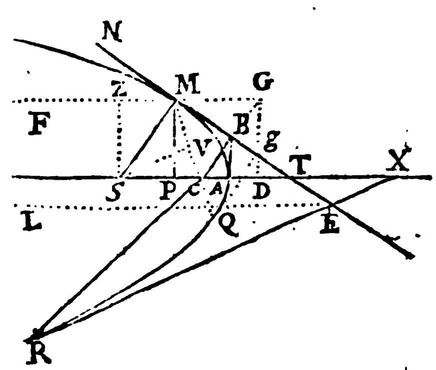
40. Resuming the Symbols (*Art.* 25.) If any Tangent (*Fig. Art.* 25.) MT be drawn touching the Parabola in M, and if AB be drawn Perpendicular to the Axis AP, and BC Perpendicular to the Tangent MT; then I say the Portion of the Axis AC will always be $= \frac{1}{4}$ the Parameter of the Axis. For $AT =$ (*Art.* 25.) AP, therefore $AB = \frac{1}{2}y$. And the Triangles PTM, ABC are Similar, therefore PT $(2x)$: PM (y) :: AB $(\frac{1}{2}y)$: $AC = \frac{1}{4}a = \frac{1}{4}$ the Parameter of the Axis of the Figure.

The Point C is call'd the Focus of the Parabola.

And it is manifest that in the Ellipse and opposite Sections, there are *two Foci*, and in the Parabola but *One*.

COROL. IX.

41. If any Line as TMN touch the Parabola in M, and if the Right-line CM connect the Focus C and the point of Contact M, and from M be drawn MF Parallel to the Axis AP, I say, the Angle $FMN = CMT$. For $PA = AT$, therefore $MB = BT$, and the Triangles CBM, CBT are Similar and Equal. And consequently, the Angle $CMB = CTM = FMN$.



2°. And $CM = CT$. That is, $AP + AC = CM$.

3°. If the Ordinate MP Co-incide with MC then $MP (= MC) = PA (= CA) + AT =$ (*Art.* 25. 40.) $\frac{1}{2}$ the Parameter of the Axis. And $AB = \frac{1}{2}MC = \frac{1}{4}$ Parameter $= AC$.

4°. If AD be taken $= AC$, and DG be drawn Perpendicular to the Axis AD, then any Line as MG drawn (from any point of the Curve as M) Parallel to the Axis AD, and Intersecting DG in G, will be equal to the Line MC drawn from the same point M to the point Focus C. For draw the Tangent MT, and the Ordinate MP, then $PA = AT$, and if to both equal things be added, *viz.* AC and AD, then $PD (= MG) = CT (= MC)$.

5°. The Right-line CB produced will pass through the point G. For if the Line BG be drawn, then $MCq - MBq = MGq - MBq$, and consequently $BC = BG$, and the Angle MBC (which is a Right-angle) $= MBG$, therefore the Lines GB, BC are in the same Streight-line.

6°. And

6°. And MC = SC; For (*Art.* 29.) PS = ½ the Parameter of the Axis = CD, and CS = PD = MG = CM.

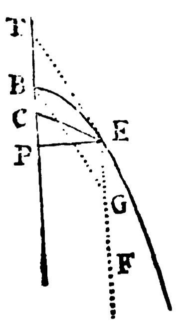
7°. If any two Right-lines ME, RE touching the Parabola in M and R Intersect each other in E, and if the Lines MC, RC be drawn from the points of Contact to the Focus C. I say, the Angle MCR = 2 MER. For CMT and CRX are Isosceles Triangles, therefore MCP = 2MTP = 2MEL and RCP = 2RXP = 2REL, and consequently MCR = 2 MER.

8°. Hence if any Line as MQ be drawn through the Focus C, and Intersect the Curve in M and Q, the Tangents Mg and Qg will Intersect each other at right angles in g. For MCP + PCQ = 2 Right-angles = 2MgQ.

9°. And because the Triangles CAB, CBM are similar, therefore CM : CB :: CB : CA, and consequently CMq : CBq :: CM : CA.

10°. And if MS be drawn Perpendicular to the Tangent MT, and SZ, SV Perpendicular to MF, MC, then the Triangles SMZ, SMV, SMP, are equal and similar, therefore MZ = MV = SP = (*Art.* 29.) ½ the Parameter of the Axis AP.

11°. If BC be the Axis of a Parabola, and EF any Diameter; and if from the point E the Right-line EP be drawn Perpendicular to the Axis; I say the Parameter (L) of the Diameter EF is = Parameter (l) of the Axis (BC) + 4BP, that is L = l + 4BP. For if ET touch the Parabola in E and intersect the Axis in T, and if BG be drawn Parallel to TE; then BG = ET, and GE = BT = BP, and TPE is a Right-angle. Therefore



l × BP = (PEq) TEq — TPq = (because BT = BP) TEq — 4BPq = BGq — 4BPq = L × EG — 4BPq = L × BP — 4BPq; and consequently l × BP = L × BP — 4BPq, and by Division and Transposition L = l + 4BP.

12°. If C be the Focus of the Parabola, and E the Vertex of any Diameter EF, then the Parameter (L) of the said Diameter EF is = four-times the distance of the Vertex E from the Focus C, that is L = 4CE; For BC + BP = CE, and L = l + 4BP = 4BC + 4BP = 4CE.

COROL. X.

42. In the Hyperbola and the Ellipsis, the Distance between either of the *Foci* and the *Center*, viz. CP is a mean Proportional between ½ the Transverse Axis CA and $\left\{\begin{array}{l}\text{CA} + \frac{1}{2}b \text{ in the Hyperb.}\\ \text{CA} - \frac{1}{2}b \text{ in the Ellipsis}\end{array}\right\}$ For,

$$CAq : \left\{\begin{array}{l} CA \times \frac{1}{2}b, \text{ or} \\ \frac{1}{4}ab, (\textit{Art. } 37.) \text{ or} \\ ax \pm xx, \text{ or} \\ \text{In the Hyperb. } AP \times BP = CPq - CAq \\ \text{In the Ellipsis. } AP \times BP = CAq - CPq \end{array}\right\} :: CA : \tfrac{1}{2}b.$$

And by Composition in the Hyperbola and Division in the Ellipsis,

$$CAq : CPq :: CA : \left\{\begin{array}{l} CA + \frac{1}{2}b \text{ In the Hyperb.} \\ CA - \frac{1}{2}b \text{ In the Ellipsis.}\end{array}\right.$$

And dividing the Antecedents by CA, we have

$$CA : CPq :: 1 : \left\{\begin{array}{l} CA + \frac{1}{2}b \\ CA - \frac{1}{2}b \end{array}\right\} \text{ that is, } CA : CP :: CP : CA \pm \tfrac{1}{2}b. \quad \text{Q.E.D.}$$

COROL.

COROL. XI.

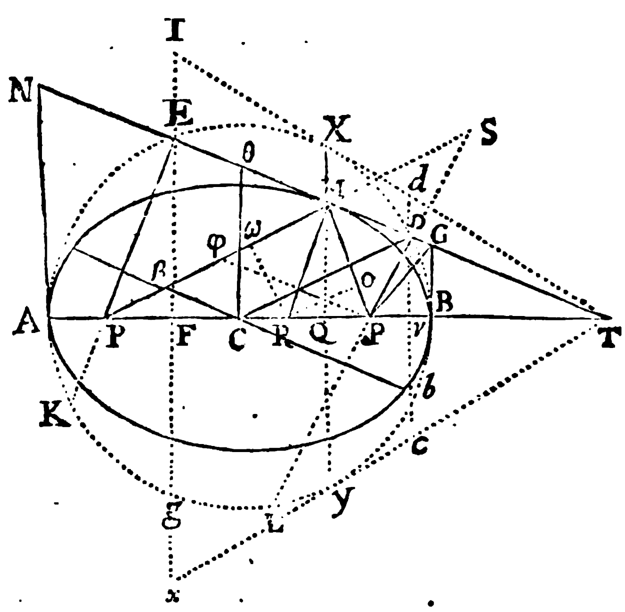
43. The Diſtance of the Focus (*Fig. Art.* 32. *and Fig. Art.* 33.) P from the Center of the Section or oppoſite Sections C is $=$ (*Art.* 42.) $\sqrt{\frac{1}{4}aa \pm \frac{1}{4}ab} = \frac{1}{2}\sqrt{aa \pm ab} = CP$. And the diſtance of the Focus P, from the Vertex A, or AP $=$ $\begin{cases}\frac{1}{2}\sqrt{aa+ab} - \frac{1}{2}a \\ \frac{1}{2}a - \frac{1}{2}\sqrt{aa-ab}.\end{cases}$ And the diſtance between the Foci, is $= \sqrt{aa \pm ab}$.

Hence in the Equilateral Hyperbola, in which $a = b$, the diſtance of the Focus P, from C the Center of the oppoſite Sections, is $a\sqrt{\frac{1}{2}} = \frac{1}{2}a\sqrt{2}$

And the diſtance between the Foci, is $= a\sqrt{2}$, and the diſtance of either Focus from the adjacent Vertex is $\frac{a\sqrt{2} - a}{2}$.

COROL. XII.

44. In the Ellipſis and oppoſite Sections, If AN, BG be drawn Perpendicular to the Axis AB, until they Interſect the Tangent NMT in N and G. I ſay, the Rectangle $AN \times BG = \frac{1}{2}AB \times \frac{1}{2}$ Parameter $= \frac{1}{4}ab$. For BT $=$ (*Art.* 32.) (BQ being $= x$) $\frac{ax}{a-2x}$; and AT $= \frac{ax}{a-2x} + a = \frac{aa-ax}{a-2x}$. If we Imagine the Fluxionary Triangle to be drawn at M, it will be ſimilar to the Triangles GBT, NAT; Therefore $\dot{x} : \dot{y} ::$ BT $\left(\frac{ax}{a-2x}\right) : BG = \frac{ax\dot{y}}{a\dot{x} - 2x\dot{x}}$



And $\dot{x} : \dot{y} ::$ AT $\left(\frac{aa-ax}{a-2x}\right)$ AN $= \frac{aa\dot{y} - ax\dot{y}}{a\dot{x} - 2x\dot{x}}$. But $\dot{y} =$ (*Art.* 32.) $\frac{ab\dot{x} - 2bx\dot{x}}{2ay}$. Therefore BG $= \frac{a^2bx - 2abxx}{2a^2y - 4axy}$. And AN $= \frac{a^3b - 3a^2bx + 2abxx}{2aay - 4axy}$, and Multiplying thoſe Analytic Values into one another, there will ariſe (after Reduction) $AN \times BG = \frac{1}{4}ab$.

COROL. XIII.

45. If a Circle deſcribed on (*Fig. Art.* 44.) AB the Tranſverſe Axis of an Ellipſis (or Hyperbola) as a Diameter, Interſect any Tangent Line NMT in E and D, and if the Lines EP and DP be drawn Perpendicular to the ſaid Tangent NMT. I ſay, they will Interſect the Axis AB in the Foci P, P: For By Similar Triangles it is, TA : TE :: AN : EP and (by the Property of the Circle) TA : TE :: TD : TB :: (by Similar Triangles) DP : BG. Therefore AN : EP :: DP : BG; and conſequently, $AN \times BG = EP \times DP =$ (*Art.* 44.) $\frac{1}{4}ab$.

And (becauſe the Chords EK, DL are Perpendicular to the Chord ED) EK $=$ DL, and (becauſe AB is the Diameter of the Circle) EP $=$ LP, and DP $=$ KP.

 There-

Therefore $EP \times DP = EP \times PK = AP \times PB = DP \times PL = BP \times AP = \frac{1}{4}ab$. And consequently, the Points P, P, are the (*Art.* 37.) Foci.

COROL. XIV.

46. The Lines (*Fig. Art.* 49.) CQ, CB, CT are continually Proportional: For, $CQ \times CT$ (*Art.* 32.) $\left(\frac{a-2x}{2} \times \frac{aa}{2a-4x}\right) = CBq\ (\frac{1}{4}aa.)$

And $AQ\ (a-x) : CQ \left(\frac{a-2x}{2}\right) :: QT$ (*Art.* 32.) $\left(\frac{2ax-2xx}{a-2x}\right) : QB\ (x.)$

And $AT \left(\frac{aa-ax}{a-2x}\right) : CT \left(\frac{aa}{2a-4x}\right) :: QT \left(\frac{2ax-2xx}{a-2x}\right) : BT \left(\frac{ax}{a-2x}\right)$

And consequently by similar Triangles $AN : C\theta :: QM : BG$.

Therefore $AN \times BG = C\theta \times QM =$ (*Art.* 44.) $\frac{1}{4}ab$.

COROL. XV.

47. In the Ellipsis and in the opposite Sections; if to any point of the Section (*Fig. Art.* 49) M the Lines PM, PM be drawn from both the Foci P, P, they will form equal Angles with the Line touching the Section in that point, that is, the Angle PME will be = PMD.

Let the Ordinate QM be produced until it cut the Circle in X and Y, and draw the Lines IXT, xYT touching the Circle in X and Y. Then the Tangents XT, YT, MT, will mutually Intersect one another in the Diameter produced in T (because the Sub-tangent QT is (*Art* 23. 32.) common to all the three). Draw the Ordinates EF and D*v*, and produce them until they Intersect the Tangents TX, TY in I, *x*, *d*, *c*. Then by the property of the Circle

$$\left.\begin{matrix} Dd \times bd \\ \text{or} \\ Dd \times Dc \end{matrix}\right\} : \left\{\begin{matrix} IE \times Ig \\ \text{or} \\ IE \times Ex \end{matrix}\right\} :: Xdq :: XIq :: \text{(by similar Triangles) } MDq : MEq$$

Again, by similar Triangles,

$Dd : EI :: TD : TE$. and

$Dc : Ex :: TD : TE$. *Ergo* $Dd \times Dc : EI \times Ex :: TDq : TEq :: Xdq : XIq :: MDq : MEq$. Therefore $TD : TE :: MD : ME$.

And by similar Triangles $TD : TE :: DP : EP$. Therefore $DP : EP :: MD : ME$. Therefore the Triangles MEP, MDP are (*Prop.* 6. *Elem.* 6.) similar, and the Angle PME is = PMD. Q. E. D.

COROL. XVI.

48. In the Ellipsis and in the opposite Sections; If from both the Foci (*Fig. Art.* 49.) P, P the right Lines PM, PM be drawn to any point M in the Section, their Sum in the Ellipsis and their Difference in the Hyperbola will always be equal to the Transverse Axis A B.

Let C be the Center of the Section and draw the Line CD, and let the right Line PD produced, cut PM (produced if need be) in S.

Then because the Angle PMD, (*Art.* 47.) SMD and MDS = MDP, and MD common to both, MS is = MP; SD = DP, and in the Ellipsis PS = PM + MP, and in the Hyperbola PS is = to their Difference.

Because

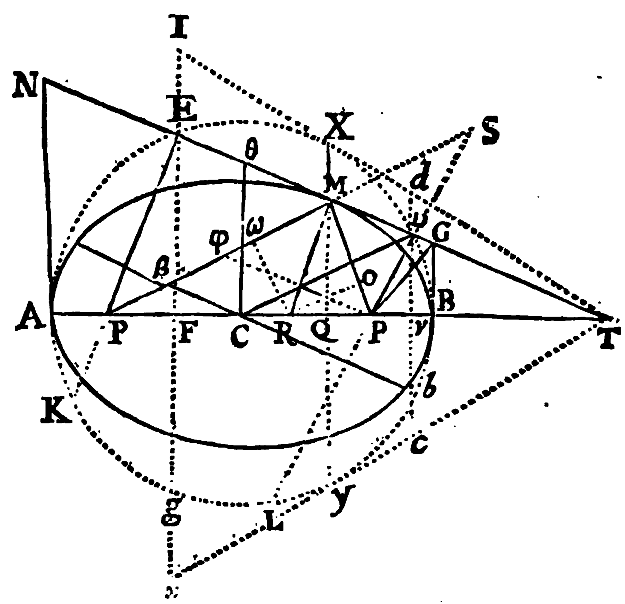
Because P P and P S are Bisected in C and D, therefore C D || P S, and P S = 2 C D = (because of the Circle) A B. Q. E. D.

If from either of the Foci, P be drawn P M to the point of Contact, and from C the Center of the Section be drawn C D Parallel to P M, until it cut the Tangent in D, then is C D = C B, and if P M; P D be produced to S, then is P D = S D and P S = A B

Being willing to avoid Tediousness, I have omitted the Schemes of the opposite Sections, which every Reader may easily Supply.

COROL. XVII.

49. If M R be drawn Perpendicular to the Tangent M T, until it Intersect the Axis in R, and if from R the Lines Rω, Ro be drawn Perpendicular to MP, MP. I say, Mω = Mo = $\frac{1}{2}$ the Parameter of the Axis: For the Angle E M P = (*Art.* 47.) D M P and R M Perpendicular to M T; therefore the Angle P M R = P M R, and (because the Angles at ω and o are right Angles and M R common) consequently, the Triangles R Mω, R Mo are similar and equal, whence Mω = Mo. Again, (because E M R is a right Angle) M R || E P, and the Triangles M P E, R Mω are similar.



Also, (because E M R = M D P) the Triangles P M R, P S P are similar.

Therefore P S : S P :: P M : M R. and

Mω : P E :: M R : P M. and by Multiplication

P S × Mω : S P × P E :: P M × M R : P M × M R, therefore P S × Mω = S P × P E = (*Art.* 48.) 2 P D × P E = (*Art.* 45.) $\frac{1}{2}$ Parameter × A B; But P S = A B, therefore Mω = $\frac{1}{2}$ the Parameter of the Axis = Mo. Q. E. D.

COROL. XVIII.

50. BQ (x) : BC ($\frac{1}{2}a$) :: QT $\left(\frac{2ax - 2xx}{a - 2x}\right)$: AT $\left(\frac{aa - ax}{a - 2x}.\right)$

And BC ($\frac{1}{2}a$) : AQ ($a - x$) :: BT $\left(\frac{ax}{a - 2x}\right)$: QT $\left(\frac{2ax - 2xx}{a - 2x}.\right)$

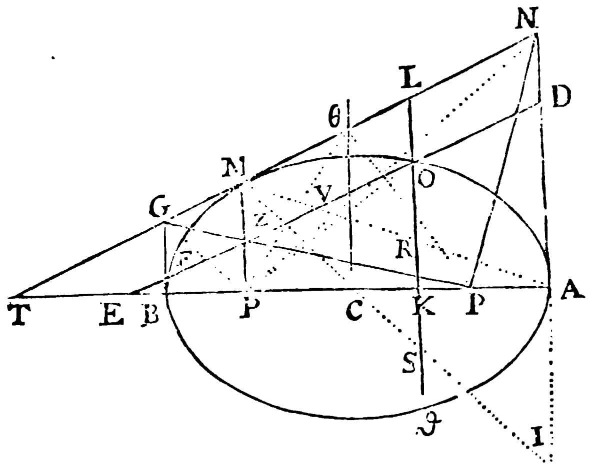
And consequently AT : BT :: AQ : BQ.

COROL. XIX.

51. If through the Center of the Section (*Fig. Art.* 49.) C be drawn the Diameter Cβ Parallel to the Tangent M T; I say, Mβ = C B. Draw Pϕ through the other Focus, Parallel to M T; Then, because P M E = (*Art.* 47.) P M D, therefore MϕP = M Pϕ and Mϕ = M P. And because Cβ || Pϕ and C P = C P, therefore Pβ = $\phi\beta$; that is, $\phi\beta$ = $\frac{1}{2}$ the difference between P M and M P or Mϕ, therefore (P M) Mϕ + $\phi\beta$ = $\frac{1}{2}$ P M + P M = (*Art.* 48.) $\frac{1}{2}$ A B = C B. Q. E. D.

COROL. XX.

52. In the Ellipsis and the opposite Sections; If A B be the Axis, and B G, A N Perpendicular thereto, Intersecting any Tangent M T in G and N, and if the Lines G P, N P be drawn to either Focus P, the Angle G P N will be a Right angle: For A N × B G = (*Art.* 37. 44) A P × P B, therefore A N : A P :: P B : B G, and the Angles comprehended N A P, P B G are Right angles, and consequently, the Triangles are similar, and the Angle P N A = B P G But P N A + A P N is = to a Right-angle = B P G + A P N, therefore G P N is a Right-angle.



And if the Conjugate Diameter be produc'd until it Intersect the Tangent in θ, then θ N = θ P = θ P = θ G; For B C = C A, therefore G θ = θ N. And if a Circle be describ'd on the Diameter G N, it will (*Prop.* 31. *Elem.* 3.) pass through P, P.

COROL. XXI.

53. If on the Focus (*Fig. Art.* 52.) P, we draw P M Perpendicular to the Axis A B, and Intersecting the Curve in M, and if the Tangent M T be drawn Intersecting the Conjugate Diameter produced, and the Perpendiculars A N, B G, in the points θ, N, G. I say, C θ = C A = C B; For B G × A N = (*Art.* 46.) P M × C θ = (*Art.* 44.) $\frac{1}{4}$ ∴ *b* = $\frac{1}{2}$ the Parameter × A C = (*Art.* 37.) P M × A C. Therefore C θ = C A = B C.

COROL. XXII.

54. If the Ordinate (*Fig. Art.* 52) P M be drawn through the Focus P, and the Lines B G, A N Perpendicular to the Axis Intersecting the Tangent M T in G and N. I say, B G = B P and A P = A N; For (*Art.* 50.) AT : BT :: AP : PB, and by Similar Triangles AT : BT :: AN : BG *Ergo ex equo*, AP : PB :: AN : BG. Therefore the Rectangular Triangles P A N, P B G are Similar, and the Angle A P N is = B P G. But the Angle G P N is (*Art.* 52) a right Angle, therefore the equal Angles A P N, B P G are each equal to 45°. and Consequently the Angle A P N = A N P, and B P G = B G P, and B P = B G, and A P = A N.

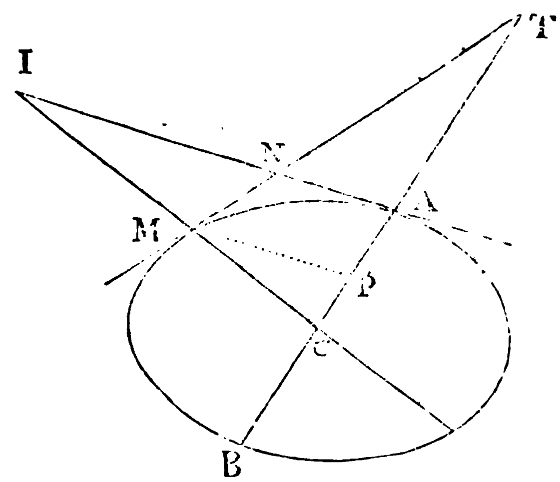
COROL. XXIII.

55. The same things being Supposed as in the precedeing *Corollary.* If A M be drawn from the Vertex A, to the point of Contact M, and if any Ordinate as L K be drawn to the Axis A B, Intersecting A M in R. I say, the right Line K P Intercepted between the Focus and the Ordinate is = L R, comprehended between the Tangent M N and A M; For, A N being ∥ R L, the Triangles M A N, M R L are Similar; therefore M N : M L :: N A : L R. also M N : M L :: P A : P K, therefore *ex equo*, N A : L R :: P A : P K; But N A = (*Art.* 54.) P A. *Ergo* L R = P K.

COROL.

COROL. XXIV.

56. The Method of determining Tangents to the Ellipsis, in respect of any Diameter being the same as when the Axis is given; in the annex'd Diagram, and in *Fig. Art.* 52. produce the Diameter MC until it Intersect the Tangent NA (also produc'd) in I. I say, the Triangles ANT, MNI are equal: For CP, CB or CA, CT are (*Art.* 46.) continually Proportional, and (by similar Triangles) CP : CA :: CM : CI. Therefore CA : CT :: CM : CI; and consequently the Triangles CMT, CAI are (*Prop.* 15. *Elem.* 6.) equal; and if to each be Added (or from each be Subtracted) the Quadrilateral Figure MCAN, the Triangles ANT and MNI will be equal



And if DE be drawn through the Point O (where LK intersects the Curve) Parallel to the Tangent NMT, then the Quadrilateral Figure DOSI will be = Triangle ADE; For the Triangle ANT is = MNI, and Subtracting from both the Quadrilateral Figure PMNA, there will remain TPM = CAI — CPM.

Then because the Triangles TPM, EKO are similar, it is TPM : EKO :: (*Prop.* 19. *Elem.* 6.) PM*q* : KO*q* :: BP × PA : BK × KA :: CB*q* (or CA*q*) — CP*q* : CA*q* — CK*q* :: Triangle CAI — Triangle CPM : Triangle CAI — Triangle CKS :: Triangle TPM : Quadrilateral Figure AKSI, therefore EKO = AKSI. And adding to both the Quadrilateral Figure ADOK, we have ADE = DOSI.

COROL. XXV.

57. The same things being suppos'd as in the precedeing *Corollary*. I say, AD*q* : AN*q* :: DF × DO : NM*q*; For △ANT = (*Art.* 56.) MNI and △ADE = □DOSI. Therefore,

△ANT : △ADE :: △IMN : □DOSI, and
∵ △ANT : △ADE :: (*Prop.* 19. *El.* 6.) AN*q* : AD*q*. Therefore
∴ △IMN : □DOSI :: AN*q* : AD*q*.

And by similar Triangles,

△IMN : △IZD :: NM*q* : DZ*q*, and
△IZD : △SZO :: DZ*q* : OZ*q*, and by Division
△IZD : △IZD — △SZO or DOSI :: DZ*q* : DZ*q* — OZ*q* = DF × DO.

Therefore *Ex æquo* △IMN : DOSI :: NM*q* : DF × DO.

And consequently,

AN*q* : AD*q* :: NM*q* : DF × DO, and inversely AD*q* : AN*q* :: DF × DO : NM*q*.

I say also, if the Line AM join the Points of Contact of the Tangents NA, NM and intersect DF in V, that then DF × DO = DV*q*.

For it was before AN*q* : NM*q* :: AD*q* : DF × DO.

And because the Triangles ANM, ADV are similar therefore (*Prop.* 22. *El.* 6.)

AN*q* : NM*q* :: AD*q* : DV*q*.

And consequently AD*q* : DF × DO :: AD*q* : DV*q*. *Ergo* DF × DO = DV*q*.

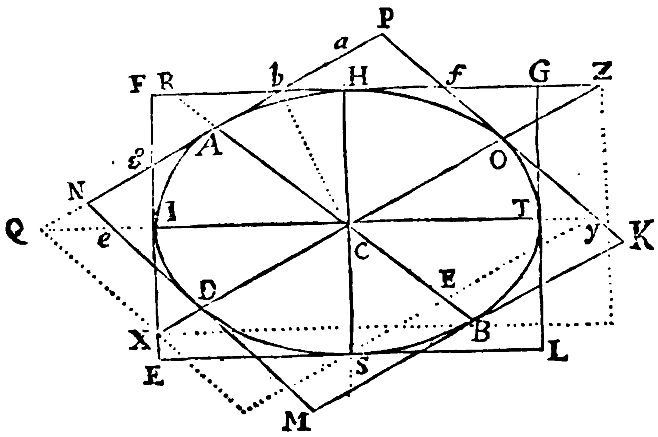
COROL. XXVI.

58. The same things being supposed as in *Art.* 55. If the right Line P O be drawn from the Focus P to the Point O (where the Line L R drawn at pleasure intersects the Curve) I say, K L is = P O.

For because the right Lines N M, N A touch the Section in M and A, and A M Connects the Points of Contact, and L *s* is drawn Parallel to the Tangent N A, therefore (*Art.* 57.) L *s* x L O = L R *q* = (*Art.* 55.) K P *q*; But L *s* x L O + K O *q* = K L *q*. Therefore K P *q* + K O *q* = K L *q* = P O *q*; and consequently, K L = P O. Q. E. D.

COROL. XXVII.

59. If a Parallelogram M N P K be described about an Ellipsis, and have its sides Parallel to the Conjugate Diameters A B, D O; and if any other Parallelogram, *v. g.* E L G F be described about the same Ellipsis, having its sides Parallel to any other two Conjugate Diameters. I say, the Parallelogram M N P K = Pgram. E L G F.



Produce I T until it Intersect P N in Q, and P K in Y; and produce D O until it intersect F E in X and F G in Z. Then the Triangle *b* A R = Triangle *b a* H And C A *a* = C H R. In like manner the Triangle C I X is = Triangle C D *e*, which is equal and similar to the Triangle C O Y.

The Parallelogram C Z *b* Q is divided into two equal Triangles by the Line C *b*, *viz.* into C *b* Z and C *b* Q; And because the Triangles *a b* H, A *b* R are equal and have each the Angle at *b* equal. Therefore *a b* : *b* A :: R *b* : *b* H, and by Composition *a* A : *b* A :: R H : *b* H. Now the Triangle C *a* A : Triangle C *b* A :: A *a* : *b* A :: R H : *b* H. And the Triangle C R H : Triangle C *b* H :: R H : *b* H. Therefore the Triangle C *a* A : Triangle C R H :: Triangle C *b* A : Triangle C *b* H. Whence it is manifest that the Triangle C *b* A = Triangle C *b* H, and because the Triangle Z C *b* = Q C *b*. Therefore Z C *b* — C *b* H = Q C *b* — C *b* A, that is C H Z = C A Q.

In the Parallelogram Y Q, the Parallelogram C A P O is a mean Proportional between the two Parallelograms Y C and C Q (because the Parallelogram Y C : Parallelogram P C :: Y O : O P :: Y C : C Q :: P A : A Q :: Parallelogram P C : Pgm. Q C,) and in the Parallelogram Z X, the Parallelogram C H F I is a mean Proportional between the two Parallelograms Z C and C X. Now the Parallelograms Q C and Z C, Y C and X C are equal (because the Triangles Z C H, Q C A are equal and the halves of the Parallelograms Q C and Z C, and the Triangles C X I, C Y O are equal and the halves of the Parallelograms X C and Y C) and consequently, the Parallelogram C P is = Parallelogram C F = $\frac{1}{4}$ the Parallelogram F L = $\frac{1}{4}$ Parallelogram P M. and the Parallelogram M N P K = Parallelogram E L G F. Q. E D.

PROP. V.

To draw Tangents to all sorts of single Geometrical or Algebraic Curves.

60. Let x represent the Intercepted Diameter, and y the Ordinate of any Algebraic Curve, and let $\frac{y\dot{x}}{\dot{y}}$ express the Value of the Subtangent in General. Assume A general Equation expressing the Nature of Infinite Sorts of Algebraic Curves, *v. g.* $fx^m + gy^n + bx^r y^s + b = 0$. In which f is the Coefficient of that Term Af-

fected with the Indeterminate Quantity x or its Powers, and m the Exponent of the said Power of x; g the Coefficient of the Term Adfected with y or its Powers, and n the respective Exponent of y. The third Term in the general Equation, represents those in any given Equation, Adfected with x and y together, or with any Rectangle under them or their Powers, b is the Coefficient of the said Term, r the Exponent of x, and s that of y. Lastly, b represents any Invariable Quantity.

Having chosen this Equation Expressing the Nature of an infinite Number of single Geometrical Curves in General; find the Fluxion thereof, *viz.*

$$mfx^{m-1}\dot{x}+ngy^{n-1}\dot{y}+rby^{s}x^{r-1}\dot{x}+sbx^{r}y^{s-1}\dot{y}=0:$$

Then by Transposition,

$$mfx^{m-1}\dot{x}+rby^{s}x^{r-1}\dot{x}=-ngy^{n-1}\dot{y}-sbx^{r}y^{s-1}\dot{y}.$$

And Dividing each side of the Equation by $mfx^{m-1}+rby^{s}x^{r-1}$

$$\dot{x}=\frac{-ngy^{n-1}\dot{y}-sbx^{r}y^{s-1}\dot{y}}{mfx^{m-1}+rby^{s}x^{r-1}}.$$

And substituting this Value of $\dot{x}$ in $\frac{y\dot{x}}{\dot{y}}$ the general Value of the Sub-tangent, we shall have the Sub-tangent $=\frac{y\dot{x}}{\dot{y}}$ equal to

$$\frac{-ngy^{n-1}y-sbx^{r}y^{s-1}y}{mfx^{m-1}+rby^{s}x^{r-1}}=\frac{-ngy^{n}-sbx^{r}y^{s}}{mfx^{m-1}+rby^{s}x^{r-1}}.$$

Which is a general Theorem, Expressing the Value of the Sub-tangent in all Geometrical Curves expressed by the foresaid general Equation.

EXAMPLE. I.

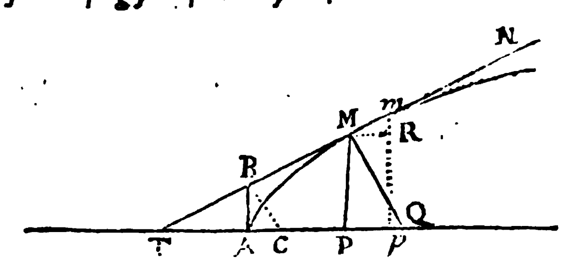
Let it be required to find the Value of the Sub-tangent PT *in the Common Parabola.*

The Equation expressing the Nature of the Parabola is $ax-yy=0$. Which being suppos'd equal to the general Equation we have:

$$ax-yy=fx^{m}+gy^{n}+bx^{r}y^{s}+b.$$

Or,

$$ax-y^{2}+\cdot\cdot+\cdot\cdot=fx^{m}+gy^{n}+bx^{r}y^{s}+b.$$



And if we Compare the Terms of the Equation Expressing the Nature of the Parabola, with the respective Terms in the general Equation *i. e.* If we Compare the Terms Affected with the same flowing Quantities, the Coefficients f, g, b, and the Exponents m, n, r, s, may be determined thus;

The Terms Adfected with x only are ax, and fx^{m}: Therefore suppose $ax=fx^{m}$, then 'tis plain that $f=a$ and $m=1$.

The Terms Adfected with y only are y^{2} and gy^{n}, which being also supposed equal, *viz.* $-y^{2}=gy^{n}$. We have $g=-1$ and $n=2$.

And because the remaining two Terms $bx^{r}y^{s}+b$ have no Terms that answer to them in the given Equation, they must be put equal to nothing, *viz.* $bx^{r}y^{s}=0$, or $bx^{r}y^{s}=0\times x^{0}y^{0}$ and $b=0$. And then $b=0, r=0, s=0$, and $b=0$.

Having thus determined, f, g, m and n, Substitute their Values in the general Value of the Sub-tangent, and reject all the Terms Adfected with the Coefficients b, r, s, so have you the Value of the Sub-tangent in the Parabola.

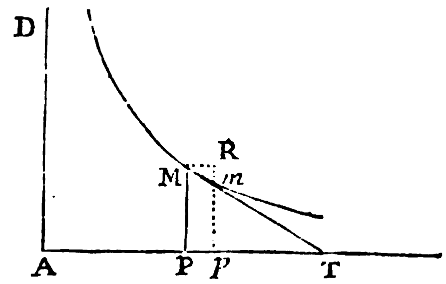
PT =

$$PT = \frac{-ngy^n - sbx^r y^s}{mfx^{m-1} + rby^s x^{r-1}} = \frac{-2\times -1\times y^2 - 0\times 0\times x^0 y^0}{1\times a\times x^{1-1} + 0\times 0\times y^0 x^{0-1}}$$

$$= \frac{2y^2}{1\times a\times x^0} = \frac{2y^2}{a} = (\text{because } ax = yy)\ \frac{2ax}{a} = 2x. \quad \text{Q. E. I.}$$

EXAMPLE. II.

If M*m be an Hyperbola between the Asymptotes* AP, AD, *and the Sub-tangent* PT *requir'd.*



Let the Equation expressing the Nature of the Hyperbola $aa = xy$ be put $= fx^m + gy^n + bx^r y^s + b$.

Then $f = 0$, $m = 0$, $g = 0$, $n = 0$, $b = -1$, $r = 1$, $s = 1$.

And consequently $PT = \dfrac{-ngy^n - sbx^r y^s}{mfx^{m-1} + rby^s x^{r-1}} = \dfrac{-1\times -1\times x^1 y^1}{+1\times -1\times y^1\times x^{1-1}} =$

$\dfrac{+1\times xy}{-1\times yx^0} = \dfrac{xy}{-y} = -x.$ Q. E. I.

EXAMPLE III.

If it be required to determine the Sub-tangent of the Curve, whose Nature is expressed by $ax^3 + cy^4 - 9x^5y - p = 0$.

Suppose $ax^3 + cy^4 - 9x^5y - p = fx^m + gy^n + bx^r y^s + b$:

Then $f = a$, $m = 3$, $g = c$, $n = 4$, $b = -9$, $r = 5$, $s = 1$.

And consequently the Value of the Sub-tangent $\dfrac{-ngy^n - sbx^r y^s}{mfx^{m-1} + rby^s x^{r-1}}$ is $=$

$\dfrac{-4\times c\times y^4 - 1\times -9\times x^5 y}{3ax^2 + 5\times -9\times yx^4} = \dfrac{9x^5y - 4cy^4}{3ax^2 - 45x^4y}.$

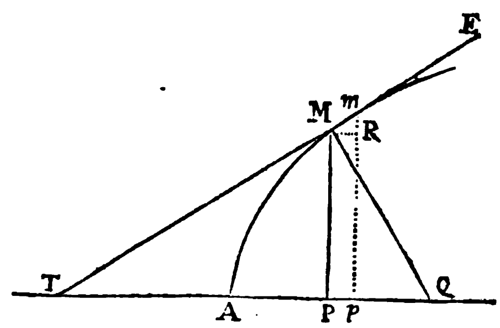
In like manner the Sub-tangents of all other simple Algebraic Curves may be determin'd.

PROP.

PROP. VI.

To deduce Universal Rules for drawing Tangents to all sorts of Geometrical Curves, when the given Equation expresses the Relation between the Ordinate and Intercepted Diameter.

61. Suppose the Intercepted Diameter $AP = x$, and the Ordinate $PM = y$; and let the Equation of the Curve be so order'd, that all the Terms (being brought over to one side) be equal to Nothing. *v. g.* If the Equation expressing the Nature of the Curve be $x^3 + yxx + lx = y^3 + yyx + by + s$. Then $x^3 + yxx + lx - y^3 - yyx - by - s = 0$. 'Tis requir'd to draw the Line MT to touch the Curve in M.



1°. Reduce the Equation to Fluxions. That is, put $(Ap =)\ x + \dot{x}$ for $(AP =)\ x$; and $(pm =)\ y + \dot{y}$ for $(PM =)\ y$, and then the Equation of the Curve will be as Follows.

$$
\begin{array}{l|l}
x^3,\quad x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3 & -yyx,\ -yyx - 2xy\dot{y} - x\dot{y}^2 \\
+yxx,\ +yxx + 2yx\dot{x} + y\dot{x}^2 & \qquad -yy\dot{x} - 2y\dot{y}\dot{x} - \dot{x}\dot{y}^2 \\
\qquad + xx\dot{y} + 2x\dot{x}\dot{y} + \dot{y}\dot{x}^2 & -by,\ -by - b\dot{y} \\
+lx,\ +lx + l\dot{x} & -s,\ -s. \\
-y^3,\ -y^3 - 3y^2\dot{y} - 3y\dot{y}^2 - \dot{y}^3 &
\end{array}
$$

2°. In this new Equation destroy all the Terms wherein neither $\dot{x}$ nor $\dot{y}$ is found, because the Sum of those Terms made up the given Equation, which was (after Transposition) equal to Nothing. Hence it is evident, that no Term not affected with $\dot{x}$ or $\dot{y}$ can be kept in the New Equation.

3°. Destroy all the Terms wherein the Powers or Rectangles of $\dot{x}$ and $\dot{y}$ are found because all such Terms compared with the others are Incomparably less than they, and consequently may be rejected. Whence 'tis manifest, that those Terms affected with $\dot{x}$ or $\dot{y}$ of one Dimension only, will remain in the New Equation. Therefore $3x^2\dot{x} + 2yx\dot{x} + xx\dot{y} + l\dot{x} - 3y^2\dot{y} - 2xy\dot{y} - yy\dot{x} - b\dot{y} = 0$. And Transposing all the Terms Adfected with $\dot{y}$ to the other side of the Equation we have,

$$3x^2\dot{x} + 2yx\dot{x} + l\dot{x} - yy\dot{x} = 3y^2\dot{y} + 2xy\dot{y} - xx\dot{y} + b\dot{y}$$

Which Equation being reduc'd to an Analogy, it will be

$$3yy + 2xy + b - xx : 3xx + 2yx + l - yy :: \dot{x} : \dot{y}$$

4°. But the Triangles MRm, TPM are similar, therefore $MR\ (\dot{x}) : Rm\ (\dot{y}) :: PT\ (t) : PM\ (y.)$

Therefore, (§. 3°) $3yy + 2xy + b - xx : 3xx + 2yx + l - yy :: t : y$.

And Reducing the Analogy to an Equation, we have $3y^3 + 2xy^2 + by - xxy = 3txx + 2tyx + tl - tyy$. Now from hence to deduce an *Universal Method* for drawing Tangents to all such Curves observe,

 5°. That

5°. That all Equations expressing the Natures of such Curves are Compounded of two sorts of Terms, *viz. Simple* (the Invariable Quantities, l, b, s not being Considered) *when the Term consists of x or y, or of the Powers of either*, as y^3, by, x^3, lx. or *Mixt when the Term consists of several Flowing Quantities*, as when x or any of its Powers is Multiplied into y or any of its Powers; as yx, yyx, y^2x^2, by^2x.

6°. That from every simple Term in the Equation of the Curve, there arises a Term of the Equation which determines the Sub-tangent. For if the Powers, &c. of $y + \dot y$ and $x + \dot x$ be Substituted for those of y and x, in the Equation of the Curve, and if the Terms of the said Powers be written in order as in common Algebra, then 'tis manifest (§. 2°, 3°.) that only the second Term will remain.

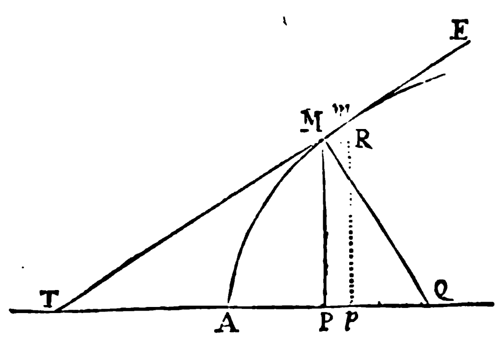
7°. That the new Term in the Subtangential Equation which arises from the simple Term y, y^2, &c. is equal to the respective Term of the Equation of the Curve, multiplied by the Exponent of the Flowing Quantity or the Number of its Dimensions.

8°. That the Terms in the Subtangential Equation affected with x, which arises from the simple Term Involving x or its Powers in the Equation of the Curve, is the same with the said respective Term multiplied by the Exponent of x, one of the Dimensions of x being destroyed and t substituted in its place. Thus if xx be a Term in the Equation of the Curve then $2tx$ is the respective Term in the Subtangential Equation.

9°. The Signs prefixt to the Terms of both Equations are the same. Hence.

General Rule,

To draw Tangents to Curves, when the Equation of the Curve consists of Simple Terms.



10°. Let the Equation Expressing the Nature of the Curve A M m be $y^4 - sy^3 + by - x^4 + lxx - mx + n = 0$, and order the same so that all the y be on the left and all the x on the right side of the Equation thus, $y^4 + by - sy^3 = x^4 - lxx + mx - n$. Multiply every Term by the Exponent of the Flowing Quantity y or x in the same respectively, and in every Term Affected with x, Substitute t for one of its Dimensions, and then the Subtangential Equation will stand thus.

$$4y^4 - 3sy^3 + by = 4tx^3 - 2ltx + mt, \text{ and}$$

consequently $PT\,(t) = \dfrac{4y^4 - 3sy^3 + by}{4x^3 - 2lx + m}$.

Again, Let this Equation $y^m = x$ (the Parameter being Supposed $= 1$.) express the Nature of all sorts of Parabola's when the Exponent m represents a Positive number, and all sorts of Hyperbola's when it represents a Negative number. Then (§. 7°.) $my^m \square x$ and (§. 8°.) $my^m = t$. But $y^m = x$. Therefore $my^m = mx = t$.

11°. And If we Consider the Mixt Terms in the Equation of the Curve, it will appear that from every one of them there will arise as many Terms in the Subtangential Equation, as there are Flowing Quantities in the respective Term. E G. Let the mixt Term be $y^m x^n$, where m and n are the Exponents of the Powers of y and x, then for (§. 1°.) y^m there will arise $y^m + my^{m-1}\dot y$, &c. and from the Term x^n there will arise $x^n + nx^{n-1}\dot x$, &c. which being multiplied together, the Product is (destroying those Terms wherein neither $\dot x$ nor $\dot y$ is found §. 2°, 3°.) $mx^n y^{m-1}\dot y + ny^m x^{n-1}\dot x$, and consequently the Terms in the subtangential Equation arising from the Mixt Term $y^m x^n$ are (§. 7°. 8°.) $my^m x^n + ntx^{n-1}y^m$. 12°. And

12°. And the Affection of those Terms is the same, as if the Equation be ordered according to §. 10. so that the same Term $y^m x^n$ stand on both sides of the Equation with contrary Signs, and y be considered only on the left side and x on the right side of the Equation, according to § 7°. 8°.

And if the said Terms be placed on both sides of the Equation, it is manifest that their Signs must be Contrary, because both the new Terms (§. 11°.) have the same Sign, and in reducing the Equation as in (§. 10.) all the Terms affected with t must be brought over to one side, according to the Common Rules of Transposition. Hence we have another.

General Rule.

To draw Tangents to Curves, when the Equation of the Curve consists of Simple or Mixt Terms.

1°. Order the Equation of the Curve so, that all the Terms wherein y or the Ordinate is found, stand on the left, and all those wherein x or the Intercepted Diameter is found stand on the right hand of the Equation; rejecting all those Terms wherein neither x nor y is found, and then all the mixt Terms will stand on both sides of the Equation with contrary Signs.

2°. Multiply every Term on the left side of the Equation by the respective Dimension of y (§. 7°. 12°.)

Multiply every Term on the right side of the Equation by the respective Dimension of x, and divide every Term by x, and multiply the Quotients by t (§. 8°. 12°.)

And thus from any Equation expressing the Relation between the Ordinate and Abscissa, we may find the Subtangential Equation, and consequently the Subtangent it self.

Example; Suppose the Equation of the Curve to be (§. 1°.) $x^3 + yxx + lx - y^3 - yyx - by - s = 0$. Then $y^3 + yyx + by - yxx$ is the left side of the Equation. Therefore the Equation will appear in this Form $y^3 + yyx + by - yxx \;\square\; x^3 + yxx + lx - yyx$ therefore the Subtangential Equation is $3y^3 + 2yyx + by - yxx = 3txx + 2txy - tyy + lt$. The same as before, and the Subtangent $PT = t = \dfrac{3y^3 + 2yyx + by - yxx}{3xx + 2xy - yy + l}$.

What I have here demonstrated of Curves Concave towards the Axis, may be applied to Figures Convex towards their Axes.

PROP. VII.

To draw Tangents to Curves, when the Equation expressing the Relation between the Ordinate and Intercepted Diameter, Involves Irrational Quantities and Fractions.

62. Suppose $AP = x$; $Pp = \dot{x}$; $PM = y$; $Rm = \dot{y}$, and let the Equation Expressing (*See the Fig. in the foregoing Page*) the Nature of the Curve AMm be

$$\frac{x}{y}\ \frac{\overline{a+bx} \times \overline{c-xx}}{\overline{ex+fxx}^2} + ax\sqrt{gg+yy} + \frac{yy}{\sqrt{bb+lx+mxx}} = 0.$$

'tis requir'd to draw the line TM, which shall touch the Curve in M; or which is the same thing, it is required to find the Proportion between the Sub-tangent PT and the Ordinate PM.

For Brevities sake, Suppose $a+bx = n$; $\overline{c-xx} = p$; $ex+fxx = q$; $gg + yy = r$; $bb + lx + mxx = s$. Then Substituting these Quantities in place of those in the first, there will arise this *Second Equation* $\frac{x}{y} + \frac{np}{qq} + ax\sqrt{r} + \frac{yy}{\sqrt{s}} = 0$, and reducing this Equation to Fluxions, there will arise this *Third Equation*,

$\dot{x}y$

$\frac{y\dot{x} - x\dot{y}}{yy} + \frac{pq\dot{n} + nq\dot{p} - 2np\dot{q}}{q^3} + a\dot{x}\sqrt{r} + \frac{ax\dot{r}}{2\sqrt{r}} + \frac{4sy\dot{y} - yy\dot{s}}{2s\sqrt{s}} = 0$. But $\dot{n} = b\dot{x}$; $\dot{p} = -2x\dot{x}$; $\dot{q} = e\dot{x} + 2fx\dot{x}$; $\dot{r} = 2y\dot{y}$; and $\dot{s} = l\dot{x} + 2mx\dot{x}$; and ſubſtituting theſe Values in the reſpective Terms of the *Third Equation*, there will ariſe this *Fourth Equation* $\frac{y\dot{x} - x\dot{y}}{yy} + \frac{bpq\dot{x} - 2qnx\dot{x} - 2pne\dot{x} - 4pnfx\dot{x}}{q^3}$

$+ a\dot{x}\sqrt{r} + \frac{2axy\dot{y}}{2\sqrt{r}} + \frac{4sy\dot{y} - lyy\dot{x} - 2myyx\dot{x}}{2s\sqrt{s}} = 0$. which is an Equation wherein the Fluxions $\dot{x}$ and $\dot{y}$ only remain, and either one or t'other will be found in every Term of the Equation, cleared of the Radical Signs and all other Incumbrances. Whence if all the Terms affected with $\dot{x}$ be kept on one ſide, and all thoſe affected with $\dot{y}$ be Tranſpoſed to the other ſide of the Equation, and if the Equation ſo ordered be reduced to Analogy, it will be,

$$\frac{x}{yy} - \frac{axy}{\sqrt{r}} - \frac{2y}{\sqrt{s}} : \frac{1}{y} + \frac{bp - 2nx}{qq} - 2pn \times \frac{\overline{e + 2fx}}{q^3} + a\sqrt{r} - yy \times \frac{\overline{l + 2mx}}{2s\sqrt{s}} :: \dot{x} : \dot{y} :: PT : PM.$$

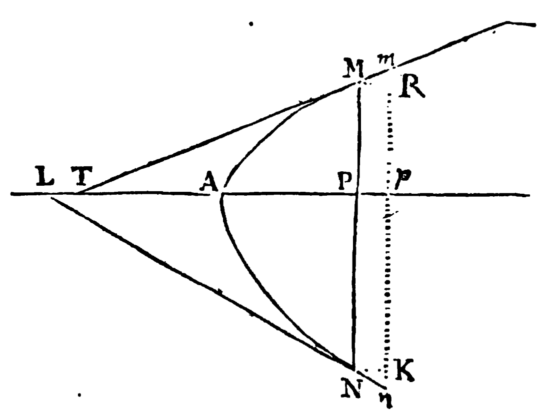
Now in this Analogy, x and y are given, and conſequently the Values of n, p, q, r, s are known, and three Terms in the Analogy are given to find the fourth Term P T. Q. E. I.

Thus I have ſhewn the Uſe of Fluxions in the Reſolution of a Queſtion otherwiſe perplext enough, that the Learner may the better underſtand how to apply the Rules which hitherto we have explained to the moſt Intricate and Perplext Queſtions of this Nature.

PROP. VIII.

Let the Curve AMm *and its Tangent* MT *be given; and let another Curve* ANn *be applied to the ſame Axis* AP; *and ſuppoſe the Relation between the Ordinates* PM, PN *to be given: 'Tis requir'd to draw the Line* NL, *which ſhall touch the Curve* ANn *in* N.

Suppoſe AP $= x$, PM $= y$, PT $= t$, PN $= u$, Pp = MR = NK $= \dot{x}$, R$m = \dot{y}$, K$n = \dot{u}$, PL $= d$.



Now becauſe the Triangles mRM, MPT are ſimilar; therefore mR $(\dot{y})$: RM $(\dot{x})$ :: MP (y) PT (t) and conſequently $\dot{x} = \frac{t\dot{y}}{y}$. Again, the Triangles nKN, NPL are ſimilar, and nK $(\dot{u})$: KN $(\dot{x})$ :: PN (u) : PL (d) therefore $\dot{x} = \frac{d\dot{u}}{u} = \frac{t\dot{y}}{y}$ and conſequently $u = \frac{tu\dot{y}}{d\dot{y}}$. But PL $= \frac{u\dot{x}}{\dot{u}}$. find the Value of $\dot{u}$ in x or the Value of $\dot{x}$ in u by help of the Equation of the Curve, and ſubſtituting the ſame in the laſt Value of the Sub-tangent, all the Fluxions will be deſtroy'd; and the Sub-tangent will be expreſs'd in known Terms.

For

For Instance, let a be an invariable Quantity, and suppose the Relation between PM and PN to be expreſs'd by this Equation $ax\ (yy) - uu = aa$, then is, $a\dot{x}$ $(2y\dot{y}) = 2u\dot{u}$, and $\frac{a\dot{x}}{2u} = \dot{u}$. Therefore the Sub-tangent PL $= \frac{u\dot{x}}{\dot{u}}$ is $= \frac{2uu}{a} =$ (becauſe $ax - aa = uu$) $2x - 2a$.

But if the Relation between the Ordinates be expreſs'd by the Letters y, u, *v.g.* $yy - uu = aa$, then we may proceed thus: $2y\dot{y} = 2u\dot{u}$ and $y\dot{y} = u\dot{u} = \frac{tuu\dot{y}}{dy}$, and dividing by $\dot{y}$, and multiplying by dy we have $dyy = tuu$, and d (= PL) $= \frac{tuu}{yy}$, and if the Curve AMm be a Parabola, then $ax = yy$, and $t =$ (*Art.* 25.) $2x$ therefore PL $= \frac{tuu}{yy} = \frac{2uu}{a} = 2x - 2a$ as before.

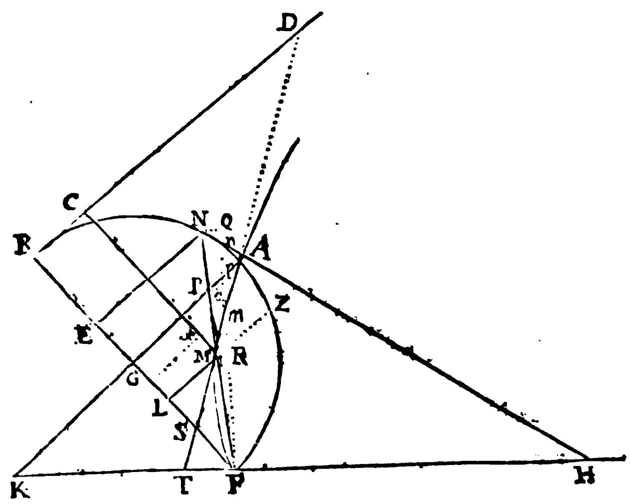
And if we ſuppoſe the Rectangle MP × PN to be always equal to the Square of a, then the Equation expreſſing the Nature of the Curve is $yu = aa$, and $y\dot{u} = -u\dot{y}$, and conſequently $\dot{u} = \frac{-u\dot{y}}{y} = \frac{tu\dot{y}}{dy}$ and d (PL) $= -t = -$ PT. and becauſe the Sign is Negative; it is evident, that as the Ordinates of the Curve AMm Increaſe; ſo thoſe of Nn Decreaſe; that is, the Curve ANn is Convex towards the Axis AP, and the Sub-tangent muſt be taken from P on the ſide of the Axis oppoſite to A.

Again, Let the Nature of the Curve ANn be expreſſed by this Equation $yu = xx$. Then $y\dot{u} + u\dot{y} = 2x\dot{x}$. But $\dot{y} = \frac{y\dot{x}}{t}$, therefore $y\dot{u} + u\dot{y}$ is $= y\dot{u} + \frac{yu\dot{x}}{t} = 2x\dot{x}$, and $\dot{u} = \frac{2tx\dot{x} - yu\dot{x}}{ty}$, therefore PL $\left(= \frac{u\dot{x}}{\dot{u}}\right) = \frac{tuy}{2tx - uy} =$ (becauſe $yu = xx$) $\frac{txx}{2tx - xx} = \frac{tx}{2t - x}$.

PROP. IX.

The Curve Line AN *and the Diameter* AP *and a Determinate Point* F *being given; Let the Curve* FMA *be describ'd, so that drawing any Diameter* FMPN, *the Relation between* FM, FP, FN *be express'd by any given Equation: 'Tis requir'd to draw the Tangent* MT *to touch the Curve* FMA *in the Point* M.

64. Draw the Line KFH Perpendicular to NF, till it Intersect the Diameter AP produc'd in K, and the given Tangent NH in H, and on the Center F with the Distances FN, FP, FM describe the little Arches NQ, Po, MR bounded by FQ infinitely near to FN. Which being done, suppose FR $= s$, FH $= t$, FM $= y$, FK $= x$, FN $= u$. Then the Triangles KFP, Pop; FMR, FPo, FNQ; HFN, NQn are similar; therefore PF (x) : FK (s) :: po $(\dot{x})$: P$o = \frac{s\dot{x}}{x}$. And FP (x) : FM (y) :: P$o\left(\frac{s\dot{x}}{x}\right)$: MR



$= \frac{sy\dot{x}}{xx}$. And FP (x) : FN (u) :: P$o\left(\frac{s\dot{x}}{x}\right)$: NQ $= \frac{su\dot{x}}{xx}$. And again, HF (t) : FN (u) :: NQ$\left(\frac{su\dot{x}}{xx}\right)$: Q$n = -\frac{suu\dot{x}}{txx}$. And lastly, R$m$ $(\dot{y})$ RM $\left(\frac{sy\dot{x}}{xx}\right)$ :: FM (y) : FT $= \frac{syy\dot{x}}{xx\dot{y}}$.

Now by help of the Fluxions of the given Equation, we may find the Value of $\dot{y}$ in $\dot{x}$ and $\dot{u}$, and therein Substituting for $\dot{u}$ its Negative Value $-\frac{suu\dot{x}}{txx}$ (because while x Increases, u Decreases), all the Terms will then be affected with $\dot{x}$, so that putting the Value of $\dot{y}$ found thus, in place of $\dot{y}$ in the Value of the Subtangent $\frac{syy\dot{x}}{xx\dot{y}}$ all the Fluxions will be destroy'd, and the Value of FT will be express'd in known Terms.

EXAMPLE.

65. Let the Curve AN be a Circle passing through the given point F (scituated so in respect of the Diameter AP, that the Line FB Perpendicular to AP, pass through G the Center of the Circle) and let PM be always $=$ PN, and draw ML and NE $\parallel$ AG, then is GL $=$ GE. Whence it is manifest that the Curve FMA is that which is now called the *Cissoide*. And because PM $=$ PN, therefore the Equation Expressing the Nature of the Curve is $u + y = 2x$, and $\dot{u} + \dot{y} = 2\dot{x}$. And (Substituting,

(Substituting $-\frac{suu\dot{x}}{txx}$ for $\dot{u}$) consequently $\dot{y} = \frac{2txx\dot{x} + suu\dot{x}}{txx}$. Whence FT $=$

$\frac{syy\dot{x}}{xx\dot{y}} = \frac{tsyy}{2txx + suu}$.

And if the point M happen to fall in the point A, then the Lines F M, F N, F P will be equal each to F A. Also F K will be $=$ F H (for then K F H is Perpendicular to A F, and the Angle K A F $=$ F A H, and F A is common to both Triangles) and

F T $= \frac{tsyy}{2txx + tuu} = \frac{x^4}{3x^3} = \frac{1}{3}x = \frac{1}{3}$ A F.

Another way.

The Tangent M T may be otherwise determined (the same way as in the Parabola) thus: Draw N E and M L Perpendicular to the Diameter F B, and find the Equation Expressing the Relation of the Abscissa F L to the Ordinate L M in this manner: Suppose F B $= 2a$, F L or B E $= x$, L M $= y$; then the Triangles F L M, F E N are similar, therefore F L (x) : L M (y) :: F E : E N :: E N $(\sqrt{2ax - xx})$: B E $(= x.)$ Therefore the Equation of the Curve is $xx = \sqrt{2axyy - xxyy}$, and $x^4 = 2axyy - xxyy$, that is, by Division

$\frac{x^3}{2a - x} = yy$ and $\frac{6ax^2\dot{x} - 2x^3\dot{x}}{\overline{2a - x}|^2} = 2y\dot{y}$, and $\frac{3ax^2\dot{x} - x^3\dot{x}}{y \times \overline{2a - x}|^2} = \dot{y}$. But L S

$= \frac{y\dot{x}}{\dot{y}} = \frac{yy \times \overline{2a - x}|^2}{3ax^2 - x^3} =$ (Substituting $\frac{x^3}{2a - x}$ for yy) $\frac{x^3 \times \overline{2a - x}}{3ax^2 - x^3} =$

$\frac{2ax - xx}{3a - x}$.

And if B D be Erected Perpendicular to B F, and M C be drawn Parallel to B F and T M produced to D, then the Value of C D, which determines the Tangent to the point M may be found in this manner: Let L Z be $= u$, B L $= 2a - x$, and C D $= t$. Then C D (t) : C M $(2a - x)$ :: xm $(\dot{y})$: xM $(\dot{x})$ whence $\dot{y} =$

$\frac{t\dot{x}}{2a - x}$. And by the property of the Circle $2ax - xx = uu$, therefore $2a\dot{x}$

$- 2x\dot{x} = 2u\dot{u}$ and $\dot{u} = \frac{2a\dot{x} - 2x\dot{x}}{2u}$. And because in the Cissoide the Lines

B L, L Z, L F, L M are continually Proportional, therefore the Equation Expressing the Nature of the Cissode is $xx = yu$, and $2x\dot{x} = u\dot{y} + y\dot{u}$, and $\dot{y}$

$= \frac{2x\dot{x} - y\dot{u}}{u}$. And because $2ax - xx = uu$ and $yu = xx$ to x, therefore, we

have this Equations $2ay - xy = xu$. Now $\dot{y} = \frac{t\dot{x}}{2a - x} = \frac{2x\dot{x} - y\dot{u}}{u}$ there-

fore $tu\dot{x} = \overline{2a - x} \times \overline{2x\dot{x} - y\dot{u}} =$ (Substituting uu for $2a - x \times x$ and xu for

$2a - x \times y$) $2uu\dot{x} - xu\dot{u}$, therefore $\dot{u} = \frac{2u\dot{x} - t\dot{x}}{x} = \frac{2a\dot{x} - 2x\dot{x}}{2u}$, and

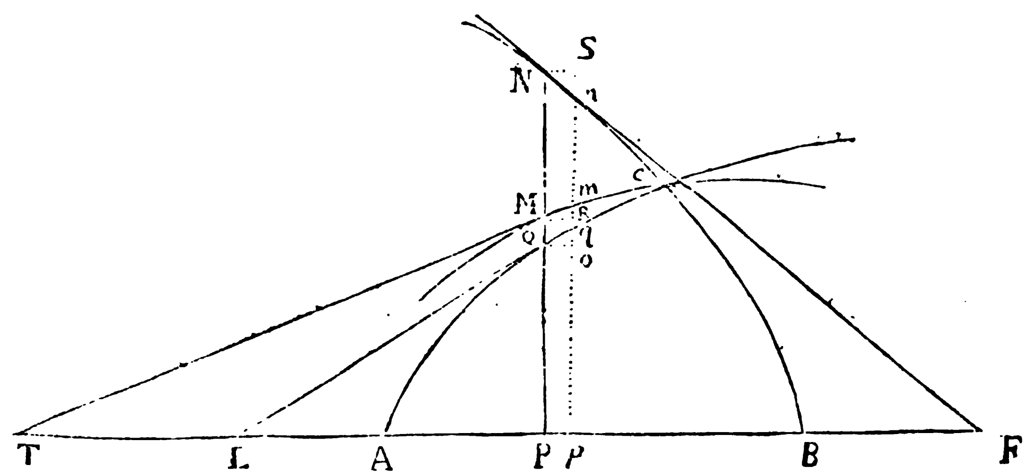
$4uu - 2tu = 2ax - 2xx$, (because $2ax - 2xx = \overline{2a - x} \times x - xx$) $uu - yu$. Therefore $2t = 3u + y$; and consequently, GD $(= t) = \frac{3}{2}u + \frac{1}{2}y$. Which admits of a most simple Construction, by taking GD $= \frac{3}{2}$ L Z $+ \frac{1}{2}$ L M.

PROP.

PROP. X.

Let the two Curves AQC, BCN *be given, and* TABF *their Diameter; and suppose the Method of drawing their Tangents* QL, NF *known, and let the Nature of the third Curve* MC *be exprefs'd by any given Relation between the Ordinates* QP, MP, NP. *It is requir'd to draw a Tangent to the Curve* MC, *which shall touch the same in any given Point,* v. g. *in* M.

66. Draw the Ordinate pm infinitely near PM, and draw NS, MR, Qo Parallel to Pp, and then we have the three little Triangles NSn, MRm, Qoq. Suppose the given Quantity $PL = s$ and $PF = t$, $PQ = x$, $PM = y$, $PN = v$. Then $oq = \dot{x}$, $Rm = \dot{y}$, Sn (*Art.* 22.) $= -\dot{v}$, (because x and y increase while v decreases) and because the Triangles LPQ, Qoq; NPF, nSN; MPT, mRM are similar, Therefore QP (x) : PL (s) :: qo $(\dot{x})$: $Qo = \frac{s\dot{x}}{x}$, and $Qo = MR$



$= NS = \frac{s\dot{x}}{x}$. Again, NP (v) : PF (t) :: nS $(-\dot{v})$: $SN = \frac{-t\dot{v}}{v}$. Therefore $\frac{-t\dot{v}}{v} = \frac{s\dot{x}}{x}$, and $\dot{v} = \frac{-sv\dot{x}}{tx}$. Lastly, mR $(\dot{y})$: RM $\left(\frac{s\dot{x}}{x}\right)$ :: MP (y) : $PT = \frac{sy\dot{x}}{x\dot{y}}$; or thus, Rm $(\dot{y})$: RM $\left(\frac{t\dot{v}}{v}\right)$ :: MP (y) : $PT = \frac{ty\dot{v}}{v\dot{y}}$, which being found, reduce the Equation of the Curve to Fluxions, and substitute $-\frac{sv\dot{x}}{tx}$ for $\dot{v}$, and find the Value of $\dot{x}$ in $\dot{y}$ and substitute the the same in $\frac{sy\dot{x}}{x\dot{y}}$. So all the Fluxions will be taken away, and the Value of the Sub-tangent will be exprefs'd in Terms altogether known.

EXAMPLE.

Let AQC be the Apolonian Parabola, and BCN a Cubical Parabola, whose Ordinates NP are in a Subtriplicate Proportion of the Intercepted Diameters BP, then 'tis manifest (*Art.* 25.) that $PL = 2\,AP$, and $PF = 3\,BP$. Now Suppose the Nature of the Curve MC to be such, that the Ordinate PM be always a mean Proportional between PQ and PN; then it is PQ : PM :: PM : PN, and consequently the Equation Expressing the Nature of the Curve MC is $yy = xv$. And finding the Fluxions of this Equation we have $2y\dot{y} = x\dot{v} + v\dot{x} =$ (because $\dot{v} = -$

And

$\left.\frac{sv\dot{x}}{tx}\right)$ $v\dot{x} - \frac{sv\dot{x}}{t} = \frac{tv\dot{x} - sv\dot{x}}{t}$, therefore $\dot{x} = \frac{2ty\dot{y}}{tv - sv}$. And $PT = \frac{sy\dot{x}}{x\dot{y}}$ $= \frac{2tsyy}{txv - sxv} =$ (Substituting xv for yy) $\frac{2ts}{t-s} = \frac{2PF \times PL}{PF - PL} = \frac{6PB \times 2PA}{3PB - 2PA}$. Q. E. I.

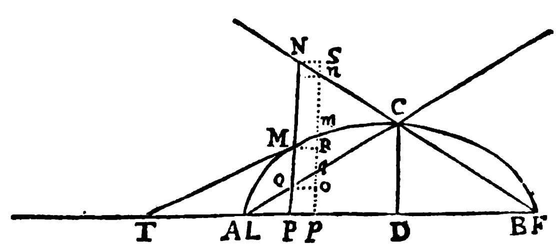
And Universally, Suppose $y^{m+n} = x^m v^n$ then $\overline{m+n} \times y^{m+n-1}\dot{y} = mv^n x^{m-1}\dot{x} + nx^m v^{n-1}\dot{v} =$ (because $-\frac{sv\dot{x}}{tx}$ is $= \dot{v}$; and consequently, the Term $nx^m v^{n-1}\dot{v}$ is $= -nx^m v^{n-1} \times \frac{sv\dot{x}}{tx} = -\frac{nsv^{n-1}vx^m\dot{x}}{tx} = -\frac{nsv^n x^m\dot{x}}{tx}$) $mv^n x^{m-1}\dot{x} - \frac{nsv^n x^m\dot{x}}{tx} = \frac{mtv^n x^{m-1}x\dot{x} - nsv^n x^m\dot{x}}{tx} =$ (Dividing every Term in the Numerator and Denominator by x) $\frac{mtv^n x^{m-1}\dot{x} - nsv^n x^{m-1}\dot{x}}{t}$.

Therefore $\overline{m+n} \times y^{m+n-1}\dot{y}$ is $= \frac{mtv^n x^{m-1}\dot{x} - nsv^n x^{m-1}\dot{x}}{t}$, and by Multiplication and Division $\dot{y} = \frac{mtv^n x^{m-1}\dot{x} - nsv^n x^{m-1}\dot{x}}{\overline{mt+nt} \times y^{m+n-1}}$.

Whence $PT = \frac{sy\dot{x}}{x\dot{y}} = \frac{\overline{mts+nts} \times y^{m+n-1}y}{mtv^n x^{m-1}x - nsv^n x^{m-1}x} = \frac{\overline{mts+nts} \times y^{m+n}}{mtv^n x^m - nsv^n x^m}$ $=$ (Substituting $x^m v^n$ for y^{m+n}) $\frac{\overline{mts+nts} \times x^m v^n}{mtx^m v^n - nsx^m v^n} =$ (Dividing Numerator and Denominator by $x^m v^n$) $= \frac{mts+nts}{mt-ns}$. Q. E. I.

COROLLARIES.

67. If the Curves AQC and BCN degenerate into right Lines, then the Curve MC will be one of the Conic Sections.

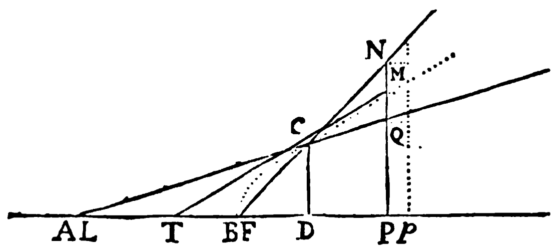


1°. If the Ordinate CD fall between A and B. And if the Angle CAB = CBA be = 45°. then the Curve MC will be a Circle. For then AD = CD = DB. And the points A, L; B, F; Coincide: Therefore $PT = \frac{2ts}{t-s} = \frac{2PB \times PA}{PB - PA} = \frac{2PB \times PA}{AB - 2PA}$, which is the Value of the Subtangent of the Circle found (*Art.* 23.) above.

2°. If the Perpendicular CD fall between A and B, and the equal Angles CAD, CBD be each less than ACD, then the Curve MC is an Ellipsis; For the Subtangent $PT = \frac{2ts}{t-s} = \frac{2PB \times PA}{AB - 2AP}$, which is the Value of the Subtangent of the Ellipsis found (*Art.* 32.) above.

3°. If the Point B be at an infinite diſtance from A, that is if BCN be parallel to APD, then the Curve MC is the Parabola: For then the Sub-tangent $PT = \frac{2ts}{t-s} = \frac{2PB \times AP}{PB - AP} =$ (becauſe PB is infinite in reſpect of AP) $\frac{2PB \times AP}{PB} = 2AP$, which is the Value of the Sub-tangent of the Parabola found (*Art.* 25.) above.

4°. But if the Perpendicular CD, fall beyond A or B, then the Curve MC will be an Hyperbola, and AB will be the Axis; For $PT = \frac{2ts}{t+s} = \frac{2BP \times AP}{AP + BP}$, which is alſo the Value of the Sub-tangent of the Hyperbola found (*Art.* 33.) above.

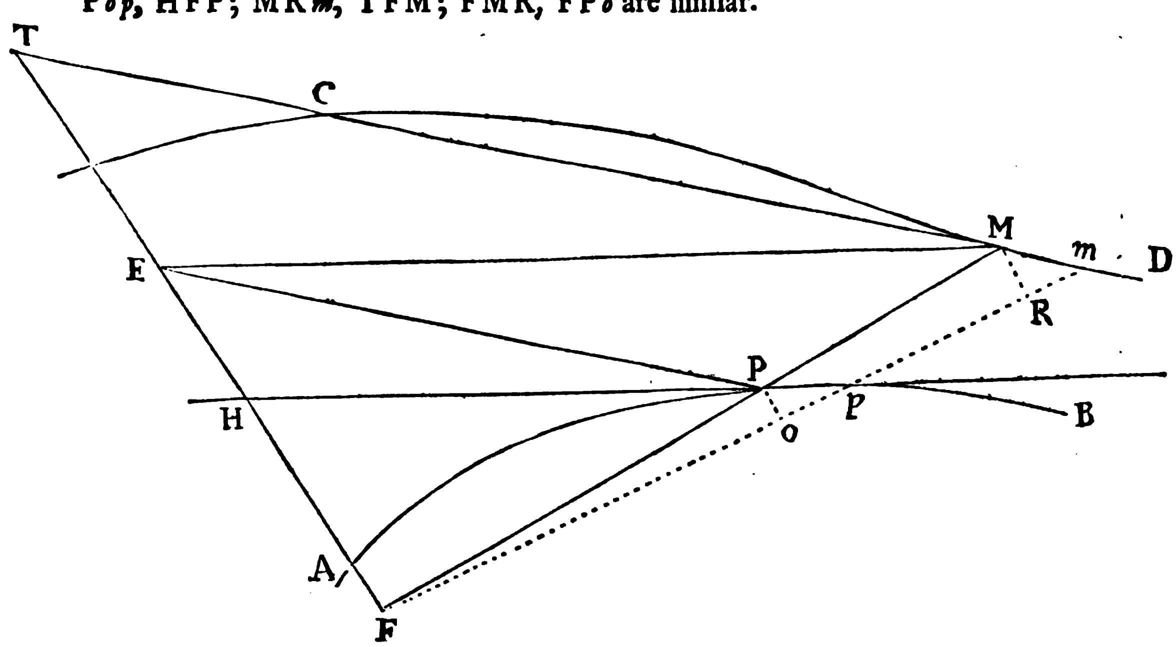


And thus it appears that all the Sections of the Cone may be deſcrib'd, by help of any two ſtraight Lines, given by Poſition.

PROP. XI.

If the Nature of the Curve Line APB, *and the Method of drawing Tangents to the ſame,* v. g. PH, *be known; and if any determinate Point* F *without the ſaid Curve be given, from which drawing the right Line* FPM *to a third Curve* CMD, *the Proportion of* FP *to* FM *be expreſs'd by any given Equation; 'tis requir'd to draw the Line* MT, *which ſhall touch the Curve in the Point* M.

68. Draw the right Line FHT Perpendicular to FM, and draw F*m* infinitely near FM, and on the Center F deſcribe the Arches MR, P*o*, then the Triangles P*op*, HFP; MR*m*, TFM; FMR, FP*o* are ſimilar.



Suppoſe the known Quantities $FH = s$, $FP = x$, $FM = y$; then ſay PF $(= x)$: FH (s) :: po $(\dot{x})$: $Po = \frac{s\dot{x}}{x}$. And FP (x) : FM (y) :: Po $\left(\frac{s\dot{x}}{x}\right)$: MR $= \frac{sy\dot{x}}{xx}$, and Rm $(\dot{y})$: RM $\left(\frac{sy\dot{x}}{xx}\right)$ :: FM (y) : $FT = \frac{syy\dot{x}}{xx\dot{y}}$, which may be clear'd of the Fluxions by help of the Equation of the Curve.

EX-

EXAMPLE.

Suppose the Curve APB to degenerate into the right Line PH, and that the Equation expressing the Relation of FP to FM is $y - x = a$ (that is to say, that PM is always equal to one and the same given Quantity a) then 'tis evident that the Curve CMD is *Nichomedes*'s *Conchoide*, and the Line MT touching the same in the point M, may be determin'd in this manner, $y = x + a$, and $\dot{y} = \dot{x}$. But FT $= \frac{syy\dot{x}}{xx\dot{y}} =$ (substituting $\dot{y}$ for $\dot{x}$) $\frac{syy}{xx}$. Whence there will arise this,

CONSTRUCTION.

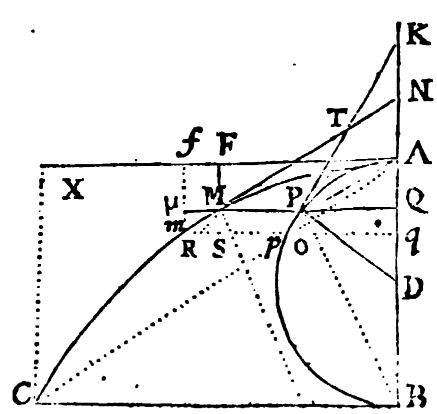
Draw ME parallel to PH, and MT parallel to PE; I say, MT will touch the Conchoide in M. For, FP (x) : FH (s) :: FM (y) : FE $= \frac{sy}{x}$, and FP (x) : FE $\left(\frac{sy}{x}\right)$ :: FM (y) : FT $= \frac{syy}{xx}$.

Hence it is evident, that PH is the Asymptote, and the Point F is the Pole of the Conchoide CMD.

PROP. XII.

If the Nature of the Curve AMm *be given, and if the intercepted Diameter* AP *be the Portion of a Curve (instead of a straight Line, as in the preceding Problems) and the Position of the Line* PK *touching the same in* P *be given: 'Tis requir'd to draw the Line* MT, *which shall touch the given Curve in* M.

69. Let AQB be the Axis of the Curve APp, and draw the Ordinate MPQ Perpendicular to AB. Draw the Tangent PT, and suppose MT Intersecting PT in T, to be the Tangent which touches the Curve AMm in the point M. Imagine another Ordinate mp Infinitely near PM. Draw MR Parallel to PT. And then Suppose the Abscissa AP $= x$ and PM $= y$. Then Pp = MR $= \dot{x}$ and R$m = \dot{y}$. and the Triangles mRM, MPT are similar, therefore mR ($\dot{y}$) : RM ($\dot{x}$) :: MP (y) : PT $= \frac{y\dot{x}}{\dot{y}}$ which is Cleared of the Fluxions, and Express'd in known Terms, by Considering the Equation of the Curve, that is by Comparing the Relation of AP $= x$ to PM $= y$, as has been done in the like Cases before, and will yet be more evident by the following



EXAMPLE.

70. If the Curve APp be a Circle, and if AP or x be to PM or y, as a is to b, then the Equation expressing the Nature of the Curve AMm will be $x = \frac{ay}{b}$, and $\dot{x} = \frac{a\dot{y}}{b}$, and consequently PT $= \frac{y\dot{x}}{\dot{y}} =$ (by substitution) $\frac{ay}{b} = x$, and the Curve AMC will be a *Semi-cycloide*; *Simple* if b be $= a$, *Protracted* if b be greater than a; and *Contracted*, when b is less than a.

And in the simple or common Cycloide MQ $=$ AP $+$ PQ, and MP $=$ AP, therefore $a = b$, and PT $= \frac{ay}{b} = y = x$. That is, PT $=$ PM.

CON-

CONSECTARIES.

1. In the Circle APB if the Chord AP be drawn, I ſay, it will be Parallel to the Tangent MT. For the Triangle MPT being an Iſoſceles Triangle, the External Angle TPQ is double the Internal Oppoſite Angle TMP; and the Angles APT, APQ ſtanding on equal Arches of the Circle are equal; Therefore the Angle TPQ is alſo double TPA. And conſequently the Angle TMP = MTP = TPA; and the Lines MT and AP are (*Prop.* 27. *El.* 1.) Parallel.

2. A Line drawn from M Parallel to PB, will be Perpendicular to the Cycloide in M.

3. Tho' the Cycloide be a *Tranſcendent Curve*, yet the *Ratio* of the Subtangent to the Ordinate may be expreſs'd in ordinary Terms; For QP : QA :: QM : QN.

4. And becauſe MN : QN :: AP : AQ, and AB : AP :: AP : AQ; therefore the Tangent MN is to the Subtangent QN as $\overline{AB}^{\frac{1}{2}}$ is to $\overline{AQ}^{\frac{1}{2}}$. And if the Tangent MN be produced until it Interſect the Baſe BC produced, and if a Line be drawn from M parallel to AB, then the Tangent will be to the Subtangent (BC being conſidered as an Axis) as $\overline{AB}|^{\frac{1}{2}}$ is to $\overline{BQ}|^{\frac{1}{2}}$.

Another way.

Suppoſe QM $= u$; QP $= y$; QK $= s$; QN $= t$; DQ $= d$; AP $= z$; AQ $= x$; Qq = Po = MS $= \dot{x}$; Sm $= \dot{u}$; $po = \dot{y}$: then it is (by ſimilar Triangles) QK (s) : QP (y) :: Po ($\dot{x}$) : po ($\dot{y}$) therefore $\dot{y} = \frac{y\dot{x}}{s}$. Again QN ($t$) : QM ($u$) :: SM ($\dot{x}$) : S$m$ ($\dot{u}$) therefore $\dot{u} = \frac{u\dot{x}}{t}$. Thirdly, DP ($r$) : PQ ($y$) :: P$p$ ($\dot{z}$) : po ($\dot{x}$) therefore $\dot{z} = \frac{r\dot{x}}{y}$.

Now let the Relation between the Curves be $a : b :: u - y : z$. Therefore $az = bu - by$. And $a\dot{z} = b\dot{u} - b\dot{y}$, and by Subſtitution $\frac{ar}{y} = \frac{bu}{t} - \frac{by}{s}$. And by Reduction $arst = bsuy - btyy =$ (becauſe $s : y :: y : d$.) $bsuy - btds$, and conſequently QN (t) $= \frac{buy}{ar + bd}$.

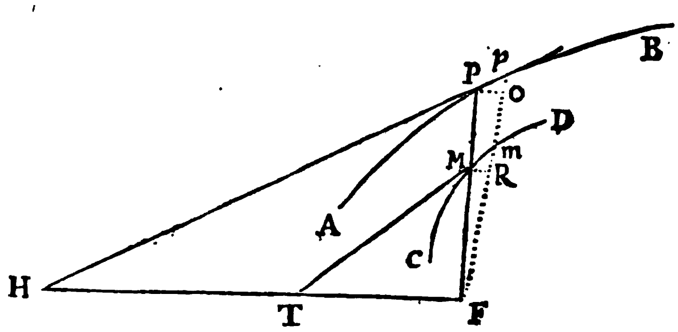
And if the Curve AMC be the *Primary* or *Simple Cycloide* then $a = b$, and $t = \frac{uy}{r + d}$. That is BQ : QP :: (QP : QA ::) QM, QN. As in the third Corollary of the preſent Propoſition.

PROP.

PROP. XIII.

The Nature of the Curve Line APB, *and the Position of the Tangent* PH *being given. Let there be another fixt and immoveable Point* F, *and also another Curve* CMD; *and let the Relation of* FM *to the Portion of the Curve* AP *be given; 'tis requir'd to draw the Line* MT *to touch the Curve in* M.

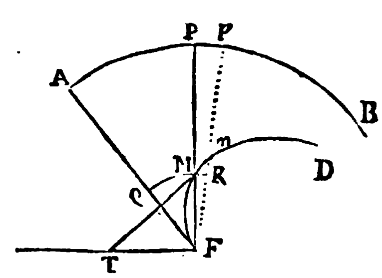
71. Draw FH Perpendicular to FP until it Intersect the Tangent PH in H, and MT the Tangent required in T, and imagine the Line Fp Infinitely near FP forming the Infinitely little Angle PFp. On the Center F with the Distances FM, FP describe the little Arches MR, Po, and then the Triangles Pop, HFP are similar; because the Angles HFP, Pop are right Angles, and the Angle HPF = HpF, their difference PFp being Incomparably less that either of them. And for the like



reason, the Triangles MRm, TFM are similar. Let the given Quantities PH = t; FH = s; FM = y; FP = x; and the Arch AP = z, and then PH (t) : FH (s) :: Pp $(\dot{z})$: Po = $\frac{s\dot{z}}{t}$, and FP (x) : FM (y) :: Po $\left(\frac{s\dot{z}}{t}\right)$: MR = $\frac{sy\dot{z}}{tx}$. Again mR $(\dot{y})$: RM $\left(\frac{sy\dot{z}}{tx}\right)$ :: FM (y) : FT = $\frac{syy\dot{z}}{tx\dot{y}}$, and consequently FT may be obtain'd in known Terms by destroying the Fluxions $\dot{z}$ and $\dot{y}$, and substituting Quantities equal to them, by help of the Equation expressing the Nature of the Curve.

EXAMPLE.

72. If the Curve APB be a Circle, and F its Center, it is evident that the Tangent PH will become Parallel and Equal to the Subtangent FH (*viz*, $t = s$) because in this Case both are Perpendicular to FP; and therefore FT = $\frac{syy\dot{z}}{tx\dot{y}} = \frac{yy\dot{z}}{x\dot{y}}$ = (putting a for x, because FP becomes an Invariable Quantity) $\frac{yy\dot{z}}{a\dot{y}}$ Now if we suppose the whole Circumference or any determinate Portion of the Circle APB = b, and if it be $b : z :: a : y$; then the Curve Line CMD, which in this Case becomes FMD will be *Archimedes's Spiral*, and the Equation expressing the Nature thereof is $\frac{az}{b} = y$, and $\frac{a\dot{z}}{b} = \dot{y}$; Therefore FT = $\frac{yy\dot{z}}{a\dot{y}}$ is = $\frac{byy}{aa}$ = $\frac{zy}{a}$. Whence we have this



CONSTRUCTION.

On the Center F with the *Radius* FM, describe the Arch of the Circle MQ, bounded at Q by the Ray FA, which joyns the points F and A. Then take FT = QM, and

and draw the Line MT; I say, it will touch the Spiral Line FMD in M: For because the Sectors FPA and FMQ are similar, therefore FP (a) : FM (y) :: AP ($\dot{z}$): MQ $= \frac{\dot{z}y}{a}$.

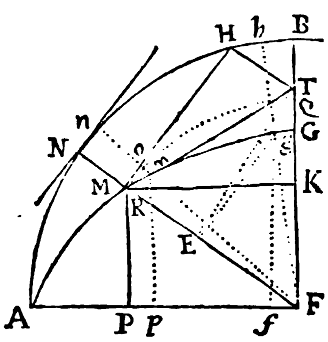
And Universally:

If it be $b^n : a^m :: z^n : y^m$. Then $b^n y^m = a^m z^n$ and $m b^n y^{m-1} \dot{y} = n a^m z^{n-1} \dot{z}$, and (dividing both sides of the Equation, by the Equation of such Curves) $m y^{-1} \dot{y} = n z^{-1} \dot{z}$, that is $m z \dot{y} = n y \dot{z}$, and $\dot{z} = \frac{m z \dot{y}}{n y}$. Therefore FT $= \frac{yy\dot{z}}{a\dot{y}}$ is $= \frac{myyz}{any} = \frac{myz}{na}$, and by help of the Equation expressing the Nature of such Curves, z may be taken away, and the Sub-tangent express'd in straight Lines.

PROP. XIV.

If the Curve Line ANB, *and the Right Lines* AF *and* FB *be given by Position; and if another Curve, as* AMG *be such, that drawing the right Line* FMN *at pleasure, and* MP *parallel to* FB, *the proportion of the Arch* AN *to* AP *be express'd by any given Equation: 'Tis requir'd to draw the Line* MT *to touch the Curve in the given point* M.

73. Suppose the Tangent requir'd MT to be drawn. Then through T the Point sought, draw TH Parallel to FM, and MRK Parallel to AF, and draw MoH Parallel to the Line touching the Curve ANB in N. Draw the Line Fmn Infinitely near FMN, and mRp Parallel to MP. Then Suppose the known Quantities FM $= s$; FN $= y$; AN $= z$; AP $= x$; MK $= u$; then the Triangles FNn, FMo; Mom, MHT; MRm, MKT are similar. Therefore,



FN (y) : FM (s) :: Nn ($\dot{z}$) : MO $= \frac{s\dot{z}}{y}$. and MR ($\dot{x}$) : Mo $\left(\frac{s\dot{z}}{y}\right)$ :: MK (u) : MH $= \frac{su\dot{z}}{y\dot{x}}$.

If the Curve ANB be the Quadrant of a Circle, and F its Center, and if it be AN (z) : AP (x) :: ANB (b) : AF (a) then it is manifest that the Curve Line AMG is what Geometers call the *Quadratrix*; And the Equation of the Curve is $az = bx$, and $a\dot{z} = b\dot{x}$, and $\dot{z} = \frac{b\dot{x}}{a}$ and $x = \frac{az}{b}$, and because FP and MK are parallel and equal, and AP $= x$; therefore $u = a - x$. Hence (in particular) MH $= \frac{su\dot{z}}{y\dot{x}} =$ (putting $a - x$ for u) $\frac{sa\dot{z} - sx\dot{z}}{y\dot{x}} =$ (because $y = a$) $\frac{sa\dot{z} - sx\dot{z}}{a\dot{x}}$ $=$ (substituting $\frac{b\dot{x}}{a}$ for $\dot{z}$) $\frac{bsa - bsx}{aa} =$ (putting $\frac{az}{b}$ for x) $\frac{abbs - asbz}{aab} =$ $\frac{sb - sz}{a}$; Which gives this easie.

CONSTRUCTION.

Draw MH (parallel to the Tangent nN, or) Perpendicular to FM and equal to the Arch MQ described on the Center F, and draw HT Parallel to FM; I say, the Line MT will touch the *Quadratrix* in M; For because the Sectors FNB and FMQ are similar, therefore FN (a) : FM (s) :: NB ($b - z$) : MQ $= \frac{sb - sz}{a}$.

COROL-

COROLLARY.

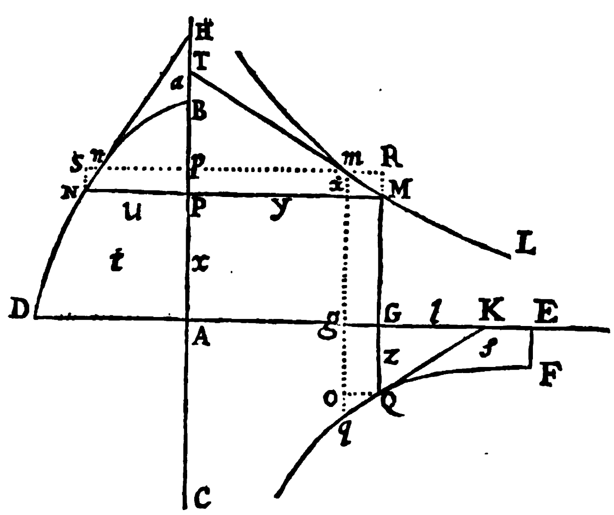
74. Hence to determine the point G, wherein the *Quadratrix* AMG intersects the Semi-diameter FB; imagine another Ray F$g$$b$ infinitely near FGB, and gf parallel to GF; then by the property of the *Quadratrix*, and the similar Triangles FBb, gfF Rectangular in B and F; it is ANB : AF :: bB : fF :: FB or AF : FG or fg. Whence it is evident that FG is a third Proportional to ANB, the Quadrant of the Circle, and AF the Semi-diameter. That is FG $= \frac{aa}{b}$, and hence we have a shorter Method for drawing Tangents to this Curve, which is this,

Having drawn TE parallel to MH, the Triangles FMK, FTE are similar, therefore MK $(a - x)$: MF (s) :: ET or MH $\left(\frac{sb - sz}{a}\right)$ FT $= \frac{bss - ssz}{aa - ax} =$ (putting $\frac{az}{b}$ for x) $\frac{bbss - bssz}{baa - aaz} =$ (dividing by $b - z$) $\frac{bss}{aa}$. Whence 'tis manifest that FT is a third Proportional to FG and FM.

PROP. XV.

Let any two Curves BN, FQ, *and their Axes* BC, ED *be given; and suppose the Method of drawing their Tangents* NH, QK *known, and let another Curve Line* LM *be such, that drawing from any Point* M *in the same, the right Lines* MGQ, *and* MPN *Parallel to* AB *and* AE, *the Spaces* EGQF *(the Point* F *being a determinate Point in* AE, *and* EF *being Parallel to* AC) *and* APND *be always in the same Proportion to each other: 'Tis requir'd to draw the Line* MT *which shall touch the Curve* ML *in the Point* M.

75. Suppose the variable Quantities AP $=$ GM $= x$, PM $=$ AG $= y$, PN $= u$, GQ $= z$, and let the Space EGQF be $= s$, the Space APND $= t$, the Sub-tangent PH $= a$, and GK $= b$. Then Pp $=$ NS $=$ MR $= \dot{x}$, Gg $=$ Rm $=$ Qo $=$ (*Art.* 22.) $-\dot{y}$, Sn $= -\dot{u} = \frac{u\dot{x}}{a}$; because the Triangles HPN, SNn are similar



$oq = \dot{z} = \frac{z\dot{y}}{b}$; because the Triangles QGK, qoQ, are also similar; NPpn $= \dot{t} = u\dot{x}$, and QGgq $= \dot{s} = -z\dot{y}$. Where it must be observ'd that the Values of Rm and Sn are Negative, because as AP $= x$ increases, PM $= y$ and PN

$= u$

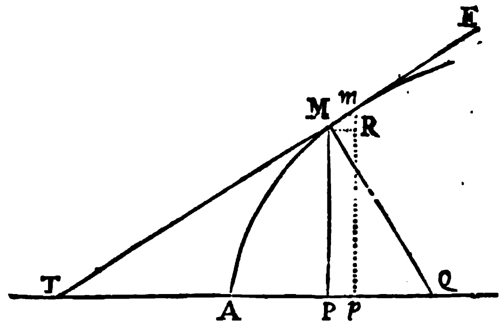
$= u$ Decrease. These things being suppos'd, take the Fluxion of the given Equation, wherein Substitute $u\dot{x}$, $-z\dot{y}$, $\frac{-u\dot{x}}{a}$, $\frac{-z\dot{y}}{b}$ in place of $\dot{t}, \dot{s}, \dot{u}, \dot{z}$, and there will arise a new Equation shewing the Proportion of $\dot{y}$ to $\dot{x}$, or of MP to PT.

76. Suppose $s = t$, then is, $-z\dot{y} = u\dot{x}$, and $\frac{-z\dot{y}}{u} = \dot{x}$; Therefore PT $\left(= \frac{y\dot{x}}{\dot{y}}\right) = -\frac{zy}{u}$. And because the Value of PT is Negative (*Art.* 27) the point T falls on the contrary side of P in respect of A the beginning of x. And if we suppose the Line FQ to be an Hyperbola, AC and AE its Asymptotes, and that GQ $= z$ is $= \frac{cc}{y}$, and that the Curve BND becomes a straight Line Parallel to AB, so that PN (u) be always equal to the Determinate Quantity c (the Parameter of of the Hyperbola) then 'tis evident that AB will become an Asymptote to the Curve LM, and the Subtangent PT $= -\frac{zy}{u}$ will be $=$ (Substituting $\frac{cc}{y}$ for z) $-\frac{cc}{u} = -c$. That is, the Subtangent of the Curve LM will be an Invariable Quantity, and the Curve LM is that which *Geometers* call the *Logarithmetic Line.*

PROP. XVI.

An Equation expressing the Value of the Sub-tangent of any Curve, in the nearest Terms being given: 'Tis requir'd to find the Equation expressing the Nature of the Curve.

77. What I mean by the nearest Terms will be best explain'd by an Example. Suppose PT $= t$, AP $= x$, PM $= y$, MT $= s$; and let the Equation expressing the nature of the Curve be $y^3 + ayy = x^3 + bxx$ then, the Subtangent TP will be $t =$ (§. 10. *Art* 61.) $\frac{3y^3 + 2ayy}{3xx + 2bx}$. Now I Call these *Terms* Expressing the Value of the Subtangent the *Nearest*, because they immediately flow from the Equation of the Curve; But if this Value of the Subtangent be Changed by applying the Equation of the Curve, *v. g.* if we put $3y^3 = 3x^3 + 3bxx - 3ayy$; and consequently $t = \frac{3x^3 + 3bxx - ayy}{3xx + 2bx}$, such I call *Remote Terms.*



Now if the Value of the Subtangent be Exprefs'd in the Nearest Terms, the Equation of the Curve may be Investigated in this manner.

Let the Curve (*Fig. Art.*77.) AM m be described, and draw MT to touch the Curve in M, then suppose the Abscissa AP $= x$; the Ordinate PM $= y$; P$p = \dot{x}$, R$m = \dot{y}$; then because the Triangles mRM, MPT are similar; therefore mR ($\dot{y}$) RM ($\dot{x}$) :: PM (y) : PT $= \frac{y\dot{x}}{\dot{y}}$. Put this Value of the Sub-tangent Equal to its Value given in the Nearest Terms; clear the Equation of the Fractions; and find the Flowing Quantity of each Term; so have you the Equation of the Curve.

EXAMPLE.

EXAMPLE I.

Let it be required to find the Equation of the Curve A M *m*, the Value of the Subtangent P T being $= \frac{2y^3}{3rr}$. The Subtangent P T is $= \frac{y\dot{x}}{\dot{y}} =$ (*ex Hyp.*) $\frac{2y^3}{3rr}$; therefore $3rry\dot{x} = 2y^3\dot{y}$ and $3rr\dot{x} = 2y^2\dot{y}$. And Substituting x for $\dot{x}$, and y for $\dot{y}$ $3rrx \,\square\, 2y^3$ and (dividing $3rrx$ by 1 the Exponent of x; and dividing $2y^3$ by 3 the Exponent of y) $3rrx = \frac{2}{3}y^3$, and $9rrx = 2y^3$, which divided by 2, we have $\frac{9}{2}rrx = y^3$, the Equation expressing the Nature of the Curve A M *m*.

EXAMPLE II.

Ler it be required to find the Property of the Curve A M *m*, the Subtangent P T being $= \frac{2yy}{r}$.

The Subtangent P T is $= \frac{y\dot{x}}{\dot{y}} =$ (by Supposition) $\frac{2yy}{r}$; therefore $ry\dot{x} = 2yy\dot{y}$, and $r\dot{x} = 2y\dot{y}$, and Substituting x for $\dot{x}$ and y for $\dot{y}$) $rx \,\square\, 2yy$, and consequently (dividing rx by the Exponent of x, and $2yy$ by 2 the Exponent of y) $rx = yy$, which shews that the Curve A M *m* is a Parabola.

EXAMPLE III.

Let it be required to find the Property of the Curve A M *m*, the Value of the Subtangent P T being $= \frac{3y^3 + 2byy}{3xx + 2ax}$

The Subtangent P T is $= \frac{y\dot{x}}{\dot{y}} = \frac{3y^3 + 2byy}{3xx + 2ax}$ Therefore $3xx\dot{x} + 2ax\dot{x} = 3y^2\dot{y} + 2by\dot{y}$, and (putting x for $\dot{x}$ and y for $\dot{y}$) $3x^3 + 2ax^2 \,\square\, 3y^3 + 2by^2$, and (dividing every Term by the Exponent of the Flowing Quantity therein) $x^3 + axx = y^3 + byy$, which Equation Expresses the Nature of the Curve A M *m*.

But because this Method depends on that Problem to find the Flowing Quantity of any Fluxion, with which the Reader is yet supposed to be unacquainted, I shall desist from prosecuting the same any further at present, and content my self to deduce the Solution of the present Proposition from the (*Art.* 61.) sixth preceding; this being nothing else but the *Reverse* of that.

78. that we may be able to proceed with the greater certainty in this inquiry, it will be necessary to observe from the forecited Place.

1°. The Subtangent t is always of one Dimension, and is Expressed by a Fraction.

2°. When the Value of the Sub-tangent is Expressed in the nearest Terms, then the Numerator of the Fraction consists only of those Terms wherein the Ordinate y, (or the Tangent s) is found.

3°. And if all the Terms of the Equation of the Curve be simple Terms, then the intercepted Diameter x never occurs in the Numerator, nor the Ordinate y, Tangent s, or Curve z in the Denominator.

4°. But if the Equation of the Curve contain mixt Terms, then both x, z, s and y may be found in both parts of the Fraction: but with this Condition, that the Fraction being reduced to an Equation, and all the Terms of the Equation being brought over to one side and every t changed into x, and every s into z, every mixt Term will be found as often as there are variable Quantities in the same. And the Coefficients

cients or prefix'd Numbers will be Equal or Proportional to the Respective Exponents of the Powers of the variable Quantities.

5°. Whence it follows that the Signs of the Terms wherein the same variable Quantities occur, are the same, after a due Division by the prefixt Numbers (or rather by the Exponents of the variable Quantities.)

Hence to resolve the Problem concerning,

The Inverse Method of Tangents.

79. 1°. Change every t into x, and every s into z (denoting the Curve) and transpose all the Terms to one side of the Equation, and diligently observe whither all the Terms are simple, or some simple and others mixt.

2°. If all the Terms be simple, divide every Term by the Exponent of the Indeterminate or Flowing Quantity in the same, so have you the Equation Expressing the Natuc of the Curve.

3°. And if there be any mixt Terms, then observe § 4. 5. *Art.* 78. And let every Term containing the same Variable Quantities be divided by the Exponent of the Power to which the Respective Flowing Quantities are advanced, so that the same Term result from every such Division, and be as often found in the Equation as it has Flowing Quantities.

4°. Retain only one of those mixt Terms which occur more then once in the Equation, and manage the other simple Terms according to §. 2°. And there will arise an Equation expressing the Nature of the Curve.

EXAMPLE I.

Suppose $t = \frac{y^3 + ayy - bby}{xx + ax + bb}$, then (by *Rule* 1°) changing t into x, and Transposing all the Terms to one side of the Equation, we have $x^3 + axx + bbx - y^3 - ayy + bby$, and because all the Terms are simple Terms, therefore (2°.) $\frac{1}{3}x^3 + \frac{1}{2}axx + bbx - \frac{1}{3}y^3 - \frac{1}{2}ayy + bby = 0$, which is an Equation expressing the Nature of the Curve, as was requir'd.

EXAMPLE II.

Let the Value of the Sub-tangent be $t = \frac{3y^3 + 2ayy - 2xyy - xxy}{3xx + 2xy + yy}$, then we have (by 1°.) $3x^3 + 2yxx + yyx - 3y^3 - 2ayy + 2xyy + xxy$, and because we have the mixt Terms $2yxx$ and yxx, also yyx and $2yyx$, each repeated twice according to the Number of the Flowing Quantities, therefore if one of them be divided by the Exponent of x, and the other by the Exponent of y (3°.) there will arise $yxx + yyx$ (by 4°.) and dividing the simple Terms by the Exponents of the Flowing Quantities in each respectively, the Equation expressing the Nature of the Curve will be $x^3 + yxx + yyx - y^3 - ayy = 0$.

EXAMPLE III.

And the Method is the same if the Curve z enter into the Value of the Sub-tangent, *v.g.* Suppose $t = \frac{6ay^3zz + 4ay^3zs + aay^4 - yxxz^3 - 3yxxzzs}{2yxz^3}$ change every t into x, and every s into z, and transpose all the Terms over to the same side of the Equation, and then we have

$$2yz^3xx + yxxz^3 + 3yxxz^3 - 6ay^3zz - 4ay^3zz - aay^4$$

Wherein

Wherein the Term $yxxz^3$ containing 3 Flowing Quantities is found thrice, and the Term ay^3zz containing two is found twice, and because those mixt Terms being divided by the Respective Exponents of the Powers of the Flowing Quantities, the same Quotient always results, it is plain that the Value of the Subtangent is given in the nearest Terms, and therefore the Equation Expressing the Nature of the Curve will be $yz^3x^2 - 2ay^3zz - \frac{1}{4}aay^4 = 0$. Or adding any determinate Quantity bb; $yz^3x^2 - 2ay^3zz - \frac{1}{4}aay^4 + bb = 0$.

80. Hence it appears that a Determinate Quantity may be added to the Equation of the Curve: which is plain from the direct method of Tangents, because then when we Investigate the Value of the Subtangent, all the Terms Consisting of Invariable Quantities are rejected or vanish; And this is sometimes absolutely necessary, *v.g.* Suppose $t = \frac{-xy}{2x+y}$. Then we have $2xx + yx + xy$, and consequently $xx + xy = 0$, and because this Equation has no true root, therefore we must add a Determinate Quantity, and then the Equation of the Curve may be $xx + xy = bb$.

COROLLARY.

81. Hence if the Value of the Subnormal (*Fig. Art.* 82.) P Q be given, the Property of the Curve may be found. For the Triangles Q M P, M T P are similar, therefore Q P : P M :: P M : P T, and if P Q be $= q$, then $t = \frac{yy}{q}$. Whence the Equation of the Curve may easily be (*Art.* 78. 79.) found

The Property of the Curve may be Investigated otherwise, thus: the Triangles m R M, Q P M are similar, therefore M R ($\dot{x}$) : R m ($\dot{y}$) :: P M (y) : P Q $= \frac{y\dot{y}}{\dot{x}}$, and putting this Equal to the Value of the Subnormal given, the Property of the Curve may be (*Art.* 77.) found.

EXAMPLE.

Suppose P Q $= \frac{aax}{2yy}$; then is $\frac{\dot{y}y}{\dot{x}} = \frac{aax}{2yy}$, and $aax\dot{x} = 2y^3\dot{y}$, and (Substituting x for $\dot{x}$, and y for $\dot{y}$) $aaxx \square 2y^4$, therefore (dividing the Terms by the Exponents of x and y Respectively) $\frac{1}{2}aaxx = \frac{1}{2}y^4$. Whence $ax = y^2$, which shews that the Curve A M m is a Parabola.

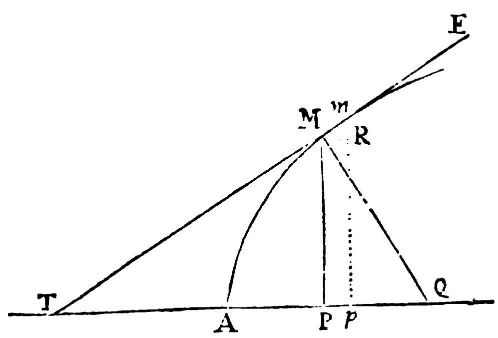
Much might be said on this Subject of Excellent Use in Mechanical Philosophy, *which the Reader may expect some hints of, as occasion offers (this place being improper for that purpose) in the sequel of this Treatise.*

PROP.

PROP. XII.

If the Right Lines TM, TA *be given by Position, and if the point* M *be given in the Right Line* TME; *Let it be required to describe the Curve* AM*m which shall touch the Right Line* TM *in the given point* M: *the Nature of the Curve* AM*m being known.*

82. Let TA produced be the Axis of the Curve required, and from the given point M let fall the Perpendicular MP on the Axis AP, and draw pm infinitely near PM, then 'tis manifest that the Particle of the Tangent Mm (*Defin.* 1) must also be a particle of the Curve required.



Suppose the Curve AMm to be described, and draw MR Parallel to Pp, then AP $= x$ is the Abscissa, PM $= y$ is the Ordinate and PT $= t$ is the Sub-tangent. And because the lines TP, PM are given by Position, Suppose PT (t) : PM (y) :: $r : s$: then it will be also $r : s$ :: (by similar Triangles) $\dot{x} : \dot{y}$. Therefore $\dot{x} = \frac{r\dot{y}}{s}$, and if the Equation Expressing the Nature of the Curve required be reduced into Fluxions, and Compared with this, the Parameter and Abscissa of the Curve, which shall touch the Right Line TM in M may be determined.

EXAMPLE I.

Let it be required to describe the Parabola AMm which shall touch the Right Line TM in M. The Equation Expressing the Nature of the Parabola is $ax = yy$. Therefore $a\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{2y\dot{y}}{a}$. But $\dot{x}$ is $= \frac{r\dot{y}}{s} = \frac{2y\dot{y}}{a}$, therefore $\frac{r}{s} = \frac{2y}{a}$, and $a = \frac{2sy}{r}$ which determines the Parameter of the Curve required. Now by the Property of the Parabola $a = \frac{yy}{x} = \frac{2sy}{r}$, and consequently $x = \frac{ry}{2s} =$ (because $t : y :: r : s$, and $t = \frac{ry}{s}$) $\frac{t}{2}$, that is AP $= \frac{1}{2}$ TP. Which agrees with the Property of the Subtangent of the Parabola demonstrated (*Art.* 25.) above.

EXAMPLE II.

Let it be required to describe the Circle AMm which shall touch the Line MT (given by position) in M. The Equation Expressing the Nature of the Circle is $2ax - xx = yy$, which being Reduced to Fluxions, we have $a\dot{x} - x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{a-x} = \frac{r\dot{y}}{s}$, Whence $\frac{y}{a-x} = \frac{r}{s}$, and $sy = ar - rx$, therefore $a = \frac{sy + rx}{r}$ $=$ (by the Property of the Circle) $\frac{yy + xx}{2x}$. And Consequently $2sxy + 2rxx = ryy + rxx$, that is $rxx + 2syx = ryy$, whence the Value of x (and Consequently that of a the Semidiameter) may be found by the solution of an Affected Quadratic Equation.

EXAMPLE

EXAMPLE III.

Let it be requir'd to defcribe an Hyperbola A M m, which fhall touch the given Line T M in a given point M. Suppofe the Tranfverfe Diameter $= a$, and the Parameter $= b$, then the Equation expreffing the Nature of the Hyperbola is $bxx + bax = ayy$. And reducing the fame to Fluxions we have

$2bx\dot{x} + ba\dot{x} = 2ay\dot{y}$, and $\dot{x} = \frac{2ay\dot{y}}{2bx + ab} = \frac{ry}{s}$. Therefore $2say = 2rbx$

$+ abr$, and by Divifion $b = \frac{2say}{2rx + ar} =$ (by the Property of the Curve)

$\frac{ayy}{ax + xx}$; Therefore $\frac{2s}{2rx + ar} = \frac{y}{ax + xx}$. And becaufe in this Equation there are two Indeterminate Quantites (*viz.* a, and x) 'tis evident that the Problem is not Limited, and that the Values of b and x will come out different according as we affume the Tranverfe Axis a.

COROLLARY.

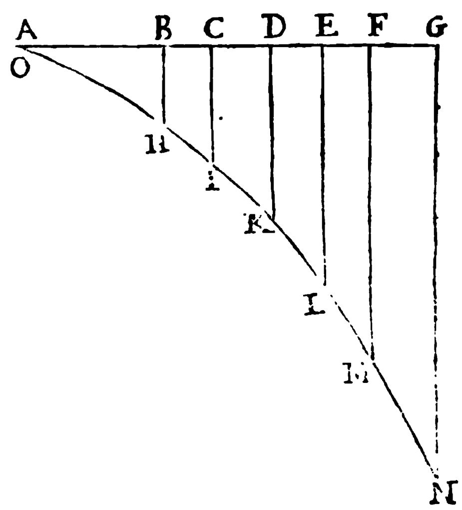
83. Hence 'tis manifeft that tho' one Line only as M T can touch a given Hyperbola in one point M, yet an infinite Number of Hyperbola's may touch a Right Line in one and the fame point.

But if the Curve A M m required to be drawn, to touch the Line T M in the given point M, be an Equilateral Hyperbola, then (*Art.* 35.) $a = b$, and the Problem will be Limited to one Particular Curve.

PROP. XX.

To Inveftigate the Equation Expreffing the Nature of any Curve, Generated by any given Proportion between its Ordinates.

84. Suppofe A D a Determinate Right Line $= a$, to which in any given Angle Apply the Right Line D K (alfo Determinate) $= b$. Then take any Number of aliquot Parts of A D, as A B, A C, or the Multiples thereof, as A E, A F; and from the points B, C, E, F *&c.* Draw the Right Lines B H, C I, E L, F M *&c.* Parallel to D K, which Keep a Conftant and Immutable Proportion in refpect of the given Line D K, and the Part of the Diameter which they cut of.



And fuppofe the faid Right Line A D to be divided into an Infinite Number of Equal Parts, and an Infinite Number of Ordinates to to be Drawn Parallel to D K all in the fame given Proportion, then it is plain that the Little Lines A H, H I, I K, K L, *&c.* will Conftitute a Regular Curve.

Now 'tis required to find the Equation of the Curve, knowing the Particular Relation of the Ordinates: that is, 'tis required to find an Equation Expreffing the Relation between the Ordinates and Intercepted Diameter.

1°. Let A K N be a Curve generated as before, and taking A B $= \frac{1}{3}$ A D $= \frac{1}{3}a$, and A C $= \frac{4}{5}a$; A E $= \frac{8}{7}a$; A F $= 2a$; and fo on at Pleafure: fuppofe the Proportion of the Ordinates to be as follows, *viz.* B H $= \frac{1}{9}b$; C I $= \frac{16}{25}b$; E L $= \frac{64}{49}b$; F M $= 4b$ *&c*, whence putting e in General for the aliquot Parts or Multiples of A D, this property of the Curve will arife from the foregoing Hypothefis, *viz.* if A B or A E be taken $= ea$, the Correfponding Ordinate B H or E L will be $= eeb$.

From which the Equation of the Curve may be Investigated thus: put AE $= ea = x$, then is $e = \frac{x}{a}$, and put EL $= eeb = y$, then is $ee = \frac{y}{b} = \frac{xx}{aa}$, and Consequently $\frac{aa}{b}y = xx$. Which is an Equation Expressing the Nature of the Curve AHN, denoting the same to be a common Parabola referr'd to the Line AF (touching the same in the Vertex A) as an Axis.

2°. Again, suppose AE $= ea = x$, and EL $= \frac{b}{e} = y$, and then $e = \frac{x}{a} = \frac{b}{y}$ and consequently $ab = xy$, and the Curve AKN is an Hyperbola (*Art.* 26.) referr'd to one of the Asymptotes AE.

3°. The same things being supposed, and AE being put $= ea = x$, and EL $= y = \sqrt{eab - eebb}$, or $yy = eab - eebb$. Then in this Last Equation Substitute $\frac{x}{a}$ for e, and there will arise $yy = bx - \frac{bb}{aa}xx$, denoting the Curve to be an Ellipsis and putting $a = b$, we have $ax - xx = yy$ expressing the Nature of the Circle.

And Universally;

4°. Let m and n Denote the determinate Indices of the Powers of e; and suppose $e^m a = x =$ AE, and $e^n b = y =$ EL, then is $e^m = \frac{x}{a}$, and $e^n = \frac{y}{b}$, and by equal Extraction $e = \frac{x^{\frac{1}{m}}}{a^{\frac{1}{m}}} = \frac{y^{\frac{1}{n}}}{b^{\frac{1}{n}}}$, and $b^{\frac{1}{n}} x^{\frac{1}{m}} = a^{\frac{1}{m}} y^{\frac{1}{n}}$, and (advancing both sides of the Equation first to the Power n, and these again to the Power m) $b^m x^n = a^n y^m$, which expresses the Nature of Paraboliform Figures, and if n be Negative; that is, if y be $= \frac{b}{e^n}$, then the Equation $b^m x^n = a^n y^m$ will express the Nature of all sorts of Hyperboliform Figures.

5°. Suppose $ea + e^2 a + e^3 a =$ AE $= x$, and $eb + e^2 b + e^3 b =$ EL $= y$, then is $e + e^2 + e^3 = \frac{x}{a} = \frac{y}{b}$, and $ay = bx$; and consequently, the Line AKN will be a right Line.

6°. Suppose AD $= a$, and DK $= b$, and AO $= r$ $(= 1)$ and suppose AE $= ea$, and EL $= \frac{b^e}{r^{e-1}}$. Hence if the intercepted Diameter x, be put $= 1a, 2a, 3a, 4a$, &c. or to $\frac{1}{2}a, \frac{1}{3}a, \frac{1}{4}a$, &c. the respective Ordinates y will be $= b, \frac{b^2}{r}, \frac{b^3}{r}, \frac{b^4}{r}$, &c. or $\sqrt{rb}, \sqrt[3]{rrb}, \sqrt[4]{r^3 b}$, &c. Now its requir'd to investigate the Relation between the Ordinate and intercepted Diameter from this known property.

Put $x = ea$, then $e = \frac{x}{a}$, and put $\frac{b^e}{r^{e-1}} = y$, then substituting $\frac{x}{a}$ in place of e, we have $b^{\frac{x}{a}} = r^{\frac{x}{a}-1} y$, which (being Reduc'd) gives $b^x = r^{x-a} y^a$, which expresses the Nature of the Curve OKN.

COROL.

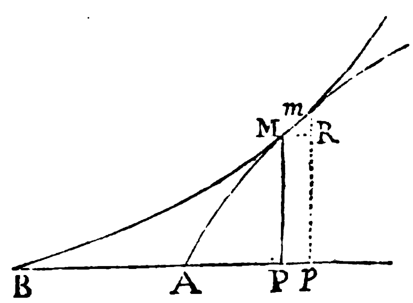
COROLLARY.

85. This Equation expresses the nature of the Logarithmetick Line, by help whereof it is an easie matter to find the Logarithm of any absolute Number and the contrary. *ex. gr.* Suppose $AO = r = 1$, $AD = a = 100000$, and $DK = b = 10$, then this Equation will arise $10^x = 1^{x-100000} y^{100000}$, and because Unity being advanc'd to any Power whatever, never alters its Value, but always remains an Unite; therefore $1^{x-100000} = 1$, and the Equation last found will become $10^x = 1 y^{100000}$. Whence, if the absolute Number y be given, its Logarithm x may be found (*& vice versa*) from the common Principles of Algebra.

PROP. XXI.

If the Curve AM*m and the Equation expressing the Nature thereof be given, and also any determinate Point* M *in the same; Let it be requir'd to describe another Curve* BM*m, which shall touch the given Curve in the given Point* M.

86. Let the given Curve AM*m* be a Circle, and suppose $AP = x$, $PM = y$, then by the property of the Circle $2ax - xx = yy$. Now let it be requir'd to describe the Parabola BM*m* to touch the Circle in the given point M.



Because the Parabola is Convex towards the Circle, the Equation expressing the Nature thereof will be (supposing $BP = t$) $tt = by$. Now the Equation of the Circle is $2ax - xx = yy$, therefore $\frac{a\dot{x} - x\dot{x}}{y} = \dot{y} =$ (because the Fluxion of t or BP is $P p = \dot{x}$) $\frac{2t\dot{t}}{b} = \frac{2t\dot{x}}{b}$, and consequently, $ab - bx = 2ty$; which being multiplied by this Equation $tt = by$, we have $at - tx = 2yy$, and $t = \frac{2yy}{a - x}$. Whence the Parameter b may also be discover'd.

But if the Parabola be Concave towards the Circle, then the Equation expressing the Nature thereof will be $bt = yy$, whence $t = \frac{yy}{2a - 2x}$, and $b = 2r - 2x$.

And in like manner, any other Curve whose Nature is express'd by a given Equation, may be describ'd, so as to touch another Curve, whose Position and Nature is given, in any given Point.

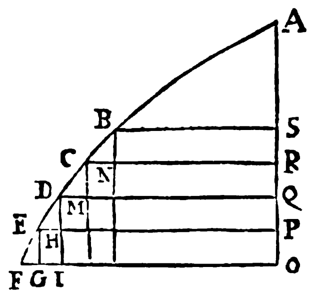
SECT. IV.

