A VIEW OF Sir *ISAAC NEWTON's* METHOD For Comparing The *Resistance* of SOLIDS.

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TO

Sir JACOB ACWORTH, *Knt.*

SURVEYER

OF

His Majesty's *NAVY.*

SIR,

YOUR Disposition to promote whatever is ingenious, justly demands the grateful Acknowledgment of all Lovers of Art: And as I have my self, in a most particular Manner, experienced your Goodness, pray Leave to lay before you, this little Treatise, on Sir Isaac Newton's *Solid of the least Resistance; which, tho' it has been handled by several able* Mathematicians, *yet, has been looked upon as a Matter of mere Speculation, 'till*

'till you were pleased (among your many successful Endeavours, to improve the Royal Navy) *in a more particular Manner to consider and reduce this to Practice: I am thoroughly sensible, to treat on the Works of the* Illustrious Author, *is a Task I am no way* equal *to; therefore beg you will pass by the Meanness of the Performance, and accept of the Endeavours of*

Sir,

Your most Dutiful and

Most Obedient Servant,

C. Robinson.

London, *April* 5. 1734.

A VIEW of *Sir* ISAAC NEWTON'S *Thirty Fourth Proposition* of the *Second Book* of his *Mathematical Principles* of NATURAL PHILOSOPHY.

THE Resistance of a Body moving in a Fluid Medium, arises partly from the Density of the Fluid in which it moves, partly from the Number of the Particles which strike the Body at the same Time, and the Angles in which they strike jointly.

The Density of the Fluid is not here considered: All the Bodies whose Resistances are here compared, being supposed to move in the same Fluid; as all in *Air*, *Water*, *Oil*, *&c.* and not One to move in *Air*, a Second in *Water*, a Third in *Oil*, *&c.*

The Number of the Particles of a Fluid which strike on Bodies moving in it, are as the Areas of the Greatest Transverse Sections of those Bodies.

The Angle in which the Particles of the Fluid strike the Body, is the Angle which the Sides or Surface of the Body make with the Direction of the

 Motion

Motion, and is called the Angle of Incidence of the Particles of the Fluid.

Thus, if the Parallelogram ADEB, *Fig.* 1. and the Triangle AVB were both to move in the same Fluid, the Number of Particles that would ſtrike on thoſe Figures would be equal, becauſe the Figures are contained under equal Baſes; but the Angle of Incidence in which they ſtrike unequal, being Perpendicular or a Right Angle on the Parallelogram, and Oblique or the Acute Angle VBE on the Triangle.

If a Curve Line FGHV, *Fig.* 2. be moved in a Fluid Medium in the Direction of its Axis AV; then ſuppoſing AB, BC infinitely Small or Fluxions of the Abſciſſa: DF, EG Fluxions of the Ordinate: FG, GH Fluxions of the Curve, the Angle of Incidence of the Particles of the Fluid in the Point H will be KHT or CTH, that is, an Angle made by the Interſection of a Tangent applied in the Point H, with the Axis AV produced in T, and is ſimilar to the infinitely Small Triangle EHG.

The Sum of the Reſiſtances on every Point of the Curve is the Reſiſtance of the whole Curve Line, and if the Fluxions of the Ordinates be every where equal, that is, if FD=GE, &c. the Number of the Particles of the Fluid which ſtrike upon every infinitely little Part of the Curve FG, GH, &c. will be equal, and the Reſiſtances of every ſuch infinitely little Part or Fluxion of the Curve will be as GE ſquared, divided by GH ſquared, or the Square of the Fluxion of the Ordinate divided by the Square of the Fluxion of the Curve, or ſuch a Proportion of the Homologous Sides of any Similar Triangle.

But, if a Solid generated by revolving the ſame Curve FGHV about an Axis be moved in a Fluid Medium in the Direction of its Axis: The Reſiſtance on every ſuch infinitely little Part of the Solid, generated by the infinitely little Triangles FGD,

GHE

GHE will not be in the ſame Proportion as on the Curve Line: For the Fluxions of the Ordinates being equal, the Number of the Particles that ſtrike on every of the Elementa or Fluxions of the Curve, conſidered as a Line is equal, but conſidered as a Solid unequal (tho' the Angle of Incidence is every where the ſame) becauſe the Surface generated by DF at the Diſtance GB or DA is greater than the Surface generated by revolving EG at the Diſtance EB or HC: the Whole of which is contained in Sir *Iſaac Newton's*

PRINCIP. MATHEM. *Lib.* II. *Prop.* 34. *Theor.* 28.

"If a Globe and Cylinder of equal Diameters be "moved with equal Velocity, in a Rare or Thin Me"dium, conſiſting of Particles equal and equally di"ſtant from one another, in the Direction of the "Axis of the Cylinder: the Reſiſtance of the Globe "would be half as much as the Reſiſtance of the "Cylinder.

"For, becauſe the Action of the Medium is the "Same on the Body (by *Cor.* 5. *of the Laws*) whe"ther the Body be moved in a Medium at Reſt, or "the Particles of the Medium, ſtrike the Body at "Reſt with the ſame Velocity: Let us conſider the "Body at Reſt, and ſee with what Force it will be "preſſed by a moving Medium.

"Let therefore ABKI, *Fig.* 3. repreſent a Sphe"rical Body deſcribed on the Center C, with the "Radius CA, and let the Particles of the Medium "fall with the given Velocity upon that Spherical "Body, in Right Lines parallel to AC: Let FB "be a Right Line of this Kind, in it let LB be ta"ken equal to the Semidiameter CB, and let BD "be drawn touching the Sphere in B, on KC and

" BD let fall the Perpendiculars BE, DL, and " the Force with which a Particle of the Medium " falling Obliquely in the Right Line FB, strikes " the Sphere in B, will be to the Force with which " the same Particle would strike the Cylinder ONGQ " circumscribed to the Sphere, with the same Axis " ACI in *b*, as LD to LB, or BE to BC. Again, " the Effect of this Force to move the Globe in the " Direction of its Incidence FB or AC is to the " Power to move the same in the Direction of its " own Determination, that is, in the Direction of the " Right Line BC in which it strikes the Globe Per- " pendicularly, as BE to BC; and by Compound- " ing these Ratio's, the Force of the Particle falling " Obliquely on the Globe in the Right Line FB, to " move the same in the Direction of its Incidence, " is to the Power of the same Particle falling per- " pendicularly on the Cylinder in the same Right " Line, as BE squared to BC squared; wherefore " if a Perpendicular *b*HE be erected to the Circular " Base of the Cylinder NAO, and *b*E be equal to " the Radius AC, and *b*H equal to $\frac{\text{BE squared}}{\text{BC}}$,

" *b*H will be to *b*E as the Effect of the Particle " on the Globe to the Effect of the Particle on the " Cylinder, and therefore, the Solid which is made " up of an infinite Number of Right Lines here re- " presented by *b*H, will be to the Solid which is " made up of an infinite Number of Right Lines " here represented by *b*E, as the Effect of all the " Particles on the Globe to the Effect of all the Par- " ticles on the Cylinder. But the former Solid is a " Paraboloid, described through the Vertex C on " the Axis CA and Parameter CA, and the latter " Solid is a Cylinder circumscribed to the Parabo- " loid: And it is known, that the Paraboloid is " half the circumscribed Cylinder. Therefore, the

" whole

" whole Force of the Medium on the Globe is half
" the Force of the same on the Cylinder. And
" therefore, if the Particles of the Medium be at
" Rest, and the Globe and Cylinder be moved with
" an equal Velocity, the Resistance of the Globe
" would be half the Resistance of the Cylinder.
" Q. E. D.

SCHOLIUM.

" By the same Method the Resistance of other
" Figures may be compared, and such may be found,
" which are best adapted to continue their Motions
" in Resisting Mediums. As if on the Circular Base
" CEBH, *Fig.* 4. which is described on the Center
" O, the Frustum of a Cone is to be constructed,
" which meets with the least Resistance of all Frus-
" tums of the same Base and Altitude, and moving
" in the Direction of its Axis towards D: Bisect
" the Altitude OD in Q, and produce OQ to S,
" that QS may be equal to QC, and S will be the
" Vertex of the Cone whose Frustum it sought.

" Hence, Seeing the Angle CSB is always Acute,
" it follows, that if a Solid ADBE, *Fig.* 5. be ge-
" nerated by the Convolution of an Elliptical or
" Oval-like Figure ADBE about its Axis AB, and
" the generating Figure be touched by three Right
" Lines FG, GH, HI, in the Points F, B and I,
" in such a Manner that GH be perpendicular to
" the Axis in the Point of Contact B, and HI, FG
" with the said Line GH contain the Angles FGB,
" BHI, 135 Degrees; the Solid which is generated
" by turning this Figure ADFGHIE about its Axis,
" CB, will be less resisted than the former Solid, if
" both move in the Direction of the Axis AB with
" the End B foremost: which Proposition, I con-
" ceive, may be useful in *Ship-Building*.

" But

" But if the Figure DNFG be a Curve of ſuch a " Kind: If from any Point N, a Perpendicular NM " be let fall on the Axis AB, and from a given " Point G be drawn a Right Line GR Parallel to a " Right Line, touching the Figure in N, and cut- " ting the Axis produced in R; MN will be to GR, " as GR cubed to 4BR×GB ſquared: The Solid " which is formed by revolving this Figure about " its Axis AB, will meet with leſs Reſiſtance, in " moving in the aforeſaid rare Medium from A to- " ward B than any other Circular Solid deſcribed " with the ſame Length and Breadth".