The Use of Fluxions

In Investigating the Area's of all sorts of Surfaces.

DEFINITION I.

If the Curve Line ABCDEF consist of an infinite Number of little straight Lines, join'd together in certain Angles in B, C, D, *&c.* then the Space AFO (the Angle AOF being a right Angle in this Case) is a *Poligon*, and if within the *Poligon*, there be drawn the right Lines EP, DQ, CR, BS, *&c.* parallel to FO, and infinitely near one another; that is, if they cut of the infinitely little Parts (equal or unequal) of the Diameter, SR, RQ, QP, PO, *&c.* they will divide the *Poligon* into the *Trapezia* DQPE, CRQD, *&c.* and the *Sum* of all the *Trapezia* will be equal to the *Infinito-lateral* or *Curvilineal Space* ABCDEFOA.



And if the said *Curvilineal Space* be divided into the *Trapezia* EGID, *&c.* by Lines infinitely near one another, and parallel to AO; then the Sum of all these *Trapezia*'s will be equal to the *Curvilineal Space* AFO.

DEFINITION II.

If AO be the Axis of the Curve ABF, and SB, RC, *&c.* Ordinates applied to the same, and if AQ be suppos'd $= x$, QP $= \dot{x}$, QD $=$ PH $= y$, EH $= \dot{y}$; then EP $= y + \dot{y}$, and the Value of the *Trapezium* DQPE $= \overline{DQ + PE} \times \frac{1}{2} QP$ $= \frac{2y\dot{x} + \dot{y}\dot{x}}{2} = y\dot{x} + \frac{\dot{y}\dot{x}}{2}$, and because the Term $\frac{\dot{y}\dot{x}}{2}$ is incomparably less than $y\dot{x}$, it may be rejected, and *the Value of the Trapezium* DQPE or (*Art.* 4) *the Fluxion of the Area* may be express'd thus $y\dot{x}$.

COROL. I.

Hence if y be put for the Ordinate of any Curvilineal Figure, and $\dot{x}$ for the Fluxion of the Abscissa, the Sum of all the Rectangles $y\dot{x}$, will be equal to the Curvilineal Space requir'd.

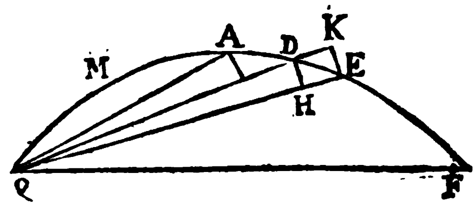
COROL. II.

If the Ordinates decrease, that is if y be $=$ PE, and $y - \dot{y} =$ QD, then the *Trapezium* DQPE $= \overline{2y - \dot{y}} \times \frac{1}{2}\dot{x} = \frac{2y\dot{x} - \dot{y}\dot{x}}{2} = y\dot{x}$.

DEFI-

DEFINITION III.

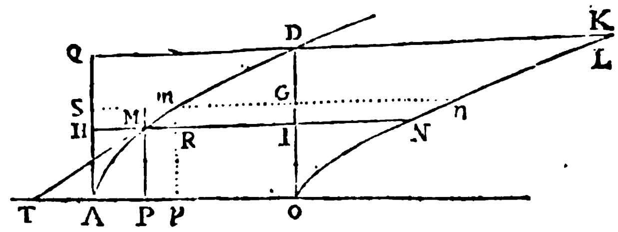
If the Curve Line QADEF be referr'd not to an Axis, but to a fixt and determinate Point Q, and if the Ordinates QA, QD, QE be drawn infinitely near one another, and the Arches DH describ'd on the Center Q, then the said Arches are taken for the Fluxions of the Absciffa, and DH is $= \dot{x}$, QD $= y$, HE $= \dot{y}$, and the Triangle QDE or *the Fluxion of the Area* QMDQ is $=$ QE $\times \frac{1}{2}$DH $= \frac{y\dot{x} + \dot{y}\dot{x}}{2} =$ (rejecting $\dot{y}\dot{x}$) $\frac{y\dot{x}}{2}$, and the whole Curvilineal Space QAF, which is equal to the Sum of all the Triangles QDE will be equal to the Sum of all the $\frac{y\dot{x}}{2}$.



PROP. I.

If the Curve AM*m*D, *the Axis* AO, *and the Ordinate* DO *be given, and if another Curve* ONLK *be such, that (if* MT *be drawn to touch the given Curve in any Point* M, *and* MRIN *be drawn Parallel to the Axis* AO, *then)* PT *be always equal* IN; *I say, the Curvilineal Space* AOD *is equal to the Curvilineal Space* DOK.

87. For since the Triangles *m*RM, MPT are similar, therefore TP (t) : PM (y) :: MR ($\dot{x}$) : R*m* ($\dot{y}$); and consequently, TP $\times$ R*m* or IN $\times$ GI ($= t\dot{y}$) is $=$ PM $\times$ MR ($= y\dot{x}$), and because there are as many Rectangles GN in the Curvilineal Space DOK, as there are Rectangles M*p* in the given Figure ADO, 'tis manifest that those Rectangles being always equal between themselves, their Sums must be equal; that is, the Figure ADO must be equal to the Figure DOK.



CONSECTARIES.

88. 1°. The Trilineal Space ADQ is equal to the Sum of all the $x\dot{y}$; For it is equal to the Sum of all the Trapezia S*m*MH, and HM $=$ AP $= x$; and SH $= \dot{y}$, and SH $\times$ HM $= x\dot{y}$.

2°. If the Curve AMD be a Paraboliform Figure, and if the Equation (*Art.* 28.) expressing the Nature of such Curves be $y^m = x$; then the Sub-tangent PT is $= mx$, and consequently, the Trilineal Space DOK is equal to all the Rectangles $mx\dot{y}$, and the Trilineal Figure ADQ is equal to the Sum of all the Rectangles $x\dot{y}$. But all the $mx\dot{y}$ is to all the $x\dot{y}$, as m is to 1; and all the $mx\dot{y}$ are equal all the $y\dot{x}$, therefore all the $y\dot{x}$ (that is the given Parabola) are to all the $x\dot{y}$ (the Complement of the Parabola to the circumscribed Parallelogram) as m is to 1; and by Composition the Parabola AMDO is to the circumscrib'd Parallelogram, as m is to $m + 1$; That is,

$\frac{m}{m+1} \times$ Parallelogram AQDO equal to the Area of the Paraboliform Figure AMDO.

Hence if the Equation of the Curve be $y^2 = x$, or $y^2 =$ (putting the Parameter $a = 1$) ax, then the Curve is a common Parabola, and the Area thereof $\frac{m}{m+1} \times$ Parallelogram AQDO is equal $\frac{2}{3}$ Parallelogram AQDO.

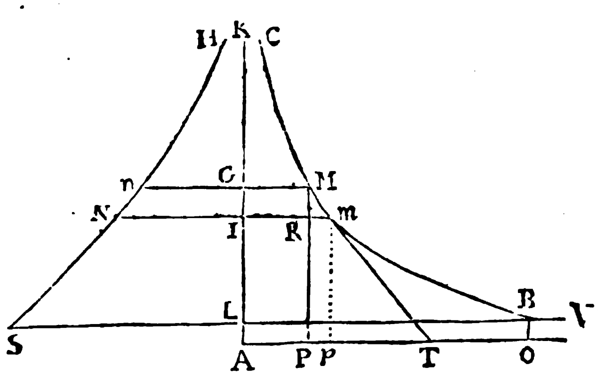
If the Equation of the Curve be $y^3 = x$; then the Space ADO is $\frac{3}{4}$ the circumscrib'd Parallelogram AD.

And if $y^{\frac{1}{3}}$ be equal x, then $y = x^3$, and the Curve AMmD becomes Concave toward its Axis; that is, it is referr'd to the Tangent AQ; and the Trilineal Space ADQ $= \frac{3}{4}$ the circumscrib'd Parallelogram AQDO.

PROP. II.

The same things being suppos'd, as in the preceding Proposition; 'tis requir'd to investigate the Areas of all sorts of Hyperboliform Figures.

89. Let the Curve CMmB be an Hyperbola or Hyperboliform Figure, and AK, AO the Asymptotes, and let the general Equation for such Curves be (*Art.* 28.) $y^m = x$ (the Parameter being $= 1$, and the Index m being Negative) then the Sub-tangent PT will be equal to mx.



And because $t\dot{y} = y\dot{x}$, if Gn be always taken equal PT, then the Rectangles NIGn $= t\dot{y}$ will always be equal to the Rectangle PRmp $= y\dot{x}$, and if this be always done, the Figure KAOBC infinite towards KC, that is, all the Rectangles $y\dot{x}$ will be equal to the Figure KLSH equal to all the $t\dot{y} =$ (because $t = mx$) all the $mx\dot{y}$. But (supposing the Figure KAOBC $= b$; and the inscribed Rectangle LBOA $= d$) the Figure KLBC will be $= b - d =$ all the $x\dot{y}$ (because IR $= x$ and RM $= \dot{y}$). Whence all the Rectangles $mx\dot{y} =$ all the $y\dot{x} = b$, are to all the $x\dot{y} = b - d$, as m is to 1, and by Division, $m : m - 1 \vdots b : d$. That is, the Figure KAOBC is to the inscrib'd Rectangle LAOB, as the Exponent of the Power of the Ordinate (m) is to the same Exponent less 1.

CONSECTARY I.

90. 1°. If m be greater than 1, then the space Indeterminate towards K may be measur'd; if m be $= 1$, then the second Term in the Analogy is equal to nothing, and consequently the Space KAOBC is infinitely extended towards K, and infinite in respect of the inscrib'd Parallelogram LO. And if m be less than 1, then the Space KAOBC is more than infinite.

CONSECTARY II.

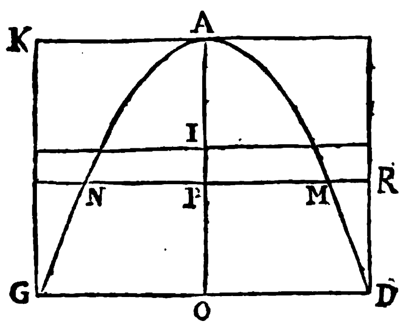
2°. The Equation expressing the Nature of the Apollonian Hyperbola is $y^{-1} = x$, or (supposing the Parameter $a = 1$) $aa = xy$, whence it appears that $m - 1$ is $= 0$, and consequently the proportion between the inscrib'd Rectangle LAOB and the said infinite Space is infinitely great. But to measure all other sorts of Curvilineal Spaces KGMC included between the Asymptotes and any Hyperboliform Curve;

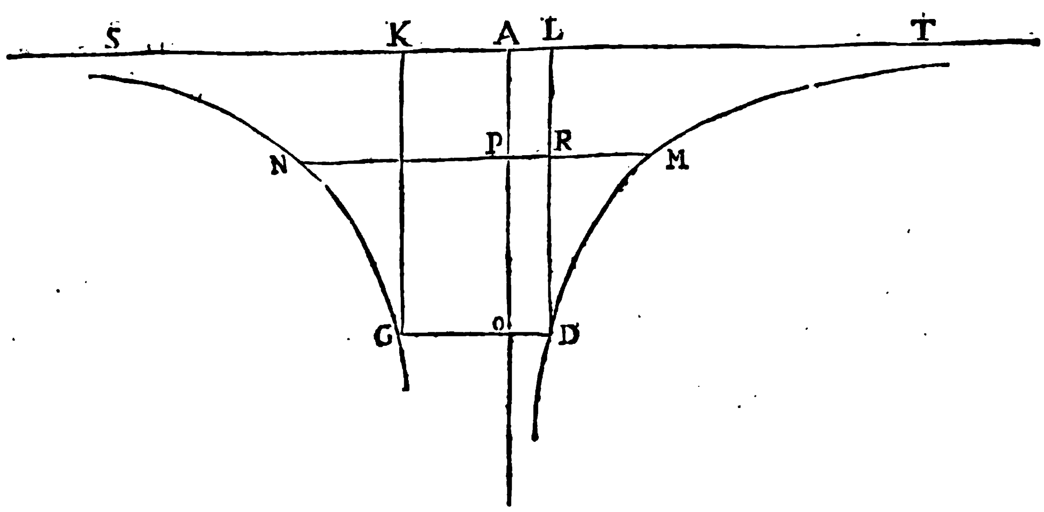
Curve; say, $m-1:1::$ Parallelogram APMG: the Area of the Curvilineal Figure KGMC: For $m:m-1::b:d$, and by Inversion and Division $m-1:1::d:b-d$.

CONSECTARY III.

3°. Hence it is also manifest, that any Parabola, or the Complement of any Parabola to the circumscrib'd Parallelogram, or an Hyperbola being given, and supposing the Ordinate PM $=y$, the intercepted Diameter AP $=x$, PR $=$ OD $=b$, the Axis AO $=c$; that all the PR or b's are to all the PM or y's, as $m\mp1$ is to m.

And if it be requir'd to find what proportion all the b's advanc'd to any Power n, has to all the y's advanc'd to the same Power n, it may be investigated in this manner. Suppose the new Curve NG to be describ'd, so that PN be always equal or proportional to $\overline{PM}|^n$ or y^n, then it is manifest that the Sum of all the y^n is equal to the Sum of all the PN or to the Curvilineal Space AOGN, and because y^n is always





equal or proportional to PN and PN becomes equal to OG, at the same time that y^n becomes equal to $b^n=\overline{OD}|^n$ 'tis likewise manifest, that the Sum of all the b^n is equal or proportional to the Sum of all the OG or the Parallelogram AOGK. Whence it appears that to investigate the Proportion of all the y^n to all the b^n is the same thing as to investigate the proportion of the Curvilineal Space AOGN to the Parallelogram AOGK. Which may be done thus; in Paraboloides and Hyperboloides, the general Equation expressing the Nature of such Curves is $y^m=x$, and consequently $y^n=x^{\frac{n}{m}}$. Now suppose $y^n=z$, then $z=x^{\frac{n}{m}}$, and $z^m=x^n$, which is an Equation expressing the Nature of a Paraboliform, or Hyperboliform Curve. Let the said Curve be ANG, and AP $=x$, AO $=c$, PN $=z$, and OG $=d$.

Then (§. 3°. *Art.* 90.) $\frac{m}{n}\pm1:\frac{m}{n}::(m\pm n:m)$ all the d: to all the z. And because z was put equal to y^n, therefore when z or PN becomes OG or d, then y^n becomes b^n; and consequently d is $=b^n$, therefore $m\pm n:m::S:b^n:S:y^n$.

Hence

Hence we may easily deduce the 64 Prop. Arith. Infinit. first discover'd by the Learned Dr. Wallis.

CONSECTARY IV.

4°. For we found before $z = x^{\frac{n}{m}}$, and it is also $m : m + n :: 1 : 1 + \frac{n}{m} :: y^n : b^n$. *In the direct Series*, and $1 : 1 - \frac{n}{m}$ :: all the y^n : all the b^n. *In the Negative Series.* Whence it is evident that if the Exponent of the Power of the intercepted Diameter x, be taken for the Index of the *Series*, it will be as 1 is to the Power of the intercepted Diameter or Index of the *Series* (because $z = y^n = x^{\frac{n}{m}}$, and consequently, $x^{\frac{n}{m}}$ represents y^n in the Dimension requir'd) *Increased by Unity*, so are all the y^n to all the b^n.

CONSECTARY V.

5°. Hitherto we have found the proportion of all the y^n or (Multiplying both by the Fluxion $\dot{x}$) $y^n \dot{x}$ to all the $b^n \dot{x}$, their absolute Value may be found thus: It was by the preceding *Corollary*, $m : m \mp n$ :: all the $z\dot{x}$: all the $d\dot{x}$; that is, so is the Space AOGN, to the Rectangle AOGK $= dc$, Therefore $\frac{mdc}{m \mp n}$ = all the $z\dot{x} = S : y^n \dot{x}$, (because $z = y^n$). But $b^n = d$, therefore $S : y^n \dot{x} = \frac{mdc}{m \mp n} = \frac{mcb^n}{m \mp n} = \frac{1cb^n}{1 \mp \frac{n}{m}}$.

CONSECTARY VI.

6°. And if we suppose the Index $\mp \frac{n}{m} = \mp \mu$, then the Value of all the $y^n \dot{x}$ is $= \frac{1cb^n}{1 \mp \mu}$; and again, If in place of b^n we substitute $c^{\mp\mu}$ (because $y^n = x^{\mp\frac{n}{m}}$, that is, when y becomes $= b$, and $x = c$, $c^{\mp\frac{n}{m}} = b^n = c^{\mp\mu}$) we shall have all the $y^n \dot{x} = \frac{c^{1\mp\mu}}{1 \mp \mu}$.

CONSECTARY VII.

7°. Hence *Mercators Lem. Prop.* 16. *Logarithmotechn.* may be deduced, upon which the Learned Dr. *Gregory*'s *Geometrical Exercise* chiefly depends. For because all the $y^n \dot{x}$ are = all the $x^{\mp\mu}\dot{x}$, it is evident that (rejecting the invariable Quantities if there be any) all the $x^{\mp\mu}\dot{x} = \frac{c^{1\mp\mu}}{1\mp\mu}$ = (by putting the greatest $x = c$) $\frac{x^{1-\mu}}{1+\mu}$. Whence we have the Demonstration of the fundamental Rule in *Summatory Arithmetick, to find the Flowing Quantity of a given Fluxion.*

CONSECTARY VIII.

8°. For instance, if the Right Line AO $= c$ be divided into an infinite Number of $\dot{x}$, the Sum of all the Rectangles contain'd under any Power of the Abscissa x, and all the $\dot{x}$ respectively, that is the Sum of all the $x^{\mp\mu}\dot{x}$, or the Flowing Quantity whereof $x^{\mp\mu}\dot{x}$ is the Fluxion, is equal to $\frac{c^{1\mp\mu}}{1\mp\mu} = \frac{x^{\mp\mu+1}}{\mp\mu+1}$ = to the Power of x

increased

increased by Unity, and divided by the new Exponent; and seeing the thread of my Discourse has led me on to this Head, I shall insist more at large on the same in the next.

PROP. III.

To find the Flowing Quantity of any Fluxion.

91. The Summing up of Infinites, or finding the Sum of all the Fluxions of an unknown Quantity, or the finding the Flowing Quantity from its Fluxion given, is not less difficult in many Cases, than the Reverse is easie. I shall begin with the easiest Examples, and proceed gradually to those that are more intricate and difficult.

EXAMPLE I.

Let it be requir'd to find the Flowing Quantity of this Fluxion $aa\dot{x}$, or $aax^0\dot{x}$; to the Index of the Flowing Quantity add 1, and then we have $aax^{0+1}\dot{x}$; divide this by the Fluxionary Letter $\dot{x}$, and by the new Index $0+1$ or 1, the Quotient aax is the Flowing Quantity of the given Fluxion.

EXAMPLE II.

Let it be requir'd to find the Flowing Quantity of $ay\dot{x}+ax\dot{y}$; the Flowing Quantity of the first Member $ay\dot{x}$ is $=axy$, and that of the second Member $ax\dot{y}$ is $=axy$; whence it is plain, that the Flowing Quantity of $ay\dot{x}+ax\dot{y}$ is (*Art.* 12. 13.) $=axy$.

EXAMPLE III.

Let it be requir'd to find the Flowing Quantity of $3xx\dot{x}$; increase the Index of the Flowing Quantity x by 1, and then we have $3x^3\dot{x}$, which divide by the new Index 3, and by the Fluxionary Letter $\dot{x}$, then the Quotient $=\frac{3x^3\dot{x}}{3\times\dot{x}}=x^3$ is the Flowing Quantity of the given Fluxion.

And Universally:

If it be requir'd to find the Flowing Quantity of $mx^{m-1}\dot{x}$, increase the Index of the Flowing Quantity x by 1, and then we have $mx^m\dot{x}$, which divide by the new Index m, and by the Fluxionary Letter $\dot{x}$, and there will arise x^m for the Flowing Quantity requir'd.

EXAMPLE IV.

Let it be requir'd to find the Flowing Quantity of $\frac{a\dot{x}}{xx}$: The Fluxion (*Art.* 16.) express'd by the other way of Notation, is $ax^{-2}\dot{x}$, and the Flowing Quantity thereof is $-ax^{-1}=\frac{-a}{x}$. Thus the Flowing Quantity of $\frac{ax\dot{x}}{x^{\frac{3}{2}}}=\frac{a\dot{x}}{x^{\frac{1}{2}}}=ax^{-\frac{1}{2}}\dot{x}$ is $=2ax^{\frac{1}{2}}$.

EXAMPLE V.

Let it be requir'd to find the Flowing Quantity of $\frac{-3\dot{x}}{x^4} = -3x^{-4}\dot{x}$. To the Index of the Power of the Flowing Quantity add 1, and divide by the new Exponent and by $\dot{x}$, the Quotient is $= x^{-3} = \frac{1}{x^3} =$ the Flowing Quantity requir'd.

EXAMPLE VI.

Let it be requir'd to find the Flowing Quantity of $\frac{x^2\dot{x}}{\sqrt{rx}}$; this Fluxion may be Expreſs'd thus $r^{-\frac{1}{2}} \times x^{\frac{3}{2}}\dot{x}$, and then the Flowing Quantity thereof is $\frac{2}{5}r^{-\frac{1}{2}}x^{\frac{5}{2}} = \frac{2\sqrt[2]{x^5}}{5\sqrt{r}}$.

EXAMPLE VII.

The Flowing Quantity of $\dot{x}\sqrt{2rx}$, or $\dot{x} \times \overline{2r}^{\frac{1}{2}} \times x^{\frac{1}{2}}$ is $\frac{2}{3} \times \overline{2r}|^{\frac{1}{2}}\, x^{\frac{3}{2}} = \frac{2}{3}\sqrt[2]{2rxxx}$, and the Flowing Quantity of $\dot{x}\sqrt{2rx - xx}$ is found by reducing $\overline{2rx - xx}|^{\frac{1}{2}}$ to an infinite Series, and Multiplying the ſame by $\dot{x}$, and then finding the Flowing Quantity of every Term.

EXAMPLE VIII.

To find the Fluent of $a\dot{x}\sqrt{ax - aa}$. In ſuch Caſes where the Fluxion is affected with a *Vinculum*; we muſt conſider whither the Fluxional Quantity ſtanding before the Radical Sign, be the Fluxion of the ſimple or compound Quantity under the ——— *Vinculum*, for in ſuch Caſes the Fluent may be found by the general Rule.

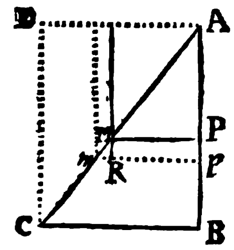
Thus in this Example I obſerve, that $a\dot{x}$ is the Fluxion of $\overline{ax - aa}$, and therefore the Fluent of $a\dot{x}\sqrt{ax - aa}$ or $a\dot{x} \times \overline{ax - aa}|^{\frac{1}{2}}$ is $\frac{2}{3} \times \overline{ax - aa}|^{\frac{3}{2}} = \frac{2ax - 2aa}{3}\sqrt{ax - aa}$.

In like manner the Fluent of $\frac{r\dot{x} - x\dot{x}}{\overline{2rx - xx}|^{\frac{1}{2}}}$ or $\overline{r\dot{x} - x\dot{x}} \times \overline{2rx - xx}|^{-\frac{1}{2}}$ or $\overline{2r\dot{x} - 2x\dot{x}} \times \frac{1}{2}\sqrt{2rx - xx}|^{-\frac{1}{2}}$ will be found (if to the Exponent $-\frac{1}{2}$ we add 1, and divide by the New Exponent $\frac{1}{2}$ and by the Fluxionary Quantity $\overline{2r\dot{x} - 2x\dot{x}}$) to be $\sqrt{2rx - xx}$.

92. These

92. *These Rules may be Demonstrated by Induction also; and because that Method by particular Instances may serve to give the Reader a clearer Notion of* Summatory Arithmetick, *I shall Explain the same in the following Examples.*

1°. In the Rectangular Triangle ABC. Suppose $AB = a$, $BC = b$, $AP = x$, $Pp = \dot{x}$, $PM = y$; then the Equation of the Triangle is $y = \frac{bx}{a}$, and the infinitely little Parallelogram Mp = to the Fluxion of the Triangle, is $= y\dot{x} =$ (by substitution) $\frac{bx\dot{x}}{a}$. And the Flowing Quantity is $\frac{bxx}{2a} =$ $\left(\text{putting } y = \frac{bx}{a}\right)$ $\frac{xy}{2}$. It remains to be prov'd that the Sum of all the $y\dot{x}$ is = to $\frac{xy}{2}$.



Compleat the Parallelogram ABCD, then it is evident that the Triangle ABC is equal to the Sum of all the $y\dot{x}$, and the Triangle ADC is equal to the Sum of all the $x\dot{y}$. But both these Triangles are equal to the Parallelogram, and each is equal to $\frac{1}{2}$ the Parallelogram, and the Parallelogram is equal to xy, therefore all the $y\dot{x}$ $= \frac{xy}{2} =$ Triangle ABC.

2°. Let AMB be a Parabola, $AP = x$, $PM = y$, the Parameter = 1, then the Equation of the Curve is $x^{\frac{1}{m}} = y$, and the Fluxion of the Parabolic Space, *viz.* Mp $= y\dot{x} = x^{\frac{1}{m}}\dot{x}$. Now it is evident that the Sum of all those Parallelograms is equal to the Parabolic Space AMBD. And the Flowing Quantity of $x^{\frac{1}{m}}\dot{x}$ is $\frac{m}{m+1}x^{\frac{1}{m}+1} =$ (putting y for $x^{\frac{1}{m}}$) $\frac{m}{m+1}xy$, which we must prove to be equal to the Sum of all the $y\dot{x}$.

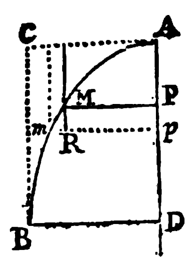
Compleat the Parallelogram ADBC, then it is manifest that the Space AMBD is equal to all the $y\dot{x}$, and the Space AMBC is equal to all the $x\dot{y}$. But by the Method of Tangents it is, $\dot{y} : \dot{x} :: y : t$, and $t\dot{y} = y\dot{x}$, and in the Parabola $t = mx$, *ergo* $y\dot{x} = mx\dot{y}$.

Whence $1 = \frac{mx\dot{y}}{y\dot{x}}$

and $\frac{1}{m} = \frac{x\dot{y}}{y\dot{x}}$

adding 1 to each side of the Equation $\Big\}$ $\frac{1}{m} + 1 = \frac{x\dot{y}}{y\dot{x}} + 1$

that is $\frac{m+1}{m} = \frac{x\dot{y} + y\dot{x}}{y\dot{x}}$



Whence

Whence $m : m+1 :: y\dot{x} : x\dot{y} + y\dot{x}$

And consequently $m : m+1 :: S:y\dot{x} : S:x\dot{y} + S:y\dot{x}$

But $S:x\dot{y} + S:y\dot{x} = xy$

Therefore $m : m+1 : S:y\dot{x} : xy$

And consequently $\frac{m}{m+1} \times xy = S:y\dot{x}$. Q. E. D.

But besides the Examples I have produc'd, there are others which occur, to which these Rules cannot be immediately applied; and that the Reader may not be at too great a loss in such Cases, I shall endeavour to assist him in that particular. But first, It will be necessary to premise this

LEMMA.

93. If a Binomial be to be rais'd to any Power, *g. v. m*, (which represents any Number, Whole or Broken, Positive or Negative) then the *Unciæ* or Numbers prefixt to the several Terms are, $1 \times \frac{m-0}{1}$. $1 \times \frac{m-0}{1} \times \frac{m-1}{2}$. $1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times$, &c. Respectively.

And if $P + PQ$ represent the Quantity to be rais'd to the Given Power; P the first Term, and Q the rest divided by that first Term, and $\frac{m}{n}$ the Exponent of that Root or Dimension, then

$$\overset{}{\overline{P+PQ}}|^{\frac{m}{n}} = \overset{A}{P^{\frac{m}{n}}} + \overset{B}{\frac{m}{n} AQ} + \overset{C}{\frac{m-n}{2n} BQ} + \overset{D}{\frac{m-2n}{3n} CQ}, \&c.$$

For instance, if it be requir'd to Extract the Square Root of $rr - xx$; that is, to raise (the Word *raise* being us'd indifferently for involving or evolving any Binomial) the Binomial $rr - xx$ to the Power or Dimension, whose Exponent is $\frac{1}{2}$, then $P = rr$, $Q = \frac{-xx}{rr}$, $m = 1$, and $n = 2$; and consequently, $\overline{rr - xx}|^{\frac{1}{2}} = r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7} -$, &c.

Let it be requir'd to raise the Binomial $a + x$ to the Power whose Exponent is m, or let m be the Index of the Root of the Binomial, which is to be Extracted. Then, $P = a$, $Q = \frac{x}{a}$, and $\frac{m}{n} =$ (n in this Case being $= 1$) m, therefore $\overline{a+x}|^m$ is $= a^m + ma^{m-} x + m \times \frac{m-1}{2} \times a^{m-2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4}x^4$, &c.

By the same Method any *Tri-nomial*, *Quadri-nomial*, &c. or *Infinito-nomial* may be rais'd to any Given Power, *v. g.* To raise the *Infinito-nomial* $a + bz + cz^2 + dz^3 +$, &c. to the Power, whose Exponent is m: in the preceding *Bi-nomial* Theorem,

Instead

Instead of x put $\overline{bz+cz^2+dz^3, \&c.}|^1$, and instead of x^2 substitute $\overline{bz+cz^2+dz^3,\&c.}|^2$, *&c.* Then it is manifest that $\overline{a+bz+cz^2+dz^3,\&c.}|^m$ is $= a^m + m a^{m-1} \times \overline{bz+cz^2+dz^3,\&c.}|^1 + m \times \frac{m-1}{2} a^{m-2} \times$ $\overline{bz+cz^2+dz^3,\&c.}|^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} \times \overline{bz+cz^2+dz^3,\&c.}|^3$ $+ m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} \times \overline{bz+cz^2+dz^3,\&c.}|^4 + \&c.$

EXAMPLE I.

Let it be requir'd to find the Flowing Quantity of this Fluxion $\dot{x}\sqrt{rr-xx}$. Reduce $\sqrt{rr-xx}$ to an (*Art.* 93.) Infinite Series, and then $\overline{rr-xx}|^{\frac{1}{2}}$ is $= r - \frac{xx}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7} - \&c.$ and consequently, $\dot{x}\sqrt{rr-xx}$ is $= r\dot{x} - \frac{x^2\dot{x}}{2r} - \frac{x^4\dot{x}}{8r^3} - \frac{x^6\dot{x}}{16r^5} - \frac{5x^8\dot{x}}{128r^7} - \&c.$ and finding the Flowing Quantity of every Term of this Series, then the Sum of all the $\dot{x}\sqrt{rr-xx}$ is $= rx - \frac{x^3}{6r} - \frac{x^5}{40r^3} - \frac{x^7}{112r^5} - \frac{5x^9}{1152r^7} - \&c.$ Q. E. I.

EXAMPLE II.

It is requir'd to find the Flowing Quantity of $\frac{rr\dot{x}}{r+x}$. It is evident from the (*Art.* 16.) Notation of Powers, that $\frac{rr}{r+x}$ is $= rr \times \overline{r+x}|^{-1}$. But $\overline{r+x}|^{-1}$ is $=$ (*Art.* 93.) $r^{-1} - \frac{x}{rr} + \frac{xx}{r^3} - \frac{x^3}{r^4} + \&c.$ and consequently $\frac{rr}{r+x}$ or $rr \times \overline{r+x}|^{-1}$ is $= r - x + \frac{xx}{r} - \frac{x^3}{r^2} + \&c.$ and $\frac{rr\dot{x}}{r+x}$ is $= r\dot{x} - x\dot{x} + \frac{x^2\dot{x}}{r} - \frac{x^3\dot{x}}{r^2} +$ *&c.* and the Flowing Quantity of $\frac{rr\dot{x}}{r+x}$ is $= rx - \frac{xx}{2} + \frac{x^3}{3r} - \frac{x^4}{4r^2} +$, *&c.* Q. E. I.

SCHOLIUM.

And if we divide the Series (*Exam.* 1.) by $\overline{rr-xx}|^{\frac{1}{2}}$ reduc'd to an infinite Series, and multiply the Divisor by the Quotient, we shall have $rx - \frac{x^3}{6r} - \frac{x^5}{40r^3} - \frac{x^7}{112r^5} - \frac{5x^9}{1152r^7} - \&c. = x + \frac{2x^3}{6r^2} + \frac{32x^5}{120r^4}, \&c.\ \overline{rr-xx}|^{\frac{1}{2}}$.

And in General, If the given Fluxion consists of Universal Exponents and Coefficients, reduce the part under the *Vinculum* to an infinite Series, which multiply by the part before the *Vinculum*, and find the Flowing Quantity of every Term; lastly, divide this last Series or the Fluent by the part under the Radical Sign Affected, with any the most convenient Exponent, and multiply the said part under the said Exponent

nent by the said Quotient, so shall you have a Series expressing the Fluent of the Given Fluxion, and readily shewing when and whither the Series consists of a finite Number of Terms or not.

94. *The Fluent of a Fluxion involving surd Quantities, may be investigated after another manner, which is sometimes preferable by much to the former: The Principles of this Method are,*

1°. Reduce the given Fluxion to its simplest Terms.

2°. Assume a new Equation Adfected with indetermin'd Coefficients, so that reducing the same to Fluxions, the Terms of this may be compar'd with those of the Given Fluxion, in order to determine the unknown Coefficients.

3°. Having determin'd the assum'd Coefficients, substitute their respective Values in the assum'd Equation, and you have the Fluent of the Given Fluxion.

Since this Method deserves the Readers Consideration, I shall endeavour fully to explain the same; and that I may not be mis-understood, I shall begin with some easie Examples.

EXAMPLE I.

Let it be requir'd to find the Fluent of $a\dot{x}\sqrt{ax-aa}$, the Fluxion reduc'd to its simplest Terms, is $a\dot{x} \times \overline{ax-aa}|^{\frac{1}{2}}$. Now suppose the Fluent of this Fluxion to be $A \times \overline{ax-aa}|^{\frac{3}{2}}$, then it is evident that the Fluxion of this Fluent must be equal to the Given Fluxion, that is $\frac{3}{2}A \times a\dot{x} \times \overline{ax-aa}|^{\frac{1}{2}}$ is $= a\dot{x} \times \overline{ax-aa}|^{\frac{1}{2}}$. Therefore (dividing by $\overline{ax-aa}|^{\frac{1}{2}}$) $\frac{3}{2}A \times a\dot{x} = a\dot{x}$, and $A = \frac{2}{3}$. Having thus found the true Value of the indeterminate Coefficient A (*viz.* $\frac{2}{3}$) in the assum'd Equation, substitute the same in place of A, and then we have $\frac{2}{3} \times \overline{ax-aa}|^{\frac{3}{2}}$ or $\frac{2ax-2aa}{3}\sqrt{ax-aa}$ equal to the Fluent of the Given Fluxion.

EXAMPLE II.

To find the Fluent of $\frac{r\dot{x}-x\dot{x}}{\sqrt{2rx-xx}}$, this Fluxion is express'd thus $\overline{r\dot{x}-x\dot{x}} \times \overline{2rx-xx}|^{-\frac{1}{2}}$. Suppose the Fluent thereof to be $A \times \overline{2rx-xx}|^{\frac{1}{2}}$, then the Fluxion of this Quantity is $\frac{1}{2}A \times \overline{2r\dot{x}-2x\dot{x}} \times \overline{2rx-xx}|^{-\frac{1}{2}} = \overline{r\dot{x}-x\dot{x}} \times \overline{2rx-xx}|^{-\frac{1}{2}}$ Therefore $\frac{1}{2}A \times 2r\dot{x}-2x\dot{x} = r\dot{x}-x\dot{x}$, and $A = 1$; and consequently, the Fluent of the Given Fluxion is equal to $\overline{2rx-xx}|^{\frac{1}{2}}$.

EXAMPLE III.

To find the Fluent of $dx^r\dot{x} \times \overline{e+fx^n}|^m$. Assume an Equation with indeterminate Coefficients, so that reducing the same to Fluxions, the Terms thereof may be compar'd with those of the Given Fluxion. Let the said Equation be $Adx^{r-n+1} + Bdx^{r-2n+1} + Cdx^{r-3n+1}$, &c. $\times \overline{e+fx^n}|^{m+1} = S : dx^r\dot{x} \times \overline{e+fx^n}|^m$.

Then,

Then,

$$\overline{r-n+1}\times Adx^{r-n}\dot{x}+\overline{r-2n+1}\times Bdx^{r-2n}\dot{x}+\overline{r-3n+1}\times Cdx^{r-3n}\dot{x}$$
&c. $\times\overline{e+fx^n}|^{m+1}+\overline{m+1}\times\overline{e+fx^n}|^{m}\times nfx^{n-1}\dot{x}\times\overline{Adx^{r-n+1}+Bdx^{r-2n+1}+Cdx^{r-3n+1}}$, &c. $=dx^r\dot{x}\times\overline{e+fx^n}|^{m}$:

Whence, supposing $\frac{1}{p}=m+1$, and putting $\dot{x}=1$:

$\overline{r-n+1}\times Adx^{r-n}+\overline{r-2n+1}\times Bdx^{r-2n}+\overline{r-3n+1}\times Cdx^{r-3n}$, &c. $\times\overline{e+fx^n}|^{\frac{1}{p}}+\frac{1}{p}\times n\times Adf\times x^r+\frac{1}{p}\times n\times Bdfx^{r-n}+\frac{1}{p}\times n\times Cdfx^{r-2n}$, &c. $\times\overline{e+fx^n}|^{\frac{1-p}{p}}=dx^r\times\overline{e+fx^n}|^{\frac{1-p}{p}}$.

And Multiplying each Side of the Equation by $p\times\overline{e+fx^n}|^{\frac{p-1}{p}}$ we have

$p\times\overline{r-n+1}\times Adx^{r-n}+p\times\overline{r-2n+1}\times Bdx^{r-2n}+p\times\overline{r-3n+1}\times Cdx^{r-3n}$, &c. $\times\overline{e+fx^n}+n\times Adf\times x^r+n\times Bdfx^{r-n}+n\times Cdfx^{r-2n}$, &c. $=dx^r\times\overline{e+fx^n}|^{\frac{1-p}{p}}\times p\times\overline{e+fx^n}|^{\frac{p-1}{p}}$.

Which being Order'd, we have

$$\left.\begin{matrix}p\times\overline{r-n+1}\times Adf\\ n\times Adf\\ \end{matrix}\right\}x^r\left.\begin{matrix}+p\times\overline{r-n+1}\times Ade\\ +p\times\overline{r-2n+1}\times Bdf\\ +\quad n\times Bdf\end{matrix}\right\}x^{r-n}\left.\begin{matrix}+p\times\overline{r-2n+1}\times Bde\\ +p\times\overline{r-3n+1}\times Cdf\\ +\quad n\times Cdf\end{matrix}\right\}x^{r-2n},\ \&c.$$

$$=dx^r\times\overline{e+fx^n}|^{\frac{1-p}{p}}\times p\times\overline{e+fx^n}|^{\frac{p-1}{p}}=p\times dx^r.$$

From which Equation the unknown Coefficient A, B, C, *&c.* may be determin'd in this manner,

$p\times\overline{r-n+1}\times Adf+n\times Adf=pd$

and Dividing by pd,

$\overline{r-n+1}\times Af+\frac{1}{p}\times n\times Af=1$,

Substituting $m+1$ for $\frac{1}{p}$,

$\overline{r-n+1}\times Af+\overline{m+1}\times n\times Af=1$.

Whence $A=\dfrac{1}{\overline{r-n+1}\times f+\overline{mn+n}\times f}=\dfrac{1}{\overline{mn+r+1}\times f}$.

Secondly,

$p\times\overline{r-n+1}\times Ade+p\times\overline{r-2n+1}\times Bdf+n\times Bdf=0$.

And by Transposition, Division and Restitution.

$\overline{r-2n+1}\times Bf+\overline{mn+n}\times Bf=\overline{n-r-1}\times Ae$.

Whence

Whence $B = \frac{\overline{n-r-1} \times Ae}{\overline{r-2n+1} \times f + \overline{mn+n} \times f} = \frac{\overline{n-r-1} \times Ae}{\overline{mn+r-n+1} \times f}$.

In like manner,

$$C = \frac{\overline{2n-r-1} \times Be}{\overline{mn+r-2n+1} \times f}, \text{ \&c.}$$

Whence it is evident that $Adx^{r-n+1} + Bdx^{r-2n+1} + Cdx^{r-3n+1}$, &c. $\times \overline{e+fz^n}|^{m+1}$ is $= \frac{d}{\overline{mn+r+1} \times f} \times x^{r-n+1} + \frac{d}{\overline{mn+r+1} \times f} \times \frac{\overline{n-r-1} \times de}{\overline{mn+r-n+1} \times f} \times x^{r-2n+1} + \frac{d}{\overline{mn+r+1} \times f} \times \frac{\overline{n-r-1} \times de}{\overline{mn+r-n+1} \times f} \times \frac{\overline{2n-r-1} \times de}{\overline{mn+r-2n+1} \times f} \times x^{r-3n+1}$, &c. $\times \overline{e+fx^n}|^{m+1} = S: dx^r \dot{x} \times \overline{e+fx^n}|^m$ Q. E. I.

In which it may be obferv'd that, the Exponents of the Terms of the Indeterminate Series before the Radical Sign, may be taken different from thofe above, provided that the Exponent of the firft Term be not lefs than $r-n+1$, and that the following Exponents proceed regularly: That the Exponents of the Terms before the Radical Sign may be continually Increafed or Decreafed by n, for in either Cafe, the Terms of the Fluxion of this affumed Equation will become Homologous to thofe of the given Fluxion: That when the Exponents Increafe regularly by n, the Fluent will Confift of a Finite Number of Terms when $\frac{r+1+mn}{-n}$ is equal to a pofitive whole number: And that when the Exponents Decreafe Regularly by n, the Fluent will confift of a finite Number of Terms, when $\frac{r+1}{n}$ is equal to a pofitive whole Number.

This General *Theorem* may eafily be applied to find the Fluent of any given Fluxion included in the General one $dx^r \dot{x} \times \overline{e+fx^n}|^m$. *v. g.* To find the Fluent of $a\dot{x} \times \overline{ax-aa}|^{\frac{1}{2}}$. I put the fame equal to the general Fluxion, *viz.*

$$dx^r \dot{x} \times \overline{e+fx^n}|^m = a\dot{x} \times \overline{ax-aa}|^{\frac{1}{2}}.$$

Then $d=a$, $r=0$, $f=a$, $n=1$, $m=\frac{1}{2}$, $e=-aa$; and if we fubftitute the faid particular Values of, d, r, f, n, m, e in the general Fluent, we fhall have,

$$\frac{d}{\overline{mn+r+1} \times f} \times x^{r-n+1} + \frac{d}{\overline{mn+r+1} \times f} \times \frac{\overline{n-r-1} \times de}{\overline{mn+r-n+1} \times f} \times x^{r-2n+1} + \frac{d}{\overline{mn+r+1} \times f} \times \frac{\overline{n-r-1} \times de}{\overline{mn+r-n+1} \times f} \times \frac{\overline{2n-r-1} \times de}{\overline{mn+r-2n+1} \times f} \times x^{r-3+1},$$

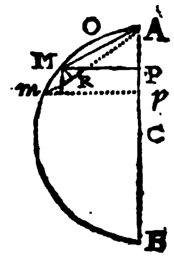
$$+ \text{\&c.} \times \overline{e+fx^n}|^{m+1} = \frac{a}{\overline{\frac{1}{2}+1} \times a} \times x^{0-1+1} \times \overline{ax-aa}|^{\frac{1}{2}+1} = \frac{2}{3} \times \overline{ax-aa}|^{\frac{3}{2}} = S: a\dot{x}\sqrt{ax-aa}. \text{ Q. E. I.}$$

I have

I have hitherto explain'd the general Methods of finding the Fluent of any Fluxion by help of Series's, and therefore shall not farther insist on these or other Methods invented for the same purpose, but refer the Reader (who desires to have a fuller Account of them) to a late Learned Treatise, writ by that Excellent Analyst G. Cheyne M. D. *and entituled* Fluxionum Methodus Inversa.

95. *Since the Business of infinite Series is sometimes tedious and too perplext, several other particular Methods have been invented to find the Flowing Quantity of a Fluxion. It shall suffice in this place to give the Reader an Idea of them, which will become more plain and familiar by several other Examples to be seen in their proper Places.*

EXAMPLE I.

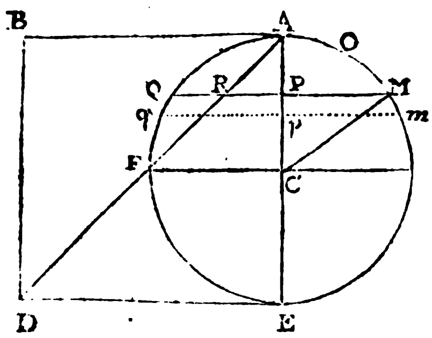


Let it be requir'd to find the Flowing Quantity of $\dot{x}\sqrt{2rx - xx}$. On the Center C, with the Radius CB $= r$, describe the Semicircle AMB, and suppose AP $= x$; then is PB $= 2r - x$, and MP $= \sqrt{2rx - xx}$, and P$p = \dot{x}$; therefore the Fluxion of the Area, *viz.* the Parallelogram Mp is $= \dot{x}\sqrt{2rx - xx}$, and consequently the Sum of all the $\dot{x}\sqrt{2rx - xx}$, that is, the Flowing Quantity of the given Fluxion is equal to the Semi-segment AMP.

EXAMPLE II.

Let it be requir'd to find the Flowing Quantity of $\frac{r x \dot{x}}{2\sqrt{2rx - xx}}$. Draw the Lines AM, A$m$ infinitely near each other, MP, mp Perpendicular to the Diameter AB, and MR Perpendicular to Am, then by the property of the Circle AM $= \sqrt{2rx}$, and Rm the Fluxion thereof is $\frac{r\dot{x}}{\sqrt{2rx}}$. Now because the Triangles APM, MRm are (the Angles AMP and MmR standing on equal Arches of the Circle) similar, it is, PM $(\sqrt{2rx - xx})$: AP (x) :: Rm $\left(\frac{r\dot{x}}{\sqrt{2rx}}\right)$: MR $= \frac{r x \dot{x}}{\sqrt{2rx} \times \sqrt{2rx - xx}}$; and consequently, the infinitely little Sector MAR $= \frac{1}{2}$ AR $\times$ MR is $= \frac{r x \dot{x}}{2\sqrt{2rx - xx}} =$ to the given Fluxion, whence it is evident that the Segment AOMA is the Flowing Quantity of the given Fluxion.

EXAMPLE III.



Let it be requir'd to find the Flowing Quantity of this Fluxion $x\dot{x} \times 2\sqrt{2rx - xx}$. On the Center C with the Radius $CA = r$, describe the Circle AFEM, and suppose $AP = x$, $Pp = \dot{x}$, $PE = 2r - x$, the circumference $AFEM = c$; then I say, that the Sum of all the $x\dot{x} \times 2\sqrt{2rx - xx}$ is $= \frac{crr}{2}$.

DEMONSTRATION.

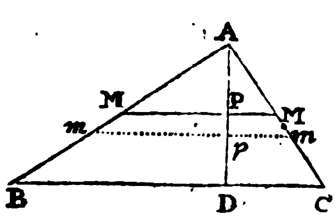
Let the Circle AFEM be the Base of an up-right Cylinder, and the Parallelogram ABDE the Section of the Cylinder through it's Axis, AB the height of the Cylinder is equal to AE the Diameter of the Base. Draw the Diagonal AD, then a Plain passing through AD, and perpendicular to the Plain BE, will divide the Cylinder in two equal parts, and cut off the Semi-quadrantal *Ungula* ADE. Now the Fluxion of this *Ungula* is equal to the Parallelogram Q*m* Multiplied into its height PR or AP (because the Angle RAP is equal to 45°.) $= x\dot{x} \times 2\sqrt{2rx - xx}$; and consequently the Sum of all the $x\dot{x} \times 2\sqrt{2rx - xx}$ is (when AP becomes = to AE or $x = 2r$) equal to the Semi-quadrantal *Ungula* $ADE = \frac{rrc}{2}$. Q. E. I.

And thus innumerable Instances might be assigned, to assist us in finding the Flowing Quantity of any Fluxion, without having immediate recourse to an infinite Series.

PROP. IV.

To find the Area of the Triangle ABC.

96. Draw AD perpendicular to the Base BC, and suppose $AD = a$, $BC = b$; draw any Line as MM (y) Parallel to the Base, and another Line *mm* infinitely near the same, and suppose $AP = x$, $Pp = \dot{x}$, then the Fluxion of the Area of the Triangle is equal to $MmmM = y\dot{x}$, and because the Triangles MAM BAC are similar, it is, $x : y :: a : b$; and consequently the Equation of the Triangle is $y = \frac{bx}{a}$ and the Fluxion of the Area $y\dot{x}$ is $= \frac{bx\dot{x}}{a}$, and the Flowing Quantity is $\frac{bxx}{2a} = \frac{1}{2}xy = MAM$, and when the Point P falls on D, then x and y becomes equal to a and b respectively, and $\frac{1}{2}xy$ is $= \frac{1}{2}ab$ equal to the Area of the Triangle ABC. Q. E. I.

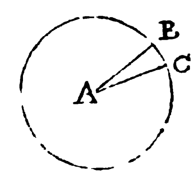


PROP.

PROP. V.

To find the Area of a Circle.

97. In the Circle ABC, draw the Rays AB, AC, infinitely near each other. Suppose $AB = r$, and $BC = \dot{x}$, then the Fluxion of the Circle is the infinitely little Sector or Rectilineal Triangle $ABC = \frac{r\dot{x}}{2}$, and the Flowing Quantity is $\frac{rx}{2} =$ (supposing the circumference $x = c$) $\frac{rc}{2}$, when the Arch BC becomes equal to the whole Periphery.



Whoso has a mind to consider this more nicely, may have recourse to the Rectification of Curves, and reduce the Periphery of the Circle to an infinite Series, &c.

CONSECTARY I.

1°. The Area of a Circle is equal to a Rectangular Triangle, whose Base is equal to the Circumference, and Altitude equal to the Semi-diameter of the Circle.

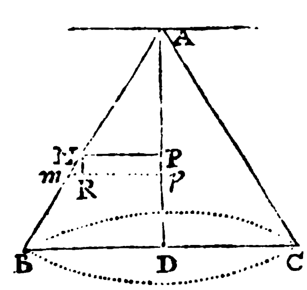
CONSECTARY II.

2°. The square of the Diameter of a Circle, is to the Area of the same Circle, as the Diameter is to $\frac{1}{4}$ the Periphery; for $4rr : \frac{rc}{2} : 2r : \frac{c}{4}$.

PROP. VI.

To find the Area of a Right Cone.

98. Let the Triangle BAC be the Section of the Cone through the Axis AD, and imagine the Axis AD to be divided into an infinite Number of equal Parts, and on every Point erect the Perpendiculars PM, *pm*. Then it is evident, that as the Rectangular Triangle ABD turns round about the Axis AD, all the Points M, *m*, describe the Circumferences, which compose the Surface of the Cone; and to find the Sum of all these Circumferences, suppose DC the Radius of the Base $= r$, and the Circumference of the Base $= c$, $AD = a$, $AP = x$, $Pp = \dot{x}$, $PM = y$; then $mR = \dot{y}$, and $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$. Then $r : c :: y : \frac{cy}{r} =$ to the Circumference describ'd by the Point M, and consequently $\frac{cy}{r}\sqrt{\dot{x}^2 + \dot{y}^2} =$ to the Fluxion of the Area of the Surface of the Cone. And because the Triangles APM, ADB are similar, it is, $x : y :: a : r$, whence $x = \frac{ay}{r}$, and $\dot{x} = \frac{a\dot{y}}{r}$, and $\dot{x}^2 = \frac{a^2\dot{y}^2}{rr}$. Therefore the Fluxion of the Area $\frac{cy}{r}\sqrt{\dot{x}^2 + \dot{y}^2}$ is $=$ (by substit.) $\frac{cy}{r}\sqrt{\frac{a^2\dot{y}^2 + r^2\dot{y}^2}{rr}} = \frac{cy\dot{y}}{r}\sqrt{\frac{a^2 + r^2}{rr}}$ =



$= \frac{c y \dot{y}}{r r} \sqrt{aa + rr}$, and the Flowing Quantity, or the Area of the Surface of the Cone is $\frac{c y y}{2 r r} \sqrt{aa + rr} =$ (because when P is in D, then $y = r$) $\frac{c}{2} \sqrt{aa + rr}$ $=$ (because $\sqrt{aa + rr}$ is $=$ A B the side of the Cone) $\frac{1}{2}$ the Periphery of the Base multiplied into the side of the Cone. Q. E. I.

CONSECTARY I.

99. 1°. The Radius of a Circle equal to the Area of a given Cone, is a mean Proportional between the side of the Cone and the Semi-diameter of the Base.

CONSECTARY II.

2°. The Surface of a Cone is to the Area of the Base, as the side of the Cone is to the Semi-diameter of the Base, for the Surface of the Cone is $\frac{ac}{2}$ (a being equal to the Side, and c equal to the Periphery of the Base) and the Area of the Base is $\frac{rc}{2}$ and $\frac{ac}{2} : \frac{rc}{2} :: a : r$.

CONSECTARY III.

3°. The Surfaces of any two right Cones are in a *Ratio Compounded* of the *Ratio* of their Sides, and the Ratio of the Semi-diameters of their Bases. Let a and b be the Sides of two Cones, and c and d the respective Peripheries of their Bases, then the Surface of one is to the Surface of the other, as $\frac{ac}{2}$ is to $\frac{bd}{2} :: ac : bd$; that is, as the Sides and the Peripheries of their Bases, or in a *Ratio Compounded* of the *Rationes* of their Sides, and the Semi-diameters of their Bases.

CONSECTARY IV.

4°. And if we suppose one Cone within another, and their Sides (or Surfaces) Parallel, the Sides of the Cones will be as the Semi-diameters of their Bases, and consequently their Surfaces will be in a Duplicate Ratio of the Semi-diameters of their Bases.

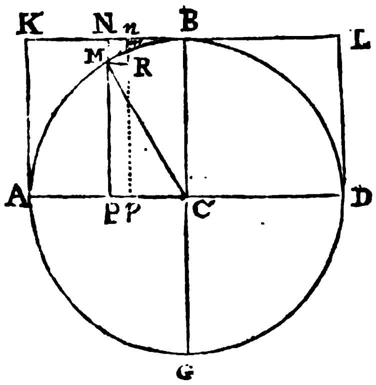
PROP. VII.

To Measure the Surface of a Sphere.

100. If the Semi-circle ABD be suppos'd to revolve about the Diameter AD, it will Generate a Sphere, and every Point in this Semi-circle will describe an intire Circle, and the Sum of all those Circles is equal to the Surface of the Sphere.

Draw

Draw any Ordinate MP $= y$, and another Ordinate mp infinitely near MP, and draw MR parallel to AD, then is $mR = \dot{y}$. Let the Radius BC or AC be equal to r, AP $= x$, and then P$p = \dot{x}$; then say, $r : c :: y : \frac{cy}{r} =$ to the Circumference describ'd by the Point M; and because the Triangles CPM, mRM are similar, it is, PM (y) : MC (r) :: MR $(\dot{x})$: M$m = \frac{r\dot{x}}{y}$; and Multiplying this Value by $\frac{cy}{r}$ (the Circumference describ'd by the Point M) the Product $c\dot{x}$ is the Fluxion of the Area of the Surface of the Sphere, and the Area it self is cx equal to the Portion of the Surface Generated by the Revolution of the Arch AM about the Diameter AD; and because x becomes equal to $2r$, when P comes to D, therefore cx is then equal to $2rc =$ to the whole Surface of the Sphere.



COROL. I.

101. The Surface of any Sphere, *viz.* $2rc$, is equal to four times the Area $\left(\frac{cr}{2}\right)$ of any of its great Circles.

COROL. II.

2°. The Area of any Segment of a Sphere, cut off by a Plain or by two Parallel Plains, is to the whole Surface of the Sphere, as the intercepted Portion of the Diameter is to the whole Diameter, and the Portion of the Surface of a Sphere Comprehended between any two great Circles, is to the whole Surface, as the Angle of Inclination between those great Circles, is to four Right-angles.

COROL. III.

3°. If a Circle be describ'd, whose Semi-diameter is equal to the Diameter of the Sphere, the Area of the said Circle is equal to the Surface of the Sphere, and if the Diameter of a Sphere be equal to the Semi-diameter of a Circle, their Superficial Contents will be equal.

COROL. IV.

4°. If a Sphere be inscrib'd in a Cylinder, whose Altitude is equal to the Diameter of the Sphere, the Area of the Cylinder $(2rc)$ is Quadruple the Area of the Base $\left(\frac{rc}{2}\right)$ and the Area of an *Hemisphere* is double the Area of the Base of the Circumscrib'd Cylinder, and the Surface of the Circumscrib'd Cylinder (without its Bases) is equal to the Surface of the Sphere.

COROL. V.

5°. If the Parallelogram AKLD be describ'd about the Semi-circle ABD, and if PM be produc'd to N, then the Surface describ'd by KN will be equal to that describ'd by the Arch AM, for the Line KL and the Semi-circle ABD describe (§. 4. *Art.* 101.) equal Surfaces, and the Surface describ'd by AM is to that describ'd by ABD or KL, as AP is to AD, and the Surface describ'd by KN is to that describ'd by KL or ABD, as KN or AP is to KL or AD, therefore the Surfaces describ'd by the Arch AM, and the Right-line KN are equal.

COROL. VI.

6°. And to find the Diameter of a Sphere, whose Surface shall be equal to the Surface of a given Cone. Let the Surface of the Cone be $\frac{ac}{2}$, (c being the Periphery of the Base and a the Side) and suppose the Semi-diameter of the Sphere requir'd be $= x$, then $r : c :: x : \frac{cx}{r} =$ to the Circumference of a great Circle of the Sphere, and consequently the Surface of the Sphere is $\frac{2cxx}{r} = \frac{ac}{2}$: Whence $x = \sqrt{\frac{ar}{4}}$ and the Semi-diameter of the Sphere is a mean Proportional between the Side of the Cone and a fourth part of the Semi-diameter of its Base.

COROLL. VII.

7°. If the Semi-diameter of a Sphere be equal to the Semi-diameter of the Base of a Cone, the Surface of the Cone is to the Surface of the Sphere, as the Side of the Cone is to four times the Semi-diameter of the Sphere. For $\frac{ac}{2} : 2rc :: a : 4r$.

COROL. VIII.

8°. If ABDG be the Base of a Cylinder, and if a Plain passing through the Diameter ACD Obliquely to the Circle of the Base, cut off a Cylindric *Ungula*, the Surface thereof, or the Portion of the Cylindrick Surface comprehended between the Circle of the Base ABD, and the said Plain, may be found thus. Let the Ratio of PM to its Corresponding Perpendicular be as m is to n, and suppose $AC = r$, $CP = x$, $Pp = \dot{x}$, $PM = y$, $Rm = \dot{y}$; then is $Mm = \frac{r\dot{x}}{y}$, and the Altitude of the infinitely little Parallelogram standing on Mm is $\frac{n}{m}y$, and consequently the said Parallelogram or the Fluxion of the *Ungular* Surface is $\frac{n}{m}y \times \frac{r\dot{x}}{y} = \frac{n}{m}r\dot{x}$, and the Flowing Quantity or the Portion of the *Ungular* Surface standing on the Arch MB is $= \frac{n}{m}rx$, and when x becomes $= r$, then the *Ungular* Surface standing on the Quadrant AMB is $= \frac{n}{m}rr = \frac{n}{m}$CK, and if the *Ungula* be a Semi-quadrantal one, then m is $= n$, and the *Cylindric* or *Ungular* Surface standing on AMB is equal to the Square ACBK, and the Surface of the whole *Ungula* is $= 2$KC $=$ AL.

COROL. IX.

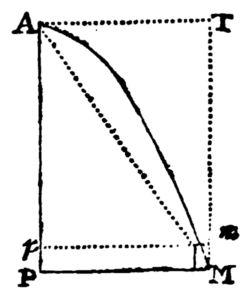
9°. The Surfaces of unequal Spheres are in a Duplicate Ratio of their Semi-diameters, for the Surfaces are proportional to the Area's of any of their great Circles, and the Area's of those Circles are (because the Area's of unequal Circles are in a Ratio Compounded of the Rationes of their Semi-diameters and Peripheries, and the Peripheries are as the Radij) as the Squares of their Semi-diameters.

PROP.

PROP. VIII.

To Investigate the Area's of all sorts of Parabola's.

102. Let AMP be a Semi-parabola, AP the Axis, and PM an Ordinate; Draw pm infinitely near PM, and suppose $AP = x$, $PM = y$, and the Parameter equal to 1, $Pp = \dot{x}$; then the Fluxion of the Area is $y\dot{x}$. Now the general Equation expressing the Nature of all such Curves is $y^m = x$; whence $y = x^{\frac{1}{m}}$, and consequently, the Fluxion of the Area $y\dot{x}$ is equal to $x^{\frac{1}{m}}\dot{x}$, and the Flowing Quantity or the Area requir'd is $\frac{m}{m+1}x^{\frac{1}{m}+1}$ = (because $y = x^{\frac{1}{m}}$) $\frac{m}{m+1}xy$.



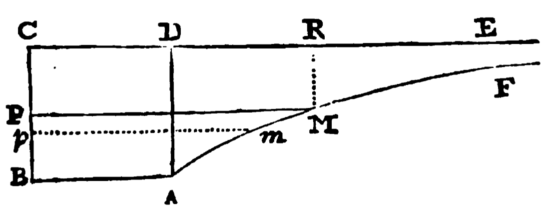
COROLLARY.

103. If m be $= 2$, then the Curve AmM is the Conic Parabola, and its Area is $= \frac{2}{3}$ AP × PM $= \frac{2}{3}$ the Circumscrib'd Parallelogram, and AmMT the Complement of the Parabola to the Parallelogram is $= \frac{1}{3}$ the Parallelogram $= \frac{2}{6}$ the Parallelogram, and the Triangle AMT is $= \frac{1}{2}$ the Parallelogram; therefore the Space Comprehended between the Curve AmM, and the Chord AM is $= \frac{1}{6}$ the Circumscrib'd Parallelogram PT, and the Space AmMT is $= \frac{1}{2}$ the Area AMP, and the Area of the Parabola is to the Circumscrib'd Parallelogram as 2 is to 3, and to the Inscrib'd Triangle as 4 is to 3.

PROP. IX.

To Investigate the Area's of all sorts of Hyperbolic Spaces.

106. Let AMF be a Semi-hyperbola between its Asymptotes CB, CE, and having drawn the Ordinates PM, pm infinitely near each other; suppose $CP = x$, $PM = y$, and $Pp = \dot{x}$; then the Fluxion of the Area is $y\dot{x}$. Now the general Equation expressing the Relation between all sorts of Hyperboliform Curves and their Asymptotes is (supposing the Parameter $= 1$, and the Exponent m Negative) $y^m = x$; therefore the Fluxion of the Area $y\dot{x}$ is $= x^{\frac{1}{m}}\dot{x}$, and the Flowing Quantity or the Area it self is $\frac{m}{m+1}xy$.



CONSECTARY I.

105. Because the Exponent m is Negative, therefore the Area is $= \frac{-m}{-m+1}xy$ or $\frac{m}{m-1}xy$. Whence it appears that if m be greater than 1, the Space ECPMF is Finite, if m be $= 1$, then it is Infinite, and if m be less than 1, 'tis more than Infinite.

CONSECT.

CONSECT. II.

2°. If the Curve AMF be the Apollonian (Equilateral when CA is a Square) Hyperbola, the Equation expressing the Nature thereof is $y^{-1} = x$ or $1 = xy$, or $aa = xy$, therefore the Area of the Hyperbolic Space CPMFE $= \frac{m}{m-1} xy$ is $= \frac{1}{0} xy$, that is, it is infinite in respect of the Parallelogram CPMR, and the Space CBAMFE is infinite in respect of the Parallelogram CBAD.

If m be $= -2$, then the Equation of the Curve $y^m = x$ is $y^{-2} = x$, or $1 = xy^2$ or $a^3 = xy^2$, and the Area of the Hyperbolic Space CPMFE $= \frac{m}{m-1} xy$ is $= 2xy = 2$ the Parallelogram CM.

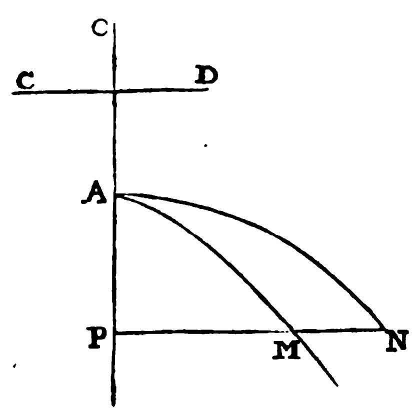
If m be $= -\frac{1}{3}$, then $y^m = x$ is $y^{-\frac{1}{3}} = x$, or $1 = xy^{\frac{1}{3}}$, or $a^4 = x^3 y$. And $\frac{m}{m-1} xy$ is $= -\frac{1}{2} xy$, which shews that the Area of the Hyperbolic Space CPMFE is more than Infinite in respect of CPMR.

CONSECT. III.

3°. $m - 1 : m ::$ Parallelogram CPMR : Space CPMFE, and by Division $m - 1 : 1 ::$ Parallelogram CPMR : Space RMFE.

CONSECT. IV.

4°. I shall here insert the Proportion between the Area of any given Hyperbola, and the Area of an Equilateral Hyperbola, describ'd to the same Principal Axis. Let AM be any Hyperbola, OA the Transverse Axis, and CD the Conjugate Axis; and let AN be an Equilateral Hyperbola describ'd to the same Principal Axis AO; then the Conjugate Axis is also equal to AO. Now by the property of the Hyperbola, the Square of any Ordinate PM is to the Rectangle OP × AP :: the Square of CD is to the Square of AO; that is, (supposing OA $= a$, CD $= b$, AP $= x$, PM $= y$, PN $= z$) $aa : bb :: ax + xx : yy$, and $aa : aa :: ax + xx : zz$. Therefore $yy : zz :: bb : aa$, and consequently, PM (y) : PN (z) :: $b : a$. That is, the Ordinates PM, PN are always in the same Proportion as b is to a; therefore the Space APM is to the Space APN as b is to a, or as PM is to PN. That is, the Spaces APM, APN are Proportional to their Altitudes PM, PN, or to the Conjugate Axis (b) and the Transverse Axis (a.)



CONSECT. V.

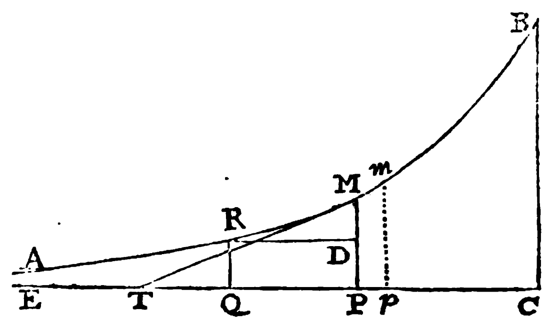
5°. In like manner, The Areas of unequal Ellipsis are in a Ratio Compounded of the Subduplicate Ratio of their Parameters, and the Sesquiplicate Ratio of their principal Axes. For Ellipses are proportional to Parallelograms Circumscrib'd about them, and the Conjugate Diameters are mean Proportionals between the Transverse Axes and the Parameters. Therefore if the Parameters be equal, the Circumscrib'd Parallelograms will be in a Ratio Compounded of the simple Ratio of the Transverse Axes and the Subduplicate Ratio of the Transverse Axes, and if the Transverse Axes be equal, the Circumscrib'd Parallelograms will be in a Sub-duplicate Ratio of the par-

Parameters; and consequently, if neither the Transverse Axes nor the Parameters be equal, the Circumscrib'd Parallelograms or the Area's of the Ellipses will be in a Ratio Compounded of the Sub-duplicate Ratio of the Parameters, and the Ses-quiplicate Ratio of the Transverse Axis.

PROP. X.

Let it be requir'd to Investigate the Area of the Logarithmetick Space AMBCE.

106. Let AMB be the Logarithmetick Line, and CE its Asymptote, and draw the Ordinates PM, pm infinitely near each other, and draw the Line MT touching the Curve in M, and Intersecting the Asymptote in T: 'Tis requir'd to find the Area of the Space AMPE Comprehended by the Curve, the Ordinate, and the Asymptote.



Suppose PM $= y$, and P$p = \dot{x}$, then the Fluxion of the Area, or the Parallelogram Mp is $= y\dot{x}$. But by the Property of the Curve, the Sub-tangent PT is equal to an invariable Quantity, *v. g.* $= a$. Therefore it is $\dot{y} : \dot{x} :: y : a$, and $\dot{x} = \frac{a\dot{y}}{y}$, and consequently the Fluxion of the Area $y\dot{x}$ is $= a\dot{y}$, and the Area it self is equal to ay.

CONSECTARY I.

107. The Space AMPE (tho' infinitely produc'd) is to the Triangle (Comprehended by the Ordinate, Tangent and Sub-tangent) MPT as 2 is to 1.

CONSECTARY II.

2°. The Space Comprehended between any two Ordinates, *v. g.* the Space RMPQ is equal to the Rectangle Comprehended under the Sub-tangent and the difference between the said Ordinates, *viz.* $=$ PT $\times$ MD.

CONSECTARY III.

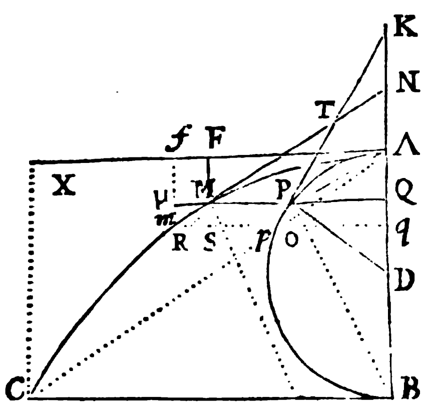
3°. The Spaces Comprehended between any two Ordinates are Proportional to the difference between them respectively.

PROP. XI.

To Investigate the Area's of Cycloidal Spaces.

108. *Among the infinite variety of Curves, none has more exercis'd the Thoughts of Geometers, than the Cycloid.* Cartesius, Torricellius *and* Robervallius *first gave the Dimensions of* Mersennus's *Cycloid; since whom Dr.* Wallis *and Mr.* Paschal, *and* Lolovera *have Written largely on this Subject, and but lately, and in our own Days, Mr.* Newton, *Mr.* Hugens *and Mr.* Romer *have Discover'd surprising Properties of a New sort of a Cycloid, having a Circular Base.*

Let AMC be a Vulgar Semi-cycloid, and the Generating Circle APB from any Point in the Ordinate, *v.g.* Q, draw QM parallel to the Base BC, cutting the Periphery of the Circle in P, make the Parallelogram AFMQ, and draw fm infinitely near FM, cutting QM produc'd in μ, and the Curve in m. Put $AB = 2r$, $AQ = FM = x$, $\mu m = \dot{x}$, $QP = y$; then (by the property of the Circle) $2rx - xx = yy$. Whence $r\dot{x} - x\dot{x} = y\dot{y}$, and $\dot{y} = \frac{r\dot{x} - x\dot{x}}{y}$; and because the Triangles DPQ, pPO are similar; therefore $PQ\,(y) : DP\,(r) :: PO\,(\dot{x})\ Pp = \frac{r\dot{x}}{y}$. Now it is the Nature of the Vulgar Cycloid that the Arch AP + the right Sine of that Arch PQ are equal to QM. Therefore it is manifest that the Fluxion of the Ordinate of the Cycloid QM, *viz.* MS is equal to the Aggregate of the Fluxions of the Arch AP, and the



right Sine PQ; that is, $mS = Pp + po = \frac{r\dot{x} - x\dot{x}}{y} + \frac{r\dot{x}}{y} = \frac{2r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$, and consequently, the Rectangle $F\mu$ is equal to $FM \times M\mu = x \times \frac{2r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}} = \frac{2rx\dot{x} - xx\dot{x}}{\sqrt{2rx - xx}} = \dot{x}\sqrt{2rx - xx}$ = to the Fluxion of the Area AMF. But the Fluxion of the Portion of the Circle $APQ = \dot{x}\sqrt{2rx - xx}$; therefore the Area AMF and the Corresponding Portion of the Circle APQ are always equal.

CONSECTARY I.

109. The Parallelogram AC is equal to the Semi-periphery $APB \times AB$ = four times the Semi-circle APBA, and the Complement of the Cycloidal Space AMCB to the Parallelogram, *viz.* AMCX is equal to the Semi-circle APBA; therefore the Area of the Semi-cycloidal Space AMCB is = to three times the Area of the Semi-circle APBA.

CONSECTARY II.

2°. The Cycloidal Space AMCB is to the Circumscrib'd Parallelogram AC as 3 is to 4.

CON-

CONSECTARY III.

3°. The Space Comprehended between the Chord A C and the Curve A M C is equal to the Area of the Semi-circle A P B. For A M C B is equal to $\frac{1}{4}$ Parallelogram A C, and the Triangle A C B is equal to $\frac{2}{4}$ Parallelogram A C; therefore the Space A M C A is equal to $\frac{1}{4}$ Parallelogram A C which is equal to the Area of the Semi-circle A P B, and the Space A M C A is equal to the Space A M C X = $\frac{1}{2}$ the inscrib'd Triangle A C B.

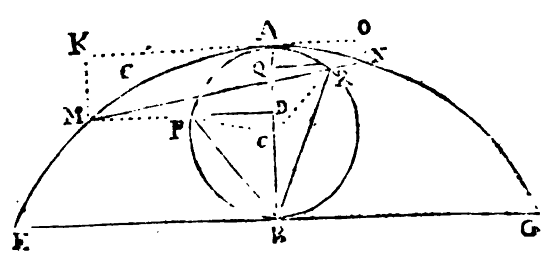
CONSECTARY IV.

4°. Though the Quadrature of the whole Cycloidal Space, or any indefinite Portion thereof depends on the Quadrature of the Circle, yet an infinite Number of Segments of the Vulgar Cycloid may be Squar'd without supposing the same.

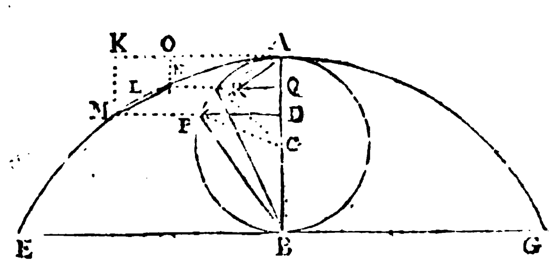
Let E A G be a Vulgar Cycloid, the Base E G, and A B the Axis, and the generating Circle A P B. I say, if the Point Q be taken at pleasure in the Axis A B, and if C D be taken equal to A Q and the Ordinates D M, Q N, and the Line M N connecting their Extremities be drawn, the Segment of the Cycloid M E N M = Rectangle Triangle P B D ± Rectangle Triangle R B Q.

Draw O A K parallel to the Base, and N O, M K parallel to the Axis A B, and draw the Radij C P, C R.

First, If the Ordinates D M, Q N be on the contrary Sides of the Axis A B. Then the Segment M e N M is equal to the Trapezium M K O N — Trilineal Figures A K M and A O N. Now the Trapez. M K O N is = $\frac{1}{2}$ $\overline{MK + \frac{1}{2} NO}$ x K O = (because N O is = A Q = C D, and K M = A D) $\frac{1}{2}$ C A x K O = $\frac{1}{2}$ C A x A K + $\frac{1}{2}$ C A x A O. And by the property of the Cycloide, $\frac{1}{2}$ A C x A K is = $\frac{1}{2}$ C A x Arch $\overline{AP + PD}$ = Sector A C P + Triangle B C P = Sector A B P. In like manner it may be Demonstrated that $\frac{1}{2}$ C A x A O is = Sector A B R. Therefore the Trapezium M K O N is equal to two Sectors P B A + R B A. But by (*Art.* 108.) the property of the Cycloid, the Trilineal Figure A K M is equal to the Segment of the Circle A D P, and the Trilineal Figure A O N is equal to the Segment A Q R. Therefore if from the Trapezium M K O N the Trilineal Figures A K M, A O N be Subtracted, and if from the Sectors P B A, R B A, the Segments A D P, A Q R be Subtracted there will remain the Segment of the Cycloid M e N M equal Triangles P B D + R B Q.

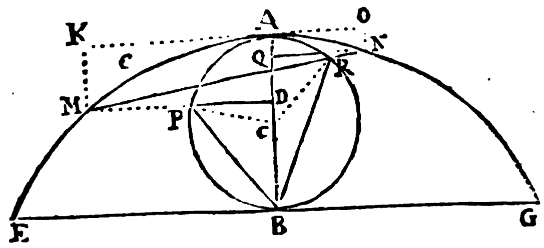


Secondly, But if the Ordinates Q N, D M be on the same Side of the Axis A B, then the Segment of the Cycloide M e N M = Trapezium M K O N + Trilineal Figure A O N — Trilineal Figure A K M. Now the Trapezium M K O N = $\frac{1}{2}$ $\overline{MK + \frac{1}{2} ON}$ x O K = $\frac{1}{2}$ C A x A K — $\frac{1}{2}$ C A x A O = Sector P B A — Sector R B A. Therefore if we Substitute the Circular Segments A D P, A Q R in the place of the Trilineal Spaces A K M. A O N, we shall have the Cycloidal Segment M e N M = Sector P B A — Sector R B A + Segment A Q R — Segment A D P = Rectangle Triangle P B D — Rectangle Triangle R B Q.



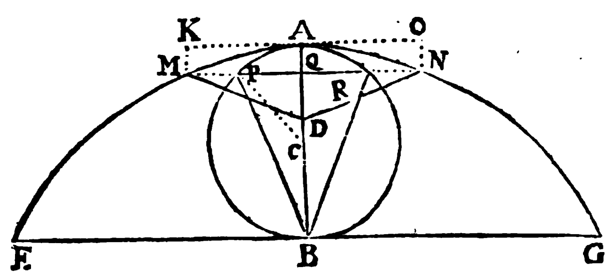
5°. If

5°. If the Points D and Q Coincide, then it is manifest that BD or BQ = $\frac{1}{2}$ BC, and the Chord MN is Perpendicular to the Axis AB, and the Segment M*e*ANM will be equal to an Equilateral Triangle inscrib'd in the Generating Circle, and Space MPBAM will be equal to three times the Area of the Triangle CPB. Which was first Discover'd by the Excellent Mr. *Hugens.*

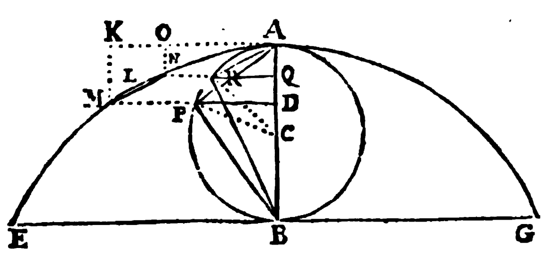


6°. But if the point D fall in the Center, then Q will be in A, and the Segment M*e*NM will Degenerate into that which the Celebrated Mr. *Leibnitz* first Squar'd without having recourse to the Area of the Circle. And the said Segment M*e*NM will be equal to the Rectangular Triangle PBD = $\frac{1}{2}$ the Square of the Radius.

7°. And to Square an infinite Number of Sectors of the Cycloid. Assume any point Q in the Axis AB, and draw the Ordinate MQN, and take CD = AQ, and draw the Lines DM, DN; then I say, the Sector of the Cycloid DMAND is equal to the Isosceles Triangle PBR. For the Sector of the Cycloid DMAD is equal to the Trapezium DMKAD — Trilineal Figure AKM. But the Trapezium DMKA is = $\overline{\tfrac{1}{2}DQ + AQ} \times AK = \tfrac{1}{2} AC \times AK = \tfrac{1}{2} AC \times \overline{\text{Arch } AP + PQ}$ = to the Sector PBA, and the Trilineal Figure AKM is equal to the Segment APQ, therefore the Sector of the Cycloid DMAD is equal to the Triangle PBQ; and consequently, the Sector of the Cycloid DMAND is equal to the Isosceles Triangle BPR.



8°. And as we have thus squar'd an infinite Number of Cycloidal Segments and Sectors, so an infinite Number of Cycloidal Zones (*viz.* Spaces Comprehended between the Portion of the Curve MN, the Portion of the Axis QD, and the Ordinates QN, DM) may be squar'd from the same Principles. For if we consider that the external Space AKM is equal to the Segment of the Circle APD, and that AP = MP, we may find the Value of any Cycloidal Space in Rectilineal Figures and Circular Segments, and therefore if it be requir'd that the said Cycloidal Space should be squarable, it is plain that those Terms which consist of Circular Segments must destroy one another, and consequently be put = o, from which Supposition the Quantities which were assum'd at first may easily be determin'd. *Ex. Gr.* Let it be requir'd to determine the Right-lines CQ, CD in the Axis AC, so that the Cycloidal Zone DMNQ be squarable. Suppose AC = a, CQ = x, CD = z, QR = p, DP = q, AR or NR = u, AP or MP = c; then the Sector ACR = $\frac{1}{2}au$, and the Sector ACP = $\frac{1}{2}ac$, and consequently the Segment AQR or the Figure AON is = ACR — CQR = $\frac{1}{2}au - \frac{1}{2}px$; And the Segment ADP or the Figure AKM is = ACP — CPD = $\frac{1}{2}ac - \frac{1}{2}qz$. But the Cycloidal Segment AQN is = QO — AON = QO — AQR = AQ × $\overline{QR + RN}$ — AQR = AQ × $\overline{QR + AR}$ — AQR = $\overline{a - x} \times \overline{p + u} - \frac{1}{2}au + \frac{1}{2}px = ap - \frac{1}{2}px + \frac{1}{2}au - xu$.



And in like manner the other Segment ADM is = $aq - \frac{1}{2}qz + \frac{1}{2}ac - zc$.

And consequently, the Zone DMNQ = ADM — AQN is = $aq - \frac{1}{2}qz - ap + \frac{1}{2}px + \frac{1}{2}ac - zc - \frac{1}{2}au + xu$.

Where

Where it appears that the four first Members consist of Rectilineal Figures only, and that the other Terms Affected with u and c hinder the Zone from being Squarable. Whence it is evident that if we suppose the Terms Affected with u and c mutually to destroy one another, then the Cycloidal Zone DMNQ will be $= aq - \frac{1}{2}qz - ap + \frac{1}{2}px$. And the remaining Terms must be $= 0$, that is, $\frac{1}{2}ac - zc - \frac{1}{2}au + xu = 0$, and if we suppose the Ratio of c to u be given (that is, as one Number is to another, that so one Arch being given, the other may be constructed Geometrically) we may destroy the Quantities c and u, and find the Relation between z and x. *v. gr.* If it be $u : c :: 1 : 2$, then the Equation $\frac{1}{2}ac - zc - \frac{1}{2}au + xu = 0$, becomes $a - 2z - \frac{1}{2}a + x = 0$; and consequently, $z = \frac{a + 2x}{4}$ and if $u : c :: 1 : 3$, then $z = \frac{2a + 2x}{6}$, or if it be $u : c :: 1 : 4$, then $z = \frac{3a + 2x}{8}$, *&c.* In the same Progression.

And to find the Value of z in other Terms; if we suppose CQ the Sine Complement of the Arch AR to be given, then CD the Sine Complement of Double, Triple, Quadruple, *&c.* that Arch may be found by common Algebra. Therefore if c be $= 2u$, then $z = \frac{2xx - aa}{a}$; if $c = 3u$, then $z = \frac{4x^3 - 3aax}{aa}$; or if $c = 4u$, then $z = \frac{8x^4 - 8aaxx + a^4}{a^3}$, *&c.* and comparing these Values of z with those formerly found, we may find the Value of x in any supposition. *v. g.* if c be $= 2u$, then $z = \frac{a + 2x}{4} = \frac{2xx - aa}{a}$; and consequently, $8xx - 2ax = 5aa$. Whence x is $= \frac{1}{8}a + \sqrt{\frac{5}{8}aa + \frac{1}{64}aa} = \frac{1}{8}a + \sqrt{\frac{41}{64}aa} = \frac{1}{8}a + \frac{1}{8}a\sqrt{41}$.

Hence it is manifest that if CQ be taken $= \frac{1}{8}a + \frac{1}{8}a\sqrt{41}$. And if the Ordinate QN be applied to the Axis in the point Q, and if the Arch RP be taken $=$ AR, and the Ordinate MPD be drawn, then the Cycloidal Zone DMNQ will be $= aq - \frac{1}{2}qz - ap + \frac{1}{2}px =$, the Rectilineal Triangles CAP + DAP − CAR − AQR.

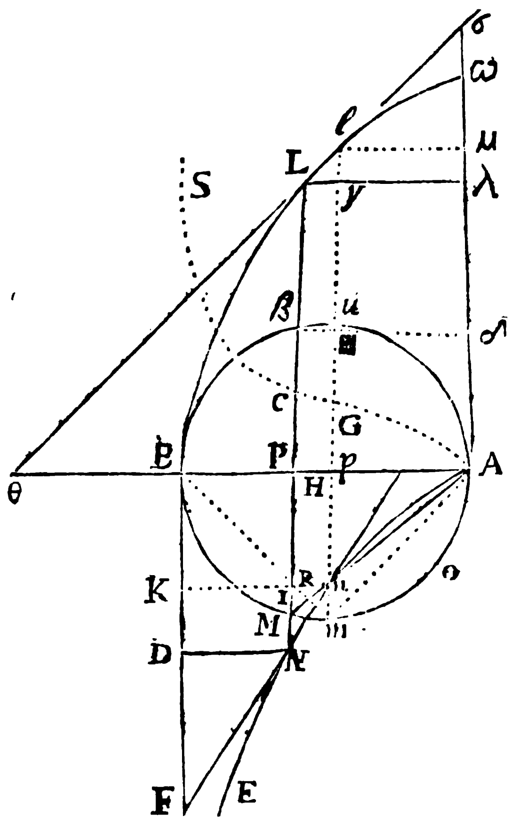
And thus an infinite Number of Cycloidal Zones may be Determin'd, which admit of a Quadrature, when the Proportion between the Arches AR, RP *is Express'd in given Numbers.*

PROP. XII.

To Investigate the Area's of Cissoidal Spaces.

110. Let ANE be a Cissoid, BF its Asymptote, and AMB the Generating Circle.

The Property of the Cissoid is such, that BP : PM :: AP : PN; and consequently (because the Triangle ABM is in the Semi-circle) PM : AP :: AP : PN. Draw another Ordinate mp infinitely near PM, and suppose AP $= x$, AB $= 2r$, PB $= 2r - x$, P$p = \dot{x}$, PM $= y$. Then PM (y) : AP (x) :: AP (x) : PN $= \frac{xx}{y}$. Now the Quadrilateral Figure PNnp or the Fluxion of the Cissoidal Space, is $=$ PN $\times$ Pp $= \frac{xx\dot{x}}{y} = \frac{xx\dot{x}}{\sqrt{2rx - xx}} =$ (because $xx = 2rx - \overline{2rx - xx}$) $\frac{2rx\dot{x}}{\sqrt{2rx - xx}} - \frac{2rx - xx}{\sqrt{2rx - xx}} \times \dot{x} = \frac{2rx\dot{x}}{\sqrt{2rx - xx}} - \dot{x}\sqrt{2rx - xx}$. Whence 'tis evident that the Flowing Quantity of the last Member is equal to the Semi-segment AOMP, and the Flowing Quantity of the first Member is Quadruple the Segment AOMA; which will appear thus,



From the point A in the Semi circle draw the Chord AM, and another Am infinitely near the same, and on the Center A describe the little Arch mR. Then (by the property of the Circle) AM $= \sqrt{2rx}$, and the Fluxion thereof RM is $= \frac{r\dot{x}}{\sqrt{2rx}}$. Now because the Triangles MPA, MRm (for the Angles MPA, MRm are right Angles, and the Angles mMA, PMA stand on equal Arches of the Circle) are Similar, it is, PM $(\sqrt{2rx - xx})$: AP (x) :: RM $\left(\frac{r\dot{x}}{\sqrt{2rx}}\right)$: Rm $= \frac{rx\dot{x}}{\sqrt{2rx}\sqrt{2rx - xx}}$, and the infinitely little Triangle mAR $= m$R $\times \frac{1}{2}$AM $= \frac{\sqrt{2rx}}{2} \times \frac{rx\dot{x}}{\sqrt{2rx}\sqrt{2rx - xx}}$ $= \frac{rx\dot{x}}{2\sqrt{2rx - xx}} =$ to the Fluxion of the Segment AOMA, and consequently, when P falls on B, and the Segment AOMA becomes $=$ AMPA or AMBA, then the Flowing Quantity of the Fluxion of the Area, that is the Area of the Cissoidal Space is equal to 4 AMBA $-$ AMBA $=$ to three times the Area of the Generating Semi-circle.

Another way.

111. Reassuming the Symbols (*Art.* 65.) it is evident that the Sub-tangent D F $= t$ is $= \frac{3}{2} u + \frac{1}{2} y$, or $3u + y = 2t$ and $3u = 2t - y$, and multiplying all the Terms by $\dot{x}$, we have $3u\dot{x} = 2t\dot{x} - y\dot{x}$, and because it is FD (t) : DN $(2a - x)$: : IN $(\dot{y})$: In $(\dot{x})$ therefore $t\dot{x} = \dot{y} \times \overline{2a - x}$; if we put PC $=$ DF $= t$, and describe the Curve A G C S, then the Trapezium CPpG will be $=$ DNnK, and all the Rectangles $t\dot{x}$ will be $=$ to the Space Comprehended between the Curve AGCS, and the right Lines AB and KB produc'd $=$ to the Cissoidal Space, or all the $\dot{y} \times \overline{2a - x}$. Now if all $u\dot{x}$ or the Semi-circle be put $= c$, and all the $y\dot{x}$ or the Cissoidal Space be put $= f$, then because it was $3u\dot{x} = 2t\dot{x} - y\dot{x}$, it will also be $3c = 2f - f = f$, and consequently, the Cissoid is triple the Semi-circle.

Another way.

112. Retaining the Symbols (*Art.* 110.) Suppose PN $= z$, then $\frac{xx}{y} = z$, and $\frac{x^4}{yy} = zz$. And substituting $2rx - xx$ for yy, and dividing by x; $x^3 = 2rzz - xzz$, and consequently, $3xx\dot{x} = 4rz\dot{z} - zz\dot{x} - 2xz\dot{z}$; and dividing by z, and substituting y for $\frac{xx}{z}$, we have $3y\dot{x} = 4r\dot{z} - 2x\dot{z} - z\dot{x} =$ (putting $b = 2r - x$) $2b\dot{z} - z\dot{x}$.

Now if we suppose the whole Cissoidal Space $= f$, and the Area of the Semi-circle $= c$, because all the $b\dot{z}$ are equal to the $z\dot{x}$ (both denoting the infinite Cissoidal Space) then will $3c$ be $= 2f - f = f$; that is, the Cissoid is triple the Semi-circle.

And to Investigate the Area of any Portion of the Cissoid.

If all the $b\dot{z}$ or all the $z\dot{x}$, be referr'd not to the infinite Cissoidal Space, but to any determinate part thereof, then let APN $=$ all the $z\dot{x}$ be $= f$, and ANDB $=$ all the $b\dot{z} =$ (supposing the Rectangle PNDB $= p$) $f + p$; and let all the $y\dot{x} =$ AOMP be $= d$, then because all the $3y\dot{x} =$ all the $\overline{2b\dot{z} - x\dot{z}}$, it is also, $3d = 2p + 2f - f$, or $3d - f = 2p$, and $3d - 2p = f$; that is, if from thrice the Area (AOMP) of the Portion of the Circle twice the Parallelogram PNDB be Subtracted, the remainder will be equal to the Portion of the Cissoidal Space APN.

And because, when P comes to B, the Rectangle PNDB becomes $= 0$, then $3d - 2p$ is $= 3d - 0 = f$, that is the Cissoid is to the Semi-circle as 3 is to 1.

113. If the Radius AH be $= r$, AP $= x$, PM $= y$, AN a Portion of a Cissoid, whose Ordinate PN $= \frac{xx}{y} = z$, and AGC a portion of a Curve, whose Ordinate PC is $= \frac{rx}{y} = b$, the proportion of the Curvilineal Space APC to the responding Segment of the Circle AOMP is requir'd; that is, the proportion of all the $b\dot{x}$ to all the $y\dot{x}$ is requir'd. The Equation expressing the Nature of the Circle is $2rx - xx = yy$, and consequently $r\dot{x} - x\dot{x} = y\dot{y}$, and multiplying by x, and dividing by y, we have $\frac{rx\dot{x}}{y} - \frac{xx\dot{x}}{y} = x\dot{y}$; that is, $b\dot{x} - z\dot{x} = x\dot{y}$, and (putting all the $b\dot{x} =$ APC $= d$, all the $z\dot{x} =$ APN $= f$, and all the $x\dot{y} =$ (suppo-

ſing P to fall between A and H) $= A\beta\delta$ the Complement of the Segment of the Circle $= b$) $f = d - b = 3c - 2p$ (putting $c =$ to the Segment A P β, and putting the Rectangle $xy = p$) whence $d = 3c + b - 2p$, and becauſe $b + c$ is $= p$) Therefore $d = 2c = p$.

And if x be $= 2r$, then p is $= 0$, and conſequently d is $= 2c$; that is, the Area of the Curviliniacl Figure d, (whoſe Ordinates is $= \frac{rx}{y}$) is double the Area of the whole Semi-circle or $= 2c$.

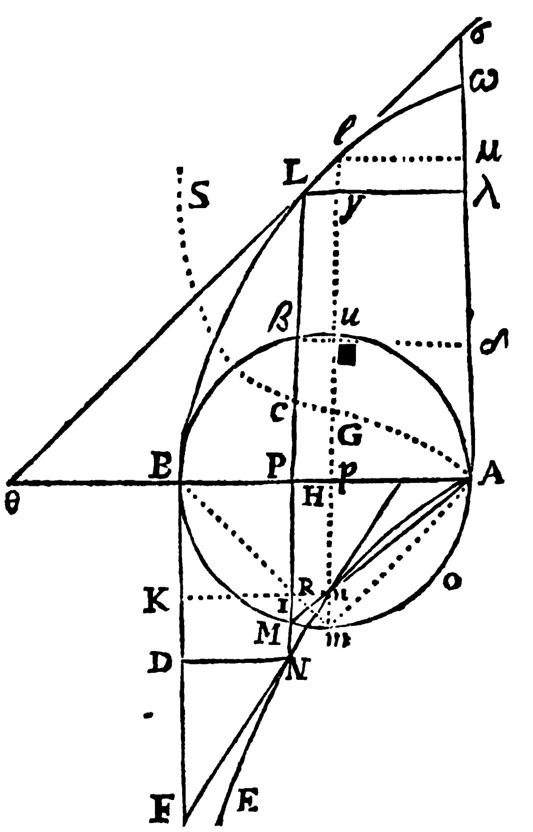
114. The ſame thing being ſuppos'd, the proportion between the Curvilineal Space APC whoſe Ordinate PC is $= \frac{rr}{y}$, and the Circle is requir'd.

The Fluxionary equation of the Circle is $r\dot{x} - x\dot{x} = y\dot{y}$, which multiplied by r, and divided by y, we ſhall have $\frac{rr\dot{x}}{y} - \frac{rx\dot{x}}{y} = r\dot{y}$, ſuppoſe all the Rectangles $\frac{rr\dot{x}}{y} =$ the Area A P C $= q$, and all the Rectangles $\frac{rr\dot{x}}{y} = 2c - p$, and all the Rectangles $r\dot{y} = n$, then will $q - n$ be $= 2c - p$, and conſequently $q = 2c + n - p$.

And extending the Equation to the whole Curvilineal Spaces, then $x = 2r$, and p and n will be $= 0$, whence $q = 2c$.

CONSECTARY I.

115. If B L ω be a Semi-cycloid Generated by the Semi-circle A β B, and if the Line θ L σ touch the Curve in L, and if H be the Center of the Circle. Suppoſe P $\theta = t$, $\lambda\sigma = s$, P L $= u$, P β $= y$, P A $= x$, the Arch B $\beta = z$, then (*Art.* 70.) $t =$ P θ is $= \frac{uy}{x}$; and becauſe the Triangles θ P L, L $\lambda\sigma$ are Similar, therefore P θ $\left(\frac{uy}{x}\right)$: P L (u) :: L λ (x) : $\lambda\sigma = \frac{xx}{y} = s$, and becauſe $\sigma\lambda$ (s) : λ L (x) :: ly ($\dot{u}$) L y ($\dot{x}$) therefore $s\dot{x}$ is always $= x\dot{u}$, and placing $\lambda\sigma$ from P to N, and deſcribing the Curve A N E, all the Rectangles $x\dot{u}$ (or the Rectangles L $\lambda\mu l$) that is, the whole Cycloidal Space is equal to all the Rectangles $s\dot{x}$ or $\frac{xx\dot{x}}{y}$ (or the Rectangles P n) that is to the whole Space F B A N E, and becauſe P N is always $=$ $\lambda\sigma = \frac{xx}{y}$, it is manifeſt that the Curve A N E is the Ciſſoid, and conſequently, the Ciſſoidal Space F B A N E is equal to the Semi-cycloidal Space B L ω A $= 3$ times the Semi-circle A M B.



2°. If

CONSECTARY II.

2°. If a simple Semi-cycloid and a Cissoid be describ'd to the same Circle, their Area's will be equal.

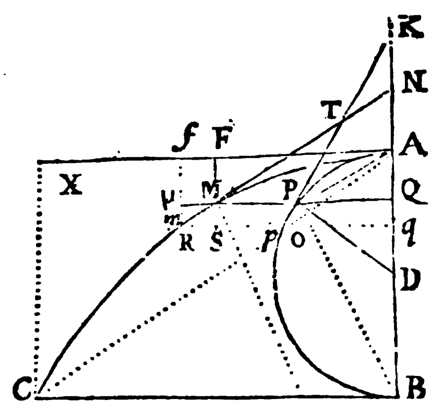
CONSECTARY III.

3°. If from any point L in the Semi-cylcloid there be drawn Lλ, LN parallel to AB, Aσ respectively, they will cut off the equal Spaces L$\lambda\sigma$, APN.

CONSECTARY IV.

4°. To find the Area's of Contracted or Protracted Cycloidal Spaces. Reassuming the Symbols (*Art.* 70.) Let the Ratio of the Arch AP to the Ordinate PM be as b is to a, and suppose AP $= z$, QP $= y$, then

PM $= \frac{a}{b}z$, and QM $= y + \frac{a}{b}z$. Then the Fluxion of the Ordinate QM, *viz.* Mμ = the Fluxion of QP + the Fluxion of PM is $\frac{r\dot{x} - x\dot{x}}{y} + \frac{a}{b}\frac{r\dot{x}}{y} = \frac{\frac{a+b}{b}r\dot{x} - x\dot{x}}{y} =$ Mμ, and the Rectangle Fμ or the Fluxion of the Area of the Complement of the Cycloidal Space to a Parallelogram is $= \frac{\frac{a+b}{b}rx\dot{x} - xx\dot{x}}{y}$.



But the Flowing Quantity of $\frac{rx\dot{x}}{y} =$ (*Art.* 110.) 2 the Area of the Semi-circle APB $= 2c$; and consequently the Flowing Quantity of $\frac{a+b}{b} \times \frac{rx\dot{x}}{y}$ is $= \frac{2a+2b}{b} \times c$. In like manner the Flowing Quantity of $\frac{xx\dot{x}}{y}$ is $= 3$ the Area of the Semi-circle APB $= 3c$; therefore the Flowing Quantity of $\frac{\frac{a+b}{b}rx\dot{x} - xx\dot{x}}{y}$ is $= \frac{2a+2b}{b} \times c - 3c = \frac{2a-b}{b} \times c =$ the Area AMCX.

CONSECTARY V.

5°. The Semi-circle APB is to the Complement of the Semi-cycloid AMCX as b is to $2a - b$.

CONSECTARY VI.

6°. And because the Parallelogram ABCX is $= \frac{4a}{b} \times c$. Therefore the Cycloidal Space AMCB is $= \frac{4a}{b} - \overline{\frac{2a-b}{b}} \times c = \frac{2a+b}{b} \times c$. And the Semi-circle is to the Semi-cycloidal Space AMCB as b is to $2a + b$.

 I might

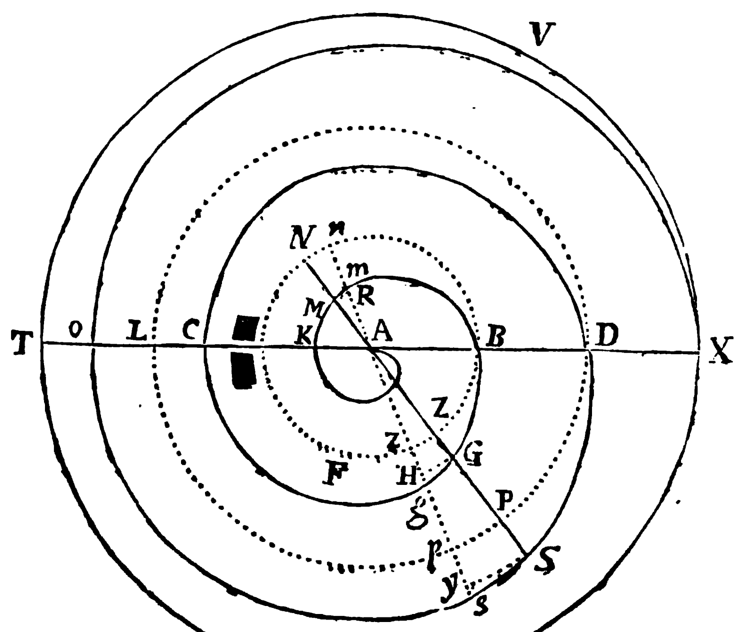
I might shew in the next Place, how to Investigate the Area's of Cycloidal Spaces, when their Bases are Arches of Circles, and when the Point which describes the Curve is taken at any Distance from the Center of the moveable Circle. But I shall refer such Speculations to a more convenient place in the Sector concerning the Rectification of Curves.

PROP. XIII.

To Investigate the Area's of all sorts of Spiral Spaces.

116. Let it be requir'd to find the Area of the Spiral Space, Comprehended between the Semi-diameter of the Circle A B, and the Spiral Line A M B.

Draw AN at pleasure, Intersecting the Spiral Line in M, and draw A n infinitely near A N, and on the Center A describe the little Arch M R. Then suppose the Circumference of the Circle BDNB $= c$, the Portion thereof BDN $= x$, N $n = \dot{x}$, A N $= r$, and the Portion thereof A M $= y$. Then because the Sectors A N n, AMR are Similar, it is, AN (r) : N n $(\dot{x})$: : AM (y) : MR $= \frac{y\dot{x}}{r}$. Which multiplied by $\frac{1}{2} y$, the Product $\frac{yy\dot{x}}{2r}$ = Triangle or Sector AMR is = to the Fluxion of the Spiral Space. Now suppose n the Exponent of the Power of the Circumference (c), and m that of the Radius (r), and suppose also that $c^n : x^n :: r^m : y^m$;



then $y = \frac{r x^{\frac{n}{m}}}{c^{\frac{n}{m}}}$, and yy is $= \frac{r r x^{\frac{2n}{m}}}{c^{\frac{2n}{m}}}$; which being Substituted for yy in (the Fluxion of the Area) $\frac{yy\dot{x}}{2r}$, it will be $\frac{r x^{\frac{2n}{m}}}{2 c^{\frac{2n}{m}}} \times \dot{x}$, and consequently the Flowing Quantity, or the Area AKMA is $= \frac{m \times r x^{\frac{2n}{m}+1}}{m+2n \times 2 c^{\frac{2n}{m}}} = \frac{m \times yy}{2m+4n \times 4}$. Therefore the

whole

whole Spiral Space AKMBA is $= \frac{m}{2m + 4n} \times rc$, becauſe then x becomes $= c$, and $y = r$.

CONSECTARY I.

117. The Spiral Space is to the Circumſcrib'd Circle as m is to $m + 2n$; that is, the firſt Spiral Space is to its Correſponding Circle as the Exponent of the Radius is to the Exponent of the Radius + twice the Exponent of the Circumference.

CONSECTARY II.

2°. If $m = 1$, and $n = 1$; then it is, $c : x :: r : y$, and the Curve AKMB is *Archimedes's Spiral*, and the Area thereof, *viz.* AKMBA $= \frac{m}{2m + 4n} \times cr$ is $= \frac{1}{6} cr = \frac{1}{3}$ the Area of the Circle BFNB.

CONSECTARY III.

3°. Hence to find a Spiral Space, which ſhall be to a given Circle in a given Proportion, *v. g.* as p is to q, we have $q : p :: m + 2n : m$; and conſequently, $q - p : p :: 2n : m$, and $\frac{q - p}{2} : p :: n : m$; whence it is evident that if p be the Exponent of the Radius, then $\frac{q - p}{2}$ muſt be the Exponent of the Circumference.

CONSECTARY IV.

4°. Imagine the Spiral Line AKMB to be continu'd from B by G, C, unto D, then is BD = AB; and to find the Area of the Space Comprehended between the ſecond Spiral Line BGCD and BD, draw the Lines AG, Ag infinitely near each other, and Interſecting the Circle BZNB in Z, Z, and on the Center A with the Radius AG, deſcribe the infinitely little Arch GH, then if AZ be $= r$, the Circumference BZNB $= c$, ZG $= y$, BZ $= x$; then is HG $= \frac{r\dot{x} + y\dot{x}}{r}$, and conſequently, the Fluxion of the Area = HG × $\frac{1}{2}$ AG is $= \frac{r + y}{2} \times \frac{r\dot{x} + y\dot{x}}{r} = \frac{rr\dot{x} + 2ry\dot{x} + yy\dot{x}}{2r} =$ (becauſe $c^n : c^n + x^n :: r^m : r^m + y^m$, and conſequently $y = \frac{r x^{\frac{n}{m}}}{c^{\frac{n}{m}}}$) $= \frac{r c^{\frac{2n}{m}} \dot{x} + 2 r c^{\frac{n}{m}} x^{\frac{n}{m}} \dot{x} + r x^{\frac{2n}{m}} \dot{x}}{2 c^{\frac{2n}{m}}}$, and finding the Flowing Quantities, the Area of the Spiral Space BAGB is

$$= \frac{r c^{\frac{2n}{m}} x + \frac{2m}{m+n} r c^{\frac{n}{m}} x^{\frac{n}{m}+1} + \frac{m}{m+2n} r x^{\frac{2n}{m}+1}}{2 c^{\frac{2n}{m}}}.$$

And the whole Space Comprehended between the ſecond Spiral Line BGCD and BD is = (becauſe then x becomes $= c$) $\frac{rc + \frac{2m}{m+n} rc + \frac{m}{m+2n} rc}{2} = \frac{4mm + 8mn + 2nn}{mm + 3mn + 2nn} \times \frac{rc}{2}$; which is a *General Theorem* for finding the Area's of an infinite variety of Second Spiral Spaces.

5°. And

CONSECTARY V.

5°. And in particular, if $m = 1$, and $n = 1$ (as in *Archimedes*'s *Spiral*) then the Area of the Spiral Space BGCDB is $= \frac{7}{3} \times \frac{rc}{2} = \frac{7}{3}$ the Area of the Circle BZNB (because the Areas of Circles are in a Duplicate Ratio of their Diameters) $\frac{7}{12}$ the Area of the Circle DPLD.

CONSECTARY VI.

6°. The Spiral Space AKMBA is $=$ (§. 2°.) $\frac{1}{3}$ the Circle BZNB, and this Circle is $= \frac{1}{4}$ the Circle DPLD, and consequently the first Spiral Space AKMBA is $= \frac{1}{12}$ the Circle DPLD.

CONSECTARY VII.

7°. The Area of the second Spiral Space BGCDB $=$ (§. 5°.) $\frac{7}{12}$ the Circle DPLD and the first Spiral Space AKMBA is $=$ (§. 6°.) $\frac{1}{12}$ DPLD. Therefore if from the second Spiral Space, the first be Subtracted, the remainder AKBGCDB $= \frac{6}{12}$ of the Circle DPLD, and the second Spiral Space less than the first, is to the first Spiral Space as 6 is to 1.

CONSECTARY VIII.

8°. And to find the Area of the third Spiral Space, *viz.* the Area of DSOXD produce AG, A*g* to S, S, and on the Center A describe the Infinitely little Arch SY, and suppose PS $= y$, then, $r : \dot{x} :: 2r + y : \frac{2r\dot{x} + y\dot{x}}{r} =$ SY. And the Fluxion of the Area, or the infinitely little Sector ASY is $= \frac{4rr\dot{x} + 4ry\dot{x} + yy\dot{x}}{2r}$ $=$ (because $y = \frac{2rx^{\frac{n}{m}}}{2c^{\frac{n}{m}}}$) $\frac{4rc^{\frac{2n}{m}}\dot{x} + 4rc^{\frac{n}{m}}x^{\frac{n}{m}}\dot{x} + rx^{\frac{2n}{m}}\dot{x}}{2c^{\frac{2n}{m}}}$. And the Flowing Quantity or the Space DASD $= \frac{4rc^{\frac{2n}{m}}x + \frac{4m}{m+n}rc^{\frac{n}{m}}x^{\frac{n}{m}+1} + \frac{m}{m+2n}rx^{\frac{2n}{m}+1}}{2c^{\frac{2n}{m}}}$.

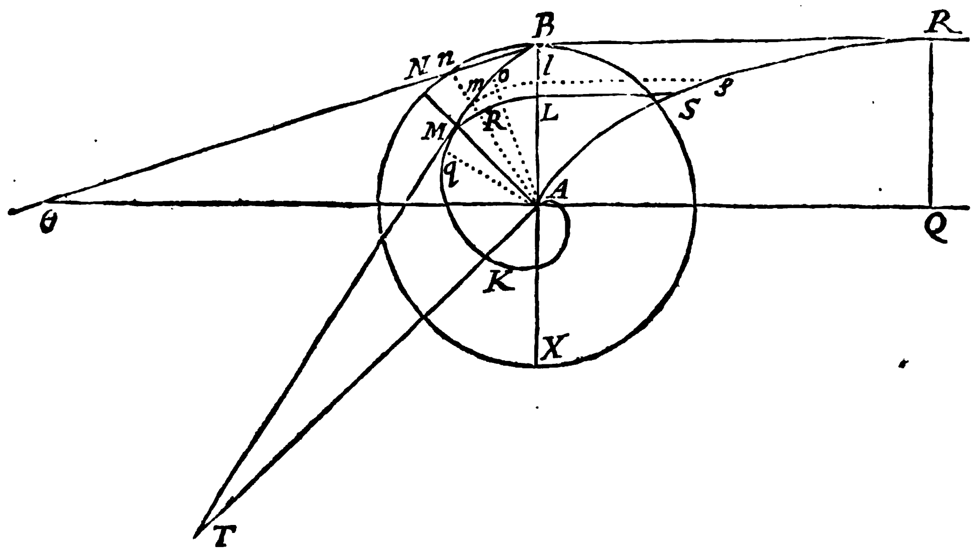
And consequently, the whole Spiral Space DSOXD will be equal to $\frac{4rc + \frac{4m}{m+n}rc + \frac{m}{m+2n}rc}{2} = \frac{9mm + 21mn + 8nn}{mm + 3mn + 2nn} \times \frac{rc}{2}$. That is, in our Hypothesis, the Spiral Space DSOXD is $= \frac{19}{3} \times \frac{rc}{2} =$ (because Circles are as the Squares of their Diameters) $\frac{19}{27}$ the Area of the Corresponding Circle XTVX.

PROP.

PROP. XIV.

To Investigate the Area's of Spiral Spaces by help of Tangents.

118. Let AKMB be a Spiral Line describ'd to the Circle BXN, and draw AMN, and A*mn*, infinitely near the same; on the Center A describe the little Arch MR, and continue the same to L, and describe also the Arch *ml*, draw AT Perpendicular to AN, and MT touching the Spiral Line in M and Intersecting AT in T. Then



suppose AB $= r =$ AN, AM $= y$, the Circumference of the Circle BXNB $= c$, the Portion thereof BXN $= x$, N*n* $= \dot{x}$, *m*R $= \dot{y}$, and MR $= \frac{y\dot{x}}{r} = \dot{z}$; Then because the Triangles *m*RM, MAT are similar, therefore *m*R $(\dot{y})$: RM $(\dot{z})$ :: AM (y) : AT (t) and consequently $t\dot{y} = y\dot{z}$. Now it is evident that all the $y\dot{z}$ are equal to twice the Area of the Spiral Space AKMBA; therefore all the $t\dot{y}$ are $=$ 2 AKMBA.

Now to find the Sum of all the $t\dot{y}$, to the point L apply the Line LS $= t =$ AT, then because L*l* is $=$ *m*R $= \dot{y}$, the Trapezium L*s* is $= t\dot{y}$, and if this be done always, and the Curve AS*s*P describ'd; then the Trilineal Space ABPS will be $=$ 2 AKMB.

Let the general Analogy expressing the Nature of Spiral Lines be, $r^m : c^n :: y^m : x^n$. Then AT (t) is $=$ (*Art.* 72.) $\frac{myx}{nr}$, and $ntr = myx$. And advancing every part of the Equation to the Power n, $n^n t^n r^n = m^n y^n x^n$, and substituting $\frac{c^n y^m}{r^m}$ for x^n, there will arise $n^n t^n r^n = \frac{m^n y^{m+n} c^n}{r^m}$, and $\frac{n^n r^{n+m}}{m^n c^n} \times t^n = y^{m+n}$.

Whence it is evident, that AL being $= y$, and LS $= t$, the Curve ASP is a sort of a Parabola, and AL will be the Intercepted Diameter, and LS the Ordinate; and to Investigate the Area of the Paraboliform Figure ASPB; BP is $=$ AO $= \frac{myx}{nr}$ $=$ (because x becomes $= c$, and $y = r$) $\frac{m}{n} \times c$, and the Circumscrib'd Parallelogram AP is $= \frac{m}{n} \times rc$, and (*Art.* 88) $m + 2n : n$ :: Parallelogram AP $\left(\frac{m}{n} rc\right)$:

Aa The

The Area of the Figure ASPB $= \frac{m}{m+2n} \times rc$. But the Area of the Paraboliform Figure APB is $= 2$ the Area of the Spiral Space AKMBA, therefore the said Spiral Space is $= \frac{m}{2m+4n} \times rc$.

CONSECTARY I.

119 If the Curve AKMB be the common Spiral Line, then $m = 1$, and $n = 1$, and the Area Comprehended under the same and the Radius AB is $= \frac{1}{6} rc = \frac{1}{3}$ the Area of the Circle BXNB.

CONSECTARY II.

2°. But if the Nature of the Spiral Line be such, that it cut the Radij of the Circle always in the same Angle, then the Triangles mRM will be always similar to one another; the Angle R being a Right-angle, and m (by supposition) being constantly the same.

Whence it appears that the Ratio of mR to RM is perpetually the same, *v. g.* as q is to p, then $p\dot{y} = q\dot{z}$ and $py\dot{y} = qy\dot{z}$, and (finding the Flowing Quantities) all the $y\dot{z}$ are $= \frac{pyy}{2q}$, and the infinite Spiral Space MKA is $= \frac{\text{all the } y\dot{z}}{2}$ $= \frac{pyy}{4q} =$ (because $p : q :: \theta A = t : AB = r$, supposing θB to touch the Spiral Line in B, and $A\theta$ Perpendicular to AB) $\frac{tyy}{4r} =$ (because $y = r$) $\frac{tr}{4}$. Whence the whole Spiral Space ABMKA is $= \frac{1}{2}$ the Triangle $AB\theta$.

CONSECTARY III.

3°. If we imagine an infinite Number of Ordinates Aq, AM, Am, Ao, &c. to be drawn, which Comprehend equal and infinitely little Angles at the Center A, its evident that the Triangles qAM, MAm, mAo are similar (because the Angles at A are suppos'd equal, and by the property of the Curve, the Angles at q, M, m, o, &c. are equal) and consequently, $Aq : AM :: AM : Am :: Am : Ao$, &c. whence it is manifest that the Ordinates are in a *Geometrical Progression*, when the Angles at the Center are infinitely little and in an *Arithmetical Progression*, and the Curve AKM is (for that Reason) call'd the *Logarithmetical Spiral*.

CONSECTARY IV.

4°. The *Logarithmetical Spiral Line* BMKA, makes an infinite Number of Revolutions before it can Terminate in the Center A.

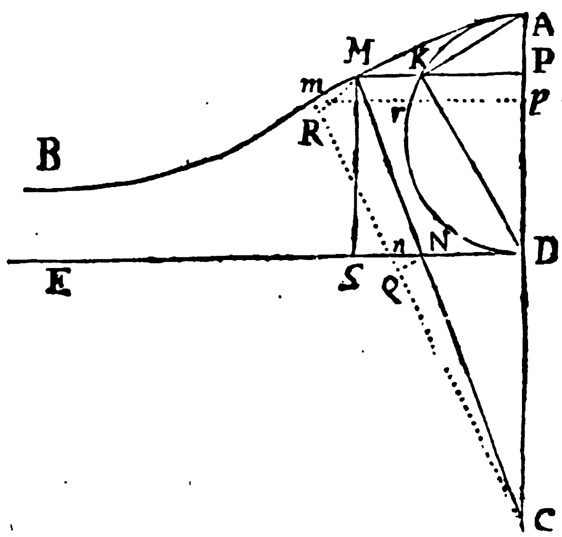
PROP.

PROP. XV.

To Investigate the Area of the Space Comprehended between the Conchoid and its Asymptote.

120. Let the Semi-conchoid AMB be deſcrib'd on the Pole C to the Aſymptote DE, and let the Nature of the Curve be ſuch, that drawing the Line CM from the Pole C to any point of the Curve M, interſecting the Aſymptote in N, the Rectangle CN × NM be always = CD × DA. 'Tis requir'd to determine the Value of the Conchoidal Space BMADE.

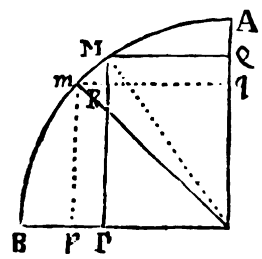
Suppoſe CD = a, DA = b, CN = x, NM = y; and draw another Line Cm infinitely near CM, and on the Center C deſcribe the little Arches NQ, MR, then is Qn = $\dot{x}$, and by the property of the Curve $xy = ab$, and conſequently, MN = $y = \frac{ab}{x}$, and CM = $\frac{ab+xx}{x}$. Now becauſe the Triangles CDN, NQn are ſimilar, it is, ND ($\sqrt{xx-aa}$) : CD (a) :: Qn ($\dot{x}$) : QN = $\frac{a\dot{x}}{\sqrt{xx-aa}}$, and (becauſe the Sectors CNQ, CMR are ſimilar) CN (x) CM $\left(\frac{ab+xx}{x}\right)$ :: NQ$\left(\frac{a\dot{x}}{\sqrt{xx-aa}}\right)$: MR = $\frac{axx\dot{x}+aab\dot{x}}{xx\sqrt{xx-aa}}$. Which being multiplied by $\frac{xx+ab}{2x}$ $\left(\frac{\text{CM}}{2}\right)$ the Product $\frac{ax^4\dot{x}+2a^2bx^2\dot{x}+a^3b^2\dot{x}}{2x^3\sqrt{xx-aa}} = \frac{ax\dot{x}}{2\sqrt{xx-aa}} + \frac{aab\dot{x}}{x\sqrt{xx-aa}} + \frac{a^3b^2\dot{x}}{2x^3\sqrt{xx-aa}}$ = to the infinitely little Sector CMR or the Fluxion of the Space CAM. Now it is evident that the firſt Member, *viz.* $\frac{ax\dot{x}}{2\sqrt{xx-aa}}$ is = CNQ = to the Fluxion of the Triangle CDN.



It remains only to find the Flowing Quantities of the other two Members, *viz.* $\frac{aab\dot{x}}{x\sqrt{xx-aa}} + \frac{a^3b^2\dot{x}}{2x^3\sqrt{xx-aa}}$ = to the infinitely little Trapezium MRQN, the Fluxion of the Conchoidal Space AMND, and conſequently the ſaid Space it ſelf.

With

With the Radius CB $= a$, describe the Quadrant CAB, and take CP $= \frac{aa}{x}$: Draw the Ordinate PM, and another pm infinitely near the same. Then MP is $=$ (because it is $= \sqrt{MCq - CPq}$) $\sqrt{aa - \frac{a^4}{xx}} = \frac{a}{x}\sqrt{xx - aa}$: And P$p$ $= m$R $=$ (the Fluxion of CP $= \frac{aa}{x}$) $- \frac{aa\dot{x}}{xx}$, and (because the Triangles MPC, mRM are similar) MR or Qq is $= - \frac{a^3\dot{x}}{xx\sqrt{xx - aa}}$, and M$m$ is $= - \frac{aa\dot{x}}{x\sqrt{xx - aa}}$, and the Flowing Quantity is equal to the Arch BM: For the Negative Sign ($-$) shews that as x Decreases $\frac{aa}{x}$ Increases; and consequently, the Arch BM and not AM must be the Flowing Quantity, because we are to find the Sum of all the $\frac{aab\dot{x}}{x\sqrt{xx - aa}}$; supposing the beginning at CN, and that we reckon to CD. Whence it is evident, that if the Arch BM be multiplied by the Invariable Quantity b, the Product is the Flowing Quantity of $\frac{aab\dot{x}}{x\sqrt{xx - aa}}$.



Lastly, To find the Flowing Quantity of the third Member, *viz.* $\frac{a^3b^2\dot{x}}{2x^3\sqrt{xx - aa}}$.

The infinitely little Space MQqm $=$ MQ $\times$ Qq $= \frac{a^5\dot{x}}{x^3\sqrt{xx - aa}}$, and the Flowing Quantity of this Fluxion is equal to the Portion of the Circle CQMB. Now if this Segment be multiplied by bb, and the Product divided $2aa$, the Quotient will be equal to the Flowing Quantity of the Fluxion $\frac{a^3bb\dot{x}}{2x^3\sqrt{xx - aa}}$, which answers to all the several Values of x from CN to CD.

And from hence I conclude, that the Conchoidal Space ADNM is equal to the Arch of the Circle BM $\times$ by the Invariable Quantity DA (b) $+$ the Portion of Circle CQMB $\times \frac{bb}{2aa}$. Q. E. I.

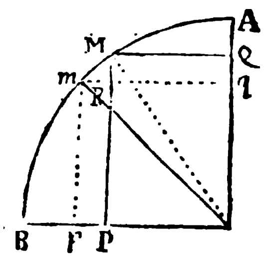
And if MN be always equal AD, then the Curve AMB will be *Nichomedes*'s *Conchoid*; and to Measure the same, suppose AD $=$ DC $= a$, CN $= x$, CM $= a + x$, Qn $= \dot{x}$; then (because the Triangles CDN, NQn are similar) it is, ND ($\sqrt{xx - aa}$) : CD (a) :: nQ ($\dot{x}$) : QN $= \frac{a\dot{x}}{\sqrt{xx - aa}}$, and (because the Sectors CNQ, CMR are similar) CN (x) : CM ($a + x$) :: QN $\left(\frac{a\dot{x}}{\sqrt{xx - aa}}\right)$: MR $= \frac{aa\dot{x} + ax\dot{x}}{x\sqrt{xx - aa}}$, which multiplied by $\frac{1}{2}$CM $= \frac{a + x}{2}$, the Product

$a^3\dot{x}$

$\frac{a^3\dot{x}}{2x\sqrt{xx-aa}} + \frac{aa\dot{x}}{\sqrt{xx-aa}} + \frac{ax\dot{x}}{2\sqrt{xx-aa}}$ is = to the infinitely little Sector MCR.

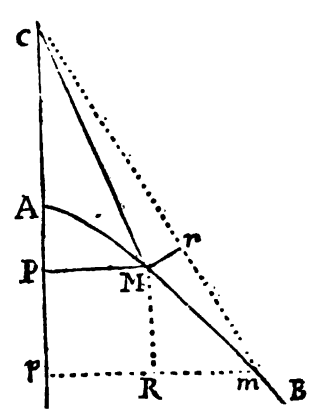
Now it is evident, that the last Member $\frac{ax\dot{x}}{2\sqrt{xx-aa}}$ is = CQN = to the Fluxion of the Triangle CDN. It remains to find the Flowing Quantities of the other two Members, *viz.*

The Flowing Quantity of $\frac{a^3\dot{x}}{2x\sqrt{xx-aa}}$ may be found thus, with the Radius CB = a, describe the Quadrant of a Circle CBA, and take CP = $\frac{aa}{x}$; Draw the Ordinate MP, and another mp infinitely near the same, and draw the Radij CM, Cm; then is MP = $\frac{a}{x}\sqrt{xx-aa}$.



And Pp or mR = $-\frac{aa\dot{x}}{xx}$. And because the Triangles CPM, MRm are similar, it is, PM $\left(\frac{a}{x}\sqrt{xx-aa}\right)$: MC (a) :: mR $\left(-\frac{aa\dot{x}}{xx}\right)$: Mm = $-\frac{aa\dot{x}}{x\sqrt{xx-aa}}$, which being multiplied by $\frac{1}{2}a$, the Product $-\frac{a^3\dot{x}}{2x\sqrt{xx-aa}}$ is = to the infinitely little Sector CMm, and consequently the Flowing Quantity is equal to the Sector CMB.

And to find the Flowing Quantity of the other Term $\frac{aa\dot{x}}{\sqrt{xx-aa}}$. Let AMB be an Equilateral Hyperbola, C the Center, and CA the Semi-axis = a, CP = x, and PM = y; draw the Ordinates PM, pm infinitely near each other, and from the Center C draw the Right-lines CM, cm. I say, the infinitely little Triangle MCm is = to the Fluxion $\frac{aa\dot{x}}{\sqrt{xx-aa}}$, and the Flowing Quantity is equal to the Space ACM; and consequently, double that Space is equal to the Flowing Quantity of $\frac{aa\dot{x}}{2\sqrt{xx-aa}}$.



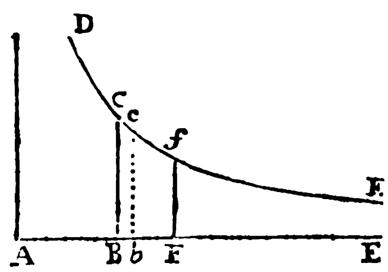
The Hyperbola is Equilateral, therefore $yy = xx - aa$, and consequently $\dot{y} = \frac{x\dot{x}}{\sqrt{xx-aa}}$. And $\overline{Mm}^2 = \dot{x}^2 + \dot{y}^2$ is $= \frac{2xx\dot{x}^2 - aa\dot{x}^2}{xx-aa}$. But CM = $\sqrt{2xx-aa}$, and the Fluxion thereof rm is $= \frac{2x\dot{x}}{\sqrt{2xx-aa}}$; therefore $\overline{Mm}^2 - \overline{mr}^2 = \overline{Mr}^2$ is $= \frac{2xx\dot{x}^2 - aa\dot{x}^2}{xx-aa} - \frac{4x^2\dot{x}^2}{2xx-aa}$ = (by reducing both to one common Denominator) $= \frac{a^4 \dot{x}^2}{xx - aa \times 2xx - aa}$; And consequently, Mr is $= \frac{aa\dot{x}}{\sqrt{xx - aa} \times \sqrt{2xx - aa}}$, which being multiplied by $\frac{1}{2}CM = \frac{1}{2}\sqrt{2xx - aa}$, the Product $\frac{aa\dot{x}}{2\sqrt{xx - aa}}$ is = to the infinitely little Triangle MCm, and the Flowing Quantity is equal to the Hyperbolick Space ACM. Therefore, &c.

Hence it's manifest, that the Conchoidal Space (tho' infinitely extended) may be Measur'd. And the like may be said of the Cissoidal Space.

PROP. XVI.

If the Relation between the Curve Line DCE, *and the Right Line* AE *infinitely produc'd, be such, that the Perpendicular* CB *being let fall from any Point of the Curve as* C, *be reciprocally as the Square, Cube,* &c. *of* AB *intercepted between* B *and any determinate Point* A *in the Right Line* AE*: It is requir'd to find the Area of the infinite Space* BCEE *Comprehended between the Right Lines* BC, BE *and the Curve* CE.

121. Suppose $AB = x$, $Bb = \dot{x}$, $BC = y$; then the Fluxion of the Area Bc is $= y\dot{x}$. But by the property of the Curve y is as $\frac{1}{x^n}$ (n being the Index of the Power of the Intercepted Diameter AB) therefore y is $= \frac{1}{x^n} \times$ into an Invariable Quantity, which suppose $= a^{n+1}$, that so the Terms may be Homogeneous, then $y = \frac{a^{n+1}}{x^n} = a^{n+1}x^{-n}$, therefore the Fluxion of the Area $= y\dot{x}$ is $= a^{n+1}x^{-n}\dot{x}$, and consequently, the Area it self or the Flowing Quantity is $= \frac{a^{n+1}x^{-n+1}}{-n+1}$ or $\frac{a^{n+1}}{-n+1\,x^{n-1}}$



= the Area lying on the other side of BC or the infinite Space BCEE, and because $\frac{a^{n+1}}{-n+1}$ is an invariable Quantity, the Area BCEE is as $\frac{1}{x^{n-1}}$. Thus if BC be reciprocally as the Cube of AB, then n is $= 3$, and $n - 1$ is $= 2$, and the Interminable Space BCEE is reciprocally as the Square of AB.

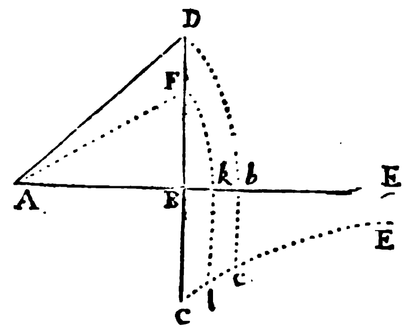
CONSECTARY I.

122. Because the infinite Space BCEE is as $\frac{1}{AB^{n-1}}$, and $FfEE$ as $\frac{1}{Ff^{n-1}}$, therefore the Space $BCFf$ is as $\frac{1}{AB^{n-1}} - \frac{1}{Ff^{n-1}}$.

CON-

CONSECTARY II.

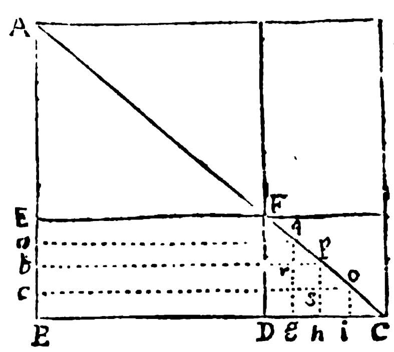
If BD be drawn Perpendicular to AE, and from every point of the Line BD, as as F, D, *&c.* be drawn Right Lines FA, DA, *&c.* to a given point A in the Line AB, and if in the Line AE we take $Ab = AD$, $Ak = AF$, and erect the Perpendiculars BC, kl, bc, &c. reciprocally as AB^n, Ak^n, Ab^n, &c. and if the Curve Line $ClcE$ be drawn, the Spaces BCEE, bcEE will be as $\frac{1}{AB^{n-1}}$, $\frac{1}{Ab^{n-1}}$ respectively, and the Space $BCcb$ will be as $\frac{1}{AB^{n-1}} - \frac{1}{Ab^{n-1}}$. For though the Curve Terminate in c, yet it may be suppos'd to be continu'd, and then the present *Corollary* is the same with § 1°.



PROP. XVII.

If the Right Line BC *be divided (at pleasure) in* D, *and if the Segment* DC *be divided into an infinite Number of equal Parts*, Dg, gh, hi, lc; *I say, all the Rectangles* $BD \times Dg$, $Bg \times gh$, $Bh \times hi$, $Bi \times iC$, *&c. are equal to* $\frac{1}{2}BCq - \frac{1}{2}DBq$.

123. On the point B erect BA Perpendicular to BC, and compleat the Square AC, draw the Diagonal AC, and DF Parallel to AB, and EF parallel to BC, then the Figures about the Diameter AC, *viz.* AF and FC are the Squares of BD and DC, therefore the Rectangles $BD \times Dg = Ea \times EF$; $Bg \times gh = aq \times ab$, $Bb \times bi = bp \times bc$, &c. And consequently, all the Rectangles (the Portions Dg, gh, &c. being infinitely little) $BD \times Dg$, $Bg \times gh$, &c. are equal to the Quadrilateral Figure BEFC = Triangle ABC — Triangle AEF = $\frac{1}{2}BCq - \frac{1}{2}BDq$.



COROLLARY.

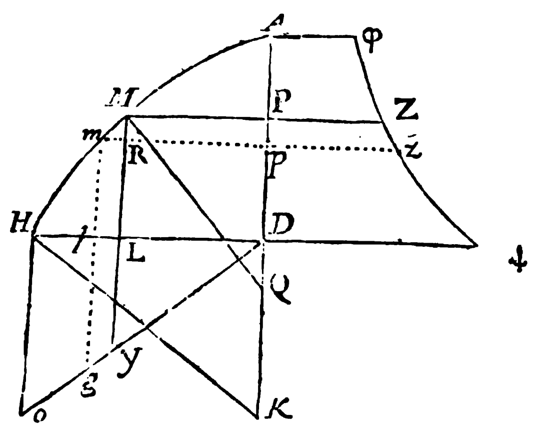
The Sum of all the foresaid Rectangles $BD \times Dg$, $Bg \times gh$, &c. is $= BD \times DC + \frac{1}{2}DCq$.

These two last preceeding Propositions are much us'd by the Incomparable Mr. Newton, Prop 79. lib. 1. Cor. 1, 2. Prop. 90. lib. 1. Prop. 51, 52. lib. 2. Princip. Mathemat.

PROP. XVIII.

Let AMH *be any given Curve,* AD *the Axis, and* HD *an Ordinate applied to the same, and let the Curve* $\varphi z \psi$ *be such, that if from any point in the Curve* AMH, *as* M *be drawn the Ordinate* MP *to the Axis* AD, *and* MQ *Perpendicular to the Curve in the Point* M, *and Intersecting the Axis in* Q, *and if* PZ *(producing* MP*) the Ordinate of the Second Curve be continually equal to the Sub-normal* PQ *of the First Curve; I say, the Curvilineal Space* AD $\varphi Z \psi$ *is* $= \frac{1}{2}$ DH*q*.

124. Let the Angle HD*o* be $= \frac{1}{2}$ a Right-angle, and divide the Axis AD into an infinite Number of equal Parts, such as P*p*, and draw *mpz* infinitely near MPZ. Draw the Lines MRLY, *mlg*, H*o* parallel to the Axis AD, intersecting D*o* in Y, *g*, *o*, and the Right Line DH in L, *l*. Then the Triangles *m*RM, MQP, are similar, therefore *m*R : RM :: QP : MP, and *m*R × MP is = RM × QP. Now *m*R is = *l*L; MP = LD = LY, and RM = P*p*, and QP = PZ, therefore L*l* × LY = P*p* × PZ; and consequently, all the Rectangles P*z*, which compose the Curvilineal Space AD$\psi\varphi$ are equal to all the Rectangles LY which compose the Triangle HD*o*, and that Curvilineal Figure is equal to this Triangle. But the Triangle HD*o* is $= \frac{1}{2}$ H*o* × HD $= \frac{1}{2}$ HD*q*. Therefore the Space AD$\psi\varphi$ is $= \frac{1}{2}$ HD*q*. Q. E. D.



COROLLARY.

125. Hence to Square any Curvilineal Space, as AD$\psi\varphi$, is the same thing as to find another Curve AMH whose Sub-normal PQ shall always be equal to PZ the respective Ordinate of the Figure to be Squar'd. For the Area APZφ is $= \frac{1}{2}$ PM*q*, and the Area ADψZφ is $= \frac{1}{2}$ DH*q*.

LEMMA.

The Relation of the Curve $\varphi z \psi$ *to the Axis* AD, *and consequently the Analytick Value of the Ordinate* Dψ *or the Sub-normal* (DK) *of the new Curve being given, to find the Relation of the new Curve* AMH *to the Axis* AD, *and the Analytick Value of the Ordinate* DH.

126. Let the Relation of the Curve $\varphi z \psi$ to the Axis AD be express'd by this Equation (supposing AD $= x$, Dψ $= y$, and DH $= z$) $y^2 = x^4 + aaxx$; then the Value of the Ordinate Dψ, reduced to its simplest Form, is, $y = x\sqrt{xx+aa}$, and therefore, by Construction DK is $= x\sqrt{xx+aa}$. Now it is requir'd to find the Relation of the Curve AMH to the Axis AD; that is, to find the Value of HD in the Terms of the intercepted Diameter, the Sub-normal DK $= x\sqrt{xx+aa}$ being given.

I. Multiply $x\sqrt{xx+aa}$, the Value of Dψ or DK by x (or any Power of x at pleasure) the Product is $xx\sqrt{xx+aa}$, and because the greatest Power without the

the Radical Sign is x^2, therefore write the same and all the inferior Terms thus, $x^2+x^1+x^0$, and to each prefix an unknown Coefficient, and then we have $bx^2+eax+ca^2$ (*a* being an invariable Quantity) which prefix before the Radical Sign in place of *x*, and suppose the Product $\overline{bx^2+eax+ea^2}\times\sqrt{xx+aa}=zz$. This is call'd the *Eminential Equation*, because it contains the Equation sought eminently.

II With the *Eminential Equation* $\overline{bx^2+eax+ea^2}\sqrt{xx+aa}=zz$, Investigate the Analytick Value of the Sub-normal DK thus: For brevities sake put $bx^2+cax+ea^2=p$, and $\sqrt{xx+aa}=q$, and then pq is $=zz$, and finding the Fluxions of both Sides of the Equation, we have $p\dot q+q\dot p=2z\dot z$. But $\dot q=\frac{x\dot x}{\sqrt{xx+aa}}$ and $\dot p=2bx\dot x+ca\dot x$, therefore restoring the Values of $p, q, \dot p, \dot q$, the Equation $p\dot q+q\dot p=2z\dot z$ will appear in this Form,

$$\frac{bx^3+cax^2+ea^2x\times\dot x}{\sqrt{xx+aa}}+2bx\times\sqrt{xx+aa}\times\dot x+ca\sqrt{xx+aa}\times\dot x=2z\dot z.$$

And reducing all the Terms to the same Denomination,

$$\frac{3bx^3+2cax^4+ea^2x+2ba^2x+ca^3}{\sqrt{xx+aa}}\times\dot x=2z\dot z.$$

Which being reduc'd to an Analogy, we have,

$$\dot x:\dot z::2z:\frac{3bx^3+2cax^2+ea^2x+2ba^2x+ca^3}{\sqrt{xx+aa}}::z:DK.$$

Therefore the Subnormal DK is $=\frac{3bx^3+2cax^2+ea^2x+2ba^2x+ca^3}{2\sqrt{xx+aa}}$ $=x\sqrt{xx+aa}$.

And clearing the Equation of Fractions and Surds,

$$\left.\begin{matrix}3bx^3+2cax^2+ea^2x+ca^3\\2ba^2x\end{matrix}\right\}=2x^3+2a^2x.$$

3°. Compare the respective Terms of these Equations, and find the Values of the unknown Coefficients; thus, $3b$ is $=2$; and consequently, $b=\frac{2}{3}$. 2°. $2c=0$, and c is $=0$, therefore all the Terms Affected with c vanish, or are equal to nothing, and enter not into the Equation requir'd. 3°. $e+2b=2$, and $e=2-2b=2-\frac{4}{3}=\frac{2}{3}$. Lastly, $c=0$.

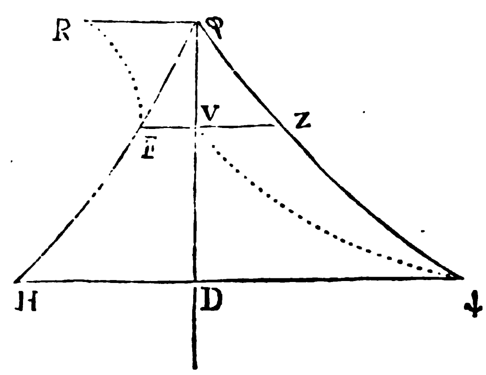
Whence the *Eminential Equation* $bx^2+cax+ea^2\sqrt{xx+aa}=zz$, becomes $bx^2+ea^2\sqrt{xx+aa}$, and substituting the Values of the unknown Coeficients, *viz.* $\frac{2}{3}$ for b and $\frac{2}{3}$ for e, we have this Equation $\frac{2x^3+2a^2}{3}\sqrt{xx+aa}=zz$ expressing the Relation between the Ordinate (DH) of the new Curve, and the Axis AD, which was requir'd.

CONSECTARY I.

127. And if we rightly consider the new Equation expressing the Nature of the Curve AMH, it will appear, that the Vertex of the Quadratrix (or the Curve) AHD does not always co-incide with the Vertex of the Curvilineal Figure AD♆Z φ which is to be Squared; But sometimes it falls above and sometimes below the same, and often it is purely Imaginary.

CASE I.

If the Curvilineal Space VD♆ be to be Squar'd, and if VD be $=x$, D♆ $=y$, and the Ordinate of the Quadratrix $=z$, and if the Relation of the Ordinate D♆ to the Axis VD be express'd by this Equation $y = x\sqrt{xx+aa}$: Then 'tis evident that when the Intercepted Diameter x is $=0$, the Ordinate y is also $=0$, and consequently, the Vertex of the given Curve is in the beginning of the Axis VD or x.



2°. The Equation expressing the Relation of the Axis of the Quadratrix VD to its Ordinate DH is $\frac{2x^2+2a^2}{3} \times \sqrt{xx+aa} = zz$. Whence to find whither the Vertex of the Quadratrix HF be in V the Vertex of V♆; we must observe, that if, when x is $=0$, z also be $=0$, they coincide, otherwise not. Therefore supposing $x=0$, the Equation expressing the Nature of the Quadratrix in the Point V is $\frac{2a^2}{3} \times a = zz$, that is $z = \sqrt{\frac{2a^3}{3}}$. Whence it appears that when x is $=0$, then the Ordinate of the Quadratrix VF is $= \sqrt{\frac{2a^3}{3}}$.

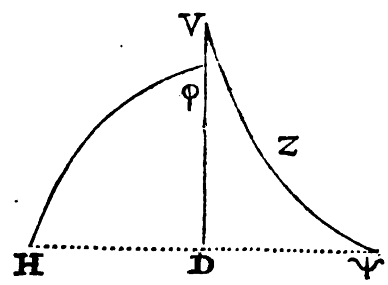
3°. To find where the Quadratrix HF will Intersect the Axis, and whether above or below V we must observe, that when the Quadratrix Intersects the Axis, then z is $=0$, whence the Equation expressing the Nature of the Quadratrix becomes $\frac{2x^2+2a^2}{3} \times \sqrt{xx+aa} = 0$, and consequently, x is $= \sqrt{-a^2}$, which is an impossible Equation, shewing that the Quadratrix HF being continu'd infinitely towards R will never meet with the Axis.

CASE II.

Let the Equation expressing the Nature of the Curve z♆ to the Axis VD be $y = \overline{x+a}\sqrt{x+a}$, then it is plain that when x is $=0$, then $y = a\sqrt{a} =$ VZ; so that the Curve to be Squar'd, Intersects not the Axis in the beginning of x (or in the Point V.) 2°. The Equation of the Quadratrix HF φ is (*Art.* 126.) $\frac{4x^2+8ax+4a^2}{5} \times \sqrt{x+a} = zz$, whence to find the Point φ where it Intersects the Axis, z is then $=0$, therefore $\frac{4x^2+8ax+4a^2}{5} \times \sqrt{x+a} = 0$; that is, $4x^2+8aax+4a^2=0$, and by equal Extraction $2x+2a=0$; that is, $x = -a =$ Vφ, and because the Value of the Abscissa is Negative, therefore the Point φ falls above V, and Vφ, V φ is $= a$. 3°. To find the Analytick Value of the Ordinate of the Quadratrix V F, when the Intercepted Diameter V D vanishes; that is, when x is $= 0$, the Equation of the Quadratrix in that Point is $\frac{4}{5}a^2 \sqrt{a} = zz =$ V F q.

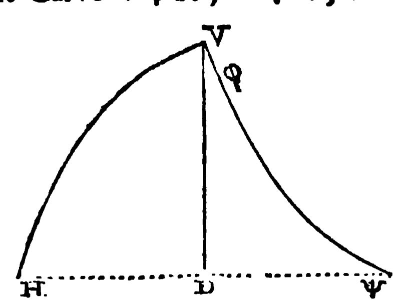
CASE III.

Let the Equation expressing the Nature of the Curve V z ψ be $y = x\sqrt{x+a}$; then it is plain, that when x or the Abscissa V D is $= 0$, the Ordinate y is also $= 0$; and consequently, the given Curve intersects the Axis in V. 2°. the Equation expressing the Nature of the Quadratrix H φ is (*Art.* 126.) $\frac{12x^2 + 4ax - 8a^2}{15}\sqrt{x+a} = zz$, and if z be suppos'd $= 0$; then $\frac{12x^2 + 4ax - 8a^2}{15}\sqrt{x+a} = 0$, and by Reduction $x = \frac{2}{3}a =$ V φ, and because the Value of x is positive, therefore the Quadratrix H φ Intersects the Axis in the Point φ below V, the Vertex of the given Curve. 3°. To find the Analytick Value of the Ordinate of the Quadratrix, when x is $= 0$, the Equation of the Quadratrix in that Point, *viz.* in V, is $\frac{-8a^2}{15}\sqrt{a} = zz =$ V F q, where it may be observ'd, that as the Abscissa x increases, the Ordinate z decreases until the Quadratrix meet the Axis in φ, and ever afterwards, the Abscissa x and the Ordinates of the Quadratrix both increase at the same time.



CASE IV.

Let the the Equation expressing the Nature of the Curve V ψ be $y = \sqrt{ax}$, then if x be $= 0$, y will also be $= 0$, and the given Curve Intersects the Axis in V. 2°. The Equation of the Quadratrix V H is (*Art.* 126.) $\frac{2}{3}x\sqrt{ax} = zz$, and if z be $= 0$, then $\frac{2}{3}x\sqrt{ax} = 0$. Whence $x = 0$ Therefore, the Quadratrix V H and the given Curve V ψ Intersect the Axis V D in the same Point V.



CONSECTARY II.

If in the Analytick Value of the Ordinate H D, any of the Members be a determinate and invariable Quantity, then the Quadratrix cannot Intersect or meet the Axis in the beginning of the Abscissa x, and if in the Analytick Value of the Ordinate D ψ, any of the Members be an invariable Quantity, then the given Curve V ψ cannot meet the Axis in the beginning of the Abscissa x. But if the indeterminate Quantity representing the Abscissa, affect all the Terms expressing the Value of D ψ or D H, then the respective Curves meet both in the same Point of the Axis, where x begins.

CONSECTARY III.

In the first Case the Quadratrix H F R never meets the Axis V D. Whence it is evident that the Area V D ψ is not $= \frac{1}{2}$ D H q; because D H answers to the whole Quadratrix R F H D φ. It remains then that $\frac{1}{2}$ D H q exceeds the Area of the Figure

V D ψ

VDψ by $\frac{1}{2}$FVq; and consequently, $\frac{1}{2}$DHq − $\frac{1}{2}$FVq = $\frac{x^2+a^2}{3}\sqrt{x^2+a^2} - \frac{a^3}{3}$ is = to the Area of the Curvilineal Figure VDψ.

CONSECTARY IV.

In the second Case, the Area of the Figure VDZψ is less than $\frac{1}{2}$DHq; for if the Quadratrix HF and the Curve ψZ be both produc'd until they Intersect the Axis in φ, then the Area φDψZφ is = $\frac{1}{2}$DHq, and the Area φVZ is = $\frac{1}{2}$FVq, therefore the Area of VDψZ is = $\frac{1}{2}$DHq − $\frac{1}{2}$FVq = $\frac{2x^2+4ax+2a^2}{5} \times \sqrt{x+a} - \frac{2a^2}{5}\sqrt{a}$.

CONSECTARY V.

In the third Case, the Area of the Figure VDψ exceeds $\frac{1}{2}$DHq; because the Quadratrix Intersects the Axis below the Vertex V; so that $\frac{1}{2}$DHq is = the Area φDψz, and $\frac{1}{2}$FVq = Area Vφz; therefore the Area VD$\psi\varphi$V is = $\frac{1}{2}$DHq + $\frac{1}{2}$FVq = $\frac{6x^2+2ax-4a^2}{15} \times \sqrt{x+a} + \frac{2a^2}{15}\sqrt{a}$.

CONSECTARY VI.

In the fourth Case, the Quadratrix Intersects the Axis in V, and consequently, the Area VDψ is = $\frac{1}{2}$DHq = $\frac{2}{3}x\sqrt{ax}$.

And thus I have briefly explain'd the Principles of another Method for squaring Curvilineal Figures, and Illustrated the same by particular Instances. I shall in the next place shew how it may be applied to Investigate the Area's of an infinite Number of Curvilineal Figures, their Nature being express'd by any one general Equation.

EXAMPLE I.

128. Let it be requir'd to Investigate the Area's of all sorts of Paraboliform Figures, whose Nature is express'd by this general Equation $a^{m-n}x^n = y^m$, the Equation reduc'd to its simplest Form, is $y = \sqrt[m]{a^{m-n}x^n}$, which being multiplied by x, the Product is $x\sqrt[m]{a^{m-n}x^n}$; therefore the *Essential Equation* is $bx + c \sqrt[m]{a^{m-n}x^n} = zz$.

2°. With this Equation Investigate the Value of the Ordinate of the Quadratrix thus, put $bx + c = p$, and $\sqrt[m]{a^{m-n}x^n} = q$. Then is $pq = zz$, and the Fluxion of this Equation is $p\dot{q} + q\dot{p} = 2z\dot{z}$. But $\dot{p}$ is = $b\dot{x}$, and $\dot{q}$ is = $\frac{1}{m} \times \overline{a^{m-n}x^n}|^{\frac{1}{m}-1} \times nx^{n-1}\dot{x} = \frac{nx^{n-1}\dot{x}}{m\sqrt[m]{a^{m-n}x^n}|^{m-1}}$. And therefore restoring the Values of $p, q, \dot{p}, \dot{q}$ in the Differential Equation $p\dot{q} + q\dot{p} = 2z\dot{z}$, there will arise,

$$\frac{nbx^n\dot{x}+cnx^{n-1}\dot{x}}{m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}+b\sqrt[m]{a^{m-n}x^n}\times\dot{x}=2z\dot{z}.$$

And reducing all the Terms to the same Denomination,

$$\frac{nbx^n\dot{x}+cnx^{n-1}\dot{x}+bm\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}\times\sqrt[m]{a^{m-n}x^n}\times\dot{x}}{m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}=2z\dot{z}.$$

That is, $\frac{nbx^n\dot{x}+cnx^{n-1}\dot{x}+bm\sqrt[m]{a^{m-n}x^n}\,\big|^{m}\times\dot{x}}{m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}=$

$$\frac{nbx^n\dot{x}+cnx^{n-1}\dot{x}+\dot{x}\times bm\times a^{m-n}x^n}{m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}=2z\dot{z}.$$

And reducing the Equation to an Analogy,

$$\dot{x}:\dot{z}::2z:\frac{nbx^n+cnx^{n-1}+bm\times a^{m-n}x^n}{m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}::z:\text{Sub-normal}$$

$$=\frac{nbx^n+cnx^{n-1}+bm\times a^{m-n}x^n}{2m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}}=\sqrt[m]{a^{m-n}x^n}.$$

And by Reduction and clearing the Equation of Surds,

$nbx^n+cnx^{n-1}+bm\times a^{m-n}x^n=2m\sqrt[m]{a^{m-n}x^n}\,\big|^{m-1}\times\sqrt[m]{a^{m-n}x^n}=$
$2m\sqrt[m]{a^{m-n}x^n}\,\big|^{m}=2m\times a^{m-n}x^n$. That is,

$$\left.\begin{array}{l} bm\times a^{m-n}x^n+cnx^{n-1}\\ nb\times \quad x^n+\quad 0\end{array}\right\}=2m\times a^{m-n}x^n.$$

And comparing the Coefficients, c is $=0$, and $bm+bn=2m$, and $b=\frac{2m}{m+n}$. Whence the *Eminential Equation* $\overline{bx+c}\times\sqrt[m]{a^{m-n}x^n}\,\big|=zz$, becomes $\frac{2m}{m+n}\times\sqrt[m]{a^{m-n}x^n}$. Therefore the Quadratrix and the Curve to be squar'd meet both in the beginning of the Abscissa, and the Area of all sorts of Paraboliform Figures is $=\frac{m}{m+n}\times\sqrt[m]{a^{m-n}x^n}=\frac{m}{m+n}xy$. Q. E. I.

EXAMPLE II.

129. Let it be requir'd to find the Area's of all sorts of Curvilineal Figures, whose Nature is express'd by this general Equation, $y=x\sqrt[m]{x+a}$, where m denotes the Exponent of any Power, whether Positive or Negative, *&c.* The *Eminential Equation* is, $bx^2+cax+daa\times\sqrt[m]{x+a}=zz$ expressing the Nature of the Quadratrix, and to determine the Values of the unknown Coefficients; suppose, for brevities sake, $bx^2+cax+daa=p$, and $\sqrt[m]{x+a}=q$. Then pq is $=zz$, and

and $p\dot{q} + q\dot{p} = 2z\dot{z}$. But $\dot{p}$ is $= 2bx\dot{x} + ca\dot{x}$, and $\dot{q}$ is $= \frac{1}{m}\overline{x+a}|^{\frac{1}{m}-1}\dot{x}$

$$= \frac{\dot{x}}{m\sqrt[m]{x+a}|^{m-1}}.$$

Therefore Subſtituting the Values of $p, q, \dot{p}, \dot{q}$, in the Equation $p\dot{q} + q\dot{p} = 2z\dot{z}$. There will ariſe,

$$\frac{\overline{bx^2+cax+daa}\times\dot{x}}{m\sqrt[m]{x+a}|^{m-1}} + \sqrt[m]{x+a}\times\overline{2bx+ca}\times\dot{x} = 2z\dot{z}.$$

And Reducing all the Terms to the ſame Denomination,

$$\frac{\overline{bx^2+cax+daa}\times\dot{x} + m\sqrt[m]{x+a}|^{m-1}\times\sqrt[m]{x+a}\times\overline{2bx+ca}\times\dot{x}}{m\sqrt[m]{x+a}|^{m-1}} = 2z\dot{z}.$$

That is,

$$\frac{bx^2+2mbx^2+cax+2mbax+cmax+cmaa+daa}{m\sqrt[m]{x+a}|^{m-1}}\times\dot{x} = 2z\dot{z}$$

And Reducing the Equation to an Analogy,

$$\dot{x}:\dot{z}::2z:\frac{bx^2+2mbx^2+cax+2mbax+cmax+cmaa+daa}{m\sqrt[m]{x+a}|^{m-1}}::$$

$$z:\text{Sub-normal} = \frac{bx^2+2mbx^2+cax+2mbax+cmax+cma^2+daa}{2m\sqrt[m]{x+a}|^{m-1}}$$

$= x\sqrt[m]{x+a}$, which being clear'd of Surds, by multiplying both ſides of the Equation by $2m\sqrt[m]{x+a}|^{m-1}$, the Equation will ſtand in this Form,

$$\left.\begin{array}{lll} bxx & +2mbax & +cmaa \\ 2mbxx & cmax & daa \\ & cax & \end{array}\right\} = 2mxx + 2max.$$

And comparing the reſpective Coefficients of both Equations,

Firſt, $b + 2mb$ is $= 2m$, therefore $b = \frac{2m}{2m+1}$.

Secondly, $c + cm + 2mb$ is $= 2m$, whence c is $= \frac{2m-2mb}{m+1} = \frac{2m}{\overline{2m+1}\times\overline{m+1}}$

Thirdly, $cm + d$ is $= 0$, whence $d = -cm = -\frac{2mm}{\overline{2m+1}\times\overline{m+1}}$.

Having thus found the Values of the Coeffieients b, c, d, (which were Indeterminate before) ſubſtitute them in place of the ſaid Indeterminate Coefficients in the *Eminential Equation* $\overline{bx^2+cax+daa}\sqrt[m]{x+a} = zz$, and there will ariſe,

$$\frac{2mxx}{2m+1} + \frac{2max-2mmaa}{\overline{2m+1}\times\overline{m+1}}\times\sqrt[m]{x+a} = zz$$

which

Which is an Equation expreſſing the Nature of the Quadratrix ſought. Whence the Area of any Curvilineal whoſe Nature is expreſs'd by the given Equation, may be found,

For inſtance, Let it be requir'd to Inveſtigate the Area of a Figure, whoſe Nature is expreſs'd by this Equation $y = x\sqrt[-2]{x+a}$ that is $y = \frac{x}{\sqrt{x+a}}$. In this Caſe the Exponent m is $= -2$, and conſequently $\frac{2m}{2m+1} = \frac{-4}{-3} = \frac{4}{3}$, and $\frac{2m}{2m+1 \times m+1} = \frac{-4}{3}$, and $\frac{2mm}{2m+1 \times m+1}$ is $= \frac{-8}{3}$, therefore $\frac{2mxx}{2m+1} + \frac{2max - 2mma}{2m+1 \times m+1} \times \sqrt[m]{x+a}$ is $= \frac{4xx - 4ax - 8aa}{3\sqrt{x+a}} = zz$, and $\frac{2xx - 2ax - 4aa}{3\sqrt{x+a}}$ is $= \frac{1}{2}zz$. Now the Quadratrix meets the Axis below the beginning of x, and when x is $= 0$, then the Square of the Ordinate of the Quadratrix is $\frac{-8aa}{3\sqrt{a}} = zz$, therefore the Area of the given Curvilineal Figure is $= \frac{2xx - 2ax - 4aa}{3\sqrt{x+a}} + \frac{4aa}{3\sqrt{a}}$. Q. E. I.

And in comparing the Coefficients, if there be more Equations or Compariſons, than is needful to determine the Coefficients, and if the Values of the ſame Coefficient come out different; that is a Sign that it is impoſſible to Square ſuch a Curvilineal Figure.

This Method may be render'd more Univerſal, if we ſuppoſe the Exponent of the Quantity prefixt to the Radical Sign, to be Indeterminate alſo, for by that means one ſingle Theorem may ſerve to Square an Infinitely-Infinite variety of Curvilineal Figures, whoſe Natures are expreſs'd by ſuch an Equation.

EXAMPLE III.

130. Let it be requir'd to find the Areas of all ſorts of Curvilineal Figures expreſs'd by this Equation $y = x^n\sqrt[m]{x+a}$. Multiply the Value of the Ordinate by x, the product is $x^{n+1}\sqrt[m]{x+a}$. Where the Indefinite Number $n+1$ is the Exponent of the higheſt Power of the Quantity prefixt to the Radical Sign. Therefore the Inferior Powers being indefinite, the *Eminential Equation* is,

$$bx^{n+1} + cx^n + dx^{n-1} + ex^{n-2} + fx^{n-3}, \&c. \sqrt[m]{x+a} = zz$$

By

By help of this Equation find the Value of the Sub-normal of the Quadratrix, and put the ſame equal to the given Value thereof, and then the Equation clear'd of Surds, will ſtand thus,

$$\left.\begin{array}{llll} \overline{n+1+m}\,b\,x^{n+1} & +nmc\,x^{n} & +\overline{n-1}\times md\,x^{n-1} & +\overline{n-2}\times mc\,x^{n-2} \\ +b & +\overline{n+1}\times mb & +n\times mc & +\overline{n-1}\times md \\ & +c & +d & +e \end{array}\right.$$

$$\left.\begin{array}{l} +\overline{n-3}\times mf\,x^{n-3} \\ +\overline{n-2}\times me \quad \&c. \\ +f \end{array}\right\} = 2u\,x^{n+1} + 2ma\,x^{n}.$$

And comparing the Reſpective Terms of this Equation, the unknown Coefficients will be determin'd as follows,

First, $b = \dfrac{2m}{\overline{n+1}\times\overline{m+1}}$

Second, $c = \dfrac{2m}{\overline{n+1}\times\overline{m+1}\times\overline{n\times m+1}}$

Third, $d = \dfrac{-n\times 2m^{2}}{\overline{n+1}\times\overline{m+1}\times\overline{n\times m+1}\times\overline{n-1\times m+1}}$

Fourth, $e = \dfrac{n\times\overline{n-1}\times 2m^{3}}{\overline{n+1}\times\overline{m+1}\times\overline{n\times m+1}\times\overline{n-1\times m+1}\times\overline{n-2\times m+1}}$

Fifth, $f = \dfrac{-n\times\overline{n-2}\times\overline{n-1}\times 2m^{4}}{\overline{n+1}\times\overline{m+1}\times\overline{n\times m+1}\times\overline{n-1\times m+1}\times\overline{n-2\times m+1}\times\overline{n-3\times m+1}}$.

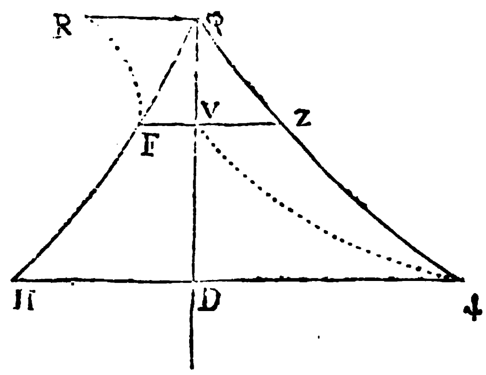
Now from the Compoſition of theſe five Coefficients it appears how all the reſt may be form'd in Infinitum; And becauſe the Progreſſion $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}$, &c. in the Numerator of the Coefficients, if n be a whole and poſitive Number, or equal to Nothing, then the Quadratrix will be an Algebraick Curve, and always there will be as many Coefficients as there are Unites in $n+1$, and as to the Signs prefix'd to the Coefficients, after the firſt two which always are Affirmative, they are Negative and Affirmative Alternately, *viz.* $+b+c-d+e-f+g-h$, &c.

And to Inſtance in particulars, if it be requir'd to find the Area of that Curvilineal Figure, whoſe Nature is expreſs'd by this Equation $y = x^{1}\sqrt[-2]{x+a}$, Then n is $= 1$, and m is $= -2$. Whence $b = \frac{4}{3}$, $c = \dfrac{-4}{3}$, and $d = \dfrac{-8}{3}$, and the Eminential Equation $bx^{n+1} + cx^{n} + dx^{n-1} + ex^{n-2} + fx^{n-3}$, &c. $\sqrt[m]{x+a} = zz$ becomes $\dfrac{4xx-4x-8}{3\sqrt[2]{x+a}} = zz$, which expreſſes the Nature of the Quadratrix, and the Area of the Curvilineal Figure may be determin'd as above.

I was more willing to treat of this Method at large, becauſe tho' the Equation expreſſing the Nature of the Curve conſiſts of Terms Compos'd of Invariable Quantities only; Nevertheleſs the Area of the Figure may be preciſely Determin'd. The admirable Aſſiſtances which we have in other Caſes, proving Defective in this: For Inſtance,

131. Let

131. Let it be propos'd to Investigate the Area of the Curvilineal Figure V D ♃ Z whose Nature is expres'd by this Equation $y = \overline{x + a}\,\sqrt{x + a}$, the Fluxion of the Area is $y\dot{x}$ or $\overline{x\dot{x} + a\dot{x}}\,\sqrt{x + a}$. Which may be cleared of the Radical Sign thus, Suppose $\sqrt{x + a} = z$, then $x + a = zz$, and $\dot{x} = 2z\dot{z}$, whence it is evident (by substitution) that $\overline{x\dot{x} + a\dot{x}}\,\sqrt{x + a} = 2z^4\dot{z}$. Now the Fluxions on each side of the Equation being equal, the Flowing Quantities must be so too. Therefore S $\overline{x\dot{x} + a\dot{x}}\,\sqrt{x + a} = \text{S}\,2\,z^4\dot{z} = \frac{2}{5}z^5 = \frac{2xx + 4ax + 2aa}{5}\sqrt{x + a}$ = to

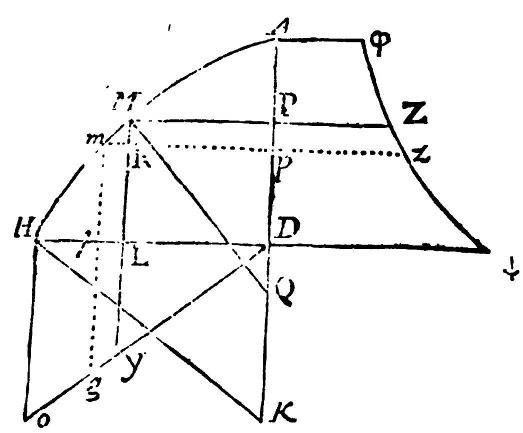


the Area of the Curvilineal Figure V D ♃ z, which Value (§ 4°. *Art.* 127.) exceeds the truth by the determinate Quantity $\frac{2a^2}{5}\sqrt{a}$.

But though we cannot find the Areas of such Curvilineal Figures by Summatory Arithmetick; yet it may be observ'd that it gives us half the Value of the Square of the Ordinate of the Quadratrix, without having recourse to Tangents or an Eminential Equation. Which is an excellent help, and exceedingly Abriviates the Work.

EXAMPLE I.

132. Let the Relation of the Curve φ Z ♃ to the Axis A D be express'd by this Equation $yy = x^4 + aaxx$. Then $y = x\sqrt{xx + aa}$, and the Fluxion of the Area is $y\dot{x}$ or $x\dot{x}\sqrt{xx + aa}$, which is clear'd of the Radical Sign thus; put $\sqrt{xx + aa} = z$, then $xx + aa = zz$; and consequently, the Fluxions of both sides of the Equation must be equal, *viz.* $x\dot{x} = z\dot{z}$. Now the Fluxion of the Area was $x\dot{x}\sqrt{xx + aa}$, and if we put $z\dot{z} = x\dot{x}$, and $z = \sqrt{xx + aa}$, the Fluxion of the Area in other Terms will be $= z^2\dot{z}$, therefore the Flowing Quantity is $= \frac{1}{3}z^3$, and reassuming $xx + aa$ for zz, and $\frac{1}{3}\,\overline{xx + aa}$



for $\frac{1}{3}z^2$, and $\sqrt{xx + aa}$ for z, we shall have $\frac{1}{3}z^3 = \frac{xx + aa}{3}\sqrt{xx + aa} = \frac{1}{2}$ the Square of the Ordinate of the Quadratrix D H; and consequently, the Square of D H is $= \frac{2xx + 2aa}{3}\sqrt{xx + aa}$; which was requir'd.

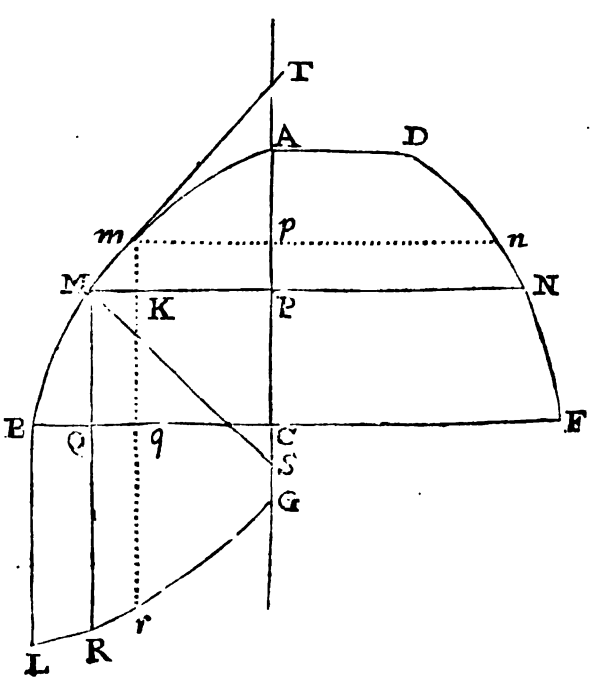
EXAMPLE II.

Let the Relation of any Curve to its Axis be expreſs'd by this Equation $x^2 \sqrt[-1]{x+a} = yy$, then $y = \frac{x}{\sqrt{x+a}}$, and the Fluxion of the Area will be $= \frac{x\dot{x}}{\sqrt{x+a}}$. Suppoſe $\sqrt{x+a} = z$, then is $x+a = zz$, and $x = zz - a$; and conſequently, $\dot{x} = 2z\dot{z}$, therefore the Fluxion of the Area $\frac{x\dot{x}}{\sqrt{x+a}}$ is $= \frac{2z^3\dot{z} - 2az\dot{z}}{z} = 2z^2\dot{z} - 2a\dot{z}$, and the Flowing Quantity is $= \frac{2}{3}z^3 - 2az = \frac{2x-4a}{3}\sqrt{x+a}$; and conſequently, the Square of the Ordinate of the Quadratrix is $= \frac{4x-8a}{3}\sqrt{x+a} = \frac{4xx-4ax-8aa}{3\sqrt{x+a}}$, which is equal to the Value of the ſaid Ordinate (*Art.* 129) found before.

PROP. XIX.

If there be three Curves AMB, DNF, *(and* AC *the Axis common to both) and* GRL *(whoſe Axis is* CB,*) and if the Relation between them be ſuch, that from any point* M *in the Curve* AMB, *drawing the Tangent* MT *Interſecting the Axis* AC *(produc'd) in* T, *and the Right Lines* MPN *and* MQR *parallel to the Axes* CB, CA; *It be always* PT : PM :: QR : PN. *I ſay, the Mixtilineal Figure* ACFD *will be equal to the Mixtilineal Figure* CGLB.

133. Let M*m* be an infinitely little Portion of the Curve AMB, and draw *mpn* parallel to MPN, and *mqr* parallel to MQR, Interſecting the Axis in the Points *q* and *p*, and the Ordinate PM in K.



Now becauſe the Triangles MK*m*, MPT are ſimilar, therefore PT : PM :: *m*K : MK. That is, PT : PM :: P*p* : Q*q*. But by ſuppoſition PT : PM :: QR : PN. Therefore QR : PN :: P*p* : Q*q*; and conſequently, PN × P*p* is = QR × Q*q*. That is, the Trapezium PN*np* is always equal to the reſpective Trapezium QR*rq*. Now the Space ACFD conſiſts of all the Trapezia PN*np*, and the Space CGLB conſiſts of all the Trapezia QR*rq*. Therefore the Mixtilineal Space ACFD is equal to the Mixtilineal Space CGLB.

CON-

CONSECTARY I.

134 If the points G and C co-incide, and if G R L be a ſtreight Line comprehending an Angle of 45°. with CB, then this Propoſition differs not from the 18th. preceeding; (that being but a particular Caſe of this) For if the Perpendicular M S be drawn, then it is P T : P M :: PM : PS :: K *m* : K M :: P *p* : Q *q* :: Q R : P N. Now P M = C Q = Q R, therefore P M (Q R) : P S :: Q R : P N; and conſequently, P S is always = P N, which is the Condition on which the forecited Propoſition is grounded.

CONSECTARY II.

If any two of theſe Curves be given, *v. g.* D N F and A M B, the third Curve G R L may be found. Suppoſe A P = z, P N = u, P M = C Q = x, P T = t, and R Q = y. Then it is (by ſuppoſition) P M (x) : P N (u) :: P T (t) : Q R $= \frac{tu}{x} = y$. Now if by help of the Equations expreſſing the Nature of the reſpective Curves, we find the Values of t and u in x, and ſubſtitute them in $\frac{tu}{x} = y$. We ſhall have an Equation expreſſing the Relation of the Curve G R L to its Axis C B. For inſtance, If the Curves A D N F, and A M B be common Parabola's, then $az = xx$, and $z = \frac{xx}{a}$; and conſequently, $t =$ P T $= 2z = \frac{2xx}{a}$, in like manner $uu = az = xx$; therefore $x = u$. Whence this Equation $\frac{tu}{x} = y$, becomes $\frac{2xx}{a} = y$; and conſequently $\frac{1}{2}ay = xx$, from whence it appears that the Curve G R L is alſo a common Parabola Convex towards the Axis C B.

And if the Curve A D N F be a Common Parabola, and A M B a Cubical Parabola, then $aaz = x^3$, and $z = \frac{x^3}{aa}$, and P T $= 3z = \frac{3x^3}{aa} = t$. Likewiſe, $uu = az$ and $auu = aaz = x^3$; and conſequently $u = \sqrt{\frac{x^3}{a}}$. Whence the Equation $\frac{tu}{x} = y$, becomes $\frac{3x^2}{aa}\sqrt{\frac{x^3}{a}} = y$; and conſequently $\frac{9x^4}{a^4} \times \frac{x^3}{a} = yy$, and by Multiplication and Diviſion $\frac{1}{9}a^5yy = x^7$, which is an Equation expreſſing the Nature of the Curve G R L.

CONSECTARY III.

And becauſe the Curve A M B may be varied infinitely, it is plain that an infinite Number of Curves G R L may be deſcrib'd, each comprehending the Mixtilineal Space Q R G C = to a given Space A P N D.

CONSECT-

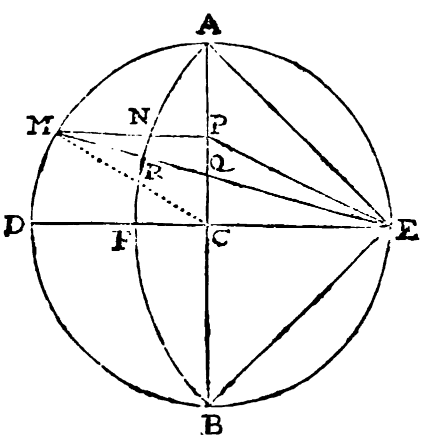
CONSECTARY IV.

If PM be $=$ PN, then PT is also $=$ QR, and the Trapezium MPpm $=$ Trapezium PNnp $=$ Trapezium QRrq; and because this Universally obtains, it follows that, if QR be always taken equal to the Sub-tangent PT, then the Mixtilineal Space QRGC will be equal to the Mixtilineal Space AMP, and the Area of the whole Figure ABC will be equal to the Area of the Figure BLGC, the points G and C Co-inciding.

PROP. XX.

To Investigate the Area of the Hypocrates's *Lunule. On the Diameter* AB *describe the Semi-circle* ADB, *and draw another Diameter* DE *at Right Angles to* AB. *Draw the Chords* AE *and* BE, *and on the Center* E *with the Radius* EA *describe the Arch* AFB. *Then the Figure comprehended between the Semiperiphery* ADB *and the Arch* AFB *is called* Hypocrates's *Lunule.*

135. Suppose the Diameter AB $= 2r$, then the Chord AE will be $= r\sqrt{2}$; and consequently, the Circle describ'd with the Radius AE is double that describ'd with the Radius CD (because Circles are as the Squares of their Semidiameters) and if we put c for the circumference of the Circle whose Radius is CD, the Area therefore will be $= \frac{cr}{2}$, and consequently, cr is $=$ Area of that whose Radius is AE. Now the Sector AFBEA being a Quadrant of a Circle, is equal to the Semicircle ADBA, and if from both the Segment AFBCA be taken away, there will remain the Lunule ABBFA $=$ Triangle AEB.

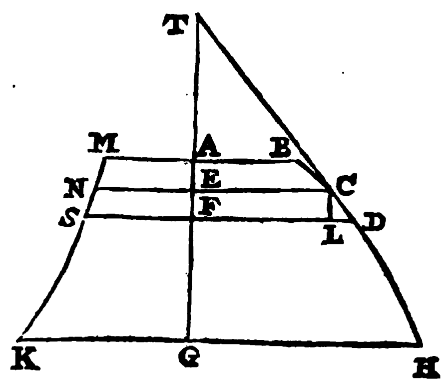


And thus not only the whole but also any part of the Lunule may be Squared, *v. g.* If the Ordinate MP, and the right Lines EP, EM be drawn, I say the Triangle EAP is $=$ the Portion of the Lunule AMRA. For the Angle AEM $= \frac{1}{2}$ ACM, therefore the Sector AER is $=$ to the Sector ACM, and the Triangles PQM, CQE are similar, therefore, MQ : QP :: QE : QC, and consequently the Triangles CMQ, PQE are (*Prop.* 15. *El.* 6.) equal. Whence the Space AQMA is $=$ Space AEPQRA, and Subtracting from both the Space AQRA which is common, there will remain ARMA $=$ Triangle AEP.

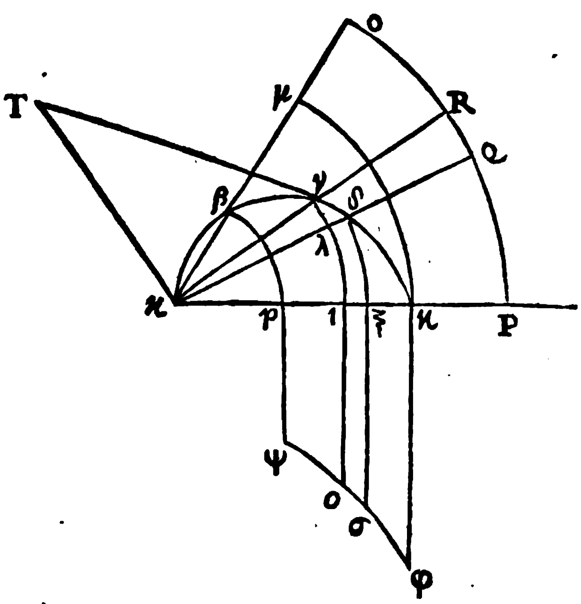
PROP. XXI.

Of the Properties of the Involute *and* Evolute (*Figures so call'd*).

136. Let the Space ABHG be divided into an infinite Number of Trapezia, and imagine the Portions of the Curve CD, and their Sines CL to be Flexible, like so many Threads, and the Ordinates AB, EC, FD, GH to be Rigid and Inflexible, then the Trapezia CEFD may be changed into the Trilineal Figures $x\gamma\delta$, *viz.* if the points E and F be suppos'd to Coincide, and if this be done in all the other Trapezia's, and if all the points of the Divisions in the Axis be suppos'd to (be contracted or) meet in G, there will be produced a new Figure $x\beta\varepsilon$, and the Point x will represent the point of Concourse, wherein all the points of the Axis A, E, F, G, *&c.* meet; and the Figure $x\beta\varepsilon$ is call'd the *Involuta* of the Figure ABHG, and this is call'd the *Evoluta* of that. Now the Properties of these Figures are,



1°. Because the Rectangle CLFE is suppos'd to be chang'd into the infinitely little Sector of a Circle $\gamma x \lambda$ this Sector is equal to half that Parallelogram the Angles at γ and λ being Right Angles, and $\lambda\gamma$ being $=$ CL, and if this be observ'd in all the rest, all the Rectangles CEFL, or the Figure ABHG is equal to twice the Sum of all the Triangles $x\gamma\lambda$ or the *Involuta* $x\beta\varepsilon$.



2°. Because by supposition CL $=$ $\gamma\lambda$, and CD $=$ $\gamma\delta$, and the Angles L and λ Right Angles; therefore the Triangles CLD and $\gamma\lambda\delta$ are similar and equal. Whence if we suppose the Angle T$x\gamma$ $=$ $\gamma\lambda\delta$, then the Triangles T$x\gamma$ and TEC will be (because γx $=$ EC) similar and equal.

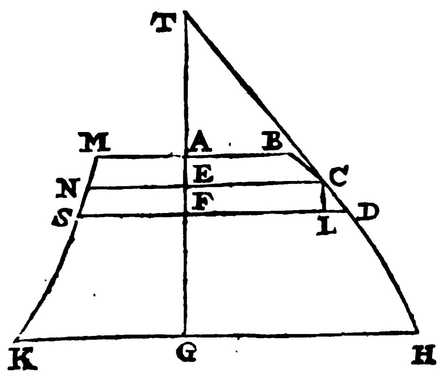
3°. The Arch $\beta\rho$ describ'd with the Radius $x\beta$ is less than the Axis AG, and the Arch $\varepsilon\mu$ describ'd with the Radius $x\varepsilon$ is greater than the said Axis AG, as is evident from the Genesis of these Figures.

Now there are two Problems, which will serve as a foundation for our Inquiries about the Properties and Uses of those Figures.

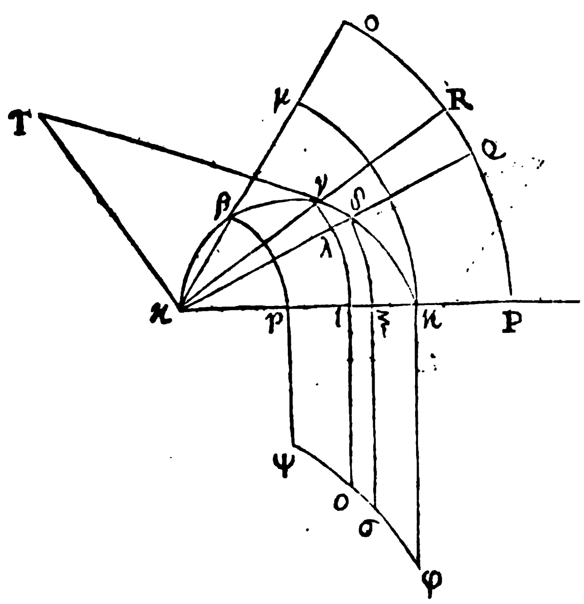
PROBLEM I.

The Involuta $x\beta n$ *being given; to find the* Evoluta ABHG.

Because from the Nature of the *Involuta* $x o$ is $= GH$, $x\delta = FD$, and $x\gamma = EC$, they are given. Now 'tis requir'd to find the length of the Axis EG correſponding to the Ordinate $EC = x\gamma$. The Inveſtigation may be thus:



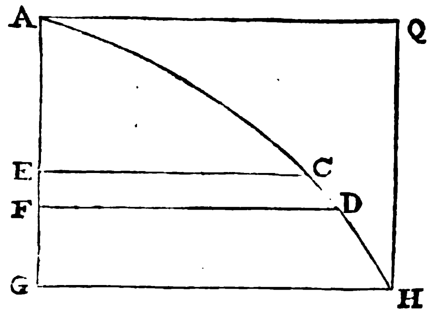
Suppoſe $Tx = TE = t$, $x\gamma = EC = y$, $\gamma\lambda = CL = \dot{x}$, $\lambda\delta = LD = \dot{y}$, $\gamma\delta = CD = \dot{z}$, $T\gamma = TC = s$. Then it is $Tx\ (t) : x\gamma\ (y) :: \gamma\lambda\ (\dot{x}) : \lambda\delta\ (\dot{y};)$ and conſequently $\dot{x} = \frac{t\dot{y}}{y}$, and $S\dot{x} = S\frac{t\dot{y}}{y} =$ all the $EF =$ Axis AG.



And becauſe y is Indeterminate, ſuppoſe $\frac{t\dot{y}}{y} = \frac{f\dot{y}}{r} = \dot{x}$. Then $f = \frac{rt}{y}$, which being always laid from ι to o, the Trapezium $\iota o \sigma \xi$ will be $= f\dot{y}$, and the Space $\iota o \varphi \varkappa$ will be $= Sf\dot{y}$, which Figure being Divided by r, there will ariſe $S\frac{f\dot{y}}{r} = S\dot{x} =$ the Portion of the Axis EG, to which in the point E apply the Ordinate $EC = x\gamma = x\,\iota$, and then the point C will be in the Evoluta. And in like manner all the reſt may be found, *v. g.* dividing the whole Figure $p\psi\varphi\varkappa$ by r, the Quotient will be $=$ AG the Axis of the *Evoluta*, to which if we apply the Ordinate AB in the point A, and make $AB = x\beta$, we ſhall have determin'd another point B in the *Evoluta*, *& ſic de cæteris*.

Hence if we reaſſume *Fig. pag.* 89 and the Symbols *Art.* 118. becauſe $t\dot{y} = y\dot{z}$, and conſequently $\dot{z} = \frac{t\dot{y}}{y}$, put the ſame $= \frac{f\dot{y}}{r}$, then in this caſe making $LS = f$, the Equation of the Curve, ASP is $rt = fy$, and Subſtituting (*Art.* 72.) $\frac{myx}{nr}$ for t, we have $mx = nf$, and advancing all the Terms to the Power n, there will ariſe $m^n x^n = f^n n^n$. And Subſtituting $\frac{c^n y^m}{r^m}$ for x^n, there will ariſe $\frac{m^n c^n y^m}{r^m} = n^n f^n$. And $m^n c^n y^m = r^m n^n f^n$, which is an Equation Expreſſing the Nature of a Paraboliform

raboliform Curve. Whence the Space A L S is (*Art.* 88.) found by saying $m+n$: n :: fy (= Rectangle A L × L S) : $\frac{nfy}{m+n}$ = $S f\dot{y}$, and consequently $S\frac{f\dot{y}}{r} = \frac{nf\dot{y}}{mr+nr} = S\,\dot{z}$ or the Length of the Axis A E, where E C = A M = y. And E F = $\dot{z}$. Now Suppose A E = b = S $\dot{z}$, then the Equation Expressing the Nature of the Curve A D H will be $\frac{nfy}{mr+mr}$

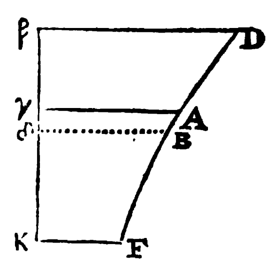


$= b$, or $nfy = \overline{m+n} \times rb$. And advancing all the Terms to the Power n, we have $n^n f^n y = \overline{m+n}^n r^n b^n$, and Substituting $\frac{m^n c^n y^m}{r^m n^n}$ for f^n, we shall have $m^n c^n y^{m+n} = \overline{m+n}^n b^n r^{m+n}$, which is an Equation Expressing the Nature of a Paraboliform Curve A C H, and therefore the greatest y = G H being $= r$, and the greatest b being = A G = $\left(\frac{nfy}{mr+nr}\right.$ = (putting $\frac{rt}{y}$ for f) $\frac{nt}{m+n}$ = (putting $\frac{myx}{nr}$ for t) $\frac{mx}{m+n}$ = (when x is $= c$) $\left.\frac{mc}{m+n}\right) = \frac{mc}{m+n}$. The Rectangle A Q G H is $= \frac{mrc}{m+n}$. And consequently the Mixtilineal Space A D H G is = (*Art.* 88.) $\frac{mrc}{m+2n}$. But this Space is the Evoluta of the Spiral Space B M K A B, therefore (*Art.* 136.) the Spiral Space is $= \frac{\frac{1}{2}mrc}{m+2n} = \frac{m}{2m+4n} \times rc$ which exactly agrees with that found (*Art.* 118.) above. I have the rather chosen to keep to the same Example, that the Reader may see how from so different Principles, the same Conclusion may be drawn.

PROBLEM II.

Another way to describe an Evoluta *to any* Involuta.

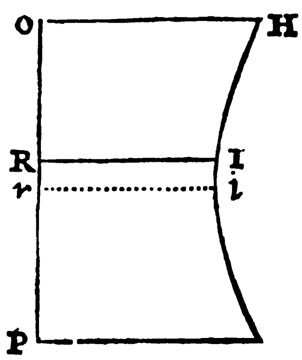
Tγ : $\gamma\delta$ :: Tx : $\gamma\lambda$, (*Fig.* 2 *in the forgoing page*) that is, $s : \dot{z} :: t : \dot{x}$, whence $\dot{x} = \frac{t\dot{z}}{s}$. Now suppose $\frac{t\dot{z}}{s} = \frac{b\dot{z}}{r}$, then will $tr = bs$. Therefore extending the Involuta into a straight Line $\beta\gamma\delta\kappa$; so that $\gamma\delta$ in both be $= \dot{z}$, apply γA $= b$, in the point γ, then A$\gamma\delta$B $= b\dot{z}$, and βDBδ is = S $b\dot{z}$, which Space being divided by r, there will arise S $\frac{b\dot{z}}{r}$ = S $\dot{x}$ = the length of the Axis (*Fig.* 1. *in the foregoing page*) A F, and making E C $= x\gamma$, the Space A B H G will be the *Evolutum* of the Space $x\beta\kappa$.



And from hence arises another Method of describing the *Evoluta* from the *Involuta* given.

With any Radius xP $= r$, describe the Arch PO, and let the infinitely little Arch QR Intercepted between $x\gamma$ and $x\delta$ be produc'd $= \dot{u}$, then because the Radij are proportional to their Arches, it is $x\gamma : \gamma\lambda :: x$R : RQ; that is, $y : \dot{x} :: r : \dot{u}$. Whence

Whence $\dot{x} = \frac{y\dot{u}}{r}$, place the Arch O R Q P in a streight Line, and apply the Line



R I $= x\gamma =$ the point γ, then is R I $\iota r = y\dot{u}$, and O $r\iota$ H is $=$ S $y\dot{u}$, which Space divided by r, will give S $\frac{y\dot{u}}{r} =$ S $\dot{x} =$ (*Fig.* 1. *page* 110.) the Axis of the *Evoluta* A E, and E C $= x\gamma =$ R I $= y$, and the Curve B D is $= \beta\delta$.

The *Evolutum* being given to find the *Involutum*. Suppose the *Evolutum* A B H G to be given, to describe the (*Fig.* 1 *and* 2. *page* 110) *Involutum* $x\beta u$. 'Tis evident, that to determine the point γ in this, answering to the point C in the *Evolutum*, the Line $x\gamma = x\iota$ is $=$ E C. But to determine the point γ we must also determine the Angle $\gamma x \iota$ or the Arch $\gamma\iota$ or P R; to do which, by the preceeding *Article* $y\dot{u} = r\dot{x}$, and $\frac{r\dot{x}}{y} = \dot{u}$. Now suppose $\frac{r\dot{x}}{y} = \frac{b\dot{x}}{r}$, then $\frac{rr}{y} = b$. Which being applied from E to N, the Trapezium N E F S $= b\dot{x}$ will be $= \frac{rr\dot{x}}{y} =$ twice the Sector of the Circle x R Q $= \frac{2r\dot{u}}{2}$; and therefore all the $b\dot{x}$ or the Space N E G K will be double the Sector x R R; and consequently, if this Space be divided by r, there will arise S $\frac{b\dot{x}}{r} =$ S $\dot{u} =$ Arch R P, which being found, the Angle R x P is also known. Whence if in the Radius x R we take $x\gamma =$ E C, we shall have the point γ in the *Involutum* Correspending to the point C in the *Evolutum*. *Et sic de Ceteris.*

S E C T.

SECT. V.

The Use of Fluxions

In the Resolution of the Questions de maximis & minimis.

THE Doctrine *de maximis & minimi s* has been applied only to those Problems, in which among an Infinite Number of Parts or Fluxions of a given Curve, the *greatest* or *least* is requir'd; but by the Industry of later Geometers it is now extended to those Problems, wherein among an Infinite Number of unknown Curves, *one is requir'd, cui maximum aliquod minimumve competat*. Which more Sublime Invention is no less useful than the Former

And because in this and the following Sections we shall have frequent use for *Second and Third Fluxions*, it will be necessary before we proceed further, to give the Reader a Notion of them.

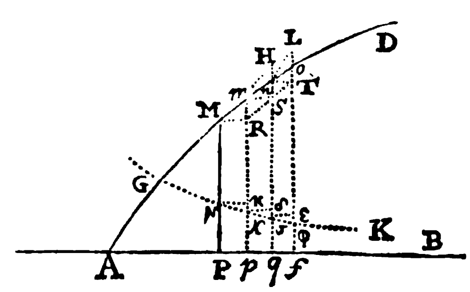
The Nature of Second, Third, Fourth &c. Fluxions.

137. I have already (*Art.* 2.) proved that Quantity is divisible in *Infinitum*, and that every such Infinitely little Particle is again Infinitely divisible; and finally, that there may be an Infinite Series of such Infinite divisions, every Term whereof will be Infinitely greater (or less) than the preceding.

I shall now apply that Doctrine more particularly to the Generation of Curves.

The Fluxions of Variable Quantities (*v. g.* the Fluxions of several Ordinates in the same Curve) are themselves Variable, because as the Terms Increase not uniformly, their Increments must needs be Unequal; and the Increments or Decrements whereby first Fluxions Increase or Decrease, are consider'd as their Fluxions: And this *Fluxion of a Fluxion* is what *Analists* Call *a Second Fluxion.*

Thus if *p m* be drawn Infinitely near P M, and M R parallel to A B, then R *m* is the (*Art.* 3.) Fluxion of the Ordinate P M, and if *p q* be = P *p*, and *q n* || *p m*, and S *m* || A B, and M H || R S, then H S = R *m*, and R *m* exceeds S *n* by H *n*, whence 'tis manifest that the Ordinates Increase unequally, and that H *n* is the *Decrement* or Fluxion of the Fluxion R *m*, that is H *n* is the *Second Fluxion* of the Ordinate P M.



In like manner if a fourth Ordinate *f o* be drawn Infinitely near *q n*, *n* T parallel to A B, and *n* L parallel to S T, then the difference between the Infinitely little lines H *n*, L *o* is the Fluxion of the Second Fluxion H *n*, or the *Third Fluxion* of the Ordinate P M.

138. And as I observ'd (*Art* 6.) before in First Fluxions; so here, Second, Third, *&c.* Fluxions, may always be express d by Finite streight Lines proportional to them or their Velocities.

Thus, if the Abscissa be suppos'd to Increase uniformly, and if *p r*, *q σ*, *f φ* be taken proportional to R *m*, S *n*, T *o &c.* And if the Curve G *μ γ σ φ* K be described, then the Ordinates of this new Curve will be proportional to the respective Fluxions of the Ordinates, of the given Curve A M *m*. And if *μ κ*, *γ δ*, *σ ε* be drawn parallel to A B, and a third Curve described, having its Ordinates proportional to *γ κ*, *σ δ*, *φ ε* the First Fluxions of the Second Curve, then the said Ordinates of the Third Curve, will be proportional to the First Fluxions of the Ordinates of the Second Curve, or to the Second Fluxions of the Ordinates of the given Curve. And in like manner a Fourth Curve may be describ'd having its Ordinates Proportional to the First Fluxions of the Ordinates of the Third Curve, or to the Third Fluxions of the Ordinates of the First given Curve, *&c.*

But if it happen that the Ordinates Increaſe uniformly, then the Curve A M *m* will be a ſtraight Line, and then there will be no *Second Fluxions*; and if the Ordinates be as the Squares of the Intercepted Diameters (as in the common Parabola Convex towards the Axis) then the Second Curve will be a ſtraight Line and the Firſt Curve or Parabola will have no Third *Fluxions*; and if the Ordinates be as the Cubes of their Intercepted Diameters (as in the Cubical Parabola) then the Third Curve will be a ſtraight Line and the Firſt Curve will have no Fourth Fluxions, *&c.*

That in the Conick Parabola there are no Third Fluxions, and in the ſaid Cubick Parabola no fourth Fluxions, *&c.* may be prov'd, if,

In the Conick Parabola,

The Intercepted Diameters be	1,	2,	3,	4,	5,	6,	7,	8,	&c.
Ordinates	1,	4,	9,	16,	25,	36,	49,	64,	&c.
Firſt Fluxions		3,	5,	7,	9,	11,	13,	15,	&c.
Second Fluxions			2,	2,	2,	2,	2,	2,	&c.
Third Fluxions				0,	0,	0,	0,	0,	&c.

In the Cubick Parabola,

The Intercepted Diameters be	1,	2,	3,	4,	5,	6,	7,	&c.
Ordinates	1,	8,	27,	64,	125,	216,	343,	&c.
Firſt Fluxions		7,	19,	37,	61,	91,	127,	&c.
Second Fluxions			12,	18,	24,	30,	36,	&c.
Third Fluxions				6,	6,	6,	6,	&c.
Fourth Fluxions					0,	0,	0,	&c.

139. As Firſt Fluxions have been noted with one Prick over the Variable or Flowing Quantity, ſo *Second, Third, Fourth, &c. Fluxions* are noted with two, three, four, *&c.* Pricks over the Flowing Quantity. Thus if P M be $= y$, then R *m* $= \dot{y}$; H *n* $= \ddot{y}$, and L *o* — H *n* or H *n* — L *o* $= \dddot{y}$.

The Powers of Second, Third, *&c.* Fluxions are noted in the ſame Manner as in common Notation, thus the Square of $\dot{y}$ is $\dot{y}^2$; the Cube of $\dot{y}$ is $\dot{y}^3$; the Square of $\ddot{y}$ is $\ddot{y}^2$; the Cubes of $\ddot{y}$ or $\dddot{y}$ is $\ddot{y}^3$, $\dddot{y}^3$, reſpectively, *&c.*

And if the Intercepted Diameters A P, A *p*, A *q*, A *f* be put equal to *x*, and the reſpective Ordinates P M, *p m*, *q n*, *f o*, be put $= y$, and the Portions of the Curve A M, A *n*, A *m*, A *o*, $= z$. Then 'tis evident that $\dot{x}$ will denote the Fluxions of the Abſciſſa, P *p*, *p q*, *q f*; and $\dot{y}$ will repreſent the Fluxions of the Ordinates R *m*, S *n*, T *o*; and $\dot{z}$ thoſe of the Curve, *viz*, M *m*, *m n*, *n o*; as has been Intimated (*Art.* 9.) above.

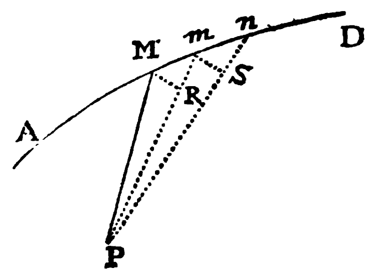
140. But to find, for Inſtance, H *n* the Second Fluxion of P M, we muſt Imagine in the Axis two Infinitely little Parts P *p*, *p q*, and in the Curve other two, *v. g.* M *m*, *m n*, in order to find the Fluxions R *m*, S *n*; and if we ſuppoſe the Infinitely little Parts P *p*, *p q*, *q f* to be equal, then its plain that $\dot{x}$ the Fluxion of the Intercepted Diameter is an Invariable Quantity in reſpect of $\dot{y}$ and $\dot{z}$; becauſe, when P *p* comes to be *p q*, it is ſtill the ſame, while R *m* which comes then to be S *n*, and M *m* which becomes *m n*, vary,

And if we ſuppoſe the Infinitely little Parts of the Curve, M *m*, *m n*, *n o*, to be equal, then $\dot{z}$ will be (*Fig. in Page.* 112.) an Invariable Quantity in reſpect of $\dot{x}$ and $\dot{y}$, or if we ſuppoſe R *m*, S *n*, T *o* to be equal, then $\dot{y}$ is an Invariable Quantity in reſpect of $\dot{z}$ and $\dot{x}$.

In

In like manner to find the Third Fluxion of P M or the Fluxion of the Second Fluxion H *n*, Imagine Three little Portions of the Absciſſa, P *p*, *p q*, *q f*; and in the Curve A M D other Three M *m*, *m n*, *n o*; and Three others in the Ordinates, *viz.* R *m*, S *n*, *o* T; then $\dot{x}$, $\dot{y}$ or $\dot{z}$ will be Invariable according as you put P *p*, *p q*, *q f*; R *m*, S *n*, T *o*; or M *m*, *m n*, *n o*, equal between themſelves. The Method is the ſame for Fourth, Fifth, *&c.* Fluxions.

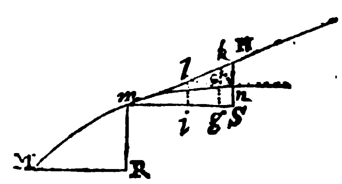
141. The like is to be underſtood of the Curve A M D, where all the Ordinates P M, P *m*, P *n*, meet in the Firſt Point P; for to find the Second Fluxion of P M, Imagine two Ordinates P *m*, P *n* to be drawn, making the Angles M P *m*, *m* P *n* Infinitely little. On the Center P with the Radius P M, P *m*, deſcribe the Arches M R, *m* S, then R *m* is the Firſt Fluxion of P M, and the difference between R *m* and S *n* is the Second Fluxion of P M; and we may ſuppoſe the little Arches M R, *m* S, or the Arches M *m*, *m n*, or laſtly R *m* and S *n* to be equal to each other Reſpectively. In the Firſt Caſe $\dot{x}$ will be Invariable in reſpect of $\dot{y}$ and $\dot{z}$; in the Second $\dot{z}$ will be Invariable in reſpect of $\dot{y}$ and $\dot{x}$, and the third $\dot{y}$ will be Invaaiable in reſpect of $\dot{x}$ and $\dot{z}$. And in this manner we may find the Third, and Fourth, *&c.* Fluxions of P M.

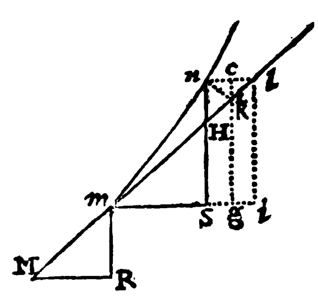


142. And here it may be obſerv'd that there are Degrees of Infinitely little Parts. Thus R *m*, for Inſtance, is Infinitely little, in reſpect of P M, and Infinitely great, in reſpect of H *n*; and the Space P M *p m*, is Infinitely little, in reſpect of the Space A P M, and Infinitely great, in reſpect of the Triangle M R *m*.

And it is alſo remarkable, that the whole Fluxion P *f* (that is ſeveral Fluxions added into one) is Infinitely little, in reſpect of A P. For every Quantity which is made up of a Finite Number of Infinitely little Parts, ſuch as P *p*, *p q*, *q f* (in reſpect of another Quantity as P A) is Infinitely little, in reſpect of the the ſame Quantity A P. Becauſe, before a Quantity Compoſed of Infinitely little Parts, can be of the ſame order with given Qnantities, the Number of their Infinitely little Parts muſt be Infinite.

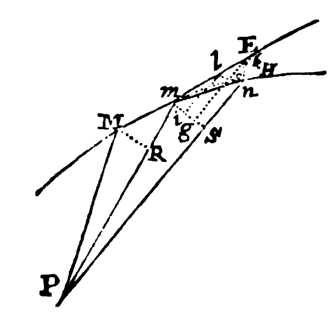
143. In Curves, wherein the Ordinates *m* R *n* S are Parallel between themſelves, produce the right Line M *m* until it Interſect S *n* in H. And on the Center M with the Radius *m n* deſcribe the little Arch *n k*, and draw the little right Lines *n l*; *l i*, *k c g*, parallels to *m* S and S *n*; this being done, if we ſuppoſe $\dot{x}$ Invariable, that is M R = *m* S; it is evident that the Triangles M R *m* and *m* S H are equal and ſimilar; and conſequently, that H *n* = *m* R — *n* S or *n* S — *m* R = $\ddot{y}$, and H *k* = $\ddot{z}$. But if we ſuppoſe $\dot{z}$ to be Invariable, that is to ſay, that M *m* = *m n* or *m k*, then 'tis evident that the Triangles M R *m*, *m g k* are equal and ſimilar. And conſequently, that *k c* = $\ddot{y}$, and S *g* = $\ddot{x}$; and laſtly, if we ſuppoſe $\dot{y}$ to be Invariable, that is, R *m* = S *n*, then the Triangles *m i l*, M R *m* will be equal and Similar, and *i* S = *n l* = $\ddot{x}$, and *l k* = $\ddot{z}$

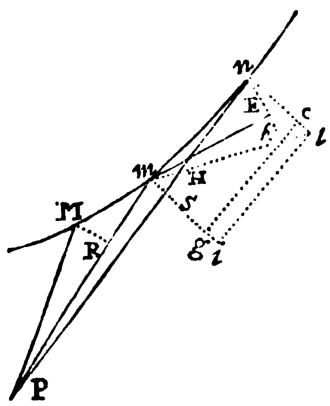




144. In

144. In Curves whose Ordinates P M, P m, P n concur in the same point P. On the Center P with the Distances P M, P m, describe the little Arches M R, m S, which we consider as Infinitely little streight Lines Perpendicular to P m and P n. On the Center M with the Radius $m\,n$ describe the little Arch $n\,k$ E, and make the Angle E m H $=$ m P n; and draw the little right lines $n\,l$, $l\,i$, $c\,k\,g$ parallels to m S and S n; then because P S m is a right Angle, the Angles P m S $+$ m P n (or E m H) that is the Angle P M E $-$ S m H is a right Angle, and the Angle P m E is an External Angle in respect of the Triangle M R m, therefore P m E $=$ M R m $+$ R M m, and Subtracting from both the right Angles M R m and Pm S $+$ E m H, we shall have R M m $=$ S m H. Hence it will follow.





1°. That if $\dot x$ be supposed Invariable, that is, if M R be $=$ m S, the Triangles S m H, R M m will be equal and similar. And then H n $=$ $\ddot y$, and H k $=$ $\ddot z$.

2°. If $\dot z$ be supposed Invariable. the Triangles $g\,m\,k$, R M m, will be similar and equal; and consequently $k\,c$ $=$ $\ddot y$, and Sg or $c\,n$ $=$ $\ddot x$.

Lastly, if $\dot y$ be supposed Invariable, then the Triangles $i\,m\,l$, R M m will be equal and similar, and consequently i S $=$ $l\,n$ $=$ $\ddot x$, and $l\,k$ $=$ $\ddot z$.

PROP. I.

To find the Fluxion of a Quantity Compos'd of Fluxions.

145. Suppose (*Art.* 143.) any one Fluxion in the given Quantity to be Invariable and all the rest Indeterminate, then find the Fluxions of all the Indeterminate Quantities, as if they were finite Variable Quantities, by the Rules deliver'd in the Second *Section*.

EXAMPLE I.

Let it be requir'd to find the Fluxion of $\frac{y\dot y}{\dot x}$. If $\dot x$ be suppos'd Invariable, then the Fluxion thereof is $\frac{\dot y^2 + y\ddot y}{\dot x}$. If $\dot y$ be Invariable, then the Fluxion of $\frac{y\dot y}{\dot x}$ is $\frac{\dot y^2\dot x - y\dot y\ddot x}{\dot x^2}$.

EXAMPLE II.

Let it be requir'd to find the Fluxion of $\frac{z\sqrt{\dot x^2+\dot y^2}}{\dot x}$. If $\dot x$ be suppos'd Invariable, then the Fluxion of the given Quantity is $\dot z \times \sqrt{\dot x^2+\dot y^2} + z\times\frac{1}{2}\,\overline{\dot x^2+\dot y^2}|^{-\frac{1}{2}}\times 2\dot y\ddot y$ divided by $\dot x = \dot z\times\sqrt{\dot x^2+\dot y^2} + \frac{z\dot y\ddot y}{\sqrt{\dot x^2+\dot y^2}}$ divided by $\dot x =$

$\dot{x} =$ (reducing the Fluxion to a Fraction of one Denomination) $\frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}$ = to the Fluxion of the given Fraction.

The OPERATION.

The Fluxion of z is $= \dot{z}$.

The Fluxion of $\sqrt{\dot{x}^2+\dot{y}^2}$ is $= \frac{1}{2} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}-1} \times 2\dot{y}\ddot{y} = \overline{\dot{x}^2+\dot{y}}|^{-\frac{1}{2}} \times \dot{y}\ddot{y} = \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$.

Therefore the Fluxion of the Numerator $z\sqrt{\dot{x}^2+\dot{y}^2}$ is $= \dot{z}\sqrt{\dot{x}^2+\dot{y}^2} + z \times \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$.

And the Fluxion of the Denominator $\dot{x}$ is $= 0$.

And the Fluxion of the Numerator Multiplied by the Denominator is $\dot{z}\sqrt{\dot{x}^2+\dot{y}^2} + z \times \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \times \dot{x}$.

And the Fluxion of the Denominator Multiplied by the Numerator is $\ddot{x} \times z\sqrt{\dot{x}^2+\dot{y}^2}$.

The Square of the Denominator is $= \dot{x}^2$.

Therefore the Fluxion of the Fraction $\frac{z\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$ is $= \frac{\dot{z}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} + z\dot{x} \times \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} - \ddot{x} \times z\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}^4}$

And because $\dot{x}$ is suppos'd invariable, therefore $\ddot{x}$ is $= 0$.

And the Term $-\ddot{x} \times z\sqrt{\dot{x}^2+\dot{y}^2}$ in the Numerator is also $= 0$.

Whence the Fluxion of $\frac{z\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$ is $= \frac{\dot{z}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2} + z\dot{x} \times \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}}{\dot{x}^2}$

$= \frac{\dot{z}\sqrt{\dot{x}^2+\dot{y}^2} + z \times \frac{\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}}{\dot{x}} = \frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}$. Q. E. I.

But if $\dot{y}$ (in the same Fraction) be suppos'd the Invariable Quantity, then the Fluxion thereof is $\dot{z} \times \sqrt{\dot{x}^2+\dot{y}^2} + z \times \frac{1}{2}\overline{\dot{x}^2+\dot{y}^2}|^{-\frac{1}{2}} \times 2\dot{x}\ddot{x} \times \dot{x} - \ddot{x} \times z\sqrt{\dot{x}^2+\dot{y}^2}$, the whole being divided by $\dot{x}^2$ the Square of the Denominator; equal to $\dot{x}\dot{z}\sqrt{\dot{x}^2+\dot{y}^2} + \frac{z\dot{x}^2\ddot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} - z\ddot{x}\sqrt{\dot{x}^2+\dot{y}^2}$ divided by $\dot{x}^2$; equal to $\frac{\dot z \dot x^3 + \dot z \dot x \dot y^2 + z \dot x^2 \ddot x - z \dot x^2 \ddot x - z \dot y^2 \ddot x}{\dot x^2 \sqrt{\dot x^2 + \dot y^2}} = \frac{\dot z \dot x^3 + \dot z \dot x \dot y^2 - z \dot y^2 \ddot x}{\dot x^2 \sqrt{\dot x^2 + \dot y^2}} =$ to the Fluxion of the given Fraction.

The OPERATION.

It is requir'd to find the Fluxion of $\frac{z\sqrt{\dot x^2 + \dot y^2}}{\dot x}$, supposing $\dot y$ Invariable.

The Fluxion of z is $= \dot z$.

The Fluxion of $\sqrt{\dot x^2 + \dot y^2}$ is $\Big\} = \frac{1}{2} \times \overline{\dot x^2 + \dot y^2}|^{\frac{1}{2}-1} \times 2\dot x \ddot x = \overline{\dot x^2 + \dot y^2}|^{-\frac{1}{2}} \times \dot x \ddot x = \frac{\dot x \ddot x}{\sqrt{\dot x^2 + \dot y^2}}$.

Therefore the Fluxion of the Numerator $z\sqrt{\dot x^2 + \dot y^2}$ is $\Big\} = z \times \frac{\dot x \ddot x}{\sqrt{\dot x^2 + \dot y^2}} + \dot z \sqrt{\dot x^2 + \dot y^2}$.

And the Fluxion of the Denominator $\dot x$ is $\Big\} = \ddot x$

The Fluxion of the Numerator Multiplied by the Denominator is $\Big\} = z\dot x \times \frac{\dot x \ddot x}{\sqrt{\dot x^2 + \dot y^2}} + \dot z \dot x \sqrt{\dot x^2 + \dot y^2}$.

The Fluxion of the Denominator Multiplied by the Numerator is $\Big\} = z\ddot x \sqrt{\dot x^2 + \dot y^2}$.

The Square of the Denominator is $\Big\} = \dot x^2$.

Therefore the Fluxion of $\frac{z\sqrt{\dot x^2 + \dot y^2}}{\dot x}$ is $= \Big\}$ $\frac{\frac{z\dot x^2 \ddot x}{\sqrt{\dot x^2 + \dot y^2}} + \dot z \dot x \sqrt{\dot x^2 + \dot y^2} - z\ddot x \sqrt{\dot x^2 + \dot y^2}}{\dot x^2}$

$$= \frac{\frac{z\dot x^2 \ddot x + \dot z \dot x \times \overline{\dot x^2 + \dot y^2} - z\ddot x \times \overline{\dot x^2 + \dot y^2}}{\sqrt{\dot x^2 + \dot y^2}}}{\dot x^2}$$

$$= \frac{z\dot x^2 \ddot x + \dot z \dot x^3 + \dot z \dot x \dot y^2 - z\dot x^2 \ddot x - z\dot y^2 \ddot x}{\dot x^2 \sqrt{\dot x^2 + \dot y^2}}$$

$$= \frac{\dot z \dot x^3 + \dot z \dot x \dot y^2 - z\dot y^2 \ddot x}{\dot x^2 \sqrt{\dot x^2 + \dot y^2}}.\quad \text{Q.E.I.}$$

EXAMPLE

EXAMPLE III.

The Fluxion of $\frac{y\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ is $=$ (supposing $\dot{x}$ invariable) $\overline{\dot{y}^2+y\ddot{y}}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ $-y\dot{y}\times\frac{1}{2}\overline{\dot{x}^2+\dot{y}^2}|^{-\frac{1}{2}}\times 2\dot{y}\ddot{y}$ divided by $\dot{x}^2+\dot{y}^2$; or $\overline{\dot{y}^2+y\ddot{y}}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}-$ $\frac{y\dot{y}^2\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ Divided by $\dot{x}^2+\dot{y}^2$, or $\frac{\dot{x}^2\dot{y}^2+\dot{y}^4+y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$. Q. E. I.

EXAMPLE IV.

The Fluxion of $\frac{\overline{\dot{x}^2+\dot{y}^2}\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$ or $\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ is (supposing $\dot{x}$ Invariable) $=\frac{3}{2}\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}\times-2\dot{x}\dot{y}\ddot{y}^2+\dot{x}\dddot{y}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}$ divided by $\dot{x}^2\ddot{y}^2$ the Square of the Denominator or $\frac{-3\dot{x}\dot{y}\ddot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}+\dot{x}\dddot{y}\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}=$ $\frac{-3\dot{y}\ddot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}+\dddot{y}\sqrt{\dot{x}^2+\dot{y}^2}|^3}{\dot{x}\ddot{y}^2}$.

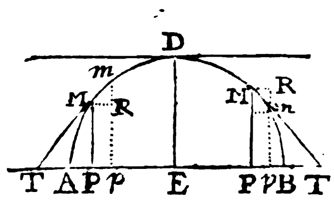
COROLLARY.

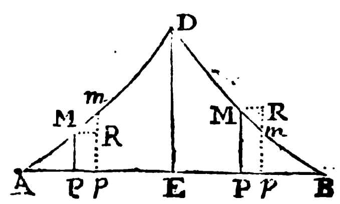
146. Hence 'tis manifest that any Fluxion in a given Quantity, cannot always be taken for the Invariable Quantity. *Ex.gr.* In the fourth Example, $\dot{y}$ cannot be taken for an Invariable Quantity, because in that case its Fluion $\ddot{y}$ would be equal to (*Art.* 3.) nothing. And the Denominator $\dot{x}\ddot{y}$ would be $=$ o. And $\ddot{y}$ could then have no place in the given quantity. And for the same reason, $\dot{z}$ or $\dot{x}$ cannot be put for an Invariable Quantity when their Fluxions $\ddot{z}$, $\ddot{x}$ are Ingredients in the proposed Quantity.

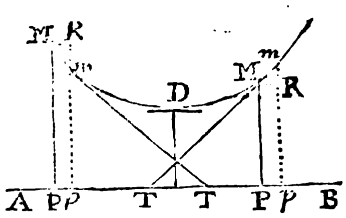
DEFI-

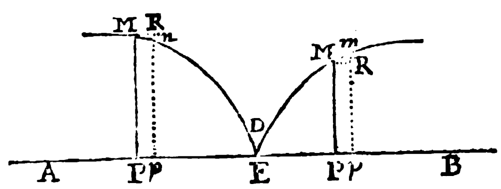
DEFINITION I.

Let the Curve M D M be given, and its Ordinates P M, E D, P M parallel between themselves. And suppose that as the Abscissa A P continually Increases, so the Ordinate P M Increases also until it come to a certain point E, in which point it is the greatest that can be applied within the given Curve, and afterwards that as it passes from E towards B it Decreases. Or on the contrary, if the Ordinate P M Decrease as the Abscissa A P Increases, until it arrive at a certain point E, (in which point it is the least Ordinate that can be applied to the given Curve) and afterwards Increase as it passes from E to B. Then the line E D is called the *greatest or least Ordinate.*









DEFINITION II.

If a Quantity such as P M componnded of one or more variable Quantities as A P, be proposed, which Increases continually as A P Increases, until it come to a certain point E, after which it Decreases, or the contrary. And if it be requir'd to find, instead of A P, the Value A E, so that E D which is composed of the same, be *greatest* or the *least* of all such like Quantities P M composed of A P, then that is called a Question *de maximis vel minimis.*

These things being premis'd, I shall endeavour in the next place, to consider the Progress of an Ordinate P M *from* A *to* B, *and to explain the various Affections of the Fluxions thereof as it moves along, and the Relation between the Fluxions of the Abscissa and them, in all the different Cases that commonly happen; and this I think, being well understood, will be sufficient to enable the Reader to resolve any Question of this Nature.*

147. When A P Increases, and also P M, then 'tis evident that the Fluxions R *m* and P *p* will be both Positive. And contrarily, if P M Decrease while A P Increases, the Fluxion R *m* will be Negative.

And it is manifest that as the Fluxions R *m*, R *m* are Positive, the Ordinates being between A and E; and Negative, the Ordinates being between E and B, so the Fluxions of the Ordinate R *m* Decrease continually from A to D, and in D the Fluxion R *m* vanishes, or is = o, and afterwards from D to B it is Negative and Increases. and thus we can easily conceive, that a Quantity which Decreases continually connot pass from being Positive to be Negative, without passing by nothing, or o.

And if we Imagine the Ordinate P M to move along the Line A B from A to B, then 'tis manifest that the Fluxion R *m* Increases continually, until the Ordinate P M become E D, after which it becomes Negative and continually Decreases. And the Fluxion R *m* from being Positive, passes by Infinity, to become Negative.

And

And to assist the imagination in this Case, (because it seems hard to conceive that a Quantity which Increases continually should pass by Infinity.) Suppose Tangents drawn to the points (*Fig.* 1. *pag.* 120.) M, D, M. Then it is evident that the Tangent in D is parallel to the Axis A B, and that the Subtangent Increases continually as the points P and M approach nearer and nearer to the points E and D, and when the point M falls in D, then the Subtangent P T will be Infinite, and when A P exceeds A E, that is when the points P and M pass to the contrary side of E D in respect of A, then the Subtangent begins to Decrease and becomes Negative. *Et e contra.*

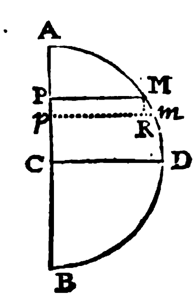
And to instance more particularly in the present Case: The *Ratio* of the Ordinate to the Sub-tangent is always as (*Fig.* 2 *and* 4. *pag.* 120.) m R is to M R: But when A P becomes = A E, the Ordinate is E D, and the Subtangent vanishes, therefore in that point, the *Ratio* of the Ordinate to the Subtangent, that is the *Ratio* of m R to M R is Infinite, and as m R passes by Infinity (in the point D,) from being Positive it becomes Negative. Or if m R be Negative, it always Increases until A P become = A E, and then it is Infinite, (in respect of the Fluxion of the Axis) afterwards it becomes Positive, and then continually Decreases.

148. *It being plain then, that a Quantity which continually Increases or Decreases, cannot pass from being Positive to be Negative; But it must pass by Infinity or Nothing,* viz. *it must pass by* 0 *when it continually Decreases, and it must pass by Infinity, when it continually Increases, before of a Negative it can become Positive, or of a Positive a Negative Quantity. This I say, being now manifest, it follows that the Fluxion of that Quantity which expresses the greatest or least* Ordinate *ought to be equal to* Nothing *or* Infinity, *and the Nature of the Curve* M D M *being given, if we find the Value of* R m, *and put the same equal to* Nothing *or* Infinity, *it will serve to discover the Value of* A E *in either of these* Suppositions. *This will appear more at large in the following Propositions.*

PROP. II.

If the Curve A M m B *be a Semi-circle,* A P = x, *the Ordinate* P M = y, *it is requir'd to find the greatest Ordinate* C D.

149. Suppose A C = a, then the Equation expressing the Relation of the Abscissa A P to the Ordinate P M is $2ax - xx = yy$, and reducing the same to Fluxions, we have $a\dot{x} - x\dot{x} = y\dot{y}$, and dividing by y, we have $\frac{a\dot{x} - x\dot{x}}{y}$ = $\dot{y}$ = (*Art.* 148.) (when P M co-incides with C D, which we suppose to be the greatest Ordinate) = 0, and multiplying both sides of the Equation by y, there will arise $a\dot{x} - x\dot{x} = 0$, and (by Transposition) $a\dot{x} = x\dot{x}$; and $a = x$. Whence it is evident that when x or A P becomes = a or A C, then C D is the greatest of all the similar Ordinates that can be applied to the same Diameter A B.



And if the Curve A M m be an Ellipsis, the Equation expressing the Nature thereof is (*Art.* 32.) $\frac{ayy}{b} = ax - xx$ (a being = A B) which being reduc'd to Fluxions, we have $\frac{2ay\dot{y}}{b} = a\dot{x} - 2x\dot{x} = 0$, because $\dot{y}$ is = 0; and consequently, the Term $\frac{2ay\dot{y}}{b}$ is = 0. Therefore $\frac{1}{2}a = x$; that is, the Conjugate Diameter is always the greatest Ordinate that can be applied to the same Diameter.

150. Let

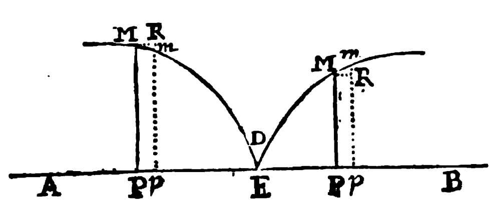
150. Let the Equation expressing the Relation between the Abscissa $AP = x$, and the Ordinate PM be $aax^3 - x^5 + bbccx = y^5$; 'tis requir'd to find the Value of x, when the Ordinate y is the greatest. If the Equation be reduc'd to Fluxions, there will arise $3aax^2\dot{x} - 5x^4\dot{x} + bbcc\dot{x} = 5y^4\dot{y} =$ (because $\dot{y} =$ (*Art.* 148.) o; and consequently, $5y^4\dot{y} =$) o, and (dividing by $\dot{x}$) $3aaxx - 5x^4 + bbcc = 0$. Whence it is easie to find the Value of x or AP, when the Ordinate PM is the greatest.

151. And if the Equation expressing the Nature of the Curve be $x^3 + y^3 = axy$, then $3x^2\dot{x} + 3y^2\dot{y} = ax\dot{y} + a\dot{x}y$; and consequently, $\dot{y} = \frac{ay\dot{x} - 3xx\dot{x}}{3yy - ax}$ = (*Art.* 148.) o. Whence $ay = 3xx$, and $y = \frac{3xx}{a}$, and substituting this Value of y in the Equation of the Curve, we have $x^3 + \frac{27x^6}{a^3} = 3x^3$, and $\frac{27x^3}{a^3} = 2$, and $\frac{3x}{a} = \sqrt[3]{2}$; that is, $x = \frac{1}{3}a\sqrt[3]{2} = AC$ the Intercepted Diameter, so that CD (drawn Parallel to PM) will be the greatest of all the Ordinates PM.

PROP. III.

If the Property of the Curve MDM *be given, it is requir'd to find* ED *the least Ordinate applied to the given Axis* AB.

152. Suppose $AP = x$, $PM = y$, a a determinate Quantity, and let the Equation expressing the Nature of the Curve MDM be $y - a = a^{\frac{1}{3}} \times \overline{a - x}|^{\frac{2}{3}}$, then $\dot{y} = -\frac{2}{3}a^{\frac{1}{3}} \times \overline{a - x}|^{-\frac{1}{3}} \times \dot{x} = \frac{-2\dot{x}\sqrt[3]{a}}{3\sqrt[3]{a - x}}$.

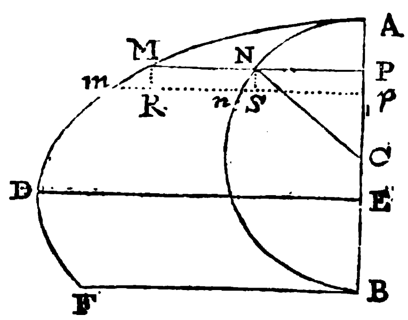


Now if we consider the Nature of the Curve MDM it will appear that $\dot{y}$ Increases as the Ordinate PM Decreases, and that in the point E, $\dot{y}$ is (*Art.* 147.) infinite in respect of $\dot{x}$, therefore I put $\frac{-2\dot{x}\sqrt[3]{a}}{3\sqrt[3]{a - x}}$ = *Infinity*. Whence 'tis evident that to make that Fraction Infinitely great, its Denominator must be Infinitely little, or nothing; therefore $3\sqrt[3]{a - x} = 0$, and $\sqrt[3]{a - x} = 0$, and $x = a$, which is the Value of AE sought.

PROP. IV.

If AMF *be a contracted Semi-cycloid, whose Base* BF *is less than the Semi-circumference of the Circle* ANB, *and whose Center is* C. *It is requir'd to find the point* E *in the Diameter* AB; *so that* ED *shall be the greatest Ordinate that can be applied to the Axis* AB.

153. Draw the Ordinate PM at pleasure, Intersecting the Semi-circle in N, and make the little Triangles MRm, NSn. Suppose AP $= x$, PN $= y$, the Arch AN $= z$, the Semi-diameter AC $= a$, the Semi-circumference ANB $= c$, BF $= b$; then (by the property of the Cycloid) it is ANB (c) : BF (b) :: AN (z) : NM $= \frac{bz}{c}$; and consequently, PM $= y + \frac{bz}{c}$, and the Fluxion thereof Rm $= \dot{y} + \frac{b\dot{z}}{c} =$ (*Art.* 148.) o, when the P falls on E the point requir'd. But the Triangles NSn, NPC are similar; therefore



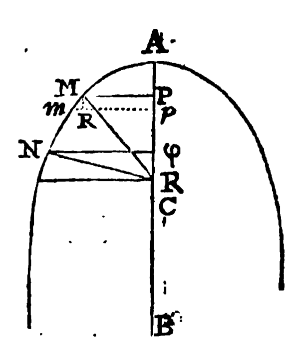
CN (a) : CP ($a - x$) :: Nn ($\dot{z}$) : Sn $= \frac{a\dot{z} - x\dot{z}}{a} = \dot{y}$; and consequently, $\dot{y} + \frac{b\dot{z}}{c} = o = c\dot{y} + b\dot{z} = ac\dot{z} - cx\dot{z} + ab\dot{z}$. Whence $ac + ab = cx$, and $\frac{ac + ab}{c} = x = a + \frac{ab}{c} =$ AE.

Whence it is manifest that if CE be taken (from C towards B) a fourth proportional to the Circumference ANB (c,) the Base BF (b) and the Radius CB (a) then E will be the point in the Axis AB requir'd.

PROP. V.

If the Nature of the Curve AMm, *the Position of the Axis* AP, *and a Determinate Point* R *in the same be given: 'Tis requir'd to find the point* N *in the Curve, so that the Right Line* RN *be the shortest Line which can be drawn from the same point* R *to the given Curve.*

154. Suppose AP $= x$, PM $= y$, AR $= c$, and PR $= c - x$; then it is evident, that PMq + PRq = RMq = $cc - 2cx + xx + yy$: It is likewise manifest, that as the point M approaches the point (N) requir'd, so the Line RM; and consequently its Fluxion Decreases, and in the point N the Fluxion of RN is $=$ o; therefore the Fluxion of the Square of RN must be $=$ o, that is $x\dot{x} - c\dot{x} + y\dot{y} =$ o, and if by help of the Equation of the Curve, we find the Value of $y\dot{y}$ in x and $\dot{x}$, and substitute the same in this Equation, it will serve to find the point φ in the Axis, in which if the Ordinate φN be applied, then N will be the point in the Curve requir'd, and RN will be the shortest Line that can be drawn from the given point R to the Curve AMR.



EXAMPLE

EXAMPLE I.

155. If the Curve AM*m* be the Common Parabola, then let the Parameter be $= b$, and the Equation Expressing the Nature of the Curve will be $bx = yy$. And reducing the same to Fluxions, we have $\frac{1}{2}b\dot{x} = y\dot{y}$. Whence we shall have $x\dot{x} - c\dot{x} + y\dot{y} = 0 = 2x\dot{x} - 2c\dot{x} + b\dot{x} = 0$. And by Transposition, and division $x = c - \frac{b}{2}$. That is Aφ = AR $- \frac{1}{2}b$; whence it is plain, that if Rφ be taken $= \frac{1}{2}$ the Parameter of the Parabola, and the Ordinate φN drawn, then RN will be the shortest Line requir'd.

COROLLARY.

If RA be $= \frac{1}{2}b$, then a Circle described on the Center R with the Radius RA, will touch the Parabola in the vertex A, and be altogether within the same. For in that case, N and A coincide, and RN or RA will be the shortest Line that can be drawn from the given point R to the Curve.

And in the present Example, its evident that a Circle describ'd on the Center R *with the Radius* RN *will touch the Parabola in* N *and in another point opposite to the same on the other side of the Axis* AP.

EXAMPLE II.

156. If the Curve AM*m* be an Ellipsis (or Hyperbola) then if the Parameter be $= b$, and the Transverse Axis AB $= 2a$; the Equation expressing the Nature of such Curves is $2ayy = 2abx \mp bxx$; which being reduced to Fluxions, there will arise $2ay\dot{y} = ab\dot{x} \mp bx\dot{x}$, and consequently $y\dot{y} = \frac{ab\dot{x} \mp bx\dot{x}}{2a}$. Therefore the general Equation $x\dot{x} - c\dot{x} + y\dot{y} = 0$, is $= 2ax\dot{x} - 2ac\dot{x} + ab\dot{x} \mp bx\dot{x} = 0$. And by Transposition and division $2ax + ab = 2ac \mp bx$, or $2ax \mp bx = 2ac - ab$. Whence $x = \frac{2ac - ab}{2a \mp b}$; and consequently R$\varphi = c - x = \frac{ab \mp cb}{2a \mp b}$ Whence arises this,

CONSTRUCTION.

Take Rφ a Fourth proportional to the Transverse Axis, AB its Parameter, and Cφ (the distance of φ from the Center of the Section) for $2a \mp b : b :: a \mp c :$ Rφ And (by Composition in the Ellipsis and Division in the Hyperbola) $2a : b :: a \mp c \pm$ Rφ : Rφ. That is, as the Transverse Diameter AB, is to the Parameter :: Cφ : Rφ.

CORROLARY.

And if AR (c) be suppos'd $= \frac{1}{2}b$, then R$\varphi = \frac{ab \mp cb}{2a \mp b}$ will become $\frac{2ab \mp bb}{4a \mp 2b} = \frac{1}{2}b$. Therefore R$\varphi$ = RA, and consequently, the points N and A Coincide, and a Circle describ'd on the Center R, with the Radius RA or RN, will touch the Hyperbola or Ellipsis in the Vertex A, and be altogether within them.

PROP.

PROP. VI.

To divide the Right Line A B, *in the point* E, *so that the Product of the Square of one of its Parts* A E, *multiplied into the other Part* E B, *be the greatest of all the products made by the Square of any one part of the Line* A B, *multiplyed into the other part.*

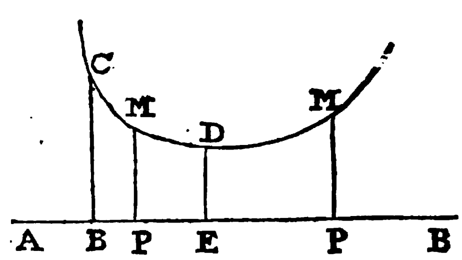
157. Suppose the unknown Quantity $AE = x$, and the given Line $AB = a$, then $AE^2 \times EB = axx - x^3$, which is requir'd to be the greatest of all the Rectangles, made by the Square of any one Part of the given Line, multiplyed into the other Part. Imagine a Curve Line M D M such, that the Relation of the Ordinate P M (y) to the Intercepted Diameter A P (x) may be expres'd by this Equation $y = \frac{axx - x^3}{aa}$ And let it be requir'd to find the Point E, whose respective Ordinate E D is the greatest that can be drawn within the Curve, and then $\dot{y} = \frac{2ax\dot{x} - 3x^2\dot{x}}{aa} = 0$; and $x = \frac{2}{3}a = AE$.

And Universally:

If it be requir'd that $x^m \times \overline{a-x}|^n$ be the *Greatest* of all such Rectangles or products (the Indices m and n representing what Numbers you please) then the Fluxion of the Rectangle is either $= 0$ or *Infinity*; $m x^{m-1}\dot{x} \times \overline{a-x}^n - n x^m \times \overline{a-x}|^{n-1} \dot{x} = 0$, and (dividing first by $x^{m-1}\dot{x}$, and then by $\overline{a-x}|^{n-1}$) $m \times \overline{a-x} - n x = 0$, that is $am = mx + nx$, and consequently $x = \frac{m}{m+n} a = AE$.

If m be $= 2$ and $n = -1$, then A E will be $= 2a$, and then the Problem is expres'd thus.

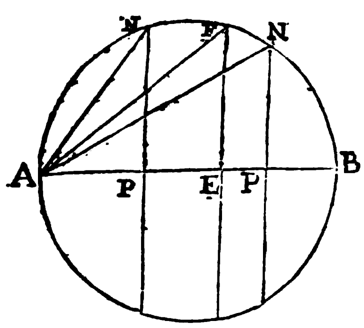
Produce the Line A B, on the side B to E, so that the Quantity $\frac{AE^2}{BE}$ be the *Least* that is possible: For then the Equation of the Curve will be $\frac{xx}{a-x} = y$, wherein, if we suppose $x = a$, then the Ordinate P M, which becomes $= BC = \frac{aa}{0}$, that is, it is Infinite, and supposing x Infinite, we shall have $y = x$, that is, the Ordinate will be Infinite also.



PROP. VII.

If an infinite Number of Cones be inscrib'd in a given Sphere; 'tis requir'd to find that which has the greatest Convex Surface.

158. The Question amounts to no more but this, to determine the point E, in the Diameter AB of the Circle AFB; so that the Rectangle comprehended under AF and the perpendicular FE be the greatest of all the like Rectangles comprehended under AN and NP. For if we imagine the Semi-circle AFB to Revolve about its Axis AB, it is evident that the Semi-circle describes a Sphere and the Rectangular Triangles AFE, ANP describe Cones Inscrib'd in the same Sphere, whose Surfaces are proportional to the respective Rectangles AF × FE, AN × NP.



Suppose the unknown Quantity AE $= x$, and AB (the Diameter of the Sphere) $= 2a$; then by the property of the Circle, AF $= \sqrt[2]{xx + 2ax - xx} = \sqrt[2]{2ax}$, and EF $= \sqrt[2]{2ax - xx}$ and AF × FE $= \sqrt[2]{4aaxx - 2axxx}$, and because this Rectangle is requir'd to be the Greatest, therefore the Fluxion thereof must be equal to 0; that is $\frac{1}{2}\,\overline{4aax^2 - 2ax^3}\big|^{-\frac{1}{2}} \times 8aax\dot{x} - 6ax^2\dot{x} = \frac{4aax\dot{x} - 3ax^2\dot{x}}{\sqrt{4aax^2 - 2ax^3}} = 0$, therefore $4a = 3x$, and $x = \frac{4}{3}a$.

PROP. VIII.

If among an Infinite Number of Parallelepipedons, each be equal to a given Cube $= a^3$; 'tis requir'd to find that which has the least Superficies, one of its Sides being $= b$.

159. Suppose x to be one of the Sides of the Parallelepipedon requir'd then the third Side $= \frac{a^3}{bx}$. And taking the Rectangles under the three Sides b, $\frac{a^3}{bx}$, x, Alternately, their Sum, *viz.* $bx + \frac{a^3}{x} + \frac{a^3}{b}$ is $= \frac{1}{2}$ the least Superficies of the Parallelepipedon sought.

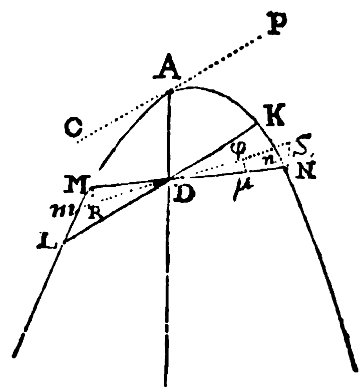
And the Fluxion thereof, *viz.* $b\dot{x} - \frac{a^3\dot{x}}{xx} = 0 = bxx\dot{x} - a^3\dot{x}$; and consequently $x = \sqrt{\frac{a^3}{b}}$. So that the three Sides of the Parallelepipedon which answer the demands of the Question are b, $\sqrt{\frac{a^3}{b}}$, and $\sqrt{\frac{a^3}{b}}$, and the two unknown Sides are now discover'd to be equal between themselves.

PROP.

PROP. IX.

Any Curve Line M A N, *and a determinate point within the same, as* D, *being given; 'tis requir'd to draw the Line* L D K, *which (of all other Lines drawn through* D) *shall cut off the least Segment* L A K.

160. Draw the Line M D N (at pleasure) through the given point D, and draw *m* D *n* Infinitely near the same; on the Center D, with the Radius D M, describe the Arches M R, $\varphi\mu$. And on the same Center with the Radius D N describe the Arch N S; then the Sectors D M R and D $\varphi\mu$ are equal, and the Sector D N S or the Decrement of the Area D N *n*, exceeds the Sector D M R, or the Increment of the Area D M *m* by the Space φ *n* N μ. Whence it is manifest that if M D N be suppos'd to move on the Center D, from N towards K, then the Decrement of the Area, will exceed the Increment, and consequently, the Space A *m n*, will be less than A M N; and when D M = D N, that is when N comes to K, and M to L, then the Decrement of the Area D N *n* will be equal to the Increment D N *m*, that is, the Absolute Fluxion of the Segment L A K will be = 0, and the Segment L A K will be the least that can be cut off by a Line passing through the given point D.



CONSECTARY.

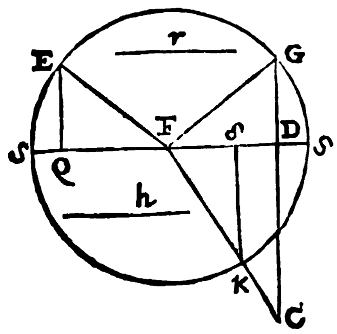
161. Hence in the Parabola, Hyperbola, and Ellipsis, if it be requir'd, to draw the Line L D K through the given point D, to cut of the least Segment L A K; through the given point D, draw the Diameter A D, and O A P, touching the Section in A, then draw L D K parallel to O A P: For A D, is a Diameter (by supposition) and the Line L D K is an Ordinate to the same; and consequently, is Bissected in D, therefore the Segment L A K is the (*Art.* 160) least that can be cut off by a Line passing through through the given point D.