The Method here given by Sir *Iſaac Newton*, extends to the Inveſtigating the Reſiſtances of Solids in general, whether they be generated by revolving the generating Lines about an Axis, or by any other known Law, that is, whether the Baſes and all the Tranſverſe Sections be Circles, Polygons, &c.

If the Tranverſe Sections be all Right-angled Parallelograms, and the reſpective Sides of all thoſe Parallelograms, parallel to one another, as in a Wedg or Figure of a Wedg-like Form, whoſe Sides, inſtead of being ſtraight, are Curved: The Reſiſtances of ſuch Solids will be, the Reſiſtances of the generating Lines multiplied into the Depth.

In Figures of this Kind, the Plains or flat Sides may be ſuppoſed to move Horizontally; however, tho this may be of Uſe in aſſiſting the Imagination, it is not any ways neceſſary, that they ſhould be confined to ſuch a Motion, but may be at Liberty to move any how, ſo it be in the Direction of the Axis of the generating Lines, and plain Superficies.

PROP. I.

To Investigate the Resistances *of* Solids *contained under* Straight Lines.

LET the Parallelogram ABCD, *Fig.* 6. be the Side of a Parallelopipedon, supposed to move Horizontally in the Direction of FK, whose Altitude AB is denoted by b, the half Breadth BK by b, and the Depth by d.

Case 1. If the abovesaid Parallelopipedon be Cyphered away from Bottom to Top, in the Right Line EB, *Fig.* 7. the Resistance of the Parallelopipedon will be to the Resistance of this Solid, as $bb+dd$ to dd, or putting $2bbd$, or the Content of the Parallelopipedon for its Resistance; the Resistance of this Solid will be $\frac{2bbd^3}{bb+dd}$, for, by the aforecited Proposition $EBq.AEq :: AB$ to $\frac{AB \times AEq}{EBq}$ = BL, whence the Resistance of this Solid will be to that of the former, as the Line BL is to the Line AB, or as the Area of the Rectangle BLNO is to the Area of the Rectangle ABOE, which (because the Slant Side EB is every where of the same Breadth) is the same Thing, as if the Parallelopipedons were compared with one another.

Case 2. If the Horizontal Plane be cyphered away to the Triangle AKD, *Fig.* 8. and not from Bottom to Top, as in the first Case, the Resistance of this Solid will be $\frac{2bb^3d}{bb+bb}$, for, as before, AKq.

AHq

AHq :: AB. $\frac{AB \times AHq}{AKq}$ = BL, or (for the Reason given in the former *Case*) the Resistance of this Solid will be to the Resistance of its circumscribed Parallelopipedon, as BL to BA.

Case 3. Let now the Parallelopipedon be cyphered away both Ways together, and the Resistance of this Piramidal Figure will be $\frac{hbd^3}{bb+dd} + \frac{hdb^3}{bb+bb}$ or half the Sum of the Resistances of the Solids in the former *Cases*: For, the Particles of the Fluid strike this Body with the same Obliquity as the former, on all its Sides respectively, and the Number of Particles that strike this Solid, will be half so many, as strike both the other Solids on their respective Sides at the same Time; or, as was observed before, the Bases of these Solids being equal, the Number of the Particles of the Fluid that strike at the same Time will also be equal.

If the Depth and half Breadth of the foregoing Solids be the same, or $b=d$ the Resistances of all those Solids will be equal. For

1st, The Resistance of the Solid in the first *Case* being $\frac{2hd^3b}{bb+dd}$ it is plain, that by substituting b for d and d for b (which, because $b=d$, does not alter the Value of the Expression) $\frac{2hbd^3}{bb+dd}$ will become $\frac{2hdb^3}{bb+bb}$ the Resistance, as in the second *Case*.

2. By

2. By Substituting as before b for d, and d for b, the Value of $\frac{bbd^3}{bb+dd} + \frac{bdb^3}{bb+bb}$ will become $\frac{2bdb^3}{bb+bb}$ or $\frac{2bbd^3}{bb+dd}$ as before, or may in this *Case* be expressed by $\frac{2bb^4}{bb+bb} = \frac{2bd^4}{bb+dd}$.

Now, if the Altitude AB or $b=66,9$, and $b=d$ $=33\frac{1}{2}$, then BL $=$ 13,3053 very near: The Solid Content of the Parallelopipedon will be 148666, and if that be put for its Resistance, the Resistance of the before-mentioned Solids will be 29567 nearly.

The Content of the Solids in the First and Second *Cases* is half the circumscribed Parallelopipedon, or 74333, by which if the Resistance be divided, there will come forth ,39776.

The Content of the Piramid or Solid in the Third *Case* is one Third of the Parallelopipedon or 49555, by which if the Resistance be divided, there will come forth ,59665: So that the Solids in the First and Second *Cases* have a greater Solid Content in Proportion to their Resistances than that in the Third *Case*.

Suppose now the Breadth to be encreased and the Depth lessened, so that the Area of the Base, and consequently their Solidities, remain the same as before, and the Breadth be in Proportion to the Depth as 3 to 1.

Put the new Half Breadth $=a$, then $3 . 1 :: 2a . \frac{2a}{3}$ the new Base: Because, the Depth of the former Parallelopipedon was $2bb$, and that of the Base here spoken of is $\frac{4aa}{3}$ therefore $\frac{4aa}{3} = 2bb$ and mul-

 tiplying

tiplying by 3, $4aa=6bb$, $2aa=3bb$, or $aa = \frac{3bb}{2}$ and by Evolution $a=\sqrt{\frac{3bb}{2}}$.

But if it were required to find the Depth, put the Depth $=e$, and then $1 . 3 :: e . 3e$ the new Breadth, and consequently, $3ee=2bb$, whence $ee=\frac{2}{3}bb$, and by Evolution $e=\sqrt{\frac{2}{3}bb}=b\sqrt{\frac{2}{3}}$.

Let this Parallelopipedon be cyphered away from Bottom to Top, as in the First *Case*, and its Resistance will be found 21111, which divided by the Solid Content gives ,284005.

If the Parallelopipedon be cyphered off as in the Second *Case*, the Resistance will then be 40339, and the Resistance divided by the Solidity ,54268.

If it be cyphered off both Ways, as in the Third *Case*, the Resistance will be (as was shewn before) half the Sum of the Resistances of the two foregoing Solids or 30725, and the Resistance divided by the Content is ,62002.

It may be observed from what has been said, that the Resistance is very much lessened when the Parallelopipedon is cyphered off as in the First *Case*, and greatly encreased in the Second, from what it was, when the Half Breadth and Depth were equal: Let us now consider what Proportion the Breadth should bear to the Depth, to make the Resistance, the least possible.

Put as before the Base $=2bb=2dd$, and let y denote the new half Breadth: Then $y . b :: b . \frac{bb}{y}$ the

new

new Depth, and the Resistance of this Parallelopipedon, when it is cyphered away as in the First *Case* from Bottom to Top, may be expressed by

$$\frac{2bbb \times \frac{b^4}{yy}}{bb + \frac{b^4}{yy}},$$ now $2bbb \times \frac{b^4}{yy} = \frac{2bb^6}{yy}$ and $bb + \frac{b^4}{yy} = \frac{b^2y^2 + b^4}{yy}$, whence $\frac{2bbb \times \frac{b^4}{yy}}{bb + \frac{b^4}{yy}} = \frac{2bb^6}{b^2y^2 + b^4}$ the Fluxion of which is $-\frac{4b^3b^6y\dot{y}}{\overline{b^2y^2 + b^4}|^2}$ which if a Minimum,

then $-\frac{4b^3b^6y\dot{y}}{\overline{b^2y^2 + b^4}|^2} = 0$, and multiplying by

$\overline{b^2y^2 + b^4}|^2$; $-4b^3b\,y\dot{y} = 0$, divided by $\dot{y}$, $-4b^3b^6y = 0$, and consequently, $-y = 0$, that is, if the Altitude and Area of the Base remain the same, the Resistance will be continually lessened by encreasing the Breadth and lessening the Depth, till the Depth vanishes.

In the Second *Case* the Resistance may be expressed

by $\frac{2bb^3y^2}{b^2 + y^2}$ the Fluxion of which is

$$\frac{4b^3b^2y\dot{y} + 4bb^2y^3\dot{y} - 4bb^2y^3\dot{y}}{\overline{bb + yy}|^2} = \frac{4b^3b^2y\dot{y}}{\overline{bb + yy}|^2}:$$ if this

be

be a Minimum, then $4b^3b^2y=0$, and consequently $y=0$, that is, the Resistance will keep in a continual Decrease as the Breadth decreases.