And if the Curve L A K be an Ellipse, then the other (Lower) Segment cut off by the same Line, L D K will be the greatest Segment that can be cut off by a Line, passing through the given point D.

PROP. X.

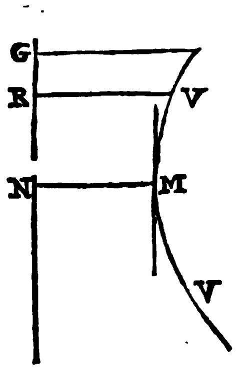
Let the two points E *and* C, *and the Right Line* S S *in the same Plain with them, be given; and let it be requir'd to find the point* F *in the Plain* S S, *so that a Body moving from* E *to* F, *with the given Velocity r, and from* F *to* C *with the given Velocity b, shall move from* E *by* F *to* C *in the* Shortest Time.

162. Let F be the point requir'd, and on the same as a Center with the Radius F E, describe the Circle E S *x* S, and on S S let fall the Perpendiculars E Q, C D, *x* ς. Then suppose the given Quantities $EQ = a$, $QD = b$, $CD = c$, $QF = x$, and then $FD = b - x$; $EF = \sqrt{aa + xx}$, and $FC = \sqrt{cc + bb - 2bx + xx}$.



Now it is manifest, that if a Body move from E to F with the Velocity *r*, and from F to C with the Velocity *b*; if those Velocities are equal, then the times will be as the Spaces describ'd E F and F C, and if the Spaces E F, F C be equal and the Velocities unequal, the times will be reciprocally as the Velocities, therefore if the Spaces and Velocities be both unequal, then the times will be in a Ratio Compounded of the direct Ratio of the Spaces and the Reciprocal Ratio of the Velocities; that is, the time which the Body takes to move from E to F will be represented by $\frac{EF}{r}$, and that from F to C by $\frac{FC}{b}$ and the sum of both $\frac{EF}{r} + \frac{FC}{b}$ or $b \times EF + r \times FC$ will represent the shortest times in which a body can move from E by F to C with the respective Velocities.

Now if we suppose the times which a Body takes to move from E to C with the Velocities *r*, and *b*, to be represented by the Ordinates R V, perpendicular to the right Line G N, then the least Ordinate N M will represent the Time, which the same Body takes to move from E by F to C.



And if we suppose $EF = \sqrt{aa + xx} =$ (for Brevities sake) $= \sqrt{m}$. And $FC = \sqrt{cc + bb - 2bx + xx} =$ (for the foresaid Reason) $= \sqrt{n}$, then we shall have this Equation $b\sqrt{m} + r\sqrt{n} = RV = y$. And reducing the same to Fluxions, we have $\frac{-b\dot{m}}{2\sqrt{m}} - \frac{r\dot{n}}{2\sqrt{n}} = \dot{y} = 0$. But $\dot{m} = 2x\dot{x}$, and $\dot{n} = 2x\dot{x} - 2b\dot{x}$, therefore $\frac{-b\dot{m}}{2\sqrt{m}} - \frac{r\dot{n}}{2\sqrt{n}} = 0$, is $= -\frac{2bx\dot{x}}{2\sqrt{m}} - \frac{2rx\dot{x} + 2rb\dot{x}}{2\sqrt{n}}$. And by Transposition, and Division $\frac{rb - rx}{FC} = \frac{bx}{FE}$. Whence it is an easy matter to find the Value of *x* or Q F.

CONSECTARY I.

163. In *Dioptricks*, if we suppose $FC = FE$ (which we may do, because the Refraction in the point F is the same, be the Line F C longer or shorter) then is $rb - rx = bx$. And consequently $r : b :: x : b - x :: QF : FD$. That is, the Sines of

of the Angle of Incidence and the refracted Angle F Q and F D, are directly as the Velocities r and b.

And if S E S be *a Medium of Air*, and S x S a *Medium of Water*; r the Velocity of a Particle of Light, moving from E to F, and b the Velocity of the same, moving from F to C, then because the Velocity of the said Particle in different Mediums is Reciprocally Proportional to the Densities of the said Mediums, it follows that the Sines of the Angle of Incidence and the Refracted Angle are Reciprocally Proportional to the Densities of the Mediums.

CONSECTARY II.

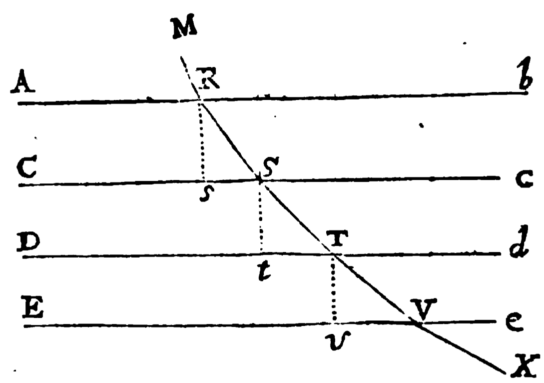
If a Radiant point E in Air, and another point C in Water be given; to find the point F, in the Surface S Q D S (dividing the Air and Water) say $r + b : b :: b : b - x ::$ Q F + F D (or Q D) : F D.

CONSECTARY III.

And in *Catoptricks*, we prove, that the Angles of Incidence and Reflection are equal: For if a Body, moving from E to F be Reflected to G, it is still in the same Medium and $r = b$, therefore F E $\times$ $b - x$ = F G $\times$ x; that is, F E : F Q :: F G : F D. Now the Sides being Proportional, and the Angles Q and D being Right Angles, it follows that the Triangles E F Q, G F D are similar, and consequently the Angles E F Q, G F D are equal; and because $r = b$, therefore $b - x = x$, and Q F = D F; that is the Sine of the Angle of Incidence is equal to the Sine of the Angle of Reflection.

CONSECTARY IV.

If Mediums of different Densities be Included between the parallel Plains A b, C c, D d, E e, and if the Densities of the Mediums decrease in any assigned Proportion, then the Ray (M R) of Incidence will be refracted from the Perpendicular, and the Particle will move from M (or R) to X (or V) in the Lines M R, R S, S T, T V, V X in the shortest time possible, and if R S, S T, T V be supposed equal, then the Sines of the refracted Angles S R s, T S t, V T v, *viz.* S s, T t, V v will be proportional to the facility, or rarity, of the respective Mediums Included between the parallel Plains, A b, C c, D d, E e. And if we suppose an Infinite Number of such Plains (between A b, and E e) parting different Mediums, whose Density decreases in any given Proportion, then the Number of equal and Infinitely little Lines R S, S T, T V will be Infinite also, and will compose a Curve, which the Particle (*v. g. of Light*) moving from R to V in the shortest Time, will describe.



And to Investigate the Property of the Curve R S T V, which a Particle of Light describes, moving from R to V in the shortest Time, through the Medium A b E e; whose Density decreases in any given Proportion, it must be observ'd that as the Density of the Medium decreases, its Rarity Increases, and that the Velocity of the Particle is Proportional to the Rarity of the Medium it moves in; that is, the Velocity of the Particle of Light as it describes the equal right Lines R S, S T, T V, is proportional to the Sines of the refracted Angles, *viz* to S s, T t, V v, &c. respectively: And that from hence it plainly appears, that the Velocity of the Particle of Light describing the Infinitely little, and equal Portions of the Curve, is always Proportional to the *Elementum* of the respective Ordinate of the said Curve.

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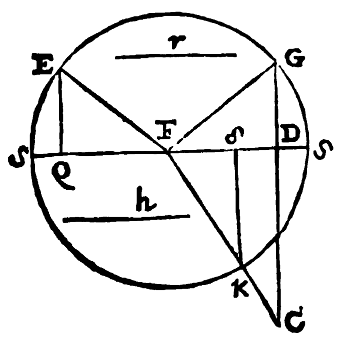

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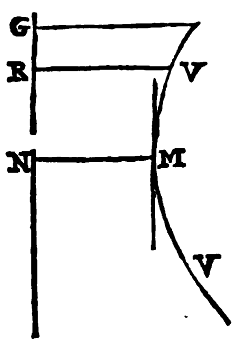

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of the Angle of Incidence and the refracted Angle F Q and F D, are directly as the Velocities r and b.

And if S E S be *a Medium of Air*, and S x S a *Medium of Water*; r the Velocity of a Particle of Light, moving from E to F, and b the Velocity of the same, moving from F to C, then because the Velocity of the said Particle in different Mediums is Reciprocally Proportional to the Densities of the said Mediums, it follows that the Sines of the Angle of Incidence and the Refracted Angle are Reciprocally Proportional to the Densities of the Mediums.

CONSECTARY II.

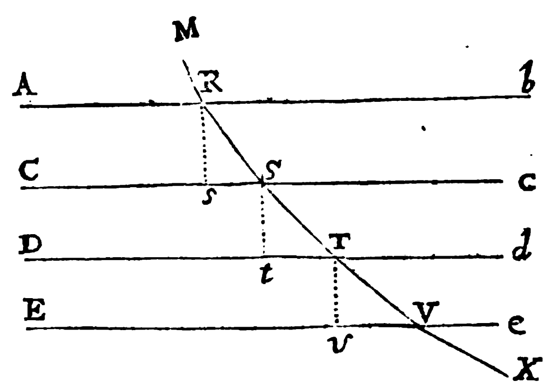
If a Radiant point E in Air, and another point C in Water be given; to find the point F, in the Surface S Q D S (dividing the Air and Water) say $r+b:b::b:b-x::$ QF + F D (or Q D) : F D.

CONSECTARY III.

And in *Catoptricks*, we prove, that the Angles of Incidence and Reflection are equal: For if a Body, moving from E to F be Reflected to G, it is still in the same Medium and $r=b$, therefore F E $\times\, b-x=$ F G $\times\, x$; that is, F E : F Q :: F G : F D. Now the Sides being Proportional, and the Angles Q and D being Right Angles, it follows that the Triangles E F Q, G F D are similar, and consequently the Angles E F Q, G F D are equal; and because $r=b$, therefore $b-x=x$, and Q F = D F; that is the Sine of the Angle of Incidence is equal to the Sine of the Angle of Reflection.

CONSECTARY IV.

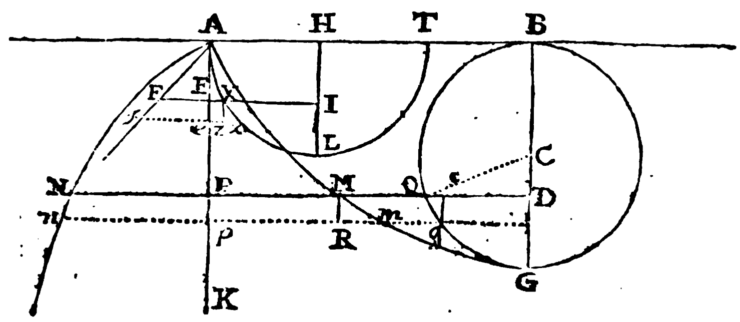
If Mediums of different Densities be Included between the parallel Plains A b, C c, D d, E e, and if the Densities of the Mediums decrease in any assigned Proportion, then the Ray (M R) of Incidence will be refracted from the Perpendicular, and the Particle will move from M (or R) to X (or V) in the Lines M R, R S, S T, T V, V X in the shortest time possible, and if R S, S T, T V be supposed equal, then the Sines of the refracted Angles S R s, T S t, V T v, *viz.* S s, T t, V v will be proportional to the facility, or rarity, of the respective Mediums Included between the parallel Plains, A b, C c, D d, E e. And if we suppose an Infinite Number of such Plains (between A b, and E e) parting different Mediums, whose Density decreases in any given Proportion, then the Number of equal and Infinitely little Lines R S, S T, T V will be Infinite also, and will compose a Curve, which the Particle (*v. g. of Light*) moving from R to V in the shortest Time, will describe.



And to Investigate the Property of the Curve R S T V, which a Particle of Light describes, moving from R to V in the shortest Time, through the Medium A b E e; whose Density decreases in any given Proportion, it must be observ'd that as the Density of the Medium decreases, its Rarity Increases, and that the Velocity of the Particle is Proportional to the Rarity of the Medium it moves in; that is, the Velocity of the Particle of Light as it describes the equal right Lines R S, S T, T V, is proportional to the Sines of the refracted Angles, *viz.* to S s, T t, V v, *&c.* respectively: And that from hence it plainly appears, that the Velocity of the Particle of Light describing the Infinitely little, and equal Portions of the Curve, is always Proportional to the *Elementum* of the respective Ordinate of the said Curve.

These things being premis'd, it remains only to find the Property of the Curve, which the Particle of Light describes, in any Hypothesis of Acceleration of Velocity; or to Trace the Path of the Particle of Light, in any Medium, whose Rarity Increases in any given Proportion.

Let A B K be a Medium Terminated by the Horizontal Plain A B, In which suppose A to be the Radiant Point. Draw A K Perpendicular to the Plain A B, and describe the Curve A N *n*, so that the Ordinate P N will always represent the Rarity of the Medium at the respective Depth A P, or the Velocity of the Particle of Light in its path in the Point M, and let A M G, be the Curve which the said Particle



describes. Suppose A P $= x$, P N $= z$, P M $= y$, P p $=$ M R $= \dot{x}$, R $m = \dot{y}$, M m $= \dot{u}$, and any Invariable Quantity assum'd at pleasure $= a$; then if M m be made Radius, the Fluxion of the Ordinate R m will be the Sine of the refracted Angle, which is Proportional to the Velocity of the Particle of Light in the point M, and because P N is Proportional to the Velocity of the Particle of Light in the point M, it follows that the Ratio between P N and R m is constant and Invariable; that is, $\frac{\dot{y}}{z} = \frac{\dot{u}}{a}$ (the Element of the Curve $\dot{u}$ being Invariable) whence $a\dot{y} = z\dot{u}$, or $aa\dot{y}^2 = zz\dot{u}^2 =$ (because M R m is a Rectangular Triangle) $zz\dot{x}^2 + zz\dot{y}^2$, and by Transposition $aa\dot{y}^2 - zz\dot{y}^2 = zz\dot{x}^2$, and by Division and equal Extraction, $\dot{y} = \frac{z\dot{x}}{\sqrt{aa - zz}}$.

And by help of this Equation the Nature of the Curve A M G *may be found in all imaginable Hypotheses.*

For Instance; if the Rarity of the Medium A B K, be in a Sub-duplicate Ratio of the Depth; that is, if the Rarity of the Medium *m* P or M, or the Velocity of the Particle of Light in M, or the Exponent of the said Velocity R *m*, or P N be as the Square Root of the Absciss A P, then the Curve A N *n* will be a common Parabola, and $ax = zz$, and $z = \sqrt{ax}$, and substituting these values in the general Equation $\dot{y} = \frac{z\dot{x}}{\sqrt{aa - zz}}$, we have $\dot{y} = \dot{x}\sqrt{\frac{ax}{aa - ax}} = \dot{x}\sqrt{\frac{x}{a - x}}$. Whence I conclude that the Curve A M G is the vulgar Cycloid; for if B G the Diameter of the generating Circle be $= a$, and if the Semi-cycloid A M G be described, the Fluxion of the Ordinate R m (*Art.* 108.) is $=$ Q q + Q S $= \frac{-a\dot{x} + 2x\dot{x}}{2\sqrt{ax - xx}} + \frac{a\dot{x}}{2\sqrt{ax - xx}}$ $= \frac{x\dot{x}}{\sqrt{ax - xx}} = \dot{x}\sqrt{\frac{x}{a - x}}$. Whence it follows that the Fluxions of both Curves being equal, the Curves must needs be one and the same.

We

We may also prove that the Curve A M G is a Cycloid, in this manner *a Priori*:

$$\dot{x}\sqrt{\frac{x}{a-x}} = \frac{x\dot{x}}{\sqrt{ax-xx}} = \frac{-a\dot{x}+2x\dot{x}}{2\sqrt{ax-xx}} + \frac{a\dot{x}}{2\sqrt{ax-xx}}.$$

Now the Flowing Quantity of $\frac{a\dot{x}-2x\dot{x}}{2\sqrt{ax-xx}}$ (supposing the said Fluxion Positive which was really Negative) is $\sqrt{ax-xx}$ or D Q, and the Flowing Quantity of $\frac{a\dot{x}}{2\sqrt{ax-xx}}$ is = Arch B Q, and therefore the Flowing Quantity of $\dot{y} = \dot{x}\sqrt{\frac{x}{a-x}}$ or P M is = B Q — D Q, therefore M D is = (assuming P D or A B = to the Semi-periphery B Q G) P D — B Q + D Q = B Q G — B Q + D Q, and consequently, G Q is = M D — D Q, and G Q = M Q. Which (*Art.* 70.) proves that the Curve A M G, is the vulgar Cycloid.

SCHOLIUM.

The Excellent Geometer M. *Jo. Bernouilli*, Professor of Mathematicks at *Groeningen*, propos'd to all the Mathematicians in *Europe*, to find the Curve, in which a heavy Body, descending by the force of its own Gravity, should move from a given Point, to another Point also given, in the shortest Time: And the same Excellent Person afterwards Demonstrated the Identity of that Curve with this, which a Particle of Light describes, in Mediums not uniform; the Rarity of the Medium affecting the same in this, that the Acceleration of Velocity does in that: For if a heavy Body descend from A, in the Curve A M, and if the Velocity be in a Sub-duplicate Ratio of the Altitude, and if a Particle of Light issuing from the Radiant point A, pass through a Medium A B K, whose Rarity Increases in a Sub-duplicate Ratio, of the Altitude or Depth, then the Velocity of the Particle of Light will be in a Sub-duplicate Ratio of the Altitude or Depth. Whence it appears, that the Velocity being the same, whither it be produc'd by the uniform Action of Gravity, or from the Rarity of the Medium, the Curve describ'd must be the same in either Supposition.

CONSECTARY V.

Make the Angle E A F = 45° 00′, and let the Ordinates E F, *&c.* represent the Rarity of the Medium at the depth A E, then in this Hypothesis, the Rarity of the Medium is proportional to the depth. Suppose A E = x, E F = z, E X = y and A X L the Curve which the particle of Light issuing from the Radiant point A describes Then $x = z$, and $xx = zz$, whence the general Equation $\dot{y} = \frac{z\dot{x}}{\sqrt{aa-zz}}$ will become $\dot{y} = \frac{x\dot{x}}{\sqrt{aa-xx}}$. Whence I conclude, that the Curve A X L is the Arch of a Circle: For if A T be taken $= 2a$, and the Semi-circle A L T be describ'd on the Diameter A T, and if H be the Center, and H L perpendicular to A T; then H I is $= x$, and I X $= \sqrt{aa-xx}$, and the Fluxion thereof zx is $= \frac{1}{2}\overline{aa-xx}|^{-\frac{1}{2}}\, 2x\dot{x} = \frac{x\dot{x}}{\sqrt{aa-xx}}$, which is the same with that found before; and consequently, a Particle of Light moving in a Medium, whose Rarity increases proportionally to the Altitude or Depth A E, or a Body descending with an uniformly Accelerated Velocity (or a Velocity proportional to the perpendicular Spaces it describes) will move in the Arch of a Circle from a given point to a given point, in the shortest Time.

CONSECT-

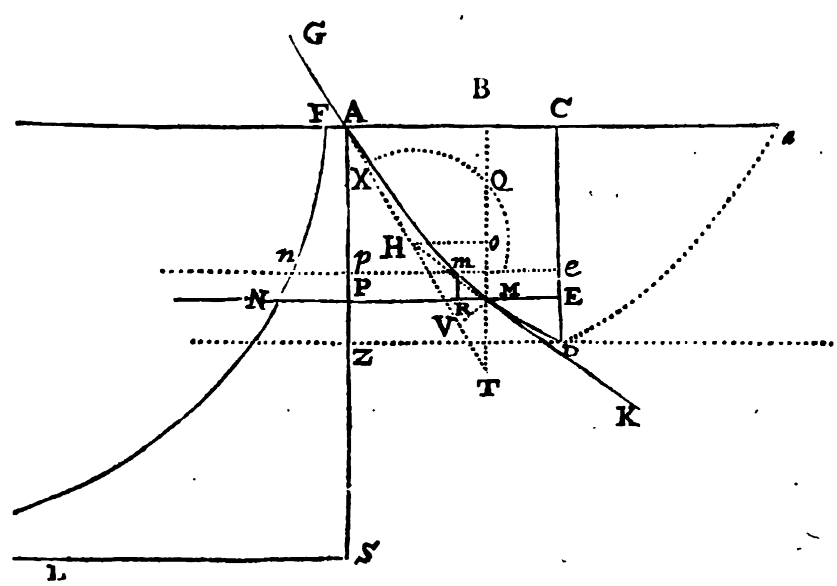
CONSECTARY VI.

In the first Hypothesis, the Rarity of the Medium being in a Sub-duplicate Ratio of the Altitudes, if A P or x be $= a$, then $zz = aa$, and $z = a$, and $\dot{y} = \frac{z\dot{x}}{\sqrt{aa - zz}}$ $= \frac{a\dot{x}}{\sqrt{aa - aa}} = \frac{a\dot{x}}{0}$. Whence at that Depth, R m the Sine of the Refracted Angle is Infinite in respect of M R, that is the Infinitely little Portion of the Curve M m will be parallel to R m, and consequently, the Particle of Light cannot descend lower in such a Medium, but in that point it will be Reflected towards the Horizontal Plain A B.

And in the second Hypothesis, the Rarity of the Medium being in a direct Ratio of the Altitudes, if A E $= x$ be $= a$, then $z = a$, and $zz = aa$, and $\dot{y} = \frac{a\dot{x}}{\sqrt{aa - aa}}$ $= \frac{a\dot{x}}{0}$. That is $z\,x$ the Sine of the Refracted Angle is Infinite in respect of X z, and consequently, the Particle of Light at that Depth (or being in the lowest point of the Circle L) will be reflected towards the Horizontal Plain A B.

CONSECTARY VII.

If A S be perpendicular to the Plain A C, and S L perpendicular to A S, and if the Curve F n N be describ'd within the Asymptotes S A, S L; so that (supposing A S $= 2a$, A P $= x$, and P N $= z$) the Nature thereof be Express'd by this Equation $\overline{2a - x} \times zz = a^3$. Then to find the Nature of the Curve of Refraction A M D,



the Ordinates P N representing the Rarity of the Medium, in the respective parallel Plains N P E, by the property of the Curve F n N, $zz = \frac{a^3}{2a - x}$ and $z = \sqrt{\frac{a^3}{2a - x}}$. Whence the general Equation $\dot{y} = \frac{z\dot{x}}{\sqrt{aa - zz}}$ will become $\dot{y} =$

$\sqrt{a^3}$

$\frac{\sqrt{a^3}\,\dot{x}^2}{\sqrt{2a-x}\sqrt{aa-\frac{a^3}{2a-x}}} = \frac{\sqrt{a^3}\,\dot{x}^2}{\sqrt{2a^3-aax-a^3}} = \frac{a\dot{x}}{\sqrt{aa-ax}}$. From whence I conclude that the Curve of Refraction A M D is a Parabola; For if A C be taken $= a$, and if the Parabola A M D be describ'd to the Focus C, then will A C be = C D, and the Parameter of the principal Axis C D will be $= 4a$, and if A P be = x, then is D E $= a - x$, and consequently E M $= \sqrt{4aa-4ax}$: Whence the Fluxion of the Ordinate E M, *viz.* R M is $= \frac{1}{2} \times \overline{4aa-4ax}|^{-\frac{1}{2}} \times 4a\dot{x} = \frac{a\dot{x}}{\sqrt{aa-xx}}$, which being the same with that formerly found, proves that the Curve A M D, which the Particle of Light describes, is the Curve of a Parabola: And if x be suppos'd $= a$, then the Equation of the Curve F N, *viz.* $\overline{2a-x} \times zz = a^3$ becomes $azz = a^3$, and consequently $z = a$, whence $\dot{y} = \frac{z\dot{x}}{\sqrt{aa-zz}}$ will be $= \frac{z\dot{x}}{0}$; that is, if A S be Bissected in Z, and the Plain Z D suppos'd to be drawn parallel to the Plain A B, then when the Particle of Light arrives in D, $\dot{y}$ will be infinite in respect of $\dot{x}$; and consequently, the Particle not being able to penetrate deeper into the Fluid, will be Reflected in D, and describe the other half of the Parabola D a.

And if x be equal o, then the Equation expressing the Nature of the Curve F N will be $2azz = a^3$, or $z = \sqrt{\frac{aa}{2}}$, whence z(A F) $= a\sqrt{\frac{1}{2}}$. That is the Velocity of the Particle of Light at its Ingress in A is as $a\sqrt{\frac{1}{2}}$.

And the Rarity of the Medium or the Velocity of the Particle of Light is reciprocally in a Sub-duplicate Ratio of its distance from the determinate Plain S L, for $\overline{2a-x} \times zz = a^3$, whence $z = \frac{\sqrt{a^3}}{\sqrt{2a-x}}$, that is (rejecting the Determinate Quantity $\sqrt{a^3}$) the Rarity of the Medium or the Velocity z is directly as $\frac{1}{\sqrt{2a-x}}$ or reciprocally as $\overline{PS}^{\frac{1}{2}}$.

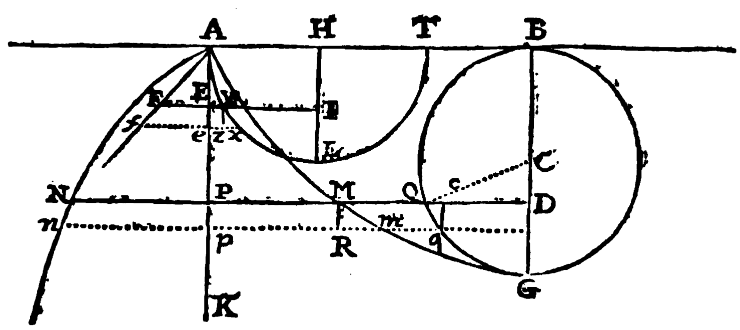
And because, the Infinitely little Portions of the Curve M m, are suppos'd equal, therefore the times of Description are reciprocally proportional to the Velocities, and because the Velocities are reciprocally as $\overline{PS}^{\frac{1}{2}}$, therefore the times are directly as $\overline{PS}^{\frac{1}{2}}$, whence if the Parabola S $\sigma\phi$ be describ'd to the Axis S A, the Ordinates P σ will represent the times which the Particle takes to describe the Infinitely little Portions of the Curve M m. And the time which the Particle takes to describe any Portion of the Curve A M D is proportional to the Area of the corresponding Portion of the Parabola S $\xi\sigma\phi$ A.

CONSECTARY VIII.

But if we suppose the Particle of Light to emerge out of such Fluids into another Fluid of an uniform Density (before it arrive at the Vertex D) then the Proportion of the Sine of the Angle of Incidence, to the Sine of the Angle of Emergence may be Investigated; for Instance, if the Rarity of the Fluid comprehended between the Plains A C, N M, be such that the Particle of Light move in the Curve of a Parabola A M, and if the Density of the Fluid below the Plain or Surface N M be uniform, then the Particle of Light will Emerge in the Point M, and continue to move in the Line M K. Through the point M draw B M T perpendicular to the Plain of Emergence, and produce the same until it Intersect the Plain of Incidence A C in B, and the Line of Incidence G A produc'd in T, and produce the refracted Ray K M until it Intersect the Ray of Incidence produc'd in H; then on the Center H, with the Radius H M describe a Circle Intersecting A T in X and V, and B M in Q. Now the Line A M being the Curve of a Parabola it is evident, that the Rectangle comprehended under T M and a given Parameter is $= A T q$. And the Line A T is bissected in H, whence if H O be drawn Perpendicular to B M, then T O will be $=$ O B, and if we add the equal Lines Q O, O M, then T Q $=$ B M; and because B M is given, T Q is also given. Now the Rectangle comprehended under Q T and T M, is to the Rectangle comprehended under a given Parameter and T M, that is, to T A q, in a given Proportion; But the Rectangle Q T M is $=$ Rectangle X T V $= HTq - HXq$ (or H M q), and A T q is to $\frac{1}{4}$ T A q $=$ H T q in a given Proportion, therefore the Ratio of $HTq - HMq$ to H T q is given, and by Division, the Ratio of H M q to H T q is given, and likewise the Subduplicate Ratio, *viz.* H M to H T is given. Whence in the Triangle H T m, because the Sides are proportional to the Sines of their Opposite Angles, the Ratio of the Sine of the Angle of Incidence H T B, to the Sine of the Angle of Emergence H M B, is given.

CONSECTARY IX.

And if the Nature of the Curve of Refraction A M G be given, then the Density of the Medium, and in what Proportion it Increases or Decreases may be discover'd. For (N°. 4.) $aa\dot{y}^2 = zz\dot{x}^2 + zz\dot{y}^2$, whence $z = \sqrt{\frac{aa\dot{y}^2}{\dot{x}^2 + \dot{y}^2}} = \frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. Now by help of this Equation and the property of the Curve of Refraction, the property of the Curve A N n, or the Relation of the Ordinate P N to the Abscissa A P may easily be discover'd, and consequently, the Proportion in which the Density of the Medium Increases or Decreases will become known.



EXAMPLE I.

Let the Curve A M G *which a Particle of Light describes in a Medium, whose Density Decreases in a given Proportion, be a Semi-cycloid; 'tis requir'd to find in what Proportion the Rarity of the said Medium Increases.*

Suppose A P $= x$, P M $= y$, B G the Diameter of the generating Circle $= a$, and suppose the Curve A N n requir'd to be describ'd, and the Ordinate P N $= z$. Now because the Curve of Refraction A M G is a Cycloid, therefore the Fluxion of the Ordinate $\dot{y}$ is $= \frac{x\dot{x}}{\sqrt{ax - xx}}$, and $\dot{y}^2 = \frac{x^2\dot{x}^2}{ax - xx}$. Whence the general Equation $z = \frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$ will become $z = \frac{ax\dot{x}}{\sqrt{ax - xx}\,\sqrt{\dot{x}^2 + \frac{x^2\dot{x}^2}{ax - xx}}} =$

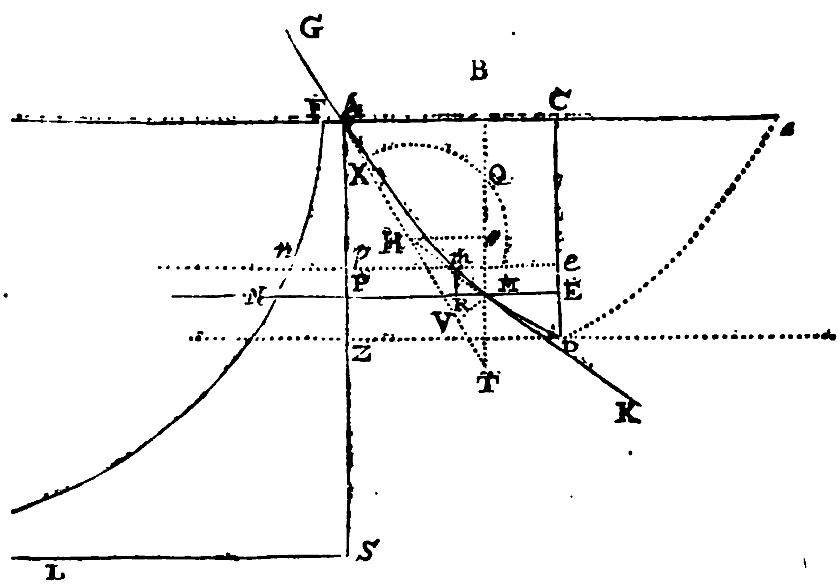
$a\dot{x}$

$= \frac{a x \dot{x}}{\sqrt{ax - xx}\sqrt{\frac{a x \dot{x}^2}{ax - xx}}} = \frac{a x \dot{x}}{\dot{x}\sqrt{ax}} = \sqrt{ax}$. Now becauſe $z = \sqrt{ax}$, it appears that the Curve ANn is a common Parabola, and becauſe the Ordinates of the Parabola PN are ſuppos'd to repreſent the ſucceſſive Degrees of Rarity; 'tis plain that the Rarity of the Medium increaſes in a Subduplicate Ratio of the Altitude AP.

EXAMPLE II.

Let the Curve of Refraction AMD *be a Parabola: 'Tis requir'd to find the property of the Curve* FN, *which determines the Rarity of the Medium in all Altitudes.*

Let the Focus of the Parabola be in the Horizontal Plane FB, in the Point C, and ſuppoſe AC $= a$, then CD the diſtance of the Focus from the Vertex is $= a$, whence (ſuppoſing AP $=$ CE $= x$, and PN $= z$) $\sqrt{4aa - 4ax} =$ EM, and MR $= \dot{y} = \frac{a\dot{x}}{\sqrt{aa - ax}}$, and $\dot{y}^2 = \frac{a^2\dot{x}^2}{aa - ax}$; and conſequently, the general Equation $z = \frac{a\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$



will become $z = \frac{aa\dot{x}}{\sqrt{aa - ax}\sqrt{\frac{aa\dot{x}^2 - ax\dot{x}^2 + aa\dot{x}^2}{aa - ax}}} = \frac{aa}{\sqrt{2aa - ax}}$ and $zz = \frac{a^3}{2a - x}$, or $\overline{2a - x} \times zz = a^3$; which Equation expreſſeth the Nature of the Curve FNn, and conſequently if AS be taken $= 2a$, and AP $= x$, the Rarity of the Medium is reciprocally as $\overline{SP}^{\frac{1}{2}}$.

CONSECTARY X.

In like manner if the Curve of Refraction be an Arch of a Circle, then the Rarity of the Medium in all Altitudes may be determin'd; And the times of deſcription will be reciprocally proportional to the Altitudes AE.

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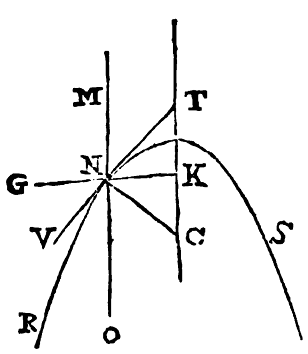
CONSECTARY XI.

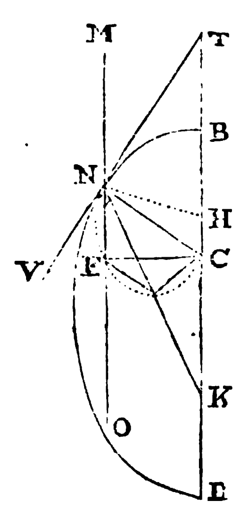
And if the Curve of Refraction be a Cycloid, then becauſe the Infinitely little portions of the Curve M *m* are equal, and the Velocities in a Subduplicate Proportion of the Altitudes; therefore the Times are reciprocally in a Subduplicate Ratio of the Altitudes, and if to the Axis A K, a Curve be deſcrib'd having its Ordinates reciprocally in a Subduplicate Ratio of the Altitudes, the Area comprehended between the ſaid Curve, and the Axis A K will repreſent the time which the Particle takes to deſcribe the reſpective Portions of the Curve A M G.

LEMMA.

164. If a Ray M N, falling on the Curve R B S, be reflected in the point N to G, and if the reflected Ray G N be produc'd towards K; and if N C, be drawn Perpendicular to the Curve in N, and through any point therein as C, be drawn C B, Parallel to the Ray of Incidence M N, Interſecting the reflected or refracted Ray N K in K, then in the firſt Caſe, K N = K C, and in the Second Caſe K N : K C :: Sine of the Angle of Incidence is to the Sine of the refracted Angle, or as *r* is to *b*.

1. In the Caſe of Reflexion; becauſe the Angle O N C = (§. 3°. *Art.* 163.) K N C, and the Angle O N C = N C K, (becauſe M O and B C are Parallels) it follows that K N C = K C N, and conſequently, K N = K C.





2. In the Caſe of Refraction; on the Diameter N C deſcribe the Semi-circle N E C, and draw the Lines C E, C F, and they will be Perpendicular to N E and N F; now the Angle C N E, is equal to the Angle of Incidence, and C N F, is equal to the refracted Angle, and C E is the Sine of that, and C F the Sine of this, therefore it is (§. 1°. *Art.* 163.) C E : C F :: *r* : *b*.

Again, the Angle C E F = C N K (becauſe both ſtands on the ſame Arch F C) and the Angle E C F = E N F = C K N; therefore the Triangles E C F, N K C are ſimilar; and conſequently, C E : C F :: K N : K C :: *r* : *b*; that is, K N is to K C, as the Sine of the Angle of Incidence, is to the Sine of the refracted Angle, or reciprocally as the reſiſtence of the Mediums.

CONSECTARY I.

165. Hence in the Parabola, If an Infinite Number of Rays M N, or O N Parallel to the Axis B C, be reflected by the Curve R N S, the Point of Concurrence will be in K, the Focus of the Parabola: For the reflected Ray N K is always = (*Art.* 164.) K C, therefore (§. 6°. *Art.* 41.) K is the Focus of the Parabola. This holds true alſo in Parabolical Conoids.

CONSECTARY II.

If B N D be an Ellipſis, B D the Diameter, and H, K the Foci; Then all Rays Iſſuing from one Focus K, and reflected in N, unite again in the other Focus H; for the Angle of Incidence K N C is = (*Art.* 47.) Angle of reflection H N C.

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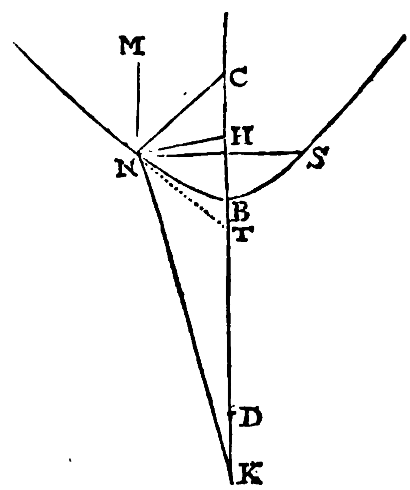
CONSECTARY III.

NH : NK :: HC : CK ; and by Composition, NH + NK : NK :: HC + CK : CK ; or (*Art.* 48.) BD : NK :. HK : CK, and by Permutation, BD : HK :: NK : CK :: *r* : *b* ; therefore if the Ellipsis be such, that it be BD : HK :: *r* : *b*, it will also be NK : CK :: *r* : *b* ; and if the Ray of Incidence MN, Parallel to BD, be refracted to K ; then it will also be NK : CK :: *r* : *b* ; and consequently, if the Transverse Axis of an Ellipsis, be to the distance between the Foci, as the Density of the Medium within the Ellipsis, is to the Density of the Ambient Medium, then if the Rays of Incidence be parallel to the Axis, all the refracted Rays will Converge to, and unite in the remotest Focus K.

And the like must be understood of Spheroides, generated by the Revolution of the Semi-ellipsis BND, about its Axis BD.

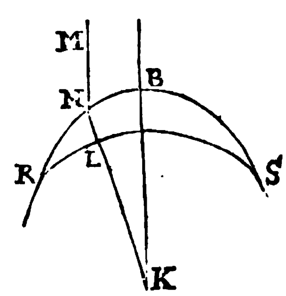
CONSECTARY IV.

In the Hyperbola let BD be the Transverse Axis, and H, K the Foci ; then (in like manner) it may be prov'd, that NK : CK :: BD : HK. Whence if by the Construction of the Hyperbola it be BD : HK :: *r* : *b*. It will be also NK : CK :: *r* : *b*. And if the Ray MN parallel to the Axis BD be refracted in N to K, it will be NK : CK :: *r* : *b*. Therefore if the Transverse Axis of an Hyperbola be to the distance of the Foci, as *r*, is to *b*, or reciprocally as the resistance of the Mediums, then all the parallel Rays MN (being refracted in N) will (after Refraction) Converge and Unite in K, the Focus of the opposite Section.



CONSECTARY. V

If RBS, represent a Portion of an Ellipsis, BK the Axis, and K the remotest Focus ; then if, on the Center K with any Radius, you describe the Arch RLS, then RBSL, is call'd a *Meniscus Speculum*, because it resembles the New Moon or a Lunule. And by help of this *Meniscus* Glass all the Rays MN parallel to the Axis BK, and Refracted in N, pass out of the Glass into the Air again in L, and unite in the Focus K, for the Refracted Ray NL converges to the Focus K, and K being the Center of the Circle RLS, the Ray NL is perpendicular to the Arch in L, and consequently passes directly on, from L to K.



CONSECTARY VI.

A Plano-convex Hyperbolical Glass NBS, unites the Rays MN parallel to the Axis BD, in K, the Focus of the opposite Section.

CONSECTARY VII.

And if K, be the Radiant point, all the Rays Issuing from the same, and refracted in N, on the Convex side of the Plano-convex Hyperbolical Glass NBS, will become parallel to the Axis BD ; and in the *Meniscus* RBSL, all the Rays Issuing from the Focus K, being refracted in N, will run parallel to the Axis KB.

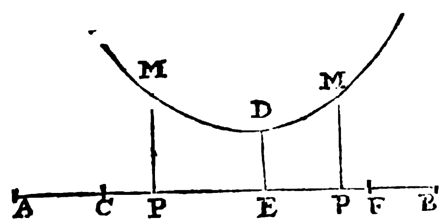
CONSECTARY VIII.

And in a Convexo-convex Hyperbolical Speculum, all the Rays Issuing from the remoter Focus of one, will Converge and unite in the remoter Focus of the other.

PROP. XI.

If the right Line A B *be divided into three Parts,* A C, C F, F B, *'Tis required to divide the middle Part* C F, *in the Point* E, *so that the Proportion of the Rectangle* A E × E B, *to the Rectangle* C E × E F, *be the least of all possible Proportions made in that manner.*

166. Suppose the given Quantities $AC = a$; $CF = b$; $CB = c$, and the unknown Quantity $CE = x$, then $AE = a + x$; $EB = c - x$; and $EF = b - x$, and the Ratio of A E × E B to C E × E F will be express'd by this Fraction $\frac{ac + cx - ax - xx}{bx - xx}$ which is requir'd to be the *least* assignable between such Rectangles. Hence if we imagine a Curve M D M to be such, that the Relation between the Ordinate P M (y) and the Abscissa C P (x) be express'd by this Equation $y = \frac{aac + acx - aax - axx}{bx - xx}$. The Question will be to find the Point E, where the Ordinate E D will be the least that can be applied to the Curve M D M; therefore, if we reduce the Equation to Fluxions, and divide by $a\dot{x}$, there will arise $cxx - axx - bxx + 2acx - abc = 0$, and finding one of the Values of the Root x, it will answer the demands of the Question.

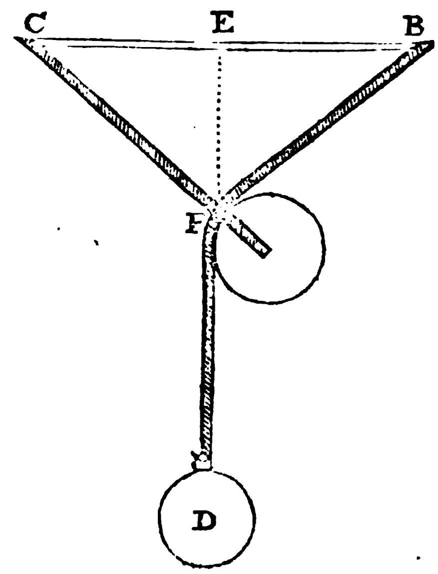


PROP.

PROP. XII.

Let CF *be a Chord, and let one end thereof be made fast at* C, *and to the other end fasten the Pulley* F, *about which suspend the Weight* D, *by the Chord* DFB, *fastning one end thereof at* B, *and let the points* C *and* B, *be in the same Horizontal Line* CB; *and suppose, both the Chords and the Pulley to be without Weight*; *'Tis requir'd to find the lowest descent of the Pulley and the Weight.*

It is evident, that the Weight D, *will descend as low as possible, below the Horizontal Line* CB; *and therefore the Line* DFE, *will represent the greatest descent of the Weight* D.



167. Suppose the known Quantities $CF = a$, $DFB = b$, $CB = c$, and the variable Quantity $CE = x$, then is $EF = \sqrt{aa - xx}$ and $FB = \sqrt{aa + cc - 2cx}$, and $DFE = b - \sqrt{aa + cc - 2cx} + \sqrt{aa - xx}$ which ought to be equal to the greatest descent of the Weight D; and consequently its Flux-

$$\frac{c\dot{x}}{\sqrt{aa + cc - 2cx}} - \frac{x\dot{x}}{\sqrt{aa - xx}}$$ is $= 0$, and

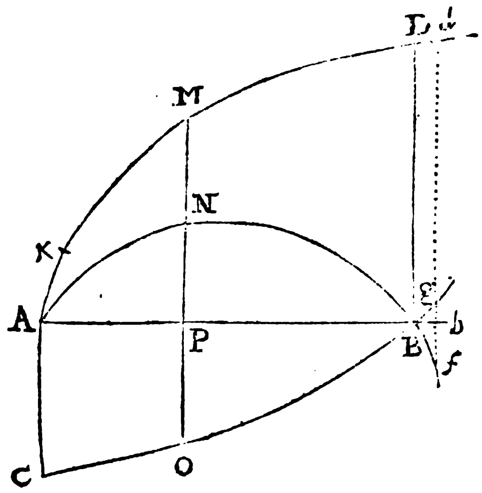
by Reduction $2cx^3 - 2ccxx - aaxx + aacc = 0$; and dividing by $x - c$, we have $2cxx - aax - aac = 0$, and consequently, one of the Values of the Root x in this Equation is equal to CE, and if EF be drawn Perpendicular to CB, it will pass by the Pulley F, and through the Weight D, when it comes to rest.

PROP.

PROP. XIII.

Let the Nature of the Curve Line AMD *be such, that the Value of the Ordinate be express'd by a Fraction; and suppose, that when the Abscissa* AP (x) *becomes* $= a$, *then the Numerator and Denominator of the Fraction become each equal to nothing: 'Tis requir'd to find the Value of the Ordinate* (PM) *in that Point.*

168. Suppose $AB = a$; and let the Curve Lines ANB, COB be applied to the same Axis AB; so that the Ordinate PN express the Numerator, and PO the Denominator of the general Fraction Expressing the Value of the Ordinate PM: *v. g.* Suppose $PM = \frac{AB \times PN}{PO}$. It is manifest that both those Curves will Intersect the Axis in the same Point B, because (by supposition) when x becomes $= a$, the Ordinates PN, PO, become each equal to nothing; now 'tis requir'd to find the Value of the Ordinate BD in that point.



Imagine another Ordinate bd to be drawn Infinitely near BD, Intersecting the Curves ANB, and COB, in the Points f and g; and then $bd = \frac{AB \times bf}{bg}$. That is, BD is $= \frac{AB \times bf}{bg}$. Now it is manifest that when the Abscissa AP becomes $=$ AB, then the Ordinates PN, PO Vanish, and that when AB becomes $= Ab$, then the Ordinates of the Curves ANB, COB become bf, bg; so that the Ordinates bf, bg, are the Fluxions of the Ordinates in B and b, in respect of the Curves ANB, COB; whence it is evident, that if we take the Fluxion of the Numerator, and divide the same by the Fluxion of the Denominator, after having supposed $x = a = Ab = AB$, the Quotient will be equal to the Ordinate bd or BD, which was requir'd.

For Instance, suppose $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$. Then it is evident that if x be $= a$, the Numerator and Denominator of the Fraction will be $= 0$. Now the Fluxion of the Numerator is $= \frac{a^3\dot{x} - 2x^3\dot{x}}{\sqrt{2a^3x - x^4}} - \frac{aa\dot{x}}{3\sqrt{ax}} =$ (when x is $= a$) $-\frac{4}{3}a\dot{x}$; and the Fluxion of the Denominator is $= -\frac{3a\dot{x}}{4\sqrt[4]{a^3x}} =$ (when $x = a$) $-\frac{3}{4}\dot{x}$, by which dividing $-\frac{4}{3}a\dot{x}$ the Quotient $\frac{16}{9}a$ is $=$ BD.

SCHOLIUM.

169. Happening to apply the Doctrine *de maximis & minimis* to Fractions, I shall here add that Problem to find the Fraction, whose Square exceeds its Cube the greatest thot can be. Suppose the Fraction to be $= x$, then $x^2 - x^3$ is $=$ *a maximum*, and consequently, $2x\dot{x} - 3x^2\dot{x}$ is $= 0$, and by Transposition and Division $x = \frac{2}{3}$. Wherefore the Square of $\frac{2}{3}$ exceeds its Cube, more than the Square of any other Fraction exceeds its respective Cube.

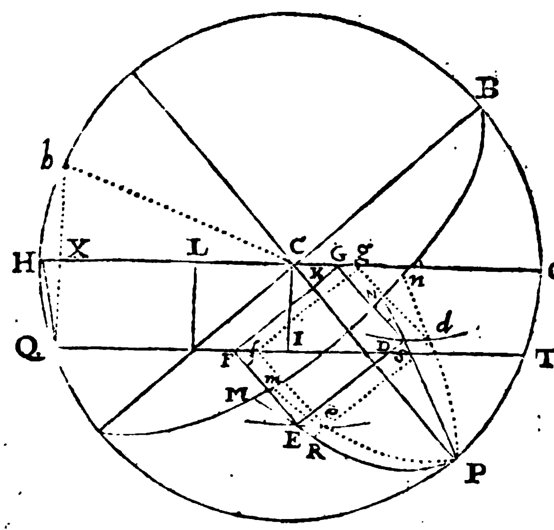
PROP.

PROP. XIV.

The Elevation of the Pole being given, to find the Day when Twilight is shortest.

170. Let C be the Center of the Sphere, A P B H the Meridian; H D *d o* the Horizon, Q E *e* T *a* parallel to the Horizon, when Twilight begins, A M N B the Equator, F E D G a Portion of the Parallel of Declination, which the Sun describes when Twilight is *shortest*, comprehended between the Plains of the Horizon and the Crepuscular Circle; P the South Pole, P E M, P D N Quadrants of Hour-circles Intercepted between the Pole and the Equator; H Q or O T an Arch of the Meridian, comprehended between the Horizon and the Crepuscular Circle, O P the Elevation of the Pole are given, and consequently, their right Sines C I = F L = Q X, and O V are also given: To find C K the Sine of the Arch E M or D N the Suns Declination when he describes the Parallel E D.

Imagine another Portion *f e d g* of a Parallel of Delination Infinitely near F E D G; and draw the Quadrants P *e m*, P *d n*; 'tis evident, that when the times which the



Sun takes to describe the Arch E D is shortest, then the Difference between the Arches M N (which is the Measure of the same) and *m n* (when E D becomes *e d*) is equal to nothing; whence it follows that the little Arches M *m*, N *n* are equal, and consequently, the little Arches R *e*, S *d* are also equal between themselves, but the Arches R E, S D being Included between the same Parallels, E D, *e d* are equal, and the Angles at S and R are right Angles, therefore the right-angled Triangles E R *e*, D S *d* (which we consider here as Rectilineal, their sides being Infinitely little) are similar and equal, and consequently, the Hypothenuses E *e*, D *d* are also equal between themselves.

The right Lines D G, E F, *d g*, *e f* being the common Sections of the Plains F E D G, *f e d g* (parallel to the Equator) with the Horizon and the Crepuscular Circle, are perpendicular to the Diameters H O, Q T; because the Plains of all those Circles are perpendicular to the Plain of the Meridian, and the Infinitely little Lines G *g*, F *f* are equal between themselves; because the opposite Sides of the Figure F *g* are parallel, whence $\sqrt{Dd^2 - Gg^2}$ or $DG - dg$ is $= \sqrt{Ee^2 - Ff^2}$ or $fe -$ F E; now it is evident that if in a Circle two Ordinates be drawn Infinitely near each other, the little Arch Intercepted between them is to their difference, as the Radius of the Circle is to the Portion of the Diameter Intercepted between the Center and the Ordinate, therefore (because of the Circles H D O, Q E T.)

CO : CG :: D*d* (or E*e*) : DG — *dg* (or *fe* — FE.)

And IQ : IF :: E*e* : *fe* — FE. Therefore

CO : IQ :: CG : IF and by composition

CO + IQ (OX) : CG + IF (LG) :: CO : CG.

And because the Triangles CVO, CKG, LFG are similar, it is,

COːCG :: OV : GK

And GK : GL :: CK : FL (QX.)

And again OV : GK :: OX : GL.

Therefore OV : CK :: OX : QX :: QX : XH (by the property of the Circle.)

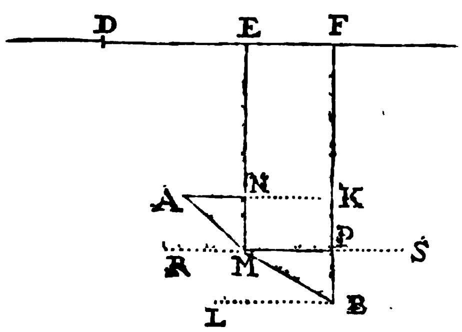
That is to say, if in the right-angled Triangle QXH you make QX Radius, then (because Astronomers suppose Twilight to begin or end, when the Sun is 18 Degrees below the Horizon, and consequently, the Arch HQ or Hβ is $= 18°$, and the Angle HQX $= \frac{1}{2}$ HCβ is $= 9°$, and HX is the Tangent of 9 Degrees;) It is *as Radius* is to the Tangent of 9°. 00′, so is the Sine of the Latitude to the Sine of the Suns Declination (South) when Twilight is of shortest continuance.

PROP. XV.

Any two points A *and* B *in the same vertical Plain being given: 'Tis requir'd to find the property of the Curve* AMB, *so that a heavy Body descending in the same by the force of its own Gravity, shall move from* A *to* B *in a shorter time than in any other Line, drawn between the said Points.*

171. I have already solv'd this curious and useful Problem from Dioptrick Principles, and have shewn its Identity with the Curve of Refraction; This Proposition shall comprize several solutions of the same Problem deduc'd from other Principles, that so it may appear that the same truths may be Investigated by very different Methods.

Let AM, MB be two Infinitely little Portions of the Curve requir'd, and suppose the Curve to begin somewhere in the Horizontal Line DF; then 'tis evident, that when the Body comes to A, with the Velocity acquir'd in its descent, it must move from A to B in the shortest time possible.



Draw the Lines AK and BL parallel to DF, and bisect BK in P, and draw SPR parallel to DF, then it remains only to find the point M, where the Line PR Intersects the Curve requir'd.

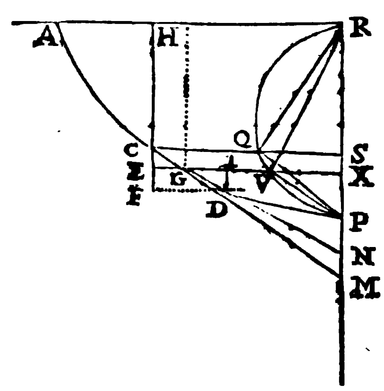
Now if we assume *Galileus*'s Hypothesis, according to which the Velocities, which heavy Bodies acquire in their descent, are in a Sub-duplicate Ratio of the Altitudes they fell from; then the Velocity of the heavy Body in the point M will be as $\sqrt{EM}$, and its Velocity in B, will be as $\sqrt{FB}$; therefore the time of its descent from A to M is (because the times are directly as the Spaces and reciprocally as the Velocities) $\frac{AM}{\sqrt{EM}}$, and the time it takes to descend from M to B is $\frac{BM}{\sqrt{FB}}$, therefore the Point M ought to be such, that $\frac{AM}{\sqrt{EM}} + \frac{BM}{\sqrt{FN}}$ be a *minimum*; whence if we suppose the points A and B to be determinate, and the Invariable Quantities BP or MN $= s$, EM $= b$, FB $= q$, and the Indeterminate

terminate Quantities A N $= u$, and M P $= z$, then $\frac{\sqrt{u^2+s^2}}{\sqrt{b}} + \frac{\sqrt{z^2+s^2}}{\sqrt{q}}$ will be a *minimum*; and consequently, the Fluxion thereof will be equal to nothing: That is $\frac{u\dot{u}}{\sqrt{b}\sqrt{u^2+s^2}} + \frac{z\dot{z}}{\sqrt{q}\sqrt{z^2+s^2}} = 0$. And because $u + z =$ to an Invariable Quantity, therefore $\dot{u} = -\dot{z}$, whence $\frac{z}{\sqrt{q}\sqrt{z^2+s^2}} = \frac{u}{\sqrt{b}\sqrt{u^2+s^2}}$. Whence 'tis manifest that $\frac{u}{\sqrt{b}\sqrt{u^2+s^2}}$ is always equal to an Invariable Quantity; now suppose the Curve A M B to have commenc'd in D, and let the Abscissa D E be $= x$, the Ordinate E M $= y$, AN $= \dot{x}$, and N M $= \dot{y}$; and let a be an Invariable Quantity, then will $\frac{\dot{x}}{\sqrt{y}\sqrt{\dot{x}^2+\dot{y}^2}}$ be $= \frac{1}{\sqrt{a}}$. Whence $\dot{x}\sqrt{a} = \sqrt{y}\sqrt{\dot{x}^2+\dot{y}^2}$, and reducing this Equation to an Analogy $\sqrt{\dot{x}^2+\dot{y}^2} : \dot{x} :: \sqrt{a} : \sqrt{y}$; but in every Curve $\dot{x}$ is to $\sqrt{\dot{x}^2+\dot{y}^2}$ as the Subtangent is to the Tangent, therefore the property of the Oligochronal Curve is, that the Tangent is to the Sub-tangent, as $\sqrt{a}$ is to $\sqrt{y}$. But this is the property of the Vulgar (*Art.* 70 N°. 4.) Cycloid, the Diameter of the generating Circle being $= a$; therefore the Curve of the swiftest Descent is a Cycloid.

LEMMA.

In the vulgar Cycloid, the Elementa of the Curve, are in a Ratio, compounded of the Direct Ratio of the Elementa of the Abscisse, and the Reciprocal Subduplicate Ratio of the Ordinates.

171. Let A C P be a Semi-cycloid, C G, G D two Infinitely Little Portions thereof; C M, G N two Tangents, R Q P the generating Circle, A the Vertex, A H the Abscissa, H C an Ordinate, C E, E F Fluxions of the Ordinates, and E G, G I Fluxions of the Abscissa.



Then

G D : G I :: G N : G X :: (*Art.* 70. N°. 4.) $\sqrt{RP} : \sqrt{RX}$, and

G I : E G :: G I : E G, and

E G : G C :: C S : C M :: $\sqrt{RS}$ ($\sqrt{HC}$) : $\sqrt{RP}$.

Therefore

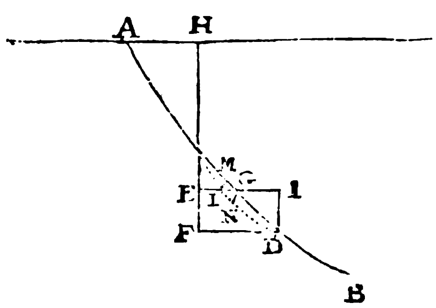
G D : G C :: G I × $\sqrt{RP}$ × $\sqrt{HC}$: E G × $\sqrt{RP}$ × $\sqrt{HE}$ (or $\sqrt{RX}$) :: G I × $\sqrt{HC}$: E G $\sqrt{HE}$ :: $\frac{GI}{\sqrt{HE}} : \frac{EG}{\sqrt{HC}}$. That is the Elementa of the Curve G D, G C are in a Ratio compounded of the direct Ratio of the Fluxions of the Abscissa G I, G E, and the reciprocal Sub-duplicate Ratio of the Ordinates H E, H C. Q. E. D.

COROL-

COROLLARY.

Hence if the Fluxions of the Abscissa EG, GI *be equal and invariable, then the Elementa of the Curve, will be reciprocally in a Sub duplicate Ratio of the Ordinates. And now to Investigate the Nature of the Curve, of swiftest Descent another way.*

173. In any Plain, inclin'd to the Horizon at pleasure, let ACB be the Curve requir'd, along which suppose a heavy Body to descend from A to B (by the force of its own Gravity) in a shorter time than it can descend in any other Curve, drawn through the given Points A and B, and in the same Plain; and let any two Points C, D, Infinitely near each other, be assum'd in the Curve, and draw the Horizontal Line AH, and CH perpendicular to the same, draw also DF perpendicular to CH, and bisect CF in E, and compleat the Parallelogram ED. 'Tis requir'd to determinate the point G in the Line EI; that is, 'tis requr'd to find the Inclination of the Portions of the Curve CG, GD to each other, so that the time of descent along CG + the time of descent along GD (which may be written thus $tCG = tGD$) shall be the shortest possible; now to do this, let any other point as L be taken between E and G, so that GL be Infinitely less than GE or LE, and draw the Lines CL, DL, and on the points C and D as Centers, describe the Infinitely little Arches LM, GN; then will $tCL + tLD$ be = (*ex natura minimi*) $tCG + tDG$; and consequently, $tCG - tCL = tLD - tDG$. This being laid down we may proceed thus:



$CE:CG :: tCE:tGC$ and also $EF:GD :: tEF:tGD$ } *Ex hyp. Gravit.*
$CE:CL :: tCE:tCL$ $\quad$ $EF:LD :: tEF:tLD$

Ergo $CE:CG - CL\ (MG) :: tCE:tCG - tCL \ldots\ EF:LD - GD :: tEF:tLD - tGD.$

and (because $\triangle MGL$, CEG are similar) $MG:GL :: EG:CG \ldots$ and $LN:LG :: GI:GD.$

Ergo, $CE:GL :: EG \times tCE : CG \times \overline{tCG - tCL} \ldots\ EF\ (CE) : LG :: GI \times tEF : GD \times \overline{tLD - tGD}.$

and consequently, $EG \times tCE : GI \times tEF :: CG \times \overline{tCG - tCL} : GD \times \overline{tLD - GD}$ :: (*ex natura minimi*) $CG:GD.$

But $EG \times tCE : GI \times tEF :: \frac{\sqrt{HC}}{EG} : \frac{GI}{\sqrt{HE}}$ *Ex hyp. Gravit.*

Therefore $CG:GD :: \frac{EG}{\sqrt{HC}} : \frac{GI}{\sqrt{HE}}$; that is, the Elementa of the Curve of swiftest Descent, are in a Ratio compounded of the direct Ratio of the Elementa of the Abscissa and the reciprocal Sub-duplicate Ratio of the respective Ordinates, and consequently, the Curve of swiftest Descent is (*Art.* 172) the vulgar Cycloid.

Having thus determin'd the Nature of the Curve of swiftest Descent, I think it will not be amiss in this place, to shew how to Investigate the Nature of the Isochronal Curve, in which a heavy Body descends equal Spaces in equal Times.

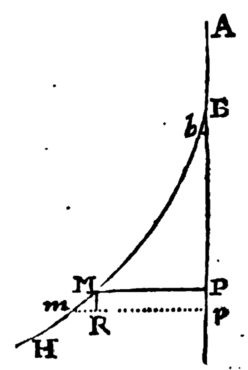
PROP.

PROP. XVI.

To Investigate the Nature of the Curve, in which an heavy Body shall descend without any Acceleration of Velocity.

The Learned G. G. Libnitz *in Novell. Rep. lit. September* 1687 *propos'd the Problem thus:* To find an Isochronal Curve, in which an heavy Body shall descend uniformly; that is, in equal times, it shall descend equal Spaces with an equable Velocity; *and the Celebrated* M. Hugens *was the first that resolv'd the Problem in* October *following, but suppress'd the Demonstration. Afterwards* M. Libnitz *himself gave the Demonstration, but suppress'd his Analysis. Lastly, the Excellent Geometer* M. Nich. Fatio Duillier *communicated his Method of Investigation, which is thus:*

174. Let ABE be a Line perpendicular to the Horizon, and suppose BH to be the Curve requir'd; then suppose an heavy Body in B, with the Velocity acquir'd since it fell from A, to continue to move from B to H, in the Curve BM, and let the Velocity acquir'd in B, be represented by $AB = a$; and suppose the Axis of the Curve BP to lie in the same streight Line with AB; then put the Abscissa $BP = x$, and the Ordinate $PM = y$, $Pp = \dot{x}$, $Rm = \dot{y}$, and then because $\dot{y}$ vanishes in B, therefore the Fluxion of the Curve in B is $= \dot{x}$; and Mm the Fluxion of the Curve in M is $= \sqrt{\dot{x}^2 + \dot{y}^2}$. Now because the Velocity in B is $= a$, therefore $\sqrt{a} : \sqrt{a + x} :: a : \sqrt{ax + aa} =$ to the Velocity of the heavy Body in P, therefore the time which the heavy Body takes to describe B*b* is $=$ (because the times are directly as the Spaces and reciprocally as the Velocities) $\frac{\dot{x}}{a}$ and the time which the Body takes to describe M*m* is $= \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{aa + ax}}$. Now these are equal by supposition, therefore $\frac{\dot{x}}{a} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{aa + ax}}$ and consequently, $aa : \dot{x}^2 :: \overline{aa + ax} : \dot{x}^2 + \dot{y}^2$; and $aa : \dot{x}^2 :: ax : \dot{y}^2$, whence $a^2\dot{y}^2 = \dot{x}^2 ax$, and $a\dot{y} = \dot{x}\sqrt{ax}$, and the Flowing Quantity $ay = \frac{2}{3} x^{\frac{3}{2}} \times a^{\frac{1}{2}} \pm q$ but in this case (where x and y vanish together) q must needs be $= 0$, therefore $\frac{9}{4} ayy = x^3$, and the Isochronal Curve is a cubical Paraboloid convex towards the Axis, and the Parameter of the Curve is $= \frac{4}{9} a = \frac{2}{4}$ AB, or AB is $= \frac{4}{9}$ of the Parameter.

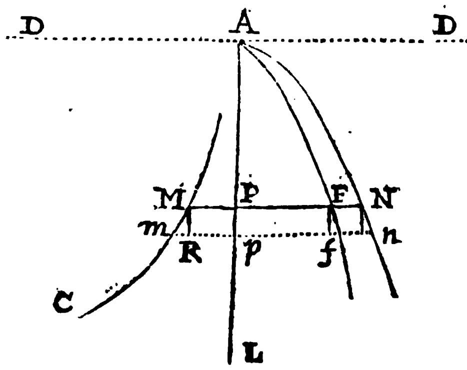


Another way.

Having thus discover'd the Nature of the Isochronal Curve in the vulgar Hypothesis of Gravity; that is, supposing the Velocity to be in a Sub-duplicate Ratio of the Altitude the heavy Body fell from; I shall now shew how to Investigate the Nature of Curves, in which heavy Bodies descending, according to any Hypothesis of Velocity, shall describe Spaces in any given Proportion to the Times.

175. Suppose DAD parallel to the Horizon; and let it be requir'd to find the Nature of the Curve MC, in which an heavy Body descending (from A) shall recede from the Horizontal Line DAD in any Proportion of the Times, and according to any Hypothesis of Velocity.

Draw the Vertical Line AL, and the Horizontal Lines MN, *mn* infinitely near each other; and describe the Curve AN*n* such, that the Ordinate PN represent the Velocity which the heavy Body has acquir'd in M (since its descent from A) and let the time which the Body takes to descend from A to M be represented by PF the Ordinate of another Curve AF*f*.



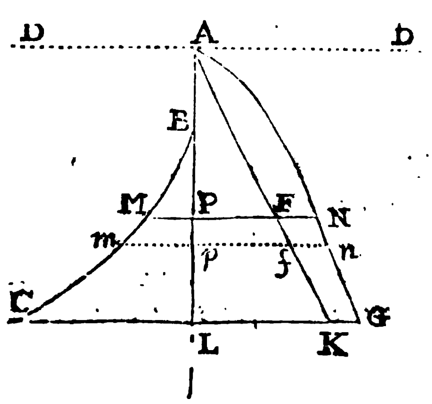
Suppose AP $= x$, PM $= y$, PN $= v$, and PF $= z$. Then, because the Times of description are directly, as the Spaces describ'd, and reciprocally as the Velocities, joyntly; therefore the Time which the heavy Body takes to describe the Particle of the Curve M*m* is $= \frac{Mm}{PN}$.

Now the Particle M*m* being Infinitely Little, the heavy Body will describe the same in an Instant, or in the Portion of Time represented by $\dot{z}$; therefore $\dot{z} = \frac{Mm}{PN} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2}}{v}$. Whence (supposing $a = 1$) we have $v\dot{z} = a\sqrt{\dot{x}^2 + \dot{y}^2}$.

And because the Curves AN*n*, AF*f* are given, therefore the Value of $\dot{z}$ in the Terms $\dot{x}$, and the Value of v in the Terms x, may be found by help of the Equations expressing the Nature of the said Curves, and substituting the said Values in the Equation $v\dot{z} = a\sqrt{\dot{x}^2 + \dot{y}^2}$, there will arise an *Equation* expressing the Nature of the Curve MC. *Which was requir'd.*

For instance, If we would apply this Theorem to the common Hypothesis of Gravity; we must consider, that in that Hypothesis, the Velocities PN are in a Sub-duplicate Ratio of the Altitudes AP, and consequently the Curve AN*n* is the common Parabola. Whence $v = \sqrt{ax}$; which Value of v being substituted in the general Equation $v\dot{z} = a\sqrt{\dot{x}^2 + \dot{y}^2}$, there will arise $\dot{z}\sqrt{ax} = a\sqrt{\dot{x}^2 + \dot{y}^2}$ or $\dot{z} = \frac{a\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{ax}}$, and if we suppose the times to be in any given Proportion to the perpendicular Descents of the heavy Body, the Nature of the Curve MC may be determin'd accordingly, in the *Gallilean Hypothesis of Gravity.*

Thus if it be requir'd that the Heavy Body moving in the Curve BMC from B to C, describe equal Spaces in equal Times; then 'tis evident that PF is always $=$ AP, and consequently the Curve AF*f* becomes a straight Line, and APF is an Isosceles Triangle. Whence $\dot{z}$ is $= \dot{x}$, and the Equation $\dot{z} = \frac{a\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{ax}}$, will become $\dot{x} = \frac{a\sqrt{\dot{x}^2 + \dot{y}^2}}{\sqrt{ax}}$, and consequently, $\dot{x}\sqrt{ax} = a\sqrt{\dot{x}^2 + \dot{y}^2}$, and Squaring both sides of the Equation, we have $ax \times \dot{x}^2 = aa \times \dot{x}^2 + \dot{y}^2$ and by Transposition, $aa\dot{y}^2 = ax - aa \times \dot{x}^2$. And by Division $\dot{y}^2 = \frac{x - a \times \dot{x}^2}{a}$, and



by

by equal Extraction $\dot{y} = \dot{x}\sqrt{\frac{x-a}{a}}$, and multiplying both Sides of the Equation by aa, $aa\dot{y} = a\dot{x}\sqrt{ax-aa}$. And finding the Fluent of both Sides of the Equation, we have $aay = \frac{2ax-2aa}{3}\sqrt{ax-aa}$, or $ay = \frac{2x-2a}{3}\sqrt{ax-aa}$, and putting $t = x - a$, there will arise $ay = \frac{2t}{3}\sqrt{at}$, or $a^2y^2 = \frac{4tt}{9} \times at = \frac{4}{9}at^3$ or $\frac{9}{4}ay^2 = t^3$. Which shews, that in this Case, the Curve B M C is a Cubical Parabola, which commences in B the point of the Axis A L, so that A B $= a$. For then B P is $= t$, and the Equation of the Curve is $\frac{9}{4}ay^2 = t^3$.

PROP. XVII.

To Investigate the Nature of the Solid of least Resistence.

Mr. Jf. Newton *(the Glory of our Age) in his Incomparable Treatise* de Princip. Math. Philos. Nat. Lib. 2. Sect. 7. *having learnedly Discours'd of the Motions of Fluids, and the Resistence of Bodies moving in them, lays down* (Lib. 2. Prop. 35. Schol) *the property of the Curve, which revolving about its Axis, generates the solid of least Resistence. the Excellency and usefulness of this Problem, has but lately engaged several* great Analysts *to consider the same more fully, and to communicate their Methods of investigating the same, because the Illustrious Author was pleas'd to conceal his own: Particularly,* The Incomparable Analysts, the Noble Marquess de l' Hospital; M. Jo. Bernouilli; M. Craig, and M. Fatio *have learnedly Treated of the same, And it is from their Excellent Essays that I have Extracted the following Solutions. In short, the Problem may very plainly be conceiv'd in these Terms.*

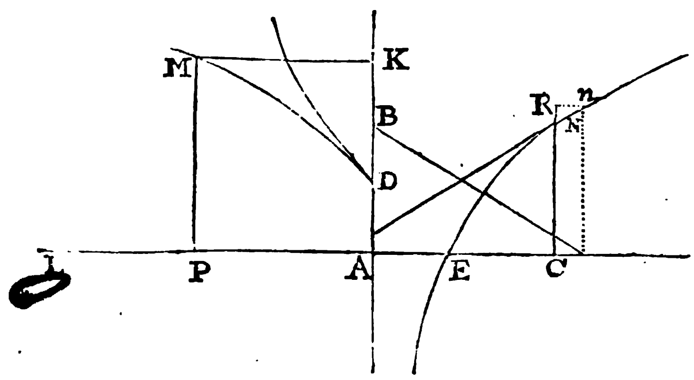
176. To investigate the Nature of the Curve D M, which being revolv'd about the Axis A L, shall describe the Surface of a Solid, which moving in a Fluid (the Particles which compose the same being at rest) from L to A according to the direction of the Axis L A, shall meet with less Resistence from the Fluid, than any other Solid, generated by a Curve describ'd to the same Axis A L, and passing through the given points D and M.

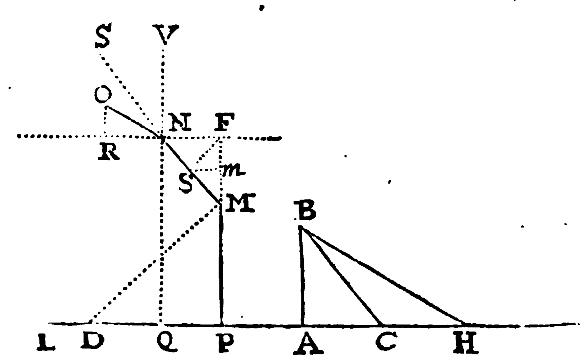
Imagine the little Lines M N, N O to be two sides of the Infinito-lateral Polygon, which constitutes the Curve requir'd: Draw M P, N Q Ordinates to the Axis A L, and draw R N F parallel to the same Axis A L, and let O R, M F be perpendicular to R N F, and M D perpendicular to the Side M N.

Then 'tis evident that if the right Lines M N, N F move in the Direction of the Axis from L towards A, that the force of Resistence of the Fluid in such a Case, is equal to the Action of the Fluid (moving in the same direction from A towards L and with the same Velocity) on the said Lines M N, M F being quiescent; draw F S perpendicular to M N, and then the Triangles F S N, F M N, P M D are similar, therefore if F N represent the Force of a Particle of the Fluid to move the Line F M, in the Direction of A L from A towards L, then F S will represent the force of the same Particle of the Fluid to move the Line M N in the Direction of M D, from M towards D; that is, the Force of the Particle to move M from F A towards L, is to the Force of the same Particle to move M N, from M toward D :: F N : F S :: M D : D P. Again, if M D represent the Force of the same Particle to move M N from M towards D, then D P will represent the Force of the same Particle to move M N in the Direction of D P from P towards D; therefore the Force of the Particle of the Fluid to move M F, from A towards L, is to the Force of the same Particle to move M N from A towards L :: $\overline{DM}^2 : \overline{DP}^2 :: \overline{MN}^2 : \overline{FM}^2$ The proportion between the Force of the Particle of the Fluid to move M F (or Q v) from A towards L, and the force of the same Particle to move M N from A towards L may

may be found thus: If F N represent the force of the Particle against Q N v in the direction from A towards L, then F S will represent the Force of the same Particle against M N s in the direction of M D; and if F S represent the force of the Particle against M N from M towards D, then m S will represent the force of the same Particle against MN in the direction of A L from A towards L; therefore the Force of the Particle of the Fluid to move M F (or Q N) from A towards L, is to the force of the same Particle to move M N from A towards L, as E N is to m S, that is as F N q is to F S q, or as M D q is to D P q.

Whence if the given right Line A B (a) represent the Velocity of the Particles of the Fluid striking against the right Lines M N, M F, then the force of the same Fluid upon the Plain describ'd by M F revolving about the Axis A L at the distance M P, and directly opposed to the motion of the Fluid, will be as the Surface describ d and Velocity Joyntly; that is as $a \times MF \times MP$, whence to find (from A towards Q) the force of the Fluid on the Surface M N; say, $\overline{MN}^2 : \overline{FM}^2 :: a \times MF \times MP : \frac{a \times \overline{MF}^3 \times MP}{\overline{MN}^2}$ = to the force (in the direction of A L from A towards L) of the Fluid on the oblique Surface describ'd by the rotation of M N about the Axis A L; or which is the same thing, the Quantity $\frac{a \times \overline{MF}^3 \times \overline{MP}}{\overline{MN}^2}$ expressing the Resistence which the same Surface moving from L towards A, suffers from the Fluid at rest.





In like manner the Resistence, which the Surface describ'd by N O revolving about the Axis A L, meets with from the Quiescent Fluid, may be represented by $\frac{a \times \overline{OR}^3 \times \overline{NQ}}{\overline{NO}^2}$.

Now if we suppose the Points M, O, and the right Line R F to be given by position, and that they are in the same Plain with the Axis A L; It remains only to determine the point N in the Line R F, so that the Surface generated by the right Lines M N, N O revolving about the Axis A L shall suffer the least Resistence.

Let the Invariable Quantities M F be $= m$; M P $= r$; O R $= n$, N Q $= q$; and the variable Quantities F N $= v$, and N R $= z$; then $\overline{MN}^2 = mm + vv$, and $\overline{NO}^2 = nn + zz$, therefore the Resistence which the Surface describ'd by the Line M N meets with, *viz*, $\frac{a \times \overline{MF}^3 \times MP}{\overline{MN}^2}$ is $= \frac{a \times m^3 \times r}{mm + vv}$, and that which the Surface describ'd by N O (revolving about the Axis A L) *viz* $\frac{a \times \overline{OR}^3 \times NQ}{\overline{NO}^2}$ is $=$ $\frac{a \times n^3 \times q}{nn + zz}$, whence it is evident (from the Nature of the Question) that the Quantity

tity $\frac{a \times m^3 \times r}{mm + vv} + \frac{a \times n^3 \times q}{nn + zz}$ ought to be a *minimum*, and (*Art.* 198.) consequently the Fluxion thereof must be $= 0$. Whence $\frac{2m^3 r x v \dot{v}}{\overline{mm+vv}^2} = \frac{-2n^3 q \times z\dot{z}}{\overline{nn+zz}^2}$. Now because $v + z$ is $=$ R F an Invariable Quantity, therefore $\dot{v} = -\dot{z}$, and consequently, $\frac{m^3 \times r \times v}{\overline{mm+vv}^2} = \frac{n^3 \times q \times z}{\overline{nn+zz}^2}$. Whence if AB ($a$) be erected perpendicular to the Axis AL, and if the right Lines BC, BH, be drawn parallel to the Infinitely little Sides MN, NO, it will be $4\overline{AB}^2 \times AC : \overline{BC}^3 :: BC : MP$; and in like manner $4\overline{AB}^2 \times AH : \overline{BH}^3 :: BH : NQ$; for because the Triangles MFN, BAC are similar, therefore $AC = \frac{an}{m}$, and $BC = \frac{a \times \overline{mm+vv}|^{\frac{1}{2}}}{m}$, whence $4\overline{AB}^2 \times AC\left(\frac{4a^3v}{m}\right) : \overline{BC}^3\left(\frac{a^3 \times \overline{mm+vv}^{\frac{3}{2}}}{m^3}\right) :: BC\left(\frac{a \times \overline{mm+vv}^{\frac{1}{2}}}{m}\right) : MP$ (r) and consequently $\frac{rm^3v}{\overline{mm+vv}^2} = \frac{1}{4}a$. In like manner, because the Triangles ORN, BAH, are similar, $AH = \frac{az}{n}$, and $BH = \frac{a \times \overline{nn+zz}^{\frac{1}{2}}}{n}$. Whence $4\overline{AB}^2 \times AH\left(\frac{4a^3z}{n}\right) : BH^3\left(\frac{a^3 \times \overline{n^2+z^2}^{\frac{3}{2}}}{n^3}\right) :: \overline{BH}\left(\frac{a \times \overline{nn+zz}^{\frac{1}{2}}}{n}\right) : NQ = q$. Whence $\frac{qn^3z}{\overline{nn+zz}^2} = \frac{1}{4}a$ and consequently, $\frac{m^3 \times r \times v}{\overline{mm+vv}^2} = \frac{n^3 \times q \times z}{\overline{nn+zz}^2}$. Which is the very same Equation that we first found.

Whence 'tis manifest, that the Nature of the Curve DM (which being revolv'd about its Axis AL, generates the Solid of least Resistence) is such, that drawing AK perpendicular to the Axis AL, and taking $AB = a$, and drawing BC parallel to any Tangent of the Curve *v. g.* in the point M, then it will always be $4\overline{AB}^2 \times AC : \overline{BC}^3 :: BC : MP$ the Ordinate passing through the point M, which is the property of the Curve that generates the Solid of least Resistence, discover'd by *M. Newton.*

And having thus discover'd this property of the Curve DM, *it may be construct by help of the Logorithmetical Line in this manner.*

In the perpendicular AK assume $AB = a$, and in the Axis AL produc'd, take $AE = \sqrt{\frac{1}{3}aa}$, and through the point E describe the Logarithmetical Line EN, and let AK be the Assymptote and $\frac{1}{4}a$ the Sub-tangent, then take AC at pleasure, which suppose $= z$, and draw CN parallel to AK, until it meet the Logarithmetical Curve in N; then if AK be taken $= \frac{aa}{4z} + \frac{1}{2}z + \frac{z^3}{4aa}$, and $AP = \frac{zz}{4a} + \frac{3z^4}{16a^3} - \frac{5a}{48} + CN$ (*viz.* $+ CN$ when $AC > AE$ and $- CN$ when $AC < AE$) and compleat the Parallelogram PK, I say, the Angle M, or the point wherein KM intersects PM will be in the Curve requir'd.

For AC being $= z$, if $AP = x$, and $PM = y$, then by the Property of the Curve AK or $PM = y$ is $= \frac{a^4 + 2aazz + z^4}{4aaz}$, and consequently, $\dot{y} = \frac{1}{2}\dot{z} + \frac{3zz\dot{z}}{4aa}$ —

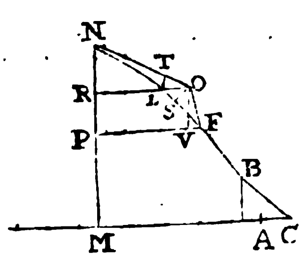
$-\frac{aa\dot{z}}{4zz}$, and becauſe B C is parallel to the Tangent in M, therefore the Triangle A B C is ſimilar to the little Triangle at M, and conſequently $a : z :: \dot{y} : \frac{z\dot{y}}{a} = \dot{x} = \frac{z\dot{z}}{2a} - \frac{3z^3\dot{z}}{4a^3} - \frac{a\dot{z}}{4z}$, and the Flowing Quantity or A P (x) is $= \frac{zz}{4a} + \frac{3z^4}{16a^3} - S\frac{a\dot{z}}{4z}$, but by the property of the Logarithmetical Line $z : \frac{1}{4}a :: \dot{z}(Rn) : \frac{a\dot{z}}{4z} = RN$, whence $S\frac{a\dot{z}}{4z} = CN$, therefore A P ($x$) is $= \frac{zz}{4a} + \frac{3z^4}{16a^3} - CN \pm$ an Invariable Quantity $\frac{Sa}{48}$, and conſequently, when C N vaniſhes, then A P or x will vaniſh alſo, therefore C M is the Curve requir'd.

COROLLARY.

Hence it appears that the Curve M D cannot approach nearer the Axis A L, then in the point D, where it Interſects the perpendicular A K; and that afterwards when A C is leſs then A E, then the Portion of the Curve D O, will be deſcrib'd Convex towards the Axis A L, ſo that the Surface of the Solid of leaſt Reſiſtence may be Concave as well as Convex towards the Axis.

To Inveſtigate the Nature of the Curve which generates the Surface of the Solid of the leaſt Reſiſtance, another way.

177. Let B F N be the Curve requir'd, which revolving about the Axis A M deſcribes the Surface of the Solid of leaſt Reſiſtence; and let the Elementa of the Ordinates M N be equal or Invariable, *viz*, N R = R P, and N L, L F the correſponding Elementa of the Curve requir'd; produce R L to O, ſo that L O be infinitely little in reſpect of R L, and draw the Lines N O, F O; then (*ex natura minimi*) the Reſiſtence which the Surface, generated by the Rotation of the Elementa of the Curve N L L F, about the Axis A M, meets with from the Fluid, is equal to the Reſiſtence which the Surface generated by N O O F, revolv'd about the ſame Axis A M, meets with from the ſame Fluid, and conſequently, the Reſiſt. of the Zone N L — Reſiſt. Zone N O = Reſiſt. Zone F O — Reſiſt. Zone F L; now reſuming the former Hypotheſis, *viz*, that the oblique Reſiſtence of N L is to the direct Reſiſtence of N R, as $\overline{NR}^2$ is to $\overline{NL}^2$, then the reſiſtence of the Elementum N L will be expreſs'd by $\frac{\overline{NR}^3}{\overline{NL}^2}$, and that of the Zone generated by N L by $\frac{MN \times \overline{NR}^3}{\overline{NL}^2}$. Whence $\frac{MN \times \overline{NR}^3}{\overline{NL}^2} - \frac{MN \times \overline{NR}^3}{\overline{NO}^2}$ is $= \frac{MR \times \overline{RP}^3}{\overline{FO}^2} - \frac{MR \times \overline{RP}^3}{\overline{FL}^2}$. Now O T, L S being the Fluxions of N L, F L, or N O, F O, they are Incomparably little in reſpect of theſe, and conſequently, $\frac{MN \times \overline{NR}^3}{\overline{NL}^2}$ —



$\frac{MN\times\overline{NR}^3}{NO^2}$ = (supposing $MN\times\overline{NR}^3 = n$) $\frac{n}{\overline{NL}^2} - \frac{n}{\overline{NO}^2} = \frac{n\times\overline{NO}^2 - n\times\overline{NL}^2}{\overline{NL}^2\times\overline{NO}^2}$

$= \frac{n\times\overline{NL}^2 + 2NL\times TO + \overline{TO}^2 - n\times\overline{NL}^2}{\overline{NO}^2\times\overline{NL}^2} = \frac{2n\times NL\times TO}{\overline{NO}^2\times\overline{NL}^2} =$

$\frac{2MN\times\overline{NR}^3\times TO}{\overline{NO}^3}$. And for the like reason $\frac{MR\times\overline{RP}^3}{\overline{FO}^2} - \frac{MR\times\overline{RP}^3}{\overline{FL}^2}$ is $=$

$\frac{2MR\times\overline{RP}^3\times LS}{\overline{FO}^3}$. Therefore $\frac{2MN\times\overline{NR}^3\times TO}{\overline{NO}^3}$ is $= \frac{2MR\times\overline{RP}^3\times LS}{\overline{FO}^3}$

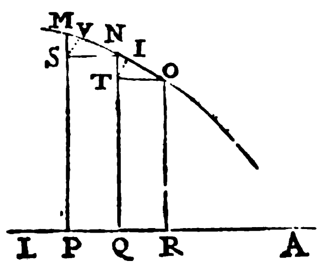
and dividing by $2\overline{NR}^3$ and $2\overline{RP}^3$, which are (*ex hyp.*) equal; we shall have $\frac{MN\times TO}{\overline{NO}^3} = \frac{MR\times LS}{\overline{FO}^3}$. And to destroy the Quantities LS, TO, which are Incomparably little in respect of the rest, I observe that, because the Triangles are similar, it is NO : RO :: LO : TO, and consequently, $TO = \frac{RO\times LO}{NO}$, and for the like reason, $LS = \frac{VF\times LO}{FO}$, and substituting these values in place of TO and LS, and dividing by LO, which is common; there will arise $\frac{MN\times RO}{\overline{NO}^4} = \frac{MR\times VF}{\overline{FO}^4}$ affected with their respective Ordinates, whence I conclude that the Nature of the Curve BFN is such, that (supposing the Elementa of the Ordinates to be equal,) if any Ordinate be multiplied by the respective Fluxion of the Abscissa, and the Product be divided by the Biquadrate of the respective Element of the Curve, the Quotient will be equal to (which is also evident in the preceeding method, supposing $m = n$) an Invariable Quantity: Assume the said Invariable Quantity at pleasure (*Servata Leg Homogeneorum.*) Then if AM be $= x$, MN $= y$, NR $=$ RP $= \dot{y}$; RO $= \dot{x}$; NO $= \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{z}$, we shall have $\frac{y\dot{x}}{\dot{z}^4}$ = an Invariable Quantity. *v. g.* $\frac{a}{\dot{y}^3}$, which being reduc'd we have this Differential Equation $y\dot{y}^3\dot{x} = a\dot{z}^4$ which expresses the Nature of the Curve requir'd. For if AB be taken $= a$, and if BC be drawn parallel to the Tangent in N, then AC is $= \frac{a\dot{x}}{\dot{y}}$, and BC $= \frac{a\dot{z}}{\dot{y}}$; whence if it be

$4AB^2\times AC\left(\frac{4aaa\dot{x}}{\dot{y}}\right) : BC^3\left(\frac{a^3\dot{z}^3}{\dot{y}^3}\right) :: BC\left(\frac{a\dot{z}}{\dot{y}}\right) : MN\ (y)$ then

$\frac{4a^3y\dot{x}}{\dot{y}} = \frac{a^4\dot{z}^4}{\dot{y}^4}$, and $4y\dot{y}^3\dot{x} = a\dot{z}^4$ and consequently $\frac{y\dot{y}^3\dot{x}}{\overline{\dot{x}^2+\dot{y}^2}|^2} = \frac{1}{4}a$, which being the same with that in the preceding Solution, shews that the property of the Curve which generates the Surface of the Solid of least Resistence is such, that drawing AB (a) perpendicular to the Axis, and BC parallel to the Tangent in any point N, it will always be $4AB^2\times AC : \overline{BC}^3 :: BC : MN$.

Another way.

178. Let A L be the Axis of the Curve M N O, and let the Ordinates M P, N Q O R be perpendicular to the Axis A L, and infinitely near one another; and suppose the Fluxions of the Ordinates, *viz*, S M, T N to be equal or Invariable, then the Resistence of the Zone M N will be express'd by $\frac{MP}{\overline{MN}^2}$, and that of the Zone N O by $\frac{NQ}{\overline{NO}^2}$ (because the Velocity of the Fluid or Solid, and the Cubes of M S and N T are equal and invariable Quantities, and may therefore be rejected.) therefore $\frac{MP}{\overline{MN}^2}+\frac{NQ}{\overline{NO}^2}$ must be a *minimum*. Now suppose



a Particle or Ray of Light, by a continual Refraction, to describe the Curve O N M consisting of the Infinitely little Lines O N, N M, so that the times which the Particle takes to describe O N, and N M, be as $\frac{MP}{\overline{MN}^2}$ and $\frac{NQ}{\overline{NO}^2}$, then 'tis (*Art.* 162. 163.) evident that their *Sum* must be a *minimum*. Now to find the times which the Particle takes to describe the Lines O N, N M, in other Terms; Draw S V perpendicular to M N, and T I perpendicular to N O, then because S M is $=$ T N, the Lines S V, T I are the Sines of the Angles O N T and S M N, that is, they are proportional to the Velocity of the Particle of Light which describes M N and O N; but because the Triangles S V M, N S M, T I N, O T N are similar, therefore the Velocity of the Particle of Light from O to N is as (T I $=$) $\frac{OT\times TN}{NO}$, and that of the same Particle from N to M is as $\frac{SN\times SM}{MN}$ or the Velocity in O N is as $\frac{OT}{NO}$, and that in N M is as $\frac{SN}{MN}$, therefore the time that the Particle takes to describe O N is as $\frac{\overline{NO}^2}{OT}$, and that which it takes to describe N M in, is as $\frac{\overline{MN}^2}{SN}$ therefore the time in which the particle moves from O to M, is as $\frac{\overline{MN}^2}{SN}+\frac{\overline{NO}^2}{OT}$, and comparing both these Proportions which determinate the times, and reducing the former to this Form $\frac{\overline{MN}^2\times SN\times MP}{SN\times\overline{MN}^4}$ and $\frac{\overline{NO}^2\times TO\times NQ}{TO\times\overline{NO}^4}$ that so there may be the same Proportion between these Terms as there is between the Terms $\frac{\overline{MN}^2}{SN}$ and $\frac{\overline{NO}^2}{OT}$ then tis evident that the multiplicators must be equal, *viz*, $\frac{SN\times MP}{\overline{MN}^4}=\frac{TO\times NQ}{\overline{NO}^4}$ that is, if any Ordinate be multiplyed be the respective Fluxion of the Abscissa, and if the Product be divided by the Fluxion of the Curve, the Quotient will always be equal to one and the same Invariable Quantity; which is the Property we took notice off in the last preceding Solution.

SECT.

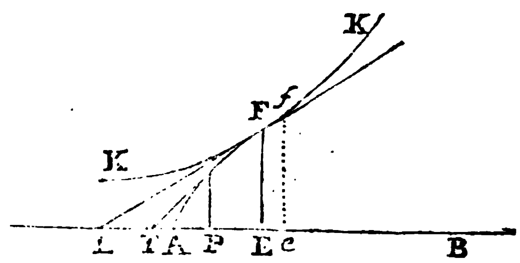
SECT. VI.

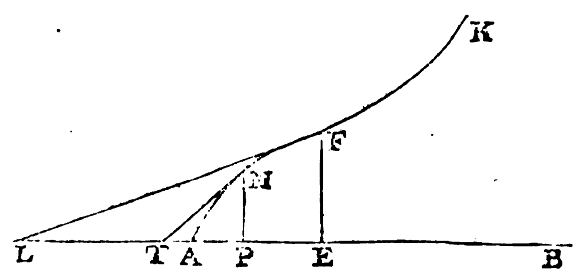
The Use of Fluxions

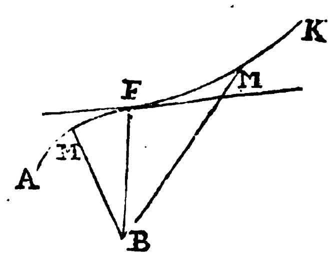
In Investigating the Points of contrary Flexion and Retrogression of Curves.

DEFINITION.

WHEN a Curve-line A F K is partly Concave and partly Convex, in respect of the Right-line A B, or in respect of the determinate Point B; the Point F which separates the Concave part of the Curve from the Convex, or which is the end of the one, and the beginning of the other, is call'd the *Point of contrary Flexion*, When the Curve is continu'd from F towards the same side as before. But when the Curve is continu'd backwards towards A, then F is call'd the *Point of Retrogression*.







179. If we suppose the Ordinate P M to move from A towards B, and consider the various Affections of the Fluxions thereof, as it moves along, it will be an easie matter to determine the point of contrary Flexion or Retrogression.

In the first place, let A B be the Diameter of the Curve-line A M K; and let the Ordinates P M, E F be parallel between themselves; and draw the Tangents M T, F L; then 'tis evident, that in Curves having a point of contrary Flexion, the Intercepted Diameter encreases continually, and the Portion of the Diameter A T Intercepted between the Tangent M T, and A the beginning of the Abscissa increases also, till the point P arrive at E, and afterwards decreases again; and hence 'tis plain, that the Portion of the Sub-tangent A T becomes a *Maximum*, when the points P and M fall in E and F.

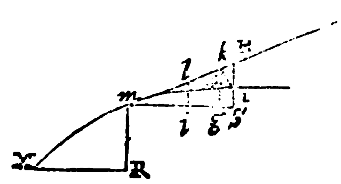
180. But when the Curve A M F is continu'd backwards from F towards A, then the Sub-tangent A T increases continually; But the intercepted Diameter increases only, until the point T arrive in L, or until the Ordinate P M co-incides with E F; and afterwards it decreases again.

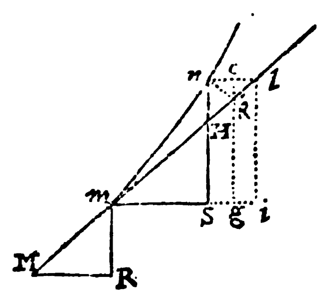
Hence to find a General Form which shall serve to Investigate the points of contrary Flexion and Retrogression.

Suppose A E $= x$, E F $= y$; then is A L $= \frac{y\dot{x}}{\dot{y}} - x$, and the Fluxion therof $\frac{\dot{y}^2\dot{x} - y\ddot{y}\dot{x}}{\dot{y}^2} - \dot{x}$ must be $= 0$, and by Transposition, and division (by $\dot{x}$, supposing $\dot{x}$ an Invariable Quantity) $\frac{\dot{y}^2\dot{x} - y\ddot{y}\dot{x} - \dot{y}^2\dot{x}}{\dot{y}^2} = 0$, and $\frac{-y\ddot{y}\dot{x}}{\dot{y}^2} = 0$, or *Infinity*; and multiplying by $\dot{y}^2$ and dividing by $-y$, we have $\ddot{y} = 0$, or *Infinity*: which for the future will serve for a General Form to find the points (F) of contrary Flexion and Retrogression; for the Nature of the Curve A F K being given, if we find the Value of $\dot{y}$ in $\dot{x}$, and again find the Fluxion of that Value (supposing $\dot{x}$, to be Invariable) we shall have the Value of $\ddot{y}$ in $\dot{x}^2$, which being put equal to nothing or Infinity, will serve in either of these suppositions, to find such a Value of A E, that the Ordinate E F shall intersect the Curve A F K in F the point of contrary Flexion or Retrogession.

181. The point A the beginning of x may be so scituated, that A L shall be $= x - \frac{y\dot{x}}{\dot{y}}$ instead of $\frac{y\dot{x}}{\dot{y}} - x$, and that A L or A E may be a *minimum* instead of being a *maximum*; but because the consequence is still the same, and that this can create no difficulty, it shall be sufficient to observe.

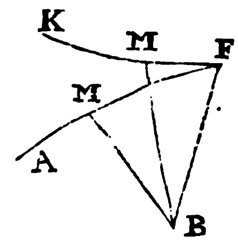
That A L can never be $= x + \frac{y\dot{x}}{\dot{y}}$; for when the point T falls on the other side of P in respect of A the beginning of x, then the value of $\frac{y\dot{x}}{\dot{y}}$ will be Negative, and consequently, the Value of $-\frac{y\dot{x}}{\dot{y}}$ will be Positive, and therefore in such a Case A E + E L is $= x - \frac{y\dot{x}}{\dot{y}}$.

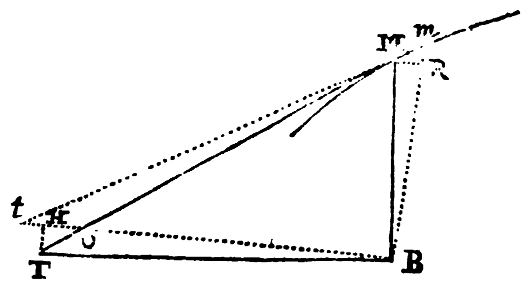




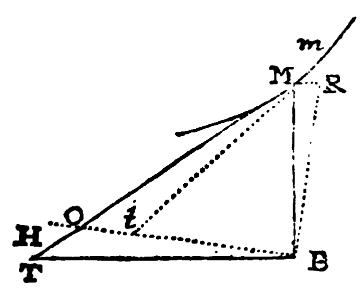
182. The point of contrary Flexion or Retrogression may be found otherwise, in this manner: It is evident that if $\dot{x}$ be supposed invariable, and that the Ordinate y be a Flowing Quantity, then S n is less than S H or R m, when the Curve is Concave towards the Axis: and S n is greater than S H or R m, when the Curve is Convex towards the Axis. Whence it follows, that the Value of H n or $\ddot{y}$ from being Positive becomes Negative in F, the point of Inflexion or Retrogression; that is $\ddot{y}$ is $= 0$, or *Infinity*.

183. And if the Curve AFK respect a single point B, then draw the Ordinates BM, BF, BM, all concurring in the given point B. Then if you draw any Ordinate as BM, and the Tangent MT intersecting BT perpendicular to BM in T, and if the point *m* be taken infinitely near to M, and the Ordinate BM, B*t* a perpendicular thereto, and the Tangent *mt* be drawn; 'tis evident (if we suppose the Ordinate BM to Increase as it comes to B*m*) that in F the Concave part of the Curve, B*t* Surpasses BO, (*o* being the point where MT intersects B*t*) and in the part of the Curve which is Convex towards B, B*t* is less than BO; whence 'tis manifest that in F the point of contrary Flexion or Retrogression, the Value of O*t* passes from being Positive to be Negative.





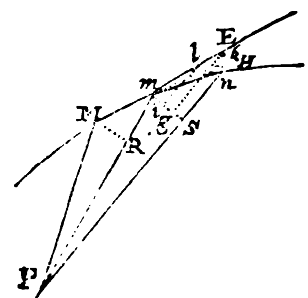
184. These things being premis'd: If on the Center B, and with the Radii BT, BM, the little Arches TH, MR be describ'd; then the Triangles MR*m*, MBT and THO are similar, and the little Sectors BMR, BTH are also similar; whence (supposing $BM = y$, $MR = \dot{x}$, $RM = \dot{y}$) $mR\ (\dot{y}) : RM\ (\dot{x}) :: BM\ (y) : BT = \frac{y\dot{x}}{\dot{y}} :: MR\ (\dot{x}) : TH = \frac{\dot{x}^2}{\dot{y}} :: TH\ \left(\frac{\dot{x}^2}{\dot{y}}\right) : HO = \frac{\dot{x}^3}{\dot{y}^2}$.

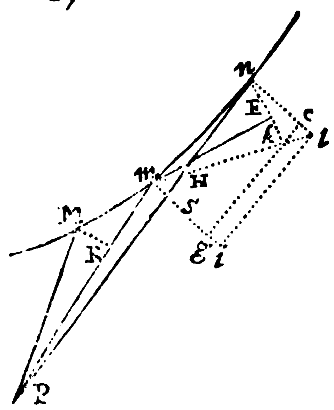


And if we take the Fluxion of BT $\left(\frac{y\dot{x}}{\dot{y}}\right)$ supposing $\dot{x}$ to be an Invariable Quantity, then is $Bt - BT = Ht = \frac{\dot{y}^2\dot{x} - y\dot{x}\ddot{y}}{\dot{y}^2}$ and $OH + Ht = \frac{\dot{x}^3 + \dot{y}^2\dot{x} - y\dot{x}\ddot{y}}{\dot{y}^2}$. Now because in the point of contrary Flexion or Retrogression, O*t* is either $= 0$, or *Infinity*, therefore in the said point, $\frac{\dot{x}^3 + \dot{y}^2\dot{x} - y\dot{x}\ddot{y}}{\dot{y}^2}$ is $= 0$, or Infinity, and multiplying by $\dot{y}^2$, and dividing by $\dot{x}$, we have $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, or *Infinity*; whence if the Nature of the Curve AFK be given, then the Value of $\dot{y}$ may be found in $\dot{x}$, and the Value of $\ddot{y}$ in $\dot{x}^2$; and if the said Values be substituted in the general form, there will remain one unknown Quantity ($\dot{x}$) and the Equation thus cleared, will serve to find such a Value of BF, that setting one foot of your Compasses in B, and with the other, at the distance BF, describing a Circle, it will cut the Curve in F, the point of contrary Flexion or Retrogression; which was required to be done.

185 And

185. And to determine the said points another way; It must be observ'd, that in the Concave part, the Angle PmE, is greater than the Angle Pmn, and contrarily, in the Convex part, the Angle PmE is less than Pmn, and consequently that the Angle $PmE - Pmn = Emn$, or the Arch En, from being Positive becomes Negative in F the point of contrary Flexion or Retrogression. And taking $\dot{x}$ for an invariable Quantity, the right angled Triangles HmS, Hnk are similar; therefore Hm $(=\dot{z})$: mS $(\dot{x})$ :: Hn $(-\ddot{y})$: $nk = -\frac{\dot{x}\ddot{y}}{\dot{z}}$; and here it must be observed, that Hn is Negative, because while Bm (y) Increases, mR $(\dot{y})$ Decreases. Now because the Sectors PmS, mEk are similar, it is Bm (y) : mS $(\dot{x})$ :: mE $(\dot{z})$: $Ek = \frac{\dot{x}\dot{z}}{y}$; and therefore $Ek + kn$ is $= \frac{\dot{x}\dot{z}^2 - y\dot{x}\ddot{y}}{y\dot{z}}$ and multiplying by $y\dot{z}$, and dividing by $\dot{x}$, we shall have $\dot{z}^2 - y\ddot{y}$, or (substituting $\dot{x}^2 + \dot{y}^2$ for $\dot{z}^2$) because of the right angled Triangle mSn, $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$, which passes from being Positive to be Negative, in the point of contrary Flexion or Retrogression.





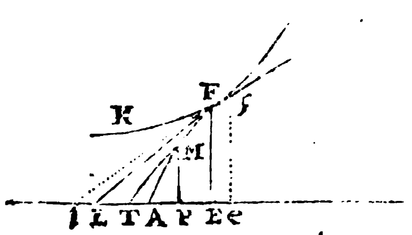
And if we suppose y to be Infinite, then the Terms $\dot{x}^2$, and $\dot{y}^2$ vanish, and are equal to nothing in respect of $y\ddot{y}$, and consequently the form $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, or *Infinity*, will become $-y\ddot{y} = 0$, or *Infinity*; that is to say, dividing by $-y$, $\ddot{y} = 0$, or Infinity; which is the form of the first case; and this ought to be so, because the Ordinates BM, BF, BM are then parallel to one another.

CONSECTARY I.

186. When $\ddot{y} = 0$, then 'tis evident that the Fluxion of AL is nothing in respect of $\dot{x}$ the Fluxion of AE; and that the two Tangents FL, fL being infinitely near each other, ought to make but one streight Line fFL.

CONSECTARY II.

And when $\ddot{y} =$ Infinity; then the Fluxion of AL ought to be infinitely great in comparison of that of AE, or which is the same thing, the Fluxion of AE (or $\dot{x}$) is infinitely little in respect of that of AL; and consequently we may draw two Tangents FL, Fl, to the same point F, comprehending the Infinitely little Angle LFf.



CONSECT.

CONSECTARY III.

In like manner, when $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, 'tis evident that, *ot* ought to be equal to nothing in respect of MR; and consequently, that the two Tangents MT *mt*, infinitely near each other, must Coincide, when the point M is the same with the point of contrary Flexion or Retrogression.

CONSECTARY IV.

And when $\dot{x}^2 + \dot{y}^2 - y\ddot{y} =$ *Infinity*, then *ot* is Infinite in respect of MR, or which is the same thing, MR is Infinitely little in comparison of *or*, and consequently the points M and *m* must Coincide; that is when the point M is the point of Inflexion or Retrogression, we may draw two Tangents through M, comprehending an Angle Infinitely little.

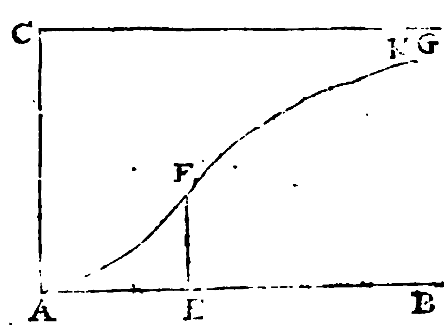
CONSECTARY V.

Hence it is evident also, that the Line which touches the Curve in the point of contrary Flexion or Retrogression, being prolonged, touches and cuts the Curve AFK in one and the same point.

PROP. I.

If the Curve Line AFK *be given, and its Diameter* AB; *and if the Relation of the Abscissa* AE *(x) to the Ordinate* EF *(y) be express'd by this Equation* $axx = xxy + aay$; *'tis requir'd to find the Value of* AE, *so that the corresponding Ordinate* EF *shall Intersect the Curve* AFK *in the point of contrary Flexion* F.

187. The Equation Curve is $y = \frac{axx}{xx+aa}$ and $\dot{y} = \frac{2a^3x\dot{x}}{\overline{xx+aa}|^2}$; and taking the Fluxion of this Quantity, and supposing $\dot{x}$ invariable, and putting the said Second Fluxion equal to nothing; we have $2a^3\dot{x}^2 \times \overline{xx+aa}^2 - \frac{8a^3x^2\dot{x}^2 \times \overline{xx+aa}}{\overline{xx+aa}^4} = 0$, and multiplying by $\overline{xx+aa}^4$, and dividing by $2a^3\dot{x}^2 \times \overline{xx+aa}$, we have $xx + aa - 4xx = 0$. And $3xx = aa$, that is x (AE) $= a\sqrt{\frac{1}{3}}$.



If we Substitute $\frac{1}{3}aa$ in place of xx in the Equation of the Curve $y = \frac{axx}{xx+aa}$, then $y = \frac{\frac{1}{3}a^3}{\frac{4}{3}aa} = \frac{1}{4}a =$ EF; so that we may determine the point of Inflexion F, without supposing the Curve AFK to be describ'd.

If AC be drawn parallel to the Ordinate EF, and equal to the given Line a, and if CG be drawn parallel to AB, it will be an Assymptote to the Curve AFK. For if we suppose x to be Infinite, then the Equation of the Curve $y = \frac{axx}{xx+aa}$ will become $y = \frac{axx}{xx} = a$. so that the Ordinate of the Curve EF cannot be $= a$ $=$ AC, before the Abscissa AE be Infinite.

 COROL.

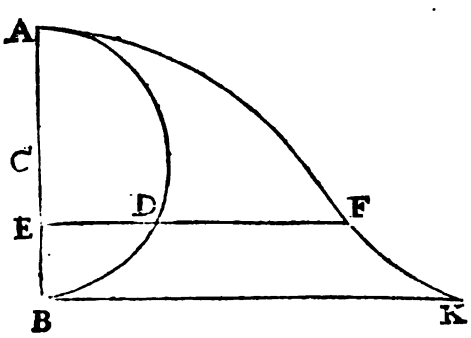
CORROLARY.

188. If the Equation of the Curve be $y - a = \overline{x-a}|^{\frac{3}{5}}$ then $\dot{y} = \frac{3}{5}\overline{x-a}|^{-\frac{2}{3}} \times \dot{x}$, and $\ddot{y} =$ (supposing $\dot{x}$ Invariable) $- \frac{6}{25}\overline{x-a}|^{\frac{-7}{5}}\dot{x}^2 = \frac{-6\dot{x}^2}{25\sqrt[5]{\overline{x-a}|^7}}, = 0.$ then $-6\dot{x}^2$ is $= 0$; which because it makes nothing for the resolution of the Question, therefore I put $\frac{-6\dot{x}^2}{25\sqrt[5]{\overline{x-a}|^7}} =$ *Infinity*; whence the Denominator $25\sqrt[5]{\overline{x-a}|^7}$ is $= 0$, and consequently, the unknown Quantity x (A E) is $= a$.

PROP. II.

If A F K *be a protracted Semi-cycloid whose Base* BK *is longer than the Semi-cumference of the generating Circle* A D B, *whose Center is* C; *'tis requir'd to find the point* E *in the Diameter* A B, *so that the Ordinate* E F *shall cut the Semi-cycloid in* F *the point of contrary Flexion.*

189. Suppose the known Quantities $ADB = a$, $BK = b$, $AB = 2r$, and the unknown Quantities $AE = x$, $ED = z$, the Arch $AD = u$; and $EF = y$; then by the property of the Cycloid $y = z + \frac{bu}{a}$, and consequently $\dot{y} = \dot{z} + \frac{b\dot{u}}{a}$; but by the property of the Circle $z = \sqrt{2rx - xx}$ and consequently, $\dot{z} = \frac{1}{2} \times \overline{2rx - xx}|^{-\frac{1}{2}} \times 2r\dot{x} - 2x\dot{x} = \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$ and $\dot{u} = \sqrt{\dot{x}^2 + \dot{z}^2} = \frac{r\dot{x}}{\sqrt{2rx - xx}}$, therefore substituting for $\dot{z}$ and $\dot{y}$ their respective Values, we have $\dot{y} = \frac{ar\dot{x} - ax\dot{x}}{a\sqrt{2rx - xx}} + \frac{br\dot{x}}{a\sqrt{2rx - xx}} = \frac{ar\dot{x} - ax\dot{x} + br\dot{x}}{a\sqrt{2rx - xx}}$ and the Fluxion thereof (supposing $\dot{x}$ Invariable) is, $\ddot{y} = \frac{brx - arr - br^2 \times \dot{x}^2}{2rx - xx \times \sqrt{2rx - xx}} = 0$, whence $brx - arr - br^2 \times \dot{x}^2$ is $= 0$, and dividing by $\dot{x}^2 \times r$, we have $bx - ar - br = 0$, and by Transposition $bx = ar + br$, and $x = r + \frac{ar}{b}$, and consequently $CE = \frac{ar}{b}$.



Hence 'tis manifest, that to have a point of contrary Flexion F, *b must be greater than a*; *for if b, be less than a, then* C E *would exceed* C B.

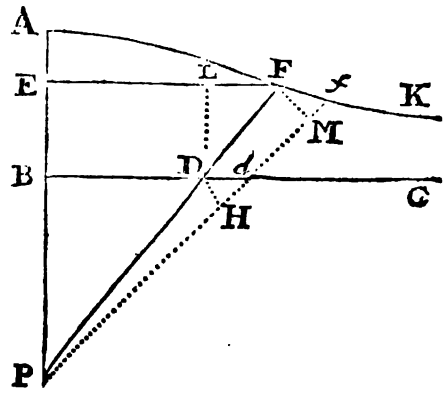
PROP.

PROP. III.

Let it be requir'd to find F *the point of contrary Flexion in* Nichomedes's *Conchoid* AFK.

190. Let BC be the Assymptote, and P the Pole of the Conchoid; then the property of the Conchoid is, that if you draw streight Lines from the Pole P to the Curve AFK, as PF, PA, then the Segments between the Assymptote and the Curve, *v. g.* AB, DF are always equal to a given Line a.

Draw PA perpendicular, and FE parallel to BC, and suppose the known Quantities $AB = FD = a$; $BP = b$: And the unknown Quantities $BE = x$, $EF = y$, and draw DL parallel to BA, then because the Triangles DLF, PEF, are similar; it is $DL\,(x) : LF\,(\sqrt{aa - xx}) :: PE\,(b + x) : EF = y = \frac{b + x\sqrt{aa - xx}}{x}$, and consequently $\dot{y} = \frac{x^3\dot{x} + aab\dot{x}}{xx\sqrt{aa - xx}}$. And $\ddot{y} = \frac{2a^4b - aax^3 - 3aabxx \times \dot{x}^2}{aax^3 - x^5 \times \sqrt{aa - xx}} = 0$, whence by Reduction there will arise $x^3 + 3bxx - 2aab = 0$, and one of the Values of the Root x will be $= PF$, which was requir'd.

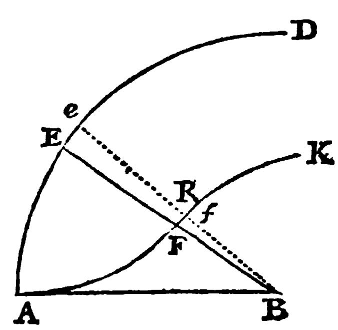


If a be $= b$, the preceding Equation will be changed into this other, $x^3 + 3axx - 2a^3 = 0$, which being divided by $x + a$, the Quotient is $xx + 2ax - 2a^2 = 0$, and consequently x is $= -a + \sqrt{3aa}$.

PROP. IV.

Let AED *be an Arch of a Circle, and* B *its Center, and let the property of the Curve Line* AFK *be such, that drawing any Ray* BFE *at pleasure, the Square of* FE *be equal to the Rectangle comprehended under the Arch* AE *and a given right Line* a. *'Tis requir'd to find the point* (F) *of contrary Flexion.*

191. Suppose the Arch $AE = z$, the Radius $BA = r$, and the Ordinate $BF = y$; then by the Property of the Curve $az = rr - 2ry + yy$, and consequently $\dot{z} = \frac{2y\dot{y} - 2r\dot{y}}{a} = Ee$; and because the Sectors BEe, BFR are similar, it is, $BE\,(r) : BF\,(y) :: Ee\,\left(\frac{2y\dot{y} - 2r\dot{y}}{a}\right) : FR = \dot{x} = \frac{2yy\dot{y} - 2ry\dot{y}}{ar}$ and the Fluxion therof (supposing $\dot{x}$ invariable) is $4y\dot{y}^2 - 2a\dot{y}^2 + 2yy\ddot{y} - 2ay\ddot{y} = 0$. And consequently $y\ddot{y} = \frac{a\dot{y}^2 - 2y\dot{y}^2}{y - a}$. Now if we substitute these Values of $\dot{x}^2$ and $y\ddot{y}$ in the general Theorem $y\ddot{y} = \dot{x}^2 + \dot{y}^2$, there will arise this Equation $\frac{r\dot{y}^2 - 2y\dot{y}^2}{y - a} = \frac{4y^4\dot{y}^2 - 8ry^3\dot{y}^2 + 4rryy\dot{y}^2 + rraa\dot{y}^2}{aarr}$ which by



Reduction

Reduction, is $4y^5 - 12ry^4 + 12rry^3 - 4r^3\overline{yy} + 3rraay - 2r^3aa = 0$; And one of the Values of the Root y will be $= BF$ requir'd.

SCHOLIUM I.

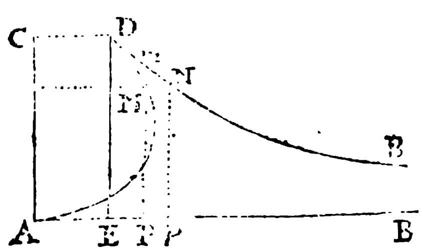
That the Curve AFK, *which we may call a* Parabolical Spiral, *has a point of contrary Flexion, may easily appear.*

For the Circumference AED not differing sensibly near A, from the Tangent in A, its plain from the Nature of the Parabola, that the Curve must be concave towards that Tangent, and that afterwards the Curvature of the circumference about its Center becoming more and more sensible, the said Curve must be concave towards the said Center B.

SCHOLIUM II.

The Points of Retrogression of Curves may be found by help of first Fluxions in this manner.

192. If the Curve AMDB be such that the Ordinates PMm intersect the same in two points M and m, then that Curve must have a point of Retrogression, *viz*, the point D; and to determine the same it must be observ'd that if (the Abscissa) x be supposed Invariable, then the Fluxion of the Ordinate when it (is greatest) passes through the point of Retrogression D, is equal to nothing; whence we may find the Value of AE the Abscissa corresponding to the same.



SECT-

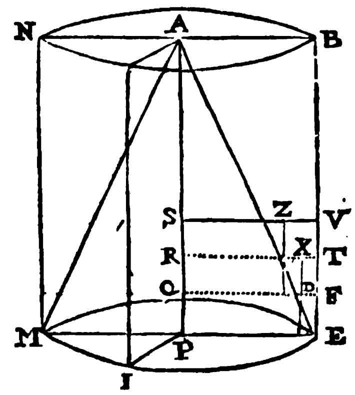
SECT. VII.

The Use of Fluxions

In Investigating the Dimensions of Solids.

The Genesis of Solids.

193. THE Rectangle ABEP consists of an Infinite Number of Rectangles QPEF, RQFT *&c.* And if we suppose $AP = x$; and the Infinitely little parts thereof, SR, RQ, QP equal to $\dot{x}$, and $PE = b$; then all the $b\dot{x}$ will be = Rectangle AE; and if we Imagine those Infinitely little Rectangles QPEF *&c.* To revolve about their respective Axes PQ, QR *&c*, each will generate an Infinitely little Cylinder, and the Sum of all those Infinitely little Cylinders will be equal to the Cylinder generated by the Rotation of the great Rectangle APEB about its Axis AP.



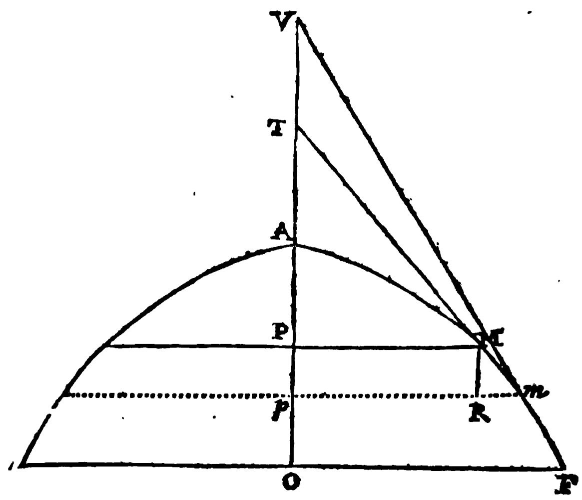
And to express the Value of those Cylinders in Analytic Terms; let the Ratio of the Radius to the circumference of any Circle be as r is to c; then because the Line PE revolves on the Point P as a Center, the Point E will describe the Periphery of a Circle $= \frac{bc}{r}$, and consequently the Area of the Circle of the Base is $= \frac{bbc}{2r}$ which being multiplyed by $PQ = \dot{x}$, the height of the Infinitely little Cylinder, the Product $\frac{bbc\dot{x}}{2r}$ is the Analytic Value thereof.

194. Let the Triangle APE be inscrib'd into the Rectangle APEB, then supposing $AR = x$, $RX = y$, $SR = \dot{x}$, $AP = d$; the Equation expressing the Nature of the Triangle will be $bx = dy$, and this Triangle will consist of all the $y\dot{x}$; but y is $= \frac{bx}{d}$, therefore the Triangle will consist of all the $\frac{bx\dot{x}}{d}$, that is, it will consist of all the Infinitely little Rectangles SRXZ, and if all these little Rectangles or the Triangle APE be turn'd about on their respective Axes SR, they will generate a Cone consisting of an Infinite number of such little Cylinders; and the Analytic Value of those Cylinders is $= \frac{cyy\dot{x}}{2r} = \left(\text{because } yy = \frac{bbxx}{dd}\right) = \frac{cbbxx\dot{x}}{2ddr}$ and the Cone AEM is = all the $\frac{cbbx^2\dot{x}}{2ddr}$ = (by finding the Flowing Quantity) $\frac{cbbx^3}{6ddr}$ = (putting $x = d$) $= \frac{cbbd}{6r}$. Whence it is evident that the Cone is to the circumscrib'd Cylinder EMNB as $\frac{cbbd}{6r}$ is to $\frac{cbbd}{2r}$, that is as 1 is to 3.

195 The

195. The Space A O F consists of an Infinite number of Trapezia P M *m p*, *p m* F O, which revolv'd about their respective Axes P *p*, *p o*, &c, generate so many Resected Cones, having their Vertices in T and V, where the Tangents or the Infinitely little portions of the Curve M *m*, *m* F produc'd, Intersects the Axis O A; and all those Resected Cones generated by the rotation of the Trapezia P M *m p*, &c about their Axes P *p*, &c. compose or constitute the Solid generated by the rotation of the Mixtilineal Figure A M *m* F O about its Axis A O.

And to find the Value of those Resected Cones, retaining the same Symbols as before, and supposing the proportion between the Radius and Circumference to be as r is to c, then the Area of the Circle describ'd by P M is $= \frac{cyy}{2r}$ which multiply-



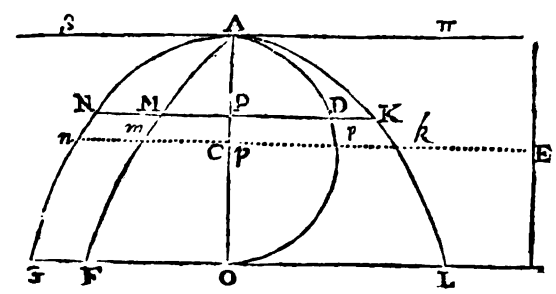
ed into $\frac{1}{3}$ P T $= t$, it will give $\frac{\frac{1}{3}tcyy}{2r}$ = to the Solidity of the lesser Cone generated by the rotation of the Triangle T P M, about its Axis T P. Again pm is $= y + \dot{y}$ and consequently the Area of the Circle generated by pm is $= \frac{cyy + 2cy\dot{y} + c\dot{y}^2}{2r}$ which being multiplyed by $\frac{1}{3}$ T $p = \frac{1}{3}t + \frac{1}{3}\dot{x}$, the product (rejecting those Terms affected with the Powers or Rectangles of $\dot{y}$ and $\dot{x}$) $\frac{\frac{1}{3}tcyy + \frac{2}{3}tcy\dot{y} + \frac{1}{3}cyy\dot{x}}{2r}$ = to the Content of the Cone generated by the rotation of the Triangle T pm about its Axis T p; from which subtracting the lesser Cone $= \frac{\frac{1}{3}tcyy}{2r}$ the remainder will be the Analytic Value of the Resected Cone generated by the rotation of the Trapezium P M mp about its Axis P p, *viz*, $\frac{\frac{2}{3}tcy\dot{y} + \frac{1}{3}cyy\dot{x}}{2r}$ = (because $t\dot{y} = y\dot{x}$) = $\frac{cyy\dot{x}}{2r} \left(= \frac{tcy\dot{y}}{2r}\right)$ Now $\frac{cyy\dot{x}}{2r}$ is = to the Cylinder generated by the rotation of the Rectangle P M R p about the Axis P p = to the Fluxion of the Conoid.

CORROLARY.

196. The Fluxion of any Solid generated by the rotation of any Curve Line about its Axis, is equal to the Area of the Base multiplyed into the Fluxion of the Axis or Abscissa.

197. If

197. If there be two Curves A M F, A N G applied to the same Axis A O; and if the Ordinate of the first Curve P M be $= y$, and that of the second P N $= z$, then will the Conoid generated by the rotation of the Figure A M F O about its Axis A O, be to the Conoid generated by the rotation of A N G O about its Axis A O as the sum of all the Cylinders $\frac{c}{2r} y y \dot{x}$ is to the sum of all the Cylinders $\frac{c}{2r} z z \dot{x}$, or (dividing by $\dot{x}$ and putting S for the Sum of all) as S $\frac{c}{2r} yy$ is to S $\frac{c}{2r} zz$, that is, as all the Circles describ'd by the Ordinates of the first, are to all the Circle describ'd by the Ordinates of the second Curve, and dividing by $\frac{c}{2r}$ as the Sum of all the yy is to the Sum of all the zz, that is, as the Sum of the Squares of the Ordinates of the first Curve is to the Sum of the Squares of the Ordinates of the Second.



198. If the Solid generated by A M F O be subtracted from the Solid generated by A N G O, then the Concave Solid generated by the rotation of the Figure A N G F M about the Axis A O is $= S \frac{czz\dot{x} - cyy\dot{x}}{2r}$.

And if two other Curves A D O and A K L be applied to the same Axis A O, and if P D be $= u$, and P K $= s$, then the Concave Solid generated by the rotation of A K L O D about A O will be $= S \frac{css\dot{x} - cuu\dot{x}}{2r}$.

Whence the Solid generated by A N G F M will be to the Solid generated by A K L O D as S $\frac{czz\dot{x} - cyy\dot{x}}{2r}$ is to S $\frac{css\dot{x} - cuu\dot{x}}{2r} :: \frac{czz - cyy}{2r} : \frac{css - cuu}{2r}$; that is as the Sum of all the Armillæ generated by the rotation of M N, $m\,n$, &c. about the Axis A O, is to the Sum of all the Armillæ generated by the rotation of D K, $d\,k$ &c. about the same Axis A O; and again, dividing by $\frac{c}{2r}$, as S $zz - yy$ is to S $ss - uu$, that is, as the Sum of the differences of the Squares of P N, P M &c. is to the Sum of the differences of the Squares of P K, P D, &c. whence making $z - y = f$, and $s - u = g$, these Solids will be to each other as $2fy + ff$ (because $\overline{z+y} \times \overline{z-y} = zz - yy = \overline{z+y} \times f = \overline{f+2y} \times f$) is to $2gu + gg$.

COROLLARY I.

199. Concave or Annular Bodies composed of Rings or Armillæ may be express'd by the differences of the Squares of their respective Ordinates; And the Fluxions of such Solids are found by multiplying the Area of any Annulus or Ring by the Fluxion of the Axis.

COROLLARY II.

If the proportion of r to s be invariable; and if it be always $r : s :: zz : yy$, then the Conoid generated by the rotation of A N G O about A O is to the Conoid generated by A M F O about A O, as r is to s.

COROL-

COROLLARY III.

And if the proportion be always $r : s :: zz - yy : uu$; Then the Annular Solid generated by the rotation of ANGFM about the Axis AO, is to the Conoid generated by the rotation of ADO about AO as r is to s; and if r be $= s$, then that Annular Solid will be $=$ to this Conoid.

COROLLARY IV.

Hence it is evident, if the Loca of any two, *v. g.* z and y be taken at pleaſure, we find that whereunto u belongs; for putting AP $= x$, ſuppoſe z to be the Ordinate of the Parabola ANGO, ſo that $2rx = zz$, and let y be applied to an Iſoſceles Triangle AFO, then y is $= x$; and becauſe $zz - yy = uu$, therefore if we Subſtitute the Values of z and y, we have $2rx - xx = uu$, whence 'tis evident that PD $= u$ is an Ordinate to the Semicircle ADO, whoſe Center is c. Therefore if a Parabolical Conoid be generated by the rotation of the Parabola ANP (whoſe Parameter is $= 2r$) about the Axis AP, and a Cone by the rotation of the Triangle AMP about the ſame Axis, the exceſs of the Parabolical Conoid (or of all the $2rx$) above the Cone (or all the xx) is $=$ to the Portion of a Sphere generated by the rotation of the Segment of a Semicircle APD about its Axis AP.

COROLLARY V.

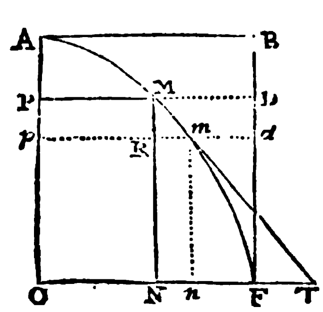
If we aſſume a fourth Figure AKLO whoſe Ordinates PK, *&c*, are always $= s$, and if we ſuppoſe $ss - uu = zz - yy$, that is, if the Annular Concave Solid generated by the rotation of AKLOD about AO be equal to the Annular Concave Solid generated by the rotation of ANGFM about AO. Then any three of theſe Quantities being given we may find the fourth.

COROLARY VI.

And if the Axis of rotation (or motion) BE be without the Ambit of the Figure LGA; the ſame proportions obtain; for putting E$n = z$, and EL $= y$; the Annular Solid generated by the rotation of the Figure LAG about the Axis BE may be expreſs'd by $S\,\overline{zz - yy} =$ (putting L$n = z - y = f$) $2fy + ff$.

200. And becauſe the Mixtilineal Plain AOF conſiſts not only of the Trapezia PMmp, but alſo of the Trapezia MNnm, if we ſuppoſe theſe Trapezia to revolve about the Axis AO, every one will generate a Tube or Cylinder hollow in the middle, and the Sum of all thoſe Tubes is equal to the Conoid generated by the rotation of the Figure AMFO abont the Axis AO.

And to find the Values of thoſe Tubes or hollow Cylinders, retaining the former Symbols, ſuppoſe PO $=$ MN $= x$; R$m =$ N$n = \dot{y}$, and $on =$ (becauſe PM $=$ ON $= y$) $y + \dot{y}$, and $mn = x - \dot{x}$; then the Rected Cone generated by the rotation of the Trapezium



PMmP about Pp is $= \frac{cyy\dot{x}}{2r}$, to which add the Cylinder generated by the Rectangle $ponm$ about AO, *viz* $\frac{cyyx}{2r} + \frac{2cy\dot{y}x - cy^2\dot{x}}{2r}$ the ſum is $= \frac{cyyx + 2cxy\dot{y}}{2r} =$ to the Solid generated by the rotation of the Pentagon PMmnO about the Axis PO, from which if we ſubtract the Cylinder generated by the Rectangle PMNO about

PO

$PO = \frac{cyyx}{2r}$, the remainder is the Solidity of the Tube generated by the Trapezium $MNnm$ revolving about PO, *viz.* $\frac{cxy\dot{y}}{r}$ = to the Fluxion or Infinitely little Increment or Decrement of the Conoid.

COROLLARY. I

201. But if $OF = AB$ be $= a$, and $AO = BF$ be $= b$, then the Fluxion of the Cylinder generated by the Rectangle $ABFO$ about the Axis AO is $= \frac{bcy\dot{y}}{r}$.

COROLLARY II.

Hence to find the Fluxion of any Solid considered as composed of Tubes or Cylindric Surfaces; multiply the circumference of any Tube by its Altitude, and that Product by the Fluxion of the Ordinate, so have you the Solidity of the Tube, or the Fluxion of the Solid.

COROLLARY III.

Let the two Figures $ANGO$, $AMFO$ be applied to the same Axis AO, and suppose $PN = z$; $PM = y$, $AP = x$; (*Fig. in Pag.* 163.) then the Solid generated by the rotation of $ANGO$ about $\beta A \pi$ drawn through the vertex A parallel to the Ordinates is $= S\frac{czx\dot{x}}{r}$ and the Solid generated by $AMFO$ about the same Axis $\beta A \pi$ is $= S\frac{cyx\dot{x}}{r}$.

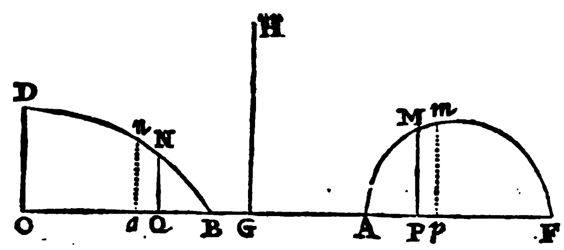
Whence that Solid is to this as $S\frac{czx\dot{x}}{r}$ is to $S\frac{cyx\dot{x}}{r}$, and dividing by $\dot{x}$, as $S\frac{czx}{r}$ is to $S\frac{cyx}{r}$ that is, as all the Cylindric Surfaces describ'd by the Ordinates PN &c, at their respective distances from the Axis of rotation, in the first; are to all the Cylindric Surfaces describ'd by the Ordinates PM &c, at their respective distances from the Axis $\beta A \pi$, in the second Solid. And dividing again by $\frac{c}{r}$; the first Solid will be to the second, as Szx is to Syx, that is, as the Sum of all the Rectangles comprehended under the Ordinates and Intercepted Diameters in the first are to the Sum of the respective Rectangles in the Second.

SCHOLIUM.

If PM or ON be $= y$, and $PO = NM = x$, $Pp = \dot{x}$, $Rm = \dot{y}$; then the Rectangle $PMNO$ or xy is the exponent of the Ratio of the Solid generated by $AMFO$ both about AO and OF, and therefore to avoid all error in such Cases; If the Solid be designed by the Rectangles xy, reduce them to their Fluxions $\dot{x}$ or $\dot{y}$; for then all the $\frac{cyx\dot{x}}{r}$ constitutes the Solid generated by the rotation of $AMFO$ about OF, and all the $\frac{cxy\dot{y}}{r}$ constitutes the Solid generated by the rotation of the same Figure $AMFO$ about AO.

202. Lastly, let the two Figures A M F and B N D be applied to equal Lines A F, B O, so that the Abscissæ A P, B Q and the Fluxions P p, Q q, be always equal, so that we may conceive them as applied to the same Axis, which ought to be carefully observ'd.

Then suppose A P $= x =$ B Q, P $p =$ Q $q = \dot{x}$, P M $= y$, Q N $= z$, A G $= b$; and B G $= d$, both determinate. And let G H be the Axis of rotation, then the Solid generated by the rotation of A M F about the Axis G H is $= S\,\frac{bcy\dot{x} + ycx\dot{x}}{r}$ and that generated by B D O about the same Axis G H is $= S\,\frac{dcz\dot{x} + czx\dot{x}}{r}$ Whence that Solid is to this, as $S\,\frac{byc\dot{x} + ycx\dot{x}}{r}$ is to $S\,\frac{dcz\dot{x} + czx\dot{x}}{r}$ or as (dividing by the Invariable $\dot{x}$) $S\,\frac{byc + ycx}{r}$ is to $S\,\frac{dcz + czx}{r}$ that is, as all the Cylindric Tubes generated by P M revolving about the Axis G H are to all the respective Tubes generated by Q N about the same Axis G H, and again dividing by $\frac{c}{r}$ as $S\,by + yx$ is to $S\,dz + xz$, that is as all the Rectangles comprehended under the Ordinates P M and Q N and their respective Distances from the Axis of rotation G P, G Q.



A General Corollary.

203. To Investigate the Dimensions of any Solid is the same thing as to find the Flowing Quantity of the Fluxion of the Solid. For the Flowing Quantity of any Fluxion is equal to the Sum of all the Fluxions, and the Sum of all the Fluxions is equal to the Solid.

PROP. I.

To Investigate the Solid content of a Cone.

204. Let the Cone be formed by the revolution of the Rectangular Triangle A D B about the Axis (*Fig.* 2. *in Pag.* 71.) A D, and let the Perpendicular M P, mp be drawn; 'tis evident that those Perpendiculars will describe Circles, and that the Sum of all those Circles is equal to the Cone.

Suppose A D $= a$, B D $= r$, the circumference of the Base $= c$, A P $= x$, P $p = \dot{x}$ P M $= y$, then is $\frac{cr}{2} =$ Area of the Base: And to find the Area of the Circle describ d by P M; say, $r : c :: y : \frac{cy}{r}$ which multiplyed by $\frac{y}{2} = \frac{1}{2}$ Radius; the Product $\frac{cyy}{2r}$ is $=$ Area of the Circle describ'd by P M, and if this Area be multiplyed by $\dot{x}$ the Fluxion of the Axis, the Product $\frac{cyy\dot{x}}{2r}$ is $=$ to the Fluxion or Element of

the

the Cone. Now by similar Triangles, $x : y :: a : r$, therefore $x = \frac{ay}{r}$ and $\dot{x} = \frac{a\dot{y}}{r}$, and consequently the Fluxion of the Solid $\frac{cyy\dot{x}}{2r}$ is $= \frac{acyy\dot{y}}{2rr}$, and the Flowing Quantity thereof is $\frac{acy^3}{6rr}$ = to the Value of that part of the Cone describ'd by the Rectangular Triangle A P M, and the Value of the whole Cone (when x becomes $= a$, and $y = r$) is $= \frac{acr}{6}$.

CONSECTARY I.

205. A Cone is to a Cylinder of the same Base and Altitude, as 1 is to 3. Hence to find the Solidity of a Cone; multiply the Area of the Base by $\frac{1}{3}$ the Altitude, the Product is the Content of the Cone.

CONSECTARY II.

This Proportion holds true also in a Partial Conversion; thus if the Rectangle A B E P be turn'd on its Axis from (*Fig. in Pag.* 161.) E to I, then the portion of the Cylinder generated by that Partial Conversion, is to the Portion of the Cone generated by the Conversion of the Rectangular Triangle A P E from E to I, as 3 is to 1.

PROP. II.

To Investigate the Solidity of a Sphere.

206. A Sphere is describ'd by the revolution of a Semicircle A M D about the Diameter A D, and if from every point in the circumference we may imagine Perpendiculars (*Fig. in Pag.* 73.) M P, *mp &c.* to the Diameter A D, they will describe each a Circle, and the Sum of all the Circles is equal to the Sphere.

Suppose the Radius P D $= r$ the Circumference $= c$, P M $= y$, A P $= x$, P $p = \dot{x}$. now to find the Area of the Circle describ'd by P M. Say, $r : c :: y : \frac{cy}{r}$, which multiplyed by $\frac{1}{2} y$, the Product $\frac{cyy}{2r}$ is = the Area requir'd. Multiply this Area by $\dot{x}$ the Fluxion of the Axis, and we have $\frac{cyy\dot{x}}{2r}$ = the Fluxion of the Sphere. But by the property of the Circle $yy = 2rx - xx$; and consequently $\frac{cyy\dot{x}}{2r}$ is $= cx\dot{x} - \frac{cxx\dot{x}}{2r}$ and the Flowing Quantity thereof is $\frac{cxx}{2} - \frac{cx^3}{6r}$ = to the Portion of the Sphere describ'd by A M P and the Value of the whole Sphere (when x becomes $= 2r$) $2crr - \frac{8crr}{6} = \frac{4}{6} crr$.

A Sphere may be consider'd as compos'd of an Infinite number of Pyramids, whose Vertex's are in the Center, and whose Bases are Infinitely little portions of the Surface of the Sphere, therefore if the Surface (which is the Base of all the Pyramids) of the Sphere, *viz.* $2cr$ be multiplyed by $\frac{1}{3}$ the Radius or common Altitude of the Pyramids, the Product $\frac{2}{3} crr$ is = to the Sphere.

CONSECT-

CONSECTARY I.

207. A Sphere is to the Cube of its Diameter, as $\frac{2}{3}$ the Circumference is to four times the Diameter; for $\frac{2}{3}crr : 8r^3 :: \frac{2}{3}c : 8r$.

CONSECTARY II.

The Sphere is to the Circumscrib'd Cylinder as 2 is to 3; for the Cylinder is $= crr$.

CONSECTARY III.

Spheres are as the Cubes of their Diameters for they are in a Ratio compounded of the Rationes of their Circumferences, and the Squares of their Diameters; that is, in a Triplicate Ratio of their Diameters.

CONSECTARY IV.

A Cone, whose Base is equal to a great Circle of the Sphere, and whose Height is equal to the Diameter of the Sphere, is to the Sphere as 1 is to 2, for the Cone is $= \frac{crr}{3}$.

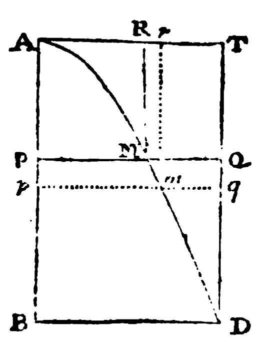
CONSECTARY V.

A Cone, a Sphere and a Cylinder of the same Height and Diameter, are as 1, 2, 3; and the Cone is equal to the Excess of the Cylinder above the Sphere.

PROP. III

To Investigate the Solidity of all sorts of Parabolical Conoids.

208. A Parabolical Conoid is generated by the revolution of the Semi-parabola ADB about its Axis AB. Draw any Ordinate $PM = y$, and pm parallel to and Infinitely near the same, and suppose $AB = b$, $AP = x$, $Pp = \dot{x}$, $BD = r$, and the Circumference describ'd by the point $D = c$; now this Solid may be consider'd as composed of an Infinite Number of Circles (parallel to the Circle of the Base) describ'd by the Ordinates as Semi-diameters; and to find the Area of the Circle describ'd by PM, say, $r : c :: y : \frac{cy}{r} =$ to the Circumference describ'd by the point M, and consequently the Area of the Circle whose Radius is $= PM$ is $= \frac{cyy}{2r}$ which multiplyed



by $\dot{x}$ the Fluxion of the Axis, the product $\frac{cyy\dot{x}}{2r}$ is $=$ to the Fluxion of the Solid; now the general Equation expressing the Nature of all sorts of Paraboloids is $y^m = x$ therefore $yy = x^{\frac{2}{m}}$, and the Fluxion of the Solid $\frac{cyy\dot{x}}{2r}$ is $= \frac{cx^{\frac{2}{m}}\dot{x}}{2r}$, and the

Flowing Quantity, or the Solid it self is $= \frac{m}{2m+4} \times \frac{cx^{\frac{2}{m}+1}}{r} = \frac{m}{2m+4} \times \frac{cyyx}{r}$ = to the Solid generated by the revolution of the Figure AMP about its Axis AP; and consequently the Value of the Solid generated by the revolution of the Figure ADB about the Axis AB is $= \frac{m}{2m+4} \times bcr$.

COROLLARY

Hence if the Equation of the Curve be $ax = yy$, then m is $= 2$; and $\frac{m}{2m+4} \times bcr$ is $= \frac{1}{4} bcr$, and consequently the Parabolical Conoid is to the circumscrib'd Cylinder (because the Cylinder is $= \frac{bcr}{2}$) as 1 is to 2; and the inscrib'd Cone is to the said Cylinder as 1 is to 3, therefore the Cylinder, the Conoid and the Cone are as 6, 3, 2.

PROP. IV.

To Investigate the Value of the Solid generated by the revolution of the Parabolic Space AMDB *about the line* DT *parallel to the Axis.*

209. Let the Ordinates PM, pm be produc'd to Q, q, then 'tis evident that the said Ordinates MP, mp &c, (*Fig. in Pag.* 168.) will describe Circular Surfaces, which will be the Elementa of the Solid whose Value is requir'd.

Suppose BD or AT $= r$, the Circmference describ'd by the Point B $= c$, AP $= x$, PM $= y$, and MQ $= r - y$; then say, $r : c :: r - y : \frac{cr - cy}{r}$ = to the Circumference describ'd by the point M. Therefore the Area of the Circle describ'd by MQ is $= \frac{crr - 2cry + cyy}{2r}$; which being subtracted from the Area of the Circle describ'd by QP or DB $\left(= \frac{rc}{2}\right)$ the remainder $\frac{2cry - cyy}{2r}$ is = to the Area of the Annulus describ'd by PM, which being multiplyed by $\dot{x}$, the Product $\frac{2cry\dot{x} - cyy\dot{x}}{2r}$ is = to the Fluxion of the Solid = (because $y = x^{\frac{1}{m}}$) $\frac{2crx^{\frac{1}{m}}\dot{x} - cx^{\frac{2}{m}}\dot{x}}{2r}$ and consequently the Flowing Quantity or the Solid generated by the revolution of the Figure APM about the Axis TQ is $= \frac{m}{m+1} \times cx^{\frac{1}{m}+1} - \frac{m}{2m+4} \times \frac{cx^{\frac{2}{m}+1}}{r} = \frac{m}{m+1} \times cxy - \frac{m}{2m+4} \times \frac{cyyx}{1}$, and the Value of the Solid generated by the revolution of the Figure ADMB about the Axis BT is = (because then x becomes $= b$ and $y = r$) $\frac{m}{m+1} bcr - \frac{m}{2m+4} bcr$.

And if ax be $= yy$, then $m = 2$, and the Value of the Solid generated by the revolution of the Parabolical Space AMDB about the Axis DT is $= \frac{2}{3} bcr - \frac{1}{4} bcr = \frac{5}{12} bcr$.

COROLLARY.

The Cylinder circumscrib'd about this Solid is to the Solid it self as 6 is to 5.

PROP. V.

To Investigate the Value of the Solid generated by the revolution of the Parabolic Space A M D B, *about the right Line* A T *which touches the Parabola in the Vertex* A.

210. Imagine the Axis AB to be divided into an Infinite Number of equal parts, and the Ordinates P M, pm, perpendicular to A B and Infinitely near one another; then 'tis evident that those Ordinates will describe Cylindric Surfaces, which will be the Elementa of the Solid whose Dimension is requir'd; Suppose A B $= r$, A T $= b$, A R $=$ P M $= x$, A P $= y$, P $p = \dot{y}$, the Circumference describ'd by the point B $= c$; then the Cylindric Surface describ'd by B D will be $= bc$: and to find that generated by P M, say, A B × B D $(br) : bc :: (r:c)$ A P × P M $(xy) : \frac{cxy}{r} =$ to the Cylindric Surface or hollow Tube describ'd by P M, which being multiplyed by P $p = \dot{y}$, the Product $\frac{cxy\dot{y}}{r}$ is $=$ to the Fluxion of the Solid; but by the property of the Curve $y = x^m$, and $y^{\frac{1}{m}} = x$, therefore the Fluxion of the Solid $\frac{cxy\dot{y}}{r}$ is $=$ $\frac{cy^{\frac{1}{m}+1}\dot{y}}{r}$ and the Flowing Quantity or the Solid it self is $= \frac{m}{2m+1} \times \frac{cy^{\frac{1}{m}+2}}{r} = \frac{m}{2m+1} \times \frac{cxyy}{r} =$ to the portion of the Solid generated by A P M, (*Fig. in Pag.* 168.) and consequently the whole Solid describ'd by A M D B is $=$ (because then $y = x$, and $x = b$) $\frac{m}{2m+1} \times bcr$.

COROLLARY I.

211. If the Equation of the Curve be $ay = xx$, then m is $= 2$, and the Value of the Solid $\frac{m}{2m+1} \times bcr$ is $= \frac{2}{5}bcr$.

COROLLARY II.

This Solid is to the circumscrib'd Cylinder as 4 is to 5, for, $\frac{2}{5}bcr : \frac{1}{2}bcr :: 4 : 5$.

COROLLARY III.

And the Solid generated by the Concave part A M D T is $= \frac{1}{10}bcr$; for the Solid describ'd by A M D B is to the circumscrib'd Cylinder as 4 is to 5, therefore the Solid generated by A M D T is $= \frac{1}{5}$ of $\frac{1}{2}bcr = \frac{1}{10}bcr$.

PROP.

PROP. VI.

To Investigate the Value of the Solid generated by the revolution of the Parabolic Space AMDB *about the Base* BD.

212. Suppose $AB = r$, $BD = b$, $AP = x$, $PB = r - x$, $PM = y$; then it is evident that the Ordinates PM describe Cylindric Surfaces or Tubes, which are in a Ratio compounded (*Fig. in Pag.* 168.) of their Rays and Heights, and consequently the Cylindric Surface describ'd by the Ordinate PM is $= \frac{cry - cxy}{r}$; and the Fluxion of the Solid is $= \frac{cry\dot{x} - cxy\dot{x}}{r}$; but by the property of the Curve $x = y^m$ and $y = x^{\frac{1}{m}}$, therefore the Fluxion of the Solid is $= \frac{crx^{\frac{1}{m}}\dot{x} - cx^{\frac{1}{m}+2}\dot{x}}{r}$, and the Flowing Quantity is $= \frac{m}{m+1} \times cx^{\frac{1}{m}+1} - \frac{m}{2m+1} \times \frac{cx^{\frac{1}{m}+2}}{r} =$ to the portion of the Solid describ'd by the Space $APM = \frac{m}{m+1} cxy - \frac{m}{2m+1} \times \frac{cxxy}{r}$ and consequently the Solid describ'd by the whole Space AMDB is $= \frac{m}{m+1} \times bcr - \frac{m}{2m+1} + bcr.$

And if the Equation of the Curve be $ax = yy$ then m is $= 2$, and the Value of the Solid generated by the Space AMDB about the Axis BD is $= \frac{2}{3}bcr - \frac{2}{5}bcr = \frac{4}{15}bcr$; and the said Solid is to the circumscrib'd Cylinder as 8 is to 15.

PROP. VII.

To Investigate the Value of the Solid generated by the revolution of the Hyperbolic Space BAMFEC *about the Assymptote* CE.

213. Let the portion of the other Assymptote BC be $= r$, and draw the Ordinates PM, pm, parallel and infinitely near each other; and suppose $PM = y$, $CP = x$, $Pp = \dot{x}$, $BA = b$; and the Circumference describ'd by the Point $B = c$; then 'tis evident that the Ordinates PM, pm (*Fig. in Pag.* 75.) will describe parallel Cylindric Surfaces; and that bc is = Surface describ'd by BA. Whence to find that describ'd by PM, say, $CB \times BA\ (br) : bc :: CP \times PM\ (xy) : \frac{cxy}{r}$ = Surface requir'd, and $\frac{cxy\dot{x}}{r}$ = the Fluxion of the Solid. Now the Equation expressing the Nature of all sorts of Hyperbola's in relation to their Assymptotes is y^m (m being a negative number whole or broken) $= x$, and consequently $y = x^{\frac{1}{m}}$ therefore the Fluxion of the Solid is $= \frac{cx^{\frac{1}{m}+1}\dot{x}}{r}$, and the Flowing Quantity is $= \frac{m}{2m+1} \times \frac{cx^{\frac{1}{m}+2}}{r} = \frac{m}{2m+1} \times \frac{cxxy}{r}$ = the Solid describ'd by the Hyperbolic Space ECPMF,

ECPMF, and consequently the Intire Solid generated by the revolution of CBAFEC is (because then x becomes $= r$, and $y = b$) $\frac{m}{2m+1} \times bcr$.

CONSECTARY.

214. In the Apollonian Hyperbola $aa = xy$, and $m = -1$, therefore $\frac{m}{2m+1} \times bcr = \frac{-1}{-2+1} \times bcr = \frac{+1}{+2-1} \times bcr = bcr$; therefore the said Solid is double the Cylinder generated by the revolution of the Rectangle AC about the Assymptote AE and thus we have the Demonstration of a *Paradox*, *viz.* that an *Infinite Space does generate a Solid of Finite Dimensions.*

PROP. VIII.

To Investigate the Value of the Solid generated by the revolution of the Hyperbolic Space BAMFEC *about the Assymptote* CB.

215. In this Genesis 'tis evident that the Ordinates PM, pm describe Circles which we may consider as the Elementa of the Solid; and if we suppose (*Fig. in Pag.* 75.) BA $= r$; the Circumference describ'd by the point A $= c$, and the Area of the Circle $= \frac{cr}{2}$; CB $= b$. then the Area of the Circle describ'd by PM will be $= \frac{cyy}{2r}$ and $\frac{cyy\dot{x}}{2r} =$ to the Fluxion of the Solid; but $xy = br$, then $y = \frac{bx}{x}$ and $yy = \frac{bbrr}{xx}$; and the Fluxion of the Solid $\frac{cyy\dot{x}}{2r}$ is $= \frac{bbcrr\dot{x}}{2rxx} = \frac{bbcrx^{-2}\dot{x}}{2}$ and the Flowing Quantity is $= \frac{bbcrx^{-1}}{-2} = \frac{bbcr}{-2x} =$ (when x becomes $= b$) $\frac{bcr}{-2} =$ to the Value of the Solid describ'd by the Space AMFECB about the Axis CB; but $\frac{cyy\dot{x}}{2r}$ is $= \frac{cx^{\frac{2}{m}}\dot{x}}{2r}$ and the Flowing Quantity is $= \frac{m}{m+2} \times \frac{cx^{\frac{2}{m}+1}}{2r} =$ to the Solid generated by the revolution of BAMP, and consequently the Solid generated by the rotation of the whole Space is $= \frac{m}{m+2} \times \frac{cb^{\frac{2}{m}+1}}{2r}$.

COROLLARY.

In the Apollonial Hyperbola $y^{-1} = x$ and $aa = xy$ and $m = -1$, therefore $\frac{m}{m+2} \times \frac{cb^{\frac{2}{m}+1}}{2r} = \frac{1}{-2} \times \frac{c}{rb} = \frac{c}{-2rb} =$ (supposing the Rectangle $rb = aa = 1$) $= \frac{bcr}{-2}$.

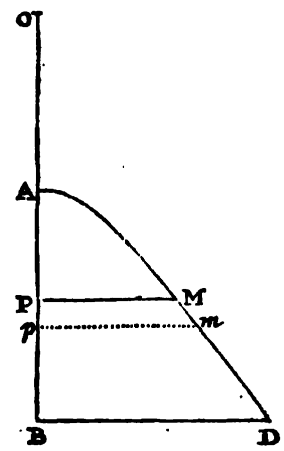
PROP.

PROP. IX.

To Investigate the Value of the Hyperbolical Conoid, generated by the revolution of the Hyperbola AMD, *about the Axis* AB.

216. Draw the Perpendicular $BD = r$, and the Ordinates PM ($= y$) and pm; now its evident that those Ordinates will describe Circular Surfaces, and that all the said Surfaces are equal to the Conoid.

Let the Transverse Axis AO be $= 2b$, $AB = d$, $OA = 2b + d$, $AP = x$, $Pp = \dot{x}$, $PO = 2b + x$; then by the preceeding Methods, the Fluxion of the Solid is $= \frac{cyy\dot{x}}{2r}$, and $rr : dd + 2db :: yy : xx + 2bx$; therefore $yy = \frac{rrxx + 2rrbx}{dd + 2db}$, and consequently the Fluxion of the Solid $\frac{cyy\dot{x}}{2r}$ is $= \frac{crxx\dot{x} + 2rbx\dot{x}}{2dd + 4db}$, and the Flowing Quantity or the Value of the Conoid generated by APM is $= \frac{crx^3 + 3rbcxx}{6dd + 12db}$, and the Value of the whole Solid generated by the Space AMDB is $=$ (because then $x = d$) $\frac{ddcr + 3bcrd}{6d + 12b}$.



CONSECTARY.

217. The Circumscrib'd Cylinder is to the Hyperbolical Conoid, as $3d + 6b$ is to $d + 3b$; for the Cylinder is $= \frac{dcr}{2}$; and if the Intercepted Diameter AB be equal to the Transverse Axis AO; then the said Cylinder is to the Conoid, as $12b$ is to $5b$, or as 12 is to 5.

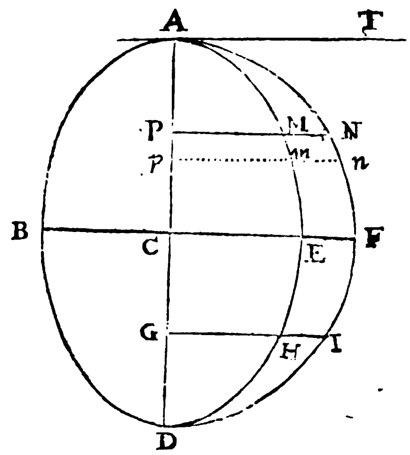
And if d be $= b$, then the Circumscrib'd Cylinder will be to the Hyperbolical Conoid, as $3b + 6b$ is to $b + 3b$, or as 9 is to 4.

And an Inscrib'd Cone is to the Hyperbolical Conoid as $d + 2b$ is to $d + 3b$.

PROP. X.

To Investigate the Value of an Oblong Spheroid, generated by the Conversion of the Semi-ellipse ABD *about its Tranverse Axis* AD.

218. Draw the Conjugate Diameter BE $= 2r$, perpendicular to AD $= 2a$, and draw any Ordinate as PM $= y$, and another mp infinitely near MP; then 'tis evident that those Ordinates will describe Circular Surfaces, which may be considered as the Elementa of the Spheroid. Suppose CP $= x$, AP $= a - x$, PD $= a + x$, then is $\frac{cyy}{2r}$ = to the Area of the Circle describ'd by the Ordinate PM. And $\frac{cyy\dot{x}}{2r}$ = to the Fluxion of the Spheroid; now by the Property of the Ellipsis, $\overline{MP}^2$ (yy) : AP × PD $(aa - xx)$ :: $\overline{CE}^2$ (rr) : $\overline{AC}^2$ (aa) therefore yy is $= rr - \frac{rrxx}{aa}$, and



consequently the Fluxion $\frac{cyy\dot{x}}{2r}$ is $= \frac{cr\dot{x}}{2} - \frac{crx^2\dot{x}}{2aa}$, and the Flowing Quantity is $= \frac{crx}{2} - \frac{crx^3}{6aa}$ = to the Value of the Solid generated by the conversion of CPME about the Axis CP, and consequently the Solid generated by ACEMA is = (because x becomes $= a$) $= \frac{acr}{2} - \frac{acr}{6} = \frac{acr}{3}$, the double whereof, *viz.* $\frac{2acr}{3}$ is = to the whole Oblong Spheroid.

LEMMA.

219. If the Circle AND be describ'd on the Axis AD as a Diameter, the Area of the Circle is to the Area of the Ellipsis, as the Transverse Axis AD is to the the Conjugate Axis BE; for AP × PD : PMq :: AG × GD : GHq : PNq : GIq. Therefore PM : PN :: GH : GI; and universally PM : PN :: CE : CF; therefore SPM (that is the Area of the Semi-ellipsis AED) is to SPN (or the Area of the Semi-circle AFD) :: CE : CF :: the Conjugate Axis of the Ellipsis to the Transverse.

Whence it universally follows also, that PMq : PNq :: CEq : CFq, and SPMq : SPNq :: CEq : CFq.

CONSECTARY I.

220. The Circumscrib'd Cylinder is to the Oblong Spheroid as 3 is to 2; for the Semi-spheroid $\frac{acr}{3} : \frac{acr}{2}$ :: 2 : 3. And because the Inscrib'd Cone is $= \frac{1}{3}$ the Cylinder, it is $= \frac{1}{2}$ the Spheroid.

CON-

CONSECTARY II.

A Spheroid is to the circumscrib'd Sphere, as the Square of the conjugate Axis is to the Square of the Transverse Axis; for if Ordinates P M, P N be drawn in the Ellipse and in the Circle, they will describe Circles which are as the Squares of the said Ordinates, that is, as the (*Art.* 219.) Square of C E is to the Square of C F; therefore all the Circles which compose the Spheroid are to all the Circles which compose the Sphere; that is, the Spheroid is to the Sphere, as the Square of the conjugate Axis, is to Square of the Transverse Axis; and any Segment of a Spheroid, is to the corresponding Segment of the Sphere in the same. Proportion.

CONSECTARY III.

A Spheroid is to the Inscrib'd Sphere, as the Transverse Axis of the Ellipsis is to the conjugate Axis; for if Ordinates be drawn to the conjugate Diameter of the Ellipsis, they will describe Cylindric Surfaces proportional to their corresponding parts in the Ellipse and in the Circle; but any Ordinate in the Ellipse, is to its corresponding Ordinate in the Circle, as (*Art.* 219.) the Transverse Axis of the Ellipsis is to the conjugate Axis; therefore the Spheroid is to the Sphere in the same Proportion.

CONSECTARY IV.

Any Portion of the Spheroid is to the corresponding Portion of the Inscrib'd Sphere, as the transverse Axis is to the conjugate Axis.

CONSECTARY V.

The Spheroid generated by the revolution of the Semi-ellipse A B D about the transverse Axis; is to that generated by the revolution of the Semi-ellipse B A E about the conjugate Axis, as the conjugate Axis is to the transverse Axis.

CONSECTARY VI.

The Spheroid generated by the revolution of the Semi-ellipse B A E, about the conjugate Axis B E, is to the circumscrib'd Cylinder as 2, is to 3.

PROP. XI.

To Investigate the Value of the Solid, generated by the revolution of the Logarithmetic Space A M B C E, *about the Assymptote* C E.

221. If we Imagine the Assymptote C E to be divided into an Infinite Number of equal parts, and an Infinite Number of Ordinates P M, $p\,m$ to be drawn, 'tis evident (*Fig. in Pag.* 77.) they will describe Parallel Circles which may be considered as the Elementa of the Solid; now supposing $CB = r$, $PM = y$, and $c =$ to the circumference describ'd by the point B, then is $\frac{cyy}{2r} =$ Area of the Circle whose Radius is $= PM$, and $\frac{cyy\dot{x}}{2r}$ is $=$ to the Fluxion of the Solid; but by the property of the Logarithmetic Line $\dot{x} = \frac{a\dot{y}}{y}$, therefore the Fluxion of the Solid

$cyy\dot{x}$

$\frac{cyy\dot{x}}{2r}$ is $= \frac{acy\dot{y}}{2r}$, and the Flowing Quantity is $= \frac{acyy}{4r}$ = to the Solid generated by the Infinite Space E P M A, and consequently (when y becomes $= r$) the whole Solid is $= \frac{acr}{4}$.

COROLLARY.

222. The Solid generated by this Infinite Space, is to a Cone whose height is = to the Sub-tangent of the Curve, and Base = to that of the Solid, as 3 is to 2; for the Cone is $= \frac{acr}{6}$.

PROP. XII.

To Investigate the Value of the Solid generated by the revolution of the Cissoidal Space E N A B F, *about its Assymptote* B F.

223. The same things being suppos'd as in (*Art.* 110.) all the Ordinates of the Cissoid P N describe Cylindric Surfaces which may be considered as the Elementa of (*Fig. in Pag.* 81.) the Solid; therefore say $r : c :: PB \times PN (x\sqrt{2rx - xx}) : \frac{cx\sqrt{2rx - xx}}{r}$ = to the Surface describ'd by P N, and $\frac{cx\dot{x}\sqrt{2rx - xx}}{r}$ is = to the Fluxion of the Solid.

And to find the Flowing Quantity; Imagine the Semi-circle A M B to revolve about an Axis parallel to the Assymptote B F, and passing through the point A; then 'tis evident that all the Ordinates P M will describe Cylindric Surfaces, and $\frac{cx\dot{x}\sqrt{2rx - xx}}{r}$ is = Fluxion of that Solid, which being the same with that of the preceding or Cissoidal Solid, 'tis evident that the Solid generated by the revolution of the Infinite Cissoidal Space, about its Assymptote is = to the Solid geneted by the conversion of the generating Semi-circle, about an Axis passing through the point A, parallel to the Assymptote B F.

Another way.

224. Retaining the same Symbols as before, B P ($\overline{2r - x} = b$) is to P M (y) :: A P (x) : P N ($= z$); therefore bz is $= xy$; that is the Rectangle B P $\times$ P N is always = the Rectangle A P $\times$ P M, and consequently the Cylindric Surface describ'd by P N, revolving about the Assymptote B F, is equal to the Cylindric Surface, describ'd by P M revolving about an Axis passing through the the point A parallel to the Assymptote B F, therefore their Sums must be equal, that is the Solid generated by E N A B F about B F, is = to the Solid generated by the Semi-circle A M B, revolving about an Axis passing through A.

PROP.

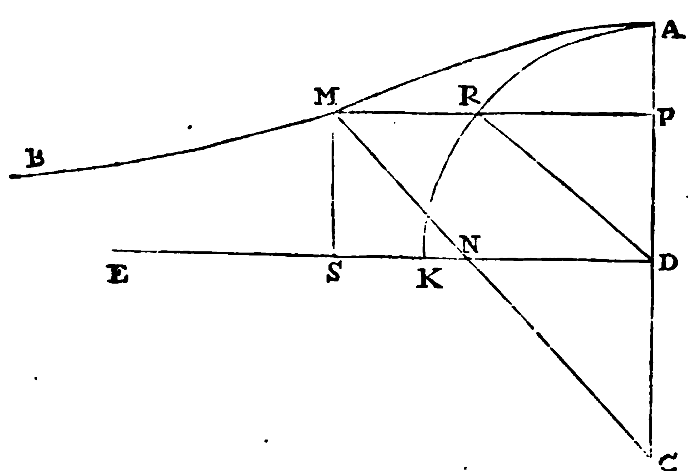
PROP. XIII.

To Investigate the Value of the Solid generated by the revolution of the Conchoidal Space BMADE, *about the Assymptote* DE.

225. Let the Conchoid AMB be such that drawing from the Pole C to any point (M) therein, the right Line CM cutting the Assymptote in N, the right Line MN be always equal to the right Line AD; to find the Value of the Solid generated by that Conchoidal Space.

With the Semi-diameter AD, describe the Quadrant AKD, and from any point in the Diameter as P, draw the Ordinate PM; then 'tis evident that these Ordinates will describe Cylindric Surfaces, which may be consider'd as the Elementa of the Solid whose Dimensions are requir'd.

Draw the Line DR and MS perpendicular to the Assymptote DE, then the Triangles NSM, DRP are similar, for the Angles MSN, RPD are right Angles, and MN is = DR, and MS = DP.



Suppose $AP = x$, $AC = a$, $AD = r$, $PC = a - x = b$, $PD = r - x = d$, $PR = y$ and $PM = z$: Then by the property of the Conchoid it is $CP\ (b) : PM\ (z) :: DP\ (d) : PR\ (= y)$ therefore by is always $= dz$, that is, the Rectangle $CP \times PR$ is always = Rectangle $DP \times PM$; now the Cylindric Surface describ'd by PM about the Axis DE, is as the Rectangle $DP \times PM$; and the Cylindric Surface describ'd by PR about an Axis parallel to the Assymptote DE, and passing through the Pole C, is as the Rectangle $CP \times PR$. Therefore the Cylindric Surfaces describ'd by the Ordinates PM, PR about their respective Axes are equal; and the Sum of all the Cylindric Surfaces describ'd by PM about DE, compose the Conchoidal Solid, and the Sum of all the Cylindric Surfaces describ'd by PR compose the Solid, generated by the Quadrant AKD, revolving about an Axis passing through C parallel to DE; therefore the Solid generated by the Conchoidal Space BADE about DE, is equal to the Solid generated by the Quadrant AKD, revolving about an Axis passing through C.

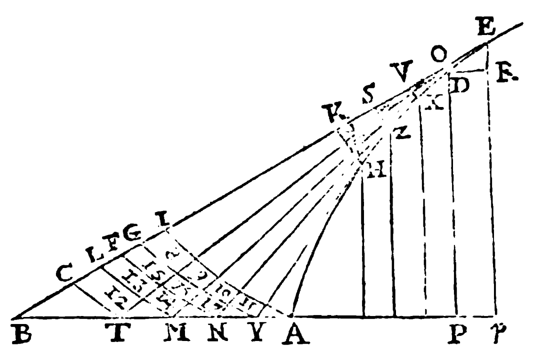
SECT. VIII.

The Use of Fluxions

In the Rectification of Curves

DEFINITION I.

LET AZXD be a Curve consisting of an infinite Number of little right Lines ED, DX, XZ, *&c.* and let the right Line BE touch the Curve in E, and suppose the other infinitely little Portions of the Curve to be produc'd, until they Interfect the Axis in the points T, M, N, R, *&c.* and assume EO = ED, and VO = D1 = DX; also SV = 31 = 2, X = XZ, and so proceed until KI be = 7, 8, = 6, 9 = 5, 10 = H, 11 = HA; then because the right Lines A11; 11, 10; 10, 9; 9, 8; 8, I, are infinitely little, and form Angles in 11, 10, 9, 8; it is evident that they will degenerate into the Curve AI, Concave towards the same part with the Curve ADE. And if we suppose a Thread to be applied to the Convex side of the Curve AZE, from A to E, and that one end being made fast in E, the other end in A be mov'd along from A towards I, so that that part of the Thread which has left the Curve, be extended at its full length, then it is also manifest that the said moveable extremity of the Thread, will describe the foresaid Curve A, 11, 10, 9, 8, 1. Now the Curve AZE is called the Evoluta, and the new Curve A9I, is said to be describ'd by Evolving the Curve AZE.



DEFINITION II.

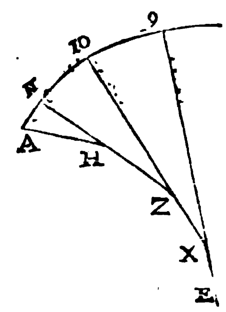
And the Portions of the Thread H11, Z10, X9, *&c.* which are extended into streight Lines, are called the Radii of the Evoluta, or the Radii of the Curvature in 11, 10, 9, *&c.*

CONSECTARY I.

226. Hence it appears that the Ray of the Evoluta *v. g.* H11, is equal to AH, the Portion of the Curve Evolved; and the Ray of the Evoluta EI, is equal to the whole Curve AZE.

CONSECTARY II.

If we imagine the Curve A Z X E to be a Polygon A H Z X E of an infinite Number of sides, then 'tis evident that A the extremity of the Thread A H Z X E will describe the little Arch A 11, on the Center H, with the Radius H A, until the Radius H A, come to be one streight Line with the infinitely little Side H Z, which is next to A H; in like manner, the extremity of the Thread will describe the Arch 11, 10, on the Center Z, with the Radius Z 11, until Z 11 come to be one streight Line with X Z, *&c.* and so on until all the Curve be Evolved. Whence 'tis evident that the Curve A 11, 10, 9, may be considered, as being composed of an infinite Number of Arches of Circles A 11; 11, 10; 10, 9; *&c.* Whose Centers are H, Z, X, *&c.*



CONSECTARY III.

Hence it appears that all the Rays of the Evoluta, touch the same as H 11 in H, 10 Z in Z, *&c.* and that all the said Rays are perpendicular to the Curve A 11, 10, 9, describ'd by Evolving the Curve A H Z X E: For, *v. g.* Z 10, is perpendicular to the same in 10, because Z 10 being produc'd, passes through Z and X, the Centers of the Arches 11, 10, and 10 9.

CONSECTARY IV.

Hence if two Curves begin in the same point A, and their Concavities look both the same way, as A E and A I, and if any Line touching the Interior Curve A E, *v. g.* D 8 be always perpendicular to the Exterior Curve A I, then the Portion of the Tangents Intercepted between the two Curves, *v. g.* D 8, will be equal to A X D the Portion of the Interior Curve Intercepted between the beginning A and the point of Contact D.

CONSECTARY V.

And because the Curvature of Circles Increase in proportion, as their Radii Decrease; it follows, that the Curvature of the infinitely little Arch A 11, is to the Curvature of the infinitely little Arch 10, 9, as X, 10, is to H 11; that is, in the Curve A 11, 10, 9, *&c.* the Curvature in 10, is to the Curvature in A reciprocally as the Rays, *viz.* as H A is to X 10, or Z 10; and in like manner, that the Curvature in 9 is to the Curvature in 10, as Z 10 is to X 9; whence it is manifest that the Curvature of the Line A 11, 10, 9, *&c.* Decreases continually in proportion to the Portion of the Curve A H Z X E, which is Evolved; so that in the point A, where the Evolution begins, the Curvature is the greatest that can be, and in 9, where the Evolution ceases, it is least.

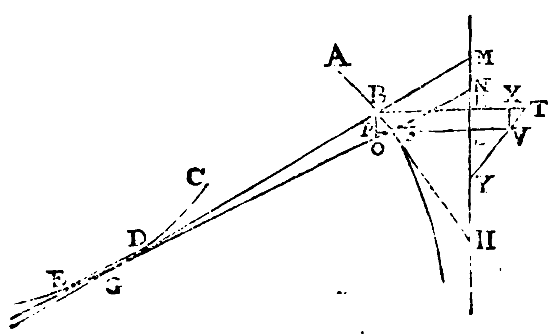
CONSECTARY VI.

Hence it appears also, that the points of the Evoluta are nothing else but so many points determined by the Intersections of the Perpendiculars to the Curve A 11, 10, 9, *&c.* the said Perpendiculars being infinitely near one another; for instance, the point X or E in the Evoluta, is determined by the Intersection of the right Lines 10 X, 9 X, which are perpendicular to the infinitely little Arches 11, 10 and 10, 9; so that if the nature and position of the Curve A 11, 10, 9, and one of its Perpendiculars *v. g.* 10 X, be given, to find the point X, where it touches the Evoluta, there is nothing else to be done, but to Investigate the point X where the Perpendiculars 10 X, 9 E, Intersect each other; for 'tis plain that that point will be in the Evoluta. PROP.

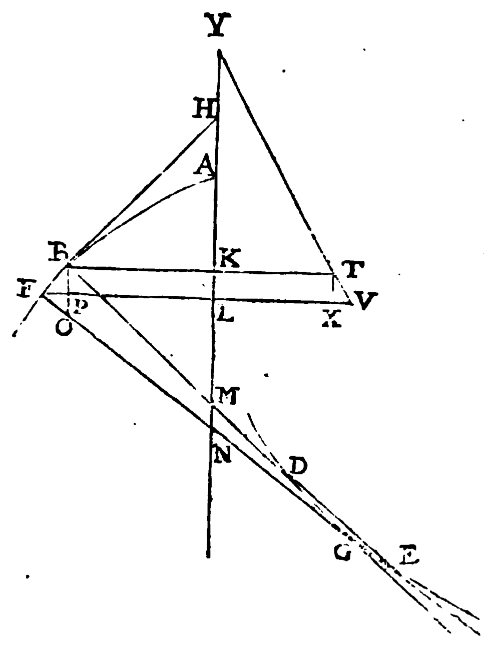
PROP. I.

Any Curve Line being given, to find the Evoluta thereof, that is, to find another by whose Evolution the given Curve shall be describ'd; and to shew that from every Geometrical Curve, another Geometrical Curve may be found, to which an equal right Line may be assigned.

227. Let A B F be any Curve, or a Portion of any given Curve Inflected one way only; and let L K be a right Line, to which all the points of the Curve are referred; and let it be requir'd to find another Curve D E, by whose Evolution the given Curve A B F may be describ'd.



Suppose the Evoluta C D E to be found; Then because all the Tangents of the Curve D E must needs be perpendicular to the Curve A B F, describ'd by the Evolution of D E; 'tis plain, that the Lines, *v. g.* B D, F E, which are perpendicular to the Curve A B F must touch the Curve C D E.



Let the points B and F be suppofed infinitely near each other; then because the Evolution begins in A, and F is the remoter point from A, therefore the point of Contact E, will be farther from A than D, and the point G, where the right Lines B D, F E Interfect each other, falls beyond the point D in the right Line B D; for 'tis manifest that the right Lines B D, F E will meet, each being perpendicular to the Concave part of the same Curve B F; and because the points B, F are suppos'd infinitely near each other, therefore the points D, G, E, are infinitely near one another; and all the three points may be taken for one. Produce B F unto H, then the right Line B H will touch the Curve in B and F; draw B O parallel to K L, and draw B K, F L perpendicular to K L, and let B O Interfect F L (produced if need be) in P; and let the points where B D, F E Interfects K L, be marked with M and N.

Now because the Ratio of B G to G M, is the same as of B O to M N; 'tis evident that if this be given, that is also given; and because the right Line B M is given in Magnitude and Position, the point G in B M produced, or the point D in the Curve (both these Coinciding) will consequently be given.

In all Geometrical Curves the Ratio of B O to M N is compounded of two Rationes, which are both given; *viz*, the Ratio of B O to M N is compounded of the Ratio of B O to B P, or of N H to L H, and of B P or K L to M N; whence 'tis evident that if those two Ratio's be given, then the Ratio of B O to M N will be given also: and that they are given in all Geometrical Curves; and consequently that Curves may be assigned to every one of them, by whose Evolution they may be describ'd; and that therefore all Geometrical Curves being describ'd by the Evolution of some Curve; and that these are reducible to streight Lines, I shall endeavour in the next place to shew by the following Examples.

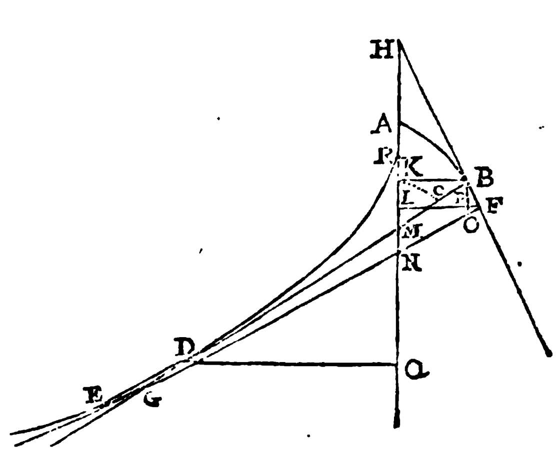
EXAMPLE

EXAMPLE I.

228. Let A B F be a Parabola, A its Vertex, and A Q its Axis; now becauſ the Lines B M, F N are perpendicular to the Curve of the Parabola, and B K, F L perpendicular to the Axis A Q, therefore (by the property of the Parabola) K M = L N is = $\frac{1}{2}$ the Parameter of the Axis; and taking away L M which is common to both, there remains K L = M N. Hence becauſe the Ratio of B G to G M is compounded of the Rationes of N H to H L, and K L to M N, and this laſt is the Ratio of Equality, therefore 'tis evident that B G : G M :: N H : H L. And by diviſion B M : M G :: N L : H L :: (becauſe the points K and L are infinitely near each other) M K : K H

Now the point B being given, the Ratio of M K to K H is alſo given, M K being = $\frac{1}{2}$ the Parameter of the Axis, and K H = 2 A K.

The Magnitude and Poſition of B M is alſo given, and therefore if we produce B M unto G, and take M G to B M, as 2 A K is to $\frac{1}{2}$ the Parameter of the Axis K M, the point G will be in the Curve R D E; and thus aſſuming ſeveral points in the Curve of the Parabola, we may determine as many points of the Curve R D E as we pleaſe; and conſequently, the Line R D E will be a Geometrical Curve, and one of the principal Properties thereof, and from which the reſt may be deduced, may be Inveſtigated.



Thus if it be required to find an Equation expreſſing the Relation of all the points of the Curve R D E, to the right Line A Q; draw the Line D Q perpendicular to A Q, and let the Parameter of the principal Axis of the Parabola be $= a$, $AK = z$, $AQ = x$, $QD = y$; then becauſe the Ratio of B M to M D, that is, the Ratio of K M to M Q is as $\frac{1}{2}a$ is to $2z$, and $KM = \frac{1}{2}a$, it follows that $MQ = 2z$. But M A is $= \frac{1}{2}a + z$, *Ergo* $AQ = \frac{1}{2}a + 3z$, and conſequently $z = \frac{1}{3}x - \frac{1}{6}a$. Moreover, becauſe $MKq\ (\frac{1}{4}aa) : KBq\ (az) :: MQq\ (4zz) : QDq\ (yy)$ therefore $yy = \frac{16z^3}{a}$ = (ſubſtituting $\frac{1}{3}x - \frac{1}{6}a$ for z $\frac{16 \times \overline{\frac{1}{3}x - \frac{1}{6}a}|^3}{a}$, and conſequently, $\frac{27}{16}ayy = \overline{x - \frac{1}{2}a}^3$. Now becauſe $\frac{1}{2}a$ is an Invariable Quantity, take $AR = \frac{1}{2}a$, and then R Q is $= x - \frac{1}{2}a$. Whence 'tis evident, that the Property of the Curve R D E, is ſuch, that the Cube of R Q, is always equal to the Square of the Ordinate Q D, multiplyed into an Invariable Quantity $\frac{27}{16}a$; and conſequently the Evoluta R D E, is a Cubical Paraboloid, and the Parabola A B F, may be deſcrib'd by the Evolution thereof; and the Parameter of R D E is $= \frac{27}{16}$ the Parameter of A B F; and the Parameter of the Parabola is $= \frac{16}{27}$ the Parameter of the Paraboloid.

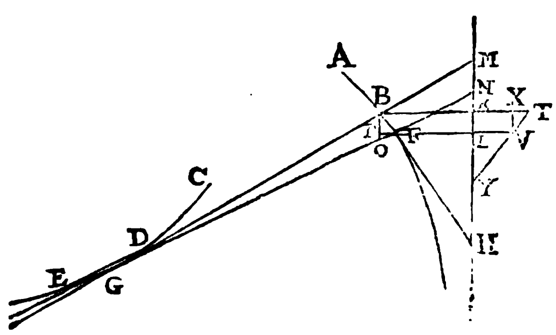
CONSECTARY I.

229. Hence if R D E be a Paraboloid and p its Parameter, and if $RQcub = QDq \times p$; then if the Tangent Q R, be produced to A, and R A be taken $= \frac{8}{27}p$, and the Evolution begin in A, then the Curve deſcrib'd by the Evolution of R D E, *viz.* A B F will be a Parabola, and the Parameter of the Parabola will be $= \frac{16}{27}p$, and the diſtance of A the Vertex of the Parabola, from R the Vertex of the Paraboloid is $= \frac{1}{2}$ the Parameter of the Parabola $= \frac{8}{27}p$.

CONSECTARY II.

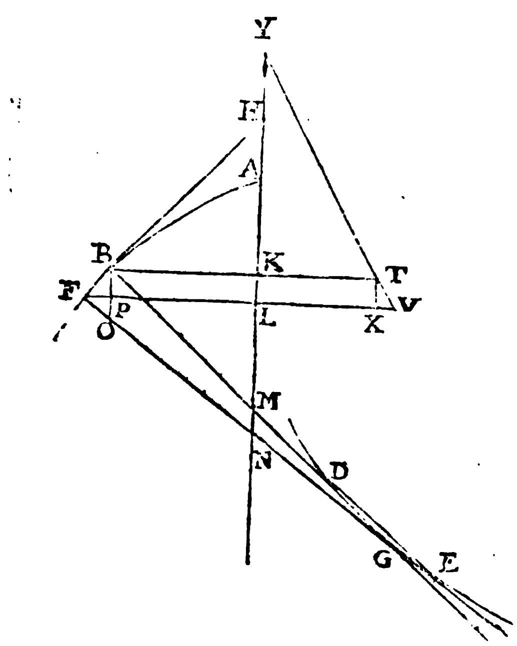
The Tangent D B, is = to the Portion of the Curve D R + A R (or + $\frac{8}{27}p$); And therefore to find the Length of the Curve D R, draw D M touching the Curve in D, and interſecting K L in M; then take M K = (becauſe M K is = $\frac{1}{2}$ the Parameter of the Parabola = A R) = $\frac{8}{27}p$; and in the point K erect the Perpendicular K B, interſecting the Tangent D M, produced in B; then 'tis evident that the point B, will be in the Curve of the Parabola A F; (although the Parabola, be not actually deſcrib'd) on the Center M, with the Radius M K, deſcribe the Arch of a Circle K C, interſecting M B in C; then is the Portion of the Paraboloidical Curve R D = D M + C B. And to find the point M, through which the Tangent D M muſt paſs, take R M = (*Art.* 25) $\frac{1}{3}$ R Q.

230. And to find the Ratio of O B to P B, or N H to H L, in all other Geometrical Curves, beſides the Parabola, is very evident and plain; it being only needful, to draw the right Line F H, to touch the Curve in the given point F, and F N, perpendicular to F H; for then N H and H L are given, and conſequently, the Ratio between them is alſo given.



But it is not ſo plain, how the Ratio of K L to M N, may become known, which nevertheleſs may always be found in manner following.

Let the right Lines K T, L V be perpendicular to K L, and let K T be = K M, and L V = L N, and draw V X (or T X) parallel to L K, and interſecting K T (or L V) in X; then becauſe the difference between L K and N M, is equal to the difference between L N and K M, or L V and K T, and the difference between L V and K T is = X V; therefore (becauſe X T or X V is = L K) N M = X V ± X T; and conſequently, if the Ratio of V X to X T be given, then the Ratio of V X to V X ± X T, that is the Ratio of V X or L K to M N will be given alſo.



And it muſt be obſerved, that becauſe K T is = K M, and L V = L N, the Locus of the Points T, V, may happen to be either a ſtreight or a Curve Line, and if it be a right Line, as it happens when A B F is a Coni-ſection, and K L its Axis; then it is evident, that the Ratio of V X to T X is given, the poſition of the Line T V being given, and the Ratio being always the ſame, the Interval K L being taken at pleaſure.

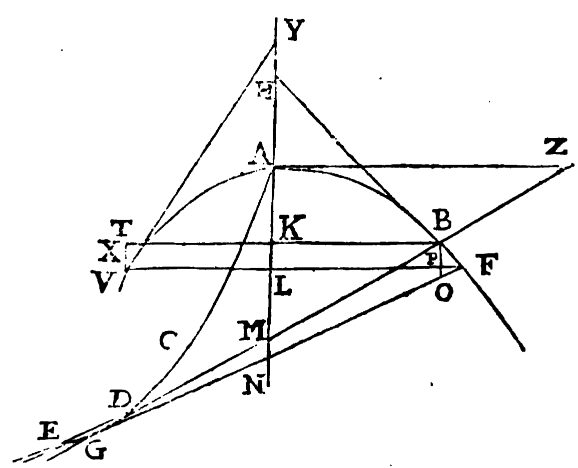
But if the Locus of thoſe Points be a Curve Line, the Ratio of V X to X T will vary, according as the Interval K L is greater or leſſer. Now in this Caſe, we muſt enquire, what the Ratio between them will be, when the Diſtance K L is infinitely little; in which Caſe, the Points B and F, and alſo V and T are infinitely near each other: And the infinitely little Line V T being drawn, will be a Portion of the Curve paſſing through V and T; and the ſame Portion V T being produced to Y, will touch

touch the said Curve in T. Now the Curve passing thro' T and V is a Geometrical Curve; and consequently the *Sub-tangent* may be determined: whence the Ratio of YK to KT, that is, the Ratio of VX to XT is given; and consequently the Ratio of LK to NM may be found as is shewn before.

It remains only to determine the Nature of the Curve TV, in order to find the Ratio between the Sub-tangent KY, and the Ordinate KT; which may be easily done, if we consider that the Nature of the Curve ABF is given, and BM is perpendicular thereto; and consequently that KM the Subnormal, or KT (y) the Ordinate of the new Curve is also given. Now from these the Nature of the Curve TV, and likewise the Ratio between the Sub-tangent KY, and the Ordinate KT may be determined. These Directions will be better understood by an

EXAMPLE II.

231. Let ABF be a Cubical Paraboloid (to which we have before assigned a Right Line equal) the Property whereof is, that the Cubes of the Ordinates KB, are Proportional to the Squares of the Intercepted Diameters AK. 'Tis required to find (or describe) the Curve CDE, by whose Evolution the Curve Line ABF shall be described.



First, The Ratio of BO to BP is easily found; for if we take $AH = \frac{1}{2}AK$, and draw HB, it will (*Art.* 25.) touch the Curve in B, and because BM is perpendicular to the Curve in B, the Lines MH and HK; and consequently their Ratio, that is, the Ratio of BO to BP is given.

Secondly, To find the Ratio of BP or KL to MN; Draw the Lines KT, LV perpendicular to KL, and equal to KM, LN; then is $VX + XT = MN$, and $KL : MN :: VX : VX + XT$. And to determine this Ratio, when the Distance KL is infinitely little, we must find the Locus, that is, we must determine the Nature of the Line which the Points T and V terminate in: To do which, Let the Parameter of the Paraboloid ABF be $= a$, $AK = x$, $KT = y$.

Then because $KH : KB : KM$ ÷÷, and $HK = \frac{1}{2}x$, and $KB =$ (by the property of the Curve) $\overline{axx}^{\frac{1}{3}}$, therefore $\frac{1}{2}x : \overline{axx}|^{\frac{1}{3}} :: \overline{axx}^{\frac{1}{3}} : \frac{2 \times \overline{axx}^{\frac{2}{3}}}{3x} = KM = KT = y$; and consequently $\frac{8 \times \overline{axx^2}}{27x^3} = \frac{8a^2x^4}{27x^3} = \frac{8}{27}aax = y^3$; whence it is evident, that the Locus of the Points T, V, is also a Cubical Parabola (of another kind;) and therefore if we take $AY = 2AK$, and draw TY, it will touch the Curve in T; therefore $VX : XT :: YK : KT$, and $VX : VX + XT :: YK : YK + KT :: KL : MN$; whence the Ratio of KL to MN is given, and the Ratio of OB to PB was found before; *ergo* the Ratio compounded of both, that is, the Ratio BD to DM is also given, and by Division, the Ratio of BM to MD, and consequently the Point D in the Curve DE.

232. Hence to construct the Curve DE: $KT = KM = y$, therefore MH is $= y + \frac{1}{2}x$, and $MH : HK :: y + \frac{1}{2}x : \frac{1}{2}x ::$ (multiplying by 2) $2y + 3x : 3x$. Again, because KY is $= 3x$, therefore $YK : YK + KT :: 3x : 3x + y$ (that is, $:: KL : MN$;) now the Ratio of BD to DM is compounded of the Rationes of BO to BP, and BP or LK to MN, that is, of $2y + 3x$ to $3x$, and $3x$ to $3x + y$,

$+y$; *ergo* BO : MN :: $2y+3x : 3x+y$: BD : MD, and by Division, $y : 3x+y$ :: BM : MD, whence arises this easie

CONSTRUCTION.

Draw AZ perpendicular to AK, and produce it until it intersect DB (produced) in Z; then because BM : MD :: $y : y+3x$; therefore BM : MD :: KM : KM + 3 AK :: MB : MB + 3 BZ; *ergo* MD is = MB + 3 BZ, whence we may easily find as many Points of the Curve CDE as we please; and any Portion of the Curve as DA, is equal in length to the Right Line DB, which meets the Paraboloid AB at Right Angles in B, and touches the Curve CD in D.

EXAMPLE III.

If the Cubical Parabola ABF *be such, that the Cubes of the Ordinates be proportional to the intercepted Diameters; it is required to describe the Curve* CDE, *by whose Evolution the said Cubical Parabola* ABF *shall be described.*

233. Let A be the Parameter of the Paraboloid ABF, the Abscissa AL $= x$, then by the property of the Curve $\overline{aax}^{\frac{1}{3}} =$ LB, and if BF touch the Curve in B, and being produced intersect the Axis in H, then (*see Fig. in the opposite Pag.*) LH is = 3 AL = $3x$; let LM = LV be $= y$.

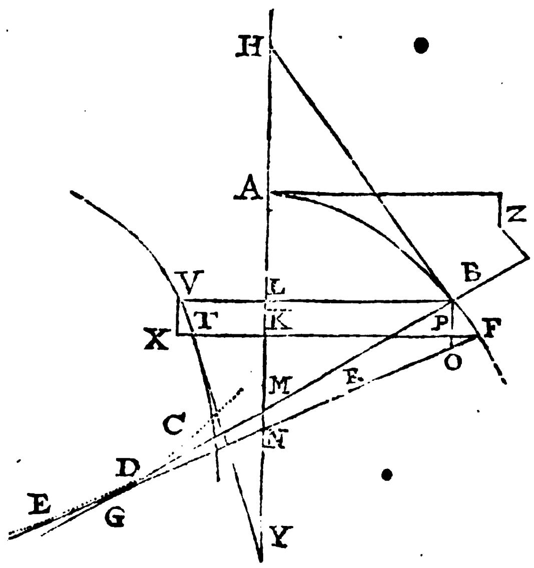
Now because HL : LB : LM $\div\!\!\div$, therefore $\frac{\overline{aax}^{\frac{2}{3}}}{3x} = y$, and $\frac{\overline{aax}^{2}}{27x^3} = y^3$, that is, $\frac{a^4}{27x} = y^3$, whence the Nature of the Curve TV is such, that the intercepted Diameters AL, are reciprocally proportional to the Cubes of the Ordinates LV, and the Convex side of the Curve TV, is towards the Axis AL; and the Equation expressing the Nature of the Curve is $\frac{1}{27}a^4 = xy^3$. Now the Ratio of BD to MD is compounded of the Rationes of BO to BP, or HM to HL, and of BP or LK to MN; but LH is $= 3x$, and LM $= y$, whence HM : HL :: $3x+y : 3x$, which Ratio is given. And to find the Ratio of LK to MN, LV is = LM, and KT is = KN (by supposition) therefore LM is greater than KM, and consequently LK is greater than MN, and LK — MN is = LV — KT = XT, whence XV — XT is = MN, and consequently LK : MN :: XV : XV — XT.

Draw TY touching the Curve VT in T, and intersecting the Axis in Y, then the Triangles TVX, TYK are similar; and by the Property of the Curve, the Sub-tangent KY is = 3AK = $3x$; whence LK : MN :: VX : VX — XT :: KY : KY — KT :: $3x : 3x-y$; *ergo* the Ratio of BO to MN is compounded of the Rationes of $3x+y$ to $3x$, and of $3x$ to $3x-y$, that is, BD : MD :: BO : MN :: $3x+y : 3x-y$, and by Division, BM : MD :: $2y : 3x-y$.

CONSTRUCTION.

CONSTRUCTION.

Because $BM : MD :: 2y : 3x - y$, therefore $BM : MD :: 2ML : 3AL - ML :: 2BM : 3BZ - BM :: MB : \frac{3}{2}BZ - \frac{1}{2}BM$, whence $MD = \frac{3}{2}BZ - \frac{1}{2}BM$; and adding to both sides of the Equation MB, we have $MD + MB = \frac{3}{2}BZ + \frac{1}{2}BM$; and if MB be bisected in R, and RD be taken $= \frac{3}{2}BZ$, the Point D will be in the Curve CDE; and in like manner, innumerable other Points of the Curve CDE may be found.



234. By this Method, the excellent Mr. *Hugens* calculated the following Table, expressing the length of the Radij of the Evoluta BD for all sorts of Paraboloides; for if the intercepted Diameter AL be $= x$, and the Ordinate $LB = y$, and the Parameter of the Curve $= a$, then,

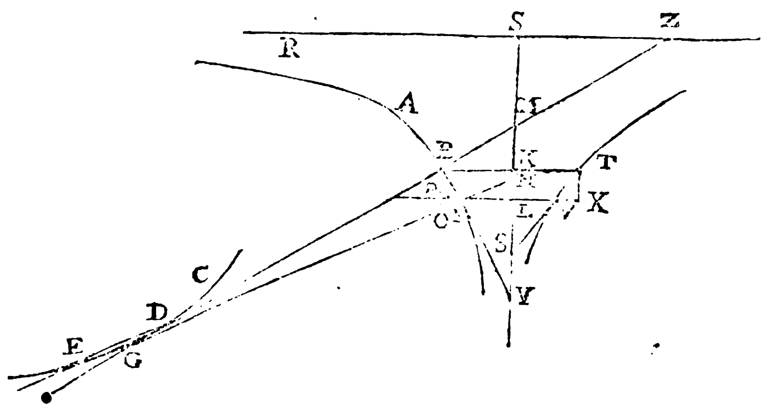
If the Equation expressing the Nature of the Curve ABF be	$ax = y^2$	Exam. 1.	then	$BM + 2BZ$	$= BD$.
	$a^2x = y^3$	Exam. 3.		$\frac{1}{2}BM + \frac{3}{2}BZ$	
	$ax^2 = y^3$	Exam. 2.		$2BM + 3BZ$	
	$ax^3 = y^4$			$3BM + 4BZ$	
	$a^3x = y^4$			$\frac{1}{3}BM + \frac{4}{3}BZ$	
	&c.			&c.	

The use of the Table is thus: If a Quadratic Parabola be propos'd, *v. g.* that in *Example* I. Then the Equation expressing the Nature thereof is $ax = y^2$; which I find in the Second Column of the Table, and right against the same, $BM + 2BZ = BD$; therefore if I assume any point as B, in the Curve, and draw BM, perpendicular to the same, and if AZ be drawn perpendicular to AM, and produc'd until it cut MB (produc'd) in Z, and BD be taken $= BM + 2BZ$, the point D. will be in the Curve CDE; which was requir'd.

The Construction of the Table is thus: Multiply BM, by the Index of the Abscissa (x) and multiply BZ, by the Index of the Ordinate y, and divide the sum of both by the Index of a, the Quotient will be $= BD$; by which easie and universal construction, the Table may be continued infinitely.

EXAMPLE.

235. Let ABF, be a Curve describ'd within the Rectangular Asymptotes SR, SK, and from any Point of the Curve, as B, draw BK parallel to SR, and suppose SK $= x$, BK $= z$, $a =$ to an invariable Quantity; and let the Equation



expressing the Relation of the Curve ABF to the Asymptote SK, be $aa = zx$, then the given Curve will be an equilateral Hyperbola; 'tis required to describe the Curve CDE, by whose Evolution the said given Curve may be described.

Let BY touch the Curve in B, and intersect the Asymptote in Y, and draw DBM, EFN perpendicular to the Curve in the Points B and F infinitely near each other, draw FL parallel to BK, and BO parallel to KL, then BD : MD :: BO : MN; and the Ratio of BO to MN is Compounded of the Rationes of BO to BP, or MY to YK, and KL to MN; now because BY touches the Curve in B, and KY is the Sub-tangent, therefore KY is $=$ (*Art.* 26.) SK $= x$; and if the Sub-normal KM be $= y$, then MY is $= y + x$, and consequently, BO : BP :: MY : YK :: $y + x : x$, and consequently, the Ratio of BO to BP is given; and to find the Ratio of BP to KL, take KT $=$ KM, and LV $=$ LN, connect the Points V and T, and draw TX parallel to KL, then to find the Locus of the Points T and V, the property of the Curve ABF is $aa = zx$, and $\frac{aa}{x} = z =$ BK, *ergo* BK$q = \frac{a^4}{x^2}$, which being divided by x, the Quotient $\frac{a^4}{x^3}$ is $=$ KM $=$ KT $= y =$ to the Ordinate of the new Curve, whence the property of the new Curve is $a^4 = x^3 y$, and consequently, if TS touch the Curve in T, and intersect the Asymptote in S, then the Sub-tangent KS will be $=$ (*Art* 28.) $\frac{1}{3}x = \frac{1}{3}$ SK; now the Triangles TVX and TKS are similar, therefore KL : MN :: TX : TX + VX :: KS : KS + KT : $\frac{1}{3}x : \frac{1}{3}x + y :: x : x + 3y$; now the Ratio of BO to MN being compounded of the Rationes of BO to OP, or MY to YK, and KL to MN; it is also compounded of the Rationes of $y + x$ to x, and x to $x + 3y$, *ergo* BO : MN :: BD : MD :: $x : x + 3y$, and by Division, BM : BD :: $2y : y + x$ :: 2KM : KM + SK :: 2BM : BM + BZ :: BM : $\frac{1}{2}$BM + $\frac{1}{2}$BZ; whence BD is $= \frac{1}{2}$BM + $\frac{1}{2}$BZ, which gives us an easie Method to construct the Curve CDE.

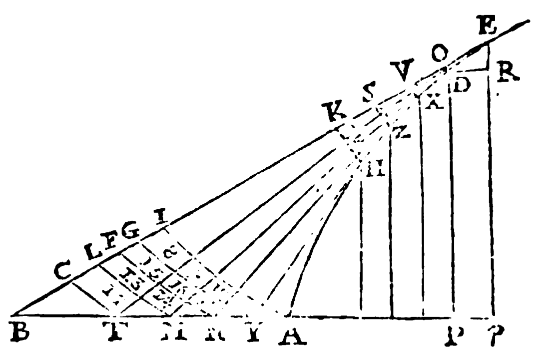
In like manner, if the Equation of the Curve ABF be $a^3 = x^2 z$, or $a^3 = xz^2$, &c. the corresponding Evoluta may be Geometrically Constructed.

PROP.

PROP. II.

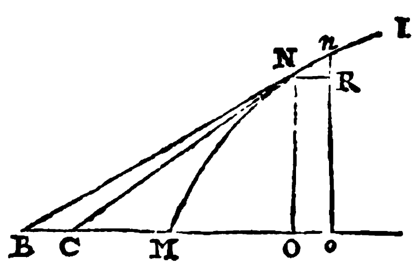
To assign a Right Line equal to the Curve Line ADE.

236. Resume the Figure in Definition I, and in the Triangle BET suppose CE = TE, and draw CT; also in the same Triangle TDM, suppose MD = 12 D, and draw M 12; and in the Triangle MXN, take X 14 = N X, and draw 14 N, and make LC = T 12, LF = 12, 13 = M 14, and so on until GI be = 15, 8 = 16, 9 = 17, 10 = Y 11. Lastly, Suppose BT, TM, MN, NV, &c. $= \dot{t}$: BC, T12, M14, &c. $= \dot{s}$, then it is evident that the Right Line BA is = to the Sum of all the $\dot{t}$, and the Right Line BI is = to the Sum of all the $\dot{s}$; and if we suppose the infinitely little Portions of the Curve DE, DX, &c. $= \dot{z}$, then because EI is = AZDE, therefore EI is = Sum of all the $\dot{z}$, and the whole Tangent BE is $= S:\overline{\dot{s} + \dot{z}}$, and the Right Line BE, will be = AHE + $S\dot{s}$ (or BI.)



Let the Absciss AP be $= x$, the Ordinate DP $= y$, Pp $= \dot{x}$, ER $= \dot{y}$; then 'tis evident that TD : DP :: DE : ER, that is, $s : y :: \dot{z} : \dot{y}$, and consequently, $s\dot{y} = y\dot{z}$.

Now to find the length of the Curve ADE, describe the Curve MNnI, and let the Ordinates thereof NO be = DP $= y$; the Fluxion of the Absciss O o $= \dot{z}$ = to the Fluxion of the given Curve; then will Rn be $= \dot{y}$ = ER, and on $= + y$, and the curvilineal Figure Mno, will be $= S y\dot{z}$; and because it is (*ex Hypoth.*) $\dot{y} : \dot{z} :: y : s$, or Rn : NR :: NO : OC, or ER : DE :: DP (= NO) : DT, 'tis plain that the Sub-tangent of this second Curve, *viz.* OC is = to the Tangent of the first Curve, *viz.* DT; and Mo the intercepted Diameter of the second Curve is $= S\dot{z}$ = the Curve Line ADE, and BM is $= S:\dot{s}$.



Hence if the Nature of the Curve MN be Investigated, the Ordinate NO $= y$, and the Sub-tangent CO $= s$ = to the Tangent of the given Curve ADE being given, the right Line Mo = to the Curve ADE may be found.

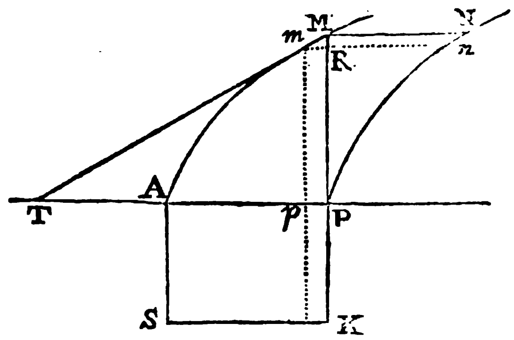
And thus the Rectification of Curves is again reduc'd to one *general Proposition*, *viz.* the Ordinate NO, and the Sub-tangent CO being given, to find the property of the Curve MNI.

Now the Quadrature of the Curves being suppos'd, the Proposition may be easily resolv'd.

For Instance, let it be requir'd to find the Equation expressing the Nature of the Curve AM, the Ordinate MP be $= y$, and the Sub-tangent PT being $= (t)$ $\frac{2y^3}{3rr}$:

Suppose

Suppose $Pp = \dot{x}$, $MR = \dot{y}$, $MP = y$, then $t : y :: \dot{x} : \dot{y} :: z : r$; and let r be an invariable Quantity, then is $r\dot{x} = z\dot{y}$; therefore applying z from M to N, there will be generated the Curvilineal Figure MPN, and if r be applied to $\dot{x}$ from P to K, then the Rectangle APKS $= rx$ will be generated; and this Rectangle AK $= S r\dot{x} = rx$ is $= S z\dot{y} =$ Space MPN, now $t\left(\frac{2y^3}{3rr}\right) : y :: z : r$, therefore $3rz = 2yy$, and $z = \frac{2yy}{3r}$, which shew that the Figure MPN, is the complement of a Parabola, and the Area thereof is $= \frac{1}{3}zy$; but all the $z\dot{y}$ or $\frac{1}{3}zy$ is $= S r\dot{x} =$



rx, therefore $z = \frac{3rx}{y} = \frac{2yy}{3r}$; whence $\frac{9}{2}r^2x = y^3$, which Equation expresses the Nature of the Curve AM, as was requir'd, that is, the Curve AM, is a Cubical Paraboloid, and consequently $TA = S : \dot{t}$ is $= 2AP$, that is, the Sub-tangent t is ($= s$ the Tangent of the given Curve) $3AP = S\dot{z} + S\dot{s} = 3S\dot{z} = 3$, the length of the given Curve; and *vice versa*, if the Equation of the Curve be $9rrx = 2y^3$, then $\frac{9rr\dot{x}}{6y^2} = \dot{y}$, and the Sub-tangent is $= \frac{y\dot{x}}{\dot{y}} =$ (by Subsist:) $\frac{2y^3}{3rr} =$ to the Value of the Subtangent given.

SCHOLIUM.

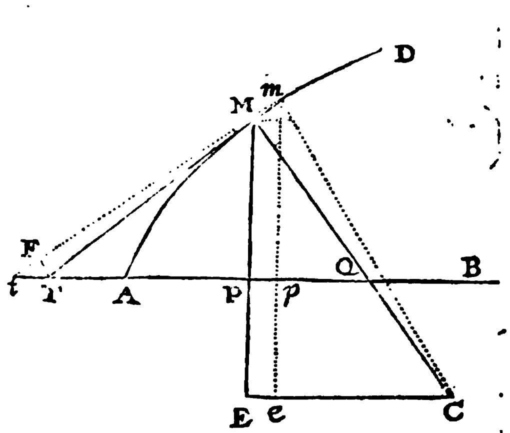
If the Value of the Ordinate and Subtangent be given, the Nature of the Curve may be investigated, without having recourse to the Quadrature of Curvilineal Spaces; *v. g.* Let it be requir'd to find the property of the Curve AM, the Ordinate being $= y$ and the Sub-tangent $= \frac{3y^3 + 2fyy}{3xx + 2rx}$, suppose AP $= x$, then the Fluxion of the Abscissa is $= \dot{x}$ and the Fluxion of the Ordinate is $= \dot{y}$, whence the Sub-tangent is $= \frac{y\dot{x}}{\dot{y}} = \frac{3y^3 + 2fyy}{3xx + 2rx}$, *ergo* $3x^2\dot{x} + 2rx\dot{x} = 3y^2\dot{y} + 2fy\dot{y}$, and finding the Flowing Quantities, $x^3 + rxx = y^3 + fyy$. Q. E. I.

PROP.

PROP. III.

The Nature of the Curve Line AMD *being given, and one of its Perpendiculars* MC; *to find the point* C, *where it touches the Evoluta of the Curve* AMD; *that is, to find the point* C, *in which the Perpendiculars* MC, *m*C *infinitely near each other, Concur.*

237. Let AB be the Axis of the Curve AMD, and the Ordinates MP perpendicular to the same, and imagine another Ordinate *mp* infinitely near MP, because the points M, *m*, are suppos'd infinitely near each other, from the point C, in which the Perpendiculars Concur; draw CE, parallel to the Axis AB, and intersecting the Ordinates MP, *mp* produc'd in E*e*; draw MR parallel to AB; then the Triangles MR*m*, MEC are similar; for the Angles EMR, CM*m* being right Angles, and CMR being common to both, the Angles EMC = RM*m*. Hence.



If we suppose AP = x, PM = y, MR = $\dot{x}$, R*m* = $\dot{y}$, ME (unknown) = z, and R*m* = $\dot{y} = \dot{z}$, M*m* = $\sqrt{\dot{x}^2+\dot{y}^2}$ then MR ($\dot{x}$) : M*m* ($\sqrt{\dot{x}^2+\dot{y}^2}$) :: ME ($z$) : MC = $\frac{z\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$

Now the point C, being the Center of the Infinitely little Arch M*m*, (as is evident from the Genesis of the Curve AMD, from the Evoluta) the Radius CM, which becomes C*m*, when EM is augmented by R*m*, will be still the same, and consequently, the Fluxion of (MC =) $\frac{z\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$ is = 0, that is, supposing $\dot{x}$ Invariable) $\dot{z} \times \sqrt{\dot{x}^2+\dot{y}^2} + z \times \frac{1}{2}\,\overline{\dot{x}^2+\dot{y}^2}\,\vert^{-\frac{1}{2}}\, 2\dot{y}\ddot{y}$ divided by $\dot{x}$, or $\dot{z}\sqrt{\dot{x}^2+\dot{y}^2} + \frac{z\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$ divided by $\dot{x}$, that is $\frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}$ is = 0.

Whence ME (z) = $\frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2}{-\dot{y}\ddot{y}}$ = (putting $\dot{y}$ for $\dot{z}$) $\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$. And because the Triangles MR*m*, MEC are similar, therefore MR ($\dot{x}$) : M*m* ($\sqrt{\dot{x}^2+\dot{y}^2}$) :: ME $\left(\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}\right)$: MC = $\frac{\dot{x}^2+\dot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}} = \frac{\overline{\dot{x}^2+\dot{y}^2}\,\vert^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

238. The Value of MC the Ray of the Evoluta, may be express'd in Terms consisting of first Fluxions only, which will easily appear thus; MC is = $\frac{\dot{z}\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$, and the Fluxion thereof is equal to nothing, that is, (supposing $\dot{y}$ Invariable)

$\frac{\dot{z}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}+z\times\frac{1}{2}\overline{\dot{x}^2+\dot{y}^2}|^{-\frac{1}{2}}\times 2\dot{x}^2\ddot{x}}{\dot{x}^2}-\frac{z\sqrt{\dot{x}^2+\dot{y}^2}\times\ddot{x}}{\dot{x}^2}=0$. And multiplying by $\dot{x}^2$, we shall have $\dot{z}\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}+z\times\overline{\dot{x}^2+\dot{y}^2}|^{-\frac{1}{2}}\times\dot{x}^2\ddot{x}-z\times\sqrt{\dot{x}^2+\dot{y}^2}\times\ddot{x}=0$. And multiplying by $\sqrt{\dot{x}^2+\dot{y}^2}$, we have $\dot{z}\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}+z\times\dot{x}^2\ddot{x}-z\times\overline{\dot{x}^2+\dot{y}^2}\times\ddot{x}=0$. And consequently $z=\frac{\dot{z}\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}}{\overline{\dot{x}^2+\dot{y}^2}\times\ddot{x}-\dot{x}^2\ddot{x}}$ $=$ (because $\dot{z}=\dot{y}$) $\frac{\dot{y}\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}}{\dot{y}^2\ddot{x}}=\frac{\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}}{\dot{y}\ddot{x}}$, and because the Triangles MRm, MEC are similar, it is, $MR\ (\dot{x}):Mm\ (\sqrt{\dot{x}^2+\dot{y}^2})::ME$ $\left(\frac{\dot{x}\times\overline{\dot{x}^2+\dot{y}^2}}{\dot{y}\ddot{x}}\right):MC=\frac{\overline{\dot{x}^2+\dot{y}^2}\times\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{y}\ddot{x}}=\frac{\overline{Mm}^3}{\dot{y}\ddot{x}}$.

$=$Now suppose $Mm=\dot{u}$, then MC is $=\frac{\dot{u}^3}{\dot{y}\ddot{x}}$, and let the universal differential Equation expressing the Nature of all sorts of Geometrical Curves be $\dot{x}=t\dot{y}$ (t being any Quantity compos'd of x or y, or both) whence $\ddot{x}=\dot{t}\dot{y}$, and consequently $\frac{\dot{u}^3}{\dot{y}\ddot{x}}$ is $=\frac{\dot{u}^3}{\dot{t}\dot{y}^2}=MC$.

239. And from this last Form, an universal Theorem consisting of Algebraic Terms only, may be deduc'd, expressing the length of the Ray of the Evoluta MC, in allsorts of Geometrical Curves.

The universal Equation expressing the Nature of all sorts of Algebraic Curves, is; $fx^m+gy^n+bx^ry^s+a=0$, in which Equation, fx^m represents all the Terms affected with x only, and gy^n, those affected with y only, and bx^ry^s represent all the Terms of the given Equation affected with x and y jointly. In the same general Equation, a is an invariable Quantity, f, g, b, are the Coefficients of the respective Terms, and m, n, r, s are the Exponents of the Powers of x and y; Lastly suppose the perpendicular to the Curve, intercepted between the Curve and the Axis $=\pi$, and the Subnormal $=z$, it is required to find an universal Theorem consisting of pure Algebraic Quantities, expressing the length of the Ray of the Evoluta MC.

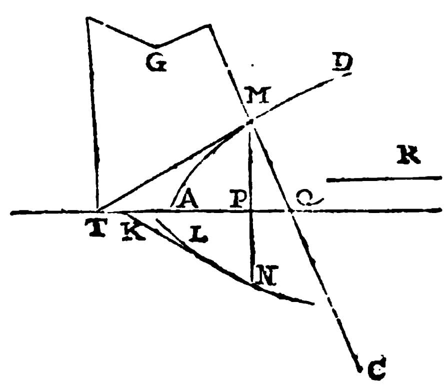
The Fluxion of the general Equation is $mfx^{m-1}\dot{x}+ngy^{n-1}\dot{y}+rbx^{r-1}$, $y^s\dot{x}+sbx^ry^{s-1}\dot{y}=0$, and by Transposition, $mfx^{m-1}\dot{x}+rbx^{r-1}y^s\dot{x}=-ngy^{n-1}\dot{y}-sbx^ry^{s-1}\dot{y}$, suppose (for brevities sake) those Quantities multiplied by $\dot{x}=p$, and those multiplied by $\dot{y}=q$, then $p\dot{x}=q\dot{y}$ and $\dot{x}=\frac{q}{p}\times\dot{y}$: in the universal Form expressing the length of the Ray of the Evoluta, *viz.* $\frac{\dot{u}^3}{\dot{t}\dot{y}^2}$ substitute the Fluxion of $\frac{q}{p}$ for $\dot{t}$ $\left(\text{because } t=\frac{q}{p}\right)$, *viz*, $\frac{p\dot{q}-q\dot{p}}{pp}$ for $\dot{t}$, or (because $p:q::\dot{y}:\dot{x}::z:y$) $\frac{z\dot{q}-y\dot{p}}{pz}$, and then $\frac{\dot{u}^3}{\dot{t}\dot{y}^2}$ will be $=\frac{pz\dot{u}^3}{z\dot{y}^2\dot{q}-y\dot{y}^2\dot{p}}$:

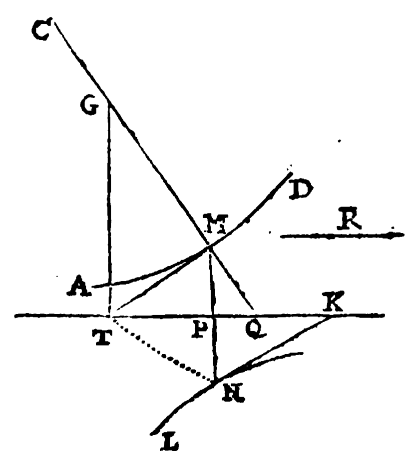
Resume

Resume the Quantities expreſs'd by p and q, and take their Fluxions, then $\dot{p} = \overline{mm - m}\, f x^{m-2} \dot{x} + \overline{rr - r}\, b x^{r-2} y^{s} \dot{x} + rsb x^{r-1} y^{s-1} \dot{y}$; and $\dot{q} = -\overline{nn + n} \times g y^{n-2} \dot{y} - \overline{ss + s} \times b x^{r} y^{s-2} \dot{y} - rsb x^{r-1} y^{s-1} \dot{x}$; and ſubſtituting in place of p, $\dot{p}$, $\dot{q}$, their reſpective Values, in the Equation $\frac{\dot{u}^3}{\dot{t}\dot{y}^2} = \frac{pz\dot{u}^3}{z\dot{y}^2\,\dot{q} - y\dot{y}^2\dot{p}}$, there will ariſe $\frac{\dot{u}^3}{\dot{t}\dot{y}^2} = \frac{mf x^{m-1} \dot{x}\dot{u}^3 \times z + rb x^{r-1} y^{s} \times \dot{u}^3 \times z}{\overline{m - m^2} \times f x^{m-2} y\dot{y}^2 \dot{x} + \overline{r - r^2}\, b x^{r-2} y^{s} y\dot{y}^2 \dot{x} - rsb x^{r-1} y^{s} \dot{y}^3 + \overline{n - nn} \times g y^{n-2} z\dot{y}^3 + \overline{s - ss}\, b x^{r} y^{s-2} z\dot{y}^3 - rsb x^{r-1} y^{s-1} z\dot{x}\dot{y}^2}$, and ſubſtituting ϖ, y and z for $\dot{u}$, $\dot{x}$, and $\dot{y}$ (becauſe theſe are always proportional to thoſe, for $\dot{u} : \dot{x} :: \varpi : y$, and $\dot{x} : \dot{y} :: y : z$) there will ariſe $\frac{\dot{u}^3}{\dot{t}\dot{y}^2}$ (MC) $= \frac{mf x^{m-1} \varpi^3 + rb x^{r-1} y^{s} \times \varpi^3}{\overline{m - mm}\, f x^{m-2} yyz + \overline{n - nn}\, g y^{n-2} z^3 + \overline{r - rr}\, b x^{r-2} y^{s+2} z + \overline{s - ss}\, b x^{r} y^{s-2} z^3 - 2rs \times b x^{r-1} y^{s} zz}$. Which expreſſes the Value of the Ray of the Evoluta in ordinary or Algebraic Terms; and the ſaid Ray may be determined by this Analogy; as the Denominator of this Fraction is to the Numerator, ſo is unity to the Ray of the Evoluta.

And this Theorem may yet be expreſs'd more ſimply, if the Fraction be divided by $\frac{\varpi\varpi}{z} = $ TQ, and then it we ſay, as the Denominator is to the Numerator, ſo is TQ to MC.

240. The Radius of the Curvature as well in Tranſcendent as Algebraic Curves, may be otherwiſe determined thus; Let the given Curve be AMD, then 'tis evident that if the Point M be given, then the Perpendicular MQ, the Tangent MT, the Ordinate MP, and the Subnormal PQ are alſo given; deſcribe the new Curve LN, ſo that the Ordinate PN be always a fourth Proportional to the Ordinate of the given Curve MP, the Subtangent PT, and any invariable Quantity R (*viz.* PM : PT :: R : PN) whence the Tangents, Subtangents, *&c.* of this new Curve will be given alſo; and if it be PT : PK :: QG : a fourth Quantity MC, then MC will be the Radius of the Curvature in M.

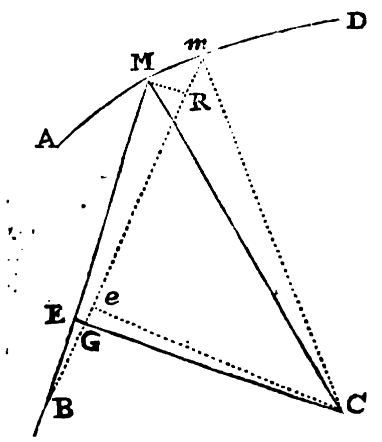




For by conſtruction MC is $= \frac{PK \times QG}{PT} =$ (becauſe the Triangles MPT, QTG,

QTG, are ſimilar) $\frac{PK \times TM \times TQ}{PT \times MP}$ = (becauſe the Triangles MTP, QMP are ſimilar) $= \frac{PK \times \overline{TM}^3}{\overline{PT}^2 \times MP}$ = (by conſtruction) $\frac{PK \times \overline{TM}^3 \times R}{PT \times \overline{MP}^2 \times PN}$ = (from the Nature of Tangents, becauſe PK : PN :: $\dot{x} : R \times \dot{t}$, and MT : PT :: $\dot{u} : \dot{x}$, and MT : MP :: $\dot{u} : \dot{y}$) $= \frac{\dot{x}\dot{u}^3 \times R}{\dot{x}\dot{y}^2 \times R\dot{t}}$ (= dividing by $R\dot{x}$) $\frac{\dot{u}^3}{\dot{t}\dot{y}^2}$ = to the Radius of the Curvature in M.

241. But if the Curve be referred not to an Axis, but to one ſingle Point; ſuppoſe all the Ordinates BM, B*m* to meet in the ſame Point B, and let the Arch M*m* be infinitely little, and draw the Perpendiculars MC, *m*C infinitely near, and mutually interſecting each other in C, the Point required. From C draw CE, C*e* perpendicular to the Ordinates BM, B*m*, and on the Center B with the Radius BM, deſcribe the infinitely little Arch MR, then the Rectangular Triangles RM*m*, EMC, BMR, BEG, C*e*G are ſimilar; therefore if we ſuppoſe BM = y, ME = z, MR = $\dot{x}$, R*m* = $\dot{y}$, M*m* = $\sqrt{\dot{x}^2 + \dot{y}^2}$, we ſhall have



CE or C*e* = $\frac{z\dot{y}}{\dot{x}}$, and MC = $\frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$; now the Arch M*m* being infinitely little, the Fluxion of MC is = 0, therefore $\dot{z}\sqrt{\dot{x}^2 + \dot{y}^2} + z \times \frac{1}{2}\,\overline{\dot{x}^2 + \dot{y}^2}|^{-\frac{1}{2}}\, 2\dot{y}\ddot{y}$ divided by $\dot{x}$, (ſuppoſing $\dot{x}$ invariable) or $\dot{z}\sqrt{\dot{x}^2 + \dot{y}^2} + \frac{z\dot{y}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$ divided by $\dot{x}$, that is $\frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2 + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}}$ is = 0. Whence $z = \frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2}{-\dot{y}\ddot{y}}$. But BM $(y) : Ce \left(\frac{z\dot{y}}{\dot{x}}\right) :: MR(\dot{x}) : Ge = \frac{z\dot{y}}{y}$ and me — ME, or R*m* — G*e* = $\dot{z}$ is $= \frac{y\dot{y} - z\dot{y}}{y}$, whence z is $= \frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2}{-\dot{y}\ddot{y}}$ is $= \frac{y\dot{x}^2 + y\dot{y}^2 - z\dot{x}^2 - z\dot{y}^2}{-y\ddot{y}}$ and conſequently $-zy\ddot{y} + z\dot{x}^2 + z\dot{y}^2 = y\dot{x}^2 + y\dot{y}^2$, and by diviſion $z = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}$.

And if we ſuppoſe the Ordinates y to be infinite, the Terms $\dot{x}^2$ and $\dot{y}^2$ will be incomparably little in reſpect of $y\ddot{y}$, and conſequently, this laſt form will coincide with that in the preceeding Caſe, which is evidently true, becauſe then the Ordinates become parallel between themſelves, and the Arch MR becomes a ſtrait Line perpendicular to the Ordinates.

And

And becauſe the Triangles M R m, M E C are ſimilar, therefore M R ($\dot{x}$) : M m ($\sqrt{\dot{x}^2 + \dot{y}^2}$) :: M E $\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - \ddot{y}y}\right)$: M C $= \frac{\overline{y\dot{x}^2 + y\dot{y}^2}\sqrt{\dot{x}^2 + \dot{y}^2}}{\overline{\dot{x}^2 + \dot{y}^2}\,\dot{x} - y\dot{x}\ddot{y}}$.

CONSECTARY I.

242. One Curve can have but one Evoluta, becauſe the value of M E or M C is but one and the ſame.

CONSECTARY II.

If the Nature of the Curve AMD be given, we may find the value of $\dot{y}^2$, and $\ddot{y}$ in $\dot{x}^2$, or the value of $\dot{x}^2$, and $\ddot{y}$ in $\dot{y}^2$, which being ſubſtituted in the preceeding forms, will give the value of M E cleared from all Fluxions, and in known and finite Quantities; and drawing E C perpendicular to M E, it will cut M C the perpendicular to the Curve in C the Point required.

CONSECTARY III.

If the Value of M E ($= \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$ in the firſt Caſe, or $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}$ in the ſecond Caſe) be Poſitive, we muſt take the point E on the ſame ſide with the Axis A B, or the point B, as I have ſuppoſed in the preceeding Calculation. Whence it is alſo evident that the Curve will be Concave toward the ſaid point or Axis; but if the Value of M E be Negative, we muſt take the point E on the ſide of the Curve oppoſite to the Axis A B, or the point B, and then the Curve will be Convex towards the Axis A B, or point B.

CONSECTARY IV.

Hence in the Point of *contrary Flexion*, which ſeperates the Concave Part of the Curve from the Convex, the value of M E of being Poſitive will become Negative, and the Perpendiculars contiguous or infinitely near each other, inſtead of converging will afterwards diverge.

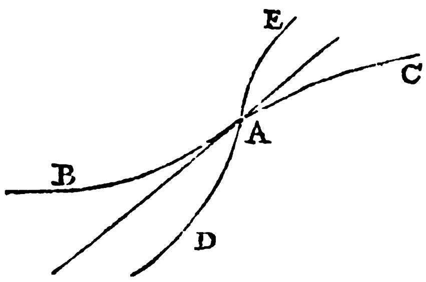
CONSECTARY V.

But this can happen only two ways, for either the Rays of the Evoluta (or the Perpendiculars) increaſe as they approach the Point of *contrary Flexion* or *Retrogreſſion*, and then they muſt at laſt become parallel, that is, the Ray or Perpendicular becomes infinite, or the Perpendiculars decreaſe, and then in thoſe Points the Ray of the Evoluta will become equal to nothing.

SCHOLIUM.

It has been thought that the Ray of the Evoluta is always infinitely great in the Point of *contrary Flexion*; but it may be obſerved, that there are infinite numbers of Curves, which in the Points of *contrary Flexion* have the Ray of the Evoluta infinitely little; and that there is but one ſort which can have the ſaid Ray infinite.

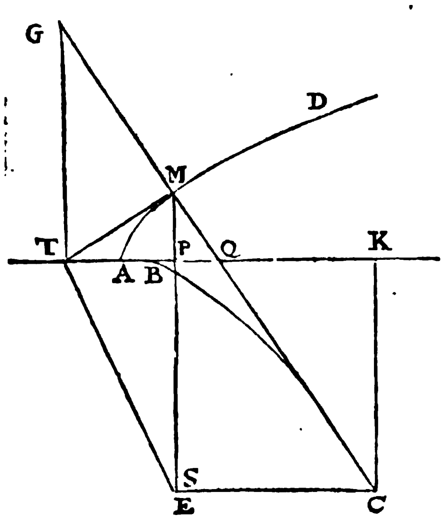
Let B A C be such a Curve, that in A the Point of *contrary Flexion*, the Ray of the Evoluta be infinite. If the Portion of the Curve B A and A C be Evolved, beginning in the Point A, 'tis evident that by such Evolution, the Curve Line D A E will be described, which will have the Point of *contrary Flexion* also in A, and the Ray of the Evoluta in that Point will be equal to nothing; and if this second Curve be Evolved, the Point of *contrary Flexion* in the third Curve will be in A, and then the Ray of the Evoluta will be $= 0$, *&c.* whence it is plain, that in all these Curves, the Ray of the Evoluta in A, the Point of *contrary Flexion*, is $= 0$.



PROP. IV.

If the Curve AMD *be a Parabola, and* AB *the Axis*, *and if* MQC *be drawn perpendicular to the Curve in* M; *'tis required to determine the Point* C *in the Evoluta of the Parabola.*

243. Suppose the Parameter of the Curve $= a$, then the Equation expressing the Nature of the Parabola is $ax = yy$, which being reduced to Fluxions, we have $a\dot x = 2y\dot y$, and $\frac{a\dot x}{2y} = \dot y = \frac{a\dot x}{2\sqrt{ax}}$, and taking the Fluxions of this last Equation (supposing $\dot x$ invariable) we have $\ddot y = \frac{a\dot x \times -\frac{1}{2}\overline{ax}^{-\frac{1}{2}} \times a\dot x}{2} = \frac{-a\dot x^2}{4x\sqrt{ax}}$, and substituting these values in place of $\dot y$, and $\ddot y$ in the general Form $\frac{\dot x^2 + \dot y^2}{-\ddot y}$, there will arise M E $= \frac{\overline{a+4x} \times \sqrt{ax}}{a} = \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$, whence there arises this



CONSTRUCTION.

From the Point T, in which the Tangent M T cuts the Axis, draw T E parallel to M C; I say, it will intersect M P (produced) in E the Point sought; for the Angles M P T, M T E being Right Angles, it is M P ($\sqrt{ax}$) : P T ($2x$) :: P T ($2x$) : P E $= \frac{4xx}{\sqrt{ax}} =$ (because $\frac{ax}{\sqrt{ax}} = \sqrt{ax}$, and $\frac{x}{\sqrt{ax}} = \frac{\sqrt{ax}}{a}$) $\frac{4x\sqrt{ax}}{a}$, and consequently MP + P E is $= \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$.

Again

Again, because the right angled Triangles MPQ, MEC are similar, therefore PM ($\sqrt{ax}$) : PQ ($\frac{1}{2}a$) :: ME ($\sqrt{ax} + \frac{4x\sqrt{ax}}{a}$) : EC, or PK = $\frac{1}{2}a + 2x$, and consequently, QK is $= 2x$.

Whence if QK be taken $= 2$ AP = TP, and KC be drawn parallel to PM, it will cut the Perpendicular MC in the Point C in the Evoluta as was required.

244. Now to find the Point B, in which the Axis A touches the Evoluta, that is, to find the Vertex of the Evoluta BC; Suppose the Point M to approach infinitely near to the Vertex A, then 'tis evident that the Perpendicular MQ will cut the Axis in B the Point required; whence in general, if we investigate the value of PQ $\left(\frac{y\dot{y}}{\dot{x}}\right)$ in x or y, and then if x or y be put $= 0$, then the Point P will fall in A, and the Point Q in B, that is, PQ will become = AB sought.

Thus in the last Example, PQ $\left(\frac{y\dot{y}}{\dot{x}}\right)$ is $= \frac{a\dot{x}y}{2y\dot{x}} = \frac{1}{2}a$, and because this Quantity $\frac{1}{2}a$ is constant and invariable, the Subnormal PQ will always be the same, whatever Point of the Curve M is in, and consequently, when M coincides with A, then P will be in A, and Q in B, and AB will be $= \frac{1}{2}a =$ PQ, that is the Vertex of the Evoluta BC, is distant from the Vertex of the Parabola, $\frac{1}{2}$ the Parameter of the Parabola.

245. And to Investigate the Nature of the Evoluta BC; suppose the Intercepted Diameter = BK = z, the Ordinate PE or KC = u, whence KC = $u = \frac{4x\sqrt{ax}}{a}$, and AP + PK − AB = $z = 3x$; and consequently $\frac{1}{3}z = x$, and substituting $\frac{1}{3}z$ for x in the Equation $u = \frac{4x\sqrt{ax}}{a}$, there will arise $27auu = 16z^3$, which expresses the Relation of BK to KC, and shews that BC, the Evoluta of the Parabola is a Cubical Paraboloid, whose Parameter is $= \frac{27}{16}$ Parameter of the Parabola.

COROLLARY.

If the Curve AMD be a Geometrical Curve, then an Equation may be found expressing the Natnre of the Evoluta BC, and the said Evoluta will be a Geometrical Curve, to any Portion whereof an equal Right Line may be assigned.

Another way to find the Length of the Ray of the Evoluta MC.

246. The general Equation expressing the Nature of all sorts of Parabola's is $x - y^n = 0$, and the universal Equation expressing the Nature of all sorts of Geometrical Curves is $fx^m + gy^n + bx^r y^s + a = 0$; and comparing the respective Terms of both Equations, we have $m = 1$, $f = 1$, $g = -1$, and $n = n$; whence the general Theorem expressing the Length of the Ray MC (*Art.* 239.) will be $\frac{\pi^3}{\overline{n - nn} \times -y^{n-2} z^3}$ = (because PT $= ny^n$, and $\frac{\text{PT} \times \text{PQ}}{\text{PM}\, q} = ny^{n-2}z$ is $= 1$) $\frac{\pi^3}{\overline{n-1} \times zz}$. Whence in the common Parabola MC is $= \frac{\pi^3}{zz}$. Which gives this

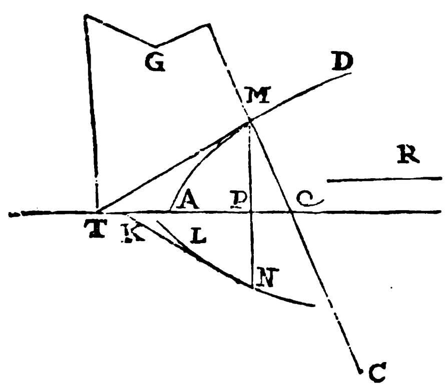
CON-

CONSTRUCTION.

Draw TG perpendicular to the Axis in the point T, and produce the perpendicular to the Curve QM, untill it interfect TG in G, and make $MC = QG$; then the point C will be in the Evoluta; for becaufe the Triangles QPM, QTG, are fimilar, and $QM = \pi$, and $QT = \frac{\pi^2}{z}$: it is QP (z) : QM (π) :: QT $\left(\frac{\pi^2}{z}\right)$: $QG = \frac{\pi^3}{zz} = MC$.

Another way.

247. This method is deduced from (*Art.* 240.) and is thus; Let the general Equation expreffing the Nature of all forts of Parabola's be $rx = y^n$, then to find the Property (or Nature) of the new Curve LN, it is by Conftruction, MP $\left(\overline{rx}^{\frac{1}{n}}\right)$: PT (nx) :: r : $PN = \frac{nrx}{\overline{rx}^{\frac{1}{n}}} = n \times \overline{rx}^{1-\frac{1}{n}} = n \times \overline{rx}^{\frac{n-1}{n}} = s$, whence the Equation expreffing the Nature of the Curve LN is $nrx = s^{\frac{n}{n-1}}$ and the new Curve is a Parabola, and the Subtangent PK is $= \frac{n}{n-1}x$, and confequently PT (nx) : PK $\left(\frac{n}{n-1}x\right)$ :: QG : $\frac{1}{n-1}$ QG = MC. That is, in the common Parabola, n being $= 2$, the Ray of the Evoluta MC is = to QG, and the Conftruction is the fame as in the preceding Article.

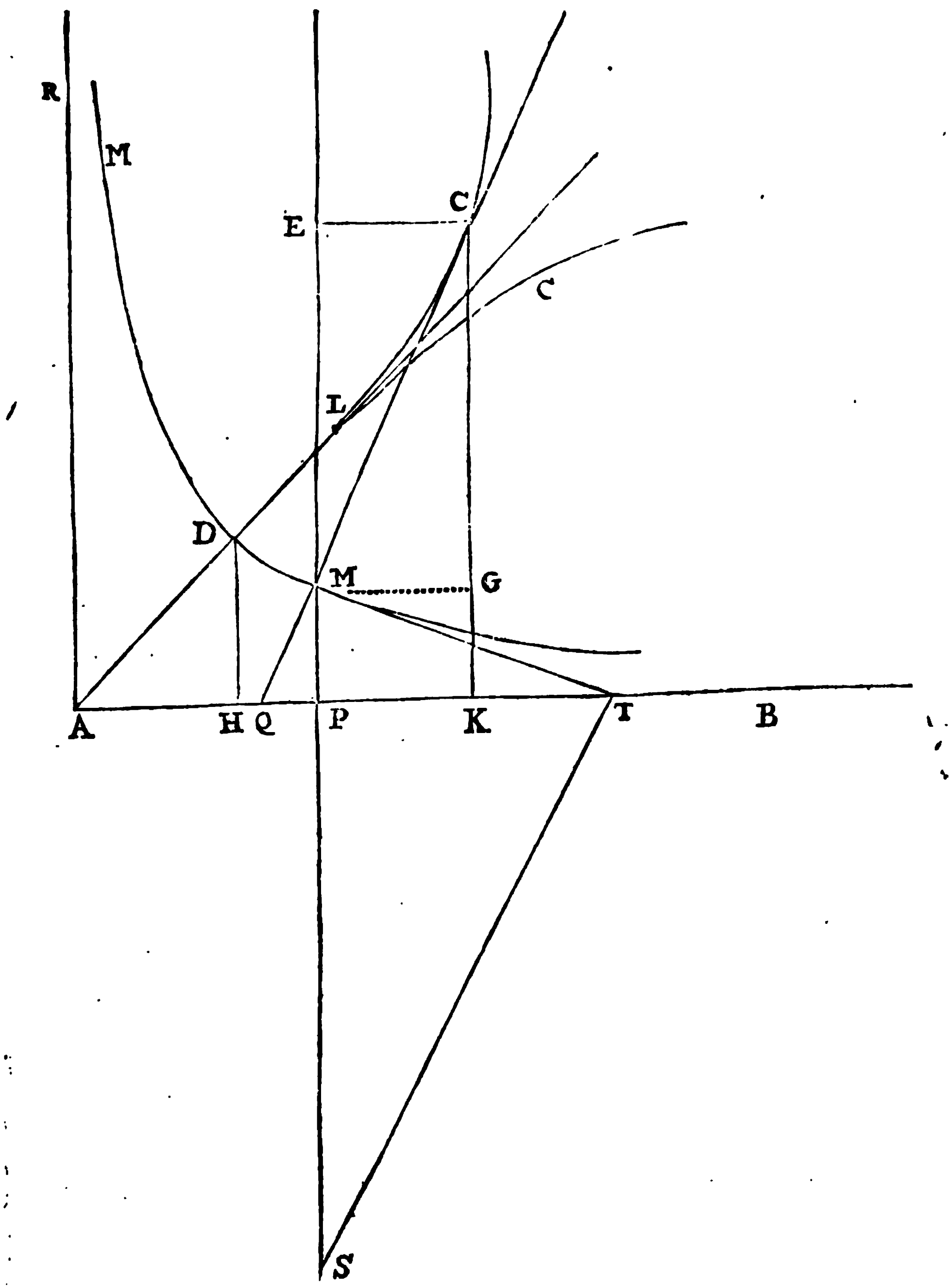


PROP.