In the Third *Case*, the Resistance may be expressed by $\frac{bb^6}{b^2y^2+b^4}+\frac{bb^2y^2}{b^2+y^2}$ or instead thereof we may use $\frac{b^4}{b^2y^2+b^4}+\frac{y^2}{b^2+y^2}$ the Fluxion of which if a Minimum, is $\frac{2b^2y\dot{y}}{b^4+2b^2y^2+y^4}-\frac{2b^2b^4y\dot{y}}{b^4y^4+2b^2y^2+b^8}$ $=0$, and dividing both Sides by $2b^2y\dot{y}$ we shall have $\frac{1}{b^4+2b^2y^2+y^4}=\frac{b^4}{b^4y^4+2b^2y^2b^4+b^8}$, or $b^4y^4+2b^2y^2b^4+b^8=b^4b^4+2b^2b^4y^2+b^4y^4$, and taking away $2b^2y^2b^4$ from both Sides of the Equation we shall have $\frac{b^8-b^4b^4}{b^4-b^4}=y^4$, or $b^4=y^4$, and by Evolution $b=y$, that is, the Resistance of this Piramidal Figure will be the least when the half Breadth and Depth are equal.

COROLLARY.

The Resistance of a right Cone is to the Resistance of its Circumscribed Piramid as the Solidity of the Cone is to the Solidity of the Piramid, and as the Surface of the Cone to the Surface of the Piramid. For,

1st,

1st, The Angle of Incidence of the Particles of the Fluid is the same on both Solids.

2d, The Curve Surface of a right Cone is equal to a Triangle whose Base is the Periphery of the Base of the Cone, and its Altitude equal to the Side of the Cone.

3d, The Surface of a right Piramid is equal to a Triangle, whose Base is the Perimeter of the Base of the Piramid, and its Altitude equal to the Perpendicular of the Isosceles Triangles that compose the same: Therefore, the Number of the Particles of the Fluid, that strike both the Solids at the same Time and with the same Angle of Incidence, are as their Surfaces.

4th, The Solidity of a right Piramid is One Third of its circumscribed Parallelopipedon, and the Solidity of a right Cone is One Third of its circumscribed Cylinder, therefore their Solidities are as their Bases; and the Area of a Square is to the Area of its inscribed Circle, as the Perimeter of the Square to the Periphery of the Circle: For, let r be the Radius of a Circle, $2r$ the Diameter, or Side of the circumscribed Square, and $8r$ the Perimeter, then if c be put for the Circumference of the Circle, it will be $8r . c :: 4rr . \frac{rc}{2}$ the known Area of the Circle.

PROP.

PROP. II.

The Altitude of a Piramid *being given, to determine the* Base, *when the* Resist-ance *shall be the least that can be in Proportion to the* Solidity.

PUT the Altitude VD$=b$, *Fig.* 9. and ED, or half the Side of the Base, $=y$.

The Resistance of the Piramid is $\frac{byy}{yy+bb}\times 4yy$, as was shewn before, or $\frac{4by^4}{yy+bb}$: the Solid Content is, $\frac{4byy}{3}$, by which, if the Resistance be divided we shall have $\frac{12by^4}{4b^3y^2+4by^4}=\frac{3y^2}{b^2+y^2}$ the Fluxion of which is $\frac{6b^2y\dot{y}+6y^3\dot{y}-6y^3\dot{y}}{\overline{b^2+y^2}|^2}$ which if a Minimum, then $6b^2y\dot{y}=0$, and dividing by $6b^2y$, $\dot{y}=0$, that is, the Base will continually decrease till it vanishes.

If the Content be divided by the Resistance, it will be $\frac{4b^3y^2+4by^4}{12by^4}=\frac{b^2+y^2}{3y^2}$ the Fluxion of which is $\frac{6y^3\dot{y}-6b^2y\dot{y}-6y^3\dot{y}}{9y^4}=-\frac{2b^2y\dot{y}}{3y^4}$, if this be a Mi-

nimum

nimum, then $-2b^2y\dot{y}=0$, and dividing by $2b^2y$, we shall have $-\dot{y}=0$, that is, ED will increase in Infinitum, to make the Quotient of the Solid Content, divided by the Resistance the least that can be.

And what has been said of the Piramid, may be applied to the Cone, or any other Piramidal Solid whose transverse Sections are regular Polygons.

PROP. III.

To Investigate the Resistance of a Paraboloid.

LET ENB, *Fig.* 10. be the Parabola given, draw FK parallel to the Axis QB, and let fall the Perpendicular NM from any Point of the Curve on the Axis QB, and draw NI Perpendicular to the Curve in the Point N, then by the aforecited Proposition of Sir *Isaac Newton*, NI*q* . IM*q* : : QB . HF, and if this be done in every Point of the Parabola, the Point H will by its Motion describe the Curve QHD, and the Resistance of the Paraboloid generated by revolving the Parabola QENB about its Axis QB will be to the Resistance of the Cylinder generated by the Parallelogram QELB in the same Rotation, as the Content of the Solid generated by revolving the Figure BQHDLB, about the same Axis QB, is to the Content of the Circumscribed Cylinder. It remains then to Investigate the Content of the aforesaid Solid: in order to which,

Put the Altitude QB$=b$, the Ordinate NM$=y$, the Subnormal IM which in this Curve is a constant Quantity $=a$, equal to half the Parameter: For, apply the Ordinate nm, infinitely near to NM; the Infinitely little Triangle rNn is similar to the Triangle

angle MNI, therefore $rN . MN :: rn . IM$, or the Subnormal $IM = \frac{MN \times rn}{rN} = \frac{y\dot{y}}{\dot{x}}$, but in the Parabola, putting p for the Parameter, $px = yy$ and $p\dot{x} = 2y\dot{y}$ therefore $\frac{y\dot{y}}{\dot{x}} = \frac{p\dot{x}}{2\dot{x}} = \frac{1}{2}p$.

By Substituting b for QB, y for NM and a for IM: instead of $NIq . IMq :: QB . HF$, we shall have $aa+yy . aa :: b . \frac{aab}{aa+yy}$ the Value of HF, according to the Property of the Parabola.

Now, if c stands for the Circumference of a Circle in general, and r for its Radius, then $4rr . \frac{rc}{2} :: 4yy . \frac{cyy}{2r}$ or $\frac{c}{2r} \times yy$ will be the Area of any particular Circle whose Radius is y.

The Base therefore of this Solid is $\frac{c}{2r} \times yy$ its Fluxion is $\frac{c}{2r} \times 2y\dot{y}$ the Fluxion of the Base, which multiplied into $\frac{aab}{aa+yy}$ gives $\frac{c}{2r} \times aab \times \frac{2y\dot{y}}{aa+yy}$ the Fluxion of the Solid.

In

In Order to find the Fluent of which, put $aa+yy=z$, then $2y\dot{y}=\dot{z}$ and by Subſtituting $\dot{z}$ for $2y\dot{y}$ the Fluxion of the Solid will be $\frac{c}{2r}\times aab\times\frac{\dot{z}}{z}$: and becauſe the Fluxion of any Logarithm is equal to the Fluxion of the abſolute Number divided by the abſolute Number: therefore the Fluent of $\frac{c}{2r}\times aab\times\frac{\dot{z}}{z}$ will be $\frac{c}{2r}\times aab\times$ Log. z, and reſtoring $aa+yy$ for z, it will be $\frac{c}{2r}\times aab\times$ Log. $\overline{aa+yy}$ but becauſe Log. $\overline{aa+yy}$ muſt vaniſh when $y=0$, therefore the true flowing Quantity or Content of the Solid enquired after will be $\frac{c}{2r}\times aab\times$ Log. $\frac{\overline{aa+yy}}{aa}$, the Reſiſtance of the Paraboloid required.

Put as before $b=66,9$ and $y=33\frac{1}{3}$, then will aa be 68,9608: The Content of the circumſcribed Cylinder will be 233525, and that of the Paraboloid 116762, for the Paraboloid is known to be half the circumſcribed Cylinder.

If the Reſiſtance of the Cylinder be put equal to its Solid Content, the Reſiſtance of the Paraboloid, will by the foregoing Theorem be found equal to 41169, which divided by the Solidity 116762 gives ,35259.

PROP. IV.

To Investigate the Resistance of a Solid *generated by revolving a* Parabola *about an* Ordinate.

LET VBD, *Fig.* 11. be the generating Curve, V the Vertex, AV the Axis, VE an Abscissa, and ED an Ordinate of the Parabola applied in the Point E, DK a Perpendicular to the Curve applied in the Point D, and let AB be the Axis of Rotation, in the Direction of which the Solid is to move from A towards B.

Put VA $=r$, VE $=x$, and the Subnormal EK (which in the foregoing Proposition was shewn to be always equal to the Semiparameter) $=a$.

Then $DKq . EDq :: AB . FG$ for the Triangle MDC is similar to the Triangle DKE.

But, by the Property of the Parabola $DEq=2ax$, and $KDq=DEq+EKq=2ax+aa$, therefore $aa+2ax . 2ax :: b . \frac{2bx}{a+2x}=FG$, and if this be done in every Point of the Curve, the Point H will describe the Curve HFL; and the Resistance of the given Solid will be to the Resistance of its circumscribed Cylinder, as the Content of the Solid generated by revolving the Curve HFL about the Axis HB to the Content of the circumscribed Cylinder: In Order to find the Content of the Solid generated by revolving HFL about its Axis HB: Let c be a Standing Expression shewing the Proportion of the Area of a Circle to that of its circumscribed Square. Now $AE=r-x$, therefore the Base of the Solid will be $4c\times\overline{rr-2rx+xx}$, the Fluxion of which is

$4c\times$

$4c \times \overline{2x\dot{x}-2r\dot{x}}$ the Fluxion of the Base, which multiplied into the Altitude FG$=\frac{2bx}{a+2x}$ gives

$4c \times \frac{\overline{4bx^2\dot{x}-4brx\dot{x}}}{a+2x} = 4c \times \frac{4bx^2\dot{x}}{a+2x} - 4c \times \frac{4brx\dot{x}}{a+2x}$ the Fluxion of the Solid.