PROP. V.

Let the given Curve MDM *be an Hyperbola between the Assymptotes* AB, AR; *'tis requir'd to determine the Length of* MC *the Ray of the Evoluta.*

248. The Equation expressing the Nature of the Curve is $aa = xy$. Whence $\frac{aa}{y} = x$, and $\frac{-aa\dot{y}}{yy} = \dot{x}$ and supposing $\dot{x}$ invariable, $\frac{-aay^2\ddot{y} + 2aay\dot{y}^2}{y^4}$



$= 0$, and consequently $\ddot{y} = \frac{2aay\dot{y}^2}{aay^2} = \frac{2\dot{y}^2}{y}$ and substituting this Value for $\ddot{y}$

for $\ddot{y}$ (*Art.* 237.) in the general Theorem $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, there will arise ME $=$ $\frac{y\dot{x}^2 + y\dot{y}^2}{2\dot{y}^2}$, and because the Triangles MQP, MEC are similar, therefore MP : PQ :: $\dot{x}:\dot{y}$ :: ME $\left(\frac{y\dot{x}^2 + y\dot{y}^2}{-2\dot{y}^2}\right)$: EC or PK $= -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$; and hence arises this

CONSTRUCTION.

Through the Point T, in which the Tangent MT intersects the Asymptote AB, draw TS parallel to MC, until it intersect MP produced in S, take ME $= \frac{1}{2}$ MS on the opposite side of the Curve, in respect of the Asymptote AB (which is instead of an Axis) because the value of ME is Negative, or take PK $= \frac{1}{2}$QT on the same side of the Ordinate with T; I say, if EC be drawn parallel, or KC perpendicular to the Axis AB, either of them will intersect the Line MC in the Point C required; for it is evident that TQ $= \frac{y\dot{y}}{\dot{x}} + \frac{y\dot{x}}{\dot{y}}$ (because $\dot{y} : \dot{x} :: y$: $\frac{y\dot{x}}{\dot{y}}$ = PT, and $\dot{x} : \dot{y} :: y : \frac{y\dot{y}}{\dot{x}}$ = PQ) and that MS is $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{y}^2}$, because MP is $= y$, and $\dot{y} : \dot{x}$ :: MP : PT $\left(\frac{y\dot{x}}{\dot{y}}\right)$ :: PT : PS $= \frac{y\dot{x}^2}{\dot{y}^2}$; whence MP + PS $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{y}^2} = 2$ ME.

249. and if we consider the Figure attentively, it will appear, that the Evoluta CLC will have a Point (L) of Retrogression, in like manner as the Evoluta of the Parabola.

Now to determine the Point L the Vertex of the Evoluta, or the Point of *Retrogression*, I observe that the Ray of the Evoluta, *viz.* DL is of all others the shortest; whence it follows that the Fluxion of $\frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}} = \frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ will be equal to nothing or infinity; and consequently (supposing $\dot{x}$ invariable) $\frac{-3\,\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} \times \dot{x}\dot{y}\ddot{y}^2}{\dot{x}^2\ddot{y}^2} + \frac{\dot{x}\dddot{y}\,\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2} = 0$, or infinity; whence dividing by $\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$, and multiplying by $\dot{x}\ddot{y}^2$ there will arise $\frac{-3\dot{x}\dot{y}\ddot{y}^2 + \dot{x}\dddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$ $= 0$, or infinity; and dividing by $\dot{x}$, $-3\dot{y}\ddot{y}^2 + \dddot{y} \times \overline{\dot{x}^2 + \dot{y}^2} = 0$, or infinity; which Equation will serve to find AH, the Value of the Abscissa (x) so that drawing the Ordinate HD, and the Ray of the Evoluta DL, the point L, will be the point of Retrogression requir'd.

Thus

Thus in our Example, $y = \frac{aa}{x}$, and $\dot y = \frac{-aa\dot x}{xx}$, and $\ddot y = \frac{2aa\dot x^2}{x^3}$, and $\dddot y = \frac{-6aa\dot x^3}{x^4}$, and substituting those Quantities in the Equation $\dot x^2\dddot y + \dot y^2\dddot y - 3\dot y\ddot y^2 = 0$, there will arise $\frac{-6aa\dot x^5}{x^4} - \frac{6a^6\dot x^5}{x^8} + \frac{12a^6\dot x^5}{x^8} = 0$, and dividing by $aa\dot x^5$, and multiplying by x^8, we shall have $-6x^4 - 6a^4 + 12a^4 = 0$, that is, a^4 is $= x^4$, and consequently $a = x =$ A H; whence it follows that the point D, is the Vertex of the Hyperbola, and that the lines A D and D L, make but one streight Line AL, which is the Axis of the Hyperbola, and the point of Retrogression L is in the said Axis, and may be determined by the foregoing General Construction.

To determine MC *the Radius of the Curvature another way.*

250. The Equation expressing the Nature of the Apollonian Hyperbola is $aa - xy = 0$, and the Universal Equation expressing the Nature of all sorts of Geometrical Curves is (*Art.* 239.) $fx^m + gy^n + bx^ry^s + a = 0$; and comparing the respective Terms of both Equations, it is plain that $b = -1$, $r = 1$, $s = 1$, whence the general Theorem (*Art.* 239.) expressing the length of M C will be $= \frac{-y\varpi^3}{2yzz} = -\frac{\varpi^3}{2zz}$, and hence we have this

CONSTRUCTION.

Take $PK = \frac{1}{2}QT$, and draw K C perpendicular to A B, until it intersect the perpendicular M C in C, I say the point C will be in the Evoluta; for the Triangles C M G, M Q P, are similar; therefore Q P (z) : Q M (ϖ) :: M G $\left(PK = \frac{\varpi^2}{2z}\right)$: $MC = \frac{\varpi^3}{2zz}$, and this Value of MC, was found to be Negative, the point G is towards the side of the Curve opposite to the Assymptote A B.

PROP. VI.

Let the general Equation $x = y^m$ express the Nature of all sorts of Paraboloides, when the Exponent m *is a Positive Number, whole or broken: And all sorts of Hyperboloides, when* m *represents any Negative Number; 'Tis requir'd to find a general Theorem expressing the Value of the Radius of the Curvature of all such Curves.*

251. Because $x = y^m$ therefore $\dot x = my^{m-1}\dot y$, and again, finding the Fluxion of this Equation (supposing $\dot x$ invariable) we have $\overline{m^2 - m}\,y^{m-2}\dot y^2 + my^{m-1}\ddot y = 0$, and dividing by my^{m-1}, there will arise $-\ddot y = \overline{m-1}\,y^{-1}\dot y^2 = \frac{\overline{m-1}\,\dot y^2}{y}$, and substituting this Value in the general Theorem $\frac{\dot x^2 + \dot y^2}{-\ddot y}$, we shall have $ME = \frac{y\dot x^2 + y\dot y^2}{\overline{m-1}\times\dot y^2}$, and consequently E C or P K is $= \frac{y\dot y}{\overline{m-1}\,\dot x} + \frac{y\dot x}{\overline{m-1}\,\dot y}$; and hence arises these General

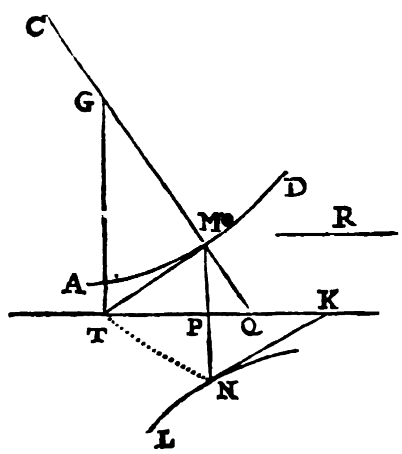
CON-

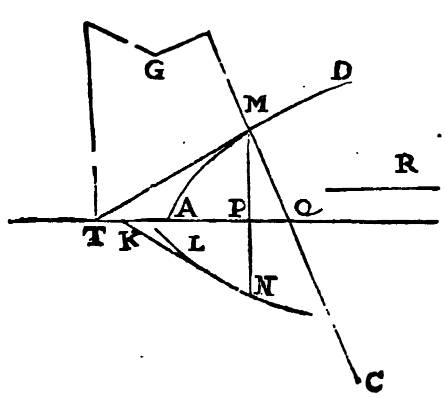
CONSTRUCTIONS.

Through the point T, in which the Tangent MT interfects the Axis, draw TS parallel to MC, until it interfect MP (produced) in S, and then take ME $= \frac{1}{m-1}$ MS (which if m be Negative or a Fraction, will be Negative) or take PK $= \frac{1}{m-1}$ TQ; then 'tis evident that if through the point E, we draw a line parallel, or through the point K, a perpendicular to the Axis AB, they will interfect MC, in the point C requir'd; for MS is $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{y}^2}$, therefore $\frac{1}{m-1}$ MS is $= \frac{y\dot{x}^2 + y\dot{y}^2}{\overline{m-1}\,\dot{y}^2}$ and QT is $= \frac{y\dot{x}}{\dot{y}} + \frac{y\dot{y}}{\dot{x}}$, *ergo* $\frac{1}{m-1}$ QT is $= \frac{y\dot{x}}{\overline{m-1}\,\dot{y}} + \frac{y\dot{y}}{\overline{m-1}\,\dot{x}}$.

Another way.

252. Let the general Equation expressing the Nature of all sorts of Paraboloides and Hyperboloides be $x = y^m$, then to find the Nature of the new Curve LN, it





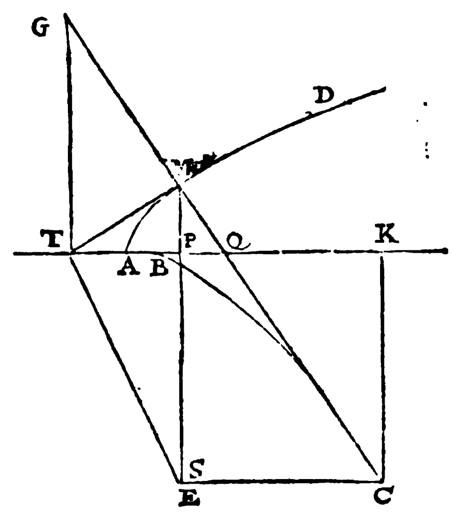
is (*ex Hyp.*) MP ($x^{\frac{1}{m}}$) : PT (mx) :: 1 : PN $= \frac{mx}{x^{\frac{1}{m}}} = mx^{1-\frac{1}{m}} = mx^{\frac{m-1}{m}} = s$, whence the Equation expressing the Nature of the Curve LN is $mx = s^{\frac{m}{m-1}}$, and consequently the Subtangent PK is $= \frac{m}{m-1}x$, whence if we say, PT (mx) : PK $\left(\frac{m}{m-1}x\right)$ :: QG : $\frac{1}{m-1}$ QG = MC = to the Radius of the Curvature in the point M, so that if QM be produced until it interfect TG in G, and if MC be taken $= \frac{1}{m-1}$ QG the point C, will be in the Evoluta of the given Curve.

CON-

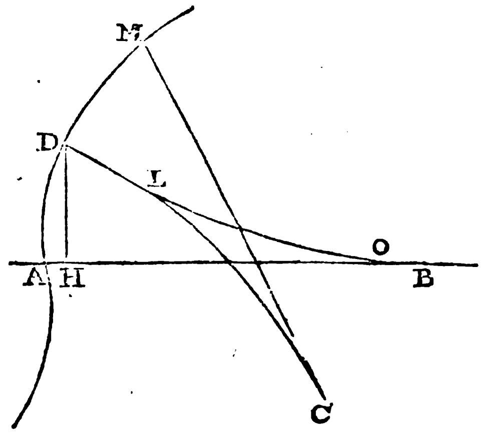
CONSECTARY I.

253. If m be Negative ($v.g.\ m = -1$, as in the common Hyperbola) as happens in all Hyperboloides, then the value of $ME = \frac{y\dot{x}^2 + y\dot{y}^2}{m-1\dot{y}^2}$ will be Negative (that is, in the Hyperbola $ME = \frac{y\dot{x}^2 + y\dot{y}^2}{-2\dot{y}^2}$) and consequently the Curve will be Convex towards the Axis, which will be the same with one of the Asymptotes. But in Parabola's in which m is a Positive Quantity, there may happen two Cases 1°, if m be less than an Unite, then they will be Convex towards their Axes, which will be a Tangent to the Curve in the Vertex, and the value of ME will be Negative, or 2° m is greater than 1, and then the Curves are Concave towards their Axes, which Axes are perpendicular to Tangents drawn through their Vertex's.

Now in this last Case to find the Point B, in which the Axis AB touches the Evoluta, that is, to find the Point B the Vertex of the Evoluta, we have $PQ = \frac{y\dot{y}}{\dot{x}} =$ (by substituting $my^{m-1}\dot{y}$ for $\dot{x}$) $\frac{y}{my^{m-1}} = \frac{y^{2-m}}{m}$, which gives us three several Cases; for 1°. m is $= 2$, which happens only in the common Parabola, and then the Exponent of y being $= 2 - m = 0$, that unknown Quantity will vanish, and consequently, PQ or AB will be $= \frac{1}{2} =$ (because the Parameter was supposed $= 1$) $\frac{1}{2}$ the Parameter of the Parabola; or 2° m is less than 2, and then the Exponent of y being Positive, it will continue in the Numerator, therefore when y vanishes, the said Fraction will be $= 0$, and consequently, the Point B in this case will coincide with the Point A the Vertex of the Curve, that is, the Vertex of the Evoluta will coincide with the Vertex of the given Curve; thus in the Quadratocubic Paraboloid ($x = y^{\frac{3}{2}}$, or $xx = y^3$) m is $= \frac{3}{2}$, and $\frac{y^{2-m}}{m}$ is $= \frac{y^{\frac{1}{2}}}{m} = PQ$,



that is, when y vanishes, PQ or AB is $= 0$, or 3°. m exceeds 2, and then the Exponent of y being Negative, y will be in the Denominator of the Fraction, which makes the Fraction infinite, when y is $= 0$, that is, the point B, or the Vertex of the Evoluta, is at an infinite distance from A, or which is the same thing, the Axis AB, will be an Assymptote to the Evoluta LO, as in the cubical Paraboloid, ($aax = y^3$) m is $= 3$, and $\frac{y^{2-3}}{m}$ is $= \frac{1}{my}$, and when y is $= 0$, then $\frac{1}{my}$ is infinite; that is, the Subnormal or the Radius of the Curvature in A, is infinite.



254. And in this last Case, it may be observed, that the Evoluta (C L O) of the Semiparaboloid A D M, has a point (L) of Retrogression, so that by Evolving the Portion L O (produced infinitely) the point D, will only describe the Determinate Portion of the Curve D A, and by Evolving the other part L C drawn out infinitely, the point D, will describe the infinite Portion of the Curve D M.

The point L, may be investigated in the same manner as in the Hyperbola.

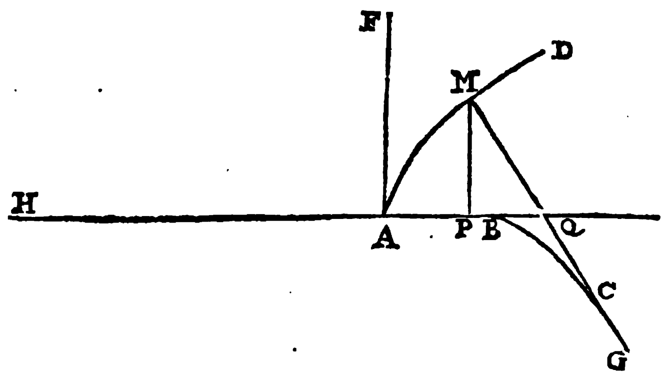
For instance, suppose $x = y^3$, or $y = x^{\frac{1}{3}}$, then is $\dot{y} = \frac{1}{3} x^{-\frac{2}{3}} \dot{x} = \frac{\dot{x}}{3x^{\frac{2}{3}}}$ and $\ddot{y} = -\frac{2}{9} x^{-\frac{5}{3}} \dot{x}^2$, and $\dddot{y} = \frac{10}{27} x^{-\frac{8}{3}} \dot{x}^3$, which Values being Substituted in the general Equation (*Art.* 249.) $\dot{x}^2 \dddot{y} + \dot{y}^2 \dddot{y} - 3\dot{y}\ddot{y}^2 = 0$, there will arise, $\frac{10}{27} x^{-\frac{8}{3}} \dot{x}^5 + \frac{10}{243} x^{-\frac{12}{3}} \dot{x}^5 - \frac{12}{243} x^{-\frac{12}{3}} \dot{x}^5 = 0$, that is dividing by $\dot{x}^5$; $\frac{10}{27} x^{-\frac{8}{3}} - \frac{2}{243} x^{-\frac{12}{3}} = 0$, or $10 x^{\frac{4}{3}} = \frac{54}{243}$, which being divided by 10, we have $x^{\frac{4}{3}} = \frac{54}{2430}$, and consequently $x^4 = \overline{\frac{54}{2430}}|^3, = \frac{157464}{14348907000} = \frac{1}{91125}$, whence x $(= AH)$ is $= \sqrt[4]{\frac{1}{91125}}$, which is the same Number that the Learned *M. Hugens* found several Years ago. *Horolog. Oscillat. Pag.* 88.

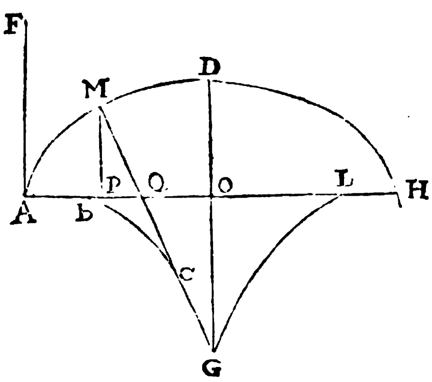
PROP.

PROP. VII.

Let the Curve AMD *be an Hyperbola or an Ellipse, whose Axis* AH *is* $= a$, *and Parameter* AF $= b$. *'Tis requir'd to describe the Evoluta* BC.

255. The Equation expressing the Nature of those Curves is $y = \frac{\sqrt{abx \mp bxx}}{\sqrt{a}}$





whence $\dot{y}$ is $= \frac{ab\dot{x} \mp 2bx\dot{x}}{2\sqrt{aabx \mp abxx}}$; and putting $p =$ to the Numerator, and $q =$ Denominator $\dot{y} = \frac{p}{q}$, and $\ddot{y} = \frac{q\dot{p} - p\dot{q}}{qq} =$ (by restoring the respective Values) $\frac{-a^3bb\dot{x}^2}{4aabx \mp 4abxx \times \sqrt{aabx \mp abxx}}$, and substituting this in the general Theorem $\frac{\dot{x}^2 + \dot{y}^2 \times \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$ (= MC) we shall have MC $=$

$$\frac{aabb \mp 4abbx + 4bbxx + 4aabx \mp 4abxx \sqrt{aab^2 \mp 4ab^2x + 4b^2x^2 + 4a^2bx \mp 4abxx}}{2a^3bb}$$

$=$ (because MQ $= \frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$ is $= \frac{\sqrt{aabb \mp 4abbx + 4bbxx + 4aabx \mp 4abxx}}{2a}$) $\frac{4\,\text{MQ}^3}{bb}$; whence arises this

CON-

CONSTRUCTION.

Find a fourth continual proportional to the Parameter b, and the Perpendicular MQ (saying, b, MQ, $\frac{MQq}{b}$, $\frac{\overline{MQ}^3}{bb}$ ∺) and multiply the same by 4, the product $\frac{4\overline{MQ}^3}{bb}$ is $=$ MC, and C is the point in the Evoluta requir'd.

Another way.

256. The Equation expressing the Nature of the Ellipse is $ayy - abx + bxx = 0$, and the Universal Equation expressing the Nature of all sorts of Geometrical Curves, is $fx^m + gy^n + bx^r y^s + a = 0$, and comparing the respective Terms of both Equations, we shall have $f = -ab$, $g = a$, $b = b$, $m = 1$, $n = 2$, $r = 2$; whence the general Theorem expressing the Value of the Radius of the Curvature MC will become (in this Case) $= \frac{\overline{-ab + 2bx} \times \pi^3}{-2az^3 - 2byyz} = \frac{\overline{ab - 2bx} \times \pi^3}{2az^3 + 2by^2z}$:

But $\frac{\text{subtang.} \times PQ}{MPq}$ is $= 1 = \frac{2az}{ab - 2bx}$ therefore MC $= \frac{\overline{ab - 2bx} \times \pi^3}{2az^3 + 2by^2z} = \frac{2a \times \pi^3}{2azz + 2byy} = \frac{4a \times \pi^3}{azz + byy} =$ (because $\frac{azz + byy}{a}$ is $= bb$) $\frac{4\pi^3}{bb} = \frac{4\overline{MQ}^3}{bb}$, whence the Construction may be the same as before.

CONSECTARY I.

257. If x be put $= 0$, then the Radius of the Curvature AB, will be $= \frac{1}{2}b$, and in the Ellipse, if we suppose $x = \frac{1}{2}a$, then the Radius of the Curvature DG, will be $= \frac{a\sqrt{ab}}{2b} = \frac{1}{2}$ the Parameter of the shortest or conjugate Diameter, whence it is evident in the Ellipse, that the Evoluta BCG, terminates in the point G, in the shortest Axis DO; but in the Hyperbola and Parabola it runs out infinitely.

CONSECTARY II.

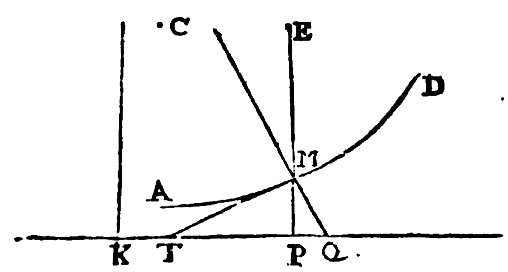
In the Ellipse, if a be $= b$, then is MC $= \frac{1}{2}a$, an invariable Quantity, whence it follows that all the Radii of the Evoluta are equal between themselves, which consequently can be but a single point; that is, the Ellipse, in such a case, degenerates into a Circle, whose Evoluta is the Center.

PROP.

PROP. VIII.

Let A M D *be the common Logarithmetic Curve, the Nature whereof is such that drawing (from any point in the Curve as* M) *the Line* MP *perpendicular to the Assymptote, and* MT *touching the Curve in* M, *and intersecting the Assymptote in* T, *the Subtangent* P T, *be always* = a *an invariable Quantity.*

258. The Subtangent PT is $= \frac{y\dot{x}}{\dot{y}} = a$, therefore $\dot{y} = \frac{y\dot{x}}{a}$, and $\ddot{y} =$ (supposing $\dot{x}$ invariable) $\frac{\dot{y}\dot{x}}{a} = \frac{y\dot{x}^2}{aa}$; but ME is $= \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} =$ by substituting $\frac{yy\dot{x}^2}{aa}$ for $\dot{y}^2$ and $\frac{y\dot{x}^2}{aa}$ for $\ddot{y}$) $\frac{-aa-yy}{y}$, and consequently E C or P K = (because T P (a). P M (y) :: P M (y) : PQ $= \frac{yy}{a}$ and M P (y) : PQ $\left(\frac{yy}{a}\right)$ :: M E $\left(\frac{-aa-yy}{y}\right)$: E C $= -\frac{aa-yy}{a}\Big)$ $= \frac{-aa-yy}{a}$ which gives this



CONSTRUCTION.

Take PK = TQ, on the same side of the Ordinate with T, (because its Value is Negative) and draw K C parallel to P M, and it will intersect the perpendicular M C in C the point requir'd; for T P is $= a$, and PQ is $= \frac{yy}{a}$, and consequently T Q is $= \frac{aa+yy}{a}$.

Another way.

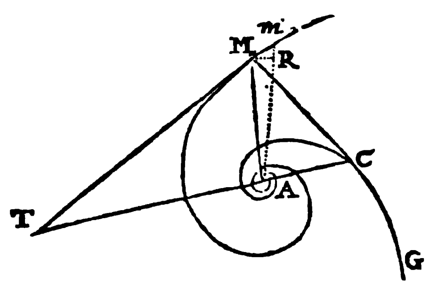
259. Let A M D be the Logarithmetic Line, then say (*Art.* 240.) M P : P T (an invariable Quantity) :: any other invariable Quantity (as P T) : $\frac{PTq}{MP}$ = P N; and describe the Curve L N, which will also be a Logarithmetic Curve, and the Subtangent P T, will be = Subtang. P K; therefore Q G is = M C = to the Radius of the Curvature in the point M; which is a remarkable property of this Curve, and is the same with that of the common Parabola, in which Q G, is always = M C, the Radius of the Curvature in M.

PROP. IX.

If AMD *be the Logarithmetical Spiral Line. 'Tis requir'd to investigate the Value of the Radius of the Curvature* MC.

260. The Nature of the said Curve is such, that drawing from any point of the Curve, as M, the right Line MA to the Center A; and the Tangent MT, the Angle AMT, will always be the same.

Because the Angle AMT or A*m*T is invariable, the Ratio of *m*R ($\dot{y}$) to MR ($\dot{x}$) is also invariable, and consequently the Fluxion of $\frac{\dot{y}}{\dot{x}}$ is $=0$, which gives $\ddot{y}$ (supposing $\dot{x}$ constant) $=0$. Now the general Theorem for such Curves is ME $=$ (*Art.* 241.) $\frac{y\dot{x}^2+y\dot{y}^2}{\dot{x}^2+\dot{y}^2-y\ddot{y}}=$ (because $\ddot{y}$ is $=0$,



and consequently $-y\ddot{y}=0$) $\frac{y\dot{x}^2+y\dot{y}^2}{\dot{x}^2+\dot{y}^2}=$ (dividing by $\overline{\dot{x}^2+\dot{y}^2}$) y, that is ME is $=$ MA, which gives this easie

CONSTRUCTION.

Draw AC perpendicular to AM, and MC perpendicular to the Curve in M, then the point C in which they mutually intersect each other, is in the Evoluta ACG.

CONSECTARY I.

261. The Angles AMT, ACM are equal (for each being added to AMC, make a right Angle) Therefore the Evoluta ACG, is also a Logarithmetical Spiral Line, and the Curves AMD, ACG differ only in position.

CONSECTARY II.

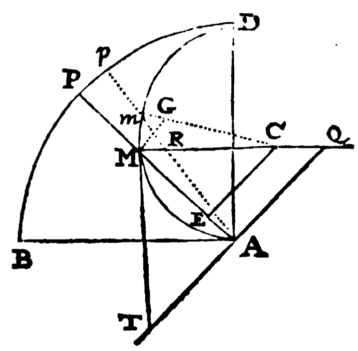
If a point C in the Evoluta ACG be given, and it be requir'd to find the Length of the Ray CM $=$ to the portion of the Curve AC (which makes an infinite number of Revolutions before it terminates in the Center) draw AM perpendicular to CA, until it intersect the Tangent CM in M; then is CM $=$ AC the portion of the Spiral Evolved; and if AT be drawn perpendicular to AM, then is the Tangent MT $=$ to the portion of the given Spiral Line AM.

PROP.

PROP. X.

If AMD *be one of the infinite sorts of Spiral Lines formed in the Sector of a Circle* BAD. *'Tis required to determine the Length of the Ray of the Evoluta* MC.

262. Suppose the whole Arch BPD $= b$, BP $= z$, AB or AP $= a$, and AM $= y$; and let the Equation expressing the Nature of the Spiral Line AMD be $y^m = \frac{a^m z}{b}$, then is $m y^{m-1} \dot{y} = \frac{a^m \dot{z}}{b}$, (because the Sectors AMR, APp are similar) AM (y) : AP (a) :: MR ($\dot{x}$) : Pp $= \dot{z} = \frac{a\dot{x}}{y}$ which being substituted in place of $\dot{z}$ in the preceding Equation, we have $m y^m \dot{y} = \frac{a^{m+1}\dot{x}}{b}$, and the Fluxion of this Equation is (supposing $\dot{x}$ invariable) $mmy^{m-1}\dot{y}^2 + my^m\ddot{y} = 0$, and (dividing by my^{m-1}) $m\dot{y}^2 + y\ddot{y} = 0$, and $-y\ddot{y} = m\dot{y}^2$, and consequently ME $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}$ is $=$ (by equal Substitution) $\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \overline{m+1}\,\dot{y}^2}$; from which Equation we may deduce this



CONSTRUCTION.

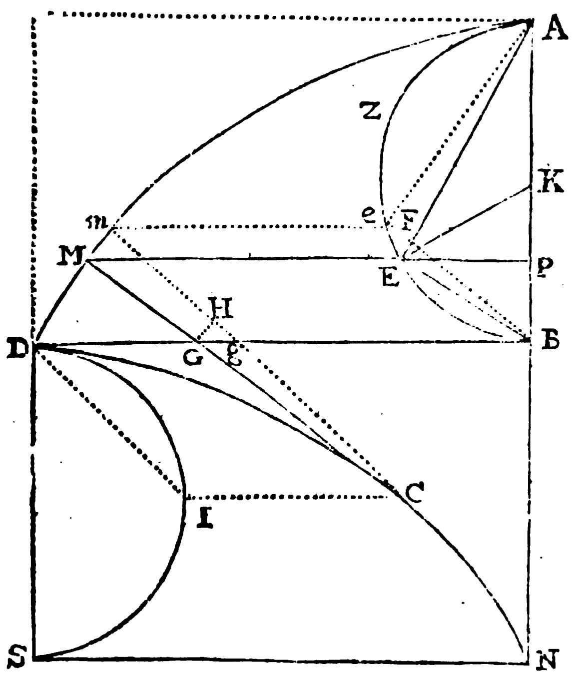
Through the Center A draw the right Line TAQ perpendicular to AM, and interfecting the Tangent MT in T, and the perpendicular to the Curve MQ in Q; then say, TA $+ \overline{m+1}$ AQ : TQ :: MA : ME, I say EC drawn parallel to TQ, will intersect the perpendicular MQ in C, the point requir'd.

For (because the Line MRG, is parallel to AQ) MR $+ \overline{m+1}$ GR ($= \dot{x} + \frac{\overline{m+1}\,\dot{y}^2}{\dot{x}}$, because MR $= \dot{x}$, and MR ($\dot{x}$) : Rm ($\dot{y}$) :: Rm ($\dot{y}$) : RG $= \frac{\dot{y}^2}{\dot{x}}$) : MG $\left(\frac{\dot{y}^2}{\dot{x}} + \dot{x}\right)$ :: TA $+ \overline{m+1}$ AQ : TQ :: AM (y) : ME $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \overline{m+1}\,\dot{y}^2}$.

PROP. XI.

If the Curve Line AMD *be a simple Semi-cycloid, whose Base* BD *is equal to the Semi-periphery of the generating Circle* BEA. *'Tis requir'd to find the Value of the Ray of the Evoluta* MC.

263. Suppose AP $= x$, PM $= y$, the Arch AE $= u$, and the Diameter AB $= 2a$, then by the property of the Circle, PE is $= \sqrt{2ax - xx}$ and by the property of the Cycloid, $y = u + \sqrt{2ax - xx}$; therefore $\dot{y} = \dot{u} + \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$; but $\dot{u} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$, therefore $\dot{y} = \frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}} = \dot{x} \times \frac{2a - x}{\sqrt{2ax - xx}} = \dot{x} \times \frac{2a - x}{\sqrt{x} \times \sqrt{2a - x}} =$ (dividing by $\sqrt{2a - x}$) $\dot{x} \times \frac{\sqrt{2a - x}}{\sqrt{x}}$; whence $\ddot{y} =$ (supposing $\dot{x}$ invariable) $\frac{-a\dot{x}^2\sqrt{x}}{xx\sqrt{2a - x}} = \frac{-a\dot{x}^2}{x\sqrt{2ax - xx}}$



and substituting this value in the general Theorem $\frac{\overline{\dot{x}^2 + \dot{y}^2}\sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$ or

$$\frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}} \text{ we shall have } \left(\text{because } \dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}\right) \frac{\overline{\frac{2a\dot{x}^2}{x}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}} = \frac{\sqrt{\frac{8a^3\dot{x}^6}{x^3}}}{-\dot{x}\ddot{y}} =$$

$$\frac{\sqrt{\frac{8a^3\dot{x}^6}{x^3}} \times x\sqrt{2ax - xx}}{a\dot{x}^3} = \frac{\sqrt{\frac{8a^3}{x^3}} \times x\sqrt{2ax - xx}}{a} = \frac{\sqrt{\frac{16a^4x^3 - 8a^3x^4}{x^3}}}{a} =$$

$2\sqrt{4aa - 2ax} =$ MC $=$ ($2\sqrt{EPq + PBq} = 2$ EB) $= 2$ MG, because MC perpendicular to the Curve in the point M is parallel to the Chord BE.

CONSECTARY I.

264. If x be supposed $= 0$, then is AN $= 2\sqrt{4aa} = 4a =$ to the Ray of the Evoluta in the Vertex A, and if we suppose $x = 2a$, then MC $= 2\sqrt{4aa - 4aa} = 0$, that is the Ray of the Evoluta in D, is equal to nothing; and in A it is equal to twice the Diameter of the generating Circle, and hence 'tis evident, that the Evoluta begins in D, and ends in N, so that BN is $=$ BA.

CONSECTARY II.

The Evoluta DCN is a Semi-cycloid equal to the given Semi-cycloid DMA: Compleat the Parallelogram BS, and on the Diameter DS describe the Semicircle DIS, and draw DI || MC || EB; then is the Angle BDI = EBD, and consequently the Arches DI, BE are equal; but EB = MG = GC; *ergo* GC = DI, and if IC be drawn, it will be equal and parallel to DG: Now by the Nature of the Cycloid DG is = Arch EB = Arch DI; therefore IC is = Arch DI, and consequently the Evoluta DCN is a Semi-cycloid, whose Base is SN = $\frac{1}{2}$ the periphery of the generating Circle DIS, that is, the Evoluta is equal to the given Cycloid, and the same with it, only placed in a contrary Position.

CONSECTARY III.

The Length of the Curve of the Cycloid DCN is = 2 AB (= twice the Diameter of the generating Circle,) and any portion of the Cycloid as DC is = 2 CG = 2 DI = twice the corresponding Chord in the generating Circle.

Another Solution.

265. The length of the Ray of the Evoluta MC, may be determined without any Calculation thus: Draw another perpendicular *m* C infinitely near the former, and another Ordinate *me* parallel to ME, and another Chord B*e*, and on the Centers C and B describe the little Arches GH, EF; then the Rectangular Triangles GH*g*, EF*e*, will be similar and equal; for G*g* is = E*e* (because BG or ME is = Arch AE, and B*g* or *me* is = Arch A*e*) and H*g* or *mg* — MG = F*e* or B*e* — BE; and (47. *Elem.* 1.) GH = EF. Now the Angle MC*m* is = EB*e* (because the perpendiculars MC, *m* C are parallel to the Chords EB, *e* B, and GH, EF, the Arches that measure those equal Angles are equal, therefore the Radii CG, EB are also equal, and consequently MG is = GC; whence 'tis evident that the Ray of the Evoluta MC is = twice the Chord BE = 2 MG.

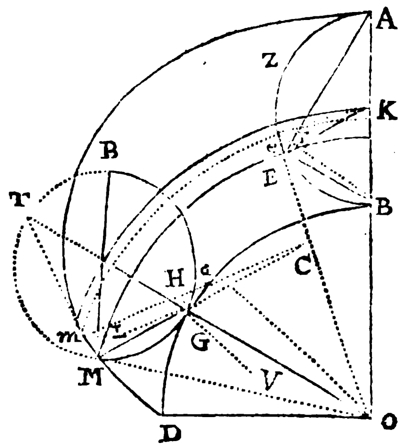
CONSECTARY.

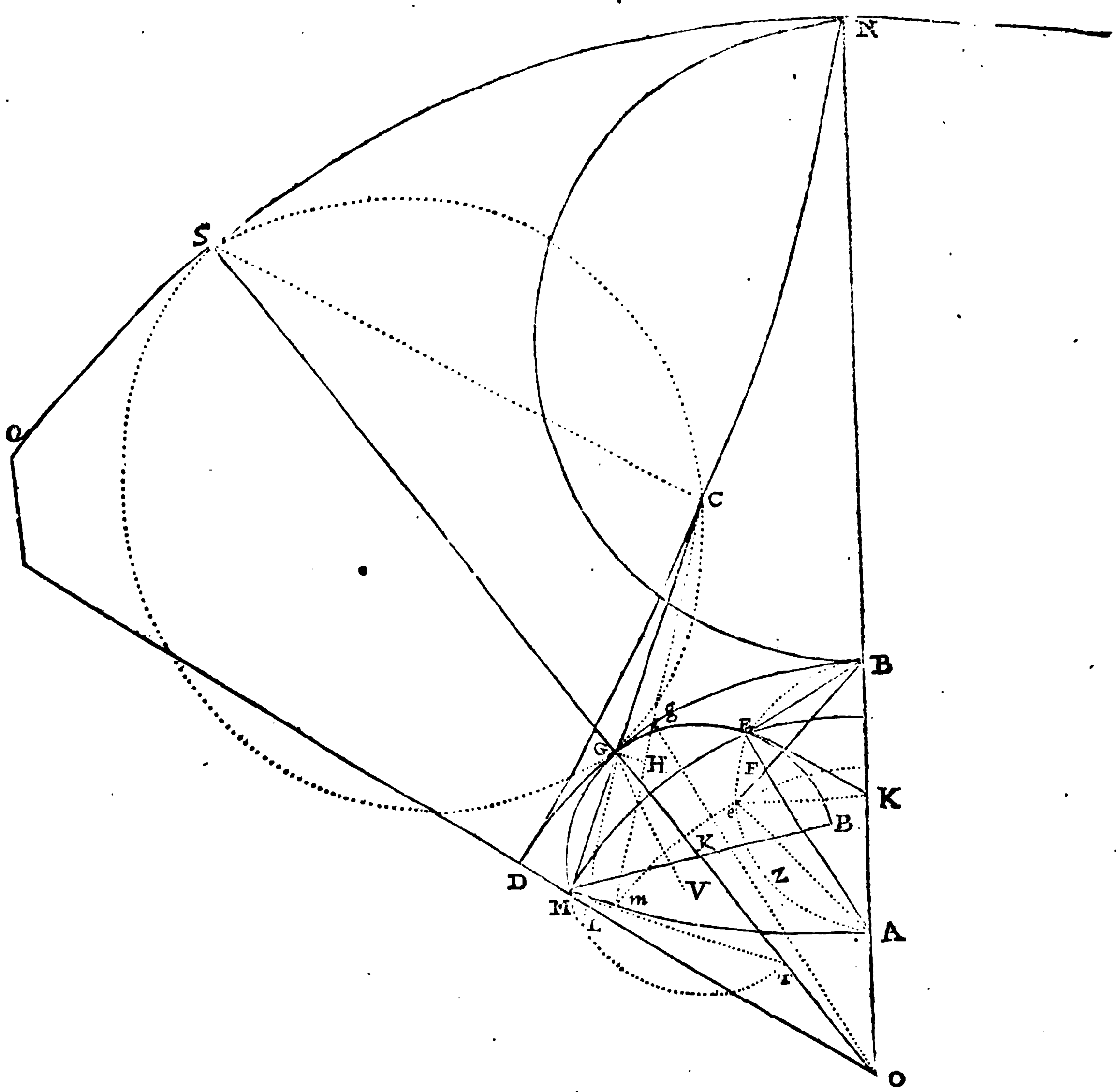
266. We have proved before that the Area of the Cycloid is triple the Area of the generating Circle, this truth may be proved from other Principles, as thus; the Space MG*gm*, or the Trapezium MGH*m* (the difference being incomparably little) is = $\frac{1}{2}\overline{Mm + \frac{1}{2}GH} \times MG = \frac{3}{2} GH \times MG = \frac{3}{2} EF \times BE$, that is the Trapezium MG*gm* is = three times the Sector EBF or EB*e*, therefore the sum of all the Trapezia, *viz.* the Cycloidal Space MGBA is equal to three times the sum of all the Triangles, *viz.* the Circular Space BEZA; and the whole Cycloidal Space AMDBA is = thrice the Area of the Semicircle AEBA.

I now proceed to investigate the Evoluta of another sort of Cycloids, having Circular Bases, and to investigate the Areas comprehended by such Curves (*Art.* 115.) which I promis'd to consider in this Section.

PROP. XII.

If the Curve AMD *be a Semicycloid describ'd by the Revolution of the Semicircle* AEB, *on the Periphery of another immovable Circle* BGD. *'Tis requir'd to describe the Evoluta of the said Curve.*





267. The movable or generating Circle, may be suppos'd to move either on the Convex or Concave side of the Periphery of the immovable Circle; and when the Semi-

Semi-circle AEB comes into the position MGB, in which position it touches the Base BD in G, and the describing point A, is in M, in the Curve of the Cycloid; then from the Genesis of the Curve I infer.

1°. The Arch GM is = Arch GD, and the Arch GD of the movable Circle is equal to the Arch GB, of the immovable Circle.

2°. MG is perpendicular to the Curve AMD, for if we consider the Semicircumference MGB or AEB, and the Base BGD, as being compos'd of an infinite Number of little streight Lines, and every one in one equal to the corresponding one in the other, 'twill be manifest that the Semi cycloid AMD is composed of an infinite Number of Circular Arches; which have for their Centers, all the points of contact G, successively; and are all describ'd by the same point M.

3°. If on O, the Center of the immovable Circle, the Concentrick Arch ME be describ'd, then the Arches of the movable Circle, *viz.* MG, EB will be equal; and also the Chords MG, and EB; and the Angles OGM, OBE will be equal between themselves; for in the Triangles OKM, OKE, the three sides of the one are equal to the three sides of the other respectively; therefore the Angles MKO is = EKO, and the Arch MG is = Arch EB, and the Chord MG is = Chord EB; and KGM is = KBE, and consequently OGM is = OBE.

These things being premis'd, Let *m*C be drawn perpendicular to the Curve AMD, and infinitly near MC, draw also another Concentric Arch *me*, and another Chord B*e*; and on the Centers C and B, describe the Arches GH, EF; then the Rectangular Triangles GH*g*, EF*e* are equal and similar; for G*g* or D*g* — DG = E*e* or the Arch B*e* — Arch BE, and H*g* or *mg* — MG is = F*e* or B*e* — BE; and consequently the little Arch GH is = Arch EF; whence it follows that the Angle GCH is to the Angle EBF, as BE is to CG: it remains therefore to find the proportion between those Angles: Which we may do in this manner:

Having drawn the Radii OG, O*g*, KE, K*e*; suppose OG or OB = b, KE = a, 'tis evident that the Angle EB*e* is = OB*e* — OBE (or OBE — OB*e*) = OG*m* — OGM (or OGM — OG*m*) = (having drawn GL, GV, parallel to C*m*, O*g*.) LGM ∓ OGV = GCH ∓ GO*g*; therefore the Angle GCH is = Angle EB*e* ± GO*g*; now the Arches G*g*, E*e* being equal, it is, GO*g* : EK*e* or 2EBF :: KE (a) : OG (b.) and consequently the Angle GO*g* is $= \frac{2a}{b}$ EBF,

and GCH is = (EBF ± GO*g*) $= \frac{b \pm 2a}{b}$ EBF; therefore GCH : EBF

(:: BE : CG) :: $\frac{b \pm 2a}{b}$: 1, and consequently the unknown Quantity CG is =

$\frac{b}{b \pm 2a}$; BE : which gives this

CONSTRUCTION.

Say, as OA ($b \pm 2a$) : OB (b) :: BE or MG : GC, then the point C, will be in the Evoluta requir'd.

CONSECTARY I.

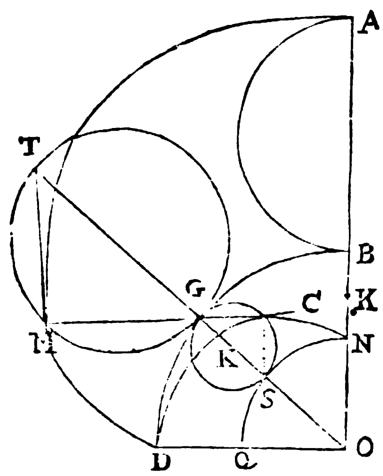
268. The Evoluta begins in the point D, and touches the Base BGD in D; for the Chord GM (the third Term in the Analogy) vanishes in that point.

CONSECTARY II.

The Evoluta DCN terminates in the point N, so that then OA : OB :: AB : BN :: OA ± AB (= OB) : OB ± BN (= ON) that is, OA, OB, ON are continually proportional.

CONSECTARY III.

If the Arch of the Circle N S Q, be described on the Center O, I say the Evoluta D C N, may be describ'd by the Revolution of the movable Circle G C S (whose Diameter is G S = B N) about the immovable Circle N S Q; that is, the Evoluta D C N is a Semi-cycloid, similar to the given Semi-cycloid A M D (because the Diameters A B, B N of the movable Circles, are proportional to the Radii of the immovable Circles O B, O N; for A B : O B :: B N : O N) and in an inverted position, having its Vertex in D; for suppose the Diameters of the movable Circles to be in O T (drawn at pleasure from the Center O) it will pass through the points of contact S and G; then if we say, A B or T G : B N or G S :: M G : G C; the point C, will be in the Evoluta (by Construction) and in the Circumference of the Circle G C S (*Prop.* 31. *Elem.* 3. *Prop.* 6. *Elem.* 6.) because the Angle G M T being a right Angle, the Angle G C S is so also; and because M G T = C G S, therefore the Arch T M (= G B) : C S :: G T : G S :: O G : O S :: G B : N S; therefore the Arch C S is = Arch N S; *ergo*, &c.



CONSECTARY IV.

Hence 'tis evident that the portion of the Curve of the Cycloid D C is = right Line C M, and consequently that D C : Tang. G C :: A B + B N : B N :: O B + O N : O N, that is, the sum (or difference) of both Diameters (of the movable and immovable Circles) is to the Semi-diameter of the immovable Circle, as D C is to the Tangent C G; for the Triangles C M *m*, C G H are similar, therefore M *m* : G H or E F :: M C : G C :: (by construct.) O A + O B ($2b \pm 2a$) : (*See Fig.* 2. *in Pag.* 210.) O B (b) and consequently the sum of all the M *m* or the portion of the Cycloid A M, is to the sum of all the E F or Chord A E or Tangent T M, as O A + O B : O B; whence 'tis evident that O B : O A + O B (= 2 O K) :: A B : A M D, and O B : 2 O K :: A B — A E : D M :: twice the Versed Sine of $\frac{1}{2}$ the Angle M K G or E K B : the portion of the Curve D M.

And because it is A M : Tang. T M :: O A + O B : O B; therefore in the vulgar Cycloid, A M : Tang: T M :: 2 : 1.

CONSECTARY V.

The Trapezium M G H *m* is $= \overline{\frac{1}{2} GH + \frac{1}{2} Mm} \times MG$, but C G $\left(= \frac{b}{b \pm 2a}\right.$ M G) : C M $\left(= \frac{2b \pm 2a}{b \pm 2a}\right.$ M G) :: G H : M *m* $= \frac{2b \pm 2a}{b}$ G H; therefore (because G H is = E F, and M G = E B) M G H *m* is $= \frac{3b \pm 2a}{2b} \times EF \times EB$, that is the Trapezium M G H *m* : corresponding Triang. E B F :: $3b \pm 2a : b$, and because the proportion universally obtains, 'tis evident that the Cycloidal Space M G B A M (*See Fig.* 2. *in Pag.* 210.) (comprehended under the right lines M G, A B, the Base G B, and the portion of the Curve A M) is to the corresponding Segment of the movable Circle B E Z A B :: $3b \pm 2a : b$, and the whole Cycloidal Space A M D B A is to the Area of the Semicircle A E B A as $3b \pm 2a$ is to b.

CON-

CONSECTARY VI.

If we imagine OB the Radius of the immovable Circle to become infinite, the Arch BGD will become a ſtreight Line, and the Curve AMD will be the vulgar Cycloid, and in this caſe, AB the Diameter of the movable Circle is $= 0$, in reſpect of that of the immovable Circle: Whence 1°. becauſe $b + 2a$ is $= b$, it is MG: GC :: $b : b$; that is, MG is $=$ GC, and conſequently if BN be taken $=$ AB; and NS be drawn parallel to BD, the Evoluta DCN will be generated by the revolution of a Circle (on the Baſe NS) whoſe Diameter is $=$ BN. 2°. the portion of the Cycloid AM is to the correſponding Chord of the Circle AE: $2b : b$, this is evident from §. 4°. 3°. The Space MGBA is to the Segment BEZA :: $3b : b$, which is alſo evident from §. 5°.

CONSECTARY VII.

The length of the Semicycloidal Curve is proportional to the Rectangle BKO, if the Semidiameter of the immovable Circle be the ſame; let BA be the Diameter of one, and Ba the Diameter of another movable Circle; and let OB be the Radius of the immovable Circle common to both, then by §. 4°.

$$OB : OA + OB :: AB : AMD$$

$$\text{And } OB : Oa + OB :: aB : amd,$$

And by proportion of Equality and Diviſion } $OA + OB : Oa + OB :: aB \times AMD : AB \times amd$

$$\text{that is, } 2OK : 2ok :: aB \times AMD : AB \times amd.$$

$$\text{Whence } OK \times AB : ok \times aB :: AMD : amd.$$

And dividing by 2, $BKO : BkO :: AMD : amd.$ Q.E.D.

CONSECTARY VIII.

Becauſe the Arches GD, GM are always equal between themſelves, it follows that the Angle DOG: Ang. GKM :: GK: OG, therefore if the point D (where the Cycloid begins) the Radii OG, GK, and the point of Contact G be given, the poſition of the point M, which deſcribes the Cycloid, is found by drawing the Ray KM, ſo that GK: GO :: DOG: GKM, and all the points of the Curve AMD may be determined Geometrically, when the proportion between the Radii OG, GK can be expreſs'd in numbers, and conſequently in that caſe, this *Cycloid is a Geometrical Curve* and the ſaid *Cycloid* is a *Tranſcendent* (or Mechanick) *Curve* when the Relation of OG to OK cannot be expreſs'd by any finite number of Terms.

CONSECTARY IX.

If in Concentric Spheres ſimilar Cycloids be deſcrib'd, their Perimeters will be proportional to the Semidiameters of the ſaid Spheres.

CONSECTARY X.

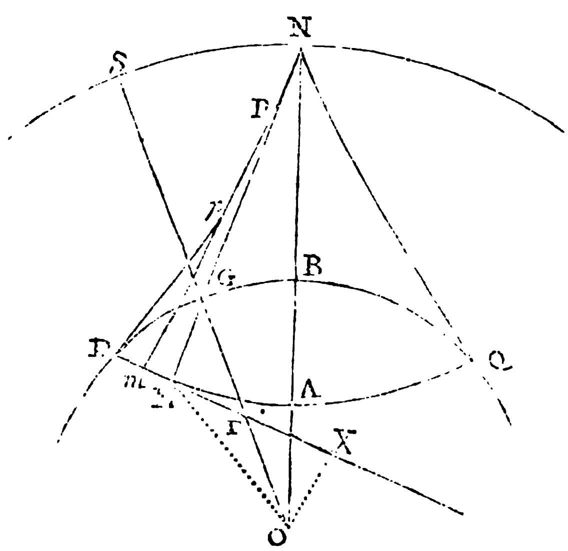
And becauſe the length of the Curve of the Cycloid AMD is proportional to the Rectangle BKO; 'tis plain, that in vulgar Cycloids, the Curve is proportional to the Diameter of the generating Circle.

SCHOLIUM.

By help of such Principles as these, the great M. Newton *has advanced several wonderful conclusions concerning the more exact measuring of time by Pendulums; as for Instance.*

269. 1°. If within the Globe B G D, the Cycloid D A Q be describ'd, being bisected in A, and terminating in the Surface of the Globe in D and Q, and if O A be produced (bisecting D Q in B) unto N, so that O A, O B, O N ∺ and the Globe N S be describ'd on the Center O, and the Semi-cycloids N D, N Q be describ'd within the said Globe; then a Pendulum suspended to the point N, and equal to N A, will Vibrate in the Cycloid D A Q, the same being described by the Evolution of the Cycloidal Cheeks N D, N Q, and thus a Pendulum may be made to Vibrate in any such given Cycloid.

2°. If the said Pendulum Vibrate in the Cycloid D A Q, by the sole force of its own Gravity, and if the force of Gravity in every point of the Curve D A Q; be as its distance from the Center O, then the Vibrations (equal or unequal) of the Pendulum, will be performed in equal times.



Let M T touch the Cycloid in M, and draw O X perpendicular to M X, then because the force of Gravity is as O M, it may be resolved into the parts O X, M X; now 'tis evident that the force O X, being parallel to the Thread P M, has no other effect but to distend the same, and is totally destroyed by its resistence, therefore the force M X only, accelerates the Motion of the Pendulum M, in the Cycloid; and the acceleration of the Pendulum in the Cycloid is always proportional to this accelerating force. Now the Triangles O X T, M G T are similar, and O T and G T are invariable Quantities, therefore M X, is always proportional to M T, and M T is proportional to the Curve of the Cycloid M A, therefore if two Pendulums N P M, N *p m* be demitted from M, *m*, at the same Instant of time, they will be accelerated in proportion to the Arches M A, *m* A, they have to describe; and consequently the portions of the Curve which they describe in the beginning of their motion, will be proportional to the Arches M A, *m* A; and the portions yet to be described or the accelerating forces will be proportional to the said Arches M A, *m* A; whence 'tis manifest that the portions to be describ'd being always in the same proportion of M A to *m* A, must vanish at the same time, that is, the Pendulums demitted from M, *m*, at the same instant of time, and descending in the Curve M A, *m* A, by the force of their own Gravity, will arrive in the point A together; and again, if we suppose the Pendules to ascend from A towards Q, with the Velocites which they have acquired *m* A, they will then be retarded e very whereby the same forces, which accelerated their Motions before, and consequently the Velocities of the Pendulums Ascending and Descending in the same Arches, will be the same, and the Arches themselves will be describ'd in the same time; whence it appears that the whole Vibrations as well as the Semi-vibrations will always be Isochronal.

3°. And if O the Center of Attraction, be supposed at an infinite distance from B, then the Curve D A Q (in which the Pendulum Vibrates) will be a vulgar Cycloid

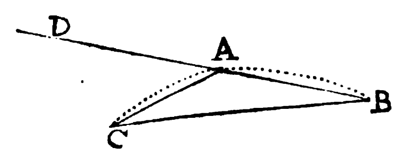
cloid, and the force of Gravity will always be the same in all places of the Curve, and the Vibrations in this also will be Isochronal; for D B Q will become a streight Line, and G T and M O will be parallel to B A, whence if M O be a determinate Quantity, and represent the force of Gravity, then M X, or M T, or M A will represent the accelerating force in the Cycloid *&c.* ***Ergo:***

The same Excellent Person has enriched this Theory with many more sublime discoveries, which for brevities sake I omit; This being sufficient to give the unacquainted Reader a Taste of the usefulness of the Doctrine concerning the Rectification of Curves.

LEMMA.

In every Triangle B A C; *if the Angles* A B C, A C B, *and* C A D *the complement of the obtuse Angle* C A B, *to two right Angles, be infinitely little; I say they are proportional to their opposite sides* A C, A B, B C.

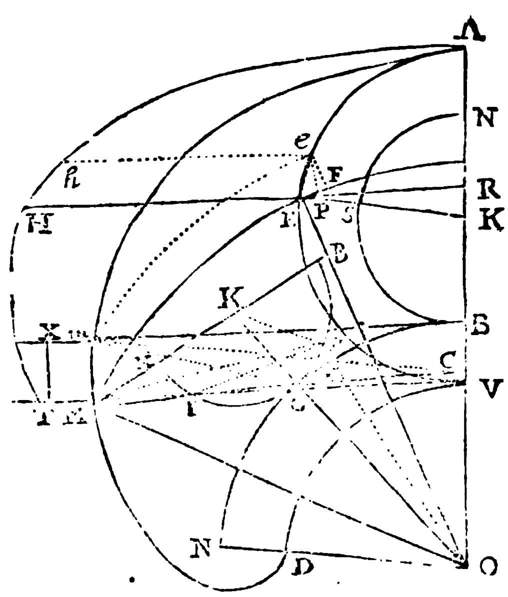
270. For if a Circle be circumscrib'd about the Triangle A B C, the Arches A C, A B, B A C, which measure double the said Angles, will be infinitely little also, and consequently they will be equal to their Chords or Subtendents.



And if the sides A C, A B, B C of the Triangle A B C be finite Quantities, 'tis plain that then the circumscrib'd Circle must be infinitely great, that so the Arches A B, A C may be infinitely little in respect of the whole Circumference.

PROP. XIII.

If A M D *be a Semi-cycloid describ'd by the Semi-circle* B S N *revolving on the immoveable Arch* B G N, *so that the Evoluta or Arches* B G, B G *be always equal to one another, and if* A *the point which describes the Curve be in the Diameter* B N *within or without the Periphery of the movable Circle. 'Tis requir'd to investigate the Value of the Ray of the Evoluta* M C.



271. Imagine another Pependicular mg infinitely near M G, intersecting M G produced in C, the point requir'd; draw the right Line G m, and take G g on the movable Circle = G g on the immovable Circle, and draw the Lines M g, I g, K g, O g; now if we consider the little Arches G g, G g as perpendicular to the Radii K g, O g, then 'tis manifest that the little Arch G g of the movable Circle falling on the Arch G g of the immovable Circle, the point M will fall on m, so that the Triangle G M g will exactly cover the Triangle G m g: whence it is evident that the Angle M G m is = g G g = (because adding to both the same Angles K G g, O G g, their sum will be equal to two right Angles) = G K g + G O g.

Now if we suppose O G = b, K G = a, G M or G m = r, and G I or I g = q, then it will be 1°. O G : G K :: G K g : G O g, and O G (b) : O G + G K (= O K = $a+b$) :: G K g : G K g + G O g or g G g or M G m = $\frac{a+b}{b}$ G K g. 2°. I g : M I :: G M g : M g I, and by composition I g + M I or M G (r) : I g (q) :: G M g

$GMg + MgI$, or $GIg = \frac{1}{2}GKg : GMg$, or $Gmg = \frac{q}{2r}GKg$. 3°. MCm or $MGm, MGm - Gmg \left(\frac{2ar + 2br - bq}{2br}GKg\right) : Gmg\left(\frac{q}{2r}GKg\right) :: Gm$ $(r) : GC = \frac{bqr}{2ar + 2br - bq}$, and consequently the Ray of the Evoluta MC is $= \frac{2arr + 2brr}{2ar + 2br - bq}$.

And if we suppose OG (b) the Radius of the immovable Circle to become infinite, the circumference BGN will become a streight Line, and the Terms $2arr$ and $2ar$, will vanish, in respect of the others, and the Value of the Ray of the Evoluta, MC, will be $= \frac{2brr}{2br - bq} = \frac{2rr}{2r - q}$.

CONSECTARY.

272. Hence to find the Area of the Cycloidal Space MGB, the Quadrature of the Circle being suppos'd, because Sectors of Circles are in a Ratio compounded of the Duplicate Ratio of the Radii, and the simple Ratio of their included Angles; it is, Angle $GMg \left(\frac{q}{2r}GKg\right)$: Angle $MGm \left(\frac{a+b}{b}GKg\right)$ :: the little Triangle (or Sector) MGg (whose Base is Gg m the movable Circle) : to the little Triangle or Sector GMm, whence the Sector GMm is $= \frac{2r}{q} \times \frac{a+b}{b}MGg =$ (supposing $MI = s$, and consequently $r = s + n$) $= \frac{2a + 2b}{b}MGg + \frac{2as + 2bs}{bq}MGg$. Now the little Triangle or Sector KGg is to the little Triangle MGg, in a Ratio compounded of the Square of KG to the Square of MG, and the Ratio of the Angle GKg to the Angle GMg, that is, as $aa \times GKg$ is to $rr \times \frac{q}{2r}GKg$; and consequently the little Triangle MGg is $= \frac{rq}{2aa}KGg$, and substituting this Value in place of the Triangle MGg in $\frac{2as + 2bs}{bq} \times MGg$, we shall have the Sector $GMm = \frac{2a + 2b}{b}MGg + \frac{\overline{a+b} \times sr}{aab}KGg$; but by the Property of the Circle, $GM \times MI$ $(sr) = BM \times MN =$ (supposing $KN = c$) $\overline{cc - aa}$, which is an invariable Quantity, and is always the same in whatsoever point of the Curve the describing point M be found, whence $GMm + MGg$ or mGg, that is the Trapezium $GMmg = \frac{2a + 3b}{b}MGg + \frac{\overline{a+b} \times \overline{cc - aa}}{aab}KGg$, now because $GMmg$ is the Fluxion of the Cycloidal Space $MGBA$ and MGg, that of the Circular Space MGB (comprehended between the right Lines MB, MG, and the Arch BG) and KGg, that of the Sector KBG; it is manifest that the Cycloidal Space $MGBA$ is $= \frac{2a + 3b}{b}MGB + \frac{\overline{a+b} \times \overline{cc - aa}}{aab} \times KGB$.

Q. E. I.

LEMMA II.

The same things being supposed; if on the Center K, *with the Radius* K A, *the Semicircle* A E V *be described, and if on the Center* O, *with any Radius between* O V *and* O A, *the Arch* E M *be describ'd, and the Radius* K S E *be drawn. I say the Arch* E M *is to the Arch* S N : : OE : O B.

273. Suppose the movable Circle B S N to come into the position B G N, then the point A which describes the Curve will be in M: connect the Centers of the generating Circles with the Line O K, which will pass through the point of contact G; then 'tis evident that the Triangles M O K and E O K are equal and similar, because the sides of one are equal to the Respective sides of the other; therefore the Angles M K O, E K O are equal; and the Arches that measure those Angles, *viz.* G N, B S and their complements to two right Angles B G, S N are also equal; and because the Angles M O K, and E O K are equal, therefore the Angle M O E is = Ang. G O B, and Arch E M: Arch G B : : O E the Radius of that : O B the Radius of this; but it has been demonstrated that the Inferior Arch G B = Superior Arch G B is = S N; therefore, the Arch E M : Arch S N : : O E : O B. Q. E. D.

CONSECTARY.

274. If the Radius O B be supposed infinite, then 'tis evident that the right Lines O B and O E will be Parallels, and the Concentric Arches V D, B N, and E M will degenerate into the right Lines V T, B X and E H perpendicular to the Axis V A, and consequently the right Line E H will be = S N, because O B and O E being infinite, are equal. Whence the Arch E M : E H : : O E : O B.

SCHOLIUM.

The Semi-cycloid A H T into which the other Semi-cyloid A M D degenerates, when the Radius O B is infinite, is the same with that generated by the Revolution of the Semi-circle B S N on the right Line B X, the describing point A, being in the Diameter B N produc'd.

PROP. XIV.

The same things being supposed, let it be requir'd to Investigate the Area of the Cycloidal Space A E M, *comprehended under the Arches* A E, E M, *and the Portion of the Cycloid* A M.

275. Imagine another Concentric Arch *m e* infinitely near to the Arch E M, and *e b* parallel and infinitely near to E H, and the Lines E F and E P, perpendicular to the Arch M E and the right Line E H (produced if need be) then are the Angles F E *e*, O E K equal, because each added to the Angle K E F makes a right Angle, and the Angle P E *e* is = complement of O K E, to two right Angles, because P E *e* + *e* E K + K E R is = two right Angles = K E R + E K R + E R K, therefore the Sine of the Angle F E *e* is to the Sine of the Angle P E *e* as the Sine of the Angle O E K is to the Sine of the Angle O K E. That is

$$Fe : Pe :: OK : OE,$$

And by the Corollary of the preceding Lemma, } $EM : EH :: OE : OB,$

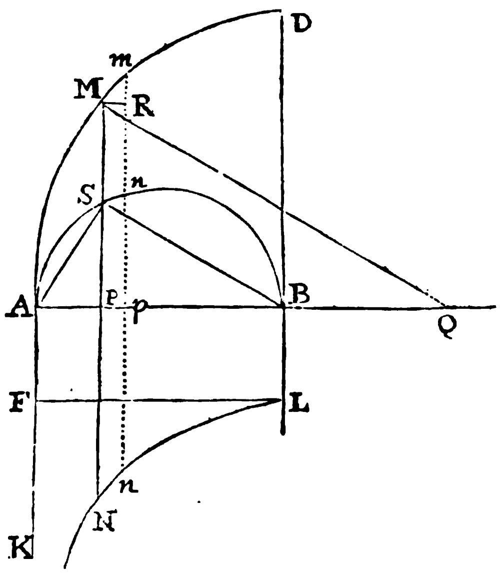
therefore $Fe \times EM : Pe \times EH :: OK : OB.$

And becauſe the infinitely little Spaces EM*me*, EH*be* are equal to the Products or Rectangles F*e* × EM, P*e* × EH reſpectively, and the ſaid Rectangles are always in the ſame proportion to one another, that is, the Space EM*me* is alway to the correſponding Space EH*be* as OK is to OB; 'tis plain that the Sum of all the EM*me*, is to the Sum of all the EH*be*, that is, the Space AME is to the Space AEH as OK is to OB.

PROP. XV.

To Inveſtigate the Relations between Curve-lines and their Axes,

276. Deſcribe the Curve AM*m* to the Axis AB, and draw the Ordinates PM, *pm* and the perpendicular MQ, and the Tangent MT; then ſuppoſe AP $= x$, P*p* $= \dot{x}$, PM $= y$, RM $= \dot{y}$, AM $= z$, M*m* $= \dot{z}$, PT $= t$, MT $= s$, PQ $= n$, MQ $= m$; then by ſimilar Triangles, $\dot{z} = \frac{m\dot{x}}{y} = \frac{s\dot{x}}{t} = \frac{s\dot{y}}{y} = \frac{m\dot{y}}{n}$.



Whence 'tis manifeſt that the Curve Line AM is $= S, \frac{m\dot{x}}{y}$, and becauſe $\dot{x}$ may be ſuppoſed always equal to it ſelf, if y were ſuch alſo, then it would be, as all the y : is to the all the m :: all the $\dot{x}$: all the $\dot{z}$ = to the Curve AM; but becauſe the Denominator y is a variable Quantity, aſſume r an invariable Quantity, and u variable, and ſuppoſe $\frac{u}{r} = \frac{m}{y}$, then $\frac{m\dot{x}}{y} = \frac{u\dot{x}}{r} = \dot{z}$; now if u be laid from P to N, and the Curve ANS deſcrib'd, then PN $(u) = \frac{rm}{y}$; and if AG be $= r$, then 'tis evident, that all the r or the Rectangle AGSB : all the u or the Space ANSB :: all the $\dot{x}$ or the Axis AB : to all the $\dot{z}$ or the Curve AM.

Now the Nature of the Curve AMD being known, and r an invariable Quantity, the Locus of u is alſo known, and conſequently the Curvilineal Space ANSB being given, the Ratio of the Curve AM to its Axis will be given alſo.

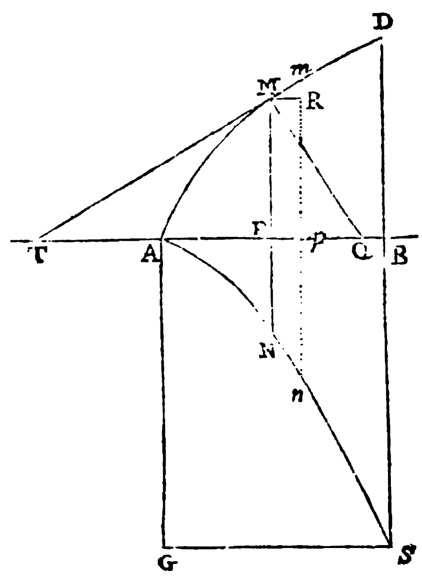
And becauſe $u\dot{x}$ is $= r\dot{z}$, therefore $\frac{u\dot{x}}{r} = \dot{z}$; that is the Curve-line AM or all the $\dot{z}$ is = to the Space ANSB divided by the invariable Quantity r.

EXAMPLE.

EXAMPLE.

Let AMm *be a Primary Cycloid, and the generating Circle* ASB, AB *the Axis, and* BD $= \frac{1}{2}$ *the Base. 'Tis requir'd to find the Ratio between the Curve-line* AMD *and the Axis* AB.

277. Let the Axis AB be $= 2r$, AP $= x$, PS $= f$, BS $= b$, MQ $= m$ perpendicular to the Cycloid in M; MP $= y$, and M$m = \dot{z}$, then the Triangles MPQ, SPB, MRm, are similar, therefore $m : y :: b : f :: \dot{z} : \dot{x} ::$ (assuming r invariable) $u : r$; whence all the $u\dot{x} =$ all the $r\dot{z}$; let PN be always $= u$, and let the new Curve be in that ways generated to find the property thereof. Because $b : f :: u : r ::$ AS : AP, therefore $uu : rr ::$ ASq : APq :: (because ASq = APq + PSq) $2rx : xx$; therefore $2r^3 = uux$, which Equation denotes LNn to be an Hyperboliform Curve, and to find the Curvilineal Space KABLn = all the $u\dot{x}$; the Equation expressing the Nature of the Curve is $2r^3 = uux$, therefore when AP becomes = AB $= 2r$, and BL $= u$, then $2r^3 = uux = 2ruu$, whence $r = u =$ BL, and consequently the Rectangle ABLF $= 2rr$, and the Curvilineal Space KABLn is $= 4rr =$ all the $u\dot{x}$; but all the $u\dot{x}$ is = all the $r\dot{z}$; and all the $\frac{u\dot{x}}{r}$ is = all the $\dot{z}$ = the Curve AMD; therefore $\frac{4rr}{r} = 4r$ is = all the $\dot{z}$ = the Curve-line AMD = twice AB the Diameter of the generating Circle.

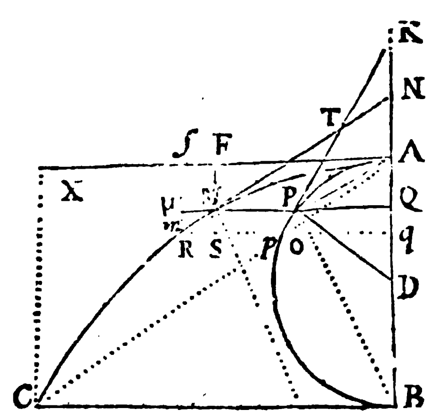


COROLLARY.

278. If all the $u\dot{x}$, or the Curvilineal Space KABLn be Squarable, then a right Line may be assigned equal to the given Curve AMD.

Another way.

279. The Proportion of the Curve-line AMD to the Axis AB may be investigated in this manner: Resume the Symbols (*Art.* 108.) Then Sm is $= \frac{2r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$ and MS $= \dot{x}$, and consequently Mmq = Smq + MSq $= \dot{x}\dot{x} + \frac{4r^2\dot{x}^2 - 4rx\dot{x}^2 + x^2\dot{x}^2}{2rx - xx} = \frac{4r^2\dot{x}^2 - 2rx\dot{x}^2}{2rx - xx}$, therefore M$m$ is $= \frac{\dot{x}\sqrt{4r^2 - 2rx}}{\sqrt{2rx - xx}} =$ (dividing by $2r - x$)



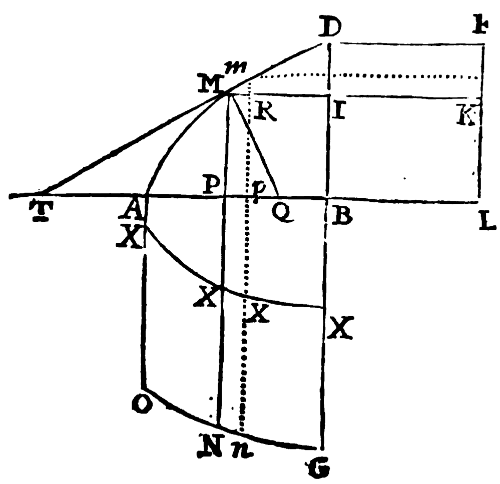
$\dot{x}\sqrt{2r}$

$\frac{\dot{x}\sqrt{2r}}{\sqrt{x}} = \overline{2r}|^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$; therefore the Flowing Quantity A M is $= 2 \times \overline{2r}|^{\frac{1}{2}} x^{\frac{1}{2}}$ $= 2\sqrt{2rx} = 2$ A P the correſponding Chord in the generating Circle, and conſequently the Entire Curve A M C is $= 2$ A B the Diameter of the generating Circle.

PROP. XVI.

Any Squarable Curvilineal Space, as A O G B, *and the Equation expreſſing the Relation of the Ordinate* P N, *to the intercepted Diameter* A P, *being given; to find the Property of another Curve* A M D, *applied to the ſame Axis* A P; *to which an equal right line may be aſſigned.*

280. Let the Curve requir'd be A M D, and ſuppoſe A P $= x$, P M $= y$, M R $= \dot{x}$, R $m = \dot{y}$, M $m = \dot{z}$, M Q $= m$, P Q $= n$; then it will be (ſuppoſing u an indeterminate Quantity) $y : m :: r : u :: \dot{x} : \dot{z}$, and conſequently $r\dot{z}$ is $= u\dot{x}$, and $u = \frac{rm}{y}$; now if u be always applied from P to N, and the Curve O N G be deſcrib'd, (including a ſquarable Space by ſuppoſ.) 'tis required to find the Relation of A P to P M, the Area and property of the ſaid Space A O G B being given.



Becauſe $y : m :: r : u$, therefore $yy : mm - yy :: rr : uu - rr$; and making $uu - rr = qq$, it will alſo be, $y : n :: r : q :: \dot{x} : \dot{y}$, therefore $q\dot{x} = r\dot{y}$; and if q be laid from P to X, and r from I to K, the Curvilineal Space A X X B will be $=$ S $q\dot{x}$, and the Rectangle D F L B will be $=$ S $r\dot{y} =$ Rectang. ry, whence the Ordinate P M or y is $= \frac{\text{S}q\dot{x}}{r}$ $= \frac{\text{Space A X X B}}{r}$. And now to reſolve the Problem.

If the Squarable Figure A O G B, and the Ordinate P N $= u$ be given, to the ſame Axis A P, apply the ſecond Figure A X X B, and let the Ordinate P X $= q$, be ſuch that $uu - rr$ be always $= qq$; then if a third Figure A M D be deſcribed, whoſe Ordinates P M or y are ſuch, that any Ordinate y multiplyed into the invariable Quantity r, *viz.* ry be always $=$ to the adjacent and correſponding Curvilineal Space A X X P $=$ all the $q\dot{x}$, then a right line may be aſſigned equal to the Curve-line A M D; and the ſaid right line will be $= \frac{\text{Space A B G O}}{r} =$ Curve-line A M D.

EXAMPLE.

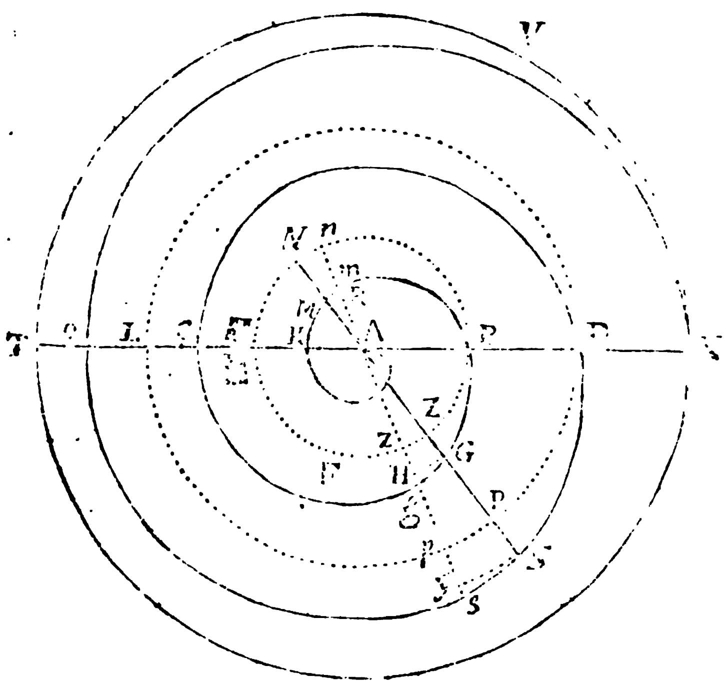
281. Let A O G B be a parabolical Trapezium, and the Equation expreſſing the Nature of the Curve $rr + rx = uu$ (the invariable Quantity r being $=$ Parameter, and to the Portion of the Axis intercepted between A and the Vertex of the given Curve) then is $qq = rx$; whence the Curve A X X B is a Parabola, and the Vertex thereof is in A; now the Area of the Parabolic Space A X X B is $= \frac{2}{3}\sqrt{rx^3} =$ all the $q\dot{x}$; and becauſe all the $q\dot{x} =$ all the $r\dot{y} = ry$, therefore

$\frac{2}{3}\sqrt{r x^3} = r y$, and $\frac{4}{9} r x^3 = r^2 y^2$, and $x^3 = \frac{9}{4} r y^2$, which is an Equation expressing the Nature of the Curve A M D, and the Curve A M D is $= \frac{2}{3} r u + x u - \frac{2}{3} r u$ divided by r, $= \frac{\overline{\frac{2}{3} r + AB} \times BG - \frac{2}{3} r \times AO}{r}$.

P R O P. XVII.

To investigate the length of Curve-lines, respecting (not an Axis, but) a certain determinate point such as Spiral lines.

282. Resume the Symbols (*Art.* 116.) and suppose the Periphery of the Circle B F N B $= c$, B F N $= x$, N $n = \dot{x}$, A N $= r$, and A M $= y$, then because the



Sectors A N n, A M R are similar, M R is $= \frac{y \dot{x}}{r}$, and if the Angle M A m be infinitely little, the Portions M R and M m will be equal; therefore the Spiral line A K M B is $= S \frac{y \dot{x}}{r}$, now the Equation of the Curve is (because $c^n : x^n :: r^m : y^m$) $y = \frac{r x^{\frac{n}{m}}}{c^{\frac{n}{m}}}$, therefore the Fluxion of the Curve $\frac{y \dot{x}}{r}$ is $= \frac{x^{\frac{n}{m}}}{c^{\frac{n}{m}}} \dot{x}$, and the Flowing Quantity or the Length of the Spiral line A K M is $= \frac{m x^{\frac{n}{m}+1}}{\overline{m+n} \times c^{\frac{n}{m}}}$; and the whole Spiral line A K M B is $=$ (because then x becomes $= c$) $\frac{m}{m+n} \times c$, and where $m = 1$, and $n = 1$; the Spiral-line A K M B $= \frac{m}{m+n} \times c$ is $= \frac{1}{2} c$ $= \frac{1}{2}$ the Periphery of the Circle B F N B.

And

And to find the Length of the fecond Spiral line *g* G C D, if the Angle G A *g* be infinitely little, then the Arches H G, *g* G will be equal, and the fum of all the infinitely little Arches H G is = to the fecond Spiral line B G C D, now H G is $= \frac{r\dot{x} + y\dot{x}}{r} = \left(\text{becaufe } y = \frac{rx^{\frac{n}{m}}}{c^{\frac{n}{m}}}\right) = \frac{r\dot{x} + rx^{\frac{n}{m}}\dot{x}}{rc^{\frac{n}{m}}}$, and the Flowing Quantity is $= \frac{rc^{\frac{n}{m}}x + \frac{m}{m+n}rx^{\frac{n}{m}+1}}{rc^{\frac{n}{m}}}$ = to the Portion of the fecond Spiral line, B G; and confequently B G C D is = (becaufe then $x = c$) $c + \frac{m}{m+n} \times c = \frac{2m+n}{m+n} \times c$.

Hence in the common Spiral line, B G C D is $= \frac{3}{2}c = \frac{3}{2}$ B Z N B = (becaufe the Peripheries of Circles are proportional to their Diameters) $\frac{3}{4}$ the Periphery D P L D.

And becaufe the firft Spiral line is $= \frac{1}{2}c$, and the fecond Spiral line $= \frac{3}{2}c$, therefore the whole Spiral line A K M B G C D is $= 2c =$ twice the circumference B Z N B = the circumference D L P D.

The Periphery of the Circle B Z N B is $= c$, and that of the Circle D P L D is $= 2c$, and the fum of both is $= 3c$, and the whole Spiral line is $= 2c$, therefore the fum of the Peripheries of both Circles is to the whole Spiral line as 3 is to 2.

The fecond Spiral line B G C D $= \frac{3}{2}c$, and the Periphery of the fecond Circle D P L D is $= \frac{4}{2}c$; therefore that is to this as 3 is to 4, and the firft Spiral line A K M B is $= \frac{1}{2}c$, and the fecond Circle is $= \frac{4}{2}c$ therefore that is to this, as 1 is to 4, and the firft Spiral line is to the fecond Spiral line as 1 is to 3.

And to find the Length of the third Spiral line D S O X, S *s* or Y S, the Fluxion of the Curve is $= \frac{2r\dot{x} + y\dot{x}}{r} = \left(\text{becaufe } y = \frac{2rx^{\frac{n}{m}}}{2c^{\frac{n}{m}}}\right) = \frac{4rc^{\frac{n}{m}}\dot{x} + 2rx^{\frac{n}{m}}\dot{x}}{2rc^{\frac{n}{m}}}$, and the Flowing Quantity thereof is $\frac{4rc^{\frac{n}{m}}x + \frac{2m}{m+n}rx^{\frac{n}{m}+1}}{2rc^{\frac{n}{m}}}$ = to the Portion of the Spiral line D S; and confequently the whole Spiral line D S O X is $= \frac{3m+2n}{m+n} \times c =$ (in the common Hypoth. fuppofing $m = 1$, and $n = 1$) $= \frac{5}{2}c$; now the firft and fecond Spiral lines are $= \frac{4}{2}c$, therefore the firft, fecond and third Spiral lines are $= \frac{9}{2}c$, and the Circle X T V ($= \frac{6}{2}c$,) is to the whole Spiral line A M B C D S O X as 2 is to 3.

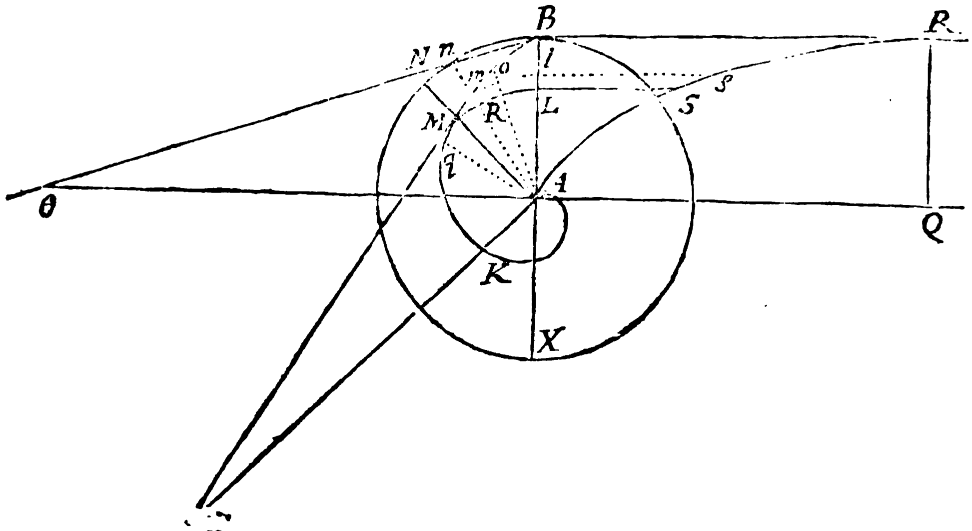
The fum of all the three Spiral lines, *viz.* the entire Spiral line A M B C D O X is $= \frac{9}{2}c$, and the fum of all the Peripheries of the three Circles is $= \frac{12}{2}c$, therefore the whole Spiral line is to all the three Circumferences, as 9 is to 12, or as 3 is to 4.

The Periphery or length of the third Spiral line being $= \frac{5}{2}c$, and the Periphery of the third Circle being $= \frac{6}{2}c$, therefore that is to this as 5 is to 6.

The firft Spiral line is $= \frac{1}{2}c$, the fecond $= \frac{3}{2}c$, the third $= \frac{5}{2}c$; the Periphery of the firft Circle is $= \frac{2}{2}c$, that of the fecond $= \frac{4}{2}c$, and that of the third $= \frac{6}{2}c$. Whence 'tis evident that in a Series of Numbers beginning with Unity and encreafing in their natural Order, as, 1, 2, 3, 4, 5, 6, *&c.* If the firft Number (or 1) reprefent the length of the firft Spiral line, the fecond number (2) will reprefent the Periphery of the firft Circle; the third (3) the length of the fecond Spiral line; the fourth (4) the Periphery of the fecond Circle; the fifth (5) the length of the third Spiral line; and the fixth (6) the Periphery of the third Circle, *&c. in Infinitum.*

283. And

283. And to investigate the length of the Logarithmetical Spiral line, resume the Symbols (*Art.* 119.) §. 2°. And suppose $M\,m = \dot{u}$, and $M\,T = \dot{s}$; then by the Property of the Curve, $\dot{u} : \dot{y} :: b : q$, and $q\dot{u} = b\dot{y}$, and finding the Flowing

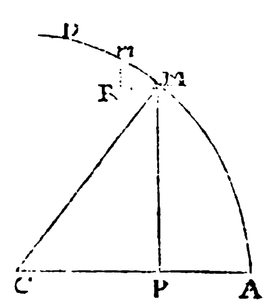


Quantities qu is $= by$; but $b : q :: \dot{u} : \dot{y} :: \dot{s} : \dot{y}$, therefore qs is $= by = qu$, and consequently $s = u$; and if u represent the infinite Spiral line, then is $\theta B = s = u =$ the infinite Spiral line B M K A; that is, the line θ B touching the Spiral line in B, is equal to the said Spiral line.

PROP. XVIII.

In the Circle A M D, *if the right Sine* PM *be given. 'Tis requir'd to investigate the length of the Arch* A M.

284. Suppose $AP = x$, $PM = y$, $AC = r$, $MR = \dot{x}$, $RM = \dot{y}$, and $Mm = \dot{z}$ = to the Fluxion of the Curve A M; by the Property of the Curve $2rx - xx = yy$, therefore $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, and (dividing by $2r - 2x$) $\dot{x} = \frac{2y\dot{y}}{2r - 2x} = \frac{y\dot{y}}{r - x}$, now $Mm\,q = MR\,q + Rm\,q = \dot{y}^2 + \frac{y^2\dot{y}^2}{rr - 2rx + xx} =$ (because $2rx - xx = yy$) $\dot{y}^2 + \frac{y^2\dot{y}^2}{rr - yy} =$ (by Reduction) $\frac{rr\dot{y}^2}{rr - yy}$;



therefore Mm is $= \frac{r\dot{y}}{\sqrt{rr - yy}}$; but $\frac{r\dot{y}}{\sqrt{rr - yy}} = \frac{1}{\sqrt{rr - yy}} \times r\dot{y} = \overline{rr - yy}|^{-\frac{1}{2}} \times r\dot{y}$; therefore if $\overline{rr - yy}|^{-\frac{1}{2}}$ be reduc'd to an (*Art.* 93.) infinite Series, and all its Terms multiplied by $r\dot{y}$, we shall have the Fluxion of the Arch AM; and finding the Flowing Quantity of every Term, there will arise a new Series expressing the Value of the Arch A M.

And in like manner if the versed Sine be given, to find the Arch; resume the Equation $2rx - xx = yy$, then $r\dot{x} - x\dot{x} = y\dot{y}$ and $\frac{r\dot{x} - x\dot{x}}{y} = \dot{y}$, now $M\,m\,q$

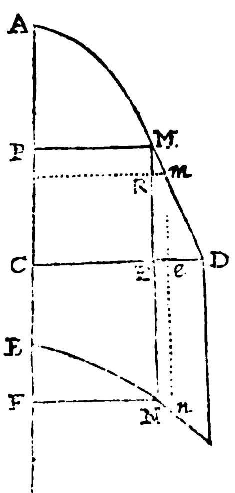
$= \dot{x}^2$

$= \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \frac{rr\dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{yy} = \dot{x}^2 +$
$\frac{r^2\dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{2rx - xx} =$ (by Reduction) $\frac{r^2\dot{x}^2}{2rx - xx}$; therefore M m is $=$
$\frac{r\dot{x}}{\sqrt{2rx - xx}} = \overline{2rx - xx}|^{-\frac{1}{2}} \times r\dot{x}$, which being reduc'd to an infinite Series and the Flowing Quantity of every Term found, we shall have a Series expressing the Value of the Arch A M.

PROP. XIX.

Let it be requir'd to investigate the length of any Arch A M, *of the Paraboloical Curve* A M D.

285. If it be requir'd to find the length of the Arch A M; suppose the Parameter of the Curve $= a$, A P $= x$, P M $= y$, then the Fluxion of the Curve M m is $=$ $\sqrt{\dot{x}^2 + \dot{y}^2}$; now by the property of the Curve $ax = yy$, whence $\dot{x} = \frac{2y\dot{y}}{a}$, and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{aa}$ and by substitution the Fluxion of the Curve $\sqrt{\dot{x}^2 + \dot{y}^2}$ is $= \frac{\sqrt{4y^2\dot{y}^2 + aa\dot{y}^2}}{a}$ $= \frac{2\dot{y}}{a}\sqrt{y^2 + \frac{1}{4}aa}$; now the Fluent of this Fluxion is equal to the Arch of the Curve A M. And to find the same,



Draw M E N parallel to A P B, and take E N $= \sqrt{yy + \frac{1}{4}aa}$ and describe the Curve B N Q, then the little Rectangle E N ne is always $= \dot{y}\sqrt{yy + \frac{1}{4}aa}$, and consequently the Space C B N E is $=$ the sum of all the $\dot{y}\sqrt{yy + \frac{1}{4}aa}$, whence if the whole Space C B N E be divided by $\frac{a}{2}$ (a being the Parameter of the given Parabola) the Quotient $= \frac{2S\dot{y}\sqrt{yy + \frac{1}{4}aa}}{a}$ is $=$ to the length of the Parabolic Arch A M.

And to find the Nature of the Curve B N Q; suppose E N $= z$, then $z = \sqrt{yy + \frac{1}{4}aa}$, and $zz = yy + \frac{1}{4}aa$; and when y is $= 0$, then z is $= \sqrt{\frac{1}{4}aa} = \frac{1}{2}a$, that is C B is $= \frac{1}{2}a$; and when z is $= 0$, then $0 = \sqrt{yy + \frac{1}{4}aa}$, and consequently $yy = -\frac{1}{4}aa$; which being an impossible Equation, shews that the nearest distance of the Curve Q N B from C D is $=$ C B $= \frac{1}{2}a$, and that afterwards the Curve Q N B recedes from D C produc'd.

Hence it is manifest that C B F may be taken for the Axis of the Curve B N Q, and because P M is $=$ F N $= y$, and E N $=$ C F $= z$, and $z = \sqrt{yy + \frac{1}{4}aa}$; therefore C F q is $=$ F N q + C B q, and F N q $=$ C F q $-$ C B q $= \overline{2\,CB + BF} \times BF$; which is the property of an Equilateral Hyperbola.

CON-

CONSECTARY.

286. Hence to find the length of any Arch (*v. g.* A M) of a Parabolic Curve A M D. Assume any point in the Axis A P, as C, and take $CB = \frac{1}{2}a = \frac{1}{2}$ th eParameter of the given Curve A M D, and supposing C B = to half the Transverse Axis, describe the Equilateral Hyperbola B N Q, and draw M N parallel to A C B; then the Parabolic Arch A M is $= \frac{2 \text{ the Space } CBNE}{a}$, and the Arch A M D is = the Hyperbolic Space C B Q D divided by half the Parameter of the Parabola A M D.

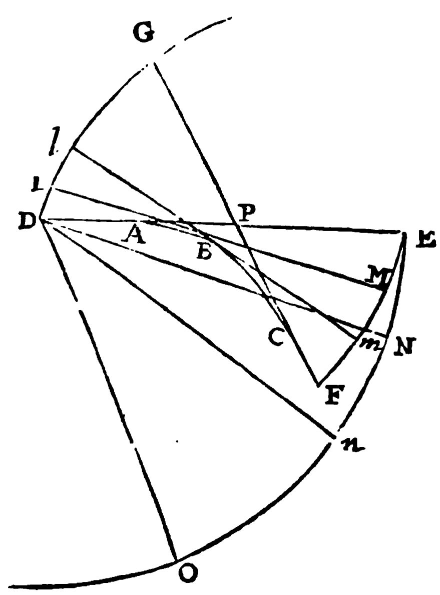
And thus it appears that the Rectification of the Curve of the Parabola, depends on the Quadrature ot the Hyperbola.

PROP. XX.

To Measure Curve Lines by Circular Lines.

The design of this Proposition is to shew, how to draw a Curve so related to any given Curve, that their sum or difference, shall be equal to a determinate Arch of a Circle.

287. Let A B C be a given Curve, on which suppose the inflexible Line or Ruler D A E to move, so that the parts of the Ruler be successively applied to the parts of the Curve; I say, the two Curves D L G, and E M F describ'd by the extreme points of the Ruler, D and E, or by any two other opposite points of the same, are equal to the Arch of the Circle E O, describ'd with the Radius D E, and subtended by the Angle E D O = Ang. E P F, formed by the two Tangents A P and C P, touching the Curve in the extreme points A and C.



Let the Ruler in any position as L B M, move into the next infinitely near *l* B *m*, and draw the Lines D N and D *n* parallel to L M and *l m*, then the Triangles L B *l*, M B *m*, and N D *n* are similar, because L M, *l m* are perpendicular to the Curves DLG and E M F, and the Angles L B *l*, M B *m*, N D *n* are equal, therefore B L : B M :: L *l* : M *m*, and by composition L M : B L :: L *l* + M *m* : L *l*; and by permutation, L M : L*l* + M *m* :: B L : L *l*, and because L M is = (*ex Hypoth.*) D N, therefore L *l* + M *m* is also = N *n*, and all the $\overline{Ll + Mm}$, or the Curves D L G and E M F taken together, are equal to all the N *n*, or the Arch of the Circle E N O.

If either of the describing points as E, be between A and D, it may in like manner be demonstrated, that the difference between the Curve D L G and E M F is equal to the Arch of the Circle E N O, describ'd by the point E.

CONSECTARY I.

Hence we have a ready way to draw an infinite number of Curves, so related to another given Curve, that any one of them added to the same, shall be equal to an Arch of a Circle.

288. For instance, if DLG be a Curve, or any portion of a given Curve, from every point thereof L, *l*, *&c.* draw perpendiculars LM, *l m*, all equal to one another; then their extremities M, *m*, *&c.* connected, will form the Curve EMF requir'd.

The Curve EMF may be describ'd more easily by the continued motion of a Thread, if we involve the Curve ABC, the Evoluta of the given Curve DLG.

CONSECTARY II.

Hence we are enabled to judge, whether a given Curve can be compared, or has any Connexion with the Dimension of an Arch of a Circle: for every Curve, whose Evoluta ABC has two parts BA, BC equal and similar, may be compared with an Arch of a Circle, for if in such a Case, CF be = AD, the Curve FME will exactly agree with DLG, and consequently DLG will be = $\frac{1}{2}$ the Arch ENO; and contrarily, every Curve generated by the Evolution of a Curve consisting of two equal and similar parts, is reducible to the Arch of a Circle.

SECT.

SECT. IX.

The Use of Fluxions

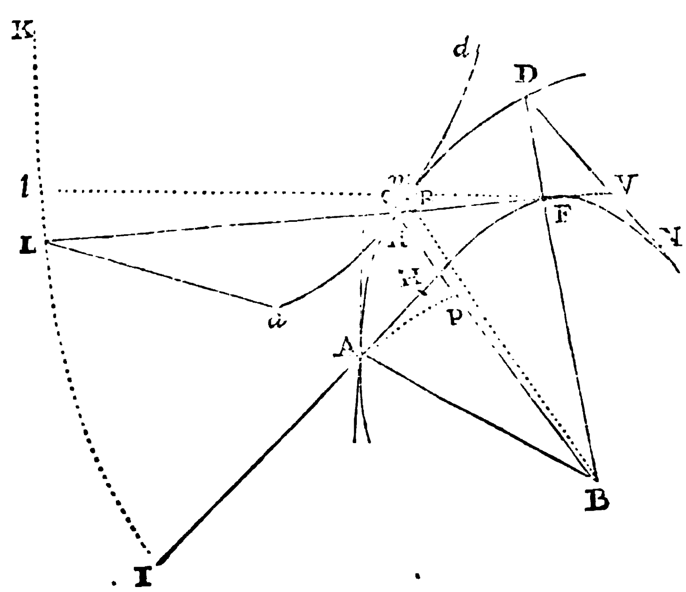
In finding Caufticks by Reflexion, to all forts of Curves.

DEFINITION.

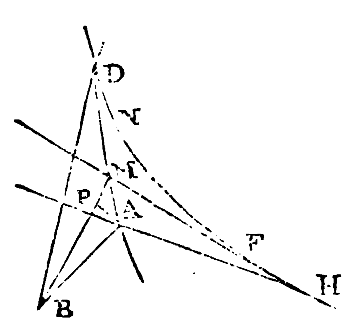
IF we fuppofe an infinite number of Rays B A, B M, B D, *&c.* iffuing from the Luminous point B, to be reflected by the Curve Line A M D, fo that the Angles of Incidence be always equal to the Angles of Reflexion; the Curve Line H F N, which touches the Reflected Rays (produced on the oppofite fide if need be) A H, MF, DN is called the *Cauftick* by *Reflexion*: hence we may eafily deduce thefe *Confectarys.*

CONSECTARY I.

289. If H A be produced to I, fo that A B = A I, and if the Cauftick H F N be Evolved, and the Curve defcrib'd by fuch Evolution begin in I, the Curve I L K will be defcrib'd, and the Tangent F L will always be = to the portion of the Cauftick F H + the right Line H I.



And if we conceive two Rays, the Incident B *m*, and the Reflected *m* F, infinitely near B M, M F, and if F *m* be prolonged to *l*, and on the Centers F, B, be defcrib'd the little Arches M O, M R, then the Rectangular Triangles M O *m*, M R *m* will be fimilar and equal; for the Angle O *m* M = F *m* D = (*ex Hypoth.*) R *m* M, and the Hypothenufe M *m* is common to both, therefore the fides O *m*, R *m* are equal between themfelves: now O *m* is the Fluxion or momentary Increment of L M, and R *m* is the Fluxion of B M, and this proportion of Equality holds in whatever point of the Curve M be taken, therefore the refpective fums of thefe Fluxions muft be equal, *viz.* M L — I A = A H + H F — M F (= fum of all the O *m*) = B M — B A (= the fum of all the R *m*) and confequently by Tranfpofition, H F (the portion of the Cauftick H F N) is = B M — B A + M F — A H.



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CONSECTARY II.

If on the Center B, the Arch A P be deſcrib'd, 'tis evident that P M is = M B — A B, and if we ſuppoſe the Luminous point B, to be at an infinite diſtance from the Curve A M D, the Rays of incidence B A, B M will become parallel, and the Arch A P will be a ſtreight Line, cutting the ſaid Rays at right Angles.

CONSECTARY III.

If we imagine the Figure B A M D to be reverted on the ſame Plain, ſo that the point B fall on I, and that the Line touching the Curve A M D in A, touch the ſame in the reverted poſition in the ſame point, and if we imagine the Curve A M D immovable, and that the reverted Curve *a* M *d* revolves on the ſame, ſo that the portions A M, *a* M be always equal between themſelves; I ſay the point B (or I, by ſuch a motion) will deſcribe a ſort of a Cycloid I L K, whoſe Evoluta is the *Cauſtick* H F N.

For from the Geneſis of the Curve, it is evident, 1°. that the Line L M drawn from the deſcribing point L, to the point of contact M is perpendicular to the Curve I L K. 2°. L *a* or I A is = B A, and L M = B M. 3°. The Angles made by the right Lines M L, B M, and the Tangent in M (common to both Curves) are equal, and conſequently if L M be produc'd to F, the Ray M F will be the reflected Ray of the Ray of Incidence B M; whence 'tis evident that the Perpendicular L F touches the Cauſtick H F N; and becauſe this holds true in whatever point of the Curve I K, we take the point L, it follows that the Curve I L K is generated by the Evolution of the Cauſtick H F N + H I.

CONSECTARY IV.

And hence it appears that the portion F H or F L — H I is = B M + M F — B A — A H, as I have already demonſtrated.

CONSECTARY V.

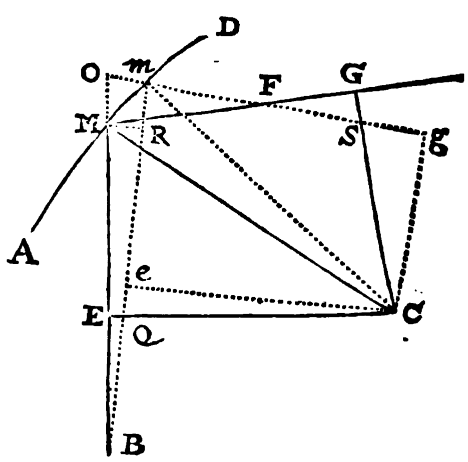
If the Tangent D N be drawn infinitely near to the Tangent M F, the points of contact N, F, and the point of interſection V will coincide; ſo that to find the point F, where the reflected Ray M F touches the Cauſtick H F N, is the ſame thing as to find the point V, in which the reflected Rays M F, *m* F (infinitely near each other) concur.

PROP.

PROP. I.

The Nature of the Curve AMD, *the Luminous point* B, *and the Ray of Incidence* BM *being given; to find in the reflected Ray* MF, *given by position, the point* F, *where it touches the Caustick.*

290. Find the of length MC the Ray of the Evoluta, to the point M, and take the Arch M*m* infinitely little; and draw the right lines B*m*, C*m*, *m*F; on the Centers B and F describe the little Arches MR, MO; and draw the perpendiculars CE, C*e*, CG, C*g*, to the Rays of incidence and reflection; and suppose BM $= y$ and ME or MG $= a$.



Then 'tis evident that the Triangles MR*m*, MO*m*, are equal and similar, and consequently MR is = MO; and because the Angles of incidence and reflection are equal, therefore CE = CG, and C*e* = C*g*, and consequently CE — C*e*, or EQ is = CG — C*g* or SG; and because the Triangles BMR, BEQ, FM*o*, FGS are similar, it is, BM + BE $(2y - a)$: BM (y) :: MR + EQ or MO + GS : MR or MO :: MG (a) : MF $= \frac{ay}{2y - a}$.

CONSECTARY I.

291. If the Luminous point B fall on the other side of the point E in respect of M, or (which is the same thing) if the Curve Line AMD be convex towards the Luminous point B, then y instead of being Positive will become Negative, and consequently MF will be $= \frac{-ay}{-2y - a} = \frac{ay}{2y + a}$.

CONSECTARY II.

If we suppose y to become infinite, that is to say, if the Luminous point B, be at an infinite distance from the Curve AMD, the Rays of incidence will be parallel between themselves, and MF $= \frac{ay}{2y \mp a}$ will become $= \frac{1}{2}a$ because a is equal to nothing in respect of y.

CONSECTARY III.

The Curve AMD can have but one *Caustick* by *Reflexion*, *viz.* HFN; for one and the same Curve can have but one *Evoluta*, and the Ray or Tangent thereof, enters into the Value MF, so that there can be but one value of MF.

CONSECTARY IV.

When AMD is a Geometrical Curve, 'tis evident that its Evoluta is so also, (because in that case, we can find the Relation between the Abscissa and Ordinate of the Evoluta) that is, all the points C, may be determined Geometrically: whence it is manifest, that all the points F of its Caustick, may also be determined Geometrically; that is, the Caustick HFN will be a Geometrical Curve.

CONSECTARY V.

A right line may be assigned equal to any portion of the said Caustick, if the given Curve A M D be a Geometrical Curve, as appears from *Art.* 209.

CONSECTARY VI.

If the Curve A M D be Convex towards the Luminous point B, the Value of M F $\left(\frac{ay}{2y+a}\right)$ will always be Positive, and consequently we must take the point F, on the same side of the Curve with the point C, in respect of M, as we have supposed in the preceeding Calculation; whence 'tis evident that the Rays of Reflection, infinitely near one another, *Diverge.*

CONSECTARY VII.

But if the Curve A M D be Concave towards the Luminous point B, the Value of M F $\left(\frac{ay}{2y-a}\right)$ will be Positive, when y exceeds $\frac{1}{2}a$; and Negative, when y is less than $\frac{1}{2}a$; and infinite, when $y = \frac{1}{2}a$; whence it is manifest, that if a Circle be describ'd, whose Diameter is $= \frac{1}{2}$ M C the Ray of the Evoluta, then if the Luminous point B. be without the Circumference of the said Circle, the Reflected Rays will Converge; if within, they will Diverge; and if the said point happen to be in the Circumference, they will be all parallel to one another.

CONSECTARY VIII.

If the Ray of Incidence B M touch the Curve A M D in the point M, then is M E (a) $= 0$, and consequently M F is $= 0$, because the Reflected Ray is in the same direction with the Ray of Incidence, and the Nature of the Caustick being such, that it touches all the reflected Rays, it follows that it must also touch the Ray of Incidence B M in M; that is, B M will be a Tangent to both Curves in the point M.

CONSECTARY IX.

If the Ray of the Evoluta M C be $= 0$, then is M E $= 0$, and consequently M F $= 0$, whence 'tis plain, that the given Curve and the Caustick make an Angle in the point M (which is common to both) equal to the Angle of Incidence.

CONSECTARY X.

If C M the Ray of the Evoluta be infinite, the little Arch M m will be a streight Line, and M F $= \frac{ay}{2y \mp a}$ will be $=$ (because M E or a being infinite, y is $=$ to nothing in respect thereof) $\mp y$; and if the Luminous point B be on the same side of the Curve with C, then the Value of M F will be Negative, and consequently the Reflected Rays will Diverge; and if the Luminous point B, be on the contrary side of the Curve in respect of C, then the Value of M F will be Positive; that is the point F will be on the same side of the Curve with the point C, and consequently the Reflected Rays, in this case also, will Diverge.

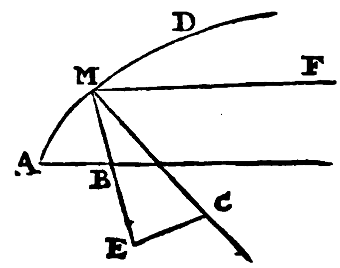
Whence it is plain, that Rays issuing from any Luminous point, and reflected by any plain Surface, will after reflexion, Diverge.

CON-

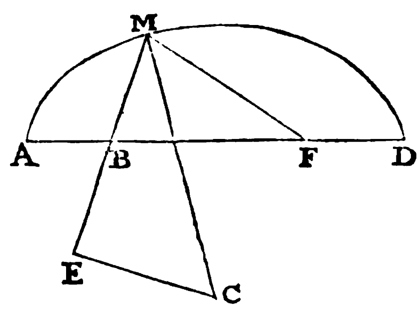
CONSECTARY XI.

If any two of the three points, B, C, F be given, the third may be found: for instance.

If the Curve A M D be a Parabola, and the Luminous point B, in the Focus, (*Art.* 41. 163. §. 3.) 'tis evident that all the Reflected Rays will be parallel to the Axis; and consequently (because the point F where two Rays of reflection intersect each other, is at an infinite distance) M F will be infinite, wherever M be taken: but M F is $= \frac{ay}{2y - a} =$ infinity: Therefore $2y - a$ (because in such cases the Denominator must be $= 0$) $= 0$, and $a = 2y$, whence if M E be taken $= 2$ M B, and the perpendicular E C be drawn, it will cut M C (the perpendicular to the Curve in M) in the point C, which will be in the Evoluta of the Parabola.



Again, if the Curve A M D be an Ellipsis, and if the Luminous point B, be in one of the Foci (*Art.* 47. 163. §. 3.) then 'tis evident that all the reflected Rays M F will meet in the other Focus F; whence if M F be suppos'd $= z$, then is $z = \frac{ay}{2y - a}$, and consequently M E $(a) = \frac{2yz}{y + z}$; but if the Curve A M D be an Hyperbola, then the Focus F, will be in the opposite Section; or on the other side of the Curve, the reflected Rays themselves will Diverge; but being produced they will unite in the Focus of the opposite Section, therefore M F will be Negative, and $-z = \frac{ay}{2y - a}$; that is M E (a) is $= \frac{-2yz}{y - z} = \frac{2yz}{z - y}$, which gives this Construction, serving also for the Ellipsis.

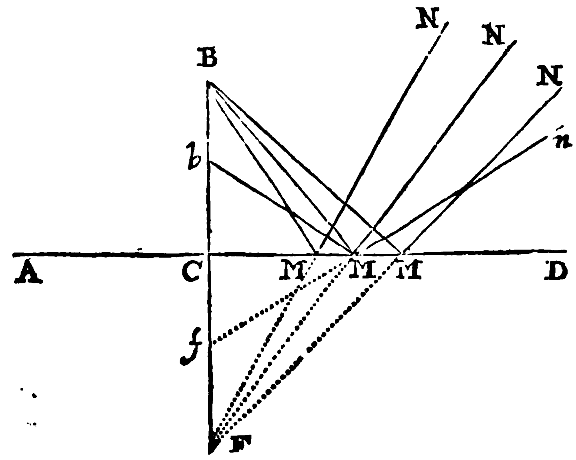


CONSTRUCTION.

Let M E be taken a fourth proportional to $\frac{1}{2}$ the Transverse Axis, the Ray of Incidence, and the Reflected Ray, and draw the perpendicular E C, it will cut M C (perpendicular to the Curve in M) in C the point in the Evoluta which was requir'd.

PROP. II.

The Radiant point B, *and the (Plain Surface or) right Line* A D *being given; to describe the Caustick by Reflection to the same.*



292. Imagine the right Line A C D to be a Curve, Concave towards B, the Radius of whose Curvature is infinite, then is a infinite, and consequently $MF = \frac{ay}{2y - a}$ is $= \frac{ay}{-a} = -y =$ B M, whence we have this.

CONSTRUCTION.

Draw B C perpendicular to A D; and in B C produced, take C F = C B; then I say all the reflected Rays M N, M N, being produced, will Converge to the point F; for the Triangles B C M, F C M, are similar and equal; therefore M B is = M F.

CONSECTARY I.

293. As in the Circle the Evoluta is contracted into one single point in the Center; so here, the Caustick by Reflexion to a streight Line, is contracted into the point F.

CONSECTARY II.

Since the Eye placed any where as in N, receives the reflected Rays M N, M N, &c. as if they issued from the Radiant point F, 'tis evident that the *Image* of B will appear in F.

CONSECTARY III.

And because the Ray of Incidence is the reflected Ray to the reflected Ray considered as a Ray of Incidence; 'tis evident, that the Rays of Incidence N M, N M, &c. Converging to the point F, will be reflected to the point B; that is, if the Rays of Incidence Converge to a point (F) beyond the Surface A D, the reflected Rays will Converge to a point on the same side with the Rays of Incidence.

CONSECTARY IV.

If A D be a plain Speculum, in an Horizontal Position, and if the Object B b be in a Vertical Position, then 'tis manifest that the Image of B will be in F, and that of b in f, &c. and consequently the Object will appear in an inverted Position, or upside down.

CONSECTARY V.

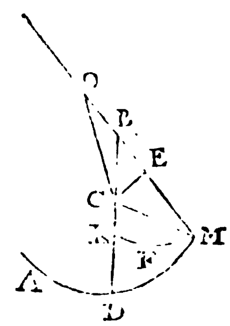
If B *b* be a radiating Plain, then 'tis evident that the Image F *f* made by a plain Speculum, is similar, and equal to the same, tho' not in a like Position; the Difference between the Object and its Image being the same as between the Image on the Seal, and that which it Imprints on the Wax.

CONSECTARY VI.

Because $BM = FM$, therefore $NM + MF$ is $= NM + MA$; that is, the Distance of the Image from the Eye, is equal to the Ray of Incidence, and the Ray of Reflexion taken together.

PROP. III.

If A M D *be an Arch of a Circle, and* B *the Luminous Point. 'Tis required to find the Points* F *in the reflected Rays, in which they touch the Caustick* F K.



294. Through the Luminous Point B, and the Center of the Arch C, draw the Right Line B C D perpendicular to the Arch in D; then 'tis manifest that all the Rays of the Evoluta of the Circle are equal between themselves, and that the said Evoluta is the Center C, whence E M is $= a$, and $BM = y$, and consequently the value of M F is $= \frac{ay}{2y - a}$; whence we have this

CONSTRUCTION.

Produce B M to O, so that $MO = 2y - a$, and take M F a fourth proportional to O M, B M, E M, then the point F will be in the Curve requir'd.

CONSECTARY I.

295. If the point M be infinitely near the point D, then BM (y) will be = BD, and $EM = a$ will be = CD, and the point K in which the reflected Ray touches the Caustick by Reflexion F K, is found; saying, $2BD - CD : BD :: CD : DK$, and by Division $BC : BD :: CK : DK$.

CONSECTARY II.

Hence 'tis evident that if A D M be a Spherical Glass, and C the Center, and B the Radiant point; all the Rays B M falling on the concave Surface of the Glass near the point D, (the Vertex of the Glass) and being Reflected, will Converge to the point K, nearly.

CONSECTARY III.

And if the Radiant point B be at an infinite distance from the Glass, then the point K, to which the Rays Parallel and near the Axis C D, Converge after Re-

 flexion

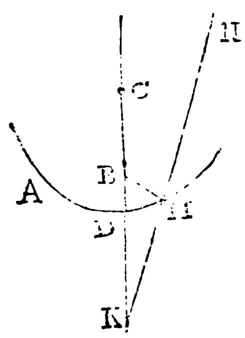
Reflexion, is the middle point between C and D, for in that case MF, or DK is $= \frac{ay}{2y-a} = \frac{ay}{2y} = \frac{1}{2} a = \frac{1}{2} EM = \frac{1}{2} CD$.

CONSECTARY IV.

If CK be $=$ DK, then Rays issuing from the point K, will be Reflected by the Spherico-concave Glass ADM, parallel to the Axis DC.

CONSECTARY V.

If the distance of the Radiant point B from D the Vertex of the Glass, be less than $\frac{1}{4}$ the Diameter of the Glass, then the reflected Rays will Diverge, and the Focus K will be on the opposite side of the Glass in respect of the Radiant point B, for MK or (supposing M but at a small distance from D) DK is $= \frac{ay}{2y-a}$; and because BD or y is less (*ex Hyp.*) than $\frac{1}{2} a$, therefore $\frac{ay}{2y-a}$ is Negative, and consequently the reflected Rays MN Diverge and the Focus K may be determined as before, *viz.* CD $-$ 2 BD : BD :: CD : DK, and by composition, CB : BD :: CK : DK.

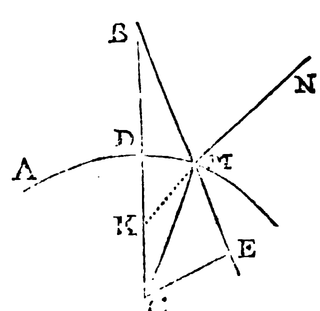


CONSECTARY VI.

Hence if Rays Converging to a given Focus K, be reflected by a Spherico-concave Glass, the Focus B, whereunto the reflected Rays Converge, may be found.

CONSECTARY VII.

If the convex Surface of the Glass ADM be towards the Luminous point B, then the Focus of the reflected Rays will be on the Concave side of the Glass, that is, the reflected Rays will Diverge, for in that case y is Negative and consequently MF or DK is $= \frac{-ay}{-2y-a} = \frac{ay}{2y+a}$, and because BM or BD is $= y$, and the Ray of the Evoluta, invariable, and ME or DC $= a$ CM therefore the Focus K may be determined, saying, 2 BD $+$ DC : BD :: DC : DK, and by Division, BC : BD :: CK : DK.



CONSECTARY VIII.

If an infinite Number of Rays NM falling on the Spherico-convex Surface of a Glass, Converge to a Focus K, whose distance DK from the Vertex is less than $\frac{1}{4}$ the Diameter of the Glass, then the Focus B to which the reflected Rays MB Converge, may be found.

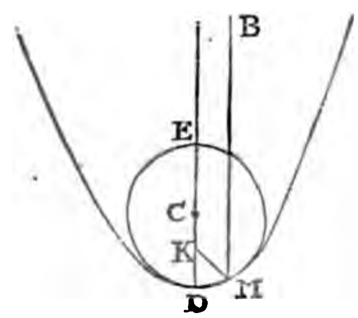
SCHOL.

SCHOLIUM I.

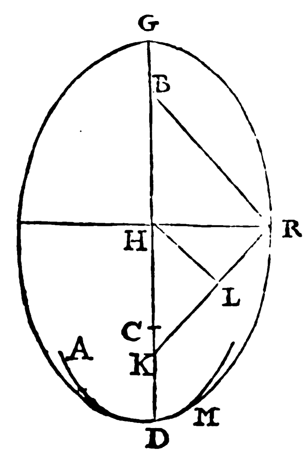
296 We have supposed the point M to be very near the Vertex of the Glass (D.) For the Eye being placed in the Axis D C, 'tis evident that those Rays only (*See Fig. in Pag.* 233.) which are Reflected from the Surface of the Glass near D, affect the Sight, and that they entring into the Pupil of the Eye more directly, and in great numbers produce the strongest and most distinct Sensations in us.

SCHOLIUM II.

297. If the Parabola DM be described to the Vertex D, Axis DE, and Parameter DE; then the Circle EMD inscribed in the same, will (*Art.* 155.) touch the Vertex of the Parabola in D; and it will be the greatest that can be inscrib'd into the Parabola, so as to touch the same in D; whence it is evident that the Curvature of the Circle and Parabola, near the point D, is the same, and consequently the Ray BM parallel to the Axis DC, being Reflected to the Focus of the Parabola K, by the Parabolic Surface, it will be Reflected to the same point K, by the Spherical Surface; and because CD is $= \frac{1}{2}$ Parameter of the Parabola, and K the Focus, therefore CK is $=$ DK, and consequently Rays parallel to the Axis and Reflected by a Spherical Surface (and by that portion thereof which is near to D) will Converge to the Focus K, so that CK shall be $=$ DK.



Again if DMR be a Spheroid, and B and K the Foci; then if B be the Luminous point, all the Reflected Rays will Converge to the other Focus K, and if on the Center C with the Semi-diameter CD $= \frac{1}{2}$ Parameter of the Ellipsis, the Sphere ADM be described (*Art.* 156.) the Curvature of both in D will be the same, and consequently Rays issuing from the point B, will be Reflected by the Spherico-concave Surface DM, to the Focus of the Ellipsis K; whence in the Spherico-concave Glasses, if the Radiant, point B be given the Focus K to which the reflected Rays Converge, may be found, saying, BC : BD :: CK : DK.



For if the conjugate Diameter HR and the right Lines BR, KR, be drawn; and HL perpendicular to KR, then LR is a third proportional to KR, and HR, and consequently is $= \frac{1}{2}$ Parameter of the Ellipsis $=$ CD : whence KL $= \frac{1}{2}$GD $- \frac{1}{2}$ Param. is $= CH = \frac{BC - CK}{2}$; again KL is a third proportional to KR, and KH; therefore.

$$DH : KH :: KH : \frac{BC - CK}{2}$$

and multiplying by 2, GD : BK :: BK : BC $-$ CK.

that is, $BD + BG : BD - DK :: BC + CK : BC - CK.$

And by composition $2BD : BD + BG :: 2BC : BC + CK.$

And dividing the Antecedent by 2, } $BD : BD + BG :: BC : BC + CK.$

And by division $BD : DK :: BC : CK.$ Q. E. D.

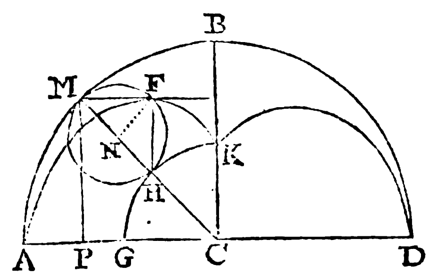
The like Coincidence might be demonstrated in the Hyperbolical Conoid; but that I shall leave to the Readers own Application.

PROP.

PROP. IV.

Let the Curve A M D *be a Semicircle,* A D *the Diameter, and* C *the Center, and let the Rays of Incidence* P M *be perpendicular to the Diameter* A D. *'Tis requir'd to describe the Caustick* A F K.

298. The Evoluta of the Circle is its Center, and MC the Ray of the Evoluta is always the same, therefore $MF = \frac{ay}{2y - a} = \frac{1}{2}a = \frac{1}{2}MP$, whence we have this



CONSTRUCTION.

Bisect the Radius CM in H, and draw HF perpendicular to MF, and the point F will be in the Caustick AFK; for the Triangles MFH, MPC are always similar, therefore MH : MC :: MF : MP.

CONSECTARY I.

299. When the point P falls in C, then the point F will fall in K, the middle point of B C.

CONSECTARY II.

The portion of the Caustick AF is $= 3MF$, for the portion AF is $= PM + MF =$ (because $PM = 2MF$) $3MF$, and the Caustick AFK is $= 3BK$.

CONSECTARY III.

If the Angle ACM be $= \frac{1}{2}$ right Angle, then is $PMC = CMF$, and the reflected Ray being all Parllel to the Diameter AD, touches the Caustick in the Supreme point F.

CONSECTARY IV.

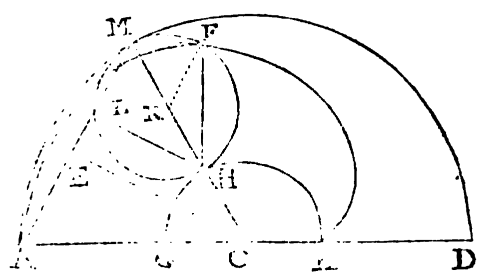
The Circle whose Diameter is MH, passes through the point F, for the Angle HFM is a right Angle.

CONSECTARY V.

The Caustick AFK is a Semi-cycloid describ'd by the revolution of the little Circle MFH on the Periphery or Base KHG; for the Circle MFH is describ'd on $\frac{1}{2}$MC, as a Diameter, and the Angle CMF is $= CMP = HCK$, and consequently the Angle HNF is $= 2HCK$, therefore the Arch HF is $=$ Arch HK, and the Curve KFA is a Semi-cycloid, whose beginning is in K, and Vertex in A.

PROP. V.

Let the Curve A M D *be a Circle,* A D *the Diameter and* C *the Center; and let the Luminous point* A *(from which all the Rays of Incidence* A M *issue) be in one of the extremities of the Diameter. 'Tis requir'd to describe the Caustick* A F K,



300. If CE be drawn perpendicular to AM, then AE will be = EM, and consequently AM (y) = $2a$; therefore $MF = \frac{ay}{2y - a}$ is = $\frac{2}{3}y$, that is, we must take MF = $\frac{2}{3}$ AM; whence 'tis evident that DK is = $\frac{2}{3}$ AD, and CK = $\frac{1}{3}$ CD.

CONSECTARY I.

301. The Portion of the Caustick AF is = $\frac{5}{3}$ AM, for the Portion AF is = AM + MF = AM + $\frac{2}{3}$ AM = $\frac{5}{3}$ AM, and the whole Caustick A F K = $\frac{5}{3}$ A D.

CONSECTARY II.

If A M be taken = A C, then the reflected Ray M F will be parallel to A D, and consequently the point F, will be the Supreme point of the Caustick.

CONSECTARY III.

If CH be taken = $\frac{1}{3}$ CM, and if HF be drawn perpendicular to MF, the point F will be in the Caustick; for drawing HL perpendicular to AM; 'tis evident that ML is = $\frac{4}{3}$ ME = $\frac{2}{3}$ AM = MF, and the Circle describ'd on the Diameter MH will pass through the point F.

CONSECTARY IV.

If another Circle KHG be describ'd on the Center C, with the Radius CK or CH, the Circle KHG will be equal to the Circle MHF, and the Caustick AFK will be a Semi-cycloid, describ'd by the Revolution of the movable Circle MFH, on the immovable Circle KHG; for the Arch HK is = Arch HF; because in the Isosceles Triangle CMA, the external Angle KCH = 2 CMA = AMF = HNF; therefore the Arches HK, HF measuring equal Angles in equal Circles must be equal.

CONSECTARY V.

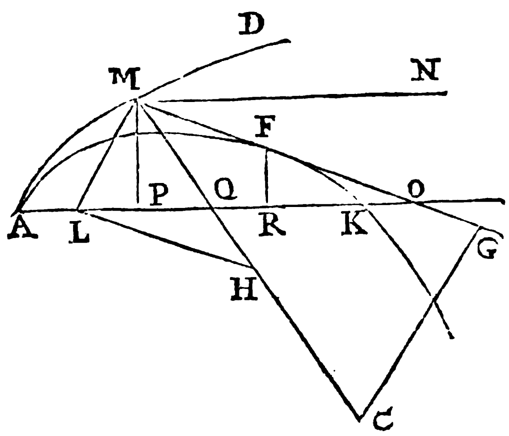
If the Radiant point A be in the Surface of a Sphere, then the reflected Rays (*viz.* those which are nearest the Axis AD) will Converge to the Focus K, distant from C the Center of the Sphere, $\frac{1}{3}$ its Semi-diameter.

And thus I think, I have briefly demonstrated the Principles of Catoptricks from one general Theorem, which I have done that the Reader may be convinced such Speculations *are of more* Universal use *than generally they are thought to be.*

PROP. VI.

Let the Curve AMD *be a Parabola, and suppose the Rays of Incidence* PM *to be perpendicular to the Axis* AP. *'Tis requir'd to describe the Caustick by Reflexion* AFK.

302. If the Ray of the Evoluta MC, and CG perpendicular to the reflected Ray MG be drawn, then 'tis evident that MF $= \frac{1}{2}$ MG $= \frac{1}{2} a$, or thus: Draw MN parallel to the Axis AP, and the right Line ML to the Focus L, then the Angles LMP, FMN will be equal (because LMQ $=$ (*Art.* 41.) QMN and PMQ is $=$ (*ex Hypoth.*) QMF) and if to both be added the Angle PMF, then LMF $=$ PMN $=$ a right Angle; now the perpendicular LH (*Art.* 291. § 11.) bisects MC in H, and LH is $= \frac{1}{2}$ MG, therefore if MF be drawn equal and parallel to LH, it will be the reflected Ray to the Ray of Incidence PM, and it will touch the Caustick AFK in F.



And to draw the greatest Ordinate FR, applied to the Caustick AFK; 'tis evident that when the reflected Ray MFG runs parallel to the Axis AP, then the Ordinate FR, will be the *greatest*; and in that Case the Angle PMQ will be $=$ PQM, and PM $=$ PQ; therefore in that point $\dot{x} = \dot{y}$.

Now let the Equation expressing the Nature of the Curve AMD be $ax = yy$, then is $\dot{y} = \frac{a\dot{x}}{2y} = \frac{a\dot{x}}{2\sqrt{ax}} = \dot{x}$, therefore $a\dot{x} = 2\sqrt{ax} \times \dot{x}$, and consequently $a = 2\sqrt{ax}$, whence $x = \frac{1}{4}a$, which shews that when the Ray of Reflection MF, runs parallel to the Axis AP, and touches the Caustick in the Supreme point, then AP is $= \frac{1}{4}$ Parameter of the Curve AMD $= \frac{1}{4}a$; that is, the point P will fall in the Focus of the Parabola, when the reflected Ray MF is parallel to the Axis; and then MP coincides with ML, LH with LQ, and MF with MN; whence 'tis evident that in that Case, MF is $=$ ML, and that if FR be drawn perpendicular to the Axis, AR, or AL $+$ MF will be $= \frac{3}{4}a$, and in this Case, the Portion of the Caustick AF is $=$ PM $+$ MF $=$ the Parameter of the Curve AMD.

And because the Caustick by Reflexion AFK may be infinitely produced beyond K, let it be requir'd, in the next place, to investigate the point K in the Axis AO, where the Caustick intersects the same.

'Tis evident in this Case that MF becomes $=$ MO, therefore the Value of MO must be investigated, and put $=$ MF; let the unknown Quantity MO be $= u$, then because the Angle PMO is bisected by the Line MQ, it is MP (y) : MO (u) :: PQ $\left(\frac{y\dot{y}}{\dot{x}}\right)$: QO $= \frac{u\dot{y}}{\dot{x}}$, and consequently OP is $= \frac{y\dot{y} + u\dot{y}}{\dot{x}} = \sqrt{uu - yy}$, and dividing both sides of the Equation by $u + y$, we have $\frac{\dot{y}}{\dot{x}} = \sqrt{\frac{uu - yy}{uu + 2uy + yy}} = \sqrt{\frac{u - y}{u + y}}$; whence $\frac{\dot{y}^2}{\dot{x}^2} = \frac{u - y}{u + y}$, and consequently $u\dot{x}^2 - u\dot{y}^2 =$

$u\dot{y}^2 = y\dot{x}^2 + y\dot{y}^2$; and MO ($u$) $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2} =$ (*ex Hypoth.*) MF ($\frac{1}{2}a$) $= \frac{\dot{x}^2 + \dot{y}^2}{-2\ddot{y}}$; therefore $-2y\ddot{y}\dot{x}^2 - 2y\ddot{y}\dot{y}^2 = \dot{x}^4 - \dot{y}^4$, and dividing by $\dot{x}^2 + \dot{y}^2$, we have $-2y\ddot{y} = \dot{x}^2 - \dot{y}^2$; that is $\dot{y}^2 - 2y\ddot{y} = \dot{x}^2$, which is a general Theorem, serving to find the point P, so that drawing the Ray of Incidence PM, and the reflected Ray MF, the same will touch the Caustick AFK in the point K, where it intersects the Axis AP. For Instance,

In the Parabola $y = x^{\frac{1}{2}}$, and $\dot{y} = \frac{1}{2}x^{-\frac{1}{2}}\dot{x}$, and $\ddot{y} =$ (supposing $\dot{x}$ invariable) $-\frac{1}{4}x^{-\frac{3}{2}}\dot{x}^2$, and substituting these Values in the preceeding Theorem $\dot{x}^2 = \dot{y}^2 - 2y\ddot{y}$, there will arise $\dot{x}^2 = \frac{1}{4}x^{-1}\dot{x}^2 + \frac{1}{2}x^{-1}\dot{x}^2$, and by Division and Multiplication $8xx = 6x$; that is, AP (x) is $= \frac{3}{4}$ the Parameter of the Curve.

PROP. VII.

The same things being supposed as before; let it be requir'd to investigate the Nature of the Caustick AFK.

303. To investigate the Nature of any Curve, is to find an Equation which expresses the Relation between the Abscissa AR, and the Ordinate RF; to do which: Suppose AR $= s$, and RF $= z$, then because, MO (u) $= \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2}$ therefore PO $=$ ($\sqrt{MOq - PMq}$) $\frac{2y\dot{x}\dot{y}}{\dot{x}^2 - \dot{y}^2}$ $=\!=$ (by the preceding *Article*.) $\frac{y\dot{y} + u\dot{y}}{\dot{x}}$; and (because the Triangles MPO, MSF are similar) MO $\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2}\right)$: MF $\left(\frac{\dot{x}^2 + \dot{y}^2}{-2\ddot{y}}\right)$ :: MP (y) : MS ($y - z$) $\frac{\dot{x}^2 - \dot{y}^2}{-2\ddot{y}}$ :: PO $\left(\frac{2y\dot{y}\dot{x}}{\dot{x}^2 - \dot{y}^2}\right)$: SF or PR ($s - x$) $= \frac{\dot{x}\dot{y}}{-\ddot{y}}$; and now we have two Equations $z = y + \frac{\dot{y}^2 - \dot{x}^2}{-2\ddot{y}}$, and $s = x + \frac{\dot{x}\dot{y}}{-\ddot{y}}$, which by the help of the Equation expressing the Nature of the given Curve, will serve to find a new Equation (cleared of the flowing Quantities x and y) expressing the Relation of AR (s) to FR (z.)

For Instance, if the Curve AMD be a Parabola, then $y = x^{\frac{1}{2}}$, $\dot{y} = \frac{1}{2}x^{-\frac{1}{2}}\dot{x}$, and $\ddot{y} = -\frac{1}{4}x^{-\frac{3}{2}}\dot{x}^2$, therefore the Equation $z = y + \frac{\dot{y}^2 - \dot{x}^2}{-2\ddot{y}}$ becomes $z = x^{\frac{1}{2}} + \frac{\frac{1}{4}x^{-1}\dot{x}^2 - \dot{x}^2}{\frac{1}{2}x^{-\frac{3}{2}}\dot{x}^2} = \frac{\frac{1}{2}x^{-1}\dot{x}^2 + x^{-1}\dot{x}^2 - 2\dot{x}^2}{x^{-\frac{3}{2}}\dot{x}^2} = x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$, and Squaring both sides of the Equation, there will arise

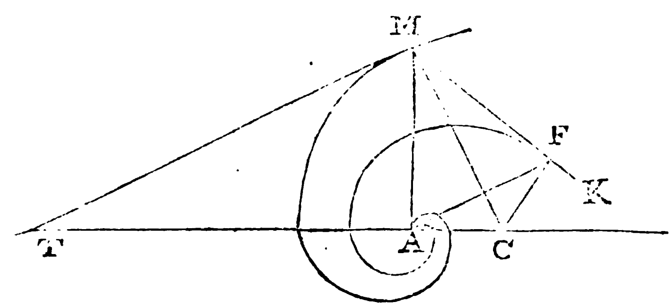
$zz =$

$zz = \frac{9}{4}x - 6xx + 4x^3$; again, $s = x + \frac{\dot{x}\dot{y}}{-\ddot{y}}$ = (by substitution) $x + \frac{x^{-\frac{1}{2}}\dot{x}^2}{\frac{1}{2}x^{-\frac{3}{2}}\dot{x}^2} = x + 2x = 3x$, whence the Nature of the Caustick A F K is express'd by this Equation, $zz = \frac{4}{27}s^3 - \frac{2}{3}ass + \frac{3}{4}aas$, and it may be observ'd that P R ($= -2x$) is always $= 2$ A P (x) because A R is $= 3x$, and this observation affords us a new Method for describing the Catacaustick A F K.

PROP. VIII.

If the Curve A M D *be the Logarithmetical Spiral, and if the Rays of Incidence* A M *issue from the Center* A. *'Tis requir'd to describe the Caustick by Reflection* A F K.

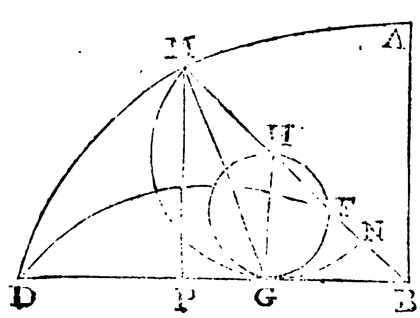
304. Draw M C perpendicular to the Curve, and A C perpendicular to the Ray of Incidence A M, then the point C will be in the Evoluta of the given Curve, and consequently A M $= y$ is $=$ a, whence M F $\left(\frac{ay}{2y - a}\right) = y$, and the Triangle A M F is an Isosceles Triangle; and because the Angle of Incidence A M T is equal to the Angle of Reflection F M S, therefore the Angle, A F M is = A M T; now the Angle A M T is invariable, by the property of the Curve; and consequently the Angle A F M is invariable, and the Caustick by Reflection A F K will be a Logarithmetical Spiral, differing from the given Spiral only in Position.



PROP. IX.

Let the Curve A M D *be a vulgar Semicycloid, describ'd by the Revolution of the Semicircle* N G M *on the right Line* B D; *and let the Rays of Incidence* P M *be parallel to the Axis* A B. *'Tis requir'd to describe the Caustick by Reflection.*

305. Because M G is $= \frac{1}{2}$ the Ray of the Evoluta, and G P perpendicular to P M, therefore M F $= \frac{1}{2}a =$ P M, whence if G F be drawn perpendicular to the Reflected Ray M F, the point F will be in the Curve requir'd.



If the Rays H M, H G be drawn from H the Center of the generating Circle, to the describing point M, and the point of Contact G, 'tis evident that H G will be perpendicular to B D, and that the Angle G M H = M G H = G M P; whence it appears that the Reflected Ray M F passes through the Center H; now the Circle whose Diameter is G H passes also through the point F, because G F H is a right Angle; therefore the Arches G N and $\frac{1}{2}$ G F, which measure the same Angle G H N are proportional to the Diameters M N, G H, of their respective Circles, and consequently the

the Arch G F = Arch G N = G B; whence it is manifest that the Caustick D F B is a Cycloid describ'd by the Revolution of the Circle G F H on the right Line B D.

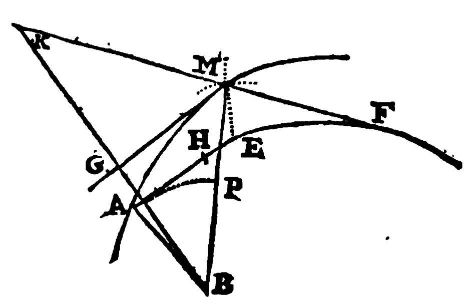
COROLLARY.

306. The Space D A B F D is equal to thrice the Area of the Circle G F H; for the Semicircle M G N is = 2 the Area of the Circle G F H, therefore the Cycloidal Space A M D B is = 6 x Area of the Circle G F H; and the Cycloidal Space B F D is = 3 the Area of the Circle G F H, and consequently the Space D A B F D is = 3 the Area of the Circle G F H; and the Curve D F B divides the Space A M D B into two equal parts.

PROP. X.

The Caustick by Reflection H F *being given, with the Luminous Point* B; *To describe an Infinite Number of Curves, such as* A M, *to which it is the Caustick.*

307. In any Tangent as H A assume the point A (at pleasure) for one of the points of the Curve A M requir'd; on the Center B with the Radius B A, describe the Arch A P, and on the same Center B, with any other Radius B M, describe another Arch: then take A H + H E = B M — B A = P M, and beginning at the point E, Evolve the Caustick H F; then the point E will describe the Curve Line E M, which will intersect the Arch of the Circle describ'd with the Radius B M, in M, one of the points of the Curve requir'd.



For A H + H E = P M, and E F = (from the Nature of Evolutions) M F, therefore P M (B M — B A) + M F = A H + H F, and consequently the Curve H F is the Caustick by Reflection to A M.

Another way.

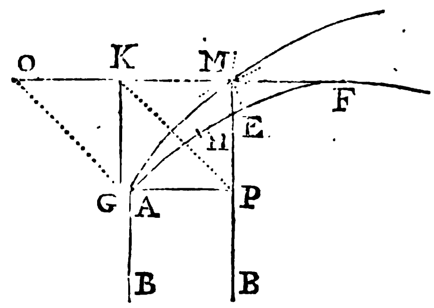
308. If B M F be a Thread whose extremities are made fast in B and F, and if the said Thread be kept at its full extent, with a Pin in M, then if the Pin be supposed to move from A to M, while the Portion of the Thread M F touches the Catacaustick in F, it will describe the Curve A M requir'd: for it is evident that P M + M F is always = A H + H F.

Another way.

309. Having drawn the Tangent H A, draw another Tangent F M at pleasure, and take F K = B A + A H + H F; draw B K, and bisect the same in G, and draw G M perpendicular to B K, and it will cut the Tangent F K in the point M requir'd: for B M + M F = B A + A H + H F, therefore the point M is in the Curve A M requir'd.

CONSECTARY I.

310. If the point B be at an infinite diſtance from the Curve A M, that is, if the Rays of Incidence B A, B M, be parallel to a given right Line; the firſt Conſtruction will alſo ſerve here, if we imagine the Arches of the Circles deſcrib'd on the Center B to become ſtreight Lines, perpendicular to the Ray of Incidence.



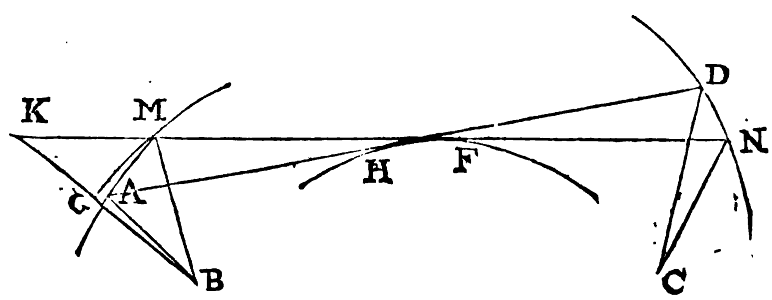
Or the Curve A M may be deſcrib'd thus: Take F K = A H + H F, then if the point M be found, ſo that drawing M P parallel to A B, or perpendicular to A P, the Lines M K, M P be equal; then 'tis evident that the point M will be in the Curve A M requir'd; for then P M + M F = A H + H F.

Now the point M may be found in this manner: Draw K G perpendicular to A P, and take K O = K G; draw O G, and draw K P ∥ O G, and P M ∥ G K, then M will be the point required; for becauſe the Triangles G K O, P M K are ſimilar, therefore P M = M K.

CONSECTARY II.

If the Curve Line D N, and the Luminous Point C be given; to find an infinite Number of Curves ſuch as A M, which ſhall make all the double reflected Rays M B, A B, Converge to a given point B.

If we imagine the Curve H F to be the Catacauſtick of the given Curve N D, C being the Radiant Point, 'tis evident that the ſame Curve H F muſt alſo be the



Catacauſtick to the Curve (A M) required, the Luminous Point (or rather the Focus to which the double reflected Rays Converge) being in B; whence F K = B A + A H + H F, and N K = B A + A H + H F + F N = (becauſe H D + D C = H F + F N + N C) B A + A D + D C − N C; whence there will ariſe this

CONSTRUCTION I.

Aſſume the point A (at pleaſure) in any of the firſt reflected Rays, for one of the points of the Curve (A M) requir'd; and in any other reflected Ray as N M, take N K = B A + A D + D C − C N: then draw B K, and biſect the ſame in G; and draw M G perpendicular to B K, and the point M will be in the Curve requir'd.

CONSTRUCTION II.

If the Cauſtick H F be drawn, the Curve A M may eaſily be deſcrib'd, if we aſſume the points B and C as two Foci, and in them fix the ends of the Thread B M N C, and with two Pins in N and M deſcribe the Curves N D, M A, ſo that the Portion of the Thread M N, A D, *&c.* always touch the Cauſtick in H and A, *&c.*

CONSTRUCTION III.

And by this Artifice, Rays iſſuing from any point may be made to Converge to any other given point, after one ſingle, double or triple, *&c.* Reflexion.

SECT.

SECT. X.

The Use of Fluxions

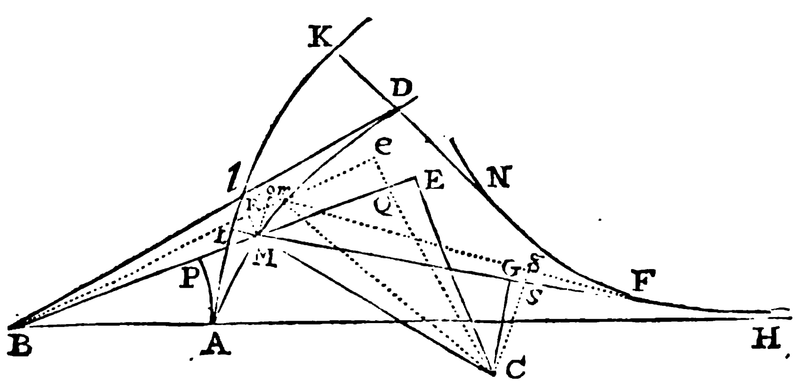
In finding Causticks by Refraction, to all sorts of Curves.

DEFINITION.

IF we imagine an infinite number of Rays BA, BM, BD, &c. Issuing from the same Luminous Point B, to be Refracted (from or to the Perpendicular MC) at the Curve AMD, so that CE the Sines of the Angles of Incidence CME be always to CG, the Sines of the refracted Angles CMG, in a given Proportion, as *m* is to *n*; the Curve Line HFN which touches all the refracted Rays is called the *Diacaustick* or *Caustick* by *Refraction*.

CONSECTARY I.

311. If the Caustick HFN be Involved, beginning at the point A, the point A will describe the Curve ALK, so that the Tangent LF + the Portion of the Caust-



ick FH, will always be equal to the same streight Line AH; and if we imagine another Tangent F*ml* infinitely near FML, and another Ray of Incidence B*m*, and if on the Centers F and B, the little Arches MO, MR be describ'd, the Rectangular Triangles MR*m*, MO*m* will be similar (because if from the right Angles RME, CM*m*, we subtract the Angle EM*m*, there will remain, RM*m* = EMC; and if from the right Angles GMO, CM*m*, we subtract the Angle GM*m*, there will remain OM*m* = GMC) to the Triangles MEC, MGC, respectively: therefore R*m* : O*m* :: CE : CG :: *m* : *n*.

And because R*m* is the Fluxion of BM, and O*m* that of LM, 'tis evident that the sum of all the R*m*, that is BM − BA, is to the sum of all the O*m*, that is ML, or AH − MF − FH :: *m* : *n*, and consequently $n \times \overline{BM - BA} = m \times \overline{AH - MF - FH}$; and by Division $\frac{n}{m} BM - \frac{n}{m} BA = AH - MF - FH$, and by Transposition, $FH = AH - MF + \frac{n}{m} BA - \frac{n}{m} BM$.

CON-

CONSECTARY II.

If the Arch of a Circle A P be describ'd on the Center B, then P M = B M — B A; and if we suppose the Luminous point B to be at an infinite distance from the Curve A M D, the Rays of Incidence B A, B M, will be parallel to one another, and the Arch A P will become a streight Line perpendicular to these Rays.

PROP. I.

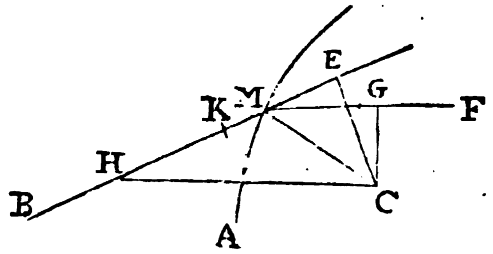
The Nature of the Curve A M D, *the Luminous Point* B, *and the Ray of Incidence* B M, *being given; to find the point* F *in the Refracted Ray* M F, *where the said Ray touches the Dia-caustick* H F N.

312. In the foregoing Figure, find the length of the Ray of the Evoluta M C, answering to the point M, and suppose the known Quantities B M $= y$, M E $= a$, M G $= b$, and the infinitely little Arch M R $= \dot{x}$.

Then because the Rectangular Triangles M E C, M R m, M G C, M O m, B M R, B Q e, are similar; it is, M E (a) : M G (b) :: M R $(\dot{x})$: M O $= \frac{b\dot{x}}{a}$, and B M (y) : B Q or B E $(y + a)$ :: M R $(\dot{x})$: Q $e = \frac{a\dot{x} + y\dot{x}}{y}$, and by the property of Refraction, C e : C g :: C E : C G :: $m : n$, therefore $m : n$ :: C e — C E or Q e $\left(\frac{a\dot{x} + y\dot{x}}{y}\right)$: C g — C G or S $g = \frac{an\dot{x} + ny\dot{x}}{my}$, and because the Rectangular Triangles F M O, and F S g are similar, it will be, M O — S g $\left(\frac{bmy\dot{x} - aan\dot{x} - any\dot{x}}{amy}\right)$: M O $\left(\frac{b\dot{x}}{a}\right)$ :: M S or M G (b) : M F $= \frac{bbmy}{bmy - aan - any}$, whence there arises this

CONSTRUCTION.

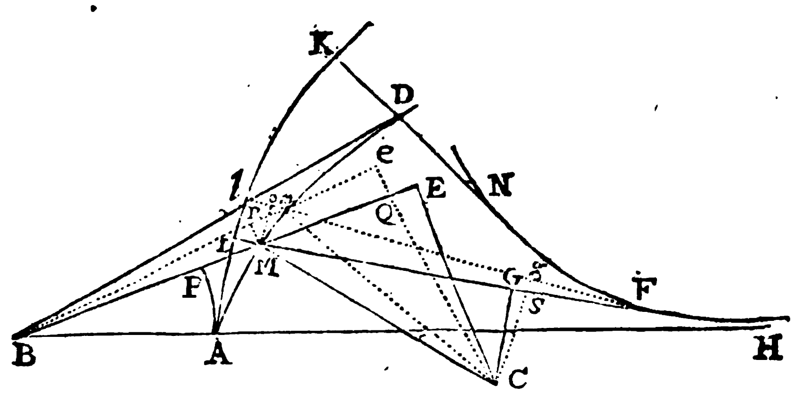
Towards M C, make the Angle E C H = G C M, and take M K (towards B) $= \frac{aa}{y}$, then I say, if H K : H E :: M G : M F; the point F, will be in the Dia-caustick H F N.



Because the Triangles C G M, C E H are similar; it is, C G : C E :: $n : m$ :: M G (b) : E H $= \frac{bm}{n}$, therefore H E — M E $= \frac{bm}{n} - a =$ H M $= \frac{bm - an}{n}$, and H M — M K = H K $= \frac{bmy - any - aan}{ny}$, and consequently, H K $\left(\frac{bmy - any - aan}{ny}\right)$: H E $\left(\frac{bm}{n}\right)$ :: M G (b) : M F $= \frac{bbmy}{bmy - any - aan}$.

CONSECTARY I.

313. If the Luminous Point B be on the side of the Curve (AMD) towards E, or which is the same thing, if the Curve be Concave towards the Luminous Point B,



then y of being Positive will become Negative, and consequently MF will be $= \frac{-bbmy}{-bmy+any-aan}$ or $\frac{bbmy}{bmy-any+aan}$, and the Construction will be the same as before.

CONSECTARY II.

If we suppose y to become infinite, then the Rays of Incidence will be parallel to one another, and $MF = \frac{bbmy}{bmy-any-aan}$ will become $=$ (because the Term aan is incomparably less than either of the other Terms bmy, or any, and consequently may be rejected) $\frac{bbmy}{bmy-any} =$ (dividing by y) $\frac{bbm}{bm-an}$; and because in this case $\frac{aa}{y}$ is $= 0$, the points M and K coincide, that is MR vanishes, and consequently the point F is found by this Analogy, HM : HE :: MG : MF.

CONSECTARY III.

The same Curve AMD can have but one Caustick by Refraction, the Ratio of m to n being given: and the Caustick is a Geometrical Curve, and may be Rectified, the given Curve AMD being Geometrical.

CONSECTARY IV.

If m be infinite in respect of n, then 'tis evident that the refracted Angle CMG is infinitely little, and consequently MF and the Ray of the Evoluta MC coincide, and the Caustick by Refraction coincides with the Evoluta of the given Curve AMD.

CONSECTARY V.

If the Curve AMD be Convex towards the Luminous point B, and if the Value of MF $\left(= \frac{bbmy}{bmy-any-aan}\right)$ be Positive, 'tis evident that the point F must be taken on the same side with the point G, in respect of M (as is suppos'd in the Calculation). But if the Value of MF be Negative, then the refracted Ray FM must be

Rrr produced

produced on the side towards B, and the point F must be taken on the same side of the Curve with B. Whence 'tis evident, that in the first case, when the Value of M F is Positive, the refracted Rays Converge, on the side of the Curve towards G (because it is on that side that the refracted Rays intersect one another, in order to determine the points F.) But in the last case, when the Value of M F is Negative, the refracted Rays Diverge, because being produc'd, they intersect one another on the same side of the Curve with B, in order to determine the points (F) of the Caustick.

CONSECTARY VI.

In like manner if the Curve A M D be Concave towards B, then is M F by *Consect.* 1. $= \frac{-bbmy}{-bmy+any-aan} = \frac{bbmy}{bmy-any+aan}$. Whence the refracted Rays, being infinitely near, Converge when the Value of M F is Negative; and Diverge, when Positive.

CONSECTARY VII.

If the Curve A M D be Convex towards the Luminous Point B, and if m be less than n, then the Value of M F $\left(\frac{bbmy}{bmy-any-aan}\right)$ is Negative, and consequently the refracted Rays Diverge. And in like manner, if the Curve A M D be Concave towards the Luminous Point B, and m greater than n, then the Value of M F is Positive, and consequently the refracted Rays Diverge (§. 5. 6.)

CONSECTARY VIII.

If the Rays of Incidence B M touch the Curve A M D in the point M, then is M E (a) $= 0$, and consequently M F $= b$; which shews that the point F will then Coincide with the point G.

CONSECTARY IX.

And if the Ray of Incidence B M be perpendicular to the Curve A M D, then the Refracted Ray M G, will Coincide with M C the Ray of the Evoluta, and the right Lines M E (a) and M G (b) will each become equal to M C; therefore M F $= \frac{bmy}{my-ny+bn}$, which will become $\frac{bm}{m-n}$, when the Rays of Incidence are paralel between themselves.

CONSECTARY X.

If the Refracted Ray M F touch the Curve A M D in the point M, then is M G (b) $= 0$, and consequently the Diacaustick will touch the Curve in the given point M.

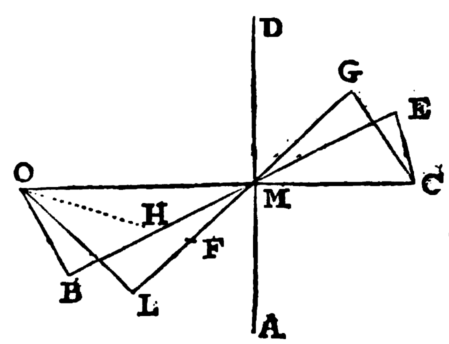
CONSECTARY XI.

And if C M the Ray of the Evoluta, be $= 0$; the right Lines M E (a) M G (b) will also be equal to nothing, and consequently M F $= 0$, and the point M will be common to the Caustick and the given Curve.

CONSECTARY XII.

If the Ray of the Evoluta C M be infinite, then the right Lines M E (a) and M G (b) will also be infinite, and the Terms bmy, any will be infinitely little in in respect of $bbmy$, aan, and consequently MF will be $= \frac{bbmy}{\mp aan}$; and because this Quantity is Negative (§ 1.) when the point F falls on the opposite side of the Curve, in respect of B; and Positive, when B and F are on the same side of the Curve; it is plain, that the point F must always be taken of the same side with the point B, and consequently, that the refracted Rays will Diverge. And in this case, it is manifest that, the Arch Mm is a streight Line. In which case, the preceeding *Construction* cannot be applied; and therefore this which serves to determine the points (F) of the Caustick HFN, when the Curve AMD becomes a streight Line, may be substituted in its room.

Draw BO perpendicular to the Ray of Incidence BM, until it intersect the right Line MC (perpendicular to AD) in O, and draw OL perpendicular to the refracted Ray MG, and make the Angle BOH = LOM. Then say, BM:BH :: ML:MF. I say the point F will be in the *Diacaustick*.



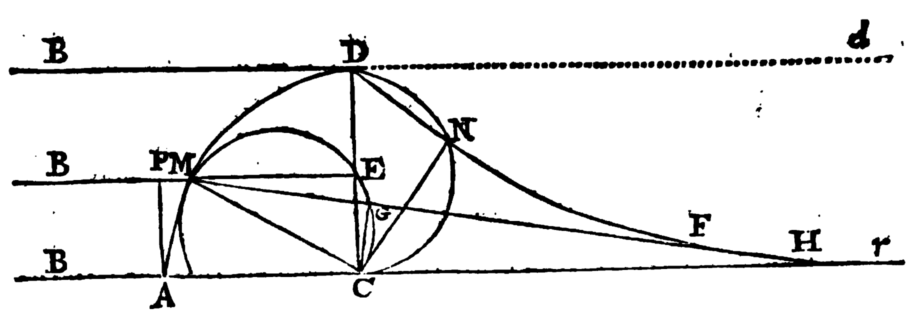
For the Rectangular Triangles MEC, MBO, MGC and MLO are always similar, how great soever MC be suppos'd; and consequently, when CM is infinite, we have this Analogy, ME (a) : MG (b) :: BM (y) : ML $= \frac{by}{a}$, and because the Triangles OLM, OBH are also similar, it is, OL : OB :: $n : m$ :: ML $\left(\frac{by}{a}\right)$: BH $= \frac{bmy}{an}$;

Whence it is evident that BM (y) : BH $\left(\frac{bmy}{an}\right)$ :: ML $\left(\frac{by}{a}\right)$: MF $= \frac{bbmy}{aan}$.

PROP. II.

If the Curve Line AMD *be a Quadrant of a Circle, and* B *the Luminous Point. 'Tis requir'd to describe the* Diacaustick HFN.

314. If AMD be a Quadrant of a Circle, then MC the Ray of the Evoluta is an invariable Quantity. Now suppose the point B at an infinite distance from AMD, then the Rays of Incidence BA, BM, BD, &c. will be parallel between themselves, and perpendicular to CD; and let the Ratio of m to n be as 3 is to 2. Then because



the Evoluta of the Circle, 'is contracted into one point C, which is the Center; it is evident that if we describe the Semi-circle MEC, on the Diameter MC, and take the Chord CG = ⅔ CE, the Ray MG will be the refracted Ray to the Ray of Incidence BM, and the point F may be found by *Art.* 313. § 2.

CONSECTARY I.

315. And to find the point H, where the Ray BA perpendicular to the Curve AMD, touches the Diacaustick HFN, we have AH $=$ (*Art.* 313. § 9.) $\frac{bm}{m-n}$ $= 3b = 3$ AC.

CON-

CONSECTARY II.

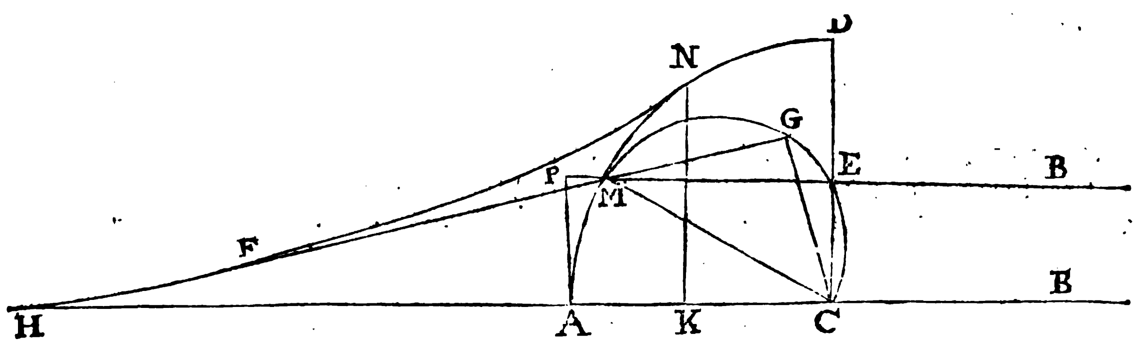
And if we describe the Semi-circle CND, on the Diameter CD, and take the Chord $CN = \frac{2}{3} CD$; it is (*Art.* 313. § 8.) evident that the point N will be in the Diacaustick.

CONSECTARY III.

And if AP be drawn parallel to CD, then the Portion of the Diacaustick FH is = (*Art.* 311. § 2.) $AH - MF - \frac{2}{3} PM$, and consequently the entire Caustick HFN is $= \frac{7}{3} CA - DN$.

CONSECTARY IV.

If the Parallel Rays BM fall on the Concave side of the Quadrant of the Circle AMD, and if the Ratio of m to n be as 2 is to 3, then on CM (the Ray of the Evo-



luta of the Circle) describe the Semi-circle CEM, and take the Chord $CG = \frac{2}{3} CE$, then will GM, produced towards F, be the Refracted Ray to the Ray of Incidence BM and the point F may be determined by *Art.* 313. § 2.

CONSECTARY V.

And to find the point H, where the Ray BA perpendicular to the Curve AMD, touches the Diacaustick; AH is = (*Art.* 291. § 9.) $\frac{bm}{m-n} = -2b$. That is, the point H falls on the Convex side of Curve AMD, and AH the distance of H from the vertex A is $= 2AC =$ Diameter of the Circle AMD.

CONSECTARY VI.

And if we suppose CG or $\frac{2}{3} CE = CM$, then 'tis manifest that the Refracted Ray MF will touch the Circle AMD in M; because the points G and M coincide. Whence it is evident, that if CE be taken $= \frac{2}{3} CD$, the point M will fall in N, the point in which the Caustick touches the Quadrant of the Circle.

CONSECTARY VII.

If CE exceeds $\frac{2}{3}$ CD, then the Rays of Incidence BM cannot (be refracted or) pass out of the *Glass* or *denser Medium*, into the *Air* or a *thinner Medium*, because it is impossible, that CG perpendicular to the refracted Ray MG, can be greater than CM, and consequently all the Rays that fall between N and D must be Reflected.

CON-

CONSECTARY VIII.

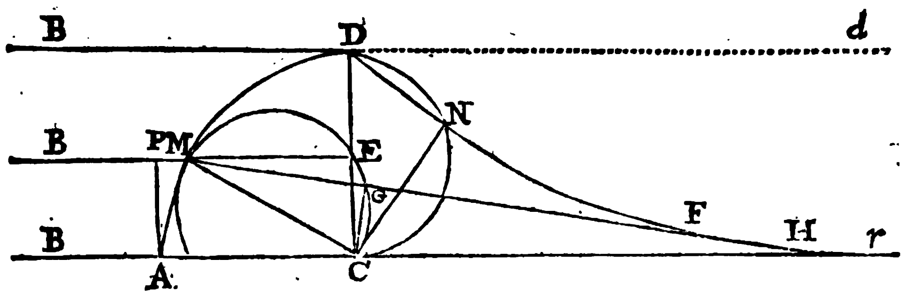
And to find the length of the Diacaustick HFN. Draw AP parallel to CD, then the Portion of the Curve $FH = AH - MF + \frac{1}{2}PM$; and drawing NK parallel to CD, the Caustick $HFN = 2AC + \frac{1}{2}AK = \frac{7-\sqrt{5}}{2}CA$.

Before I leave this Proposition, I think it will not be Improper to shew how

The Doctrine of the Foci *of* Spherical Glasses *of all sorts, expos'd either to Diverging, Converging, or Parallel Rays, may be deduc'd from the Principles here deliver'd.*

CONSECTARY IX.

If AMD*d* be Revolv'd about the Axis AC, the Quadrant AMDC will generate an Hemisphere, and if this Hemisphere infinitely produc'd towards *d*, *r*, be suppos'd of Glass; and the ambient Fluid, Air. Then all the Rays BM, parallel to and infi-



nitely near the Axis AC, will (being refracted at the Spherico-convex Surface of the Glass) Converge to the point H, three Semi-diameters of the Sphere distant from the Vertex A.

CONSECTARY X.

And if an infinite Number of Rays Diverge from the point H, distant three Semi-diameters from the Spherico-concave Surface of the Air, and be Refracted at the said Spherico-concave Surface; they will (after Refraction) run parallel to the right Line HA, drawn through the Luminous Point H, and C the Center of the Sphere.

CONSECTARY XI.

If an infinite Number of Rays BM, be parallel and infinitely near to the Axis AC, and passing out of a *Medium* of *Glass* into a *Medium* of *Air*, be refracted at the Spherico concave Surface of the *Air*, they will Converge to the point H, whose Distance from the Vertex A is = the Diameter of the Sphere.

CONSECTARY XII.

If the Ray H M Diverge from the point H (in *Air*) distant one Diameter from the Vertex of the Sphere of Glass, be refracted at the Spherico-convex Surface of the Glass, all the refracted Rays will run parallel to the Line H C, drawn through the Luminous Point H, and C the Center of the Sphere.

CONSECTARY XIII.

In a *Medium* of Glass, if the Rays of Light be parallel to the Axis, and if they be refracted at the Spherico-convex Surface of the *Air*, the refracted Rays will Converge to a point in the Axis (and in the *Medium* of Glass) distant one Diameter of the Sphere from the Vertex of the Glass.

CONSECTARY XIV.

In a *Medium* of *Air*, if an infinite Number of Rays Converge to a point (beyond the Spherico-convex Surface of the *Air*) distant one Diameter of the Sphere from the Vertex, and be refracted at the Spherico-concave Surface of the Glass, the refracted Rays will run parallel to the Axis of the Sphere.

CONSECTARY XV.

In a *Medium* of *Air*, if an infinite Number of Rays parallel to the Axis, be refracted at the Spherico-concave Surface of Glass, the refracted Rays will Converge to a point in the *Air*, (and in the Axis) whose Distance from the Vertex is equal to three Semi-diameters of the Sphere.

CONSECTARY XVI.

In a *Medium* of Glass, if an infinite Number of Rays Converge to a point distant three Semi-diameters beyond the Spherico-convex Surface of *Air*, and be refracted at the said Spherico-convex Surface of *Air*, they will after Refraction, run parallel to the Axis.

SCHOLIUM

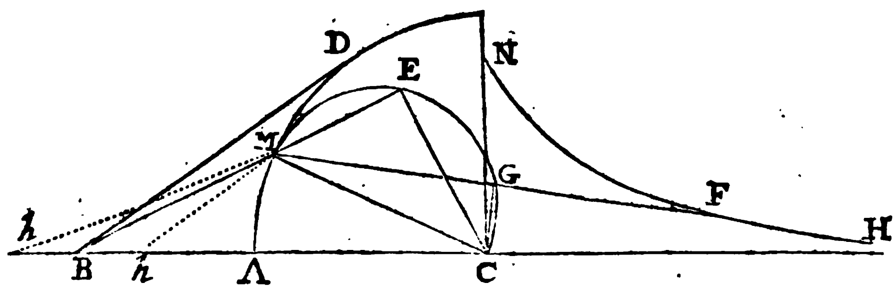
In Dioptricks, it is necessary only to consider those Rays, which can enter into the Eye in any given Position; and therefore as we have supposed the Eye posited in the Axis H C, *so the Demonstrations can strictly agree with none but those Rays that are infinitely near the said Axis: But in Practice they may be allowed a greater Latitude; and the same Proportions may be used, even when the Point* M *is at some little Distance from the Vertex* A, *without producing any sensible error.*

Before I proceed further on this Subject, it will be requisite to premise another.

PROP. III.

If the Curve AMD *be a Quadrant of a Circle, as before, and* C *the Center, and if the Rays of Incidence* BA, BM, BD, *&c. Diverge from a given point* B. *'Tis required to describe the Caustick by Refraction* HFN.

316. Through the Luminous point B, and C the Center of the Circle draw the indefinite Line BCH, and let the Sine of the Angle of Incidence be to the Sine of the refracted Angle, as m is to n; then because the Center C is the Evoluta of the Circle, the Line MC will be a Ray of the Evoluta.

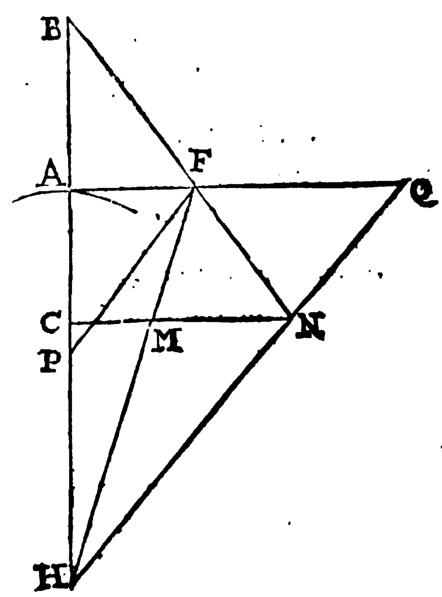


On the Diameter MC describe the Semi-circle MEC, and produce BM the Ray of Incidence, until it interfect the same in E, draw the right Line CE, and take $CG = \frac{n}{m} CE$, then it is evident that MG will be the Refracted Ray to the Ray of Incidence BM, and the point F in the Caustick HFN may be found by taking $MF = (Art.\ 312)\ \frac{bbmy}{bmy - aan - any}$.

And to find the point H, where the Ray BA perpendicular to the Curve AMD touches the Diacaustick HFN, we have $AH = (Art.\ 313.\ \S 9.)\ \frac{bmy}{my - bn - ny}$; whence arises this

CONSTRUCTION.

To find the point H; through the points A and C, draw the right Lines AQ and CN parallel to each other; and from the Luminous point B, draw BN at pleasure Interfecting AQ in F; then take CM : CN :: $n : m$, and draw FM, and produce the same to H; I say H is the Point requir'd.

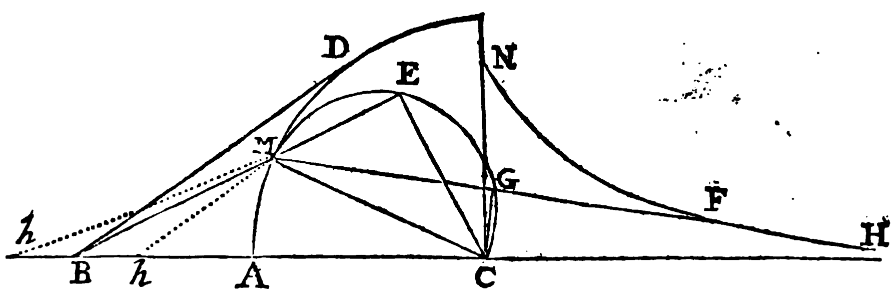


For if the right Line HN be produced to Q, and if FP be drawn parallel to QH, then BC $(y+b)$: BA (y) :: CN (m) : AF $= \frac{my}{b+y}$, and CM (n) : CN (m) :: AF $\left(\frac{my}{b+y}\right)$: AQ $= \frac{mmy}{bn+ny}$, and AQ $-$ CN $\left(\frac{mmy - bmn - mny}{bn+ny}\right)$: AQ $\left(\frac{mm}{bn+ny}\right)$ :: AH $-$ CH or AC (b) : AH $= \frac{bmy}{my - ny - bn}$.

CON-

CONSECTARY I.

317. If B be a Radiant point, A M D a Lens Spherico-convex towards B, and H the Focus, whereunto the refracted Rays M H Converge, then the Resistences of the Mediums are Reciprocally in a Ratio compounded of the Ratio of B C to C H,



and of the Ratio of A H to A B; for the Ratio compounded of the Rationes of B C $(b+y)$ to C H $\left(\frac{bny+bbn}{my-ny-nb}\right)$ and of A H $\left(\frac{bmy}{my-ny-nb}\right)$ to A B (y) is as $bbmy+bmyy$ is to $bbny+bnyy$, that is, as m is to n, or as the Sine of the Angle of Incidence is to the Sine of the Refracted Angle, or (*Art.* 162) reciprocally as Densities of the Mediums.

CONSECTARY II.

In a Medium of Air, if the Rays of Incidence B M, B D Diverge from the point B, and be refracted in M, at the Spherico-convex Surface of a Medium of Glass, then is A H $= \frac{3by}{y-2b}$, and consequently the Focus H may be found by this Analogy $\frac{1}{3}$ B C $-$ A C $\left(\frac{y-2b}{3}\right)$: A C (b) :: B H $-$ A H or B A (y) : A H $=$ $\frac{3by}{y-2b}$.

CONSECTARY III.

If the Foci B and H, and A the Vertex (or Pole) of the Glass, be given; the point C, and consequently A C the Semi-diameter of the Refracting Sphere may be found, by this Analogy; 3 B H $-$ A H $\left(\frac{3yy}{y-2b}\right)$: A H $\left(\frac{3by}{y-2b}\right)$ :: B C $-$ A C (y) : A C $= b$.

CONSECTARY IV.

If the Foci B and H, and C the Center of the Refracting Sphere be given, the Semi-diameter of the said Sphere may be found, by this Analogy; 3 B H $-$ B C : B C :: A H $-$ A C or C H : C A.

CONSECTARY V.

In a Medium of Glass, if the Rays of Incidence H M Diverge from the Radiant point H, and be refracted in M, at the Spherico concave Surface of Air, the refracted Rays M B will Converge to the Focus B, whose distance B C from C the Center of the Refracting Sphere is a fourth proportional to $\frac{1}{3}$ A H $-$ A C, A C, and C H;

CH; for $\frac{1}{2}$ AH $-$ AC $\left(\frac{2bb}{y-2b}\right)$: AC (b) :: CH or AH $-$ AC $\left(\frac{2by+2bb}{y-2b}\right)$: BC $= \frac{2bby+2b^3}{2bb} = b+y$.

CONSECTARY VI.

In a Medium of Air, if the Rays of Incidence BM be Refracted in M at the Spherico-convex Surface of a Medium of Glass, then is AH $= \frac{3by}{y-2b}$; and if we imagine C the Center of the Sphere to be at an infinite distance from A. (the Pole of the Glass) then the Refracting Surface will be a Plain, and AH will become $= -\frac{1}{2}y = Ab$, whence 'tis manifest that b the Focus of the Refracted Rays is on the same side of the Plain with the Radiant point B; the Focus may be found by this Analogy, as the Sine of the Refracted Angle (2) is to the Sine of the Angle of Incidence (3) so is BA (y) to A$b = \frac{1}{2}y$.

CONSECTARY VII.

In a Medium of Glass, if the Rays of Incidence BM be Refracted in M, at the Spherico-convex Surface of Air, then is AH $= \frac{2by}{-3b-y} = -\frac{2by}{3b+y}$, and H the Focus of the Refracted Rays is on the same side of the Refracting Surface with the point B, and the said Focus H may be found; saying, 3 AC + AB ($3b+y$) : 2 AB ($2y$) :: AC (b) : AH $= \frac{2by}{3b+y}$.

CONSECTARY VIII.

In a Medium of Glass, if the Rays of Incidence BM be Refracted in M, at the Spherico-convex Surface of Air, then is AH $= -\frac{2by}{3b+y}$; and if the Center C be at an infinite distance from the Vertex A, the Refracting Surface will become a Plain, and AH $= -\frac{2}{3}y = Ab$; and the Distance of the Focus b from A is to AB, as 2 is 3; whence in this Case the Focus of the Refracted Rays falls between A and B, and in the Case of §. 6. the Focus b falls beyond the Radiant point B.

CONSECTARY IX.

In a Medium of Air, if the Rays of Incidence HM Diverge from the point H, and be Refracted in the point M, at the Spherico-concave Surface of Glass, then the distance of the Focus of the Refracted Rays from A (the pole of the Glass) is $= \frac{3by}{y+2b}$, and consequently the point or Focus, unto which the Refracted Rays Converge, is on the same side of the Curve with the Radiant point H, and may be found; saying, AH + 2 AC ($y+2b$) : AC (b) :: 3 AH ($3y$) : the distance of the said Focus from A $= \frac{3by}{y+2b}$.

CONSECTARY X.

In any Lens or Prospective Glass, the Nature of the Glass and the Position of the Radiant point being given, the Focus whereunto the Refracted Rays Converge, may easily be found. For 1°. find the Focus of the Rays Refracted at their Entrance into the

the Glaſs; if the Surface of the Glaſs, expoſed to the Radiant point be plain, or if the Surface of the Glaſs be Spherical, and the Rays of Incidence parallel to the Axis of the Glaſs; or if the Surface of the Glaſs be Spherical, and the Rays of Incidence either Converge or Diverge. And as we thus find the Focus of the Rays Refracted at the Surface of the Glaſs, expoſed to the Radiant point, ſo in like manner may we find the Focus, called the Focus after Emerſion, of the Rays Diverging from this Focus, and Refracted by the ambient Medium, as they paſs out of the Glaſs, *&c.* if there be many Glaſſes we may proceed in ſuch manner from one to another

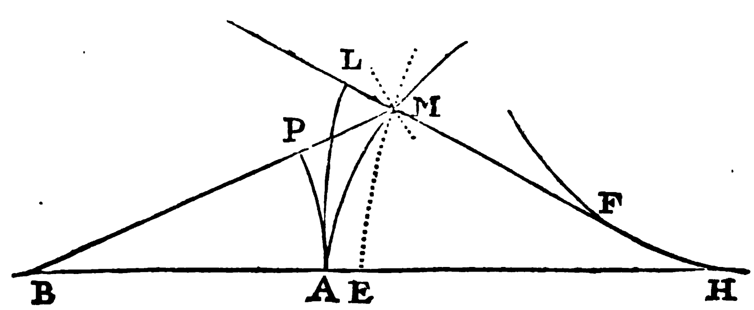
SCHOLIUM.

Many more Corollaries might be deduced from the foreſaid Principles which for brevities ſake, I have here omitted; for he that will conſider all the Caſes, ariſing from the diverſities of Mediums, their various and different Poſitions in reſpect of one another (or in reſpect of the Refracting Surface) the different Poſitions of the Refracting Sphere in reſpect of the Radiating point, and even the greater or leſſer diſtance of the ſaid Radiating point from the Refracting Sphere may eaſily extend this Speculation to an infinite Number of Caſes, which I have not mention'd.

PROP. IV.

The Cauſtick by Refraction H F, *the Luminous point* B, *and the Ratio of* m *to* n *being given; to find an infinite Number of Curves, ſuch as* A M, *to which the given Curve* H F *ſhall be the Diacauſtick.*

318. Draw any Tangent at pleaſure as H A, and aſſume any point therein as A, for one of the points of A M the Curve requir'd; on the Center B with the Radius B A, deſcribe the Arch A P, and with any other Radius B M, deſcribe another ob-

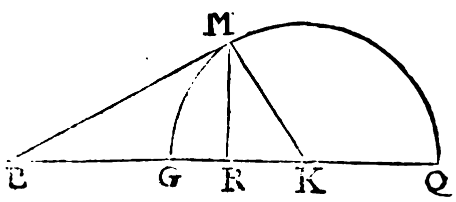


ſcure Arch; then take $AE = \frac{n}{m} PM$, and deſcribe the Curve E M by involving the Cauſtick H F, until it cut the Arch deſcrib'd with the Radius B M in M; then (by Conſtruction) PM or $BM - BA : AE$ or ML or $HA - FM - FH :: m : n$, and conſequently $FH = HA - FM + \frac{n}{m} BA - \frac{n}{m} BM$ (*Art.* 311. §. 2.) and the point M is in the Curve required.

Another

Another Solution.

319. In any other Tangent FM find the point M, so that $HF + FM + \frac{n}{m} BM = HA + \frac{n}{m} BA$, in this manner: Take $FK = \frac{n}{m} BA + AH - FH$, and in the Line FK find the point M, so that $MK = \frac{n}{m} BM$, then M will be the point requir'd.



Now this may be done by describing the Curve GM such that drawing from any point thereof M, the Lines MB, MK, to the points B and K, they shall always be to each other as m is to n.

Draw MR perpendicular to BK, and suppose the known Quantity $BK = a$; the indeterminate Quantities $BR = x$, $RM = y$; then because the Triangles BRM, KRM, are Rectangular, therefore $BM = \sqrt{xx + yy}$ and $KM = \sqrt{aa - 2ax + xx + yy}$, and to answer the Demands of the Problem it is; $\sqrt{xx + yy} : \sqrt{aa - 2ax + xx + yy} :: m : n$, whence there arises this Equation $yy = \frac{2ammx - aamm}{mm - nn} - xx$, which shews that the Locus of the point M is in the Periphery of a Circle, whence there will arise this

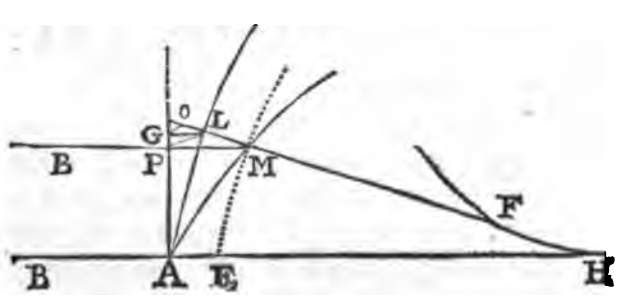
CONSTRUCTION.

Take $BG = \frac{am}{m+n}$, and $BQ = \frac{am}{m-n}$, and on the Diameter GK describe the Semi-circumference GMQ, I say it will be the Locus required; for because $QR = BQ - BR = \frac{am}{m-n} - x$, and $RG = BR - BG = x - \frac{am}{m+n}$, by the property of the Circle $QR \times RG = MRq$, and consequently $yy = \frac{2ammx - aamm}{mm - nn} - xx$.

COROLLARY.

If the Rays of Incidence BA, BM, *be parallel to a right Line, whose position is given; the first Solution will always serve, or in place of the second, the following.*

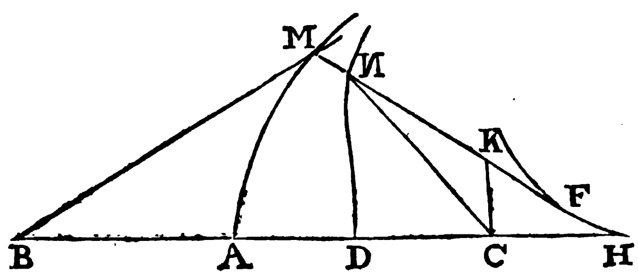
320. Take $FL = AH - HF$, and draw LG parallel to AB and perpendicular to AP; then take $LO = \frac{n}{m} LG$, and draw LP parallel to OG, and PM parallel to GL, then 'tis evident that M is in the Curve requir'd; for $LO = \frac{n}{m} LG$, and $ML = \frac{n}{m} PM$.



PROP. V.

The Curve Line A M, *the Luminous point* B, *and the Ratio of m to n being given; to find an infinite Number of Curves, such as* D N, *which shall refract the refracted Rays* M N, *and make them Converge to any given point* C.

321. If we imagine the Curve F H to be the Caustick by Refraction to A M, the Radiant point being in B, 'tis evident that the same Curve F H ought to be the Caustick by Refraction to the Curve D N, the Luminous point being in C, therefore (*Art.* 311. § 2.) $\frac{n}{m}$ B A + A H = $\frac{n}{m}$ B M + M F + F H; and N F + F H − $\frac{n}{m}$ N C = H D − $\frac{n}{m}$ D C; and consequently $\frac{n}{m}$ B A + A H = $\frac{n}{m}$ B M + M N + H D − $\frac{n}{m}$ D C + $\frac{n}{m}$ N C; and by Transposition $\frac{n}{m}$ B A − $\frac{n}{m}$ B M + $\frac{n}{m}$ D C + A D = M N + $\frac{n}{m}$ N C; whence we have this



CONSTRUCTION.

In any Refracted Ray A H, take the point D at pleasure, for one of the points of the Curve D N required, and in any other Refracted Ray as M F, take M K = $\frac{n}{m}$ B A − $\frac{n}{m}$ B M + $\frac{n}{m}$ D C + A D; and find the point N by *Art.* 319. so that N K be = $\frac{n}{m}$ N C; then the point N will be in the Curve D N requir'd.

General Consectary.

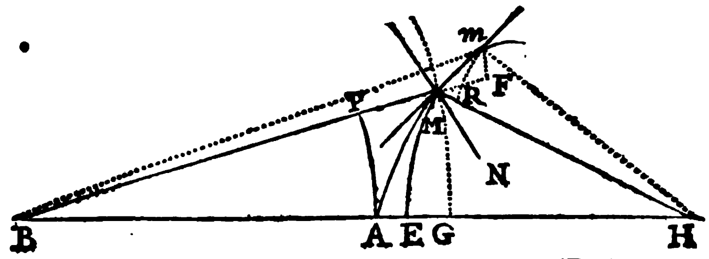
322. It is evident that the same Curve has but one Evoluta, one Caustick by Reflexion, and one Caustick by Refraction, when the Luminous point and the Ratio of *m* to *n* is given; and that when the said given Curve is Geometrical, they are so also; and consequently may be Rectified, and any Curve may be an Evoluta, Caustick by Reflexion or Refraction to an infinite Number of Curves.

SCHOLIUM.

Before I conclude this Section, I think it will not be amiss to acquaint the Young Student further, that it may be requir'd to construct Curves, which shall make Rays, issuing from any given point Converge to another given point, after one single Reflexion or Refraction; and because the Speculation thus enlarged may be of more universal use: I shall briefly shew how such Curves may be describ'd,

323. Let

323. Let B be the Luminous point, from which the Rays B M, B m Diverge; and suppose the Diverging Rays to be Refracted at the Curve A M m, and that all the refracted Rays M H, m H Converge to any point H; let the Rays B M, B m be infinitely near each other, and draw the Tangent M m, and MN perpendicular to the same in M; and let the Sine of the Angle of Incidence F M N be to the Sine of the Refracted Angle H M N as m is to n; draw M R, M F, perpendicular to the Ray of Incidence B M F,



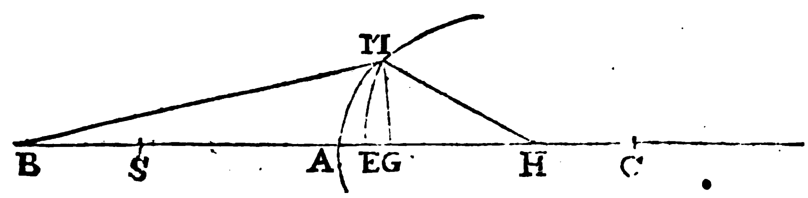
and the refracted Ray MH, then the Angle M m F is = Angle of Incidence F M N, because each being added to the Angle m M F makes a right Angle and (for the like reason) M m R is = the Refracted Angle H M N; therefore if m M be made Radius, M F will be equal to the Sine of the Angle of Incidence, and M R = Sine of the Refracted Angle; and consequently M F : M R :: m : n; that is, the Increment of the Ray of Incidence (B m — B M) is always to the Decrement of the Refracted Ray (H M — H m) as m is to n; and consequently the Sum of all the Increments of the Rays of Incidence, or B M — B A or M P = A G, is to the Sum of all the Decrements of the Refracted Rays, or H A — H M or A E, as m is to n, whence if both the Foci B and H, and A the Vertex of the Curve A M requir'd, be given, the Curve A M may be describ'd thus.

Take A G at pleasure, then say $m : n :: AG : \frac{n}{m} AG = AE$, then on the Center B with the Radius B G, describe the Arch B M G, and on the Center H, with the Radius H E describe another Arch E M, intersecting G M in M, and the point M will be in the Curve A M requir'd.

324. And because the Velocities of the Particle of Light, before and after Refraction (*Art.* 163. § 1°.) are proportional to the respective Facilities of the Mediums, and that the times are directly as the Spaces, and reciprocally as the Velocities; the times which the particle of Light takes to describe B A and A H, may be represented by $BA \times n + AH \times m$, and the times which any other particle of Light takes to describe B M and M H, will also be represented by $BP \times n + PM \times n + MH \times m = BA \times n + PM \times n + AH \times m - AE \times m =$ (because $PM : AE :: m : n$, and $PM \times n = AE \times m$) $BA \times n + AH \times m$; whence it is manifest, that particles of Light Diverging from B and Converging to any given point H, will describe the Lines B A, AH; B M, M H; B m, m H, &c. in equal times: and hence we have another Method for describing such Curves.

 Make

Make $AC = \frac{m}{n} AH$, and having describ'd on the Center B any Arch MG intersecting AH in G, on the Center H, and with the Semi-diameter $HE = \frac{n}{m} GC$, des-



cribe another Arch EM intersecting MG in M; I say the point M is in the Curve requir'd.

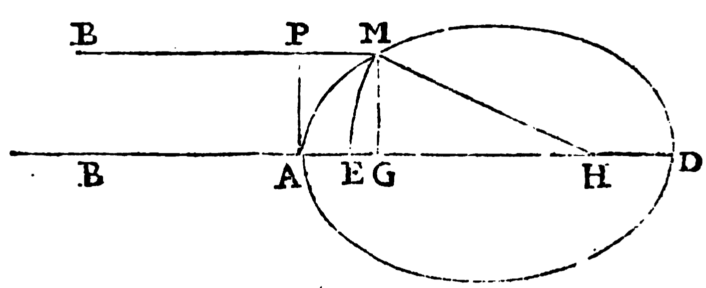
For if AB represent the time that the Particle takes to move from B to A, then $AC = \frac{m}{n} AH$ will represent the time the Particle takes to move from A to H, and consequently BC will represent the whole time the Particle takes to move from B to H; in like manner, BM or BG will represent the time which the Particle takes to move from B to M, and GC (being by Construction $= \frac{m}{n} MH$) will represent the time from M to H, and consequently BC will represent the time which the Particle of Light takes to describe $BM + MH$; therefore the point M is in the Curve requir'd.

325. The point M may also be found, if AB be divided in S, so that $AS = \frac{n}{m} AB$; and if on the Center H with any Radius, the Arch ME be describ'd, intersecting AH in E, and on the Center B with the Radius $AG = \frac{m}{n} SE$, another Arch GM be describ'd intersecting the first Arch in the point M requir'd.

326. It is also manifest, that if m be $= 3$, and $n = 2$, then $AG = \frac{3}{2} AE$ and the Curve AMm is one of *Cartesius's Ovals*; and if we suppose the point B or H to be at an infinite distance from A, or both to be on the same side of the Curve AMm, then we shall have all those Figures which the forenamed Excellent Person has treated of (in Relation to Refractions) in his Geometry and Dioptricks.

For instance, if the Rays of Incidence BM be parallel to one another, then the Arch GM will become a streight Line perpendicular to the Axis AH, and the Curve AMD may be constructed by any of the preceeding Methods.

And the said Curve AMD will be a perfect Ellipsis, describ'd so that the Transverse Axis AD is to the Distance between the Foci, as (supposing $m = 3$ and $n = 2$)

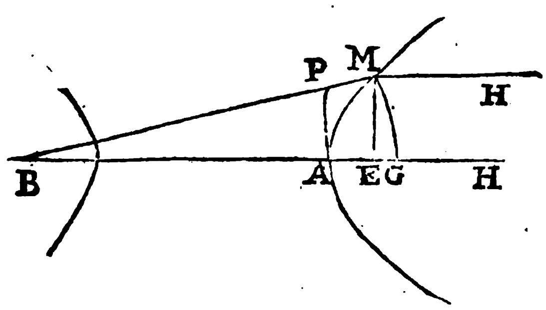


3 is to 2; that is, as the Sine of the Angle of Incidence is to the Sine of the Angle of Emergence: for if we suppose the known Quantity $AH = a$, the perpendicular $AP = MG = y$, $PM = AG = x$, and $GH = a - x$, then $MH = $

$\sqrt{aa}$

$\sqrt{aa-2ax+xx+yy}$; now the Nature of the Curve is such that $\frac{2}{3}$ PM + MH = AH, whence we have this Equation $\frac{2}{3}x+\sqrt{aa-2ax+xx+yy}=a$, and by Transposition, $\sqrt{aa-2ax+xx+yy}=a-\frac{2}{3}x$; and by Involution $aa-2ax+xx+yy=aa-\frac{4}{3}ax-\frac{4}{9}xx$, and consequently $yy=\frac{2}{3}ax+\frac{5}{9}xx$, and $\frac{9}{5}yy=\frac{6}{5}ax-xx$; that is, if AD be made $=\frac{6}{5}$ AH, then the the Rectangle AGD is $=\frac{9}{5}$ the Square of the Ordinate GM; whence it is evident that the Curve AMD is an Ellipsis, and the Transverse Axis AD is to the Parameter as 9 is to 5, and consequently the Square of AD is to the Square of the Distance between the Foci, as 9 is to 9 − 5, that is as 9 is to 4; and the Transverse Axis AD is to the Distance between the Foci, as 3 is to 2.

And if we suppose the point H to be at an infinite distance from A, then the Curve AM will be an Hyperbola, and the Luminous point B will be in the Focus of the opposite Section; or if we suppose the Rays of Incidence HA, HM parallel to one another, to be Refracted at the Concave Surface (so that the Sine of the Angle of Incidence be to Sine of the Refracted Angle as 2 is 3) of an Hyperbola AM, they will Converge to the Focus of the opposite Section; and the Transverse Axis is to the distance between the Foci, as the Sine of the Angle of Incidence (2) is to the Sine of the Angle of Emergence. (3).



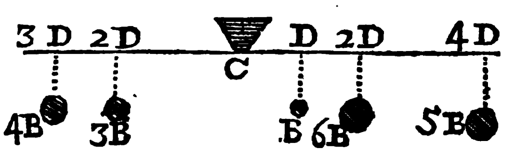
SECT.

SECT. XI.

The Use of Fluxions

In Investigating the Centers of Gravity of Lines, Surfaces and Solids.

313. *The consideration of the Center of Gravity, has always been esteemed one of the Noblest Speculations in Geometry, and the Dignity of the Subject has so influenced the Minds of the best and latest Geometers, that their productions have been answerable; and indeed since the admirable Discovery of a* more Sublime Analysis, *the advancements this way have been such, that we can hardly expect greater, the whole being reduced to one* general Proposition, *which depends on a few simple Mechanick Principles, such as,*



I. 3 D, C, 4 D, Represent a Ballance, and C the point of Suspension, and if the Burdens 4 B and 6 B, be so applied to the Ballance, that their Masses be reciprocally proportional to their distances from C the point of Suspension; the Burdens will rest in *Equilibrio*, or exactly poize each other; thus if D represent the Distance of B from the point of Suspension, 3 D that of 4 B, and 2 D that of 6 B, then the Burdens 4 B and 6B mutually poize each other, for 4 B : 6 B :: 2 D : 3 D.

II. The Momentum of any Burden is equal to the Rectangle comprehended under its Velocitiy, and the Quantity of matter in the same; thus the Momentum of the Burden 6 B is $= 6\,B \times 2\,D$, its distance from the point of Suspension $= 12\,BD$.

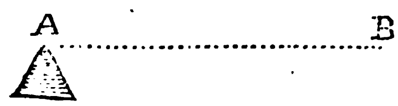
III. And if the Momentum and Weight of any Burden (or Quantity of matter in the same) be given, the Distance of the point of Application from the point of Suspension is found by dividing the *Momentum* of the Burden by the Weight or Quantity of matter in the same; thus if a given Burden be 6 B, and the Momentum thereof $= 12\,BD$, then the distance of the point of Application from the point of Suspension is $= \frac{12\,BD}{6\,B} = 2\,D$.

☞ IV. If several Burdens be Suspended on each side of the point of Suspension; multiply every Burden by its respective distance from the point of Suspension, then if the Sum of all the Rectangles on one side be equal to the Sum of all those on the other, the Burdens will be in *Equilibrio*; if not, that side will preponderate whose Sum is the greater; thus in the Example, the Sum of all the Rectangles towards the left hand of C is = (the Sign $-$ denoting towards the left hand, and the Sign $+$ denoting towards the right Hand) $-18\,BD$, and the Sum of all the Rectangles towards the right Hand of C is $= +33\,BD$, whence it is evident that the preponderancy is towards the right Hand, and is $= 15\,BD$ = to the Momentum of all the Burdens.

V. The Momentum of all the Burdens being $= +15\,BD$; and the Sum of all the Burdens $= 19\,B$; it is evident that if that be divided by this, the Quotient $\frac{15}{19}\,D$ = (*Princip.* 3.) the *distance of the common Center of Gravity* of all the Burdens from the point of Suspension C.

VI. And if the Burdens B, 6 B and 5 B be suspended on the same side of C, then the sum of their Momenta is = 33 BD, and the Sum of the Burdens is = 12 B, therefore a Burden = 12 B, and suspended $\frac{33}{12}$ D distant from C the point of Suspension, will gravitate in the same manner as the seperate Burdens do now at their respective distances; that is the said point is the Center of Gravity of the Burdens, for the *common Center* of *Gravity* of many Burdens is that point in which all their Forces unite, and whereat if they be all joyntly suspended, they will produce the same effect as before they did seperately.

VII. If we suppose the Line A B to be suspended at A, and if the Line be divided into an infinite Number of heavy points; it is evident, that the points, the further they are from A, the more they Gravitate, and the Momentum of every point is equal to the Rectangle comprehended under its distance from A the point of Suspension, and it self or Unity; and consequently the *Momentum* of all the *Points* is = to all the said Rectangles; and if the said *Total Momentum* be divided by the *Total Gravity* of all the points; that is, by the Gravity of the whole Line A B, the Quotient will be equal (*Princip.* 3, and 6.) to the distance of a certain point from A, at which if all the points be suspended, their Momentum will be the same as it is now; that is, that point will be the *common Center of Gravity* of the Line A B.



VIII. If a Line or a Plain or a Solid be divided into two halves, by a Line or by a Plain, so that all the parts in one Segment be equal to the respective parts of the other, and equidistant from the said Line or Plain, then 'tis evident that the Center of Gravity of all such Figures, must be in that Line or Plain. Hence it naturally follows that,

328. To find the Center of Gravity of any Line, Plain, or Solid; imagine Lines to consist of an infinite number of Points, Plains, of an infinite Number of Lines, and Solids of an infinite Number of Plains; and suppose all the said parts to be suspended to the same Arm of a Ballance common to all; and let the point of suspension be in the Extreme point of the Line, edge of the Surface, or Surface of the Solid; find the Sum of the Momenta of all those parts, which divide by the Sum of the Weights, or the Weight of all the parts; the Quotient is the distance of the Center of Gravity of the Line, Plain, or Solid, from the point or Axis of Suspension.

PROP. I.

To find the Center of Gravity of a Line.

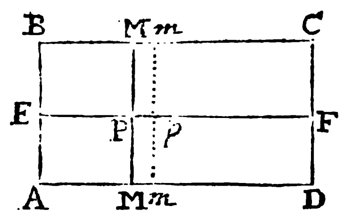
329. Let the Line A B be $= x$, and one of the parts thereof, infinitely little b B $= \dot{x}$, then the Momentum of the Portion B b is $= x\dot{x}$ (by *Princip.* 2.) that is, $x\dot{x}$ is the Fluxion of the Moments, and the Flowing Quantity or the Sum of all the Moments is $= \frac{xx}{2}$

which being divided by the Sum of the Weight $s = x$, the Quotient $= \frac{1}{2}x$, is the distance of the Center of Gravity of the Line A B, from A the point of Suspension; that is, the Center of Gravity of A B is in the middle point between A and B.

PROP. II.

To find the Center of Gravity of the Parallelogram ABDC.

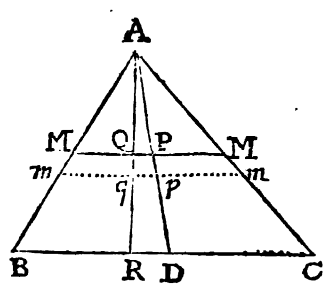
330. It is evident (by *Princip.* 8.) that the Center of Gravity of the Parallelogram AC must be in the Line EF, which divides the same into two equal parts, which is further confirmed, because the Centers of Gravity of all the Lines MM, *mm*, are in the points P, *p*, wherein the Line EF bisects them; now supposing the point of Suspension in E, and AD or EF $= a$, EP $= x$, P*p* $= \dot{x}$, AB or MM $= b$, the Momentum of the infinitely little Parallelogram M*m*, or the Fluxion of the Moments is $= bx\dot{x}$, and the Flowing Quantity or the Sum of all the Moments is $= \frac{bxx}{2}$, which being divided by $bx =$ Parallelogram BM, or the Sum of all the Weights; the Quotient $\frac{1}{2}x$ is $=$ to the distance of the Center of Gravity of the Parallelogram AM from E, that is $\frac{a}{2}$ is $=$ to the distance of the Center of Gravity of the whole Parallelogram AC from E, because then x becomes $= a$.



PROP. III.

To find the Center of Gravity of any Triangle.

331. Let any Triangle as ABC be given, and from the Vertex A draw the Line AD ($= a$) dividing the Triangle into two equal parts; then 'tis evident (by *Princip.* 8.) that the Center of Gravity of the Triangle must be in that Line; draw the Line AR ($= b$) perpendicular, and MM (y) parallel to the Base BC $= b$; and draw *m m* parallel and infinitely near to MM; and suppose AP $= x$, AQ $= z$, and Q*q* $= \dot{z}$; then Supposing the point of Suspension in A, the Fluxion of the Weights; that is, the infinitely little Parallelogram M*m* is $= y\dot{z}$; and because the Triangles AQP, ARD are similar; it is, $z : x :: b : a$, and $z = \frac{bx}{a}$ and $\dot{z} = \frac{b\dot{x}}{a}$, therefore the Fluxion of the Weights $y\dot{z}$ is $= \frac{by\dot{x}}{a}$, and because the Triangles MAM, BAC are similar; y is $= \frac{bx}{a}$, and consequently $\frac{by\dot{x}}{a}$ is $= \frac{bbx\dot{x}}{aa}$, which multiplyed by AP $= x$, the Product $\frac{bbxx\dot{x}}{aa}$ is $=$ to the Fluxion of the Moments; and the Fluent or the Sum of all the Moments is $= \frac{bbx^3}{3aa}$; which being divided by the Sum of the Weights, or the Triangle AMM $= \frac{bbxx}{2aa}$; the Quotient $= \frac{2}{3}x$ is $=$ to the distance of the Vertex A from the Center of Gravity of the Triangle AMM; and when P falls in D, then x is $= a$, and the distance of the Center of Gravity of the whole Triangle ABC from the Vertex A is $= \frac{2}{3}a$.



COR-

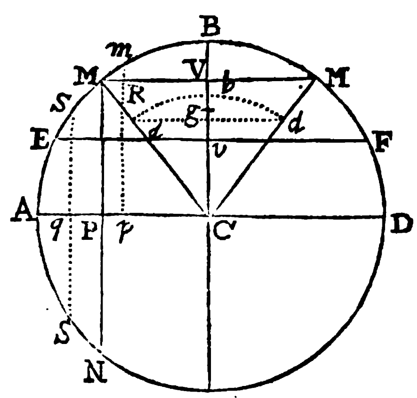
COROLLARY.

332. If the Triangle A B C be an Isosceles Triangle, then the Lines A D and A R will coincide, and the Distance of the Center of Gravity of the Isosceles Triangle, from the Vertex A is $= \frac{2}{3}$ the perpendicular let fall from the said Vertex on the Base.

PROP. IV.

To find the Center of Gravity of an Arch of a Circle.

333. Let the Arch of the Circle EBF, whose Center of Gravity is requir'd, be less than the Semi-circumference; then 'tis evident, that the Center of Gravity of that Arch, must be in the Ray which bisects the same: For if an infinite Number of Chords be drawn Parallel to EF, they will divide the whole Arch EBF into an infinite Number of equal Arches, which we may consider as so many Weights applied to the Extremities of the Chords drawn parallel to EF, and bisected by the Ray CB, which also bisects the given Arch.



And to find the Center of Gravity of the Arch EBF, in the Ray CB; let the Diameter AD be drawn parallel to the Chord EF, and draw the Ordinate MP, and another Ordinate mp infinitely near the same, and draw the Radius CM. Then suppose $AD = 2r$, $EF = 2a$, the Arch $EBF = 2c$, the Ordinate $MP = y$, $CP = x$, $Pp = \dot{x}$; the Arch $BM = z$, and $Mm = \dot{z}$. Then if we suppose AD to be the Axis of Suspension, the Distance of the infinitely little Arch Mm from the same is $= PM = y$, and consequently the Momentum thereof is $y\dot{z}$. Now because the Triangles CPM, MRm are similar, $\dot{z}$ is $= \frac{r\dot{x}}{y}$, therefore $y\dot{z}$ is $= r\dot{x} =$ to the Fluxion of the Moments, in respect of AD the Axis of Motion. And consequently the Sum of all the Moments is $= rx$, which being divided by the Sum of all the Weights $BM = z$, the Quotient $\frac{rx}{z}$ gives the Distance of the Center of Gravity of the Arch BM from the Axis of Motion AD, and when the Arch BM becomes $= BE$, then $z = c$, and $x = a$, and $\frac{rx}{z} = \frac{ra}{c} =$ to the Distance of the Center of Gravity of the Arch EBF from the Axis of Motion AD.

CONSECTARY I.

334. Hence 'tis evident, That the distance of the Center of Gravity of an Arch from the Center of the Circle, is to the Radius of the Circle, as the Chord of that Arch is to the Arch, for $\frac{ar}{c} : r :: a : c :: 2a : 2c$.

CON-

CONSECTARY II.

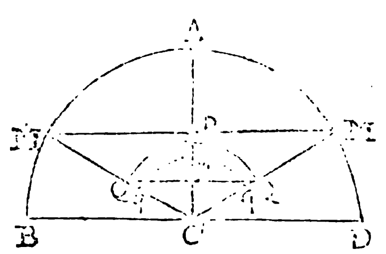
If the Ratio of the Chord of an Arch to the Arch it self be given, the Center of Gravity of that Arch may be found; and contrarily, if the Center of Gravity of an Arch of a Circle be given, the Ratio between the Chord and the Arch, and consequently the Ratio between the Diameter and Semi-circumference may be found, which compleats the Quadrature of the Circle.

PROP. V.

To find the Center of Gravity of the Sector of a Circle.

This might be perform'd by Cor. Prop. 3, *and* Prop. 4. *But I chuse rather to do it independently of either; in this manner:*

335. Let MCM be the Sector of a Circle, whose Center of Gravity is requir'd. Draw the Line CA bisecting the given Sector; then 'tis evident that the Center of Gravity of the Sector must be in the same. On the Center C describe any Arch QPQ, and draw the Chords MM, QQ. Then suppose the Radius CQ $= x$, and draw another Arch qpq infinitely near QPQ, and then Q$q = \dot{x}$.



Now (*Art.* 331.) the Momentum of any Arch MAM in respect of the Axis of Motion BD, is equal to the Radius of the Arch multiplied into the Chord of the Arch, therefore the Momentum of one Arch is to the Momentum of another Arch, in a Ratio compounded of the Rationes of their Semi-diameters and Chords; that is, (because the Chords are as the Radii or Semi-diameters) in a Duplicate Ratio of their Semi-diameters. Whence $\overline{CM}^2$ (rr) : $\overline{CQ}^2$ (xx) :: Moment of MAM $(2ar)$: Moment of QPQ $= \frac{2axx}{r}$, which being multiplied by $\dot{x} = $ Qq, the Product $\frac{2axx\dot{x}}{r}$ is equal to the Moment of the infinitely little Annulus QPQqpq, and the Flowing Quantity is $= \frac{2ax^3}{3r} =$ to the Sum of all the Moments of all the Annuli that compose the Sector QCQ. Which being divided by $\frac{cxx}{r} =$ the Value of the Sector QCQ, the Quotient $\frac{2ax}{3c}$ is $=$ to the Distance of the Center of Gravity of the Sector QCQ from the Center C, and when Q falls in M, then x will become $= r$; and $\frac{2ax}{3c} = \frac{2ar}{3c} =$ to the Distance of the Center of Gravity of the Sector MCM from the Center C.

CONSECT-

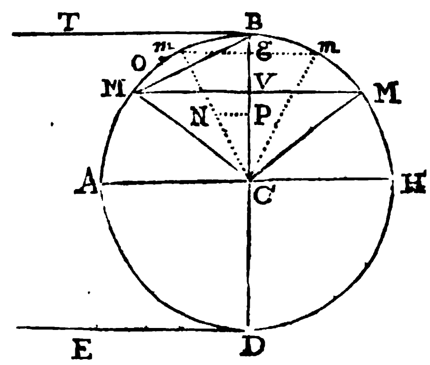
CONSECTARY I.

536. The distance of the Center of Gravity of any Sector of a Circle from the Center is to $\frac{2}{3}$ parts the Radius, as the Chord of the Arch is to the Arch it self: for $\frac{2ar}{3c} : \frac{2}{3}r :: a : c :: 2a : 2c.$

CONSECTARY II.

Hence if the Center of Gravity of the Sector of a Circle MCM, and of the Triangle MCM be given, the Center of Gravity of the Segment MAM may easily be found: for if the Momentum of the Triangle be Subtracted from the Momentum of the Sector, the remainder is the Momentum of the Segment MAM, which being divided by the Segment MAM, the Quotient is the distance of the Center of Gravity of the said Segment from the Center C.

For instance, the distance of the Center of Gravity of the Sector MCMB, from the Axis of Motion DE is $= r + \frac{2a}{3c}r = \frac{3c+2a}{3c}r$, and the distance of the Center of Gravity of the Sector MCMB from the Axis of Motion BT is $= r - \frac{2a}{3c}r = \frac{3c-2a}{3c}r$.



The Sector MCM is $= rc$, therefore the Momentum thereof in respect of DE is $= \frac{3c+2a}{3c}r \times rc = \frac{3c+2a}{3}r^2$, and the Momentum of the same Sector, in respect of the Axis of Motion BT is $= \frac{3c-2a}{3c}r \times rc = \frac{3c-2a}{3}r^2$.

The Center of Gravity of the Triangle MCM being in CV, is distant from C, $\frac{2}{3}CV = \frac{2}{3}r - \frac{2}{3}v$ (supposing BV = o) and the Area of the Triangle is $ra - va$, whence the Momentum of the Triangle MCM in respect of DE is $= \overline{\frac{5}{3}r - \frac{2}{3}v} \times \overline{ra - va} = \frac{5ar^2 - 5rva - 2rva + 2av^2}{3} = \frac{5ar^2 - 3avr - 4avr + 2av^2}{3}$ $= \left(\text{because } 2vx = a^2 + v^2\text{, and } vr = \frac{a^2+v^2}{2}\right) \frac{5ar^2 - 3avr - 2a^3}{3}$, and the Momentum of the Triangle MCM in respect of BT is $= \overline{\frac{1}{3}r + \frac{2}{3}v} \times \overline{ra - va}$ $= \frac{ar^2 + avr - 2av^2}{3} = \frac{ar^2 - 3avr + 4avr - 2av^2}{3} = \frac{ar^2 - 3avr + 2a^3}{3}$.

And the Momentum of the Sector *Minus* the Momentum of the Triangle = to the Momentum of the Segment MBM, in respect of DE, is $= \frac{3c+2a}{3}r^2 - \frac{5ar^2 - 3avr - 2a^3}{3} = cr^2 - ar^2 + avr - \frac{2}{3}a^3$, and in respect of BT the Momentum of the said Segment MBM is $= \frac{3c-2a}{3}r^2 - \frac{ar^2 - 3avr + 2a^3}{3} = cr - ar^2 + avr - \frac{2}{3}a^3$.

And the Momentum of the Semi-segment M B V in respect of $\left\{\begin{matrix} DE \\ BT \end{matrix}\right\}$ is $\left\{\begin{matrix} \frac{1}{2}cr^2 - \frac{1}{2}ar^2 + \frac{1}{2}avr + \frac{1}{3}a^3 \\ \frac{1}{2}cr^2 - \frac{1}{2}ar^2 + \frac{1}{2}avr - \frac{1}{3}a^3 \end{matrix}\right\}$ and dividing the Momentum of the Segment MBM in respect of DE, by the Segment it self, that is dividing $cr^2 - ar^2 + avr + \frac{2}{3}a^3$ by $cr - ra + va$, or dividing $3cr^2 - 3ar^2 + 3avr + 2a^3$ by $3cr - 3ra + 3va$, there will arise $r + \frac{2a^3}{3cr - 3ra + 3va}$ for the distance of the Center of Gravity of the Segment M B M, or the Semi-segment M B V from D E; and the distance of the Center of Gravity of the said Segment or Semi-segment from the Center C is $= \frac{2a^3}{3cr - 3ra + 3va}$, and the distance of the Center of Gravity of the said Segment or Semi-segment from B T is $= r - \frac{2a^3}{3cr - 3ra + 3va}$ and when V comes to D, then $a = 0$, and consequently $r - \frac{2a^3}{3cr - 3ra + 3va}$ is $= r =$ to the distance of the Center of Gravity of the Circle B M D M or the Semi-circle B M A D from B T.

CONSECTARY III.

And to find the distance of the Center of Gravity of the Sector M D M B from D E or B T.

The distance of the Center of Gravity of the Triangle M D M from D E is $= \frac{4}{3}r - \frac{2}{3}v$, and the distance of the said Center from B T is $= \frac{2}{3}r + \frac{2}{3}v$, which distances being multiplyed into the Area of the Triangle M D M $= 2ar - va$, the Products $\left(\frac{8ar^2 - 4avr - 4avr + 2av^2}{3} = \frac{8ar^2 - 8avr + 2av^2}{3} = \right) \frac{8}{3}ar^2 - \frac{8}{3}avr - \frac{2}{3}a^3$, and $\left(\frac{4ar^2 - 2avr + 4avr - 2av^2}{3} = \frac{4}{3}ar^2 + \frac{2}{3}avr - \frac{2}{3}av^2 = \right) \frac{4}{3}ar^2 - \frac{2}{3}avr + \frac{2}{3}a^3$, are equal to the Momenta of the said Triangle M D M in respect of D E and B T respectively.

Now the Momentum of the Segment M B M in respect of $\left\{\begin{matrix} DE \\ BT \end{matrix}\right\}$ is $\left\{\begin{matrix} cr^2 - ar^2 + avr + \frac{2}{3}a^3 \\ cr^2 - ar^2 + avr - \frac{2}{3}a^3 \end{matrix}\right\}$ to which add the Momenta of the Triangle M D M in respect of D E and B T, then the Momentum of the Sector M D M B in respect of $\left\{\begin{matrix} DE \\ BT \end{matrix}\right\}$ will be $\left\{\begin{matrix} cr^2 + \frac{2}{3}ar^2 - \frac{1}{3}avr \\ cr^2 + \frac{1}{3}ar^2 + \frac{1}{3}avr \end{matrix}\right\}$ respectively.

And if the Momentum of the Sector M D M B, in respect of D E, *viz.* $cr^2 + \frac{2}{3}ar^2 - \frac{1}{3}avr$, be divided by the Area of the said Sector, *viz.* $cr + ar$, the Quotient $r + \frac{2ar - av}{3c + 3a}$ is = to the distance of the Center of Gravity of the Sector M D M B (or of the Semi-sector M D A) from the Axis of Motion D E.

And the distance of the Center of Gravity of the said Sector M D M B from the Axis of Motion B T is $= r - \frac{2ar - av}{3c + 3a}$.

And the distance of the Center of Gravity of the Sector M D M B above the Center C is $= \frac{2ar - av}{3c + 3a}$.

CONSECTARY IV.

The Triangle BMV is $= \frac{1}{2}av$, and the Distance of the Center of Gravity thereof from BT is $= \frac{2}{3}v$, and from DE $= 2r - \frac{2}{3}v$, and therefore the Momenta of the said Triangle BMV, in respect of BT and DE are $\frac{1}{3}av^2 =$ (because $2rv - aa = v^2$) $\frac{2}{3}avr - \frac{1}{3}a^3$, and $avr - \frac{1}{3}av^2 = \frac{1}{3}avr + \frac{1}{3}a^3$, respectively.

And subtracting the Momenta of the said Triangle BMV from the respective Momenta of the Semi-segment BMV, there will remain the Momenta of the Segment BOMB, in respect of $\left\{\begin{matrix} BT \\ DE \end{matrix}\right\}$ viz. $\left\{\begin{matrix} \frac{1}{2}cr^2 - \frac{1}{2}ar^2 - \frac{1}{6}avr \\ \frac{1}{2}cr^2 - \frac{1}{2}ar^2 + \frac{1}{6}avr \end{matrix}\right\}$ which being divided by the Area of the Segment BOMB $= \frac{1}{2}cr - \frac{1}{2}ar$, there will come in the Quotient, the Distance of the Center of Gravity of the Segment BOMB, from $\left\{\begin{matrix} BT \\ \\ DE \end{matrix}\right\}$ viz. $\left\{\begin{matrix} r - \dfrac{av}{3c - 3a} \\ r + \dfrac{av}{3c - 3a} \end{matrix}\right\}$, and the Distance of the said Center from AC is $= \dfrac{av}{3c - 3a}$.

CONSECTARY V.

And to find the Distance of the Center of Gravity of the Sector MCB from the Axis of Motion BD; bisect the Arches BM, BM, in m, and m, and draw the Chord $mgm =$ the Chord MB, which suppose $= d$, then $mg = \frac{1}{2}d$, and the Center of Gravity of the Sector MCB is in Cm, and the Distance thereof from the Center C is $=$ CN $= \dfrac{2dr}{3c}$, and because the Triangles CNP, Cmg, are similar; therefore Cm (r) : mg $(\frac{1}{2}d)$ :: CN $\left(\dfrac{2dr}{3c}\right)$: NP $= \dfrac{d^2}{3c} = \dfrac{2vr}{3c} =$ to the Distance of the Center of Gravity of the Sector MCB from the Axis of Motion BD.

CONSECTARY VI.

And the Distance of the Center of Gravity of the Quadrant BAC from the Axis of Motion BC is $=$ (because v becomes $= r$,) $\dfrac{2r^2}{3c} =$ (supposing C $=$ the whole Periphery of the Circle $= \dfrac{8r^2}{3C}$.

And the Distance of the Center of Gravity of the Semi-circle BAD from the Axis of Motion BD is $=$ (because v becomes $= 2r$, and $c = \frac{1}{2}$C) $\dfrac{8r^2}{3C}$.

CONSECTARY VII.

The Momentum of the Sector MCB in respect of BD is $= \frac{1}{2}cr \times \dfrac{2vr}{3c} = \frac{1}{3}vr^2$, and the Momentum of the Triangle MCV in respect of BD is $= \frac{1}{3}a \times \overline{\frac{1}{2}ar - \frac{1}{2}av} = \frac{1}{6}ra^2 - \frac{1}{6}va^2$, and consequently the Momentum of the Semi-segment BMV in respect of BD is $= \frac{1}{3}vr^2 - \frac{1}{6}ra^2 + \frac{1}{6}va^2$. Which being divided by $\frac{1}{2}cr - \frac{1}{2}ar + \frac{1}{2}av$, the Area of the Semi-segment MBV, there will arise $\dfrac{\frac{1}{3}vr^2 - \frac{1}{6}ra^2 + \frac{1}{6}va^2}{\frac{1}{2}cr - \frac{1}{2}ar + \frac{1}{2}av}$, for the Distance of the Center of Gravity of the Semi-segment BMV from the Axis of Motion AD.

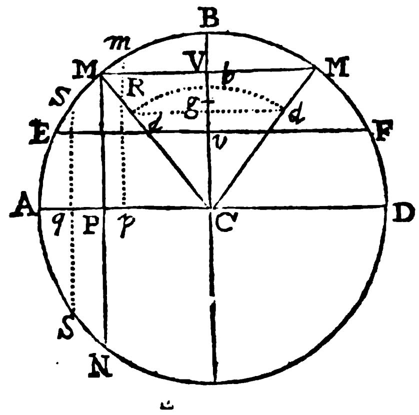
CON-

CONSECTARY VIII.

In similar Sectors the Center of Gravity divides the Rays in which they are Scituated, in the same proportion.

CONSECTARY IX.

And because Portions of Spherical Surfaces are Generated by the Semi-revolution of an Arch about the Semi-diameter bisecting the same, I chuse here to determine the Centers of Gravity of all Spherical Surfaces *v. g.* If the Center of Gravity of the Spherical Surface MAN be requir'd, it is evident, that it must be somewhere in the Sector NAMC, which bisects the same, and it must be in the Line AC which bisects the Sector; and in the point q, which bisects AP, because all the infinitely little Annuli on each side of q, are equal and Equi-distant from q; and if the Center of Gravity of the Spherical Surface generated by the Revolution of the Arch MB about the Diameter AD be requir'd; by a like Argumentation it may be prov'd to be in the middle point between P and C, and the Center of Gravity of the Surface of an Hemisphere is Distant from the Center $\frac{1}{2}$ the Semi-diameter of the Sphere.



PROP. VI.

To find the Centers of Gravity of all sorts of Paraboliform Figures.

337. Let the general Equation expressing the Nature of all sorts of Paraboliform Figures be $y^m = x$, then is $y = x^{\frac{1}{m}}$, and the Fluxion of the Weights $y\dot{x}$ is $= x^{\frac{1}{m}}\dot{x}$, which multiplied by AP or x (the Distance of the Weight Mm from the Axis of Motion AT) the Product $x^{\frac{1}{m}+1}\dot{x}$ is $=$ to the Fluxion of the Moments, and consequently the Sum of all the Moments in the Parabolic Space MAM is $= \frac{m}{2m+1}x^{\frac{1}{m}+2}$. Which being divided by the Sum of all the Weights $\frac{m}{m+1}x^{\frac{1}{m}+1}$; the Quotient $\frac{m+1}{2m+1}x$ is $=$ to the Distance of the Center of Gravity of the Parabolick Space MAM from the Vertex A, and consequently when x becomes $= b$, then the Distance of the Center of Gravity of BAB from the Vertex A is $= \frac{m+1}{2m+1}b$.

CONSECTARY I.

338. Hence in the common Parabola, $yy = x$, and $m = 2$, therefore $\frac{m+1}{2m+1}b = \frac{3}{5}b = \frac{3}{5}$ AD $=$ to the Distance of the Center of Gravity of the Parabola BAB from the Vertex A.

CONSECTARY II.

And in the Cubical Parabola, where $y^3 = x$, and $m = 3$, and $\frac{m+1}{2m+1}b = \frac{4}{7}b = \frac{4}{7}AD$; and if m be $= 4$, the said distance will be $= \frac{5}{9}b = \frac{5}{9}AD$.

CONSECTARY III.

If m be $= \frac{1}{2}$, then the Equation expressing the Nature of the Curve is $y^{\frac{1}{2}} = x$; or $y = xx$; whence the Line A T which touches the Vertex, becomes the Axis of the Curve, and $\frac{m+1}{2m+1}b$ is $= \frac{3}{4}b =$ to the distance of the Center of Gravity of the Concave Space A B T from the Axis of Motion A D, taken in the Axis A T.

PROP. VII.

To find the Centers of Gravity of all sorts of Hyperbolic Spaces, comprehended between the Curve and the Asymptotes.

339. The same things being supposed as in *Art.* 104. 105. the Fluxion of the Weights is $= x^{\frac{1}{m}}\dot{x}$, which being multiplied by x or C P = to the distance of the infinitely little Parallelogram P m from the Axis of Motion C E, the Product $x^{\frac{1}{m}+1}\dot{x}$ is the Fluxion of the Moments, and the Flowing Quantity or the Sum of all the Moments is $\frac{m}{2m+1}x^{\frac{1}{m}+2}$; which being divided by the Sum of all the Weights $\frac{m}{m+1}x^{\frac{1}{m}+1}$; the Quotient $\frac{m+1}{2m+1}x$ is = to the distance of the Center of Gravity of the Hyperbolic Surface E C P M F from the Axis of suspension C E; and when P falls on B, then C P = C B = b, $\frac{m+1}{2m+1}x$ is $= \frac{m+1}{2m+1}b$.

And because in such Cases the Value of m is Negative, therefore the distance of the Center of Gravity from C E the Axis of Motion is $= \frac{-m+1}{-2m+1}b = \frac{m-1}{2m-1}b$.

And if m be $= -1$, then the Figure is the common Hyperbola, and the distance of the Center of Gravity from C E the Axis of suspension is $= \frac{0}{1}b$; that is, the distance of the said Center from C E is infinitely little.

And if the Equation of the Curve be $y^{-3} = x$, then $m = -3$ and $\frac{m-1}{2m-1}b$ is $= \frac{2}{5}b =$ to the distance requir'd,

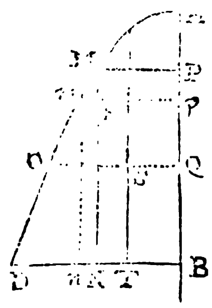
And if the Nature of the Curve be expressed by $y^{-2} = x$, then $m = -2$, and $\frac{m-1}{2m-1}b$ is $= \frac{1}{3}b =$ to the distance of the Center of Gravity from C E.

N B. *In Calculating the Centers of Gravity of Hyperbolical and Parabolical Surfaces, we suppose similar and equal Spaces on each side of the Line, wherein the Center of Gravity is found.*

PROP. VIII.

To find the Center of Gravity of a Semi-parabola.

340. Let A M D B be a Semi-parabola, then if A Q be taken $= \frac{2}{5}$ A B, and Q O drawn parallel to B D it is evident that the Center of Gravity of the Semi-parabola must be in (Art. 338.) Q O: it only remains to find the distance of the said Center from A B, which may be done thus: Imagine A B to be the Axis of Suspension; and draw the Ordinate M P $= y$, and $m\,p$ infinitely near and parallel to the same; and draw the Lines M N, $m\,n$ parallel to A B; then suppose A B $= b$, D B $= r$, A P $= x$, P $p = \dot{x}$, and M R or N $n = \dot{y}$; then the infinitely little Parallelogram M n, or the Fluxion of the Weights is $= b\dot{y} - x\dot{y} =$ (because by the property of the Curve $ax = yy$) $b\dot{y} - \frac{y^2\dot{y}}{a}$, which being multiplyed by y, the distance



from the Axis of Suspension, the Product $by\dot{y} - \frac{y^3\dot{y}}{a}$ is $=$ to the Fluxion of the Moments; and consequently the Sum of all the Moments is $= \frac{byy}{2} - \frac{y^4}{4a}$; which being divided by the Sum of all the Weights $by - \frac{y^3}{3a}$; the Quotient $\frac{12abyy - 6y^4}{24aby - 8y^3}$ is equal to the distance of the Center of Gravity of the Segment A M N B from A B.

And when x becomes $= b$, and $y = r$, then $\frac{12aby^2 - 6y^4}{24aby - 8y^3}$ will be $=$ $\frac{12aby^2 - 6axy^2}{24aby - 8axy} = \frac{3}{8}r =$ to the distance of the Center of Gravity of the Semi-parabola A B D from the Axis of motion A B.

Whence if B T be taken $= \frac{3}{8}$ B D, and T S be drawn parallel to A B, it will intersect Q O in S, the Center of Gravity requir'd.

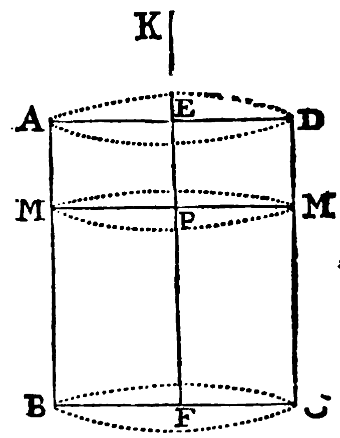
And in like manner, the Center of Gravity of the Semi-hyperbolic Space (if it has one) comprehended between the Curve and the Asymptote, may be investigated, which I leave for the Readers Exercise.

PROP.

PROP. IX.

To investigate the Center of Gravity of a Cylinder.

341. It is evident that the Center of Gravity of a Cylinder is in the Axis E F; suppose then EF $= a$, EP $= x$, $\frac{cr}{2}$ = the Area of one of the Circles of the Cylinder; and AD the Axis of Suspension, then is $\frac{cr\dot{x}}{2}$ = to the Fluxion of the Weights, which multiplyed by x or E P, its distance from the Axis of Suspension, the Product $\frac{crx\dot{x}}{2}$ is = to the Fluxion of the Moments, and consequently the Sum of all the Moments is = $\frac{rcxx}{4}$, which being divided by $\frac{crx}{2}$ = to the Sum of all the Weights, the Quotient $\frac{x}{2}$ is = to the distance of the point requir'd from E.

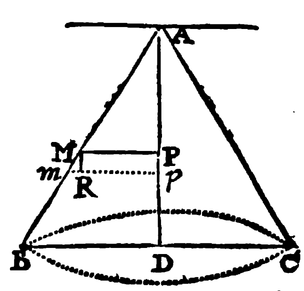


And when x becomes $= a$, then is $\frac{a}{2}$ = to the distance of the Center of Gravity of the whole Cylinder from the Axis of Motion A D.

PROP. X.

To find the Center of Gravity of a Cone.

342. The same things being supposed as in *Art.* 204. the Fluxion of the Weights (supposing the Axis of suspension to pass through the Vertex A, and to be parallel to the Base B C) is $= \frac{crx^2\dot{x}}{2aa}$; and the Fluxion of the Moments is $= \frac{crx^3\dot{x}}{2aa}$; and consequently the Sum of all the Moments is $= \frac{crx^4}{8aa}$; which being divided by the Sum of all the Weights $= \frac{crx^3}{6aa}$; the Quotient $= \frac{3}{4}x$ is = to the distance of the Center of Gravity of the Portion of the Cone generated by the Triangle A M P, from the Vertex A; and consequently the Distance of that of the whole Cone from the Vertex A is $= \frac{3}{4}a = \frac{3}{4}$ A D.



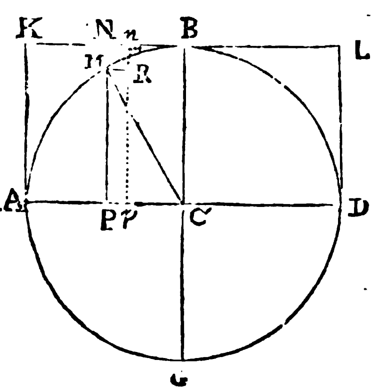
And by a like Method the distance of the Center of Gravity of a Pyramid from the Vertex A is $= \frac{3}{4}a = \frac{3}{4}$ the perpendicular or Axis of the Pyramid.

PROP.

PROP. XI.

To investigate the Center of Gravity of a Sphere, or of any Segment of a Sphere.

343. If the Line A K touching the Sphere in A, be the Axis of Suspension, then 'tis evident that the Center of Gravity of the Sphere, or of any Segment thereof, must be in the Diameter A D perpendicular to the said Axis A K.



Now the same things being supposed, as in *Art.* 206. the Fluxion of the Weights is $= c x \dot{x} - \frac{c x^2 \dot{x}}{2 r}$, and the Fluxion of the Moments is $= c x^2 \dot{x} - \frac{c x^3 \dot{x}}{2 r}$, and consequently the Fluent or the Sum of all the Moments is $= \frac{c x^3}{3} - \frac{c x^4}{8 r}$; which being divided by the Sum of all the Weights $= \frac{c x x}{2} - \frac{c x^3}{6 r}$ the Quotient $= \frac{8 r x - 3 x x}{12 r - 4 x}$ is = to the distance of the Axis of Motion A K from the Center of Gravity of the Portion of the Sphere describ'd by the Revolution of A M P about the Axis A P.

And the distance of the Center of Gravity of the whole Sphere from A is $= r =$ to the Semi-diameter of the Sphere; because then $\frac{8 r x - 3 x x}{12 r - 4 x}$ will become $= r$.

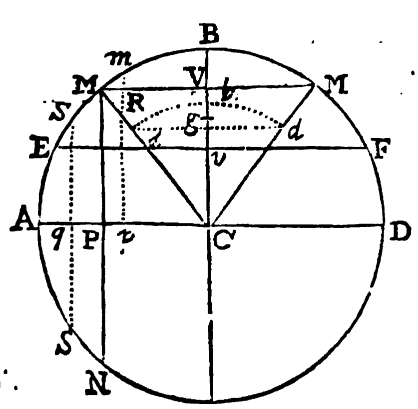
And the distance of the Axis of Suspension A K, from the Center of Gravity of the Hemisphere generated by the Revolution of the Quadrant A B C is $= \frac{8rx - 3xx}{12r - 4x}$ $=$ (because $x = r$) $\frac{5}{8} r$.

CONSECTARY I.

344. Hence the Center of Gravity of any Portion of a Sphere cut of by any two parallel Plains, may be found.

CONSECTARY II.

And to find the Center of Gravity of the Sector of a Sphere; let it be requir'd to find the Center of Gravity of the Spherical Sector M C M; 'tis evident that this Sector is composed of an infinite Number of Cones or Pyramids, whose Bases compose the Spherical Surface M B M, and whose common Vertex is in C; now the Center of Gravity of every Pyramid is distant from its Vertex $\frac{1}{4}$ of its altitude; therefore if the Spherical Surface *d b d* be describ'd with the Radius $= \frac{3}{4}$ C B, 'tis manifest that the Centers of Gravity of all the infinitely little Pyramids, which compose the Sector, are in that Surface (*d b d*) and that all the parts of the same are equally gravitated or loaded with their respective Pyramids, and hence it is plain that the Center of Gravity of the



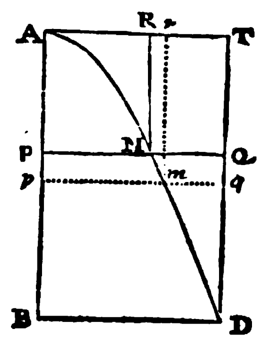
Spherical

Spherical Sector M C M is the same with that of the Surface dbd and the Center of Gravity of the Surface dbd is in g, the middle point between v and b, therefore the Center of Gravity of the Sector of the Sphere M C M is in g; and because C b is $= \frac{3}{4}$ C B, therefore the Versed Sine bv is $= \frac{1}{4}$ the Versed Sine B V; and $bg = \frac{1}{2} bv$ is $= \frac{1}{8}$ B V, therefore C B $- bg = \frac{3}{4}$ C B $- \frac{1}{8}$ B V $=$ C g is $=$ to the distance of the Center of Gravity of the Spherical Sector M C M from C the Center of the Sphere; and consequently the distance of the Center of Gravity of an Hemi-sphere from the Center C is $= \frac{3}{8}$ C B.

PROP. XII.

To Investigate the Centers of Gravity of all sorts of Parabolical Conoids, generated by the Revolution of any Parabolic Space about its Axis.

345. The same things being suppos'd, as in *Art.* 208. the Fluxion of the Weights is $= \frac{c x^{\frac{2}{m}} \dot{x}}{2r}$, and the Fluxion of the Moments is (supposing the Tangent A T to be the Axis of Suspension) $= \frac{c x^{\frac{2}{m}+1} \dot{x}}{2r}$; and consequently the Sum of all the Moments is $= \frac{m}{4m+4} \times \frac{c x^{\frac{2}{m}+2}}{r}$. Which being divided by the Solid or the Sum of all the Weights $= \frac{m}{2m+4} \times \frac{c x^{\frac{2}{m}+1}}{r}$; the Quotient $= \frac{2m+4}{4m+4} \times x$ is $=$ to the distance of the Axis of Suspension A T from the Center of Gravity of the Parabolical Conoid, generated by the Revolution of A P M about A P; and consequently the distance of the said Axis from the Center of Gravity of the whole Solid generated by the Conversion of A D B about A B is $= \frac{2m+4}{4m+4} \times b$.



Hence if the Equation of the Curve be $ax = yy$, as in the common Parabola, then m is $= 2$, and the distance of the Center of Gravity of the said Parabolical Conoid from the Vertex A $= \frac{2m+4}{4m+4} b$ is $= \frac{8}{12} b = \frac{2}{3}$ the Altitude of the Solid.

PROP. XIII.

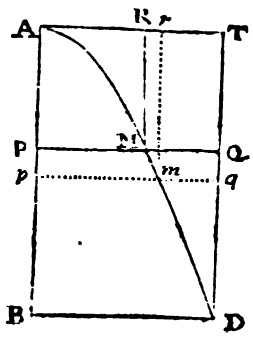
To investigate the Center of Gravity of any Portion of a Parabolical Conoid, cut off by one or more Plains passing through its Axis.

346. Its evident that those Solids are composed of Sectors which are in the same proportion with the Circles that compose the entire Solid; and consequently the distance of the Center of Gravity of the Portion of the Solid from the Tangent A T considered as the Axis of Suspension, will be the same with that of the Center of Gravity of the entire Solid, that is $= \frac{2m+4}{4m+4} b$.

And the Center of Gravity of such Solids must be in the Plain passing through the Axis A B, and bisecting the Sector of the Base, because that Plain will also bisect all the other Sectors, and consequently the Center of Gravity of the Soli is in the common Section of two Plains, one whereof Bisects all the Sectors, and the other passes through the point m the Axis determined by the preceeding Proposition, and runs parallel to the Plain of the Base.

It remains only to determine the distance of the said Center from the Axis of Suspension AB, *which may be performed in this manner:*

The same things being supposed as in *Art.* 208. the Fluxion of the Weights (supposing the Arch of the Sector of the Base $= c$, and the Radius of the Base BD $= r$) is $= \frac{cyy\dot{x}}{2r}$, and because this Fluxion is an infinitely thin Sector of a Circle, the distance of the Center of Gravity thereof from the Axis of Suspension AB is = (supposing the Chord $= g$) $\frac{2gy}{3c}$, and consequently the Fluxion of the Moments is = (multiplying $\frac{2gy}{3c}$ by $\frac{cyy\dot{x}}{2r}$) $\frac{gy^3\dot{x}}{3r} = \frac{gx^{\frac{3}{m}}\dot{x}}{3r}$, and the Sum of all the Moments is $= \frac{mgx^{\frac{3}{m}+1}}{3m+9\times r}$, which being divided by the Sum of all the Weights $= \frac{m}{2m+4} \times \frac{cx^{\frac{2}{m}+1}}{r}$, the Quotient $\frac{2m+4}{3m+9} \times \frac{gx^{\frac{1}{m}}}{c} = \frac{2m+4}{3m+9} \times \frac{gy}{c} =$ to the distance of the Axis of Suspension AB from the Center of Gravity requir'd.



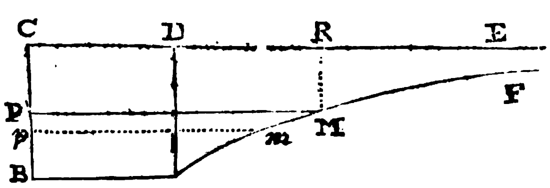
Hence if the Equation of the Curve be $ax = yy$, then the distance of the said Center of Gravity from AT is $= \frac{3}{5} b$, and the distance of the same from AB is $= 1\frac{1}{5}\frac{gr}{c}$.

And if the Equation of the Curve be $aax = y^3$, then the distance of the Center of Gravity from AT is $= \frac{4}{7} b$, and the distance of the same from AB is $= \frac{5gr}{9c}$.

PROP. XIV.

To investigate the Centers of Gravity of Solids generated by the Revolution of Hyperbolic Spaces about one of their Assymptotes BE.

347. The same things being supposed as in *Art.* 213. the Fluxion of the Weights is $= \frac{cxy\dot{x}}{r}$, and because the Centers of Gravity of the Cylindric Surfaces which compose the Solid, divide them in the middle, we may take $\frac{1}{2} y$ for the distance of the Fluxion of the Weight from the Axis of Suspension CB; and consequently the Fluxion of the Moments is $= \frac{cyyx\dot{x}}{2r} = \frac{cx^{-\frac{2}{m}+1}\dot{x}}{2r}$, and the Fluent or the Sum of all the Moments is $= \frac{m}{4m-4} \times$



$\times \frac{cx^{-\frac{1}{m}+2}}{r}$, which being divided by the Solid or the Sum of all the Weights $\frac{m}{2m-1} \times \frac{cx^{-\frac{1}{m}+2}}{r}$ the Quotient $\frac{2m-1}{4m-4} \times x^{-\frac{1}{m}} = \frac{2m-1}{4m-4}\, x\, y$ is $=$ to the distance of the Axis of Suspension C B from the Center of Gravity of the indeterminate Solid, and $\frac{2m-1}{4m-4} \times b =$ to the distance of the said Axis from the Center of Gravity of the Solid.

COROLLARY.