To find the Fluent of $\frac{4bx^2\dot{x}}{a+2x}$ put $a+2x=z$; then

$x=\frac{z-a}{2}$, $4xx=zz-2za+aa$ and $\dot{x}=\frac{\dot{z}}{2}$, whence

$\frac{4bx^2\dot{x}}{a+2x} = \frac{bz\dot{z}}{2} - ba\dot{z} + \frac{baa\dot{z}}{2z}$ the flowing Quantity of which is $\frac{bz^2}{4} - baz + \frac{baa}{2} \times$ Log. z, and restoring $a+2x$ for z, it will be $\frac{b}{4} \times \overline{a-2x}|^2 - ba \times \overline{a+2x} + \frac{baa}{2} \times \text{Log.}\frac{\overline{a+2x}}{a}$.

And to find the Fluent of $-\frac{4brx\dot{x}}{a+2x}$, put as before $a+2x=z$, then $x=\frac{z-a}{2}$ and $xx=\frac{zz-2za+aa}{4}$,

 whence

whence $2x\dot{x} = \frac{z\dot{z} - \dot{z}a}{2}$ and $4x\dot{x} = z\dot{z} - \dot{z}a$, by which Substitution, the Term $\frac{4brx\dot{x}}{a+2x}$ will become $-br\dot{z} + \frac{abr\dot{z}}{z}$ the Fluent of which is $-brz + abr \times \text{Log.}\ z$, and restoring $a + 2x$ for z we shall have $-br \times \overline{a+2x} + abr \times \text{Log.}\ \frac{\overline{a+2x}}{a}$.

The Aggregate of all which is $\frac{b}{4} \times \overline{a+2x}|^2 - ba \times \overline{a+2x} - br \times \overline{a+2x} + \overline{\frac{baa}{2} + abr \times \text{Log.}\ \frac{\overline{a+2x}}{a}}$ or $\frac{baa}{4} + abx + bxx - aab - 2abx - abr - 2brx + \overline{\frac{baa}{2} + abr \times \text{Log.}\ \frac{\overline{a+2x}}{a}} := -\frac{3aab}{4} - abx + bxx - 2brx - abr + \overline{\frac{baa}{2} + abr \times \text{Log.}\ \frac{\overline{a+2x}}{a}}$.

If $x = 0$, we shall have $-\frac{3aab}{4} - abr$, but if

$x = r$

$ED = \frac{bx^{\frac{1}{2}}}{r^{\frac{1}{2}}}$ the Ordinate or Altitude : Put as before $2x\dot{x} - 2r\dot{x}$ for the Fluxion of the Base which multiplied into $\frac{bx^{\frac{1}{2}}}{r^{\frac{1}{2}}}$ the Altitude gives $\frac{2bx^{\frac{3}{2}}\dot{x}}{r^{\frac{1}{2}}} - \frac{2brx^{\frac{1}{2}}\dot{x}}{r^{\frac{1}{2}}}$ the Fluent of which is $\frac{\frac{4}{5}bx^{\frac{5}{2}}}{r^{\frac{1}{2}}} - \frac{\frac{4}{3}brx^{\frac{3}{2}}}{r^{\frac{1}{2}}}$

If $x=0$, the whole will vanish, but if $x=r$, we shall have $\frac{\frac{4}{5}br^{\frac{5}{2}}}{r^{\frac{1}{2}}} - \frac{\frac{4}{3}br^{\frac{5}{2}}}{r^{\frac{1}{2}}} = \frac{4}{5}br^2 - \frac{4}{3}br^2 = -\frac{8}{15}br^2$ which deducted from o, gives $\frac{8}{15}br^2$; but br^2 will in this Case stand for the circumscribed Cylinder: therefore the Content of the Solid enquired after will be $\frac{8}{15}$ of the Circumscribed Cylinder: and in Proportion to the Paraboloid as 16 to 15.

If this Solid be of the aforesaid Dimensions, its Solidity will be 124546, and its Resistance divided by the Solidity ,44443.

PROP.

PROP. VI.

To Investigate the Resistance of a Spheroid.

PUT the Altitude QB$=b$, *Fig.* 13. the Abscissa MB$=x$, and the Ordinate NM$=y$.

Let p stand for the Semiparameter, then by the Property of the Ellipsis $yy=\frac{p}{b}\times\overline{2bx-xx}$, and by Evolution $y=\frac{p^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\overline{2bx-xx}|^{\frac{1}{2}}$ the Fluxion of which is $\dot{y}=\frac{p^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\frac{b\dot{x}-x\dot{x}}{\overline{2bx-xx}|^{\frac{1}{2}}}$, then $nr.r$N::NM. MT, Or $\dot{y}.\dot{x}::y.\frac{x\dot{y}}{\dot{y}}$ the Subtangent, and by Substituting the Values of $\dot{y}$, $\dot{x}$ and y, we shall have $\frac{p^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\frac{b\dot{x}-x\dot{x}}{\overline{2bx-xx}|^{\frac{1}{2}}}.\dot{x}::\frac{p^{\frac{1}{2}}}{b^{\frac{1}{2}}}\times\overline{2bx-xx}|^{\frac{1}{2}}.\frac{2bx\dot{x}-x^2\dot{x}}{b\dot{x}-x\dot{x}}=$ $\frac{2bx-xx}{b-x}$ the Subtangent; but $2bx-xx=\frac{b}{p}\times yy$, and if we put $\frac{b}{p}=e$, then $2bx-x=eyy$, or

xx—

$xx-2bx=-eyy$; by adding bb to both Sides of the Equation, $xx-2bx+bb=bb-eyy$, and by Evo-volution $b-x=\overline{bb-eyy}|^{\frac{1}{2}}$ whence $\frac{2bx-xx}{b-x}=$ $\frac{yye}{\overline{bb-eyy}|^{\frac{1}{2}}}$ the Square of which is $\frac{y^4e^2}{bb-eyy}$, then NMq $+MTq=NTq$, or $\frac{y^4e}{bb-eyy}+yy=\frac{y^4e^2+y^2b^2-ey^4}{bb-eyy}$ and $NTq.NMq::QB.HF$, or $\frac{y^4e^2+bbyy-ey^4}{bb-eyy}$ $.yy::b.\frac{b^3y-yeby^4}{y^2e^2+bbyy-ey^4}=\frac{b^3-eby^2}{y^2e^2+bb-ey^2}$.

Now, if we call $\frac{c}{2r}yy$ the Base of the Solid, the Fluxion of the Base may be called $2y\dot{y}$, which multiplied into $\frac{b^3-eby^2}{y^2e^2+bb-ey^2}$ gives $\frac{2b^3y\dot{y}-2eby^3\dot{y}}{y^4e^2+bb-ey^2}$ the Fluxion of the Solid generated by revolving the Curve QHL about the Axis QB, the Solidity of which expresses the Resistance of the Spheroid.

Put $e^2-e=m$, then the Fluxion of the Solid will become $\frac{2b^3y\dot{y}-2eby^3\dot{y}}{bb+my^2}=\frac{b^3\times 2y\dot{y}}{bb+my^2}-\frac{eb\times 2y^3\dot{y}}{bb+my^2}$;

To

To find the Fluent of $\frac{b^3 \times 2y\dot{y}}{bb+my^2} = \frac{\frac{b^3}{m} \times 2y\dot{y}}{\frac{bb}{m}+yy}$, put $\frac{bb}{m}+y^2=z$, then $yy=z-\frac{bb}{m}$ and $2y\dot{y}=\dot{z}$ whence $\frac{\frac{b^3}{m} \times 2y\dot{y}}{\frac{bb}{m}+yy} = b^3 \times \frac{\dot{z}}{z}$ the Fluent of which is $b^3 \times$ Log. z, and reſtoring $\frac{bb}{m}+yy$ for z, it will be $\frac{b^3}{m} \times$ Log. $\overline{\frac{bb}{m}+yy}$, but becauſe when y vaniſhes, the whole Expreſſion ſhould vaniſh, the true Fluent will be

$$\frac{b^3}{m} \times \text{Log.} \frac{\frac{bb}{m}+yy}{\frac{bb}{m}}.$$

And to find the Fluent of $-\frac{eb \times 2y^3\dot{y}}{bb+my^2} = -\frac{\frac{eb}{m} \times 2y^3\dot{y}}{\frac{bb}{m}+yy}$, put as before $\frac{bb}{m}+yy=z$, then $yy=z-\frac{bb}{m}$ and $y^4=z^2-2z \times \frac{b^2}{m}+\frac{b^4}{m^2}$, $4y^3\dot{y}=2z\dot{z}-2\dot{z}$

$\times \frac{b^2}{m}$, or $2y^3\dot{y} = z\dot{z} - \dot{z} \times \frac{b^2}{m}$, whence $-\frac{\frac{eb}{m} \times 2y^3\dot{y}}{\frac{bb}{m} + yy}$

$= -\frac{\frac{eb}{m} \times z\dot{z}}{z} + \frac{\frac{eb}{m} \times \frac{b^2}{m} \times \dot{z}}{z}$ the Fluent of the former Term $-\frac{eb}{m} \times \dot{z}$ is $-\frac{eb}{m} \times z = -\frac{eb}{m} \times$

$\frac{bb}{m} + y^2 = -\frac{eb}{m} \times \frac{bb + my^2}{m}$, $= -\frac{eb^3}{m^2} - \frac{eby^2}{m}$.