348. Hence in the Apollonian Hyperbola $y^{-1} = x$, or (supposing the Parameter $a = 1$) $aa = xy$, m is $= 1$, and consequently $\frac{2m-1}{4m-4} \times b$ is $= \frac{1}{0} \times b =$ to the distance of the Center of Gravity of the whole Solid from the Axis of Motion B C; which in this Case is infinite, and shews that such a Solid has no Center of Gravity.

And as we proved (Art. 214.) *in another Place, that an infinite Space may generate a Solid of finite Dimensions; so it now appears that a certain sort of Solids have no Centers of Gravity, which is another Paradox equally strange and no less true than the former.*

PROP. XV.

To Investigate the Center of Gravity of the Hyperbolical Conoid, generated by the Revolution of the Hyperbolic Space A M D B.

349. The same things being suppos'd as in *Art.* 216. the Fluxion of the Weights is $= \frac{crx^2\dot{x} + 2bcrx\dot{x}}{2dd + 4ab}$; and if the Axis of Suspension be suppos'd parallel to the (*See Fig.* 1. *in pag.* 173.) Base B D, and to pass thro' the Vertex A, then the Fluxion of the Moments is $= \frac{crx^3\dot{x} + 2bcrx^2\dot{x}}{2dd + 4db}$; and the Flowing Quantity or the Sum of all the Moments is $= \frac{crx^4}{8dd + 16db} + \frac{bcrx^3}{3dd + 6db} =$ (multiplying the Numerator and Denominator of the first Member by 3, and those of the second by 8) $\frac{3crx^4 + 8bcrx^3}{24dd + 48db}$. Which being divided by the Sum of all the Weights $= \frac{crx^3 + 3bcrx^2}{6dd + 12db} = \frac{4crx^3 + 12bcrx^2}{24dd + 48db}$, the Quotient $\frac{3xx + 8bx}{4x + 12b}$ is $=$ to the Distance of the Axis of Suspension from the Center of Gravity of the Solid generated by the Revolution of the Hyperbolic Space A M P about the Axis A D; and consequently $\frac{3dd + 8bd}{4d + 12b}$ is $=$ to the Distance of the said Axis of Suspension from the Center of Gravity of the whole Solid; because then x is $= d$.

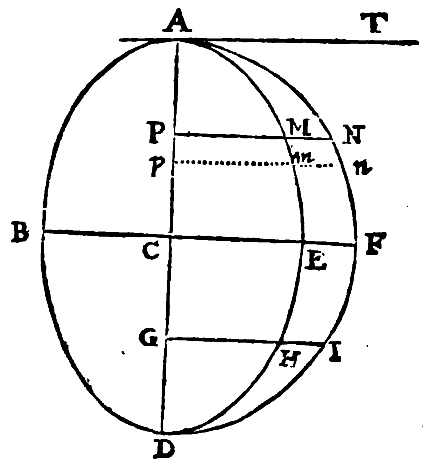
And hence it is manifest that the Distance of the Center of Gravity of this Solid from the Vertex thereof, is to its Axis, as $3d + 8b$ is to $4d + 12b$.

PROP.

PROP. XVI.

To Investigate the Center of Gravity of the Spheroid generated by the Revolution of the Semi-ellipsis AMD *about the Axis* AD.

350. The same things being suppos'd as in *Art.* 218. The Fluxion of the Weights is $= \frac{cyy\dot{x}}{2r}$, and by the property of the Ellipsis, $yy : 2ax - xx :: rr : aa$, therefore $yy = \frac{2arrx - r^2x^2}{aa}$, and consequently $\frac{cyy\dot{x}}{2r}$ is $= \frac{2carx\dot{x} - crx^2\dot{x}}{2aa}$, and if we suppose the Axis of Suspension to pass through the Vertex A, then the Fluxion of the Moments is $= \frac{2carx^2\dot{x} - crx^3\dot{x}}{2aa}$; and the Flowing Quantity or the Sum of all the Moments is $= \frac{crx^3}{3a} - \frac{crx^4}{8aa} = \frac{8carx^3 - 3crx^4}{24aa}$, which being divided by the Sum of all the Weights $= \frac{12carxx - 4crx^3}{24aa}$; the Quotient $\frac{8ax - 3xx}{12a - 4x}$ is $=$ to the Distance of the Center of Gravity of the Portion of the Spheroid generated by the Conversion of AMP about the Axis AD, from the Vertex A; and $a =$ to the Distance of the Center of Gravity of the whole Solid or Spheroid from A.



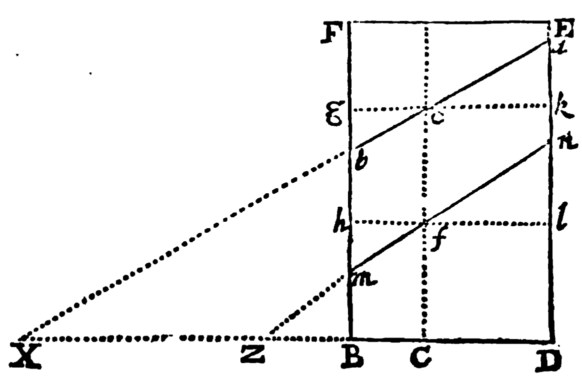
Now that the Usefullness of this Doctrine may in some measure appear, I shall shew the Application thereof in the Mensuration of Surfaces and Solids; which will serve as a Taste to judge of the Excellency of the Subject, and how much it deserves our Study.

PROP.

PROP. XVII.

If a Cylinder, Prism or any other Prismatic Solid, be erected on any Plain, and if another Plain be supposed to cut the same obliquely in respect of the Plain of the Base; the Solid comprehended between those two Plains is equal to a Solid of the same Base, and whose Altitude is equal to the Altitude of that Line which is drawn from the Center of Gravity of the Base, and equally inclined to the same, with the Solid.

351. Let BD represent the Plain of the Base (which we may suppose either Rectilineal, Curvilineal or Mixt) and let the Eye be supposed in the same (produced) at an infinite distance; and suppose B D E F to be a Cylinder, if the Plain of the Base be a Circle, or any other Prismatic Solid; let C be the Center of Gravity of the Base, and draw C *e* parallel to D E; and suppose this Solid to be cut off by another Plain *b e d* inclined to the Plain of the Base, and intersecting the same produced in the right Line X; I say the Solid B D *d b* comprehended between those Plains, is equal to a Solid of the same Base B D and Altitude C*e*; that is, the Solid B D *d b* is = to the Solid B D *k g*.



Suppose the Plain X B D parallel to the Horizon and the Line X (for in this position of the Eye, the Projections of such Lines are points) the Axis of Motion; then 'tis evident that all the points B, C, D, ponderate in proportion to their distances from the Axis of Motion; that is, the Momenta of the points B, C, D, are as X B, X C, X D, *&c.* and because C is the common Center of Gravity of all those points, therefore the Sum of all their Momenta at their respective distances from X is equal to the Momentum of all the said points, suspended in the point C; and consequently all the X B, X C, X D, *&c.* are equal to as many times X C; but the right Lines B *b*, C *e*, D *d*, *&c* are proportional to X B, X C, X D, *&c.* therefore all (12. *Elem.* 5.) the B *b*, C *e*, D *d*, *&c.* are equal to as many times C *e*; that is, the Frustum or Ungula B D *d b* is = to the Prismatic Solid B D *k g*.

CONSECTARY I.

352. If another Plain *m f n* cut the foresaid Prismatic Solid, either above or below the Plain *b d*, then the portion of the Solid comprehended between those two Plains, *viz.* *m n d b* is equal to the Solid comprehended under the Plain of the Base B D and the right Line *f c*.

CONSECTARY II.

Hence all Plains howsoever inclined, which passe through the same point *e*, in the Line C *e* (which passes through the Center of Gravity of the Base, and is parallel to the sides B F, D E) cuts of Segments or Ungula's, equal between themselves, and to the Prismatic Solid B D *k g*.

CONSECTARY III.

And hence it follows that the Surface of the Ungula B D *d b* (excluding the Bases) is equal to the Surface of the Prismatic Solid B D *k g* (excluding its Bases also) for if we suppose C not the Center of Gravity of the Base, but that of the Perimeter of the Base; then it is evident that all the points of the Perimeter B, D, *&c.* ponderate in proportion to their distances from X the Axis of Motion; and because C is the Center of Gravity of the Perimeter, therefore all the Lines, X B, X D, &c. are equal to as many times X C; and by similar Triangles, the Lines X B, X D, X C, &c. are proportional to B *b*, D *d*, C *c*, &c. therefore all the B *b*, D *d*, &c. are equal to as many times C *c*; that is, the Surface of the Frustum or Ungula B D *d b* is euqal to the Surface of the Prismatic Solid B D *k g*.

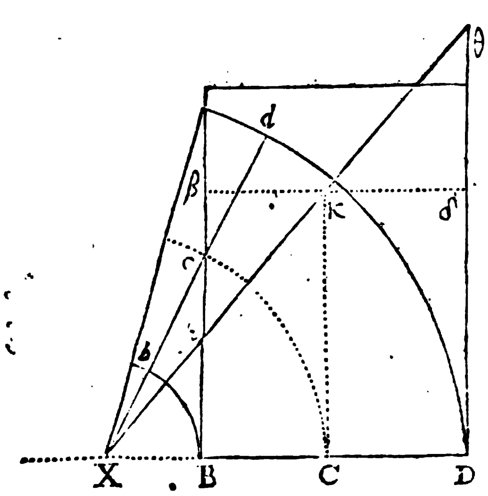
CONSECTARY IV.

And the Surfaces of all Ungula's cut of by Plains passing through the same point *c* in the Line C *c*, are equal between themselves, and to the Surface of the same Prismatic Solid B D *k g*, and the Surface of the Portion of the Solid comprehended between any two Plains *v. g r.* between *hd* and *m n*, is equal to the Surface of a Solid whose Base is = B D, and whose Altitude is = *f c*.

PROP. XVIII.

If a Line or Surface, Rectilineal or Curvilineal, Revolve uniformly about an Axis in the same Plain; the Surface or Solid generated by that Motion, is equal to a Parallelogram or Parallelepipedon, whose height is equal to the Periphery describ'd by the Center of Gravity, and whose Base is a Line or Parallelogram equal to the Line or Surface given.

353. Let B C D represent any Rectilineal Figure, and suppose the same to revolve about the right Line X in the same Plain, and by such Motion to describe the Solid B D *d b* (represented in this projection by a Plain) and suppose C the Center of Gravity of the Base to describe the Arch C *c*, or if B C D be a Line, then suppose it to describe the Surface B D *d b*.



On the same Base B C D raise the Solid or Superficies, whose Altitude C *κ* is equal to the Arch C *c*, describ'd by C the Center of Gravity, and let the Plain X *κ* cut off the Ungula B D θ *ι*.

Now because the similar Arches B *b*, C *c*, D *d*, are proportional to the Radii X B, X C, X D. &c. and these are proportional (by similar Triangles) to B *ι*, C *κ*, D θ, &c. therefore these right Lines are proportional to those Arches: and because by Construction C *κ* is = C *c*, therefore all the right Lines B *ι*, D θ are equal to the Arches B *b*, D *d*, &c. respectively; that is, the Figure B D *d b* (whether Solid or Superficial) is = to the Ungula B D θ *ι*, which is (by the preceeding Proposition) = to the Figure B D *d b*.

CONSECTARY I.

354 Hence if the distance of the Center of Gravity of the Line or Solid B C D, from the Axis of Motion X, and the Magnitude of the Line or Surface be given; the Value of the Surface or Solid generated by a Total or partial Conversion may be found, and in general, any two of the three being given, the third may be found.

CONSECTARY II.

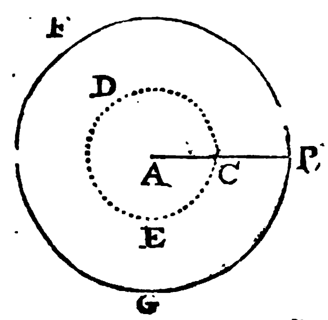
Equal Lines or Surfaces revolving at unequal Distances, generate Surfaces or Solids proportional to the Distances of their Centers of Gravity from the Axis of Motion; and unequal Lines or Surfaces whose Centers of Gravity are equidistant from the Axis of Motion, generate Surfaces or Solids proportional to the generating Lines or Surfaces; and if neither the Surfaces (or Lines) nor the Distances of their Centers of Gravity from the Axis of Motion, be equal; the Surfaces or Solids generated by them, are to one another in a Ratio compounded of the Ratio of the Lines or Surfaces, and the Ratio of the Distances of their Centers of Gravity from the said Axis of Motion.

For Instance, the common Parabola is to the circumscribed Parallelogram, as 2 is to 3, and the Distance of the Vertex of the Parabola from the Center of Gravity of the Parallelogram is $= \frac{1}{2}$ the Axis, and the Distance of the said Vertex from the Center of Gravity of the Parabolic Space is $= \frac{3}{5}$ the Axis; therefore the Distance of *that* is to the Distance of *this* as 5 is to 6; and if we suppose both Spaces to revolve about a Line touching the Vertex of the parabola, and parallel to its Base, then the Solid generated by the Parabolic Space, will be to the Solid generated by the Parallelogram in a Ratio compounded of the Rationes of 2 to 3, and of 6 to 5, that is, as 12 is to 15, or as 4 is to 5.

Whoso has a mind to see more of this Subject, may consult that Learned Treatise written by the Excellent Dr. Wallis, *and Inscrib'd* de Calculo Centri Gravitatis.

CONSECTARY III.

If the right Line A B revolve (in the same Plain) on the point A as a Center, then the point B will describe the Periphery of a Circle and the Line AB will describe a Circular Surface BFGB; and if C be the Center of Gravity of the Line A B, it will describe the Periphery CDE = (because AC is = CB) $\frac{1}{2}$ BFG; and the Area of the Circle B F G B will be = A B × Periphery CDE = $\frac{1}{2}$ A B × Periphery BFG; whence 'tis manifest that the Area of any Circle is = to the Area of a Rectangular Triangle, whose Base is equal to the Periphery, and Altitude equal to the Radius of the Circle.

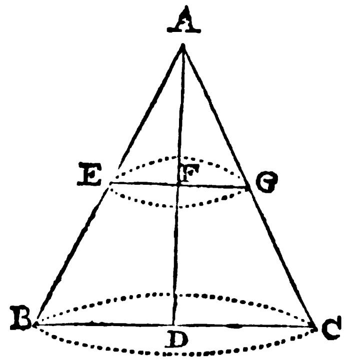


CONSECTARY IV.

The Area of the Annulus comprehended between the Peripheries B F G and CDE is = to the Rectangle comprehended under CB and the Periphery describ'd by the Center of Gravity of CB; thus if A C be = C B, then the Area of the Annulus EDCBFG is = CB × $\frac{3}{4}$ B F G = $\frac{1}{2}$ AB × $\frac{3}{4}$ B F G, and the Area of the Circle B F G B is = $\frac{1}{2}$ A B × $\frac{4}{4}$ B F G; therefore the Area of the Annulus E D C B F G is = $\frac{3}{4}$ the Area of the Circle B F G B; and consequently the Area of the Circle C D E C is = $\frac{1}{4}$ the Area B F G B = $\frac{1}{3}$ the Area of the Annulus E D C B F G.

And hence is appears that Circles are in a Duplicate Ratio of their Diameters.

CONSECTARY V.

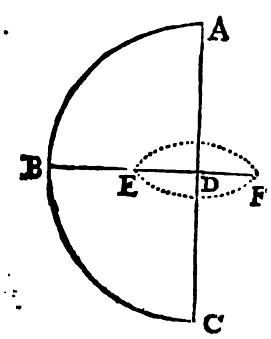


If the Rectangular Triangle A B D revolve about the Axis A D, then the Line A B will describe the Conic Surface A B C; and E the Center of Gravity of the Line A B will describe the Periphery E G = (because A E = $\frac{1}{2}$ A B, and consequently F E = $\frac{1}{2}$ B D) $\frac{1}{2}$ the Periphery B C; whence the Conic Surface (excluding the Base) A B C is = A B × E G = $\frac{1}{2}$ A B × B C; that is, the Surface of the Cone A B C is = to a Triangle whose Base is equal to the Periphery of the Base B C) and Altitude equal to the Side of the Cone (A B.)

CONSECTARY VI.

The Surface of a Cylinder is to the Surface of its Base, as the height of the Cylinder is to the Radius of the Base.

CONSECTARY VII.



If the Semi-circle A B C revolve about the Diameter C A, it will generate the Surface of a Sphere; and if E be the Center of Gravity of the Periphery A B C, it will describe the Periphery E F; whence the Surface of the Sphere describ'd with A B C is = ABC × E F; and if ABC be suppos'd $= c$, and AD $= r$, then DE will be = (*Art.* 333.) $\frac{2rr}{c}$, and E F $= 4r$, and consequently A B C × E F is $= 4r \times c = 2r \times 2c = \frac{4}{2} r \times 2c$ = 8 times the Area of the Semi-circle A B C; that is, the Surface of a Sphere is equal to four times the Area of one of its great Circles.

CONSECTARY VIII.

And if E be the Center of Gravity of the Space A B C, then the Semi-circle A B C will describe a Sphere, and E the Center of Gravity will describe the Periphery E F; and the Sphere generated by the Revolution of A B C about the Axis A C will be = to the Area of the Semi-circle A B C × into the Periphery E F; whence if A D be $= r$, and the Quadrantal Arch A B $= c$, then D E will be $= \frac{2rr}{3c}$, and consequently the Periphery of the Circle E F will be $= \frac{4}{3} r$, which being multiplyed by rc = to the Area of the Semi-circle A B C; the Product $\frac{8rrc}{3} = \frac{4}{3} rr$ × the Periphery of a great Circle of the Sphere = to the Solidity of the Sphere; that is, the Sphere is to the circumscrib'd Cylinder as 2 is 3.

CON-

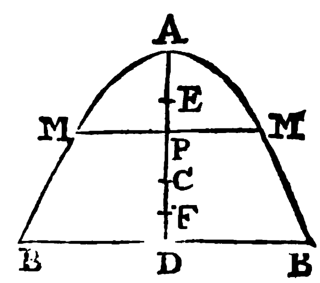
CONSECTARY IX.

And if the Parabolic Space ADB revolve about the Axis AB, the Parabolical Conoid (generated by such a Motion) is = Area ADBA × the Periphery of the Circle describ'd by the Center of Gravity S = (the same Symbols being retain'd, as in *Art.* 340.) $\frac{2}{3} br \times \frac{1}{8} c = \frac{6}{24} brc = \frac{1}{4} brc = \frac{1}{2}$ the circumscrib'd Cylinder.

PROP. XIX.

The Center of Gravity of any Figure, Plain or Solid being given, with that of one of its Parts; to find the Center of Gravity of the other Part.

355. In the Parabolical Conoid, formed by the Semi-revolution of the Parabolic Space BAB about the Axis AD; let C be the Center of Gravity of the whole Conoid, and E that of the Segment or Portion MAM; 'Tis requir'd to find the Center of Gravity of the Portion BMMB.



Suppose the whole Conoid to be suspended by the point C, and E the Center of Gravity of MAM, and F that of BMMB. Then 'tis evident that the Segments being suspended by the point C, are in *Equilibro*; and consequently the Distance between C and the particular Centers of Gravity E and F, are reciprocally proportional to their Masses. That is, as the Segment BMMB : Segment MAM :: EC : CF, and F is the Center of Gravity of the Segment BMMB. Q. E. I.

SECT. XII.

The Use of Fluxions

In Investigating the Centers of Percussion of Lines, Surfaces and Solids.

356. IN Calculating the Centers of Gravity, we suppose the Figures to be simply suspended to a Point or Axis; but in order to Calculate their Centers of *Percussion*, they are suppos'd actually to revolve about a Point or Axis; and as in that Case, we consider the simple *Momenta*, so in this we consider them also, only with *Velocity* superadded. And as the Sum of all the simple *Momenta*, on every side of the Center of Gravity are equal; so the Sum of all the Forces on every side of the Center of *Percussion* must be equal. *Whence the Center of Percussion of a Body in Motion, is that Point wherein all the Forces of that Body are consider'd as united into one: So that the Force of Percussion in that Point, is greater than in any other.* And the Center of *Percussion* is the same in respect of the *Forces*, as the Center of Gravity is in respect of the *Weights*: That is, one is the Center of the Moments or Efforts, as the other is of the Center of the Weights, and as we find the Center of Gravity by dividing the Sum of all the *Momenta* by the Sum of all the *Weights*; so to find the Center of *Percussion*, there is nothing to be done, but to divide the Sum of all (the Fluxions of the *Forces*, which are equal to all) the Rectangles comprehended under the *Momenta* and their respective *Velocities* (or Lines proportional to them) by the Sum of all the Moments.

And it may be observ'd, That the Center of *Percussion* may be found in the same manner as the Center of Gravity is found; if we suppose the Weights to be encreas'd in proportion to their *Velocities* at the *Instant of Percussion*; and find the Center of Gravity of the Weights so encreas'd.

And because (when a Figure revolves about an Axis) the *Velocities* of the Parts, are proportional to their Distances from the said Axis; therefore to encrease the *Weights* in proportion to their *Velocities* at the Instant of *Percussion*, is to multiply every individual Part (or Moment) by its Distance from the Axis of Motion; and if we take the Sum of all the Rectangles for the Sum of all the *Weights* at the Instant of *Percussion*, then the Center of Gravity of the said *Weights* will be the same with the Center of *Percussion*. Whence in General,

To find the Center of Percussion.

357. *Multiply all the Infinitely little Parts which compose the Figure, by the Squares of their Distances from the Point or Axis of Suspension, and divide the Sum by the Sum of all the Weights multiplied into the Distance of their Center of Gravity, from the Point or Axis of Suspension; the Quotient is the Distance of the Center of* Percussion *from the said Point or Axis.*

PROP

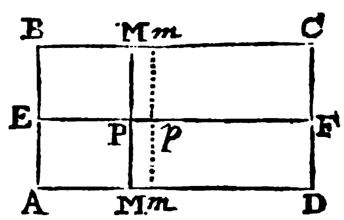
PROP. I.

To find the Center of Percussion of a Line, suspended by one Extremity, about which as a Center it is suppos'd to move.

358. Let the given Line AB be suspended by the point A; If we suppose this Line to be divided into an infinite Number of equal Parts, 'tis evident that they will describe Concentric and similar Arches of Circles in the same time, which are proportional to their Rays or Distances from the point A. Now their Velocities are proportional to the Arches they describe; that is, their Velocities are proportional to their Distances from the point of Suspension A; and if we suppose the Line $AB = x$, $Bb = \dot{x}$, then the Fluxion of the Moments is $= x\dot{x}$, which being multiplied by x (representing the Velocity of the Particle Bb) the Product $xx\dot{x}$ is the Fluxion of the *Forces*: and the Sum of all the *Forces* is $= \frac{x^3}{3}$, which being divided by the Sum of all the Moments $\frac{xx}{2}$, the Quotient $\frac{2}{3}x$ is $=$ to the Distance of the point A from the Center of Percussion of the whole Line AB.

CONSECTARY.

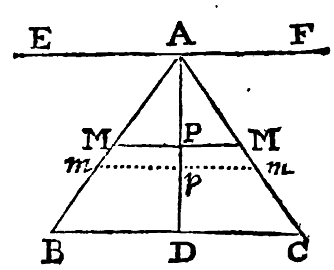
359. In the Parallelogram AC, the Center of Percussion is in the right Line EF, which divides the Parallelogram into two equal Parts, and if the point of Suspension be suppos'd in E, the Distance of the said Center from E is $= \frac{2}{3}$ EF.



PROP. II.

To find the Center of Percussion of the Isosceles Triangle ABC, *revolving about the right Line* EF *parallel to the Base, and passing thro' the Vertex* A.

360. Draw the Line AD perpendicular to the Base BC, then the Center of Percussion is in the same. Draw the Lines MM (y) and mm parallel to the Base, and infinitely near one another; and suppose $AD = a$, $BC = b$, $AP = x$, $Pp = \dot{x}$; then the Fluxion of the Weights Mm is $= y\dot{x} =$ (because $a : b :: x : y$.) $\frac{bx\dot{x}}{a}$, and the Fluxion of the Moments is $= \frac{bx^2\dot{x}}{a}$; and the Fluent or the Sum of all the Moments is $= \frac{bx^3}{3a}$, and the Fluxion of the Forces is $= \frac{bx^3\dot{x}}{a}$, and the Sum of all the Forces is $= \frac{bx^4}{4a}$, which being divided by the Sum of all the Moments $\frac{bx^3}{3a}$, the Quotient $\frac{3}{4}x$ is $=$ to the Distance of the Center of Percussion of the Triangle AMM from the Vertex A; and when x becomes $= a$, then the Distance of the Center of Percussion of the Triangle ABC from the Vertex A is $= \frac{3}{4}a = \frac{3}{4}$ AD.



* PROP.

PROP. III.

To find the Center of Percussion of the Isosceles Triangle ABC *revolving about it's Base* BC.

361. The same Symbols beimg retained as in the precedent *Art.* the Fluxion of the Weights is $= y\dot{x} = \frac{bx\dot{x}}{a}$ and because DP is $= a - x$, the Fluxions of the Moments is $= bx\dot{x} - \frac{bx^2\dot{x}}{a}$, and the Fluent or the Sum of all the Moments is $= \frac{bxx}{2} - \frac{bx^3}{3a}$, and if the Fluxion of the Moments $bx\dot{x} - \frac{bx^2\dot{x}}{a}$ be multiplyed by $a - x$, the Product $abx\dot{x} - 2bx^2\dot{x} + \frac{bx^3\dot{x}}{a}$ is the Fluxion of the Forces, and consequently the Sum of all the Forces is $= \frac{abxx}{2} - \frac{2bx^3}{3} + \frac{bx^4}{4a}$, which being divided by the Sum of all the Moments $\frac{bxx}{2} - \frac{bx^3}{3a}$, the Quotient $\frac{6aa - 8ax + 3xx}{6a - 4x}$ is = to the distance of the Center of Percussion of the Space AMM from the Axis of Motion BC; and when x becomes $= a$, then $\frac{6aa + 3xx - 8ax}{6a - 4x}$ becomes $= \frac{1}{2}a$; that is, the distance of the Center of Percussion of the whole Triangle ABC from the Axis of Motion BC is $= \frac{1}{2}$ the Perpendicular AD.

PROP. IV.

To Investigate the Centers of Percussion of all sorts of Parabolic Spaces, revolving about an Axis parallel to the Base and passing through the Vertex.

362. The same things being supposed as in *Art.* 337. the Fluxion of the Moments is $= x^{\frac{1}{m}+1}\dot{x}$; which being multiplyed by x, the Fluxion of the Forces is $= x^{\frac{1}{m}+2}\dot{x}$, and consequently the Sum of all the Forces or the Fluent is $= \frac{m}{3m+1}x^{\frac{1}{m}+3}$; which being divided by the Sum of all the Moments $\frac{m}{2m+1}x^{\frac{1}{m}+2}$ the Quotient $\frac{2m+1}{3m+1}x$ is = to the distance of the Vertex A, from the Center of Percussion of the indeterminate Space MAM; and consequently when x is $= a$, the distance of the Center of Percussion of the whole Parabolic Space BAB, from the Vertex A is $= \frac{2m+1}{3m+1}b$.

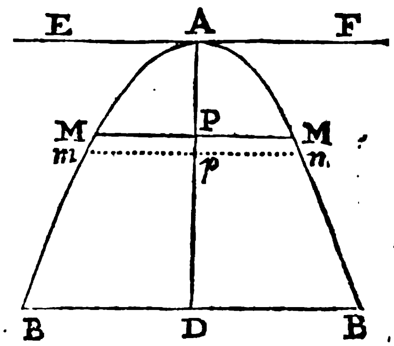
Hence in the common Parabola, in which $ax = yy$, m is $= 2$ and $\frac{2m+1}{3m+1}b$ is $= \frac{5}{7}b =$ to the distance of the Center of Percussion of the said Parabolic Space from the Vertex A.

PROP.

PROP. V.

To find the Center of Percussion of all sorts of Parabolic Spaces, revolving about their Bases.

363. It is evident that the Fluxion of the Weights Mp is $= y\dot{x} = x^{\frac{1}{m}}\dot{x}$, which being multiplied by $b - x$ the Distance thereof from the Axis of Motion, the Product $bx^{\frac{1}{m}}\dot{x} - x^{\frac{1}{m}+1}\dot{x}$ is equal to the Fluxion of the Moments: and the Sum of all the Moments is $= \frac{m}{m+1}bx^{\frac{1}{m}+1} - \frac{m}{2m+1}x^{\frac{1}{m}+2}$; now if the Fluxion of the Moments $bx^{\frac{1}{m}}\dot{x} - x^{\frac{1}{m}+1}\dot{x}$ be multiplyed by $b - x$, the Product $bbx^{\frac{1}{m}}\dot{x} - 2bx^{\frac{1}{m}+1}\dot{x} + x^{\frac{1}{m}+2}\dot{x}$ is = to the Fluxion of the Forces; whence the Sum



of all the Forces is $= \frac{m}{m+1}bbx^{\frac{1}{m}+1} - \frac{2m}{2m+1}bx^{\frac{1}{m}+2} + \frac{m}{3m+1}x^{\frac{1}{m}+3}$, which being divided by the Sum of all the Moments $\frac{m}{m+1}bx^{\frac{1}{m}+1} - \frac{m}{2m+1}x^{\frac{1}{m}+2}$. The Quotient

$$\frac{\overline{3m+1}\times\overline{2mm+m}\,bb - \overline{3m+1}\times\overline{2mm+2m}\,bx + \overline{2m+1}\times\overline{mm+m}\,xx}{\overline{3m+1}\times\overline{2mm+m}\,b - \overline{3m+1}\times\overline{mm+m}\,x}$$

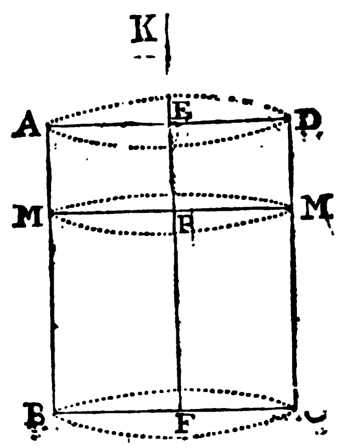
is equal to the Distance of the Center of Percussion of the Space M A M, from the Axis of Motion B B; and when x becomes $= b$, then the distance of the Center of Percussion of the whole Parabolic Space B A B, from the Axis of Motion B B is $= \frac{2m^3}{3m^3+mm}b = \frac{2m}{3m+1}b$.

Whence in the common Parabola, where m is $= 2$, the distance of the Center of Percussion of the Space M A M from the Axis of Motion B B is $= \frac{35bb - 42bx + 15xx}{35b - 21x}$, and the distance of the Center of Percussion of the whole Parabolic Space B A B, from the Axis of Suspension B B is $= \frac{4}{7}b = \frac{4}{7}$ the Perpendicular A D.

PROP. VI.

To find the Center of Percussion of a Cylinder, suspended by one of its Extremities, about which it is supposed to Revolve.

364. Let the Cylinder E F be suspended by the Extremity E, about which it is supposed to Revolve; 'tis evident that the Velocities of all the infinitely little and equal parts are proportional to the Spaces (the times being equal) which they describe; that is, proportional to their distances from the point of suspension. Suppose then E F $= a$, E P $= x$, the Circumference of the Base $= c$, and the Semi-diameter of the Base $= r$, then (*Art.* 341.) the Fluxion of the Moments is $= \frac{crx\dot{x}}{2}$, and the Fluxion of the Forces is $= \frac{crx^2\dot{x}}{2}$; and the Sum of all the Forces is $= \frac{crx^3}{6}$; which being divided by the Sum of all the Moments $\frac{crxx}{4}$, the Quotient $\frac{2}{3}x$ is $=$ to the distance of the point E from the Center of Percussion of the indeterminate Portion of the Cylinder A M M D, and $\frac{2}{3}a =$ to the distance of E, from the Center of Percussion of the whole Cylinder A B C D.



PROP. VII.

To find the Center of Percussion of a Cylinder, revolving about the point R, in the Axis produced.

365. Suppose R F $= a$, R E $= b$, E P $= x$, P p $= \dot{x}$, R P $= b + x$, and E F $= a - b$; then the Fluxion of the Moments is $= \frac{bcr\dot{x} + crx\dot{x}}{2}$, which being multiplyed by $b + x$, the Product $\frac{bbcr\dot{x} + 2bcrx\dot{x} + crxx\dot{x}}{2}$ is $=$ to the Fluxion of the Forces; and the Sum of all the Forces is $= \frac{bbcrx}{2} + \frac{bcrxx}{2} + \frac{crx^3}{6}$. which being divided by the Sum of all the Moments $\frac{bcrx}{2} + \frac{crxx}{4}$, the Quotient $\frac{6bb + 6bx + 2xx}{6b + 3x}$ is $=$ to the distance of the Axis of suspension (passing through the point R, and parallel to A D) from the Center of Percussion of the portion of the Cylinder M A D M; and when x becomes $= a - b$, then the distance of the point R from the Center of Percussion of the whole Cylinder is $= \frac{2aa + 2ab + 2bb}{3a + 3b}$.

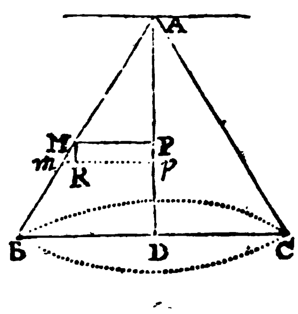
COROLLARY.

366. Hence if R E represent a Mans Arm, and E F a Staff or Cane, it is easie to determine the Point therein, which will strike with the greatest Force.

PROP.

PROP. VIII.

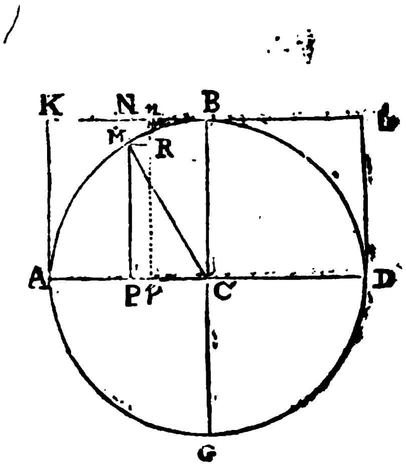
To Investigate the Center of Precussion of a Cone, Revolving about an Axis passing through the Vertex A, *and Parallel to the Base* BC.



367. The same things being suppos'd as in *Art.* 342. The Fluxion of the Moments is $= \frac{c r x^3 \dot{x}}{2 a a}$, and the Fluxion of the Forces is $= \frac{c r x^4 \dot{x}}{2 a a}$, and the Sum of all the Forces is $= \frac{c r x^5}{10 a a}$, which being divided by the Sum of all the Moments $= \frac{c r x^4}{8 a a}$, the Quotient $\frac{4}{5} x$ is = to the Distance of the Axis of Motion from the Center of Percussion of the Portion of the Cone generated by the Triangle APM; and $\frac{4}{5} a$ for the Distance of the said Axis from the Center of Percussion of the whole Cone.

PROP. IX.

To find the Center of Percussion of a Sphere, in respect of the Axis of Motion AK, *perpendicular to the Diameter* AD.



368. The Fluxion of the Moments is $= c x x \dot{x} - \frac{c x^3 \dot{x}}{2 r}$, and the Fluxion of the Forces is $= c x^3 \dot{x} - \frac{c x^4 \dot{x}}{2 r}$, and the Sum of all the Forces is $= \frac{c x^4}{4} - \frac{c x^5}{10 r}$; which being divided by the Sum of all the Moments $\frac{c x^3}{3} - \frac{c x^4}{8 r}$ the Quotient $= \frac{30 r x - 12 x x}{40 r - 15 x}$ is = to the Distance of the Axis of Motion AK from the Center of Percussion of the Portion of the Sphere generated by APM, and consequently the Distance of A from the Center of Percussion of the whole Sphere is $= \frac{4}{5} r$.

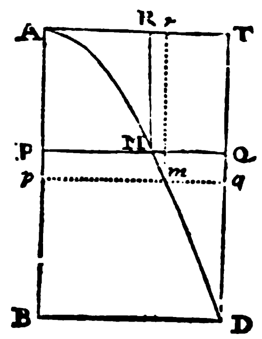
COROLLARY.

Hence 'tis easie to find the Center of Percussion of an Hemisphere, or of any Segment of a Sphere, in respect of the Axis of Motion AR, or in respect of any other point in the Diameter AD produc'd.

PROP. X.

To Investigate the Centers of Percussion of all sorts of Parabolical Conoids, in respect of AT *the Axis of Motion.*

369. The same things being suppos'd, as in *Art.* 345. the Fluxion of the Moments is $= \frac{c x^{\frac{2}{m}+1} \dot{x}}{2r}$; and consequently the Fluxion of the Forces is $= \frac{c x^{\frac{2}{m}+2} \dot{x}}{2r}$, and the Flowing Quantity, or the Sum of all the Forces is $= \frac{m}{6m+4} \times \frac{c x^{\frac{2}{m}+3}}{r}$. Which being divided by the Sum of all the Moments $= \frac{m}{4m+4} \times \frac{c x^{\frac{2}{m}+2}}{r}$; the Quotient $\frac{4m+4}{6m+4} \times x = \frac{2m+2}{3m+2} x$ is $=$ to the distance of the Axis of Motion AT from the Center of Percussion of the Indeterminate Portion of the Conoid, generated by the Revolution of the Space APM about the Axis AP; and $\frac{2m+2}{3m+2} b =$ to the Distance of the said Axis of Motion from the Center of Percussion of the whole Conoid.



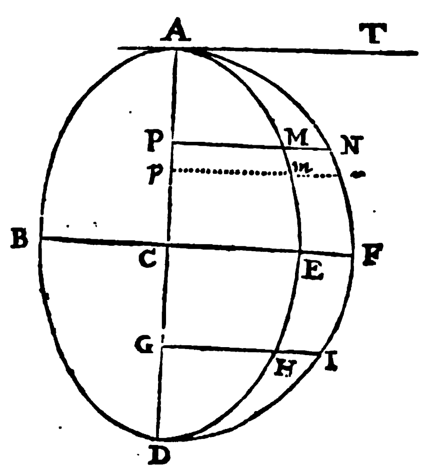
Hence in the common Parabola $ax = yy$, we have $m = 2$, and consequently, $\frac{2m+2}{3m+2} b = \frac{3}{4} b = \frac{3}{4}$ the Axis AB.

And if the Equation of the Curve be $aax = y^3$, then $m = 3$ and $\frac{2m+2}{3m+2} b$ is $= \frac{8}{11} b$.

PROP. XI.

To Investigate the Center of Percussion of the Elliptical Spheroid ABDE, *in respect of the Axis of Motion* AT, *passing through the Vertex* A, *and at right Angles to the Axis* AD.

370. The same things being suppos'd as in *Art.* 350. The Fluxion of the Moments is $= \frac{2acrx^2\dot{x} - crx^3\dot{x}}{2aa}$, and consequently the Fluxion of the Forces is $= \frac{2acrx^3\dot{x} - crx^4\dot{x}}{2aa}$, and the Fluent or the Sum of all the Forces is $= \frac{crx^4}{4a} - \frac{crx^5}{10aa}$, which being divided by the Sum of all the Moments $= \frac{crx^3}{3a} - \frac{crx^4}{8aa}$; the Quotient $\frac{30ax - 12xx}{40a - 15x}$ is $=$ to the distance of the Axis of Motion AT from the Center of Percussion of the Indeterminate Portion of the Spheroid generated by AMP. Whence the Distance of the said Axis AT from the Center of Percussion of the whole Spheroid is $= \frac{6}{5} a = \frac{6}{5}$ AC.



SECT.

SECT. XIII.

The Use of Fluxions

In Investigating the Centers of Oscillation.

371. THE Learned *Mersennus* so highly esteem'd this piece of Mathematical Philosophy, and thought it a Subject of so much difficulty; that he propos'd the Consideration thereof to the most Celebrated Geometers of his Age. And to find the Centers of *Oscillation* of all Figures, appear'd so intricate a Business, that the Illustrious *Cartesius* could only find them in some particular, and those the most easie Cases. Yea the Excellent Mr. *Hugens* was so discouraged in his first attempts (as he himself witnesseth) that he gave up the Cause, if not as desperate, yet until a more lucky turn of Genius should offer. Which accordingly happen'd, for afterwards he not only resolv'd all *Mersennus*'s Problems, but even others much more difficult.

But now that we are taught (in resolving Problems) to contemplate things in their first Principles, and to follow the steps of Nature in our Inquiries: The Difficulties which appear'd insuperable to former Ages, are easily remov'd. Which will further appear in handling the present head.

I. The Nature of a Pendulum (I presume) is obvious to every Reader: A Line, a Plain, or a Solid, suspended either mediately (by a Thread, *&c.*) or immediately to an immovable Point or Axis, and Vibrating by the sole Force of its own Gravity, generally going under that Denomination.

II. And those Pendulums, which have the same Length, and are agitated by the same Force, describe equal Arches in equal Times; that is, their respective Vibrations will be *Isochronal & vice versa.*

III. Hence if the solid ABDE, be suspended to the Axis AT in A, then the Vibrations of the Solid will be *Isochronal* to those of the simple Pendulum, whose length is equal to the Distance of the Axis of Motion AT, from the Center of Percussion of the Solid, and which is agitated by a Force equal to that of the Center of Percussion. For it is evident, that all the Forces of the Solid are (as it were) united in the Center of Percussion: And if we suppose the Solid to be contracted into that point, and the same Force to remain, then 'tis manifest that both Pendulums will be simple ones; and because they are equal in Length, and agitated by the same or equal Forces, their Vibrations must be *Isochronal.*

IV. That point of the Figure wherein all the Forces are united is but one, and consequently one only simple Pendulum can be made, whose Vibrations (the Forces in both being equal) shall be *Isochronal* to those of the whole Solid, and because the point in the Figure, wherein all the Forces are united, determines the length of the simple *Isochronal Pendulum*; and is that wherein all the Figure is suppos'd to be contracted with all the Forces, while it Vibrates; therefore it is called the *Center of Oscillation of the Figure.*

V. Hence it is manifest, that the Centers of *Oscillation* and *Percussion* in every Figure are the same; and the Investigation of that is in every respect the same with that of this.

General Conſectaries.

372. If a Line AB be ſuſpended by the point A, and be ſuppos'd to Vibrate; the Diſtance of the Center of Oſcillation from A the point of Suſpenſion is $= \frac{2}{3}$ AB; and hence the Reaſon appears, why the Vibrations of a Rod or Virga, ſuſpended by one end, are Iſochronal to thoſe of a ſimple Pendule, whoſe length is $= \frac{2}{3}$ parts the length of the Rod.

CONSECTARY II.

If the Iſoſceles Triangle ABC, be ſuppos'd to Vibrate about an Axis, paſſing through the Vertex A, and in the ſame Plain with the Triangle; then the Diſtance of the Center of Oſcillation of the Triangle from the Vertex A is $= \frac{3}{4}$ the perpendicular AD. In like manner, the Centers of Oſcillation of the Cylinder, the Sphere, all ſorts of Parabolic Spaces, and Parabolical Conoids, *&c.* are inveſtigated and determined in the reſpective Examples of the preceeding Section: And by the ſame Method, the Center of Oſcillation of any Figure may be inveſtigated.

SECT.

SECT. XIV.

The Use of Fluxions

In Astronomy.

THE Study of the Heavenly Motions is so sublime a piece of Humane Knowledge, that the most learned Men of all Ages have apply'd themselves to the same. But notwithstanding this, the Systems they have fram'd, and the Causes they have assign'd for the Celestial Motions are, for the most part, so inconsistent with Reason and so remote from Truth, that it is strange they should have been (for so many Ages) so universally receiv'd.

It would be improper in this place to trouble the Reader with an Account of the *Astronomy* of the *Ancients*, and by what means they endeavour'd to salve the Celestial appearances.

I shall only tell him, that tho' the Croud of Ancient Astronomers seem to have had but very confus'd Notions concerning those upper Regions, yet some there were of a more refin'd Genius, that doubted not to assert even that System which now a-days passes for the best; such as *Pythagoras* and *Philolaus*, who asserted the Motion of the Earth and Stability of the Sun.

But this seemed so incredible to the wisest of Succeeding Ages, that not being able to render any Solid Reason for this Opinion, besides the Authority of an Illustrious Sage, it was exploded out of all places, which seemed most to countenance Ancient Learning, and scarce allowed a place in the rank of Possibilities.

Thus the *Ancient Pythagorean System of the World* lay slighted and neglected, untill the excellent *Copernicus*, well perceiving the defects and inconsistencies of all the other Hypotheses, restored it to Light.

This Infant of Ages thus again inspired with a new Breath, quickly became known to *Urania's* Sons, and among all those that approved of the same, none was more eminent than the great *Kepler*; he not only defended the *Pythagorean System* of the World, but advanced yet further, and found that every Primary Planet describ'd an Elliptic Orbit about the Sun, placed in one of the Foci; and that the Areas, which every Planet describes about the Sun, are proportional to the times of Description. This excellent Person observed also, that their Periodic times are in a Sesquiplicate Ratio of their mean distances from the Sun: Which law the Satellites observe also, in respect of their Primary Planets, as the *Learned Cassini* observes.

These are discoveries wholly unknown to the Ancient Astronomers, and whereof even their first Authors could give no demonstrative account; they saw that these proportions were confirmed by the most accurate observations, but to demonstrate the truth of them *à priori*, was a task too hard to be attempted by any unacquainted with a Method somewhat resembling the great and infinite Architect: I mean the Method or Analysis of Infinites.

And since his days, several other great Persons have appeared, who have made incredible advances this way.

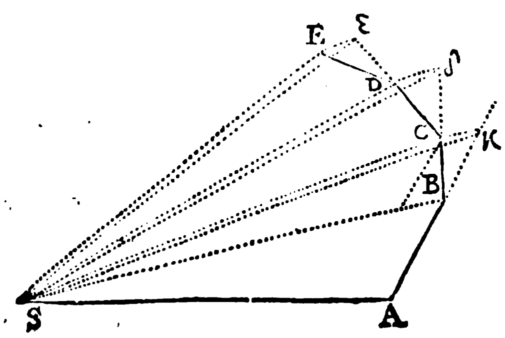
I shall confine my self at present to some of those which have been made by help of Fluxions.

PROP.

PROP. I.

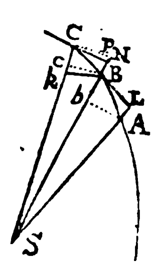
The Area's which Bodies Revolving about an immovable Center describe by Rays drawn to the same, are proportional to the times of description, and are all in the same immovable Plain.

373. Let the Time be divided into equal parts, and suppose in one of them, a Body describes the Space A B (by a power which it has to move in the right Line A *x* from A towards *x*) in the next Moment of Time, if nothing hindred, it would move from B to *x*, describing the Line B *x* equal to A B, so that drawing the Rays A S, B S, *x* S to the immovable Center S, the Areas A S B and B S *x* described, would be equal: But when the Body comes to B, let a Force in S attract the same, and by one single but strong Impulse, make the Body deviate from the right Line B *x*, and move in the right Line B C; draw *x* C parallel to B S, intersecting B C in C, then at the end of the second Moment of Time, the Body will be found in C, in the same Plain with the Triangle A S B; joyn S C, and the Triangle S B C, because of the Parallels S B and C *x*, will be equal to the Triangle S B *x*, and consequently it will also be equal to the Triangle S A B: In like manner, if the Central Force (or *Vis Centripeta*) act successively in C, D, *&c.* and make the Body in successive Moments of Time describe the Lines C D, D E, *&c.* they will be in the same Plain, and the Triangle S C D will be equal to the Triangle S B C, and S D E will be equal to S C D = S B C. Whence it is manifest that a Body revolving about an immovable Center in an immovable Plain, describes equal Areas in equal Times; and by composition, the Area S A C S is to the Area S A E S, as the Time which the Body takes to describe *that*, is to the Time it takes to describe *this*.



Let the Number of the Triangles be encreased, and their Breadth diminished *in infinitum*, then the Perimeter ABCDE will be a Curve Line, and consequently the *Vis Centripeta* which perpetually draws back the Body from off the Tangent of this Curve, acts continually; and the Areas S A C S, S A E S proportional to the Times of their description, will also in this Case be proportional to the same Times. Q. E. D.

CONSECTARY I.



374. If a Body revolving in the Curve A B C, be attracted by a Central Force in S, and if the Body describe the infinitely little portions of the Curve A B and B C in equal Times, then the infinitely little Triangles A S B, B S C will be equal: and if on the Center S, and with the Radii S A, S B, the little Arches A *b*, B *c*, be describ'd, then the Triangle S A B or S A *b* is = $\frac{1}{2}$ S A × A *b*, and the Triangle S B C is = $\frac{1}{2}$ S B × B *c*, therefore it is, $\frac{1}{2}$ S A : $\frac{1}{2}$ S B :: S A : S B :: B *c* : A *b*; that is, the infinitely little Arches A *b*, B *c*, are reciprocally proportional to the Radii S A, S B.

DEFINITION. I.

The Center of Attraction is that point to which the Revolving or moving Body is attracted or impelled by the Force or Impetus of Gravity; Thus the Sun is such in the respect of the Primary Planets, and the Earth in respect of the Moon.

DEFI-

DEFINITION. II.

Paracentric Motion of Impetus is so much as the revolving Body approaches nearer to or receeds farther from the Center of Attraction; thus if S be the Center of Attraction, and if a Body in A move to B, then S B — S A = B *b*, is called the Paracentric Motion of that Body.

DEFINITION. III.

Circular Velocity of a Body is measured by the Arch of a Circle; thus if a Body in A move to B, or *b*, its Circular Velocity is measured by the Arch of the Circle A *b*, describ'd on the Center of Attraction S, and the Circular Velocity of a Body moving from B to C is measured by the Circular Arch B C.

DEFINITION. IV.

Conatus Excussorius, is measured by a Line let fall from a point infinitely near to another point, perpendicular to a Line drawn to touch the Curve in that other point, whence it is manifest that the *Conatus Excussorius Circulationis*, or *Conatus Centrifugus* may be express'd by P N the Versed Sine of the Angle of Circulation C S N (or by *c k*, because the difference between the Radii S C, S B is incomparably little) for the Versed Sine is equal to a perpendicular let fall from one end of the Arch to a Tangent drawn to the other end of the Arch.

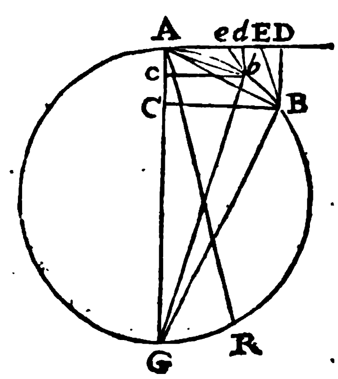
DEFINITION. V.

Solicitatio Paracentrica Gravitatis vel Levitatis, or the *Paracentric Solicitation* of Gravity or Levity is express'd by the right Line A L, drawn from the point A, parallel the Ray S B (infinitely near S A) untill it intersect the Tangent B L.

LEMMA I.

The Versed Sines of infinitely little Arches are in a duplicate Ratio of the Chords of the said Arches.

375. Let the right Line A D touch the Circle A B G in A, then D A B is the Angle of Contact; Let AB be an infinitely little Arch, A B the Chord and A C the versed Sine thereof, I say A C or B D is as the Square of A B; that is, if another infinitely little Arch A *b* be taken, then the Versed Sine A *c* (or *b d*) : Versed Sine A C (or B D) :: A *b q* : A B *q*.



Draw the Diameter AG, and draw the Lines G B, G *b*; then by the property of the Circle, we have AB*q* = AC × AG and A *b q* = A G × A *c*; whence it is, A B *q* : A *b q* :: A C × A G : A *c* × A G :: A C : A *c* :: B D : *b d*.

Now when the points B, *b*, are infinitely near the point A, then the Chords A B, A *b*, are equal to the Arches A B, A *b*, and consequently the Versed Sines A C, A *c*, or the Subtences of the Angle of Contact B D, *b d*, are in a duplicate Ratio of the Conterminal Arches A B, A *b*.

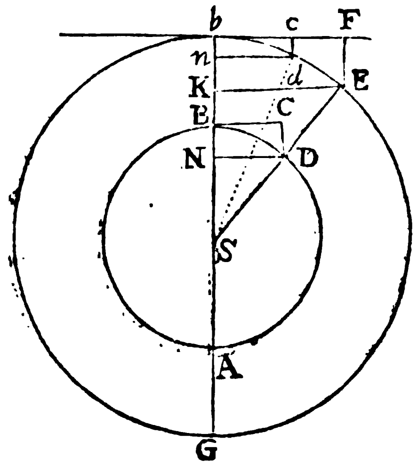
And if the the Lines B E, *b e*, subtend the Angle of Contact D A B, and be parallel to any Line (less than the Diameter A G) drawn within the Circle, as A R, then the Lines B E, *b e*, will be as the Squares of the Conterminal Arches A B, A *b*, for B D : *b d* :: B E : *b e* :: A B *q* : A *b q*.

CORROLLARY.

376. The Subftences of the Angles of Contact of Curves, whofe Curvature in the point of Contact is the fame with that of fome Circle (or when the difference between them is incomparably little) are in a duplicate Ratio of the Conterminal Arches.

LEMMA. II.

377. In unequal Circles ABD, GbE, if the infinitely little Arches BD, bd be equal, then the Verfed Sines BN, bn of thofe Arches will be reciprocally proportional to the Radii SB, Sb.



Produce SD unto E, and draw EF parallel to Sb, and draw the Lines DN, EK, dn perpendicular to SB.

Then it is, $bK : BN :: Sb : SB$.

And $bn : bK :: bdq : bEq$.

But $bE = \frac{Sb}{SB} BD$.

Therefore, $bn : BN : Sb \times BDq : \frac{Sbq}{SB} BDq$.

That is, $bn : \frac{bq}{Sb} : \frac{bdq}{SB}$.

And fuppofing $bd = BD$,

We have $bn : BN :: SB : Sb$. Q.E.D.

PROP. II.

The Conatus Centrifugi *(or* Vires Centripetæ*) of Bodies Revolving in equal Circles, with an equable Motion, are in a duplicate Ratio of their Velocities.*

378. The *Conatus Centrifugus* is equal to the Verfed Sine of the Angle of Circulation, and the Verfed Sines of Arches infinitely little are in a duplicate Ratio of the Chords of thofe Arches; that is, in a duplicate Ratio of the Arches themfelves, and the Velocities (the times being fuppofed equal) are as the Arches; therefore the *Conatus Centrifugi* are in a duplicate Ratio of the Velocities.

CONSECTARY I.

379. If two Bodies B, b, revolve in unequal Circles, ABD, GbE, and defcribe the Areas SBD, Sbd; then the *Conatus Centrifugi* (or *Vires Centripetæ*) DC, dc, will be in a Ratio compounded of the duplicate Ratio of the Velocities Directly, and the fimple Ratio of the Radii Inverfely.

For if the Radii be equal, the *Conatus Centrifugi* are as the Squares of the Velocities; and if the Velocities be equal, the *Conatus Centrifugi* are reciprocally as the Radii; therefore if neither the Radii nor the Velocities be equal, the *Conatus Centrifugi* are in a Ratio compounded of the Rationes of the Squares of the Velocities directly, and of the Radii Inverfely.

This *Corrollary* is demonftrated more univerfally, in one of the fteps of the Second Lemma; for it is there, $bn : BN :: \frac{bdq}{Sb} : \frac{BDq}{SB}$.

CON-

CONSECTARY II.

And if the Bodies B, *b*, describe the equal Areas BSD and *b*S*d* in equal times (that is if $SB \times BD = Sb \times bd$, then $bd : BD :: SB : Sb$) then the Velocities BD and *bd* will be reciprocally as the Radii, and the Squares of the Velocities will be as the Squares of the Radii Inversely, whence the proportion $bn : BN :: \frac{bdq}{Sb} : \frac{BDq}{SB}$ will become $bn : BN :: \frac{SBq}{Sb} : \frac{Sbq}{SB} :: \overline{SB}|^3 : \overline{Sb}|^3$, that is the *Conatus Centrifugi* are reciprocally in a Triplicate Ratio of the Radii.

CONSECTARY III.

If the Velocities be directly as the Radii, then the Periodic Times will be equal, and the Analogy $bn : BN :: \frac{bdq}{Sb} : \frac{BDq}{SB}$ will become $bn : BN :: Sb : SB$; that is, the *Conatus Centrifugi* are proportional to the Radii.

CONSECTARY IV.

If the the Bodies B, *b*, describe the Arches BD, *bd* in equal-times, then the Periodic time of *b* will be to the Periodic time of B, as $\frac{Sb}{bd}$ is to $\frac{SB}{BD}$; because the Times are directly as the Spaces and reciprocally as the Velocities; and because bn: BN $(:: dc : DC) :: \frac{bdq}{Sb} : \frac{BDq}{SB} :: \frac{SB}{BDq} : \frac{Sb}{bdq} ::$ (multiplying by $SB \times Sb$) $\frac{Sb \times SBq}{BDq} : \frac{SB \times Sbq}{bdq}$. Therefore the *Vires Centripetæ* are in a Ratio compounded of the Rationes of the Radii directly, and the Squares of the Periodic Times Inversely.

CONSECTARY V.

And if the Squares of the Periodic Times be as the Radii, that is if $\frac{Sbq}{bdq} : \frac{SBq}{BDq} :: Sb : SB$, then it will be $bn : BN :: \frac{Sb \times SBq}{BDq} : \frac{SB \times Sbq}{bdq} ::$ (by substitution) $Sb \times SB : Sb \times SB$; that is, the *Vires Centripetæ* are equal; and because $\frac{Sb}{bdq} = \frac{SB}{BDq}$, therefore $\sqrt{Sb} : \sqrt{SB} :: bd : BD$; that is, the Velocities are in a Subduplicate Ratio of the Radii. *Et vice versa.*

CONSECTARY VI.

And if the Squares of the Periodic Times be as the Squares of the Radii, that is if $\frac{Sbq}{bdq} : \frac{SBq}{BDq} :: Sbq : SBq$, then it will be $bn : BN :: \frac{Sb \times SBq}{BDq} : \frac{SB \times Sbq}{bdq}$ :: (by substitution) $Sb \times SBq : SB \times Sbq :: SB : Sb$; that is the *Vires Centripetæ* (or *Conatus Centrifugi*) are reciprocally as the Radii; and because (in this Supposition)

SB*q*

$SBq \times \frac{Sbq}{bdq} = Sbq \times \frac{SBq}{BDq}$, therefore $BD = bd$; that is the Velocities are equal: *Et Vice versa.*

CONSECTARY VII.

If the Squares of the Periodic Times be as the Cubes of the Radii, that is, if $\frac{Sbq}{bdq} : \frac{SBq}{BDq} :: \overline{Sb}|^3 : \overline{Sb}|^3$. Then it will be $bn : BN :: \frac{Sb \times SBq}{BDq} :$ $\frac{SB \times Sbq}{bdq} :: Sb \times \overline{SB}^3 : SB \times \overline{Sb}^3 :: SBq : Sbq$; that is, the *Vires Centripetæ* are reciprocally in a duplicate Ratio of the Radii; and because $\frac{SB}{bdq} = \frac{Sb}{BDq}$, therefore $\sqrt{SB} : \sqrt{Sb} :: bd, BD$; that is, the Velocities are reciprocally in a Subduplicate Ratio of the Radii: *Et vice versa.*

SCHOLIUM.

And because it is found by observation, that the Squares of the Periodic Times of Planets, are as the Cubes of their distances from the Sun, and that in equal Times they describe equal Areas about the Sun; therefore it is manifest, that the Sun is the Center of all the Planetary Motions, and that the *Vis Centripeta* (or Force of Gravity) of one Planet is to the *Vis Centripeta* of another Planet, as the Square of this Planets distance from the Sun is to the Square of that Planets distance from the Sun.

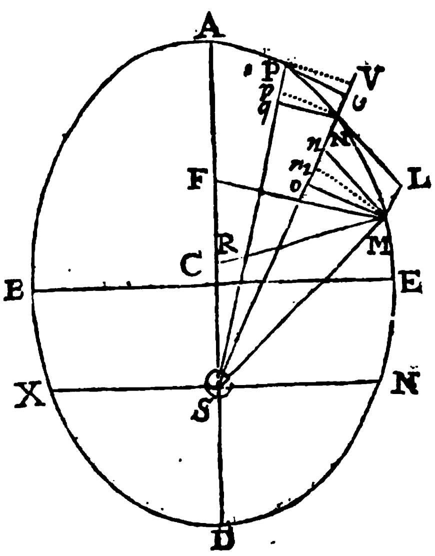
It is also evident that these Planets which are nearest the Sun move swiftest, for the Velocity of one Planet is to the Velocity of another Planet as the distance of this Planet from the Sun is to a mean proportional between the distances of this and that Planet from the Sun.

PROP.

PROP. III.

If the Areas which a Body, revolving about an Immovable Center, describes by Rays drawn to the said Center, be proportional to the Times of description; the Elementum or infinitely little Increment or Decrement of the Paracentric Impetus is equal to the difference or Sum of the Paracentric Solicitation (Solicitation *of Gravity, or the impression made by the Action of Gravity or Levity, or any such like Cause) And twice the* Conatus Centrifugus, *viz. to the Sum, if it be the Solicitation of Levity; or to the Difference, if the the Paracentric Solicitation arise from the Action of Gravity.*

380. From the points P and M, draw the Lines P *v*, M *o* perpendicular to S N; then because the Triangles P S N, N S M are equal, (the times being supposed equal) therefore (because the Base S N is common to both) the Altitudes P *v*, M *o* are equal; take N *n* = L M, and draw M *n* parallel to L N; then the Triangles P N *v*, M *n o* will be equal and similar, and P N = M *n*, and N *v* = *o n*; again in the right Line S N (produced if need be) take S V = S P, and S *m* = S M, then is N V the difference between the Radii S P and S N, and N *m* is the difference between the Radii S N and S M; now N V is = (N *v*) *n o* + V *v*; and N *m* is = N *n* + *n o* — *o m*, therefore N V — N *m* = V *v* + *m o* — N *n* = to the differentio-differential, or infinitely little Increment or Decrement of the Paracentric Velocity, = 2 *m o* — N *n* (because V *v* and *m o*, the Versed Sines of two Angles and Radii, whose difference is incomparably little, are equal) now the difference between the Radii S P, S N, and S N, S M, expresses the Paracentric Velocity, and their difference again, is the infinitely little Increment or Decrement of the said Paracentric Velocity; and *m o* or V *v* is equal to the *Conatus Centrifugus Circulationis*, and N *n* is = to the Solicitation of Gravity; therefore the Elementum of the Paracentric Velocity is equal to the difference between twice the *Conatus Centrifugus* (2 *m o*) and the simple Solicitation of Gravity (N *n*) or (which may be proved in like manner) to the Sum of twice the *Conatus Centrifugus*, and the simple Solicitation of Levity.



CONSECTARY I.

381. Hence it appears, that if the Solicitation of Gravity prevail, then N V — N *m* will be Negative, that is N *m* will be greater than N V, and the Descensive Paracentric Velocity increases, and the Ascensive decreases. But if twice the *Conatus Centrifugus* prevail, then N V — N *m* will be Positive, and the Ascensive Paracentric Velocity increases, and the Descensive Decreases.

CONSECTARY II.

If the Elementum or infinitely little Increment or Decrement of the Paracentric Velocity be given, the Solicitation of Gravity or Levity may be found; for the *Conatus Centrifugus* is always given (*Art.* 379. § 2.) it being constantly in a Triplicate Reciprocal Ratio of the Radii.

PROP. IV.

The Angles which a Planet describes about the Sun, in equal times, are Reciprocally in a duplicate Ratio of the Radii.

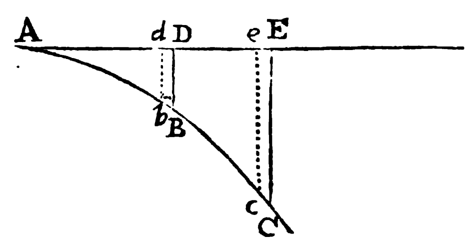
382. The Circular Velocities are in a Ratio compounded of the Rationes of the Angles and Radii, Joyntly; therefore the Angles are in a Ratio componnded of the direct Ratio of the Circular Velocities, and the reciprocal Ratio of the Radii; but because in equal times, the Areas are equal (*Art.* 379. N. 2.) therefore the Circular Velocities are reciprocally as the Radii, and consequently, the Angles are reciprocally in a duplicate Ratio of the Radii.

And such are the apparent Diurnal Motions of the Planets observ'd from the Sun (for days, in such Cases, are parts of Time little enough, especially in Planets more remote from the Sun) which are almost reciprocally as the Squares of their distances from the Sun; so that a Planet, in a given Element of Time, describes but the fourth part of that Angle, which it would describe at half its present Distance from the Sun.

LEMMA III.

The Spaces which a Body describes in the beginning of its descent, are in duplicate Ratio of the Times.

383. Let the right Line A E be divided into an infinite number of equal parts *d*D, *e*E, &c. representing equal Moments of Time, and draw the Ordinates D B, E C, &c. proportional to the Velocities of the heavy Body, at the end of the Times represented by A D, and A E, and describe the Curve A B C; now because the Space which a Body describes is proportional to the Time of description and the Velocity joyntly it is evident, that the Space which the heavy Body describes in the Moment of Time D *d* is proportional to the Rectangle D *b*, and the Space which the same heavy Body describes in the Moment of Time E *e* is proportional to the Rectangle E *e*; whence the Space which the Body describes in the Time A D, is to the Space it describes in the Time A E as the Curvilineal Space A D B is to the Curvilineal Space A E C, but when the Body begins to descend, the Ordinates D B, E C are indefinitely near the point A; in which case, the Trilineal Figures A D B, A E C, become rectilineal similar Triangles, the indefinitely little portions A B, B C being in the same streight Line: now the Areas of similar Triangles are in a duplicate Ratio of the Homologous Sides; that is, the Area A D B: Area A E C : A D *q* : A E *q*, therefore the Spaces which a heavy Body describes in the beginning of its descent are in a duplicate Ratio of the Times. Q. E. D.

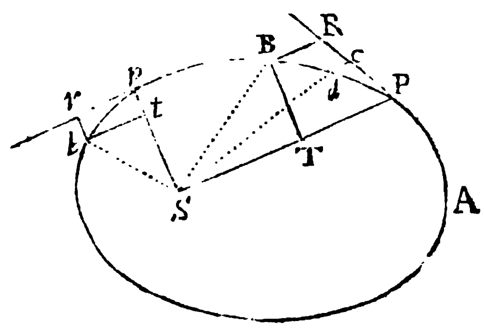


PROP.

PROP. V.

If a heavy Body revolving in the Periphery of a Curve, about an immovable Center, describe Areas proportional to the Times. 'Tis required to find the Law of the Vis Centripeta *tending to the said Center.*

384. Suppose a Body P to be projected in the Line P R from P towards R, and let the Body at the same time be attracted by a Force in S, so that by a Motion compounded of the projectile and attractive Forces, it describe the Curve A P p; and let the Line P R touch the said Curve in P; draw S P, and assume any point B in the Curve indefinitely near P; and draw B R parallel to S P, and BT perpendicular S P; assume another point p in the Curve; and draw S p, the Tangent pr and rb parallel, and bt perpendicular to S b, and suppose the Body describes the Arches P d, pb in equal Times; and draw dc parallel to S P, then the Ratio of the *Lineola Nascens* BR to the *Lineola Nascens* br, is compounded of the Rationes of B R to dc, and of dc to br: but (*Art.* 375. 376.) B R is to dc as the Square of the Arch P B is to the Square of the Arch P d; and because the Arches P B, P d are indefinitely little, they are proportional to the Triangles P S B, P S d; (*Art.* 383.) that is, they are proportional to the Times the Body takes to describe them, or to the Times which the Body takes to describe the Arches P B, $p\,b$ and consequently B R is to dc as the Square of the time which the Body takes to describe the Arch P B, is to the Square of the Time that it takes to describe the Arch $p\,b$; again, because P d and $p\,b$ are supposed to be describ'd in equal Times, therefore dc is to br, as the *Vis Centripeta* in P is to the *Vis Centripeta* in p; whence it is evident that B R is to br in a Ratio compounded of the Rationes of the Squares of the Times in which the Arches P B, $p\,b$ are describ'd, and of the *Vis Centripeta* in P to the *Vis Centripeta* in p; that is, (because the Times of describing the Arches P B, $p\,b$, are proportional to the Triangles P S B, p S b, or to the Rectangles S P $\times$ B T, S $p \times bt$.)



$$BR : br :: V \times SPq \times BTq : v \times Spq \times btq.$$

And by Division $\frac{BR}{SPq \times BTq} : \frac{br}{Spq \times btq} :: V : v.$

$$\text{Or } \frac{Spq \times btq}{br} : \frac{SPq \times BTq}{BR} :: V : v.$$

That is, the *Vis Centripeta* in P is as the Solid $\frac{SPq \times BTq}{BR}$ Inversely.

385. This may be more briefly demonstrated thus: If the Times be equal, B R is as the *Vis Centripeta*, and if the *Vis Centripeta* be given, then B R (*Art.* 373. 383.) is as the Square of the Times; and if neither the Times nor the *Vis Centripeta* be equal, then B R is (supposing V = to the *Vis Centripeta* in P, and T = to the Time of description) as V T², therefore V is as $\frac{BR}{T^2}$; and because the Time is as the Area P S B or as the Rectangle S P $\times$ B T, therefore V is as $\frac{BR}{SPq \times BTq}$ directly, or as $\frac{SPq \times BTq}{BR}$ Inversely. Q. E. D.

COR-

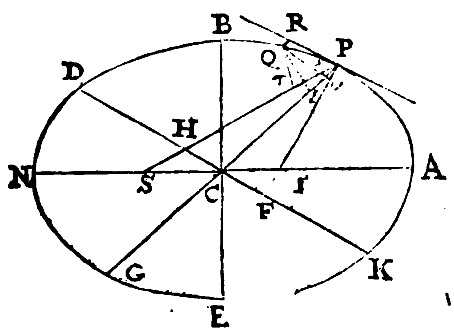
COROLLARY.

386. Hence if any Figure, as APp be given, and the point S to which the *Vis Centripeta* tends; then the Value of the Solid $\frac{SPq \times BTq}{BR}$ may be determined from the Nature of the Figure; and consequently the Law of the *Vis Centripeta*, which is reciprocally as the said Solid, may be found.

PROP. VI.

If a Body Revolve in the Periphery of an Ellipsis. 'Tis requir'd to find the Law of the Vis Centripeta, *tending to the Focus of the Ellipsis.*

387. Let ABD be the Ellipsis, and S the Focus, to which the *Vis Centripeta* tends. Draw the Axis AN, and the Conjugate Diameter BE; draw the Line PR touching the Curve in any point (P) and draw the Diameter PG, the Conjugate Diameter DK, PF perpendicular to DK, and Qv parallel to PR: draw SP intersecting DK in H, and intersecting Qv in x; and draw QR parallel to SP. Then (*Art.* 51.) PH = AC; Draw QT perpendicular to SP; and suppose the Parameter of the Axis $\left(\frac{2BCq}{AC}\right) = L$. Then,



$L \times QR : L \times Pv :: QR : Pv :: Px : Pv :: PH : PC :: AC : PC.$

$L \times Pv : Gv \times Pv :: L : Gv.$

$Gv \times Pv : Qvq :: CPq : CDq.$

And because Qvq is $= Qxq$, when Q is infinitely near P.

$Qxq\ (= Qvq) : QTq ::$ (by similar Triangles) $HPq\ (= ACq) : PFq ::$ (*Art.* 60.) $CDq : CBq$.

And multiplying the respective Terms of these Analogies into one another, there will arise this,

$L \times QR : QTq :: L \times AC \times CPq : Gv \times CBq \times CP.$

That is, $L \times QR : QTq :: 2BCq \times CPq : Gv \times BqC \times CP.$

And $L \times QR : QTq :: 2PC : Gv.$

But when the point Q is indefinitely near P, then $2PC = Gv$.

Whence $L \times QR = QTq$.

And multiplying both sides of the Equation by $\frac{SPq}{QR}$, we shall have $L \times SPq = \frac{SPq \times QTq}{QR}$. (*Art.* 386.) Therefore the *Vis Centripeta* is reciprocally as $L \times SPq$; and because L is a determinate Quantity, therefore the *Vis Centripeta* is reciprocally as the Square of (SP) the Distance of the Body in P from the Center of Attraction S. Q. E. I.

COR-

COROLLARY. I.

388. The Parameter of the Axis (L) is $= \frac{QTq}{QR}$.

COROLLARY. II.

If the Center of Attraction S, and the adjacent Vertex N, be suppos'd immovable, and if the other Foci I approach nearer and nearer to S, and at last coincide with the same, then the Body will revolve in the Periphery of a Circle, and the Law of the *Vis Centripeta* will be the same as in the Ellipsis.

COROLLARY. III.

If the Vertex's A and N be given, and if the Focus I coincide with A, and the Focus S coincide with N, then the Ellipsis A P N will become a streight Line, coinciding with the Diameter A N, and the Body will move in the same, without any Attraction from without the Line.

COROLLARY ·IV.

If the Vertex N, and the (Focus of the Ellipsis, or) Center of Attraction S be given, and if the other Focus I be at an infinite Distance from S, then the Ellipsis N P A will degenerate into a Parabola, and the *Vis Centripeta* in P will be as the square of the Distance S P Inversely.

COROLLARY. V.

The same things being suppos'd, if the Focus I be at more than an infinite Distance from S; that is, if it fall on the contrary side of N in respect of S, then the Body will move in the Curve of an Hyperbola, and the *Vis Centripeta* will be reciprocally as the Square of its Distance from the Focus S.

COROLLARY. VI.

If the Focus I and the Vertex A be given; and if the Center of Attraction S be suppos'd at an infinite Distance from I, then the Curve A P will be a Parabola, and the *Vis Centripeta* will be the same in every point of the Curve; and contrarily, if a Body moving at first in a streight Line, be attracted to a Center at an infinite Distance from the same, then that Body will move in the Curve of a Parabola, and the Center of attraction will be in the Axis of the Parabola, at an infinite Distance from the Vertex.

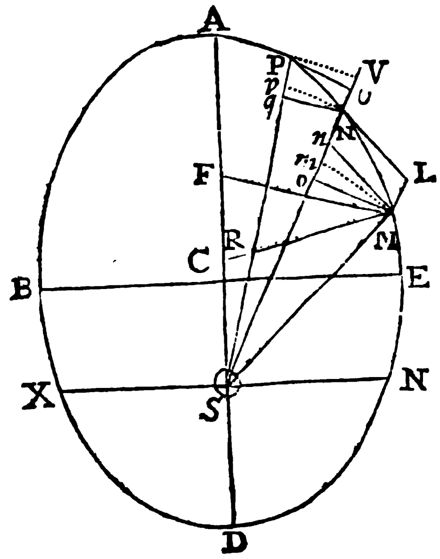
SCHOLIUM.

It may be observ'd that the Paracentric Solicitation of Gravity, *and the* Vis Centripeta, *are Terms signifying the same thing.*

PROP. VII.

The Solicitation of Gravity, or Vis Centripeta *of a Planet, is to the* Conatus Centrifugus *of the same Planet, as its present Distance from the Sun, is to* $\frac{1}{4}$ *the Parameter of the Planetary Ellipsis.*

389. Suppose SM = D, and L = to the Parameter of the Axis, and let $t \times L$ be a constant Plain, equal to twice the Elementary Triangle MSN; then the Arch M*m* is $= \frac{t \times L}{D}$, and M*mq* is $= \frac{t^2 \times L^2}{D^2}$,



and $mo = \frac{Mmq}{2\,SM} = \frac{t^2 \times L^2}{2\,D^3}$ = to the *Conatus Centrifugus.*

Again, the Solicitation of Gravity is as D^2 Inversely, or (*Art.* 374.) as $\overline{Mm}|^2$, or $\frac{t^2 \times L^2}{D^2}$ directly, or as (dividing by the Invariable Quantity $\frac{1}{2}$ L) $\frac{2 \times t^2 \times L}{D^2}$ directly. Whence 'tis evident that the Solicitation of Gravity is to the *Conatus Centrifugus*, as $\frac{2 \times t^2 \times L}{D^2}$ is to $\frac{t^2 \times L^2}{2\,D^3}$, or as D is to $\frac{1}{4}$ L, and because $\frac{1}{4}$ L is an invariable Quantity. The Rationes of the Solicitation of Gravity to the *Conatus Centrifugus* are proportional to the Distances of the Planet from the Sun.

PROP. VIII.

The greatest Ascensive or Descensive Paracentrick Velocity of a Planet, is when the Distance of the Planet from the Sun is equal to $\frac{1}{2}$ *the Parameter of the Axis of the Ellipsis.*

390. Draw S W perpendicular to the Axis AD, I say the greatest Paracentric Velocity is in W or X. For the Solicitation of Gravity is to the *Conatus Centrifugus*, as D is to $\frac{1}{4}$ L; and the Solicitation of Gravity is to twice the *Conatus Centrifugus*, as D is to $\frac{1}{2}$ L; and because S W = D is = $\frac{1}{2}$ L, therefore in the point W (or X) the Solicitation of Gravity is equal to twice the *Conatus Centrifugus*; and (*Art.* 380.) consequently the Fluxion of the Paracentric Velocity is = o: Whence it is evident, that if on S as a Center, a Circle be describ'd with a Radius = $\frac{1}{2}$ the Parameter of the Axis, it will cut the Orbit of the Planet in two points W and X, in which the greatest Paracentric Velocity happens.

COROLLARY.

391. The *Conatus Centrifugus* of Receding from the Sun, is always less than the Solicitation of Gravity. For the Solicitation of Gravity is always to the *Conatus Centrifugus*, as the Distance of the Planet from the Focus is to $\frac{1}{4}$ part of the Parameter of the Axis; and in the Ellipsis, the Distance of a Planet from the Focus, is always greater than $\frac{1}{4}$ part the Parameter of the Axis. Therefore, &c.

PROP.

PROP. IX.

The Impetus *which a Planet acquires (during the whole time of its Motion) by the continu'd Attraction of the Sun, are proportional to the Angles of Ci culation; that is, as the Angles of apparent Motion from the Sun.*

392. I say, The *Impetus* which a Planet acquires, as it moves from A to P, is to the *Impetus* which it acquires, moving from A to M, as the Angle A S P is to the Angle A S M; For the Increments of those Angles (*Art.* 382.) are Reciprocally as the Squares of the Radii or Distances; that is, (*Art.* 386.) as the Solicitations or Impressions of Gravity: Therefore the Sum of *these* is proportional to the Sum of *those*; that is, the Sum of all the *Impetus* or Impressions of Gravity acquir'd from A to P, is to the Sum of all the Impressions of Gravity acquir'd from A to M, as the Angle A S P is to Angle A S M.

COROLLARY.

393. Hence in the point W (in which an Ordinate to the Axis drawn through the Focus S, interfects the Ellipsis) the *Impetus* which a Planet has acquir'd since it descended from the Aphelion, is equal to half the *Impetus* acquir'd from the Aphelion to the Perihelion; and in the said point W, the Distance of the Planet from the Sun is = ½ the Parameter of the Axis of the Figure.

And the *Impetus* which a Planet, describing any Arch of its Orbit, acquires, is to the *Impetus* acquir'd in a Semi-revolution, as the Angle of apparent Motion is to two right Angles; and here is mean'd the *Impetus* impress'd by Gravity or Attraction, simply consider'd by themselves, the contrary *Impetus* arising from the *Conatus Centrifugus* not being consider'd.

PROP. X.

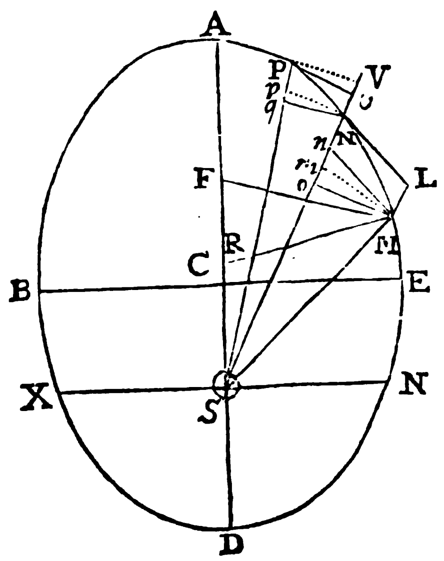
To Explain the Motion of a Planet through the whole Revolution, and to shew how a Planet approaches to, and again recedes from the Sun, Alternis Vicibus.

394. If a Planet be at its greatest Digression from the Sun, or in the Aphelion A, the *Conatus Centrifugus*, and the Solicitation of Gravity are less than if it were nearer to the Sun. But at that Distance, *viz.* in the Aphelion A, the Solicitation of Gravity is greater than twice the *Conatus Centrifugus* (because S A the Distance of the Aphelion from the Sun is greater than ½ the Parameter S W) therefore the Planet will descend towards the Sun in the Curve Line A P M D, and (*Art.* 380.) the descensive *Impetus* will continually increase, as in heavy accelerated Bodies, so long as the Solicitation of Gravity is stronger than twice the *Conatus Centrifugus*: For the descensive Paracentric Motion increases, as long as the Solicitation of Gravity is greater than twice the *Conatus Centrifugus*; and therefore the descensive Paracentric Motion will increase (although the infinitely little Increment of the Paracentric Motion decrease at the same time) until the Planet arrive at W, in which point the Solicitation of Gravity is equal to twice the *Conatus Centrifugus*; and consequently the Paracentric Velocity is greatest in W, when the distance of the Planet from the Sun is equal to ½ the Parameter of the Orbit, afterwards, altho' the Planet continues to approach nearer and nearer to the Sun, untill it come to D, yet the Paracentric Velocity Decreases; for the Solicitation of Gravity is to twice the *Conatus Centrifugus*, as the Distance of the Planet from the Sun, is to ½ the Parameter of the Orbit; and consequently all the while the Planet is in describing the Portion of the Orbit W D X, twice the *Conatus Centrifugus* is greater than the Solicitation of Gravity; and from W to D the Paracentric Velocity Decreases; which it continues to do, until the Centrifugal Impressions collected into one, from the Aphelion A, precisely consumes all the impressions of Gravity collected into one, from the Aphelion A; or until the Centrifugal Impetus be equal to the Centripetite Impetus. Now this happens in the Perihelion D, where the Paracentric Velocity Vanishes, and in which the *Conatus Centrifugus* and Solicitation of Gravity are equal and contrary, so that the Planet cannot pproach nearer the Sun, than it is in the point D. After-

Afterwards, the Motion being continu'd: As the Planet has hitherto approached to, so now it begins to recede from the Sun in the Focus S, and endeavours to move from D by X towards A. For twice the *Conatus Centrifugus*, which began to exceed the Solicitation of Gravity in W, continues to prevail from D to X, and therefore, seeing the Planet begins to move (as it were anew) from D to X, (the former contrary Impetus mutually destroying each other) the Centrifugal Paracentric Velocity Increases from D to X, but the Increment thereof, or the Impression Decreases, until the Planet arrive in X, where the Solicitation of Gravity is equal to twice the *Conatus Centrifugus*; therefore the greatest Centrifugal Paracentric Volocity is in X; from X to A, the Solicitation of Gravity prevails above twice the *Conatus Centrifugus*; and consequently, the Centrifugal Paracentric Velocity Decreases, until the Planet arrive in the Aphelion A, in which point the *Conatus Centrifugus* and Solicitation of Gravity become equal and contrary, and consequently mutually destroy each other: and thus the Planet returns to A, from whence it departed, and begins and finishes new Revolutions successively, and without interruption.

CONSECTARY I.

395. Hence we have six remarkable points in the Elliptic Orbit of a Planet, *viz.* four Obvious, A the Aphelion, D the Perihelion; E and B the mean distances (for S B or S E is $= \frac{1}{2}$ the Transverse Axis A D, and consequently an Arithmetical Mean between S A and S D the greatest and least Digression of a Planet from the Sun) and two more, *viz.* W and X, being the extremities of the Parameter of the Orbit applied to the Axis in the Focus S, in which points happen the greatest Ascensive or Descensive Paracentric Velocity.



CONSECTARY II.

The Impetus which a Planet acquires by the Action of Gravity from A to W is equal to half the Impetus which it acquires in its descent from A to D, and the Impetus acquir'd from A to W is = to that acquir'd from W to D; for the Impetus are proportional to the Angles of apparent Motion, and the Angles A S W and W S D are right Angles.

CONSECTARY III.

Hence to determine the Species of the Planetary Ellipsis; the Focus of the Ellipsis S is given; and the point A where the Planet is when the Sun begins to attract it, being supposed at the greatest distance of the Planet from the Sun, the remoter Vertex of the Ellipsis is also given, and the proportion between the Solicitation of Gravity, or force of Gravity, wherewith the Sun begins to attract the Planet in A, and the *Conatus Centrifugus* in the same point A being known; the Principal Parameter of the Orbit W X, or an Ordinate applied to the Axis in the Focus S, may be found. For, S A (given) is to S W ($= \frac{1}{2}$ the Parameter of the Orbit) as the Force of attraction in A is to twice the *Conatus Centrifugus*, and if $\frac{1}{4}$ the Parameter be subtracted from S A, the greatest distance of the Planet from the Sun, the remainder will be to S A, as S A is to S D; therefore A D the Transverse Axis of the Ellipsis is also given: whence the Planetary Ellipsis may be describ'd.

CON-

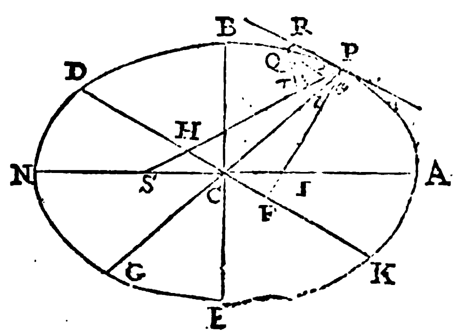
CONSECTARY IV.

A Planet will describe a Circle when the Solicitation of Gravity, and twice the *Conatus Centrifugus* are equal at the beginning of the Attraction, for in that Case they will remain equal, there being no Cause to make the Planet approach nearer to or recede farther from the Center of Attraction, about which it Revolves; but when in the beginning the Force of Attraction and twice the *Conatus Centrifugus* are unequal (provided the simple *Conatus Centrifugus* be always less than the Attraction) then the said Planet will describe an Ellipsis; and if the Force of Attraction prevail, the point where the Motion begins, is the Aphelion; or if twice the *Conatus Centrifugus* prevail then the said point is the Perihelion.

PROP. XI.

If several Bodies revolve about a common Center, and if the Vires Centripetæ *be reciprocally as the Squares of their distances from that Center; then in Ellipses, the Squares of the Periodic Times will be as the Cubes of the Transverse Axes of the Ellipses.*

396. Reassume the Symbols in *Art.* 387. then the Parameter of the Axis of the Figure L (*Art.* 388.) is $= \frac{QTq}{QR}$, when the point Q is infinitely near P, and if the Times be equal, then QR is directly as the *Vis Centripeta*, or reciprocally as (the Square of the distance) S P *q*; therefore L is as QT*q* × SP*q*; that is, the *Latus Rectum* (L) is as the Square of the Area QT × S P; and the Area QT × SP, or $\frac{1}{2}$ QT × S P is in a Subduplicate Ratio of the Parameter (L)



And if the Periodic Times be equal, the Areas of the Ellipses, are in a Subduplicate Ratio of the Parameters; and if the Parameters be equal, the Areas are proportional to the Periodic Times; and if neither the Parameters nor the Periodic Times be equal, then the Areas of the Ellipses are in a Ratio compounded of the Subduplicate Ratio of the Parameters, and the simple Ratio of the Periodic Times; therefore the Periodic Times are in a Ratio compounded of the direct Ratio of the Areas and the reciprocal Subduplicate Ratio of the Parameters. Now the Areas of unequal Ellipses, are (*Art.* 105. N°. 4.) in a Subduplicate Ratio of the Parameters, and the Subduplicate Ratio of the Cubes of the Transverse Axes joyntly. Therefore the Periodic Times are in a Ratio compounded of the Subduplicate Ratio of the Parameters directly, the Subduplicate Ratio of the Cubes of the Transverse Axes directly, and the Subduplicate Ratio of the Parameters inversely; that is, the Periodic Times are in a Subduplicate Ratio of the Cubes of the Transverse Axes, and consequently the Squares of the Periodic Times are proportional to the Cubes of the Transverse Axes. Q. E. D.

COROLLARY.

397. The Squares of the Periodic Times of Bodies revolving in Ellipses are as the Cubes of their mean Distances from the (Focus of the Figure or) Center of Attraction.

SECT. XV.

To find the Fluxions of Logarithms and of Powers when the Exponents are Flowing Quantities. To Construct Exponential Curves and Determine their Tangents.

HIHERTO we have consider'd *Algebraic* and *Transcendent Curves*; there is yet another sort of Curves, which partakes of the Nature of both, *viz. Exponential Curves*. For such Curves may be said to partake of the Nature of *Algebraic Curves*, because they consist of a finite Number of Terms, tho' the Terms themselves be indeterminate; and they may be said to partake of the Nature of *Transcendent Curves*, because they cannot be Algebraically Constructed.

When I shew'd how to find the Fluxions of all sorts of Powers, their Exponents were suppos'd Invariable Quantities, and I acquainted the Reader at the same time, that that speculation might be extended to Powers when the Exponents themselves are Flowing Quantities.

How to handle Equations when the Exponents are Variable Quantities; and to draw Tangents to Curves express'd by such Equations, has been reckon'd one of the abstrusest points in the sublimer parts of Geometry. Of all Exponential Curves the Logarithmetical is the most simple: And because the properties thereof are now generally known, I shall shew how by help of this Curve, the *Fluxions of Flowing Powers* may be found; and also, how all *Exponential Curves* may be Constructed, and their Tangents Determin'd.

LEMMA I.

Quantities continually proportional are proportional to their Differences.

398. I say, if a, b, c, be continually proportional, then they are proportional to their Differences.

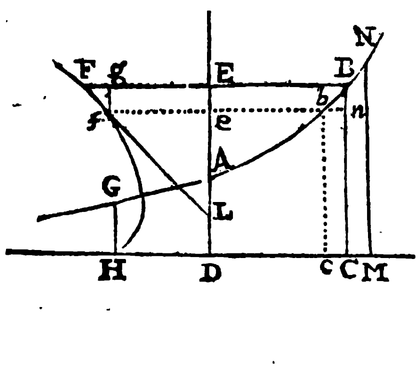
For by supposition, $a : b :: b : c$,

And by Division of proportion, $a : a - b :: b : b - c$.

LEMMA II.

The Sub-tangent of the Logarithmetic Curve is an invariable Quantity.

I have already Demonstrated this property of the Logarithmetic Curve in Art. 76. *But because that depends on the Quadrature of the Hyperbola, and therefore not so proper for our purpose, I now prove the same another way.*



399. Let Mm be the Logarithmetic Curve, AP the Axis, and PM an Ordinate; Then it is evident from the Nature of the Curve, that if the intercepted Diameters AP, Ap, Aq, &c. be in an Arithmetical Progression, the Ordinates PM, pm, qn, &c. will be in a Geometrical Progression; that is, they will be continually proportional, and consequently they will be (*Art.* 398.) proportional to their Differences.

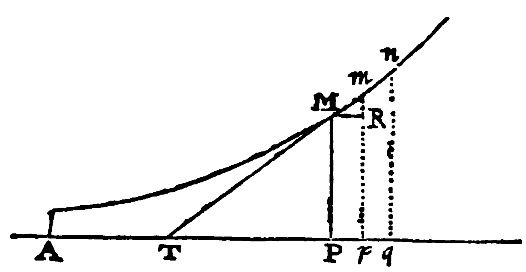
Suppose

Suppose $AP = x$, $PM = y$, the Sub-tangent $PT = t$; $Pp = \dot{x}$, and $Rm = \dot{y}$, then $\dot{y} : \dot{x} :: y : t$, and $t = \frac{y}{\dot{y}} \dot{x}$. But $\dot{x}$ is Invariable, and the Ratio of y to $\dot{y}$ is Invariable, therefore $\frac{y}{\dot{y}} \dot{x}$ is an Invariable Quantity, and consequently the Sub-tangent PT is Invariable. Q. E. D.

PROP. I.

One and the same Indeterminate Quantity, may be an Ordinate in the Logarithmetical Curve, and at the same time, the Abscissa of another Curve.

400. Let DE be the Axis of the Curve HGF, and let DC perpendicular to DE be the Axis of the Logarithmetic Line AB; produce the Ordinate FE, until it cut the Logarithmetic Line AB in B; then if the perpendicular BC be drawn, it will be equal to the Abscissa DE; but BC is an Ordinate to the Logarithmetic Curve; therefore it is evident, that the Abscissæ (DE) of any Curve (HGF) may be Ordinates not only in the Logarithmetic Curve, but also in any other Curve; and that in such manner, that the Relation between DE and EF be always the same, whither DE increase equally or unequally: So that supposing the infinitely little Increments (Cc) of the Axis equal, yet the Fluxion Fg may be to the Fluxion Ee; that is, the Ordinate FE may be to the Sub-tangent EL always in the same general Relation.



PROP. II.

If any Equation, as $x^v = y$, be propos'd, then I say $v \times lx = ly$, understanding by lx, ly, &c. the Logarithms of x, y, &c.

401. For the Logarithm of the Square, Cube or Biquadrate, &c. of any Number, is equal to twice, thrice or fourtimes, &c. the Logarithm of the Root. Therefore Universally, the Logarithm of x^v is $= v \times lx$; but $x^v = y$, therefore the Logarithm of x^v is equal to the Logarithm of y, that is $v \times lx = ly$.

PROP. III.

The Logarithm of any absolute Number is equal to the Sum of all the Fluxions of the same absolute Number, divided by it self, v. g. The Logarithm of x is $= S \frac{\dot{x}}{x}$

402. Suppose $AD = 1 =$ Sub-tangent of the Logarithmetic Curve AB, and let BC be an absolute Number, then 'tis evident that DC is the Logarithm of BC. Now $Bn : Cc :: BC : AD$ (= Sub-tangent of the Curve = 1.) Therefore $AD \times Bn = Cc \times BC$, that is $\frac{Bn}{BC} = Cc$.

Whence

Whence if BC be suppos'd $= x, y$ or z, &c. and DC $= lx$, ly, or lz, &c. then is B$n = \dot{x}$, and C$c = \overline{\dot{lx}}$, and consequently $\frac{\dot{x}}{x}$ is $= \overline{\dot{lx}}$. Now if the Fluxions be equal, then the Flowing Quantities must also be equal, that is S $\frac{\dot{x}}{x} = lx$.

COROLLARY.

403. The Fluxion of any Logarithm however Compounded is equal to the Fluxion of the Corresponding absolute Number, divided by the said absolute Number. For if AB be a Logarithmetic Curve, and the Sub-tangent = AD = 1, and if BC $= x$ be an Ordinate to the Curve AB or an Abscissa to any other Curve, then DC will be $= lx$, and B$n = \dot{x}$, and C$c = \overline{\dot{lx}}$, and consequently $\overline{\dot{lx}} = \frac{\dot{x}}{x}$. That is the Fluxion of the Logarithm of x is = the Fluxion of x divided by x.

Hence to find the Fluxion of any Logarithm.

Let it be requir'd to find the Fluxion of the Logarithm of $\sqrt[2]{xx+yy}$; the absolute Number is $= \overline{xx+yy}|^{\frac{1}{2}}$.

And the Fluxion thereof is $= \frac{1}{2}\,\overline{xx+yy}|^{-\frac{1}{2}} \times \overline{2x\dot{x}+2y\dot{y}} = \frac{x\dot{x}+y\dot{y}}{\sqrt{xx+yy}}$.

Which being divided by the absolute Number $\sqrt{xx+yy}$, the Quotient $\frac{x\dot{x}+y\dot{y}}{xx+yy}$ is $= \overline{\dot{l\sqrt{xx+yy}}}$, and thus we may find the Fluxion of any Logarithm.

More EXAMPLES.

To find the Fluxions of the Logarithms of all sorts of Powers.

To find the Fluxion of the Logarithm of $x+1$.

The absolute Number is $x+1$.

And its Fluxion $\dot{x}$.

Which being divided by $x+1$.

The Quotient $x+1)\ \dot{x}\ (= \frac{\dot{x}}{x+1}$ is $= \overline{\dot{lx+1}}$.

To find the Fluxion of the Logarithm of rx^3+2x^2.

The Fluxion of rx^3+2x^2 is $= 3rx^2\dot{x}+4x\dot{x}$.

Which being divided by the absolute Number rx^3+2x^2.

$$rx^3+2x^2)\ 3rx^2\dot{x}+4x\dot{x}\ \left(\frac{3rx^2\dot{x}+4x\dot{x}}{rx^3+2x^2}\right.$$

The Quotient $\frac{3rx^2\dot{x}+4x\dot{x}}{rx^3+2x^2} = \frac{3rx\dot{x}+4\dot{x}}{rxx+2x}$ is = to the Fluxion of $l{:}\ \overline{rx^3+2x^2}$.

To

To find the Fluxion of $l: \frac{1}{rx^3+x^2}$.

The absolute Number is $\frac{1}{rx^3+x^2} = \overline{rx^3+x^2}|^{-1}$.

And the Fluxion thereof $-\overline{rx^3+x^2}|^{-2} \times \overline{3rx^2\dot{x}+2x\dot{x}}$.

$$= \frac{-3rx^2\dot{x}-2x\dot{x}}{\overline{rx^3+x^2}|^2}$$

Which being divided by the absolute Number $\frac{1}{rx^3+x^2}$.

$$\frac{1}{rx^3+x^2}\Big)\ \frac{-3rx^2\dot{x}-2x\dot{x}}{\overline{rx^3+x^2}|^2}\ \Big(\frac{-3rx^2\dot{x}-2\dot{x}x}{rx^3+x^2}$$

The Quotient $\frac{-3rx^2\dot{x}-2x\dot{x}}{rx^3+x^2} = \frac{-3rx\dot{x}-2\dot{x}}{rxx+x}$ is = to the

Fluxion of the Logarithm of $\frac{1}{rx^3+x^2}$.

The Fluxion of $l: \overline{xx+ee}|^n$ is $= \frac{n \times \overline{xx+ee}|^{n-1} \times 2x\dot{x}}{\overline{xx+ee}|^n} = \frac{2nx\dot{x}}{xx+ee}$.

To find the Fluxions of the Square, Cube, &c. *of any Logarithm:* i. e. *To find the Fluxions of all sorts of Powers to which the Logarithms of all sorts of Quantities can be rais'd.*

To find the Fluxion of $l: \overline{x+e^n}|^2$.

Multiply $l: \overline{x+e^n}|^2$ by 2, the Exponent of the Power,

And we have $2 \times l: \overline{x+e^n}|^2$.

Lessen the Exponent by 1, and we have $2 \times l: \overline{x+e^n}|^{2-1} = 2 \times l: \overline{x+e^n}|^1$.

Which multiplied by $\frac{\dot{x}}{x+e^n}$ the Fluxion of $l: \overline{x+e^n}|$,

There will arise $2 \times l: \overline{x+e^n}| \times \frac{\dot{x}}{x+e^n} = \frac{2 \times l: \overline{x+e^n}|\dot{x}}{x+e^n}$ = to the

Fluxion of $l: \overline{x+e^n}^2$.

To find the Fluxion of $l: \overline{x+a^m}|^m$.

Multiplying the given Quantity by the Exponent m, we have $m \times l: \overline{x+a^m}|^m$.

Subtracting 1 from the Exponent m, $m \times l: \overline{x+a^m}|^{m-1}$.

Which being multiplied by the Fluxion of the Root $\frac{\dot{x}}{x+a^m}$.

The Product $m \times l: \overline{x+a^m}|^{m-1} \times \frac{\dot{x}}{x+a^m}$.

Is equal to the Fluxion of the Logarithm of $\overline{x+a^m}$ rais'd to the Power whose Exponent is m.

To find the Fluxion of $x^n \overline{l:x}|^m$.

The Fluxion of x^n is $= n x^{n-1} \dot{x}$.

And the Fluxion of $\overline{l:x}|^m$ is $= m \times \overline{l:x}|^{m-1} \frac{\dot{x}}{x} = m x^{-1} \overline{l:x}|^{m-1} \dot{x}$.

Therefore the Fluxion of $x^n \overline{l:x}|^m$ is $= m x^{n-1} \overline{l:x}|^{m-1} \dot{x} + n x^{n-1} \overline{l:x}|^m \dot{x}$.

To find the Fluxion of $\overline{a+x}|^n \times l:\overline{x+a^m}|^m$.

The Fluxion of $\overline{a+x}|^n$ is $= n \times \overline{a+x}|^{n-1} \dot{x}$.

And the Fluxion of $l:\overline{x+a^m}|^m$ is $= m \times l:\overline{x+a^m}|^{m-1} \times \frac{\dot{x}}{x+a^m}$.

Therefore the Fluxion of $\overline{a+x}|^n \times \overline{l:x+a^m}|^m$ is $=$

$$n \times \overline{a+x}|^{n-1} l:\overline{x+a^m}|^m \dot{x} + m \times \overline{a+x}|^n \times \overline{x+a^m}|^{-1} l:\overline{x+a^m}|^{m-1} \dot{x}.$$

These Principles being laid down, I come next to Treat of Exponential Equations.

There are several degrees of *Exponential Quantities*, and the *lowest Degree* is, when the Exponent consists of ordinary indeterminate Quantities, as y^m, x^n, supposing m and n to be simple indeterminate Quantities.

An *Exponential Quantity of the second Degree*, is when the Exponent it self is an Exponential Quantity, as y^{m^n}, and if an Equation consists of several Exponentials of different Degrees, then the Equation or the Curve whose Nature it expresses, takes its Name *a Potiori.*

PROP. IV.

To find the Fluxion of any Exponential Quantity of the first Degree.

402. Let it be requir'd to find the Fluxion of y^n; suppose $y^n = r$, (*Art.* 401.) then $n \times ly = lr$, and finding the Fluxions of each side of the Equation, by the Common Rules, we shall have $n \times \dot{\overline{ly}} + ly \times \dot{n} = \dot{\overline{lr}}$. (*Art.* 403.) But $\dot{\overline{ly}} = \frac{\dot{y}}{y}$, and $\dot{\overline{lr}} = \frac{\dot{r}}{r}$. Therefore by equal Substitution, $ly\dot{n} + \frac{n\dot{y}}{y} = \frac{\dot{r}}{r} =$ (because $r = y^n$) $\frac{\dot{r}}{y^n}$; *Ergo*, $\dot{r}$ or the Fluxion of the Exponential Quantity y^n is $= y^n ly\dot{n} + ny^{n-1}\dot{y}$.

Thus far of finding the Fluxions of Logarithms when the exponents are Invariable.

And this is the first Special Rule for all Exponentials of the first Degree: For we may suppose indeterminate Quantities Compounded at pleasure, to be in place of y and n.

EXAMPLE.

Let it be requir'd to find the Fluxion of the *Compound Exponential Quantity* $y^n x^m$; first the Fluxion of $y^n x^m$ is $=$ (by the Common Rules) $= \dot{\overline{y^n}} \times x^m + \dot{\overline{x^m}} \times y^n$, but the Fluxion of y^n is $= y^n ly\dot{n} + ny^{n-1}\dot{y}$; and the Fluxion of x^m is $= x^m lx\dot{m} + mx^{m-1}\dot{x}$, and therefore by equal Substitution, the Fluxion of the Compound Exponential Quantity $y^n x^m$ is $= x^m y^n ly\dot{n} + nx^m y^{n-1}\dot{y} + y^n x^m lx\dot{m} + my^n x^{m-1}\dot{x}$.

PROP

PROP. V.

To find the Fluxion of any Exponential Quantity of the Second or any higher Degree.

403. Let it be requir'd to find the Fluxion of y^{n^m}; suppose $y^{n^m} = r$, then is $n^m ly = lr$, and taking the Fluxions of each side of the Equation by the Common Methods, we have $n^m \dot{\overline{ly}} + ly \dot{\overline{n^m}} = \dot{\overline{lr}}$. Now because $\dot{\overline{n^m}} = n^m ln \dot{m} + mn^{m-1} \dot{n}$, and $\dot{\overline{ly}} = \frac{\dot{y}}{y}$; therefore, by equal Substitution, we have $\frac{n^m \dot{y}}{y} + n^m ly ln \dot{m} + mn^{m-1} ly \dot{n} = \frac{\dot{r}}{r} =$ (because $r = y^{n^m}$) $\frac{\dot{r}}{y^{n^m}}$, and consequently $\dot{r}$ or the Fluxion of y^{n^m} is $= n^m y^{n^m - 1} \dot{y} + mn^{m-1} y^{n^m} ly \dot{n} + n^m y^{n^m} ly ln \dot{m}$. Which is the Rule for Exponential Quantities of the second Degree.

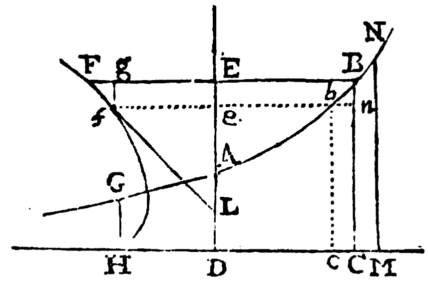
And he that shall have occasion for the Fluxions of compound Exponential Quantities of the second Degree, or for the Fluxions of simple or compound Exponential Quantities of the third Degree, *&c.* may easily find them, if he perfectly understands what is already written.

It remains to apply this Doctrine to Exponential Curves, and to shew how such Curves may be constructed, and their Tangents determin'd.

PROP. VI.

Let it be requir'd to construct the Curve, whose Nature is express'd by this Exponential Equation $x^x = y$.

404. Take the Logarithm of both sides of the Equation, and then we have $xlx = ly$: now if we suppose the Logarithmetic Line A B to be drawn, and its Subtangent = the first Ordinate AD = 1, and DE or BC be $= x$, then is DC $= lx$, and because $xlx = ly$: therefore 1 (= AD = Subtang.) : x (BC or DE) :: lx (DC) : ly = DM, and consequently MN is $= y$; and if MN be laid from E (the end of the Abscissa DE) to F, the point F will be in the Curve required. And thus the other points of the Curve HGF may be determined.



PROP. VII.

If the Nature of the Curve HGF *be express'd by the Equation $x^x = y$, let it be required to draw the Line* FL *to touch the Curve in* F.

405. Suppose the Tangent FL (being drawn) to intersect the Axis DE in L; the Equation of the Curve is $x^x = y$, therefore $\dot{y} = xx^{x-1}\dot{x} + x^x lx \dot{x} = x^x \dot{x} + x^x lx \dot{x} =$ (substituting y for x^x) $y\dot{x} + ylx\dot{x}$; therefore $y + ylx : 1 :: \dot{y} : \dot{x}$

:: y :

:: y : Subtangent EL $= \frac{1}{1 + lx}$, and consequently if EL be taken a third proportional to $\overline{AD + DC}$, and AD, then FL will touch the Curve HGF in F, for $1 + lx : 1 :: 1 : \frac{1}{1 + lx} =$ EL; and thus the Exponential Curve HGF may be constructed, and the Subtangent expres'd in ordinary Terms.

PROP. VIII.

Let it be required to Construct the Curve, whose Nature is express'd by this exponential Equation $a^x = y$.

406. Take the Logarithms of both sides of the Equation, and then $x\,la = ly$; now if we suppose DE $=$ CB $= x$, AD $= 1$, and $la =$ the Logarithm of an invariable Quantity, it will be 1 (AD) : x (BC) :: $la : ly =$ DM, and consequently NM is $= y$; therefore if MN be applied to the Axis DE from E to F, the point F will be in the Curve requir'd.

And to determine the Subtangent to this Curve, the Equation of the Curve is $a^x = y$, and finding the Fluxions of each side of the Equation, we have $x\,a^{x-1}\dot{a} + a^x\,la\,\dot{x} = \dot{y}$. But a is an Invariable Quantity, and $\dot{a} = 0$, therefore $x\,a^{x-1}\dot{a}$ is $= 0$, and consequently $a^x\,la\,\dot{x} = \dot{y} =$ (because $a^x = y$) $y\,la\,\dot{x}$, and resolving this Equation into an Analogy, $\dot{y}$ (Fg) : $\dot{x}$ (Ee) :: $y\,la : 1$ :: (by similar Triangles) y : Subtangent EL, which consequently is $= \frac{y}{y\,la} = \frac{1}{la}$ = to an invariable Quantity.

COROLLARY.

407. Hence it is manifest that the given Equation ($a^x = y$) expresses the Nature of the Logarithmetic Curve it self.

SCHOLIUM.

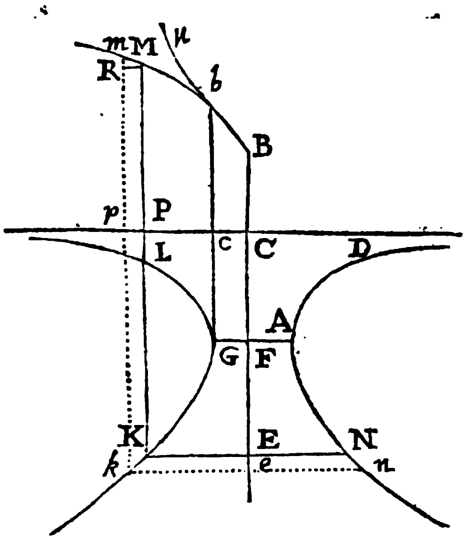
The Fluxion of the foresaid Equation, may be found more easily thus: $x\,la = ly$, and $\dot{x}\,la = \dot{\overline{ly}} = \frac{\dot{y}}{y}$, therefore (by multiplication) $y\,la\,\dot{x} = \dot{y}$.

PROP.

PROP. IX.

The same things being supposed, as in Art. 176. *Draw the Axis* C P, *and describe the Curve* B M *m*: *Suppose* C P $= x$, *and* P M $= y$, R $m = \dot{y}$, P $p = \dot{x}$, *and* M$m = \dot{z}$. *Then if the Nature of the Curve* B M *m be express'd by this Equation* $y\dot{y}^3\dot{x} = a\dot{z}^4$. *'Tis requir'd to construct the same.*

408. The Equation expressing the Nature of the Curve is $y\dot{y}^3\dot{x} = a\dot{z}^4$, and because $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$ therefore $\dot{z}^4$ is $= \dot{x}^4 + 2\dot{x}^2\dot{y}^2 + \dot{y}^4$, and the Equation of the Curve will be $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$.



Now to clear the Equation of Fluxions; Suppose $\dot{x} =$ (m being an indeterminate Quantity) $\frac{m\dot{y}}{a}$, then (by substitution) the Equation of the Curve will become $\frac{my\dot{y}^4}{a} = \frac{m^4\dot{y}^4}{a^3} + \frac{2m^2\dot{y}^4}{a} + a\dot{y}^4$, and dividing by $\dot{y}^4$, $\frac{my}{a} = \frac{m^4}{a^3} + \frac{2m^2}{a} + a$, or $aamy = m^4 + 2a^2m^2, + a^4$; whence $y = \frac{m^3}{aa} + 2m + \frac{aa}{m}$, and finding the Fluxions of each side of the Equation, we have $\dot{y} = \frac{3m^2\dot{m}}{aa} + 2\dot{m} - \frac{aa\dot{m}}{m^2}$ and multiplying both sides of the Equation by $\frac{m}{a}$, we have $\dot{x}$ $\left(\text{or } \frac{m\dot{y}}{a}\right) = \frac{3m^3\dot{m}}{a^3} + \frac{2m\dot{m}}{a} - \frac{a\dot{m}}{m}$, and finding the Fluent of each side of the Equation there will arise $x = \frac{3m^4}{4a^3} + \frac{m^2}{a} - lm$.

And thus we have found the Values of the Co ordinates x and y, in Terms involving one indeterminate Quantity (m) only, whence the Curve B M may be constructed thus: Produce B C (infinitely) towards E, and take C E $= m$, and to the Axis C E apply the Ordinate E N $= y = \frac{m^3}{aa} + 2m + \frac{aa}{m}$, and describe the Algebraic Curve D A N; in the same point E, apply another Ordinate E K $= x = \frac{3m^4}{4a^3} + \frac{m^2}{a} - lm$, and describe the Transcendent Curve L G K; then the Lines E N, E K will be the Co-ordinates of the Curve B M, whence if the Line K P be drawn parallel to E C, and produced to M so that P M be $=$ E N; then the point M will be in the Curve B M m, which was required.

409. And

409. And to find whether the Curve D A N touches the Axis C E or not, the Value of the Ordinate E N (y) is $= \frac{m^3}{aa} + 2m + \frac{aa}{m}$ and $\dot{y}$ is $= \frac{3m^2\dot{m}}{aa} + 2\dot{m} - \frac{aa\dot{m}}{m^2}$, which being put $= 0$, we shall have $\frac{3m^2\dot{m}}{aa} + 2\dot{m} - \frac{aa\dot{m}}{m^2} = 0$, and consequently $3m^4 + 2a^2 \times m^2 = a^4$; whence, if m or C F be $= a\sqrt{\frac{1}{3}}$, then the Ordinate F A will be the least that can be applied to the Curve D A N; in like manner, when the Fluxion of the Ordinate E K is $= 0$, then the Value of m will be $= a\sqrt{\frac{1}{3}}$ as before, and F G will be the least Ordinate that can be applied to the Curve L G K; that is, both Curves begin to recede from the Axis C E at the same time; and the least Ordinate F A (y) $= \frac{m^3}{aa} + 2m + \frac{aa}{m}$ becomes $=$ (because $aaym = m^4 + 2aam^2 + a^4$, which, substituting $a\sqrt{\frac{1}{3}}$ for m, becomes $ya^3\sqrt{\frac{1}{3}} = \frac{1}{9}a^4 + \frac{2}{3}a^4 + a^4$, and consequently $y = \frac{\frac{16}{9}a}{\sqrt{\frac{1}{3}}} =$) $\frac{16}{9}a\sqrt{3}$; and if we suppose $a =$ Unity so that la be $= 0$, then the least Ordinate F G $= \frac{3m^4}{4a^3} + \frac{m^2}{a} - lm$, will be $= \frac{5}{12}a -$ Log. $a\sqrt{\frac{1}{3}}$: and because the Logarithm of $a\sqrt{\frac{1}{3}}$ is Negative (the absolute Number being less than Unity) therefore it is manifest that $\frac{5}{12}a -$ Log. $a\sqrt{\frac{1}{3}}$ is a Positive Quantity.

And hence it appears, that if G cb be drawn parallel to C F, and if bc be taken $=$ F A, then the intercepted Diameter C c is $= \frac{5}{12}a -$ Log. $a\sqrt{\frac{1}{3}}$, and the Ordinate cb is $= \frac{16}{9}a\sqrt{3}$, and the Curve M b is nearest to the Axis C P in the point b, and afterwards the Portion of the Curve $b\mu$ will be describ'd convex towards b M; and and the point b is called the point of Retrogression.

410. And because the Curve L G K is an exponential Curve, it will not be amiss to shew how the exponential Equation expressing the Nature thereof, may be investigated.

The Ordinate E K or x is $= \frac{3m^4}{4a^3} + \frac{m^2}{a} - lm$, and multiplying both sides of the Equation by $4a^3$, there will arise

$$4a^3x = 3m^4 + 4a^2m^2 - 4a^3lm,$$

and by Transposition, $4a^3lm = 3m^4 + 4a^2m^2 - 4a^3x$.

And because the first part of the Equation is a Logarithmetical Quantity, multiply all the Terms of the Equation by any given Logarithm, which for uniformities sake, suppose la, (a being supposed = to some certain Number, and not to Unity; in this Case, else la will be equal 0) that so the other part may also be a Logarithmetical Quantity, and then $4a^3 la\, lm = \overline{3m^4 + 4a^2m^2 - 4a^3x} \times la$, and because the absolute Numbers of equal Logarithms are equal, therefore (taking the absolute Numbers of the Logarithms of each side of the Equation) $m^{4a^3 la} = a^{\overline{3m^4 + 4a^2m^2 - 4a^3x}}$ which is a Transcendent Exponential Equation expressing the Nature of the Curve L G K. Q. E. I.

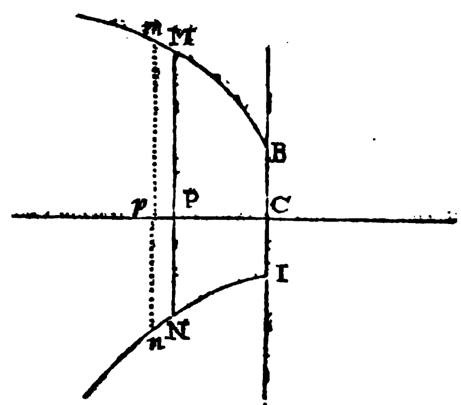
PROP.

PROP. X.

If an Equation in Fluxions expressing the Nature of any Curve be given; and if either of the Indeterminate Quantities be wanting. 'Tis requir'd to Construct the said Curve.

That the usefulness of the Method made use of in the preceding Proposition for constructing the Curve b M m, may more plainly appear, I have added this general Proposition, in which it is proposed to construct any Curve whose Nature is express'd by an Equation in Fluxions, be the Fluxions $\dot{x}$ and $\dot{y}$ or either of the Indeterminate Quantities x or y, or the Powers of any of them combin'd among themselves at pleasure. The Method is this.

411. If x be wanting, then suppose $\dot{x} = \frac{m\dot{y}}{a}$, and substitute this Quantity in the given Equation, in place of $\dot{x}$; then it is manifest that the new Equation may be divided by some Power of $\dot{y}$, so that there will remain an Equation in ordinary Terms, between m and y, expressing the Nature of an Algebraic Curve; in which y is equal to the Absciss, and m equal to the Ordinate, and the Curvilineal Space being divided by a, the Quotient will be equal to x the other Coordinate of the Curve to be constructed.



EXAMPLE.

412. Let the Equation expressing the Nature of the Curve BM be (supposing $CP = y$, and $PM = x$) $y^3\dot{x}^5 + aay\dot{y}\dot{x}^4 = a^3\dot{y}^5$. 'Tis requir'd to construct the same: suppose $\dot{x} = \frac{m\dot{y}}{a}$, then by equal Substitution, $\frac{y^3m^5\dot{y}^5}{a^5} + \frac{ym^4\dot{y}^5}{aa} = a^3\dot{y}^5$, and dividing by $\frac{\dot{y}^5}{a^5}$, there will arise $y^3m^5 + a^3ym^4 = a^8$, which is an Equation in ordinary Terms, and if CP be $= y$, and PN $= n$, and the Relation between them express'd by this Equation, the Curve IN n will be an Algebraic Curve; and if the Space CPNI be divided by a, the Quotient will be = PM the other Co-ordinate of the Curve to be constructed. For $3m^5\dot{y}y^2 + 5y^3m^4\dot{m} + a^3m^4\dot{y} + 4a^3ym^3\dot{m} = 0$, and $\dot{y} = \frac{-5y^3m^4\dot{m} - 4a^3ym^3\dot{m}}{3m^5y^2 + a^3m^4} = \frac{-5y^3m\dot{m} - 4a^3y\dot{m}}{3m^2y^2 + a^3m}$ and consequently $\frac{m\dot{y}}{a} = \dot{x} = \frac{-5y^3m\dot{m} - 4a^3y\dot{m}}{a \times \overline{3my^2 + a^4}}$, and P n the Fluxion of the Area $= m\dot{y}$ is $= \frac{-5y^3m\dot{m} - 4a^3y\dot{m}}{3my^2 + a^4}$; whence PM or x is = the Area CPNI divided by a.

FINIS.