And the Fluent of the latter Term $\frac{\frac{eb}{m} \times \frac{b^2}{m} \times \dot{z}}{z} =$

$\frac{eb^3}{mm} \times \frac{\dot{z}}{z}$ is $\frac{eb^3}{m^2} \times$ Log. z, and by restoring $\frac{bb}{m}$

$+ yy$ for z, the Fluent will be $\frac{eb^3}{m^2} \times$ Log. $\frac{\frac{bb}{m} + yy}{\frac{bb}{m}}$

The Aggregate of all the Terms of the flowing Quantity is $\frac{b^3}{m} \times$ Log. $\frac{\frac{bb}{m} + yy}{\frac{bb}{m}} + \frac{eb^3}{m^2} \times$ Log.

$\frac{\frac{bb}{m} + yy}{\frac{bb}{m}} - \frac{eby^2}{m} - \frac{eb^3}{m^2}$, but because the whole

Expression

Expreſſion muſt vaniſh when $y=0$, the true Fluent or Reſiſtance is $\frac{c}{2r}\times\overline{\frac{b^3}{m}+\frac{eb^3}{m^2}\times\text{Log.}\frac{bb+myy}{bb}-\frac{eby^2}{m}}$

for $\frac{\frac{bb}{m}+yy}{\frac{bb}{m}}=\frac{bb+myy}{bb}$.

If we put as before, $b=66{,}9$, $y=33\frac{1}{2}$, then $p=$ 16,6085 nearly, and the Reſiſtance will come out 65603: the Solidity of the Spheroid is 155683, by which the Reſiſtance being divided gives ,42138.

If the Spheroid degenerates into a Sphere, then $e=1$ and $m=0$, now the Fluxion of $bb+myy$, is $2my\dot{y}\times$ $\overline{bb+myy}|^{-1}$ & $\overline{bb+myy}|^{-1}=\frac{1}{bb}-\frac{myy}{b^4}+\frac{m^2y^4}{b^6}-\frac{m^3y^6}{b^8}$

which multiplied into $2my\dot{y}$ gives $\frac{2my\dot{y}}{bb}-\frac{2m^2y^3\dot{y}}{b^4}+$ $\frac{2m^3y^5\dot{y}}{b^6}-\frac{2m^4y^7\dot{y}}{b^8}$ the Fluent of which is $\frac{my^2}{bb}-$ $\frac{m^2y^4}{2b^4}+\frac{m^3y^6}{3b^6}-\frac{m^4y^8}{4b^8}$ &c. the Log. of $bb+myy$, which multiplied into $\frac{b^3}{m}-\frac{eb^3}{m^2}$ gives $by^2-\frac{my^4}{2b}$

 +

$+ \frac{m^2y^5}{3b^2} - \frac{m^3y^8}{4b^5}$ &c. $+ \frac{eby^2}{m} - \frac{ey^4}{2b} + \frac{mey^6}{3b^3} -$ $\frac{em^2y^8}{4b^5}$, but when $m=0$ it will be $by^2 + \frac{eby^2}{m} -$ $\frac{ey^4}{2b}$ because the other Terms will all vanish, and subtracting $\frac{eby^2}{m}$ it will be $by^2 - \frac{ey^4}{2b}$, and when $y=1$ and $e=1$, we shall have $\frac{c}{2r} \times \frac{b^3}{2}$ the Resistance or half the circumscribed Cylinder.

PROP. VII.

To Investigate the Resistance of an Hyperbolical Conoid.

LET QENB, *Fig.* 14, be the generating Hyperbola, QB$=b$ the Axis of the given Solid: and put the Transverse Diameter of the Hyperbola $=2d$, the Semiparameter $=p$, and let y denote an Ordinate and x the Abscissa proper to it.

By the Property of the Hyperbola $yy=\frac{p}{d} \times \overline{2dx+x^2}$ by Evolution $y=\frac{p^{\frac{1}{2}}}{d^{\frac{1}{2}}} \times \overline{2dx+xx}|^{\frac{1}{2}}$ the Fluxion of which of which is $\dot{y} = \frac{p^{\frac{1}{2}}}{d^{\frac{1}{2}}} \times \frac{d\dot{x}+x\dot{x}}{\overline{2dx+xx}|^{\frac{1}{2}}}$ then $nr \,.\, rN$ :: NM . MT, or substituting their Values, $\frac{p^{\frac{1}{2}}}{d^{\frac{1}{2}}} \times \frac{d\dot{x}+x\dot{x}}{\overline{2dx+xx}|^{\frac{1}{2}}} \,.\, \dot{x} :: \frac{p^{\frac{1}{2}}}{d^{\frac{1}{2}}} \times \overline{2dx+xx}|^{\frac{1}{2}} \,.\, \frac{2dx\dot{x}+x^2\dot{x}}{d\dot{x}+x\dot{x}} = \frac{2dx+xx}{d+x}$ the Subtangent. And if we put $\frac{d}{p}=e$, then $yye=2dx+xx$ and $\frac{2dx+xx}{d+x}=\frac{yye}{d+x}$, by adding dd to both Sides of the Equation $dd+2dx+xx=dd+yye$, and by Evolution $d+x=\overline{dd+yye}|^{\frac{1}{2}}$ therefore $\frac{2dx+xx}{d+x}=\frac{yye}{\overline{dd+yye}|^{\frac{1}{2}}}$, the Square of which is $\frac{y^4e^2}{dd+yye}$. and $\frac{y^4e^2}{dd+yye}+yy=\frac{y^4e^2+d^2y^2+ey^4}{dd+yye}$, the Square of the Tangent: Now $NTq \,.\, NMq$ :: QB . FH, or substituting their Values, $\frac{y^4e^2+d^2y^2ey^4}{dd+yye} \,.\, yy :: b \,.\, \frac{bd^2y^2+bey^4}{y^4e^2+d^2y^2+ey^4}=\frac{bd^2+bey^2}{y^2e^2+d^2+ey^2}$, and putting $e^2+e=m$ it will be $\frac{bd^2+bey^2}{d^2+my^2}$:

If

If we put $\frac{c}{2r} \times y\dot{y}$ for the Base of the Solid; because $\frac{c}{2r}$ is a standing Expression, shewing the Ratio of a Circle to a Square whose Side is the Radius of the Circle: the Fluxion of the Base may be called $2y\dot{y}$ which multiplied into $\frac{bd^2+bey^2}{d^2+my^2}$ will give $\frac{bd^2 \times 2y\dot{y}}{d^2+my^2} + \frac{2bey^3\dot{y}}{d^2+my^2}$, the Fluxion of the Solid whose Solidity expresses the Resistance of the given Solid.

In Order to find which, put $d^2+my^2=z$, then $y^2 = \frac{z-d^2}{m}$, $2y\dot{y} = \frac{\dot{z}}{m}$, $y^4 = \frac{zz-2d^2z+d^4}{m^2}$, $4y^3\dot{y} = \frac{2z\dot{z}-2d^2\dot{z}}{m^2}$ and $2y^3\dot{y} = \frac{z\dot{z}-d^2\dot{z}}{m^2}$, by substituting which in the Fluxion of the Solid, we shall instead of the first Term $\frac{bd^2 \times 2y\dot{y}}{d^2+my^2}$ have $\frac{bd^2 \times \frac{\dot{z}}{m}}{z} = \frac{bd^2}{m} \times \frac{\dot{z}}{z}$ the Fluent of which is $\frac{bd^2}{m} \times$ Log. z, and resto-

ring

ring d^2+myy for z we shall have $\frac{bd^2}{m} \times \text{Log.} \overline{\frac{d^2+myy}{d^2}}$ because the Log. of z must vanish when $y=0$.

And instead of the second Term $\frac{be \times 2y^3\dot{y}}{d^2+myy}$, there will arise $\frac{bez\dot{z}}{m^2z} - \frac{bed^2\dot{z}}{m^2z}$, the Fluent of which is $\frac{bez}{m^2} - \frac{bed^2}{m^2} \times \text{Log.}\ z$, and restoring d^2+myy for z, we shall have $\frac{bed^2}{m^2} + \frac{beyy}{m} - \frac{bed^2}{m^2} \times \text{Log.} \overline{\frac{d^2+myy}{d^2}}$. The Aggregate of all which is $\overline{\frac{bd^2}{m} - \frac{bed^2}{m^2}} \times \text{Log.} \overline{\frac{d^2+myy}{d^2}} + \frac{beyy}{m} + \frac{bed^2}{m}$.

But, because when y vanishes the whole Expression must vanish with it, the true Fluent will be $\overline{\frac{bd^2}{m} - \frac{bed^2}{m^2}} \times \text{Log.} \overline{\frac{d^2+myy}{d^2}} + bey^2$ which Expression multiplied into $\frac{c}{2r}$ gives the Resistance required.

If

If $b=d$, or the Altitude be equal to half the transverse Diameter as in the Spheroid it will then be

$$\frac{b^3}{m} - \frac{eb^3}{m^2} \times \text{Log.}\, \frac{bb+myy}{bb} + \frac{bey^2}{m}.$$

Now, if we put $b=d=66{,}9$, and $y=33\frac{1}{3}$, then $p=5{,}5361$: And the Resistance will come out 38058; The Solidity of this Hyperbolical Conoid is 10378[illegible], by which the Resistance being divided will give ,36668.

PROP. VIII.

To Investigate the Frustum of a Cone, which shall meet with less Resistance, than any other Frustum of the same Base and Altitude.

PUT OC$=r$, *Fig.* 15. OD$=b$, and OS$=x$; Then CS$q=xx+rr$, and OS (x) . OC (r) : : DS $(x-b)$. DF $=\frac{rx-rb}{x}$. Again, if CSq $(xx+rr)$. OCq (rr) : : OD (b) . AE$=\frac{rrb}{xx+rr}$.

Then the Resistance of this Frustum generated by revolving OCFD about the Axis OD may be expressed by the Solidity generated in turning the *Fig.* OIHAED about the said Axis OD.

The

The Content of the Cylinder generated by the Parallelogram AEDG in this Rotation about OD may be represented by $\frac{r^4b}{xx+rr}$, and that of IHGO in the same Rotation by $\frac{br^2x^4-2r^2x^3b^2+b^3r^2x^2}{x^4+r^2x^2}$: the whole Resistance of the Frustum will therefore be $\frac{r^4b}{xx+rr}+\frac{br^2x^4-2r^2x^3b^2+b^3r^2x^2}{x^4+r^2x^2}=$ $\frac{r^4bx^2+br^2x^4-2r^2x^3b^2+b^3r^2x^2}{x^4+r^2x^2}$ equal to $\frac{r^4b+br^2x^2-2r^2xb^2+b^3r^2}{xx+rr}$, which by the Nature of the Question is to be a Minimum.

The Fluxion of the Numerator $r^4b+br^2x^2-2r^2xb^2+b^3r^2$ is $2br^2x\dot{x}-2b^2r^2\dot{x}$, which multiplied into the Denominator x^2+r^2 gives $2br^2x^3\dot{x}-2b^2r^2x^2\dot{x}+2br^4x\dot{x}-2b^2r^4\dot{x}$.

And the Fluxion of the Denominator x^2+r^2 is $2x\dot{x}$, which multiplied into the Numerator $r^4b+br^2x^2-2r^2b^2x+b^3r^2$, will produce $2br^4x\dot{x}+2br^2x^3\dot{x}-4b^2r^2x^2\dot{x}+2b^3r^2x\dot{x}$; the Fluxion therefore of the Resistance $\frac{r^4b+br^2x^2-2r^2xb^2+b^3r^2}{x^2+r^2}$, is $\frac{2b^2r^2x^2\dot{x} - 2b^2r^4\dot{x} - 2b^3r^2x\dot{x}}{\overline{x^2+r^2}|^4} = 0$, if a Minimum,

whence $2b^2r^2x^2\dot{x} - 2b^2r^4\dot{x} - 2b^3r^2x\dot{x} = 0$, and dividing by $2b^2r^2\dot{x}$, we shall have $x^2 - r^2 - bx = 0$, or $x^2 - bx = r^2$, and adding $\frac{1}{4}bb$, or the Square of half the known Coefficient b, to both Sides of the Equation, it will be $x^2 - bx + \frac{1}{4}bb = r^2 + \frac{1}{4}bb$, and by Evolution $x - \frac{1}{2}b = \overline{r^2 + \frac{1}{4}bb}|^{\frac{1}{2}}$, now $QS = x - \frac{1}{2}b$, and $CQ = \overline{r^2 + \frac{1}{4}bb}|^{\frac{1}{2}}$ whence follows that easy Construction, given by Sir *Isaac Newton*;

Bisect the Altitude OD in Q, and produce OQ to S, so that QS may be equal to QC, and S will be the Vertex of the Cone whose Frustum is sought.

Now, if $b = 66,9$, and $r = 33\frac{1}{2}$, as in the Solids whose Resistances we have compared before, then OS=80,673 FD=5,6909 AE=11,4216, and the Resistance of the Frustum will be 41045, and its Solid Content 93435, by which the Resistance being divided will give 43929.

The Resistance of the inscribed Cone generated by revolving the Triangle OCD about the Axis OD is represented by the Solidity of the Cylinder generated by the Parallelogram KEDL in the Same Rotation, in which the Altitude KE is equal to $\frac{OD \times OCq}{CDq} = 13,3053$, whence the Resistance of the Cone will be 46444: its Solidity is one Third of the circumscribed Cylinder, or 77841, by which the Resistance being divided will give ,52729.

And

And the Resistance of this Cone, to the Resistance of that Frustum of a Cone of the same Base and Altitude, is as 46444 to 41045, and its Solidity to the Solidity of the Frustum no more than as 77841 to 93435.

And from this Construction of the Frustum of the Cone of the least Resistance (as Sir *Isaac Newton* observes) it follows: That

If a Solid be generated by revolving any Curve of an Elliptical or Oval-like Figure DNFB, *Fig.* 16. about its Axis CB: and another Solid be generated by revolving DNFGB about the same Axis, in which QB is Perpendicular to the Axis, and FG be a Tangent to the Curve in the Point H, making the Angle FGB 135 Degrees, the Resistance of the latter Solid will be less than that of the former: And this will appear from the Consideration of the foregoing Construction of the Frustum of the Cone which meets with the least Resistance: For, if QB were infinitely small, it is plain that $FQ = QP$, or the Angle $FPQ = 45$, or Angle $FGB = 135$ Degrees.

Let the Curve DNFB be an Ellipsis: Then (as was observed before) the Resistance of the Spheroid may be expressed by the Content of the Solid generated by the Rotation of the Curve LMC about the same Axis; and that of DNFGB by the Content of the Solid generated by LMEO in the same Rotation: where universally $ME = QF - GB$, and $GE = EO = \frac{1}{2} BC$, because, $FPq = 2FQq$, and by the foregoing Method of comparing the Resistances of the Solids it will be $2FQq \,.\, FQq :: CB \,.\, \frac{1}{2}CB = MS$.

Let

Let as before, b denote the Altitude CB, p the Semiparameter, y an Ordinate, and put $\frac{b}{p}=e$.

The Subtangent will be $\frac{y^2e}{\overline{bb-eyy}|^{\frac{1}{2}}}$ as was shewn before, in the Investigation of the Resistance of the Spheroid; and in this Case $\frac{y^4e^2}{bb-eyy}=yy$, and dividing by yy we shall have $\frac{y^2e^2}{bb-eyy}=1$, whence $y^2e^2=bb-eyy$, or $y^2e^2+y^2e=bb$, and dividing by e^2+e, $y^2=\frac{b^2}{e^2+e}$, but $e^2+e=\frac{b^2}{p^2}+\frac{b}{p}$ equal to $\frac{b^2+pb}{p^2}$ whence $\frac{b^2}{e^2+e}=\frac{p^2b^2}{b^2+pb}=\frac{p^2b}{b+p}=y^2$ as above, and by Evolution $\frac{pb^{\frac{1}{2}}}{\overline{b+p}|^{\frac{1}{2}}}=y=QF=QP$.

And to find the Value of $QB=x$; $\frac{b^2}{e^2+e}=\frac{2bx-xx}{e}$ or $\frac{b^2}{e+1}=2bx+xx$, or which is the same, $xx-2bx=-\frac{b^2}{e+1}$, and by compleating the Square, $xx-2bx+xx=bb-\frac{b^2}{e+1}$; by Evolution

$b-x$

$$b-x=\overline{bb-\frac{b^2}{e+1}}\Big|^{\frac{1}{2}} \text{ or } b-\overline{bb-\frac{b^2}{e+1}}\Big|^{\frac{1}{2}}=x: \text{ But}$$

$$bb-\frac{b^2}{e+1}=\frac{bbe}{e+1}=\frac{bb\times\frac{b}{p}}{\frac{b}{p}+1}, \text{ now } bb\times\frac{b}{p}=$$

$$\frac{b^3}{p} \text{ and } \frac{b}{p}+1=\frac{b+p}{p}, \text{ therefore } \frac{bbe}{e+1}=\frac{b^3}{b+p},$$

whence $b-\overline{bb-\frac{b^2}{e+1}}\Big|^{\frac{1}{2}}=b-\overline{\frac{b^3}{b+p}}\Big|^{\frac{1}{2}}=x=QB$ the Abſciſſa.

Now, if we put $b=66,9$, and $p=16,6085$ as before, then QF $=14,8654$, GB $=8,8775$, and QB $5,9879$: by the Help of which the Reſiſtance of the Fruſtum of the Cone generated by FGBQ in its Rotation about its Axis will be found to be 31527.

And the Reſiſtance of the Solid generated by FBQ in the ſame Rotation will be found 32989, which ſubtracted from 65603 the Reſiſtance of the whole Spheroid, leaves 32614, to which if we add 31527 we ſhall have 64141, the Reſiſtance of the Solid generated by revolving the Figure DNFGBC about the ſame Axis CB: which is to the Reſiſtance of the Spheroid, as 64141 to 65603, or as 64 to $65\frac{1}{2}$, nearly.

Let the given Curve DNFB, *Fig.* 17. be a Parabola, put the Semiparameter $=a$, and the Altitude CB$=b$ as before: then, if the Angle FPQ be an Angle of 45 Degrees, FQ$=$QP$=a$, and BP$=$QB$=$GH $=\frac{1}{2}$ QF, whence the Reſiſtance of the Fruſtum of the

the Cone generated by the Rotation of FGBQ about its Axis, as represented by the Solid generated by the Rotation of BSMEOC about the same Axis, may be expressed by $c \times \frac{1}{2} b \times yy \times \frac{yy}{4}$ (c standing for the Ratio of the Periphery of a Circle to its Diameter) $= c \times \frac{1}{2} b \times \frac{5yy}{4} = c \times \frac{5}{8} byy = c \times \frac{5}{8} baa$, or $\frac{5}{8}$ the Resistance of a Cylinder whose Radius of the Base is SB, or the Semiparameter.

And if we put CB=66,9, CD=33½, as before, then $y = a =$ FQ $= 8,3042$; $yy = aa = 68,9608$, and the Resistance of the Frustum of the Cone generated by FGBQ in its Rotation about its Axis will come out 9058: And the Resistance of FBQ in the same Rotation will be found 10044: the Resistance of the whole Paraboloid generated by CDNFB was shewn before, to be 41169, from which if we deduct 10044, there will remain 31125, to which if we add 9058, we shall have 40183, the Resistance of the Solid generated by revolving the Figure CDNFGB, about the same Axis CB, which is to the Resistance of the Paraboloid, as 40183 to 41169, or 40 to 41 nearly. And this Proposition, Sir *Isaac Newton* says, he supposes may be useful in *Ship-Building*: but, with this Remark, that

If the Figure DNFG, *Fig.* 18. be a Curve of such a Kind: if from any Point N, a Perpendicular NM be let fall on the Axis CB, and from a given Point G be drawn a Right Line GR, parallel to a Right Line touching the Figure in N, and cutting the Axis produced in R: MN will be to GR, as GR cubed

to

to $4BR \times GBq$: the Solid generated by revolving this Figure about its Axis CB will meet with less Resistance than any other Circular Solid of the same Base and Altitude: and here it naturally follows,

PROP. IX.

To Investigate the Property of a Curve, *which revolved about its Axis, generates a* Solid *of the least Resistance.*

IT was observed in the foregoing Proposition, that when the Altitude of the Frustum of a Cone of the least Resistance becomes infinitely short, the Side of the Cone will make an Angle with the Axis of 45 Degrees, and consequently a Tangent applied in the Head of the least Ordinate of the Curve sought will cut the Axis produced in the same Angle.

Let CD, *Fig.* 19. be the Axis of the Curve, IE a Right Line Parallel to the Axis, and the Points K and D given; draw the Ordinates KC, NM and FD infinitely near to each other, put the Ordinate NM $=y$, and KI $=d$ invariable, and let the Variable Quantities be CM$=x$, HF$=e$, and the least Ordinate DF$=a$.

Now, if the Force of the Fluid striking directly on IK at the Distance IC or NM be called dy, then $d^2+x^2 \,.\, d^2 :: dy \,.\, \frac{d^3y}{d^2+x^2}$, the Resistance on the Surface generated by the infinitely little Part of the Curve KN in its Rotation about its Axis.

And

And becaufe HN=HF; $2ee \,.\, ee :: ne \,.\, \frac{ne}{2}$, the Refiftance generated by NF, and the Refiftance of DF will be $\frac{nn}{2}$; the Refiftance therefore of the contiguous Particles of the Solid generated by KNFD in its Rotation about its Axis may be reprefented by $\frac{d^3y}{d^2+x^2}+\frac{ne+nn}{2}$, which is to be a Minimum, its Fluxion therefore is $-\frac{2d^3yx\dot{x}}{\overline{d^2+x^2}|^2}+\frac{\dot{n}e+n\dot{e}-2n\dot{n}}{2}=0$, but CD is a conftant Quantity, therefore $-\dot{x}=\dot{n}$; and CD $-x=$ NM $-n$, therefore $\dot{x}=\dot{n}$, whence $\frac{\dot{n}e+n\dot{e}+2n\dot{n}}{2}=\frac{\dot{x}e-n\dot{x}+2n\dot{x}}{2}=\frac{\dot{x}e+n\dot{x}}{2}$, and then the Fluxion of the Refiftance will be $-\frac{2d^3yx\dot{x}}{\overline{d^2+x^2}|^2}+\frac{\dot{x}e+n\dot{x}}{2}=0$, and dividing by $\dot{x}$; $\frac{2d^3yx}{\overline{d^2+x^2}|^2}+\frac{e+n}{2}$, Or $\frac{4d^3yx}{\overline{d^2+x^2}|^2}=e+n$, and becaufe e is infinitely fmall

in

in respect of n, $\frac{4d^3 y\dot{x}}{\overline{d^2+x^2}|^2}=n$, $4d^3 y\dot{x}=\overline{d^2+x^2}|^2 \times n$, or

$4d^3\dot{x} . \overline{d^2+x^2}|^2 :: n . y$, but $\dot{x}$ is the Fluxion of the Axis, and $\dot{d}$ the Fluxion of the Ordinate: therefore if GB$=a$, and BR$=z$, be the Legs of a Triangle similar to the Fluxionary Triangle IKN, it will be $4a^3z . a^4+2a^2z^2+z^4 :: n . y$, and y equal to

$\frac{a^4+2aazz+z^4}{4aaz} \times \frac{n}{a}$, whereby it is plain that Sir

Isaac Newton's Property of this Curve makes the least Ordinate $n=a$ the Parameter, and then it will be, MN to GR as GR cubed, to 4BR×GB squared.

And tho' a Curve described by this Property will approach nearer to the Axis, than the Point where a Tangent makes an Angle with the Axis of 45 Degrees, and recedes from it again, becoming then convex towards the Axis, yet after it has passed that Point it will not generate a Solid of the least Resistance; because it has lost one of its chief Properties.

Besides, if DNGB, *Fig.* 20. were the Curve, then DNFB was shewn to resist less than DNGB, and the Frustum of the Cone generated by NFB in its Rotation about its Axis MB, would resist less than the Frustum generated by revolving NHB about the same Axis in which MC=CB and CS=CN, which is absurd: because, this was shewn to resist the least of all Frustums of the same Base and Altitude.

If we put x for an Abscissa of the Curve, its Fluxion $=\dot{x}$, y an Ordinate, its Fluxion $\dot{y}$, GB$=a$, and BR$=z$, as before, then in the similar Triangles IKN and BGR, it will be GB . IK :: BR . IN, or

$a \,.\, \dot{y} :: z \,.\, \dot{x} = \frac{z\dot{y}}{a}$ but $y = \frac{a^4 + 2aazz + z^4}{4aaz} = \frac{a^2}{4z}$ $+ \frac{z}{2} + \frac{z^3}{4aa}$, the Fluxion of which is $\dot{y} = \frac{3z^2\dot{z}}{4aa}$ $+ \frac{\dot{z}}{2} - \frac{aa\dot{z}}{4zz}$, whence $\dot{x}$ or $\frac{z\dot{y}}{a} = \frac{3z^3\dot{z}}{4a^3} + \frac{z\dot{z}}{2a} -$ $\frac{a\dot{z}}{4z}$ the flowing Quantity of which is $x = \frac{3z^4}{16a^3} + \frac{z^2}{4a}$ $- \frac{a}{4} \times$ Log. z, but when $x=0$ then $z=a$, which being substituted for z, we shall have $\frac{3a^4}{16a^3} + \frac{a^2}{4a} -$ $\frac{a}{4} \times$ Log. $a = \frac{7}{16}a + \frac{1}{4}a \times$ Log. a, which being subtracted from $\frac{3z^4}{16a^3} + \frac{zz}{4a} - \frac{a}{4} \times$ Log. z, leaves $\frac{3z^4}{16a^3} + \frac{zz}{4a} - \frac{7a}{16} - \frac{a}{4} \times$ Log. $\frac{z}{a}$, the Abscissa required.

And putting GB$=a=1$, it will be $4z \,.\, 1+z^2 ::$ $1+z^2 \,.\, y$ the Ordinate: and $16 \,.\, 3z^2+4 :: z^2 \,.\, 4th$, from which fourth Term subtracting $\frac{7}{16} + \frac{1}{4}$ Log. z leaves

leaves the Abſeiſſa proper to the Ordinate found as above.

PROP. X.

To Investigate the Reſiſtance of a Solid *of the leaſt Reſiſtance turned about its Axis.*

LET QENGB, *Fig.* 21. be the generating Figure, draw KF parallel to the Axis QB, meeting with the Curve in N, draw NM perpendicular to the Axis, and NI perpendicular to the Curve, and the Force with which a Particle of the Fluid ſtriking directly on the Perpendicular Head of the Cylinder in the Point F, is to the Force with which it would ſtrike obliquely on the given Solid in the Point N, as NIq to NMq, or becauſe GR is parallel to a Tangent applied in the Point N; as GRq to GBq, and putting BR$=z$, GB$=a$, and conſequently GR$=\overline{zz+aa}|^{\frac{1}{2}}$, and taking any determinate Quantity as the Altitude QB$=b$, as before, it will be GRq . GBq : : QB . HF, or $zz+aa$. aa : : b . $\frac{aab}{zz+aa}$ and if this be done in every Point of the given Curve, the Point H will deſcribe the Curve OHD, and the Reſiſtance of the given Solid will be to the Reſiſtance of the circumſcribed Cylinder, as the Content of the Solid generated by revolving the Figure QOHDLB about the Axis QB, is to the Content of the Circumſcribed Cylinder.

Put as before $HF = \frac{aab}{zz+aa}$, then $KH = b - \frac{aab}{zz+aa} = \frac{zzb}{zz+aa}$, the Fluxion of which is $\frac{2aabz\dot{z}}{\overline{zz+aa}|^2}$, and put the Baſe $= \frac{c}{2r} \times yy$, but by the Property of the Curve $y = \frac{z^4+2aazz+a^4}{4aaz}$, whence $\frac{c}{2r} \times yy = \frac{c}{2r} \times \frac{\overline{z^4+2aazz+a^4}|^2}{16a^4zz}$ which multiplied into $\frac{2aabz\dot{z}}{\overline{zz+aa}|^2}$, gives $\frac{c}{2r} \times \frac{2aabz\dot{z}}{\overline{zz+aa}|^2} \times \frac{\overline{zz+aa}|^4}{16a^4zz}$ or becauſe $\frac{c}{2r} \times b$ is conſtant, it may be omitted, and then it will be $2aaz\dot{z} \times \frac{\overline{zz+aa}|^2}{16a^4zz} = \frac{\dot{z}}{8aaz}$ multiplied by $\overline{z^4+2aazz+a^4} = \frac{z^3\dot{z}}{8aa} + \frac{z\dot{z}}{4} + \frac{aa\dot{z}}{8z}$, the Fluxion of the Solid; the flowing Quantity of which is $\frac{z^4}{32aa} + \frac{z^2}{8} + \frac{aa}{8} \times \text{Log.}\frac{z}{a}$ to which if we add the Cylin-

der

der generated by the Parallelogram ADBL, in the ſame Rotation, which in this Caſe is $\frac{\overline{zz+aa}|^4}{16a^4z^2} \times \frac{aa}{zz+aa}$ or $\frac{\overline{zz+aa}|^3}{16aazz} = \frac{z^6+3a^2z^4+3a^4z^2+a^6}{16a^2z^2} = \frac{z^4}{16a^2} + \frac{3z^2}{16} + \frac{3a^2}{16} + \frac{a^4}{16z^2}$; we ſhall have $\frac{3z^4}{32aa} + \frac{5z^2}{16} + \frac{3a^2}{16} + \frac{a^4}{16z^2} + \frac{aa}{8} \times \text{Log.}\ \frac{z}{a}$, but when $z=a$, it will be $\frac{3a^2}{32} + \frac{5a^2}{16} + \frac{3a^2}{16} + \frac{a^2}{16} + \frac{aa}{8} \times \text{Log.}\ 1$, or $\frac{21a^2}{32}$ which ſhould be a^2, but is deficient $\frac{11}{32}a^2$, therefore $\frac{3z^4}{32aa} + \frac{5z^2}{16} + \frac{17a^2}{32} + \frac{a^4}{16z} + \frac{aa}{8} \times \text{Log.}\ \frac{z}{a}$ is the true Fluent, which multiplied into $\frac{c}{2r} \times b$, gives the Reſiſtance enquired after.

And putting $b=66,9$ and $=33\frac{1}{3}$, then $z=12$, and the Reſiſtance of this Solid will be 37265, the Solidity of the circumſcribed Cylinder being put for its Reſiſtance.

PROP.

PROP. XI.

To Cube the Solid *of the least Resistance.*

PUT the Radius of the Base NM$=y$, *Fig.* 22. its Diameter $2y$, and let c be a standing Expression, shewing the Proportion of the Area of a Circle to its circumscribed Square: the Area of the Base will then be $c \times 4yy$, but because c is a standing Quantity we may omit it, and call the Base $4yy$, then if the Altitude MB of the Solid be x; its Fluxion $\dot{x}$ multiplied into the Base will give $4y^2\dot{x}$ the Fluxion of the Solid.

But by the Property of the Curve, putting BR$=z$,

$y = \frac{z^4 + 2aazz + a^4}{4aaz}$, whence $4yy$ will be equal to

$\frac{z^8 + 4aaz^6 + 6a^4z^4 + 4a^6z^2 + a^8}{4a^4z^2}$, and $\dot{x}$ equal to

$\frac{3z^4\dot{z} + 2az^2\dot{z} - a^4\dot{z}}{4a^3z}$, whence $4yy\dot{x}$ will become

$$\frac{3z^{12}\dot{z} + 14a^2z^{10}\dot{z} + 25a^4z^8\dot{z} + 20a^6z^6\dot{z} + 5a^8z^4\dot{z} - 2a^{10}z^2\dot{z} - a^{12}\dot{z}}{16a^7z^3}$$

the Fluxion of the Solid; by Reduction equal to

$$\frac{3z^9\dot{z}}{16a^7} + \frac{7z^7\dot{z}}{8a^5} + \frac{25z^5\dot{z}}{16a^3} + \frac{5z^3\dot{z}}{4a} + \frac{5az\dot{z}}{16} + \frac{a^3\dot{z}}{8z} -$$

$a^5\dot{z}$

$\frac{a^5\dot{z}}{16z^3}$ the Fluent of which is $\frac{3z^{10}}{160a^7}+\frac{7z^8}{64a^5}+\frac{25z^6}{96a^3}+\frac{5z^4}{16a}+\frac{5az^2}{32}-\frac{a^3}{8}\times\text{Log.}\,\frac{z}{a}+\frac{a^5}{32z^2}$,

Now, when $z=a$, the whole, which should vanish, will be $\frac{3a^3}{160}+\frac{7a^3}{64}+\frac{25a^3}{96}+\frac{5a^3}{16}+\frac{5a^3}{32}+\frac{a^3}{32}$, or $\frac{3a^3}{160}+\frac{7a^3}{64}+\frac{25a^3}{96}+\frac{30a^3}{96}+\frac{15a^3}{96}+\frac{3a^3}{96}=\frac{3a^3}{160}+\frac{7a^3}{64}+\frac{73a^3}{96},=\frac{853}{960}\times a^3$, which being subtracted from the flowing Quantity found as above, there will remain $\frac{3z^{10}}{160a^7}+\frac{7z^8}{64a^5}+\frac{25z^6}{96a^3}+\frac{5z^4}{16a}+\frac{5az^2}{32}+\frac{a^5}{32z^2}-\frac{853a^3}{960}-\frac{a^3}{8}\times\text{Log.}\,\frac{z}{a}$, the true Fluent, which multiplied into c will give the Content of the Solid.

And putting $b=66,9$ and $y=33\frac{1}{3}$ as before, then $z=12$, and the Content of this Solid will be 106360, by which the Resistance 37265 being divided, we shall have ,350376.

PROP.

PROP. XII.

The Altitude and Base of a Cone *being given, to describe a* Trapezium, *whose Altitude and Solidity generated by its Rotation about its Axis, shall remain the same as in the* Cone, *and its Resistance be the least of all* Solids *of the same Altitude and Solidity, generated by any* Trapezium *whatever in its Rotation about the same Axis.*

LET the right angled Triangle ABC, *Fig.* 21. represent the given Cone, and ABFD the Trapezium, put the Axis AB=b, AC=p, AD=y, and EB=x.

If the Solidity of the Cone generated by revolving the Triangle CAB about the Axis AB be called $\frac{ppb}{3}$: The Content of the Solid generated by revolving the Figure ABFD about the same Axis will be $yyb-yyx+\frac{yyx}{3}$, and if this be equal to the Solidity of the Cone, then $\frac{ppb}{3}=yyb-yyx+\frac{yyx}{3}$ and

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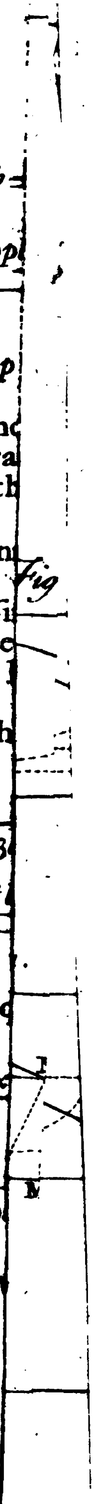

$36b^4p^4x^2\dot{x}-16b^3p^4x^3\dot{x}+2b^4p^6\dot{x}=0$, and dividing by $2b^3p^4\dot{x}$ it will be $-9b^2x+18bx^2-8x^3+p^2b=0$, or $8x^3-18bx^2+9b^2x=p^2b$, and dividing by 8, x^3-

$\frac{9}{4}bx^2+\frac{9}{8}b^2x=\frac{p^2b}{8}$, which Cubic Equation being solved will give the Value of x.

And if we put $AB=b=66,9$, and $AC=p=33\frac{1}{3}$; as before, then $EB=x=46,1$, $EF=y=26,17$, and $HB=16,308$: So that the Resistance of the Cone being 46444, as was shewn before; the Resistance of the Solid generated by revolving this Trapezium ABFD about the Axis AB will be 35101 nearly. And this may serve as a general Hint, that the Resistances of Solids depend much upon their greatest transverse Sections, and it is probable from this, some useful Observations may be made.

FINIS.